TEACHERS' KNOWLEDGE OF STUDENT THINKING AND THEIR INSTRUCTIONAL PRACTICES IN ALGEBRA

by

AYHAN KURSAT ERBAS

(Under the Direction of James W. Wilson)

ABSTRACT

This study was designed to address and contribute to our emerging knowledge and understanding of teachers' knowledge of student difficulties and related issues with instructional practices in algebra. The perspectives suggested by the constructs of teaching in context, cognitively guided instruction, and pedagogical content knowledge influenced the theoretical orientations of the present study. Since "knowing" is a variety of separate entities, I found the distinction of knowing as *knowing-about* (i.e., *knowing-that*, *knowing-why*, *knowing-how*) and *knowing-to* useful to look at different degrees of knowing. Qualitative case study research design and methodologies were used in generating data collected from two inservice mathematics teachers of first year algebra (one eighth-grade and one ninth-grade teacher) who were selected purposefully. Data collection strategies included conducting audio-recorded semi-structured interviews, making video-recorded classroom observations, and collecting archival documents. Data stories about each case included thick descriptions of each participant's beliefs, knowledge, and practices concerning student thinking. Findings revealed that even though both teachers presented an awareness and recognition of students' thinking and difficulties in terms of "knowing-that," their knowledge in terms of "knowing-why" and "knowing-how" was narrow and even problematic in some cases. Such insufficient knowledge might have limited the teachers' pedagogical content knowledge of student thinking in terms of "knowing-to" and hampered the teachers when acting in the moment. Issues other than conceptual, cognitive, and epistemological problems characterized both teachers' knowledge of and beliefs about general sources of students' difficulties in terms of "knowing-why." Those issues were: lack of arithmetical and geometrical knowledge, lack of motivation, lack of experience with nontraditional curricula, lack of practice in similar type of problems, carelessness, and inability to understand and apply definitions. However, the teachers were able to give explanations for students' difficulties and mistakes in specific concepts they were teaching. Both case studies revealed that textbook dependence was central to the teachers' practices at different stages of instruction such as when planning lessons, assigning homework, or assessing students' learning. This dependence served as a blocking factor for teachers in trying to get more elaboration and knowledge of student thinking. Commonalities and differences arise for individual reasons for textbook dependence.

INDEX WORDS: algebra, mathematics education, teacher knowledge, beliefs, student thinking, student difficulties, inservice mathematics teachers, qualitative research, case study

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DEDICATION

To

my parents Memet and Guldane,

my lovely wife Betul,

my teachers and friends,

for their constant support, encouragement, and wisdom.

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CHAPTER 1: INTRODUCTION

A beginning is a promise for an ending. More important, how you begin is not only a promise for an end but also how it does end. I do not claim my beginning for this report is from every aspect a “good” one. I can only say it is a significant one for me. I have spent 20 years of my life as a student in educational institutions. Although I have been through a lot of problems and difficulties, mathematics has always been my favorite subject. I am not so sure that my mathematics teachers had any effect on this preference. Actually, they were the least favorite teachers I had. I cannot explain exactly why that was so, but I can say that my mathematics teachers were all alike in the sense that I never felt they were actually helping me with my difficulties and understanding. Grades were never a problem. I was an A student. I guess that was enough for my parents and teachers. What was true for me was also true for my classmates. I am not quite sure how many hands were raised when the teacher asked whether there was anything that was not understood, even though almost all of the class did not understand what was going on. It seems to me that the main problem was the lack of communication. Whenever I (or any of us) asked about something that the teacher was trying to teach, the answers were simply a repetition of what was written on the blackboard like a broken tape player. I do not want to be unfair, but I was learning not my mathematics but the teacher’s mathematics.

My love for mathematics eventually led me to a mathematics education department. I expected to become a teacher who would not make the same mistakes my teachers did. I have not had much of a chance to test this goal except on several part-time teaching occasions.

Although I consciously tried to avoid presenting mathematics in a traditional way, I am sure I did so and unconsciously imitated my teachers.

As a young researcher, I would like to do something to reveal the mystery behind what is going on in a mathematics classroom in terms of teacher knowledge, goals, and beliefs; of teachers' actions (or instructional practices), and of the role of student thinking in that context. As a core topic in mathematics, algebra has become my target. I investigated student difficulties in elementary algebra in the research for my Master of Science degree (Erbas, 1999). One of the interesting but quite expected results was teachers' unawareness of student difficulties. In fact, it seemed students and teachers were speaking different languages. Because this result was drawn from some questionnaires, I needed to conduct further research to understand why students have difficulties in algebra and whether teachers are aware of them. More important, gaining that understanding is a personal goal for my life spent in mathematics classrooms.

Problem Statement and Research Questions

> Since Algebra surpasses all human subtlety and then the clarity of every mortal mind, it must be accounted a truly celestial gift, which gives such an illuminating experience of the true power of the intellect that whoever attains to it will believe there is nothing he cannot understand.
>
> *Girolamo Cardano* (1501–1576)

Not everybody is as excited about algebra as Girolamo Cardano, a key figure in the history of algebra, was. For many people, algebra is a challenge and a gatekeeper. Because the role of algebra in mathematics is more than just being a branch of mathematics in its own right, the difficulties encountered in teaching and learning algebra can lead to impediments in the learning of higher level mathematics. Experience and research have shown that the road to algebra is never as smooth as one might wish. Algebraic tasks are difficult to learn and teach. Among all other variables that might make the learning of algebra problematic, the teacher, as the key figure in this process, warrants study to prevent, lessen, or otherwise deal with the difficulties students encounter. This study, therefore, was designed to address and contribute to our emerging knowledge and understanding of teachers' conceptions of student difficulties and related issues with instructional practices in algebra.

I observed and interviewed one eighth-grade teacher and one ninth-grade teacher of Algebra 1 to investigate and understand their knowledge of student thinking and instructional practices in algebra. The following research questions guided the study:

- What is the nature of teachers' professional knowledge about student thinking in Algebra 1? How is this knowledge grounded?
- How does student thinking and knowledge of it inform teachers' instructional practices?
- What are teachers' beliefs about student thinking in Algebra 1?

Rationale for the Study

The rationale for this study consists of several themes. I first draw upon the importance of algebra and its role in the school curriculum and society. Then I turn my attention to the existence of students' difficulties with algebra. This topic leads to the gaps in research among student difficulties, teacher practices, and teacher knowledge in algebra. The importance of algebra, the existence of student difficulties, and the lack of research on algebra teachers create a need to investigate teachers' pedagogical content knowledge, particularly their knowledge of student thinking as a crucial part that deserves further attention and research.

Importance of Algebra

> The real importance of algebra, and of mathematics in general, is not that it has enabled man to solve this problem or that, but that it has given man a new outlook on the universe. From the time of

> Galileo onward, mathematics has encouraged man to look at the universe with the continual question: “Exactly how much?”
>
> *Isaac Asimov* in “Realm of Algebra”

The nature and role of algebra in school mathematics have been the focus of extended discussion in recent years. National reports focused on the presumed crisis in education, particularly in mathematics and science (e.g., National Commission on Excellence in Education, 1983; National Council of Teachers of Mathematics, 1980), and the poor performance of U.S. students in international studies such as the Second International Mathematics Study (SIMS), which was conducted in 1981-1982, were major inspirations for many reform efforts during the 1980s and 1990s. Although they were diverse with respect to their focus (e.g., new curricula, teacher enhancement), many curricular efforts were started that proposed to change the face of mathematics education to prepare U.S. students for a new century. Past experience and research have shown that many adults and students suffered from math phobia and math avoidance; international comparisons showed that the United States lagged behind its economic competitors; national assessments indicated low levels of mathematics achievement and a big gap in achievement between racial and economic classes. The number of students entering schools and their need to take mathematics for the demands of the workplace and effective citizenship are increasing. Computing/Electronic/Digital technology has increasingly become an integral part of daily life. Technology has become so demanding that it has changed society and its components including mathematics itself. The change is so fast that we need more and more people who have problem-solving capacities to solve the unfamiliar and non-routine problems that occur as a result. These trends show that we need to redefine our educational philosophies and practices to improve in mathematics education.

Within this context, algebra has found a place and has played an important role in the reform efforts. "Algebra for All" has become a slogan for the importance and increased access to algebra by all students. The National Council of Teachers of Mathematics (NCTM), through its publications such as *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Algebra for Everyone* (Edwards, 1990), and *Algebra for the Twenty-First Century* (NCTM, 1993), and *A Framework for Constructing a Vision of Algebra* (Burrill et al., 1998), pointed out the importance of increased algebra access and outlined a set of content and instruction expectations. By taking the position that algebra is for all, the NCTM has called for a complete rethinking of the meaning and teaching of school algebra and advocated that the notion of algebra needs to be expanded to include a range of mathematical activity. NCTM's messages on algebra reform continued in *Principles and Standards for School Mathematics* (NCTM, 2000) with the following emphases:

- "Algebraic competence is important in adult life, both on the job and as preparation for postsecondary education. All students should learn algebra." (p. 36)
- "In middle grades, the majority of instructional time would address algebra and geometry." (p. 30)
- "By the end of secondary school, they should be comfortable using the notation of functions to describe relationships." (p. 38)

The value associated with algebra and the reasons for studying it cannot be disregarded. Ever since the invention of symbolic algebra by Vieta, algebra has provided humanity the power to operate with concepts at abstract levels and then apply them. As Asimov pointed out in the quotation on pp. 3-4, it is this power that enabled us to understand more about ourselves and the universe we live in. The value of algebra is summarized in *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) as follows: "Algebra is the language through which most of mathematics is communicated. It also provides a means of operating with concepts at an abstract level and then applying them, a process that often fosters generalizations and insights beyond the original context" (p. 150). Furthermore, in *Principles and Standards for School Mathematics* (NCTM, 2000) algebra is considered as an essential component of contemporary mathematics and its applications in many fields. Thus, studying algebraic concepts is considered to provide a foundation for developing higher-order thinking and problem solving abilities. Without algebra, not only advancement into most areas of mathematics, such as analytic geometry, trigonometry, combinatorics, analysis, and statistics, but also the study of other disciplines requiring mathematical abstraction and modeling such as science and engineering beyond the descriptive stage are limited if not impossible. Research has shown that students who take algebra by the eighth or ninth grade are more likely to take higher mathematics courses and pursue higher education than those who do not (Cooney & Bottoms, 2002; Riley, 1997). Moreover, taking algebra benefits students by increasing their achievement regardless of their prior mathematical skills—low-achieving or high-achieving (Gamoran & Hannigan, 2000). And conversely, placing students in lower-level classes does not benefit them (Hoffer, 1992).

The value of algebra goes beyond academic study. Algebraic literacy is also a medium for social justice and job and later opportunities. According to Moses and Cobb (2001), algebra is "the new civil right," and it is "the key to the future of disenfranchised communities." Supporting the claims of Moses and Cobb from a different angle, in an article in the *Wall Street Journal*, Kronholz (1998) reports that "youngsters who take algebra tend to go to college, research shows, and low-income youngsters who take it are almost as likely to go to college as middle- and upper-income kids. The gap in test scores between students in private school and those in public school largely disappears if they take upper-level math courses, beginning with algebra" (Poisoned Attitudes section, ¶ 2). Furthermore, according to the National Research Council (1989) report *Everybody Counts*, over 75 percent of all jobs require proficiency in elementary algebra and geometry either as a prerequisite to a training program or as part of a licensure examination. In an increasingly technological society and workplace, the importance of algebra as a language of mathematics (NCTM, 1989) keeps the doors open to opportunities for those who would like to pursue them.

Student Difficulties in Algebra

> The teacher pretended that algebra was a perfectly natural affair, to be taken for granted, whereas I didn't even know what numbers were. Mathematics classes became sheer terror and torture to me. I was so intimidated by my incomprehension that I did not dare to ask any questions.
>
> *Carl Jung* (1875–1961)

The difficulties encountered in teaching and learning of meaningful algebra lead to impediments in the learning of higher-level mathematics. Despite its importance, however, algebra has been a stumbling block in school mathematics. Research on children's interpretation and understanding of algebra reveal that there are many conceptual difficulties (Booth, 1984, 1988; Clement, Lochhead, & Monk, 1981; English & Halford, 1995; Herscovics, 1989; Kieran, 1989, 1992; Küchemann, 1981, Leinhardt, Zaslavsky, & Stein, 1990; MacGregor & Stacey, 1993, 1997a). Students often leave an algebra course with a feeling that they have been taught some abstract system that has no meaning and purpose, not to mention that it is disconnected from what they already know and learned in arithmetic or any other mathematics class (Herscovics, 1989; Kieran, 1992; Sfard, 1995). Students have been taught the syntax of a language without the semantics (Herscovics, 1989). Sfard (1995) claims that most students view "algebraic expressions as meaningless symbols governed by arbitrary established transformations" (p. 30). In other words, they know the rules of the grammar, but they do not understand the meaning.

Gap in Research

> Research in mathematics education has two main purposes, one pure and one applied: Pure (Basic Science): To understand the nature of mathematical thinking, teaching, and learning; Applied (Engineering): To use such understandings to improve mathematics instruction.
>
> *Alan H. Schoenfeld* (2000)

Even and Tirosh (1995) comment that “research on learning and learners, and research on teaching and teachers have been following separate tracks for a long time” (p. 3). Despite the importance of algebra and the knowledge of difficulties and misunderstandings students have in algebra accumulated from research over the last three decades, there has been little change in the ways of teaching and learning algebra. This lack of change points to a disconnection between research on students’ learning and actual classroom practice (ICMI Working Group on Teachers’ Knowledge for Teaching Algebra, 2001; Kieran, 1992). We know very little about the issue of what is happening in the classrooms despite extensive debates on algebra (RAND Mathematics Study Panel, 2003). As Kieran (1992) notes, there is an “enormous gap in the existing literature on teaching regarding how algebra teachers interpret and deliver that content” (p. 356) and “the research community knows very little about how algebra teachers teach algebra and what their conceptions are of their own students’ learning” (p. 395). Schoenfeld (2000) made it clear that we need to understand the nature of teacher knowledge and teaching of algebra as well as student thinking in algebra to improve algebra instruction. This argument underscores the need to examine teachers’ practice and their knowledge of student thinking in algebra (ICMI Working Group on Teachers’ Knowledge for Teaching Algebra, 2001).

Teacher Knowledge of Student Thinking in Algebra

> When it comes to algebra we have to operate with *x* and *y*. There is a natural desire to know what *x* and *y* really are. That, at least, was my feeling. I always thought the teacher knew what they were but wouldn't tell me."
>
> *Bertrand Russell* (1872–1970)

Considerable attention is being directed by researchers and policy makers to the notion that teachers need to understand what students know and how they think about a particular concept or problem situation in order to help move their understanding forward (Carpenter & Fennema, 1991; Fennema & Carpenter, 1996; National Board for Professional Teaching Standards, 1998; NCTM, 1991, 2000; Shulman, 1986). Matz (1980) comments that students' errors with algebraic algorithms are often due to their learning or constructing the wrong idea, not because of failing to learn a particular idea. Furthermore, students bring informal and self-constructed techniques into algebra classrooms where more formal methods are developed (Booth, 1988). It is widely accepted that teaching should recognize, value, and incorporate students' prior knowledge and intuitive or informal solution methods with a combination of opportunities for student interaction and discussion (Boaler, 1998; Booth, 1988; Fennema, Franke, Carpenter, & Carey, 1993; Filloy & Rojano, 1989; Swafford & Langrall, 2000; Thompson, 1988). Teachers need to develop a knowledge of mathematics for teaching with an understanding of the mathematical processes and concepts, of the relationships between different domains of mathematics, and of students' ways of thinking and their mathematical background (Fennema & Franke, 1992; Ma, 1999, Conference Board of the National Sciences, 2001).

Ball (1997) describes the complexity of what teachers need to know as follows:

> What [teachers] need to know cannot be fully specified in advance. Instead, across multiple intellectual, personal, and cultural divides, teachers must work to see and hear students' flexibility in the moment and over time. As they work to help their students develop understanding of content, teachers must listen to students — making use of their knowledge of the content while not being limited by it. (p. 775)

Ball points out the contextual nature of instruction and the need for listening to students and using their mathematical thinking in instruction. Knowledge of student thinking is considered an important part of the knowledge base required for teaching (Bromme, 1994, 1995; Shulman, 1986, 1987). Shulman (1986) proposed that knowledge of student thinking— particularly knowledge of common conceptions, misconceptions, and difficulties that students encounter when learning particular content and strategies to respond to those— are important components of pedagogical understanding; thus it is an essential part of the knowledge base required for successful teaching. He emphasized that a research-based knowledge of students' misconceptions, their influence on learning, and instructional strategies to overcome and transform students' initial conceptions "should be included at the heart of our definition of needed pedagogical knowledge" (Shulman, 1986, p. 10). Furthermore, knowledge of students' prior knowledge and common difficulties was also emphasized as an essential part of effective teaching practices throughout reform documents of the NCTM (1991, 1995, 2000) and the NBPTS (1998) in which teachers' ability to evaluate and foresee related student thinking were attributed to successful instruction. Given the importance of algebra, investigating teacher knowledge of student thinking is critical to understanding and building upon the knowledge base required for proficiency in teaching and filling the gap between student learning and teaching, which in return informs teacher education.

Research shows that students' conceptual thinking and skills can be improved by implementing reform-based instructional programs (Hiebert, 1999). However, transforming teacher practice from a traditional approach to a reform-oriented approach is difficult (Hiebert, 1999; Russell, 1996; Schifter, 1993; Shepard, 2000). In her survey of the research on arithmetic-based learning, Fuson (1992) concludes that students can "learn much more than is presented to them now if instruction is consistent with their thinking" (p. 118). An approach to teaching mathematics in which taking knowledge of children's thinking as the center of instructional decision-making is envisioned in Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), which has been shown to be effective in changing and improving instruction in elementary arithmetic and geometry (Fennema & Carpenter, 1996; Lubinski & Fox, 1998; Swafford & Jones, 1997; Vacc & Bright, 1999). By providing research based-knowledge about the development of children's mathematical thinking through professional development activities, CGI changed teachers' expectation of their students mathematical understanding, paralleled the students' and teachers' learning and transformed teachers into learners (Franke & Kazemi, 2001).

One can assume that the success obtained with arithmetic can be replicated with algebra. We still need to understand, however, how teachers develop their knowledge of students' thinking as well as need more effective models of what constitutes teacher knowledge with respect to student thinking in particular domains of mathematics such as algebra. This need becomes particularly important considering that, reinforced by No Child Left Behind legislation, many U.S. high schools now require students to demonstrate solid proficiency in algebra before they can graduate as a result of public pressure for higher standards and associated accountability systems (RAND Mathematics Study Panel, 2003).

CHAPTER 2: CONCEPTUAL AND THEORETICAL ORIENTATION

In this chapter, I discuss the literature and philosophies I have used to conceptualize teacher knowledge and beliefs in relation to student thinking. Those philosophies reflect how I see my study as fitting into a broader literature of research on or with teachers.

A Conceptual and Theoretical Framework

> Without theory, practice is but routine born of habit. Theory alone can bring forth and develop the spirit of inventions.
>
> *Louis Pasteur* (1822–1895)

One of my major dilemmas throughout my graduate education has been choosing a theoretical perspective. I have been bombarded with discussions of the importance of theories, and the fact that I need to have and state *a priori*, *macro* and *mid-level* theories. Often the most important gap in a research study is the lack of a theoretical standpoint. Especially in an era of paradigm wars, this meant so much to those people. I usually found myself resisting those stands and describe myself as atheoretical. It took me quite a while to progress toward a coherent understanding of the importance and meaning of a theory in life and academia. I came to realize that my understanding of *situating yourself in a theory* is also situated in such a way that I often experience shifts from one theory to another. The need for theories, however, is vital to identify and explain the key processes and acts in complex events of life such as learning and teaching.

In this study, my conceptual, philosophical and theoretical understanding have been shaped by the perspectives suggested in Teaching-in-Context (Schoenfeld, 1998), Cognitively Guided Instruction (Carpenter & Fennema, 1991; Fennema & Carpenter, 1996), and Pedagogical Content Knowledge (Shulman, 1986, 1987).

Figure 1 presents a visual summary of the framework for this study. The model is a modified version of Teaching-in-Context (Schoenfeld, 1998) with a focus on students' thinking. The model shows teachers' knowledge, beliefs, and goals as main factors affecting what teachers do and why they do it and student thinking as a central unification theme for all three. Although there are many components forming or consisting of teachers' knowledge, beliefs, and goals, I focused on those are related to student thinking. Student thinking as the intersection of three components in the model represents this point. Equally important to beliefs, knowledge, and goals, as Schoenfeld (1998), points out, is the context that activates the valued knowledge, belief, or goal. Naturally, the current context is affected by past contexts and experiences. In the rest of the chapter, I first give a rationale for focusing on student thinking and then present an extended overview of my understanding and conceptual orientation of the model in terms of teacher knowledge, beliefs, and goals.

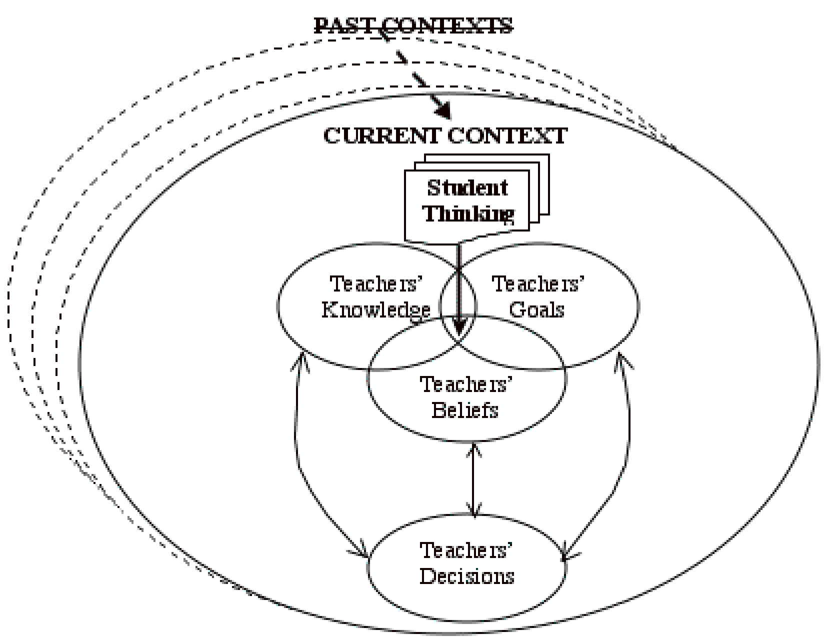


Figure 1. General model for conceptual, philosophical, and theoretical perspectives.

Focus on Student Thinking

> An understanding of how students learn specific mathematical concepts is important, including an awareness of common misconceptions and a familiarity with strategies for helping students describe and reconsider their understandings. (NCTM, 1995, p. 52)

Bromme (1994, 1995) points out that perceptions of students' understanding provide a rich setting for the application of professional knowledge. He criticized research reports in which teachers observed their students during lessons with a focus on remembering particular, individualistic strategies and difficulties. The reports presented a negative picture—showing very little knowledge of students and often underestimating teachers' professional knowledge. His research revealed that teachers do not focus on individual student errors instead they concentrate on the entire flow of the lesson from a "whole class" or "collective student" perspective. He thus proposed that one should look at a lesson as a whole and look at teachers' knowledge and perceptions with respect to problems of and progress in understanding, which in return would give better indications of teacher knowledge.

Shulman (1986) states that content knowledge is the domain in which research on learning and research on teaching are most closely related. According to NCTM's (1991) *Professional Standards for Teaching Mathematics*, knowing general principles of learning and teaching is not enough because it does not involve research-based knowledge of students' thinking and the mathematics they know. "Teachers need opportunities to examine children's thinking about mathematics so that they can select or create tasks that can help children build more valid conceptions of mathematics" (p. 144).

Several programs have attempted to enhance teacher knowledge of student thinking so that teachers can improve the quality of their mathematics instruction. CGI seems to be one of the promising programs, with lots of success stories (Fennema & Carpenter, 1996; Lubinski &

Fox, 1998; Swafford & Jones, 1997; Vacc & Bright, 1999). Focusing on teachers' pedagogical content knowledge, CGI has the goal "to help teachers develop an understanding of their own students' mathematical thinking and its development and how their students' thinking could form the basis for the development of more advanced mathematical ideas" (Fennema & Carpenter, 1996, p. 404).

Carpenter and Fennema (1991) emphasize that the approach CGI brings is not an attempt to provide a prescription or a series of procedures for instruction, but it is an attempt to help teachers to improve their own instructional decisions using knowledge from cognitive science. Thus, the premise behind CGI is the perception that

> The teaching-learning process is too complex to specify in advance, and as a consequence, teaching essentially is problem solving. Instruction must necessarily be mediated by teachers' decisions, and we can ultimately bring about the most significant changes in instruction by helping teachers to make more informed decisions rather than by attempting to program them to perform in a particular way. (pp. 10-11)

Student knowledge and instructional goals should be the bases for teachers' instructional decisions. Teachers' knowledge of content (i.e., its difficulty, its depth, multiple ways to represent it, etc.) is important in this decision-making process. In addition to assess their students' thinking, teachers should know the general stages that students pass through while learning the concepts and procedures in a specific domain or topic. The most fundamental principle of all is that "instruction should be appropriate for each student" (p. 11). In other words, the task, concepts, procedures should be all meaningful to students. Furthermore, instruction should be designed to emphasize the relationships among concepts, procedures, and problem solving and allow students to construct their own knowledge with understanding. Therefore, promoting teachers' content knowledge and pedagogical content knowledge in terms of the "knowing that" and "knowing why" (Even & Tirosh, 1995; Shulman, 1986) aspects of student thinking seems to be promising for more informed instructional decisions and thus quality mathematics education.

The research on CGI was mostly done on elementary school mathematics: Fennema and Carpenter (1996), on change in the beliefs and instructions of word problems of 21 primary grade teachers; Lubinski and Fox (1998), on one pre-service teachers' knowledge of division of fractions; Vacc and Bright (1999), on 34 elementary pre-service teachers' beliefs about teaching and learning mathematics; Swafford and Jones (1997), on how enhancing teachers' knowledge of geometry and their knowledge of research on students' cognition in geometry affects their instruction. All these studies provided evidence that a consideration of students' thinking was related to a change in instruction. Considering that there is a considerable number of studies on students' cognition in algebra, one would expect that the results obtained from elementary school teachers teaching arithmetic can be replicated with teachers of algebra in middle school and secondary school.

Errors and Misconceptions in Student Thinking

Chiu, Kessel, Moschkovich, and Muñoz-Nuñez (2001) define a conception as "an idea that is stable over time, the result of a constructive process, connected to other aspects of a student's knowledge system, robust when confronted with other conceptions, and widespread (i.e., appearing in more than one instance or problem solver). A conception is not merely a response to a question, but is a definite idea" (p. 219). Following this definition, I define a misconception as a conception that contradicts its counterpart, the concept, within the domain in which it is defined. In contrast, an error is an incorrect application and conclusion of mathematical expressions or ideas. According to Payne and Squibb (1990), errors can be classified as "slips" or "mistakes." The distinction between those two lies in the intentions of the actor: “If you intend to perform the appropriate action but fail to do so, then you have slipped; if you formulated the intention incorrectly, then you have made a mistake” (p. 465). A central claim in the literature is that many errors can be explained by the students’ mental representation and application of a faulty procedure, called *mal-rule* (Payne & Squibb, 1990), arising in the context of purely formal manipulation of symbols in the absence of any semantic realization of these manipulations. M*(N + P) = M*N + P is an example of a mal-rule. To exemplify the distinction between errors and misconceptions, consider the following cases: *Letters have place value* (Chalough & Herscovics, 1988; Herscovics, 1989; Perso, 1992) is a misconception. However, an answer “z = 8” to the question “if 1yz = 138 and y = 3, what is the value of z?” contains an error, namely that the answer is 8. Similarly, *a “+” or “-” sign must produce closed answer: conjoining open expressions* (Booth, 1988; MacGregor & Stacey, 1997a, 1997b) is also a misconception. However, $3x + 2 = 5x$ or = 5 is an error. As in these cases, errors usually are not random but reflections of students’ misconceptions regarding the actions.

Olivier (1992) interprets slips, errors and misconceptions in the following domain:

> I distinguish between slips, errors and misconceptions. Slips are wrong answers due to processing; they are not systematic, but are sporadically carelessly made by both experts and novices; they are easily detected and are spontaneously corrected. Errors are wrong answers due to planning; they are systematic in that they are applied regularly in the same circumstances. Errors are the symptoms of the underlying conceptual structures that are the cause of errors. It is these underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors that I shall call misconceptions. (p. 195)

According to Olivier, students’ erroneous thinking is an important part of learning process. Similarly, Nesher (1987) describes the students’ errors and misconceptions as the students “expertise, his contribution to the process of learning,” while she discusses the role of students in a learning situation according to their contribution of expertise. She builds an argument on the contribution of performance errors to the process of learning, indicating that errors and misconceptions do not originate in a consistent conceptual framework based on earlier acquired knowledge but rather are usually an outgrowth of an already acquired system of concepts and beliefs wrongly applied to an extended domain. Any future instructional theory will have to change its perspective from condemning errors into one that seeks them. Also, a good instructional program should predict the types of errors and purposefully allow them in the learning process. Teachers should be aware of their students' possible errors and misconceptions and incorporate them into their instructional considerations since they cannot fully predict the effect of the student's earlier knowledge system in a new environment.

Issues in Teacher Knowledge

Forms of Knowing

"Knowing" is a variety of separate entities. In their review of what "knowing" means from different authors' perspectives, Mason and Spence (1999) particularly focus on teacher knowledge and characterize it as a dynamic and evolving phenomenon. They mainly put forward three types of knowing which constitute *knowing-about* the subject: "knowing-that (factual), knowing-how (technique and skills), and knowing-why (having a story in order to structure actions and from which to construct actions)" (p. 135). However, *knowing-about* is not sufficient to respond in particular situations. Thus, an active knowledge, *knowing-to*, is desired that presents itself at the moment it is needed. In other words, *knowing-to* requires "relevant knowledge to come to the fore so it can be acted upon" (p.139). Mason and Spence say that *knowing-to* should be distinguished from other kinds of knowledge because its absence disables teachers (or learners) from responding and acting at the moment. And it is the knowing-to act that brings knowledge and practice together so the knowledge can be demonstrated to be useful or not. Mason and Spence also distinguish between understanding and acting. They claim that understanding does not require knowing-to act or vice versa. The knowledge might be inert rather than active as required by knowing-to act.

Renkl, Mandl, and Gruber (cited in Mason and Spence, 1999) gave three reasons to explain why knowledge is inert: "*metaprocess* (disturbance to accessing what is needed), *structure deficit* (aspects missing in what is known), and *situatedness* (mismatch between current situation and previous situations)" (p. 141). According to Mason and Spence, although the situation activates acts, those activations are based on personal psychology. Further, they claim that forms of justification and confirmation of knowing may lead to inert or active knowledge.

> *Knowing* which is confirmed only in external sources is not only much robust but also less likely to be brought to mind than knowing which can be constructed from structural elements, since it is those structural elements which are most likely to be triggered by novel situations. By contrast, sources of confirmation of *knowing-to* act begin and stay with personal judgment of appositeness of what came to mind, in the light of the consequences of actions, augmented by external suggestions for modification, praise, or criticism, of those actions. (p. 142)

Usually a sudden change in focus of attention marks knowing-to, and it is not apparent to the knower in the moment. Further, it is also difficult for one to be aware that one does not know to act in the moment. Someone whose awareness extends beyond the awareness of the observed, however, can recognize when the observed do not know-to act.

Knowledge for Teaching

Some of the major roles for mathematics teachers are defined in *Professional Standards for Teaching Mathematics* (NCTM, 1991) as follows:

- Setting goals and selecting or creating mathematical *tasks* to help students achieve these goals;
- Stimulating and managing classroom *discourse* so that both the students and the teacher are clearer about what is being learned;
- Creating a classroom *environment* to support teaching and learning mathematics;
- *Analyzing* student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions. (p. 5)

Fulfilling these roles is assumed to be the major part of becoming a good mathematics teacher. Students have the opportunity to learn mathematics that is meaningful and structural, as envisioned by NCTM (1989, 2000). Those high expectations characterize mathematics teaching as challenging. It demands:

- Knowing mathematics and school mathematics,
- Knowing students as learners of mathematics,
- Knowing mathematical pedagogy. (NCTM, 1991, 2000)

Shulman (1986) distinguishes three types of content knowledge domains relevant to teaching: *subject matter content knowledge*, *pedagogical content knowledge*, and *curricular knowledge*. Within each one, three types of knowledge categories are defined: *propositional knowledge*, *case knowledge*, and *strategic knowledge.* In a later discussion of teacher knowledge, Shulman (1987) lists the following components of teacher knowledge: subject matter content knowledge, pedagogical content knowledge, knowledge of other related content, knowledge of curriculum, knowledge of learners, knowledge of educational aims, and general pedagogical knowledge.

In his discussion of pedagogical content knowledge, Shulman (1986) listed *pedagogical content knowledge* (PCK) as an ignored but important category of knowledge. He conceptualized PCK as moving beyond mere understanding of the content knowledge and looking at it for teaching:

> Within the category of pedagogical content knowledge includes, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. (p. 9)

Shulman (1987) elaborated more on the concept of pedagogical content knowledge:

> Pedagogical content knowledge is the special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding.

> Pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue (p. 8)

> Pedagogical content knowledge is the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful yet adaptive to the variations in ability and background presented by the students. (p. 15)

In Shulman's views, PCK is mainly based on content knowledge and its transformation into understandable forms for students. Since PCK is combination of both content knowledge and pedagogical knowledge, the definition of PCK, by its nature, has ambiguities because it is hard to determine where one ends and the other begins.

In his historical review, Bullough (2001) discusses the treatments of ideas of PCK before Shulman and PCK's role in efforts to professionalize teaching and teacher education. He cites Gary Fenstermacher, who commented that it is probably the difficulty of differentiating and analyzing the epistemological frames of content knowledge and pedagogical knowledge that makes PCK. Further, the practical and formal aspects of PCK are mentioned as other sources of confusion. Although formal elements can be stated as propositional knowledge and thus can be taught in both preservice and inservice teacher education, practical aspects of PCK are context dependent. So preservice teachers may be learning something that will not come across in the context they will eventually be teaching in and causing the knowledge and ideas become "inert."

After Shulman, various authors (in different fields of education) used slightly different definitions of PCK. Bromme (1994, 1995) looks at the concept of teacher knowledge from a mathematics education perspective and elaborates more on Shulman's categories. He defined PCK as "the specialized didactic knowledge of teachers that illuminates the connection between subject matter and classroom know-how" (Bromme, 1995, ¶ 1). By differentiating the content and its philosophy between these in the discipline and these used for teaching purposes, he adds philosophical and psychological dimensions to knowledge categories in his topology of teacher knowledge. He identified five areas of knowledge that a mathematics teacher needs: "knowledge about mathematics as a discipline; knowledge about school mathematics; the philosophy of school mathematics; general pedagogical (and psychological) knowledge; and subject-matter-specific pedagogical knowledge." *Subject-matter-specific pedagogical knowledge* in Bromme's topology can be considered as corresponding to Shulman's PCK. Bromme (1994) defines it as:

> It is integrated knowledge cross-referring both pedagogical knowledge and the teacher's own experience to the subject-matter knowledge. This integration is exhibited, for instance, when the logical structure of the subject matter is reshaped into a temporal sequence. Further, it consists in changing the structuring and relative weight of concepts and rules; something that is of central importance from the viewpoint of mathematical theory may be accorded less weight from the perspective of teaching. (¶ 10)

In Bromme's view, Subject-matter-specific pedagogical knowledge must be developed through individual experiences. Bromme (1995) points out that PCK as a psychological construct should be differentiated from PCK as a didactic concept.

Critiques of Shulman's Analysis

Shulman's work is not without its critics and limitations. In a philosophical paper, "Two Problems with Teacher Knowledge," Orton (1993) suggests that there are conceptual problems associated with teacher knowledge and that Shulman's "pedagogical content knowledge" needs further analysis and discussion. Orton thinks that underlying research efforts to build a knowledge base lies in two political and social forces: one is to professionalize teaching and the other is to evaluate teaching. He suggests two areas of challenge in identifying a knowledge base for teaching; the "tacit problem" and the "situated problem":

> [The] "tacit problem" is that teacher knowledge appears to be primarily a form of knowledge how. In other words, the most credible justification for a teacher's knowing is the fact that he or she can do something in the classroom (e.g., create situations that

> enable students to learn)... [The] “situated problem” is that teacher knowledge is deeply dependent on particular times, places, and contexts, and lacks the general character of knowledge in mathematics, physics, or even psychology. Thus, it is difficult to formulate criteria which can be used to explain how a piece or instance of teacher knowledge might be justified. (¶ 2)

The situated problem, according to Orton, is methodological, and it belongs to a general discussion in educational research about the nature of justification and truth. It develops the sense that certain texts such as a video transcription, field notes, a narrative, a series of scores on an observation instrument, or some other artifact can be taken as a representative of teacher knowledge. Then the problem becomes one of interpretation of how the text demonstrates teacher knowledge. In this sense, the solution to the situated problem requires an acceptance or willingness to discuss whether a given artifact shows teacher knowledge and, if it does, to seek an interpretation. The tacit problem, on the other hand, is more complex and challenges the idea of a knowledge base for teaching. Assuming that good teachers have an inclination, skill, or “know how” to be effective (and that cannot be reduced to “know that”), the tacit problem arises from the argument that

> One still needs to make an argument connecting the causes of a complex skill, which is a psychological or descriptive matter, and the reasons why a complex skill is judged to be effective, which is an intellectual or logical matter. Such arguments are exceedingly difficult to make, and one might suppose, in the spirit of Ryle, that they rest on a categorical mistake of confusing intellectual and psychological representations of knowledge. (Orton, 1993, “Teacher Knowledge as Tacit (Whether Situated or Not)” section, ¶ 10)

In his response to Orton (1993), Ennis (1993) does not agree that the tacit problem is a theoretical problem, because there is propositional knowledge related to knowledge how and this and know-how itself add to the knowledge base for teaching and its roles. Further, Ennis argues that most teacher knowledge is general, and so Orton’s situated problem is not valid. Using nonauthentic tests in high-stakes cases or overgeneralizing without enough prudence should be taken into consideration in all cases.

Wortham (1998) gives a brief summary of a discussion between Shulman (1987) and Sockett (1987) in the *Harvard Educational Review* in 1987 in which Shulman argued the role of knowledge in good teaching whereas Sockett reacted that knowledge and skills cannot be the major agencies for good teaching because of moral action involved in particular cases. According to Wortham, Shulman and Sockett agreed on the fact that good teachers use "judgment" to react to particular situations in the classroom; however, they differ on the nature of judgment. For Sockett, teachers judge and respond to situations involving moral aspects, and propositional statements cannot capture this. That is why teacher judgment cannot be abbreviated as propositional knowledge or principles for practice. On the other hand, while recognizing the moral dimensions in some aspects of teaching, Shulman claimed that knowledge and rational thinking are at the heart of teacher judgment and classroom practice.

Taking Sockett's side, Wortham (1998) takes a critical approach to Shulman and tries to argue that knowledge and action cannot be extricated easily; they are entangled in verbal practice. He argues that a dialogic approach evinces knowledge and action permeating the classroom, whereas knowledge-based approaches are not sufficient to describe the complicated nature of social interactions in a classroom.

In his response to Wortham (1998), Duncan (1998) points out the importance of tacit knowledge as a part of teaching. Contrary to Sockett and Wortham's agreement about the harm done by isolating systematized knowledge from practice because of losing something in the analysis, Duncan supports Shulman's articulation of tacit knowledge. Because teaching is a complex phenomenon, it is hard to define. It takes time to learn it and becomes second nature, which also makes it hard to define and articulate. Although teachers need to learn what works best for them— something that can never be learned through a set of procedures or guidelines— a well-vocalized set of guidelines and well-documented research are a vital part of teacher education programs. And when learning how to teach, teacher should focus on things that can be vocalized in a systematic way.

Teacher Knowledge and Student Thinking

Knowledge of student thinking is an important part of the knowledge base required for teaching (Bromme, 1994, 1995; NCTM, 1991, 1995, 2000; NBPTS 1998; Shulman, 1986, 1987). Shulman (1986, 1987) identifies knowledge of specific learning difficulties and student conceptions as a crucial part of PCK for teachers.

> [It] also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If those preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners, because those learners are unlikely to appear before them as blank states. (Shulman, 1986, p. 9-10)

For effective teaching, teachers are expected not only to know the concepts with which students have the most difficulty but also to seek ways to reveal those misunderstandings so that they can act accordingly.

> They need to know the ideas with which students often have difficulty and ways to help bridge common misunderstandings.… Effective teachers know how to ask questions and plan lessons that reveal students' prior knowledge; they can then design experiences and lessons that respond to, and build on, this knowledge. (NCTM, 2000, p. 17)

> They are aware of the preconceptions and background knowledge that students typically bring to each subject and of strategies and instructional materials that can be of assistance. They understand where difficulties are likely to arise and modify their practice accordingly. (NBPTS, 1998, p. 2)

In setting national standards to certify deserving teachers, the NBPTS (2003) policy statement *What Teachers Should Know and Be Able to Do* characterizes awareness of student thinking as a part of subject-specific knowledge:

> Subject-specific knowledge also includes an awareness of the most common misconceptions held by students, the aspects that they will find most difficult, and the kinds of prior knowledge, experience and skills that students of different ages typically bring to the learning of particular topics. (NBPTS, 2003, Proposition #2, ¶ 7)

Further, NBPTS (1998) attributes successful instruction to teachers' ability to evaluate and foresee related student thinking:

> Teachers' instruction succeeds, in part, because of their ability to assess and anticipate students' difficulties, understanding, and misconceptions, and yet build on students' strengths. They view instruction and the learning process from a student's perspective and are thereby able to understand a student's misconception, identify the underlying rationale, and clarify the student's thinking. (p. 15)

Teacher Beliefs

Beliefs are "mental constructs that represent the codifications of people's experiences and understandings" (Schoenfeld, 1998, p. 21). Although I set my main goal in this study as an exploration of teacher knowledge, I also wanted to look at teachers' beliefs, at least in a general sense, because a belief "speaks to an individual's judgment of the truth or falsity of a proposition" (Pajares, 1992, p. 316) and "unexplored entering beliefs may be responsible for the perpetuation of antiquated and ineffectual teaching practices" (p. 328). As Schoenfeld (1998) says, "People's beliefs shape what they perceive in any set of circumstances, what they consider to be possible or appropriate in those circumstances, the goals they might establish in those circumstances, and the knowledge they might bring to bear in them" (p. 21). After exploring the effects of teachers' knowledge of mathematics, Ernest (cited in Pajares, 1992) concluded that teachers who have similar knowledge may teach in different ways because of the effect of their beliefs on their decision making. In general, beliefs have a strong impact on teachers' actions (Cooney, 1985; Fennema & Franke, 1992; Thompson, 1992; Vacc & Bright, 1999).

The line between beliefs and knowledge might be fuzzy, and teachers often treat their beliefs as their knowledge (Pajares, 1992; Thompson, 1992). The distinction between both has been mentioned and analyzed in detail (see Pajares, 1992; Thompson, 1992). In a short summary, it can be said that beliefs might have different degrees of conviction and so are open to change and may not need to be verified, unlike the case of knowledge (Thompson, 1992).

"Beliefs cannot be directly observed or measured but must be inferred from what people say, intend, and do—fundamental prerequisites that educational researchers have seldom followed" (Pajares, 1992, p. 207). Schoenfeld (1998) warns that one should distinguish between beliefs that are *professed* and beliefs that are attributed to underlie actual behavior. Teachers do not always act as they say or say as they act (Thompson, 1992). A contradiction may exist between what teachers state they believe and what they do (Cohen 1990; Cooney, 1985). As an explanation for that contradiction, Schoenfeld (1998) suggests the effect of the current context that activates and prioritizes the beliefs, knowledge, and goals.

Although it may not be possible to know what teachers truly believe, Schoenfeld (1998) suggest looking at the following beliefs affecting teachers' actions in any study of teaching:

- beliefs about the nature of subject matter (in general and with regard to the specific topics being taught);
- beliefs about the nature of the learning process (both cognitive and affective);
- beliefs about the nature of the teaching process and the roles of various kinds of instruction;
- beliefs about particular students and classes of students. (p. 23)

For the purposes of this study, I consider those suggestions in terms of student thinking and related teacher beliefs affecting their knowledge, goals, and decisions.

Teacher Goals

A goal is something one intends to achieve in a period of time, whether long, medium or short. According to Schoenfeld (1998), what teachers do and how they do it at any stage of instruction are shaped by their *currently active, high priority* goals. Exploring teachers' goals as well as knowledge and beliefs is essential as they are a reflection of their beliefs and knowledge. According to Bransford, Brown, and Cocking (2000), examining teachers' goals are necessary as they are connected to teacher knowledge and beliefs.

> Teachers' ideas about mathematics, mathematics teaching, and mathematics learning directly influence their notions about what to teach and how to teach it—an interdependence of beliefs and knowledge about pedagogy and subject matter (e.g., Gamoran, 1994; Stein et al., 1990). It shows that teachers' goals for instruction are, to a large extent, a reflection of what they think is important in mathematics and how they think students best learn it. Thus, as we examine mathematics instruction, we need to pay attention to the subject-matter knowledge of teachers, their pedagogical knowledge (general and content specific), and their knowledge of children as learners of mathematics. Paying attention to these domains of knowledge also leads us to examine teachers' goals for instruction. (p. 164)

Schoenfeld (1998) points out four issues central to the discussion of goals as a factor in teachers' decision making. The first one is the necessity of making a distinction between *attributed* and *professed* goals, which in the case of beliefs and attribution should be made with as much triangulation as the researcher can provide. The second issue is that goals can occur in different sizes and activation levels for varying amounts of time. The third issue is the possibility that goals can be *emergent* as well as *pre-determined*. The fourth issue is that there is no basic correspondence between actions and goals; teacher can have a flock of active goals with various levels of activation.

CHAPTER 3: REVIEW OF LITERATURE

This chapter constitutes the literature review for this dissertation study in which I observed and interviewed two teachers of Algebra 1 to investigate and understand their knowledge of student thinking and their instructional practices in algebra.

Teachers' Conceptions and Knowledge of Student Thinking in Algebra

Teachers' Conceptions in Algebra

Although a rich and extensive body of research focuses on students' understanding and learning in algebra, few studies have investigated teachers' conceptions in algebra, and those studies typically have investigated the concept of function held by practicing teachers (Hitt, 1994) and prospective teachers (Bishop & Stump, 2000; Even, 1993; Stump, 2001; Wilson, 1994).

Hitt (1994) studied teachers' concepts of functions to investigate whether teachers had misconceptions about the construction of functions. Using questionnaire data collected from 117 mathematics teachers, he found that teachers had a tendency to think in terms of continuous functions with little proficiency in constructing such functions. They barely considered the alternative, discontinuous functions. Hitt concluded that historical perspectives could be effective in making clear the continuity of functions.

Even (1993) studied prospective secondary mathematics teachers' content knowledge and pedagogical content knowledge of the concept of function. She collected questionnaires from 152 prospective teachers in their last stage of formal education to capture teachers' subject matter knowledge about functions and their analysis of students' mistaken solutions. She then interviewed ten other pre-service teachers coming from diverse backgrounds as a follow up. She found that the pre-service teachers did not have a modern concept of function. Those who used a modern definition of a function did not use the same modern definition when approaching student difficulties. Even's elaboration at this point is relatively short. She gave an example of a teacher who used a "1-1 mapping" type of definition for a function but confronted with student difficulty changed it to generating a second set of numbers from a starting set by using four operations, assuming students would not understand what a mapping was. Furthermore, Even found that although a few students used the "vertical line test" in defining a function, many provided it as a rule to students in order to get the right answer even without understanding it. Another reason the preservice teachers for provided the line test was that their teachers had taught them in the same way when functions were taught. Even recommends that pre-service teachers be given more quality (not quantity) of subject matter preparation aligned with reform oriented views of teaching and learning.

Wilson (1994) examined the evolving knowledge and beliefs of a pre-service secondary mathematics teacher in a mathematics education course that emphasized pedagogical connections between multiple representations of the function concept. Although the prospective teacher's understanding of the function concept had changed significantly during the semester, her anticipated approach to teaching did not because of her insufficient content knowledge and narrow views of mathematics. Wilson suggested that in the preparation of pre-service teachers, mathematics content and pedagogy should be integrated to allow pre-service teacher to reflect upon a deeper understanding of the connection between multiple presentations (i.e., symbolic, graphical, and tabular) of functions.

Bishop and Stump (2000) used a framework with categories of procedural and conceptual perspectives to examine 32 pre-service elementary and middle school teachers' conceptions of algebra at the beginning and end of two separate algebra classes. They found the majority of the pre-service teachers gave either a non-algebraic or a procedural definition of algebra. Although more pre-service teachers with a conceptual perspective stated a problem-solving position, at the end of the semester more took a generalization view. Moreover, the pre-service teachers expressed a narrow understanding and appreciation of the generalization perspective in algebra. Bishop and Stump suggested that even after two courses emphasizing conceptual approaches to algebra, many of the pre-service teachers did not have an understanding of what distinguished arithmetic from algebra, and a majority of those who did had a procedural perspective of algebra.

Stump (2001) examined the development of three pre-service teachers who participated in a secondary mathematics methods course and then taught a basic algebra course regarding their knowledge of students' difficulties with slope and their representations for teaching slope. Stump collected the data from written assignments, interview transcripts, and transcripts of the algebra lessons. His findings suggested the teachers developed sensitivity to students' procedural and conceptual knowledge of slope. They also extended their knowledge of representations for teaching the concept of slope and provided meaningful problem-solving activities for their students. The biggest difference among the teachers was in their conceptions of slope as a measure of steepness and as a measure of rate of change. Furthermore, they also differed in using real-world situations: One used them to engage students in problem solving; another used them as illustrations of slope. Also, whereas one teacher focused on physical situations, another stressed functional situations. Stump concluded that the pre-service teachers' limited exposure to nontraditional curriculum materials might be the reason they had for developing a weak knowledge of slope in real-world settings. Greater exposure to those materials might further develop their pedagogical content knowledge. The results showed their ability to develop worthwhile lessons involving slope.

Impacts of Teachers' Conceptions in Algebra

Insufficient subject matter knowledge is a concern since it could hinder the teaching and learning of algebra. As Even and Tirosh (1995) put it, "Insufficient subject-matter knowledge, on the part of teachers, does not seem to be a sporadic, infrequent phenomenon, but rather a widespread one whose consequences for the actual teaching should be investigated" (p. 6). Stein, Baxter, and Leinhardt (1990) investigated the effects of limited subject-matter knowledge in a case study. They questioned a fifth-grade teacher, highly recommended as an excellent mathematics teacher with 18 years of experience, on his mathematical knowledge and educational ideas concerning the concept of function. Observations of videotaped recordings of his teaching revealed a connection between his subject-matter knowledge and his teaching. The teacher presented a limited interpretation of function as a calculating rule. He did not express any modern notion of a function even in simple terms; for example, as a mapping of quantities upon one another or one element in the domain being assigned to a corresponding element in the range. Although the teacher's limited concept of function did not lead to a false statement about functions in the classroom, it weakened development of the subject matter in class in three ways: "(a) the lack of provision of groundwork for future learning in this area, (b) overemphasis of a limited truth, and (c) missed opportunities for fostering meaningful connections between key concepts and representations" (p. 659).

Chazan (1999) reflected on his subject-matter knowledge and the ways it influenced his Algebra 1 instruction to illustrate the ways in which subject-matter knowledge might differ in the resources it provides for the support of student-centered instruction. He argued that a function-based approach to teaching algebra provided him essential resources for encouraging and supporting student investigation, whereas these were missing when he taught with a textbook that used a traditional number- and symbol-based approach. Chazan argued that the function-based approach to algebra allowed him to help students value what the course was about and how it was related to the world around them (p. 143). Furthermore, his use of various canonical representations of functions supported students with important resources for independent problem solving (p. 143).

Llinares (2000) examined the relationship between pedagogical content knowledge and the dilemmas posed when a secondary mathematics teacher with 17 years of experience in Spain taught the concept of function. Llinares conducted videotaped classroom observations and contextual and biographical interviews in which the teacher was asked to classify and analyze textbook problems, analyze student answers to some hypothetical questions about the function concept, analyze critical incidents encountered in the videotapes, and answers questions about her planning of lessons involving functions. The study revealed the interrelation between teacher subject matter knowledge and knowledge of how students "manage" to learn mathematical topics. In other words, the teachers' knowledge of mathematics supported her knowledge about how children understand mathematical concepts and procedures. The teacher's knowledge of functions enabled the teacher to emphasize that her students should build a meaning for functions from object and process perspectives and a link between algebraic and graphical representations of a function. Llinares found that the teacher had the knowledge about students' conceptions of functions, what made the learning of the concept of function easy or difficult, and necessity of taking advantage of this knowledge in teaching functions by providing tasks that might reveal difficulties and create discourse. The teacher presented a flexible notion of teaching functions and algebra by using multiple representations and linking geometric and algebraic concepts. Llinares suggests that the teacher's flexibility was supported by three components of her knowledge:

> (1) knowledge about mathematical content (key ideas of the concept of function in relation to the educational level in which is it found, curricular goals);
> (2) knowledge of different modes of representation and the role that may be played by the translation processes in the learning of some mathematical concepts (Mathematics-specific pedagogical content knowledge);
> (3) knowledge about the pupils' understanding of the mathematical concept of function (knowledge of mathematics learner). (p. 55)

In contrast, although the teacher's conception of learning processes was aligned with constructivist perspectives, it created dilemmas in the teaching context such as an understanding-memorization dilemma that arose from a discussion of the difference between a problem and an exercise and their place in teaching. Llinares concludes that mathematics-specific pedagogical content knowledge stems from a "cognitive integration" of knowledge of mathematical content, knowledge of multiple representations, and knowledge of students.

Even, Tirosh, and Robinson (1993) studied an expert and two novice teachers to examine differences in the connectedness of their instruction on equivalent algebraic expressions by analyzing lesson plans, observations, and post lesson interviews. Even et al. argued that teachers should help the learner construct interconnections between concepts, representations, topics, and procedures for conceptual understanding. Their results showed that only the expert teacher used both lessons and content connectedness to guide the teaching by thoughtfully building in connections between lesson segments and series of lessons. They concluded that teachers need to have a deep and comprehensive knowledge of and about mathematics to make connections and also need to develop their pedagogical content knowledge.

Research on teachers' use of reform-based algebra curricula suggests that teachers do not implement curriculum materials in a straightforward manner (Haimes, 1996; Lloyd, 1999). Haimes's (1996) findings suggest that teachers' knowledge and beliefs affect both the content and the instruction, influencing both what they teach and how they teach. Haimes studied a veteran Australian algebra teacher's implementation of a reform-oriented curriculum in which she implemented a function approach to introductory algebra. The study illustrated how teachers do not necessarily follow the intended curriculum and suggested that the impact was minimal. The teacher presented a limited view of mathematics and a limited appreciation of the idea of teaching algebra for developing problem solving and algebraic reasoning. She envisioned the goal of algebra at this level as the development of skills necessary for further study of mathematics. Even though the curriculum suggested ways of approaching the topic, the teacher simply showed procedures in an effort to cover the curriculum. Her practices presented a teacher-centered rather than a student-centered pedagogy. Haimes suggested a further examination of the difficulties facing teachers in reforming traditional practices. Professional development experiences providing meaningful support materials and training are needed if successful implementation of the approach is desired and teachers are to see algebra as more than a collection of procedures.

Lloyd (1999) studied two veteran high school mathematics teachers' conceptions of a reform-oriented mathematics curriculum, Core-Plus mathematics, units including functions and algebra, in terms of its cooperation and exploration components. Lloyd found that the teachers differed in their interpretation and implementation of the curriculum although both valued the theoretical aspects of the curriculum that emphasized solving problems with cooperation and exploring problems situated in real-world settings. Although one teacher perceived the problems as open to student interpretation (i.e., open-ended) and overly challenging, the other one found the problems too structured. Each teacher attributed difficulties with students' cooperative work to the amount of structure and direction offered by the problems, indicating that both teachers were more concerned with the structure of the problems than students' responses and thinking. In one case, the teacher responded to student questions directly instead of asking them to reflect and make decisions by themselves. Moreover, both teachers expressed concerns about classroom authority and management. Lloyd was particularly puzzled by the fact that the culture of the mathematics departments had shaped the teachers' perceptions and practices in such a way that they did not change the curriculum problems and activities to better suit their personal goals. Lloyd concluded, "When teachers feel restricted in their attempts to make adjustments to the curriculum, they may face difficult challenges in adapting the curriculum to suit the needs of their students and best fit their own goals and strengths" (p. 246). She also suggested that the curriculum materials themselves posed a constraint to the teachers: "When a reform-minded teacher uses traditional materials in the classroom, he or she may be afforded more room for personalization because the goals of the materials are so different from his or her own goals" (p. 246). Lloyd concluded that through professional activities teachers should be assisted in making sense of deviation between the reform curriculum and their previous experiences as classroom teachers to better prepare themselves, and interact with the curriculum dynamically.

Teachers' understanding of the mathematical content of an innovative reform-oriented curriculum influences the implementation and related results in the classroom. Teachers who have limited mathematical understanding face difficulties when they attempt reform (Heid, Blume, Zbiek, & Edwards, 1999), whereas teachers who have robust mathematical understanding can create opportunities to highlight important connections, meaningful representations, and meaningful discussions (Lloyd & Wilson, 1998).

Heid et al. (1999) examined three high school mathematics teachers' use of interviews to understand their students' mathematical understanding as they learned to use task-based interviews and technology-intensive algebra curricula. The instruction was organized around the concept of function and mathematical modeling. They studied the teachers' mathematical understanding, knowledge of technology, and perceived importance of curriculum topics; their views of knowing mathematics; their perceptions of students' characteristics and needs; and teachers' perceptions of interviewing and the role of questioning in interviews. Each of the three teachers had a different lens to define how they saw interviews and how they viewed the understanding that those interviews revealed. One teacher had as her lens the self-concept image for the mathematical concepts she was assessing. The second one used as a lens the mathematics curriculum that was the goal of her instruction, and the third's lens was the set of skills students should be able to perform. The teachers' views of learning and knowing mathematics affected their approaches to interviewing; none of the teachers indicated any belief that learning is an individual product of personal experience and ideas. Each teacher used interviews to compare student's understanding with some standards to test whether students knew what they had been taught. The follow-up questions the teachers used did not address understanding or probe for the deeper conceptual understanding seen by the researchers. The teachers used follow-up questions when they wanted students to consider certain features of a concept, to direct students towards the correct answer, and to keep the interview moving along and going smoothly.

Lloyd and Wilson (1998) used the case of an experienced high school teacher to study the teacher's content conceptions and link those conceptions to their role in the teacher's first implementation of the Core-Plus Mathematics curricular materials during a 6-week unit on functions. The teacher's flexible content conceptions allowed him to emphasize on conceptual connections, powerful representations, and meaningful discussions in his implementation of the unit. Furthermore, the teacher's robust and integrated understanding of the function concept, graphical proficiency, and personal focus on patterns of covariation in particular enabled him to effectively implement the curriculum material. Lloyd and Wilson suggested the teacher's case provided evidence that an insightful understanding of concepts can be isolated from traditional conceptions about how to teach them and can furthermore lead to the development of new pedagogical content conceptions.

Teachers' Knowledge of Student Thinking

Falkner, Levi, and Carpenter (1999) found that when they presented the problem "8 + 4 = □ + 5" to a six-grade teacher to give her students, she replied, "Sure, I will help you out and give this problem to my students, but I have no idea why this will be of interest to you" (p. 232). Finding that all 24 of her students in one class answered 12, she gave the problem to all 145 sixth-grade students in the school and found that those students thought that either 12 or 17 was the answer (p. 232). This example is an indication of teachers' limited knowledge of and about student thinking in algebra.

Teachers rarely explicitly study or give attention to students' conceptions and ways of thinking in mathematics. Therefore, they are not knowledgeable about these topics. Research on teachers' reactions to student thinking and conceptions has shown that the majority of teachers judges the students' answers only in terms of being right or wrong and provides the students with the teacher's own explanation for the right answer. Many teachers make no attempt to understand the sources of students' responses nor could explain why students would respond in certain ways when they were directly asked (Even & Markovits, 1995; Even & Tirosh, 1995). Even and Markovits found that teachers do not pay much attention to students' reasoning and knowledge construction; instead, they try to explain students' reasoning and difficulties from their own perspectives.

Teachers' knowledge of students' thinking can be revealed and improved through an insightful review and consideration of students' work (Hallagan, 2003; Miller, 1992; Miller & England, 1989). Miller and England (1989) and Miller (1992) studied the effect of the use of impromptu writing by algebra teachers. Miller and England (1989) reported the effect of regular reading of student writing on three experienced (20, 18, and 2 years) in-service teachers' awareness of student difficulties in and attitudes toward algebra. Contextual, instructional, reflective, and miscellaneous writing prompts were developed, and students were required to write clearly about their understanding of a mathematical concept, skill, or generalization as well as their feelings about how the class was going for a minimum of 5 minutes on at least 4 out of every 5 instructional days. The teachers were required to spend time reviewing the students' writing each week. The findings revealed that teachers gained insight into their teaching of algebra, could identify misconceptions and patterns of errors, and retaught the problematic concepts immediately without waiting until after an examination, and could observe the gaps in student performance. The two more experienced teachers pointed out that the less experienced one gained more about teaching and learning of algebra and students through reading students' writing than they would have in several years. Furthermore, Miller and England found that students' and teachers' attitudes toward each other and about the teaching and learning of algebra were improved.

In another analysis of the same research project, Miller (1992) examined the benefits of using impromptu writing by teachers in first- and second-year algebra classes to examine what the teachers could learn about their students' understanding of mathematics by reading their written responses to writing prompt. Miller also studied how their instructional practices were affected as a result. The teachers were surprised by students' writings in which they were not able to articulate concepts that the teachers assumed the students understood. Furthermore, the teachers changed their instructional practices in at least five ways: reteaching immediately, delaying an examination because of a revealed lack of understanding, designing a review based on information from student writing, initiating private discussions with individuals who had misconceptions, and using prompts during a lesson to ascertain understanding. Moreover, the teachers realized they needed to be very explicit and provide examples when everyday language was used in a mathematical context. In one case, talking to a student whose writing was interpreted as indicating a misunderstanding revealed that the student had misrepresented in writing what she meant. Thus, Miller recommended that teachers be sensitive to students' ability to express themselves in writing.

In a dissertation research study, Hallagan (2003) studied middle school mathematics teachers' models of their students' algebraic thinking. She used model-eliciting activities to disturb and reveal teachers' thinking through "Ways of Thinking" sheets based upon students' responses to selected middle school algebraic tasks from the Connected Mathematics Project. Three male teachers with teaching experience ranging from approximately 10 to 15 years and two female teachers with teaching experience ranging from 1 to 18 years from two urban middle schools participated in the study. Data were collected from classroom observation, artifacts of model-eliciting activities, semi-structured interviews, and team discussions. The findings suggested that five aspects are central to teachers' models of student responses to tasks concerning equivalent expressions and the distributive property. "Teachers recognized that students: (a) tended to conjoin expressions, (b) desired a numerical answer, and (c) had difficulty writing algebraic generalizations. In addition, teachers identified that (d) visual representations were highly useful as instructional tools. And finally, (e) the teachers in this study needed more experience in analyzing and interpreting student work" (p. 256). The findings were consistent across the Ways of Thinking sheets, library of student works, individual and team interviews, and classroom observations.

A detailed analysis focusing on teachers' knowledge of a particular students thinking and teaching approaches in algebra was given by Tirosh, Even, and Robinson (1998) in their investigation of novice and expert teachers' awareness of students' tendency to conjoin algebraic expressions (i.e., $3x + 2 = 5x$ or $= 5$) and the attributed sources for it when teachers' aware of the situation. For all the four teachers, the main teaching approach in some versions was "collecting like terms." Although the two novice teachers used that method as soon as they introduced the topic of simplifying algebraic expressions, one of the experienced teachers, in contract, introduced like and unlike terms and gave students enough time to practice and master "collecting like terms" before she introduced the topic of simplification. Tirosh et al. saw this approach (i.e., skills presented to students as rituals to be practiced) as problematic. It is also closely related to the teachers' judgments of importance of skills in mathematics. Another approach, used by one novice teacher, was called the fruit salad approach (i.e., using fruit names to interpret an algebraic expression such as interpreting 4a as four apples). Tirosh et al. saw this approach problematic in most ways, because it may lead to some other wrong generalizations while it would be useful in some cases.

Affect of teachers' conception of students thinking was studied on a large scale in Nathan and Koedinger (2000a, 2000b). Suggesting that teachers' beliefs about student thinking affect their instructional practices, Nathan and Koedinger (2000a) studied a group of mathematics teachers' (*n* = 67) teaching 7th through 12th grades and mathematics education researchers' (*n* = 35) views and beliefs on difficulty levels of a set of algebra problems. Contrary to teacher's expectations, students found algebra story and word problems less difficult than solving a symbolic-equation problem. Having found that high school teachers inaccurately predicted high school students' problem solving performance and thus misjudge their symbolic- and verbal-reasoning abilities, Nathan and Koedinger concluded that those teachers were dominated by *symbol-precedence view* of student mathematical development, "wherein arithmetic reasoning strictly precedes algebraic reasoning and symbolic problem solving develops prior to verbal reasoning" (p. 209). According to Nathan and Koedinger, the results suggested a significant level of agreement between teachers' views and the organization of algebra textbooks.

Nathan and Koedinger (2000b) designed a parallel study where they had a larger group of in-service teachers from elementary, middle and high school. The purpose of the study was to investigate the accuracy of teachers' beliefs about relative difficulty of solving various algebra problems and how teachers' judgments were influenced by their general beliefs about mathematics teaching and student learning. There were 105 research participants from 2nd grade to 12th grade teachers all from the same district enrolled in an obligatory school district-sponsored workshop on primary and secondary mathematics teaching. For all grade levels, teachers ranked arithmetic problem as more difficult than matched algebra problems. Moreover, on average, teachers ranked verbal problems as more difficult than symbolic problems and word-equation problems as harder to solve than story problems (p. 223). On the other hand, high school teachers were least accurate in their prediction of student performance in problem solving whereas middle school teachers were the most accurate in their predictions based on student data from authors' previous studies with students. Nathan and Koedinger found that, in general, high school teachers were least likely to agree with reform views expressed in the survey in contrast to elementary school teachers who expressed the strongest agreement and middle school teachers falling midway on each construct. Although majority of the teachers participated in the study expressed reform-based views of mathematics learning and instruction, they did not seem affected by these particular beliefs when they were asked to judge how students would perform on a set of algebra and arithmetic problems. As in Nathan and Koedinger (2000a), a symbol-precedence view was concluded to be the dominant belief affecting teachers' views of problem difficulty. Grade level was concluded to be more important factor than symbol-precedence in determining problem difficulty. Since formal representations and solution methods are increasingly emphasized from elementary to middle to high school, and since high school teachers tend to have greater expertise in their content areas, Nathan and Koedinger concluded that they were least aware of difficulties of their novice students and they were usually the ones to least agree with reform views.

The present literature suggests and supports the existence of strong relation between content knowledge and pedagogical content knowledge. This is more apparent in the case of knowledge of student thinking. Teachers having more strong and conceptual knowledge of mathematics are more likely to provide insight into student difficulties than ones who have not. Studies found teachers would have the same type of misconceptions or concept deficiencies that students might have. This means they would not recognize and attempt to solve it once students are present. In the worst case, they even may transmit the same ill conceptions to pupils. Pre-service teachers, in general, present inadequate or lack of knowledge of student thinking. This would probably be due to not having worked with students or not having taught the subject. It may also reflect the mathematics they typically study in preservice courses. Thus, prospective teachers need to experience actual student work in various ways in their mathematics and mathematics education courses. In-service teachers in several studies (e.g., Stump, 2001) were found to present the knowledge of student difficulties and particular mistakes in terms of knowing-that, however, they fail to give meaning to it in terms of sources of those conceptions, which is knowing-why perspective of professional knowledge. Furthermore, those studies suggest the need for further study of teacher knowledge of student thinking and its affect.

Teaching and Learning of Radical Expressions

Farah-Sarkis (1993) examined categories of misconceptions appeared frequently in 160 eleventh grade students' written tests on powers and radicals. Results showed that 35% or more of the students gave incorrect answers to most of the test items dealing with radicals. Students particularly had difficulties with problems contained a negative number under the radical. Student interpreted higher order roots as square roots resulting in incorrect answers (e.g., $^4\surd 16 = 4$). Many students considered that it was impossible to take third degree root of a negative number (e.g., $^3\surd{-27}$). Furthermore, the negative number under the square root was also caused incorrect interpretations and results. Many students responded $\surd(-4)^2$ as –4 indicating that students possibly as a result of a wrong definition $\surd a^2 = a$. This situation has been identified as one of the serious issues related to teaching and learning of radicals in literature.

Kepner (1974) discussed the handling and mishandling of the definition $\surd a^2 = |a|$. Kepner mentioned that in order to argue the feasibility defining $\surd a^2$ as $|a|$, teachers generally use an example like "$\sqrt{(-2)^2}$ = ?" and get students recognizing that both 2 and –2 are the solutions leading to a case that is not mathematically practical. Kepner further explained that after a while using this definition within a set of exercises, teachers usually proceed to perform all the calculations as $\sqrt{a^2} = a$ assuming that all the variables are positive until the end of chapter review where they use the previous definition again. Kepner argued that this practice inculcates a false sense of security in both students and teacher. They begin to think that $\sqrt{a^2} = a$ is valid for every real number a, constantly abusing the real definition. To make his argument, he presented two examples of some of the gross errors he encountered even in textbooks resulting from the misuse of this definition where one assumes a as positive without any reason. The first example is the derivation of the quadratic formula by completing to a square of quadratic equation. A second example concerned the derivation of the relation between sum and product of roots of a quadratic and its coefficients. In both cases, Kepner argued that students and teachers might easily assume a variable as positive without any reason and continue to use $\sqrt{a^2} = a$ instead of considering $\sqrt{a^2} = |a|$ and the possibilities that $a < 0$ or $a > 0$.

Abramovitz, Berezina, and Berman (2002) discussed the importance of using the correct definition $\sqrt{a^2} = |a|$ in their reflection upon their teaching mathematics to engineering students in Israel. They discussed presenting wrong proofs to students or provoking them to make mistakes and then explaining the mistakes could be effective in improving students' understanding and may prevent them from making similar ones in the future. One of the mistaken claims they mention was 10 = 0 with a proof followed as $(-5)^2 = 25 = 5^2 \Rightarrow \sqrt{(-5)^2} = \sqrt{5^2} \Rightarrow -5 = 5$ and thus 10 = 0. Obviously the mistake in the proof is the claim that $\sqrt{(-5)^2} = -5$. Abromovitz, Berenzina, and Berman reported that when they provided this example to high school students in an enrichment program and first-year engineering students, the advanced students and the enrichment program students did not have a problem in finding the mistake while the less advanced students needed some explanations. Eventually, once understood, all the students realized the importance of the concept of absolute value in "$\sqrt{a^2} = |a|$."

Atherton (1971) reflected on his experience as a high school teacher with solving absolute-value equations. He indicated students' difficulty in attempting to solve inequalities and absolute value problems. Students can sometimes get half of the answer but they have more difficulty when the numbers in absolute-value symbols are negative. He made suggestions on a method of solving absolute value problems promoting an appreciation of the domain concept.

Barnard (2002) suggests that at the center of various errors is the inability to conceive the objects of manipulation (e.g., $2x + 1$, a^2, $\sqrt{t^2+1}$) as meaningful things in their own right and look for a closure. Such failure to accept might obviously result in providing $(a + b)$ as an answer to $\sqrt{(a^2 + b^2)}$. Barnard further mentions that $\sqrt{a} + \sqrt{b} = \sqrt{(a + b)}$ is a common mistake. He recommends that teacher may show it does not work with numbers (i.e., a numerical example). He also notes, however, that this remedy may not be long-lasting if the abstract expressions are not meaningful to the student.

Comiti and Grenier (1995) argued that teachers plan their lesson considering the images they have of their discipline, the content to be taught, the transfer of knowledge and their students' ways of learning. However, in practice, classroom interaction can take different turns because of unforeseen events resulting in not being able to apply the original plan. One of those unforeseen events can occur if teacher considers that his/her students acquired a certain knowledge he/she tried to transmit but in fact it has not been acquired. Comiti and Grenier investigated two teachers' practices in teaching of square root in two classes of 14-15 year olds in order to present different types of events unplanned by the teachers. They reported that the first teacher followed an arithmetic introduction and presumed that students would produce pairs of numbers (a, b) where b is positive and equal to a^2 using the definition of the square of a number and the properties of multiplication and sign rule in integers. Although the teacher planned 10 minutes introduction and reminding about properties of squares, she ended up spending more than half an hour to convey students that although $-(1)^2 = -1$, which came from one of the students, was a correct representation, it had nothing to do with her question. Comiti and Grenier characterized this situation as “a split situation” where “the student does not understand what the teacher expects of him, and the teacher does not understand the answer the student gives” (p. 20) and teacher tried to obtain the answer she was expecting from the “good” student who functioned in the same situation as the teacher did. Comiti and Grenier reported that the second teacher followed a geometrical introduction in which two exercises were considered to get students to investigate that when the side lengths of a square is doubled, its area is not twice as much, but four times as much and the area of a square of a side length a is equal to a^2. Interpreting the problem as drawing the desired object as accurately as possible, students functioned in a context of drawing and measurements instead of geometric constructions, which was what the teacher desired. Thus, Comiti and Grenier interpreted this situation as another example of “a split situation.”

Teaching and Learning of Graphing and Graphs

The traditional algebra curriculum is organized around the concept of equation and equation solving. Within this context, the concept of functions has been perceived as more abstract and introduced toward to the end of curriculum. In recent years, a function-centered approach to algebra is gaining recognition and respect (NCTM, 1989; NCTM, 1995; Thorpe, 1989). In this approach, the concepts of function and relation are perceived the central organizing themes for theory, problem solving, and techniques in algebra (Fey, Heid, Good, Sheets, Blume, & Zbiek, 1995). A functional approach to algebra highlights the importance of the graphs, graphing and functions in teaching and learning of algebra.

Leinhardt, Zaslavsky, and Stein (1990) reviewed research and theory related to teaching and learning in functions, graphs, and graphing at ages between 9 and 14. After reviewing a vast number of research literature, Leinhardt et al. (1990) categorized misconceptions and difficulties related to graphs and graphing under five subheadings: (a) what is and is not a function; (b) correspondence; (c) linearity; (d) continuous versus discrete graphs; (e) representations of functions; (f) relative reading and interpretation; (g) concept of variable; and (h) notation. I summarized research findings reported by Leinhardt et al. (1990, p. 30-45).

What is and is not a function: Students tend to accept only graphs showing an apparent or straightforward pattern as graphs of functions. Often students desire a linear pattern in a graph in order to represent a function. Even though they can tolerate different patterns other than linear they still demand some sort of "reasonableness" (e.g., constantly decreasing or increasing, symmetry, etc.). Students are also found to have difficulties in deciding whether a graph represents a function even though they may know the exact formal definition of a function, which may indicate the influence students' concept of a function from their experience as only patterned graphs represents functions. At this point, Leinhardt et al. points out that this tendency is consistent with historical approach to what a function is because the modern set-theoretic definitions expands the historical definition of functions that did not include discontinuous functions, functions defined on split domains, functions with finite number of exceptional points that may not necessarily follow regular, symmetrical, or easily recognizable patterns. Another misconception about functions is that functions must consist of changeable, variable quantities causing students to not accept constant functions as functions at all. Moreover, students may also equate the concept of dependency with the concept of causal connections that may come from overdependence on intuitions.

Correspondence: Students often demand to see a one-to-one correspondence before they are satisfied with the conditions have been met for a functional relationships. Leinhardt et al. suggested that this might be a consequence of instructional sequencing emphasizing always one-to-one functions like linear functions and lack of additional examples showing many-to-one correspondence. Another difficulty involving correspondence in functions is inability to distinguish between many-to-one and one-to many functions.

Linearity: Students are attracted to linearity in a variety of situations. They tend to define a function as a relation whose graph is a linear pattern. When they are asked to generate examples of graphs of functions passing through two points, they tend to respond with a linear graph. This produces a similar case when more than two points are given. Students often connect each two consecutive points by a straight line to produce graph of a function and confirm that it would be the only graph that would pass through the points when they are asked. Tendency to think linear about functions can also be observed when students are specifically told to produce the equation for a graph of a parabola passing through three labeled points. Some students use only two points to find the equation by calculating the slope value and inserting it as the leading coefficient in the equation. Leinhardt et al. suggested that students' tendency to think linear might have been a consequence of "connect the dots" type of activities students used to do in preschool where the aim was to connect the adjacent dots with straight lines to produce a final picture hidden in the dots. They also highlighted that students' tendency to overgeneralize the linear properties and apply them in wrong contexts is a common source of errors such as reciprocal, cancellation and generalized distribution errors in algebra. So, this overgeneralization is also manifested in graphs and functions with tendency to think functions and their graphs as linear.

Continuous versus discrete graphs: Students may have difficulties associated to the meaning of “uninterrupted” line or curve of a continuous graph. Students may tend to see only the marked points on graphs and disregard existence of the others, unmarked ones. Even though they may be aware of the existence of many others between two points on a graph, they can still think the total number of points in terms of the physical constraints of actually drawing the points and consider that the number of points between two marked points have to be bounded by the dimensions of size and space. On the other hand, students may also attempt to connect discrete points when it is inappropriate to do so like in a scattergram because they may think that’s how a graph should look like. Thus, students may tend to focus on the individual points on most cases and see lines as a connecting function rather than having meaning.

Representations of functions: Students may have difficulties with different representations of function including ordered pairs, equations, graphs, and verbal description and translating from one another, which may involve different psychological processes in different directions. For different age groups, when the function is familiar, movement from graphical representation to symbolic, equation, representation of a function can be more difficult since it requires recognition of patterns than the reverse, graphing an equation, which is the most common practice, involves pretty straightforward steps such as generating ordered pairs, plotting them on a Cartesian grid, and connecting them with a line. For the function less familiar, translation is both directions can be equally hard. Translations may even be harder in cases involving constant functions or quadratic functions of which one of the coefficients is missing (i.e., zero).

Relative reading and interpretation: When examining graphs representing a situation, students usually concentrate on a single point or group of points rather than focusing on more general features of the graph such as its shape, intervals of increases and decreases, etc. A possible cause may be the traditional instruction focusing on graphing from a table of ordered pairs and asking to look up specific pieces of information. Students may often limit their focus to a single point instead of a range of points or an interval that might be more appropriate. Ambiguous questions involving when and where asking for a specific point or region an information is correct triggers the tendency even more and leads students to give the easiest answer, a single point even though the when question might be answered by an interval or region of values. Another version of this “interval/point confusion” is “slope/height confusion” that is to confound the slope of a graph with the maximum or the minimum value and do not know that the slope of a graph is a measure of rate. “Iconic interpretation,” interpreting graphs of situations as literal pictures rather than as symbolic representations of them, is prevalent with secondary school students who were found to interpret distance/time graphs as the paths of actual journey such as “climbing a mountain,” “going upwards and then downwards,” “going north, then east then north again.” Microcomputer-based Laboratories (MBLs) were shown to help students to learn that a graph is not a picture and overcome the slope/height confusion mentioned above.

Concept of variable: A summary of research dealing with students’ learning of concept of variable and related errors and misconceptions has been mentioned in “Students Errors & Misconceptions in Elementary Algebra” section of this chapter.

Notation: Students often have difficulties related to the notational systems integral to both the graphical and algebraic symbols used to represent functions. When constructing graphs, students have difficulties with the notion of interval scale and coordinates. Some students, for example, think that it is meaningful to construct different scales for the positive and negative parts of the axes while some thinks that the scales on the x and y-axes have to be identical even if it blurs the relationship. On the other hand, some students do not understand the effect of changing the scale would have on the appearance of the graph.

Hadjidemetriou and Williams (2000) reported about the results of a pilot study conducted in UK investigating students' graphical understanding, errors and misconceptions held by 50 10th grade mathematics pupils obtained by a test constructed from a set of graphical problems and interviews with selected pupils. They also studied 4 teachers, 3 of whom were teaching the 10th grade classes and the 4th was a retired teacher involved in mathematics education research, in order to investigate teachers' knowledge of their students' understanding through interviews and questionnaires. Results showed that student data contradicted the teachers' prediction that items dealing with concept of the rate of change would be difficult. Although the teachers indicated an awareness of the slope-height confusion found with the students and mentioned in the literature, they presented lack of methods and strategies to address the issue. The teachers' approaches to errors and misconceptions regarding "slope" tended to focus on the calculation part rather than interpretation difficulties. In a follow up study, Hadjidemetriou and Williams (2001a; 2001b) studied 425 9th and 10th grade students from seven schools to investigate their understanding of graphical literacy and their 12 mathematics teachers about their knowledge of their students' errors and misconceptions in graphical reasoning. The researchers found that the teachers' prediction of item difficulties was significantly incorrect on a number of items. They suggested this might be a consequence of overestimating the level of knowledge needed to answer a question correctly. The teachers' knowledge of students' misconceptions asked in the questionnaire varied greatly from one or two of them mentioning only one to two of the teachers mentioning all but one of the seven misconceptions. The researchers suspected different data sources related to whether the teachers' knowledge was tacit or explicit. Possible misconceptions were listed in the questionnaire so that teachers could check if they thought their students would have them. The researchers concluded with confirming the existence of a gap between students' difficulties and teachers' perception of these difficulties. Hadjidemetriou and Williams (2002) elaborated on whether teachers' knowledge of their students error and misconceptions and compared with respect to data sources as interviews and/or questionnaires. They proposed that teachers' knowledge seemed to be sensitive to whether data came from the questionnaire or the interview and suggested that much of the teachers' knowledge is tacit and revealed by an example question. Explicit knowledge, on the other hand, was suggested in interviews without prompts. They also reported that although the twelve teachers had moderate difficulty in identifying the slope-height confusion and problem with order of coordinates, they had least difficulty in identifying the graphs-as-picture and misreading of scale problems.

Student Difficulties in Algebra

It is important to be knowledgeable about student difficulties, errors, misconceptions as indicators of their thinking in algebra along since it informs teachers' knowledge of students thinking. In this section, thus, I begin with a short review and discussion on current issues and problems related to learning of algebra at primary and secondary schools. This discussion, which begins with what algebra is and then continues with epistemological and cognitive approaches to student difficulties and ends with a summary of research findings about student difficulties, errors and misconception in elementary algebra concepts, informs and forms a backdrop for teacher knowledge and research findings came out of this dissertation research.

Nature of Algebra

Meaning of algebra as part of curricula, textbooks and ideas about what constitutes to algebraic thinking may vary across time and culture. Bednarz, Kieran, and Lee (1996) present four approaches to goals and content of the school algebra curriculum. The four aspects of algebra, each providing a different emphasis for an algebra course, are: algebra as the expression of generality; algebra as a problem-solving tool; algebra as modeling, using multiple representations; algebra as the study of functions.

In the twenty-first century, the image of algebra is that it is the discipline involving manipulation of symbols, solving equations, and simplifying expressions involving symbols. However, algebra is more than mere manipulations of symbols. It is about “relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change” (NCTM, 2000, p.37). There are various definitions and approaches to what algebra is. In Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989), algebra was described as “the language through which most of mathematics is communicated” (p. 150). This definition was restated later as “algebra is about abstract structures and about using the principles of those structures in solving problems expressed with symbols” (NCTM, 2000, p. 37). According to NCTM (1998), “Algebra can be seen as a language—with “dialects” of literal symbols, graphs, tables, words, diagrams, and other visual displays.… Algebra is a way thinking about and representing many situations. It has a language and syntax, along with tools and procedures, that promote this thinking and modeling” (p. 163). Like NCTM, Sharma (1988) defines algebra as the symbolic language of real numbers. In this context, understanding algebra is to know the relationships among the language elements and where to use these relationships in solving problems; and doing algebra is manipulating the symbols to produce other relationships and solutions to examples, exercises, and problems.

Usiskin (1988) looks at the issue from a utilitarian perspective and approaches to algebra from what do we do with it. He thinks, "Purposes for algebra are determined by, or are related to, different conception of algebra, which correlate with the different relative importance given to various uses of variables" (p. 11). He talks about for conceptions of algebra: algebra as generalized arithmetic in which variables can be considered as pattern generalizers; algebra as a study of procedures for solving certain kinds of problems in which variables appears as either unknowns or constants; algebra as the study of relationships among quantities in which variables vary; and algebra as study of manipulating and justifying structures in which "faith is placed in properties of the variables, in relationships between *x*'s and *y*'s and *n*'s, be they addends, factors, bases, or exponents. The variable has become an arbitrary object in a structure related by certain properties. It is the view of variable found in abstract algebra" (p. 16).

The nature of algebra has epistemological roots and has been at the center of discussions and definitions of what algebra is. Kieran (1990) talks about a serious debate among British mathematicians in the first half of the 19th century. One side took the position that algebra is universal arithmetic (or generalized arithmetic). In this sense, algebra deals with quantities and operations on them. Its rules are defined by the properties of quantitative arithmetic. The other side took the position on the debate that algebra is purely a system of arbitrary symbols and it's essentially ruled by arbitrary principles. Kieran finds the former problematic because it disrupts the legitimate use of negative, irrational, and imaginary numbers due to the fact that they can't be interpreted as measures of quantity. Wheeler (1996) approaches the problem as algebra being an "extension" or "completion" of arithmetic. It looks at the problems that arithmetic excludes but cannot handle by itself. For Wheeler, arithmetic cannot live without the help of algebra because it needs real numbers for functioning. He warns that an approach to algebra as generalized arithmetic may not be appropriate from a school pedagogy perspective. Because algebra has its own roots not necessarily deductible from arithmetical relations. Existence of certain continuities between algebra and arithmetic makes the conflict on the definition and extend of algebra (Wheeler, 1996).

Epistemological Approach to Student Difficulties

Many educators and students have observed that students often leave an algebra course with a feeling they have been taught some abstract system that has no meaning. A perspective for examining the difficulties encountered in the learning of algebra is an epistemological one. Herscovics (1989) states that students have been taught the syntax of a language without the semantics. In other words, they know the rules in the grammar, but they do not understand the words. Kieran (1992) also approaches the problem epistemologically and outlines the historical development of algebra in three stages: The first stage is "rhetorical algebra" in which no symbols were used at all. This period is usually timed up to Diophantus (ca, A.D. 250). The characteristic of this period was no use of symbols or special signs to represent the unknown quantities, but use of verbal interpretations in solving particular types of problems that needs algebraic thinking. The second stage is "syncopated algebra" in which some abbreviations for the frequently recurring quantities and operators were used starting with Diophantus and lasted until the end of 16th century. Within this period, the role of letter(s) was very situational and did not considered to be used to expressing the general. The third stage is "symbolic algebra," which is the algebra that we use today. This period initiated by Vieta (1540–1603) when he started to use letters to refer to given quantities. This enabled him to express general solutions and use of letters to state rules about numbers and operations. Considering the historical aspects, we may divide the algebra into "classical algebra" where the main purpose is solving equations or finding the unknown value in practical problems and "abstract/modern algebra" which is the study of abstract objects such as groups, rings, and fields. Kieran proposes that the problem with modern algebra is that we impose symbolic algebra on students without taking them through the stages of rhetorical and syncopated algebra.

Like Kieran, Anna Sfard (1991, 1995) and Sfard and Linchevski (1994) outline a framework with regard to how people construct abstract concepts and meanings in mathematics or algebra. The framework is based on the theory that the development of algebraic understanding in the individual follows a parallel path that can be observed in the historical development of algebra. She suggests "difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians" (Sfard, 1995, pp. 15-16). Sfard (1991) presents an ontological-psychological approach along with analyses of different mathematical definitions and representations, which led her to two kinds of complementarity. The first one is that abstract notions can be conceived in two ways: *operationally*, a process to be carried out, or *structurally*, as objects that are unique and posses certain the properties. She expresses that "whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential, and detailed" (p. 4). Sfard points out that operational and structural conceptions are not mutually exclusive in nature. One can observe both interplaying in a single mathematical activity. Within this framework, Sfard describes the structural understanding as what Skemp (1987) calls as "relational understanding," or knowledge of what and why to do, both rules and reasons. However, she also emphasizes that "reasons without rules" can be as dangerous and inappropriate as "rules without reasons." The second complementarity is between historical and psychological development of concepts. Based on this historical analysis of concept formation (the transition from computational operations to abstract objects), Sfard identifies a theoretical model for concept formation characterizing the development of mathematical understanding that suggests the following cyclic pattern:

Operations/processes on objects → **Interiorization** → **Condensation** → **Reification** → New object.

In the first stage, interiorization, some process is performed on a familiar object. This can be associated by rhetorical algebra. The second step is condensation. The process is refined and made more manageable, as in syncopated algebra. At this point, a given process is perceived as a whole without associating it to any process. In the third stage, reification, it is necessary turn computational operations into permanent object-like entities. In other words, one has to move from operational or computational orientation to a structural orientation. Sfard emphasizes that the reification is not easy to access. It requires the ability to see the familiar in a totally new way. In other words, within the cycle of concept formation, "the lower-level reification and the higher-level interiorization are prerequisite for each other" (p. 31). Sfard distinguishes between those three processes as "whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into objects, into a static structure" (Sfard, 1991, p. 20). This three-stage schema is presented in a hierarchical sense, in which "… one stage cannot be reached before all the former steps are taken" (p. 21). In other words, absence of a structural conception may not allow further developments.

Sfard's theory of reification seems to have promising potentials to study concept formation, thus students' difficulties and misconceptions in algebra as well as implications for teaching. An application of this theory to algebra curriculum should fallow the path drawn by Sfard's that historical and epistemological development of algebra from rhetorical to symbolic must be reproduced in the individual to achieve understanding of algebra. One of the unique features of Sfard's theory is the argument that a person could hardly arrive at a structural conception without previous operational understanding since the structural approach is more abstract than the operational. The reform movements in the past decade put the idea or conception that the operational part is not important, or at least not as much as, the structural part is. As a result, a less emphasis on the operational part (e.g., symbol manipulation) is always dictated. As Sfard approaches algebra both constitute the two inseparable parts of the same coin. She suggests that algebraic thinking has its origin in operational processes and evolves in a sequence that goes on towards gaining generality and structure: from rhetorical to syncopated algebra, from this to Vieta's symbolic algebra, and finally abstract algebra.

Sfard's theory suggests theoretical explanations for students' difficulties stated elsewhere (e.g., Booth, 1984, 1988, 1989; English & Halford, 1995; Herscovics, 1989; Herscovics & Linchevski, 1994; Kieran, 1989, 1992). Based on Sfard's theory, one may suggest that students often use rhetorical or syncopated algebra rather than symbolic algebra. Thus, this puts them in an operational paradigm since it is the symbolic algebra, which usually leads to a structural understanding. Moreover, this steady position they are in affects their perception of algebra as a computational process rather than a static construct in which it has a logical, meaningful embodiment. She claims that most students view "algebraic expressions as meaningless symbols governed by arbitrary established transformations," and suggests, "a student may become quite skillful in manipulating such mathematical objects as number, function, or algebraic expression even without reifying them (converting them to object-like entities)" (Sfard, 1995, p. 30, 35).

Difficulty of access to reification also has explanations on cognitive obstacles encountered (such as seeing $x + 8$ as the process not an object, seeing equal sign as a prompt of action rather than signifying a balance between two expression) is explained by students difficulty in reification. As Rojano, (1996) suggests "lesson from history has implications for the teaching in the sense that the potential of dominating algebraic syntax will not be appreciated by students until they have experienced the limits of the scope of their previous knowledge and skills and start using the basic elements of algebraic syntax" (p. 62).

Cognitive Approach to Student Difficulties

Traditionally, the algebra curriculum has been organized around the concept of equation and methods for solving equations. Instruction has focused on the mastery of symbolic manipulation skills. Within this tradition, students often develop what Skemp (1978) calls an "instrumental understanding" of algebra: "It is what I have in the past described as 'rules without reasons,' without realizing that for many pupils *and their teachers* the possession of such a rule, and the ability to use it, was what they meant by 'understanding'" (p. 153). Lee (1996) suggests that mathematics is a culture and algebra is a "mini-culture" in it. In this context, students' entry into algebraic culture is analogous to "cultural shock." This analogy is particularly true considering that inconsistencies between arithmetic and algebra are the main considerations for the obstacles students encounter (Booth, 1984, 1988, 1989; English & Halford, 1995; Herscovics, 1989; Herscovics & Linchevski, 1994; Kieran, 1989, 1992).

With the development of cognitive psychology, the leading domain to perceive and explain the student difficulties has become "cognitive obstacles" such as the tendency deceptive intuitions, tendency to generalize, and obstacles caused by natural language. One theoretical framework to understand cognitive obstacles in the learning process is drawn by Herscovics (1989) from Piaget's theory of equilibration:

> This process of equilibration involves not only *assimilation*—the integration of the things to be known into some existing cognitive structure—but also *accommodation*— changes in the learner's cognitive structure necessitated by the acquisition [of] new knowledge. However, the learner's existing cognitive structures are difficult to change significantly, their very existence becoming cognitive obstacles in the construction of [a] new structure. (p. 62)

From this perspective, cognitive obstacles are the natural consequences of the process of accommodating the elements (i.e., beliefs, way of communicating, etc.) of the new culture.

The notion of equality and the equation sign, the different uses of letters in arithmetic and algebra, as well as the procedural nature of arithmetic versus the relational nature of algebra are some of the differences between the cultures of arithmetic and algebra. Among others, students' perceptions about the use of letters in algebra are one of the key factors in understanding the elements of the algebraic culture. Different uses of letters are also among the sources of difficulty in learning algebra (Küchemann, 1981; Wagner & Parker, 1993). In general, the view that algebra is "generalized arithmetic" or "completion of arithmetic" (Usiskin, 1988) reminds us of the importance of justification and generalization in algebra. However, overgeneralization, or generalization of rules in the wrong context, is one of the frequent and persisting difficulties in learning algebra.

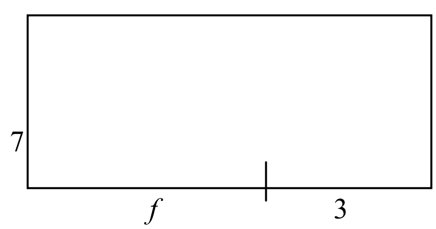
As a cognitive obstacle, misgeneralization and justification are among major sources of students' difficulties in algebra, as Maurer (1987) observed, "The research brings Good News and Bad News. The Good News is that, basically, students are acting like creative young scientists, interpreting their lessons through their own generalizations. The Bad News is that their methods of generalizing are often faulty" (pp. 165-166). For the question "Is the statement $(2x + 1)/(2x + 1 + 7) = 1/8$ definitely true?/possibly true?/never true?" Lee and Wheeler (cited in

Kieran, 1990) reported that most of the 354 students chose "definitely true" for two distinct reasons: Either the $2x$ s were cancelled, or cross-multiplying led to an equation that was solved or at least appeared to be solvable. Barnard (1989) found that 40.7% of the subjects selected $1/y + 1/x$ when they were asked to simplify $x/(x + y) + y/(x - y)$. Similar results were reported for students' simplification of $(a^2 + a)/a$ as a^2. Margilues (1993) and Parish and Ludwig (1994) reported that most students commit errors of the types $3/a + 3/b = 3/(a + b)$, $(a + b)/(c + d) = a/c + b/d$, $(a + b)/b = a$, $(x + 2)^2 = x^2 + 4$, $(a - b)^2 = a^2 - b^2$. Provided with $5/(2 - x) + 5/(2 + x) = 4 \Rightarrow 5(2 + x) + 5(2 - x) = 4 \Rightarrow 5(2 + x + 2 - x) = 4 \Rightarrow 5(4) = 4 \Rightarrow 20 = 4$, Lee & Wheeler (as cited in Kieran, 1990) reported that over half of the students appeared to accept the validity of the line $20 = 4$; more than half of these accepted the entire work, the remainder indicating a problem elsewhere. Only about 20% indicated, implicitly or explicitly that $20 = 4$ was unacceptable. As in this case, cross multiplication (i.e., $a/b \div x/y = ay/bx$) seems to be causing some mis-generalizations (Laursen, 1978) such as $a/b + x/y = ay/bx$; $a/b - x/y = ay - bx$; $a/b \cdot x/y = ay/bx$.

Another perspective about student difficulties in algebra has linguistic origins or approaches. Many students have difficulty formulating linear algebraic equations from information presented in words (Clement, 1982; Clement, Lochhead, & Monk, 1981; MacGregor & Stacey, 1993, 1997a, 1997b; Rosnick, 1981). *Syntactic translation*, in which translation of a statement of English into an equation occurs by replacing key words by mathematical symbols sequentially from left to right, is accepted as a procedure frequently used by students for formulating equations from natural language expressions and is thought to be an important cause of errors, particularly "the reversal error" (Clement et al., 1981). Many students including college level may write $6S = P$ instead of $S = 6P$ for so called the Student-Professor problem (Clement, 1982). Although syntactic translation was the first source blamed for reversed equations, Clement et al. (1981) observed that students frequently use another approach to writing equations. In this approach, which Clement et al. called “static comparison” and Herscovics (1989) called “semantic translation,” the equation is used to represent an association of related groups (i.e., students and professors) rather than of equal numbers. MacGregor and Stacey (1993) reported that the majority of secondary school students did not use a syntactic translation procedure for writing simple linear algebraic equations; instead, they tried to express the meaning of the statement and wrote incorrect equations. MacGregor and Stacey suggested that the majority of the incorrect equations, particularly the reversal equations, are consequences of cognitive models attempting to represent compared unequal quantities. In a different cultural context, Mestre (1989) reported that many Hispanic students have misconceptions in solving the Student-Professor problem (Clement, 1982), such as “6S = 6P and 6S + P = T,” that originate in language differences. He concluded that differences in language caused Hispanics to commit the same types of errors as Anglos, but with a higher frequency.

Students Errors & Misconceptions in Elementary Algebra

Students have serious difficulties with fundamental concepts of algebra. A simple question like “What is the area of the following figure?”



would result in erroneous answers such as *7f3* or *f21* or *21f* or *f + 21* by forty-two percent of the 13- year-olds responded with (Booth, 1984). The following sections summarize the research literature about some of the common misconceptions and errors in introductory algebra. I should need to remark that the ones reported here is just a glimpse of what is presented in the research literature and restricted to variables, literal terms and basic equations. In other words, this is just the tip of the iceberg compared to all concepts of algebra.

Leitzel (1989) pointed out that "the concept of variable is more sophisticated than we often recognize and frequently turns out to be the concept that blocks students' success in algebra" (p. 29). Several misconceptions students commonly held mentioned in research literature on students' conceptions of variables and meanings of letters in algebra. First one is "*letters have no meaning in mathematics"* (Küchemann, 1981; Perso; 1992). Many students believe that mathematics consists of numbers, and letters do not have a place in that numerical world, they belong to literacy. This leads to interpret an algebraic statement as being "nonsense" and attempt to either guess an answer or to leave out the question containing a variable or letter. A second one is "*letters represent numerical positions in the alphabet"* (Perso, 1992)**.** Some students believe that each letter can only represent one number resulting in the belief that alphabet symbols correspond to the numerical positions in the alphabet (i.e., $a = 1$, $b = 2$, $g = 7$, etc.). A third one is "*letters are sequential, as in the alphabet"* (Perso, 1992). Some students bring the notion that letters have unique position (i.e., sequential) in the alphabet into algebra and do not see that they are arbitrary and not related to other letters. Perso (1992) reports that, for example, the question "if $a = 3$ and $c = 5$, what is the value of b?" has resulted in a large proportion of students aged 13 through to 17, giving '4' as the answer. A fourth one is "*Letters stand for objects; they are labels"* (Booth, 1984, 1988; Küchemann, 1981; Perso, 1992). Many students in algebra see a letter standing for an object, or acting as a label, which are the common uses of letters in arithmetic. Examples: As in the formulas: V as volume, L as length, W as width, H as height as in the formulae V = L×W×H. As abbreviations for words: 5*m*; the '*m*' stands for 'meters', 3*c*; the '*c*' stands for 'cents', $4a + b$: the '*a*' stands for 'apples' and the '*b*' stands for 'bananas.' A fifth one is "*a letter standing alone equals 1"* (Küchemann, 1981; Perso, 1992). A possible source of this misconception is students' tendency to think existence of one-to-one correspondence between letters and numbers and so believing that any letter standing alone must equal '1.' A sixth misconception concerning letters is "*each letter has a unique value"* (Booth, 1984; Perso, 1992; Wagner & Parker, 1993). Within this misconception, it is believed that different letters must stand for different numbers; if a letter equals '3' in one expression, then it must equal '3' in all other expressions. For example, Küchemann (1981) reports that 89% of the 13-year-olds tested were not successful with the following question- - a question that, according to Küchemann, required the concept of generalized number in order to be answered correctly: $L + M + N = L + P + N$ is true- - always/never/sometimes. A seventh one is "*letters do not behave as numbers"* (Perso, 1992). Many students believe that same letters can have different values within the same expression/equation possibly because of the belief that letters behave as words. For example, in the sentence "he is a student" various names rather than one to make the sentence true or false can replace the 'he.' However, the some students may see that 'x' in the equation $x + 5 = x + x$ may take different values such as 5 for the second x and anything for the first x.

Another set of misconceptions students often have in elementary algebra stems from their lack of acceptance of closure in algebra unlike. One such misconception, for example, is "*an '=' sign means action, not equivalence*" (Kieran, 1981; Perso, 1992; Sharma, 1988). Many students hardly believe that an open expression such as '$2y + 3$' or '$3 + x$' as a final/valid answers. They think that an equal sign must produce a single entity, i.e., without signs such as '+' and/or '-.' To them, an operation symbol followed by an equals sign means 'do something' until getting a single entity like in arithmetic.

Students may also have misconceptions concerning the order of operations in algebra. Some students think "*order of operations is unimportant*" (Perso, 1992). They tends to neglect the order of operations learned in arithmetic and will instead try to perform in any order which appears possible. Furthermore, many students tend to think that the written sequence of operations determines the order and think "*working in mathematics is always left to right*" (Perso, 1992). Contrary to arithmetic, many students do not see a need for brackets in algebra and they often tend to ignore it which lead to errors like $3\times(a + b) = 3a + b$. This may lead to misconception "*parentheses don't mean anything in algebra"* (Perso, 1992).

Students may also have misconceptions related to solving equations. For example, a common misconception is that "*inverse operations are not necessary*" (Perso, 1992). Students often do not know which operation is the reverse (i.e., undoes) of another operation. Moreover, during equation solving process, in attempting to remember the rules such as "change side, change the sign" and "do the same to both sides" without an understanding of why the rules work cause confusions and distortions leading students to think that "*reverse operation is used on the other side of an equation, not the same operation* (Kieran, 1988, 1989; Perso, 1992, Wagner & Parker, 1993). On the other hand, "*numbers, variables and signs are detached*" (Perso, 1992) is a misconception that can cause difficulties in solving equations. Some students see that although some signs are attached to numbers (or variables) but not the others. For example, they can give meaning to '-' in front of –2 but not to '×2' or '÷2.' They saw the latter as detached. Moreover, some students, when attempting to solve equations, transpose only the literal part of the term and leave the coefficient behind. For example, they solve the equation $4x = x + 3$ as $4 - 3 = x \div x$. these types of errors in solving equations was described as mal-rules or bugs in the research literature.

Sleeman (1984) and Payne and Squibb (1990) reported empirical studies of elementary algebra errors in the subject of linear equations in one unknown. The errors are diagnosed using the mal-rules (or bugs), student's mental representations and applications of faulty procedures. Some of the mal-rules they found are: $M*x = N \rightarrow x = M/N$; $M (Nx \pm P) \rightarrow M*Nx \pm P$; $M (Nx \pm P) \rightarrow M*Nx \pm M \pm P$; $M (Nx \pm P) \rightarrow Nx \pm M * P$; $Mx \pm N = Px \pm Q \rightarrow Mx + Px = N + Q$; $Mx = N \rightarrow x = M - N$; and $M*x = N*x + P \rightarrow x + x = M + N + P$. Sharma (1988), on the other hand, summarized several types of errors in solving linear equations in a more general sense as:

- *Arithmetic:* basic facts, defective algorithm, wrong operation, and wrong order of operation.
- *Properties of Numbers:* associative, distributive (sign error in distribution), commutative, operational/ruled based.
- *Procedural:* misuse of the property of equality, + property of =, × property of =, coefficient error +, coefficient error wrong operation (wrong inverse operation).
- *Conceptual:* order of operations (opposite sign), like terms, zero annexation, misunderstanding constants as variables.
- *Mechanical/Perceptual:* careless/random, incomplete operations.

CHAPTER 4: METHODOLOGY

Design of the Study: A Qualitative Approach

One needs to get close to people (e.g., talking to people, observing them in their day-to-day lives) in order to understand the way they think, learn what and how they know about the world around them, how these definitions are constructed and come to play out in different contexts (Bogdan & Biklen, 1998). I have planned and followed a qualitative research design envisioning such an approach to generate data relevant to teacher knowledge and beliefs.

The characteristics of qualitative research are frequently presented as dichotomies or contrasts to more traditional quantitative approaches (Bogdan & Biklen, 1998; Denzin & Lincoln, 2000; Glesne, 1999; Lincoln & Guba, 2000). Also called interpretive research, naturalistic research, phenomenological research [although this can mean a specific kind of qualitative research as used by some], descriptive research, *qualitative research* is an *umbrella term* for research strategies that share certain general logic and theoretical perspectives (Bogdan & Biklen, 1998; Glesne, 1999). Bogdan and Biklen (1998) identify distinguishing characteristics of qualitative research under five strands, "naturalistic; descriptive data; concern with process; inductive; and meaning" (p. 4). Qualitative research stressing the socially constructed reality studies phenomena in their actual settings to make sense of, or interpret, the meanings people bring to them (Bogdan & Biklen, 1998; Denzin & Lincoln, 2000). Qualitative research encourages emergent (as opposed to predetermined) designs in which researchers focus on process rather than simply on products or outcomes. Researchers "do not search out data or evidence to prove or disprove hypotheses they hold before entering the study; rather, the abstractions are built as the particulars that they have been gathered are grouped together" (Bogdan & Biklen, 1998, p. 6). The researcher acts as the "human instrument" (Patton, 1990) of data collection. In other words, "qualitative researchers set up strategies and procedures to enable them to consider experiences from the informants' perspectives" (Bogdan & Biklen, 1998, p. 7).

Participant Selection

According to Patton (1990), qualitative researchers neither work (usually) with populations large enough to make random sampling meaningful, nor is their purpose that of producing generalizations (pp. 28-29). Rather, qualitative researchers tend to select each of their cases purposefully. Thus, purposeful sampling, which seeks information-rich cases that can be studied in depth, is the dominant strategy in qualitative research.

> "The logic and power of purposeful sampling lies in selecting information-rich cases for study in depth. Information rich cases are those from which one can learn a great deal about issues of central importance to the purpose of the research." (Patton 1990, p. 169)

Patton identifies and describes sixteen types of purposeful sampling, including *extreme or deviant case sampling*; *typical case sampling*; *maximum variation sampling*; *snowball or chain sampling*; *confirming or disconfirming case sampling*; *politically important case sampling*; *convenience sampling*; and others (Patton, 1990, pp. 169-183). Among all, according to Lincoln and Guba (1985), the most useful strategy for the naturalistic approach is maximum variation sampling which "aims to capture and describe the central themes or principal outcomes that cut across a great deal of participant or program variation" (Patton, 1990, p. 172). *Maximum variation sampling* potentially yields detailed descriptions of each case, in addition to identifying shared patterns that cut across cases. In this study, that continuum was obtained by choosing two teachers representing varying characteristics regarding to demographics (e.g., gender, experience, ethnicity, education, etc.) and institutional backgrounds (e.g., curriculum used, district, etc.) of the two teachers such as program/curriculum used, gender (M or F), experience (years in professions), and education (BS, MS, or PhD).

On the other hand, Goetz and LeCompte, as cited in Merriam (1988), use the term *criterion-based sampling* instead of *purposeful sampling*. They mean that researcher finds a sample that marches to the criteria, bases or standards identified to be included in the study. Although the sampling methods they suggested have similarities with those of Patton (1990), among others, they listed one particular method that I found useful in my selection of participants and gaining access to their contexts. This was *reputational-case selection* in which cases are selected based on the recommendations of experienced experts in an area.

For the purposes of this study, my criteria, bases, or standards to include a teacher in the study were simple. I wanted to investigate inservice mathematics teachers who are expert or have experience in teaching not only mathematics in general but also algebra in particular. The definition of expert or experienced teacher is troublesome. My approach to those terms was in terms of years in teaching the subject and having a good reputation as a model teacher in the school, parental, and possibly research community. I believed I would get more insightful data from such participants. Another criterion, also a great challenge, was that not only the participant but also the school administration and parents of the students in the class had to agree to classroom videotaping. The latter was not an easy task in terms of getting the Institutional Review Board (IRB) for Research Involving Human Subjects and all parents' approval even if the teacher agreed with it. Thus, my research proposal went through a special screening to make sure that it was ethical in terms doing research with human subjects and required getting parental consent forms from all students so that I would videotape. Furthermore, I wanted to investigate an Algebra 1 context because as a common course in middle and high school it would give me the flexibility to broaden my research and participant contexts. One more criterion for selection was that the participants had to be in a close driving distance to where I was living because of consecutive daily observations I planned to conduct.

Being a foreigner to K-12 communities and school district administrations, I used networking in order to identify and locate a group of possible participants. With the help of the project manager I was working with at the university, I was directed to a nearby school district and its high and middle school where she had research experience and set a continuous relationship. Upon her recommendation of me and my research to the superintendent and school principals, I send e-mails to those persons requesting to do research in the schools with an attachment explaining my research purposes and procedures I planned to fallow. After I got their permissions, I identified three possible participants, who were teaching algebra one in the middle school, upon recommendation of the same colleague and made contact with them via e-mail. I expressed my interest in doing research within their algebra one classrooms along with a brief summary of my research purposes and procedures as written in my research proposal. With e-mail trafficking and personal site visits, I was able to recruit one of them, Ms. Sands (pseudonym), who agreed to work with me and was able to get parental consent forms from all students. The other two teachers were unable to participate because of their busy schedule at the moment. On the other hand, in a similar way, I had another possible participant in the high school but I had to drop her because we were unable to get back parental consent forms signed except for a few students. After that, I turned my attention to other school districts. One possibility was the teacher I worked with in my pilot study, which I will talk about it in data collection section of this chapter. Although the teacher expressed great interest in the study, I was unable to get the district's approval to gain access to the site simply because of the district's unique policy of approving and supporting certain kind of research studies. Upon recommendation of my major professor, I approached to a high school teacher, Mr. Casey (pseudonym), teaching algebra one in a nearby school district. Getting his agreement to be a participant in the study, I followed the same procedures (i.e., getting the district's, school's and parental approvals). Copies of participant teacher and parental consent forms are available in Appendix A.

Participants and Research Contexts

Ms. Sands

Ms. Sands was a white female in her mid-forties who had experience in teaching mathematics for about 10 years by the time of data collection. Although she has a bachelor's degree in music education and she first searched for a music teaching position, the only position she could find was a mathematics position and "it seemed to fit OK" at the time. Having a computer programming course in undergraduate school which was counted as a mathematics class was the basis of her T-4 certification she obtained at the time. She began teaching as a middle school teacher. She describes those days as "the days of grouping" when she mostly did remedial work because she had "the low groups." She left teaching to raise her children after her first-born and did not teach for about 16 years. She came back to teaching in 1996 as an algebra teacher and has been teaching algebra and pre-algebra since then. Shortly after her return to teaching, she received her Master of Education degree in middle grade education with a concentration in mathematics from a reputable southern university. She had algebra and geometry for middle school courses when she was studying for her Master of Education degree. She also took mathematics courses such as number sense and geometry through a local Regional Education Service Agency (RESA).

Rural County Middle School (RCMS) where Ms. Sands has been teaching algebra since her return to teaching is a small town middle school. It is the middle school of four public schools of a small district. The mission of the Rural County School System was “to ensure that all students will be successful in their learning and personal development through a system characterized by extensive community and parental involvement, quality resources, an exemplary staff, a safe and caring environment, and a challenging, personalized, educational program encompassing advanced technology.” Furthermore, one of the principles noted on the walls of the Rural County Middle School (RCMS) was to move willingly to a *no excuse environment* where there is: “no excuses for not knowing what children know and need to know; no excuses for not being prepared to deliver quality instruction; no excuses from them for lack of achievement” School envisioned that those can be attained by “examining existing data to see where we stand; using data to diagnose what children know now; planning action based on that knowledge; measuring progress OFTEN; and refusing to let ANY of our children miss out.” It was also envisioned that 90% of Rural County students would be prepared for algebra one by grade eight.

According to the state’s office of student achievement statistics, there were 263, 252, and 257 students enrolled in grades 6, 7, and 8 respectively during the fall semester of 2001-2002 academic year. The school composition was 64% white, 34% black, and 2% Hispanic; 41% of the students were eligible for free or reduced meals and 16% of the students were with disabilities. The school had a zero percent dropout rate in 2002-2003 school year (and only 0.2% in the previous year).

According to the 8th Grade-Georgia Criterion-Referenced Competency Tests (CRCT), the school’s progress/performance in mathematics during the last three years has been as follows:

36, 51, 14 percents in 2002-2003; 37, 49, 13 percents in 2001-2002; 43, 48, 8 percents in 2000-2001 where the percentages mean "does not meet", "meets", and "exceeds" respectively in each year. It seems that they are getting better even though it is slow. Considering the grade level progress for the same student body the CRCT scores for 8^{th} graders in 2002-2003, 7^{th} graders in 2001-2002 and 6^{th} graders in 2000-2001 are as follows 36, 51, 14 percents; 24, 66, 10; and 36, 49, 15 where the percentages mean "does not meet", "meets", and "exceeds" respectively for each year. It seems that students made better progress in 7^{th} grade compared to 6^{th} grade but then it went down to 6^{th} grade level in their 8^{th} grade.

In Rural County Middle School (RCMS), the decisions for who should and can take algebra in 8th grade are made during 7th grade when students were taking pre-algebra. Two factors in the decision process were the student's yearly average, which was desired to be an A, and the teacher recommendation even though they did not have a direct authority in decisions. Even if a student had a yearly average of B, he/she could get into algebra-1 in 8th grade if the student had the "work ethic" and his/her teacher recommended it.

The mathematics curriculum in Rural County Middle School (RCMS) was *College Preparatory Mathematics* which is "a middle grades and secondary mathematics program that integrates basic skills and topics with conceptual understanding and problem solving strategies to achieve a complete and balanced mathematics curriculum" as defined in the CPM web site (http://www.cpm.org/about/what.html). The emphases of the CPM are on the process of evaluating a problem, devising possible solutions, and transforming these solutions to a mathematically correct form, getting a correct solution, and evaluating the validity of an answer. The CPM curriculum advocates uses of a variety of teaching methodologies, including lecture, class discussions, manipulatives, and structured study teams. CPM also features a spiraling curriculum that allows a topic to be introduced and mastered across longer periods of time than "traditional" textbooks typically support. Fundamental skills and procedures are practiced as exercises over multiple weeks and integrated into application problems. During class, teachers are expected to provide "much-needed information", instruction, and support for students but allow them to work through problems and questions in a study team environment. Teachers are expected to answer student questions in a challenging and motivating manner so that students develop and test solutions themselves. Teachers also direct the discussions to summarize lessons, interact with the groups, and lecture based on observed needs of the students. Teachers teaching is described as "prescription in nature, not "one size fits all."" On the other hand, teachers are also expected to attend a hands-on preparation session prior to teaching CPM first time and attend workshops during the first year of implementation. As Ms. Sands stated, it was second year of implementing this new curriculum and students seemed to be happy with, in that she did not hear much of *whining*, at least as much as she did in the first year of implementation.

Mr. Casey

Mr. Casey was a 50+ years old white male who had experience in teaching mathematics for about 11 years. He had been teaching mathematics in the state of Georgia through his entire teacher career. He had his Bachelors of Science (BS) degree in mathematics from a university in the West and Master in Science (MS) degree in mathematics education from a southern university (the same one Ms. Sands attended). At the time of my data collection, he was working towards his Specialist in Education (EdS) degree in the same university he got his Master of Education degree. He was also waiting to hear about his application for National Board certification, which he received during the next academic year.

Teaching was not Mr. Casey's first career choice. When he graduated in 1980 he was working in a family company and doing carpentry work. Having kids and spending more hours away from home made him want to change careers so he could spend more time with his family. He wanted to get a master's degree in education and start teaching. He chose mathematics because there was a large need for teachers in this area and having a degree in the subject affected his decision. When he had his first teaching position, he was "basically get told what to take [teach]" (I1) and it was algebra. However, he considers himself as a mathematics teacher; not particularly algebra:

> "When I came to *Suburban County* they said: 'oh, so you're a freshman algebra teacher!' and I said, 'well, no actually I'm a math teacher and I'll teach anything that you have'. You know, any course you offer." (I1)

Mr. Casey thinks that what makes who he is a combination of his education, his personality, a "patient guy" enjoying working with the kids, experience and a good family life. He expressed that his background in mathematics allowed him to be confident in a classroom and talk about mathematics. On the other hand, he doesn't think that his MS in mathematics education fully prepared him for the classroom even though it helped him with the content knowledge and provided him the student teaching opportunity that he needed. He stated that he gets along with students and administration very well. He thinks that kids like to come to his class partly because of his knowledge of the content, knowledge of pedagogy, and his personality.

> "From the University you get a research perspective, there's things that you don't think of or they force you to look at you know how kids learn or how they acquire their knowledge and you find it sometimes to difficult you know why should I need to know these stuff but then you can actually see some of it in practice so it's a good idea the exposure was a valuable experience. I wouldn't say it's like any other kind of job where if you're an accountant you know they teach you how to do your accounting and you out and you account. The interaction with the students and the way that all of their

> personalities come together make classrooms you know such a diverse dynamic that it's … it all it makes you part of who you are and so." (I1)

Mr. Casey defines his role as an algebra-1 teacher as he is the "dispenser of information"

> "On for the algebra-1 classes I am mostly the dispenser of information that it's hard to keep the kids focus. We do some discovery activities but for the most part it's a- I deliver the information and they're supposed to process and they get chance in class to work some of the problems and, Uhm, demonstrate their mastery of our objectives." (I1)

He further expands his role as "a peace keeper, a father, a friend." (I1)

Suburban County High School (SCHS) where Mr. Casey has been teaching at is described as a rural public high school located in close proximity to a major research university in a southern town. It is the only high school in the county. According to the state's office of student achievement statistics, there were 472 students [of 1,720 in total] enrolled in 9th grade during the spring semester of 2003. During the 2002-2003 academic year, the racial distribution of student body consisted of 90% white, 5% black, 3% Hispanic, and 2% Asian students. The percentages of white and black students were significantly different than those of state's average, which were 52% and 38% respectively. Only 8% of the students were eligible for free or reduced meals. This was significantly below the state's average of 45%. The percentage of students with disabilities was 8%. The school had a 2.4% dropout rate consecutively in 2002-2003 academic year and the year before.

According to Georgia High School Graduation Test (GHSGT) results, the school's student performance in mathematics during the last three years was as follows: 3, 30, 67 percents in 2002-2003; 2, 27, 71 percents in 2001-2002; 4, 44, 52 percents in 2000-2001 where the percentages means "fail (i.e., below 500)", "pass (i.e., between 500-534)" and "pass plus (i.e., above 535)" respectively in each academic years.

Every student graduating from SCHS has to take algebra at some point in their study. For a college prep diploma or dual seal (e.g., college prep and technical prep), every student has to have four mathematics courses that are algebra-1 and above. For a technical prep only diploma, one has to have three mathematics credits and algebra-1 has to be one of them. Mr. Casey expressed that this has been the same for generations; it is a "kind of odd" but "hard to change" situation. Each of the two main feeder middle schools they have has one class where students take algebra in 8th grade. Those students receive high school credit for algebra-1 and take geometry as freshman. The school has a large population of freshman and there are usually eleven sections of algebra-1. So, most of fourteen mathematics teachers in the school gets at least one section to teach. He explained that according to latest statistics they had approximately 48% failure rate for the first year algebra students and this is "unacceptably high" for him and the administration. So, they decided to divide the algebra textbook and made it two semester courses. However, since they're on a block schedule, the state doesn't allow giving credit for both courses; they can only have one algebra credit. So, they decided to make the first semester course a mathematics elective course and call it "Concepts of Algebra Advanced" covering the first six chapters of the book and cover chapters seven through twelve in the second semester course as "Algebra-1" which is where they get the credit for algebra. By doing so, they found out that the failure rate dropped about 30% for the entire school. The failure rate for two or three sections of advanced algebra-1 consisting of students who either took pre-algebra in 8th grade or algebra but want to repeat for some reason (e.g., low grades) was even less than this. However, he thinks that students in those classes are in upper 25%. For Mr. Casey, the failure rate for regular algebra-1 is still not as low as the school would like. Crowded classes makes it hard to try nontraditional instruction like guided discovery, which may make mathematics more enjoyable and may let students better master the subject and further drop the failure rate.

The textbook Mr. Casey was using was described as a student centered program by the publisher. It was claimed that the style of the book would lead students to discovery of mathematical ideas, communicate what they'd learned and apply it. It required both individual and group work to solve problems. Furthermore, the book focused on real-life situations to help students appreciate the role of mathematics in their lives.

Data Collection

I generated the data trough audio-recorded semi-structured interviews, video-recorded classroom observations, and evaluation materials (i.e. the tests, quizzes etc.). A description and rationale for each of the methods will be given below in this section.

Pilot Study

From my previous experiences, I think that unexpected is expected and extraordinary is ordinary in research. This is more than true for qualitative research. So, to inform, inspire and strengthen the design of my qualitative research, I ran a pilot study during the fall semester of 2000-2001 academic year. I interviewed a middle school algebra teacher on one occasion for about an hour and then observed his classroom on two different occasions. I furthermore collected available documents about him, the school and the mathematics curricula. The information gathered, the overall experience and insight from implementing these interviews, observations and archival analyses assisted me in the formulation of questions for my interview guide, and research problems. An experimental writing of this pilot study analysis can be found in Appendix B.

Interviews

Kvale (1996) defines the interview as “a conversation that has a structure and a purpose” (p. 6). Although it is not problem-free in nature (see Scheurich, 1995), the interview is a rich site for constructing knowledge from a qualitative research perspective (Kvale, 1996). It enables “interchange of views, *inter view*, between two persons conversing about a theme of mutual interest” (p. 14). Patton (1990) talks about the existence of three types of qualitative interviewing: informal, conversational interviews; semi-structured interviews; and standardized, open-ended interviews. The informal interviews and standardized interviews are also called unstructured interviews and structured interviews respectively. Semi-structured interview is defined as “an interview whose purpose is to obtain descriptions of the life world of interviewee with respect to interpreting the meaning of the described phenomena” (Kvale, 1996, p. 6), and was used to gather data for the purposes of this study.

In a series of semi-structured interviews, I gathered data about participant teachers’ backgrounds (experience, education, etc.), their knowledge and beliefs about what algebra and mathematics is (i.e., content perspective), about how algebra is learned and should be taught, about how algebra curriculum should be organized, and lessons prepared and delivered. Furthermore, student thinking and difficulties in algebra and teachers’ knowledge, beliefs and approaches to those issues were in the focus of interviews. I interviewed each participant at the beginning, middle and the end of the study. By doing so, I also intended to study the change in teachers’ beliefs [if there were any] and action toward student thinking and difficulties as their course moves along. Each interview was audio taped with participants’ permission.

I interviewed Ms. Sands in three occasions during her preparation hours in the afternoons when she was available. The first interview took place on the first day of observations and lasted about an-hour-and-a-half. I had the second interview lasted about an hour after the eight observation/lesson halfway through the unit. Finally, the last interview lasted about an hour after the unit was ended and she graded student test for the unit. The questions for each interview with Ms. Sands are summarized in Appendix C.

I interviewed Mr. Casey in four occasions during his preparation hours in the mornings when he was available. The first interview took place before the third observation and lasted about an-hour-and-a-half. The second interview lasted about an hour before the ninth observation/lesson. The last interview lasted about an hour after the unit was ended and he graded student tests for the unit. The questions for each of the three interviews with Mr. Casey are summarized in Appendix C. I had a fourth interview that I did not include for this dissertation report. It was mainly students' difficulties with the concept of function that was the next chapter after radical expressions. I planned to use it along with the additional observations I mention later in the next section about videotaped observations.

Classroom Observations and Videotaping

Teacher and student behaviors are non-random reflections of certain reasons, beliefs and values (Marshall & Rossman, 1999). Thus, the classroom observations, for me, were a place to validate the information provided by interviews and documents as well as a means to seek explanation for the nature of interaction within the context of the real setting, the classroom. The main focus of the classroom observations was the teachers' interpretation of topics/concepts, questions to students, responses to students' questions, and dealing with students' correct and incorrect thinking. Furthermore, the classroom observations also helped me to understand the teachers' instructional practices in algebra and what conditions affect those. More specifically, I mainly tried to observe the context of teacher delivery of subject matter, which includes teacher approach to the subject matter, questioning the students, and the nature of feedback provided to students.

I was positioned as an observer rather than a participant observer. By “being there” (Wolcott, 1999), I did not intend to participate in anything but just watch and make sense of people’ verbal and non-verbal behaviors (Marshall & Rossman, 1999). Thus, I was a participant as observer by being there (Wolcott, 1999).

I observed both teachers and their classes from start to end of a unit. I video recorded all of the observations, which allowed me easy reviewing and observing the same events for any number of times and get a more detailed account of what’s going on in the class and exact transcriptions of the classroom discourses.

Even though it has methodological challenges and requires labor-intensive work, video provides benefits beyond the difficulties (Stigler & Hiebert, 1997). Based on their experience with TIMSS Videotape Classroom Study, Jacobs, Kawanaka, and Stigler (1999) conclude that using video data in educational research has significant benefits and potential that is just being realized (p. 723). Because,

> “Video data provide the kind of detailed permanent real-time records of behavior that enable researchers to detect patterns and to code a variety of characteristics reliably within and among the tapes.” (Jacobs, Kawanaka, & Stigler, 1999, p. 723)

I used 8-mm videotapes to record classrooms. Although I used a digital-8 camcorder in the case of Ms. Sands since her class period was about 50 minutes and it was enough for a single tape in digital format, I had chosen to use an analog camcorder and analog recording in the case of Mr. Casey because he had block-scheduled classes lasting about 90 minutes, which would require two tapes recorded in digital format causing a jump in the wholeness of the lesson recorded.

Rather than depending on the internal microphone of the camcorder, to get a better quality audio, I used a wireless lapel microphone attached only to the teacher in each case. Because of the technical difficulty and expense of using multiple wireless and/or stationary microphones, I did not use any other external microphone for students. Thus, audio from students was recorded through teacher microphone. In some cases this caused poor audio quality and hearing later in the transcription and analysis of the video segments. Using quality headphones and tuning the audio, I tried to minimize the effect of this problem.

I located the camera for recording at the back of each classroom as shown in Figure 2 and Figure 3. In choosing a place for the camera, I tried to have an angle maximizing my view of whole classroom, students and teacher, yet I tried to minimize my presence in front of the students in order not to get their attention and disturb their focus. I used a videography tripod allowing me to have smooth rotation, quick attaching and removing the camera, good height and stability. The camera was not moved; it stayed stationary during each of the observed lessons for both teachers. Except for a few occasions, I arrived to the research site (i.e., the classroom) before the class to set up the camera and wireless microphone and begin recording while students were coming into the class. I also tried to keep recording until after students had left the class and/or the teachers turned off the wireless microphone. I wanted to capture revealing remarks (if there were any) before and after the class.

I did not have serious technical issues during videotaping processes in either case mainly due to my readings on the technical aspects of videotaping (e.g., Ratcliff, 2004; Roschelle, 2000). However, I would say that I mainly practiced and got my lessons from the first case, Ms. Sands, by actually doing it. The most critical learning was to make sure that I had the timestamp feature turned on and actually had it recorded, which I failed to do on several occasions in the first case study. Later when I was watching and digitizing the tapes into a computer format I solved this problem by releasing and viewing the timestamp from the LCD screen of the camcorder because the tapes already had the time and date stamps recorded as a hidden information and could be reviewed directly from the camcorder as a feature of it.

I believe that my presence in the classrooms classroom had no significant observer effect on what the students do. As Mr. Casey expressed it in during the second interview, my presence got students excited the first day thinking that they were going to be in TV. But, they had gotten used to the fact that I was in the class videotaping and become less aware of it. They stayed pretty much the same way they were before. To my observations, this was the same for the students in Ms. Sands's case. Mr. Casey answered the students' curiosity about the purpose of videotaping and my presence in the class as:

> "It's not a college course, no. It's research. It's all- it's- the reason we got taped so that they can review and see what happened. What's happening is what's important, not the picture of it. So, they'll just use it to collect all the information they need. ... Try and figure out how we can teach teachers to teach. Preservice teachers to be good inservice teachers." (Mr. Casey, 04/30/2003, Lesson 11, 1:52-1:53)

As far as the change in teacher behaviors is concerned, the presence of the camera and a researcher affected teachers to be more careful about what they were doing and try to make it better. I do not think it changed much of what they should be doing. This was evident when Mr. Casey explained how the presence of camera had changed his behaviors a little bit in a positive way.

> "You know I have to be, uhm, always careful what I say you know trying to be politically correct and not have gender bias and those kind of things. But knowing that someone else is watching is, I think, it's a good thing … well I just- you know- I work a little bit harder all the camera is on ☺☺☺ I mean as far as getting around and make sure I see everybody not to give you an impression that I just go sit down when the camera is not here. But, you know extra couple minutes and you know looking over the stuff and trying to anticipate questions and that kind of things so that it flows little better." (Mr. Casey, I2)

<u>*Observing Ms. Sands*</u>

I had observed and videotaped Ms. Sands's classroom during the fall semester of 2002-2003 academic year for a period of sixteen lessons in which the third unit in the textbook called "The Burning Candle: Patterns and Graphs" (Sallee et al., 1998, pp. 90-121) was covered. I also observed and videotaped the classroom on the day they had the end of unit test. In addition, I observed and videotaped a morning tutoring session she regularly held early in the mornings before the homerun. The class met every morning between 9:50 and 10:40 weekdays. During the time that the unit was covered, there were a few days that Ms. Sands did not have the classes. She had substitutes doing vocational guidance and career counseling for students in those days. I did not observe these days.

It was written in the introduction of the chapter that previous units covered organizing data and patterns with the use of tables and in this new chapter, students were going to discover how to represent this information on graphs. The purpose of the unit was to give students opportunity to

- use patterns and organized data tables to draw graphs and solve problems.
- explore and use the xy-coordinate system.
- explore families if equations and their graphs, with a primary focus in linear and quadratic functions.
- begin writing algebraic expressions to describe the rule that governs tables of input and output values. (Sallee et al., 1998, p. 92)

The unit consisted of eighty-nine individually written problems, named as BC1 through BC89 covering tasks for the new and previously learned topics and concepts. In Table 1, I summarized the times and dates of videotaped observations, brief description of the content of each lesson, its duration, and the homework assigned afterwards.

Table 1

Videotaped observations summary for the case of Ms. Sands

Observation Date	Lesson Content	Duration (/min)	Homework
21-Oct-2002	BC1: Algebra Walk: Understanding Input/Output Relationships	38	BC2-BC7
22-Oct-2002	BC1 (Continued) & BC2	44	BC3
23-Oct-2002	BC8	43	BC9-BC18
24-Oct-2002	BC13, BC17, BC18, BC19, BC20, BC21	44	BC22-BC27
25-Oct-2002	Silent Board Game; "Tables and Graphs" Toolkit from Student Workbook on p.96	38	Syllabus, Toolkit
28-Oct-2002	Silent Board Game, BC28-BC36	41	BC28-BC36
29-Oct-2002	BC31; BC37-BC44	42	BC37-BC44
30-Oct-2002	BC40, BC43; BC45-BC54	42	BC45-BC54
5-Nov-2002	BC55: $-x^2$ vs. $(-x)^2$	41	BC55-BC63
7-Nov-2002	BC56, BC61, BC62	44	BC64-BC66
8-Nov-2002	BC-64: Discussing $1/x$	42	BC64: ($1/x$, all they can think about 0)
13-Nov-2002	BC66: Graphing $y=2^x$, $y=\sqrt{x}$, $y = 1/x$,	43	BC66-BC71
14-Nov-2002	Square root of a negative number, Cellular Phone Activity (instead of BC72)	44	Cellular Phone Planner
15-Nov-2002	Cellular Phone Activity-Day2; BC66	42	Write the business letter to conclude which plan is better under what circumstances
18-Nov-2002	BC79, BC80, BC81: Unit Three Summary	43	BC82-BC88
19-Nov-2002	BC87a, BC87d, BC85a, BC82, BC83, Distributive Property	42	BC73-BC78
20-Nov-2002	Unit 3 Chapter Test	42	-
19-Nov-2002	Morning Session: Mainly discussing BC-86a and BC-86b.	14	-

She had 28 students in the classroom, 15 girls and 13 boys. She grouped them in fours so that there are two boys and two girls in each group except for one of the groups where there is one boy and three girls. The physical setting of the class consisted of an overhead projector and its screen, a television mounted to the ceiling, two white boards mounted to the walls, two book shelves, a file cabinet, and a couch. There were also five computers; one on teacher workstation and four on the back of the room used for the remedial classes. The physical setting and my position during observations are shown in Figure 2.

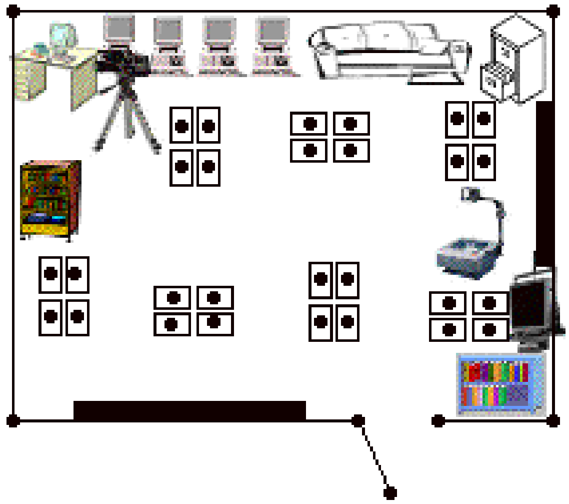


Figure 2. Observation and classroom setting in the case of Ms. Sands

Observing Mr. Casey

I observed and videotaped Mr. Casey's classroom during the spring semester of 2002-2003 academic year for a period of seventeen lessons each of which lasted about 90 minutes since the school has a block scheduling policy. I have excluded the last five lessons covering the chapter 12, "Relations and Functions," of the textbook from my analysis for this report considering that I only observed Ms. Sands's classroom for a unit and it was my plan to observed and analyze a unit. I observed the additional lessons simply due to having an opportunity to have some additional data for a further research paper on a topic that I am interested in, relations and functions. Furthermore, I did not see any reason to include the additional chapter instead of the one that I started, which was chapter 11: "Radical Expressions and Equations" (Smith, Charles, Dossey, & Bittinger, 2001) lasted twelve lessons excluding the end of unit test that I did not observe. The class met every afternoon between 1:50pm and 3:20pm weekdays.

As it was stated on the teacher edition of the textbook, the purpose of the unit was to get students learn about real numbers including square roots and cube roots; simplify radical expressions and to multiply, divide, and subtract them; solve problems using the Pythagorean Theorem and solve radical equations. In the student edition, it was stated that at the end of the unit students would learn:

- How to find square roots,
- How to simplify radical expressions,
- How to use the Pythagorean theorem to find missing lengths of a right triangle,
- How to solve radical equations. (Smith et al., 2001, p. 480)

The unit consisted of nine sections: real numbers; radical expressions; simplifying radical expressions; multiplying radical expressions; dividing and simplifying; addition and subtraction; the Pythagorean theorem; using the Pythagorean theorem; and equation with radicals. In the teacher edition of the textbook, the pacing was suggested as five days to cover the nine sections of the unit within a course with block scheduling. Each section first briefly introduced the topic, and then presented several solved examples for different types of possible problems followed by *Try This* questions asking to solve similar problems to the examples presented. After introducing each topic/concept that way, the book presented an extensive list of exercises under *A*, *B*, and *Challenge* sections with items increasing in level of difficulty to address "all ability levels" and "allow teachers to easily craft just the right assignment for individual students". Mr. Casey assgined homework problems among these exercises. Section A exercises were described as "practice by examples" targeting all students to give opportunity to review skills and concepts with items of mere calculation asking to, for example, find square roots, simplify, multiply/divide/add/subtract and simplify, etc. The section B exercises were as "Apply Your

Skills” and aimed to check the understanding of skills and practice for connecting skills to solve problems. The last section of exercises were called “Cahllenge Problems” and offered students little bit more open-ended problems and tasks to build on prior understanding and develop crtical thinking. In Table 2, I summarized the times and dates of videotaped observations, brief description of the content of each lesson, its duration, and the homework assigned afterwards.

Table 2

Videotaped observations summary for the case of Mr. Casey

Observation Date	**Lesson Content**	**Duration (/min)**	**Homework**
15-Apr-2003	Section 11-1: Real Numbers	97	Section 11-1, 1-40 on p.485
16-Apr-2003	Section 11-2: Radical Expressions	94	Section 11-2, 1-30 on p. 489
17-Apr-2003	Section 11-3: Simplifying Radical Expressions	95	Section 11-3, 9-44 on p. 493.
21-Apr-2003	[Review from practice book for next day’s quiz]	97	Preparation for Quiz
22-Apr-2003	Quiz-1; Section 11-4: Multiplying Radical Expressions	95	Section 11-4, 2-40 (even numbers) on p. 496
23-Apr-2003	Section 11-5: Dividing and Simplifying	95	Section 2-12 (even numbered), 13-18 on p. 501
24-Apr-2003	Section 11-6: Addition and Subtraction	96	Section 2-12 (even numbered), 13-18 on p. 505
25-Apr-2003	[Review from practice book for next day's quiz]	96	Skill practice 31: 7-42 on p. 39, Skill practice 32, 1-12 on p. 40
28-Apr-2003	Section 11-7: The Pythagorean Theorem	95	Section 11-7, 1-18 on p.510
29-Apr-2003	Section 11-7: The Pythagorean Theorem [Distance Formula]	85	Section 11-8: 1-6 on p. 513 and 1-9 on p. 515-516
30-Apr-2003	Section 11-8: Using the Pythagorean Theorem [Applications]	95	Section 11-9: 7-24 (even) on p.521, chapter Wrap-up 50-64 (even) on p. 527
1-May-2003	Section 11-9: Equations with Radicals	95	Chapter Assessment 1-36 on p.528

He had 20 students in the classroom, 9 girls and 11 boys. He organized the classroom so that students would sit in five columns each consisting of four students. Since I positioned myself at the very end corner of the class, on of the students moved to another place making one of the columns consisting of five students and leaving one consisting of three. The physical setting of the class consisted of an overhead projector and its screen, a television mounted to the ceiling, two boards mounted to the walls, one bookshelf, a teacher desk and a computer which is used by the teacher. The physical setting and my position during observations are shown in Figure 3.

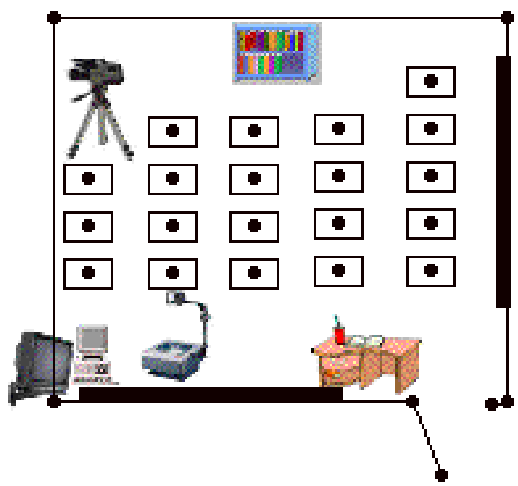


Figure 3. Observation and classroom setting in the case of Mr. Casey

Documents

Archival data was a secondary (and supportive) source of data for the study. As Hill (1993) pointed out, "In archival work, what you can find determines what you can analyze, and what you analyze structures what you look for in archival collections." (p.6) I wanted to overview and examine documents related to the teachers, their students, and the school. Thus, I searched for documents under three categories: School related, teacher related, and student related. School related materials were mission statement; school web site; newspaper releases about the school, federal and state education department information and statistics about the schools. The purpose in this analysis was to understand the school context better since I believed that school atmosphere could play certain roles on teacher and student attitudes, beliefs and successes. For the teacher related materials, I mainly had teacher prepared lesson plans in algebra. Even though I asked questions about lesson planning and got verbal information during the interviews, I wanted to see, to what extend, the teachers reflected their knowledge and beliefs, particularly about student understanding of algebra, in their lesson preparation. However, I was not able to obtain any written lesson plan because neither of the teachers had any articulated and organized written plan that they could have shared with me. I did not want to force teachers to do so considering the busy schedules they already had and the time they spared for me. Since I obtained considerable amount of data (to learn about the teacher) from the Internet web sites including previous course websites, local newspaper archives and school website, I wanted to follow the same strategy to obtain information about the teachers as additional or confirmatory to what I already had from interviews and observations. For student related documents, I collected graded copies of quizzes and unit tests for the whole class from each teacher. I used the end of unit test questions and student answers to prompt questions during the last interview with each teacher. I used quizzes to refer back and related teacher comments made; (to me) during interviews and (to students) in videotaped lesson observations. I collected artifacts such as student worksheets, graded quizzes and tests, and textbooks to make sense of discussions, conversations or some other details missing on the videos. Furthermore, those documents were also used during the interviews to open up conversations on curriculum, students' progresses and difficulties during the units covered and in algebra in general.

Data Analysis

Data analysis in qualitative inquiry is both an iterative and an on-going process. With rigorous standards, data and interpretations are continuously checked with respondents who have acted as sources, differences of opinions are negotiated and outcomes are agreed upon, understood and reflected. Glesne (1999) defines qualitative data analysis as “working with the data, you describe, create explanations, pose hypotheses, develop theories, and link your story to other stories” (p. 130) Although there are various ways of analyzing data in a qualitative research, it becomes a common sense in qualitative inquiry that data analysis is a continuous task right from the beginning of research. According to LeCompte (2000), the first step in data analysis should be identification of bias since it affects what the researcher sees and interprets. I took this point into consideration in my data collection and analysis. This issue is explained later in the *researcher biases and assumptions* section. I considered and performed analyzing the data I collected under four categories: data management, description, analysis, interpretation and representation. I consider these as categories (not steps or stages) based on my experience. Because, I often found myself cycling around those processes and even crossing boundaries of each if there was any in reality. I shall now briefly describe my experience and methodology in each category.

Data Management: Handling and keeping track of data as study moves along is the initial step in data analysis. This issue is referred as “managing data” by Dey (1993, p. 75), “tidying up” by LeCompte (2000, p. 148), and “early data analysis” by Glesne (1999, p. 131). I started analyzing the data by managing what I was collecting: interviews, documents and videotaped observations. I first transcribed the interviews recorded in audiotapes and transformed them into text files that could be stored and reviewed in digital formats.

I followed a similar approach with the videotapes I recorded during the classroom observations. I used a Macintosh based computer with iMovie and QuickTime software to review and digitize the contents of the tapes into movie files. During the first review of the videotapes, I created lesson graphs of each observation or lesson where I outlined the main events, flow of the lesson, basically what is happening in the lesson with the time codes and comments that I would think of at the moment for guiding me through later in analysis and synthesis. During that process, I also did some transcription of the portions of the lessons that I thought important and would use it during analysis and writing. Since computerizing videos was very time consuming process with amateur equipment, I decided not to digitize every lesson in its full length; I only digitized the portions and created mini video clips that I considered important in terms of my research questions, were related to what Ms. Sands told during the interviews. However, I decided to digitize each tape in full length in Mr. Casey's case. The basic factor in this decision was seeing the easiness of reviewing and transcribing the information in digital format. I thought this was necessary not only for writing this dissertation but also for any future need to go back to raw data and review for another type of analysis that I would do for a research paper.

Once I completed digitizing the tapes, I was ready to transcribe the clips or portions of the videos. The first challenge was to use a systematic notation or transcribe it conventionally. So, I used the conventions explained in TIMSS 1999 Video Study Transcription/Translation Manuals (Jacobs et al., 2003). A summary of those conventions was provided in Appendix D. The second challenge was to control or get the QuickTime to behave/act like a transcriber so that I can play, pause, rewind, forward, play from a few second back, and etc. with keyboard shortcuts. To solve this issue I have found and used a software called *Keyboard Maestro* along with an Apple script containing a small OS X programming to control the QuickTime. During all those processes, I was also accumulating, all my notes, artifacts, interview and video transcripts, video clips, lesson plans to categorize/organize the data into file folders on the computer and as tangible texts.

Description: Using data from multiple sources (interviews, observations, and artifacts) I articulated a "thick" and rich description of the persons and context they are in terms of school and curriculum. I also consider the lesson graphs as a part of description process since they describe what was in each observation.

Analysis: During this process, I manipulated the data by creating and applying abstract categories and then use those categories to compare, contrast, sort, and refine distinguishable thoughts, behaviors, and events. To facilitate those processes, I first tried to use a computer-assisted data analysis program called *Tams Analyzer* (Weinstein, 2002), available for Macintosh OS X operating systems. I used it with the interview data for Ms. Sands's case. I later decided that I did not need to use it simply because I did not have many individual interviews that I could not handle without using it.

In order to make connections among stories, the *Interpretation*, analysis began with identification of the themes emerging from the raw data, a process sometimes referred to as open coding (Strauss & Corbin, 1990). During analytic coding, I identified and tentatively named the conceptual categories into which the phenomena observed would be grouped. The goal was to create descriptive, multi-dimensional categories, which formed a preliminary framework for analysis. Words, phrases or events that appeared to be similar were grouped into the same category. These categories was gradually modified or replaced during the subsequent stages of analysis that followed.

As the raw data were broken down into manageable chunks, I devised an "audit trail"; that is, a scheme for identifying the data chunks according to their speaker and the context. I used "the voice" in the text; that is, participant quotes that illustrate the themes being described. The next stage of analysis involved re-examination of the categories identified to determine how they are linked. The discrete categories identified in open coding are compared and combined in new ways as I begin to assemble the "big picture." During this time, the initial categories identified were revised, leading to re-examination of the raw data. The next step was to translate the conceptual model into the story line that will be read by others. Ideally, I wanted the report to be a rich, tightly woven account that "closely approximates the reality it represents" (Strauss & Corbin, 1990, p. 57).

Interpretation: Informed by descriptions and analysis it is the stage where I made inferences by connecting the data with the theoretical structure that frames the study.

Representation: The writing up of the study and the cases in particular was considered in a formal or traditional dissertation format. I tried to enrich this formal structure by writing up realistic, confessional, and impressionist tales (Van Maanen, 1988). My role in the writings was somewhere between *translator/interpreter* and *transformer* (Glesne, 1999, Glesne & Peshkin, 1992). I did not only want to understand the others' world and then to translate the text of lived actions into a meaningful account, but also I wanted my readers, as they read, to identify with the problems, worries, joys, and dreams that are the collective human lot.

Validity, Reliability, and Trustworthiness

Many qualitative researchers have been criticized by critics who have claimed that the research is too subjective, or the number of cases is too small, or that mere talking is never a scientific method (Merriam, 1995). These are obviously the questions of validity and reliability, which address *trustworthiness* of a qualitative study. Today, it is widely accepted by academia that qualitative research is no less rigorous than more traditional forms of inquiry in terms of validity and reliability.

According to Lincoln and Guba (1985), trustworthiness addresses the question: "How can an inquirer persuade his or her audiences that the research findings of an inquiry are worth paying attention to?" (p. 290). Rigorous standards are generally applied to quantitative research such as truth-value or validity (internal validity), applicability or generalizability (external validity), consistency or reliability and neutrality or objectivity. According to Lincoln and Guba (1985), comparable rigor is often maintained through standards imposed on qualitative research including credibility, transferability, dependability and conformability. In general, eight verification procedures are used to build the trustworthiness in qualitative research: prolonged engagement and persistent observation, triangulation, peer review and debriefing, negative case analysis, clarification of researcher bias, member checking, rich/thick description and external audit (Glesne, 1999; Lincoln & Guba, 1985; Merriam, 1995).

When evaluating the validity and reliability in qualitative study, it is important to remember the assumptions of qualitative inquiry and kind of questions it can/may answer (Merriam, 1995). In quantitative inquiry, internal validity refers to the extent to which the findings accurately describe reality. Lincoln and Guba (1985) state that "the determination of such isomorphism is in principle impossible" (p. 294), because one would have to know the "precise nature of that reality" and, if one knew this already, there would be no need to test it (p. 295). The qualitative researcher, on the other hand, assumes the presence of multiple realities and attempts to represent these multiple realities adequately. Credibility becomes the test for this. In this study, a variety of techniques were employed to ensure credibility of data and data analysis. Through prolonged engagement with the participants, both with interviews and in their classrooms, an attempt was be made to establish and maintain an effective vehicle for obtaining and processing reliable information. Furthermore, I maintained credibility through on-going dialogue with participants, numerous observations, and by the use of member checks. The participants, my major professor, and I scrutinized the data and their interpretations. Member checking with the two participant teachers confirmed that data stories were representative of their beliefs and nature of events that took place during the observations and interviews:

> "I think you captured my feelings and the class expectations in a positive manner that reflects how things really were in that class." (Ms. Sands, e-mail conversation, June 9, 2004)

> "Overall the description of the class and my personal philosophies are fairly accurate." (Mr. Casey, e-mail conversation, June 10, 2004)

On the other hand, triangulation, to seek multiple and comparative opinions about the same topic or issue, was another means of strengthening data collection and analysis through confirmation and completeness. Furthermore, triangulation provided the convergence, inconsistencies and contradictions in the data and/or phenomenon under study (Mathison, 1988). Denzin (cited in Mathison, 1988) has identified four types of triangulation: data triangulation (including time, space and person), investigator triangulation, methodological triangulation, and theory triangulation. Additionally, two more types of triangulation can be added to the list: interdisciplinary triangulation, and analysis triangulation. In this research, I tried to use all of these types:

By source - data was collected from different resources (e.g., documents, video-recorded classroom observations, interviews) and sources (e.g., teachers and students)

By methods - different data collection strategies were used such as individual interviews, video-recorded classroom observations, and analyses of related documents;

By Investigator - peers and major advisor were consulted time to time to analyze the data,
By theories - multiple theories and perspectives were addressed in literature review process (see Chapter 2) and considered during data analysis and interpretation.

Consistency, which may also be known as reliability, is dependent upon stability, consistency and predictability (Lincoln & Guba, 1985, p. 296). In quantitative inquiry, reliability refers to "the extent to which one's findings will be found again. That is, if the inquiry is replicated, would the findings be the same?" (Merriam, 1995, p. 55) Considering the instability of human behavior and social factors, qualitative researchers seek a means for taking into account both factors of instability and factors of phenomenal or design induced change. Dependability in interpretive research is often accomplished using an audit trail (Lincoln & Guba, 1985, p. 382-385) in which the researcher maintains a log containing personal notes, which allow for reflection upon what happens in relationship to personal values or perceptions. In addition to triangulation and peer examination, the audit trail used in this study included raw data, data reduction and analysis procedures, data reconstruction and synthesis, and processing notes.

Applicability, or external validity for the quantitative paradigm, is measured by the degree of transferability. Small sample size and non-random sampling (mostly purposeful) in a qualitative study leads to a notion of non-generalizability across different settings. However, the purpose of qualitative research is "to understand the particular in depth, rather than finding out what is generally true of many" (Merriam, 1995, p. 57). Thus generalizability is interpreted differently in qualitative research. Three concepts, *working hypotheses, concrete universals* and *reader/user generalizability*, are considered as bases for interpretation of generalizability of findings of qualitative research (Merriam, 1995). In that context, interpretations are neither value nor context free. Thus, as long as similar context is provided, there is no reason not to generalize (or transfer) research findings in other types of situations. Moreover, “the general lies in the particular” (Merriam, 1995, p. 58). Most of the time, we make general conclusions from similar particulars. On the other hand, generalizability is useful to the readers of the research findings. In other words, it is up to the reader to decide the extent of generalizability of the research findings, not up the researcher (Merriam, 1995). For this reason, I provided the readers with *thick descriptions* (See Geertz, 1973) to enter the research context.

Qualitative research, which relies on interpretations and is admittedly value-bound, is considered to be subjective. In the world of conventional research, subjectivity leads to results that are both unreliable and invalid. There are many researchers, however, who call into question the true objectivity of statistical measures and, indeed, the possibility of ever attaining pure objectivity at all (Lincoln & Guba, 1985). Lincoln and Guba (1985) choose to speak of the “confirmability” of the research. In a sense, they refer to the degree to which the researcher can demonstrate the neutrality of the research interpretations, through a “confirmability audit.” In this study, confirmability was maintained by providing raw data that can be traced to original sources and by describing how the data is to be interpreted and placed into categories or conclusions (Lincoln & Guba, 1985, p. 384-385).

Researcher Biases and Assumptions

Objectivity vs. subjectivity is an old debate, possibly aged with the history of science. Epistemologically, objectivity holds the position that truth is independent of the observer. On the other hand, the subjectivity refers to the view that the knowledge and truths are based on the perceptions, reactions, beliefs and values of the observer making statements. The old positivist/quantitative concern for the subjectivity is that it may decrease the validity and reliability in the study. On the other hand, naturalistic/qualitative inquiry sets that subjectivity in a qualitative research does not always compromise those issues, or in other words, it does not cut its usefulness. According to Peshkin (1988), subjectivity is virtuous in terms of researcher's unique contribution to the data collected and thus it must be present and searched for. I believe that objectivity is not possible without first accepting our subjectivity. A study could be on the path of objectivity only if the researcher accepts and presents the subjectivity in it. I am a non-American, white, middle class, 28-year-old male and here I had to look at my preconceptions and attempt to gain clarity of vision.

My first bias would/should be my personal reasons to choose the research problem that I introduced at the background section of the rationale in the first chapter. I consider this one as one of my biases. My second bias that I bring into study is my different country, culture, etc. orientation. Although I have been reshaped as a thinker, educator, and human being during my doctoral study in the United States, I am still a minority in this country and I am not very well into the American culture and educational system. Thus, my original culture and nationality still affect my perceptions. My third bias should be my personal vision of education, teaching and learning in general and in mathematics in particular. I believe that my perceptions and mechanics of sense making are bounded by the frame personal vision of teaching and learning constructed for me. I always look at related events behind those eyeglasses. My fourth and last bias is that I had little experience in teaching [mathematics] at K-12 institutions. So, not being in the shoes of those two teachers that I worked with probably limits my understanding of teachers' thinking and some contextual factors surrounding and affecting it. However, I believe that prolonged engagement with the teachers and presence in their classes enabled me to minimize this and be emphatic.

Using video in qualitative research may bring some biases along with it. Roschelle (2000) cautions that the gains that video brings to research can cause researcher to miss the limitations of the medium and have serious misconceptions. He lists four possible misconceptions that researchers should be aware of so that they can control the biases of videotaping: video captures what an observer would see, video has no point of view; video captures context; a research video is like a research paper. I have considered those limitations and common misconceptions of using video in my research to reduce possible biases it brings. As Roschelle (2000) puts it “video is a constructed artifact, not an "objective capture" or a "holographic memory" of the original happenings” (p. 709-710). The truth is probably the otherwise. While I was focusing on one thing, I was probably de-emphasizing or neglecting something else.

CHAPTER 5: THE CASE OF MS. SANDS

Ms. Sands's Beliefs About Mathematics and Algebra

The older Ms. Sands got, the more she thought mathematics as "really a problem solving kind of a thing" (I2) and she wished she had learned mathematics this way. In the mathematics to which she had been exposed, the kinds of problem solving students had to do was pretty basic, involving only procedural operations and their different combinations: "There is not much more to do but add, subtract, multiply and divide; they just come up in different situations" (I2). She thought problem solving was contextual and one would need to have a purpose or a real life problematic situation involving mathematics in order to be motivated for solving the problem. Nobody would just go home and sit down and divide fractions or do some other computation; there had to be a purpose like putting up some recipe so that it would give an actual meaning and make each problem worthy of solving. She would rather want to solve or do something with a problem involving a floor plan than solving a page full of multiplication. She had found the latter boring and having seen the problem solving approach she understood why the computation alone was boring. Even though she did not conceive the basic operations such as addition, subtraction, multiplication, and division as problem solving in its entirety, she considered learning those basic skills as necessary for knowing how to solve problems. For her, if one did not know what to do instinctively, or what kind of operations he/she needed to solve a problem, it meant he/she did not have basic information for problem solving. Even though she was not certain about it, she also thought one could learn to solve problems while doing it: they might both go simultaneously. On the other hand, although Ms. Sands thought of mathematics as problem solving, she also thought of it as sequential, meaning that one had to have a foundation to be able to do the next step or something more advanced: "It's so sequential in mathematics anyway, you need a foundation in order to go to something else" (I2).

Ms. Sands's beliefs about the nature of algebra were parallel to those of mathematics. For Ms. Sands, algebra was very different than what it had used to be, as she knew it as a student. Algebra, as she believed, was problem solving oriented through thinking, which had meaning and there was a reason for knowing it. However, algebra she had been taught was consisted of manipulation of symbols and series of computations only:

> Algebra is a lot different now than the way I knew it. When I learned it algebra was all computation. It was go home and do these twenty problems and tomorrow you gonna get twenty-five more and then the next day you have fifteen. It was all computation practice and this [CPM Algebra] is not like that at all. (I1)

It was evident that Ms. Sands's views were greatly affected by the CPM curriculum she was using and its problem solving approach through which a big idea was developed in a unit.

> I think there I think they are putting together a reason for knowing it, doing it the way we do. Although they don't have the old way to compare it to like we do. But I like the way this takes the problem solving approach where the *x* and the *y* [to students] may not be as obvious as it is to me. Because I know where they going with the problem. (I1)

These words also described what she meant by the problem solving, that is-- the situation where the answer was not obvious and it was a part of a bigger problem or its solution constituted to a bigger meaning. She also saw problem solving as putting algebra into a context to create a reason as a motivational force to do the operations (i.e., addition, subtraction, etc.).

Ms. Sands believed even though algebra was usually considered as being about *x* [i.e., symbols], she thought it was not the way to think about what algebra was. For her, algebra was not just about symbol manipulation. Algebra was a process of thinking through a problem situation, where there was something one did not know how to do, in order to find a logical way to solve it. She thought *x* would be handy for constructing equations to solve problems. However, it did not always have to be that way; problems could be solved without having *x* in them.

> You know it's always- you know- it's always about *x* but it's not always about *x*. It's always just there is something you don't know how to do and finding a logical way to solve it. Because we can make equations out of all of this but it didn't necessary and I think that's probably the good about doing it this way that there were no *x*'s in the probability and percent problem. It was all guess and check. There was a prerequisite knowing what percent means to doing problem but after that it was all guessing and they didn't know until now, now in this chapter they know what they're doing is trying to find what one *x* was and related to something else that they know and we didn't even use any of that in this and we were still doing it. But it was a way of thinking to make them, uhm, looking possibilities and what makes sense. (I3)

On the other hand, Ms. Sands thought that "algebra is broad" (I3) and students would need to know how broad it was as early as possible. This belief was represented when she valued having the last three problems in the end of unit test as about probability, arithmetic, and geometry. She pointed out that they were going to have similar problems in the semester test as well because "it is not enough just to know how to do one kind of problem and be good at it" (I3). As she thought, small pieces of knowledge would always lead to something else and require solving a bigger problem. She thought that finding the area of a particular figure, for example, would not stop there; it would be the reason for applying this knowledge to solve a problem like *the kitchen floor*, which they did in the previous chapter. When I asked her about how algebra would be different, if it was, from arithmetic and/or geometry from content perspective, she expressed that she used to think they were all separate but as she had to teach more and more algebra she started thinking that "they are all *so* connected" and "it makes much more sense when you make those connections" (I1). Even though she might not see it in everything that she did, when she talked about, for example, graphing and equations, she found the connection between algebra and geometry (I3). She also pointed out that it would be hard to make certain connections like the distributive property and an area model for it if one was not coming from such a background (i.e., where one does not solve problems involving those situations). On the other hand, she was not sure that students would be aware of the connections between algebra and geometry.

> I don't know. They are not ready to make those connections. I don't know what that is. I don't know if you have to reach maturity to understand some of those things. (I1)

She suggested that connections would not be obvious unless the content was presented from both perspectives at the same time to allow comparison. For example, teaching an algebraic manipulation involving the distributive property would not guarantee transferring it to an area model until seeing both at the same time and making the comparison.

> When I think of doing the distributive property in an area model it's very interesting. But since they haven't seen at the other way they don't see it that way. They don't see it from as many sides as I do then I didn't see it as an area model ever before…until now. I don't know why! I guess it was just never presented to me that way and so it didn't occur to me. And I didn't think deeply enough about it to come up with all my own. (I1)

Ms. Sands's Beliefs About and Practices in the Teaching and Learning of Algebra

Ms. Sands was very enthusiastic about teaching and learning of algebra. For her, algebra was like "little puzzles" and "they can be fun" to learn (I1). As a teacher, Ms. Sands defined her role as facilitator of students thinking. She did not think that she was the sole interpreter of the content and the source of knowledge. She often preferred to put students in pairs and let them talk to each other and learn from each other. She thought that her role required asking questions in order to make students think. As she believed that algebra was thinking process rather than meaningless manipulation of symbols, she often emphasized the importance of thinking rather than computations in learning algebra.

Responses such as “The answer is not the important thing” and “I really want to see what you were thinking in the middle” were not rare to hear as a response to “So, what is the answer?” in Ms. Sands’s algebra class. For Ms. Sands, it was how to solve or start the problem that was important, not making some computation errors and getting a wrong answer. That is why she demanded to see the students’ work and solutions on the paper instead of scratch paper. A statement like saying “I tried but couldn’t get” was not enough for her, as she believed “Trying means there’s something on your paper!” [Ms. Sands, 10/22/2002, Lesson 2, 10:08] By doing so, she could see if all or some of students were thinking the wrong way so she could put those back in track. She thought students could arrive at the right answer by “really thinking” or using some thought processes instead of computation since without reasoning or using some thought processes computation would not be effective. This did not mean, however, that she did not value computation at all. She thought students needed to be good at “main computation problems” so they could use computation when they needed it. It was her responsibility to promote such a student centered learning without directly telling them when and how to use something.

> I think it’s my job to ask them questions that make them think instead of telling them the right answers. So, that they arrive at the right answer through some taught process instead of mainly the computation process. I think the algebra students in 8th grade ought to be really good at main computation problems so that they know when to use them. I think it’s my job to … help them think through when they should use something and not tell them when to use something. (I1)

When I asked her about why she would value thinking process more than the answer because usually what mattered or was important in mathematics classrooms was having the (correct) answer, she agreed this was the general notion in mathematics classrooms and she used to think so too: “I used to think it was great when I can just go home and grade papers and check off answers” (I1). She obviously did not think that way any more but she was not sure exactly what kind of experiences she had experienced to change her thinking and it was something she wondered about herself. She believed she had seen many students who were not learning and who could not do something and getting them to explain what they were doing and thinking was the only way to help them understand their difficulties.

> I wonder that myself. I guess because I saw too many people not understanding why they did something and sometimes it's easy to get an answer but if you really want to find out why somebody can't do it you have to know what they did to get there. (I1)

It was not until she started getting student work with so many things wrong that she became interested in seeing student work or thinking processes. She could not identify why they were doing it wrong and she thought that seeing their work would be an answer:

> I did it this way because you know whatever it was they were trying to solve but I didn't used to think that way until I start to get so many things that were wrong and I didn't know why they were wrong because I never could see their work. And I decided well if I know that they are doing the right thing we can solve the little things but we can't solve the big things if we don't know they started a problem. That was big step for me though. (I1)

Ms. Sands expressed that her change came soon after she started teaching algebra. As she elaborated on her first year of teaching, she had observed students having difficulties she would not anticipate even on more straightforward concepts. That triggered her try to get students to express their thinking to find out not only why and if they were having difficulties but also how and if they were finding the right answers in different ways than she would do.

> The first year I kind of just I taught it. Stood right there and taught. And I guess after that first year then- maybe the next year too- I started to figure there has to be a little bit more of this and just getting the right answer. They kind come a time the old way like when we would start doing things like equations of lines and slope and when they were really started to bug down and I wasn't really sure why they're bugging down because it was just arithmetic, right? You put this in and you solve the equation and draw it and there it is and it was easy. But they could not necessarily put the equation and the picture together. And that's probably what did it for me when I think back on it. Not knowing myself when I was in 9th grade what the equation and the picture had to do with each other either. Now it's just so clear. I don't know why that changed but I knew that I had to do something to figure out what they were thinking. And if they never wrote it down

> and they could not explain it they have a really hard time verbalizing. They can write it down better than they can explain. So if they write it down I'll be able to tell where they went it wrong and if they didn't go wrong they're going to get the right answer anyway. I mean they'll always come with some answer. And if they get the right answer… it helps me to figure out what they did if they did it a different way than I would have done it. They can still do it and get the right answer. They learn some of that from each other. (I1)

She also did not think having the right answer or wrong answer was enough if there was no explanation about how it was found. This would not suffice for her expectations as she believed that one should be able to explain how she/he had arrived at the answer, especially if it was correct.

> I don't know I don't know if I can put my finger on why that happened but I know that it didn't make sense to me to put answers on a piece of paper. They couldn't explain to me how they got that answer. Maybe that's part of it. Well, how did you, if it is right you know it right how did you get it? Well I don't know. Well where is your work? Well I throw it away. And maybe it was partly just their habits made me to change my mind too because I wanted to see where it came from even if that was right. Because they could not tell me how they got a right answer. Much less how they got a wrong answer. It's really hard to tell how you get a wrong answer but if you can explain how you got your right answer you should be able to. I don't know maybe it's a combination of those little things that made me changed my mind about what their paper should look like…. But I am serious. Very serious about it. (I1)

She also mentioned that helping her own children and observing their struggles with mathematics might have something to do with her change as well.

Ms. Sands also thought, "If I can learn to ask the right questions and not tell them everything, they'll learn more of it" (I1). She did not think, however, that she was doing this (i.e., asking the right questions) as well as she could and improving it was her big issue. The main reason for this was lack of knowledge of knowing how to do it. As she thought back in time, she recognized that she had not learned algebra from a problem solving perspective and began teaching as she had been taught. Thus, she described herself a learner in the process of algebra as problem solving.

> I think it comes from probably the lack of … teaching how to teach math. Because I haven't been taught how to teach math. I think the way we teach to them now in problem solving is … the best way for me to learn how to do it. Since I didn't learn how to do it that way for the first time myself back in high school. I didn't learn to do this either. (I1)

Ms. Sands saw teaching algebra classes as a challenge. The main reason, as she expressed, was the school was trying a new curriculum that was very different than the previous curriculum in terms of demanding students to really think. Students were not used to this in their mathematics curriculum:

> My pre-algebra classes are a challenge…partly because of our new curriculum, and thinking aspects. I think it goes along with Algebra too though; they are not used to really thinking as hard as they used to. (I1)

For Ms. Sands, teaching algebra was not much different than teaching mathematics in general. She thought, however, that there were reasons for knowing algebra and if students did not see a reason to know, they would tune out. She considered algebra as something of practical use in life whether we were conscious about it or not:

> When I think about their parents saying you know I never use algebra and I think did you choose a cell phone plan? If you did like it or not you're using algebra because you're trying to figure out the best way to solve this problem even though you didn't take out paper and pencil and work equations you were still thinking in your mind "OK, it's gonna cost me this much for month and this much per minute and that's all algebra. And that's what I want them to do... to be able to figure out. It's just a way of solving problems. It's not going home and you know trying to figure out your bank balance using algebra. It's not that it's the way of thinking about solving problems just you run into everyday when making choices. Because any time you make a choice you have to weigh something against something else. And balance is out. You solve the equation so. (I1)

Her comments also highlighted the parents' role as an important element in teaching algebra or in education general. As she pointed out, for many parents algebra was something one would never use or had very little use for in everyday life. As she explained, the main reason was that they thought algebra was paper and pencil and they did not ever think it was problem solving.

This notion was not particular to algebra. Parents usually thought the same way about mathematics in general and algebra in particular because it was a part of mathematics.

> I think they believe it's math, just regular math. They don't understand the letters. It's all letters. Not as in word problems but it's x's and y's and a, b, c. That's the way they see it. And of course we taught algebra that way, which makes it interesting to wonder why people wanted to teach it. If we're taught the old way, why do you want to do it this way? It makes so much more sense. (I1)

Ms. Sands frequently expressed that learning algebra would demand a certain prerequisite knowledge base, particularly some important concepts of arithmetic such as fractions, decimals and percentages: "when I think about what they should know you know they should just…really be well familiar with things like fractions, decimals and percents" (I1). Her comments pointed to her views of arithmetic as a required knowledge base to learn algebra. Furthermore, she pointed out that the current students had taken pre-algebra, and saw some of the arithmetic concepts such as proportion, percent, ratio, and she was hoping they had seen enough of those concepts in some other background so they could get through the algebraic ideas. Although there were things they did not even cover in 6th grade at that point in their learning, she thought the students were doing fine and she had the confidence they were smart enough to catch on without having to re-teach most of the topics.

Ms. Sands saw herself pretty flexible in her instruction. As long as "they are on the way to learning" she did not mind if they would not get as far as she wanted. She was well aware that there were many things to cover. But, getting them covered in a hurry without understanding was not going to help with further learning because "they [the students] just won't understand what comes in two months if they do not get the foundation where we are now." If the students would understand the mathematics concepts when they were introduced, it could make learning later concepts easier. Ms. Sands's relaxed and flexible notion about time management and emphasis on quality teaching and putting student learning before all was evidently affected by the schools philosophy on this issue. The principal had made it clear that "you don't have to cover everything but what you do cover you need to cover well."

Problem Solving in Teaching and Learning of Algebra

Ms. Sands believed in problem-solving approach and it was what she was supposed to do in her classroom: "I have felt like problem-solving approach is what we are supposed to do because we made the change from the traditional curriculum in the first place to problem solving" (I2). She expressed that teaching and learning through worksheets would not satisfy students' curiosity or need for knowing it. In this sense, a problem-solving approach applied to real life situations would give meaning to learning and motivate students for wanting to learn more of it.

> I do wanted to be more fun then sitting around, doing worksheets all the time. Some people like them but most people really don't. I think there is turning point where they really wanna know why you have to do it. And if the problems are meaningful and I think most of these are, then they'll see the need to do it and then they'll want to do more of it. (I3)

She also felt the same for herself: "I think if I had learned algebra this way; it would have been a lot easier; there is a reason for knowing it" (I1).

On the other hand, she did not think her students had understood the value of a problem solving approach or the types of problems they had been working. She was hopeful they would realize the importance of problem solving in the future (maybe in 10 years) and be glad "they learned to solve problems that may have meant something to what they have to actually do" (I2). Even if they might not remember the problem or the picture on the textbook in the future, solving the type of problems in the CPM curriculum would contribute to their development as problems solvers in such a way that they could recall back the skills and information when they needed it.

> If they have to paint a room, they may not go back in their mind to the picture on that page but they will- it'll be in there somewhere. There is a hook in their brain somewhere where they're putting this problem so that they can pull it back up when they need to know what to do when they have actually to solve that problem. (I2)

On the other hand, she did not think that problem solving was working very well for some students who did not know what to do when they saw a problem due to lack of skills. That is why she had the extended learning time and doing remediation where she believed she was basically doing skills. She considered the degree of lack of skills problematic because they were still counting on fingers in eighth grade and she thought that was not normal; there had to be a level of proficiency and counting fingers did not count as proficiency to her. She did not know where the problem started but did not want to be blamed or would not put the blame on anybody below her either. She thought previous teachers were as frustrated as she was with the level of proficiency with their students.

> I would like to end it but that's not always possible. I just, I don't like sending kids to the high school who can't find the percent of the something. Then we may all go shopping they'll all see 25% of and I just think that they should know what that means. And I don't wanna be blamed because I didn't teach it to them. (I2)

She believed teachers would like to pass the blame on down to previous teachers as they found that students could not solve a problem right away. Although she described the remediation class as a "very, very frustrating time of the day," she considered it necessary so that "all the middle schools have to have the extra amount of time to make sure that remediation is being done" (I2).

Textbook in Teaching and Learning of Algebra

Ms. Sands found the sequences of topics appropriate and believed in CPM's approach to teaching algebra. Although she had not read the other textbooks of the CPM sequence, such as algebra-2 and geometry, in detail, she thought she "kind of" had the big picture of the whole CPM approach from the table of contents. She was not sure, however, of where it would lead nor would she have a real big picture without seeing how it was presented in classroom, but she would guess where it was supposed to go from the algebra she knew.

Ms. Sands's beliefs about the nature of mathematics as sequential as it would require a knowledge base and her trust on CPM's problem solving approach affected how she looked at the textbook. Although the school administration emphasized that the textbook was not the curriculum and she thought so as well, she felt she could only work within the order the book had provided since it was providing a foundation by following a spiral approach building little by little. She could pick a skill students were missing and work on it until they were comfortable with it although she did not think she needed to do so because she did not assign problems where the students did not have the right skills. She thought she would need to do this in the next chapter, which was about equations, because she knew that "they do have a little uncertainty about integers and what to do with them" (I2).

Ms. Sands could not afford not to follow the textbook, as she needed to read it ahead so that she could keep up with where the problems or ideas were leading the students.

> I really have to read ahead. It's really hard not to depend on that book because it's so problem solving oriented you don't want to tell them how top do problem of they're supposed to figure it out. And the book tells you that they're not supposed to get this now but they're gonna get it later. It is really hard not tell them what they're supposed to get. (I1)

Her expressions suggested a dilemma. She wanted to tell or explain to the students about where they were headed with the problems but she could not because "the book" told her the students were supposed to get it later, not at the moment. This created a dilemma of *to tell, or not to tell* and "it's a hard thing to get over telling too much" (I1). This dilemma was also affecting how she was helping and interacting students in the class. The only things she had done to help them was to clarify what the problem was asking them to do. She tried to do this very carefully because she did not want to "tell too much" but she felt "it is hard not to tell" (I1). She liked doing algebra and she wanted her students to feel the same way about it. However, she thought that if she told too little her students would not notice her enthusiasm or think that she "hated" algebra, which "would not be true anyway" (I1).

She described the CPM approach as doing small pieces, problems, without a bigger picture in mind or a reason for doing it. When I expressed my confusion about how this was different than doing computations as a response, she first presented an approach from a traditional text perspective as follows:

> I think when we're teaching it from a traditional text before we had, uhm, a lesson on putting points on the coordinate plane and we would do that for the day. The next day we might move into how do points make a line and we might learn to draw some those on you know put points on the coordinate plane and connect them to make the line then we might learn *x*'s and *y*'s and simple equations if you evaluate this equation for *x* being 3 then you can find that what the *y* is sort of what we did today and you just plot those points on there they do make a line and they start to see that this happens then we are going to learn what slope is next day and it's all just very sequential. (I1)

She continually maintained this was different in the new text (i.e., the CPM) as CPM would begin a chapter with a big problem (but they did not solve it immediately) and all the little problems after that were steps to give students the skills to solve the big problem.

> This way they learn more. They learn more of the big picture but they don't see the big picture. They have to later on some other units find this "Aha!" "Oh, I get it!" When we're just finished the unit 2 that we did- was it unit 2? Unit 1, about the kitchen floor. And the big problem at the very beginning of the chapter was to find the surface area of someone's floor and how she is going to put tiles down and they had had no clue what that meant. But as we worked all these you know hundred problems getting toward that end when we got to it I told them now go back and look on page whatever to the problem as it was first introduced to you and go back to what we're doing today or even what we did yesterday and they said "oh, it's the same problem. I get it!" And they have to figure out everything that we had done between zero and one hundred set them up to this problem. (I1)

As I summarized and asked her if she was saying "when you start the unit it doesn't make a lot sense to students but at the end it really does", she was not sure about it. She thought students got something out of it but they still might not see that the pieces along the way were pieces of a big problem:

> I think it does but I don't they have… I don't think they are quite savvy enough to put together with the fact that all the little pieces are leading them to the big problem. (I1)

She believed the reason for this was students' lack of experience with this new approach and she hoped t they would start making connections as they were more experienced with the approach.

> I think they will get there but they don't have enough experience with it yet just after two and a half months of using this type of …trying to say "approach." But once they have done it once they have figured out the beginning problem is the same as the ending problem they'll hopefully, by the time they get to the end, remember that that's what they were started to learn from the beginning and all the little things that we did took them there. I hope! (I1)

Being able to see the big picture presented in the textbook was one of Ms. Sands's major difficulties as students did not always see the mathematics and where they were going with it whereas she did: "I can see when I do it. I can see the math its in it and they don't necessarily see it. And that's the main difference right now between them and me. I know where they're going but they don't" (I1). This knowledge of "big picture" became the difference between her and the students as she felt "Right now I am the only person who knows what is going on!" (I1)

Technology in Teaching and Learning of Algebra

Ms. Sands attended to a professional development workshop for developing teachers' mathematical knowledge using technology and was familiar with commonly used technological tools in mathematics education such as Graphing Calculator (or NuCalc), spreadsheets, and Geometer's Sketchpad (GSP). She did not, however, have any of these software tools on the four computers in the classroom or anything that would substitute. Furthermore, she thought it would take a lot of training to make students how to use, for example GSP, before they could do it. She suggested she could have used computer technology by putting a problem like x^2 or $x^2 + 2$ and moving the vertex up and down on the y-axis and letting the students observe it was still the same shape. She also mentioned she had graphing calculators available and "it's probably something they could explore" but "there is a lot of little lesson things I think they have to learn before they can use it" although the students knew some basics about graphing calculators. She mentioned that students did well the previous year when one graduate student from the nearby university came and helped with one of her classes about the functions of basic buttons and how to change scales in graphing calculators. Talking about the parabolas made her wonder if students would have the same type of difficulty in case they were provided with graphing calculators. She thought that although graphing calculators had some limitations such as limited screen size and scope, it could make students' understanding of a line and/or parabola more concrete in terms of continuity and infiniteness.

> It [Graphing Calculators] would probably make a little concrete to know that you can have bigger *x* values and see a bigger graph but your line or your parabola is going to fill up the screen no matter what and you have to realize that it's still infinite and I don't think that they are quite that abstract yet. Maybe a few! (I2)

It was evident she had a fundamental knowledge of or idea about appropriate and possibly effective use of technology of teaching algebra. Her view of the textbook or how she saw it seemed to be the primary reason she did not use such an approach: she could not situate it in the curriculum/textbook she was following: "I'm not sure that's where this goes and I haven't seen any reference to graphing calculators with this book" (I2).

Ms. Sands's Classroom Culture and Knowledge of Students

Ms. Sands described her algebra-1 classroom culture as "more fun" compared to other classes she was teaching. The students were ones with whom she could have a "real conversation," who would accept challenge, and who had a basic knowledge of arithmetic. All of these factors helped move the class smoothly.

> It's more fun because they understand the conversation that they can you can have a real conversation with kids who understand basic arithmetic and make them use it in different ways. Or make them decide how to use it. It's much different than trying to teach how to turn a decimal into a percent because it's so structured. There is a lot less structure and they also move faster. You don't get bugged down on a topic for too long like you do in my pre-algebra classes. They seem to really want be challenged and want to go. (I1)

When I asked about what would be the reasons for this and suggested several possibilities such as age, experience, maturity, and/or mathematical maturity, she agreed it may be something to do mathematical maturity but "they are not really any more mature than anybody else" (I1). She thought it rather had to do with their problem solving ability and their proficiency with arithmetic:

> It maybe some mathematical maturity they are not really any more mature than anybody else. Truly they are not. For some reason they have this problem solving ability the others don't have. And they learned to do arithmetic and it fit like a globe and they really understand what to do. But this is also a challenge for them. It's a good challenge for them. But the others I think they don't get because they have some deficiencies that haven't been found yet, which hopefully the extended learning time classes can fix to some extent. But these kids know… know what to do with the numbers. And they are probably having more fun doing it. And that's why it's a more fun class. Even for me. (I1)

Ms. Sands stated that her algebra-1 students liked to ask questions and they knew what questions they wanted to be answered -- "they want to know why something happens, which is sometimes not always known" (I1). Her pre-algebra students, on the other hand, always wanted her to tell them the answer. For the algebra-1 students she pointed out that they needed to figure it out together and she found it "kind of neat the first time you taught out of the book because you don't always have all the answers" (I1). She praised her algebra students in that they usually had more than one way of getting somewhere with the solutions and this was something she really liked -- "it helps somebody whose not sure because one of those ways just might fit the way they think" (I1).

Ms. Sands's knowledge of her students and their characteristics was well formed. Since it was a small community and "everybody knows everybody," Ms. Sands knew some of the students and their personalities. She did not know some of the students at all but she tried to get to know them better and see where they would fit as group members. She was also aware that some students had a lot of family and other problems outside school.

In forming a group, she would look for people who "will do things" (i.e., those who are confident) and she would pair them up with students who were less confident about doing algebra. She neither wanted to have a group of four students where nobody would talk, nor have a group where everybody would want to be "the boss" because they would never agree. She also considered the cultural and gender issues when she was forming the groups. She did not want to end up with a group of all boys where the discussion was about the football or something it should not be. Grouping was a big thing but she found it easy with 28 students in her class although though she had more girls than boys. At the end, she tried to change groups about every unit so that they can work with different peers all course long.

When Ms. Sands had a new student in her class, she would put the student with a group working well to see how they would work with him or her. Usually the new students did not share too much. That is why she would put a new student with students who were working hard, trying to make better grades, and have more understanding in what they were doing. The new students would then feel more comfortable with people and get acclimated into the new situation.

Ms. Sands thought that groups were functioning well throughout the unit. She had tried to change the group compositions in this unit so that the students could work with different situations. When forming groups, she was careful not to get either "too many strong people working together" or "too many weak people working together." In several groups she had strong people but she did not think that they were “very vocal about what they know and they won’t necessarily share or not confident to share” (I2). On the other hand, Ms. Sands’s knowledge of each of seven groups in this unit and how they were thinking and functioning suggested an insightful knowledge of student characteristics and needs.

> Yeah this group right over here has a lot of lot of talkers in it and they ask good questions they want good answers. They’re not just satisfied with you know, “Can you put x in it and make this parabola go further?” They wanna know if you can go further with it. This group over here is pretty weak, they work pretty hard but they don’t ask enough questions of each other- they need me more than they need each other. They do have two people in this group who are pretty sharp but they don’t have much confidence. This group over here, uhm, has really two thinkers in it but all of them talk to each other; they talk really well about the problems and they usually come up with solutions on their own and don’t have too many questions. If they do I think it’s just reassurance because they really do good job. This boy who sits right here is really on top of things and he is keeping his group going really well. He, uhm, there is a girl who has been missing all week here all week and I worry about that. She is good and I think she’ll ketch up real well but I don’t want it to be a problem for her. The other two- let me think who sits right here. I can’t remember who sits right here right now but I know that this guy works really well even by himself. Over here there are some weaker math students. There is one who does a good job and understands a lot. He’s the one who usually ask questions. Well he does ask the questions but the other questions that I get form this group is usually easy ones and I really had ask them questions back. But two that are sitting right here, this boy and this girl, she really got on top what they’re doing today better than anybody in her group and once she understood it she probably took care of these other two sitting here. Uhm, in this one, … (   ) bored. They are pretty average group. Nobody in there is probably the leader. I don’t think they have a leader. They probably need one. But they do- they do OK. They’re maybe a little shy and they really wanna know how to do something but they’re not really sure what to ask sometimes. Back in that corner, that last group has that one girl who got the x^2 figured out before she had drawn it and she is the really hard worker in that group and there are couple in that group that kind of play and then one that kind of hanging on to what everybody else is doing. So that one girl is probably the big leader in that group and she is probably doing everything and so- I think that’s everybody [i.e., all the groups]” (I2)

In general, she identified two groups that could function independently, having “all the answers” and how to get to it but wanting confirmation on their answers to make sure they were on the right track. The other five, as she thought, were the ones requiring more of her attention. She also thought that some of the groups just needed “a pat on the back” and wanted to hear frequently that they were doing fine.

Ms. Sands’s Planning and Preparation for Lessons

The way Ms. Sands was planning for algebra class was a little different than her pre-algebra. Although she taught the same pre-algebra curriculum the year before it was only her first year teaching algebra with this new curriculum and textbook. Thus, she was using the textbook as the primary resource in planning for her lessons and she needed to “read, read, read” (I1) as she felt that she “had to be on top of knowing” (I2). She needed to stay ahead of students so that she could anticipate where they might have trouble.

> When you teach something you’ve never seen before you just have to stay ahead on your toes so that you can anticipate their questions or notice when they’re having a problem what you have to do to make solving this problem easier. (I2)

As she stated, sometimes she was right with her predictions and sometimes not. There were times she was surprised because students had trouble with a particular problem although she did not think that they would. She was aware that she and her students did not see a problem in the same way and their understanding and approaches would differ. In those cases, she wanted students to talk to her about what they did. On the other hand, she found several ambiguous questions in the book that she did not understand what was being asked or how the answers written in the teacher version were found. She thought her students were going to find those problems ambiguous too and she wanted to be prepared for clarifying it for students even though she did not tell them what to do. Since CPM was originated as a California project, she considered the differences between thinking and talking styles of Georgians and Californians as a possible source of ambiguous problems in the book. She considered the possibility that Californians might phrase a question a little bit different than people do in Georgia.

On the other hand, although the teacher version of the textbook included solutions for all of the problems, she wanted to try to solve them all by herself and there had been times that she "slipped." She thought this was important because, otherwise, she would just skip a problem by just looking at its answer and thinking that "Ok, I got that one". But students might have really tripped up on that problem and she might not know what exactly or where they were having trouble with it. Thus, by solving the problems first by herself, she hoped to find the pitfalls students were going to find and prevent it before it would happen. Even if she was not able to do that, she at least hoped to know what the cause was so that she might help them in fixing their mistakes.

Ms. Sands's strategy in planning for lessons did not change through the unit. At the halfway point, she was still just trying to stay ahead. Although she had read the whole book, that just gave her an overview. She felt that students did exactly what she asked them to do so far in the unit. She had not bogged down in this unit although she had in the previous one. Students seemed to be handling this unit pretty well and she was just staying ahead of them. It was easy to plan because there had been fairly similar topics for the last three or four days. She needed to stay ahead probably faster than usual because of "the change in what they're supposed to do" (I2) as they were moving from lines to "curves or curve lines." If she had not read ahead to make sure she was clear on what was going on she would stumble.

Ms. Sands's Assessment Strategies

For Ms. Sands, the purpose of a test was to see if students could do whatever it was she asked them to do. Thus, what was going to be asked on the test was not a secret to keep from students. She was willing to let students know about the kind of questions she expected them to do.

> I'm working to decide what questions you need to ask. Basically what's on your test so you know what to do. I don't think tests are big secret. Right? If I want you to get me something, I need to tell you what to do, right? That's the whole purpose of a test. Can you do what I asked you to do. [Ms. Sands, 11/19/2002, Lesson 16, 10:04am]

Ms. Sands did not have a particular criterion to select items to ask on the quizzes and tests. She did not prepare her own questions either. She used the test that came with the book and she was happy with them so far. She considered those tests as really good representations of what students needed to know and do. She thought that so far the tests were very similar to what they had done in the class, not something completely different. She mentioned an example of an item in the previous test where students saw the exact same problem two days before the test and she knew that, it was OK to ask it. Because there were still students who missed the problem "for some reason." As they get further into the course, however, she wanted students to have "a little bit more transfer: Can you use this in a completely different situations than one you've seen before" (I1).

Ms. Sands's dependence on the textbook continued in her assessment strategies and items. The end of test called "Unit 3: Individual Test" came out of CPM's Assessment Handbook for Mathematics-1. To her, it measured what students supposed to learn in the unit. It contained eight problems covering various topics from algebra, arithmetic, and geometry: reading points on a graphs, graphing equations, describing shapes of graphs for given equations, distributive property, combining like terms, guess and check tables, order of operations and area of geometrical figures such as rectangle, square and circle. The test contained items from both unit three and previous units. So, it could be considered an assessment of students' conceptions up to unit four in algebra-1 within Ms. Sands's classroom context. She was very satisfied with the test as it was. If she could go back, she would not change it or prepare a better one asking the right questions. She thought it was a fair test and there was nothing tricky about the questions.

On the other hand, Ms. Sands had been thinking about assessment a lot and she would change the assessment if she had seen a need for doing it so. She did not know, however, how she would do assessment differently. She considered a “big combination project that goes with the unit” would be a good idea for a different assessment strategy if she could find one. She pointed out that she had changed the project at the end of unit three from the burning candle problem to the cell phone problem because she did not like the former. She had been using it for the last five years and decided that she could get the same results with using the cell phone problem. She thought that students liked the problem and really understood what they were supposed to do. To her, this was an evidence that students knew more than they could show and they were basically careless when committing mistakes and errors like they did when combining like terms. Furthermore, she stated she would like to give more projects to students if she could find the right ones to measure the same kinds of things that a test would. She thought the cell phone problem would be an example of that and she even counted it like a test and gave grades because she was pleased with the product students gave at the end and she rewarded them for it. She stated she would like to see topics in algebra applied to something -- not just think of them as a bunch of computations. She felt students tend to think of mathematics consisting of computation problems, just do the problem and be done with it. It would have nothing to do outside of the classroom. However, she thought that students enjoyed the cellular phone problem: “Somebody even said, "This algebra is fun!"” (I3).

As much as she wanted to make everything students did in algebra fun, she thought that having some struggles would also be good:

> Maybe some of it should be painful. And if you struggle through something you’ll spend more time trying to get it right. And some of them will do that with algebra, they will. I do it with science. That’s my struggle so I know where it is. But I don’t want them to struggle to the point of failure either. I really want them to get it and if I knew of other

> ways to assess drawing lines on graphs I probably would do it. At least to make it more interesting and I think all the problems in this particular series have been way more interesting than anything than we have ever done. (I3)

She thought that although as 13 or 14 year-old children students did not see the value of doing activities such as "The Cellular Plan Problem," they would eventually realize the value of it and make connections when they saw similar situations where they needed it.

> I hope that they'll see the value of this when they need to. They'll remember when they're in geometry or algebra-2, "I remember doing something like this. Now let's see what was it? We did the cell phone problem." and then they have started thinking about it and talking about it, things will come back to them that they really know and they can make that connection that this is really it does make math to do this so that they don't think that it's just a bunch of games. ☺☺☺ (I3)

She expressed that being a 8th grade teacher she did not have the opportunity to observe the growth of their students as much as a 6th grade teacher, who could watch them as they moved up to 7th and 8th grade, or a 9th grade teacher, who could see them for the next three years for how they changed and mature with their understanding.

Ms. Sands's Beliefs About Middle School Students' Thinking and Difficulties in Algebra 1

Sources of Student Difficulties

Procedural learning detached form conceptual understanding was an issue that Ms. Sands identified as a significant difficulty in students' learning of algebra. As she put it, "sometimes in algebra they know what to do with them but they still aren't real sure what they mean." For example, "they [students] can name all the parts, they can- you know, they can tell you some rules but then when they do it they don't do what they say they do" (I1). Lacking a conceptual knowledge, students even could not skillfully apply procedures even though they can tell you what should be done from their memory.

Ms. Sands identified two sources for student difficulties and mistakes in algebra. First, she attributed students' difficulties to lack of practice in solving similar types of problems. She thought she needed to make sure that students would solve enough problems of similar types so they can understand the concept.

> I think, uhm, I probably didn't write these kinds of problems out enough for them to really see what it means and that maybe my problems since I have been doing this is not to take enough time for the concept like that to be sure that they can do it. (I3)

Second, carelessness and acting without reasoning no matter what she does was a second source of problem that she identified. She did not have obvious solutions for it.

> Of course they are lazy and they will not write all of their steps out no matter what I tell them. "Oh, I got it. I can do it." Well I really want to see it. And then they'll hurry through it and they make some of those careless mistakes so they don't I don't think they see how much help is supposed to be to do it so that they get the pattern in their mind. (I3)

Lack of firm arithmetical knowledge was one of the first things Ms. Sands identified about students' difficulties in algebra-1. She mentioned, for example, students had problems with subtracting numbers involving zeros and dividing numbers when there should be a zero on the quotient as they would bring down the next number but they would not put the zero as 35 instead of 305. She described the situation as tough and stated she "I need to figure out why that is even though ☺ it's 4th or 5th grade" (I2) level subject. Although she probably needed a certain kind of scope and sequence to see where things should be mastered, at the moment she only knew that there were kids who could not do it. Similarly, she pointed out decimals and particularly fractions as problem areas. This added to her confusion since these were introduced way back in elementary grades. She frequently acknowledged that students always had trouble with transferring basic addition of fractions to situations where fractions had letters. She felt the lack of certain arithmetic knowledge for the students in the class as a stumbling block for their learning and urged students to come and ask questions early in the morning before homerun.

> Listen, listen. You have spent a lot of time on this. Some of us have issues. Some of us some real issues about positive and negative, where things go on the number line, what is a fractions. We have some big issues. If you don't know what something is, you need to be here at 7:30 to ask that question. [Ms. Sands, 10/23/2002, Lesson 3, 10:45am]

If she would do go back and re-teach some concepts again, she would do it briefly for simple things like a basic addition of fractions, like 1/2 + 1/3, involving algebraic symbols or letters. For her, "it's just a fraction concept problem." Even after "being there", she did not know exactly what the depth of the problem was because "they [students] don't describe it very well themselves what they don't understand." She considered students' problems with decimals small and could be overcome without much difficulty compared to fractions.

> They can work out the bugs with the decimals but fractions seem give them a feet. And, uhm, I don't know where that comes from. I don't know if it comes form lack of experience or lack of understanding because I think this is probably introduced in 4th or 5th grade that they don't have a real good concept of what the fraction really is. And then when we go and change it and call it a ratio and then when we go in change it probability; and we keep writing things the same way they really aren't sure what we're talking about. So I think they need something, some firm ground in what fractions and ratios are. And I am not really sure they have that even when they come to algebra. (I1)

Her explanations suggested an awareness of certain factors and elements affecting students' understanding. Although she was not sure if the problem with understanding the fractions was due to lack of experience or a conceptual issue, her explanations pointed more of a conceptual problem. As she pointed out the use of same representation (i.e., something over something) with different meanings (e.g., ratio, fraction, probability) a source of confusion as students could not differentiate between different contexts. This situation about fraction concept caused difficulties particularly when they were learning the slope concept. Students could not see the slope as a ratio since it looked like a fraction.

> I know that we are going to have that issue when it comes to slope because I've seen that one before. And you know rise over run is, it looks like a fraction so they are not sure what that means: rise over run. And they are not really sure what to do with it. We haven't hit that in this class yet but from past years I know that that was a real issue when the slope was a fraction or looked like a fraction. They could not see it as a ratio. (I1)

Being able to see the connections between algebra and other areas like geometry was better for Ms. Sands because she had more to draw on when explaining concepts or ideas. It also presented a struggle, however, and became a source of difficulty for students due to lack of geometry knowledge:

> I noticed in this CPM Algebra [that] there is a lot of geometry tied into the algebra and they had a little problem with that. Because I think they haven't spend a lot of time with geometry. They have the basic idea of things like area but…they have to do a lot of practice before they really sure what that algebra problem means in terms of area. So, I think that's something that they probably need little bit of before they come in. (I1)

The struggles also presented delays, as students needed more time to revisit some background knowledge in geometry. The Kitchen Floor problem in the previous chapter, for example, had taken two days to complete due to lack of the concept of a floor plan although it was planned as a daylong problem. She did not want to go further before they got it since students did not understand. She thought students would never understand the problem at the end if they did not understand what finding the area of a wall was in the first place. In general, her comments indicated an awareness of a conceptual difficulty of students as not being able to contextualize or transfer a meaning for algebra in different settings like geometry. More specifically, students would have a hard time in making a connection between algebraic procedures and their meanings or applications in geometry. For example, they might not able to contextualize a meaning for $3(1 + e)$ as the area of a rectangle that has 3 as its width and $1+ e$ as its length. For Ms. Sands, the connection between distributive property and the area model was from abstract to concrete. She was able to transfer this abstract knowledge to a more concrete phase with the area model because she had seen the abstract "in the old days" and having been exposed to area model. For students, however, it was the other way around. Since they had learned the distributive property with the area model, they needed to make it abstract. She was not sure students could make the connection.

> Since they didn't learn it the old way I am still wondering probably how they're gonna make this more abstract in their mind since they do see a picture of area, it's more concrete and I don't know yet how they are gonna make the transfer in something more abstract. It's common. And I know it's common. But I think it's something I am just gonna have to watch for and see if they're really making that connection so that they can do it without the picture. (I1)

For example, she thought she needed to spend more time with modeling distributive property in order to get students understand what it means even though she recognized that it might be hard to model every situation like $-4y(2y - 7)$ (I3).

When I asked her about what kind of experiences students needed to have to help them in making connections between concrete and algebraic/symbolic representations, she commented that:

> Maybe if they had to design their own pictures. If they saw a distributive problem and they would have to draw their own representation of it- … get them away from having to have one. If they had to, you know, produce it themselves. I am not sure about that… because I haven't tried it. (I1)

She had a strategy to act but she was not sure about its effectiveness because she had not tried that way yet. From her expressions, it was obvious she knew it was a common problem and she needed to watch for it to make sure the students could do it without the area model. She believed students should eventually be able to do the distributive property with algebraic expressions without the area model even though it was very helpful.

> They just need practice at that but the picture helps them a lot and they haven't graduated from getting away from the picture yet. (I1)

Another source of difficulty was the absence of multiplication sign when doing distributive property with algebraic expressions:

> There have been times when we just had a number problem using the distributive property and they would get little confused about what to do because they're not totally sure about multiplication when there is no sign. (I1)

Recognizing Students' Thinking and Difficulties

Ms. Sands asked "why" questions a lot in the classroom to get students elaborating on how they knew that they were right and it made sense.

> Ms. S: [to a student] What did you do to get that answer?
> Ms. S: Why did you push that button? That's the first thing you need to know: why are you pushing that button.
> Ms. S: That's not your fraction button, that's the fraction button. That might help to get a better answer but whatever answer you get Brandon if it makes sense or not.
> Ms. S: No asking me is not makes sense. You need to know whether it makes sense. Does it fit to your pattern? [Ms. Sands, 10/23/2002, Lesson 3, 10:29am]

On the other hand, if they were working through a problem as a whole class and nobody seemed to be having trouble with it, she would go on. However, if it were a "particularly deeper, rich problem" she would go into the details of why a particular strategy would work or not. Since an understanding of whether students were having trouble with a particular topic was fundamental in deciding to move on or not, I was curious about how she would know if things were going smoothly or not with the kids. She sometimes asked them to their face like "Do you understand?" She thought that if students understood, they would say so. She also thought, however, sometimes even though they did not understand they would say they did because they did not want to say otherwise. She found this situation "tricky." She did not always know if they were telling the truth or they were embarrassed to ask for an explanation.

This was not the situation with her algebra-1 class. She was certain that they would ask her if they really did not understand or they would ask somebody to do it for them. If students had done similar kinds of problems before and had enough experience with them, they were sure of what they did and what they understood. If they were not, she expected them to come back to morning session next day with that problem. She always asked students to circle the problem they were having trouble with in the homework and do it anyway because “it’s better to try something and get stumped and do nothing” (I1). She thought some of the students, “at least half a dozen of them,” really did that (i.e., circle the problem) when they needed it even though it was not every night.

Ms. Sands’s view of student difficulties with a particular idea or problem was shaped by the CPM’s content organization with a spiral approach. She seemed to think students would eventually resolve their difficulties and reach to an understanding after having been exposed to the same topic in a gradually expanding manner towards a big idea. It was not clear, however, what she thought about how students’ lack of understanding in small pieces would constitute to problems in the big picture. When she observed students having difficulties with a particular problem or idea, her planning strategy was to “read ahead” to see if it was going to be presented later. She thought students “shouldn’t necessarily understand everything the first time it’s presented because it always, it builds on some basic thing” (I2). She pointed out, for example, that the current chapter did not mention “curve lines” before it covered straight lines. Further, the students had not formally been introduced to equations and solving them symbolically. Instead they were presented guess and check tables that she thought would be “the big lead out into solving equations formally.” Those topics were coming in the next chapter and the purpose of this chapter was to prepare students for it. She indicated she “had let it slip” to the students that what was coming after this was to know the purpose of the equation even though they had not seen any of it yet.

Helping Students with their Difficulties

One of Ms. Sands's big questions was "how do you get in somebody's head so that you can find that what their misconception is?" She believed she could turn them around if she could find what it was. She gave an example that one of her students thinking, "one eightieth is the same as eighty over one." Even though the other students tried to help him and showed him that they even did not look the same, he would not budge. No matter what, he maintained his argument, which was "well one times eighty is still eighty and eighty times one is still eighty." Ms. Sands observed that he could not understand that it was not multiplication: "The representation of that fraction is division, it's not multiplication so they are not equal." It was like "pulling a teeth" for Ms. Sands and she did not understand why he could not see it because everybody else in the class could. Although she doesn't know how to get there to make him see it, she was optimistic when she added: "Not yet!"

Ms. Sands thought she needed to keep learning to ask questions and to know enough about what she was trying to get them know as parts of her process in getting into her students' minds. She particularly emphasized she needed to know enough about what she wanted her students to know so that she could "get deeper" into questioning to find the foundation or the source of difficulties She needed to practice asking questions. It was hard work and she did not know what to try. She thought it had to happen as students were working because there was no set of questions she could ask. She needed to listen them talk because she believed that "if they are thinking aloud and if they are really and truly thinking out loud you can catch what they're doing wrong." If they were not really on task, however, it was hard for her to find out what their problem was because it was hard to understand what they were saying or what they meant.

After a deep inhale, “I don’t know!” in a helpless tone was the answer I got when I asked Ms. Sands about what she could do to help student overcome their difficulties. She found this a difficult question to answer and all she could think of upfront was the obvious one; “just doing it for them...I think I can’t just tell them what to do. I can’t do that” (I1). She could not make people care about doing it neither and it was a frustrating thing for her. She recognized that it was a difficult age with those kids, especially the ones in pre-algebra. They were also easily frustrated with things they did not know and they did not know the questions to ask about it. This made them get mad and refuse the help from her. In return, she was patient with her students. She tried “not to get mad back at them when they get mad” with her and help them even though they did not want it. Though it was a part of her job, Ms. Sands stated that she did not like “trying to figure out what their real deficiencies are because it’s so hard to do, to go back and back and back until to find out what it is they don’t know how to do so that you can try to help them to fix it.” Although she did not have that problem with her algebra class, she found herself doing this a lot especially with her remediation class where she was mostly “trying to find out what it is they don’t know so that I can find some way to teach it the right way without doing it for them.”

Ms. Sands looked for groups having the most trouble when she was walking among the groups. If the group members were working and not having trouble she would continue to look until she found somebody who needed her help or asked group members and they could not come up with anything. She asked them what they have tried to solve the problem. If they were doing something right, she would ask them what was wrong with it or why they were not satisfied with their answer. She thought this would be a good thing because “if they are not sure their answer makes sense or is reasonable then they are not asking themselves good questions.”

In any case, what students usually wanted to know was that if they were on the right track on the first place. She also acknowledged that if she did not "hit the right group" and she would miss the one that really needed her.

After reviewing the results of the end of unit test, Ms. Sands decided - she would talk more to the students who had difficulties and misconceptions to make sure they had help to overcome their problems. She thought she needed to check particularly the lower achieving students and pair them up with someone catching onto topics quickly so that they could have a chance to talk to somebody and ask questions. Also, she thought that she needed to slow down a little bit and make sure students understood; otherwise, they would have harder time and more difficulties when they started more advanced topics like quadratic equations. There were little things that she considered she did too fast. She wanted to identify those so that she did not do it again. Writing notes on her book like "spend more time here" was an idea she thought would achieve that purpose.

Ms. Sands saw early morning sessions before homeroom as a way to get a "little one-to-one" help to students who needed it. It was not rare to hear her in class reminding students about attending early morning sessions. The early morning sessions were the time for asking questions to get help about homework or especially getting assistance with the things they lacked background on because she could not spend much from the usual class time to revisit the topics they should have learned a long time ago.

> If you didn't understand number three, you should have been here this morning! … I saw you this morning but you didn't come in! [Ms. Sands, 10/22/2002, Lesson 2, 10:07am]

> This isn't the time to teach you how to multiply fractions. This isn't the time to teach what a positive and negative is. You've done all before whether or not you've learned is another issue. If you haven't learned it, you got to learn it. Otherwise this won't make a bit of sense to you. …Some of you already all beyond this lesson and some of you

> haven't gotten there yet. We have to deal with that, ok? We need to deal with this at 7:40. [Ms. Sands, 10/23/2002, Lesson 3, 10:45am]

Her starting point for help in early morning sessions was to have them show her what they did and reread the problem. She found it amazing that "nine times out of ten is rereading the problem" (I2) that would answer their own question. Thus, she often suggested students to reread the problem in order to understand so that they could solve it.

> Ms. S: Why didn't you do number three? Did you read it? You must have not read it, where is yours? [to another student] … What is this? Why nobody is doing number three? What is this?
> [Several students shouted like—"I did number three"]
> Ms. S: (after Annie said that she did it) Exactly, because all Annie had to do it read again to herself and she says—"Ohh, I get it!" [Ms. Sands, 10/22/2002, Lesson 2, 10:05am]

If they really needed help, on the other hand, she would sit down with them and got into one-to-one interaction. Usually it was the students themselves who would come up with a solution or explanation. She thought they usually did not need her and that was a "funny" (I2) thing for her. She noted it was always the algebra class and almost always the "higher level" (I2) students who would come to ask questions. She also noted students attending regularly were usually the ones who were not confident and did not ask a lot of questions in the class. She also suggested that those who had difficulties or questions might also stay for the tutoring sessions in the afternoons. None of her algebra-1 students had done it. She was not surprised or disappointed by this because she did not think they really needed it. She thought peer help, talking to each other would be more beneficial and they could get a lot more information.

Issues in Helping Students with Their Difficulties

Ms. Sands identified big class sizes as one of the problems when helping students to overcome their difficulties. As a school system they tried to lower the class sizes so that there would at most 28 students in each class, which she thought as better than 30, the maximum size required in the state of Georgia. She thought 28 was not a lot if things worked well. In case of a struggle, however, it was still a lot of kids to deal with because it would make it impossible to talk all of them if it was needed.

This issue became more critical in remediation and tutoring sessions. She had twenty students in her remediation class and six students in her last tutoring session. She thought this was too much because she could not give individual attention to everybody in an hour the way she wanted to do without someone being left out. It would take a lot of one-to-one interaction to get somebody talk to her so that she could catch something in their explanation of some topic that was wrong and attempt to remedy it. For this reason, even though it was tedious for her to read them, she thought it was one of the good things about having students write reflections like they did in pre-algebra at the end of every investigation/problem or some other situation explaining why something worked or the reasoning when trying to solve a problem. It was also difficult, however, for her to get students to put their thinking or reasoning into words. She described this as

> It's like pulling teeth for some of them. Some of them will write a book. They will flip their paper over and keep writing that they understand. And those that don't understand may write a sentence or two or they may just copy some numbers and never explain. And those are the ones that it is hardest to get inside their head to have them explain something to you because they are not real sure what to do and to me that's what the remediation should be all about. Get inside their heads; find out what their misconception is or why do they have trouble. (I2)

In order to find out student difficulties by getting inside of their thinking, she wanted to make her mission to sit with somebody everyday for about 20 or 30 minutes and got him/her talk to her about whatever the problem he/she was working on. She had dilemmas with this intention in her remedial class, however, as she attempted to find the neediest or tried to adjust the level of class:

> But then it would take me a month to sit everybody. ☺ Who do I pick first? Who is struggling the hardest? Who is the lowest that I need to push up or who is the highest so

> that I can get him out and make my class smaller, who do you sit with? If I pick the lowest one, the highest ones, they may get it and they'll probably get it. But if I want my highest achievers to get my class lower, I need to be sitting with them too. So it's a double edge sword you don't really know what to do one day to the next. (I2)

Trying to find the neediest students was a big problem for her. Even though she identified who they were, they were usually not on the same page. She pointed out that usually if they were needy in mathematics they were needy in reading but the reverse might not be the usual case. As a school, they tried "teach to the lowest people in your class" (I2) approach in the last several years but she did not think that it was the right thing to do. She wanted to help everybody but it was not that simple. She thought the decision was a hard one:

> It's hard not to teach to the lowest because if you don't teach where they are, they will never get it. If you teach too high, there is a lot of people that won't get it. (I2)

However, she taught with the lowest in mind, then:

> The highest kids are going to get bored. The middle kids will get bored but they'll get something out of it. But the lowest kids if you don't spend a lot of time with the lowest kids they're still not gonna get it. If you teach to the low, you never accelerate them up to where they should be. You're always working down there at the bottom and not making progress. Because you can't teach enough stuff to get people to where they ought to be and you can't pull anybody more than a year along if you only have a year to do it... If you teach to the middle, you still have children who won't get it. And you want everybody to get it. It's tough. It's a hard decision. (I2)

She also tried to homogenize groups according to their level or specific difficulties and had them do different things. However, she thought that is also problematic for her.

> If they were really having trouble with that zero division I would have a group of kids doing that until they are good at it and I'd bump them up to whatever came next that they could not do as a regular thing not as remediation but and I would have kids another place working on the regular book doing all of that because they could. And it draw me crazy because I may have four groups of things going on at the same time and I can't give them all my attention. (I2)

Another issue was how big the student's difficulties were and if the difficulties could be resolved in a year within a 30 minute time period every day. She did not think one could make two years of gain in a remediation class and she was not sure that remediation would solve such a big problem either. She considered student's motivation or self-confidence, however, as a possible a key for success in remediation. She thought the ones t in remediation knew why they were there and they did not see themselves ever getting out. So, she tried to get somebody out the first nine weeks of the class so they would see they could do the work and get out of it if they wanted to do it badly enough. Thinking that "if you don't ever start you don't ever do it" (I2) she graduated two students from her remedial class. She decided that they don't need for too much remediation because they worked well and made B's. Given the example of two students just did, she believed they could all do it.

Ms. Sands's Knowledge of and Approaches to Students' Thinking in Unit 3: Burning Candle

Ms. Sands thought that the goal of the Unit Three was not to learn how to graph points because it was a prerequisite for this unit. The real goal was to get students to see the relation between tabular and graphical representations of an equation through seeing that table consisted of values satisfying the equation:

> I think what they supposed to get from it is; there is a relationship between this table and this picture of the table and that the thing that's gonna be added on to it is that this line is an equation and all those *x*s and *y*s are the *x*s and *y*s that you replace in the equation to make the equation true. (I2)

She often emphasized the importance of graphing in algebra to motivate students to learn as much as possible: "Oh the graphing is very important, it's algebra. Once you graphed it all, you're gonna be expert on graphing and it's gonna feel so good about it" [Ms. Sands, 11/05/2002, Lesson 9, 10:10am]. She thought the algebraic representation was going to be added to tabular and graphical relationships later on in the next unit where students were going to be solving equations symbolically although she was not sure of it and expressed that she needs to look further into the next unit to see if it was as she thought. On the other hand, in terms of students' goals in this unit, Ms. Sands thought that "they are just doing the work and they're noticing patterns in tables" (I2).

In Unit Three, they had done some generalizations with patterns, which led them into guess and check tables and she believed her students did a good job of it. Even though solving equations was the next chapter in the book, she believed that guess and check tables prepared students for solving equations as well. She further elaborated on how guess and check tables worked in problem situations in which the base of a rectangle was given in terms of its height with the perimeter of the rectangle and students were asked to find the base and the height:

> I really didn't see the relevance for a while until they had a test question about perimeter of a triangle and the dimensions of the triangle were all in terms of an *x*, nothing was given to them but the perimeter and they have, you know this one is 2 times then this one is the length of the first side plus 3 and they really had to think and figure out and what that was and use guess and check table you know they would have it in like three tries and some of them had it in two tries because they knew what they were looking for; they just haven't formally written down as *x* and *x* plus 2 and 2*x* yet, which is what I do when I solve it but they haven't gotten to the point where they know they can manipulate it with a symbol. They are just really guessing, they are really guessing. But they know what to guess, they are knowing to guess well if it isn't 6 and that was too low let me try 8 and they can get there… there are other two sides but guessing. When 1 doesn't work and it's too low pick a higher number and if it's too high back up a little bit and try again and that's gonna lead them to solving equations but they don't know it yet. (I1)

As she indicated, Ms. Sands did not realize the connection or how guess and check strategy would be useful in forming and solving equations until she saw a similar problem in which the base and the height of the rectangle was given in terms of *x* as a test question. When she was mentioning this, she was aware that the same problem they did with guess and check is going to appear later in the unit about forming and solving equations. As she commented, she was fond of the CMP's spiral curriculum approach: "I like the way the spirals back to things they've had to use before so they have to remember everything" (I1). She made a comparison with their old textbook where the guess and check tables were only one lesson and as she expressed "This what we're going to learn today and ☺ you'll have it forever" (I1).

As Ms. Sands identified, the big misconceptions students had in the unit were combining like terms, how to write coordinate pair, how to do the distributive property and what it means, and possibly some issues with positive and negative numbers and operations with those even though she was hoping they would know that from last year since they did a lot of those with CMP curriculum. She also pointed out that "Their [Students'] concept of variable is not really firm yet" (I1). Particularly she mentioned that students had trouble with using letters different than *x* as a variable; "If you change *x* to a *b*, they are not really sure what to do with that" (I1). For this reason, she rarely use *x* and *y* so that "they'll see that a variable can be whatever we say it is" (I1). On the other hand, she stated she had not done that yet this year because she had not seen a lot of problems with the variables yet. She stated that there were still several chapters until they would start solving equations formally.

She observed students were little bit confused by the term "rule". She wanted them first to understand what a rule was and make a concrete sense of it before they gave a formal name for it. She was trying to stay away from terms like functions and "*y* equals" equations. However, she encouraged students to write their "rules" as y first because it would be better and she had a reason but would tell it later not to confuse them [Ms. Sands, 10/29/2002, Lesson 7, 10:35am]. On the other hand, the textbook used the term function as "Graphs of Non-Linear Functions" in a section subtitle in Unit Three but this did not come as a question in the class. She thought using those terms would scare them and if they saw what it means they could give it a name. Thus, they started by using a rule to find out what *y* was in a table and then they had the table and tried to go from a missing place in the table to its solution. She found students weren't sure what to do in this process. She was proud that when they got to hang of it, however, they were able to solve some difficult problems like the ones involving negative numbers. Also, she noted students still had problems with negative numbers although they spent time to extenuate the negative in the Connected Mathematics program last year in pre-algebra. Even though they were not solid on "positive and negative integer rule", they did a great job after they figured out what the rules were. She thought that students difficulties with finding the rule with guess-and-check tables was part of their learning process and they would eventually lead students into conceptual understanding so that they could understand and solve equations without the "pictures" and guess-and-check tables.

> Doing the guess and check tables, they really hate it and after they done a few of them they began to get really good at it and then when guess and check tables becomes equations they didn't want to do equations. So I'm hoping that it's just gonna- it's a phase ☺☺ they going through and it will ware of and they start to not need pictures and guesses anymore and they'll understand what really happens when you solve an equation- I hate to say rules but there are rules and there are steps that you take to solve one and I think the pictures and the things help them but I think sometimes- I think more people have got it than half got it and go on. (I3)

Issues with Graphing: Graphs as Finitely Connected Points and Graphs as Pictures

One problem Ms. Sands had noticed with the students' graphing approaches was that they were connecting consecutive points rather than smoothing or giving it a curve when they were graphing the parabola. This tendency was also true for graphing a line after plotting the points. She pointed out to students that their graphs should be smooth, not like broken segments, as she observed them making segmented graphs. For example, she suggested to a student who obviously was connecting the dots as she went to "Draw the points first and then connect them" [Ms. Sands, 10/30/2002, Lesson 8, 10:34am]. She made similar comments later on to another table/group because their graph looked choppy. She told them the graph was supposed to be smooth even though she was not expecting it to be perfect.

Ms. Sands suggested an interesting explanation that brought out the role of students' prior experiences and knowledge to this context. She felt students' prior experiences with finding hidden shapes by connecting numbered dots common in dot-to-dot games coloring books might have a significant effect on students graphing.

> We have this game of playing dot-to-dot, you were you are on a, it's usually like in coloring books and you find a dot and you connect one to two and two to three and that's the way they are drawing their points and connecting them instead of drawing all the points like I suggested and finding the line or the curve whichever they would a draw a dot, draw their next point and connect them so their picture of the graph didn't look like it probably should have as if the computer baby had drawn it. Because that is what they used to it: drawing one point and connecting it to the next point without seeing the big picture first. (I2)

Such an approach and thinking might lead to misconceptions that graphs only consisted of finite number of points and disregarding other values *x* (or another variable defined in context) could take in the domain. She thought students needed to consider all of the points to visualize the whole graph/curve than just the individual points constituting a table defined by a rule. On the other hand, Ms. Sands disregarded or did not notice or bring up any issues regarding the language the textbook used in asking students to make a graph after completing tables with a rule. Students were often asked to "plot and connect the points" after completing the table for a given rule through the unit (e.g., BC-12, BC-25, BC-28, BC-40, BC-45). This type of instruction might have misdirected students. Also, Ms. Sands's use of certain vocabulary such as "to draw pictures in math" might have misled students to think that graphing is drawing pictures mathematically: "I'm glad that it's easy. It's fun *to draw pictures* in math, isn't it?" [italics added] [Ms. Sands, 10/30/2002, Lesson 8, 10:34am]

One of the problems where students' understanding of graphs as pictures was apparent was BC-37 where students were asked to copy and complete the following table (see Table 3) and then describe in words what the rule they used.

Table 3

College Preparatory Mathematics (CPM) Algebra-1 Problem BC-37

x	2	0	-3	1/2	-1	0.3		-1/3		-11	-2	*x*
y	5	1	-5	2			7		10			

One of the sub-questions was "What does the graph represent?" In the introduction, it was evident some of the students thought the purpose doing tables and graphs was to lead them to "graphing shapes." It was not clear what this would mean to the students since Ms. Sands did not push this idea forward and ask what students were thinking.

> [Ms. Sands, 10/29/2002, Lesson 7, 10:19am]
> Ms. S: Today we're gonna start 37. Some of you already started with 37. I didn't put up the silent board game today. I thought maybe you've had enough of it. Guess what 37 is?☺
> Ss: ( )
> Ms. S: It's another table. Why do you think you're doing so many tables and graphs?
> Ss: ( )
> Ms. S: You'll have to do them it later on. Why do you think you're doing so many of them now?
> Ss: [it's hard to understand what students are saying when all of them talks at once!]
> Ms. S: Do you think this may lead to something else?
> Ss: Yes.
> Ms. S: Like what?☺
> Ss: ( )
> Ms. S: *Graphing shapes*? Maybe. [Italics added]

While students started doing BC-37, Ms. Sands was walking over to tables/groups and answering their questions. Some of her comments to students having problems were:

> Ms. S: Check your signs.
> Ms. S: See if it works for another pair. …Make sure that it works for all pairs. … Maybe you don't need the calculator, maybe you don't [he was doing simple multiplication]. Maybe putting the signs in the calculator messing you up. [Ms. Sands, 10/29/2002, Lesson 7, 10:22am]

They tried to use their calculators to do computations but Ms. Sands commented as "Your calculator is just a tool and you don't need it for stuff like this." [Ms. Sands, 10/29/2002, Lesson 7] Since they were supposed to look at the pattern and find the rule, she wanted them to think rather than be busy with the calculator believing all they had to do were simple manipulations.

Towards to the end of the class, one of the students, Brook, was asking Ms. Sands about what the graph represented. She asked him to *connect the dots* first and then she asked him about what he thought it represented. It was strange to hear that she was using a vocabulary that might lead to a notion of graphing as connecting dots as in dot-to-dot puzzles as she pointed it above as a tendency she observed among students. It was the vocabulary, however, the textbook was using in the problem statement and she was probably imitating or rereading this portion from the problem statement. Brook thought was a hill that the graph represented. Ms. Sands seemed surprised with this result because a plausible answer to this question would be "graph of a line"— the line being $y = 2x + 1$ in this case. Brook perceived her surprised tone and commented that it was a deep hill. Understanding what he was thinking, Ms. Sands commented that the graph looked steep. After Brook added another comment that it might represent a building and seeing she was waiting for him to add more, Ms. Sands clarified that she was asking him to think mathematically and see the mathematical representation. She was aware that Brook was thinking the situation pictorially, not mathematically. Brook then tried to look at the graph as a line or line segment. She wanted him to think more on the issue since Brook wanted to put "line" as his answer. She encouraged him to feel free to look at it from different angles and write it if he saw something else rather than just a line. Based on the previous suggestion she made about looking at the situation mathematically, she wanted him to stay in this perspective although she did not make any direct comment on whether it was right or wrong to think of a graph as a picture.

> [Ms. Sands, 10/29/2002, Lesson 7, 10:44am]
> Ms. S: So what does the graph represent?
> B: (   ) … A hill.
> Ms. S: A hill? [She shakes her head suspiciously]
> B: (Deep hill)
> Ms. S: It's pretty step. [She agreed with him]
> B: No (   ) I don't know, a building? [upon a questioning look from Ms. S]
> Ms. S: No, not, mathematically what does it represent?
> B: A line. ... A line segment.
> Ms. S: No, not a segment. [She shakes head to disagree]
> B: But it (   ) the other segment.
> Ms. S: So why is it not a segment?
> B: Because it can't go (   ) with them.
> Ms. S: Yeah, and it will. Only your paper stops. [She was referring to extending line infinitely and paper had a boundary]
> Ms. S: What do you think? [to another student on the same group]
> Ms. S: If you want to put line, put line [As an answer to Brook whom to put "line" as an answer.]
> B: (   )
> Ms. S: Think deeply Brook. Deeply... If it only represents a line, that's fine with me. If you see more than that, write more than that.

The notion of graphs as pictures was also apparent in BC-61 when one of the students, Veronica, voluntarily tried to sketch a graph on the board in order to represent the height of a burning candle compared to time. First, she had difficulty labeling the vertical and horizontal axes as height and time respectively. She did the opposite. Ms. Sands guided her to check the problem statement where it was stated that vertical axis was given first (i.e., the height of the candle). She overcame the difficulty and correctly labeled the graph with the help from other students.

> [Ms. Sands, 11/07/2002, Lesson 10]
> Ms. S: Height of a burning candle compared to time. [She was reading the problem to student]
> V: See, I had showed (   ) because I wanna- because a candle it goes- it (   ) gone away. Candle kinda melts and (   ) gone.

Ms. S: Well, label them first and then you can think about how you will draw it. The height of a burning candle-
V: So the height is down here? [she pointed horizontal portion in the graph as the height]
Ms. S: Compared to time?
V: Where the height (   )? [She labeled the horizontal axes as height]
Ms. S: What did the direction say?
V: Let me see, I forget. [She went back to table to check the problem statement]
Sn: The height of a burning candle compared to time.
Sn: Vertical axes is given first [that's what the problem statement said]
Ms. S: The vertical axis is given first. The height of a burning candle compared to time.
Sn: [she is back] So the height is right here. [She correctly labeled vertical axes as height]
Ms. S: Ok.
V: [She labels the horizontal as time]
Ms. S: Ok, do you agree with her labels?
Ss: [Yes, Yes I do, etc.]

Veronica had a hard time conceptualizing what she needed to do. She wanted to draw a graph like (1) in Figure 4, which was the same graph the class did previously for the total cost of a gasoline compared to the number of gallons of gasoline one bought. Ms. Sands tried to guide her to visualize what would happen in an actual situation of burning a candle.

V: Uhm, see, I ain't- I could (   ) like [she pointed that she would draw something like (1) in Figure 4]
Ms. S: Well, explain the situation. What happens to a burning candle?
V: It stops at the- during burning all the wax?
Ms. S: Well, describe a candle.
V: (   ) a big big wax in the end and-
Ms. S: How big? [She opened her hands to illustrate a possible height]
V: How about that big [she was satisfied with the height Ms. Sands was showing with her hands]
Ms. S: That big?
V: A little taller. All they depends on what size.
Ms. S: Ok, here is the candle [She opened her hands vertically as if it was a candle] Light it, figuratively. [She lighted it figuratively]
Ms. S: What happens?
V: It melts down.
Ms. S: Well, you make something happen.
V: [She gently pushed Ms. Sands's top hand down to illustrate that it was melting]
Ms. S: What should this graph look like?
V: I think it's going- I feel like a little line like this. I'll show you.
Ms. S: You better blow it out of my hands catch on fire! ☺ [She was still holding her hands and keeping pushing the top down as if the candle was melting].
V: Oh [She blew it out] [She was also drawing a graph looked like (2) of Figure 4]

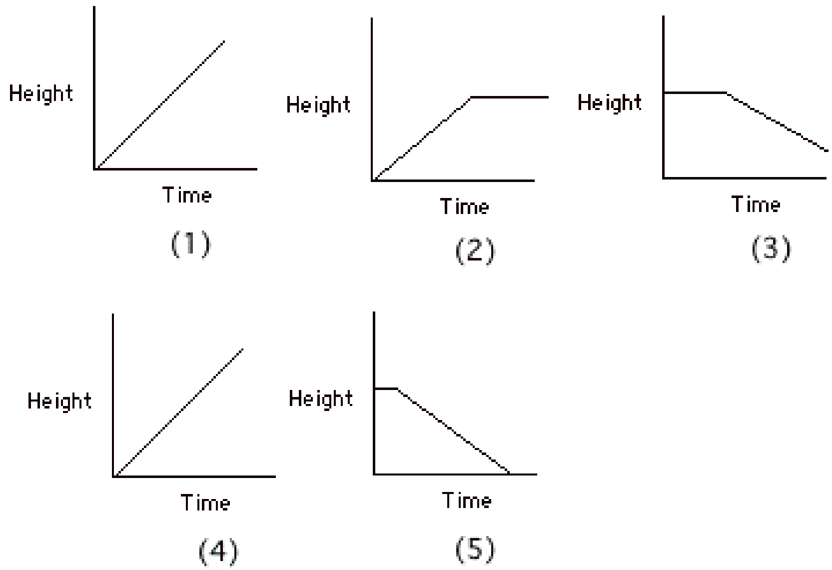


Figure 4: Sequence of Veronica's graph of the height of a burning candle compared to time

After Ms. Sands tried to get Veronica to see that the candle had an initial height or tallness that decreased as it melted down, Veronica drew a graph looked like (2) in Figure 4. As soon as she finished with the drawing, all of the students started talking at once. It was impossible to hear who was saying what. They were most probably disagreed with her graph. One of the students commented that the candle would get bigger in this graph.

> V: It seemed that they got bigger? [She seemed to hear the student's comment and she deleted the graph]

The same student pointed the vertical axis as a starting point.

> V: So it goes like this? [She drew a graph looked like (3) in Figure 4]

Seeing this, the same student commented that the time started at zero and Ms. Sands agreed with it.

> V: So, it goes like up? [She went back to her initial idea and drew a graph looked like (4) in Figure 4]

At this point, several students disagreed with her saying "no, no" and pointing that this way it would go bigger.

> V: I don't know how to make this graph [She seemed frustrated]
> Ms. S: [She opened her hands vertically as she did previously and started bringing the top closer to the bottom as if the candle was burning down]
> V: Ok, we [are] going down. [She still seemed confused]
> Ms. S: What's the y-axis?
> V: [She showed the vertical]
> Ms. S: How tall is this? [by pointing to the origin, she was asking the height of the candle at time 0]
> V: [She pointed the horizontal axis]
> Ms. S: It's nothing!
> V: Zero?
> Ms. S: It's zero tall. Well, show me where the tall candle is.
> V: Right? [She pointed the origin] It's like- it's already tall so it's up here somewhere [pointing somewhere above the horizontal axis], ain't it?
> Ms. S: [She confirmed her thinking with shaking head]
> S: And, [Ms. Sands opened her hand to show a candle with a height again] then it gets smaller [looking at Ms. Sands's illustration. And Ms. Sands started to push the top hand towards the bottom] So …
> Ms. S: So, what should the line- where should the line start?
> V: Probably over here since it's already tall. [pointing a height]
> Ms. S: If it's tall. [agreeing with her]
> V: And it go [she drew (5) in Figure 4, a similar to (3)]

Having a little portion outside the vertical axis lying horizontally caused confusion among the students and they wanted to know why she did that. Ms. Sands suggested a point, which Veronica accepted as a possible reason for why she did it even though her thinking might have been different as she would possibly thought the initial portion as a marking for the initial height of the candle starting to melt down after it is lit.

> Sn: Why did she make the hat?
> Sn: Why did she make the little one (   )
> V: I don't (   )
> Ss: (   ) [commenting on the extra portion of the graph]
> Ms. S: Maybe it was tall before it's lit.
> V: Yeah, you see.
> Sn: Yeah, maybe before it starts to melt.
> Ms. S: That's good.

Ms. Sands seemed satisfied with the end product even though it was not correct with the extra portion students had objected to. It was not clear if the students accepted Ms. Sands's suggestion and made sense of why they would have the extra portion in the graph so it would be the right graph to represent the height of a burning candle.

The next task, BC-61, was about the height of an unlit candle compared to time and Ms. Sands would have a chance to talk about a possible explanation to clarify why they would have an extra portion in the graph at the beginning. However, Ms. Sands did no such attempt. The whole episode suggested Veronica's thinking to graph the height of a burning candle compared to time was guided a pictorial thinking representation of the situation rather than thinking of it in terms of a functional relationship and its graph as a set of points showing the height at a certain time. Ms. Sands's effort to pictorially represent the situation to help Veronica led her thinking towards a pictorial representation rather than a functional relationship between height and time as the candle burned down.

Scale and Graphing

One of the problems Ms. Sands gave importance was BC-31 (See Figure 5). Ms. Sands thought this type of problem was important because if students saw problems like that "it'll stop them from doing the same thing" (I2). It'd remind them to think about their own scaling and keep it consistent when they drew their own graphs. She hoped "that's the purpose of that type of problem" (I2). Ms. Sands thought it took "a lot longer" for students to figure out this problem than she expected. She was disappointed because "it was only a scale problem on the graph" (I2) where "the graph, the chart, everything was already written and they had to figure out what was wrong with the line that was drawn in as an example" (I2).

After Cheryl completed her table and graph for the equation y = -x – 1, a member of your team thought her graph (shown at right0 didn't look correct. Cheryl needs your help to find her mistake.

x	y
-3	2
-2	1
-1	0
0	-1
1	-2
2	-3
3	-4

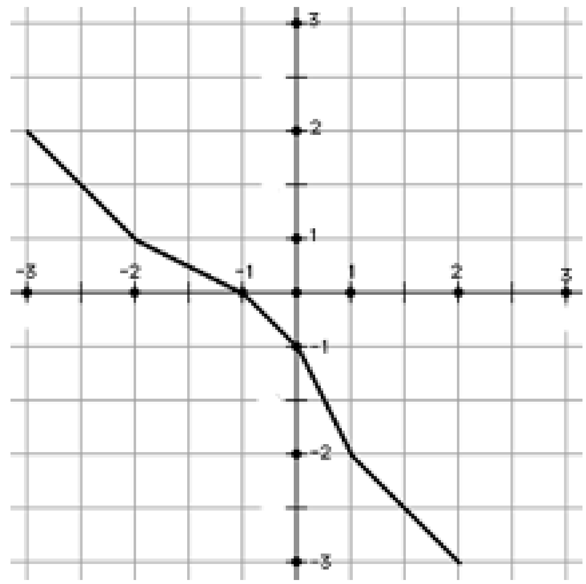


Figure 5: College Preparatory Mathematics (CPM) Algebra-1 Problem BC-31

Ms. Sands thought students knew the table and the equation was right and so the graph was supposed to be right but it had a bump in it instead of a straight line. She did not think they even saw that the problem was about scaling; they were not looking for scale. Even though it was only two students, she was happy that finally somebody did figure out what was wrong with the graph, which was a scaling mistake. When I asked her to elaborate on why she was surprised students had more difficulty than she had anticipated, she thought of a particular student in the class and said "I probably didn't spend enough time with him to figure out why he didn't understand it first" (I2). She thought "what he was trying to do was to look at the table and make sure that the points were all graphed correctly" (I2) and she noticed he had found that they were when she was talking to him. She further explained that one of the students who came up with an answer and put it on the board had difficulties explaining his answer and she thought that was because he labeled it wrong.

> I noticed he was having a lot of trouble explaining why even though he knew why and when he would draw it, he drew up on the board too and he was trying to explain why the half was there and he had his half instead of one and a half he put one half where two things should have been separated by the same scale factor and he knew what he was talking about but he had it labeled wrong and that was gibing him some difficulty explaining what he meant. (I2)

Johnny, the student to whom Ms. Sands was referring, asked about this problem the next day when they had it as homework. She did not ask the rest of the class what they were thinking or what it was that they were having difficulty with. She let Johnny go to the board to draw the units on the *x*-axis similar to the original figure in the problem statement so that 1/2 is not exactly at the point the half the distance between 0 and 1 (See Figure 6). He tried to say the mistake in the problem was the scaling and that was why the line on the graph was broken.

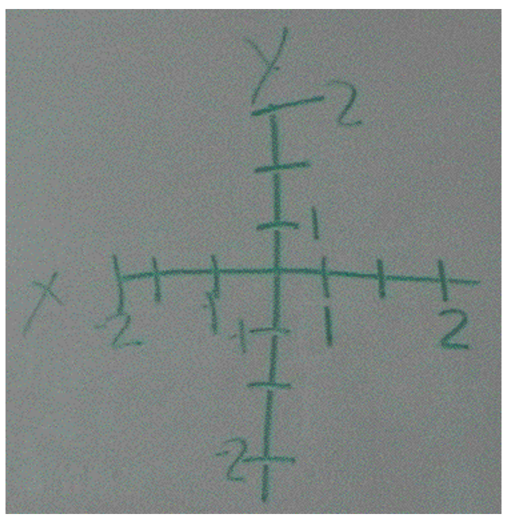


Figure 6. Johnny's figure for CPM Algebra-1 problem BC-31

After she listed to what the student was trying to say, Ms. Sands clarified his explanation that it was the length of units and thus different scales causing the problem. She wanted students to particularly compare the distance between 0 and 1 and then 1 and 2 so that they could see the difference in unit lengths.

[Ms. Sands, 10/29/2002, Lesson 7, 11:11am]
Ms. S: So the distance between 0 and 1 is not the same distance between 1 and 2.
Ss: ( )
Ms. S: Didn't you see it? Look at your book… Just look at your book. The distance between 0 and 1 is not the same as the distance between 1 and 2. Aren't they supposed to be the same?
Ss: Yes.
Ms. S: 0 and 1 is the same distance as 1 and 2…. On this graph it is not. There are some issues with the graphs that you drew and it's probably related to the table that you made. If you're getting weird lines you all… if you're getting weird lines something is wrong with your table. Should you be getting line that zigzag all over the places? No. If you're putting more than one line on a graph you need to label them so you know which one it is. What do I mean label the line? What do I mean label the line? [She scrolled through the book to find the page talking about what it means to label a graph] What is it mean to label the line? ... What is more important listening? [She showed the page] What is it that more important? Why do you have to show every y and just listen and here's why mine has a problem. Why is it important to label your line?
S: So, you know which one…
Ms. S: So you know which one it is! Duh! … That's a duh question. Look at this thing that you're supposed to put in your toolkit and look at how the line is labeled. What is it labeled with?
Sn: Labeled with the rule.
Ms. S: Labeled with your rule. Your y equals rule; whatever *y* is. That rule that you figured out that goes on your line.
Sn: Do we do that to every one?
Ms. S: Do it to every one.
Sn: Even if it is one?
Ms. S: Even if it is just one… If you don't label your lines, they don't mean anything to you. And I told you that there were things about this equation that there were going to be important to you later and you'd know what a line would look line just by looking at its equation, if you don't have the equation on your line, it's not gonna ever make sense to you. Yes… Do not forget. You need to label the *x* and *y*-axes. You need to label a point with its coordinates on it. If you put a point, know what it is. Put the equation and rule on your line so that you know what it is. Yeah, it'll take ten seconds longer to do each graph like this. It's not asking too much of your time, is it?

In the discussion, Ms. Sands particularly emphasized that the units used on both axes should be same for all values. She pointed out they should not get any zigzag or broken lines and if they do they needed to check their tables. She did not elaborate on why they needed to check their tables, however, or what was the relation between issues of scaling and tables. She probably tried to point out the need to check the correctness of *y* values computed with a rule. On the other hand, she also highlighted the need for labeling even though she did not make the relationship between labeling and scaling clear. Her explanations included clues about what it might be. Since students would have more than one lines producing broken, zigzagged lines in the same graph in cases of piecewise defined functions, she might have wanted them to be aware of this fact. Thus, labeling each piece or each line in such a case would allow them to see that each piece belonged to a different equation. It was nice to see such an approach even though it was not clear and probably students only understood it as a reminder of the necessity of labeling for some reason. Ms. Sands probably did not want to elaborate on this point thinking that it is a topic of future and she did not want to confuse students.

After discussing about students' difficulties in BC-31, Ms. Sands realized that she should have stayed enough with somebody to find out how they really solved the problem because she only knew their beginning approach in which they were looking to see if the points were drawn correctly. She had been pleased with students' progress for the last couple days in filling a table, graphing points, and finding patterns and rules. She was particularly pleased that they did a good job figuring out what the rule was, even with some equations not so obvious like the ones including $-x$. She thought she needed to spend more time with "slower students" to make sure they were aligned with the rest of the class.

Domain and Range of a Parabola: Does this work for any value of x?

In BC-45, students were asked to complete the table using the rule $y = x^2$ for given -4, -3, -2, -1, -0.5, 0, 0.5, 1, 2, 3, 4 as x values and then plot and connect the points. They were then asked to think about the following questions: "(a) Is this graph a straight line? Describe the result; (b) Have we seen this shape before? Look back on your previous graphs. What feature of this rule, $y = x^2$, determined by this shape?; (c) Can any value for x be used? Explain; (d) Can any y-value result from this rule?; (e) Use the graph to approximate the x-value(s) when y = 5; (f) Use your calculator to solve part (e) more accurately."

In her mid-chapter evaluation, Ms. Sands thought what they did so far in the unit was not new for the students; they were familiar with graphing points and knew how to put points on the coordinate system. They did not know much about the equations and lines, however, and she thought they should not anyway. Connecting points and making a line and then changing from straight lines to parabolas were something new for them. When they were graphing $y = x^2$, she was surprised that one of girls in the class recognized that the pattern was going to make a "V shape" because she did not expect anybody to recognize before fully plotting all the points.

> She knew that it was gonna make the V shape because she saw the pattern form the middle out, she was getting the same y-values and she knew what it was going to look like before she drawn it. And she was pretty sure of that. And I thought that was very unusual because I wasn't sure if anybody would see it until they drew it. (I2)

According to Ms. Sands, students were particularly having difficulties not with the shape but what would happen if x were changed to something not in the table/chart part of the problem. She did not think students saw the infinity of the problem although they saw it better with lines because "they know more about lines than curves" (I2). She did not think they could see that the parabola extent to the infinity. Although she did not know where and what taught them about curves, she knew in the previous year they had a unit called "Variables and Patterns" in which they did graphs and things similar to what they were doing now and maybe that was why they did well so far.

Students were not real sure about it when Ms. Sands asked if the graph of $y = x^2$ would work for any value of x other than the ones included in the table given. One of the reasons, as she thought, was the concrete and limited boundary of a piece of paper since it would allow to graph only a number of (integer) values of x. Thus, students did not realize that the parabola goes on forever beyond values could written on a piece of paper.

> The paper is so concrete they don't realize this parabola goes on forever. Somebody over here did finally! But this group was having trouble realizing that just because there wasn't room on their paper doesn't mean that you can't find $x = 5$ and graph way up here off the paper. It does exist but they couldn't see it. (I2)

Furthermore, she also suggested that the construction of tables for a few number of values x, before they even graphed it, might have contributed to the problem.

> They weren't sure that x could be any value because that's all the table said and they didn't think beyond the table or outside the box to think that they were other possibilities. (I2)

Ms. Sands's points were evident in her conversation with a group of students having problems in conceptualizing if they could use any value for x.

> [Ms. Sands, 10/30/2002, Lesson 8]
> Ms. S: Come over here and ask your question.
> Ss: It's the same question.
> Ms. S: It's the same question? You all have the same question?
> Sn: Yeah. I don't know.
> Sn: Yes.
> Ms. S: Well, they say that you did.
> Sn: We all (confused) means by any value for x could be used.
> Ms. S: Can any value for x be used?
> Sn: What's that-
> Ms. S: What are your x values?
> S: -1, -2, -3, -0.5-
> Ms. S: Can you use any value for x?
> Ss: Yes.
> Sn: Use it for what?
> Sn: Any number?
> Sn: Yes.
> Ms. S: Can any x value be used?
> Ss: Yes.
> Sn: Because it could be any number.

It was most probable the student's agreement with the possibility of using any number for x came from her concept of variable, that x in $y = x^2$ was a variable and it could be any number because it varies. It was not obvious whether Ms. Sands saw the student's answer might be related to her concept of variable rather than thinking of x in the context and possible values as domain of $y = x^2$ rather than thinking of x alone as a variable. From this point forward, she tried to direct students to see if any value for y could result from the rule. The students thought the direction of graph would change for different values of x and she encouraged them to explain what they meant and how the change might be shown. She tried to illustrate all y values would be positive since they were taking the square of the x each time. She also indicated the symmetry in the points (i.e., x,y pairs) but it was not clear if the students saw where she was trying to lead them.

Ms. S: But, what would happen to your ( )
Sn: It would change ( )
Sn: It didn't ( )
Ms. S: It go different way than the way it was supposed to- how was it supposed to go?
Ss: ( ) [they possibly though that it would go towards down based on the discussion followed]
Ms. S: What if x was 5?
Sn: Like 5-squared.
Ms. S: What if x was –5?
S: 25.
Ms. S: What if x was 10.
S: 100.
Ms. S: -10?
S: 100.
Ms. S: What would your graph look like?
S: Different, changed.
Ms. S: How different? Where is your graph?
S: [she showed her graph]
Ms. S: How would it change?
S: It would go different way.
Ms. S: Which way?
Ss: ( )(negative)
Ms. S: Show me (5, 25).
S: [she drew the point on the graph] That's ( )
Ms. S: So, it went down?
S: No, it went- no.

One of These Points Is Not Like the Others – Outliers in a Graph

BC-40 was one of Ms. Sands's favorite problems. She liked it because there was an extraneous point somewhere that did not belong to the pattern and students had to figure out why it was there. She indicated she would never have thought to ask such a question. She thought this was a good problem because if students had such points (i.e., ones that were out of the pattern) when they graphed, they could recognize the situation and check to see if they made an error in their rule or table so they could fix it.

In class, one of the students, Jake, was having difficulty with BC-40 and asked for help. She asked him if he had done his chart correctly, pointing out that he needed to check his calculations and oversee some values. Obviously she thought the problem was a matter of miscalculation or mis-plotting, assuming Jake had a conceptual way to attempt the problem or had some thoughts about it.

> [Ms. Sands, 10/30/2002, Lesson 8, 10:07am]
> Ms. S: So what's wrong with 40? Jake, you got the chart right?
> Jake: (   )
> Ms. S: Is there a point that obviously shouldn't be there?

To illustrate her comment about the point that does not fit to the pattern, Ms. Sands prepared an overhead to demonstrate BC-40. She drew the axes and asked about the points so that she could plot them. She then asked how they would attempt to connect the points. The pattern was a parabola and an outlier point (see Figure 7). She asked Jake about if his graph looked anything like this. Josh explains that he plotted the points and then connected them.

> Ms. S: Well, I would not attack these kinds of problems as one of those things as dot-to-dot puzzles. If you plot all your points and then look for a pattern you'll see, you saw this even before I drew lines, you saw the (-4, -2)- I don't know (-2,-2) [She looked for one of the points from the book]. Oh, where is 42, that's 40. Anyway, it's (-3, -2). Ok, anything else? Are you having problems with plotting points? Drawing lines, except drawing them at the wrong time? I'd plot them all first then I, uhm, see what line fits. [Ms. Sands, 10/30/2002, Lesson 8]

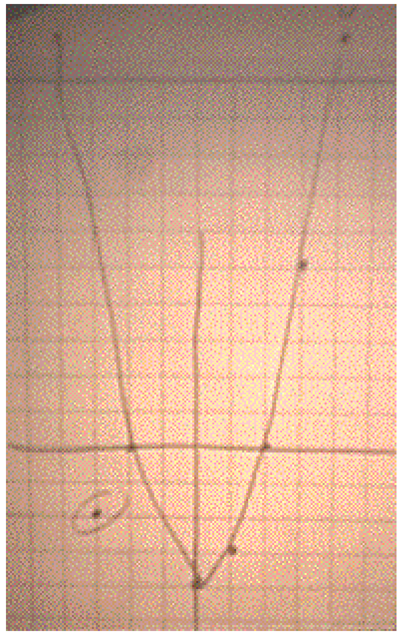


Figure 7. Ms. Sands's graph for CPM Algebra-1 problem BC-40

She emphasized they needed to look at the problem from a pattern perspective so they could see that all but one point would fit in. Knowing students' tendency to graph by connecting individual points consecutively and probably assuming and/or observing that Jake was connecting all the points without looking at the general pattern, she warned the students about linking points to the draw the graph as if it was a dot-to-dot puzzle.

The difference Between $-x^2$ *and* $(-x)^2$

Students' difficulties with squaring a negative number was first revealed when they were trying to construct a table of (x,y) pairs for graphing $y = x^2$. Ms. Sands identified calculator use as one source of the problem. When she was explaining to one of the students who was getting difficulty squaring –4 with the calculator, her first reaction was that she loved difficulty and the problem was with the calculator since they needed to put -4 into parentheses and then square it with scientific calculators. That's why she was getting opposite of 16 as she entered $-4^2 = -16$.

She suggested the student "You have to be very careful with your calculator; that's why you don't need it." [Ms. Sands, 10/30/2002, Lesson 8, 10:22am] She then explained the difference between -4^2, $-(4)^2$, and $(-4)^2$ on the board to that group. Soon after, she had a similar discussion with another group of students who were getting $-4^2 = -16$ from the calculator. One of the student's calculator was giving 16 even he had entered it as -4^2. JP commented both of the calculators were scientific calculators but they were operating differently and this one probably knew what the student was thinking, "It got a sixth sense.☺" [Ms. Sands, 10/30/2002, Lesson 8, 10:28am] Because of the confusions they were getting from the calculator, she told another group that they should trust their knowledge, not to the calculators: "You need to trust yourselves not to the calculators! ... Trust yourselves." [Ms. Sands, 10/30/2002, Lesson 8, 10:33am]

During the second interview, Ms. Sands indicated she was surprised students were using their calculators to find squares of numbers. She found this unnecessary especially after all the discussions in the class "about the square of *x* if it is negative and why their calculator gave them something they just did not understand why they got a negative number" (I2). She thought using calculators in this situation got their focus on thinking why their calculator gave errors rather than focusing on the problem: "They are having to really think why does my calculator do that if I put in –4 squared, why do I get –16 because it doesn't look right" (I2). She believed that if they were not using calculators and directly taking square of a negative by multiplying itself, they would know better what x^2 meant: "If they would not use their calculator they would know what $(-4)^2$ was if they're actually squaring -4 not looking for the opposite which is what their calculator did." (I2)

A more extensive discussion about the differences between $-x^2$ and $(-x)^2$ and what would happen when x was a negative or positive number took place in problem BC-55, which was the introductory investigation in "exploring parabolas" section in unit three. It asked students to consider the expressions $-x^2$ and $(-x)^2$ to answer several sub-questions in order to decide if they were the same and what they could conclude about both. Discussion began with a question from a student asking how they would proceed if they were trying to find square of $-x$ when x was a negative number.

> [Ms. Sands, 11/05/2002, Lesson 9, 10:20am]
> Ms. S: Number 55.
> S: I have a question.
> Ms. S: Okay.
> S: When your rule is like negative x square whatever, if your number is already negative, do you keep it negative?
> Ms. S: That's kind of all 55 is about. I'm glad you asked that.
> S: Because that's confused me.
> Ms. S: Because 55 is gonna make you think about that very thing.

After this brief introductory conversation, she went into summarizing the problem BC-55 and what it was asking them to find. From their previous experience, a student suggested $-x^2$ and $(-x)^2$ were different things to manipulate in the calculators. She agreed with him even though some students objected that they were not different. She let the first student explain why he was thinking they were different.

> Ms. S: Ok, we had some questions on some of our tables and I think even as far as back in Wednesday. We weren't sure what to do with rule when we have $-x^2$ and your x is negative, what do you do? So look at (a). Look at (a) and what it says: Consider the expressions … well it looks like $-x^2$ and $(-x)^2$. We'll think about if there is a difference between those two things. [She wrote $-x^2$ and $(-x)^2$ on the board].
> S: They are different in calculators.
> Ms. S: Huh?
> S: They are different formulas on the calculators.
> Ms. S: Yeah, they are.
> Sn: No. (They don't)

Ms. S: So Brook, what is different.
B: That one has break in an order of operations of ( ) before x square. ( ) what's inside.

As the student said it was related to the order of operations, she did not push this idea forward by going into the details of order of operations right at the moment and end the discussion. This was probably because she had other things to say and/or illustrate and did not want to end the discussion without getting students to understand why the problem was an issue of order of operations.

She found the problem interesting because they were not going to deal with exponential rules until algebra-2 but they needed to solve this problem with their knowledge in some other way. She started showing the difference between those two expressions by illustrate the cases for a particular value of x, -3.

Ms. S: Well, there are some interesting things about this problem. If fact talking to somebody in the high school you're not gonna learn all the exponent rules until algebra-2 which is unusual because I have always kind of did it in algebra-1 but the thing is we got to figure out what this means. These really say two different things. If your x is a negative number like say –3 [She wrote $x = -3$ on the board] and if we want to use this rule [She pointed out $(-x)^2$] what would we do? John?
J: ( )
Ms. S: Say that again.
J:( )
Ms. S: Well, I'm wondering why you don't know what we were doing. … Say it nice and loud to the camera. You didn't pay attention.
J: [He was shaking his head to agree]
Ms. S: That's what I was thinking. So, maybe you should. It will help you in the long run. So, Alice T, what do you think this might be with this rule down here?
A: [Silence]
Ms. S: Not sure? That's fair enough. Margaret.
M: Could it mean like we had another number. It could mean that other number times negative x then you square it together or something that you multiply it and you square it.
Ms. S: Well, what are you gonna multiply?
M: Like your number was 2 [Ms. S said "Ok" and wrote $x = 2$ under $x = -3$] and then you multiply- I don't know. Ok. [She walked to the board] You could multiply number 2 [She pointed in front of $(-x)^2$] by –3 [pointed out inside the parentheses] and we square it [showing the 2 in the exponent].
Ms. S: We don't have anything up here [pointing in front of $(-x)^2$].

Sn: I have a question.
M: But if you did-
Ms. S: But we don't. We don't in this example. We're not gonna make it real complicated. We're just want to know what this means [pointing out $(-x)^2$].

At first, Ms. Sands thought Margaret was thinking another value, a positive value for x rather than what she first suggested (i.e., –3) and expected Margaret to find the value of $(-x)^2$ for $x = 2$. For some reason, however, Margaret was bringing an additional number, 2, and suggesting she could multiply it with -3 (i.e., 2 times –3). As soon as she understood that Margaret was bringing 2 to the situation in a different way than having it as a particular value for x, Ms. Sands rejected the idea, not because it was wrong but it was not the particular example that they were working on. Even though she probably did not want to take students focus away from $(-x)^2$ by going one step further in a variation it involved, she could have accepted such a number making a different case like $2(-(-3))^2$. What Margaret was suggesting was problematic from two perspectives: (1) She was suggesting to multiply 2 and –3 without considering the negative sign in $-x$, probably associating it with the negative in –3; (2) She was not considering the role of squaring that should come before multiplying it with 2 in the order of operations. What Ms. Sands did after this discussion, however, was to start from what she initially thought about Margaret's suggestion (i.e., $x = 2$) and ask another student what would happen if they have $x = 2$. She probably thought that starting with a negative number like -3 might not a good way to start and easiest would be to start with a positive value like 2.

Ms. S: What if x is 2 Curly?
C: Oh, would you put that number, whatever your x number is inside the parentheses and then square it.
Ms. S: Well, how do you mean? Like that [She wrote $(-x)^2 = (-2)^2$].
C: Yeah.
Ms. S: Just replace x with the 2 and leave minus sign where it is.
C: Uh-huh.

After she had the type of response she was expecting, Ms. Sands went back to the case of x = -3. The same student suggested replacing –3 with 2 in the expression and Ms. Sands wrote this in the table $(-3)^2$. This divided the students into two groups, those who thought that it should have been positive and those who thought the otherwise and satisfied with $(-3)^2$. She let one of the students explain why it should have been positive. He tried to explain it should have been -(-3) producing a positive number as she tried to explain it that having double negatives should result in positive.

> Ms. S: What if x is -3? How am I gonna put it in here?
> Ss: (   )
> C: You put 3 where the 2 is. [She wrote $(-3)^2$]
> Ss: ([some thought it would make positive and some others thought it would make negative])
> Sn: ([asking if it would make positive or negative?])
> Ms. S: That's- that's my question. All right Brook.
> B: All right. The negative- put the negative number in there. When you subtract- when you like multiply a negative from a negative, you get positive. But you have a negative in there; you have to change it to positive. (   ) I mean (   ) you had a positive number there and positive times (   ) it gives you- it gives you negative, so it would be a negative.

Since Brook was having difficulty in explaining that negative times negative makes positive, and she knew that having difficulty with sign rules would be a general case for most of the students, Ms. Sands suggested another perspective to look at –x. She suggested to perceive it "opposite of x" rather than negative x, which she realized as a source of problem.

> Ms. S: Uh-huh. Think of it this way. This says [pointing –x] not x squared //but opposite.
> Ss: //The negative.
> Ms. S: Don't think about negative. That stuff is gonna confuse you. That's why you don't know your sign rules very well. This is the opposite of x [pointing –x], isn't it? It's not x, it's the opposite. So if it's not 2, it's negative 2. Right? It's the opposite of x. If x is negative, then we want the opposite of -3, don't we? Is that help?
> Ss: [uh-huh and no's]
> Ms. S: Think of this is the opposite of the whatever the number is. … It's not negative x. That doesn't mean much to us because x isn't represented by a number. What are you doing now John?
> J: (   )

Ms. S: Whatever it is, put it up and pay attention. Yes ma'am.
Sn: ( ) about the parentheses wouldn't it be the same thing?
Ms. S: We're not there yet. Hold up that thought, we're going there. Does this right? [She pointed out $(-3)^2$]
S: No, it should be positive.
Ms. S: If *x* is -3, we want the opposite of it.
Sn: (Maybe regular 3)
Ms. S: No, that's regular 3. ☺
S: No positive three.
Ms. S: [She corrected it as $(+3)^2$]

It looked like some students were not satisfied with Ms. Sands's first explanation about why and how they should see $-x$ as opposite of x, not negative x. It was not clear, however, if they were satisfied after the second explanation since there was an interruption followed by a question about parentheses that Ms. Sands postponed responding. She realized with the comment about opposite of –3 as regular 3 that the student meant positive and let him correct himself.

In continuation, Ms. Sands seemed satisfied with $(+3)^2$ and stated that the rest (i.e., squaring) was a simple multiplication, which she was sure they all knew. However, some students were still not satisfied with the expression they found previously. At this point, Ms. Sands asked them to elaborate on why it was not making any sense. One question came about where she got 2 suggesting that she was still at a place trying to make sense of why they were using particular numbers while they had x in the expression. Another question was about the function of using parentheses in the expressions. Ms. Sands elaborated on it since they would need it to make the order of operations clear. She did not approach it from a conceptual perspective trying to explain it that it gives a different meaning like it told them to take the square of what is inside, which is the opposite of *x*.

Ms. S: Yeah and we know how to square the numbers, that's not the problem. What we're having problems with is the signed numbers which you should have done it in ( ) negative last year. Because squaring on this doing multiplying and I mean that's just multiplication from what you've already learned or heard [She showed quota-o-quota with her hands and fingers]. Ok? Does this expression make sense to you?

Ss: [Yes and No's]
Ms. S: No? Tell me why it doesn't make sense to you?
Sn: Why do they use parenthesis?
Ms. S: Can I answer hers first? I made it up [to another student asking "Where did you get the 2 from?"] I just made up the numbers so that we can figure out how to put them in place of the *x*. It came from right here [She pointed her head with her finger] It's a dangerous place to be but-. So, why is it in parenthesis?
S: Yeah, why do they- ( ) to use parentheses?
Ms. S: No, they try to help you with parenthesis! [One of the students in the back said order of operations] The parenthesis is there to help you because you do know the order of operations, true?
S: Yes ( )
Yes, all that. So if you're looking at the parenthesis, it's a helper. It's not supposed to confuse you. It's supposed to tell you what's in there- that number. Ok?
S: Ok.
Ms. S: So no parenthesis there, we have to think differently.
S: Ok.

One of the students tried to explain her understanding of opposite *x* as only changing the signs from + to – or vice versa and that it confused her. Upon asking and getting no feasible response from the student about why it was a source of confusion, Ms. Sands suggested the locations of positive and negative of a number as placed on a number line in order to explain the relation between changing signs and meaning of opposite of a number. However, the same student still thought that opposite of *x* was negative. It looked like the real problem was that she did not have a well developed understanding of an algebraic expression and the role of variables in them. In this case, it was the meaning of $(-x)^2$ and the role of *x* as a variable in it.

Ms. S: Fun time. You got a question? No tell me or ask me [she wanted to show it on the paper].
Sn: Well, if all you do is like change the signs? Like is it- uhm, is it –*x* and you changed the -3 to +3?
Ms. S: You want the opposite of *x*. This tells you I want the opposite of *x*, is that confuse you?
Ss: (Yeah).
Ms. S: Why?
Sn: I don't know.
Ms. S: Well, we got *x* and –*x* [She wrote those to board]. Are they the same thing? 2 and –2, they are opposites, aren't they? *x* and -*x;* they're opposites. *y*, -*y,* opposites. 14, -14, opposites [She opened her hands to show that opposites were on opposite sides] Zero and

there are two numbers exactly same place on the number line from the zero. So we want the opposite of x.
S: That would be negative.
Ms. S: Depends on what x is and that's our problem. We're not sure what to do with it when we don't know what x is. If x is the 2, we want the opposite of x, which is -2. But if x is a negative number we want the opposite of that x, which makes it positive, ok?

In the follow up, one another student asked if he needed to square once or twice. Ms. Sands did not wonder what he was thinking to lead to such a question. She rather answered it shortly as saying he needed to take the square once. He might have been considering that 2 in the exponent meant take the square twice, thinking the value of the number in the exponent would determine the number of times to take the square. On the other hand, some other students still had problems of having 2 and/or –3 as particular examples.

Ms. S: Yes ma'am.
Sn: Uhm, is that right there where it says parentheses $-x$ squared- uhm, $-x$ person and (   ) with the square thing. Would you only square it once or you square it twice?
Ms. S: You want all that in a parenthesis. You want all that stuff you just said on the parenthesis.
S: (   )
Ms. S: If it looks like that? [She pointed out $(-x)^2$]
S: Uh-huh. (   ) would you only square it once?
Ms. S: Yes, yes. All right.
S: Because also all you did negative negative two, right? So, you just changed it to a –2 to the square. But you already had 3. You made of (   )
Ms. S: I made both of those numbers [i.e., 2 and –3] up.
S: Ok, ok ☺☺:
Ms. S: I made them my little example. I thought it up myself because it deals with number 55, Ok? All this is about number 55, Ok?

Ms. Sands did not go into talking about $–x^2$. She wanted students to figure out if $(-x)^2$ and $-x^2$ were the same things. At that point, another student brought up the initial discussion where they would have a number in front of $(-x)^2$ and asked if he needed to multiply first and then take the square root. Ms. Sands reminded about the order of operations dictated them consider parentheses and exponents before doing the multiplication. This confused some of the students and required Ms. Sands elaborate on order of operations with $3\cdot(2 –3^2)$ as a different example first and then $[2(4 + 3) + 6(2 + 1)] + 4$ as a more basic case to review how the order of operations required them to act.

Sn: Ok, so is there that where (   )
Ms. S: I want you to decide if they are the same things. [i.e., $(-x)^2$ and $-x^2$]
Sn: (   )
Ms. S: Say that again.
S: [He went to board] Say you have this $-x$ squared. You have like a 2 before [He put 2 in front of $(-x)^2$], would you multiply these two [pointing 2 and $-x$] and then square that answer?
Ms. S: No.
S: Or do you square this [pointing $-x$]
Ms. S: You do your parenthesis and your exponents first before you do any multiplication first.
S: Because of order of operations?
Ms. S: Because of the order of operations. Exactly. Yes [to a student]
Sn: There were something- (   ) I'm confused.
Ms. S: While you're finding it, did you understand what John was asking?
Ss: ([No and yes])
Sn: You square your number before you multiply it.
Ss: (   )
Ms. S: We still have to follow the order of operations. If there happens to be a 2 or some number outside which looks like the distributive property, right?
Ss: Uh-huh, right.
Ms. S: Only there is nothing to really distribute but in here we're not adding stuff. We still have to do our parenthesis first and our exponent before we do any regular multiplication. So, the 2 is not the part of this square business at all.
Sn: So you do that and then square.
Ms. S: If the 2 is in here, it is and that's something else.
Sn: But you square and then you multiply.
Ms. S: Square first and then multiply that result back to. Yes.
Sn: What about, uhm, BC53, 2 minus 3-squared [he was referring another problem: $5\cdot4-3\cdot(2-3^2)$]. It was- it was in the front. Which would you do first?
Ms. S: What do you always do first?
S: You do the square.
Sn: Inside parenthesis.
Ms. S: I see it.
Sn: So you do the-
Ms. S: This is 53(a) if you want to look at it. You always- what do you first?
Sn: You do the square.
Sn: (   ) exponent or multiplication.
Sn: No, if the exponent is in the parenthesis, what would you do?
Ms. S: What's the exponent on?
Sn: 3.
Ms. S: Well, then should you do 2 minus 3 and then square it?

Sn: No, you're supposed to do your exponents.
Ms. S: You have to do that exponent first and do 2 minus 9. Because the 2 is not part of the square business. If the square were on the outside of that parenthesis, if it's out here, yeah you do what's in there and then square it, ok? Yes ma'am.
Sn: What kind of- this is kind of- it's on the subject but if can I come to board I'm just gonna ask it.
Ms. S: Ok.
S: You have little bracket thing look like half a square (   ) ok, on my worksheet either a long time ago, I don't know like eighteen pages of that.
Ms. S: 18 pages of that! I gave you something like that?
S: Well, you were out and Mr. X.
Sn: No, it was like three or four pages.
Ms. S: I wouldn't do that to you either. ☺
Sn: Anyway.
Ms. S: Anyway.
S: Problems in them like they were in the parentheses then the parentheses ended and they still went on. What would you do then?
Ms. S: What's your order of operations tell you to do?
S: Well, parentheses but that's (   ) like parenthesis.
Ms. S: Inner most parenthesis first.
Sn: (Is that just to say except parenthesis)
Sn: So those are parentheses too? [Referring to the brackets [] that Ms. S was drawing on the board to illustrate and example where they would have more than one parenthesis]
Ms. S: They act like parentheses, yes.
Sn: So, those are the same as parentheses?
Ss: (   )
Ms. S: You gonna do this first, four plus three. [pointing 4 + 3 in [2(4 + 3) + 6(2 + 1)] + 4] ok? [then they solved the rest of it together]
Ms. S: Just stay in your brackets till you worked all out.
Sn: Would you put your answer in brackets?
Ms. S: No, I am thinking (   ) what's in the bracket because it's a grouping symbol.
…
Ms. S: You always do what's in inner most first. Of course this was easy because it was all numbers. But you're gonna do the same thing when it's not all numbers.

After discussing the order of operations, Ms. Sands went on to explain the second case, $–x^2$. She first pointed out that $–x^2$ did not have any parentheses to make a point that exponents come first. She then introduced writing $–x^2$ as $-1{\cdot}x^2$ to clear the role of "-" in $–x^2$. This caused confusion for some students.

[10:36am]
Ms. S: Now, talk about this [she showed $–x^2$] so you can do 55. In fact, you've got 55 to,

uhm, yeah 63. 63, I didn't say 73 [She disagreed with some of the students who are saying 73]. Let's talk about what this one means. And we've talked about this before. Some of us have talked about this before. We need to decide if this expression says the same thing as this one [pointing $-x^2$ and $(-x)^2$]. Does it have any parenthesis?
S: No.
Ms. S: So what should you do first?
Ss: Where?
Sn: You should, uhm-
Ms. S: We should do the exponents first. How many x^2s do I have?
Ss: One
[Ms. S put 1 between – and x. i.e., $-1x^2$] So what I have is –1 times x^2. That's how many-
Sn: Where the 1 come from?
Ms. S: I asked you how many $-x$ squares I have and you said 1.
S: Ok.
Ms. S: So I wrote it up.
Sn: Is it just an example or is that, I don't. I thought you said-
MS. S: [Ms. S seems confused] It's not what x is. It's how many x squares I have.
S: Oh, so like you got in there-
Ms. S: That's not what x is, it's how many of them I have.
S: (   )
Ms. S: It's the big x^2 and I have one of them. If I had 3 of them it would be $-3x^2$, right?
S: Ok.
Ms. S: If I have 3 of these, negative three of them. They'd be what was the back side of that blue? [Ms. S made her fingers at 90 degree and connected them to form a rectangle with three sides. She imitated the algebra tiles to show $-3x$] The red was always positive right? I think those were blue.

Ms. Sands had to remind about algebra tiles so that students could make sense of $-3x^2$. At this point one student asked about what would be the case if they had negative exponents. She did not go into any explanation since it was a topic related to exponential rules, which she previously indicated they would learn in algebra-2 and indicated that all of the exponents they were using were positive. She then requested them to compute the value of $(-x)^2$ for $x = 2$ and emphasized the order of operations and role of parentheses. After that, she re-stated why she wrote $-x^2$ as $-1x^2$ and asked them to compute its value for $x = -3$ and arrived at a conclusion that while $(-x)^2$ always produced a positive answer, $-x^2$ produced a negative one.

Ms. S: Yes ma'am.
Sn: Oh, what happens if there was exponent (   )
Ms. S: If the exponent is negative?
S: [silence]
Ms. S: If the exponent is negative?
S: Yes
Ms. S: We're not going there right now. I don't think, not today. Not today. So, we're talking positive exponents here, ok. Not negative exponent today. They're all positive. So, if x-squared- I have one negative x-squared or one opposite x-squared and x is 2. What should I do?
Sn: -1 times 2 and then square it.
Sn: Negative, ok, if it is negative x times x would be one-
Ms. S: x is 2.
Sn: Oh, x is 2. Then I would multiply it with –1.
Ms. S: That's what I would do. Thank you. Had to follow the order of operations. I don't have any parenthesis… I told you that 1 is gonna be a very good friend of yours for a life, there it's, you forgot about it. That's what makes this different from this [pointing $–x^2$ and $(-x)^2$]. Because we don't have any parenthesis to tell us that we want the opposite of x squared, depending on what x is. So we're gonna use this imaginary 1 [pointing at $–1x^2$] that's very real now. Don't forget about it. We have to square the x, whatever it is, and multiply, because multiplication comes later on in the order of operations, by -1. So if x is –3. I replace my x with –3. I have –3 squared which is …
Sn: 9.
Ms. S: 9 times –1 is –9. So all of these answers came up to be what? [Pointing $(-x)^2$]
Sn: Negative.
Ms. S: All positive. This answer [pointing $-x^2$] is always come out …
Sn: Negative.
Ms. S: There's the difference. Look at 55. You got some explaining to do in #55.

After the discussion ended, students started doing BC55-BC63 as their homework. When Ms. Sands visited one of the groups, it seemed the students had some struggles with the generalization she made at the end that $-x^2$ would always produce a negative value. She explained again that x^2 was positive whether x was negative or positive and –1 times a positive number would always be negative. Although some students seemed to realize what their thinking lacked and got the idea, some others seemed to have a hard time accepting this generalization. Ms. Sands's questions to students thinking $-x^2$ would also produce positive values did not reveal clues about why they were thinking that way because of their inability to explain their thinking. Interruptions from other students did not let Ms. Sands to pursue to get their explanations.

> [11:42]
> Ms. S: How are we doing? [to a group]
> S: Didn't you say when was on without the parenthesis it always come out negative?
> Sn: No.
> Ms. S: Well, do you think it'll always be negative?
> Ss: No
> Ms. S: What happens when you square a number?
> Sn: It becomes greater or the opposite.
> Ms. S: If you square a positive number is it always positive?
> S: Yes.
> Sn: No
> S: No, not every-
> Sn: Yeah, it is different. Is it negative?
> Ms. S: If the number is a positive number and you square it, always positive.
> Ss: Yes.
> Ms. S: Ok. If a number and square it, is it always positive?
> S: yes. [Others: silence except him who says yes too]
> Ms. S: What happens when you square a negative number? Always positive. If you always multiply positive number times –1, what are you gonna get?
> Sn: Positive.
> Sn: Positive
> Ms. S: Did you hear what I said? If you always multiply a positive number with –1, what are you gonna get?
> S: Oh, negative.
> Ms. S: Always?
> Sn: No, all the time
> Sn: Always.
> Sn: I don't think all the time?
> Ms. S: Why don't you think all the time?
> S: Because, uhm, negative …
> Ms. S: Are you gonna get a negative always? Always, as far as you know.
> Sn: Ok, so that one would be positive and that one would be negative?
> Ms. S: Uh-huh.
> Ms. S: Do you know why?
> S: Yes. Because this is gonna be squared and negative, uhm, no, yeah. I understand about ( ).
> Sn: May I ask a question?
> Ms. S: Certainly.
> S: About that –1, it is there to help you, right? But when we got the *x*, it is ( ) because you- … ok I know what I am doing. I am mixing it up with the other.
> Ms. S: Ok.
> S: I'm sorry.

Ms. S: So, you're ok?
S: Yes. Ma'am.

Ms. Sands found it challenging to help or teach them without interrupting their current level of knowledge dictated by the curriculum. Within this frame, she reminded them about the exponential laws from discussion about $-x^2$ versus $(-x)^2$ and imaginary numbers in the classroom.

> There are certain things I would like to be telling them for instance some exponent rules that I know aren't here yet, I know where they come and I'm kind of used to doing that and I really don't want to give anything away but at the same time I don't wanna tell them so much that they are too overwhelmed. So I think I need a lot of time teaching this way so that I know where it's really going and I don't get too much away so that they're confused and you know when we talked about imaginary numbers, they just had to know and I told them and it's kind of like opening Pandora's box. You know, you gotta know and then once you know you can't get it out of your head. And you don't really need it right know so. (I3)

She was also frustrated that the curriculum did not teach and students were expected to understand exponential laws until algebra-2. This was something she was not used to it because she always taught exponents in algebra-1. She thought she could take an aside and show exponential laws for couple minutes, but she did not want to destroy their thinking about it so they could learn the rules later by discovering some properties about exponents themselves. She also did not see a need for knowing about exponential rules/laws to solve the problem: All they need to know was taking squares, which they already knew, and how it would affect the graph.

> I think all were really trying to get out so far is the quadratic thing, not really the *x* to the 10 times *x* to the 14 and what is that mean. I don't think we're going for those kinds of exponent laws and I think we were just dealing with squares so far and to see what squares do and how they change the line. So I don't know if I really need to go into all that exponent stuff and they really do know how square numbers. But the other stuff I don't think I need yet and I think that's in their curriculum later on because what to do when you multiply powers and take a power to a power and I don't need to do that yet. (I2)

These lines suggests that Ms. Sands had an insightful idea about what the problem required and what students needed to know without affecting students' thinking with something they did not need to solve the problem. This also showed that Ms. Sands's decision making was not blindly based on following the textbook. It was judgment and made by evaluating what was appropriate or not.

Non-Linear Functions and Understanding 1/x

Problem BC-64 required to use the $1/x$ key on a calculator to find the results of substituting each of the following values for x: (a) 2; (b) 3; (c) 0.5; (d) -0.75; (e) 100; (f) –100; (g) 0.01; (h) 1/2, (i) -2 1/2. As a follow up, in BC-65, they were asked to think and elaborate on what they had observed in BC-64 when $x > 1$, $0 < x < 1$, $x < 0$, $x = 1$ and $x = 0$. Those two problems intended to prepare students to graph $y = 1/x$ in BC-66.

Ms. Sands began to the problem with asking students how they could make 6/2 as a multiplication. She rejected the students' vocabulary "6 over 2" because it "is not a math word", it should be "6 divided by 2" [Ms. Sands, 11/08/2002, Lesson 11, 10:09am]. She then asked them to put 0/1, 0 divided by 1, in the calculator and asked them what they got and what was the multiplication problem. She wrote $1 \cdot 0 = 0$ and asked them if they and their calculators would agree with it. She then wrote 1/0 on the board. Students stated they were getting 'error' messages but Ms. Sands asked them keep trying. Some students commented that zero had no value and she disagreed with them as responding, "zero has value!" Discussion went on and some students made the same comment that zero had no value, no factors, etc. She insisted, saying zero had value and they needed to get over it. Discussion went on with zero had no value; zero had no factors, etc.

The lesson went on as Ms. Sands asked them to find the $1/x$ button on their calculators. Using this button, she wanted them to put the operations in BC64 into calculator and write down the results. Students started doing it by using the $1/x$ button. She walked among the groups and talked to individuals. Some of the students did not know how to use 1/*x* button and she asked them to figure it out: sometimes the number first (*x* =?) and then the button or vice versa. If they did not have the 1/*x* key, she asked them use just plain old division. Most of the students had it as secondary button. She showed that fact as a "secret!" After the students were done, she began the group discussion by asking about the function of 1/*x* key in their calculators.

> [Ms. Sands, 11/08/2002, 10:30am]
> MS. S: What did you notice, what does this button do? Does it half the number?
> S: No, ( )
> Ms. S: Fourths the number? What do you think it does?
> Sn: ( )
> Ms. S: Yeah that is what it does…1 is divided by *x* and *x* is the number that you put in. That number hit that button and it does what; it does division.

Ms. Sands then guided students into BC-65 first to discuss the variation in 1/*x* as *x* was increasing. Students could not see the general pattern as decreasing and described the variation in 1/*x* as it was changing while *x* took different values even though Ms. Sands pointed out that the answer was "decreasing". She skillfully tried to manipulate students' thinking so that they would see that when *x* was a number bigger than 1, the value of 1/*x* was smaller than 1. But some students were claming the opposite.

> [10:31am]
> Ms. S: Look at 65 [i.e., BC-65] and see what happens as *x* increases. *x* increase, what happens? The answer decreases. Why?
> Ss: ( )
> Ms. S: Because you're dividing what by more? You're dividing 1 by more, more being a bigger number. Are all the numbers here do they get bigger as you go?
> S: They change.
> Ms. S: They change, yeah…. What kind of conclusion can you make? … Are you saying if the number is bigger than one you get a smaller number?
> Sn: ( )
> Ms. S: If the number is smaller than 1 you get a smaller number. Wow!
> Sn: ( )
> Ms. S: If it's like 0.01. Is that one of your examples. You got 100. That's bigger than 1…

She restated the question as taking the focus to BC-64 and asking what they did notice about the values of 1/*x* when *x* was bigger than 1. She got confirmation that when $x > 1$, the value of 1/*x* becomes less than 1 and less than *x*.

> Ms. S: What do you notice about your results when *x* is greater than 1?
> Ss: ( )
> Ms. S: Then your result is less than the *x*, isn't it? Is it less than 1? Or was it just less than the *x*? (*a*) was 2 right? (*a*) was 2, wasn't it? You divided 1 by 2 and you got .5. Is that less than 2?
> Ss: Yeah
> Ms. S: Is it less than 1? Ok. You had another problem that was *x* =100 right? Well, when you put that in the calculator what did you get? .01,… is that less 100?
> Ss: Yeah
> Ms. S: Is that less than 1?
> Ss: ( )
> Ms. S: Are they all less than 1?
> Ss: ( )
> Ms. S: What Rose?
> Rose: ( )
> Ms. S: If it was a whole number, the answer less than 1. Ok, the *x* being a whole number. That's what you're saying. …What do you notice when *x* is greater than 1? The answer is less than *x*.
> Ss: //( )
> Ms: //and less than 1?
> Ss: ( )
> Ms. S: (h)? What's (h)? Oh, h is one-half. That's not what (a) is about. (a) is about *x*'s that are greater than 1. If *x* is greater than 1, what is *y*?
> S: Less than 1
> Ms. S: Less than one? What is our conclusion for (a)?
> S: ( )
> Ms. S: Less than *x* and less than 1?
> Ss: ( )

It was interesting to see Rose making a generalization that if *x* was whole number 1/*x* was less than 1. She was probably affected by the fact that in BC-64, all of the numbers greater than 1 were whole numbers. However, I do not think Ms. Sands saw this judgment based on her response and accepting it as a plausible conclusion.

After getting a closure and a conclusion on part (a), she moved to part (b) by asking them what would happen to 1/*x* as x is between 0 and 1. Although the original statement of the problem did not include a functional approach or notation to $1/x$ as $y = 1/x$, Ms. Sands referred to $1/x$ as y at the beginning and changed it to $1/x$ after using it twice. This notational reference gave a clue that she knew where the problem was taking students or what concept it was trying to build, non-linear functions. It was also interesting to see Ms. Sands using '1 over x' although she did not want students use it since it was not mathematical in this situation.

> Ms. S: If x is less than 1 and greater than 0, what happens to y?
> Ms. S: You get a number between 0 and 1?
> S: No, (   )
> Ms. S: It's asking you x, excuse me and listen. If x is between 0 and 1, what's y? Or $1/x$ what's one over x if x is between 0 and 1? A whole number? Always? I mean all the ones you did was greater than 1? Look at them. …Because some of you didn't write down what you put in, you just wrote down the answers. See you don't have a comparison on your paper. If x is between 0 and 1 what is 1 over x? What can you conclude? … Is [it] greater than 1? 1 over x is always greater than 1! …What's the biggest fraction you can think of between 0 and 1?
> S: (   )
> Ms. S: Ninety-nine hundredths! Put that in your calculator. Put that in your calculator!
> Ss: (   )
> Ms. S: .99 is fine or .999 is fine. Any of them is fine.
> Ss: (   )
> Ms. S: Is that bigger than 1? … Is it greater than 1? Shhh, you don't all have to report what you got. I think we're getting about same thing. Is it greater than 1? So what can you conclude? If x is smaller than 1 or between 0 and 1 we're not talking about the negative numbers here. Then 1 divided by x is, more than 1?
> Ss: (   )

Later, she got students to notice that when $0< x <1$, $1/x$ was bigger than one. It also was nice to see Ms. Sands bringing fractions to illustrate a number less than but very close to 1 so that they could see the result is always greater than 1 no matter which number x is between 0 and 1. However, Ms. Sands neither made them try a number very close to 0 nor discussed the change in $1/x$ as x was taking numbers from 0 to 1 or vice versa.

After completing part (b), she then moved part (c) to discuss the case when x was less than 0. She briefly discussed what happened to $1/x$ when x was a negative number. Ms. Sands actually used the term "pattern." It looked like she was not referring to a general trend in $1/x$ as x

was decreasing and less than 0 since the only conclusion the class got out of it was that $1/x$ was a negative number when x was less than 0. On the other hand, Ms. Sands noticed that division of positive and negative numbers and sign rules was an issue with the students and she wanted them get the idea permanently that division of positive and negative always produced a negative number.

> Ms. S: Ok, Ok. …What do you notice about the results when x is less than 0? Which ones were less than 0? (d), (f), (i). Look at (d), (f), (i). What happens to your answers when x is less than 0? They were also negative; they were still less than 0! What's less than 0?
> Ss: ( )
> Ms. S: It's Ok. (d), (f), and (i) are negative, they're negative values for x. So is the 1 over x is always less than zero? Look at (d), (f), and (i) and see if there's a pattern there. They're not negative?!
> Ss: ( )
> Ms. S: Why is that? Think about your signs rule. Shh, Jake is talking. Say it again.
> Sn: ( )
> Ms. S: What you were dividing positive and negative, what do you get? … You always get negative. Not a positive and a negative. And is kind of for addition. It's not true. Multiplying and dividing positives and negatives, what do you get? You always get negative. Don't you remember you're getting negative? But you didn't make any generalization last year, did you? You should be doing today right now this minute. You should be doing this to your head so that it sticks even when you go to Trig[onometry], you'll remember this. Right. It's important.

In continuation, Ms. Sands asked about the cases of $x = 1$ and $x = 0$. The former was not a problem whereas the latter was. Students tried to answer 1/0 by using their calculators and got "error" messages in return. Ms. Sands did not want them to use their calculators since they would not work. But she tried to get students to think about why they were getting error messages. Some of the students believed they could divide 1 and 0 even though somehow their calculator was not doing it. They might have reminded about their problem with the calculators when they were doing $(-x)^2$ and $-x^2$ and the special way of entering it to different calculators or getting the error messages when they were taking the square root of a negative number. So, there could be a way to get calculators around this problem and make the division possible and get rid of the error messages. It did not look like Ms. Sands was aware of any possible effect of students' past experiences with calculators and error messages they got in the past. On the other hand, some of the students suggested that zero had no value so that is why they were getting error messages or the division was not possible. Ms. Sands tried to convince students that zero was a number and had value by pointing out its role in place value.

> Ms. S: All right, (d). What happens if x equals to 1? … x is 1. So 1 divided by x is still 1. We have calculators drop today. They're not going to work. What happens x is 0? Now why is that you're getting error when you're getting this? [She pointed out $1/x$ on the board]
> Ss: ( )
> Ms. S: Why is that?
> Sn: ( )
> Ms. S: Well, 1 is bigger than 1/2 and you can do that!!! I don't get it. Who said, Teresa?
> Sn: ( )
> Ms. S: What's that got to do with it?
> S: ( )
> Ms. S: Don't tell anybody of that. How old are you?
> S: ( )
> Ms. S: What happens when you were one zero? But you can't be if zero is a number. You can't be 10 if zero is not a number. So don't go there. Zero is a number otherwise you can't be 10 years old. And I know if you're going to be 20 someday, zero has to be number. Ok, Brad what do you think.
> Sn: ( )
> Ms. S: That's for sure.
> Sn: ( )
> Ms. S : No I can't agree with that either. I can't agree with that. I agree with you [She pointed Brook] I don't agree with you [She showed Annie whom was just talking to her] Finish what you were saying.
> Sn: ( )
> Ms. S: So you think it can be done but calculator won't do it. Ok, Annie, say what you said again so I can be agree with you.
> Sn: ( )
> Ms. S: I'm not telling you are wrong, I just disagree with you. …So the blue calculator, the Casio, the TI are all making mistakes. Is that what you're saying?
> S: ( )
> Ms. S: Of course they do.
> S: No, I have common sense.
> Ms. S: You're saying and I agree with that almost a hundred percent. So all of you are saying, rise your hand if you're saying you can't divide by 0 on the calculator because it's making an error… you believe you can divide by 0 and the calculator just wont do it… ok, put your hands down and raise if you don't have clue on what's going on. [8-10

students raise their hands] No you can't raise your hand twice, Ok?
Sn: ( )
Ms. S: Then how come when you divide 0 by 1 you get 0?
Ss: ( )
Ms. S: Why will it do this but don't do this.
Ss: ( )
Ms. S: Why can't 1 go into 0?
Ss: ( )
Ms. S: It is. It went into 0 times [Talking about 0/1 and$1 \cdot 0 = 0$]
Ss: ( )
Ss; ( )
Ms. S: Why can't 0 go into 1?
Ms. S: ( )
Ms. S: Don't say it doesn't have value!
Ss: ( )
Ms. S: Well, 1 doesn't have as much value as 2 but it goes, doesn't it? Tell you something. Tell you something really profound. I didn't say profane, I said profound.

They could not reach to a conclusion about possibility of dividing 1 with 0 even though Ms. Sands tried to her initial idea of writing division as fraction as illustrated 0/1 as $1 \cdot 0 = 0$ and tried get students to see that there was no number that would give 1 when it's multiplied by 0. Towards to the end of the class, one student told Ms. Sands that he did not learn anything that day. She tried to convince him that talking about 0 was a "cool" thing to do. She asked all of the students to think about 0 and write down anything they know about it. I was not clear if she was giving this homework so that they would accept that zero was a number and had value or understand the role of 0 in mathematics general or to reach a conclusion about $1/x$ when $x = 0$.

[10:49am]
Ms. S: You haven't learned anything?
[to a student]
S: I haven't learned anything today.
Sn: ( )
MS. S: Well I think it's cool to sit down and talk about 0. I want you to think all you can think about 0. Excuse me, [Daily announcements came] Not until I gave your homework. Your homework, Ms. Hanna will be here on Monday and Tuesday so we'll not be back to this until Wednesday. I want you to write everything you know about 0. Just write down anything you know about 0. And I want your papers back right here. [i.e., the tests]…

Ms. Sands did not use the little figure showing a basic sketch of $y = 1/x$ printed next to the problem statement in BC-65. On the other hand, graphing $y = 1/x$ was one of the tasks in BC-66, the next investigation. For this reason, she might just have wanted to leave the graphing and discussion until that problem, not wanting to distract students' thinking, as she believed happened in some other cases. They did not have any other discussion about this except a comment Ms. Sands made on the next time [11/13/2003, 10:04am] that they had a lot of discussion on last class about $1/x$ and she hopes that they had thought about 0 because it definitely was worth discussion.

Ms. Sands's Approaches to Students' Thinking in the End of Unit Test

The highest score on the end of unit test was 94, and the lowest was 13 that she considered as "just not knowing anything: getting lucky on one problem maybe" (I3). She expressed that she was disappointed with the results in general because she did not think anybody would fail except for maybe a few of them. In the evaluation of the test results, she expressed that when she looked at the results she saw "some good thinking" but she also saw "typical 8th grade carelessness" (I3). She did not think there were any questions they could not understand, so she attributed students' wrong answers in the test to their carelessness in general rather than lack of understanding.

Although she was not a big believer of it, test anxiety was another perspective Ms. Sands brought to why students would have low performance or mistakes on the test. Since the students were not doing the test together or doing it at home like a group test or homework, she thought some of them might have gotten anxious about a test and could not do it in a particular time frame. That is why she let students correct their mistakes, hoping they would not do it again. She claimed she tried not to make the test so stressful that students would get scared of it. She would rather let them know what problems they were going to be seeing in the test beforehand so they would not worry if they knew how to do it because that was what mattered for her.

She described the first problem as a place where she saw a lot of careless mistakes. In the first question, a line was drawn on a grid coordinate system without labels and its equation and students were asked to find the corresponding *y*-coordinate for a given *x*-coordinate value (or vice versa) and coordinates where the line crosses *x* and *y* axes. She indicated one could clearly tell where it crossed each of the axes and she knew "very well that they know how to read where the points are even though the point may not be clearly labeled" (I3). At this point, she thought there was no excuse for missing that problem and it was "unforgivable" to make mistakes on "which is the *x* and which is the *y*" (I3). She explained that when the students got their test back she had discussed this point with them before they corrected it and they were disappointed about how could they missed that problem. She thought it was because they did not read the problem and think about it.

> They're very careless mistakes because they know how to read a point on a graph and they simply didn't do it even though the point isn't labeled they did have to read the point and write the point and some of them would put *y* before *x* which they know is not right. But they were not using the whole brain I think. (I3)

As it was implied, she thought having a grid without the labels for points might have been part of the difficulty but she thought it was part of the deal and they should know how to read the coordinates of those points. She also expressed that some of the students were reversing the order in (x,y) pair when they were writing their answers. She further thought that maybe she did not emphasize enough to write it alphabetically (i.e., first *x* and then *y*) to remind the order of pair (x,y). She thought thinking alphabetically was the logical way to look at (x,y) pair and students would think that way too. But she realized it was not.

> It may be that I didn't emphasized enough that you know write alphabetically x, y. Like you know. It doesn't make sense to write y, x because that does not feel right and I guess I probably assume that if after I said something once like that it would stick but obviously what logical to me isn't logical *yet* in their mind. (I3) [Italics added]

She thought this issue was something she had to notice from now on and need to talk about it and let students red the test over again and ask what it was that they did not understand. She identified the problem in this situation as not being really engaged in the problem and thus making careless mistakes.

The second question on the test was about graphing the equation $y = -3x + 2$ by using the given table and completing y values for corresponding x values provided. She felt she knew students would have issues with the positive and negative x (i.e., $+x$ and $-x$). Referring back to a previous discussion about the difference between x and $-x$ they had in the class, she thought it affected their thinking while solving this problem: "I know that was the discussion that probably took some people's brains somewhere else that they really didn't need to go but we did it and we have get over it" (I3). She thought they really used that discussion to solve this problem even though it was the coefficient of x that was negative, not the x itself. Even though most of the students got this problem right, she identified having the coefficient of x as negative instead of having $-x$ as being a source of confusion that led students to end up making V-shaped lines (or two lines crossing) out of the equations instead of a straight line, which she thought they were supposed to know it as a fact. She indicated the students who were confused during the discussion of x and $-x$ had the same confusion in this problem and probably did not read the question or the equation carefully to really understand what it said. They were thinking that when they had a negative x value they needed to change it to a positive (as referring back to $-x$) and then multiply it with -3 again in the equation.

> Most of them did number two pretty well but we did have issues with that $-3x + 2$ because they were thinking that if x is -3 in the table then I got to change it to +3 in the equation where x was not "negative" it wasn't the opposite of x, it was you know -3 times $-3x$ in the table and they were totally confused like a handful of them. Totally confused. (I3)

The third problem asked for describing the shape of each of the equations $y = 2x + 4$ and $y = x^2 - 1$ without drawing their graphs. She thought students would transfer what they had learned from the class project and the posters they did about describing what the graphs of equations would look like when they had different powers of x such as 1, 2, 3, and 1/2. However, she did not think it was carried over too well. Even though students knew $y = 2x + 4$ was linear, they could not really write it down properly:

> Straight lines rising in a constant…nothing! ☺☺ Didn't finish that. You know here is a straight line; it would be a linear… didn't finish that. It would be a line or line that's diagonal. I saw a lot of diagonal answers, which I accepted that because I knew what they meant even though it wasn't really the way I wanted them to describe it. (I3)

On the other hand, she thought some of the students did not remember what happens when "something is squared" (I3). She commented that some of the students would draw the "picture" of what it would look like. She accepted as a correct answer those like a "V" shaped graphs because she knew what they meant.

The fourth item in the test presented an interesting case because it was one of the review problems when they covered this topic in the previous unit. Even though it was not as much disappointment as the first item where students could not "read the line on the graph"; students' performance in this problem was a disappointment to her. The item was about using the distributive property and rewriting four algebraic expressions: $5(x + 2)$, $3(y - 1)$, $2x(3 - x)$, and $-4y(2y - 7)$. Although students did well with the first two expressions, she described student responses to the third and the fourth one as a "mess" (I3). She had noticed that some of the students were having problems to remember what to do with the distributive property ever since chapter two. She thought students simply had not mastered the distributive property since it was one the new concepts in the previous chapter and thus they would need more practice. She thought - the main problem was that students did not remember that each term inside the parenthesis had to get multiplied but they "forget to multiply" with the second term inside the parenthesis. She described this situation as something she had seen ever since she started teaching algebra and she was getting out of ideas about how to make it clear. As frustrated as she was, the only thing she could think of was to get them to remember what they need to do: "I don't know maybe they are just gonna have to remember it" (I3). She had tried different ideas to make the distributive property clear to students. For example, she used the analogy of one person distributing certain amounts of items to everyone in a group. Yet, forgetting to multiply with the second term phenomena persisted and she did not know why.

> Well the thing I used the first time is to remember if I distribute papers to the class I'm giving paper to every one so you have to give this five to everyone whose in the class which is what is in the parenthesis and they always give it to the first term and forget the second term. It's like a universal problem. I mean why they won't do that is still a mystery to me. (I3)

On the other hand, she pointed out that $-4y(2y - 7)$ presented some other issues like the lack of simplification in the answers.

> The last one the (d) one seem to be an issue because of that of that –4y times -7 and I had some answers with...uhm, you know "$-8y^2 - (-)$" and I just wanted them to simplify didn't count it wrong because it didn't wrong but it was not in the simplest form we could have had. (I3)

After I pointed out that some students had forgotten to multiply y's and write $-8y$ or $8y$ or not y^2, she indicted she saw this situation a lot and it had happened with $2x$ and x in $2x(3 - x)$ as well. She further added that they had also omitted y when they multiplied $-4y$ and -7 and just wrote 28 as if there was no y or did not mean anything. She considered these as mistakes due to carelessness. She believed they knew how to do better because when she looked at their corrections, they were generally right. She thought that the problem was they would not check or "really" check their answers and mathematics because they would think they did it right in the first time. When she asked students if they had checked their work, most of the time they would say, "yes" even though they really did not. When they would say "no, I didn't" they would go back but what they would really do was to read what they just wrote or looking over to make sure they had answered all the questions. She did not think they would read their arithmetic, for example. She thought they would catch their mistakes if they did. She mentioned the difficulty of getting students to really go back and check their arithmetic and see their mistakes unless it was directly pointed out to them what their mistakes were. At the end, she did not know how to get students to overcome this problem because she was the same way when she was a student; she was never checking papers, thinking she did it right the first time unless her mistakes were pointed to her and then she would realize what mistakes she had made. She still found it a mystery why students did not go back and proofread in mathematics like they would in proofreading of words. She considered it as a general problem with students in mathematics

> I think it is particularly bad and it's a problem in math because you just don't go back and redo the math in your head. If you do you'll find it. I think it's carelessness. (I3)

On the other hand, she also found it difficult to proofread one's own work; that is why she thought there should be people paid to proof read proofread because they could see errors that other people would write. When I further pointed out one of the students who changed y in $-4y$ to x and wrote $-4x \cdot 2y$ and got $-8xy$, Ms. Sands described the student as a "sharp" student who made a good grade on the test but she did not know why the student did that particular action. She explained the student was very fond of using x's and "she thinks x is what we use" because, in just about everything else we had been using x. Ms. Sands felt this might have something to do with the student changing y to x. On the other hand, she did not think that the student used it as a times symbol. This thought triggered her for a close look at the answer and on a second thought she suggested she actually used x as a multiplication symbol as a reminder to herself.

> I think she used it as a multiplication symbol as a reminder to herself because she really wrote it out what the distributive property means in that second line that she wrote and she forget the y. (I3)

Looking back to her answer in $2x(3 - x)$ and seeing that she actually multiplied $2x$ and x and get x^2, she suggested that this case was simply an error of forgetting on her side because she knew what to do.

> I think she would have got it right because she does see right. She knew what to do. Because she did multiplied $2x$ times x and get an x^2 instead of just an x. I think she drops the y even though when she supposed to do after she has written it herself and she didn't go back to the problem and look at it again. I think that's a middle error on her part I think she knows what to do. (I3)

The fifth problem was about simplifying algebraic expressions by combining like terms in: (a) $(7x + 3) - (2x + 1) = ?$; and (b) $(5x^2 - 3x + 4) + (6x^2 + 3x + 6) = ?$

Ms. Sands pointed out that most of the students got this question, particularly part (a), right. She indicated the students who did not get the part (b) mostly did not "see" that $-3x$ and $+3x$ would cancel each out and thus they could be simplified. To illustrate common student mistakes in this problem, she particularly pointed out one of the papers as an example in which the student had conjoined algebraic expressions.

> This particular person didn't because I think what he did was he took all the terms and added them altogether. Yeah, he did. He took $7x + 3$ and made it 10 or $10x$ and then he subtracted $2x + 1$ which made $3x$ and he got $7x$. So he didn't understand the combining like terms and he also missed (b). (I3)

She was able to explain what the student did to get such a wrong answer but she could not explain why the student did it other than he did not understand combining like terms. I commented that this observation was interesting and asked if she had seen this lot. She replied she had seen it in the beginning but not by the test time. She previously introduced the idea of using algebra tiles to do algebraic manipulations in the beginning of the course because students were not sure about it. She explained how she used it to explain why $x + x$ was not equal to x^2 and how it could be used with algebra problems involving area.

> In the beginning they are not sure and that's why we used the tiles. We used the x^2 s and the big squares and the x is where the rectangles and the constants were the small squares and I would put them up on the overhead and we this does not look like this, does it? This x and this x don't make a square so they are not x^2 when you combine them. So, I know that there a lot more area type problems like that coming up when they start multiplying binomials, they are not there yet but I know that the algebra tiles were coming into play when that happens so. (I3)

She further explained that they also used it to solve simple equations but sometimes it also confused students.

> We have used the concept of the algebra tiles just to solve simple equations now because they would draw the x's as the rectangles and the constants just one unit square and that helped them with their equations up to a point but the pictures sometimes confused them. (I3)

She explained how she emphasized the difference between unlike terms and variables by using the *fruit salad* metaphor/approach:

> I have always told them x and y's aren't the same. It's like apples and stakes ☺☺ and they're just not alike and they are not even both fruit they're different. And you can't put them together unless you're cooking something. We're not cooking. If we have a $7x + 3$: we've got 7 apples, 3 stakes and that does not make salad or anything. It's just; it's apples and stakes so you don't put them together. So this is disappointing that he would go ahead and put them together after all that. I don't think I used it quite like that but you know we did talk about apples and oranges and making fruit but I need to make it even more drastic than that so that they don't think of them as something you can put together in their mind. You know I can put them together make a salad because we're not cooking ☺ (I3)

When she saw students still doing the same mistakes, she thought that she "have to do a better job of making sure that they know the difference between what is a different term you can put together and what's not" (I3).

Curious about student conceptions of variable, I asked her if she had ever observed students replacing x or a variable or unknown with some number and make an alphabetical match like 1 for a, 2 for b, etc. instead of accepting the variables as they are; letters. She commented that she had not seen it. She was very much surprised when I explained that those situations were reported in research. She further commented that she had noticed, however, that students had a hard time accepting to associate x with different values (i.e., solutions) when solving equations.

> I have noticed that when we first started this next chapters on equations they always want them to be x and they have always want, what x was 2 in the first equation, why isn't not 2 in the next equation? (I3)

To her root of the problem was the term "variable" and the meaning associated with it; students did not have it and they thought that algebra was just about x.

> I have seen that a lot and we have to talk about the root of the word "variable." It's not the same every time we use it. If I use an equation or use an a, "no, we want it to be an x." "no it's gonna be an a. a is a very good letter and we can use it just like any other variable. We can use h. It doesn't matter." The idea is this is a number you don't know yet and it doesn't matter what it is. You can pick q; you can r; it can be z. But they still wanted to be x. I think they have in their minds that x is algebra. If it's not an x, well, you have to change it. And I think that's kind of funny but-… (I3)

I further asked if she had observed, particularly in pre-algebra, students thinking algebraic letters; as unit of measures they used to do in arithmetic like $8m$ for 8 meters or $5l$ for 5 liters; or as labels for quantities of some objects like $8b$ for 8 bananas or $5a$ for 5 apples. Although she did not see the former situation, she thought it was possible that there would exist students thinking in the latter way. Even though she was not sure about it and expressed a need to think, she thought maybe students needed to think that way or call it something (e.g., $8b$ as 8 bananas) so that they would not combine things that did not belong together.

The problem six was about using guess and check table to solve the following problem: "*A jar contains blue, red, and green marbles. There are twice as many blue marbles as green and the probability of selecting a red marble is 30% (0.3). If there are 300 marbles total, how many blue and green marbles are there. What is the probability of selecting a blue marble? a green marble?"* For Ms. Sands, this was a real problem where they had to know percents. She thought this was a reasonable expectation. They really did not have to know much about percent to get the problem started. If they knew the meaning of 30% as 30 out of a 100 and that they were trying to find out 30% of 300, they would get the problem started and the rest would flow easily because "you have twice as many as one as the other and you have already figured out how many that color you have" (I3). If they did not know how to find 30%, however, then the problem would become a dead end. During the test, there were students asking her what 30% means and she expressed that she would not tell. Instead, she would ask them questions like what percent meant so that they would make a connection and think it was 30 out of a 100. If she heard that, she would say "yeah, go ahead" nothing more. Otherwise it'd be her doing the problem, not them.

This problem became a "nightmare" for students although she did not think it would be a hard one for them. She only thought it would be the one taking the most of their time and become a "stumbling block". She thought she was right on the prediction because a lot of students were saying they were stuck and she thought they needed to move on and come back to it later because it was taking their time.

> I had a lot of people asking I don't know how to find 30 percent, how do I do that? Well I'm not gonna tell you how to find 30 percent but I will ask you what is percent mean so they will start thinking on their own. (I3)

The problem eight asked students to list the sub-problems needed to find the area of a shaded region formed by removing a circle with radius 3 and a 2 by 2 square from a 12 by 8 rectangle. The problem specifically asked not to find the area. Ms. Sands considered this problem as another stumbling block for students that she did not expect. She claimed most of the students did the mathematics (i.e., found the areas and made some calculations) without any label but what they did was not listing the sub-problems, which was what the problem asked for.

> I had a lot of people who would do the math and not even label what it is they were doing which to me is not doing the sub-problem and you know I would get the answer anyway and I can understand the need to find the answer because do the problem but don't find the answer just doesn't feel right ☺ So you know I'm not gonna count off if they find the area because it might help them if their answer is reasonable but you know I got a lot of papers with just numbers and I would write a note on it "how am I supposed to follow this, you didn't tell me what you were doing" so I would count it wrong; just for not following directions: list the sub-problems. I mean if it's not an organized list and I don't know what you are doing it's not gonna help me fallow your thinking and some of them have a list. They really listed what they were doing or even if they did the math you know here is the paper where I just have 2 times 2 is 4, 3 times 3 is 9 you know that doesn't mean anything to me. I know what it is but I don't know that you know what it is because you didn't tell me what you were doing. So I counted this problem wrong if that's all they did is a bunch of arithmetic problem, not even tell me what they were doing. (I3)

As she expressed, she knew that students calculated the areas instead of listing the sub-problems because they could not accept the lack of a numerical answer when they could find it. Furthermore, their past experiences with similar type of problems led them to calculate the areas and find the numerical answer because that was what they used to do. Ms. Sands thought students knew what she wanted to know but they just did not do it and that was just carelessness. They just looked at the figure without thinking about the instructions and attempted to solve it thinking that they had seen that type of problem and they knew what to do with it, to find the area. Furthermore, none of the students had asked her about what it meant during the test although they had asked about some others items like the problem six.

CHAPTER 6: THE CASE OF MR. CASEY

Mr. Casey's Beliefs about Mathematics and Algebra

Mathematics, for Mr. Casey, was one of the core subjects people think important to learn. He thought that mathematics was also "one of the pure forms of science that we have" (I2). That was because "mathematics is concrete" and "when there is a rule, it's a rule"; "there is very few exceptions to the rules and the properties that we have in mathematics" (I2). Mr. Casey saw mathematics as a tool to model real life situations to predict and solve problems.

> [Mathematics is] a way to model things. The way that you can find out or predict what may or may not happen or- you know- to prove or disprove- you know- statements and it's a way to represent the physical world without actually having to- you know- make every possible prototype. (Mr. Casey, I2)

On the other hand, Mr. Casey described algebra as "generalized arithmetic" (I1). What he meant with *general* was that "there is an option; you have range or values that might work" (I1). This perception of generality was further explained in his argument for solving algebraic and arithmetical problem solving. He thought that in arithmetic there was an expression and simplifying gave an answer whereas in algebra the question was usually imagining or abstracting from arithmetic and looking for possible values, which meant there was more than one answer. Mr. Casey's perception of symbols in algebra was quite interesting. In a lesson, he told the students who would think that algebra was all about symbols, that "Algebra is full of all kinds of symbols and the symbols aren't- that's not the important part. The important part is the properties that you apply to those symbols" (Mr. Casey, 4/16/2003, 2:36). His view of symbolism showed his understanding of algebra as generalized arithmetic in a different way—

that is, using and generalizing/formulizing properties of arithmetic with symbols in a new context, which is algebra.

Mr. Casey thought algebra as a “brake” in the school mathematics and an “administrative discontinuity” after arithmetic because it was not until algebra that mathematics curriculum was divided as pre-algebra, algebra-1, algebra-2, geometry, etc. For him, algebra was interconnected throughout the mathematics curriculum and thus it was no different that other parts of mathematics. The basics learned in algebra (e.g., properties of equations, manipulation of formulas) served as a foundation for topics learned in higher mathematics (e.g., the notion of functions, exponentials, and logarithms) and the mastery of those basics ensure success with more complicated concepts and topics:

> Algebra is just- it’s interconnected. It’s wiped throughout the curriculum but when- it’s- you have to have the foundation for the higher math; formulas, you know we start out with simple linear equations and maybe get to quadratics and algebra one. But then when you go on to algebra two and calculus, pre-calculus and things beyond that the nature of the functions is more complicated and so what you learn in algebra sets the foundation; if you can manipulate simple formulas and distance and rate and time. And then when it get a little bit more complicated or they’re using things beside x and y; if you have a good mastery of that you’ll be more successful with the other more complicated formulas. (I1)

Mr. Casey stated that when students did not make sense of a problem he needed “that different way to look at things” (I1). In this sense, he saw integration of algebra and geometry was useful “to look at things from two perspectives in a course” (I1). For example, in learning relations often graphing was needed as a different way to look at it and he thought that graphing was geometry and thus it is connected to algebra. Thus, algebra and geometry could not be separated in that respect.

On the other hand, he considered geometry as being different than other parts of mathematics including algebra in terms of its content. Unlike algebra, geometry introduced and required the idea of a formal proof [of theorems and problems], which he thought as the biggest separation between algebra and geometry because algebra involved very little proof, just informal reasoning as far as formal mathematics considered.

> In algebra- uhm- we might demonstrate the reasons: why- you know- do it inductive type of demonstration that like we did yesterday in class where we say why can't you take the square root of negative but there is nothing formally done. And then geometry, you do- you have to write a proof whether it's a- you know- whether it's two-column proofs or paragraph proof or something like that so. ... But then as far then we get to algebra-2, it's taken what you know in algebra and again we go then just quadratics to cubics or logarithms or square roots or we're looking at functions that have different degree than just first or second degree than we do in algebra. (I1)

Mr. Casey's Beliefs About and Practices in the Teaching and Learning of Algebra

Mr. Casey believed that students would learn better when they were actively involved in learning process: "I think they learn better when they actively engaged in the discussion or the use of manipulatives or being able to go to the board" (I2). For him, *ideally* the best way to teach algebra was getting students to "work using technology or through activities and sort of try guided discovery" (I1). Mr. Casey had only one computer, the teacher workstation, in his classroom. However, he asked all of the students to have their own calculators and he kept separates for those who did not have. He thought that students saw calculators as "the dispenser of right answers or at least what enables them to get the right answers" (I1). He, however, wanted them to go beyond this view as he thought calculators as a "discovery type tool or investigative tool" (I1). In this chapter, the difference between his and the students' views and uses was clearly separated. He expressed that he tried to integrate these more and more in the curriculum, as he got more experienced. For the past several years, he had tried to change the way he taught to a state where students could get more input from him not just the rule and a set of thirty problems they could do with it. He tried to get students involved in the discussions and keep their minds active by asking questions like "How should I do this?" or "What do we need to solve this kind of problem?" and requesting help with statements like "I need your help you guys do the calculator work and check me out on this kind of a thing" (I2). Moreover, purposely letting students go down the wrong path, allowing them to make mistakes and then leading guided discovery and formalize at the end was another way for him to get students take active participation in what was going on. However, *in reality* there were barriers to implement what he thought best for students. He felt that existence of a certain set of objectives mandates him what to cover and how to cover so that

> When they go to their next course, in the state of Georgia at least, uhm, they'll have seen or been exposed to or had the opportunity to master the objectives that are supposed to allow them to continue their studies. (I1)

He expected all of his students "to do their best ... do enough to earn their credit so that they don't have to do it all again" (I1). He saw the situation as "a mixed bag". On one hand, he would like to "be able to down and do all the things that would make math part of their own"; on the other hand, he was up against "the expectations of the administration and the teacher they're gonna get next" (I1). Furthermore, crowded classes did not allow him to use *ideal* ways of teaching algebra.

> I have had as many as 30 kids in class and when you have 30 freshmen in your class it's hard to do the guided discovery activity learning that would make math more enjoyable for them and possibly allow them to better master their objectives. (I1)

Algebra for All and Challenges

Mr. Casey thought "algebra for all is unrealistic; ... algebra is for anyone who wants it" (I1). For him, in a sense, *algebra for all* was like "leading a horse to water; you can't make him drink" (I1). He identified that getting students motivated and want to learn as the hardest obstacle he had in a context consisted of inclusion students, kids where severe learning disabilities like ADHD; "for some people they won't get it" (I1). Having reflected upon his past when attending to school was a privilege for many who felt that it was their obligation to do well for this reason. He thought this was not the case for the kids in classrooms of this era.

Although Mr. Casey saw algebra as a mathematical content serving as "a stepping stone" (I1) for studying higher mathematics, his students did not necessarily see it "as a foundation for higher education" (I1). He claimed that they rather would like to see the utility of it: "They have to find some use for it in their life" (I2). He thought that concepts learned in algebra do not make sense to students if they are not connected to real world problems or situations. One of the questions students asked him most frequently was "Where am I ever gonna use this?" (I2) They wanted to know, for example, where they were ever going to use the square root of seventeen ($\sqrt{17}$). He stated that the answer to such a question was "probably never" unless they went onto higher education and for higher mathematics. But, an answer such as the following would make a case for utility of algebra in real life:

> I used the example of being able to square up a foundation if you graduate high school and never go to college if you want to build a platform or a deck for your kids to play basketball you want to be square you know you can use a lot of the mathematics you are taught in high school to be sure that you are actually building a square box and that it's leveled. (I2)

Moreover, he usually explained to his students, who always wanted to know *why they have to do what they have to do*, that algebra was "an exercise for mind" (I1) allowing to learn or be a better thinker:

> How to learn other subjects and learn how to learn about life for you get new definitions and new properties and you try and used those in context of solving problems and so my idea of algebra is training people to be better thinkers as much as it's being able to manipulate equations and do the things that they call objectives for algebra. (I1)

Mr. Casey's Classroom Culture

Mr. Casey defined his classroom culture as a series of "routine" practices. He started the lesson by getting students check their homework from the answers placed on the overhead so that they could "come up with the questions that" they "have that are gonna help" him "understand why" they "made a mistake if in fact" they "made any" [Mr. Casey, 4/17/2003, Lesson 3)]. He would walk among them to check if they were doing it or had it done. He considered this first part of the lesson as participatory from students' perspectives and it would not necessarily be used as a learning tool: "The homework is a participation sort of a thing. They have to [have] done the work if they get the right answers or not. It's not really important if we can use that as a learning tool" (I1). After students finished checking their homework, he would answer the questions from the homework. He would sometimes collect the homework after checking was done and other days he would just walk through the room and check it. After finishing the homework check, he tried to do new material and discuss it. After that he would assign the homework for next day and gave time in class to do their work until the principal comes on with the announcements that would end the day. After two or three lessons they had "a quiz or some type of assessment to find out what they need to do; if everybody is lost they stop and reinvestigate" (I1).

Interaction with Students

Mr. Casey set a close, friendly yet serious interaction with the students so that they would feel comfortable coming to class and talking to him.

> I try and set myself apart because I want to be friendly towards them but I don't necessarily want them to be my friends. But I want them to feel, I want them to be comfortable to talk to me and I think that from what I have, students have told me what their parents that communications that I get from parents is that most of the students like come into the class. Uhm, they like being part of it. Uhm, I always say hello or good morning in the outside and I try make some kind of personal contact with most of the kids at least every other day. (I1)

He used small chats in the hall before the bell rings to make personal contacts with students to tell them "they're not doing what they should be doing or make suggestions that would help their studies" (I1).

<u>Interaction Among Students</u>

Although students interacted and got along with each other pretty good on daily subjects, getting them concentrate on algebra and involve in the discussion was a challenge for Mr. Casey.

> They interact pretty well but it's not in the context of algebra. It's somewhat but mostly their interactions are with note passing and note writing. And, uhm, when we discuss our mathematics and we're trying- I'm trying them involve them in the discussion. ... So you know the interactions between them is I have to really work to get them to interact in a discussion of mathematics where they would interact with discussion of DUI yesterday had a DUI thing here so that they wanted to talk about decapitation you know among themselves where we were trying to get them to talk about algebra and this you know the wild factor isn't quite the same. (I1)

He tried to set a comfortable classroom atmosphere in which students are friendly and respects each other's ideas:

> In class I tell them at the beginning here just couple words that- here's couple words that they're just unacceptable and two of those are "shut up" and "stupid" and so we don't accept, uhm, negative vocabulary if we can avoid it. (I1)

> A lot of times they think that being the loudest and the most interruptive is the best way to be heard and trying to win their arguments through intimidation and then we back up discuss should work and how people really work together to come up with better ideas and not just to discount what someone says because you don't like them or you know that you just broke up with the hour of that goes. (I1)

Mr. Casey's Lesson Preparation

Mr. Casey described a basic lesson as "just the next section in the textbook" and the learning goals as "part of the package" they bought. He thought that "it's probably selling the kids short saying that they can't discover what you want them discover in the time you have for them" a lot of what students got in terms of content was "delivered to them" and they were "asked to practice" what was shown them. He considered Georgia Quality Core Curriculum

(QCC) Standards as not only dictating to cover certain objectives but also how they covered it. He agreed that his instruction was basically QCC and textbook driven since they had so many sections they needed to try and stay aligned with them. He expressed that he would use supplementary material or activities if he thought he was "moving along and at a decent enough pace and all of the classes stay pretty much aligned" (I1). However, even though he thought he should, he did not use manipulatives like algebra tiles; making a personal decision considering the management factor with the current group of students in the class

He got prepared for his daily lessons during his preparation time in the morning. He stated that he had three folders for every class he had taught: "a folder with the notes, a folder with the tests and quizzes, and a folder with resources." Although he had written plans [probably from over the years], they usually sat in his briefcase; he mostly used his notes collected over the years as a main resource in preparation for lessons. If he thought he needed to do an extra or supplementary activity, which was not in the textbook, he got those from his resource folder. To prepare for a lesson, he would basically look at the section from the book he was going to cover that day, and then look back at his notes from prior years to remember what kind of difficulties and other issues were associated with the lesson if there were any. Once he identified where the students had mistakes in the past, he would try to lead students away from the same kind of mistakes with a statement warning that the situation might be tricky made to get students attention about possible mistakes. However, "they all get there and they get an answer" reflecting the same mistakes since they tended to go straight to the answer before thinking and planning for solution. To illustrate his points, he suggested the following situation: In solving $|x - 2| \leq -1$, students tended to attempt solving the problem by breaking it into two cases to find two x values even though they all knew that absolute value could never be negative. He sated he knew by experience that students were going to do this but instead of telling them the problem had no solution he let them proceed and do it so he could ask them to check their answers by plugging the *x* values they found and let them realize it did not have a solution because absolute value would never be negative.

Mr. Casey was fond of asking questions, which he thought as the best way to teach when he could not start a lesson by an application. As he stated, he often asked questions of the following types: “what do we need to solve this problem? How far can we go with what we know?” Once they discussed the application or problem, they formalized the topic by definitions and properties inherent in the section. This was usually followed by individual study of practice or homework problems. Once in two or three lessons, instead of focusing on one particular lesson, he would go back to the *skill practice workbook* of the textbook and do a few more problems from each section covered in order to put pieces together. He also used this to assess or see what gaps needed to be filled in. Usually a chapter ended after eight to ten sections.

Mr. Casey expressed a flexible view for the role of lesson plans in his instruction. Although he usually tried to follow the lesson plan and the direction he had in mind, hoping that “all these good stuff that’s going to happen”, in some circumstances (e.g., he gave a quiz and only two students passed) he had to back to re-instruct with different resources, problems, or more practice and reassess students learning. During instruction he often went off the subject to respond student questions by doing a different activity or group discussions.

The Textbook

Mr. Casey was using a textbook (Smith, Charles, Dossey, & Bittinger, 2001) picked and ordered by the mathematics department of the school in order to have a specific book for all sections of algebra one. He expressed that he did not like the textbook and would prefer to use a different one because the department had picked one that had the most "drill and practice" (I1) in it. He did not think the current book gave students much opportunity to try and explain what they were doing. If it was his decision, he would have picked one that had more applications, discussions, better examples, and open ended problems asking for reasoning and explanations like the one they were using for advanced algebra. He thought students would not appreciate a book such as the one in his mind because they "want just have a problem and get one answer and have it be right or wrong" (I1). He thought the book they were using did not have open-ended problems so it would not challenge student thinking. He, however, would like to see more of open-ended type of problems like whether "zero is positive or negative" that they discussed in the class, which was significant in terms of Mr. Casey's ability to recognize student thinking. The discussion took place when one of the students objected to Mr. Casey's conclusion that "it doesn't matter whether you take a negative number or whether you take a positive number and square it, you always gonna get positive" [Mr. Casey, 04/16/2003, Lesson 2].

> Mr. C: What do you do when you multiply negative times negative?
> Sn: (A positive)
> Mr. C: A positive. And what do you get when you multiply positive times positive?
> S: (positive)
> Mr. C: Positive. So whenever you square a number that's a kind of special thing about squaring it doesn't matter what it is, what kind of number positive or negative that you square you always gonna get a positive number, ok? Zero is a special exception.
> Mr. C: Have we ever figured out that's positive or negative again?
> Ss: (   )
> Mr. C: Does anybody know if zero is positive or negative?
> Sn: // It's neither, it's neutral.
> Sn: //No it's neutral.
> Ss: (   )
> Mr. C: Well, I don't know either that's something- I'll tell you I don't- I don't know.
> Sn: I think it's a positive number because I (doesn't say positive or negative)
> Mr. C: Well, there you have it. See! Now- that's-
> Ss: What if it gets the negative or positive or negative zero?
> Sn: (   ) Then how can it be just a positive one.
> Sn: No, look you're doing negative one, you never say negative zero. You never get positive one.

Ss: (   ) [everybody was commenting on]
Mr. C: All right now, hold off for a second, everybody be quiet. Just a minute! His idea is that whenever a number is positive there is no sign attached to it, ok? So if you read this number right here [he pointed out $14^2 = 196$ written on the board] you don't say positive fourteen, you say fourteen. Here [he pointed out $6^2 = 36$ written on the board] you don't say negative six, you say six. So, Jerry's idea is that because you don't say positive zero or negative zero you just say zero that when he is suggesting is that then zero would be considered positive because it's the kind of number that you don't say anything in front of. Now, whether that's right or whether it's wrong; I don't know, but it's certainly makes sense the way he try to explain it.

It was obvious from Mr. Casey's first question that the class had discussed the same question previously and it was left unanswered. His comment that he did not know the answer opened up a discussion and students started to reason on why they would think in a particular way. Even though the one student was not very explicit on why he thought that zero was positive, Mr. Casey was able to identify his thinking and transform it to a more explicit form for all students. Mr. Casey could not make a conclusion about whether zero was negative, positive or neither even though he found the student's explanation, about why zero would be positive, meaningful as it was stated. Thus, it seemed like the discussion never ended in some of the students' minds possibly because of leaving the question unanswered after the discussion. When they were working on the homework problems several students asked Mr. Casey if zero (0) was real:

S: Is zero a real number?
Mr. C: Is zero a real number? Yes. Any number that's on the number line you've been using since whenever they started talking about number lines is a real number. So like pi [π], 29, or .045: Those are all real numbers. [Mr. Casey, 04/16/2003; 2:41pm]

In his response, he did not ask the student why he would think zero might not be a real number, instead he directly confirmed and explained why by making connection to number line: everything on it was real and so was zero (0) since it was on it.

Mr. Casey described how the textbook categorized homework problems under sections A, B, and C. According to the pacing guide made by the department, teachers did not generally use the sections B and C. Some even never used those sections. He, however, liked to use them because they made student to "think critically" even though there was "so little response" from the students. He also thought that "a high quality algebra book" would promote those types of problems instead of giving priority to questions that were like in section A. Although he thought this way, however, in classroom practice he chose to ask questions from section A thinking that test questions were from usually that section. For example, the first day of the unit (4/15/2003, 2:45pm) he responded to a student asking about the graph of an absolute value equation as they should not worry about this section because test items came from the section A of the homework and doing it should be enough for the test. Furthermore, his practice in assigning the homework during the chapter represented the department's tendency of assigning problems from section A.

Although Mr. Casey stated that once a textbook was determined by the department he could use it as a guide, he felt that he had to follow it for three reasons: difficulty of trying something other than drill and practice; meeting the state and national objectives; and possible parental and administrative pressure. Even though "it's way too much drill and practice" (I3), he thought that it would make sense to do *mass practice* considering the levels of students in the class: "if we try to do things more on a intellectual level, it would be more difficult" (I3). His use of the items in section A of exercises suggested in the textbook as homework problems confirms his struggle and perception. On the other hand, Mr. Casey suggested he would use materials from outside the textbook to support and supplement the textbook material.

> When we do differences of squares we're trying to factor different subsquares so the activity is this paper folding thing that I do so I ran that off and that's not in the book and we spend time trying to get this again it's kind of one those geometry ideas where you start off with a certain area and you cut out a little square and turn it around and you can see then how it factors and so that's supplementary to what our textbooks says. (I1)

On the other hand, Mr. Casey thought that following the book was a way to make sure students were meeting *the objectives* stated in Georgia's Quality Core Curriculum (QCC) because the book was written with Georgia in mind. He could follow which objectives he was covering by looking at the corresponding QCC standards listed for each lesson. Furthermore, the book was supposed to be aligned with the NCTM's standards as well. Although he found this claim was legitimate in some places, he thought otherwise in general because it was "not very constructivist;" possibly as a consequence of considering that NCTM's standards presented a constructivist perspective for him. A lot of times the book just made a point in a reference next to a challenge problem to state that it reflected the ideas emphasized in NCTM documents. He thought that was just a strategy to sell the product rather than a true reflection of NCTM's philosophy.

Also, Mr. Casey was hesitant to not follow the textbook because He thought he would get calls from parents and administration asking him how come he was doing other things or was far ahead in the book or section.

> I don't have to follow the textbook but if I don't then I get calls from parents and administration says; "How come you're so far ahead?" or "You're doing this other stuff!" and- uhm, matter of fact I have in another class they were complaining that we were so far ahead; I had a parent called and say that I was driving their kids to hard and then we are two chapters of the other algebra classes. And of course they don't- that we skipped one of the chapters and go back to it because I like the sequencing better. Ok, so instead of calling me and asking what's going on and they just call up and complain and then it all comes down. (I1)

Assessment of Students' Learning in Algebra

Mr. Casey's basic assessment methods for Algebra One consisted of quizzes and tests. He did not assign projects. Additionally, students got graded three times a semester from the notebook they kept to which he gave a particular emphasis. He thought students needed to do their work neatly and get the notebook organized so that they had reference to help them through “the difficulty times”. He also thought that “the log book for the class” in which everyday a different student would take notes for the whole class so that somebody absent could get the notes and the assignment from the class was another form of assessment teaching them how to keep/take notes in mathematics. He had a form to guide them on what to put into the log such as writing definitions, properties, and examples. Although he got students to explicitly read from the textbook (e.g., reading a polynomial) and made “a daily grade informal assessment of their reading” in his more advanced classes, they did not read a lot in Algebra 1 because there was not much in the textbook for the students to read.

Mr. Casey prepared tests as “representative of most of the work we’ve done in class” (I3) from homework problems that “most people had difficulty with” but “truly assess the understanding of the concept” (I1). He suggested the absolute value problem $|x\text{-}2| \leq \text{-}1$ as an example because “it discusses the nature of absolute value” (1) and students still had difficulties even after spending almost two days discussing it in class and doing it as homework. For him a good test question would be the one he had seen it “multiple times” because it assesses “mastery of particular objective” (I1). In searching for a test item, he usually looked at the logbook to see the questions students asked him to work for them in class, and then did as homework and were asked to solve in a quiz but they came back and asked again and again. He thought that “these problems are supposed to mirror what they’re doing” (I3). Although he did not ask the exact same questions students had seen previously, he made slight changes with keeping the steps for solving identical. He also tried to ask questions as simple as possible for the sake of testing the ideas and concepts even though there were more complicated ones in the textbook. He usually included a bonus item bringing several concepts together. On the other hand, he was flexible in asking students to write their answers in a specific format. He expressed that although “some teacher says that it has to be a mixed number or a proper fraction or a decimal or what," for him it did not matter. They would write their answers anyway they wanted as long as it was correct.

Mr. Casey's argument for choosing the items for the end of unit assessment test was consistent with his previous statements about basis of choosing a test question:

> They've seen all those problems: They've- they- I either use them as a demonstration before the homework or they got it in the homework and they asked me about them or they were on a quiz and someone asked me. But I try and pick questions that they've seen repeatedly. (I3)

One of the items in the unit test was to simplify $2\sqrt{3} - 3\sqrt{(1/3)}$, which was the modified version of $3\sqrt{2} - \sqrt{(1/2)}$ they solved in the class prior to test. He decided to ask this type of a question because he discussed it on different days and occasions, asked in the quiz and he still had students asking him how to do it:

> We worked it Thursday and we worked it like on Monday when we went through that section and they quizzed over it and then this question came up to one that 3 radical 2 and right before the test. I worked it again and I said: now watch and be mindful; be careful; this problem one of these kinds is on your test. And uhm we did it right there on the side board and they said ok. They asked me you know why then is- when this is divided by 3, how come you make it a third and add the things and you know I explained to them all over again that it's really- it's coefficiency, pull it out and treat the radical like a variable and it becomes a problem and even I put it in parentheses that using distributive property that you know this times radical 2 and this times radical 2 well you put those coefficients together and there was probably ten of them that came up and ask me: how do you this again? ☺ (I3)

On the other hand, he asked $\sqrt{(100x^3)}/\sqrt{(25x^5)}$ in order to see if students were solving it in the way they had been doing in the class:

> We've discussed over and over again that you make it one fraction and you simplify the fraction first instead of trying to take the square roots or you could try to square roots but it's easier you end up with 4 over *x* square and then you got 2 over *x*. (I3)

Individual difficulties that appeared in the test scores and still persisted after the test had no affect on Mr. Casey's instruction for the whole class. He thought he could offer help for remediation before and after school for those students. He would also go back and adjust the student's score slightly provided that she worked on some similar extra problems that he provided after the test. He would not insist on doing it, however; it was the student's responsibility to decide if he or she needed a better grade like A and B and needed some help, redo and work harder; or satisfied with a C. As a class, on the other hand, they could not stop; they had to move on, keep going unless it was a group of students having difficulties with a particular content. In that case, he would then stop and reinstruct by first diagnosing the problem and its causes. He would discuss the situation with the students in order to identify the mistakes, what was causing the error in the concept development or arithmetic, and what was stopping them from getting the right answer. He would also discuss monitoring techniques they could do on their own. For example, he thought that really quick use of FOIL to multiply back would lessen the mistakes in factorization by making sure the expression was factored properly.

Mr. Casey's Beliefs About High School Students' Thinking and Difficulties in Algebra 1

Sources of Student Difficulties in Algebra 1

Mr. Casey considered all of his students in the class as smart and capable of mastering the objectives of the course and earning good grades. However, he identified the main sources of high failure rate or student difficulties in algebra as; lack of desire, goals, and intrinsic and extrinsic motivation.

> I think the main problem with the people that aren't mastering the objectives is desire. Because they're all smart enough. They just don't try hard enough or don't care I guess it's the best way to put it. (I3)
>
> I don't see any, well not any but I say 4 out of the 20 kids in the kids are intrinsically motivated, you know, they want to be successful, they worked hard at it, uhm, so I- uhm, there is a lack of motivation and, uhm, they are all capable. ... I don't think it has to do with their mental capability or their intelligence because most of them are you know ok with in that aspect so it has to do with other factors. (I1)

He believed learning was a matter of choice and desire: if students wanted to succeed or had learning goals even if it was for getting a good grade, then they could learn, understand the concepts and had fewer difficulties. His example of a student represented this point:

> Kevin, who has a test average of 50, realizes that if he doesn't pass last three tests that he as no chance at all to pass in the class and so he buckles down for the ten days that we're in this chapter and scores 88 on his test. Other test he has had as lowest 8 out of a 100. So it's a matter of, the way I see it, it's a matter of desire. (I3)

Three main sources for the lack of motivation emerged from Mr. Casey's thinking. First of all, he identified that most of the students lacked motivation influenced by their parent's beliefs/views about algebra.

> When I call their parents, their parents says: "Well, I wasn't never any good at it either!" which enables them to just say: "Well, if you weren't good at it I don't have to be good at it!" and they don't. (I1)

As a second source, he thought that students lacked motivation because they did not see a use for learning algebra and resisted to learn it.

> Probably they don't see the usefulness of it, the utilitarian aspect of algebra when am I ever gonna use this, why do you make me learn this stuff? When you are all up against that then that help, increases your failure rate because they see no use for it. (I1)

He often found himself confronted by the questions about why they had to learn algebra or where they were going to use it. Two such events occurred on 04/16/2003 when Mr. Casey finished finding the domain of $\sqrt{(t^2 - 5)}$ and on 04/28/2003 when Mr. Casey's lesson about rationalizing denominators was interrupted when one of the students asked about why they had to learn that. The dialogue between Mr. Casey and the student in the first episode was as following:

> [*04/16/2003, after Mr. Casey finished finding the domain of* $\sqrt{(t^2 - 5)}$]
> S: ([he wanted to learn when/where they will use this information])
> Mr. C: Let's say that you're out and you're gonna pour a slab in your backyard so that you can skateboard back there and be better at it. //And we -
> S: //([he talked about hiring somebody to do it for himself])
> Mr. C: Yeah, you get a- hire the construction worker but how do you know if he is using the right amount of concrete? How can you be sure that-

S: (   )
Mr. C: You just want to ask that question without getting an answer, right?
S: (   )
Mr. C: ... If you wanted to make sure that the guy poured a square slab you can use this stuff right here because what you wanna do is you go to one of the corners and you measure this, how far this would be t and you come up here and measure this t and this number right here should be $t\sqrt{2}$ [he drew a square with t as one of the sides and the diagonal measure up to $t\sqrt{2}$] and if that's true you measure the sides because you can have something that has four equal sides that looks like that. Or if you want to skate on something that looks like this [he drew a diamond shape or kite]
S: (   )
Mr. C: So the question is you know- you don't even know where you're gonna be in five years, ok. And most people now that entering the professions they start off with this grand idea that they're gonna work somewhere and have this job and everything is gonna be rosy and they find out that in five years they wanna change careers. Or in another five or six years they do it again so don't short count yourselves. I mean, it's mathematics that allows you to solve all those kinds of problems and if you can learn it then it doesn't matter what you decided to do you're gonna be that much better off for it. So if you're searching for the meaning and the reason you should be doing it, it's the same reason that your granddad and his granddad, their granddads and all the way back to Euclid did it. Train your mind. It's mental exercise and it's good for you because that way less (   ) girl on your brain and makes you better think it.

In the second incident, Mr. Casey was talking about rationalizing denominators when a student asked about the purpose in adding x and y in algebra:

[04/28/2003]
S: I got a question?
Mr. C: Does that anything to do with the rationalizing denominators?
S: It's got something to do with math.
Mr. C: Ok.
S: All right. (   ) how come like you know so many different ways to add like x and y. Why would you wanna add x and y? Why would you add two unknowns? Like, like, when you're gonna be in the supermarket and like you- you find something that's called exposed.
Mr. C: No, it's more like this: You're gonna run a production of a play.
S: I don't wanna do that.
Ss: (   )
Mr. C: Oh, ok. You're gonna sell cars and trucks.
S: Sure (   )
Mr. C: In order to maintain your business you know you gotta sell- you gotta sell altogether- you gotta sell at least thirty thousands dollars worth of cars and trucks.
S: Ok.
Mr. C: So, if you sold ten cars, how many trucks would you have to sell? And you go one step beyond that to linear programming you find out what maximizes your profit. And

there is two variables; you got cars and trucks. You gotta sell them both and both have some profit associated with them. So, you wanna able to determine you know, how many cars and trucks you have to sell in a month not only to meet your overhead but you have projected profits. You know, you can't just stay in business and break even every month. Because that's not what businesses is about. Businesses is profit. You know, so you want to make money. When you do these kinds of things and the variables when you just used two variables and in most businesses there is more than just two, ok? So they can model their business mathematically and find out how many of each thing to they gotta sell. And not only that- that's just where- if a sales example. Production is even more complex because you have more factors like labor and then parts that you have to put together and machines that you have to keep running. And all that stuff has to work together in order for everything to work and you can use math to model the situation and find out what the minimums are. So, my brother for example is the district manager of Home Depot in southern California and in springtime they sell a gazillion pants. So they gotta make sure that they have enough people staffed in that department to make sure that they're moving the plants in and out because they go bad so fast if you only get them on the ground and water them. So they sit down at their meetings in January and discuss you know like the- how many plants they're gonna get. How many people do they need to sell every thousand plants that they order? And then now they got and they gotta figure in their water bills. Because while they have the stuff in stock, they have to water and take care of it, which requires labor. Ok, and labor is one of the most expensive things you have because you don't just pay people their wage, you gotta pay them benefits and health and all of that stuff. So, instead of actually waiting to see what happens and hope you make money, they model all that stuff mathematically and look at their models and see if what they have in mind makes sense for them. And if doesn't, they're not doing it because they're not in the business to loose money.
S: Ok, but then like, why didn't they teach all of these stuff when we are younger. We are now if they taught us this thing when we were younger wouldn't it be just as smart when we are younger than we are now?
Mr. C: I don't- not sure your intelligence changes. But your knowledge base does. The longer you live the more things you've experienced and easier it is for you to relate //things together.
S: // But like why don't you- like I was- like (   ) they taught us adding and subtracting and multiplication. Why didn't they just-
Mr. C: Shh, what a rot (   ) to use.
S: Why didn't start teaching us this kind of stuff? Why don't we have to keep going (   ) all the completion of-
Mr. C: Because they have volumes of volumes of volumes of educational research, and there is a theory that a lot of people subscribed to that says that at certain stages of maturity in human beings, you're not able to abstract thought. I have to be concrete. You don't understand the idea that there might be fifteen of whatever you're thinking of on the other side of the wall. It's hard to imagine what that's like. Or have (   ) be there or actually say that some variables gonna represent a quantity you don't know. And they say well you know and a certain stage in your development you can't see past that idea that you can- you can actually talk about stuff that's not physically present. And so a lot of the way that they write curriculum and they design their elementary school, I don't know

> how it works in Australia [student was originally from Australia] or Japan or Germany but, uhm they think that you're not ready for it. And why do they think that, it's because of all the research that they've done over the years.
> S: But, like with dogs; if you teach it like to do something when it's young it usually remembers it. But if you teach it to do when it gets older it's- it gets stupider. I mean forgets it like- one of my (   ) told uhm (   ) everybody-
> Mr. C: You're not comparing yourself to a dog, are you?
> S: No, but I'm just saying, right.
> Mr. C: Because the size of your brain is probably four times in size of a dog brain.
> ...
> Mr. C: It's pretty obvious you didn't like my answer but that's the answer.
> ...
> 2:58
> Mr. C: The next time Jerry asks me why you gotta learn this stuff, the answer will gonna be because if it was good enough for the kids that were taking algebra thirty-five thousand- thirty-five hundred years ago, it's good enough for you.

He identified students having problems with the meaning of algebra and thus resisting learning as a psychological problem beyond his capacity and knowledge to deal with. He thought somebody other than himself needed to deal with those problems. He expressed that what he could do to solve their difficulties was a tough question since what he could do to help was limited to "try and show them alternative ways of looking at things, looking a different problems, different perspectives" (I1). However, he could not force them to do anything such as making them to do their homework, wanting to pass, and even making them to "sit down for 30 minutes straight" (I1).

A third reason for high failure rate in algebra "has to do with sociology." He though that at this age students had social issues going on with their life that they valued more than algebra and took their focus away from it.

> There is all the other things that happen like I have had girls coming here and their boyfriends whose only being the boyfriends for two weeks just broke up with them. Ok, so they laid their head down and they cry for 45 minutes. And so that the, all the other stuff that goes on in their life is more important than algebra or more important their school. So it's hard to get through all of those other things for them to concentrate focus on algebra. (I1)

For Mr. Casey, weak mathematical background was another source of student difficulties in algebra. He particularly thought lack of proficiency with arithmetical or number operations and calculator dependence for doing these were issues limiting creative and independent thinking.

> They have a weak math background. There are still kids in my class that can't add fractions without a calculator, unit fractions; a half and a third without a calculator they can't add it and most of them will pick up a calculator, multiply nine times seven which I think it limits their ability does it free. Their minds aren't free enough. They don't try and think creatively as much as they should. And I don't know why! (I1)

In the unit observed, "Radical Expression and Inequalities", he identified the following questions among the most frequently asked by students: "What is the perfect square factor in a number? How do you prime factor it?" (I2) He thinks a lot of times some students were not even sure where to start and the ones who knew usually started with looking at possibilities of division by two, three, and so forth. On the other hand, he thought that students had most difficulty with rational expressions in algebra. They could simplify without much problem because they could factor pretty well; but it was the adding them that caused most of the problem even though they revisited adding fractions of all kinds before they started this chapter.

Although Mr. Casey did not think that differences between algebra and arithmetic constituted as a main source of problems in algebra, lack of arithmetical proficiency and persisting difficulties were carried over to algebra as a blocking factor in successful learning of higher mathematics. Even though he did not know why exactly it was so, he used the following analogy about learning to swim in a pool and then swim in an ocean to illustrate his perception:

> Some people learn how to swim in a swimming pool and then when you go to the ocean if you have a well developed sense of how you can swim then you can you know that the currents and the waves don't, it's not that big of a problem. But if you haven't learn how to swim well you know you go to pool little bit you know you slash around and do your thing but then you go try and swim in an ocean it's very intimidating, it's a very challenging-. (I2)

Learning arithmetic first and trying algebra after it was just like learning to swim in a pool and then going to swim in the ocean. If students did not learn their arithmetic well enough or could not do it without help [of a calculator, for example] then they would not have the self-confidence that they could do some higher order mathematics like algebra.

> When you have to pick up a calculator to multiply 7 times 3, it interrupts some of the thought processes and I don't think necessarily that it has to do that but I think that the kids perceive it that way that you know how if you can't multiply without a calculator how can you think about some of these other things. The kids, some of them with the lowest averages can think better. They have you know better thought processing skills or more imagines with mathematics but they can't formalize or they can't write it down for you. It's something that happens just up here that they're not really aware of what's happening. (I2)

He also thought that not being strong in arithmetic or number sense would slow students somewhat because "they've been told or they perceive that difficulty in that area is limiting their intelligence so they don't have the idea that they could think ... or develop strategies ... to work with more complicated things" (I2).

Mr. Casey felt the quality and the form of arithmetic/mathematics students experience in middle school constituted to the problem. He thought that "students have been put the sleep in middle school" and "work-sheeted to death" (I1) by a mathematics that was not challenging and not adding to their current level of mathematical knowledge.

> I think one of the reasons for so many people fail and so hard teach algebra is that they haven't done anything new for a couple of years, two or three years if they are on this track that we're talking about where you're taking algebra in the 9th grade. (I1)

> I think in my own personal opinion is that they've been put to sleep in middle school. I don't see the middle school mathematics curriculum as challenging as or even moving forward from what they've learned when people get out of the 5th grade. When they're done, they learned their multiplication you know they have done a lot of activities and discovering and their numerology, their numbers sense should be in place [but it is not] you know they learn how to do fractions, add fractions and all that stuff and then in 6th and 7th and sometimes in 8th grade they have just work sheeted to death. (I1)

He thought that middle school curriculum, except for the gifted programs, did not offer new knowledge to students. It was just a continuum of what they had been learning through elementary school. As a consequence, this created mentally inactive students hard to get back active in 9th grade. Thus, he thought that algebra should be taught in 8th grade after having pre-algebra in 7th grade. That's why he thought the students in advanced algebra-1 were more active because they either had taken pre-algebra or algebra in 8th grade.

Mr. Casey considered students' biggest difficulty in algebra as "not being able to use their definitions" (I3). Students were "not really strong in begin able to learn a definition and apply it: ... if you have this particular thing then this is what you do and there are more steps that the problem involves" (I3). They could not make connections by starting first with the definition and then trying to build on it. He made this point to students as well in the first day of the unit when he was presenting alternative definitions for square root. He suggested that learning of a definition would mean to memorize it, not necessarily comprehend. Mathematically speaking, he suggested that definitions were mathematical conventions to operationalize the idea and one would just need to accept it as it was. Once it was learned, it would be applied to situations where needed:

> Mr. C: No, no, no. *Definitions aren't something that you have to comprehend. It's something that you learn. It's something that gets memorized. It's a definition.* We are saying: we're gonna be using this thing, this thing is a square root and here's the definition. Once you learned the definition you can use it in other places. That's where the learning comes in- that's where the comprehension has to be. It's to be able to understand which of the definitions and properties you have to use in a situation you need. [italics added] [Mr. Casey, 04/15/2003, Lesson 1]

Furthermore, he pointed out that students would give up easily even the problem was only three or four steps. He found this situation "disappointing and shocking" considering that "we go through all that those steps and it just it means practically nothing to them" (I3). As a solution he thought, "if there was a way to guided them trough the discovery, maybe they'll grasp more"; however, he had "never seen an activity that can help you understand a definition" (I3).

Knowing and Recognizing Student Difficulties

Mr. Casey did not have an easy answer for how he would know a particular student's difficulties. He mostly came to know thorough direct communication or one-to-one interaction. He could basically see which students were having difficulties by "walking around and looking at their paper or when they raise their hand and say that I don't get this; what's going on" (I1).

Purposefully letting students make mistakes on problems (e.g., let them come up with 2 as square root of -4) and then helping them see why it should have been different or corrected was another way of Mr. Casey's strategy to identify student difficulties. However, he often found himself going back and explaining the same things to individual students working on the very same problem(s) that they just discussed and explained as a whole group. For example, he had to explain "at least fifteen times" why they needed to have the absolute value when taking the square root [of a variable] once they started to do their assignment even though he just explained it. He expressed that they had this in their notes and could have used them; but "they take their notes and put them away because they think they don't need any help doing their work" (I1).

Mistake vs. Misconception

Mr. Casey differentiated between a mistake and a misconception. He explained the difference through an example where he described a mistake as an arithmetical error:

> If it's arithmetic; when people take 3 – (-2) and get one, that's just a mistake, that's arithmetic. Uhm, when they take the square root of 16 and they write radical 4, that's a misconception. (I1)

In case of a plain mistake, arithmetic error such as above, he thought that it would be relatively easy to deal with it: "you just tell ☺ look you gotta be more careful with your arithmetic but if you don't understand what we're doing let's discuss it you know what do you think should happen or what is going on here?" (I1) However, if it was a misconception such as above, they "have to go back and discuss what it means" (I1) through the definitions (e.g., what a square root is). He thought that dealing with misconceptions was "a matter of time and practice and corrections and more time and practice and corrections" (I2). He also expressed his desire to find some manipulative to serve as a visual or concrete aid to help students see more than the abstract idea so they could understand how a definition works. He considered a manipulative as "anything that's colorful and bright and they can put their hands on" (I2) like algebra tiles used when factoring polynomials.

Because of his experience in teaching for a long time, Mr. Casey felt pretty confident in identifying misconceptions or where students went wrong provided they showed their work to him. It was not his policy, however, to force students to show him their work. He thought that if he did "there will be three or four more kids failing because they just they don't want to do that; I'm not doing, you can't make me do" (I1). Instead, he encouraged students to show their work so he could better able to help them, make sure they were mastering the objectives, and give some partial credits based on the evidence that they got the concept right but only made arithmetical mistakes due to "pushing wrong button on the calculator" or "going fast". Because he thought that "they've mastered the concept but they haven't mastered the monitoring skills to eliminate some silly careless error." He personally liked it when students show their work because it gave him "a better idea you know of what's going wrong or if there is anything going wrong". It also enabled him to recognize if somebody had a unique way of solving a problem that he did not think of it so he could put it in his notes to be used as a source to get his points across in another class or another time (I1). However, he left the choice of showing the work to student, as it was evident in a short classroom episode:

> S: Do we have to show our work like that one?
> Mr. C: Well, I- yeah it's pretty simple.
> S: So, no?
> Mr. C: Well, if you need to show your work. If you don't, don't. (04/16/2003, 2:43)

Mr. Casey's Knowledge of and Approaches to Students' Thinking and Difficulties in Radical Expressions

Mr. Casey stated that he did not know exactly where the concept of radical expressions would fit into algebra and mathematics curriculum in general. He thought of it as "a basic concept" and "another technique for solving equations or finding relations" (I2). Students needed to study radicals as a part of their foundation for higher mathematics as well as for their utilitarian uses exercising the mind to apply the topic in new ways.

> It goes back to the same idea that you just trying to learn new things and apply them in different ways as a mental exercise to train your mind to think logically and so that the simple study or radicals that we do in this class is it has its foundation or it is foundation for mathematics but at the same time the utilitarian uses that you're doing mental exercise you know learning new things and apply them in new ways. (I2)

He thought that the study of radicals was particularly helpful in mathematics and its applications where irrational numbers involved:

> When you try to find the zeros of polynomials or you talk about like space travel or you know linear programming and- uhm- quadratics- you know- projectiles or microwaves you know making sure that they bounced into the parabola or how they collect all of that stuff you know the solutions are irrational a lot of times. So it allows you to help to find zeros of polynomials and give you the idea of just working with radicals. (I2)

After his assessment of the first three sections in the unit, Mr. Casey thought the class was going quite well based on the results showing one of the best averages so far in the course; nearly everyone was passing. He thought the path the students went through in finding squares, how to add them together, and evaluating and simplifying radicals with making mistakes and then discover what was wrong with the situation and try to put the properties together enabled them to do well so far in the unit (I2).

Meaning of Square Root

In the first lesson of the unit, Mr. Casey began the introduction of the square root concept with asking students to memorize squares of numbers up to 20 and cubics up to 10. He commented to the students that they were going to build upon this information. He then started with asking squares of some numbers and he wanted to get the answers real quick expecting students immediately recall their squares:

> [Mr. Casey, 04/15/2003, Lesson 1]
> Mr. C: Let's see. Neo, what's a 13 squared?
> N: ...
> Mr. C: Shh. [be quiet--to students]
> N: ... [several students raises their hands to answer the question]
> Mr. C: ... You got stuck on that one, ok. But now this is what I mean, I don't mean you looking your head and you go: [he tuned his head towards the ceiling and pointed his finger to the air and started doing an imaginary operation] one, three, one, three, three times three is nine and ... You know That's not what I meant by memorizing. I meant [snapped his fingers], recall, instant recall.

He continually got the answer from another student and continued to ask couple more. He then asked a different type of question:

> Mr. C: Nina, is 27 a square?
> N: (I don't think so)
> Mr. C: I don't think so either. What's the biggest square smaller than 27?
> N: (26)
> Mr. C: That's right. The biggest square smaller than 27.

At that point, he made a transition to new unit and introduced the idea of square roots. He gave two definitions of square root. The first one, "c is the square root a, if $c^2 = a$" came from the textbook. The second one was "$\sqrt{a}$ is one of two equal factors, where product is a" or shortly "one of the two equal factors" (I2). He liked the latter better in terms of easiness and direct translation for computation of square root of a number:

> Mr. C: The whole idea now is that we're gonna learn about square roots, ok? And there are two definitions that I'm gonna give you. One I'm not gonna ask you to write down because I don't like it. And the other one that I'm gonna ask you to write it down because I think it's a better definition. It says the same thing but it's not quite so complicated. Ok, in your textbook they said that *c*- I better read it because I don't- I didn't memorize this one. It says number *c* is the square root of *a*, if c^2 is equals to *a*. Can you guys work with that real easy. [italics added]
> Ss: (   )
> Mr. C: Wait a minute; I'll write it down for you. It says *c* is the square root of *a*, if $c^2 = a$. [italics added]
> …
> Mr. C: This is the definition of the square root that I like and it works well with the way that you take square roots later in your life so this is what you want to write down: The square root of *a*, [he wrote $\sqrt{a}$ on the board] this is what it looks like, is one of two equal factors whose product is *a*. You don't like that one either, do you? I would write that one, yeah, that's the definition we want to use. [italics added]
> Ss: (   )
> Mr. C: ☺ If you can understand the first one and you like it better, use it. It's ok. I mean I know them both but-... The first one is pretty good for asking questions but actually trying to compute square roots-...

According to the results of the quiz that he gave [between the first and the second interview], he thought that for the most part students understood what a square root was because they did score pretty well. However, after the end of chapter test, he stated that a number of students still did not understand what a square root really was and he considered it as the biggest problem with them: "They take the square root of 81 and they'll get 9 and then they'll go on and take the square root of 9 and end up with an answer of 3. Or they'll do square root of 169 and put radical 13" (I3). He was disappointed in seeing the same mistake persisting even "after two weeks of continuous talking about this and drill it; ... one of two equal factors, how to write the radicand as a product of primes and then grabbing one for every two you see" (I3).

Mr. Casey addressed this problem as one of the misconceptions students had in radicals and algebra one: "the biggest misconception is that they still don't know what a square root is"

(I3). He thought that the lack of square root concept led them "invent new mathematics without proving it" (I3). He pointed to such mathematics when students were taking square root of a number. They continued to use/put the radical symbol around the root of the number (e.g., $\sqrt{4} = \sqrt{2}$) even though the radical should have gone. In cases where the square root was also a perfect square, they wrote the numerical equivalent as their final answer (e.g., $\sqrt{81} = \sqrt{9} = 3$). In other words, they continued to take the square root as far as they could. Across several interviews, he mentioned that "this phenomenon" occurred continually even though he repeatedly explained them that it was not true. This was also a common question that students ask frequently asked: "Why isn't the square root of 169 radical 13?" (I2)

He addressed this issue to the whole class when he was simplifying $\sqrt{(36x^5y^8)}$:

> [Mr. Casey, 04/22/2003, Lesson 5]
> Mr. C: Some people are taking square root and they're putting a radical with it. I saw on the quizzes that I collected. The very first problem was the square root of 169 [he writes $\sqrt{169}$] and somebody wrote the answer was square root of 13 [writes $\sqrt{13}$]
> S: It wasn't me.
> Sn: Yes, I didn't do that.
> Mr. C: That's not what you're supposed to do. //Because
> Sn: //13 times 13
> Mr. C: This [pointing 169 under radical] right here is 13 times 13, that's right. And the *definition* ... the definition of a square root is one of two equal factors whose product is the radicand. So, if this is a 13, 13 under there then the answer is just one of those so it should look like that [writes $\sqrt{13.13} = 13$] [italics added]
> S: I do that.
> Mr. C: With the same thing this *x*- 36 is the square root is 6. Not radical 6, but plain ordinary 6.

He thought students who had this misconception had a lack of understanding of what a square root really meant so he "have to go back and discuss what it means ... whole definition again of what a square root is" (I1). He thought that "if they knew for sure in their own mind that square root is one of equal factors then they'll be done at 9 but they're not; they think that they got to do more ... so either they put radicals where they don't belong or they don't put them where they do belong" (I3). When he was dealing with the misconception stated above he tried to explain the definition visually by prime factoring the radicand and circling the pairs and taking one of the pairs out summarized as "just factor it, circle and take one out" (I2). Other than this, he did not think there were appropriate visual ways or manipulatives like algebra tiles that could assist him explaining the definition. When I reminded him about using a geometrical approach in which he could use area of a square to model/explain square roots, he did not think it was a good way to follow because it did not contain the features of a manipulative he has in mind: "It's not bright; it's not colorful; it's not something it's gonna wild them into thinking to get it right. I mean it's a nice approach but it's not that- … what are call; a gameboy(ish) kind of a thing were it grasps their interest in" (I2).

Below is a short episode where Mr. Casey had a small talk with one student who thought that $\sqrt{64}$ was equal to 8^2.

> [Mr. Casey, 04/21/2003, Lesson 4, 2:18]
> S: Sir. (   )
> Mr. C: That's the answer, yes I do, 8.
> S: Is it just 8 or 8^2?
> Mr. C: Not 8^2. Because, the square root of 64 is not 8^2. You know the definition of a square root says one of the two equal factors.
> S: Uh-huh.
> Mr. C: So what are the two equal factors that multiply- then multiply and they give me 64.
> S: (   )
> Mr. C: So one- it's 8 times 8. One of those is the square root. Ok, not 8^2, not radical 8; just plain ordinary 8.
> S: All right.

The student asked Mr. Casey to confirm his answer since he was not sure if it was 8 or 8^2. Mr. Casey confirmed the student's initial answer, 8, as correct. Without asking him why he would think that the result was 8^2, Mr. Casey disconfirmed the correctness of it as an answer along with another possible idea (i.e., the radical 8 as an answer) and immediately reminded him about the definition of square root as one of two equal factors. He then restated that 8 was the answer because it was one of the two equal factors convincing the student about the correctness of the answer.

Later on in the same day, he was checking answers of another student who was trying to take square root of 1000 as 500.

> [Mr. Casey, 04/21/2003, Lesson 4, 2:52pm]
> Mr. C: This can't be right. Look at- on your calculator do 500 squared. [He watched while she was doing it on the calculator]... No, not times 2. Squared.
> S: Oh, that's not what I wanted. Because (   )
> Mr. C: Yeah. So you're supposed to- but look- you're supposed to write this as first of all it says factor, right? So a thousand is 100 times 10, right? What's a square root of a hundred?
> S: 10.
> Mr. C: So, this is 10√10. ...
> Mr. C: Now, 50. So you're not just dividing by it 2. You're trying to find the square root. So you gotta factor so it should look like this. You should have 2 times 5 times 5, would you agree that's 50?
> S: Uh-huh.
> Mr. C: And then add an *a*, right? And two of them inside makes one of them outside. That's not right. Again, see you take 60 divided by 2. That's not what square root means. So, you're not doing those right. You need to change all those.

In his reaction, Mr. Casey first indicated that the answer had to be mistaken and then he asked the student to take square of 500 [expecting that student would notice why her answer was wrong according to the definition of the square root] by using her calculator. He warned her that she was multiplying it by 2, not squaring. His further explanations indicated that he was able to identify what the student was doing (i.e., dividing by two instead of taking one of the equal factors as the square root) and she did the same thing in some other computations. Although he noticed that she had a problem with the meaning of square root, he never tried to ask and reveal what she was thinking and why about meaning of square root and how to compute it.

Why $\sqrt{x^2} = |x|$*?*

During the first interview, Mr. Casey expressed that he did not consider having the absolute value when taking square root as much of a difficulty for students. However, he expressed that he did not know how for that particular lesson [the first lesson of the chapter] he could have slowed it down so that he could show multiple ways to do it. His introduction for *Why* $\sqrt{x^2} = |x|$ in class began He began with writing $(-8)^2$ on the board and asking students to simplify or finding what the value of the expression was. Several students gave a negative answer. Eventually he was able to get 64 as the answer and explained that it is because –8 times –8 is 64. Continually he wrote down $(-2)^2$, $(6)^2$, $(-11)^2$, $(14)^2$ one at a time and asked students about the answer without using their calculators. After completing all these, he stated:

> [Mr. Casey, 04/16/2003, Lesson 2]
> Mr. C: Ok, so now, what I want you to understand and what I hope you can realize is look at all these numbers over here [showing the list on the board]
> S: They are all even.
> Mr. C: No they're not.
> Sn: They're all positive.
> Mr. C: They're all positive. Ok, so it doesn't matter whether you take a negative number or whether you take a positive number and square it, what do you always gonna get?
> Ss: Positive.

After a brief discussion about whether zero was negative or positive, he moved on to his point:

> Mr. C: Let's say that we took the square root of 64 [He wrote $\sqrt{64}$ on the board].
> Ss: (Eight) (negative eight)
> Mr. C: Oh, so we don't know what number we started with, do we? This right here, this is 8 or it's –8 [writes $\sqrt{64} = 8$ or -8]
> Sn: (   )
> Mr. C: Because either one of them work. I don't know- the question was why do you have to have both of them and that's because if you take either one of those and multiply it by itself you get 64.
> Mr. C: Now, ..., how much is this? [He wrote "$(x)^2 =$"]
> Sn: (x squared)
> Mr. C: x^2. [he wrote $(x)^2 = x^2$]

How much is this? [he wrote $(-x)^2 =$]
Ss: x squared.
Mr. C: x^2. [he wrote $(-x)^2 = x^2$]
Mr. C: Now, do this? [He wrote "$\sqrt{x^2} =$"] John?
Sn: x and $-x$.
Mr. C: [He wrote x and $-x$] Hmm? How do we know which one it is?
Sn: We don't.
Mr. C: We know.
Sn: ( )
Mr. C: Well, no we know.
S: (either or or)
Mr. C: Which means we don't ☺ or we do. You are arguing with yourself again. You have to start picking a side, one side or the other and just argue for that one unless someone contains your mind and so I have changed it. But to argue both side all at once is very confusing. Well, the book has a way to deal with it. We are mostly interested in ... what the book says if we want the principal root one thing we can do to be sure is uhm let this thing [i.e., $\sqrt{x^2}$] be absolute x [He wrote $|x|$] and then we can't be wrong. And this will give us the principal root [He wrote "principal root" on the board]. Ok, so when you wanna take the square roots and you're working with variables you're looking for the principal root and so they write it the square root of x^2 is the absolute value of x. And that gives you the principal root. Because do you know is this negative 8 or positive 8 [He pointed x in $\sqrt{x^2} = x$ or $-x$ on the board]
Ss: ( )
Mr. C: I don't know either. ... So this needs to go in your notes [He circled $\sqrt{x^2} = |x|$ and "principal root"]: square root of x squared is absolute x. We're concerned with the principal roots most often and so that's why we have to have the absolute value so we make sure we get the positive one.
Sn: ([asking if they could use $\sqrt{64} = |8|$)
Mr. C: Oh, yeah, I could if I'm looking for the principal root that what I- but I don't need because I can just write down 8 because I can see that 8 is the positive one. When I have a variable I'm not sure whether it's the positive or the negative one. So when I want the principal one I got to take absolute value, ok?

As the discussion suggested, Mr. Casey pointed the textbook as the authority to decide on whether it was negative or positive x as the square root of x^2. Following this introduction, he introduced impossibility of taking square root of a negative number:

Mr. C: Let's try this thing. We know that- we got all this stuff. What is [He wrote $\sqrt{-4}$] that?
Ss: ( )
Mr. C: Oh, wait a minute. 16, 2, 4, and impossible. So we have several choices. Danny, why do you think it's impossible? [He skipped asking about the other choices]
S: ( )

Mr. C: Yeah, because- that's right. So this is not real [He wrote "not real" under $\sqrt{-4}$]. There is no real number answer for that problem. So you can't take the square root, you can't take the square root of a negative number until you get to algebra-2. And there they're gonna teach you there is this other set, it's a bigger set than the real numbers in which the square roots of negatives are actually in the set.

Upon a question, Mr. Casey then mentioned about imaginary numbers and *i* but skipped to give more details since it was beyond this course. However, he used this to ask and show the square root of a negative number squared.

Mr. C: The square root of negative two squared [He wrote $\sqrt{(-2)^2}$]. What is that?
Ss: (4) (2)
Sn: ( ) then it's square root of four and square root four is two [$\sqrt{(-2)^2} = \sqrt{4} = 2$]
Mr. C: ... [He wrote down $\sqrt{(-2)^2} = |-2|$]
Ss: ([surprises and laughs])
Mr. C: What is the absolute value of negative 2?
Sn: Two.
Mr. C: That's why we have to take the absolute value. Yeah it's gonna be positive, we get the principal root and that's the one we're looking for. [He gave voice to Jerry]
Sn: Negative two square is //four [He explained his suggestion $\sqrt{(-2)^2} = \sqrt{4} = 2$]
Mr. C: //four uh-hu. Take the square root of that it's two or negative two and since we always want the principal root we would forget the negative and just take the two, ok?. But generally when you take the square root of something squared like you will see this a thousands times [He wrote $\sqrt{(-2)^2} = x$]. We are not sure whether this number is positive or negative because this has two roots but we take the principal one by taking the absolute value in the answer so there you have it.

Students seemed to understand the point in having the absolute value when Mr. Casey wrote $\sqrt{(-4z)^2}$ and asked a student about what should be done next. Upon the students' comments he wrote $\sqrt{(16z^2)}$ and then $|4z|$ and then he took 4 out because it was positive and left *z* inside because they did not know what its sign was: $4|z|$. He then moved to domains of radical expressions and evaluating expressions. Later during the time students were working on their homework, it was apparent that some students did not very well understand the need for absolute value.

[Mr. Casey, 04/16/2003, Lesson 2, 3:04pm]
S: For *t* squared I'm still a little ( ). Ok, I mean- [he sounded very frustrated]
Mr. C: No, all you gotta do is take the square root of *t* squared.
S: It's *t*.

Mr. C: It's not the- why did you put the radical over it?
S: I don't know
Mr. C: But it's not just *t*. It has to be the absolute value of *t*.
S: Then why is-
Mr. C: Because you don't whether *t*- *t* squared- like if you took -3 and square it what do you get?
S: ... [silence]
Mr. C: Negative three squared, what do you get?
S: ... [silence]
Mr. C: Use your calculator. -3 [he started entering -3 into his calculator]... squared ... Ok. Try it again. Clear that out [he probably entered it as -3^2 and get -9 instead of 9] Open the parentheses ... Negative three. Close parentheses, square it. You get nine right. Ok, now hit square root nine. ... Square root, you got to make parentheses. ... Square root nine.
Mr. C: Now you start with -3 and you ended up with +3. How does that happen?
S: Because you...
Mr. C: Because it takes the square root, see. And you're not real sure whether *t* is positive three or negative; that's why you take square root. You guarantee that you get the principal root so whenever you take the square root of something squared you want to put in the absolute value so the answer is absolute *t*.
S: So the number nineteen ... Uhm.
Mr. C: Well now you know what the square root of nine is- the principal root of nine is what?
S: Three.
Mr. C: And then but the principal root of *x* squared is -
S: *x*.
Mr. C: Not just *x*, it's?
S: Absolute *x*.
Mr. C: Absolute *x*. That's right.

At the beginning of the conversation, Mr. Casey responded to student's difficulty by giving direction for what he needed to solve the problem. He then realized the student made another mistake and asked why he wrote $\sqrt{t}$ when he was taking square root of t^2. However, Mr. Casey did not push to get what the student was thinking when he said he did not know. Without further trying to get why he was objecting to that idea, Mr. Casey also went on trying to convince student that $\sqrt{t^2}$ was not just *t;* it was absolute *t* since, for example, square roots of –3 squared and +3 squared was both positive three. He went on until student explicitly stated that square root of a variable squared was absolute value of the variable. Shortly after this conversation, he had a similar conversation with two other students who had the same difficulty of leaving the square root of a variable without the absolute value. He followed the same path to convince the students that the answers should have included the absolute value around the variables as he did with the previous cases.

> [Mr. Casey, 04/16/2003, Lesson 2, 3:06pm]
> Mr. C: Problem 24, you're taking the square root of –3 squared.
> S1: Right.
> Mr. C: Ok, so what did you get for an answer up here?
> S1: 2*a*.
> Mr. C: It can't be just 2*a*. Because look, down here watch this: If you take negative three and you square it, what do you get?
> S1: 6. Square...
> Mr. C: Negative three times negative three.
> S2: It's nine.
> Mr. C: You get nine.
> S1: Oh. ☺
> Mr. C: Now, if you take the square root of nine, what do you get?
> S1: Three.
> Mr. C: Oh. But look, it was negative three. So the idea is that we always got to get the principal root so any time we take the square root of a variable squared, the answer is absolute *x*.
> S1: So it is absolute-
> Mr. C: So this up here should be $4|d|$. And 20 would be $2|a|$ and you only need absolute on the variables.
> S1: Ok.
> Mr. C: Ok, now for this problem it'd be well
> S1: Nine.
> Mr. C: Yeah
> S1: Absolute *d*.
> Mr. C: Not nine, it's three.
> S1: Oh, three, I'm sorry. ☺☺
> Mr. C: Ok, now this one what's that gonna be?
> S1: Uhm, ... absolute *x* and nine.
> Mr. C: Well though because you're adding see over here we're multiplying so we can take the number out. It's gotta be absolute *x* plus three. [$|x + 3|$]

The next day during the first interview, Mr. Casey expressed his concern with students understanding of why they needed to use absolute value with the variables even though they were giving the right answers.

> I don't think yesterday's lesson was- I … even know they were shaking their head and they were telling me the right answers to the questions [but] I'm not sure they understand why you need the absolute value on the variable and I showed it three different times and four or five more times individually to students when we talked about it. Uhm, so as far as them grasping the understanding of the concept that the idea of why you have to have the absolute value if you want to principal root versus you know if you just tell them if it's a variable you gotta have this. Ok that they can understand and it's pretty simple formula gets the right answer but I don't think they understood why it has to be that way. The day before their problem you know one of the things that they're asking if it's rational or irrational, they don't remember some kids you see that they can't remember from one day to next what an integer is or a whole number and so I think the lessons went well, I mean they get the concept most of them were real happy with number that they got correct on when they checked their homework which means that they're able to do the problems. Uhm… we'll see. You know when I assess that and ask them we get irrational or rational I'll find out if they have that concept or not and uhm I think that most of them discovered that uhm you have to take, take the absolute value when you're doing that but if you asked them why- if you asked them, you know, at the bottom of it next quiz I get, I'll ask them to explain why and we can see, uhm, what they write and go as I hate to be, uhm, negative but I'm pretty sure that what we're going to get is gonna be pretty shallow. And I don't know how important it is that they really really understand that whole thing, uhm, most of the teachers we just say forget that absolute value, we're just, just know that it's gonna be positive even though you never really know. (I1)

As he expressed, Mr. Casey was aware that students were just following the rule, which was to have the absolute value when there was a variable included in the radical expression, to get the correct answers. He suspected, however, that they really had an understanding of why they needed to have the absolute value. He expected to confirm his worries in the quiz by asking why they would need to have the absolute value. He was not very optimistic about student answers, however, expecting they would be superficial. At the end of his comments, Mr. Casey presented a dilemma, understanding of why there was a need for absolute value (i.e., conceptual understanding) versus being able to apply the absolute value as a rule and get the correct answer (i.e., procedural understanding). Having tried to explain the why part to whole class and then individually at several occasions and still observing the same problem repeated, he could not decide if the former was really important if they were able to use the rule and get the correct answer. As he indicated, for most teachers, the situation would be easily resolved by *forgetting* about the absolute value and possibility of having a negative root. Confirming Mr. Casey's worry about students' understanding of the need for absolute value, the same day students asked him about one of the homework problems they had difficulties with, "Simplify $\sqrt{(-7)^2}$"

> [04/17/2003, 2:00]
> Mr. C: Ok. So, now, what questions from the homework do you have for me?
> S: 8.
> Mr. C: Holly.
> Sn: (Number twenty one)
> Mr. C: Number 21 [It was "Simplify $\sqrt{(-7)^2}$"]. Uhm, take negative seven and square it, what do you get?
> Ss: (Forty-nine)
> Mr. C: Forty-nine. What's the square root of forty-nine?
> Ss: (seven)
> Mr. C: Or?
> Ss: (   )
> Mr. C: So in that case when we're doing this thing we need to use 7. We want the principal root, that's how you do it.

Right after this conversation and a couple minutes of talk concerning discipline in the class, Mr. Casey decided that they got the concepts so they could move on. Although he suggested in the interview that he would ask students why they need to take absolute value in taking square roots of variables in the next quiz, possibly because of the above question from the homework, Simplify $\sqrt{(-7)^2}$, he asked students to turn homework questions over and answer this question: "Explain why $\sqrt{x^2} = |x|$". Some students asked if they could use their notes. He responded that they would first write without the notes and if they wanted, then they could look at their notes and rewrite it. After about 4-5 minutes, he collected the responses and started to read them aloud.

> [Mr. Casey, 04/17/2003, Lesson 3, 2:07pm]
> Mr. C: This person says: 'it's because *x* is *x* squared' principal root.'
> S: (What? Is he dumb?)
> Mr. C: Why would you make- what you done is you've attached a value judgment to someone's answer.
> S: It's not (   )
> Mr. C: And, uhm, It's not explain that the whole meaning but it's a pretty good idea.
> S: *x* squared is *x*.

Mr. C: No. It says because x is x squared' principal root.
S: It's squared.
Sn: I have to (   )
Mr. C: This one just writes it down- just like I have it up there [He pointed the sentence "Explain why $\sqrt{x^2} = |x|$" from the board]. So that person didn't have a lot to say.
Mr. C: Here it says 'because the absolute x will-... because number is beyond x' oh "will be" I'm sorry. "Because absolute x will be numbers beyond x." Uhm, I don't know about that.
Mr. C: It says, "Because if you get the squared root of x squared, it's absolute x. This is because the absolute value of x times x is x squared."
Ss: //(   ) [some laughs]
Mr. C: //Not necessarily.
Mr. C: What if x is negative then the absolute value of x times x is not x squared. It's the opposite of x squared. Right?
Ss: Right. [In a very weak tone]
Mr. C: Well, let's- this is a nice problem to investigate. Let's say if $x = -3$ [He wrote "if $x = -3$" on the board] then absolute x times x is equal to and this would be -9 [He wrote $|x|\cdot x = -9$] And that's not- let's see that's not x squared. But it's- it's getting there. That's the whole idea. This is the mystery that we're involved with because if we take uhm x squared we don't know whether x is positive or negative. We do know that this is equal to x times x [writes $x^2 = x\cdot x$] right? That much we know for sure. If x is negative or x is positive we're always gonna end up with positive result but if you work this thing the way we're doing it yesterday we say if x squared- what we did to x equals -3 then x equals 9 but the square root of x^2 is 3. Or it could be -3. But when we're simplifying these things we want the principal root and so that's why we take absolute value: just in case the answer happens to be negative.
Mr. C: "Because you don't know if the variable x is positive or negative so it's a safe way using either way- you know it's a safe way to say either- wait a minute. So it's safe to say either using absolute value." I don't understand that.
Sn: (   )
Mr. C: Well, you guys need to measure- never mind. Major (???ship)
S: (But it says you get the result)
Mr. C: "Because x squared can be negative three squared but the answer has to be the number an absolute value." So anyway, it's some pretty good stuff.
Mr. C: "Because x is square root of x^2 it's a principal root." Well, it's got- anyway. [He stopped reading after this]

During the second interview, he commented that 90% of the responses students gave on their homework to the "why do you have to have the absolute value of x when taking $\sqrt{x^2}$?" as very close to what they had discussed in the class as the reason of why you need that absolute value; "because you never sure whether the variable is positive or negative" (I2). Even though he thought students understood what they were doing so far based on those answers, later on in the unit, somehow students were still having the same difficulty. One such incident happened when Mr. Casey had asked to simplify $\sqrt{42x^4}/\sqrt{7x^2}$. After he let students work on it for couple minutes, as suggested by one of the students, he wrote the whole operation under one radical to simplify and by the help of students he was able to get $\sqrt{6x^2}$. Then the problem was revealed when he tried to simplify it further.

> [Mr. Casey, 04/23/2003, Lesson 6]
> Mr. C: We're not done yet because remember they always have to be in simple form.
> Ss: (   )
> Mr. C: What is the square root of *x* squared?
> Ss: *x*.
> S: and radical six (   ).
> Mr. C: Ohh, tell me again.
> Ss: (   )
> Mr. C: Stop, shh, shh.
> Sn: The value of *x*.
> Mr. C: Six we know is not a perfect square and the factors are three and tow so you can't do anything with that. But, tell me again. What's the square root of *x* squared.
> Sn: (   )
> Mr. C: //No.
> Sn: //Absolute *x*.
> Mr. C: Stop. Absolute *x*. Thank you. That's what I'm looking for. This simplifies to absolute *x* square root 6 [He wrote $|x|\sqrt{6}$]
> Ss: Ahh, yey.

At this point an interesting conflict occurred when one of the students asked if they had to use the absolute value. Even though he thought that they had to use it, he wanted to check the textbook for the solution. Observing that the authors did not have the absolute value written around the variable "for some reason," he decided that it was no longer important for him whether they used it or not.

> [Mr. Casey, 04/23/2003, Lesson 6]
> Sn: Do you have to do the absolute part?
> Mr. C: Do you have to do the absolute part?
> Ss: (   ) [they were speaking at once]
> Mr. C: Uhm, it's my opinion that yes you do but let me see what the author said. ... If I

> can find it. what was- that was. Oh, no, they said they dropped their absolute value bars for some reason. I don't really care.

The situation probably added to students' general confusion and problems about having absolute value when variables were involved in radicals. Whether they needed to use absolute value or not become a hard decision to make. This was apparent when one of the students asked Mr. Casey about his confusion because of the instructions about variables being nonnegative and if he needed to use the absolute value. Mr. Casey responded that:

> Don't worry about absolute value. There was a day that we did absolute value. Don't worry about it. Don't worry- don't worry about it. They tell you- they tell you that the variables are non negative which means you can forget about the absolute value. [Mr. Casey, 05/01/2003, Lesson 12, 2:36pm]

Even though Mr. Casey's argument for not worrying about the absolute value was well suited to the context that the variables were nonnegative, it sounded like the use of absolute value was a temporary situation and it remained in the past. However, it seemed to solve students' difficulty with the concept of having absolute value.

Finding Domains of Radical Expressions

Finding domains of radical expressions (i.e., range of values making the radical real value) was problematic for students as recognized by Mr. Casey. He noticed that most of the students had made the same types of mistakes in their homework papers and expressed his concern by explaining the whole idea again in the class with $\sqrt{(t^2+5)}$. He tried to get students understand the idea/concept before they urge themselves to find the answer.

> [Mr. Casey, 04/17/2003, Lesson 3, 1:54 -1:59pm]
> Mr. C: Ok, now. There is something that we need to get straight before we go any further because I noticed that on your papers a lot of you have the same mistake and it comes with this idea right here where it asked you to find the value- the values of t that make that [He pointed $\sqrt{(t^2+5)}$ which he just wrote on the board] a real number. Yes.
> S: (Answer is)
> Mr. C: No, Uh-huh, no sir. I don't want the answer, the answer- everybody has got an answer. What I want to know is to be begin with what kind of number does it have to end

up being under that radical, let me finish my question, in order for to be a real number? Paul?
Sn: (   )
Mr. C: Does zero count?
S: (Yeah) It does.
Me. C: It does- it could be zero under here. Anything; it can be this. What happens when we find some value for t and we put it in here and we square, we add five to it; this number has to be greater than or equal to zero. So, we're trying to find numbers that make that true. Ok. Not only trying find them, we have to state it and we call that the domain. Ok? So, what you can look at this idea is on this number line [he drew a number line on the board]. Now remember yesterday we developed this technique that if this has to be greater than or equal to zero we come over here and you just set it greater than or equal to zero [he wrote $t^2+5 \geq 0$] and you solve this inequality and you get t squared greater than or equal to negative five [he wrote $t^2 \geq -5$]. By subtracting five from both sides you end up with that. So, here's zero right here and here's negative five right here [he pointed 0 and -5 on the number line].
Sn: (And you have to put the little) (   )
Mr. C: Not yet, I haven't taken any square roots. //I- When-
S: //(   )
Mr. C: But see you're confusing the whole, the concept. You're taking the next part instructions and putting in with this problem and I'm not there yet. What I'm doing, what I'm trying to get across is they're asking us to find values for t that makes this real. Ok, they're not asking us to find a square root yet, that's the next part, ok? So, We come over here, we know that this [He pointed $\sqrt{(t^2+5)}$] in order for to be real this has to be greater than or equal to be zero. Ok, so we find out what values makes that true. Now, if you took this [He pointed $t^2 \geq -5$] one step further, you'll be taking the square root of a negative number and you know that can't happen. Ok so what we have is zero [0] is right here and here is -5 [He pointed 0 and -5 on the number line] so now the question is if you pick any number on this number line pick any number you want and square it- take any number you want and square it, where is it gonna fall on this number line?
Sn: (   )
Mr. C: Yeah, it's gonna go from right here to this direction, right? [He drew a filled circle above 0 and made an arrow towards positive infinity] If we square it. Now, what numbers then can I use?
Sn: All the positive numbers
Mr. C: No, what about- //what's- what's
Ss: //( )
Mr. C: What's this number right here? [He marked a negative number less than -5]
Sn: It's (negative) 10.
Mr. C: Ok, negative ten. If you square it what do you get?
S: Something (less)
Ss: Hundred, yeah (   )
Mr. C: So in any case whatever number we pick we square it, it's gonna be bigger than or equal to zero, right? And if you take a number that's bigger than or equal to zero and add five, is that still gonna be positive?
Sn: (Not)

> Mr. C: If you take a positive number and add five, will the result be positive?
> Ss: Yeah.
> Mr. C: Ok, so the idea here is that in order to find out what values make this [i.e., $t^2 + 5$] greater than or equal to zero we can see that it doesn't matter what number we pick so our answer in this case should look like this: all real numbers [He wrote a double backed R] and you guys have seen that before I hope.
> Ss: (   )
> Mr. C: It's a double back R. Ok. We use Z to represent the integers, right? What letter do you use to represent natural numbers?
> Sn: R.
> Mr. C: [He had a disappointed face]
> S: Q.
> Mr. C: Natural numbers?
> Ss: N.
> Mr. C: N, thank you. And rational numbers, what do you use?
> Ss: Q.
> Mr. C: Irrational?
> Ss: (R)
> Mr. C: Real numbers?
> Ss: R, backed.
> Mr. C: Double back R for real number, ok? So the answer to that question and I know that at least five or six of you back there I saw it you didn't have; you left it blank or you got it wrong. You have to understand why it's all real numbers. Because no matter what number you take and you square it, it's gonna be positive. And if you add five to a positive number, it's still gonna be positive that makes this thing- this whole thing [He circled $\sqrt{(t^2+5)}$ on the board] a real number.

During second interview, Mr. Casey expressed his concern that students were still not where he wanted them to be. He was particularly concerned that his mistake he made in the class and corrected later would have some negative impact. He wanted to go back and recover the topic during the review before the test.

> The part about what you know what values for a variable would make a radical a real number. I think they're still just little shaky on that part. It was because the mistake that I made that I corrected, you know, the next day and I don't know yet, uhmm, that they have the idea of what, you know, some of them do what makes that expression a real number and we'll have to go back and make sure cover that again when we review later this week, uhmm, so they are little shaky on that. (I2)

The mistake that Mr. Casey was referring back happened while he was solving a problem on finding the domain of a radical expression, $\sqrt{(x^2 - 5)}$, as an exercise before students had a quiz.

His solution was $x \geq \sqrt{5}$. He corrected this the next day as solving $|x^2| - 5 \geq 0$ which would have given $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$. He gave full credit for those who had solved the way he showed before the quiz even though what they did was partial in terms of an answer.

> [Mr. Casey, 04/23/2003, Lesson 6, 1:58 -2:04pm]
> Mr. C: Yesterday, before you started taking your quiz I showed you this problem [He wrote $\sqrt{(x^2 - 5)}$] and the instructions were that you were supposed to find all the values of x that make that expression a real number. Phil, what did I showed you how to do?
> S: x squared minus five is greater than or equal zero.
> Mr. C: [He wrote $x^2 - 5 \geq 0$]
> S: Add five to other side.
> Mr. C: You know what I forgot?... Well we can do this [i.e., $x^2 - 5 \geq 0$] /Well x squared is greater than or equal to five [He wrote $x^2 \geq - 5$]. And so then we got x is greater than or equal to the square root of negative five [$x \geq \sqrt{-5}$].
> Sn: This is all (real)?
> Mr. C: No I'm sorry. What am I thinking? Greater than the square root of five [He erased $\sqrt{-5}$ and wrote $\sqrt{5}$]. What I forgot and I just gave you guys credit anyway if you put that down for an answer that's what I showed you how to do. You got credit for that. But it's only partially correct. Because if we were to do this on the number line what I really need is this [He drew a number line and takes the x^2 into absolute value symbols in the inequality]. Because if I pick a value for x that's over here, this is the square root of five [He marked the origin and another point, $\sqrt{5}$]. If I pick any number bigger than that and square it and subtract five, this is a real number, ok? But on this other side over here is negative square root five [He marked another point on the number line, $-\sqrt{5}$] and if I picked a value over here like negative six, ok? Negative six doesn't fall in this range [He pointed out $x \geq \sqrt{-5}$] ... right?
> Sn: Right.
> Sn: No.
> Mr. C: It doesn't because negative six is not bigger than square root five but if I put negative six in for x right here [He pointed $\sqrt{(x^2 - 5)}$] the answer is square root twenty-nine. That's a real number, isn't it?
> Sn: Uh-huh.
> Mr. C: So, I really needed to solve this equation here [He corrected $x^2 - 5 \geq 0$ as $|x^2| - 5 \geq 0$ and then wrote $|x^2| \geq 5$] and so we get two cases: x squared is greater than or equal to five or opposite of x squared is greater than or equal to five [He wrote $x^2 \geq 5$ or $-x^2 \geq 5$] and when I solve both of those then I get this interval where I have uhm that x is less than or equal to negative square root five or x greater than or equal to square root five [He wrote $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$] and this is the real, true, whole and complete answer and I'm sorry. But I didn't count you off because I didn't even notice until I started grading the quizzes and then I thought 'Oh my goodness gracious. Look what I've done!' I've eliminated a whole bunch of numbers that make that a real number. Whole bunch of variable- values of x that would make that expression a real number so be careful Listen to your teachers but make- you know, try and independently verify that the information

you get is true and correct. So, that I had to clear up, that was on your quiz most everybody got this part right here [He pointed $x \geq \sqrt{5}$ on the solution] so you got full credit. The real answer is all of the stuff in the box [He pointed the $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$ inside a box].
Sn: I have a question.
Ss: ( )
Mr. C: That was- uhm, what number that was Eve?
S: Seven.
Mr. C: Seven.
Sn: It is up to ( ) number wasn't for.
Sn: It was all real numbers.
Mr. C: No, it's not all real numbers. If x is less than this negative square root five or bigger than square root five, this [He pointed $\sqrt{(x^2 - 5)}$] is a real number. It's like back in what we did in absolute value. You know we had the and's and the or's as a conjunction and a disjunction. So, this solution said what numbers make real is a disjunction and in symbolic form, it's this right here [He pointed the $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$]. Any number that's over here on the number line that's all of these numbers out this way [He highlighted the numbers less than of equal to $-\sqrt{5}$]. All of those numbers and all of these numbers [He highlighted the numbers greater than of equal to $\sqrt{5}$] make that expression a real number.
Sn: Vow, so.
Sn: Holly told me this. But I guess she was wrong. She said that uhm, everyone that's squared is always and the big one real number.
Sn: //That's a ( ).
Sn: //And less than ( ).
Mr. C: No, no no. I said if you had something like this; x squared plus a number. Excuse me!
Sn: Like number six.
Mr. C: Like number six. If you have x squared plus a number, then the answer is all real numbers. If it's a minus a number then it's gonna be a disjunction and it's gonna look like this.
Sn: Ok.
Mr. C: Ok. So. I'm glad we got that cleared up. And on your test, I'm gonna make sure that you understand this concept right here so be sure that you break it into two cases and get the entire range of values that makes it a real expression, ok? So, I doubt I'll come up again as a warm up in the next couple of days just to make sure you understand what I'm talking about. Brad, did you get what I'm talking about up here?
Sn: ( )
Mr. C: Uhm, no, you're doing what we've done.
Sn: ( )
Mr. C: Oh. ... If it comes up again you can ask me again later.
Ss: ( )
Mr. C: That was it. The bad news was I cheated you out of half of the numbers that make uhm that a real- that expression a real number.

In Mr. Casey's explanation, one thing in particular caught my eyes. It was the line where he made a generalization about the cases when they had x squared plus a number versus x squared minus a number. Even though Mr. Casey was talking about the cases $\sqrt{(x^2 + a)}$ and $\sqrt{(x^2 - a)}$ in particular, students might remember and implement the same generalization in situations where there would be additional terms beside $x^2 \pm a$. From that aspect, Mr. Casey's generalization would be problematic and would have negative impact on meaning. This is, however, not to say that it was a wrong thing to do. Maybe, it should have done with some other examples emphasizing what he really meant and where those generalizations would be usefully and appropriate to apply. Mr. Casey seemed relaxed thinking that he had clarified a mistake and completed a part of knowledge students missed. However, he was going to be sure if this was the case in the test. He would also check their understanding as a warm up problem in the next days. In the end of unit test, he asked two questions regarding to finding domains of each radical expression a real number, $\sqrt{x}$ and $\sqrt{(x\text{-}5)}$. Although most of the students had correct answers for those items, several students gave answers such as "all real numbers", "Any number", "all perfect squares". Several students also used absolute value brackets around the variables or numbers when they stated their answers. This suggested that students had problems with the concept of radical root and the need for absolute value.

Simplifying Radicals

Before they started simplifying radicals, Mr. Casey pointed out that it was going to be a topic where students would have difficulty. Up to this point in the curriculum students were asked; to simplify radicals that were perfect squares, or to determine whether the result was a real, rational or irrational number. However, when they came across questions like simplifying $\sqrt{112}$, they tended to make a decimal approximation without noticing the existence of perfect squares they could get out. In a previous class students had "to take numbers and write them in prime factored form and they got the idea that you have to have an even number of factors to get a square root" and he thought they did well with prime factoring the numbers. However, when he asked them: "now take one of those numbers if you can and multiply, make it a product of a perfect square as large as possible times another number"; students were giving answers like three times seventy-five. His strategy to deal with this situation was that:

> I can show them you know either you prime factor it and if you have even number you can take the square root out or you can write this number as product of a perfect square, the largest one that you can find and another number and they just take you know take the square root of the perfect square, it's easy to do and then leave what's under the radical there. (I1)

Mr. Casey's statement about students' tendency to look for decimal approximation instead of simplifying played a role in his introduction of the topic. He wanted to get students to notice that finding the value of a radical with non-perfect square radicand with calculator may not be appropriate and introduce the rational and irrational numbers within that context.

> [Mr. Casey, 04/17/2003, Lesson 3, 2:16pm]
> Mr. C: If we are to simplify for example one hundred and twelve [he wrote "Simplify √112] we want to simplify this thing. Ok? What we want to do is we want to take any perfect square factors- any perfect square factors we want to be able to take them out from under the radical. And anything that's nit a perfect square, factors that don't make squares have to stay underneath the radical. That's it means to be simplified. You taking out all the perfect square factors. Now, try this on your calculator. Try square root of 112. Yes.
> Ss: (   )
> Mr. C: What?
> Ss: ([they wanted to barrow calculator])
> Mr. C: Yes. ... What was that number rational or irrational.
> S: You can divide it by four.
> Mr. C: Is that- Wait a minute I don't wanna divide it by four. Four is not (   )
> Ss: (   )
> Mr. C: Yeah if it was rational we could be able- it would terminate or repeat in decimal form- terminate or repeat in decimal form or we would be able to write it as the ratio of two integers. So, the – it's irrational which means we can't write it because irrational numbers never stops. It's like pi(π) you know and we don't want to spend our time right now numbers like that so instead of trying to write out a gazillion, trillion, million

numbers we just leave it under the radical. The smallest part that is. Heidi?
Sn: (   )
Mr. C: ... Before I go any further is there anybody else that needs a calculator? Come and get them.
Sn: ([want to ask a question])
Mr. C: I'm ready almost. What's your question?
S: They say that-
Mr. C: They? Wait a minute, wow wow wow. Who are they?
S: Investigators.
Mr. C: Investigative people, ok. Like Colombo?
S: They say this the numbers like never end or repeat but any numbers that they can use really, uhm, one (   ) zero through like nine over to the any number that they use
Mr. C: But-
S: Listen (   ) a combination of way- combination that they can run out eventually. See eventually they has to be a way they reach.
Mr. C: Well, they who are doing the investigation to begin with 0 through 9 are called digits. And we put digits in certain places in a number to represent some value, ok? So when we're talking about decimals we start you know we're going this way from the decimal point and it's tens, hundreds, thousands, ten-thousands and all like that. ... Since, oh I guess the Egyptians or maybe even before them when they started building their pyramids they started running into this idea of some numbers can't be expressed as the ratio of two integers, ok? So for 3000 years or more people have been trying to find like if pi(π), for instance, terminates or repeats. And they haven't been able to do it. So, they employ this thing they called deductive logic and it says and they can prove that the square root of two irrational using a proof- using logic proof by contradiction. So, it does never terminate or it never repeats. And at this stage in your career I think realizing that there are some rational and irrational numbers is a good thing and then as you advance if you can proof that pi [π], for example, terminates or starts repeating, you would be world famous that would be your ticket to mathematical-
S: (   )
Mr. C: Yes, you would. Because you would be able to publish a result and people would be coming to you thinking that you're really smart guy and they want you on their team. Maybe work for RAIN Corporation or anywhere they trying to develop new technologies or new ways of looking at mathematics. And there is lots of- there is money in there I mean lots and lots of. It's certainly enough to keep you fed and clothed. So, we're going back here to this idea.

Mr. Casey thought that it was important for students to know the list of perfect squares to be successful in simplifying radicals. This was an essential part of his methodology in simplifying radicals. Even though it was simple task that he gave students as their first assignment in the class, Mr. Casey expressed that they did not learn their squares as if they were resisting. He thought it was one of the stumbling blocks for students in simplifying the radicals. After he enhanced his introduction of irrational numbers with a little bit historical background, Mr. Casey moved to a formal a definition or explanation for what simplifying radicals means where his emphasizes on perfect squares he previously expressed had meaning.

> Mr. C: Paul thinks that it's easy to do this. And I'm going to have to agree with him but I want to see his way of doing it first. And then because lot of times you all have better ways than I do anyway. ... Uhm I don't think it's well developed in him yet ... Are you ready yet? [He tried to get students be quiet and listens for a few seconds]
> S: When I see the answer that they gotten I know how to multiply they do everything backed out to get the and then the number you start with but I don't know how to take the number you start with and get down to the way that they have their answers.
> Mr. C: Ok, I think the key- the key to your dilemma is contained in this word right here
> S: What is that? What is that?
> Mr. C: Factor [He pointed out the word "factor" in "Means pull out all perfect square factors from under the radical"].
> S: Factor.
> Mr. C: Factor. So now
> S: Like a trick, like a trick. Ok, I had it. [He was very excited] I got it.
> Mr. C: Now.
> S: Twelve. Is this three factors in twelve: 2, 2 and 3 right?
> Mr. C: //No, no, no, no.
> S: //( )
> Mr. C: 12, 12. What are you doing with 12. That's a 112.
> S: Never mind, never mind I got it. I got it.
> Ss: [laughs]
> S: No no no. I'm not trying talk about that. I was doing the one.
> Sn: What page you're on.
> S: 493 and 743.
> Mr. C: Hold on a minute. Back up just slow down for a second. Ok. Mark. Quit trying steal my spotlight, ok?
> Ss: Say yes sir.
> S: Yes, sir.
> Mr. C: We are who we are, ok? But I want your attention now. The idea is factoring. Now, on your sheet you have one twelve in factored form so you there is no mystery where these numbers are coming from. I'm sorry you don't have one yet. I'll try and find you one in a minute. So, 112 is equal to 2 times 2 times 2 times 2 times 7 [He wrote 112 = 2.2.2.2.7]. That's problem number two.
> Sn: I know how you get it.
> Mr. C: Ok, we'll- then I'm gonna be quiet. I'm gonna just say one more thing: Look down at the bottom where you see problem number two at the bottom and you can see 112 written in a different way [He wrote 112 = 16.7] ok. Eve, share your wisdom.
> S: Ok, this is what I think- this is what I think you may need to do. Well, in the back- in the back of the book
> Mr. C: Ah, that's it, right there. You're right. You shouldn't have said that.

S: What is there? (   )
Ss: (   )
Mr. C: Just a minute, just a second. I don't- you guys, quit looking in the book. Stop looking in the book right now. Problem we are working on what we are trying to simplify is the square root of 112. So if you're looking at anything, it should be looking at your sheet. The sheet you already done or then thinking about what's going on there. So I don't want to square root of 12 because the number I'm working with is the square root of 112. So let's keep our focus-
S: The answer, you really don't look.
Mr. C: What do I look for?
S: To get this, you add 6 + 6.
Mr. C: Plus? No no no! That doesn't make factors. 6 plus 6 doesn't make factors.
S: (   ) get it.
Mr. C: Sorry, that we're not talking about- we're talking about numbers you multiply together to get 12, ok? You have to do multiply together. Now, look here. Then if we wanted the square root of this, it would be the same, this is 112- this is 112. We have this square root here, ok? Or we can take this square root and this square root. [He pointed out 112 = 2.2.2.2.7 and 112= 16.7]
Sn: Mr. Casey, Mr. Casey.
Mr. C: Yes John.
S: Could you just write that 2 times 2 times 2 times 2 is, uhm, 2 to the 4.
Mr. C: You could. I don't know if it would help you any. That's why I left it that way. It's not incorrect to do that. But I don't know that's gonna get you closer to be able to simplify these things.

In the dialogue, it was apparent that students had problems carried from arithmetic related to factorization and factors of a number. Mr. Casey recognized this as one source of their difficulty in understanding the idea of simplifying. On the other hand, students were trying to follow the procedures and the answers from the back of the book instead of focusing to understand what Mr. Casey had to say.

At this point in the discussion, one of the students claimed that he had the answers. Mr. Casey asked him not to tell the answer but explain how he did get it.

Mr. C: Brad, do you have an idea?
Sn: I have the answers.
Mr. C: You do? [with a surprised tone]
S: Yes.
Mr. C: Ok, well then don't tell me the answer. Tell me how did you get your answer?
S: All right. I took 112 and made it into the 2 to the fourth and 7.
Mr. C: Yeah.

S: Uhm, I took- I took the square root of 2 to the fourth and left 7 under the radical.
Mr. C: And what is the square root of 2 to the fourth.
S: 2 squared.
Mr. C: And that is?
S: (   )
Mr. C: Yeah, 2 squared.
S: Uhm, two, two.
Ss: Four.
S: 2 squared is 4.

Satisfied with the student's thinking, Mr. Casey used the students' idea to introduce the product property of radicals.

> Mr. C: Right, ok. Yeah, so, good thinking. Ok, what he did and he didn't know about it yet but what he just described for you was the property of square roots and here's how it is: This is the property; it says that if you have the square root of *a* times *b* [He wrote "Property $\sqrt{a \cdot b}$ = "] and in this case this would be 2 to the fourth and this would be seven [He wrote $\sqrt{2^2 \cdot 7}$] so he had the square root of this product. It's equal to the product of the square roots which is square root a times square root b [$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$] and what BRAD did was he took the square root of 2 to the fourth and times the square root of seven [writes = $\sqrt{2^2} \cdot \sqrt{7}$].

After this statement, some students still seemed to have difficulty grasping the procedures and accepting to have a radical expression as a final answer. This would mean that students' understanding of a radical with non-perfect square radicand as irrational numbers and meaning of square root was troublesome.

Mr. C: John.
Sn: I have another question.
Mr. C: That's why you have raised your hand.
S: Yeah. 16 times 7 equals whatever and then square root whatever is and that'd be it.
Mr. C: Well, 16 times 7 is 112. And we already determined that that answer is irrational so we don't get a number that's easy the work with.
S: Then you go- you go to the four, square root it.
Mr. C: Right. What-
S: Then-
Mr. C: But see we already know what this is [He pointed $\sqrt{2^2}$] right, right?
Sn: No. Six. Sixteen.
Mr. C: Oh, what's the square root of 16.
Ss: Four.
Mr. C: Ok. [He wrote $4.\sqrt{7}$] Look you've accomplished your task. It says here
Sn: Can't square root 7, right?

Mr. C: No, because how many digits the square root of seven have?
S: I don't know. A lot.
Mr. C: ☺ Infinitely many.
Sn: 2.645.
Mr. C: No, no no. No, no, no. That's just what it shows there. Remember, if it's not a perfect square, it's irrational and the decimals never stop. So, it has infinitely many digits. All right.
Sn: Well, I am trying to (   ) delete that little line on top of the seven.
Mr. C: Oh, wait a minute; this [He pointed the radical sign, √] isn't called the line. What's this called?
Sn: A square root.
Sn: Radical.
Mr. C: Yes, it's a radical, not a square root. It's radical.
Sn: Delete the radical (   ) on top of the seven.
Mr. C: Yes.
S: How come? [He was still thinking that they needed to remove the radical sign probably because he was thinking that they needed to after all they took the square root]
Mr. C: Because we didn't take the square root of seven.
S: All right.
Mr. C: Ok. We- because basically we go back to our definition of what it mean to simplify a radical expression and what it means to simplify is that if there any perfect square factors in the radicand, under the radical, you have to take those out. That's what we're trying to do. And in order to do that more effectively, we can use this property right here [He pointed the product property written on the board] So Brad successfully separated his numbers, 112 into 16, which we know the square root of, times 7, which we will just write radical 7 because we can't take the square root of that. We can get, oh Casey quit it. I'm giving you guys valuable pieces of information and you're playing foams on your calculator and that's even well it's stopping you being as successful as you wanna be. John. ... Thumbs you know you can tell people playing games on calculators when all they uses their thumb. That's what I mean. That's- you wouldn't call that a game? What would you call it?
S: Press a button?
Ss: ☺☺
Mr. C: Wasting time like a mindless.
S: It could (   ) his thumb.
Mr. C: What was your question?
Sn: (   )
Mr. C: Yeah but- excuse me. There is a big difference between listening right and comprehending what you hear. I mean you can hear things, you can listen, you can- you know it takes more than just sound waves bouncing off your ear to understand math. Yes.
Sn: If the number- if the number wasn't 7, if it was a number that could be square rooted, right.
Mr. C: Yeah.
S: Could you just times the two together or you have to do it that way and squared it out and-
Mr. C: Well, it would depend on whether the product itself you know when you multiply

them together if you can recognize that as a square root or not.
S: Not, like if [He stood up and went to the board] If this was *a* instead and you made it like. //2 times 2-
Mr. C: //Yeah but if that's-
S: And would you just got (buzzing sound) and write that out and square root.
Mr. C: ... Yes.
Sn: If you were looking for the answer the book says 16*a*. But 2 times 2 times 2 times 2 times *a*, right?
Mr. C: No, I wouldn't because ( ), it's not factored.
S: They're not if it was 16*a*.
Mr. C: 16*a*! Oh, yeah, yeah.
S: You do. I mean come out as 4 and then.
Mr. C: Radical *a*. Yeah.

After finishing the first way of simplifying, which was using the product property of radicals, he started doing it another way and asked students if they were going to like it better. He wrote $\sqrt{}(243t^3)$ on the board and asked what it means to simplify. He eventually got "taking square factors out" from one of the students. He factored out everything under the radical and wrote $\sqrt{}(243t^3)$ as $\sqrt{}(3.3.3.3.3.t.t.t)$.

[Mr. Casey, 04/17/2003, Lesson 3, 2:52pm]
Mr. C: Now, whenever you have two of the same factors under the radicals- Jenny, tell him what is a definition of a square root.
S: Any number that a square root.
Mr. C: Square root?
S: A number that can ( )
Mr. C: No.
S: Oh, that's a square. The square root...
Mr. C: Paul, do you remember what a square root is?
Sn: ( ) a square root is?
Mr. C: Yeah.
S: Something like that.
Mr. C: Prog what is it?
Sn: It can like
Ss: ( ) (times by itself)
Mr. C: Ok.
Sn: Not consecutive.
Mr. C: Not consecutive. That means add one to the next one. The previous one. Ok, definitions- boys and girls! It's very important that you understand the definitions are not up for an individual interpretation. A definition is a meaning for a word that never changes. Unless there are three, four meanings for the same one like (Ra???)
S: Ra? What is that?

> Mr. C: Well it can be iron or it can be some emotional state that caused by stress. Ok, what I told you the definition was is if- a definition of a square root of a number- the definition of a square root of a number is one of two equal factors whose product is the radicand. That's what a square root is. Ok, so if I have two of these here [He circled the first pair of threes on the board] one of those is a square root. So, I put it out here, ok? These are gone [He crossed the pair inside the radical] Now, if I have these two, one of them is a square root [He crossed the next pair and wrote one of the threes out]
> Sn: Another one!
> Mr. C: I have two of these [t] one of them is a square root [He wrote one t out and crosses both]. Does this have a pair? [He pointed the remaining 3]
> Sn: Uh-huh.
> Mr. C: No. Where is the pair?
> S: Up there.
> Mr. C: Up, here [He pointed the $\sqrt{(243t^3)}$ in the previous step or in the original statement]
> S: Yeah.
> Mr. C: No, this number is right here. I just rewrote it. So the answer to that is no. Does this have a pair? [He pointed the remaining t]
> Sn: No.
> Mr. C: Then these have to stay. In order to take them out of here they have to have a pair. An equal factor. So now I take this and this [He pointed $3.3.t$ that he took out] and what doesn't have a pair stays. So the answer is $9t\sqrt{(3t)}$ and this is the simple version.
> Ss: (   )
> Mr. C: That's is an elementary way of doing it, yes.
> Sn: I like it better the other way though.
> Mr. C: Then do it the other way. It's faster and quicker. Some people don't see it.

With this second example, Mr. Casey tried to establish a way of simplifying by using the definition of square root. He thought would be more meaningful for most of the students, expecting that they all knew what the square root was. This was probably why he started with asking the definition or meaning of square root. It turned out that they did not have an explicit definition for square root as he wanted. He expressed that they were interpreting the definition of square root and it was not appropriate to do since he believed that definitions were not open to individual interpretations.

Adding and Subtracting Radicals

Mr. Casey considered students' tendency to add radicals with different radicands by writing it as a sum under a single radicand (i.e., $\sqrt{a} + \sqrt{b} = \sqrt{(a + b)}$ as a misconception. He stated that when students add radicals, for example, $\sqrt{3}$ and $\sqrt{5}$, they "just add the 3 and the 5 and take radical 8 and be done with it" (I3). Although he expressed that "that one won't be too hard" to deal with during the second interview, the problem was more persistent and difficult to remedy than he thought. Even though he told students that the radicands had to be the same in order to be added, they would still continue to do it. He thought of this as a case where students "invent, or make up, new mathematics without proving it" (I3). He did not know where the problem came from because he thought that "it's nothing I demonstrated it and it's so contrary to the definition" (I3). However, he identified lack of square root concept or understanding the definition of it as the root of the problem. He also considered that students often had hard time to accept lack of closure: "they think that they got to do more" (I3).

Students' difficulties in adding radicals with both different and same radicands had been at the scene since the day of Mr. Casey's introduction of adding radicals when he started with an example, find $4\sqrt{3} + 3\sqrt{3}$. At the beginning of the lesson, Mr. Casey summarized what they had done so far such as multiplying and dividing radicals and then he asked students what they thought about should be next. Upon a suggestion of "times them" Mr. C commented that it was multiplying and they already done that one. Another student suggested addition. He agreed with that gladly and explained that they were going to add the subtract radicals on that day. One of the students, Paul, suggested that adding radicals would be just like adding two fractions where they would need to have denominators same and then add. Mr. Casey disagreed expressing that they were not doing fractions and they would add on one level (i.e., no denominators) even though it was not when they encountered addition of radicals involving fractions.

> [Mr. Casey, 04/24/2003, Lesson 7, 2:18pm]
> Mr. C: Today, we have to be able to add radicals together.
> Paul: So, basically ... almost like we're gonna have to do that on two fractions and then we're gonna have to make common bottom terms and then add them together.

Mr. C: No, no. We're gonna keep them on one level. We're not doing fractions right now. ... But maybe we'll. Yes, we might.
Paul: Oh, we might get that.
Mr. C: Yes.
Paul: Radical four plus radical four and that would be four. Because that's two plus two.
Mr. C: Yes.
Paul: That's basically ( )
Mr. C: Yeah but not quite like that. But close, that's very close.
Mr. C: [He wrote $4\sqrt{3} + 3\sqrt{3}$ on the board] Ok, take out notepaper. Take out your notepaper. And you write down section nine, I mean- I'm sorry. It's eleven six.
Sn: ( )
Mr. C: Eleven six. Nobody is absent [to the student who was keeping the class log that day]
S: Adding
Mr. C: Adding and subtracting radicals.
Ss: ( ) [noise from students]
Mr. C: Ok, should I throw- that's not right [to Paul].
Paul: Yes, it is. I know how to ( ).
Mr. C: It's not seventy-five.
Paul: ( ) it is. [He approached to Mr. Casey]
Mr. C: Ok-
Paul: Because what ( )
Mr. C: Just a minute. If you'll be four- think about it for a second I'm gonna ask you what you did and we're gonna discuss it, ok?
Paul: ( ) ok.
Mr. C: Yes, yes. ... [He waited for silence and commented on students' misbehavior].
Sn: What's the section called? [It was the student keeping the class log]
Mr. C: It's adding and subtracting radicals. [He then continued to try getting students silent and behaved]

Before he started with Paul's idea he had to deal with some discipline issues and disturbance that was a common problem in the class. Paul's idea was that the result was 75, which was initially rejected by Mr. Casey. He then asked him how he got this result.

Mr. C: It's always something. If you would change your focus, ignore him and think about adding radicals, I think your life would be a lot better. Ok, so, now, Paul has an idea of what he thinks the sum is. Paul, what do you think the sum is?
Paul: Seventy-five.
Mr. C: He thinks it's seventy-five. Now, what I'm gonna do is I'm going to put a big red circle around this because I don't know if it's true or not [He then wrote $4\sqrt{3} + 3\sqrt{3} = 75$ and puts a red circle around 75]
Paul: But you have to ( )
Mr. C: Yes, that would be a good idea. But, now, Paul, how did you get that answer?

Paul: Because I know how to, uhm, ( )
Mr. C: No, no, no. Not, ok, procedures please.

It was interesting to see Mr. Casey did not immediately disregarded Paul's result and suggested that it needed further details or methods of finding. It was an act of valuing and respecting students' thinking. However, it was interesting to see that Mr. Casey was interested in Paul's procedures more than what he was thinking or why he did what he did when he interrupted Paul's explanation of his reasoning. This was probably because Mr. Casey wanted to see the gaps in his erroneous thinking in his explanation of procedures or he thought his explanations were irrelevant in the situation or at the moment. But, in some point, I expected to see him asking the *why* question after Paul explained his procedures of getting the answer as 75 but he did not. Mr. Casey was patient when he was trying to fully understand how he got 75. His explanation revealed that Paul thought the solution as $4\sqrt{3} + 3\sqrt{3} = 4 \cdot 4 \cdot 3 + 3 \cdot 3 \cdot 3 = 75$. Mr. Casey's first reaction was to ask what happened to the radicals and decided that they needed to come up with some other solution better than this because it was not true.

Paul: I know we have a number outside of the radical that if you do it elementary way it helps because you have a pair of one. So, you don't ( ) you had to double back. ( )
Mr. C: So, yeah, if I put this four back in here, I'd have two factors of four. Is that what you telling me?
Paul: Yeah. Times ( )
Mr. C: Ok, wait a minute, so it looks like this. Uhm, three times four times four [He wrote $\sqrt{(3.4.4)}$], right?
Sn: Yes.
Mr. C: That's what you're saying? Because we can do that of course. That's how this four got out to begin with, right?
Paul: No, no, no.
Mr. C: Oh, that's not what you were doing.
Paul: Look-
Mr. C: You guys need to be quiet. And if you want to join in the conversation raise your hand and if you don't pay attention because this is where the learning is taking place.
Paul: When you multiplied out you don't keep the radicals. When you multiply out you get rid of the radical. That's how you multiply that back out. [He seemed very confident]
Mr. C: So, what am I multiplying?
Paul: Four the radical three is the same thing as four times four times three [i.e., [$4\sqrt{3}$ =

4.4.3]
Mr. C: No, it's not.
Sn: Because two-
Mr. C: Because four times four times three is sixteen times radical three [He wrote 16√3]
Sn: But then when-
Sn: Two times two times-
Paul: I'll do sixteen times three.
Mr. C: Forty-eight.
Sn: I was writing-
Mr. C: Ok, wait a minute now. Let's just think about it.
Mr. C: Well, you want me to just multiply four times four times three. What happened to the radical? If you got- if you get rid of the-
Paul: All right, when you write out four times four times three [Mr. C wrote 4.4.3] If you do this the elementary way. You have a pair of four. (Tickle) the four. You put the core on the outside the radical and three [Mr. C circled the pair of fours]. And three does not have a pair so you that on the inside [Mr. C wrote 4√3]. Now what do you do to multiply that back out? You have to double the four which will be sixteen times three which will be forty-eight [Mr. C wrote 16.3].
Mr. C: What happened to the radical? You squared? You, so- so you squared this number to get this number to get sixteen and you squared this to get rid of the radical.
Paul: You don't square the three. The three doesn't not have a pair so it's-
Mr. C: Well, then it can't come out from the radical. What happened- my question is then where does the radical go? And that, that-
Paul: (   )
Mr. C: ☺ Yeah, it's- it disappears, right? So, I think we need to come up with something better. Because- [interruption/distraction from the students]
Mr. C: Paul, you know the words- we have properties to deal with radicals, ok? If we have two of the same thing under the radical then we can take one out of and drop the radical. But if that's not the case like you pointed out very carefully and very adequately, this does not have a pair and it must remain under the radical. So, if we were going to do anything if we wanted to multiply these two together we would need to take and approximation for the square root of three and then multiply them. But that's not how we add radicals. So, let's try something else. Brad, do you got an idea?

Paul's thinking of adding radicals had problems in terms of the meaning of radicals. Although, to my observation, he did not have a problem of getting square roots or simplifying them given a radical, he had a misunderstanding in reversing the procedures. Namely, he would probably simplify √48 as 4√3 but he thought reversing the procedure would get rid of the radical sign. Understanding his procedures in a complete form, Mr. Casey corrected and emphasized that the radical should have stayed. He also suggested that Paul's method (i.e., getting everything back under the radical sign) might not be the correct way to start with because writing it under the radical did not guarantee that $\sqrt{48}$ and $\sqrt{27}$ were perfect squares; they had to find an approximation and then add, which were not what they were looking for. Deciding that they needed another idea, Mr. Casey asked Brad.

> Brad: Yes, sir. Uhm, if- put the four inside of the radical, it makes it two fours.
> Mr. C: Yes.
> Brad: And, uhm, [Mr. C wrote $\sqrt{(4.4.3)}$], and then on the other one, you do the same thing and make three (   ) three threes.
> [He wrote $\sqrt{(3.3.3)}$]
> Brad: And you multiply the see together and put the radical together.
> Mr. C: Ok.
> Brad: And you take the, uhm, square root of what's (   )
> Mr. C: Now, when I put these [points out $\sqrt{(4.4.3)}$ and $\sqrt{(3.3.3)}$] together, do I put a plus sign between them? Or do I put a multiply between them?
> Brad: You put the multiply sign between them.
> Mr. C: Oh, but I'm supposed to add them. I'm not supposed to multiply them.
> Brad: Do I?
> Mr. C: Yeah, it says right here. I want- I want to add these two things together.
> Brad: You add the uhmm-
> Mr. C: Ok, this is forty-eight and this is nine right or twenty-seven, right? So, add those together what do you get?
> Brad: Uhm, ...
> Mr. C: Seventy-five? [He wrote $\sqrt{48} + \sqrt{27} = \sqrt{75}$]
> Sn: Seventy-five yeah.
> Ss: (   ) [talking all together]
> Mr. C: Now, what is this?
> Paul: That's some nasty-gusty.
> Mr. C: No, it's not.
> Ss: (   ) [talking all together]
> Mr. C: [He wrote $\sqrt{(25.3)}$] twenty-five times three, right? Which is five square roots of three [He wrote $5\sqrt{3}$].
> Brad: Yes.
> Ss: Uh-huh.

Brad suggested that $\sqrt{(4\cdot4\cdot3)} + \sqrt{(3\cdot3\cdot3)} = \sqrt{(4\cdot4\cdot3 + 3\cdot3\cdot3)} = \sqrt{75} = 5\sqrt{5}$. Mr. Casey suggested students to use their calculators and find an approximate answer for the result (i.e., $5\sqrt{5}$) and sum $4\sqrt{3} + 3\sqrt{3}$ by calculating each separately.

> Mr. C: Ok, now, keep that in mind. So, he thinks that it's five square root of three. Take your calculator and get a decimal for this for me.

Sn: I got 6 (   ) 254 (   )
Mr. C: No, no, no, no. That's not right. I want you to multiply on your, please, please. This is what I want. Don't push buttons until you find out what it is that I want. I want you to find out what five times the square root of three is. So, you have to do five times square root three, ENTER. Yes, Heidi.
Sn: ([asking for a calculator])
Mr. C: No, you can barrow one.
Sn: [She approached to Mr. C. with his calculator in his hand] Mr. Casey, it'll just be through this. It's exactly the same as you could do that.
Mr. C: Yeah.
S: And you said it was wrong.
Mr. C: What is it? Eight point six, six [i.e., 8.66]
S: Yeah.
Mr. C: Ok. ... Get the calculator please and go- Uhh [to another student]
[Students came and borrowed calculators]
Mr. C: John, sit down please. ... John, sit down please.
Mr. C: Now, lets do this. Do- Find on your calculator- find out what this number is? So, it's four times square root of three. Do that on your calculator. ...
Paul: Six point nine, eight,
Mr. C: Ok.
Paul: two [continued]
Mr. C: That's enough. Six point, nine, two, eight. Ok, do this on your calculator [points $3\sqrt{3}$]
Sn: Wait, what?
Mr. C: I want you to multiply three times the square root of three on your calculator.
Paul: Five point one, nine, six.
Mr. C: [writes 1.196] Ok, this number right here [points 6.928 + 5.196]- this number is bigger than this number [He wrote > 8.66 that John found.]
Paul: You sure about that? Yeah, yeah, yeah.
Mr. C: It's bigger than that number. So, there is problem because if I do this two things individually and add them together I get something that it's like twelve, ok? If I put these things together- if I add them together and do it I get 8. There is a problem there. Those numbers should be the same, shouldn't they?
Paul: Yeah, shouldn't they?
Mr. C: They should. I think they should.
Sn: I think so.
Mr. C: So, there is something wrong.

Within the above discussion Mr. Casey respected students' thinking without first saying that it was wrong. He used and followed the student thinking by correcting it along with highlighting the points where it had flaws. The calculator activity helped Mr. Casey to challenge Brad's thinking by showing that what they started with was not the same thing as what they ended up.

This really convinced students that there had to be a problem they needed to know. Setting up this interest with students, Mr. Casey introduced an alternative way of thinking, his approach to the sum of radicals and presented a "fruit salad approach" where he suggested considering √3s as something else like oranges.

> Paul: What are doing it wrong?
> Mr. C: Let's try this then. Let's treat this as something that we can't mess with. We know that that's rational. We know- I mean it's not rational. It's irrational. The square root of three.
> Paul: If you add those together that equals nine through the radical bar and you can- that's-
> Mr. C: No, no, no. Three and three it's six.
> Paul: Oh, go ahead.
> Mr. C: But, besides that lets treat this [He pointed 4√3 + 3√3] something that we don't- it's- it's irrational, right? It means we keep on going and going and going. Well, let's say that this thing that we're dealing with here, we know it's a radical three, but lets treat it like it's something else like an orange.
> Sn: Why?
> Mr. C: Because if you had four oranges- if you had four oranges and you added three more oranges how many oranges would you have all together?
> Sn: (Three and three)
> Mr. C: Four plus three.
> Ss: Seven.
> Mr. C: You have seven oranges, ok? I think that you should add these numbers together so I have four of these things and then I have three more of them that gives me seven. Ok? Three plus four is seven. So, I think that when you add these two things together what you get is seven square roots of three. Now, add these two numbers together please on your calculator for me real quick.
> Sn: What number?
> Mr. C: It's six point nine, two, eight plus five point one, nine, six.
> Ss: ([talking all together])
> Mr. C: Stop. Everybody be quiet. Six point point nine, two, eight plus five point one, nine, six. What did you get?
> Sn: One two and one, two, four.
> Sn: Twelve and one, eight, four, four.
> Sn: Twelve, four. One, two, four.
> Mr. C: Oh. That's the thing about calculators. If you push the right buttons this- if you all push the same buttons you all should get the same answer. Now, what I want you to do on your calculator is do seven times the square root of three.
> Sn: It's one (   )
> Ss: No, it's twelve point one, two, four.
> Mr. C: Ah-ha. Now, look. Now, they are the same.

Ss: ([talking all together])
Mr. C: The answer would be seven square roots of three. Ok.

By the fruit salad approach with that Mr. Casey tried to get students to see $\sqrt{3}$ as an object, rather than something to be calculated. He found the result as $7\sqrt{3}$ (i.e., four oranges added to three oranges) and asked students to confirm that the initial addition had the same decimal value as what they found as the result. Still not satisfied with this finding, one of the students asked why the result was $7\sqrt{3}$. As a response to this *why* question, Mr. Casey generalized the idea that the radicands had to be the same in order to be added and gave another example to illustrate his definition or rule for adding radicals.

Sn: Why?
Mr. C: Because when you add radicals together first of all the index- the index has to be the same. Now, so far the only index we're working with is two and we don't write it because we're working with just square roots. So, all the square roots- square roots are all the same. But, in order to add radicals together- in order to add the radicals that radicands have to be the same.
Sn: Ok, so, if they're not?
Mr. C: If they're not the same then we can't add them together. And we have to leave it as with the plus sign between them. So, for instance, if I wanted to add three square roots of five minus two square roots of five [He wrote $3\sqrt{5} - 2\sqrt{5}$ on the board] what would I get?
Ss: (   )
Sn: One square root of five.
Mr. C: One square root of five. That's right. [He wrote $=\sqrt{5}$]

At this point one of the students wondered what would the case if they were trying to add radicals with different radicands.

Sn: So, what do you do when they don't have the same number under-
Mr. C: Ah, good question. What would I do if I try to add the square root of three plus the square root of five? [He wrote $\sqrt{3}+\sqrt{5}$]
Sn: Divide or multiply?
Mr. C: You can't. It says add. They want you to add those together. So, here is the-
Sn: (   )
Mr. C: No, square root of three plus the square root of five. Are the radicands the same?
Sn: No.
Sn: I don't-
Ss: ([talking at once])

Mr. C: No, that's the radicand and that's the radi- they're not. So this is equal to the square root of three plus the square root of five. They can't be added together.
Sn: You can add.
Sn: Is it, is it-
Mr. C: You can't add radicals together.
Sn: Is there- is there- when there be a (   ) one in front of both of those?
Mr. C: There would. It's not imaginary. It's actually you can put it there. But, still if the radicands aren't the same, you can't add them together. We found that out so we tried it a couple of different ways but it didn't work. I mean- you gotta- the radicand have to be the same.
Sn: So, what would be the answer of uhm, do we have a symbol or something?
Mr. C: No, this is what I put for the answer; the same thing I started with. They cannot be added. Casey.

Mr. Casey insisted that if the radicands were not the same then they could not be added. It was interesting that students had a hard time understanding the statements "can't be added if the radicands are not the same" as an answer and looked for a way or a special symbol to state the final result or make a closure. Mr. Casey suggested the answer would be the same thing he started with. Mr. Casey's statement "can't be added if the radicands are not the same" would have lead to a wrong generalization because it did not emphasize simplifying radicals first and then looking if the radicands were the same. Luckily, this point was made, even though it was not explicitly expressed, when one of the students asked about a case of adding radicals with different radicands but one could be simplified.

Sn: Ok, so, if it says square root of twenty-seven minus two times and radicand three.
Mr. C: [He wrote $\sqrt{27} - 2\sqrt{3}$] that's radical three, like that?
S: Yeah.
Mr. C: Ok, can you add those together?
Sn: You can factor out if you want.
Mr. C: Now factor out. Simplify. Right, you can. And in that case you would because you can recognize the twenty-seven is not in simple form. Ok, what's the simple version-
Sn: Radicand three.
Sn: Three times-
Mr. C: Two radical three. Ok, now we simplified radical twenty-seven before. What do you think it is?
Sn: It's a three- three-.
Mr. C: Three radical three. So, this number we replaced with three radical three then we can subtract two radical three because look the radicands are the same now [He wrote

3√3 - 2√3]. ...And this is again radical three. [He wrote = √3] And there is the answer. So, sometimes you have to simplify before you add these things together. But, even if- if you got done simplifying and the radicands are different you're done. You gotta stop. Because in order to add them together the radicands must be the same.
Sn: What happens to-
Mr. C: So, oh, yeah. I'm sorry go ahead.
S: I don't understand. I don't understand. I don't- Three minus two one. I guess I was wrong.
Mr. C: Just, when these things are the same Eve you can treat it like an *x*. Three *x* take away two *x*; one *x*. Only they are not *x'*s. They're radical threes. Ok, so a radical with one particular radicand in order to combine it with others like- they gotta be like radicals- like radicals have the same radicand.

The case stated so far was one of the incidents Mr. Casey and I found interesting. He thought that when he wanted to try his idea of adding radicals together Paul was going so fast they could not track what he was doing until he got him slow down. Once they understood what he was trying to do, they were able to show by using their calculators that his strategy would not work: "the only way would be the take the square roots and then add those square roots together if you want a decimal approximation" (I3). He also found that some students who are usually quiet were helpful with the calculator activity.

After this incident, Mr. Casey wrote another problems, 5√8 + 15√2, and students solved it as 5√8 + 15√2 = 5.2√2 + 15√2 = 10√2 + 15√2 = 25√2. After that he responded to one of the students asking why the exponents of the radicals added up like $x + x = 2x^2$.

[Mr. Casey, 04/24/2003, Lesson 7, 2:30 pm]
S: Why don't you-
Mr. C: Why don't I what?
S: Why don't you add the- add the ...
Mr. C: Radicals?
S: Radicals.
Mr. C: I did.
S: It equals ( )
Mr. C: No, it equals radical two. How much is this? [He wrote $x + x$ =]
S: Two *x*.
Mr. C: Two *x*. Well, you just asked me how come it's not two *x* squared. Because when you add, the exponent don't change.
S: I know-

Mr. C: That's- well, no but it's the same approach. The answer to the question why this is an *x* squared is the same reason why I don't do anything with that.
S: Right.
Mr. C: Because it identifies a particular kind of a radical. So, I have ten of one kind plus fifteen of that same kind and that gives me twenty-five of that one kind of radical; not the- you know-
Mr. C: //you're not multiple-
S: In //the- if you add the-
Mr. C: Yes, but you don't- that's multiplying and we're not multiplying, we are adding.
S: I did two radicals together.
Mr. C: Oh, really. So, but if you add radical two plus radical two, what you get is two radical two. You don't get four.
S: Alright, two point eight, two, eight, four and that doesn't (   )
Mr. C: ... We're not.
S: That's bigger.
Mr. C: Yeah, but we're not finding decimal approximations. We're not doing that.

The student was adding not only the coefficients 15 and 10 but also the radicands and get $25\sqrt{2}^2$ or $25 \cdot 2^2$. She was probably trying to apply a misunderstanding from the generalization that if the radicand were the same, which it was in this case, she could add the radicals. So, she was probably trying to add the radicands as well. Mr. Casey responded by resembling the situation to adding two same things like *x* and *x*. He repeated his approach trying to convince the student who insisted in the same way of thinking. When Mr. Casey suggested that $\sqrt{2}$ plus $\sqrt{2}$ was $2\sqrt{2}$, not $\sqrt{4}$ or 4, she tried to find a decimal approximation to support the later somehow with probably imitating the experience they had with $4\sqrt{3} + 3\sqrt{3}$ previously. But, interestingly, Mr. Casey did not confirm the suitability of using the decimal approximation in this case.

Students' difficulty with adding the radicals was apparent to Mr. Casey from students' answers he got from assessments. In evaluation of results of second quiz, he pointed out in the class that students had serious problems with the following problem, "Add or subtract: $3\sqrt{18} + 2\sqrt{32} - 5\sqrt{50}$". Only eight out of twenty students had the correct answer, $-8\sqrt{2}$, for the problem. He thought that students would not have much trouble since he chosen the numbers accordingly.

[Mr. Casey, 4/29/2003, Lesson 10, 2:22pm]
Mr. C: Ok, now, the last problem. This problem was another one that caused a great deal of gripe. I picked numbers so that when you simplified, all of them had the same radicand. So, I was hoping then you'd be successful in adding them together and some of you were and some of you weren't. You have three radical eighteen plus two radical thirty-two minus five radical fifty [He wrote $3\sqrt{18} + 2\sqrt{32} - 5\sqrt{50}$]. So this is three and then you have nine times two; plus two and this is sixteen times two; minus five and this is twenty-five times two [He wrote $3\sqrt{(9.2)} + 2\sqrt{(16.2)} - 5\sqrt{(25.2)}$]. I'm putting in all the steps. Now, some of you prefer to prime factor all the numbers and then circle to them and put one out upfront and that's ok. But, I'm not doing that, I'm recognizing now my perfect squares to doing working with them so much. Now, the square root of nine is three. So, here we have three times three and the square root of two [$3.3.\sqrt{2}$]. And here we have four times two times four and the square root of two [$2.4\sqrt{2}$]. And here we get negative five and times five and that's square root of two [$-5.5\sqrt{2}$]. So, this is nine square roots of two. Plus eight square roots of two minus twenty five square roots of two. Now, all the radicands are the same. And what we want to do now is add the coefficients. So, this is nine plus eight minus twenty-five square roots of two. and again your addition let you down. There were really strange numbers people got.
S: Hey, I got that one-
Mr. C: Some people multiplied two times four and got six. You know, I can't help you there.
Ss: ( )
Mr. C: No you can talk to me about it. Go ahead.
S: You said since they were all the same...
Mr. C: Radicands?
S: Radicands. You said that ok, you said that we can add them like ( ) we did on that one.
Mr. C: Yeah, but what-
S: So why-
Mr. C: No, when the radicands are all the same what you do is you add the numbers in front. But you gotta keep the radicand, the radical.
Sn: I don't ( )
Sn: No, with the- ok. Ugh.
Mr. C: What step did you go wrong?
S: The uhm, you don't ( ) ☺☺
Mr. C: No. [He went to students and picked up the quiz to see her solution]
Mr. C: Oh, there it is. You have- you multiplied two times four and got sixteen and I'm not sure how that happens. [Possibly she multiplied $2.4\sqrt{2}$ as $2.4.2 =16$] Right there you have two times four. And right there you got sixteen.

After he explained how they could have solved the problem, he expressed that some of the students had arithmetical errors that he could not help with. Later, one of the students had a misunderstanding about adding the radicals when the radicands were same. She was misinterpreting the addition of radicals with the same radicands as adding the radicands also. After explaining that only the coefficients in front of the radical sign should be added and the radical should be kept as it was, Mr. Casey's went on to see in which step she had gone wrong and found out the problem as a mistaken result of a multiplication for which he neither did have an explanation (other than pointing out the erroneous arithmetic) nor did ask to learn for why she did.

The day before the unit test, Mr. Casey expressed his worries about students' understanding of adding radicals. He thought that students were missing an opportunity to ask questions about the things they did not understand. Only two persons out of eighteen were successful in adding radicals in last quiz and he would solve some addition problem if they wanted him to do.

> [Mr. Casey, 05/01/2003, Lesson 12, 2:17pm]
> Mr. C: You guys are making a serious mistake. Because in that review, there were problems where you were supposed to add radicals and on the last test two out of eighteen of you were able to add radicals successfully. And now, I'm here offering an opportunity to review how to add radicals I guess on those homework problems you didn't have any trouble adding those radicals together?
> Sn: I do.
> Ss: ([talking at once])

And as one student admitted immediately, students had problems in adding radicals. This was proven and observed in the end of unit test they took the day after. In his evaluation of the unit test results, Mr. Casey identified writing the answers in simplest form as an issue in adding radicals. He observed that although most of the students were able to write their answers in the simplest form for questions such as finding $10\sqrt{5} - 4\sqrt{5}$ and/or $\sqrt{48} + 5\sqrt{27}$, they were not able to do the same in the problem $2\sqrt{3} - 3\sqrt{(1/3)}$ even though they were able to rationalize the denominator. He identified the problem as lack of understanding in adding fractions:

> When they had to add radicals they simplified, most of them did, when they had a fraction as a coefficient again that's not very many people got that question right because they could not seem to … they can't add fractions, they don't know what it means. (I3)

Furthermore, he also pointed out that students misinterpreted the statement "the radicands have to be the same in order to be added" and thought that √48 and √27, for example, could not be added together since the radicands were not the same even though they could be added together after simplifying them. His point was valid in several incidents as presented above as well.

Multiplying and Dividing Radicals

He was surprised that in the unit test most of the students got the problem $\sqrt{(7a^2b)}\sqrt{(42a^3b^2)}$ right and missed √25√3 by writing the answer as √75. He describes this situation as "pretty crazy" (I3). After the quiz between first and second interview, Mr. Casey thought that students had an acceptable level of understanding in properties of radicals such as multiplying and simplifying radicals.

> They have a fairly good grasp of the properties of the radicals as far as the multiplying and dividing. Well the dividing we're going to check today but simplifying and finding square roots I think that they understand what's going on. (I2)

After the interview he gave the quiz and the next day when he was returning the graded test he made a comment about second question. They had just skipped FOIL and produced their own methods to multiply [$\sqrt{(a - b)}\sqrt{(a + b)}$].

> Mr. C: Ok, problem two [$\sqrt{(a - b)}\sqrt{(a + b)}$]. You're supposed to multiply those things together. Some of you have decided you don't like the FOIL when you're supposed to multiply binomials so you just make up your own rules and you multiply in a variety of ways. The answer is the square root of *a* squared minus *b* squared. [Mr. Casey, 4/29/2003, Lesson 10, 2:12pm]

Student answers where Mr. Casey referred as "made up rules" $\sqrt{(a - b)}\sqrt{(a + b)}$ were as follows:

$\sqrt{(a^2 - b^2)} = ab\sqrt{(a - b)}$; $\sqrt{(a^2 - b^2)} = ab$; $\sqrt{a^2} - b^2$; ab; $a + b$; $a - b$; $a^2 - b^2$; $a^2 - ab - b^2$

From the answers, it was obvious that most of the students were able to multiply (a - b) and (a + b) and get a^2 -b^2 under the radical. However, it looked like they had a problem of accepting this as the final answer and tried to take the square root of a^2 -b^2. In doing so, they misgeneralized the situations where they could take square root of combined terms and got mistaken answers in the forms stated above. Thus, contrary to what Mr. Casey, who was thinking that students had problems in multiplying by using FOIL and making up their own rules, students' difficulty was with the concept of square root and the role of factorization in it. Furthermore, as he commented in students' difficulties in adding and subtracting radicals, students often had hard time to accept lack of closure thinking, "they got to do more" (I3).

Rationalizing the Denominator

Although students were telling Mr. Casey they understood the topics, particularly rationalizing the denominator in radical expressions, he was not sure of that and he would see if it was true in the quiz they were going to have. He thought that students were divided into two groups when they wanted to rationalize the denominator; some students could see that "they need that smallest integer to make that denominator square" (I2) and the others wanted to use the same number. He evidently favored the former group for ease of simplification and preventing certain problems since the second group had to simplify the fractions but they left it as it was without any simplification. For example, they would write that $\sqrt{5}/\sqrt{12} = (\sqrt{5}.\sqrt{12})/(\sqrt{12}\sqrt{12}) = \sqrt{60}/12$ and left the answer as $\sqrt{60}/12$ without going further for simplifying it as $\sqrt{15}/6$. He identified "forgetting all about the criteria for simplified radical" (I2) as a common mistake.

"Why can't you have a radical in the denominator?" (I2) was a question Mr. Casey expected students would ask him during the unit but they did not. He thought this was probably because of how they had been trained. In case someone would ask, he could only tell "it's so that you can get practice of multiplying radicals or something" (I2). The answer to such a question was not a good one for him and so he was glad that no one had asked yet because "an answer is the answer" (I2) and there was no reason for having $1/\sqrt{3}$ rationalized especially with the computer and calculator technologies that would calculate the result if a decimal approximation was needed. He could not think of a reason mathematically either. Because, when he compared the situation with the one in abstract algebra he had taken in college; where it was not a problem to leave radical in the denominator. He expressed his confusion [and possibly disappointment] with the situation to his students as well in a lesson involving simplification of division of radicals.

> [4/28/2003, Lesson 9, 2:06-2:12]
> S: Radical thirty-six over radical one hundred. [They were looking at problem $\sqrt{(36/100)}$ and Mr. Casey commented that this was where they used quotient property]
> Mr. C: Yeah, radical thirty-six-
> S: So it's six over ten?
> Mr. C: Yes, but we don't leave it six over ten, do we? We make it three over five because we can't have fractions that aren't simple. Six over ten is equivalent but you have to take the two out of both of those numbers and it's three out of five when you get all of done.
> Sn: ( ) think square root of ( ) over ( ) simplified?
> Mr. C: Yeah. Yeah because whenever you have a fraction as an answer and it's not in simple form, it's not completely correct. Uhm, I guess. We- we discussed that earlier today about-... You know how you can't have a radical in the denominator?
> Sn: Yeah.
> Mr. C: That's the- one of the rules in this textbook and it's part of the idea that has uhm it's one of the Quality Core Curriculum objectives for algebra one is to be able to rationalize denominators. But when you get to college, it's not a requirement anymore.

After the end of chapter test, he believed that probably half of the class would still use the same radicand to multiply the denominator instead of using the smallest number and then leave it without simplifying (e.g., $1/\sqrt{8} = \sqrt{8}/(\sqrt{8}\sqrt{8}) = \sqrt{8}/8$) thinking that everything was good once the radical had gone. He attributed part of the problem to students' weakness in simplifying fractions and the idea of a common factor in the numerator and the denominator that were never well developed conceptually.

CHAPTER 7: FINDINGS AND IMPLICATIONS

In the previous two chapters I presented the analyses of two case studies. I begin this chapter with findings and discussions of the study that is followed and concluded with the generalizations and lessons learned from the study as I present implications for curriculum development and policy, teacher education, and future research.

Summary and Findings

This study was designed to address and contribute to our emerging knowledge and understanding of teachers' conceptions of student difficulties and related issues with instructional practices in algebra. I have observed and interviewed one eighth-grade teacher and one ninth-grade teacher of Algebra 1 to investigate and understand their knowledge of student thinking and instructional practices in algebra. The following research questions guided the study:

- What is the nature of teachers' professional knowledge about student thinking in Algebra 1? How is this knowledge grounded?
- How does student thinking and knowledge of it inform teachers' instructional practices?
- What are teachers' beliefs about student thinking in Algebra 1?

My initial interest for the study grew out of my master of science theses (Erbas, 1999), where I found that students had many misconceptions even about simple algebraic concepts. I also found that teachers were unaware of student difficulties. I needed to conduct further in-depth research to understand the nature of teacher knowledge of and beliefs about student thinking and how they would affect instruction. I thought that further understanding in this area would influence teacher education and algebra instruction. Because, the often stated importance of algebra in today's highly technological society and its role in the school curriculum, the existence of student difficulties, and the lack of research on algebra teachers bring need for investigating teachers' pedagogical content knowledge, particularly their knowledge of student thinking and its role in instruction.

For the purposes of this study, my conceptual, philosophical and theoretical understanding have been shaped by the perspectives suggested in Teaching-in-Context (Schoenfeld, 1998), Cognitively Guided Instruction (Carpenter & Fennema, 1991; Fennema & Carpenter, 1996), and Pedagogical Content Knowledge (Shulman, 1986, 1987). Since "knowing" is a variety of separate entities, I found Mason and Spence' (1999) distinction between knowing as *knowing-about* (*knowing-that*, *knowing-why*, and *knowing-how*) and *knowing-to* useful to look at different degrees of knowing. Among these, knowing-that and knowing-why was previously mentioned and suggested by others (Shulman, 1986; Even & Tirosh, 1995; Even, Tirosh, & Robinson, 1993) in order to look at teachers' knowledge of particular difficulties and misconceptions students would have learning a topic.

Qualitative case study research design and methodologies were used to generate data to address the research questions. Two mathematics teachers (one eighth-grade and one ninth-grade teacher of Algebra 1) were purposefully selected by networking with experienced experts who recommended them based on their reputation and other. Data collection strategies included: three audio-recorded semi-structured interviews for each participant; video-recorded classroom observations (sixteen lessons of 50 minutes each in one case and twelve lessons of about 90 minutes each in the other case); and evaluation materials (i.e., tests, quizzes etc.). I analyzed the data under four categories: data management, description, analysis, and interpretation and representation.

Data stories were written about each of the two teachers. Thick descriptions included the teacher's beliefs about the nature of mathematics and algebra, the teacher's beliefs about and practices in teaching and learning algebra, the teacher's general knowledge and beliefs about students' thinking and difficulties in algebra, and the teacher's knowledge and instruction in particular algebra concepts that students were observed and presented with.

Some important findings of the study were the following:

- Even though both teachers showed an awareness and recognition of students' thinking and difficulties in terms of "knowing-that" (Even & Tirosh, 1995; Mason & Spence, 1999; Shulman, 1986), their knowledge in terms of "knowing-why" and "knowing-how" was narrow and even problematic in some cases.
- Issues other than conceptual, cognitive, and epistemological problems characterized both teachers' knowledge of and beliefs about general sources of students' difficulties in terms of "knowing-why." Those issues were: lack of arithmetical and geometrical knowledge base, lack of motivation, lack of experience with nontraditional curricula, lack of practice in similar type of problems, carelessness, and inability to understand and apply definitions. However, they were able to give explanations for students' difficulties and mistakes in specific concepts they were teaching.
- Both teachers expressed similar reform oriented, beliefs about the nature of algebra and how it is connected to other parts of mathematics such as arithmetic, geometry and statistics (Bednarz et al., 1996; NCTM, 2000). They saw algebra not simply as the manipulation of symbols but rather as a tool to model and solve real-life problems

(Bednarz et al., 1996; Davis, 1989; NCTM, 1998, 2000; Sharma, 1988; Thorpe, 1989; Usiskin, 1988). Furthermore, both teachers thought the basics learned in algebra (e.g., properties of equations, manipulation of formulas) and arithmetical foundation were an essential foundation for topics learned in higher mathematics and that mastery of those basics ensures success with more complicated concepts and topics. However, both teachers were differed in terms of instructional practices and how their beliefs were manifested in them.

- Both case studies revealed that, for certain reasons, textbook dependence was central to these teachers' practices at different stages of instruction such as planning lessons, assigning homework, and assessing students' learning. This dependence seemed to have a negative effect on the teachers' acquisition and elaboration about knowledge of student thinking.

Discussions of Findings

Teachers' Knowledge of Student Thinking

"Knowing-that": How Teachers Came to Know Students' Thinking and Difficulties

Both teachers indicated that they mostly came to know students' difficulties through direct communication or one-to-one interaction. They both also expressed that having students show their work in detail was important in identifying student difficulties. Observations and interviews suggested that both teachers acquired their knowledge of student thinking in terms of "knowing-that" as they teach. In other words, they recognized particular difficulties as they happened. In certain cases, both teachers presented approaches suggesting they had previous knowledge of the issue. However, this knowledge, again, was from their classroom experience. There was no indication of building such a knowledge domain by gaining access to external resources (e.g., research findings, journal articles for practitioners, discussion with colleagues) or education.

Awareness of several student difficulties was central to Ms. Sands's knowledge of students' thinking in terms of "knowing-that" in the context of patterns and graphs. She recognized that the big misconceptions students had in the unit were in combining like terms (Booth, 1984, 1988; Hallagan, 2003; MacGregor & Stacey, 1997a, 1997b; Tirosh et al., 1998), how to write coordinate pairs (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990), how to use the distributive property and what it meant (Booth, 1984, 1988; Kieran, 1992; Sharma, 1988; Sleeman, 1984; Payne & Squibb, 1990), which she had observed ever since she started teaching algebra; and possibly some issues related to the meaning of positive and negative numbers and operations with them (Sharma, 1988). She also thought students' concept of variable was not really firm yet (Booth, 1984, 1988; Küchemann, 1981; Leinhardt et al., 1990; Leitzel, 1989, Perso, 1992; Sharma, 1988). In particular, students had trouble using letters different from x as a variable and were not really sure what to do if she used b, for example, instead of x. One problem Ms. Sands was aware of with the students' graphing approaches was that they wanted linearity in their graphs (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990). They would connect consecutive points when they were graphing the parabola. She was also aware that students had the notion of graphs as pictures of actual events or subjects (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990) as some students thought graphs might represent a hill, a building, or a series of pictures of a burning candle. On the other hand, in graphing a parabola from a given set of points, Ms. Sands knew that students would have difficulty in seeing the graph as extending infinitely in both directions. Furthermore, she noticed that students had difficulties understanding the meaning of $-x$ when x was a negative number. She also recognized that students would not accept the lack of a numerical answer (e.g., $7 + 2x$) when they could find one even it was not mathematically correct (e.g., $9x$ or 9) (Barnard, 2002, Booth, 1984, 1988; Kieran, 1992; MacGregor & Stacey, 1997a, 1997b).

Since the curriculum was new to her, Ms. Sands was trying to get more familiar with the textbook and the problems by reading and solving them herself so she could anticipate what students might have trouble with. Sometimes she was right with her predictions and sometimes not. At times she was surprised because students had trouble with a particular problem although she did not think that they would. She was aware that she and her students did not see a problem in the same way and that their understanding and approaches would differ. That was why she wanted students to her what they had done. She thought that if students understood, they would say so when she asked, "Do you understand?" However, she was also aware that even they did not understand sometimes they would say they did. So, she did not always know if they were telling the truth or were embarrassed to ask for an explanation. For this reason, getting into students' thinking was important for Ms. Sands, who believed she could help them if she could find their difficulties and misconceptions. She asked "why?" questions a lot in the classroom to get students elaborating on how they knew that they were right and that their ideas made sense. She thought she needed to keep learning to ask questions as a part of her process in getting into her students' minds. She also emphasized that she needed to know enough about what she wanted her students to know so that she could get deeper into questioning to find the source of their thinking and difficulties. She also needed to listen to students talk because she believed that if they were thinking aloud she could catch what they were doing wrong. If they were not really on task, however, it was hard for her to find out what their problem was because it was hard to understand what they were saying or what they meant. Even though she recognized that it was a part of her job, she did not like trying to figure out what students' real deficiencies were because going back and back until finding out what it was they did not know how to do was a very hard thing for her to do.

Mr. Casey's knowledge of students' thinking in terms of "knowing-that" consisted of an awareness of several difficulties students had concerning radical expressions. He was aware that when students were taking square root of a number, they would continue to put the radical symbol around the root of the number (e.g., $\sqrt{4} = \sqrt{2}$) even though the radical symbol should have gone away. In cases where the square root was also a perfect square, they wrote the numerical equivalent as their final answer (e.g., $\sqrt{81} = \sqrt{9} = 3$. Moreover, he noticed that students did not understand very well the need for absolute value in $\sqrt{x^2} = |x|$ (Abromovitz, et al., 2002; Even & Tirosh, 1995; Kepner, 1974). He was aware that to get correct answers, students were just following the rule (Herscovics, 1989), which was to have the absolute value when there was a variable included in the radical expression. He recognized that finding domains of radical expressions (i.e., ranges of values making the radical value real) was problematic for students (Atherton, 1971). Furthermore, Mr. Casey anticipated that simplifying radicals was going to be a topic where students would have difficulty because when students came across questions like simplifying $\sqrt{112}$, they would make a decimal approximation without noticing the existence of perfect squares as factors. In adding radicals, Mr. Casey recognized that students tended to add radicals with different radicands by writing them as a sum under a single radicand (i.e., $\sqrt{a} + \sqrt{b} = \sqrt{(a + b)}$) (Barnard, 2002). He also considered that students often had hard time accepting lack of closure, feeling they had to do more (Barnard, 2002, Booth, 1984, 1988; MacGregor & Stacey, 1997a, 1997b). Furthermore, he also recognized not having a simplified answer as a final result was common mistake students made in rationalizing denominators.

Because of his long experience in teaching, Mr. Casey felt pretty confident in identifying misconceptions or where students went wrong provided they showed their work to him. It was not his policy, however, to force students to show him their work, because he thought there would be more students failing due to the pressure if he did. He personally liked it when students showed him their work because it gave him a better idea of what was going wrong. It also enabled him to recognize if somebody had a unique way of solving a problem that he did not think of it so he could put it in his notes to be used as a source to get his points across in another class or at another time. He also used his notes from prior years to remember what kind of difficulties and other issues were associated with the lesson if there were any. Once he identified where the students had mistakes, errors, and misconceptions in the past, he would try to lead students away from the same kind of fallacies with a statement warning that the situation might be tricky made to get students' attention about possible mistakes. He thought, however, that students would always find an answer reflecting the same difficulties since they tended to go straight to the answer before thinking about and planning for a solution. Thus, he recognized students' errors and misconceptions as a valuable contribution to his instruction and students' learning (Nesher, 1987; Olivier, 1992), as he purposefully let students make mistakes on problems. Using guided discovery, he would formalize the solution after students are done to help them to see why their answers or solutions should have been different or corrected. This strategy also helped him identifying students' misconceptions and getting them to take active participation in what was going on.

"Knowing-Why": Teacher Knowledge of Sources of Student Thinking and Difficulties in Algebra

The findings of this study suggested that the teachers were more concerned with factors other than the nature of algebra itself and/or conceptual difficulties associated with it as fundamental sources of student difficulties in algebra as it was suggested in literature (e.g., Booth, 1984, 1988; Clement, et al., 1981; English & Halford, 1995; Herscovics, 1989; Kieran, 1989, 1992; Küchemann, 1981, Leinhardt et al., 1990; MacGregor & Stacey, 1993, 1997a). Both teachers thought of factors such as students' lack of arithmetical and geometrical knowledge base, lack of motivation, lack of experience with nontraditional curricula, lack of practice in similar type of problems, carelessness, and inability to understand and apply definitions as general sources of students' difficulties in algebra. These views also indicated inadequate level of pedagogical content knowledge these two teachers possessed in this context.

Even though there were contextual and demographic differences between the two teachers, they both indicated that the lack of certain knowledge base and the lack of motivation were two main barriers to success in Algebra 1. Although both teachers mentioned the lack of arithmetic proficiency and its persisting difficulties were carried over to algebra as a blocking factor in successful learning, only Ms. Sands talked about geometry since the curriculum she was using emphasized connections between algebra and geometry. Both teachers' emphasis on lack of arithmetical proficiency as a barrier for learning algebra suggested that the teachers believed arithmetical understanding precedes algebraic understanding, despite research studies have shown it should not be necessarily true (Kieran, 1992; Nathan & Koedinger, 2000a, 2000b). Another source of difficulty both teachers expressed was lack of motivation. Mr. Casey considered all of his students in the class as smart and capable of mastering the objectives of the course and earning good grades. However, he identified the main sources of high failure rate or student difficulties in algebra as; lack of desire, lack of goals, and lack of intrinsic or extrinsic motivation. Both teachers identified three main sources for the lack of motivation. First, most of the students lacked motivation influenced by their parent's beliefs about algebra. Second, students lacked motivation because they did not see a use for learning algebra and resisted to learn it. The third reason was related to sociology as students at this age had social issues with their life that they valued more than algebra and took their focus away from it.

Other than lack of knowledge base and lack of motivation, the two teachers expressed other reasons for why students would have difficulty in learning algebra. Ms. Sands thought the current curriculum (i.e., the College Preparatory Mathematics) was a source of struggles and a challenge for the students and herself because of their transition from a traditional approach where the emphasis was on computations and procedural learning to a more reform-based approach where the emphasis was on the problem solving and conceptual learning. Furthermore, students' lack of experience with the curriculum did not allow them to see how individual problems in a unit evolved around a bigger problem and constituted a bigger concept. She believed students would make sensible connections not only among the problems but also between algebra and other areas of mathematics as they become more familiar with the CPM philosophy and its problem-solving approach. Furthermore, she identified two sources to what she attributed students' difficulties and mistakes in algebra. On one hand, she thought one source was students' lack of practice in solving similar types of problems that she did not provide. On the other hand, carelessness and acting without reasoning no matter what she did were a second source of problems but for which she did not have obvious solutions. She believed students know more than they could show and that most of their mistakes were due to the carelessness. In contrast, Mr. Casey considered students' biggest difficulty in algebra as their inability to learn a definition and apply it. He thought students could not make connections by starting first with the definition and then trying to build on it. He suggested to students that learning of a definition means to memorize it, not necessarily comprehend it. Mathematically speaking, he suggested that definitions were mathematical conventions to operationalize the idea, and one would just need to accept them as they are.

In some cases, both teachers had context specific explanations for why students would do certain mistakes or have certain approaches. For example, Ms. Sands expressed that students prior experiences with finding hidden shapes by connecting numbered dots that was common in dot-to-dot games in coloring books might have an significant effect on students' tendency to look for linearity in graphs and thus connecting consecutive points. This brought up the role of students' prior experiences and knowledge to students' ways of graphing in this context (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990). Ms. Sands's knowledge of why in this context recognized students informal and self-constructed techniques into algebra classrooms where more formal methods are developed (Booth, 1988). Thus, her instruction recognized, valued and incorporated students' prior knowledge and informal solution methods with a combination of opportunities for student interaction and discussion (Boaler, 1998; Booth, 1988; Fennema, et al. 1993; Filloy & Rojano, 1989; Swafford & Langrall, 2000; Thompson, 1988). On the other hand, there were also incidents where students had difficulties because of their prior experiences that Ms. Sands did not recognize. During the discussion of dividing 1 by 0, although some of the students suggested that zero had no value so that was why they were getting error messages on the calculators or the division was not possible, some of the students believed that they could divide 1 and 0 even though somehow their calculator was not doing it. They probably were reminded about their problem with the calculators when they were doing $(-x)^2$ and $–x^2$ and the special way of entering it to different calculators or getting the error messages when they were taking the square root of a negative number. So, there could be a way to get calculators around this problem and make the division possible and get rid of the error messages. It did not look like Ms. Sands was aware of any possible effect of students' past experiences with calculators and error messages they encountered in the past.

As Matz (1980) pointed out, students' errors with algebraic algorithms are often due to learning or constructing the wrong idea, not because of failing to learn a particular idea. However, in this study the teachers often attributed students' failing to learn a certain concepts or carelessness as a source of a difficulty or errors. Such an example included both participant teachers' reasoning about students' mal-rules (Payne & Squibb, 1990; Sleeman, 1984).

Ms. Sands considered students' response for $2x(3 - x)$, and $-4y(2y - 7)$ as indicating a state of confusion even though they did well with $5(x + 2)$ and $3(y - 1)$. She thought this confusion occurs because the students simply had not mastered the distributive property and committed mistakes due to carelessness. This view contrasts with that of Sleeman (1984) and Payne and Squibb (1990), who interpreted such errors as mal-rules (or bugs), student's mental representations, and applications of faulty procedures.

One of the test questions about multiplying radicals was significant in terms of Mr. Casey's perception of some of the errors, mistakes as "made up rules" as in the case of adding and subtracting radicals where students wrote $\sqrt{a} + \sqrt{b} = \sqrt{(a + b)}$ (Barnard, 2002). In a similar situation, he thought students had just skipped the FOIL and made up their own rules or produced their own methods to multiply $\sqrt{(a - b)}\sqrt{(a + b)}$. Contrary to what Mr. Casey thought, students' difficulty was with the concept of square root and the role of factorization in it causing mental representations and applications of faulty procedures (Payne & Squibb, 1990; Sleeman, 1984). Furthermore, this was also a case where students had a hard time to accept a lack of closure, thinking they had to do more (Barnard, 2002, Booth, 1984, 1988; MacGregor & Stacey, 1997a, 1997b).

"Lack of understanding" was the kind of explanation both teachers favored as a reason that students would make certain mistakes or make up rules. For example, while Ms. Sands interpreted students' tendency to conjoin algebraic expressions (Booth, 1984, 1988; Barnard, 2002; Hallagan, 2003; MacGregor & Stacey, 1997a, 1997b; Tirosh et al., 1998) as a place where they showed lack of understanding of combining like terms, Mr. Casey thought it was the lack of understanding of square root concept causing to invent new mathematics without proving it in cases like $\sqrt{4} = \sqrt{2}$ or $\sqrt{81} = \sqrt{9} = 3$.

Both teachers expressed concerns about calculator dependence as barrier limiting creative and independent thinking and conceptual learning. Ms. Sands often discouraged students for use of calculators and trust their knowledge unless they were suggested to use in a problem. For example, she thought that using and depending on the calculators in students' attempt to understand the meaning of x^2 moved their focus away from the problem, as they were more concerned about why their calculators gave errors. She believed that if they were not using calculators and directly taking the square of a negative by multiplying itself, they would know better what x^2 meant. Similarly, Mr. Casey thought lack of proficiency with arithmetical or number operations and calculator dependence for doing these were issues limiting creative and independent thinking.

"Knowing-How": Teachers' Ways of Dealing with Students' Thinking and Difficulties

Both teachers differed in their focuses when dealing with student difficulties probably because of how they organized their classroom for teaching and learning. In Ms. Sands's case, usually, the whole group was her attention rather than the individual. She looked for groups having most trouble when they were working on their assignments. If the group members were working and not having trouble she would continue to look for until she found somebody who needed her help or asked group members and they could not come up with anything. She thought that she needed to check particularly the lower achieving students and pair them up with someone learning up topics quickly so that they could have a chance to talk to somebody and ask questions. She would talk more to the students who had difficulties and misconceptions to make sure they did not have problems. In most cases, what the students wanted to know was whether they were on the right track on the first place. She usually started helping by asking questions concerning what they had tried to do to solve the problem. If they were doing something right, she would ask them what was wrong with it or why they were not satisfied with their answer. Usually it was the students themselves who would come up with a solution or explanation. She found it amazing that most of the times letting students reread the problem would reveal the answer to their own question. Thus, she often suggested that students reread the problem in order to understand it so that they could solve it.

Individual difficulties had no affect on the pace of Mr. Casey's whole class instruction. He thought he could offer help for remediation before and after school for those students. He usually spent the time when students work on their homework in the class to monitor and help students having difficulties. If it was a group of students having difficulties with a particular content, he would then stop and reinstruct by first diagnosing the problem and its causes. He would discuss the situation with the students in order to identify the mistakes, what was causing the error in the concept development or arithmetic, and what was stopping them from getting the right answer. He would also discuss monitoring techniques they could do on their own. For example, he thought that really quick use of FOIL to multiply back would lessen the mistakes in factorization by making sure the expression was factored properly. His differentiation between misconceptions as conceptual issues and mistakes characterized as arithmetical errors affected how he would act in either case. He thought it would be relatively easy to deal with a plain mistake or arithmetic error because it would only require warning students that they needed to be more careful. If it was a misconception, on the other hand, they had to go back and discuss what it means through the definitions. He thought that dealing with misconceptions was a matter of time, practice, and corrections. He also considered finding some manipulative to serve as a visual or concrete aid to help students see more than the abstract idea so they could understand how a definition would work.

Both teachers presented approaches in trying to help with certain student difficulties that would be considered problematic in the research, as they might lead to other erroneous thinking in the long term. Such a case appeared in the teachers' use of "fruit salad" analogy or metaphor when they tried to show adding algebraic and radical expressions by representing mathematical objects (numbers and variables) with fruit names such as apples, oranges, and bananas (physical objects) in an attempt to show like and unlike terms. Although neither of the teachers saw this as potentially questionable, beyond causing potential erroneous thinking with variables, research studies pointed such approaches as problematic since they might obstruct the transition from procedural to conceptual thinking (Kieran, 1992; McGregor & Stacey, 1997; NCTM, 2000).

In Ms. Sands's case, as a response to students' tendency to conjoin algebraic expressions (Booth, 1984, 1988; Barnard, 2002; Hallagan, 2003; MacGregor & Stacey, 1997a, 1997b; Tirosh, et al., 1998), she emphasized the difference between unlike terms and variables by using the *fruit salad* analogy or approach (Tirosh, et al., 1998) where she labeled x and y as apples and stakes. She thought root of the problem was the term "variable" and the meaning associated with it. However, this problem often referred in research literature as inability to conceive the objects of manipulation (e.g., $2x + 1$, a^2, $\sqrt{t^2+1}$) as meaningful things in their own right and look for a closure (Barnard, 2002, Booth, 1984, 1988; MacGregor & Stacey, 1997a, 1997b). Within this context, when I asked if she had observed, particularly in pre-algebra, students thinking algebraic letters as unit of measures as they used to do in arithmetic like 8*m* for 8 meters or 5*l* for 5 liters; or as labels for quantities of some objects like 8*b* for 8 bananas or 5*a* for 5 apples (Booth, 1984, 1988; Küchemann, 1981; Perso, 1992), she thought it was possible that there would exist students thinking in the latter way even though she did not see the former situation. Although she was not sure about it and expressed a need for thinking, she thought maybe students should think that way or call it some objects (e.g., 8b as 8 bananas) so that they would not combine things that did not belong together. She was not aware that *fruit salad* analogy using variables in literal terms as labels were problematic and may bring more harm than benefit in the long term even though it would be a quick solution at the moment as the research literature on students' thinking in algebra pointed out (e.g., see Booth, 1984, 1988; Küchemann, 1981; Perso, 1992; Tirosh, et al., 1998). In Mr. Casey's case, he introduced an alternative approach to the sum of radicals where he suggested considering $\sqrt{3}$'s in $4\sqrt{3} + 3\sqrt{3}$, for example, as something else like oranges. Although he was trying to get students to see $\sqrt{3}$ as an object, rather than something to be calculated by this "fruit salad approach" (Tirosh, et al., 1998), Mr. Casey, like Ms. Sands, was not aware that his *fruit salad* analogy might be problematic in different contexts and may bring more harm than benefit as the research literature on students' thinking in algebra pointed out (e.g., see Booth, 1984, 1988; Küchemann, 1981; Perso, 1992; Tirosh, et al., 1998).

As a cognitive obstacle, misgeneralization and justification are among major sources of students' difficulties in algebra, As Maurer (1987) observed students had tendency to generalize by interpreting their lessons through their views, however, their methods are often faulty. Several incidents highlighting the importance of being careful about making generalizations happened in this study. Although it was the teacher who made generalizations in Mr. Casey's case, it was a student in Ms. Sands's case.

Probably affected by the fact that all of the numbers greater than 1 were whole numbers in the set of values for x given in the problem, one of the students made an overgeneralization (Barnard, 1989; Hallagan, 2003; Kieran, 1990, 1992; Laursen, 1978; Parish & Ludwig, 1994; Margilues, 1993) that if x was a whole number $1/x$ was less than 1. Based on her response and accepting it as a plausible conclusion, however, it did not look like Ms. Sands realized the origin of this observation and possible deficiencies with it, if there was any.

In his explanation of finding domains of radical expressions, encouraged by the students' thinking, he made a generalization about the cases that if they had x squared plus a number as the radicand, then the answer was all real numbers, whereas if it was x squared minus a number, then the answer would be a disjunction. Even though he was talking about the cases $\sqrt{(x^2 + a)}$ and $\sqrt{(x^2 - a)}$ in particular, students might remember and implement the same generalization in situations where there would be additional terms beside $x^2 \pm a$. From that aspect, Mr. Casey's generalization would be problematic and would have negative impact on meaning leading to overgeneralizations. This is, however, not to say that it was a wrong thing to do. Maybe, he should have presented some other examples emphasizing what he really meant and where those generalizations would be usefully and appropriate to apply. Several students' answers to finding domains of $\sqrt{x}$ and $\sqrt{(x\text{-}5)}$ as "all real numbers," "any number," "all perfect squares" in the end of unit test were suggesting such a point. Several students also used absolute value brackets around the variables or numbers when they stated their answers. This suggested that students had problems with the concept of radical root and the need for absolute value. Another incident of generalizing happened when Mr. Casey was talking about addition of radicals. As a response to a student unsatisfied asking *why* the result was $7\sqrt{3}$ ($= 4\sqrt{3} + 3\sqrt{3}$), Mr. Casey generalized the idea that the radicands had to be the same in order to be added and gave another example to illustrate his definition or rule for adding radicals. When students asked about adding radicals with different radicands such as $\sqrt{5}$ and $\sqrt{3}$, they had a hard time understanding and accepting Mr. Casey's statement "can't be added if the radicands are not the same" as an answer and looked for a way or a special symbol to state the final result or make a closure (Barnard, 2002, Booth, 1984, 1988; MacGregor & Stacey, 1997a, 1997b) rather than thinking the answer as the same thing they started with. Mr. Casey statement "can't be added if the radicands are not the same" would have lead to a wrong generalization (Barnard, 1989; Hallagan, 2003; Kieran, 1990, 1992; Laursen, 1978; Parish & Ludwig, 1994; Margilues, 1993) because it did not emphasize simplifying radicals first and then looking if the radicands were the same. Later in the same class, one student's suggestion, which was to add not only the coefficients 15 and 10 but also the radicands in $10\sqrt{2} + 15\sqrt{2}$ and get $25\sqrt{2^2}$ or $25{\cdot}2^2$, pointed out possible consequences of having such generalizations misunderstood. She was probably trying to apply a misunderstanding from the generalization that if the radicands were the same, which it was in this case, she could add the radicals and so she was probably trying to add the radicands as well as the coefficients.

"Knowing-to": Acting to Help with Students' Thinking in the Moment

As Mason and Spence (1999) discussed *knowing-about* (i.e., *knowing-that*, *knowing-why*, and *knowing-how*) is not sufficient to respond in particular situations. An active knowledge, *knowing-to*, is desired to present itself in the moment bringing the relevant knowledge so that it can be acted upon when it is need. It is the knowing-to act that brings knowledge and practice together so the knowledge can be demonstrated to be useful or otherwise. In this study, teachers presented limited knowledge in terms of *knowing-to* when they were dealing with student thinking in the class.

Mason and Spence (1999) argued that although the situation activates acts, those activations are based on personal psychology and forms of justification and confirmation of knowing may lead to inert or active knowledge. Such a situation was apparent with Ms. Sands's approach to students' thinking of graphs as pictures of actual events or subjects (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990). Even though she was aware and observed this notion, she did not make any direct comment on whether it was right or wrong to thinking a graph as a picture. She encouraged students to feel free to look at a graph from different angles and write if they saw something else rather than just a mathematical figure. Even though this was an act of valuing students' alternative thinking, the long term effects of such approaches. In her approach, her beliefs about not to give direct feedback about what to do (not that she did not know) because of her beliefs about the textbook and its spiral approach that she thought students would come to know certain things step-by-step and student should be the one to discover it. In other words, students should not necessarily understand everything the first time it was presented because it was always built on some basic thing. So, when she observed students having difficulties with a particular problem or idea, her planning strategy was to read ahead to see if the same topic or concept was going to be presented later. In case of Mr. Casey, his beliefs lack of concept of square root as the common sources of students' difficulties in radicals and using the definition of square root in responding and helping to students with difficulties. Two incidents were significant in terms of characterization of Mr. Casey's recognition of and dealing with students' thinking related to concept of square root. Upon a question from a student requesting help to confirm his answer since he was not sure if it was 8 or 8^2, Mr. Casey confirmed the student's initial answer, 8, as correct. Without asking him why he would think that the result was 8^2, Mr. Casey disconfirmed the correctness of it as an answer along with another possible idea (i.e., the radical 8 as an answer) and immediately reminded him about the definition of square root as one of two equal factors. He then restated that 8 was the answer because it was one of the two equal factors convincing the student about the correctness of the answer. Later on in the same day, he was checking answers of another student who was trying to take square root of 1000 as 500. In his reaction, Mr. Casey first indicated that the answer had to be mistaken and then he asked the student to take square of 500 [expecting that student would notice why her answer was wrong according to the definition of the square root] by using her calculator. His further explanations indicated that he was able to identify what the student was doing (i.e., dividing by two instead of taking one of the equal factors as the square root) and she did the same thing in some other computations. Although he noticed that the student had a problem with the meaning of square root, he never tried to ask and reveal what and why the student was thinking about the meaning of square root and how to compute it.

Even though Mason and Spence argued *knowing-about* would not be sufficient for *knowing-to*, the cases in this study suggested that having knowledge in terms of *knowing-that* and *knowing-why* in particular concept played an important role in preventing or leading to *knowing-to*. An example to the former can be found in Ms. Sands's case. During a discussion aimed to promote students' awareness for domain and range in of domain and range of $y = x^2$, even though students were not real sure about it when Ms. Sands asked them if it would work for any value of x, their agreement with the possibility of using any number for x came from their un-contextualized concept of variable, that is-- a variable could be any number because it varies so does the x in $y = x^2$ as a variable (Booth, 1984, 1988; Küchemann, 1981; Leinhardt et al., 1990; Leitzel, 1989, Perso, 1992; Sharma, 1988). It looked Ms. Sands did not recognize this thinking as related to their concepts of variable as they saw x as an independent variable rather than thinking it in the context as a domain of $y = x^2$. When she tried to direct student to observe if any value for y could result from the rule, affect of concept of variable was apparent as they thought the direction of graph would change for different values of x.

On the other hand, Ms. Sands's knowledge of student tendency to graph by connecting consecutive individual points came into action as a warning point or preventive measure as a part of knowing-to in a problem where there was an extraneous point somewhere in the graph that did not belong to the pattern and students had to figure out why it was there. She emphasized that they needed to look at the problem from a pattern or holistic perspective so that they could see all but one point would fit in (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990). Probably because of assuming or observing that the student was connecting all the points without looking at the general pattern, she warned all students about connecting points to draw the graph as if it was a dot-to-dot puzzle (Leinhardt et al., 1990).

An example to how *knowing-about* would enhance teachers' knowledge in terms of *knowing-to* was evident in Mr. Casey's case. Because of his knowing-about students' tendency to add radicals with different radicands by writing it as a sum under a single radicand (i.e., $\sqrt{a} + \sqrt{b} = \sqrt{(a + b)}$) (Barnard, 2002). He successful transferred this knowledge into his instruction attempting to solve this misconception and get students in the right direction. He valued students' thinking as a result of his knowledge in terms of knowing-to as he knew how he could use it as a powerful demonstration of his points. When they were adding $\sqrt{48}$ and $\sqrt{27}$, he did not disregard the students' solutions even though he was interested in the student's procedures more than in what the student was thinking or why he did what he did probably because of trying to see the gaps in the student's erroneous thinking in his explanation of procedures. During the discussions, Mr. Casey respected students' thinking without first saying that it was wrong. He used and followed student thinking by correcting it along with highlighting the points where it had flaws. Aligned with the students' thinking, the calculator activity helped Mr. Casey to challenge the proposed ideas by showing what they started with was not the same thing as what they ended up. Even though this remedy would not be long-lasting if the abstract expressions were not meaningful to the students (Barnard, 2002), it convinced students that there had to be a problem with their thinking and they needed to reconsider their approaches.

Textbook Dependence in Algebra Instruction

Textbook dependence was central to the teachers' practices at different stages of instruction during this study. Although both teachers were similar in terms of beliefs and views they expressed about mathematics, algebra, and teaching and learning of them, they were different in their instructions as they both followed textbooks in different ways. In other words, contradictions existed between what teachers stated they believe and what they did. This finding is consistent with research documenting that teachers' professed beliefs do not always match their instructional practices (e.g., Cohen 1990; Cooney, 1985; Fennema & Franke, 1992; Haimes, 1996; Pajares, 1992; Raymond, 1997; Schoenfeld, 1998; Thompson, 1992; Vacc & Bright, 1999). Table 4 shows comparison of both teachers in terms of some of their important views and beliefs and the corresponding actions in practice or instruction.

The observations in this study confirmed the suggestions in literature that textbooks serve as a major source of content and instructional activities (Ball & Cohen, 1996; Nathan & Koedinger, 2000a, 2000b). Even though both teachers possessed reform-oriented beliefs about the nature of algebra and its teaching and learning and had indicated beliefs that the textbook is

Table 4

Comparison of Ms. Sand's and Mr. Casey's beliefs and actions

	Ms. Sands		Mr. Casey	
	Beliefs	**Action**	**Beliefs**	**Action**
Mathematics	▪ problem solving ▪ meaningful applications and modeling of real life situations	▪ problem solving ▪ textbook problems ▪ teacher produced real life problems	▪ concrete, well-defined, form of science ▪ a tool to model real life situations to predict and solve problems	▪ traditional textbook problems ▪ application problems from the textbook
Algebra	▪ problem solving ▪ part of general mathematical sense and connected to arithmetic, geometry and statistics	▪ problem solving ▪ connections to geometry and other parts of mathematics	▪ generalized arithmetic ▪ no different that other parts of mathematics	▪ generalized arithmetic ▪ connection limited to application problems
Teacher	▪ guide and facilitator of student thinking ▪ learner in the process	▪ guide and facilitator of student thinking thorough questioning ▪ learner in the process	▪ guide and facilitator	▪ holder of the information
Instruction	▪ guiding through asking questions ▪ group work ▪ problem solving ▪ discussions	▪ discussions ▪ group work and interaction ▪ guiding through asking questions ▪ problem solving	▪ use of technology for problem solving ▪ through activities and guided discovery ▪ open-ended problems	▪ teacher centered ▪ traditional textbook problems ▪ mass practice ▪ calculator use to make calculations and check answers.
Textbook	▪ not the curriculum ▪ BUT had to be followed to be consistent with the general curriculum and some other reasons.	▪ curriculum ▪ source for planning lessons, assigning homework, and assessing students' learning	▪ insufficient to promote applications, open-ended problems, discussions, and better examples ▪ a guide and outside materials can be used to support and supplement ▪ BUT had to be followed for some reasons.	▪ authority for content ▪ source for planning lessons, assigning homework, and assessing students' learning

not the curriculum and they should use it as a guide, their dependence on the textbook was strong and made sense to them as they supported it by several reasons. As opposed to Barko and Shavelson (cited in Nathan & Koedinger, 2000b), who expressed a concern that textbooks and teacher manuals may interfere with teachers' decision-making processes and serve as a pedagogical expedient for novice teachers, this study showed that it may also be the case for experienced teachers as long as they make sense of such a dependence. The textbook she was using matched with Ms. Sands's beliefs and it was easy to depend on the textbook for her. However, it was a mismatch, creating a dissonance (Nathan & Koedinger, 2000a, 2000b) for Mr. Casey who would still made sense of dependence on the textbook for some reasons.

In Ms. Sands's case, although the school administration emphasized that the textbook was not the curriculum and they had positive attitudes and policies promoting students' learning as their priority in education, she did not feel like she could teach or go out of the textbook. She believed she had to follow the textbook to be consistent with the general curriculum. Since it was problem solving oriented, she did not want to go out of the textbook and teach them something they needed to figure out by themselves. Her beliefs about nature of mathematics as sequential as it would require a knowledge base as well as what she thought of the CPM's problem solving approach affected how she looked at the textbook and become very fond of it. Thus, textbook dependence had mostly positive interactions with her beliefs about teaching and learning of algebra since she saw the textbook as a means of promoting an atmosphere to practice her beliefs.

In Mr. Casey's case, the textbook was contrary to his beliefs and he was dissatisfied with it since it did not give students much opportunity to try and explain what they were doing. It also, in his view, lacked applications, discussions, better examples, and open ended problems that would challenge students' thinking and ask for reasoning and explanations for the concepts they were learning. Despite his dissatisfaction with the textbook, he thought once a textbook was determined by the department he could use it as a guide and he would use materials from outside the textbook to support and supplement the textbook material. He felt he had to follow it for three reasons: (1) difficulty of trying things more on a conceptual level other than *mass practice* or drill-and-practice with the group of students he had in the class; (2) meeting the state (Georgia Department of Education, 1997) and national (NCTM, 1989, 2000) objectives, as the state objectives, in particular, mandated him what to cover and how to cover; and (3) possible parental and administrative pressure if he do something outside the book or curriculum.

The culture of the mathematics department was very much influential on Mr. Casey's practices. He did not change the curriculum problems and activities to better suit his personal beliefs and goals even though he had reform oriented perception of algebra and had beliefs and goals in teaching and learning of it. This was unlike what Llyod (1999) proposed, that reform oriented teachers using traditional materials may afford more room for personalization. Mr. Casey was unable to do that because of the reasons he expressed for depending on the textbook. Previous research suggests that teachers do not necessarily follow the intended reform based curricula and the impact becomes minimal because of their traditional beliefs affecting both the content and the instruction, influencing both what they teach and how they teach (Cooney, 1985; Fennema & Franke, 1992; Haimes, 1996; Schoenfeld, 1998; Thompson, 1992; Vacc & Bright, 1999). The cases in this study, however, suggest that having a reform oriented mind may not itself be enough for effective implementation of reform curricula unless they are supported to isolate certain external and internal concerns and bring students' thinking forward in the sequence of importance in instruction. Otherwise, meeting those concerns becomes priority in teaching and learning rather than students and their thinking as the focus of the teaching and learning process. Even though it was probably unintentional, evidently this was the case as a consequence of textbook dependence and the reasons behind it for both teachers.

Textbook dependence seemed to serve as a barrier for the teachers in this study to acquire, process, and deal with knowledge of their students' learning. It often hindered the teachers' knowledge about students' thinking and/or getting this knowledge. For Ms. Sands, for example, it was her responsibility to assists students with their thinking process in solving a problem without directly telling them what the answer was. This notion affected how she approached understanding and responding to student thinking as she did not want to go out of the textbook and teach them something they needed to figure out by themselves. She felt it was hard to get over telling too much when she was teaching or helping them with their problems. She seemed to think students would resolve their difficulties eventually and reach to an understanding after having been exposed to the same topic in a gradually expanding manner towards a big idea.

Two examples of textbook dependence from two cases are significant to examine as they show how textbook dependence may have served as a barrier for the teachers in this study to acquire, process, and deal with knowledge of their students' learning and lead teachers unintentionally ignoring and even reinforcing students' difficulties. The first example is from Ms. Sands's practice. Even though she had a through knowledge of students' tendency to desire for linearity in their graphs (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990) and she was also aware that students had the notion of graphs as pictures of actual events or subjects (Herscovics, 1989; Kieran, 1992; Leinhardt et al., 1990), she disregarded or did not notice and/or bring up any issues regarding the language the textbook was using when asking students to graph after completing tables with a rule. They were often asked to "plot and connect the points" after completing the table for a given rule through the unit. Probably just imitating the textbook, she also used a similar vocabulary, "*connect the dots*," on several occasions. This type of instruction might have misdirected students (Hallagan, 2003) acting to perceive graphing as drawing in order to find a certain picture in it. It was evident that some of the students thought doing tables and graphs was in order for leading them to "graphing shapes." However, it was not clear what they would mean since Ms. Sands did not push this thinking or idea forward and ask what they were thinking. Her use of certain vocabulary such as "to draw pictures in math" influenced by the textbook might also have misled students to think that graphing is drawing pictures mathematically.

The second example is from Mr. Casey's practice. He noticed students did not very well understand the need for absolute value in $\sqrt{x^2} = |x|$ (Abromovitz, et al., 2002; Even & Tirosh, 1995; Kepner, 1974) and he had a dilemma as he did not know what he should pursue as he confronted by the situation; understanding of why there was a need for absolute value versus being able to apply the absolute value as a rule and get the correct answer. Having tried to explain the why part to whole class and then individually at several occasions and still observing the same problem repeated, he could not decide if the former was really important if they were able to use the rule and get the correct answer. He thought for most teachers, the situation would be easily resolved by *forgetting* about the absolute value and possibility of having a negative root. Even though it looked like he desired a conceptual understanding, he used the textbook as a reference when students had to decide if absolute value was needed in $|x|\sqrt{6}$ as a result of $\sqrt{42x^4}/\sqrt{7x^2}$. Observing that the authors did not have the absolute value written around the variable "for some reason," he decided it was no longer important for him whether students used it or not. The situation probably added to students' general confusion and problems about having absolute value when variables were involved in radicals (Kepner, 1974). Whether they needed to use absolute value or not become a hard decision to make. This was apparent when one of the students asked Mr. Casey about his confusion because of the instructions about variables being nonnegative and if he needed to use the absolute value. Mr. Casey responded that they did not need to worry about the absolute value anymore and they could forget about it because the book indicated the variables were non-negative. Even though Mr. Casey's argument for not worrying about the absolute value was well suited to the context that the variables were nonnegative and seemed to fulfill students' desires, it was problematic as it was sounded like the use of absolute value was a temporary situation and it remained in the past. This would lead to other problems in the future as a result of devaluing the importance of having absolute value such as the ones presented by Abromovitz, et al. (2002), Even and Tirosh (1995), and Kepner (1974). On the other hand, textbook dependence as the way it was used in this context served as an answer favoring a rule or a procedural approach to his dilemma of *instrumental* versus *relational* understanding (Skemp, 1987). As Sfard (1991) reminded us, operational and structural conceptions are not mutually exclusive in nature and one can observe both interplaying in a single mathematical activity. However, *reasons without rules* can be as dangerous and inappropriate as *rules without reasons* (Sfard, 1991).

A blind dependence on the textbook in different stages of instructions may bring harm rather than good. A recent review of algebra textbooks by the American Association for the Advancement of Science (AAAS, 2000) found that seven of the twelve textbooks evaluated by Project 2061 were considered adequate, however, not one was rated highly. Five textbooks, including three that are widely used in American classrooms, were rated so inadequate that they lack potential for student learning. Two findings from the review are: no textbook does a satisfactory job of providing assessments to help teachers make instructional decisions based specifically on what their students have--or have not—learned; and no textbook does a satisfactory job of building on students' existing ideas about algebra or helping them to overcome their misconceptions or missing prerequisite knowledge. As the AAA pointed out, algebra textbooks may not be sufficient in many ways. Furthermore, discussion between Even and Tirosh (1995, 1997) and Goel and Robillard (1997) suggested teachers should be more careful and not solely rely on the information provided by textbooks since different books might offer different perspectives for same concepts or definitions.

Implications and Suggestions

This study looked at teacher knowledge in two existing classrooms. It was a study to understand processes teachers in real classes acquire, process, and deal with knowledge of their students' learning. The findings of this study have implications for teaching mathematics, learning to teach mathematics, and future research.

Implications for Teaching Mathematics and Algebra

Teaching mathematics should incorporate students' thinking (Boaler, 1998; Fennema, et al., 1993; Filloy & Rojano, 1989; Swafford, & Langrall, 2000; Thompson, 1988) and help them overcome their misconceptions or missing prerequisite knowledge. Teachers need to be sensitive to their students' thinking and develop ways in which to confront certain misconceptions and difficulties other than continuing with the correct solution or expecting students would solve it in the future as they develop their knowledge. While doing this, furthermore, instruction should not include approaches and vocabulary identified as problematic as they might promote or lead to other erroneous thinking and difficulties. Such examples may include, but not limited to, vocabulary like "connect the dots" and "mathematical pictures" as they were mentioned in Ms. Sands's case and approaches like "fruit salad" analogy as it was appeared and mentioned in both cases.

The focus on the instruction should be on meaningful development of important algebraic or mathematical ideas, not promote procedural proficiency (Edwards, 2000; Hiebert & Carpenter, 1992; Koehler & Grouws, 1992; Skemp, 1978; Woodbury, 2000). Quality mathematics instruction should be designed in a way that it emphasizes the relationships among concepts, procedures, and problem solving and allow students to construct their own knowledge with understanding. Instruction should support students' thinking by emphasizing the mathematical and practical meanings of ideas, including how the idea, concept or skill is connected in multiple ways to other mathematical ideas as well as to real life where it is possible. Thus, a classroom context should be provided students to construct meaning and make connections within not only algebra and mathematics but also across other disciplines. Teachers should help the learner construct interconnections between concepts, representations, topics, and procedures for conceptual understanding (Even, et al., 1993). Teachers should use their students' thinking and knowledge construction to guide the teaching by thoughtfully building in connections between students' thinking and the concepts to be learned. They need not only try to understand sources of students' responses but also attempt to predict and incorporate those into the lesson segments and series of lessons. Teachers should keep in mind that students' errors with algebraic algorithms are often due to learning or constructing the wrong idea, not because of failing to learn a particular idea (Matz, 1980). They should try to explain students' reasoning and difficulties from students' perspective, not from their own perspectives. This would allow creating learning and teaching atmosphere in which ideas are supported and students can feel free to think creatively.

As these two teachers pointed out, students need a reason to learn algebra. However, Mr. Casey's case suggested it is not an easy task. Students should be provided with meaningful tasks that develop critical thinking skills and enhance appreciation for the usefulness of algebra. As Davis (1989) cautioned these mathematical tasks should be of a genuine nature, not the artificial contrived word problems found in many texts. As Thorpe (1989) pointed out a topic should not be included unless it has at least one of *intrinsic value*, *pedagogical value*, or *intrinsic excitement or beauty*. On the other hand, as Mr. Casey suggested, providing or having a reason to learn a particular topic is part of conceptual understanding and lack of it promotes procedural learning as doing mathematics for the sake of applying rules and getting an answer. However, providing a meaning for some tasks seems to be challenging even for the teachers. "Why can't you have a radical in the denominator?" was a question Mr. Casey expected students would ask him but they did not. In case someone would ask, he could only tell "it's so that you can get practice of multiplying radicals or something." The answer to such a question was not a good one for him and so he was glad that no one had asked yet because he thought there was no reason for having $1/\sqrt{3}$ rationalized especially with the computer and calculator technologies that would calculate the result if a decimal approximation was needed. Comparing the situation with the one in abstract algebra he had taken in college where leaving radicals in the denominator was not a problem, he could not think of a reason mathematically either. He expressed his confusion [and possibly disappointment] with the situation to his students as well in a lesson involving simplification of division of radicals. As he shared Mr. Casey's dilemma, Davis (1989) stated that it is assumed that students would understand what is meant by "simplifying" in general even though a question of which one of, for example, $1/\sqrt{2}$ or $\sqrt{2}/2$ is simpler depends on what one plan to do next with the result. Thus, asking to learn a concept or perform a task like simplify an expression without a motivation preceding it would not make sense from student's point of view.

Since both teachers in this study expressed concerns about calculator dependence in the way it was used by the students in their class as a source of student difficulty as it may become a barrier limiting creative and independent thinking and conceptual learning, addressing uses of technology in teaching mathematics and algebra as an implication needs for further attention, even it is not a primary purpose and a finding in this study. Mathematics instruction and teachers should see and use technology (not necessarily limited to calculators) not as a barrier to students' conceptual learning but a door opening to quality mathematics education if it is used intelligently (NCTM, 2000). Technology would be a powerful tool in assisting students in problem solving by allowing for multiple representations, creativity in problem solving techniques, looking at data in a variety of ways, and seeing the viability of their answers, which may lead to experience different ways of thinking, develop better insights and understandings of problem situations, and increase comprehension about mathematical concepts (Erbas, Ledford, Orrill, & Polly, 2004; Erbas, Orrill, Polly, & Ledford, forthcoming; Glazer & Erbas, 2003; NCTM, 2000). Good technology use would allow students opportunities to concretize abstract principles through multiple representations and then to take what may seem more concrete to a higher level of abstraction. It would provide an avenue for curiosity and fosters the mathematical disposition to engage in problem-solving activities. When supported by the teacher, technology tools would provide students with opportunities to investigate and manipulate mathematical situations in order to observe, experiment, and make conjectures based on patterns, relationships, tendencies, and generalizations. Research tells us students can learn new skills and concepts without sacrifice of traditional pencil and paper skills by solving problems (Heid, 1996; O' Callaghan, 1998).

Implications for Teacher Education

Knowledge of student thinking is considered as an important part of knowledge base required for teaching (Bromme, 1994, 1995; Shulman, 1986, 1987). Teachers need to understand what students know and how they think about a particular concept or problem situation in order to help move their understanding forward (NBPTS, 1998; NCTM, 1991, 2000; Shulman, 1986). Shulman (1986) proposed that knowledge of student thinking, particularly knowledge of common conceptions, misconceptions, and difficulties that students encounter when learning particular content, and strategies to respond/address those are important components of pedagogical understanding; thus it is an essential part of knowledge base required for successful teaching. For achieving quality mathematics instruction, thus, it is necessary for teachers to acquire, conceptualize, and use knowledge of student thinking, as they need to develop robust pedagogical content knowledge (Shulman, 1986, 1987). They should not only know the concepts and ideas students have most difficulty but also they need to seek for ways to reveal those misunderstandings so they can modify their approaches and act accordingly (NBPTS, 1998; NCTM, 2000). As Olivier (1992) points out, students' erroneous thinking is an important part of learning process since they form part of the individual's conceptual structure that influence, mostly in a negative way, the learning of new concepts. From this perspective, knowing how to ask questions and planning lessons to reveal students' prior knowledge and anticipate where difficulties are likely to arise are necessary domains of teacher pedagogical content knowledge. Like Chazan (1999) discussed based on his experience in teaching algebra using various curricula, it was observed in this study that reform based approaches to algebra with problem solving and functional approaches seems more promising to provide teachers with essential resources for acquiring, encouraging and supporting students thinking that may not be easy with a traditional number and symbol manipulation approach. Furthermore, such reformist approaches would allow teachers what the algebra is about and how it is related to the world around them.

As the findings of this study indicate, even experienced teachers have insufficient knowledge of student thinking (and thus pedagogical content knowledge). Even though their knowledge in terms of "knowing-that" (Even & Tirosh, 1995; Mason & Spence, 1999; Shulman, 1986) was present in most cases, "knowing-why" and "knowing-how" (Even & Tirosh, 1995; Mason & Spence, 1999; Shulman, 1986) was limited and even problematic in some cases. Such insufficient knowledge limits teacher pedagogical knowledge of student thinking in terms of "knowing-to" and disables teachers acting at the moment (Mason & Spence, 1999). As they develop classroom experience they also develop awareness (i.e., knowing-that) towards student thinking, particularly their difficulties. However, understanding the sources (*knowing-why*), developing techniques and skills to respond (*knowing-how*) and bringing all this relevant knowledge to act upon (*knowing-to*) may not easily develop with the experience and are shaped by the curriculum or their belief systems. However, teacher knowledge is dynamic and evolving. Thus, teacher education programs should aim to furnish teachers with knowledge of student thinking in both aspects of knowing: knowing-about (knowing-that, knowing-why, knowing-how) and knowing-to (Even & Tirosh, 1995; Mason & Spence, 1999; Shulman, 1986). In doing this, "teachers need opportunities to examine children's thinking about mathematics so that they can select or create tasks that can help children build more valid conceptions of mathematics." (NCTM, 1991) As Tirosh, Even, and Robinson (1998) argued, however, it is not sufficient to present teachers with a list of common student misconceptions, rather, "a main objective may be to raise their general sensitivity to students' ways of making sense of the subject matter and the instruction." (p. 62) From this point of view, *Cognitively Guided Instruction* (CGI) (Fennema &

Carpenter, 1996) model for professional development activities should be developed for algebra teachers in order to help them develop an understanding of students' algebraic thinking, its development and how it may form the basis for the development of more advanced algebraic ideas. Similar models should be developed for in-service mathematics teachers as well. Content and pedagogy courses for pre-service mathematics teachers should integrated and allow reflection upon a deeper understanding of subject to be taught (Wilson, 1994) and aim to promote awareness and understanding of student thinking. Even though content knowledge is necessary and helpful to recognize what a student is trying to say or where the flows in his or her thinking are, understanding its sources and responding to it require more than content knowledge. Both pre-service and in-service teacher education should attempt to increase teachers' knowledge of general stages students pass through during the event of learning algebra (or mathematic) concepts and procedures in specific domains or topics. Such knowledge should be supported by examples from theoretical and empirical research literature. Such courses should extend teachers' knowledge of representations for teaching particular concepts and help them provide meaningful problem solving activities for their students (Stump, 2001). However, instructors of such courses and professional development activities should be sensitive to the levels of participants, as they may need to simplify the language and interpretations of findings from such literature. While developing this knowledge, pre- and in-service teachers should be given opportunities to assess actual students' thinking (Hallaghan, 2003; Miller, 1992; Miller & England, 1989). For in-service teachers, their actual classrooms and students would be the place for making observations and evaluations of students' thinking. Pre-service teachers, on the other hand, should be encouraged and given opportunities (in addition to their student practice) to involve in actual classrooms to observe and analyze student thinking.

Many studies currently available in the literature have the analysis of student thinking as their focus. These studies present students' mathematical activities and thinking behind them. To facilitate teachers' understanding of students' mathematical thinking most effectively, the discussion of these case studies should include a focus on making sense of the students' thinking behind the activities. It would also be useful to engage participants as learners of the same mathematical task and reflect about how their students would see and approach the same task- in other words, try to see the task through their students' eyes. If teachers are given opportunities to test their insights generated in the discussions with students in real classroom or in their own classes, understanding students' mathematical thinking as a domain of knowledge can be furthered.

Watching prerecorded classroom videos or descriptions of student thinking from literature in order to observe and analyze student thinking should be an integral part of pre-service teacher education courses and professional development activities for in-service teachers. Once they start to develop knowledge base for recognizing, understanding and responding student thinking, pre and in-service teachers can use video to record their practice in order for self-reflection and improvement. Furthermore, insightful review and consideration of student works such as homework, tests, quizzes (Hallagan, 2003; Miller, 1992; Miller & England, 1989) would be a way to raise teachers' sensitivity to student thinking and look for ways to reveal and incorporate ideas into their instruction. Teachers in this study seemed be more concerned and reflective about students thinking as they considered and talked about students' difficulties during the instruction and in their assessments of students understanding. This enabled them to think about nature and sources of students' difficulties as they elaborated on students' answers and strategies in certain problems and concepts. If teachers become more concerned about their students' thinking in their instructions and make such reflections an important habit, they would improve their knowledge and potentially their instructions and students' learning by making more informed decisions.

Prospective and inservice teachers must leave their programs and professional development activities with an awareness of students' mathematical thinking and an ability to acquire, process and respond to it. Teacher education programs must reflect on whether they provide their students with the best education possible and increase their knowledge of student thinking. Educating preservice and inservice teachers to work in mathematics classrooms where student thinking is the focus of instruction may be a challenge, but it is also an exciting opportunity to learn and build new approaches to reach all students mathematically.

This study highlights some differences in the nature of first-year algebra courses in middle school and high school and possible implications for teaching and learning of mathematics and teacher knowledge. In making "algebra for all" a reality, more and more school districts start to offer algebra to all students in middle schools. In this context, the nature of middle school algebra curriculum becomes an important issue. Kilpatrick, Swafford, and Findell (2001) argue that offering algebra through quadratics as first-year algebra courses to everyone in middle school is not a promising way to develop algebraic proficiency in students. Middle school algebra curriculum should be different than that of high school. Algebra in middle school should be offered in a way that "algebraic ideas are developed in a robust way and connected to the rest of mathematics" (p. 420). Although both algebra curricula in this study were similar in terms of topics covered, they were different in sequence, organization, and approaches to topics. In the middle school case, the course promoted a functional approach to algebra (Fey et al., 1995; NCTM, 1989, 1995; Thorpe, 1989). Thus, some of the topics considered advanced and harder for students in the high school algebra curriculum (e.g., graphing, functions, and quadratics) and left towards to the end in the sequence were covered or started to develop earlier in the middle school case. It seemed that flexible content structure of the middle school algebra curriculum aiming to develop some important ideas of algebra would provide a better context for the teacher to promote and reinforce student thinking and develop professional knowledge.

The findings of this study suggest existence of possible differences between middle school and high school mathematics teachers in terms of attending and knowledge of students' thinking. As Nathan and Koedinger (2000a, 2000b) pointed out high school teachers may have lower expectations from their students and does not value their thinking as much as their colleagues in middle school. This was evident in Mr. Casey's perception of some of the students' errors and/or informal methods as "made up rules" and his instruction, which mainly consisted of drill-and-practice rather than conceptual activities as a consequence of his low expectation from the students. Ms. Casey, on the other hand, tried to promote a conceptual learning environment, respected students' thinking even they were wrong, and had higher expectations from her students as she thought they know more than they can show to her. Nathan and Koedinger suggested that high school mathematics teachers may hold beliefs that cause them to systematically misjudge students' algebraic reasoning abilities and less likely to agree with reform views than middle school teachers. One explanation they provided is that middle school mathematics teachers may have more opportunities to observe students' transition from arithmetic to algebra as they start their formal training in algebra. High school teachers, on the other hand, are often focused on using formal methods and exclude other methods, like ones invented by students. High school teachers are also have more expertise in their content areas and thus are least aware of difficulties their novice students may have.

Ms. Sands's involvement in a new curriculum would be an explanation for why she tried to attend students' thinking more than her high school colleague. This made the context Ms. Sands worked in dynamic rather than static. She saw herself a learner in the process of the curriculum change as she examined the new content, thought about how to implement it effectively, and anticipate students' interaction and reaction to the new materials. The nontraditional nature of the new curriculum, support from the colleagues and the school administration while the school was going through the change provided Ms. Sands opportunities to develop more pedagogical content knowledge as she tried to implement the new materials in appropriate ways and learn more about students' thinking in the process. Mr. Casey, on the other hand, worked in a pretty static context in terms of curriculum and delivery of the content. He accustomed to the content of a typical first-year algebra course and characteristics of the students taking it over the years and thus might have developed beliefs and practices that he may perceive more appropriate to work with rather than the ideal ways he talked about.

Implications for Future Research

Some suggestions and implications for further research stemmed from this study in the field of mathematics education. First of all, I propose further studies should study teachers' beliefs and knowledge of students' thinking in particular concepts and attempt to improve it before the instruction takes place in order to study the affects of such a knowledge on improvement of instruction, change in beliefs and possible affect on students' achievements. These studies should particularly focus on experiences best facilitating teachers to improve their knowledge and gain experience in recognizing, analyzing and responding appropriately. Existence of vast literature concerning students' learning and difficulties in important concepts of algebra should be used in such ways to improve algebra education, which becomes a greeter concern in a highly technological society where algebraic skills are most valued for modeling and problem solving. Furthermore, as I conduct and analyzed videotaped observations I came to awareness that such studies are also valuable in terms of observing and understanding students' learning in various contexts that may not be reported in current mathematics education literature. Such an example manifested in my attempt to locate literature on students' understanding of radicals not studied in detail so far. Even though some of the difficulties students had in radicals observed in this study were explainable by certain trends from the general literature on students' understanding of algebra, the observations pointed out the need for further study of student conceptions in this area.

Further research is warranted on the same series of lessons in the CPM curriculum that was concern to this study. Since the study was conducted with the teacher, Ms. Sands, who was implementing the lesson for the first time, further study is needed how her knowledge evolves over the years as she becomes more familiar with the curriculum, textbook and students' learning in such a nontraditional, reform influenced environment. On the other hand, further study of other teacher, Mr. Casey, is also needed in environments where he would feel he could teach the way he believes. Such a study would allow comparing various contexts for the same teacher and reveal information on how teachers deal with students' thinking in different environments, most importantly the one they feel safe to act the way they believe and know best.

Further research studies may seek for answers to the following question: "Why are there pronounced differences between middle school and high school mathematics teachers' attendance and knowledge of students' thinking?" Also, the difference between high school and middle school mathematics teachers' pedagogical content knowledge and its development might be a point of interest for further research. The effects of involvement in nontraditional, reform oriented curriculum changes in development of teachers' pedagogical content knowledge should be studied further. In this context, one hypothesis that might be investigated could be "If teachers involve in new curriculum, they would give more attention to student thinking and develop more pedagogical content knowledge." Furthermore, studies that are interested in studying middle and high school algebra classroom and teachers should be sensitive to the differences between algebra curriculums without assuming that they are same or similar in nature and include this as a variable in the design.

This exploratory study into inservice teachers' beliefs and knowledge of student thinking suggest that further studies of teacher knowledge of students thinking are needed with multiple teachers teaching the same concepts. Such research would provide valuable insight into factors affecting teachers' beliefs and knowledge student thinking and how it may have developed. Such studies are also needed for generalizing the observations made in this study over large populations of teachers. Since reform oriented practices often emphasize to attention to students' thinking, studying reform-oriented classrooms deserves further attention. Longitudinal studies should study affects of nontraditional practices and algebra curricula on teacher knowledge of students' thinking as they change within the context and affect on student thinking and learning of algebra. Moreover, further studies focusing on greater number of teachers teaching the same content topic across traditional and nontraditional algebra classrooms would provide insight into the affects of different contextual factors on teachers' beliefs and knowledge and the role they are playing in recognizing and valuing students' thinking in the instruction.

Although the focus of this study was limited to teachers' pedagogical content knowledge in terms of student thinking, it is often suggested that poor content knowledge may limit teachers' recognition of and insight into their students' thinking (Ball, 1997). Thus, teachers' content knowledge in various domains of algebra (not only in functions that is what usually studied and reported in research literature) should be studied further along with its relation to knowledge of student thinking and instructional decisions in future studies.

This study has also methodological implications for the future research studies involving classroom observations. As a research methodology, I found video taping very useful in this research study even though there were limitations and hardship associated with it. It allowed reenacting the classroom events and discussions selectively focused on during the time of observations and produced data possibly richer than what it might have been from field notes or audio taped observations. However, for future research studies, I recommend myself and other researchers to use additional microphones located at different spots of the classrooms in order to get a quality audio from the students. Even though it might be expensive and not feasible in various contexts (e.g., one time observations), using multiple microphones so students can be heard clear enough is worthy. It may be a necessity for long-term commitments, particularly if it is day-to-day basis. On the other hand, analyzing the videotapes in a daily or timely manner to extract clips and episodes interest to the research and researcher(s) and using them as focus of interviews with the teachers may reveal crucial data related to their knowledge, beliefs, goals, instruction, and the decisions taking place in the context. Since it allows observing themselves in a different dimension, such a methodology would also serve as a stronger and beneficial source of self-reflection for teachers than trying to remember about the incidents reflect on it. Such a self-reflection about their thinking and actions may be one of the few things we as researchers can offer to teachers or participants for involving in a research.

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APPENDIX A: TEACHER AND PARENTAL CONSENT FORMS

Participant Teacher Consent Form

I, _______________________________________ agree to participate in the research study titled, "Teachers' Knowledge of Student Thinking and Instructional Practices in Algebra," which is being conducted by Mr. Ayhan Kursat Erbas, from the Department of Mathematics Education at the University of Georgia under the direction of James W. Wilson (The Department of Mathematics Education, Phone #: 706-542-4194). I understand that this participation is entirely voluntary and I can withdraw my consent at any time without penalty. I can ask to have all of the information about me returned to me, removed from the research records, or destroyed.

The following points have been explained to me:

1. The reason for the research is to investigate and understand the relationship between the professional knowledge, beliefs, and goals of teachers with a focus on student thinking and how those affect their instructional decisions.

2. The benefits that I may expect from this research are (1) the opportunity to think and reflect on my knowledge (pedagogical and content), beliefs, and instructional practice in algebra and (2) the opportunity to share that information with others.

3. If I volunteer to take part in this study, I will be asked to be involved in the following things and brief descriptions, types of the questions, and rationale for each of these are explained to me by the researcher before the study: (1) contextual and biographical interviews; (2) video recorded classroom observations; (3) interviews on teacher planning and assessing of algebra lessons; (4) interviews to analyze hypothetical situations that described students' answers to questions about some Algebra topics; (5) interviews following the critical incidents observed during the classes.

4. No risk of psychological, social, legal, economic or physical discomfort, stress or harm are foreseen except for some rare stress or discomfort when the researcher observe the classrom or ask questions during interviews.

5. No future risks are foreseen.

6. The results of this participation will be confidential and will not be released in any individually identifiable form without my prior consent, unless otherwise required by law. My name and any details that might identify me will be changed in any written reports to protect confidentiality, and tapes of the interview and classroom observations and interview transcripts will be destroyed upon the completion of dissertation and related articles by January 1, 2005.

7. The investigator will answer any further questions about the research, now or during the course of the project and can be reached at (706) 714-3298 (mobile) or (706) 542-4177 (work)

I understand the procedures described above. My questions have been answered to my satisfaction, and I agree to participate in this study. I have been given a copy of this form.

___________________________________ ____

Signature of Researcher Date Signature of Participant Date

Questions or problems regarding your rights as a participant should be addressed to Chris A. Joseph Ph.D. Human Subjects Office; The University of Georgia; 606A Graduate Studies Research Center; Athens, GA 30602-7411; Telephone 706-542-3199; E-Mail Address IRB@uga.edu

Parental Consent Form for Videotaped Classroom Observations

I agree to allow my child ___________________________ to take part in a study titled, “Teachers' Knowledge of Student Thinking and Instructional Practices in Algebra,” which is being conducted by Mr. Ayhan Kursat Erbas, from the Department of Mathematics Education at the University of Georgia under the direction of James W. Wilson (The Department of Mathematics Education, Phone #: 706-542-4194). I do not have to allow my child to be in this study if I do not want to. My child can stop taking part at any time without giving any reason, and without penalty. I can ask to have the information related to my child returned to me, removed from the research records, or destroyed.

The following points have been explained to me:

1. The reason for the study is to investigate and understand the relationship between the professional knowledge, beliefs, and goals of mathematics teachers with a focus on student thinking and how those affect their instructional decisions in Algebra 1.

2. The researcher hopes to learn something that may help to educate mathematics teachers better so they can help other children learn math better in the future.

3. If I allow my child to take part, my child will be involved in video recorded classroom observations that focus will be the teachers' interpretation of topic, and teacher-student interactions; questions to students, answers to students' questions, and dealing with students’ difficulties. Those observations will help to understand the teacher's instructional practices and the context of teacher delivery of subject matter.

4. The research is not expected to cause any psychological, social, legal, economic or physical discomfort, stress or harm. My child can quit at any time. My child’s grade will not be affected if my child decides to stop taking part.

5. The results of my child's participation will be confidential and will not be released in any individually identifiable form without my prior consent, unless otherwise required by law. The tapes of the classroom observations will be kept in a secured location and will be destroyed upon the completion of research report.

6. The investigator will answer any further questions about the research, now or during the course of the project and can be reached at (706) 714-3298 (mobile) or (706) 542-4177 (work)

I understand the study procedures described above. My questions have been answered to my satisfaction, and I agree to allow my child to take part in this study. I have been given a copy of this form to keep.

_________________________________ _______________________________

Signature of Researcher Date Signature of Parent or Guardian Date

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Questions or problems regarding your child's rights as a participant should be addressed to Chris A. Joseph, Ph.D. Human Subjects Office, Institutional Review Board, Office of the Vice President for Research, University of Georgia, 606A Boyd Graduate Studies Research Center, Athens, Georgia 30602-7411; Telephone (706) 542-6514; E-Mail Address IRB@uga.edu.

APPENDIX B: A PIECE OF EXPERIMENTAL WRITING FROM THE PILOT STUDY

Those Who Know, Can Teach

> Since Algebra surpasses all human subtlety and then the clarity of every mortal mind, it must be accounted a truly celestial gift, which gives such an illuminating experience of the true power of the intellect that whoever attains to it will believe there is nothing he cannot understand.
>
> Girolamo Cardano (1501-1576)

> Flexibility is a great thing in your teaching of mathematics and teaching of anything.
>
> Mr. Algebra Teacher (from interview transcripts)

Not everybody is as excited as Girolamo Cardano, a key figure in algebra and mathematics, was about algebra. For many people, algebra is a challenge and a gatekeeper. Successful completion of an algebra course is considered not only a prerequisite for further study in mathematics and other subjects but also a door opening to many jobs and later opportunities. However, personal experiences and research have shown that the road to algebra is never as smooth as one may wish: algebraic tasks are somehow difficult to learn and teach. Bertrand Russell once said, “When it comes to algebra we have to operate with x and y. There is a natural desire to know what x and y really are. That, at least, was my feeling. I always thought the teacher knew what they were but wouldn't tell me.” Bertrand Russell’s struggle with most fundamental ideas or concepts of algebra is in fact so common that it is often expressed as a source of jokes by cartoon characters.

As a core topic in mathematics and society, algebra has become my target. I investigated student difficulties in elementary algebra for my Masters in Science dissertation several years ago. One of the interesting but quite expected results was teachers’ lack of awareness of the students' difficulties. It looks to me that students and teachers talks in different languages. Since this result is drawn from some questionnaires, I need to conduct further research in order to understand why students have difficulties in algebra and whether teachers are aware of the difficulties and the possible reasons for the difficulties. Pursuing such a research study, if I cannot throw light on people' experience in algebra classroom, I at least aim to understand my life spent in mathematics classrooms, where no hands were raised when the teacher asked if there was anything not understood (although maybe only a few in the class were able to follow what was going on); where whenever I or my classmates asked about something that the teacher was trying to teach, the answers were simply a repetition of what was written on the blackboard (like a broken tape player); where I felt that I was learning not my mathematics but the teacher's mathematics.

I am now close to finishing my PhD in Mathematics Education in the United States as a part of my long-term goal: educate the educators, raise the teachers. Although the ultimate goals and outputs are somehow the same, being an international student, I am well aware of the fact that the mathematics curriculum, particularly the algebra curriculum is diverse and different from some other parts of the world (At least in terms of who takes it and when they take it). My first real, firsthand viewing sightseeing of an American school happened in the fall of 2000, when I was trying to do a mini study for the requirements of a qualitative research course I was taking. This paper is about the middle school I have visited (the High Hill Middle School) and the story of the mathematics teacher (Mr. Algebra Teacher) teaching algebra in that school as a result of my interview with him, a few observations of his algebra classroom, and documentary analysis about the school and the teacher.

Meet the School

Located on the eastern suburban side of a university town and situated in close proximity to a busy road, the High Hill Middle School (HHMS) is one of four middle schools in the city, which has a population of approximately 100,000, a third of whom are students at the university. HHMS had a student population of about 600 by the time I visited. Mainly from the effect of the international community at the university, the school has a diverse population of students. Students from various parts of the world such as Japan, Korea, India, Europe, and Mexico can be found in the school. The school site houses three buildings and is adjacent to a public park that includes a soccer field. In front of the school and closest to the road is a forested area that consists mainly of pine trees. There is also an open, grassy area that includes a parking lot directly in front of the school and behind it. Right in front of the school are a few planted trees and several bushes.

By the time I visited HHMS, it had been under renovation for several months. So, if you went there for a visit at that time, you would probably have difficulty in finding the school office, which used to be in the main school building, because of the construction. After you parked your car (probably on grass because of the full parking lot), you might see the sign showing the direction to the school office if you looked around carefully. If you walked a few feet straight ahead on the way to the office, you encounter pre-fabricated sheds right behind the school building. If you got the impression that these are for construction workers' use, you were wrong! These were the temporary/new classrooms for the students until they got back their old ones. If you listened carefully, you might hear high-pitched voices of little humans. If you did not believe that, you could stay until the break and see them coming out. The office was a big shed among the others. A smooth ramp led to the entrance. Before you went in, you might notice the big chains and the lock on the door. These were probably for the safety of the office to protect it from walkers at night (?!!!) A smiling lady, probably a receptionist or a secretary, right behind the door would welcome you with her head and a smile while speaking on the phone. While you waited for her to finish the conversation, you might want to look around with curiosity. The first thing your eyes might capture might well be a bunch of clothes lying on the floor near the entrance. I wondered what those were— probably lost or forgotten clothes from the students. If you moved your eyes up from the floor, you might notice the half-glass walls separating several private offices. When the secretary finished her conversation, she asked how she could help. You were there to see Mr. Algebra Teacher. If you came before the half hour (e.g., 08:30, 09:30), you would probably be asked to wait until the break bell rings. You had better sit on the coach and wait until the bell rings. While sitting, you would probably see several students going in and out of the offices. You would just observe and try to understand the characteristics of the school population: black, white, Asian, Mexican, etc. Wait with a little bit of both fascination (of observing) and regret for coming early! Finally, after the bell rang, the secretary would try to reach Mr. Algebra Teacher over walkie-talkie: great technology to find somebody in a huge and crowded place like a school! Soon a tall, brunette, and half-grey-haired man in his mid-40s would show up at the door. You could easily see the notches that the years had put on his face. He is the teacher we were waiting for. After a short welcome, he offers to go to a more suitable place to speak. While you were walking inside the maze-like corridors of the school building, you would introduce yourselves. You would stop in front of a huge room. There would be several bookshelves located at the center, which might lead you to think that it was the school library. In the corner you might also see a group of students with their teachers in a class.

From the level of the noise, you would understand that you were right behind the current construction area. You sit and begin to chat.

Meet the Teacher

Mr. Algebra Teacher was born and raised in an English-speaking foreign country in the Caribbean, where he also spent almost 20 years of his teaching career. He has taught calculus, advanced algebra, and trigonometry at the high school level and something at the college level. He moved to the United States in 1990. Shortly after his arrival, he was offered an opportunity to teach algebra and accelerated mathematics by the principal of his current school. In his terms, this was the only job going in the town at that time. Although he had taught many years at the high school level, he accepted the job to teach in a middle school for a year and then would try to get a job in a high school. But circumstances did not let him do that, and he was still teaching at a middle school and was happy about it. In his terms, he must be doing a good job because the parents want him to remain. His educational background is quite colorful. He has a bachelors degree in mathematics as well as a postgraduate diploma in mathematics education from the University of City X. In addition, he has a masters degree as well as a specialist degree in mathematics education from the University of City Y. One can get a first impression of the quality of his teaching from the words of the school principal about him: “If I had a math teacher like that, I might be a physicist or winner of a Nobel Prize by now.”

His reputation goes beyond the walls of his school. His career is full of recognition and honors from the school, the district, and the nation. He was accorded not only the Teacher of the Year for his school, but also the Teacher of the Year in his county. He was also involved in several projects at the university. He is a well-recognized mathematics teacher in the county and that’s not all. He was among only 53 middle school mathematics teachers nationwide in contention for the Presidential Award for Excellence in the Teaching of Mathematics and Science. He got his statewide award two weeks ago. With a humble tone, he says that he does not think he stands a chance (at winning the national award). But, he also adds that he also did not think he stood a chance to be selected at the state level.

Of course, as with every success story, this has its own roots in experience, dedication and hard work. Although his school is ranked 372 among 410 middle schools in the northeast part of the state according to the reading and mathematics scores of students taking the 1999 Iowa Test of Basic Skills, his Mathcounts team, a group of students from the school who compete statewide in mathematics, has significant achievements. As he mentions, they swept the state competition last year against schools across the state considered to be top-notch academies that recruit their students.

Mr. Teacher arrives at the school at 6:40 a.m. each school day to coach the seventh- and eighth-grade *Mathcounts* team, which arrives at 7 a.m. for drills before classes begin at 8:30. A student on the school’s mathematics team says, “He gets here at 6:30 every morning…. He's always coming up with new material for us to challenge our learning.”

Meet the Class

If you would like to visit Mr. Teacher’s classroom, you are more than welcome. After long walk among corridors, you may reach his algebra classroom. He shares this classroom with another teacher because of the shortage of classrooms. In his algebra classroom this day, there were five boys and fifteen girls, of whom there were two black boys, two black girls, two Indian girls, two Asian girls, three white boys, and six white girls. Although algebra is an eighth-grade course, he has students not only in eighth grade but also seventh grade. He uses an overhead projector instead of board to present his lesson. The classroom has three old-fashion Macs and a new PC. The wall holds several posters such as posters for Texas Instrument graphing calculators. At the back of the classroom are bookshelves that mostly hold extra textbooks.

You can easily see that Mr. Teacher's classroom management and discipline are one of his secrets. He did not start the class until he got total silence. Although he looks strict, his students seem to understand the idea behind it. One of his students says, "He's a great teacher—he's strict, but that's the way teachers should be. And behind the strictness, there's a warmth about him that people don't see." According to Mr. Teacher, one of his successes is the flexibility in his planning. Although he has no particular written plan, he tries to be flexible enough to change plans mid-stream if circumstances require.

Teaching and Learning of Algebra

According to Mr. Teacher, algebra is represents a gateway to many things, and doing algebra in middle school enables students to have access to calculus before they leave high school. However, in his terms, "Whether that is a good thing or bad thing I am not sure." At this point he notes the existence of "a huge gap" between the courses pre-algebra and algebra, "It is a superior type of child who will attempt to do algebra in middle school— a child who is going to be pretty smart, a child who is going to be able to see beyond." This is certainly a challenge to middle school teachers who only have middle school certification when they face a group of students asking questions whose content goes beyond the normal algebra course. In other words, according to Mr. Teacher, most middle school teachers have a lot of difficulty teaching algebra because of their inadequate content knowledge. Some measures of sophistication were required to teach successfully. He continues the difficulties involved in teaching algebra as follows:

> Some difficulties also arise because of the types of students we teach. We are talking about the brightest students at that each level in the district. And these kids ask a lot of questions [that] sometimes go far beyond the scope of the Algebra 1 course. Most of the people [who are] certified [to teach] middle school under the present curriculum do not

> have the type of the capabilities to meet/provide satisfactory answers to those questions that kids [raise]. And so there, that [represents] the difficulty. Right now they are working on the middle school curriculum, trying to put a little bit more iron [into] it so that they can provide a basis for meeting the needs of those teachers.

Asked how algebra should be taught, Mr. Teacher answered as follows:

> Any mathematics has got to be a combination of many things. [You've] got to be able to invite the kids [to] explore their mathematics, to make conclusions based on their investigations. They have got to be able to use technology…notions of technology in their incubations as far as mathematics is concerned. The teacher has got to provide, [a] wide range of problems. Meaningful problems and explore these kids' problem solving; sophisticated problem solving at that, too. So, to me mathematics is, teaching algebra is a combination of many things which the teacher has got to use with judicious timing if he is gonna be effective.

He perceived the teacher's role in teaching hard topics such as algebra as going beyond just following the textbook:

> We as teachers have got to provide bridges to gaps in what the books do. It is too often we as the teachers we allow the books to become the curriculum. *It's not*. Right. We have to provide information some time, strategies some time. And sometimes go outside of the book where find material to find bridges for kids so that they can comprehend the stuff better. So if you are interested in what you are doing, and if you really want to your kids to do really well, you got to be prepared to do stuff like this. Recognize some difficulties and be able to work on them.

Helping Students Having Difficulties in Algebra and other Mathematics

About helping students with difficulties in algebra, Mr. Teacher went beyond schooling and formal teaching duties. In a poetic language (as I perceive), he spoke:

> These are very young kids you have to remember.
> Some of them have some personal problems,
> Which impact their performance in class.
> You want to find to out a little bit about their background.
>
> What is happening at home.
> Establish a relationship with the parents or guardians;
> Try to see how you can get an environment at home,
> Which facilitates the learning of mathematics.
>
> Some of them their backgrounds are very weak.
> Some of the basic things they need to know.

In those circumstances it is pointless going on
That's the background work is zero.

You can do either of two things:
You can spend some time in class,
and do some of the background work
Or if the difficulty only seems to pertain to very few students,
Try to get those some out of school,
work with them and bring that background
that they are missing into focus.

Work on it, helping them to overcome whatever the difficulties [are]
that they have that might affect their learning of a new concept.

Experienced Teacher, Textbooks, and Preparing for a Lesson

Mr. Teacher is very opposed to the common myths of being experienced. To him, people claiming to have 30 years of experience have nothing but one year of experience 30 times over because they teach the same way year after year. He believes that the textbook is not the curriculum; the teacher himself or herself is the curriculum. The teacher should not be dependent on the textbook. He or she should be able to draw examples from real life and use mathematics to model them. Mr. Teacher also believes that the important thing in students' work is the appropriate reasoning to be seen in it, not just the correctness of numerical calculations:

There is one question *I love to ask*.
And only three letters in [it]: Why?
I like to have my children like to dig deep
    into their understanding of the stuff.

"Why" is a popular question for me.
I mean I ask the child about a question and tells us
    what he thinks might be a solution for that question, and I say "Why?"
I am interested in seeing how children think.

Their answers may not be numerically correct,
    but if they can provide to me a logical framework
    as how they move from the individual statements they have
What do I know of this particular question?

Can they arrange those statements
    or those ideas
    or those pieces of information they have?

Can they arrange them now in a logical sequence?
Can they arrive at a valid conclusion
    using these initial statements about the questions?

And my job is to guide them along that part.
    analyzing the question
    getting the necessary bits of information
    and facilitating the process of arranging them in a [sequence]
    to arrive at a valid conclusion.
And validity is what I am more interested in,
    more than what might be numerically correct.
Valid reasoning is kind of what I am interested in.

Mr. Teacher's general philosophy of teaching is quite representative of that of many teachers:

Teaching is like interacting with the future. It helps me to keep my own perspectives fresh and vibrant, but at the same time permits me to contribute to the development of the leaders of tomorrow.

Maybe there is more to talk about, and a lot to see, but the day has other responsibilities to carry on. We left Mr. Teacher in his classroom waiting to respond students' questions after class if they have any. He is there for our children, for our future. He is one of us. He just has the joy of *raising the future* in his small garden. You people, do you have the same joy? Thank you very much Mr. Algebra Teacher. Thank you very much those who can teach.

Methodological Reflection

Through this data story, I have tried to write a descriptive piece centered on an algebra teacher whom I interviewed once and observed (in his classroom) on two occasions. In a single participant case study report, I aimed to reflect a thick description of the teacher and his surroundings (i.e., this school). The piece I wrote here is mainly based on the interview, archival data I collected, and my personal observations. I did not include data about his teaching of specific (algebra) topics that I observed during the classroom observations. Since I did not have multiple interviews or a lot of fieldnotes, my data analysis for this paper was not as extensive as one would expect for a multiple cases or multiple interviews, observations, and so on. I did not need to do a systematic coding and categorizations by hand because I was able to conceptualize and organize all of the data in my head. I was able to go and pick up a related piece of data and plug it into the piece that I want to write. I think what I have experienced in such a data analysis is an example of one's being immersed in data, living with it, and talking with it. Of course, in my case this happened because there was fairly a small amount of data. I am not sure if I would do the same without standard methods of data analysis (e.g., coding and categorizing). Probably not! I need to add one more thing, which is my attempt to represent some of the data as poetry for the paper. I find this experience significant because this is the first time I ever tried that. It was a little bit odd too. I am not sure that what I call "poetic" is poetic at all, though I feel it is. Certainly being a bilingual and having English as a second language, my cultural and linguistic interpretation and sense of poetry are different than that of one who has English as one's first language.

In terms of ethics and politics of writing, I would say that my biggest concern was whether what I wrote was really a representation of the teacher and the context he lived in. I feel uncomfortable not because I feel guilty of misrepresenting or highlighting my ideas about his remarks but because I did not send the paper to him and let him read and reflect on what I wrote about him. I am comfortable that what I wrote, particularly about the teacher, is totally based on the interview and observation data. However, I am also conscious of the fact that the things I wrote about him, the school, and the order of things are very subjective. After all, I asked questions about what I found important. At this point, I certainly feel that I (and everyone too) should do a member check before publishing any writing that includes information concerning a participant. I think this is especially important in descriptive studies where the focus is on the thick description of events, people, and their experiences. Another ethical dilemma I have concerns the anonymity of the subject and the setting since I included detailed information about the school and the teacher. I guess most of the people in my department would recognize the participant when they read this paper. And I am not sure whether I need to cut out many details about him like his nomination for presidential teaching award. I guess I have to decide whether those details are necessary to make my point. On the other hand, politically I have concerns about what kind of impression or impact my writing will make. Probably these concerns are also related to ethics. I have to ask the question: Would my participant be recognized by the audience, and if it so, would he be harmed as a consequence? After all, I still need to know who my audience is. Is it me? Or is it everyone who reads an education column in a newspaper that I would write for?

APPENDIX C: INTERVIEW GUIDES

Ms. Sands: Interview-1

- Would you tell me about yourself just a little bit? Your background in mathematics as a mathematics teacher. Your background in algebra in particular.
- How about your background as a middle school teacher? Your educational background and how you decided to become a teacher.
- How is it different teaching algebra than teaching math?
- Would you tell me about yourself as a math teacher in general? Who are you in the classroom? How would you describe yourself as a math teacher?
- What do you think that what mathematics is, what algebra is if it is different than what mathematics is?
- How could you do it better? Or what kind of things would help you to make it better?
- How is it different than doing a computation and not having the whole picture?
- Would you tell me what do you think about the differences among algebra, arithmetic and geometry from content perspective? Do you see all them separate or somehow tied together?
- What kind of differences it makes for you and for students in terms of learning and teaching of algebra.
- What kind of experiences do you think would help students to make the connection between concrete and the symbolic representations?

- Do you have a special way of teaching algebra than just teaching mathematics? I mean should it be different than just any other subject or is it the same way that you teach mathematics?
- How would you define your algebra classroom culture? Is it different than your other…
- What can you cay about the interaction the communication between you and your students in the classroom and outside the classroom, mathematically speaking?
- What made you changed [from not caring about what students thinks rather than the answer to the other way around]? What was the experience?
- What do you think about student difficulties in algebra-1, in general? Would you be more specific what are the areas students have most difficulty?
- How about some particular areas specific to algebra like what *x* and *y* means or what about all those letters appearing they didn't have any experience you know which is having letter instead of numbers.
- How about, I mean, they are using equations or solving equations a lot how about their difficulties in that?
- As a teacher, what can you do to help students having particular difficulties in general? What do you do? What are your general strategies?
- Those are the things you can do to help them. What do you think about the opposite? I mean what are the things that you can't do to help them?
- How is it possible to help a student if he/she can't explain himself/herself or know what exactly he/she doesn't know?
- How to get into students into students' minds? What kind of practices you are hoping to employ or would work?

- How do you do listen [what a student is trying to say] in a classroom with 28 students?
- When you are grouping students do you have something in your mind particularly? What kind of strategies you're using in grouping.
- When you have new students in your classroom that you have met first time, what kind of things that you do or ask to learn about their mathematical problems.
- What kinds of things affect your classroom instruction, your practice?
- Would you tell me little bit about how you prepare for a lesson, for the next day?
- Do you have some particular criteria to select which questions to ask in the classroom?
- How do you understand if it's going smoothly? For example, being student in the classroom in the old days, I remember that teacher ask this question did you understand or do you understand, did you all understand and nobody would say a thing and teacher I guess just assumes that everybody understood and moves on. So, how do you understand that students really understand something?
- Do you have particular criteria to select items to ask on the test or quiz? Do you prepare your own questions?
- Would you make a brief evaluation of today and your plan tomorrow? What was the general purpose in today's activity?
- Do you think the book gives the teacher details about you know the purpose in doing particular activities or is it something the teacher has to figure out?

Ms. Sands: Interview-2

- Would you make an evaluation of the unit so far from last Monday through today? How was it? Did you satisfy your goals and did you find anything interesting in terms of student learning?

- Do you really believe in this curriculum?
- If you teach a topic today and see some problems, how this affects your planning for the next day? Next time?
- What else did you see in terms of student difficulties in this unit?
- You said that students still had the difficulty of figuring out that it was just a simple scaling problem? Why do you think they had it? Can you elaborate a little bit on why do you think they should do it quickly instead of having too much difficulty? Do you have a general idea about what was it that they didn't understand?
- What do you think about how does the unit connected to the other units? What is the general purpose, goal of the unit?
- What do you think the student perspective is? I'm sure they're just doing the moment but…
- Thinking about the limitations of just paper and pencil, do you think technology working on the computer or graphing calculator would solve that kind of difficulties? What's your stand on using technology?
- What were the typical [student] questions from your perspectives in this unit?
- Do you think the group work so far is working out well for this unit and why? Would you give some specific examples from the groups?
- Is your planning changing day by day or do you plan and act pretty much in the same fashion? This could be from beginning of the semester not just for this unit. Is it the same for pre-algebra or any other math class that you have been teaching?
- How flexible is the school administration on following the curriculum? Do they give you room for flexibility?

- Would you tell me little bit more about your description of what mathematics is because I hear you're saying, "Math is sequential anyway"? So what is mathematics to you?
- Would you describe me your vision for how students learn best? What should be the environment, what should be the strategies in general?
- What are the pedagogical issues for students learning best?
- Would you describe a typical morning or helping, tutoring session when a kid comes with a question at hand?
- What are your plans until the end of the unit from now on?

Ms. Sands: Interview-3

- What do you think what will happen in a case where one takes the CPM algebra and takes a geometry class with a traditional curriculum?
- Why do you think having algebra in geometry so important?
- When you look at the test results, what do you see in terms of student thinking? I mean not just as a score like 70 out of 100…
- What kind of ideas did you use before to make this concept clear to students?
- What do you think what is going on here [in this particular answer]?
- What is the highest score and the lowest score?
- When you say I'm really disappointed on this test do you mean that they did a lot of careless mistakes or do you think that they have serious issues? And why do you think so in general?
- What kind of questions students were asking during the test?
- What do you think the value of including those types of problems [statistics and geometry] other than algebra items here in an algebra test?

- After seeing results would you like to change the problems little bit especially number six and eight?
- On problem five, combining like terms, what kind of mistakes and errors do you observed?
- When you say they combine 7x and 3 and write 10x, the observation that you made is really an interesting one. Do you see this lot? Or it is…
- Do you observe students replacing x or any other unknown with some numbers instead of having staying leaving it as an x? Or for example if you have *a* and *b*, do they like alphabetical matching like a is the first term in the alphabet and so it should be 1 and b is the second term so it should be 2. Do you observe those kinds of things?
- Do students try to label in algebra? Like if you gave 8*b* and/or 8*a*, do they say like 8 bananas and/or 8 apples?
- Seeing the results if you had a chance what would you do when you go back and teach those things again, would you change anything? Use some other approaches?
- In overall, if there are any, what do you the misconceptions are for this unit or so far in algebra?
- Any last comments on anything; students, policy, curriculum, assessment, just anything?

Mr. Casey: Interview-1

- Would you tell me about yourself? Your education as a mathematics teacher, your background in algebra…
- How did you decided to become a mathematics teacher? Particularly teaching algebra?
- Would you tell me about yourself as a mathematics teacher in general? I mean who are you in the classroom?

- Would you tell me about what do you think about algebra as a mathematical content?
- Do you think algebra is somehow different than other part of mathematics?
- What are the continuities and discontinuities between geometry, algebra and arithmetic from your perspective? How about the content perspective?
- If you need to define what algebra is, how would you define it? What is it to you?
- Would you tell me your ideas about how to teach algebra?
- How would you define your algebra classroom culture?
- What can you say about the interaction, the communication between you and your students in the classroom or outside the classroom? How about the interaction between themselves?
- What do you think about students' difficulties in algebra 1?
- In general why do they fail [in algebra] this much? What are factors affecting it?
- Do you think those differences like you know algebra is kind of different than arithmetic, jumping from arithmetic to algebra makes it different.
- What do you think the challenges are in terms of *algebra for all*?
- What can you do as a teacher to help students to solve their difficulties and what can't you do?
- What are the particular concepts that they have most difficulty like the absolute value thing you just mentioned?
- Would you tell me a little bit about what circumstances affect your classroom instruction, classroom practices?

- Would you tell me how you prepare for a lesson? What are your goals, objectives, representations used, and types of tasks and concepts kind of things? Is it basically QCC and textbook driven kind of?
- How about technology?
- How about your curriculum in general? Like you basically use a textbook; do you like the textbook or would you use another one different than this?
- Once you select the textbook, do you have to follow it?
- How do you apply your lesson in the classroom? Once you made a plan how do you apply it? Do you have a written plan?
- You said that I look at my old notes from previous years and used the word, "students' difficulties". How much of this affects your planning?
- Tell me about you assessment strategies. What do you use: tests, projects, and journals? How often assess students and do you prefer particular kinds of assessments?
- What are your criteria to decide which questions to ask in the classroom or in the quiz or test?
- How do results of a test or any assessment strategy help you to go further?
- Do you see any difference between mistake and a misconception? Are they same or?
- How confident do you feel in identifying misconceptions in algebra-1?
- In a quiz, test or homework, do you ask students to provide detailed solutions or any solution is acceptable to you if they provide the right answer?
- For the unit that you're teaching right, simplifying radicals, what kid of misconceptions do you think had in the past and would have in this classroom?

- What do you think for the last two days and what do you foresee for the coming days in terms of the evaluation of the class so far.
- What are you going to do today?
- What makes you who you are right now, I mean is it your experience or is it the graduate education that you're getting or you got so far, what is it?

Mr. Casey: Interview-2

- Would you make and evaluation of the unit so far? Did you achieve or satisfy your goals, did you find anything interesting in terms of student learning?
- What are the common mistakes or misconceptions that you observed so far?
- What do you think you or teachers can do to solve that kind of misconceptions?
- You said "I wish I had some kind of manipulatives that I can show it them." What kind of manipulatives?
- Would you locate the concept of radical expression or radicals in algebra or in mathematics, why is that important if it is? How is it connected to other concepts?
- You mentioned about the utilitarian view of mathematics. What is your concept of mathematics; what's mathematics to you?
- What are typical student questions in the unit so far?
- You have expressed concerns with students' arithmetical knowledge and competency. Do you think the biggest part in students' difficulties in algebra comes from arithmetical difficulties?
- Would you describe your vision of how students learn best algebra?
- You gave a quiz last week. How did you find that about, about the results? Did you get surprised, what did this quiz tell you?

- What are your plans for the rest of the unit?
- Do you have any other comments on the unit, on teaching or student learning?
- Why do they fear about word problems?

Mr. Casey: Interview-3

- When you look at the test results for Chapter 11, what do you see? Except the numbers like he got 17 or …
- What do you think the main problem is in those difficulties? Why they can't learn them, why they can't understand, grasp the concept?
- What do you think how this is going to affect the next step I mean the learning of functions or learning of any other mathematical concept?
- How did they do in the application problems?
- What's the maximum and minimum for the test?
- During the quizzes I often see you know lots of kids asking questions. What kind of questions did they ask during the test?
- If you go back and reteach the unit, what would you do; would you change anything?
- In overall, what do you think the misconceptions or common errors for the unit or so far in algebra?
- What do you mean when you say: "they try to reinvent some new mathematics?"
- How do you find the approach to this concept of the textbook?
- Is there anything else you would like to say for the chapter? The assessment, policy, content, just anything.
- Based on your experience what are the some misconceptions or student difficulties for the unit? What are your plans to deal with those issues?

APPENDIX D: TRANSCRIPTION CONVENTIONS

[Conventions and explanations presented here are adopted from TIMSS 1999 Video Study Transcription/Translation Manuals (Jacobs et al., 2003) in transcribing videotaped classroom discussions and audio taped interviews]

Common Backchannels

Throughout their talk, the speakers made use of "backchannel" devices, “discourse markers,” and “hesitation indicators” to show that they were, for example, paying attention (e.g., mm hm, uh huh), agreeing (e.g., yeah, yep), hesitating (e.g., uh, mm), showing surprise or displaying some new understanding (e.g., ah ha, oh, ah). In order to be relatively consistent throughout the transcription, these were transcribed as follows:

- Ah; Uh; Um, Oh
- Ah-ha; Uh-huh; Nn-hnh; Mm-hm
- Yeah; Yep
- Okay

Identifying the Speaker

In a normal situation, there was one teacher and a group of students in a classroom who are doing most of the talking. A radio microphone that was wired to the teacher captured the teacher’s voice. The microphone sent a mono-signal to the camera. This radio mike also captured the voices of students who were a few feet away from the teacher. Internal microphone on the camera captured the voices of students who were farther away from the teacher.

Speaker Codes

The following five speaker codes had been developed to try to deal with, as reasonably as possible, the discourse of the classroom:

Ms. S: Teacher-- Ms. Sands
Mr. C: Teacher—Mr. Casey
S: Single student
Sn: Student-new. A single student whose identity differs from the last student to speak
S?: When the identity of the student (whether the speaker is S or SN) is unclear.
Ss: Multiple students, but not the entire class.
E: Entire class (or sounds like the entire class): used to indicate choral.
O: Other; used to indicate speech by a non-member of the class, such as school personnel, office monitors, or talk from public address systems

While it is generally easy to distinguish the teacher's voice from that of the students', it is not always possible to distinguish between individual student voices. In several occasions, attempts were made to track the voice of any individual student in the ongoing discourse (e.g., marking S1: Student 1; S2: Student 2; S3: Student 3, etc. or with the initial of their names as altered and appeared in the text).

Transcription Conventions

Overlapping Speech

In the transcription system adopted here, moments of overlapping speech will be indicated by double backslashes (//) to indicate where the overlap begins, as in the following example:

Ms. S: Uh-huh. Think of it this way. This says [pointing $-x$] not x squared //but opposite.

Ss: //The negative.

The double backslashes always came in a set. The point where the two speakers began to talk over each other was always marked in both the overlapped turn (the one who began speaking before being overlapped) and in the overlapping turn (the one who began speaking during the talk of another).

Entering a New Speaker Code: Lengths of Pauses and Activity Shifts

Pauses: If a pause lasted longer than three seconds, a new speaker code was entered, even when the same speaker resumed speaking. Organizing the speech in this manner allowed timecodes to correspond with the actual utterances in the video in case it was needed.

Activity Shifts: Several activities occur in the classroom, for example: the teacher's lecture, student group work, question and answer, students at the blackboard, etc. If such a shift occurred during an utterance, a new speaker code was entered.

Punctuation, Diacritical Marks, and Other Conventions

The use of punctuation marks (such as a comma, period, colon, etc.) in the transcription of videotapes in this study followed the normal rules used in written English. In addition the following conventions will be used as part of the transcribing system:

PROPER NAMES: Proper names of teachers, students, schools, and locations were changed due to confidentiality issues. Even though transcripts used real names, which were in all CAPS, data stories changed the name using the same first letter, if it was possible, and deleted all caps.

- (HYPHEN): A hyphen indicated that a speaker had "cut-off" (or self-interrupted) his/her speech.

Ms. S: It's a dangerous place to be but-. So, why is it in parenthesis?

? (QUESTION MARK): A question mark indicated that the utterance was to be understood as a question (usually determined through intonation), as in the following.

Ms. S: No? Tell me why it doesn't make sense to you?

Sn: Why do they use parenthesis?

Ms. S: Can I answer hers first?

. (PERIOD): A period marked the end of a phrase, a sentence, or a turn at talk that was NOT to be understood as being a question.

S: That would be negative.

OTHER PUNCTUATION MARKS: Other punctuation marks such as commas, exclamation points, semi-colons, and colons were used when appropriate.

Ms. S: 18 pages of that!

MIMICS: Happy faces (i.e.,☺) were used to describe the tone of the conversation or the atmosphere.

. . . (THREE DOTS): A series of three dots separated by a blank space before and after, was used to indicate a pause.

S: Because, uhm, negative …

(   ): Empty parentheses indicated that some speaker had spoken, but the words cannot be made out.

Mr. C: This can't be right. Look at- on your calculator do 500 squared. [He watched while she was doing it on the calculator] ... No, not times 2. Squared.

S: Oh, that's not what I wanted. Because (   )

(a word or words): Word(s) surrounded by parentheses indicated that the transcriber had made a best guess at what the speaker had said, but cannot guarantee it.

Mr. C: Number 21 [It was "Simplify $\sqrt{(-7)^2}$"]. Uhm, take negative seven and square it, what do you get?

Ss: (Forty-nine)

Mr. C: Forty-nine. What's the square root of forty-nine?

Ss: (seven)

Capital: Capital letters was only used with proper nouns (names, cities, countries, languages, etc.), at the beginning of a new turn at talk, or after a period or question mark.

Mr. C: Yes, you would. Because you would be able to publish a result and people would be coming to you thinking that you're really smart guy and they want you on their team. Maybe work for RAIN Corporation or anywhere they trying to develop new technologies or new ways of looking at mathematics.

ALL CAPS: When speakers referred to points, lines, angles, etc. by their alphabetical label, these labels were transcribed in capital letters, even if it would otherwise appear as a lowercase letter.

[Transcriber's Note]: In cases where a bracketed comment was essential to the understanding of an utterance, the transcriber entered a note, in brackets. Non-verbal actions though to be important understanding the flow of the discussions was particularly indicated in notes.

Mr. C: I have two of these [*t*] one of them is a square root [He wrote one *t* out and crosses both]. Does this have a pair? [He pointed the remaining 3]