A SYNTHESIS OF THE COMBINATORIAL REASONING AND PROPORTIONAL REASONING STUDIES IN TERMS OF PIAGET'S DESCRIPTION OF DEVELOPMENT STAGES

by

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(Under the Direction of Leslie P. Steffe)

ABSTRACT

The purpose of this research was to synthesize the combinatorial reasoning and proportional reasoning studies. Throughout the age range that Piaget found that formal reason emerges, have other researchers found that combinatorial reasoning and proportional reasoning synchronously emerge within these same age ranges?

INDEX WORDS: combinatorial reasoning; proportional reasoning; Piaget; development stages

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DEDICATION

I dedicate this study to my father, my mother, and my brothers.

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TABLE OF CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

Page

CHAPTER 1

INTRODUCTION

In his book *The Art of Conjecturing*, Bernoulli defined combinatorics as the art of enumerating all the possible ways in which several things can be combined, transposed, or joined with each other to make sure that nothing has been omitted that can contribute to one's purpose (Sylla, 2006). Combinatorics is the mathematics of counting (Hart, 1992). It concerns the existence, enumeration, analysis, and optimization of discrete structures (Brualdi, 2010).

Discrete mathematics investigates the settings in which functions are defined by discrete or finite sets of numbers, such as positive integers (Dossey, 1991), and combinatorics is an important area of discrete mathematics: "[A]s an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum" (National Council of Teachers of Mathematics [NCTM], 2000, p. 31). According to Kapur (1970), combinatorics is an essential component of discrete mathematics, and as such, it plays an important role in school mathematics. He explained his reasons as follows:

- Since combinatorics is independent of calculus, it has suitable problems for all grades; challenging problems can be presented to students in order for them to discover the need for the creation of more mathematics.
- Combinatorics can be used to train students in the concepts of enumeration, making conjectures, generalizations, and optimizations and engaging in systematic thinking; it can

help in the development of many concepts, such as mapping functions and equivalence relations.

- The applications to many fields can be indicated.

In addition to combinatorial reasoning, proportional reasoning is a pivotal concept in school mathematics as well (Lesh, Post & Behr, 1988). Proportional reasoning has been described as "the capstone of elementary school mathematics and the gateway to higher mathematics, including algebra, geometry, probability, statistics, and certain aspects of discrete mathematics" (Kilpatrick, Swafford & Findell, 2001, p. 242).

Background and Rationale

According to Piaget and Inhelder (1975), combination is an operation requiring the coordination of different series or correspondences, and permutation is an operation requiring an arrangement that references a mobile and reversible system. In the case of combination, for example, finding how many ways three red hats can be placed on the heads of five people, there are two series involved, the five people and the three red hats. How these series are coordinated, of course, is paramount, but the example is meant to illustrate that such a coordination of two distinguishable series is involved. Further, in the case of a permutation of five people, there is a selection of any one of the five people for the first position in the permutation, which already involves the concept of variable, or a mobile system. After the first selection, a second selection is similarly made and coordinated with the first selection to produce 5 x 4 paired elements. Reversibility is implicit in producing the 20 paired elements because each time a second selection is made, a return to the first five selections must occur in order for the given second selection to be paired with the totality of the first selection. Reversibility is even more prominent upon the third selection. Both permutation and combination are operations on operations, which means that a current way of operating takes previous ways of operating as givens in operating. Formal thought is characterized by these second power operations. As reported by these authors, a *structured whole* depends on establishing a combinatorial system that links "a set of base associations or correspondences with each other in all possible ways so as to draw from them the relationships of implication, disjunction, exclusion, etc." (Inhelder & Piaget, 1958, p. 107). Since combinatorial reasoning and proportional reasoning are key points in Piaget's theory of cognitive development, particularly in the formal operational stage, I am interested in these two types of reasoning jointly.

Besides having an important role to play in cognitive development, mathematics education researchers have found that combinatorial problems (Maher & Martino, 1996; Martino & Maher, 1999) and proportional problems (Fisher, 1988; Fujimura, 2001; Noelting, 1980a, 1980b) may promote students' reasoning and generalization processes. Likewise, combinatorics and proportions comprise a rich structure of powerful principles that underlie several areas of the curriculum, such as counting, computation, fractions, ratios, and probability (English, 1993, 1996, 2005; Lobato & Ellis, 2010). Additionally, combinatorial problems (Kapur, 1970) and proportional problems (Fischer, 1988) may promote students' reasoning and generalization processes.

Moreover, some researchers claimed that both combinatorial reasoning (English, 1991) and proportional reasoning (Lamon, 2007) are specifically fertile fields for mathematics education research. Even though combinatorial reasoning is regarded as very valuable to the mathematics education of students, there are only a small number of studies in this area. One of the principal reasons I chose to investigate studies concerning combinatorial reasoning is that it is, more or less, an untapped field of research in mathematics education even though it has a major role in formal reasoning. In fact, no reasons are given in the scholarly research as to why combinatorial reasoning is not addressed in more studies. Furthermore, in spite of its role in formal reasoning, it is still a neglected topic in mathematics curricula.

In spite of the scarcity of research on combinatorial reasoning, studies have indicated that most students experience combinatorics as a very difficult method of reasoning (Batanero, Navarro-Pelayo, & Godino, 1997; Eisenberg & Zaslavsky, 2004; English, 1991). Kapur (1970) emphasized the importance of combinatorial reasoning and insisted that combinatorial mathematics has an important role in school mathematics. Similarly, it has been found that proportional reasoning (Boyer, Levine & Huttenlocher, 2008; Lamon, 1993; Tourniaire, 1986) is very a difficult method of reasoning. In a way similar to Kapur's (1970) view of combinatorial reasoning, Watson and Shaughnessy (2004) stated that proportional reasoning is "fundamental to problem solving across the curriculum" (p. 104). My personal knowledge of the difficulties in teaching and learning combinatorics and proportions, along with their reputations as difficult mathematical topics to teach and to learn, was another reason for my interest in researching combinatorial and proportional reasoning.

I already indicated that combinatorial reasoning and proportional reasoning play an important role in Piaget's theory of cognitive development. According to Piaget and Inhelder (1958), both types of reasoning are indications of the formal operational stage. In this study, I focused my research on these two types of reasoning not only at the formal operational stage but also at other developmental stages. I am interested in how combinatorial reasoning and proportional reasoning are related and if these types of reasoning exhibit any juxtaposition.

Research Question

The overriding question that guided my study is:

Throughout the age range that Piaget and Inhelder (1958) found that formal reason emerges, have other researchers found that combinatorial reasoning and proportional reasoning synchronously emerge within these same age ranges?

As a starting point, I will define principles that are critical for combinatorial reasoning, namely, the fundamental principle of counting. The fundamental principle of counting is explained as follows: "If one thing can be accomplished in n_1 different ways and after this a second thing can be accomplished in n_2 different ways, … , and finally a *k*th thing can be accomplished in n_k different ways, then all *k* things can be accomplished in the specified order in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ different ways" (Spiegel, Schiller & Srinivasan, 2000, p. 9). The multiplication principle is a short version of the fundamental principle of counting:

Let S be a set of ordered pairs (a, b) of objects, where the first object a comes from a set of size p, and for each choice of object a there are q choices for object b. Then the size of S is p × q: |S| = p × q. (Brualdi, 2010, p. 28)

In this study, by multiplicative reasoning, I am referring to the participants' *units coordinating activity*. Olive and Steffe (2010) explained *units-coordinating* as "a multiplication scheme that gets its name from the coordination of, to the observer, two composite units of units where one composite unit is inserted into each unit item of the other composite unit" (p. 91). *Composite unit, scheme,* and *unit* are crucial terms in this definition; thus, I prefer to give their definitions, too. Steffe (1994) defined the concept of a unit as "an entity that is treated as a whole" (p. xvii) and that of a composite unit as "a unit that itself is composed of units" (p. 15). Piaget (1980) defined a scheme as "all action that is repeatable or generalized through application to new objects" (p. 24). As a general example of units coordinating, consider a unit of four and a unit of nine as two composite units. When a composite unit of four is inserted into each of the nine units of one, nine units of four are produced, which is also a composite unit that yields the product 9 × 4. To apply this concept to combinatorics, consider a deck of 26 cards numbered from 1 to 26 and a die. A person draws a card from the deck and tosses the die, resulting in a card/die combination. How many different combinations are possible if all cards are drawn? There are 26 different possibilities for drawing a card and there are 6 possible outcomes for a die roll. Thus, there are 26 × 6 different possible combinations that exist. In this example, a composite unit of 6 is inserted into, or paired with, each of the 26 units of one to produce 26 × 6 different ways. "Inserted into" is understood as the composite unit of six is taken as a unit using the unit comprised by the unit established to conceive of "any card" as an entity.

With the idea that students who can formulate "mathematical ideas recursively have an advantage when they learn many of the applications of mathematics" (Cornell & Siegfried, 1991, p. 154) as a basis, recursive reasoning of the participants was another concept that formed my study. Graham (1991) defined recursion as "another technique used to solve problems when trying to describe future results by looking at previous step(s)" (p. 25). This study adopted Olive and Steffe's (2010) concept of *units coordinating activity* and Graham's (1991) definition of recursion to help describe a student at the formal operational stage. The student at the formal operational stage reasons recursively and forms pairs from single elements, triplets from couples, quadruplets from triplets, quintuplets from quadruplets, and so forth, in combinatorial problems.

The concept of a units coordinating activity corresponds to understanding the meaning of the sign of multiplication and the meaning of numerals. Both the signs of multiplication and the numerals and operating with them recursively are required in the formal operational stage. When students understand the meaning of each numeral, why they multiply these numerals, and how they operate recursively, they can reason systematically for finding all possible outcomes. By all possible outcomes, I mean the structured whole, and forming the structured whole is required for formal thought. Systematic thinking is also required for reasoning at the formal operational stage. The students who have not processed those operations have not reached that stage yet. Here is the diagram of the theoretical framework in Figure 1.

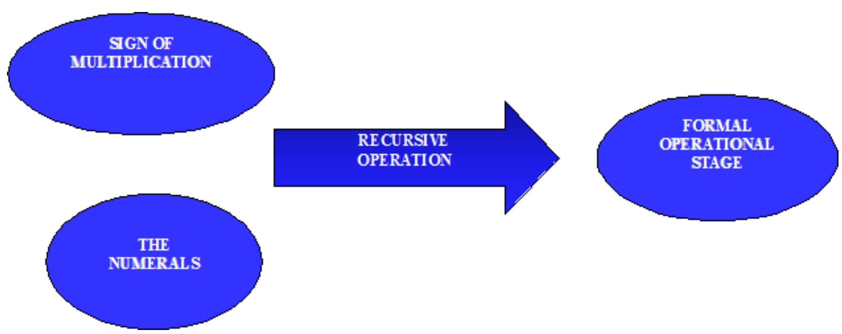

Figure 1: Diagram of Theoretical Framework

Based on the theoretical framework, performing recursion requires "jumping right into a typical case, supposing that you know how to treat a previous case, and working your way down and back" (Maurer & Ralston, p. 202). One way to find the possible outcomes for small numbers of elements in combinatorial problems is to simply list the possible outcomes. However, when the number of elements increases, it becomes difficult to list all of the possible outcomes. Students at the formal operational stage can list the number of outcomes for smaller numbers; they can shift to using the multiplication principle for larger numbers and also use the number of outcomes for the next step. That is, they are capable of operating recursively; they can use the results of the pairing of two elements as input for pairing them with the additional elements in a sequential pattern. Students at the formal operational stage can produce pairs from single elements, triplets from pairs, quadruplets from triplets, quintuplets from quadruplets, etc., in the combinatorial problems.

Recursive Reasoning and Multiplicative Reasoning in Combinatorial Problems and Proportional Problems

For proportional problems, additive reasoners cannot see the relation between two relationships multiplicatively. Let us consider missing value problems: For finding the missing value in (a : b) (c : x); where a, b, and c are given values and x is the unknown, students who reason multiplicatively can use two possible strategies.

The first strategy consists of understanding the relationship between the first and second quantities in each ratio (or how a relates to b) and the relationship between that ratio and c : x. Students who use multiplicative reasoning can see the relationship between a and b, use that relationship as an input, and use it for finding the relationship between c and x. Using the relationship between a and b as an input and using it for finding the relationship between c and x requires recursive reasoning.

The second strategy consists of understanding the relationship between the comparable quantities in each ratio or how a relates to c and using that to determine how b relates to x. Students who use multiplicative reasoning can use the relationship between a and c and use that relationship as an inpu, and use it for finding the relationship between b and x. Similarly, using the relationship between a and c as an input and using it for finding the relationship between b and x requires recursive reasoning.

For combinatorial problems, additive reasoners cannot take all the possible outcomes for small numbers as an input and use these outcomes to find possible outcomes for larger numbers or more complicated problems. Students who can reason multiplicatively can construct an ordered set such that this ordered set could be filled with any order.

Following is the explanation of recursive reasoning and multiplicative reasoning in combinatorial problems. Let us consider arranging 3 people in a row. The first person was labeled with “1,” the second person with “2,” and the third person with “3” as a starting point.

Students who use multiplicative reasoning can put person 1 at the beginning and keep this person in the first slot and switch the persons 2 and 3. These students can get a pairing of the singleton unit in the first place with an ordered set. Figure 2 is the demonstration of triples starting with person 1.

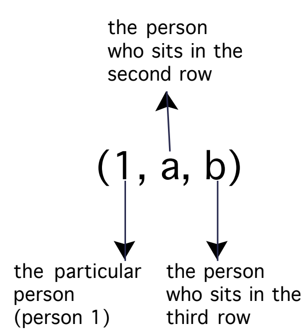

Figure 2: Triples Starting with a Particular Element

In this case, an ordered set is a composite unit of two abstract units that implies the two specific orderings of 2 and 3. 1 is the particular element, and (2, 3) and (3, 2) are the two strings of blocks. Thus, the pairing of the singleton unit in the first place with an ordered set implies two ordered strings of blocks. Students who reason multiplicatively consider 2 possible outcomes for starting with person 1. Similarly, they reason that they can get the same number of possible outcomes for starting with person 2 or person 3. Thus, these students can reason finding 6 possible outcomes multiplicatively.

CHAPTER 2

METHODOLOGY

As an initial step, I searched the University of Georgia Library's multi-search catalog using "combinatorial reasoning" as the search term, and this search yielded 312 results. However, some of the results were multiple references to the same study. Thus, I excluded these duplicates, resulting in 150 studies.

Next, I searched the University of Georgia Library's multi-search catalog using "proportional reasoning" as the search term, and this search yielded 2076 results. After including the term "mathematics", the number of results was reduced to 1219. As in the case of the search using the term "combinatorial reasoning," some of the results were multiple references to the same study. Excluding these duplicates resulted in 717 studies. Because this amount of studies was still too large, I revised my strategy. Given that the number of combinatorial reasoning studies was reasonable for my purposes, I decided to reduce the number of proportional reasoning studies to match that of the combinatorial reasoning studies. To achieve this reduction, I examined the references lists of the proportional reasoning studies in order to identify those studies that were the most closely related to my interest.

The final total amount of studies was approximately 400 studies on combinatorial reasoning and proportional reasoning. I categorized most of the studies as either a combinatorial reasoning study or a proportional reasoning study, although there were some studies that were both combinatorial and proportional reasoning studies. I determined the category for a study based on the content of the problems the students were asked to solve. If the problems involved combinatorial reasoning, the study was placed into the combinatorial reasoning category. Similarly, if the problem involved proportional reasoning, the study was placed into the proportional reasoning category. However, some studies included both combinatorial reasoning and proportional reasoning problems; these studies were placed into both categories. Next, for each category I determined whether or not a study made meaningful reference to Piaget. By "meaningful reference," I mean reference to Piagetian concepts, tasks, or claims related to combinatorial reasoning or proportional reasoning. Studies that made no meaningful reference to Piaget were excluded.

For the next step, I divided the studies into the following Piagetian categories: stage 1 (pre-operational), stage 2 (concrete), and stage 3 (formal). Studies with students from age 6-7 years old were placed into stage 1, those with students from 8-10 years old were placed into stage 2, and those with students from age 11-12 years old and older were placed into stage 3. Studies that had students from multiple age groups were placed into every applicable stage. I purposefully categorized the studies based on the ages of the students rather than on the researchers' assessments of their performance to determine if the selected studies confirmed that students performed as described by Piaget. Thus, if the students in a study were at stage 3 in terms of assessed performance but younger than 11 years old, that study would be categorized as stage 1 or stage 2 depending on age.

Copper's (1988) taxonomy of the literature reviews helped me place my study in the taxonomy and understand the types of characteristics on which my study focused. He divided the characteristics of literature reviews into six categories: (1) focus; (2) goal; (3) perspective; (4)

coverage; (5) organization; (6) audience. Following is the explanation of the subcategories of each characteristic and how the taxonomy was applied to this study.

According to the taxonomy of Cooper (1988), literature reviews can focus on (a) research findings; (b) research methods; (c) theories; (d) practices or applications. The cognitive development theory of Piaget and the findings of the studies are the prior focus of this study. The studies that made meaningful reference to Piaget were used.

Based on the taxonomy of Cooper (1988), the goal of literature reviews can be (a) integration—integration is also categorized into three subcategories as generalization, conflict resolution, and linguistic bridge building; (b) criticism; (c) identification of central issues. The goals of this study were integration, especially generalization, and criticism. Studies were categorized based on the age of the students and this study investigated the similarities and differences essential for performing combinatorial reasoning and proportional reasoning based on the reviewed literature and made generalizations for both types of reasoning. The study explored if the studies confirmed Piaget's claims or contradicted Piaget's claims. Moreover, it critically examined the elements that might affect the way researchers labeled the students (e.g., the difficulty of the tasks, the usage of manipulatives, the information that was given to make that judgment).

In Copper's (1988) study, reviewers' perspectives were categorized as (a) neutral and (b) espousal of position. In the former perspective, "[t]he interpretations are presented in a fashion similar to that employed by the original authors, and an attempt is made to ensure that all sides are represented" (p.110), whereas in the later perspective, "[t]he reviewer plays a role of an advocate, mustering the evidence so that it presents his or her contentions in the best possible light" (p. 110). In this study, I attempted to review the studies neutrally.

Also, Cooper (1988) divided the coverage of the review into four subcategories: (a) exhaustive; (b) exhaustive with selective citation; (c) representative; (d) central or pivotal. In exhaustive reviews, the reviewers try to include almost all the existing studies and make the conclusions in the basis of all relevant studies that is "within the limitations of the author's definition of the area" (p. 114). In exhaustive with selective reviews, the reviewers also make the conclusions based on the entire literature but with including only a selected group of studies. In representative reviews, reviewers use representative samples. In central or pivotal reviews, reviewers select only key studies. Based on the taxonomy, I used exhaustive with selective reviews for combinatorial reasoning studies and central or pivotal reviews for proportional reasoning studies.

Moreover, Cooper (1988) put the organization of literature reviews into three subcategories: (a) historical; (b) conceptual; (c) methodological. I organized the studies methodologically and categorized the studies based on the age of the students in the studies. Cooper (1988) classified the audiences of the literature reviews into four subcategories: (a) specialized scholars; (b) general scholars; (c) practitioners or policy makers; (d) general public. This study is written for specialized scholars, general scholars, and practitioners or policy makers. I explain the categorization process of the studies in the following section based on the Cooper's taxonomy.

Table 1
The Cooper's Taxonomy and Its Application to the Study

Characteristic	Categories	The reviewer's Preference
Focus	Research Outcomes Research Methods Theories Practices or Applications	Research outcomes Theories
Goal	Integration a) Generalization b) Conflict Resolution c) Linguistic Bridge-building Criticism Identification of Central Issues	Integration (a) Generalization Criticism
Perspective	Neutral Representation Espousal of Position	Neutral Representation
Coverage	Exhaustive Exhaustive with Selective Citation Representative Central or Pivotal	Exhaustive with Selective Citation Central or Pivotal
Organization	Historical Conceptual Methodological	Methodological
Audience	Specialized Scholars General Scholars Practitioners or Policy Makers General Public	Specialized Scholars General Scholars Practitioners or Policy Makers

CHAPTER 3

LITERATURE SYNTHESIS

Previous studies approach combinatorial reasoning and proportional reasoning with many different objectives and emphases; however, much of this research overlaps in sundry ways. To organize these studies into more manageable categories to inform my current research, I organized the literature on combinatorial reasoning into three parts: studies at stage 1, studies at stage 2, and studies at stage 3. Because I am interested in cognitive developmental stages of the participants, I explored the literature based on Piagetian combinatorial developmental stages.

Combinatorial Reasoning Studies

Studies at Stage 1

In Piaget and Inhelder's (1975) study, in the combination of colored counters task, piles of colored counters were put on a table, and children were asked to make as many different pairs of colors as possible. Except in certain cases, they did not conclude that counters of the same color, such as red and red, were a pair. Children at Stage I (6 – 7 years), empirical combinations, found some of the possible pairs by trial and error. Children's methods consisted of making pairs independent of each other, and there was not a systematic prediction of the composition of pairs. This first stage is the preoperational stage in Piaget's cognitive development stages.

In addition to their 1975 study, Piaget and Inhelder conducted another study (1958) on combinations. In their study, in the combinations of colored and colorless chemical bodies task, children were asked to combine chemical substances among themselves. There were four similar flasks which contained colorless, odorless liquids: (1) diluted sulphuric acid; (2) water; (3) oxygenated water; (4) thiosulphate, and a smaller flask, labeled *g*, which had potassium iodide in it. If liquid *g* was added into a mixture of 1 and 3 (1 + 3), the mixture liquid turned yellow. If water (2) was added into the mixture, 1 + 3 + *g*, there was no change. If thiosulphate (4) was added into the mixture, 1 + 3 + *g*, thiosulphate bleached the yellow mixture of the 1 + 3 + *g* liquid. There were two glasses; one of them contained a mixture of 1 and 3 (1 + 3), and the other one contained water (2). The liquid *g* was added into both mixtures, the 1 + 3 liquid and the 2 liquid, in front of the children, and the children were asked to note the different reactions. Then, the experimenter asked children to produce a yellow colored liquid by using the liquids, 1, 2, 3, 4, and *g,* or any of the five flasks that they wanted. Children at Stage I, empirical associations and precausal explanations, were limited to randomly pairing two elements at a time.

In addition to combinations problems, Piaget and Inhelder's (1975) study included permutation tasks as well. In the permutations of colored counters task, children were given two counters of different colors, A and B, and were asked to show in how many different ways two counters could be arranged. Then, children were given three counters of different colors, A, B and C, and were asked to show in how many different ways two counters can be arranged. If children could find the six possible permutations, they were asked to find permutations of four counters. Children at Substage I-A had difficulty finding a systematic way to produce all of the possible permutations for the three elements. They were not capable of understanding that several permutations can be constructed with the same elements. Children at Substage I-B started to make some permutations and discovered some regularities.

In Piaget and Inhelder's (1975) study, in the card arrangement task, they studied "arrangement" as opposed to "permutation" or "combination" in that repetition was allowed. A deck of 78 cards was arranged into three decks of 26 cards; each card in the first deck was numbered with "1," each card in the second deck was numbered with "2," and each card in the third deck was numbered with "3." For children who did not know to make numbers with 2 cards out of 3 cards that had digits on them, they used another deck of 78 cards and the deck was arranged into three decks of 26 cards as they did before. For the second deck of 78 cards, each card in the first deck of 26 cards displayed a locomotive; each card in the second deck displayed a railroad passenger car, and each card in the third deck displayed a freight car. The experiment was divided into three parts. In the first part, either deck of 78 cards was placed on the table. Children were asked how many different two-digit numbers (or how many different pairs of cards of railroad cars) they could construct. In the second part, children were asked to draw two cards from the shuffled deck; predict which cards they received, and then record what they received. In the third part, the experimenter and each child analyzed the child's record. After children realized that the number of cards they drew from each deck of 26 cards was not equal in their records, they were asked whether the inequality of the distribution would possibly increase or decrease when they have more cards, such as a basket full of these cards to draw from. Children at Stage I could not make systematic arrangements; their arrangements were empirical. Also, they were not capable of understanding the random mixture; they constructed their own arrangements and denied the role of chance in the arrangements. For instance, children at that stage might think that they could only construct 11, 22, and 33 by using each card in the first deck that was numbered with "1," each card in the second deck was numbered with "2," and each card in the third deck was numbered with "3;" they did not consider all other options and believed that a hidden order existed in the shuffled cards. To summarize, Piaget and Inhelder claimed that children in the pre-operational period did not have a systematic method for solving combinatorial problems.

Taking Piaget's studies and building on them, English did several studies with young children. English (1991, 1993) mainly focused on young children's counting problems. Different from Piaget and Inhelder, she did not examine a cognitive structured whole or combinatorial system that "links a set of base associations or correspondences with each other in all possible ways so as to draw from them relationship of implication, disjunction, exclusion, etc." (Inhelder & Piaget, 1958, p. 107). However, she did find that young children's combinatorial problem solving capacities are better than what Piaget and Inhelder claimed. In her study, English (1991) investigated the combinatorics strategies of 50 students aged between 4 years 6 months and 9 years 10 months. The students were asked to dress a toy bear with a colored top and a colored pair of pants or a colored top and skirts with different numbered buttons for the purpose of finding all possible outfits for the toy bear. She listed the students' strategies into 6 categories: random selection of items with no rejection of inappropriate items; a trial-and-error procedure with random item selection and rejection of inappropriate items; an emerging pattern in item selection, with rejection of inappropriate items; a consistent and complete cyclical pattern in item selection, with rejection of inappropriate items; emergence of an odometer pattern in item selection with possible item rejection; and a complete odometer pattern in item selection, with no rejection of items. Because some strategies included cyclical patterns and also constant and pivotal items, English used the term odometer with regard to the pattern of the strategies. It was found that students shifted their strategies both within the one task and between tasks. It was also found that there was a relationship between the age and the level of sophistication of the students' strategies. Younger students used ineffective strategies and showed insufficient improvement over the set of tasks. Moreover, it was found that using manipulative materials was helpful to students in adopting effective strategies at an age earlier than that claimed by Piaget.

English (1993) advocated that concrete-operational children are able to use a systematic method to solve two and three-dimensional combinatorial problems if they have appropriate learning opportunities. One of the reasons that English (1993) extended her combinatorial reasoning studies was because combinatorial reasoning is crucial in Piaget's formal operational stage. She developed her previous study by exploring the combinatorial reasoning of 96 students between 7 and 12 years old by examining their strategies for the dressing of the toy bears task. The goal of the students in this follow up study was to find all possible combinations of colored tops and bottoms (tops × bottoms) that was assigned to younger children or colored tops, bottoms, and tennis rackets (tops × pants × tennis rackets) that was assigned to older children. Students' strategies were categorized as two-dimensional strategies and three-dimensional strategies. Two-dimensional strategies referred to students' strategies for finding all possible outcomes of colored tops and bottoms, whereas three-dimensional strategies referred to students' strategies for finding all possible outcomes of colored tops, bottoms, and tennis rackets. Moreover, two-dimensional strategies are listed into 5 subcategories as a trial-and-error approach (Strategy 1), transitional between the trial and error and odometer pattern approaches (Strategies 2 and 3) and the odometer pattern approach (Strategies 4 and 5). Similarly, three-dimensional strategies are listed into 5 subcategories as the trial-and-error approach (Strategy 6), the use of both a systematic procedure and a trial-and-error approach (Strategies 7 and 8), and an odometer pattern approach (Strategies 9 and 10). It was found that students were more successful on two-dimensional problems than three-dimensional problems. Piaget and Inhelder (1975) claimed that students at the concrete operational stage do not use systematic methods entirely; however, this study claimed that students at concrete operational stage can use systematic methods if they were provided appropriate learning conditions.

Studies at Stage 2

In Piaget and Inhelder's (1975) study, in the combinations of colored counters task, children at Stage II (8-11years) searched for a system. Children at this stage started to make systematic quantifications, but they did not use exhaustive procedures and could not find all possible pairs. When children at this stage used six colors (which were called A, B, C, D, E, and F) and made pairs, some children at this stage had the idea of juxtaposition according to the way in which they paired the colors, such as AB, BC, CD, DE and EF. However, the rest of the pairs were made empirically. Some other children at this stage used juxtaposition by making symmetrical pairs, such as AB then FE, BC then ED, and finally CD. But, the rest of the pairs were made empirically. Some other children at Stage II that were making pairs, such as AB, AC, AD, AE, AF and then BC, BD, BE, BF, were close to Stage III. Nevertheless, they did not keep making pairs by using the symmetrical pairs as previous group of children did. Stage II children tried to make connections between pairs; however, they still did not complete making pairs in a systematic way, because they shifted back and forth between juxtaposition (AB, CD, or AB, BC, CD) and symmetry. This second stage is the concrete operational stage in Piaget's cognitive development stages.

In Piaget and Inhelder's (1958) study, in the combinations of colored and colorless chemical bodies task, children at Substage II-A, multiplication of factors by "g," were limited to adding liquid *g* to all of the other bottles. Children at Substage II-B, multiplicative operations with the empirical introduction of *n*-by-*n* combinations, found several combinations by trial and error, but they did not have a systematic method.

In the permutations of colored counters task of their study, Piaget and Inhelder (1975) explained that children at Substage II-A could discover a procedure for three elements, whereas children at Substage II-B could anticipate the same possibility for four elements, but there was still not a systematic way to find four permutations. Using the card arrangement task of the same study, they found that children at Stage II started to make systematic arrangements and understand the role of chance in the arrangements. The discovery of systematic arrangements was empirical, and there were not systematic arrangements and an understanding of the role of chance for large numbers. To summarize, Piaget and Inhelder claimed that children in the stage of concrete operations started to use systematic methods, but there was not a fully systematic method until the stage of formal operations.

Scardamalia (1977) explored information processing capacity and horizontal *décalage* of 15 participants from 8-10 years of age, 15 participants from 10-12 years of age, and 10 adults. The term horizontal *décalage* was used by Piaget and it refers to "the asynchronous emergence of various manifestations of the same cognitive structure: for example, the appearance of conservation of weight after conservation of substance" (Scardamalia, 1977, p. 28). Both horizontal *décalage* and the term information processing capacity are related to the demand factors of the tasks. Students were asked combinatorial reasoning tasks that had card problems and differed based on the number of dimensions and the number of variables in each dimension. There were several dimensions, and the color (blue, green, red, and yellow), shape (square, rectangle, diamond, and rhomboid), and the type of lines (thick vertical, solid and dotted, and thick and horizontal, solid and dotted) were some of the dimensions of the cards. Students could trade in as many cards as they wanted; however, they needed to always have four cards in their hands, and they needed to have one card from each dimension. So, whenever they traded a shape card, they needed to pick another shape card so that they still had cards from all dimensions. Students were asked to find all possible sets of four cards with a strategy so that they did not have the same set twice. Scardamalia (1977) found that adults struggled more on developing an effective and consistent strategy than children; however, there was not enough information about why that was the case. He also found that adults did worse on easier tasks, whereas they did better on difficult tasks. These findings contrast with Piaget's claims because in Piaget's studies, the older students performed better than younger students.

Drawing on Piaget's cognitive development theory and Scardamalia's (1977) description of odometer pattern and multidimensional tasks, English (1996) investigated the combinatorial reasoning of 9-year-old high achieving and low achieving students in school mathematics. Students were asked two-dimensional and three-dimensional problems that included dressing toy bears and finding all possible combinations of colored tops and bottoms (two-dimensional problems) and, also, all possible combinations of colored tops, bottoms, and tennis rackets (three-dimensional problems). Both two-dimensional and three-dimensional student strategies were categorized as a non-planning stage, a transitional stage, and an odometer stage. Non-planning stage students used a trial-and-error approach. Transitional stage students started to construct a pattern, but they could not continue to use the pattern. They switched from using a pattern to a trial-and-error approach. Odometer stage students used an odometer pattern that refers to selecting an item of one type and holding it constant and changing items of other types systematically. English (1996) found that students' achievement in school mathematics does not always show their capability of solving new problems. Additionally, she noticed that when students were challenged with a difficult concept, they need some time to understand these concepts. If enough time and opportunities were not provided, students might construct inadequate models. These findings are consistent with Piaget's findings.

Besides English (1993, 1996), White (1984) also framed her study by using Piaget's theory and Scardamalia's (1977) study. She used Piagetian colored token problems and explored the combinatorial reasoning of 56 students in the second through fifth grades. Pretests and posttests were administered, and the problems on the tests required finding all possible pairs of some colored tokens, such as finding all possible pairs of four colors or finding all possible pairs of six colors. It was found that children at the pre-transitional stage for pretests performed at formal operational stage for posttests. Namely, White (1984) stated that good performance on combinatorial problems could be based on the information-processing demands of the tasks rather than children's cognitive capacity. Different from Piaget and Inhelder's studies, both Scardamalia's (1977) and White's (1984) studies indicated that if concrete operational stage students were asked combinatorial problems that had appropriate information-processing demands for them, they could solve these problems systematically.

Studies at Stage 3

In Piaget and Inhelder's (1975) study, in the combinations of colored counters task, children at Stage III (after 11-12 years), the discovery of a system, started to discover a system such that no pairing was skipped and arrived at methodical and complete combinations. This third stage is the formal operational stage in Piaget's cognitive development stages.

In Piaget and Inhelder's (1958) study, in the combinations of colored and colorless chemical bodies task, children at Substage III-A, formation of systematic *n*-by-*n* combinations, used a systematic method to find all possible combinations of the five liquids. Compared to

Substage III-A, children at Substage III-B, equilibration of the system, used more systematic methods especially for the proofs.

In the permutations of colored counters task of their study, Piaget and Inhelder (1975) found that children at Substage III-A could generalize partial systems, whereas children at Substage III-B could generalize systematically. Using the card arrangement task of the same study, they claimed that children at Stage III could understand the system of arrangements and the laws of the random mixture of large numbers. Children at this stage knew that if there were many cards to draw from, the number of drawings from each deck of cards numbered “1,” “2,” and “3” would become almost equal, and the inequality of the distribution would decrease. Substage III-A and Substage III-B should be distinguished during Stage III. In Substage III-A, children could discover all arrangements of 3 and 4 elements and arrange them in pairs that reflect the law of square, n^2. However, they still did not understand the reason for these computations. In Substage III-B, children could generalize and understand why constructing arrangements with repetitions gave the formula n^2.

There is some disagreement, however, concerning what Piaget and Inhelder claimed about children’s combinatorial reasoning. The most well-known disagreements are those of English (1991, 1993) and Fischbein (1975). According to Fischbein (1975), Piaget and Inhelder’s studies (1958, 1975) had some problems. First, there was not enough information about the percentage of the participants that were capable of using systematic methods at the formal operational stage. Next, even though the formal operational stage was categorized as 12-15 years of age, participants at the formal stage before the age of 13 actually did not give satisfactory answers. Also, before the ages of 14-15, the participants in the formal operational stage did not use systematic methods for permutation problems. Thus, Fischbein claimed that during the stage of formal operations (12-15 years), children's intellectual capacities required for combinatorial operations were still developing and the development was not completed at this stage. Fischbein (1975) suggested that without appropriate teaching and guidance, children in the formal operational stage may not have reached full combinatorial reasoning capacity yet.

There were two studies in which the researchers worked with middle school students as teacher-researchers (Shin & Steffe, 2009; Tillema, 2007). Shin and Steffe (2009) investigated two seven graders' enumerative combinatorial reasoning considering additive and multiplicative reasoning through a year-long teaching experiment. They examined these reasoning of students through enumerative combinatorial problems and defined enumerative combinatorial problems as "counting problems" (p. 170) such as the coloring a window problem, the two-digit number problem, and the card arrangement problem. Additive enumeration, multiplicative enumeration, and recursive enumeration were the different enumeration types that were discussed. In coloring a window problem, students were asked to find all possible ways to paint four windows with two colors. In this problem, students were able to use additive enumeration. In two-digit number problem, students were asked to find all two-digit numbers from 10 to 90. At first, neither student could find all possible outcomes without having a table or writing all possible outcomes. According to Piaget, students at formal operational stage age reason multiplicatively. However, this study found that even though the students were at the age where they should be at the formal operational stage, they still could not reason multiplicatively. Moreover, they found that these two students had not constructed the concept of a slot that was described as "abstracted unit" (p. 176). This finding is consistent with Piaget and Inhelder's (1958) experiment on colored liquids. Although the children in the Shin and Steffe (2009) study were at the age where one might expect to observe multiplicative reasoning, that they did not engage in that kind of reasoning is compatible with the findings of Fishbein (1975, 1988). In the card arrangement problem, students were asked to find the number of pairs that could be made from 52 cards. Similar to the previous problem, students could not reason at the formal operational stage. Shin and Steffe (2009) claimed that the students' units- coordinating operations could help them construct their enumerative combinatorial counting. They suggested that permutation problems of more than five elements included more than recursive multiplicative enumeration; that is, "the concept of a program of multiplicative operations" (p. 174). In summary, the findings of this study challenge the age at which Piaget and Inhelder (1958) claimed that combinatorial reason emerges and showed that students at the ages where formal operational reasoning should emerge could not reason multiplicatively or recursively fully.

In his dissertation, Tillema (2007) investigated how three eighth graders produced an algebraic symbol system through their symbolizing activity. Piaget's radical constructivism theory and Piaget's (1958) combinatorial reasoning problems helped him frame his study. He asked multiplicative combination problems such as the three-card combination problem, the coin problem, the outfits problem, the handshake problem and the flag problem. Piaget's distinction between the concrete and formal operational stages for combinatorial problems helped Tillema (2007) understand students' multiplicative reasoning. He used this distinction as a basis for forming students' algebraic reasoning and provided some connections between algebraic reasoning and combinatorial reasoning. Tillema (2007) was specifically interested in the symbolized aspects of multiplicative and quantitative ways of operating; the changes in students' multiplicative and quantitative reasoning while interacting with a teacher-researcher; the mental imagery and operations that students demonstrated in the context of solving quantitative problems; the students' notation function in the process of constructing algebraic symbol system;

the methods students used in the context of their notating activity; and finally, the role of social interaction in the process. He suggested a number of ways in which the students used their symbolizing activity that seems to reside in the province of constructing an algebraic symbol system. The findings of Tillema's (2007) study demonstrated that understanding permutation problems could be challenging for students. This finding was consistent with Piaget's claim because Piaget claimed that students need to wait until the age of 11 to understand permutation.

Eizenberg and Zaslavsky (2003) studied cooperative problem solving in combinatorics. They investigated the inter-relations between the control process and successful solutions. The control process was how the participants kept track of what they were doing. They worked with 14 undergraduate students who had taken at least one combinatorial course and found that the students who worked collaboratively gave more correct solutions then the students who worked individually. In another study, Eizenberg and Zaslavsky (2004) focused on verification strategies, which most students struggle with in combinatorial problems. Again, they studied 14 undergraduate students and found five different verification strategies considering these students' methods. Students who used the first strategy, reworking the solution, basically checked their answers. Students who used the second strategy, adding justification to the solution, used justifications to support their solutions. Students who used the third strategy, evaluating the reasonability of the answer, looked at what they found and reasoned whether the result was possible to get or not by estimating. Students who used the fourth strategy, modifying some components of the solution, either altered their representations or applied smaller numbers and used the same solution process that they used earlier. The students who used the fifth strategy, using a different solution method and comparing answers, used a totally different method to solve the same problem compared to what they had done before. Their findings "support the assertion that combinatorics is a complex topic – only 43 of the 108 initial solutions were correct" (p. 31).

In his dissertation, Panapoi (2013) investigated how two pairs of seventh-grade students construct the multiplicative principle using combinatorial problems. A constructivist teaching experiment was conducted, and by using cards, students were asked to make pairs, triplets, quadruplets, and quintuplets considering if order mattered or did not matter or with/without a replacement. Besides the card activity, students reasoning on tossing a coin, rolling a die or two dice, and coloring models of floor plan problems were explored. Based on the findings, Panapoi (2013) claimed that children who were able to take two levels of units as a given is not enough for the construction of the multiplication principle. A student who was able to take three levels of units as a given was able to construct the multiplication principle. Because combinatorial reasoning also requires abstract and advanced thinking and the multiplicative principle plays an important role in combinatorial problems, constructing three levels of units in combinatorial problems is consistent with Piaget's claim that combinatorial reasoning was one of the key principles in formal thought.

Analysis of the Combinatorial Reasoning Studies

The distinction between listing the possible outcomes of an experiment in activity and mentally arranging the possible outcomes in anticipation can be used to account for what seem to be discrepancies between the studies of English and Piaget and Inhelder. In her 1991 study, English used a toy bear that was to be dressed with several colored tops and colored pairs of pants. This led to her observation of what she called the odometer principle that others have referred to as holding "initial marks constant" (Shin & Steffe, 2009, p. 6). That is, for a given colored top, a child "runs through" the colored pairs of pants pairing each pair of pants with the selected colored top, then selects another colored top and again "runs through" in activity the colored pairs of pants, etc. Although this odometer principle certainly constitutes a coordination of the colored tops and the colored pairs of pants, the coordination is made in activity the results of which are available to the child only after the activity is completed. Further, the nature of the results were not made clear by English. That is, did the children who used the odometer strategy consider the results as a structured unity containing composite units whose elements were pairs of outfits? Or, were the results experiential results that, if enumerated, would need to be reproduced either experientially or mentally and counted? That is, did the children who engaged in the odometer strategy regard the results as a multiplicative structure or did they regard the results as a collection of countable items that could be reproduced and counted? A third possibility would be if the children regarded the results as ephemeral and not subject to counting.

Based on children's use of the odometer strategy alone, it cannot then be said that the work of English stands in contrast to the work of Piaget and Inhelder. Even in the case where there were three separate collections from which to choose—tops x pants x tennis rackets—it is possible to organize the choices systematically according to the odometer principle prior to constructing the multiplicative principle (Shin & Steffe, 2009). The students in the Shin & Steffe (2009) study were at the age level (13 years of age) where one would expect formal reasoning to emerge, but their difficulty in constructing the fundamental principle of counting is compatible with the findings of Panapoi (2013) and Tillema (2007) that reasoning with three levels of units is fundamental in the construction of the multiplicative principle but that it does not guarantee it. These studies, when coupled with the findings of Fishbein (1975, 1988) and Eizenberg and Zaslavsky (2003, 2004), point to the scenario that combinatorial reasoning for a majority of students, and hence, formal operational thinking, is a much more protracted construction than envisioned by Piaget and Inhelder (1958).

Proportional Reasoning Studies

In this section, proportional reasoning, another characteristic of formal thought, will be discussed. Piaget and Inhelder (1958) stated that understanding only the relationship between two objects is not sufficient for proportional reasoning; such reasoning requires second-order thought. By second-order thought they mean comprehending the relationship between two relationships.

Siegler and Vago (1978) conducted a study on children's understanding of the concept of fullness. One of the reasons that they were interested in this proportionality concept was because understanding proportionality had an important role to play in Piaget's formal operational thought, and preadolescents were not capable of understanding proportional concepts. They performed six experiments and investigated 6-and 10-year-old children's proportional reasoning. Researchers used a number of one-quarter, one-half, three quarters or entirely full glass beakers with different heights and diameters. Next, children were asked to label what portions of each glass was full and label them as one-quarter, one-half, three quarters or entirely full. Then, they were asked to compare two beakers in terms of their fullness. Students' responses varied depending on whether they compared heights, volumes, or proportions. Students who used proportionality rules looked at the proportion of the filled part to the empty part of the beakers. The experiments showed that most of the young children gave their decision on fullness based on the height of the beakers, whereas most of the older children gave their decision on fullness based on the volume of the beakers. The study found that it was very challenging to invent the proportionality rule for 10-year-old children. This finding gave the idea that students at the age of 10 did not reason proportionally and did not reach the formal operational stage. Thus, the findings confirm Piaget's development stages on proportional reasoning.

Drawing on Piaget's studies on proportional reasoning, Noelting (1980a, 1980b) conducted studies in the development of proportional reasoning and concept of ratio. In part 1 of his study (Noelting, 1980a), Noelting conducted the Orange Juice Experiment and asked 23 items to 321 students between 6- to 16-years-of-age. Students had a number of orange juice and water glasses and were asked to compare the relative taste of orange for two orange juice and water mixtures. Using Piaget's chronology of development, items were grouped based on their difficulty. Stage 1 students made the comparison of the relative taste of orange in orange juice and water mixture based on the number of orange juice glasses. Stage 2 students used one-one compensation and made their choice based on the residue. Following is the description of stage 2 students reasoning for comparing the relative taste of orange between the mixture of 4 glasses of orange juice and 2 glasses of water and the mixture of 2 glasses of orange juice and 1 glass of water. The mixture with 4 glasses of orange juice and 2 glasses of water consists of the mixture of 2 glasses of orange juice and 2 glasses of water and also 2 glasses of orange juice. Similarly, the mixture of 2 glasses of orange juice and 1 glass of water mixture consists of the mixture of 1 glass of orange juice and 1 glass of water and also 1 glass of orange juice. In this case, the mixture of 2 glasses of orange juice and 2 glasses of water in the first mixture and the mixture of 1 glass of orange juice and 1 glass of water in the second mixture are one-one compensation. For the relative taste of orange, students at Stage 2 looked at the residue. They considered 2 glasses of orange juice as a residue for the first mixture and 1 glass of orange juice as a residue for the second mixture. Because 2 glasses of orange juice is more than 1 glass of orange juice, Stage 2 students thought that first mixture had a stronger orange taste than the second mixture. This comparison is similar to the description of Piaget's additive strategy. Stage 3 students used the ratio of orange juice and water and compared these two ratios. This comparison is similar to the description of Piaget's multiplicative strategy.

In part 2 of Noelting's study (1980b), the stages in part 1 were explained and analyzed. In part 2, he explained symbols and the terms that he used in part 1. Moreover, he explained problem solving strategies at each stage in detail and categorized the concept of ratio as ratio within a concept (within-state ratios) and ratio between concepts (between-state ratios). In orange juice and water mixture problem, within-state ratio is the ratio between orange juice and water in each mixture, whereas between-state ratio is the ratio between the number of glasses of orange juice between each mixture or the ratio between the number of glasses of water between each mixture. Additionally, the structure of items at each stage in terms of between-state ratios and within-state ratios was explained. Passing from one stage to another required constructing more advanced schemes, and Noelting called this process adaptive restructuring, which is similar to Piaget's increasing equilibration. In terms of the development of stages, Noelting found two types of changes: qualitative changes between stages and quantitative changes within a stage. Piaget also focused on qualitative and quantitative changes in proportional problems and similar to Noelting, he claimed that "qualitative operations are inadequate to establish the law" (Piaget, 1958, p. 172). Thus, both of the researchers claimed that for proportional problems not only qualitative reasoning but also quantitative reasoning was required. Piaget generally focused on students' performance based on the stages and different from Piaget, Noelting's study focused on the structure of the problems and how the structure of the problem could affect students' performance. Both researchers claimed that the complexity of students' strategies changed based on their stages.

Furthermore, Steffe and Parr (1968) investigated the development of the concept of ratio and fraction of students at fourth, fifth, and sixth grade. Besides these concepts, they also explored the proportional reasoning of these students and constructed four pictorial level tests and two symbolic level tests. The researchers found that there was little correlation between students' performances on symbolic level data and their performances on ratio and fractional data. Additionally, they found that the pictorial proportional problems that were presented as a ratio were easier than the ones that were presented as a fraction. In their study, Steffe and Parr (1968) applied an intelligence test and found that high intelligence children performed better than low intelligence children in both pictorial and symbolic problems. Students' performance depending on their intelligence was different from Piaget's categorization of students' performance based on age group range. They also found that when pictorial data was not helpful to solution, the data was not meaningful mathematically for students. Thus, the proportional problems could be very challenging for fourth-, fifth-, and sixth- grade students except high intelligence six graders. This finding is compatible with Piaget's claim that proportional reasoning requires advanced reasoning and could be challenging for students.

Jeong, Levine and Huttenlocher (2007) claimed that students were more successful when engaging with proportional problems involving continuous quantities than discrete quantities and examined students' proportional reasoning in the context of continuous and discrete quantities. They questioned Piaget's claim that students' proportional reasoning did not develop until the age of 11. Sixty students from six-, eight-, and ten-years of age from Korea were asked a variation of Piaget and Inhelder's (1975) marble task. Students were given a donut shaped figure with blue and red regions and were asked to compare the regions. The sizes of the donuts were different and questions varied based on three conditions: continuous, discrete adjacent, and discrete mixed condition. In the continuous condition, different sized donuts were divided into two pieces with different portions and one part was shaded and the other part was not. In the discrete adjacent condition, different sized donuts were divided into multiple equal parts and a number of adjacent parts were shaded. In the discrete mixed condition, different sized donuts were divided into multiple equal parts and a number of non-adjacent parts were shaded. Among all conditions, discrete mixed condition was the closest task to Piaget's marble task. They found that students performed better with continuous quantities than discrete quantities. Also, students mostly used erroneous counting strategies with discrete quantities and consistent counting strategies with continuous quantities.

Lesh, Post, and Behr (1988) considered proportional reasoning as a capstone for elementary mathematics, a cornerstone for advanced mathematics, and consisting of both qualitative and quantitative reasoning. Solving proportional reasoning problems does not mean that students use proportional reasoning; however, proportional related problems require using proportional reasoning. Thus, they preferred to utilize proportion related problems. They categorized proportion related problems into 7 types: missing value problems; comparisons problems; transformation problems; mean value problems; proportions involving conversations from ratios, to rates, to fractions; proportions involving unit labels as well as numbers; and between-mode translation problems. As a capstone of elementary school mathematics, Lesh et al. (1988) discussed transitions from pre-proportional reasoning to proportional reasoning. The first reasoning involves additive reasoning, whereas the second reasoning involves multiplicative reasoning. Lesh et al. (1988) described the transition from pre-proportional reasoning to proportional reasoning different from Piaget et al. (1968). According to Piaget, proportional reasoning is a global ability; however, according to Lesh et al. (1988), this reasoning refers to the gradual development of local competence and "[p]roportionality is initially mastered in small and restricted classes of problems settings. Competence is then gradually extended to larger classes of problems" (Lesh et al., 1988, p. 103). Similar to Piaget's development stage sequence, Lesh et al. described five stages in students' reconceptualization cycles. Students using the first conceptualization stage used additive reasoning only. Students using the second conceptualization stage started to use multiplicative reasoning for understanding the relationship between two items. Students using the third conceptualization stage recognized a pattern and replicated that pattern. This stage is still pre-operational stage. Students using the fourth conceptualization stage used multiplicative proportion; however, this conceptualization was based on sampling from a biased subset of information. Students using the fifth conceptualization stage used multiplicative proportion such that the proportion was based on using a systematic information procedure. Different from Piaget's stages, this conceptualization stages were drawn on the gradual development of the same students. In Lesh et al.'s (1988) study, the stages were categorized based on the gradual development of students' reasoning over a long period of time. In Piaget's studies, the stages were categorized based on students' general and global reasoning not based on the evolution of same students' reasoning. Thus, Lesh's categories were based on "gradual increase in local competence" (p. 116).

Analysis of the Proportional Reasoning Studies

The studies confirmed that proportional reasoning requires advanced reasoning (Noelting 1980a, 1980b; Lesh et al., 1988). Some studies (Lesh et al., 1988; Noelting, 1980a, 1980b; Steffe & Parr, 1968) investigated not only proportional reasoning problems but also proportion related problems (i.e., fraction and ratio). Lesh et al. (1988) focused on the use of proportion related problems, whereas Steffe and Parr (1968) focused on the performances of students from two different school systems.

Researchers found that recognizing the relationship between two relationships is challenging for students. The challenges that were the focus of each study varied. In Siegler and Vago's (1978) study, in terms of the concept of fullness of two glasses of water, some students judged the concept based on one variable (height or volume), and the challenge was thinking about the proportion of the empty and full parts. Ten-year-olds struggled more on inventing a proportionality rule whereas 7-year olds did not have as much difficulty in learning proportionality. The finding contrasts with Piaget's findings based on the developmental stages. However, Siegler and Vago (1978) compared the invention of proportional problems for 10-years-olds and the difficulty of learning proportional reasoning for 7-years-olds. So, they did not compare the students based on the same criteria (i.e. comparing both 7 and 10-years-olds in terms of their invention of proportional reasoning problems or comparing both age groups in terms of their learning difficulty). Thus, making comparison based on different criteria is a limitation for this study for the purposes of this thesis. They also found that 10 years-olds did not reason proportionally yet, and this finding is compatible with Piaget's claim that proportional reasoning does not emerge for a majority of students until the age of 11 or 12.

Noelting (1980a, 1980b) also used liquids as Siegler and Vago (1978) did. However, Noelting had two types of liquids (orange juice and water) and he explored within-state ratios and between-state ratios. In terms of both within-state ratios and between-state ratios, the relation between discrete quantities, in this case number of classes, was discussed. Similar to Piaget (1958), Noelting (1980a, 1980b) made the distinction between the concrete operational stage and the formal operational stage in proportional reasoning in that students at the concrete stage used an additive strategy, whereas students at the formal operational stage used a multiplicative strategy.

Different from Noelting (1980a, 1980b), Jeong et al. (2007) used discrete objects. In their study, the size of the donut figure and the proportion of red and blue parts of these figures were different. The relationship between quantities varied among continuous, discrete, and discrete mixed. This study provided some insights into proportional reasoning of Korean students. Because the study was conducted with Asian students, culture or curriculum might be a factor in the differences of the performances between Asian and Caucasian students. Although the researchers were American and Korean, they did not discuss the culture factor in their study.

Lesh et al.'s (2007) description of types of proportional reasoning tasks was helpful to understand proportion related concepts. Different from Piaget's categorization, Lesh et al. used a longitudinal study to categorize students' conceptualizations. This way, the effect of instruction, the usage of manipulatives or other visuals, and some other factors could be explored. In Lesh et al.'s research method, the progress of students could be monitored, whereas in Piaget's research method, the progress of students was only predicted. Both Lesh et al. (2007) and Piaget (1958) were not clear about how students in the concrete operational stage shifted to the formal operational stage. What triggered multiplicative reasoning was a crucial question to investigate, but the researchers in neither study provided any insights into this question. Investigating the shift from additive to multiplicative reasoning could also help to see the relation between multiplicative reasoning and proportional reasoning. In terms of teaching strategies, neither of these scholars was specific about what was necessary to reason multiplicatively.

CHAPTER 4

CONCLUSION

Abstract thinking is crucial in both proportional reasoning and combinatorial reasoning. In both types of reasoning problems, the relationship between two abstract units is essential. In combinatorial reasoning problems, students at the formal operational stage are capable of constructing a composite unit from the two abstract units. Students at this stage can take all the possible outcomes of a certain number of elements as an input and use it for finding all the possible outcomes of problems with more number of elements. Moreover, students at this stage can extend this strategy for other complex combinatorial problems.

Likewise, in proportional reasoning problems, students at the formal operational stage are capable of recognizing the relationship between two relationships. The relationship between two relationships refers to second order thought. Similar to combinatorial reasoning, the relationship between two abstract units is also crucial for second order thought of proportional reasoning.

Researchers agreed that both proportional reasoning and combinatorial reasoning require advanced thinking and are more complicated than Inhelder and Piaget (1958) described. The age range that was required for reasoning combinatorially and proportionally is compatible with Piaget's age range. Inhelder and Piaget's (1958) description of performing at formal operational stage for these two types of reasoning is necessary but not sufficient to complete formal reasoning.

Generalization and recursive reasoning are important concepts in both proportional reasoning and combinatorial reasoning. Making generalizations requires operating recursively and operating with three-levels of units. Using recursion and units coordinating activity are necessary for both combinatorial reasoning and proportional reasoning. However, there was not enough information in the literature for me to determine if recursion and units coordinating activity are sufficient for both types of reasoning. For future research, the impact of recursion and units coordinating can be explored.

Moreover, the key points that also need to be investigated are the use of recursion in combinatorial and proportional problems. For future studies, the relationship between students' recursive reasoning and multiplicative reasoning in combinatorial and proportional reasoning can be investigated. Moreover, how students' units coordinating activity is related with their recursive reasoning in both types of problems is worthy of study. Also, what triggers students' recursive reasoning in multiplicative reasoning can be explored.Recursive reasoning, multiplicative reasoning, units coordinating activity, proportional reasoning, combinatorial reasoning, formal operational stage, and abstract thinking are all interrelated. The effect of each one can be explored by investigating their interrelationships.

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