INCHUL JUNG
Student Representation and Understanding of Geometric Transformations with Technology Experience
(Under the direction of JAMES W. WILSON)

The primary purpose of this study was to investigate how and to what extent 'representations' affect the students' understanding and the growth of understanding in a technology [GSP]-based collegiate mathematics classroom. There are three themes in the framework of the study to support this purpose: 1) technology in mathematics education; 2) images on computer screen - visualization and representation; 3) understanding and growth of understanding.

The following three research questions guided this study: 1) How do students present each component of representations when they study 'transformations' in a technology [GSP]-based classroom? If there is any difference between the first and second presentation for each component, how are they different?; 2) How and to what extent do representations affect the students' understanding and the growth of understanding in a technology [GSP]-based classroom?; 3) What types of benefits and obstacles are there when students study 'transformations' in a technology [GSP]-based classroom?

This study was conducted during spring semester of 2001. A qualitative methodology study was used, especially case study, which is the most powerful and appropriate method for intensive examination (Goetz & LeCompte, 1984). Two college students were purposefully chosen based on three criteria and voluntarily participated for this study. Data were collected from descriptive notes, reflective notes, archival data, interviews, and concept maps. The collected data were analyzed using 'constant comparison method' described by Corbin and Strauss (1990) along with analytic induction.

Findings indicated unbalanced students' representations of what they learned in the classroom. Pictorial representation was dominant and verbal representation came along with it. However, written representation was often avoided unless participants were specifically asked to use it. Students' growth of understanding in transformations was based on a) their understanding basic concepts, b) the applicability of these basics, and c) their mathematical understanding of the given situations. If paper and pencil were the main tools for this course, written representations might play a major role in the whole process. Due to the complexity of mathematical contents, 'don't need boundary' in Pirie and Kieren's model (1994) was not significantly featured in this study, whereas 'folding back' was.

There were obstacles and benefits with technology experiences. As for obstacles, first, images on computer screen restrict learners' logical thinking; second, images on computer screen might lead learners nowhere or to misconception. As for benefits: first, students could quickly make a solid conjecture with accurate constructions with technology; second, the dynamic function of technology plays a significant role; third, well-constructed figures lead learners to efficient problem solving and psychological relief.

Finally, this study suggests that further researches with various perspectives relating to technology should be conducted for students' better understanding of mathematics so that we can use technology that we have more efficiently. Further, we need to develop a theory or a tool to recognize students' understanding and growth of understanding.

INDEX WORDS: Technology, Mathematical Understanding, Growth of Understanding, Visualization, Representation, Transformation, Geometry, Qualitative Research, Case Study, Teaching, Learning,

STUDENT REPRESENTATION AND UNDERSTANDING OF GEOMETRIC TRANSFORMATIONS WITH TECHNOLOGY EXPERIENCE

by

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In loving appreciation of

my wife, Jungae Myung; my mother, Soonja Choi; my first son, Sion Jung;
my second son, Jinyoung Jung; my daughter, Joy Jung.

That which we persist in doing becomes easier for us to do,
not that the nature of the thing itself is changed,
but that our power to do is increased.
- Heber J. Grant -

There is no substitute for hard work.
- Thomas Edison -

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CHAPTER 1

INTRODUCTION

Emergence of the problem

The origin of this study stems from my deep interest about the use of technology in mathematics education. I had never been taught any mathematics using any type of technology (even a simple calculator) through my life up to the point of becoming a graduate student and had never taught mathematics using any type of technology as a teacher in a secondary school nor as a tutor up to this point. The only tools that I used were ruler, compass, protractor, and abacus, each for just a short period of time. I recall chalk and blackboard or pencil and paper whenever I think of studying mathematics, regardless of grade or subject. Thus, I had always thought that technology, such as calculators or computers, could play an important role only in architecture or a modern industry, but not in education. I had never thought of the use of technology in learning and teaching mathematics. Maybe I could not imagine how it might look to study mathematics using technology.

Nowadays as we can easily see around us, the range of technology, especially computers, is getting wider and wider. The educational field is not an exception. Not only is the number of computers at school quickly growing, but also the recognition about the use of technology is rapidly increasing. In addition, the internet has accelerated the use of technology in mathematics education. As a consequence, people recognize that it is wise to reform mathematics curriculum to take advantage of technology (National Council of Teachers of Mathematics [NCTM], 1989) and that "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, 2000, p. 24). The influence of computer is getting more and more prominent. The rapid development of technology in quantity and quality seems to force people to pay more attention to the use of technology in mathematics classroom.

Among the many aspects of technology in mathematics, I was deeply interested in investigations of how and to what extent the representational mode of computers affects the students' initial understanding and the growth of their understanding. Along with this, I investigated which component of representations students present when they study 'transformations in geometry' in a technology-based classroom and how each component works. Further, I examined the obstacles and benefits with the use of technology in mathematics classroom. This study was conducted in the unique environment where there was no specific textbook used. Instead, the computer software, *The Geometer's Sketchpad* [GSP], (Jackiw, 1991) was the main tool and the instructor's personal notes were the main source of information for the course. In addition, the materials built on the instructor's personal website were always available to students and they were encouraged to visit the useful websites as much as they could. I adopted qualitative methods - especially, case study with two participants - for a closer examination.

Theoretical backgrounds

Technology in mathematics education

Introducing a new tool into mathematics classrooms brings changes with it. This new tool changes the nature of classroom environment or the way students learn mathematics or interaction between students and teachers. Those changes may be due to the structure of classroom with technology which is different from a traditional classroom or the role of technology which did not exist before in the classroom. People naturally became curious about how and to what degree technology effects mathematics education.

Thus, many researchers have investigated, with various perspectives and various themes, the effects of technology in mathematics education.

Some researchers (e.g., Heid & Zbiek, 1995; Hollar & Norwood, 1999; Lynch, Fischer, & Green, 1989; O'Callaghan, 1998) focused on a computer or calculator-intensive approach to algebra. Under the umbrella of algebra with technology, Heid and Zbiek (1995) found that "CIA [Computer-Intensive Algebra] curriculum engages students and teachers in the exploration, discussion, and understanding of mathematics in newly meaningful ways" (p. 656). Hollar and Norwood (1999) found that students in the graphing-approach curriculum as a group did not significantly differ from those students who were traditionally taught, whereas O'Callaghan (1998) found that students using the CIA curriculum significantly improved their mathematical attitudes. Lynch, Fischer, and Green (1989) concluded that "computer-intensive algebra curriculum required new ways to organize, deliver, and evaluate mathematics instruction, as well as new environments and roles for students and teachers" (p. 694). In addition, they found that students improved their problem-solving skills, broadened their concept of mathematics, and developed an appreciation for the purpose of algebra.

In addition to this, the areas where technology was used are very diverse. There are many researchers who investigated the influences of technology (e.g., Ayersman, 1996; Christmann, Lucking & Badgett, 1997; Hollar & Norwood, 1999; Ruthven, 1990; Thompson, 1992) and some did make contributions towards providing a big picture of technology in mathematics education (e.g., Connors, 1997; Kaput, 1992; Kaput & Thompson, 1994). Some researchers investigated problem solving skills with technology experiences (e.g., Casey, 1997; Blume & Schoen, 1988; Hatfield & Kieren, 1972; Nolan, 1984), teacher education with technology (e.g., Borba, 1995; Bornas, Servera, & Llabrés, 1997; Forman, 1997; Maddux, 1997a; Zbiek, 1998), teaching and learning under technology environment (e.g., David, 1994; Hannafin, Hill & Land, 1997; Hannafin & Land, 1997; Hill & Hannafin, 1997; Land & Hannafin, 1997; Noss, 1988; Pokay &

Tayeh, 1997; Schwarz & Hershkowitz, 1999; Wright, 1997). Some researchers used computer games or world wide web as an instructional tool (e.g., Bright & Harvey, 1984; Flake, 1996; Maddux, 1997b; Starr, 1997).

The degree of influence on mathematics education might be slightly different from study to study, but almost all the studies show that the use of technology is a positive influence on students' performances, understanding, and problem solving skills. However, these studies have not shown us enough information about how and why the technology would be used was helpful. Most studies showed a simple aspect of the use of technology. There is no strong consensus on the effect of technology use as yet (Hollar & Norwood, 1999).

On the other hand, Zheng (1998) raised a warning that if the technology is misused, it can be detrimental to students. For example, he said that "in answering $16^{-1/2}$ =?, many students responded 0.25, instead of 1/4. The decimal answer, as the students admitted, came from using calculators" (p. 3). Later, he was able to notice that some of these students did have difficulties in understanding negative and rational exponents and the rules of exponents. He claims that understanding in this situation is "not very desirable from the cognitive perspective" (p. 3). Johnson (1997) was also surprised to recognize how unprepared we are for the use of technology in mathematics classroom. In order to alert the teachers about mathematical learning, Shank and Edelson (1989/90) said that "many of the instructional techniques currently in use are there because technology makes them easy to employ, not because they are educationally sound" (p. 20). Thus, it is questionable whether we are knowledgeable enough to fully take advantage of the technology that we have. I hope this study can contribute to helping us prepare for the efficient use of technology.

What is it like when technology is introduced into a mathematics classroom? Here technology means computers unless it is otherwise specified from now on. The first aspect that we can think of is that the mathematics classroom may become more student-centered rather than teacher-centered (Hannafin, 1992; Hannafin, Hall, Land, & Hill, 1994; Hannafin, Hannafin, Land, & Oliver, 1997; Hannafin, Hill, & Land, 1997; Hannafin & Land, 1997). In a student centered environment the power in the classroom is distributed from teachers to students to some extent. Students have their own time as an individual or as a group. I also assume that the amount of time that students spend with objects they created on their screen also increases. Students have more opportunities to explore problems in depth with their own objects rather than when they simply follow as a teacher leads. The objects that students concentrate on in their screen will be words and phrases if it is a writing class, equations and graphs if it is algebra, and geometric figures if it is geometry. Pilot studies (Jung, 2000a; Jung 2000b) show that it was especially prominent that students' eyes were fixed on the computer screen in geometry class. When GSP was used in the computer laboratory where each student could individually use a computer, I recognized that students spent most of their time looking at images they themselves created on the computer screen. Interestingly, each student had slightly different figures from each other's for the given mathematical situation than those provided by the teacher. From this, we can infer that each student's understanding of mathematics might vary depending on the figures they construct.

Images on computer screen - Visualization & Representation

For more than 100 years, mathematics educators have been interested in the visualization of mathematical ideas (Bishop, 1989). Bishop also pointed out that visual presentation offers a powerful introduction to the complex abstractions of mathematics. Nowadays, the rapid development of technology has increased the range of aids to visualization enormously. Zimmerman and Cunningham (1991) defined visualization as "the process of producing or using geometrical or graphical representations of mathematical concepts, principles, or problems, whether hand drawn or computer-generated" (p. 1). Schnotz, Zink, and Pfeiffer (as cited by Nemirovsky & Noble, 1997) described visualization as "a process of structure-mapping of a visuo-spatial configuration onto a mental model" (p. 1) in an information-processing type of description of the process. On the other hand, Zazkis, Dubinsky, and Dautermann (1996) defined visualization as “an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses” (p. 441). I view visualization as including both the process of constructing or modifying geometrical, or graphical images for the given mathematical problems, concepts, principles, or figures and the process of connecting the internal knowledge or understanding with the images constructed to confirm, refine, and extend pre-existing knowledge or understanding (Jung, 2000b). That is, not only physical construction but also mental activities are the main elements of visualization.

“The computer is thus a very powerful tool in the area of visualization development” (p. 13) said Bishop (1989). Also, visualization plays an important role in delivering abstract concepts to students. The dynamic function of computer software makes its use especially powerful, at least in geometry (Bennett, 1997; Cuoco & Goldenberg, 1997; de Villers, 1998; Finzer & Bennett, 1995; Giamati, 1995; Goldenberg & Cuoco, 1998; Hannafin, Burruss, & Little, 2001; Olive, 1998). On the other hand, Yerushalmy and Chazan (1990) claimed that students looked at the diagrams differently from the way their teacher intended. This is directly related to a student-centered learning environment. There is a computer screen where students can easily fix their eyes on the static or dynamic figures and explore mathematical problems more actively and vividly. Students often walk around their own world for the given mathematical situation. Nowadays, calculators also have graphic functions in addition to simple and repeated numeric computation. If computer software (GSP, the main tool of the course) creates images on the computer screen, then the process of creating images may be the main material for students to work with. Many researchers (Ben-Chaim & Lappan, 1989; Bishop, 1989; Eisenberg & Dreyfus, 1989; Hershkowitz, 1989; Kiser, 1990; Presmeg, 1986a; Presmeg, 1986b; Presmeg, 1989; Smith, 1996) made efforts toward figuring out the meaning and the role of visualization in mathematics education. Although their work related several obstacles in understanding visualization, they found that visualization played an important role psychologically and mathematically in mathematics education. Then, what is the relationship between visualization and representation?

From reviewing various literature, I was able to recognize the difference between visualization and representation. Visualization deals mainly with graphical or geometrical images externally produced for the given mathematical situations and connections internally formed with previous knowledge or understanding, whereas, visualization is contained as a proper subset within representation. Davis, Young, and McLoughlin (1982) defined representation as "a combination of something written on paper, something existing in the form of physical objects and a carefully constructed arrangement of idea in one’s mind" (p. 54). As a compatible notion of their definition, Janvier (1987) viewed that representation consists of three components: symbols (something written on paper), real objects (physical objects), and mental images (arrangement of idea in one’s mind). If we match ‘real objects’ with ‘images externally reproduced’ and ‘mental images’ with ‘connections internally formed,’ the third component, ‘symbols,’ is left as the factor to make the meaning of representation broad so that the notion of representation includes that of visualization (Figure 1.1).

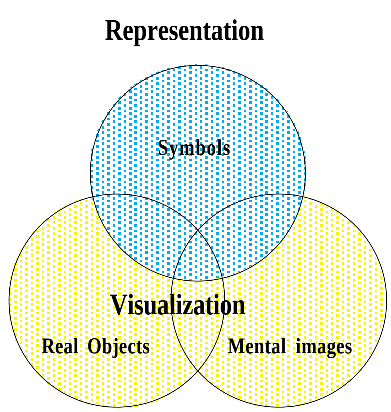


Figure 1.1 Representation contains visualization

Visualization, consisting of external objects and mental objects, works as part of representation. Studying mathematics with computers necessarily implies working with objects on the computer screen mentally or physically. The first notion that we can think of in studying mathematics with computers will be visualization. In other words, students will look at the images on the computer screen created either by themselves or somebody else. At the same time, they will think about those objects in their mind along with further exploration. However, in addition to objects, because the symbols enhance concepts (Hiebert & Lefevre, 1986) and help us mathematically to communicate better (NCTM, 1989), we can extend the notion of visualization to a more comprehensive notion, which is representation containing visualization as a part in it. Thus, I approached this study with representation in mind and visualization as the main part of representation.

Understanding and growth of understanding

One of the most important ideas accepted in mathematics education is that students should *understand* mathematics. Understanding is an important element of almost all the studies in mathematics education: assessment (Berenson & Carter, 1995; Stallings & Tascione, 1996; Stix, 1994); classroom environment (Henningsen & Stein, 1997); culture or race (Fuson, Smith, & Lo Cicero, 1997; Ladsin-Billings, 1997; Sleeter, 1997); curriculum development (NCTM, 1989); problem solving (Baranes, Perry, & Stigler, 1989; Mayer, 1985; Paul, Nibbelik, & Hoover, 1986); teacher education (Barnett, 1991; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Lehrer & Franke, 1992); teaching and learning (Ausubel, 1968; Hiebert & Carpenter, 1992; Linchevski, & Kutscher, 1998); technology (Hart, 1982; Land & Hannafin, 1997; O'Callaghan, 1998); representation (Janvier, 1987; Kiser, 1990; Nemirovsky & Noble, 1997). Eisenhart et al. (1993) nicely and strongly suggested how important 'understanding' is in mathematics education:

> Teaching mathematics for understanding is one of the hallmarks of current reform efforts in mathematics teacher education. Numerous commissions

> (Cockcroft, 1982; Collins, 1988; Howson & Wilson, 1986; Mathematical Sciences Education Board, 1991) and professional organizations (e.g., Mathematical Association of America, 1991; National Council of Teachers of Mathematics [NCTM], 1989) have called for teachers to devote more time and attention to developing students' understanding of mathematics. But teaching mathematics for understanding is an extremely complex process (Hiebert, 1986), and the mathematical and pedagogical skills and knowledge needed are considerable (Ball, 1991; McDiarmid, Ball, & Anderson, 1989). (pp. 8 – 9)

Also, Hiebert and Carpenter (1992) claimed that "the goal of research and implementation efforts in mathematics education has been to promote learning with understanding" (p. 65). Knowledge, when it is understood with strong and various connections to preexisting knowledge, plays a crucial role in preserving mathematical knowledge (Hiebert & Carpenter,1992). This is also closely related with Ausubel's (1963) assimilation theory for cognitive learning, that is, meaningful learning. So, what is 'understanding'? What does it mean to understand mathematics? There are many researchers who have tried to figure out the meaning of understanding (Brownell & Sims, 1946; Buxton, 1978; Byers & Herscovics, 1977; Hannafin & Land, 1997; Haylock, 1982; Hiebert & Carpenter, 1992; Janvier, 1987; Pirie & Kieren, 1989; Pirie & Kieren, 1994; Schoenfeld, 1992; Skemp, 1978; Skemp, 1987; Wiggins, 1993). To understand mathematics "is a much more complex question than might appear at first sight. Probably this is the reason why there have been few recent attempts to deal with it in any sort of depth" (Byers & Herscovics, 1977, p. 24).

Since the article by Richard Skemp (1978), *Relational understanding and instrumental understanding*, it is noticeable that many researchers have tried to make the meaning of understanding clear and precise in mathematics education. Almost all of their works in describing the meaning of understanding contain references to mental and physical activities in connection with past experiences. It is also continually ongoing rather than static at a certain place, i.e. it is dynamic. Janvier (1987) suggested the following features of understanding:

1. Understanding can be checked by the realization of definite mental acts. It implies a series of complex activities.
2. It presupposes automatic (or automatized) actions monitored by reflexion and planning mental processes. Therefore, understanding cannot be exclusively identified with reflected mental activities on concepts.
3. Understanding is an ongoing process. The construction of a ramified system of concepts in the brain is what brings in understanding. Mathematics concepts do not start building up from the moment they are introduced in class by the teacher. This well-known tenet is not easily nor often put into practice in day-to-day teaching.
4. Several researchers attempt to determine stages in understanding. We incline to believe that understanding is a cumulative process mainly based upon the capacity of dealing with an "ever-enriching" set of representations. The idea of stages involves a unidimensional ordering contrary to observations. (p. 67)

Along with these features above, this study was based on three external components of understanding: symbolizing, visualizing, and verbalizing (Figure 1.2). Because symbols enhance concepts of students (Hiebert & Lefevre, 1986) and visual or mental presentation, e.g. figures or verbalizing, offer a powerful introduction to the complex abstractions of mathematics (Bishop, 1989), all three components can work together for better understanding. At the same time, however, each component can stand itself or one specific component can possibly be strengthened with the help of the others. These three components of understanding are also in line with those of representation.

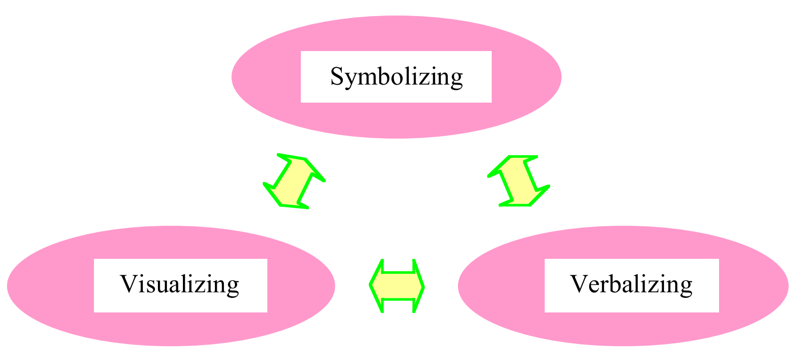


Figure 1.2 Three components of understanding

Assuming that understanding is a dynamic ongoing process, I would like to examine how understanding grows rather than simply describe what mathematical understanding is. Although many researchers were interested in describing the meaning of understanding and understanding is one of the most important topics in mathematics education, there are very few works done about describing the growth of understanding (Pirie, 1988; Pirie & Kieren, 1989; Pirie & Kieren, 1992; Pirie & Kieren, 1994).

I adopted Pirie and Kieren's model (1994), 'growth in mathematical understanding' (Figure 1.3), for this study. This model consists of eight levels and each level has a distinct feature, which will be explained in detail later. Those potential levels or distinct modes are as follows: primitive knowing, image making, image having, property noticing, formalising, observing, structuring, and inventising (Pirie & Kieren, 1994). Their model can be applied to any level of knowledge and to any age in the learning process. Pirie "observed understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories" (Pirie & Kieren, 1994, p. 165). They stress that they do not see the growth of understanding as a monodirectional process. Rather they view it as back and forth movement between levels.

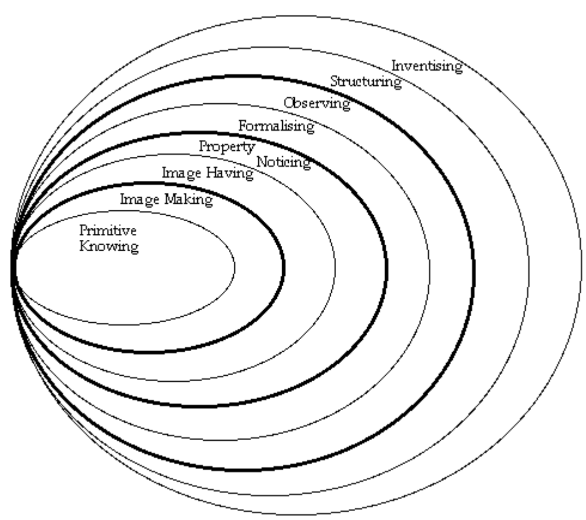


Figure 1.3 Model for the growth of understanding (Pirie & Kieren, 1994, p. 167)

There are two crucial features in Pirie and Kieren's model: 'don't need boundaries' and 'folding back' (Figure 1.4). The former means that a learner is able to operate the mathematical situation without reference to basic concepts. For example, suppose that a learner can construct an image in his mind (image having) for the mathematical situation. Then, he does not need actions to create an image (image making). Bold rings in their model represent 'don't need boundary.' Whereas, the latter means when a learner faced with a problem or question at any level, which is not immediately solvable, one needs to *fold back* to an inner level in order to extend one's current, inadequate understanding. A learner can do 'folding back' at any level, maybe the 'folding back' is necessary for an advanced, deep, and refined understanding. Detailed explanations follow in the next chapter.

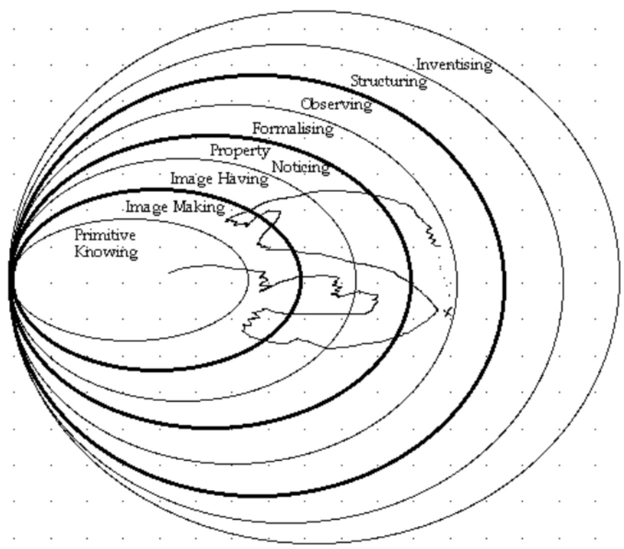


Figure 1.4 Folding back (Pirie & Kieren, 1994, p. 186)

Three pilot studies

Pilot study I

The first pilot study (Jung, 1999) was conducted in a technology-based college geometry classroom (Figure 1.5) where *The Geometers' Sketchpad* [GSP] was used throughout the semester instead of any textbook. The classroom had enough computers so that every single student could use one individually, and it was equipped with a projector for the computer and for transparencies.

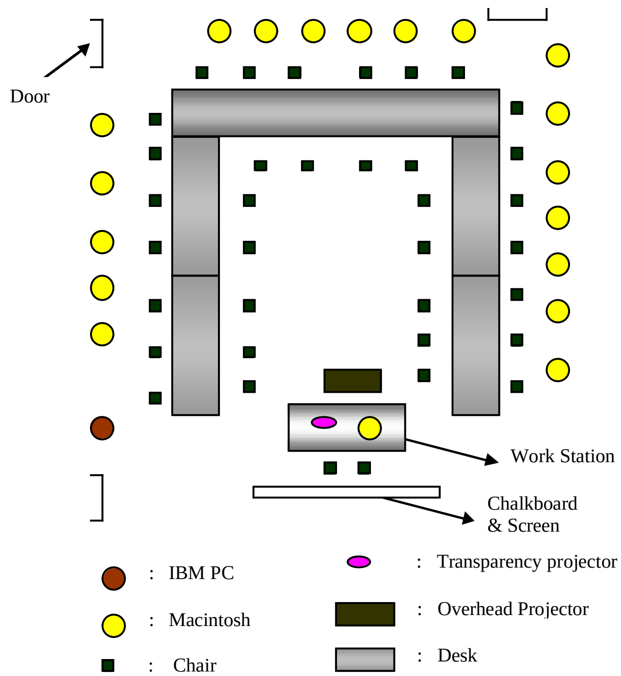


Figure 1.5 The structure of the computer laboratory

I attempted to investigate how GSP helped two college students to better understand mathematical concepts in the uniquely structured setting. Two participants volunteered for this study. Although there were some obstacles found such as 'proficiency with technology for doing mathematics' and 'access to technology after school,' I was able to find the potential positive powers of technology from this study. One of the findings is that participants, Simon and Doris, were hesitant to lean on GSP in the beginning of semester. This was their very first time to learn mathematics in a technology-based environment. They were not used to using computers for studying mathematics and were not skillful enough to explore or study mathematics in this study. Simon said that "It always starts with an idea in my mind" (Jung, 1999, p. 5) and Doris said that "As long as I know mathematical concepts by hand, GSP is very helpful in the sense of quickness and easiness" (Jung, 1999, p. 5). But, as the semester was moving along, they naturally came to use GSP rather more than pencil and paper.

The second finding was that GSP helped students to enjoy geometry, confirm their understanding, and investigate further. Simon, who quickly picked up the use of technology, said that "If I draw a triangle with my hands, it looks like a square. GSP is quicker and easier. Since GSP is graphically powerful, I can easily verify that I know. It's just fun to play with it" (Jung, 1999, p. 6). The more skillful he became with GSP and the more geometry he learned, the more he relied on GSP as the semester was going along. Doris, although she was a bit slower in learning technology, said that "I never played computer games before. So, it took a long time to learn how to use GSP. But, once I understood concepts and knew how to use GSP, GSP was very powerful and helpful. It was worth it to spend time for learning GSP. I enjoyed the second semester a lot more that the first semester" (Jung, 1999, p. 6).

Concluding this study, I raised the question that the proficiency might possibly be a crucial factor for students' understanding because I observed a bit of difference in terms of proficiency between two participants: one was more skillful with GSP than the other. The skillful participant had a tendency to use technology more often than the less skillful one. This triggered the following study.

<u>Pilot study II</u>

Thus, I conducted another research study (Jung, 2000a) from the perspective of 'students' proficiency' with technology [GSP] in the same setting (Figure 1.5) with the first pilot study. This study closely examined 1) how they overcome the difficulties of inexperience with GSP; 2) if there is any change in their view of mathematics in light of their use of technology; 3) what the instructor should consider about students' inexperience with GSP. Two college students, Tiffany and Mark, who had no experience with GSP, were selected. Interestingly, both of them had a lot of experience and a good bit of knowledge about computers in general. Tiffany used a computer since she was 4 years old and played a lot of games using computers. Mark also used it a lot due to his past and current job. Their past experiences and general knowledge about computers helped them quickly learn how to use GSP to some extent.

They labeled themselves as those who learn software by themselves in most cases. Tiffiany said that "I play around with it until I find it. If I play around enough, I figure it out" (p. 6). In the case of Mark, he said that "In the beginning, GSP bothered me. I refused to touch it because I just want to do it by paper .... He [instructor] showed me how to use it. I went back and practiced it. Oh, I can do this now! Now, I do my homework, first time I do it in GSP" (p. 7).

For the second question of this study, there were a lot of changes positively in favor of technology. Unfortunately, Mark dropped the course due to personal reasons, but he said that he enjoyed using GSP in studying geometry and that he would go back to school as soon as possible to be a mathematics teacher. Tiffany also enjoyed the course throughout the semester. Her responses about the use of technology shifted from 'I agree' to 'I strongly agree' in many items of the questionnaire (Appendix D). She had no strong feeling about the use of technology in the beginning of semester. But, the growth of her skills with GSP and geometrical understanding led her to rely on and to favor technology. Moreover, besides GSP, she liked the Internet because she could get a lot of sources for her study. Also, the instructor put notes on his website so that students could refer to them.

Both participants seemed to have no difficulty in learning how to use GSP. Rather, they had a hard time with mathematical understanding in general. When they had stable understanding of geometrical concepts, GSP was very powerful for them to explore mathematical problems even more deeply than expected by the instructor. If mathematical concepts were beyond them, GSP was not very much helpful at least in the beginning.

The first thing that the instructor pointed out about the course was the class size. There were 28 students at the time I was observing his class. He recommended that there should be about 18 students or so in this classroom environment. Although he taught only one course during that semester, he had to spend most of his time preparing lessons. First of all, he had to design the course almost from scratch. It was first time he taught geometry in this unique setting, without even a textbook. Moreover, there were various levels of skills with technology among the students, ranging from those who never heard of GSP to those who are very skillful. One other problem he found during the course was that the class did not go as he intended. His idea was to approach the mathematics theoretically, but students were not ready to do that. He felt that he lost half of students at some point. Thus, he had to stop using technology for a while and went back to traditional way in teaching mathematics, which means simply teaching mathematics with chalkboard and chalk without students' individual exploration in the classroom.

Although there were various levels of proficiency with technology among the students and trials and errors with the course, students reached a reasonable level so that they could do their own work using GSP for the given task. Their past experiences with computers were the intermediate tools that helped them to overcome inexperience with technology. Tiffany felt very comfortable with technology and was able to solve most of the problems using GSP as the semester went along. She said that it would be difficult to study geometry without GSP. But, understanding mathematical concepts seemed to be more important rather than just knowing how to use GSP. I realized that because people are exposed to a society where various technologies are everywhere, the proficiency of using any software did not seem to be the main issue. Inexperience with technology was not the main obstacle to their studying geometry.

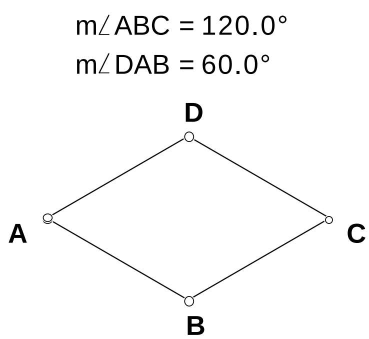
From this study, I noticed an interesting phenomenon. I found out that each participant had his/her own world, i.e. computer screen, for the given mathematical problem. That is, the problem posed was exactly the same, but the pictures on each screen were slightly different from person to person. Especially because they were required to use GSP throughout the semester, they spent most of their time looking at the images they constructed on the screen. Maybe, they were forced to look and use images on the screen. The quickly, easily, and accurately drawn figures on screen gave good opportunities and provided environments for students to explore problems in various ways and deeply. The following study more closely examined these issues.

Pilot study III

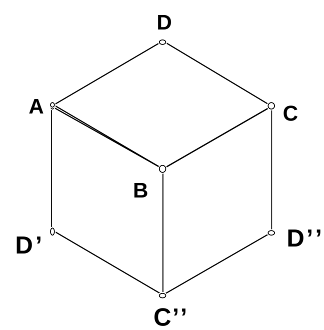
The research question in this pilot study was introduced from the second pilot study (Jung, 2000b). Technology in geometry generally means computers rather than hand-held calculators. In this case it was GSP. In a traditional and teacher-centered classroom, students look at the figures that the teacher constructs on the board, while in a technology-based classroom students create their own figures and deal with those for the given mathematical situation or refer to those figures created by the teacher. Students' thinking processes are stimulated by the images on screen to some extent. The purpose of this third pilot study was to find out 'to what extent and how computer screen images impact on students' understanding of geometry in a technology-based classroom.'

I selected two participants for a more intensive and in-depth study, but one dropped out right before the first interview. Thus, I ended up with one participant, Melissa. She was a very active junior in her class and had a hard time with mathematics in general from elementary up to high school. She said that “When I was in elementary school, they always put me ahead in math. And I never understood why and I never understood math. And this kept going through high school” (p. 8). But, she also said that “Obviously I am not dumb in math. I am capable of understanding algebra and trig” (p. 8). From the experience of being a substitute teacher for a while, she decided to be a mathematics teacher.

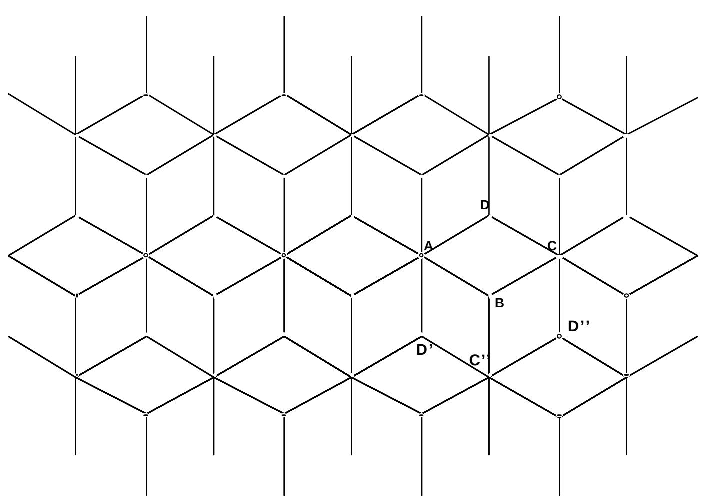
Data were collected in the same site (Figure 1.5) as the first and second pilot studies. The first finding of this study was that visualization coupled with previously fixed images could be an obstacle to student's understanding. When she was working with 'P3-type wallpaper pattern' to find rotations, she was confused with two dimensional space and three dimensional space. She had no confusion with the initial generator (Figure 1.6). But, for the plane generator (Figure 1.6), it was seen by her to be just like in three dimensional space. Thus, looking at the completed P3-type (Figure 1.6), she could not see 120° rotations. They were 90° rotations to her. "I just see it in 3-D for some reason. I don't know why." said Melissa (p. 13).



Initial generator of P3-type



Plane generator of P3 type



Completed P3-type of wallpaper pattern

Figure 1.6 The process of constructing P3 type

As in the case of the second pilot study, Melissa also showed that mathematical understanding and knowledge were much more fundamentally needed in visualizing mathematical situations. She was able to create all types of wallpaper patterns using GSP. But, her lack of mathematical understanding and knowledge hindered her from seeing mathematical properties for the constructed figure. She knew that all triangles in the modified P3-type of wallpaper pattern (Figure 1.7) were equilateral triangles. When she tried to find centers of rotation, she easily noticed a center of rotation at the vertex of each triangle, but could not recognize the center of rotation at the centroid of each triangle. The constructed figure possibly tricked her eyes to be fixed on the vertices of the triangles. She finally made a conjecture that the center of a triangle [centroid] could be a center of rotation. But, she could not support her conjecture mathematically. Simply she used her insight for this. Thus, she became dubious of the conjecture that she made when further questions were asked.

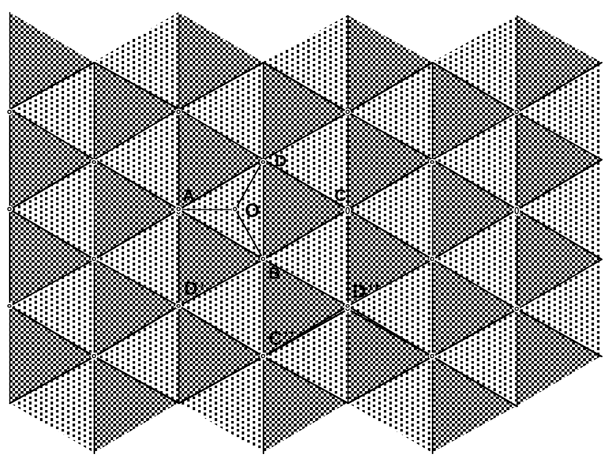


Figure 1.7 Modified P3-type of wallpaper pattern

Finally, visualization helped Melissa to recognize new properties. She constructed a PG-type wallpaper pattern using two different transformations, glide reflection and translation, in GSP. It was beyond her imagination to create a PG-type wallpaper pattern in her mind, but, construction using GSP helped her to overcome this difficulty. Moreover, she was able to recognize new properties from this wallpaper pattern which she did not expect during the process of construction. "The neat thing about that is I did not make this with any rotation. I made it with reflection and transformation. So, the rotation is just the symmetry that this one has" said Melissa (p. 18).

In order to explore symmetries of wallpaper patterns, visualization was critically important to Melissa throughout the semester. She also strongly agreed that she understood better because of GSP. Especially, the dynamic function of GSP appealed to the usefulness of new technology for learners and teachers. Melissa seemed to be helped by GSP and to enjoy GSP:

> I think I understand geometry better because of GSP. Umm. Being able to move things is so important. I mean you can move it and see if it doesn't work there or look at if it still works here or what happens if I shift this point around. Just being able to see things move makes all the difference in the world. Cause when you're checking something on paper, you know, you draw this way. You draw equilateral triangle and you draw an isosceles triangle and try right triangle. And it takes forever. And it's not always accurate. You can't always get exact, you know, you are just sketching. (p. 20)

Significance of the study

Technology is not only for those professionals who work in science or industry anymore. The development of technology in both quantity and quality has grown by geometric progression. "According to the President's Committee of Advisors on Science and Technology (1997), a ratio of 4 to 5 students per computer represents a reasonable level for the effective use of computers within schools" (Rowand, 1999). Data from National Center for Education Statistics [NCES] (Bare, Foundation, & Meek, 1998) show approximately 6 students per computer in public schools. In addition, about 35 percent of public schools were connected to the Internet in 1994 but, in the fall of 1998, 89 percent of public schools were connected to the Internet (Rowand, 1999).

Now we have decent level of facilities with technology in elementary and secondary schools, but it is questionable whether we take advantage of the technology that are available to us. I think it is time for us to build up more substantial and practical foundations for the use of technology in the mathematics classroom so that technology can contribute to students' better understanding. Computers create new challenges for educators because they alter the classroom "ecology" (Borba, 1995, p.334). Johnson's (1997) surprise to recognize how unprepared we are gives us a message about the use of technology in mathematics classroom. Shank and Edelson (1989/90) also pointed out technology is not being used as it is supposed to be. I believe that this study can contribute to the current and future preparation of a teacher in ways that promote how we can use technology for teaching and learning mathematics with students' better understanding.

The most important two components of this study are 'growth of understanding' and 'representation' in a technology-based classroom. I agree with many other researchers who claim that 'understanding' is a major goal in mathematics education. Many researchers put forth efforts to make sense of the meaning of understanding. This study tried to describe how representation affects students' growth of understanding. Along with the growth of understanding, because students will lean upon the world that they create using technology to some extent, I see the significance of this study how representation consisting of symbols, real objects, and mental object affects the growth of student's understanding. Although it has been a long time since people have been interested in representation (especially visualization which is the main part of representation) the importance of representation is becoming greater than ever before in the situation where we are equipped with technology. I believe that it is the right time to investigate the growth of understanding with the perspective of representation in a technology-based classroom.

I also believe that this will contribute to the discipline of mathematics education. Technology is now provided in the educational field. However, it is not extensively used by teachers and students in teaching and learning mathematics. There are currently many teachers who were trained in a traditional way and have taught mathematics that way for a while. The new technology tools can be easily mistreated against our intention or may be used in a very limited sense. This study investigates the growth of students' understanding, which is the extension of the major goal of understanding, in mathematics education with the lens of representation. Because there was no textbook used throughout the course that I observed for the investigation, this study was conducted in a very unique situation to closely examine the impacts of representation on students' growth of understanding. I suspect that there are many teachers and learners who want to use technology more efficiently so that they can teach and learn mathematics with better understanding. I also suspect that there are mathematics educators and administrators who have been interested in understanding the role of technology in mathematics education. I sincerely hope that this study can contribute to the mathematics education society as well as to the people who are interested in the wise, effective, and successful use of technology in mathematics classroom.

Research questions

The primary purpose of this study is to know how and to what extent 'representations' affect the students' understanding and the growth of understanding in a technology [GSP]-based classroom. Based on this purpose, the following research questions guided this study:

1. How do students present each component of representations when they study 'transformations' in a technology [GSP]-based classroom? If there is any difference between the first and second presentation for each component, how are they different?

There are three components in representation: symbols (written), real objects (pictorial), and mental objects (verbal) (Janvier, 1987). I asked participants to answer the diagnostic test (Appendix E) in the beginning and at the end of the study, compared their presentations in terms of these three components of representations in detail, and analyzed how each component is different.

2. How and to what extent do representations affect the students' understanding and the growth of understanding in a technology [GSP]-based classroom?

I investigated how students' understanding grows based on Pirie and Kieren's model (1994), 'growth in mathematical understanding' (Figure 1.3). Assuming that understanding is not linearly or unidimensionally developing, I used this model in order to describe the students' transition in terms of understanding. I also examined which components of representation were adopted and what were the roles of those components for the transition. The 'folding back' and 'don't need boundaries,' which are the crucial features in their model, were closely examined when transition happened. Along with this transition, it was also closely observed which components of representation were adopted and what were the roles of those components for the 'folding back' and ' don't need boundaries.'

3. What types of benefits and obstacles are there when students study 'transformations' in a technology [GSP]-based classroom?

I closely investigated what kinds of obstacles and benefits exist for the students' growth of understanding. Further, I examined how we can take advantage of benefits and overcome those obstacles.

Summary and overview

As the introductory chapter, I described the big picture of this study. My deep impression, interest, and enthusiasm led me to conduct this study. The rapid emergence of technology in quantity and quality made me feel the necessity of doing this study so that we can be better ready for more efficient use of technology in our mathematics classrooms. The following notions triggered this study: the necessity of the use of 'technology' in schools; the importance and significance of 'representation' with the use of technology; 'understanding' accepted as the major goal in mathematics education; 'the growth of understanding' extended from the recognition of the importance of understanding. In addition, three pilot studies helped me to refine and update my specific research area with appropriate research questions. I believe that this study will contribute to teachers, learners, and mathematics educators to some degree. The purpose of this study is to know how and to what extent 'representation' affects the students' understanding and the growth of understanding in a technology [GSP]-based classroom.

In the following chapter, I review various literature relating to the following elements in mathematics education: The nature of understanding; Skemp's model of understanding and those related; Some more models of understanding; The growth of understanding; Conceptual and procedural knowledge; Visualization; Technology in mathematics education, Influences or impacts of using technology; Teaching and learning with technology; Geometry in mathematics education and van Hiele model, Logo and dynamic geometry; World wide web as instructional tools. These important elements were somehow closely related with this study. In chapter 3, I describe the most appropriate methodology adopted to answer the research questions - qualitative methodology: case study. Two participants were selected based on three criteria. There were five different methods used for data collection: descriptive notes, reflection notes, documents, interviews, and concept maps. For the data analysis, 'constant comparison method' was used. In chapter 4, I described findings based on the three research questions proposed above. In chapter 5, I give an overall description of this study and provide the conclusions and implications of this study.

CHAPTER 2

REVIEW OF RELATED LITERATURE

This chapter is a review of literature relevant to theoretical backgrounds and the research questions of this study. The main portion of this study concerns representation and the growth of understanding in a technology-based environment. My deep interest in how technology can be used in mathematics education for better learning with understanding triggered this study. Especially, among many modes that technology has, figures on computer screen which stimulates students' thinking and understanding were the main interest for this study. Since understanding is one of the most important themes in mathematics education (Eisenhart, et al, 1993; Hiebert & Carpenter, 1992; NCTM, 1989; Skemp, 1978), this chapter examines a range of literature on mathematics understanding and the growth of mathematics understanding. Particular attention is given to the literature on the role of visualization in mathematics understanding and the impact of technology use in mathematics learning with understanding.

The nature of understanding

Learning with understanding has been recognized as one of the most important areas in mathematics education research. We have a reasonable description of mathematics understanding although there is not an agreed upon precise definition.

I would like to start this section with the general concepts and essential characteristics of understanding addressed by Brownell and Sims (1946, pp. 28-43).

First, we may say that a pupil understands when he is able to act, feel, or think intelligently with respect to a situation. The term, 'situation,' is used to mean any set of circumstances to call for an adjustment, in other words there is "no obvious way" (Mayer, 1985, p. 123) or "the direct route" (Kilpatrick, 1985, p. 3) in resolving the circumstances. A situation may be equated with 'problem,' which "occurs when you are confronted with a given situation - let's call that the *given state* - and you want another situation - let's call that the *goal state* - but there is no obvious way of accomplishing your goal" (Mayer, 1985, p. 123). Second, rather than being all-or-none affairs, understandings vary in degree of definiteness and completeness. It is generally assumed that the completeness and definiteness of our understandings may vary directly with the amounts and kinds of experiences we have had. And the degrees, qualities, and kinds of understandings manifested by the several members of a group of children may vary for the given situation. Third, the completeness of understanding to be sought varies from situation to situation and varies in any learning situation with a number of factors. In order to understand anything in the sense of completeness, one is required to have a thorough grasp of its function, structure, and incidence. Fourth, typically, the pupil must develop worth-while understandings of the world in which we live as well as of the symbols associated with this world. Fifth, most understandings should be verbalized, but verbalizations may be relatively devoid of meaning. Sixth, understandings develop as the pupil engages in a variety of experiences rather than through doing the same thing over and over again. Seventh, successful understanding comes in large part as a result of the methods employed by the teacher. Eighth, the kind and degree of the pupil's understanding is inferred from observing what he says and does with respect to his needs.

Skemp's model of understanding and those related

Skemp (1978) was one of forerunners of more intensive and active studies about figuring out "What does it mean to understand mathematics?" (Byers & Herscovics, 1977, p. 24), which is a much more complex question to answer than might appear at first. Skemp gave credit to Stieg Mellin-Olsen for distinguishing two meanings of understanding using two terms, 'relational understanding' and 'instrumental understanding.' Skemp (1978) himself described 'relational understanding' as "knowing both what to do and why" (p. 9) and 'instrumental understanding' as "rules without reasons" (p. 9). Earlier, he did not regard instrumental understanding as understanding.

In his article 'relational understanding and instrumental understanding,' however, Skemp (1978) proposed three advantages that instrumental understanding might provide. First, "within its own context, *instrumental mathematics is usually easier to understand*; sometimes much easier" (p. 12). For example, some topics which are difficult to understand relationally such as multiplying two negative numbers, subtraction of two negative numbers, or division of two fractions, are easily understood within a short period of time in an instrumental way. Second, "*the rewards are more immediate and more apparent*" (p. 12). It is good to have a feeling of success after students get a right answer, which must not be underrated. Those students who consider themselves as 'thickos' need this kind of feeling to restore the self-confidence. Third, "Just because less knowledge is involved, *one can often get the right answer more quickly* and reliably by instrumental thinking rather than the relational" (p. 12). It is recognized that sometimes "even relational mathematicians use the instrumental thinking" (p. 12).

Skemp also proposed the four advantages that relational understanding might provide. First, "*It is more adaptable to new tasks*" (p. 12). Skemp observed that after students learn how to multiply two decimals by dropping the decimal points, multiplying it as whole numbers, and re-inserting the decimal points, they were able to apply what they understood to the division of two decimals without any difficulty and confusion. This phenomenon leads to the following advantage, "*It is easier to remember*" (p. 12). We can see the paradox here. It is true that it is harder or takes more time to learn mathematics in the perspective of relational thinking, i.e. "It is certainly easier for pupils to learn that 'area of a triangle = 1/2 base × height' than to learn why this is so" (pp. 12-13). But, it is not desirable if students have to learn separate rules for the areas of triangles, squares, rectangles, parallelograms, trapezoids, etc. Ideas from understanding a particular topic may turn out to be the potential power to understand several different related topics so that students can easily understand or remember new information. Third, "*Relational knowledge can be effective as a goal in itself*" (p. 13) and fourth, "*Relational schemas are organic in quality*" (p. 13). Connecting the third and fourth points, he claims that "if people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring a new territory in search of nourishment" (p. 13).

Byers and Herscovics (1977) proposed some different types of understanding that are neither instrumental nor relational understanding. 'Intuition', generally equated with 'guessing', is often mentioned and used for the non-deductive mathematical thinking. Byers and Herscovics described intuition as the immediate relation to given problems as if they were very familiar. There are other characteristics of intuition: "implicit perception of the problem as a whole, reliance on imagery, the absence of well-defined steps, little awareness of the solving process and difficulty in reporting it" (p. 25).

Another type of understanding Byers and Herscovics recognized is that "the mathematics we teach is a cultural product developed by generations of mathematicians. The development of this product has been inseparable from the development of mathematical symbolism and notation" (p. 25). Students' understanding may come from intuition but it is eventually the result of analytic thinking. The understanding in this case is mainly focused on representation and on logical reasoning. They also recognize that "the understanding of mathematics deepens with the acquisition of new mathematical knowledge. Moreover, the relative importance of the various types of understanding changes as the students' understanding deepens" (p. 27). As a result they suggest the four different types of understanding of mathematics as follows:

> Instrumental understanding is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
>
> Relational understanding is the ability to deduce specific rules or procedures from more general mathematical relationships.
>
> Intuitive understanding is the ability to solve a problem without prior analysis of the problem.
>
> Formal understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. (p. 26)

Unlike Byers and Herscovics (1977), Backhouse (1978) considers Skemp's two meanings of understanding sufficient. In 1987, Richard Skemp finally suggested three kinds of understanding, which are similar to Byers and Herscovics' (1977), but with a small change:

> Instrumental understanding is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
>
> Relational understanding is the ability to deduce specific rules or procedures from more general mathematical relationships.
>
> Formal [= Logical in my table] understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. (p. 166)

On the other hand, Buxton (1977) suggests four levels of understanding. The first level is 'rote' which is the purely instrumental. In this level, the information is imprinted on students' mind and is reinforced through constantly repeated experiences. If students were given a "7x9?", they answer 63 without going through thinking process. Next, it is 'observational' which is slightly deeper than purely instrumental but not fully relational. In this level, students "perceive a relation or pattern, that both serves as a mnemonic and makes a more general statement" (p. 36). It is not clear if anything has been surely understood although students feel more satisfied and confident than pure instrumental understanding. The third level is 'insightful' which is certainly relational. It is characterized by "Oh, I see...." At this level, students feel that they understand not only how a specific concept works but why. The final level is 'formal' which is "only appropriate after insightful or relational understanding is achieved and at a stage in the student's development where some idea of the need for and the nature of proof is accepted" (p. 36).

Some more models of understanding

Haylock (1982) defines understanding something as "to make (cognitive) connections" (p. 54). He also claims that the more connections a learner can make between the new experience and previous experiences, the deeper the understanding is. The new experience sometimes connects previously unconnected experience. If this happens, drastic advance in understanding is expected. However, if students fail to make connections between the new experience and previous experiences, the new experience will be presented as an isolated chunk and float around in an unstable situation, which is called 'rote learning.' If that is the case, it is doubtful whether concepts learned through rote learning can be fully used whenever they are needed. Probably, it is hard to keep the value of new experience, and furthermore it will be lost forever at some point.

The delicate issue of Haylock's model is how we recognize the evidence of student's understanding, i.e. something that indicates that students made connections. Haylock regards this task as one of the most important tasks for the teacher because the teacher can reward and therefore encourage students. Haylock suggests four components should be considered for identifying important connections in mathematics: words, pictures, concrete situations, and symbols (Figure 2.1). It is assumed that students can demonstrate some degree of understanding and that they can make a suitable connection between categories. Thus any arrow in the figure may suggest means of assessing an aspect of understanding for the given mathematical ideas.

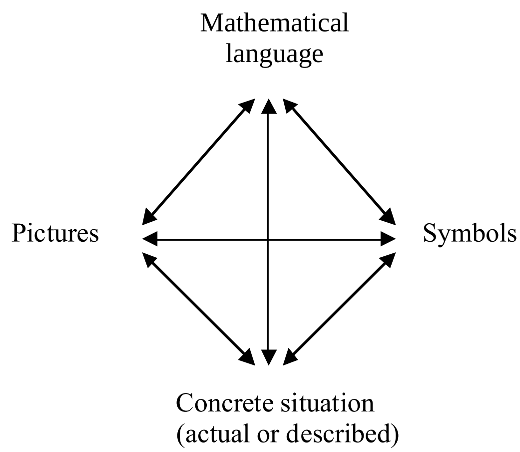


Figure 2.1 Four components of identifying connections

For example, "*Write in figures four hundred thousand and seventy-three*" (p. 55) can be accepted as an item to assess the connection from mathematical language to symbols. The next picture problem (Figure 2.2) can be regarded for the connection from pictures to symbols. The problem 'to write a story' for a calculation like 84 ÷ 28 will be the item for the connection from symbols to a concrete situation.

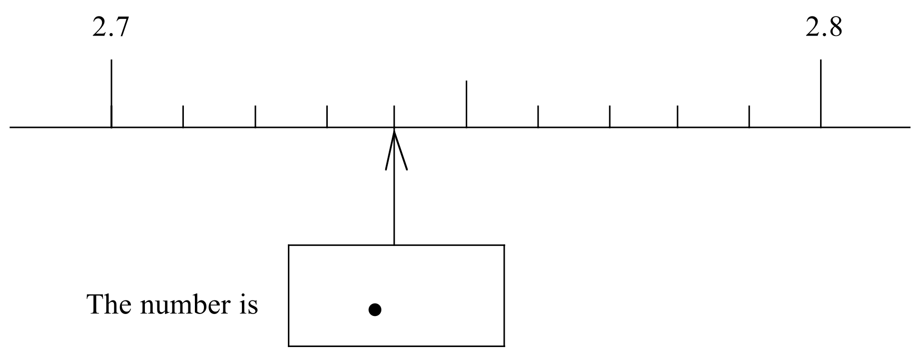


Figure 2.2 Connection form pictures to symbols

Wiggins (1993) defines understanding as the ability to use knowledge "wisely, fluently, flexibly, and aptly in particular and diverse contexts" (p. 207). Hiebert and Carpenter (1992) define understanding based on the way information is represented and structured. "A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections" (p. 67). They claim that understanding grows as the networks within students' mind become larger and better organized.

> Understanding increases as networks grow and as relationships become strengthened with reinforcing experiences and tighter network structuring .... Growth can be characterized as changes in networks as well as additions to networks. ... Ultimately understanding increases as the reorganizations yield more richly connected, cohesive networks." (p. 69)

It is argued that understanding is "neither inherently hierarchical nor the product of incremental teaching methods, but a natural consequence of curiosity, experience, reflection, insight, and personal construction" (Hannafin & Land, 1997, p. 181) and "involves continually modifying, updating, and assimilating new with existing knowledge. It requires evaluation, not simply accumulation" (p. 189). Pirie and Kieren (1989) summarized understanding as follows:

> Mathematical understanding can be characterized as levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication .... Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without. (p. 8)

The growth of understanding

Hiebert and Carpenter (1992) explained the increase of understanding in terms of networks constructed within students' mind. They said that students' understanding grows as the organizations of networks become larger and the depth of connections is getting strengthened. Understanding is expected to be "constantly modified and refined as a result of successive experiences and reflections" (Hannafin & Land, 1997, p. 189) rather than static or accomplished with one time of experience or reflection. Even the newly understood concept may be lost forever if they do not make any connection with previous ones (Haylock, 1982).

Pirie "observed understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories" (Pirie & Kieren, 1994, p. 165). Pirie and Kieren (1994) differentiated the growth of understanding into eight potential levels or distinct modes: primitive knowing, image making, image having, property noticing, formalising, observing, structuring, and inventising. Especially, their model can be applied to any level of knowledge and to any age in the learning process, which is a critically important feature of their model. Pirie and Kieren's model is not for defining the concept of understanding. Rather, their model is for explaining the growth of learner's understanding.

‘Primitive knowing’ starts the process of coming to understand, “Primitive here does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding. It is what the observer, the teacher, or researcher assumes the person doing the understanding can do initially” (p. 170). The first recursion occurs when the learner begins to form images out of this doing, called ‘image making.’ In this level, "the learner is asked, through specific tasks, to make distinctions in her previous abilities, using them perhaps under new conditions or to new ends" (Pirie & Kieren, 1992, p. 246). And the 'image' is not strictly restricted to pictorial representation. It can be anything which can convey any kind of mental representation. In the next level, ‘image having,’ action-tied images are replaced by a form for the images, which frees a person’s mathematics from the need to take popular actions to create images. The image itself within a person's mind can be used without going through a specific construction in mathematical knowing. “A fourth level or mode of understanding occurs when one can manipulate or combine aspects of ones images to construct context-specific, relevant properties" (Pirie & Kieren, 1994, p. 170), which is called ‘property noticing’. This includes noticing distinctions, combinations, or connections among images, predicting how they are interrelated, and presenting those relationships noticed.

In formalising, “the person abstracts a method or common quality from the previous image dependent know how which characterized her noticed properties” (pp. 170-171). That is, a person in this level consciously thinks about the noted properties and possibly abstracts the common features. This person now has class-like mental objects which is free from meaningful images. "A person who is formalising is also in a position to reflect on and coordinate such formal activities and express such coordinations as theorems called ‘observing’” (p. 171). ‘Structuring’ occurs when one attempts to think about one's formal observations as a theory, which means “that the person is aware of how a collection of theorems is inter-related and calls for justification or verification of statements through logical or meta-mathematical argument” (p. 171). This level can be thought of setting one's thinking in an axiomatic structure and this mode of mathematical understanding can also be used for formal proofs.

Finally, the outermost level, which is the eighth in their model, is called ‘inventising.’ The learner at this level has “a full structured understanding and may therefore be able to break away from the preconceptions which brought about this understanding and create new questions which might grow into a totally new concept” (p. 171). In other words, one can go freely and imaginatively beyond the bounds of a fully structured understanding in understanding mathematics. Pirie and Kieren stressed that they do not see the growth of understanding as a monodirectional process, rather as back and forth movement between levels. Thus, they characterize understanding as a dynamic and organizing process.

Especially, there are two crucial features in Pirie and Kieren’s model: ‘Don’t need boundaries’ and ‘Folding back.’ In mathematics, a learner is able to operate at a symbolic level without reference to basic concepts. They named this phenomenon 'Don't need boundaries', which means in order to "convey the idea that beyond the boundary one does not need the specific inner understanding that gave rise to the outer knowing. One can work at a level of abstraction without the need to mentally or physically reference specific images" (p. 173). For example, when a learner has an image of a mathematical idea, he does not need actions or the specific instances of image making. There are three ‘don’t need boundaries’ illustrated by bold rings: between image making and image having, between property noticing and formalising, and between observing and structuring. (Figure 1.3) The 'Don't need boundary,' however, does not mean that one can not return to the specific background in spite of one's necessity to go back to previous levels. Quite to the contrary, one can freely return to the previous level from any level whenever he wants to, which is the core of 'Folding back.'

The next feature is ‘folding back’ which is more crucial and vital to the growth of understanding and reveals the non-directional nature of understanding mathematics. (Figure 1.4)

> When faced with a problem or question at any level, which is not immediately solvable, one needs to *fold back* to an inner level in order to extend one’s current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings. Continuing with our metaphor of folding, we can say that one now has a ‘thicker’ understanding at the returned-to level. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. (p. 173)

Thus, their model is not static. According to their model, a learner can do ‘folding back’ at any level, maybe the ‘folding back’ is more necessary for an advanced, deep, and refined understanding.

They mention another feature of their model, which is the structure within levels themselves. They believe that each level except primitive knowing consists of "a complementarity of *acting* and *expressing* and each of these aspects of the understanding is necessary before moving on from any level" (p. 175). The growth of understanding occurs from going through acting and then expressing. Here, acting encompasses all previous levels of understanding and on the other hand expressing shows distinct features of the particular level. In other words, "acting can encompass mental as well as physical activities and expressing is to do with making overt to others or to oneself the nature of those activities" (p. 175). They chose the verbs, reviewing / doing, saying / seeing, recording / predicting, justifying / applying, prescribing / identifying, and proving / conjecturing as labels for the acting/expressing complementarities within the image making, image having, property noticing, formalising, observing, structuring and inventising respectively.

Conceptual and procedural knowledge

Wilson, Shulman, and Richert (1987) hypothesized that teachers draw from the following seven domains of knowledge as they plan and implement instruction: "knowledge of subject matter, pedagogical content knowledge, knowledge of other content, knowledge of the curriculum, knowledge of learners, knowledge of educational aims, and general pedagogical knowledge". Since this study is closely related to subject matter knowledge in conjunction with conceptual and procedural knowledge, I would like to describe subject knowledge and conceptual and procedural knowledge. Brown and Borko (1992) used the Shulman and Grossman's definitions of subjective knowledge.

> Subject matter knowledge consists of an understanding of the key facts, concepts, principles and explanatory frameworks in a discipline, known as *substantive knowledge*, as well as the rules of evidence and proof within that discipline, known as syntactic knowledge. (Shulman & Grossman, 1988) (p. 211)

In mathematics, substantive knowledge in general includes mathematical facts, concepts, and computational algorithms, whereas, syntactic knowledge encompasses an understanding of the methods of mathematical proof and other forms of argument used by mathematicians (Brown & Borko, 1992). Ball (1988) developed a conceptual framework for exploring teachers' subject matter knowledge in the area of mathematics. She claimed that understanding mathematics for teaching requires both 'knowledge *of* mathematics' and 'knowledge *about* mathematics'. Knowledge of mathematics is very closely related to substantive knowledge. She argued that teachers must have "knowledge of mathematics characterized by an explicit conceptual understanding of the principles and meaning underlying mathematics procedures and by connectedness - rather than compartmentalization - of mathematical topics, rules, and definitions" (Brown & Borko, 1992, p. 212) in order to teach mathematics effectively. On the other hand, knowledge about mathematics is related to syntactic knowledge. This includes an understanding of the nature of knowledge - "where it comes from, how it changes, how truth is established, and what it means to know and do mathematics" (p. 212). Although the concepts of substantive and syntactic knowledge are not exactly synonyms with conceptual and procedural knowledge respectively, there is at least one common element that they share. That is, substantive and conceptual knowledge are based on the conceptual understanding of the principals to some degree, whereas syntactic and procedural knowledge are based on somewhat procedural understanding leaning on rules and syntax.

Conceptual and procedural knowledge are not always distinct, instead they appear to be mixed in most cases. But each knowledge has unique characteristics so that we can differentiate conceptual knowledge from procedural knowledge to some degree. Hiebert and Lefevre (1986) defined conceptual knowledge in the following way:

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other

> pieces of information. The development of conceptual knowledge is achieved by the construction of relationships between pieces of information. (pp. 3 - 4)

Conceptual knowledge is mainly interested in building relationships among different pieces of information. Information means newly or early constructed knowledge. The newly constructed knowledge can be the part of network of all knowledge, establishing connections with preexisting knowledge. Or, without new knowledge, there can be made connections among already existing knowledge in learner's mind. Although new knowledge is not added in this case, building a relationship among preexisting knowledge can contribute to learners' advancing knowledge. Thus, there are two different establishments of connections among different pieces of knowledge.

According to Hiebert and Lefevre, one is called the 'primary' level and the other is the 'reflective' level. In the primary level, "the relationship is constructed at the same level of abstractness (or at a less abstract level) than that at which the information itself is represented" (p. 4). For example, they used the decimal numbers. When students learn about decimal numbers, they learn that 1) the position values to the right of the decimal point are tenths, hundredths, and so on; 2) when they add or subtract decimal numbers, they line up the decimal points so that they can add appropriately.

In the reflective level:

> Relationships are less tied to specific contexts. They are often created by recognizing similar core features in pieces of information that are superficially different. The relationships transcend the level at which the knowledge is currently represented, pull out the common features of different-looking pieces of knowledge, and tie them together" (p. 5).

As an example of the reflective level, the learners recognize that they are adding same size pieces when adding common fractions by recalling that they lined up decimals so that tenths and hundredths, etc. can be added appropriately.

On the other hand, Hiebert and Lefevre (1986) defined procedural knowledge in the following way:

> Procedural knowledge of mathematics encompasses two kinds of information. One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols. (pp. 7 - 8)

Procedural knowledge consists of two distinct parts: One part is composed of formal language (or symbol representation system) of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks. The first part includes a familiarity with symbols used to represent mathematical ideas and an awareness of syntactic writing rules. For example, those who posses these aspects can recognize that the expression 3.5 / Δ = 2.71 is syntactically acceptable (although they may not know the "answer") and that 6 + = Δ 2 is not acceptable (p. 6). Thus, knowledge of the symbols and syntax of mathematics implies an awareness of surface features, not a knowledge of meaning. The key feature of the second part is that the procedures are executed in a predetermined linear sequence. That is, the n^{th} step must come right after the $n\text{-}1^{th}$ step is performed. The sequential nature of these procedures sets this aspect of knowledge apart from other forms of knowledge.

Constructivists view that knowledge is not fixed. Rather it is individually constructed. Thus, understanding is derived through experiences. Further, Hannafin and Land (1997) claimed that students-centered learning environments emphasize concrete experiences that serve as catalysts for constructing individual meaning. Also, "the belief of constructivism is that all knowledge is necessarily a product of our own cognitive acts" (Confrey, 1990, p. 108). Teachers, parents, or peers might provide a learner with appropriate aids for the construction of knowledge. The learner might be provided with either procedural or conceptual sense. Eventually, the learners are constructing their knowledge through individual experiences.

In summary (Figure 2.3), knowledge consists of two parts: conceptual knowledge and procedural knowledge. Conceptual knowledge can be viewed as a network of knowledge connecting all pieces of information, whereas, procedural knowledge can be viewed as the construction of each separated piece of information. There are two different levels in conceptual knowledge: primary and reflective levels. The first is about constructing a relationship at the same or less abstract level, and the second is about a relationship created by recognizing similar core features in pieces of information that are superficially different. Likewise, there are two distinct parts in procedural knowledge: 'syntax-based' and 'rule-based' development. The first is to recognize whether the given mathematical syntax is acceptable or not regardless of students' ability to solve that problem. The second is the strictly sequential procedures for solving problems. Finally, the knowledge described above is constructed from individual or group experiences.

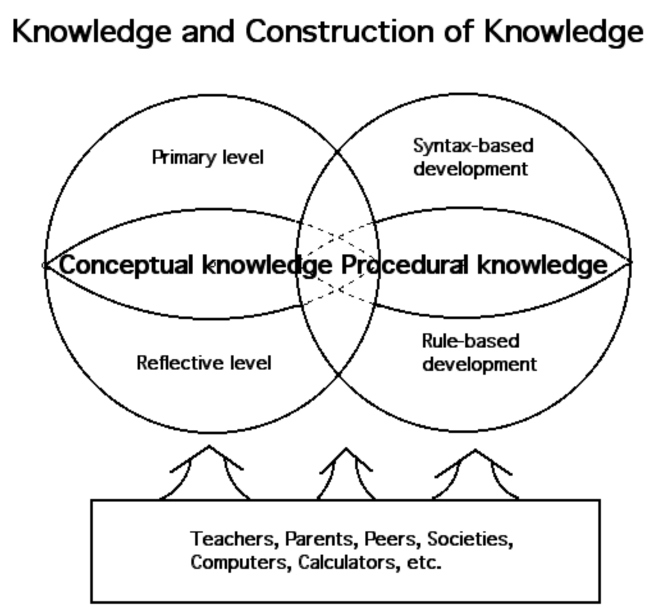


Figure 2.3 Knowledge and construction of knowledge

Visualization

Zimmerman and Cunningham (1991) define visualization as "the process of producing or using geometrical or graphical representations of mathematical concepts, principles, or problems, whether hand drawn or computer-generated" (p. 1). Schnotz, Zink, and Pfeiffer (cited in Nemirovsky & Noble, 1997) in which they "describe visualization as 'a process of structure-mapping of a visuo-spatial configuration onto a mental model,' in an information-processing type of description of the process" (p. 101). On the other hand, Zazkis, Dubinsky, and Dautermann (1996) define visualization as "an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses" (p. 441). But, this definition does not restrict visualization to learner's mind nor some external medium, but defines visualization as the means from travelling between these two (Nemirovsky & Noble, 1997). For any purpose (Jung, 2000), visualization includes both the process of constructing or modifying geometrical, or graphical representations for the given mathematical problems, concepts, principles, or figures and the process of connecting the internal knowledge or understanding with the images constructed to confirm, refine, and extend pre-existing knowledge or understanding. That is, not only physical construction but also mental activities are the main elements of visualization.

To better understand the meaning of visualization, I would like to give a brief description of the terms often used in describing visualization. Presmeg (1986a, 1986b) described various terms used in order to relate visualization in her studies.

Visual image. According to Presmeg (1986b), "a visual image is a mental scheme depicting visual or spatial information" (p. 42). Furthermore, She classified the images used by the visualizers into the following five categories: 1) concrete imagery (pictures-in-the-mind), 2) pattern imagery (pure relationships depicted in a visual-spatial scheme), 3) memory images of formulae, 4) kinaesthetic imagery (imagery involving muscular activity), and 5) dynamic (moving) imagery. This definition is broad enough to include kinds of imagery which depict shape, pattern, or form. It also allows for the possibility that verbal, numerical or mathematical symbols may be arranged spatially to form the kind of numerical or algebraic imagery (Presmeg, 1986a).

Visual and nonvisual methods of solution of mathematical problems. A visual method of solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed. A nonvisual method of solution is one which involves no visual imagery as an essential part of the method of solution. "Following Krutetskii's [1976] model, the position is taken that all mathematical problems involve reasoning or logic for their solution. Beyond this requirement, the presence or absence of visual imagery as an essential part of the working determines whether the method is visual or nonvisual" (Presmeg, 1986b, p. 42).

Mathematical visuality (MV). "A person's mathematical visuality is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods" (Presmeg, 1986b, p. 42).

Visualisers and nonvisualisers. "Visualisers are individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods. Nonvisualisers are individuals who prefer not to use visual methods when attempting such problems" (Presmeg, 1986b, p. 42).

Visual presentations. "A visual presentation is a way of teaching which involves formation and use of visual imagery by the teacher or pupils or both" (Presmeg, 1986b, p. 42).

Teaching visuality (TV). "A mathematics teacher's teaching visuality is the extent to which that teacher used visual presentations when teaching mathematics" (Presmeg, 1986b, p. 43).

Visual and nonvisual teachers. "A visual teacher is a teacher of high teaching visuality. A nonvisual teacher is a teacher of low teaching visuality" (Presmeg, 1986b, p. 43).

Technology in mathematics education

Technology has been and is currently a topic for various research studies in mathematics education. Since Papert (1980), Logo has been used for over two decades. Many other tools such as Calculators (Ruthven, 1990; Zheng, 1998), Computer-intensive algebra [CIA] (Lynch, Fischer, & Green, 1989; O'Callaghan, 1998), Computer-assisted instruction [CAI] (Christmann, Lucking, & Badgett, 1997), Computer microworld [e.g. TIMA: Bars (Olive & Steffe, 1994), JavaBars (Biddlecomb & Olive, 1999)], and dynamic geometry software [The Geometer's Sketchpad (Jackiw, 1991) and Cabri Geometry (Laborde, Baulac, & Bellemain, 1988-1997)] have been developed and used.

NCTM (1989) claims that "technology, including calculators, computers, and videos, should be used when appropriate" (p. 67). It also says that "calculators must be accepted at the K-4 level as valuable tools for learning mathematics. Calculators enable children to explore number ideas and patterns, to have valuable concept-development experiences, to focus on problem-solving processes, and to investigate realistic applications" (p. 19). However, NCTM (1989) gives us a warning that "calculators do not replace the need to learn basic facts, to compute mentally, or to do reasonable paper-and-pencil computation" (p. 19). The use of "technology has dramatically changed the nature of the physical, life, and social sciences" (NCTM, 1989, p. 3). Furthermore, technology and its format "free students from tedious computations and allow them to concentrate on problem solving and other important content. They also give them new means to explore content" (NCTM, 1989, p. 67).

> Technology also blurs some of the artificial separations among topics in algebra, geometry, and data analysis by allowing students to use ideas from one area of mathematics to better understand another area of mathematics. ... As some skills

> that were once considered essential are rendered less necessary by technological tools, students can be asked to work at higher levels of generalization or abstraction. Work with virtual manipulatives (computer simulations of physical manipulatives) or with Logo can allow young children to extend physical experience and to develop an initial understanding of sophisticated ideas like the use of algorithms. Dynamic geometry software can allow experimentation with families of geometric objects, with an explicit focus on geometric transformation. Similarly, graphing utilities facilitate the exploration of characteristics of classes of functions. (NCTM, 2000, pp. 26-27)

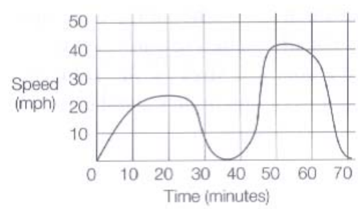
Influences or impacts of using technology

O'Callaghan (1996) investigated whether Computer-intensive algebra [CIA] students develop a richer understanding of the function concept than Traditional algebra [TA] students using CIA curriculum which was developed in the late 1980s under the direction of James Fey and M. Kathleen Heid in response to calls for reform in teaching and learning of mathematics. As a function-oriented curriculum, CIA "is characterized by (a) a problem-solving approach based on the modeling of realistic situations, (b) an emphasis on conceptual knowledge, and (c) the extensive use of technology" (p. 21). Based on a problem-solving environment, the function model that he framed consists of four components: modeling, interpreting, translating, and reifying.

> In this innovative curriculum, the students are presented with question-rich situations that they can explore under a variety of conditions and modeling assumptions. The activities usually require the students to solve problems, often with the help of computer tools (e.g., symbol-manipulation programs), and to describe their methods of solution. The goal is to help student discover important ideas and the relationships among those ideas in algebra. (Fey, 1992) (p. 22)

This study focused on three classes: the experimental class (CIA) taught by the researcher, a traditional class (TA1) taught by the researcher, and a traditional class (TA2) taught by another instructor. Attitude scales, questionnaires, and tests on functions were given to all students at the beginning and at the end of the semester. Interviews were also conducted with six students from CIA and six from traditional classes.

Table 2.1 Sample questions from the functions tests (p. 30)

Component	Question
Modeling	A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following. a. The total weigh of the truck if the number of boxes is 75. b. The number of boxes if the total without of the truck is 6,740 pounds. c. Using *W* for the total weight of the truck and x for the number of boxes, write a symbolic rule (or equation) that expresses the weight as a function of the number of boxes.
Interpreting	The graph below gives the speed of a cyclist on his daily training ride. During his ride, he must climb a hill where he pauses for a drink of water before descending. Use this graph to answer the following questions as accurately as possible.  a. Find the speed when time equals 25 minutes. b. Find the time when speed equals 30 mph. c. During what time intervals was the speed increasing? d. During which 10-minute interval did the speed decrease the most? e. When was the cyclist at the top of the hill?
Translating	A roast is taken from the refrigerator and put into an oven. The following is a table of its temperature, in degrees (*D*), recorded at different times during the first 120 minutes (*m*). M 0 10 20 30 60 120 --- D 50 100 140 170 200 220 Draw a graph of this situation.
Reifying	A small company determines its contribution to charity (*C*) by its profit (*p*), which is dependent on the number of items (n) sold according to the following formulas: $C = .10(p - 1000)$ and $p = 100n - n^2$ a. What will the company contribute to charity if it sells 50 items? b. Write a formula expressing *C* as a function of n.

Also, here is an example for interviews, which is analogous to the problems on the function tests (p. 29):

> The table below gives the average price of a new home in a small town, USA, for every 2 years since 1980:

Year	Price
1980	$50,000
1982	$53,000
1984	$57,500
1986	$62,000
1988	$67,000
1990	$70,000
1992	$74,000

> Probes
> a. What would you predict for the price in 1995?
> b. When would you predict the price to be $100,000?
> c. Write the equation expressing price as a function of the year.
> d. Draw the graph of the data and the equation.
> e. Ask about annual rate of change, slope, and intercept.

The results say that "CIA students demonstrated a significantly better conceptual knowledge of functions in their overall scores" (p. 33). But they failed to outperform the traditional students in reifying. The results of interviews indicate that CIA students were "clearly superior to the TA students at interpreting graphs demonstrated an excellent understanding of and appreciation for the main topics of their course, where the TA students did not seem to be very clear on what the course was all about" (p. 35). Although CIA students demonstrated a better understanding than the traditional students did, "they appeared to be somewhat less skillful at symbol manipulation" (p. 35). Also, "the overall attitudes of the CIA students toward mathematics showed significant improvement, and their levels of anxiety in relation to the subject were significantly reduced as a result of their CIA experience" (p. 37).

Following O'Callaghan's study (1998), Hollar and Norwood (1999) extended his "CIA study by using the same framework to investigate a different curriculum, specifically a graphing-approach curriculum employing the TI-82 graphing calculator" (p. 221). The purpose of their study is to see if O'Callaghan's results on the four components of his function test and on the attitude scales would hold with a different curriculum incorporating graphing calculators. The reification component was of special interest. A balance of graphing calculator and traditional algebra work was found in the textbook that was used in treatment groups. This includes exploration and discovery examples to help students look for patterns and make discoveries. Whereas, the textbook used in control group emphasized memorizing isolated facts and procedures, and becoming proficient with paper and pencil calculation although it covered the same topics as the experimental text. Students in treatment group were allowed to use calculators in class, and for homework exercises and tests, but not for the O'Callaghan Function Test or the traditional final examination.

The following table (Table 2.2) about O'Callaghan Function Test reveals that "the experimental classes had a significantly better overall understanding of functions that the control classes had" (p. 224). The experimental classes had higher means for each of the four components as well as for the total score on the function test.

Table 2.2 Function posttest mean scores (p. 223)

		Experimental		Control	
Component	Maximum Score	M	SD	M	SD
Modeling	7	4.32	1.65	3.33	1.64
Interpreting	11	7.46	1.92	5.90	2.21
Translating	9	5.05	2.26	3.64	2.21
Reifying	10	4.20	1.89	2.74	0.29
Total	37	21.02	5.87	15.62	4.70

Although the experimental group overall had a slightly higher mean score on departmental final examination, no significance was found between the scores of treatment and control groups. The attitude survey showed that "students in the experimental class had slightly more positive attitudes than their counterparts in the control class" (p. 224). But, interestingly, on the reification component, Hollar and Norwood found a significant difference between the graphing-approach group and the control group, whereas O'Callaghan found no difference. Concerning traditional algebraic skills, no significant differences were found between two groups. For the attitudes toward mathematics over the course of the semester, O'Callaghan found that students using CIA curriculum significantly improved their attitudes, but Hollar and Norwood found that students in experimental group were not different from students in control group. As yet, there is no consensus on the effect of technology use on attitude. As a last, they conclude the study with the following:

> It is important to study how technology positively and negatively affects the development of both structural and procedural concepts. (p. 225)

On the other hand, there are some concerns about the use of technology in the mathematics classroom. Unlike the investigation of impacts of using simple calculators in the past, today the problems are getting more complicated as more advanced calculators (e.g., graphing calculator) and computers are easily available and widely used. Zheng (1998) proposed the following questions to be answered: 1) Do our students know how to use calculators properly?; 2) Is it possible that students will easily develop some misconceptions when they use calculators?; 3) Will using the technology foster some particular problem solving behaviors that are desirable?; 4) Are there conflicts between mathematics presented on calculators and mathematics learned from other sources (p. 2). He made the following comments that are enough to bring people's attention about the use of technology;

> Because calculators are generally numerical in nature, students may not acquire solid conceptual understanding. Their view of mathematics will probably be more procedural and accordingly their problem solving skills may be limited.

> The development of their structural view about mathematics could also be hindered. Moreover, because of it [*sic* its] design, a calculator may deliver misleading information and create confusion in learning notation. There is no cure but adaptation. Simply prohibiting the use of calculators has been proved to be a failure. The focus ought to be when, where, and how to use the device. Students should be informed very clearly that calculator is only one of many tools used to learn mathematics and it is a miracle only to a certain extent. The teacher is responsible to show student both positive (quickness, multiple representations, etc.) and negative (possible distortion even deception, self-contained rules, etc.) aspects. It will be especially beneficial to demonstrate how the technology is capable and fascinating in one case and incapable and helpless in another. It is also useful to engage students in solving problems in different environments, technologically rich or not, and demonstrate how one problem can be interpreted in very different cognitive challenges. (p. 9)

Battista (1988) examined some of the ways in which educational technology is inadequate. In his article, *Mathematics and technology: A call for caution*, he argued that "if we are not careful, current naivete and attitudes about the use of calculators and computers in mathematics instruction may lead to frustration and eventually rejection of the wonderful tools that technology will make available to us in the future" (p. 31). He also claimed that technological limitations are dictating pedagogy rather than supporting it because designers too often "search not for the best way to teach, but for *some* instructional idea that can be easily programmed on a particular computer" (p. 32). Battista asserted the need of software (e.g., such as Logo & the *Geometric Supposer*) that encourages learners to explore mathematics problems and allow creative teachers to exercise their imaginations in their instruction.

Although Battista believed that technology has great potential for enhancing mathematics instruction and much to offer mathematics educators, if people's mindset is not organized toward technology in mathematics, it can become extremely dangerous because "it tends to greatly overestimate the quality of the materials produced. Pride in accomplishment tends to obscure the pedagogical compromises that were made because of technological limitations" (p. 32). He warned us with a caution that "unrealistic appraisals of the benefits of educational technology are almost certain to lead to disaster" (p. 32). As a last, he suggested several steps that must be undertaken to ensure the future health of technology in mathematics teaching;

> - We must demand that the manufacturers of technology respond to the needs of education.
> - We must inform the educational community and the public about the pace of progress in technology and the impact that this pace has on the educational use of technology.
> - We must make a commitment to education technology, not only in terms of equipment budgets, but in terms of teacher training.
> - Last, and most important, we must anticipate the dangerous attitudes that may lie ahead of us on the evolutionary road of technological innovation in education. We must be realistic in assessing the current use of technology and forthright in admitting its limitations. (p. 33)

Teaching and learning with technology

Over the recent years, there has been widespread interest in creating new learning systems that differ from those of traditionally directed instruction, especially in relation to the use of technology. One product from people's efforts for creating new learning systems is 'open-ended learning environments (OELEs).' As Hannafin et al. (1994) suggested, the best way to understand the concept of open-ended learning might be to contrast it with that of directed learning. (Table 2.3)

> Directed learning involves the systematic acquisition and retention of externally-defined knowledge and skills. Directed learning emphasizes the designer's ability to assist the individual in discerning "important" from "unimportant" information, and the individual's capacity to acquire, recall, and demonstrate knowledge and skills in accordance with external requirements. An individual is a successful directed learner when he or she is able to utilize the strategies and features provided by a designer to acquire prescribed knowledge and skill to acceptable levels.

> Open-ended learning refers to processes wherein the intents and purposes of the individual are uniquely established and pursued. It involves the individual determination of what is to be learned, how it is to be learned, when (or if) learning goals have been met, and what (if any) subsequent steps might be taken, Any number of things might be candidates for individual learning; the intents and goals of different learners would likely vary substantially. In effect, the fundamental difference between directed and open-ended learning is in who determines what is to be learned and what steps are taken to promote learning. (p. 48)

Table 2.3 Distinctions between directed and open-ended learning environments (Hannafin et al., 1997, p. 95)

Directed Environments	Open-Ended Environments
Driven by specific externally-generated objectives	Focus on the relevant processes associated with problems, contexts, and content
Teaching and learning is "bottom up"; basics first	Engage in complex meaningful problems, linking content and concepts to everyday experience
Structured algorithmic approaches designed to convey discrete identifiable body of knowledge	Design approaches centered around "wholes" - exploring higher-order and concepts, flexible understanding, and multiple perspectives
External conditions and content believed to activate internal processes	Rich experiences where learners deploy diverse personal knowledge and use tools to augment thinking
Problem-solving skills broken down and taught via directed approaches	Problem solving processes emphasized with opportunities to manipulate, interpret, and experiment
"Information" is identified and taught to learners	Knowledge evolves as understanding is continually modified, tested, and revised
Mastering content and objectives is paramount	Thinking processes more important than discrete content outcomes
Learning is externally-driven via explicit activities and practice	Learners evaluate own needs, make decisions, and take responsibility for learning
Learning contexts structured according to task, objectives, and prerequisites	Learning contexts embedded in authentic problems where "need to know" is naturally generated

Mastery is achieved by reducing or eliminating errors	Mastery might be riddled with misconceptions; deep understanding rooted in initial, often flawed, beliefs

OELEs were designed to "support individuals' efforts to understand that which he or she determines to be important" (Hannafin, Hall, Land, & Hill, 1994, p. 48), to support higher-order cognitive skills such as identifying and manipulating variables, interpreting data, hypothesizing, and experimenting (Roth & Roychoudhury, 1993)" (p. 50), and to provide "activities to support the individual's efforts to mediate his or her learning" (Hannafin, 1992, p. 51) and "interactive, complimentary activities that enable individuals to address unique learning interests and needs, study multiple levels of complexity, and deepen understanding" (Hannafin & Land, 1997, p. 168). In OELEs, students judge "what, when, and how learning will occur - beliefs influenced heavily by constructivists, who emphasize the importance of constructing personal meaning (Guba, 1990)" (Hannafin, et al., 1994, p. 49). Furthermore, Hannafin, et al. claimed that

> to support and cultivate mathematics, learners must be empowered in the use of their own strategies rather than being required to supplant them with those of the instructor or designer. The challenge in learner-centered environments is to determine how best to support and guide these highly individualized processes without imposing unnecessary, potentially conflicting, external structures (p. 49).

Perspectives of OELEs based on individuals' experiences and judges are consistent with those of constructivists' perspective, which is that reality is not definitive and objective, but rather subjective and a personal by-product of individual experiences. Thus, students' understanding grows as they are actively involved and engaged in various activities and builds upon their own naive intuitions and modes by questioning limitations and refining them. (Table 2.4) The ecology has been changed since the emergence of OELEs. The leadership and the power in the classroom have been shifted from teachers to students to some degree. The students' role in OELEs changed significantly. "The need for individual mediation (e.g., setting meaningful goals, identifying appropriate resources, evaluating relevance, monitoring comprehension) is critical. Students must be empowered and supported while making purposeful use of the tools, resources, and activities in the environment - skills for which many students are ill-equipped" (Hannafin et al., 1997, p. 97). The proponents of OELEs support that learning is most effective when it evolves from rich, hands-on, concrete experiences with realistic and relevant problems through students' thorough meditation. However, only "few students presently possess the skills to handle independently the substantial self-monitoring requirements" (p. 97). This shows that there are things to be done for the full implementation of OELEs. From the aspect of teaching in OELEs, appropriate feedback is critical for the success. "The teacher facilitates the student's goals. Teachers identify and acquire needed resources, create, refine, and extend problem contexts, while providing "human resources""(p. 97).

Table 2.4 Assumptions and underlying beliefs of OELEs (p. 96)
[Adapted from Hannafin, Hall, Land, and Hill (1994)]

Assumption	Beliefs
Understanding is individually mediated	• Utilize unique sense-making capabilities • Learners can be guided to make effective choices • Need to learn how available tools and resources can be used to support and manage learning
Activities focus on cognitive process and not solely the verifiable products of learning	• Manipulate processes via system-supplied tools and resources • Use of cognitive and metacognitive strategies • Integrate content with relevant higher-order processes
Qualitatively different learning processes require qualitatively different teaching methods	• Didactic instruction is just one method for learning, not the only or most important • Conceptual understanding requires varied representations and activities • The potential for complex understanding increases as the environment becomes richer

	and more engaging
Understanding is more critical than simply knowing	• Understanding transcends the information given • Wisdom cannot be "told" • Learners hold intuitive theories that must be acknowledged, questioned, and built upon
Context and experience are critical to understanding	• Cognitive processes and learning contexts are inextricably connected • Recognize implications of concepts within problems and scenarios • Learning requires immersion in concrete authentic experience

There are five foundations in which learning environments are rooted: psychological, pedagogical, technological, cultural, and pragmatic. Direct instruction environments generally are based on foundations that are consistent with objectivists, designers-centered perspectives, whereas, student-centered learning environments' foundations reflect a more user-centered view about the nature of knowledge and the role of the learner. Although both are rooted in psychological foundations, they take different approaches (Hannafin & Land, 1997, p. 172). It is assumed that those five foundations are functionally integrated in learning systems in general. As illustrated in Figure 2.4, "each foundation should interact to some degree with all others, indicating mutual interdependence" (p. 178). As foundations become increasingly or decreasingly interdependent, the portion of intersection increases or decreases accordingly. "The more complete the coincide, the better integrated the foundations; the better integrated the foundations, the greater the probability of success in the setting for which the learning environment is designed. In practice, the larger the coincidence among foundations, the better aligned the learning system's underlying psychological, pedagogical, technological, cultural, and pragmatic factors" (p. 178). However, complete alignment is relatively rare.

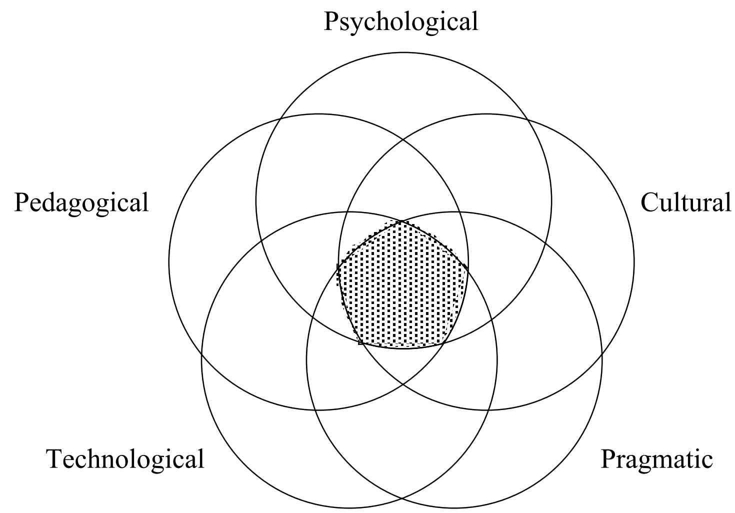


Figure 2.4 A conceptual representation of a balanced, integrated technology-enhanced student-centered learning environment (p. 179)

Technology has been expected to impact the way we teach mathematics and what we should teach in the mathematics classroom, and it has opened the door to a wealth of applications that were not feasible in the past. Pokay and Tayeh (1997) gave some practical suggestions on how to integrate exploratory applications into the mathematics classroom based on a first-year college geometry course focusing on shapes in two and three dimensions, similar and congruent figures, spatial visualization, transformations, and measurement. Problem solving, open-ended questions, hands-on activities, and cooperative groups, both in the computer lab and in the regular classroom setting, were used in the course.

Pokay and Tayeh were able to categorize the following six themes (pp. 118-120) when students pretended the computer was a real person. First, the focus of student learning changed over time. The main emphasis was on geometry and the general introduction to the computer. As time progressed, the emphasis changed to the new software. Students reported uncomfortable feeling and inadequacy in addition to an inability to communicate with the computer and feeling out of control. Thus, although the emphasis was on geometry throughout the semester, students' attention and energy were diverted as they worked with new tools, GSP. A second theme was the focus on communication, or lack of communication, between the students and the computer. Students were not very successful in having a good communication with computer, whereas they had only a limited communication with computer. When they did something wrong, computer did not tell them anything about how to fix it except error message, which made students feel frustrated.

Third, students expected the computer to provide answers to problems. One student complained of not just handing them the answers to the questions, which was a cause of frustration. It was not until they became comfortable with the computer that it became a useful tool in the learning process. Fourth theme was the issue of control. Students in the beginning not only felt out of control but, felt that the computer was in control of them. But, later, the computer was in less control and instead became a partner in solving geometry problems.

Fifth, when students were asked to explain their relationship with the computer, most students described the computer as an acquaintance or a tool at the beginning of the semester. However, the computer and students have become pretty good buddies. As the semester went on, most of relationships became friendships. Finally, when advantages of using GSP for learning geometry were mentioned, some students focused on the computer, while others focused on geometry. They suggested the following:

> When integrating into a course, the teacher needs to be aware of the difficulties this will present to the student. Suggestions include giving the student more "play time" with the technology in order to become more comfortable. This will reduce anxiety while allowing the student to become more proficient before major assignments are undertaken. In addition, students can be provided with some structure for initial exploration, perhaps in the form of a checklist of proficiencies needed to begin their work. The teacher needs to recognize that initial discomfort is a natural part of this type of learning process and needs to communicate this to students. Finally, since the goal of integration is to focus on problem solving and higher level thinking, as well as the processes used assessments need to be

> developed and implemented that tap these outcomes. Only in this way can the teacher monitor student learning and the effectiveness of the instruction. (p. 123)

Geometry in mathematics education and van Hiele model

The 1989 version of standards emphasizes the importance of geometry from grades K-4 because "geometric knowledge, relationships, and insights are useful in everyday situations and are connected to other mathematics topics and school subjects" (NCTM, 1989, p. 48). In geometry, spatial visualization, which is "building and manipulating mental representations of two- and three-dimensional objects and perceiving an object from different perspectives (NCTM, 2000, p. 41), is "necessary for interpreting, understanding, and appreciating our inherently geometric world" (NCTM, 1989, p. 48). Here is the geometry standard that the 2000 version of standards suggests: Instructional programs from prekindergarten through grade 12 should enable all students to -

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations;
- use visualization, spatial reasoning, and geometric modeling to solve problems. (p. 41)

Sherard (1981) suggested the seven reasons to the questions, "Why is geometry a basic skill? Why is it important that our students acquire some basic facts and understandings about geometry during their mathematics experiences in school? Why is some knowledge of geometry essential for a person to function effectively in modern society?" (p. 19) Geometry is a basic skill, because

1. it is an important aid for communication. If we are to communicate to others the location, size, or shape of an object, geometric terminology is essential.

2. it has important applications to real-life problems.
3. it has important applications to topics in basic mathematics.
4. it gives valuable preparation for courses in higher mathematics and the sciences and for a variety of careers requiring mathematical skills.
5. it provides opportunities for developing spatial perception.
6. it can serve as a vehicle for stimulating and exercising general thinking skills and problem-solving abilities.
7. there are cultural and aesthetic values to be derived from the study of geometry. (See Sherard, 1981, pp. 19-21)

The two Dutch educators, Pierre M. van Hiele and his wife, Dina van Hiele-Geldof, developed a model, called the van Hiele model, that reflects a student's level of geometry learning. "The van Hiele model of geometric thought emerged from the doctoral works of Dina van Hiele-Geldof (1984a) and Piere Marie van Hiele (1984b)" (Crowley, 1987, p. 1). After Dina died shortly after her dissertation, Pierre clarified, amended, and refined the theory. Except for the Soviet Union that revised their geometry curriculum to conform to the van Hiele model, this model did not get much international attention. Since "a North American, Izaak Wirszup (1976), began to write and speak about the model" (p. 1), the interest in van Hiele's contributions "has been enhanced by the 1984 translations into English of some of the major works of the couple (Geddes, Fuys, and Tischler, 1984)" (p. 1).

The van Hiele model consists of five levels, which assumes the existence of unique characteristics in each level (Crowley, 1987; Hoffer, 1981). Also, van Hiele identified several general properties that characterize the model. (Table 2.5) "These properties are particularly significant to educators because they provide guidance for making instructional decisions" (p. 4).

<u>Level 0. Visualization or Recognition</u>. Students are aware of space only as something that exists around them. They can identify and name geometric figures. But, their concepts about geometric figures are viewed as a whole, i.e., as a physical appearance, rather than as having parts, properties, or attributes. Students in this level are

expected to learn geometric vocabulary, identify specified shape and a given figure, and reproduce it. For example, students can "recognize a picture of a rectangle but likely will not be aware of many properties of rectangles" (Hoffer, 1981, p. 13).

Level 1. Analysis or Descriptive. Students analyze properties of geometric figures from observations, experiences, and experimentation. They also begin to discern the characteristics of geometric figures. These emerging properties and characteristics are used to classify geometric figures. Hence, geometric figures are recognized as having parts, properties, or attributes and are recognized by them. Students in this level may "realize that the opposite sides and possibly even the diagonals of a rectangle are congruent but will not notice how rectangles relate to square or right triangles" (Hoffer, 1981, p. 14). Thus, students possibly can generalize the class of rectangles, but relationship between properties cannot be explained yet.

Level 2. Informal deduction or Ordering. Students logically order figures and understand interrelationships between figures and the importance of accurate definitions. Students can establish the interrelationships of properties both within figures and among figures. They are ready to deduce properties of a figure and understand class inclusion. Definitions are meaningful and informal arguments can be followed. However, students in this level do not comprehend the significance of deduction. Students in this level will "understand why every square is a rectangle but may be able to explain why the diagonals of a rectangle are congruent" (Hoffer, 1981, p. 14).

Level 3. Formal Deduction or Deduction. Students can understand the significance of deduction, interrelationships, and the role of postulates, definitions, theorems, and proof. Students in this level can construct, not just memorize, proofs; the possibility of developing a proof in more than one way is seen; the interaction of necessary and sufficient conditions is understood; distinctions between a statement and its converse can be made. Students will "be able to use the SAS postulate to prove statements about rectangles but not understand why it is necessary to postulate the SAS condition (Byrkit, 1971; Krause, 1975) and how the SAS postulate connects the distance and angle measures" (Hoffer, 1981, p.14).

Level 4. Rigor. Different postulated systems can be established and then, those systems can be analyzed and compared. "At this stage the learner can work in a variety of axiomatic systems, that is, non-Euclidean geometries can be studied and different systems can be compared. Geometry is seen in the abstract" (Crowley, 1987, p. 3).

Table 2.5 Properties of the model [Adapted from Crowley (1987, p. 4)]

Properties	Description
Sequential	As with most developmental theories, a person must proceed through the levels in order. To function successfully at a particular level, a learner must have acquired the strategies of the preceding levels.
Advancement	Progress (or lack of it) from level to level depends more on the content and methods of instruction received than on age: No method of instruction allows a student to skip a level; some methods enhance progress, whereas other retard or even prevent movement between levels.
Intrinsic and extrinsic	The inherent objects at one level become the objects of study at the next level.
Linguistics	Each level is assumed to have its own linguistic symbols and its own systems of relations connecting these symbols.
Mismatch	If the student is at one level and instruction is a different level, the desired learning progress may not occur.

As van Hiele asserted above, since the progress in the model heavily depends on instruction received rather than on age, "the method and organization of instruction, as well as the content and materials used, are important areas of pedagogical concern" (Crowley, 1987, p. 5). Based on this, van Hiele proposed the five sequential phases of the learning process, which are another important aspect of the van Hiele model:

inquiry/information, directed orientation, explication, free orientation, and integration (Crowley, 1987, p. 5).

Phase 1. Inquiry / Information. Students and the teacher are engaged in conversation, experiments, and activities about the objects of study. Questions are given to students and observations are made from the interactions between students and the teacher. For example, a teacher asks students such as: what is a square or what is a rectangle? These activities have two important goals: 1) the teacher learns students' prior knowledge; 2) students learn what direction they will take.

Phase 2. Directed Orientation. Students explore the topic of study from materials or activities sequentially and carefully prepared by the teacher. Structure of those activities is gradually revealed to students, thus materials and activities are designed to be short to get specific responses from students. For example, activity can be "to build a rhombus with four right angles, then three right angles, one right angle, ..." (p. 5).

Phase 3. Explication. Based on students' previous knowledge, they express in words their views observed through materials and activities. The teacher's role becomes minimal but the teacher encourages students to use accurate and appropriate terms or language. Students begin to construct the system of relationships from this stage. By continuing the rhombus example in phase 2, "students would discuss with each other and the teacher what figures and properties emerged in the activities above" (p. 5).

Phase 4. Free Orientation. More complex and open-ended questions with many steps are given students, which can be solved in various ways. Students gain more knowledge and experiences from finding their own alternative solutions or just simply resolving them. And many relations built on their minds become more explicit to students. For example, after folding a piece of paper in half, then in half again (Figure 2.5), imagine what type of figure you would get if you cut off the corner of this folded paper. "Justify your answer before you cut. What types of figures do you get if you cut the corner at a 30^{o} angle? At a 45^{o} angle?" (p. 6).

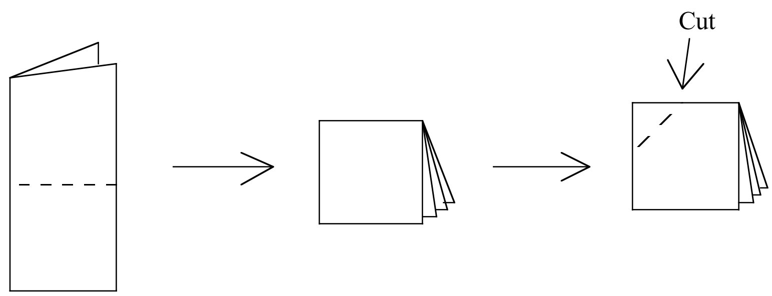


Figure 2.5 Folding a piece of paper and cutting it off

Phase 5. Integration. Students focus on a general review and summarization of what students learned with an overview of networks of objects and relations. The teacher assists students with global surveys of what they learned. "The properties of the rhombus that have emerged would be summarized and their origins reviewed" (p. 6).

After students reach the fifth stage, they reach the new level of thought. This replaces the old and students are ready to repeat the phases of learning for the next level.

Computer software: LOGO and dynamic geometry

The wide spread of computers, with which students create geometric figures and explore them by themselves, contributed to the use of technology in geometry class. After Papert (1980) developed "turtle geometry" as part of the Logo computer language about two decades ago, Logo was used for learning and understanding geometry by researchers (Battista, 1987; Clements, Battista, Sarama, Swaminathan, & McMillen, 1997; Johnson-Gentile, Clements, & Battista, 1994; Noss, 1987; Olive, 1991). The turtle, the cursor on screen, is students' body and helps them build relational understanding (Skemp, 1978) of geometry. The actual outcome of programming created by students may be different from what they expected. This difference is from "bugs" in their program. The debugging process can lead them to think about their own understanding. This reflective thinking process is considered to be necessary for building relational understanding.

Johnson-Gentile et al. (1994) suggested the following benefits that to which the Logo computer environment could contribute. First, the computer may act as a transition device to more abstract setting. Students' symbolic representations can develop from their own activities rather than being imposed by the teacher. Thus, Logo mediates between thought and action by providing a linked, external, and symbolic representations. Second, the Logo motions environment is more precise than paper-and-pencil and manipulatives, and can provide a concrete justification for the use of precise and formal language. Third, because the computer environments are designed for the quick and easy manipulation of geometric ideas, students can test the ideas for themselves on the computer, and receive feedback regarding discrepancies between what they expected and what their actions produced graphically. Fourth, the computer does exactly what it is told, without any mistake - whereas with paper and pencil, students may decide correctly what need to be done and yet perform the operation incorrectly or vice versa. Fifth, there is evidence that working within such a Logo environment may help students become explicitly aware of, and thus progress beyond, their mathematical intuitions, thus facilitating their transition from the visual to the descriptive/analytic geometric thinking in the van Hiele hierarchy (Johnson-Gentile et al., 1994, p. 123).

After the appearance of Logo, a new type of software was developed, which is more dynamic than Logo and has come to be referred to collectively as dynamic geometry [DG]. The term, *dynamic geometry*, was first used by Nicholas Jackiw and Steve Rasmusseen as the trademarked phrase and "entered the literature as a generic term because of its aptness at characterizing the feature that distinguished DG from other geometry software: the continuous real-time transformation often called "dragging"" (Goldenberg & Cuoco, 1998, p. 351). King and Schattschneider (1997) explain the 'dynamic' as follows: "*Dynamic* is the opposite of "static." *Dynamic* also connotes action, energy, even hype. Dynamic geometry is active, exploratory geometry carried out with interactive computer software" (p. ix). The dragging mode in dynamic geometry allows users to freely move the parts of constructed figures. As these parts are moved, all relationships that were specified as essential constraints of the original construction are maintained so that users observe originally constructed figures to the altered conditions. There are several different types of dynamic geometry currently used: "Geometer's Sketchpad (1991, 1995), Cabri (Baulac, Bellemain, & Laborde, 1992, 1994), Geometry Inventor (Brock, Cappo, Dromi, Rosin, & Shenkerman, 1994), and, in partial way, SuperSupposer (Schwartz & Yerushalmy, 1992)" (Goldenberg & Cuoco, 1998, p. 351).

I would like to look at the following fallacious theorem which has been in the literature for a long time before pointing out the features of dynamic geometry software.

> **Theorem**. *All triangles are isosceles*.
> *Proof*: Let ABC be a triangle with *l* the angle bisector of A, *m* the perpendicular bisector of BC cutting BC at midpoint E, and D the intersection of *l* and *m*. From D, draw perpendiculars to AB and AC, cutting them at F and G, respectively. Finally, draw DB and DC. Figure 1 shows a sketch of the situation.
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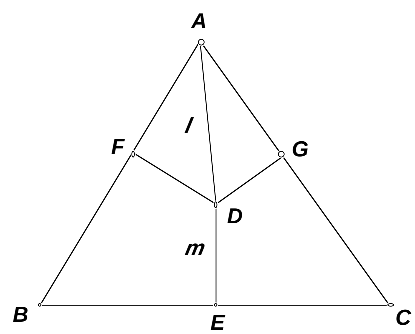
> 
>
>
> Figure 1
>
> ΔADF ≅ ΔADG (aas), so AF ≅ AG and DF ≅ DG.
> ΔBDE ≅ ΔCDE (sas), so BD ≅ CD.
> This implies ΔBDF ≅ ΔCDG (hyp.-leg), so FB ≅ GC.
> Thus AB ≅ AF + FB = AG + GC ≅ AC, and so ΔABC is isosceles. QED
> (King & Schattschneider, 1997, p. ix)

Obviously, there is nothing wrong with this proof in a logical sense. But, although the sketch is plausible, unfortunately, the point D, the intersection of *l* and *m*, is never inside ΔABC if the figure is accurate. The accurate construction shows that D' is the correct intersection of *l* and *m*. (Figure 2.6) This fallacious example has been offered not to do drawing of any figures at some point. Quite to the contrary, King and

Schattschneider argued that "this example offers compelling reasons to emphasize the need for accurately constructed figures, rather than quick sketches, especially when they are to be used in deductive arguments" (p. ix). Northrop (1944) put it in this way, "It shows how easily a logical argument can be swayed by what the eye sees in the figure and so emphasizes the importance of drawing a figure correctly, noting with care the relative positions of points essential to the proof" (p. 101).

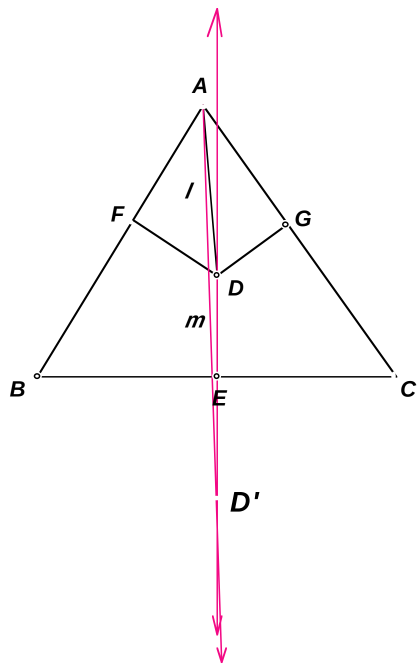


Figure 2.6 Accurate construction of fallacious theorem

To the question "What is dynamic geometry software good for?", King and Schattschneider (1997) addressed the features of dynamic geometry as follows.

Accuracy of construction. "Dynamic geometry software provides an *accurate constructor* for any ruler-and-straightedge construction in Euclidean geometry, and configuration that can arise by applying affine transformations (isometries and dilations) to a Euclidean construction, or any locus of an object (or set of objects) that arises when part of a construction is moved along some path" (p. xi).

Visualization. "As a demonstration tool in the classroom, dynamic geometry software can help students *see* what is meant by a general fact" (p. xii). Morrow (1997) claims that "Dynamic geometry software can be used to aid the process of visualization in all mathematics classes, not just in the study of geometry. Students can construct, revise, and continuously vary geometric sketches" (p. 47). Furthermore, Cuoco and Goldenberg (1997) propose three claims relating to visualization:

> Claim 1. By allowing students to investigate continuous variation directly (without intermediary algebraic calculation), dynamic geometry environments can be used to help students build mental constructs that are useful (even prerequisite) skills for analytic thinking.
>
> Claim 2. Dynamic geometry software can be used to help students develop a much broader collection of techniques for visualization continuous variation and the behavior of functions.
>
> Claim 3. Dynamic geometry environments can encourage students to develop mental images of functions that are especially suited to analysis. (pp. 34-35)

Exploration and Discovery. Students in a traditional learning environment are told definitions and theorems and assigned problems and proofs without having opportunities or experiences to discover the geometric relationships and to invent any mathematics in general. Whereas, dynamic geometry software is designed to implement exploration and discovery although the type of tasks may be various - from guided to totally open-ended. Garry (1997) observed that "Students could test their own mathematical ideas and conjectures in a visual, efficient, and dynamic manner and - in the process - be more fully engaged in their own learning" (p. 55).

Proof. "While dynamic geometry software cannot actually produce proofs, the experimental evidence it provides users with produces strong conviction which can motivate the desire for proof" (King & Schattschneider, 1997, p. xiii). de Villers (1997) claimed that it is not difficult to solicit students' curiosity to prove why the explored geometric conjectures in geometry are true. He also mentioned that "to present the fundamental function of proof as explanation and discovery requires that students be introduced early to the art of problem posing and allowed sufficient opportunity for exploration, conjecturing, refuting, reformulating and explaining ... Dynamic geometry software strongly encourages this kind of thinking" (p. 23) (Figure 2.7).

Transformations. "Dynamic geometry can *transform* figures in front of your eyes. Isometries and similarities are important examples of functions. In witnessing the action of these transformations moving and scaling figures, students see that functions are not synonymous with symbolic formulas" (King & Schattschneider, 1997, p. xiii). Transformations facilitate the Euclidean constructions based on the construction rules set up around 300 B.C.

Loci. "It is very virtually impossible for most people to imagine a point moving in a configuration (in which other several parts may also may be moving) and be able to describe the locus of the point's path as it travels. Dynamic geometry software, with its built-in feature to trace the locus of any specified object is ideally suited to show how a locus is generated and to reveal the shape of its traced path" (p. xiii). Due to limitation of describing the traced path, this subject has been avoided in most geometry textbooks.

Stimulation. "Dynamic geometry software's special features of dragging, animation (of points on line segments or on circles), tracing loci, and random point generation provide many opportunities to simulate a surprising variety of situations" (p. xiv).

Microworlds. "Dynamic geometry software produces an environment in which Euclidean geometry can be explored" (p. xiv).

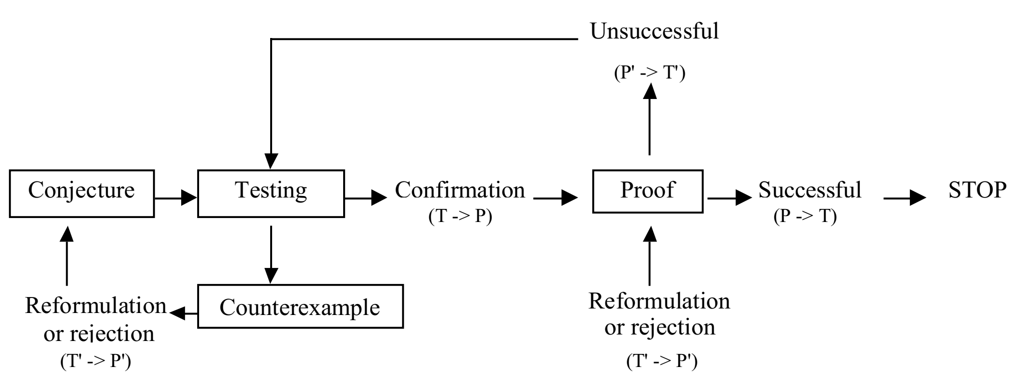


Figure 2.7 Outline of from conjecturing to proof

World Wide Web as instructional tools

The World Wide Web, called WWW, is "a network of networks that has a body of software and a set of protocols and conventions in common" (Flake, 1996, p. 89). It uses hypertext and multimedia techniques and became user-friendly with the development of hyperlinking systems, which makes the web easy for anyone to browse, roam, and make contributions.

The Internet began in 1969 through the Department of Defense with four universities: the University of California at Los Angeles, the Stanford Research Institute, the University of California at Santa Barbara, and the University of Utah. In 1972, E-mail was first invented to send a mail through a network and in 1973, transatlantic connections were established to England and Norway. In 1982, the first definition of an "internet" was instituted as a linked set of networks. By 1984, there were about 1,000 hosts and by 1991, 100,000 hosts (Figure 2.8). Figure 2.9 shows evidence of the rapid growth in the number of WWW servers. The potential of WWW has been recognized as a major force in our society. Perhaps the biggest question that we need to consider seriously is "How do we want it to grow and help become a major contributor to our educational system?" (p. 91).

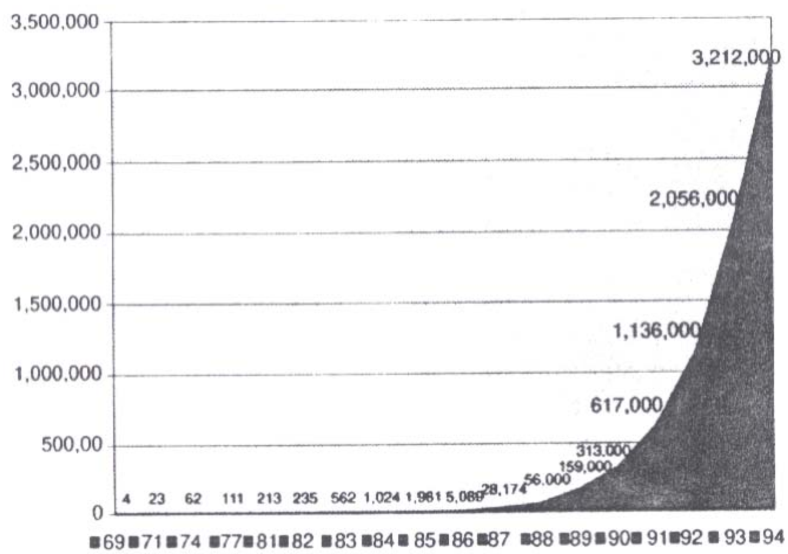


Figure 2.8 Growth in number of Internet hosts from 1969 to 1994 (p. 90)

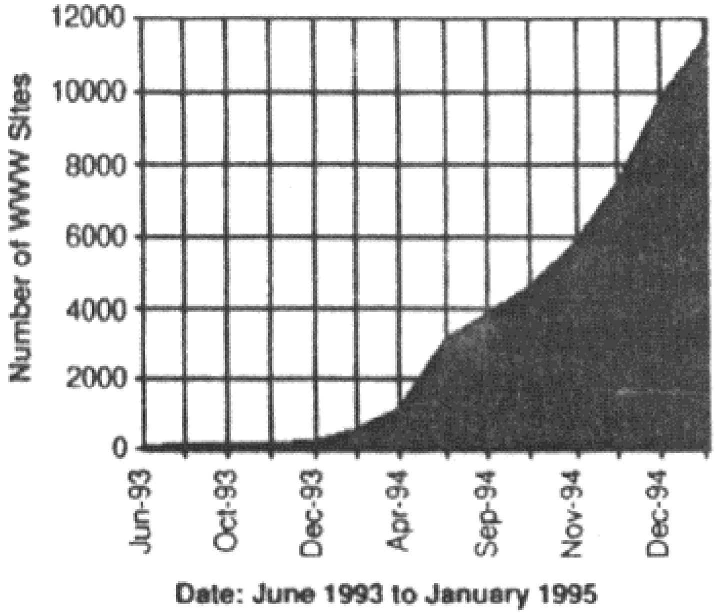


Figure 2.9 Growth in number of WWW servers, June 1993 to January 1995 (p. 91)

Flake (1996) summarized several very important educational opportunities that the World Wide Web provides us. First, students have access to a wide range of knowledge and information. But because those knowledge and information may be out of date or irrelevant, students need to learn to develop skills in gaining knowledge. Second, students can develop in socially relevant ways. The WWW can become a social environment for learning because students not only can learn knowledge and information from others, but also can gain ideas and insights by looking at other reports on understandings and investigations beyond the boundary of theirs. Third, the WWW provides a vehicle for the path to the newly developed high technology. As the technology is greater, people need to be in touch with other people. Fourth, as people explore the WWW, expectations will continue to rise, and the materials on the WWW will continue to get better and better.

Starr (1997) claimed that there are three keys to the educational value of the World Wide Web that are important to the instructional designer:

> The first of three keys to the educational value of the Web is hyperlinking that allows user control of information, the most basic level of interactivity. The

> designer can create links independent of a rigid hierarchy to allow, for example, the user to seek elaboration of the hyperlinked word(s), choose which topic to view, or follow different pathways through the program.
>
> The second key is delivery of multimedia with graphical browsers, perhaps the most important feature of the Web for instruction, enabling the widespread and inexpensive distribution of excellent images.
>
> The third key is the support for high levels of interactivity, with program response to the user to implement features such as searches, scoring of test answers, and simulations based on user input. The cross-platform nature of the Web assures facile widespread distribution of programs and enables seamless integration of lessons from widely dispersed sites. Worldwide programs can be updated or expanded by simply revising or adding files to a single computer. (p. 11)

On the other hand, Flake (1996) mentioned several concerns about the WWW although she acknowledged the great potentials of the WWW. First, there is no censorship. Students can access some inappropriate materials which might cause to waste their time and energy. Those also can hurt students psychologically and mentally. A second concern is that there is no systematic information system. Too much information can bring confusion. Hence, it becomes a major challenge to locate appropriate materials within a reasonable time. The last potential problem that she recognizes is that because WWW provides a dynamic process of updating and modifying materials, an active address one week may lead to a blind alley the next. In addition, different browsers may display the screen differently for the same information or the speed may vary browser to browser as well as designers may have difficulty in locating appropriate materials due to the copyright issues (Starr, 1997). Flake (1996) claimed that WWW provide "a very open approach to education where students no longer are dependent upon their teacher or a textbook as their sole source of information. Students may learn a variety of topics in a number of ways .... WWW also holds the potential of raising the standards of education" (p. 100).

CHAPTER 3

RESEARCH METHODOLOGY

This chapter describes the research design, procedure, participants and their backgrounds, methods of data collection, data analysis, setting of this study, and limitation of methodology. Considering the purpose of this study and research questions proposed above, a qualitative methodology study is the most appropriate for this study. In particular, the case study method was used. The case study provides some of the most useful methods available in educational research due to its flexibility and adaptability to a range of contexts, processes, people, and foci (Macmillan & Schumacher, 1993). This study focused on very specific case with two participants. The participants were above average in their mathematics performances and were undergraduates, juniors around age 20 ~ 21. Further, this study tried to investigate the growth of understanding of 'transformations in geometry' with technology experience. Many researchers (Goetz & LeCompte, 1984; McMillan & Schumacher, 1993; Merriam, 1988; Merriam & Simpson, 1995; Wiersma, 1991) put their efforts to clarify the concepts of case study. "Case studies are differentiated from other types of qualitative research in that they are intensive descriptions and analyses of a *single unit* or *bounded system* (Smith, 1978) such as an individual, program, event, group, intervention or community" (Merriam, 1998, p. 19). As Stake (1995) explained, "it is the case we are trying to understand" (p. 78).

Merriam (1988) characterized four essential properties of a qualitative case study as followings: 1) *Particularistic*: Case studies focus on a particular situation, event, program, or phenomenon; 2) *Descriptive*: The end product of a case study is a rich, "thick" description of the phenomenon under study; 3) *Heuristic*: Case studies illuminate the reader's understanding of the phenomenon under study, which can bring about the discovery of new meaning, extend the reader's experience, or confirm what is known; 4) *Inductive*: Case studies for the most part rely upon inductive reasoning for the formulation of generalizations, concepts, or hypotheses from an examination of data (pp. 11-13). Also, Goetz and LeCompte (1984) claim that case study is the powerful and appropriate research method "for intensive, in-depth examination of one or a few instances of some phenomena" (p. 46). Thus, case study is

> a particularly useful methodology for exploring an area of a field of practice not well researched or conceptualized. In-depth describing and understanding of a phenomenon are needed before generalizations can be made and tested. Case study, which has as its purpose the description and interpretations of a unit of interest, can result in abstractions and conceptualizations of the phenomenon that will guide subsequent studies. (Merriam & Simpson, 1995, p. 112)

Procedure

The course that I observed was MATH 5210 - Foundations of Geometry II taught at a major university located in the southeastern United States in Spring 2001. The main topic I observed for this study was 'transformations in geometry.' Two participants were selected based on three criteria (to be discussed later in detail) for this study. Under their permission, they were observed and interviewed on a regular basis. I asked them to sit side by side during the class so that I could observe them at once during the period of investigation. The class met twice a week, Tuesday and Thursday. Each class lasted for 75 minutes. Along with these regular classes, interviews were regularly conducted with each participant.

The first and the last interviews were different. In the first interview, I started with a discussion of the outline (Appendix A) of this study, and explained the purpose and process of this research study in detail. Secondly, we talked about the general background of each participant, of course separately, using an interview protocol (Appendix C). Participants were asked to fill the technology questionnaire (Appendix D). Thirdly, a diagnostic test about 'transformations in geometry' was conducted (Appendix E). Last, I asked them to make a concept map for 'transformations in geometry' using the basic concepts (Appendix F) I provided. The basic concepts were first selected by me, then revised by the instructor, and finalized by the instructor and me. The participants were allowed to add more concepts as they wanted.

At the twelfth week, the last interview was conducted to make another concept map with the updated list of concepts about 'transformations in geometry,' which included every concept that the course covered up to the moment. A close comparison was performed between the concept maps made in the beginning of this study and that in twelve weeks to find some clues about their understanding and the growth of understanding. The updated concepts were provided in the same manner as it had been done in the previous interview (Appendix G).

Five different types of methods were used for data collection: descriptive notes from classroom observations, reflective notes, archival data, modified interviews consisting of formal and informal interviews, and the concept maps. Each interview was audiotaped and videotaped. Modified constant comparison method was used for the analysis of collected data.

Participants and their backgrounds

Two participants, Abbey and Emily, were chosen on the basis of the following three criteria: 1) students in MATH 5210 who have successfully completed MATH 5200 with above average performance; 2) students who are able to express what they are doing; 3) undergraduate, junior, age 20 ~ 21. In order to select participants, I discussed selecting participants with the instructor based on those criteria and approached students who fit to the criteria. Because the course was the second semester, I needed some advice and help form the instructor to select students who fit to these criteria. I explained the purpose of this study and the process with the outline (Appendix A) to each student and finally selected two participants who volunteered and fit to the criteria.

The process of selecting participants was conducted with researcher's intention, which is called *purposive* (Chein, 1981) or *purposeful* (Patton, 1990). "Purposeful sampling is based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned" (Merriam, 1998, p. 61). Patton (1990) argued that "the logic and power of purposeful sampling lies in selecting *information-rich case* for study in depth" (p. 169). For the purposeful sampling, it is essential to have specific criteria so that the researcher can appropriately choose people or sites to be investigated. LeCompte and Preissle (1993, p. 69) preferred the term *criterion-based selection* to the terms *purpose* or *purposeful* sampling.

Both participants had similar mathematical backgrounds and experiences with transformations in geometry. As you can read from the table (Table 4.1) in the following chapter, they had many things in common: mathematical backgrounds, their goals to be a teacher, experiences and skills with GSP, knowledge with transformations, grades of mathematics, attitude toward mathematics and technology, etc. Also, another two tables in the following chapter show participants' responses to technology questionnaire (Appendix D) given to them in the beginning and at the end of this study (Tables 4.2 & 4.3).

With two participants who exactly fit to the category of this study and are very active in the academic sense, I was able to secure multiple and common evidences from observing students, who are in a typical age as undergraduates. Also, they were in the similar situation by their performances of the previous course so that this study can have more intensive interpretations and understandings for the specific case.

Data collection

Five different methods were used to collect data for this study. These various data collection methods allowed the researcher to investigate 'student representations and understanding of geometric transformations' and 'the growth of understanding' with technology experience in various perspectives: observing, reflecting, examining documenting, and interviewing. Two of five are field notes: one is ‘descriptive notes’ from classroom observation; the other is ‘reflective notes’ for each class activity and discussion (Creswell, 1998, p. 128). The other two are archival data and modified interviews (informal and formal). The last is a concept map.

The 'descriptive notes' from classroom observation consists of contents for each class, classroom activities, questions posed by the instructor, responses of participants to instructor’s questions, participants' accomplishment in the class, and assignments. It is basically to describe what is happening in the class. Right after each class and formal or informal discussion inside and outside classroom, 'reflective notes' were recorded, which contains analysis and interpretation of what I observed. This consists of researcher's reflections rather than just descriptions of the class. In addition, this included possible questions that could be asked during interviews. Participants’ body language such as nodding, shaking, frowning, etc. were very important sources about what to ask for the coming interview. For example, I prepared questions in order to find out what was the meaning of the frowning face during the class when the instructor explained this or what was the meaning of shaking, or which activity was most helpful to you for better understanding or which activity was least helpful.

For archival data, there are four different types of sources: participants’ class notes, mid-term examinations, e-mails sent to me, and assignments they turned in. Because all these documents were personal belongings, I used those materials with their permission for investigation and report.

The fourth for data collection is ‘modified interview,’ which is the main source for data of this study. I categorized modified interview into two different types: informal and formal. ‘Informal interview’ means all kinds of conversations between participants and the researcher outside class, and limited conversation during the class activity. It could happen right before class began or right after the class was over. Also, it could happen in the hall by accident, during lunch time, in the library, etc. As long as the conversation was somehow related with this study, it was used as a part of data.

‘Formal interview’ means the interview that was conducted at the appointed time between each participant and the researcher in a separated place outside the class. We set up the formal interview on a regular basis: it was usually Tuesday for Abbey and Thursday for Emily. Each interview lasted about an hour or so and was designed to be semistructured (Merriam, 1998). This formal interview had two aspects for data collection. The one was to listen to what they were able to understand contents covered druing the class and finish assignments. During this time, participants mainly explained to the researcher about what they understood and accomplished. In addition to participants' explanation, the researcher asked questions prepared in the descriptive and reflective notes during and after classroom observations. These notes were used for rich conversation with each participant during each interview. The other was first to listen to what they could not understand and accomplish. Then, the researcher tried to find out what hindered them from understanding a specific topic or accomplishing their assignments through discussion or sort of tutoring. This process gave the researcher a chance to find clues how and to what extent the representation affects students' understanding and growth of understanding either negatively or positively. Thus, the formal interview was sort of research-intended free tutoring. In this setting, participants lead some portion of interview and made contributions to the substantial formal interview. This was just neither listening nor conversation. Teaching was partly included in the formal interview. This is why I named this interview ‘modified interview.’ During the formal interview, one video camera and one tape recorder were set up (Figure 3.1) for securing records. The audiotapes recorded were more helpful in transcribing conversation (Appendix H) of each interview than video tapes.

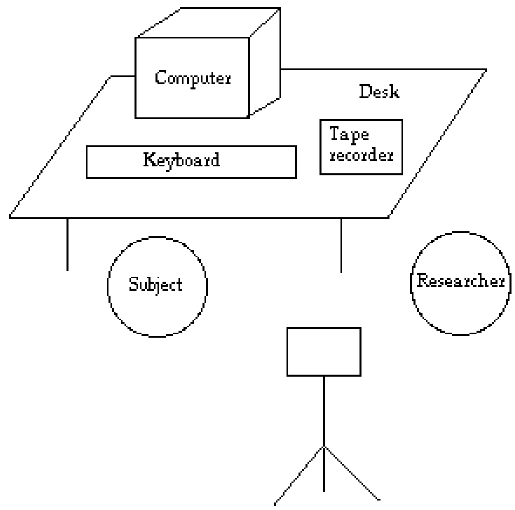


Figure 3.1 Structure of formal interview site

The last method was a concept map. Each participant was asked to make two concept maps: one was in the beginning and the other at the end about 'transformations in geometry.' 'Concept maps' which is a metalearning strategy based on the Ausubel-Novak-Gowin theory of meaningful learning consists of concepts and propositions. Novak (1980) defines 'concept' as "the regularity in events or objects designated by a sign or symbol" (p. 283) and 'proposition' as "specific relationship between two or more concepts" (p. 283). Concept maps have been called the 'window to the mind' of the students we teach because of seeing in (by the teacher and other students), seeing out (by the student), and reflecting on one's own perceptions (by everybody). Furthermore, concept maps facilitate a sharing of meaning unhampered by any lack of verbal skill. Thus, consequently both teacher and students are able to judge with some degree of clarity how well they themselves have grasped a particular concept and how well their colleagues or classmates have done so (Malone & Dekkers, 1984, p. 231). Concept mapping is "particularly suitable for use as an interview technique in research on mathematical knowledge development" (Hasemann & Mansfield, 1995, p. 69) as well as an instructional tool. Also, concept maps have been used to identify weaknesses in student's understanding (Hasemann & Mansfield, 1995).

Concept maps in this study were used to recognize the occurrence of participants' understanding and growth of understanding, and the extension of their knowledge about transformations in geometry. The role of a concept map was not to explain or interpret participants' understanding or the growth of understanding, but it was the tool to help the researcher notice the transitions that students made over the investigation period. Thus, the concept maps were the auxiliary method to recognize understanding and especially the growth of understanding in addition to observation, notes, archival data, and modified interviews.

Data analysis

The data collected from those five different types above were analyzed using 'constant comparison method,' which was developed by Glaser and Strauss (1967) as the means of developing grounded theory. As the basic strategy of the constant comparative method which is compatible with inductive concept building orientation of qualitative research, this method has been adopted and developed by many researches (Corbin & Strauss, 1990; Creswell, 1998; Goetz & LeCompte, 1984; LeCompte & Preissle, 1993; Merriam, 1988). Moreover, the constant comparison method of data analysis is widely used in all kind of qualitative studies.

"The constant comparative method involves comparing one segment of data with another to determine similarities and differences" (Merriam, 1998, p. 18). Data are grouped together based on a similar dimension and each group is tentatively given a name. As data collection goes on, that name is refined and updated and data are regrouped in an appropriate place so that it finally becomes a category. The constant comparison method is an ongoing process throughout the study to find patterns from collected data. In some sense, data collection and analysis cooperate with each other for better investigation and go together at the same time.

I began the analysis with analytic induction which "involves scanning the data for categories of phenomena and for relationships upon an examination of initial cases, then modifying" (Goetz & LeCompte, 1984, pp. 179-180). After scanning, I found relationships and categories and refined them by referring back to previously analyzed data in conjunction with newly collected data. Going over descriptive notes, reflective notes, and archival data, and transcribing an interview as it was being conducted made it possible to do constant comparison method. As Patton (1990) pointed out, "by using a combination of observations, interviewing, and document analysis, the fieldworker is able to use different data sources to validate and cross-check findings" (p. 244). Merriam (1998) explained that "in case studies, communicating understanding - the goal of the data analysis - is linked to the fact that data have usually been derived from interviews, field observations, and documents" (p. 193). Because I selected the one case with two participants having similar backgrounds, this analysis was "within-case analysis" (Merriam, p. 194).

The following coding system played a major role in analyzing collected data and keeping track of evidences for students' transitions in understanding and growth of understanding. This coding system was the guideline to categorize phenomena and relationships.

Table 3.1 Data coding system

Index	Elements	Code	Description
Pirie & Kieren's Model for the Growth of Understanding	Primitive Knowing	PK	It does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding.
	Image Making	IM	The learner begins to make form images out of doing understanding.

	Image Having	IH	Action - tied images are replaced by a form for the images, which frees a person's mathematics from the need to take popular actions to create images.
	Property Noticing	PN	This includes noticing distinctions, combinations, or connections among images, predicting how they are interrelated, and presenting those relationships noticed.
	Formalising	FO	The learner abstracts a method or common quality from the previous image dependent know how which characterized one's noticed properties
	Observing	OB	A learner reflects on and coordinates such formal activities and express such coordinations as theorems.
	Structuring	ST	A learner is aware of how a collection of theorems is inter-related and calls for justification or verification of statements through logical or meta-mathematical argument.
	Inventising	IN	The learner has a full structured understanding and may therefore be able to break away from the preconceptions which brought about this understanding and create new questions which might grow into a totally new concept.
Two features of Pirie & Kieren's Model	Don't Need Boundaries	DN	To convey the idea that beyond the boundary a learner does not need the specific inner understanding that gave rise to the outer knowing.
	Folding Back	FB	When faced with a problem or question at any level, a learner need to fold back to an inner level in order to extend ones' current, inadequate understanding.
Three components of Representation	Written Representation	WR	The mathematical symbols presented by a learner.
	Pictorial Representation	PR	The physical objects created by a learner.
	Oral Representation	OR	The learner's words without written and pictorial representation.
Positive or Negative effects with Technology	Benefits	BE	Positive effects on students' growth of understanding with technology experiences
	Obstacles	OT	Negative effects on students' growth of understanding with technology experiences

Settings

Course chosen for the study

The course chosen for this study was MATH 5210 - Foundations of Geometry II taught in Spring 2001 (Monday, January 8 - Friday, April 30 for 15 weeks). The instructor has taught geometry for many years. Only in the past three years, however, has he taught geometry in a technology-based classroom. Students were required to take MATH 5200, Foundations of Geometry II, prior to MATH 5210. Axiom systems in geometry, trigonometry, analytic geometry, and Euclidean geometry were the main topics in MATH 5200 and the main topics in MATH 5210 were transformations in geometry and Platonic solids. The instructor used *The Geometer's Sketchpad* [GSP] (Jackiw, 1991) as a teaching and learning tool throughout both courses. GSP was the main technology tool throughout the semester and used by the instructor in his demonstrations and lectures as well as by the students for homework and explorations. The instructor posted a website with his lecture notes, personal notes, hints, and information for the course (Appendix I) as well as assignments (Appendix J).

The design of the geometry sequence, MATH 5200 and MATH 5210, relies in a fundamental way on the software used, The Geometer's Sketchpad and Kaleidomania. The overall goal of these two courses was for the student to develop geometric understanding that is both visual (experimental) and logical (based on definitions and proofs). An important aspect of such understanding was the ability to explain geometric concepts using diagrams and words.

Transformations in geometry, the topics observed for this study, were taught for basic understanding of transformations and classification of transformations (translation, rotation, dilation, reflection, and glide reflection) during the first four and one-half weeks. After that, the extension and application of transformations were the main topics for the next five and one-half weeks. At this second stage, students were introduced to wheel symmetry (called point symmetry), strip symmetry (called line symmetry), and wallpaper symmetry. Along with this introduction, they discussed seven types of strip patterns and 17 types of wallpaper patterns. Especially, students were introduced to Conway notation and Kaleidomania (Lee, 1999) for wallpaper patterns. For the next three weeks, the main topic was the 'platonic solids' and the last two weeks were used for students' presentations of their final projects. Students had two mid-term examinations (Appendices K & L), but no final examination. The final projects were in lieu of a final.

GSP, Kaleidomania, the instructor's personal website, and e-mail were significantly useful tools for this course. Sources from the Internet were also used. Regular office hours were set up twice per week in the computer laboratory, which is the classroom where MATH 5210 was taught, and in the instructor's office. Also, students were allowed to make an appointment other than office hours if they needed help.

Classroom

There were 20 computers along three walls in the computer laboratory: nineteen Macintoshes (G3's, G4's, & power PC's) and one Windows PC (Figure 1.5). In the middle there were six tables and some chairs around each table. The projector connected to the computer for the instructor's use projected on the front wall so the instructor could show students what he was doing during class or show materials he had prepared for the class. There was an another projector for the projection of transparencies. Students liked this classroom because each student could have his/her own computer for class activities and there was enough space for class discussion. Students had a folder on a server that they could access through the network at any time from any computer in the classroom. Thus they did not have to carry diskettes containing their works with them, another benefit of technology.

There were 14 students in the class and eight of them were undergraduate students in MATH 5210 and six were graduate students in MATH 7210. Students were encouraged to have group discussions for classroom activities and assignments. They were asked to write individual reports using their own words.

Teaching style

When the instructor taught this course in the past, class met three times a week and each class lasted 50 minutes. From this experience, he found that 50 minutes was too short for class activity, discussion, and a lecture, especially if students had individual exploration time. Thus, he designed this course to meet twice a week for 75 minutes per each class. He started out with a short introduction for the topic of each class and demonstration with GSP prepared for the class almost every class period. Sometimes he began with proofs, but not very often. His introductory lecture and demonstration led to discussion as a whole group. Then, he threw out a question, generally an open question, so that students could figure it out by themselves. At the end of each class, he spent some time for discussion as a whole group again to finally wrap up what was done during the class. This was the typical format of each class.

For example, after he briefly introduced the topic 'transformations in geometry' and had a brief discussion about it, he let students play around with GSP to investigate how they could use GSP for transformations and had them report about what they investigated, which is the first assignment (Appendix J). He did not teach very much how to use GSP. Actually, the students already knew a lot about GSP because they had used GSP in MATH 5200 and other classes. He pointed out what to focus on for doing transformations, e.g. defining data for each transformation. The wrap-up discussion at the end of each class helped students to be ready for accomplishing an assignment.

He sometimes spent a whole class on a proof. In these rare cases, he used only transparencies or the whiteboard and students did not use the computer. In addition to the regular assignments, there was a project instead of a final examination. Students were allowed to do projects individually or as a group. The topics for projects were restricted to something related with geometry. He asked students to prepare a pre-proposal by

March 1, 2001 and a full proposal by March 20, 2001. The due date for the final project was April 24, 2001. Before students conducted their own projects, the instructor had various class activities to illustrate ideas for projects. The two mid-term examinations that tested knowledge about transformations in geometry and platonic solids, and these projects gave students a chance to be more creative and constructive with their understanding of geometry.

Time line

This study was conducted during the first 12 weeks out of 15 weeks in Spring, 2001. 'Transformations in geometry' was the main topic for ten weeks and two more weeks were mainly for interviews. There were also two mid terms: one was in the 7^{th} week and the other was in the 13^{th} week. The second examination was performed after the study of transformations was over. But, because half of the second examination was about transformations, I included the second examination as part of my data. Here is the time line for the events related with this study.

Dates	**Events**
Tuesday, January 9	First class. Introduction of the course and transformations in geometry. Playing around with GSP individually or as a group. First assignment was given.
Wednesday, January 10	First interview with Abbey and Emily - to find out their general backgrounds and make a concept map with basic concepts about transformations.
Thursday, January 11	Discussion about the first assignment and about the transformations in depth. Second assignment was given.
Tuesday, January 16	Second interview with Abbey. First informal interview with Abbey before the class starts. Products of transformations was introduced. Third assignment was given.

Thursday, January 18	Second interview with Emily. Discussion about products of transformations in depth and proof of them. Isometry game.
Tuesday, January 23	Third interview with Abbey. Proof of isometry. Isometry is bijection.
Thursday, January 25	Fourth assignment was given. Could not be observed due to researcher's emergency in the family.
Tuesday, January 30	The Triangle theorem. The classification theorem.
Thursday, February 1	Third interview with Emily. Understanding different coordinates for isometry. Linear transformations.
Monday, February 5	Fourth interview with Emily.
Tuesday, February 6	Fourth interview with Abbey. Matrix for transformations. Affine transformations ($F(X) = MX + B$, where $\det(M) \neq 0$). Center for rotation when $F(X) = MX + B$.
Thursday, February 8	Plane symmetry - Wheel symmetry, Strip symmetry, Wallpaper symmetry. Introduction of Kaleidomania. Generators. Fifth assignment was given.
Friday, February 9	Fifth interview with Emily.
Tuesday, February 13	Multiplication table of types and proofs of the product of types in the strip patterns.
Wednesday, February 14	Fifth interview with Abbey.
Thursday, February 15	Sixth interview with Emily. Starting with 3. (a) of homework 5 to give hints. Seven types of strip patterns.
Tuesday, February 20	Sixth interview with Abbey. Starting with 3. (a) for a full solution. Spent the whole class period with proofs.
Thursday, February 22	First mid-term.

Tuesday, February 27	Review of strip patterns. Generators of symmetry. Generating set. Fundamental region of pattern or unit of pattern. Sixth assignment was given.
Thursday, March 1	Generating set. Fundamental region. Lattices of wallpaper groups. Translation unit.
Monday, March 5	Spring break starts for a week.
Tuesday, March 13	Informal interview with Emily.
Thursday, March 15	In depth discussion about lattices of wallpaper patterns and fundamental region. Seventh assignment was given.
Friday, March 16	Seventh interview with Abbey and Emily.
Tuesday, March 20	Discussion about assignment 6. The relation between fundamental region and a wheel pattern. Being able to tell the pattern of 17 types.
Thursday, March 22	Using Kaleidomania for creating wallpaper patterns, finding fundamental regions, lattices, and symmetric groups. Figuring out Conway notations.
Monday, March 26	Eighth interview with Emily.
Tuesday, March 27	Eighth interview with Abbey. New topics began - Platonic solids.
Tuesday, April 3	Ninth interview with Abbey.
Thursday, April 5	Ninth interview with Emily.
Thursday, April 12	Second mid-term.

Limitation of methodology

The methodology selected for this study may have elements that possibly limit this study: language, gender, cultural background, and race. Because the researcher's first language is not English, there might be some miscommunication between the researcher and the participants whose first language is English. But I believe the video and audio tapes of interview, researcher's field notes, and students' notes will make clear the things which are possibly ambiguous. The conversation between each participant and the researcher throughout this study might be affected by gender (female vs. male) and cultural background (oriental and occidental).

The other factor is from the process of data collection. First, because this formal interview is somewhat like a tutoring format, participants might depend on me for help. In other words, they might just come to an interview for help without doing their works at all. But, I clearly and manifestly told them in the beginning of this study that they would basically lead the interview. To avoid this type of invalidity, I asked them to present topics they were able to understand and topics they were not able to understand for the first part of each interview. This request made each interview more vivid and active.

Second, when I engage participants to discuss topics they could not understand, I might induce or force them to use representation consisting of written, oral, and pictorial expression for understanding. In addition, during the interview, I might use different approaches or styles from those of instructor’s when I try to help them understand topics. Then, it would be hard to decide if their growth of understanding is from either different teaching styles, approaches, or repetitive learning. To avoid this kind of invalidity, the discussion was focused on finding how representation played a role in their understanding and the growth of understanding at the time of interview rather than on helping them use representation. Further, teaching styles and approaches should be similar to what instructor used during the class so that the other factor can not play a major role for investigating participants. Nonetheless, the aspect of repetitive learning does not seem to be avoided.

Also, participants might be affected by environmental atmosphere. For example, they might feel more comfortable or uncomfortable in one-on-one situation rather than in a big group of class. I will ignore this case because this study was designed to have an interview one-on-one situation and adopt case study investigating a single unit-student.

As a last, the possibility of biased description by the researcher is expected. Because I am interested in representations, understanding, and growth of understanding with technology experiences, I might mainly focus on these three aspects throughout this study. Especially, as a person who has been interested in 'technology in mathematics education,' I might try to let representation using technology stand out in my study. In order to overcome this weakness, using the words of participants (*emic*) helped me to be away from this sort of bias in data collection and analysis. Also, by providing multiple and common evidences, e.g. "triangulation" (Merriam, 1998, p. 204), for the phenomenon, the bias can be overcome. We can expect more persuasiveness from multiple and common evidences supporting a specific interpretation than we can from a single evidence. In addition, I used 'member checking' to make sure that I reported participants' real intentions. For example, although I used participant's direct words from an interview, it might not be their real intention. Also, I might misinterpret their intentions according to my bias or desires. 'Member checking' helped me make myself less biased in reporting.

CHAPTER 4

RESULTS AND FINDINGS

This study investigated the impacts on students' growth of understanding when they study transformations in a technology-based classroom. This study examined how students present their understanding: written, pictorial, and verbal representations. In addition, obstacles and benefits with the use of technology were examined. The coding scheme (Table 3.1) was used throughout the study in analyzing the data and it helped the researcher see the flow of students' understanding and categorize phenomena.

Introducing participants

Getting to know Abbey

It was Fall, 2000 when I met Abbey for the first time. At that time, I was the instructor of the course, EMAT 4680 - Technology and Secondary School Mathematics, which Abbey was taking, and it was the first year when she began her major, mathematics education, as a junior. There were two sections for EMAT 4680, which is required for the major. She was taking several courses in that Fall semester, one of them was EMAT 4680 and one of them was MATH 5200 - Foundations of Geometry I, which is prerequisite course for taking MATH 5210 - Foundations of Geometry II - in the following semester. She always worked hard and was very much active and positive in class activities in my class. She loved mathematics and showed strong enthusiasm about mathematics all the time. Further, she made a decent grade in EMAT 4680 as well as in MATH 5200.

[1]R: Do you like mathematics?

A: Yes, I do.

R: Why?

A: Because it's fun and challenging and it's kind of like mystery. You have to solve it and you will come to an answer. Not always in theory. But, you know ....

R: Is there any moment that you really hated mathematics?

A: Hated? No, I just get upset sometimes when it's really difficult that I can't figure it out. But, I never hated.

R: Would you tell me about your experiences with mathematics? Can you think of the moment that you liked mathematics and didn't like mathematics?

A: I remember I was in an eleventh grade and I decided to be a math teacher because, my math teacher, he brought all different subjects into math. It was pre-calculus. And we had to write papers in that class. I thought that was so strange at first. But, it made me like dig deeper into mathematics and understand how it's connected to all the other subjects and studies. And he made it fun. Like some teachers can just, you know, kill you as math. Just write problems upon the board. But, he made it fun and interesting. But, then I realized at one point I didn't like math. I almost changed my major last year when I was in MATH 3200. It was the first theory based class. So, it scared me. I can't do theory. That was a big turning point.

R: How about right now? Do you feel all right?

A: Yes, sir.

R: Are you ready to move on? Do you like mathematics in general through all your life?

A: When I can remember, always loved math and I did well in English but it wasn't just my favorite subject.

R: So, it was a scary year, last year.

A: Right.

R: Other than that, generally you like mathematics. Can I ask you the grades about math in general if you don't mind?

A: Yea, about A, B.

[1] R stands for Researcher and A stands for Abbey.

Further, her leadership in class activities and group projects was outstanding. I found out later that she was the president of sorority. GSP was used to some extent in EMAT 4680 and was a main tool extensively used in MATH 5200 throughout the semester. Abbey used calculators a lot in high school but not GSP. Abbey had one semester of experience with GSP from EMAT 4680 and MATH 5200 before this study was conducted. Despite of this short period of experience, she felt very comfortable and skillful with GSP.

R: O.K. Would you tell me about yourself?

A: O.K. I am Abbey. I am a junior here. And I spend a lot of time at school. It seems like. Also, I live at my sorority house and I am the president. So, I have a lot of work to do when I get home. Umm, what else? I love sports.

R: What kind of sports?

A: I like basketball. My sister is a basketball coach. And I love my family and friends. I like spending time with them.

R: How long did you use The Geometer's Sketchpad up to this point?

A: Just one semester. Last semester [Fall, 2000].

R: So, for about 5 or 6 months?

A: Right.

R: Before that, you never touched it at all?

A: I remember when I was in tenth grade geometry, we did go to the computer lab and use sort of software, but I don't remember if it was GSP or not. I don't think it was.

R: O.K. What year was that?

A: It was 96.

R: Now, do you think you are skillful with GSP?

A: Yes, sir.

R: How do you know that you are skillful with GSP? What made you think you are skillful?

A: I had to work with it a lot in two classes [EMAT 4680 & MATH 5200] last semester and so most of my homework for basically those classes were on that. I think you know I did pretty well on that. That made me feel like that I knew GSP pretty well.

R: Good. Is it O.K. if I interpret your response in this way: when you do your homework, you have no big problem with the use of GSP. But, you might have a problem with mathematical concepts rather than GSP.

A: Right. Ha ha ha!!!

Getting to know Emily

Emily was enrolled in the same courses as Abbey in Fall, 2000. But, she took EMAT 4680 from another instructor. So, I did not meet her until I contacted her for this study in Spring, 2001. I was able to recognize her by face when I first approached her in order to ask for being a participant. She was also a junior at that time. Emily had many things in common with Abbey. Emily also loved mathematics, especially calculus and geometry, always worked hard, and was very active and positive in class activities and projects. She always showed a strong passion about mathematics. Further, she made a decent grade in EMAT 4680 as well as in MATH 5200 as Abbey did. She has been doing martial arts for 15 years. She spends a lot of time with martial arts as a coach and as a competitor in the tournaments.

[2]R: Would you tell me about yourself in general who you are, what you do, and so on?

E: I am originally from the North, Connecticut. I like math and want to be a math teacher for secondary math education and I like math a lot because I enjoy math puzzles and trying to figure things out. And I like a lot of sports.

R: What kind of sports do you like?

E: Marshal arts.

R: Marshal arts?

E: 15 years now.

R: Wow! In what area?

E: Now, Taekwondo and taking full content kick boxing of American Karate and coaching Karate demonstration team.

R: Do you like mathematics?

E: Yes.

R: Why do you like mathematics?

[2] R stands for Researcher and E stands for Emily.

E: Like I said, I love puzzles and I just realized that about a month ago. And I never really knew why I like mathematics so much, not necessarily like Rubics cube and stuff like that. But even just, you know, you're given just a paper puzzle thing to do or, you know, a word puzzle or something. I just love figuring out. I think of mathematical proofs, especially geometric proofs and all puzzles and you're given a piece of information. You have to figure it out how to get from one point to the other. I like to have a right answer. You get a good feeling.

R: If you don't mind, would you tell me about your grades of mathematics in general?

E: In high school, I always in the top level for all the math classes and I got A's and one or two B's in some other very ... the AP that are like advance placement classes. They were pretty difficult. I got B's on those but they are counted A's because of the weight on the class. In here, A's or B's it depends on. The class even if I worked hard and gotten B. I knew that I learned a lot.

R: Which course is your favorite course in mathematics?

E: That I have had so far? Or ….

R: Or, in general in mathematics.

E: Calculus, but more and more than like in geometry.

R: Is that because of the course or is that because of the [instructor] …?

E: It's because of the course I think.

R: Not because of the teacher?

E: Uh huh (meaning no). For 5200, the instructor has helped me to understand even though I understood. And freshman year high school last time I had it. I wasn't really interested cause I never really liked it too much and I didn't spend a lot of time. All the rest of classes were algebra and calculus. That's why I like those more but since these proofs class 5200, I have enjoyed geometry a whole lot more.

Although she sometimes struggled with mathematics, she was usually able to get through everything for herself. When she meets mathematical problems she can not understand, or when she can not move on further, she first tries to figure out by herself. Then, she will go to either a friend or a teacher for help later. She almost never goes to teacher until she has attempted the problem.

E: I like ... umm ... I enjoy sitting and trying to figure out stuff on my own. If I have questions, I usually hesitate to ask. One of my friends asks a teacher beforehand and I go and help a lot. I think that helps me if I work on it first to know exactly where I am stuck. So, once I go to a teacher or ask a friend or something that right one day, you know, get to the point. And I can realize what I did. So, it helps me to learn that way instead of just going to someone right away. I had a friend who we did almost all of our works together because problems we're assigned in that third semester calculus [Third semester in college]. We were very difficult and we spent many many hours just sitting there and working on stuff and this one problem. We worked forever. And we finally figured out the answer and we were like... We both have figured it out parts of it and we finally got to the end and we were like so excited during the whole day.

She had great teachers when she was a sophomore in high school and a sophomore in college who greatly influenced her desire to become a mathematics teacher in secondary school. She recalls that almost all her mathematics teachers were excellent except one instructor in college.

E: I've had some really wonderful teachers in high school. That's what pretty much why I want to teach math because they were very helpful and hopefully now I understand things. And they are available mostly for extra help and are very willing to help me to learn something if I have problems with anything. And they made it very fun and exciting and interesting although we didn't do stuff on the computer. Umm ... we still did a lot of hands on projects and that helped to be able to understand math and realized that it was more than just pencil and paper all the time.

R: Ah, that kind of experience brought you here. Did you have any math teacher who affected you negatively?

E: No, I had had all great teachers.

R: Even including instructors in college?

E: I had one real bad teacher, sophomore year, here.

R: What subject was that?

E: It was second semester Calculus. He just couldn't teach stuff. He'd go over things but he wouldn't go in depth and whenever we try to ask a question, he wasn't very willing to help us understand and it seems like and so very difficult. I didn't do nearly as well in that class as I did in all my other classes because it was very stressful to try out everything when he wasn't teaching very well.

R: When was the best time with mathematics teacher?

E: Probably, my sophomore in high school and also my sophomore first semester in college here. I had very good teachers then.

R: What was that course?

E: Umm ... It was a third semester calculus. And it was a very hard class but my teacher was very good and so, I learned a whole lot and it was very challenging.

As for experiences with GSP, Emily also began to use GSP since the previous semester when she took EMAT 4680 and MATH 5200 although she had used calculator a lot in high school. She had a strong confidence with the use of GSP and felt very comfortable and skillful with GSP. However, at the same time she admitted that GSP is good to supplement, but can not be substitution for learning mathematics. She recognized GSP as a very useful tool in exploring conjectures.

R: How many years did you use GSP?

E: Ah, just since last semester.

R: Before then, you never touched it?

E: Never touched it.

R: Never heard of it?

E: Never heard of it.

R: Do you think you are skillful with GSP?

E: Uh, yea. We used it a whole lot last semester.

R: Why do you think you are skillful with GSP? What made you say so?

E: I've gotten pretty decent grade [in MATH 4680 & MATH 5200]. We're given a whole lot of assignments especially in 5200 that used GSP. So if you used it every single day in class throughout semester and I know that we're given something to construct whatever that is, I've usually been able to do it pretty efficiently.

R: That's why you can say so. Is it O.K. if I interpret your response this way: you can do pretty much everything whatever you want to do with GSP?

E: Most of them.

R: As long as you understand mathematical stuff.

E: Right.

R: But, what happens if you don't understand mathematical stuffs even with GSP?

E: You can't understand what's going on when you actually…

R: So, GSP is a little bit far away from mathematics in that situation.

E: I think it's good to supplement the mathematics that you learned but not used as substitution. Specially, you can't use GSP to prove things. You can only explore your conjectures.

R: So, at this time you have no problem with the use of GSP.

E: Yeah. Right.

R: When you learn how to use GSP, what was the main source for learning how to use it? From instructor, friend, a book, or yourself?

E: Not from the book. Not normally except for a little bit but translations and stuff recently. Umm.... some of that was from instructors but once they gave the basic commands, it was just up to classmates and I to play around with it so. But pretty much half and half then between myself and just exploring it on my own and then with friends from the class. If we had a question about something, we can ask each other. You know, make sure we understood how to do it.

The table (Table 4.1) below is the summary of general background of Abbey and Emily. There are many things in common as for their general backgrounds in terms of mathematics and experiences with GSP. Both of them work hard in study and are very active in what they are involved in. Their attitudes are very positive and passionate in general.

Table 4.1 General backgrounds of Abbey and Emily

Items \ Names	Abbey	Emily
Birth Place	South East	North East
Current grade	Junior	Junior
Extra curricular activity	President of Sorority	Coach of Karate demonstration team of University.
Sports	Basketball. Her sister is a basketball coach.	Marshal arts - Taekwondo, Kick boxing and American Karate for 15 years.
Experiences	• Just touched software for	• Never touched and heard of GSP

with GSP	geometry when high school student, but could not remember if it was GSP or not. • One semester (Previous semester with Math 5200) • Learn GSP from working with a group. 70% this way and 30% from instructors.	before the previous semester. • One semester (Previous semester with Math 5200) • Learn GSP almost by herself from playing around with it.
Skillfulness with GSP	• Skillful with the use of GSP - graded by herself (She feels comfortable with GSP in general) • Spent very much and intensive time with GSP in MATH 5200 and EMAT 4680	• Skillful with the use of GSP - graded by herself (She feels comfortable with GSP in general) • Spent very much and intensive time with GSP in MATH 5200 and EMAT 4680. But GSP is for supplement, nor for substitution of mathematics.
Experience with mathematics	• Decided to be a math teacher when in eleventh grade - affected by the pre-calculus teacher • He brought everything into math and made mathematics fun and interesting. • Does not like theory at all. So, almost changed her major last year because of the theory-based class. • Always loved mathematics because it is fun and challenging. • Had a moment of getting upset when mathematics is very difficult but not very often. Never hated mathematics. • Never had a bad teacher but a lot of teachers that were really boring. And this affected her negatively. • Favorite subject was Calculus. But after she learned a lot of geometry, Geometry became her favorite subject.	• Decided to be a math teacher when a high school student • He was very helpful and made mathematics fun, exciting and interesting. • Had a really bad teacher in the second semester of Calculus, sophomore year at college. • Had a wonderful teacher in her sophomore year of high school and also in her sophomore year of college. • Like mathematics because she loves puzzles, proofs, etc. Enjoy sitting and trying to figure out stuff on her own. But, hesitate to ask. Ask help from friend before teacher. • Except physics applications, able to understand most of mathematics. • Experience of excitement during the third semester calculus with a friend after spending hours. • Favorite subject in math: Algebra, Calculus, and Geometry.
Knowledge of Transformations	• Have heard of translation, reflection, rotation, and dilation in high school. But, never heard of glide reflection. Do not remember what she learned about transformations at all.	• Dealt a little bit of transformations when a freshman year of high school. Do not remember what she learned about transformations at all.

	• Just learned it for two periods last semester in a different course - just went over basics of transformations. Did not spend enough time to know well.	• Just learned it for two periods last semester in a different course - just went over basics of transformations. Did not spend enough time to know well.
Grades of Mathematics	Almost all A's, but some B's	Almost all A's, but rarely B's
Understanding mathematics.	Generally understand mathematics with appropriate assistance.	Generally understand mathematics with appropriate assistance.

Responses of Questionnaire

The following two tables (Tables 4.2 & 4.3) are participants' responses to the questionnaire (Appendix D) given to them in the beginning and at the end of the study. The questionnaire that they filled in the beginning and at the end of the study was exactly the same. However, they could remember little of how they responded to the questionnaire in the beginning of this study when they filled the questionnaire at the end of this study.

Both Abbey and Emily thought that although they loved to use various types of technology such as graphing calculators, GSP, Excel, internet, etc., in teaching and learning mathematics, students should first learn most mathematics by hand. Both of them claimed that technology should be supplementary in teaching and learning mathematics to reinforce students' knowledge for a greater understanding in the long run and to accent the learning process. In addition to reinforcement of knowledge, technology could help students further and extend their knowledge through exploring concepts. However, technology should not replace teaching and learning mathematics. In their view, technology is a very useful tool to better understand and to further their understanding of mathematical concepts.

Table 4.2 Responses of Abbey to technology questionnaire

Items \ Names	First	Second
1. High school mathematics students should be allowed to use calculators	Only after learning procedures by hand	Only after learning procedures by hand
2. High school mathematics teachers should be proficient in the use of - Calculators - and computers	 I strongly agree I strongly agree	 I strongly agree I strongly agree
3. If I were teaching a high school mathematics class, I would feel comfortable using - a spreadsheet - Geometer's Sketchpad - or the Internet	 I agree I strongly agree I agree	 I agree I strongly agree I agree
4. I would feel comfortable taking a high school mathematics class to work in the computer lab	I strongly agree	I strongly agree
5. I can foresee taking my high school mathematics students to the computer lab	Several times a week	Several times a week
6. Thinking back to my high school mathematics classes, I wish I had had access to - Geometer's Sketchpad - spreadsheets - and the Internet	 I strongly agree I agree I agree	 I strongly agree I strongly agree I strongly agree
7. I ____________ get frustrated when I am using computers.	Seldom	Seldom
8. I ____________ find myself helping others with technology-related questions or problems.	Seldom	Often or Seldom
9. I feel like I understand mathematics better when I explore it - with computer technology - or with graphing calculators	 I agree I agree	 I agree I am not sure

	First writing responses	Second writing responses
10. Describe the ideal environment under which you would like to learn mathematics.	I enjoy the classroom in which we have MATH 5210, where we have a desk in the middle of the classroom and then the computers are on the outskirts of the room so we can use them whenever we need them.	I love the classroom that we use now because there are tables for the students to sit at, but then they can turn around & work on the computers too. (Figure 1.5)
11. Describe the ideal environment under which you would like to teach mathematics.	I would love to have a classroom where every student had their own computer (or at least enough computers for every student) that was equipped with GSP, Excel, and other math programs.	In the situation above, or where each student has their own lap top with GSP and Excel on their computers.
12. In your opinion,	It means using technology to help	Technology should "accent" the

what does the phrase "learning with technology" mean?	and aid in the learning of mathematics. You don't depend on calculators, computers, etc to totally teach a concept in math b/c students may become too dependent on them.	learning process, not replace the teaching process.
13. In your opinion, what does the phrase "teaching with technology" mean?	The same thing I stated in last question. Teachers use technology to help teach mathematics, but not depend on it too much, to where students really don't understand the concept.	The teacher using technology means they use computers to aid in teaching. They don't just sit the kids in front of the computer.
14. In your opinion, what does the phrase "understanding with technology" mean?	I like this phrase better, because you are using technology to further your understanding of mathematical concepts. You can explore concepts more using calculators and computers which will help you understand more thoroughly.	Once you get & understand a concept, you use technology to reinforce the concept.

Table 4.3 Responses of Emily to technology questionnaire

Items \ Names	First	Second
1. High school mathematics students should be allowed to use calculators	Only after learning procedures by hand	Occasionally
2. High school mathematics teachers should be proficient in the use of - Calculators - and computers	 I strongly agree I agree	 I strongly agree I strongly agree
3. If I were teaching a high school mathematics class, I would feel comfortable using - a spreadsheet - Geometer's Sketchpad - or the Internet	 I agree I strongly agree I agree	 I agree I strongly agree I agree
4. I would feel comfortable taking a high school mathematics class to work in the computer lab	I strongly agree	I agree
5. I can foresee taking my high school mathematics students to the computer lab	Several times a week	Several times a week
6. Thinking back to my high school mathematics classes, I wish I had had access to - Geometer's Sketchpad - spreadsheets - and the Internet	 I strongly agree I agree I agree	 I strongly agree I strongly agree I strongly agree
7. I ____________ get frustrated when I am using computers.	Seldom	Seldom
8. I ____________ find myself helping others with technology-related questions or problems.	Seldom	Seldom

9. I feel like I understand mathematics better when I explore it - with computer technology - or with graphing calculators	 I strongly agree I strongly agree	 I strongly agree I strongly agree
	First writing responses	Second writing responses
10. Describe the ideal environment under which you would like to learn mathematics.	For me to learn mathematics, I would have definitely wished that we had access to the computer labs and GSP in high school to explore mathematical ideas and conjectures.	I would definitely like to be able to incorporate technology in my learning, using Spreadsheet, GSP, Graphing calculators (both hand-held and on the computer), and the internet. Using technology allows me to get a clearer picture of the ideas behind the mathematics and look at the solutions in many different ways for better understanding.
11. Describe the ideal environment under which you would like to teach mathematics.	Of course, I would love to have the students be willing to learn. Even if they did not understand the problems in class, as long as they were willing to ask for extra help and try their best, that is all that is important.	Again, I would like to teach mathematics using as much technology as possible. Today, we live in a technological age, and students need to be proficient in the use of all kinds of technology. I would like for student to think independently, but still be able to engage effectively in cooperative learning. Students should feel free to ask questions and not feel intimidated.
12. In your opinion, what does the phrase "learning with technology" mean?	In my opinion, I believe that students should first learn most mathematics by hand, but have access to technology to supplement their learning. On the other hand, I believe that students can learn same things better by exploring on the computer first, and then discussing in class later. Either way, technology can be greatly beneficial to students, but should not substitute classroom learning.	I believe that this phrase is trying to emphasize that technology is there to help supplement your learning. Therefore, the main concepts may be first discussed in lecture or other form of classroom instructions and then possibly, the students can use technology for reinforcement of the main ideas.
13. In your opinion, what does the phrase "teaching with technology" mean?	"Teaching with technology" means to use technology to supplement classroom learning. Technology can help students to achieve a better understanding of many mathematical ideas if they are able to experiment on their own.	This phrase means that again, you may use technology to help aid in teaching of the basic math concepts, but it should be used as a supplement and not as a full substitute compared to lecture and classroom notes. Technology may help to reinforce ideas taught in the classroom.
14. In your opinion, what does the phrase "understanding with technology" mean?	Understanding with technology means to have a grasp of how technology I used. I believe the phrase means to use technology as a tool for exploration, but not as a	This phrase means that students can use technology as a method of seeing multiple solutions to the same problem, and therefore, reinforce their knowledge for a

	means of proving conjectures.	greater understanding in the long run. The way that one student learns will differ from another, and so the more ways you can show a concept to your students, the more students you will have that will actually understand a particular concept.

Two diagnostic tests and concept maps

Participants were given the same diagnostic tests (Appendix E) in the beginning and at the end of this study. This test was given simply in order to find out not only how much they can express the specific concepts (translation, rotation, reflection, & glide reflection) relating to transformations in three different representations, but also to find out the changes of their expressions about the same concepts in terms of representations. When they were given the first diagnostic test, they were able to express something relating to pictorial and verbal representation, but they were not able to express these concepts using mathematical symbols. So, I asked them to write what they could using their own words rather than using mathematical symbols. For the second diagnostic test, they were able to use mathematical symbols. Although dilation was one of transformations on the diagnostic test, I dropped it because the concept of dilation was discussed only very briefly in the class and was not used subsequently.

At the end of first and second diagnostic test, I asked the participants to make a concept map of transformation. Because they had made a concept map in the previous semester, I did not explain about how to make a concept map. Instead, I provided participants with concepts relating to transformations (Appendices F & G). The participants were also allowed to add any additional concepts as they made their concept maps.

Abbey's representations and a concept map

Written representation

Abbey mainly leaned on and was guided by the pictures she drew in order to express each transformation. She completed pictorial representations first and looked at the pictures in order to describe it. Her expression of each transformation was simply the description of what is happening when doing the transformation based on specific pictures. She could not express any transformation using mathematical symbols in the first diagnostic test. She presented her knowledge and understanding about transformations using pictures and her own words. All the pictures she drew were exactly representing each transformation: translation, rotation, reflection, and glide reflection. Her works showed that she possessed a bit of understanding of transformations, but did not have the mathematical symbols in describing each transformation. Her descriptions were generally limited to the specific situation she presented in her pictures. Abbey had not encountered transformations these in her mathematics studies. Her experiences with transformations consisted of just two class periods in the previous semester from EMAT 4680. But, she held a pretty good understanding of transformations except glide reflection.

In the second diagnostic test, Abbey's expression about translation was very generalized and firmly represented the exact concept of translation (Table 4.4). She could exactly tell the relation between transformation and translation or rotation: "Translation is one type of transformation where the original figure is moved ..." and "Rotation where the original figure is rotated ...." Now she had appropriate mathematical symbols such as 'T' for transformation, (a , b) for translation vector, $\cos\theta$ or $\sin\theta$ for rotation, etc. in expressing translation and rotation.

Table 4.4 Abbey's first & second diagnostic test - Written representation

Transformation\Test	First diagnostic test	Second diagnostic test
Translation	To translate ΔABC by using the marked vector DE (from D to E is the direction), the resulting figure is ΔA'B'C' (Refer to Figure 4.1).	Translation is one type of transformation where the original figure is moved in the same direction & same length as the translation vector that is given. Translation vector is (a, b) T (x,y) = (x+a , y+b).
Rotation	In the first example I had triangle ABC and rotated it 270^o around the fixed point I. The result of the rotation is ΔA'B'C' (Refer to Figure 4.3). In the second example I used a fixed point D which is on the original triangle DEF. Then I rotated the triangle 90^o about point D and got triangle D'E'F' (Refer to Figure 4.3).	Transformation where the original figure is rotated by a given angle. Given angle θ, T (x,y) = (cosθ, sinθ).
Reflection	The line segment *l* is marked to be the reflection line. So, A'B'C' is the result of reflecting ΔABC across line *l* (Refer to Figure 4.5).	She knew what it is, but could not recall, especially using mathematical symbols.
Glide Reflection	She had no clue about Glide reflection.	She knew what it is, but could not recall, especially using mathematical symbols.

<u>Pictorial representation</u>

The following two pictorial representations (Figures 4.1 & 4.2) of Abbey's are slightly different. The pictorial representation of the second diagnostic test is more exact and clear. For example, in the first one (Figure 4.1), she just presented 'the translation vector,' which is the defining data of translation, with the simple segment. There is length, but no direction. We can only tell the direction of the translation vector by looking at labels of the original image and the image translated. On the other hand, in the second (Figure 4.2), she clearly presented the translation vector using an arrow. Further, she added the relation between the given translation vector and segments connecting corresponding points in terms of length. If she added that the segments connecting corresponding points are parallel to the given translation vector: $\vec{V} // \overrightarrow{AA'}$, $\vec{V} // \overrightarrow{BB'}$, and $\vec{V} // \overrightarrow{CC'}$ (Figure 4.2), it would be perfect.

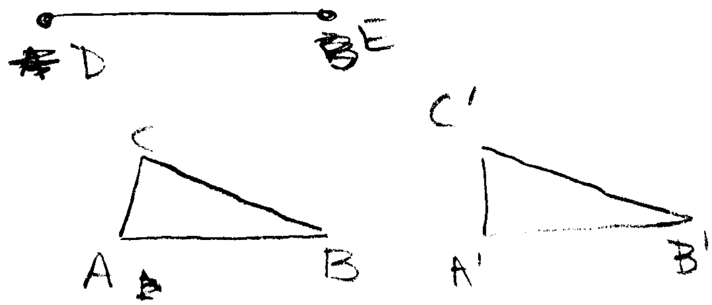


Figure 4.1 Abbey's pictorial representation of translation in the first diagnostic test

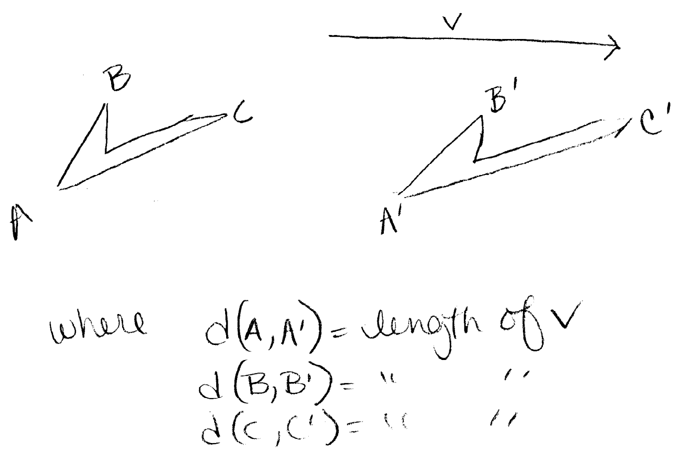


Figure 4.2 Abbey's pictorial representation of translation in the second diagnostic test

In discussing rotation, in the first diagnostic test Abbey presented two different situations depending on the location of centers for rotation, but she did not present any angle for rotation (Figure 4.3). The defining data in rotation consist of two elements: a center and an angle. On the other hand, in the second diagnostic test she presented exactly two defining data for rotation including the specific direction of the angle in rotation. This shows that she had very clear pictorial representation about rotation. If she presented the relations between the given angle and the angle made by the original figure, center, and rotated image, i.e., ∠EFG = ∠ACA' = ∠BCB' = ∠CCC', where the 'C' in the middle is the center for each angle (Figure 4.4), it would be better.

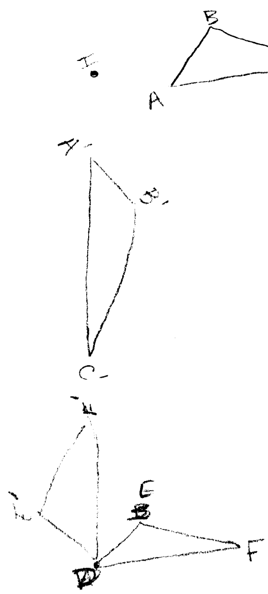


Figure 4.3 Abbey's pictorial representation of rotation in the first diagnostic test

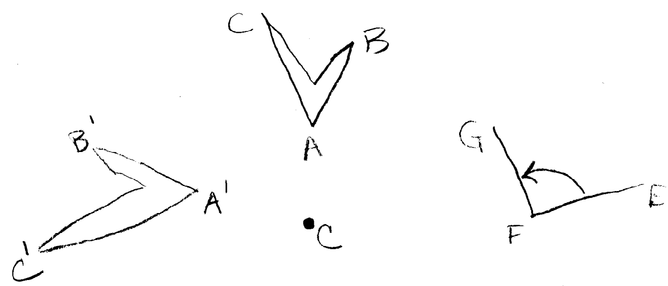


Figure 4.4 Abbey's pictorial representation of rotation in the second diagnostic test

The reflection evident in the two pictorial representations Abbey presented was very similar on the two diagnostic tests (Figures 4.5 & 4.6). The only difference was that she added the label of the line as 'mirror *m*' instead of just '*l*' in the second test. The figure in the first diagnostic test is a triangle, but that in the second is a pentagon, which is more efficient in working with orientation. Giving a name to the line for the mirror, which is the defining data for reflection, shows that she kept in mind the importance of defining data in a reflection. She did not mention the relation between the mirror and segments connecting corresponding points in both.

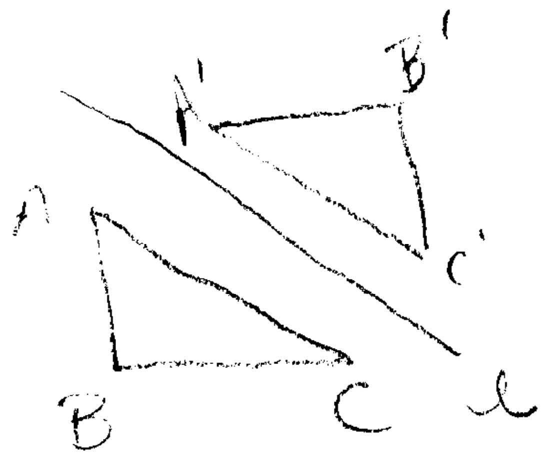


Figure 4.5 Abbey's pictorial representation of reflection in the first diagnostic test

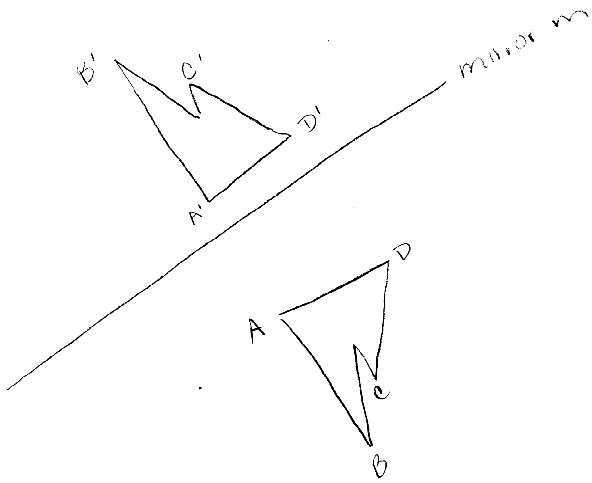


Figure 4.6 Abbey's pictorial representation of reflection in the second diagnostic test

Abbey did not know anything about glide reflection at the time when she was given the first diagnostic test. But, she expressed the pictorial representation very well in the second diagnostic test (Figure 4.7). Especially, she presented the intermediate transformation of glide reflection, i.e., reflection in this case using dotted line. Two defining data for glide reflection (translation vector for translation and mirror for reflection) were clearly presented. But, there was no critically important relation between the translation vector and the mirror, whose relation is parallel.

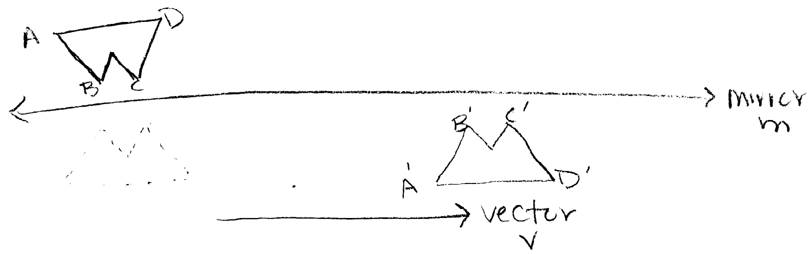


Figure 4.7 Abbey's pictorial representation of glide reflection in the second diagnostic test

Verbal representation

Abbey's verbal representations for the first diagnostic test (Table 4.5) seem to present the outline of each transformation rather than to give the essence of it. She seemed to express her verbal representation just like she was physically doing the transformation in her head. So, her verbal presentation was to describe what is happening while doing transformation. The verbal presentation was rather based on one specific case performed in her head. In the case of translation, she included the concept of defining data and the relation between the defining data and those segments connecting corresponding points. However, for rotation and reflection, although I was able to catch what she was trying to present, considering the essence of the concept of each transformation, her presentation was not enough for those who do not know transformation well to grasp the full concept of each transformation.

On the other hand, Abbey's verbal representations in the second diagnostic test (Table 4.5) were much more precise and informative than those in the first. In the second test, she seemed to verbally present each transformation based on written representation, mathematical symbols, of each transformation. She also technically presented the role of the defining data for each transformation well.

Table 4.5 Abbey's first & second diagnostic test - Verbal representation

Transformation\Test	First diagnostic test	Second diagnostic test
Translation	When you have a figure and a marked vector going from point A to point B, your original figure will be moved in the same direction and same length as vector AB.	Translation is a transformation using translation vector, and it's moved in the direction of the vector at the given length of the vector.
Rotation	You have to have a marked point to rotate the figure about. It can be on the figure and not on the figure. And then, you rotate that in many degrees according from the point marked.	Rotation is a transformation where the figure is rotated around a center point that can be on the figure or off the figure. It's rotated through a given angle. And orientation is preserved.
Reflection	The original figure is reflected about a line segment or a line and it's almost like a mirror image.	Reflection is a transformation which the original figure is reflected cross a given mirror. You create, if it's A , A'. A' is the reflection of A then the line segment AA' is perpendicular to the mirror.
Glide Reflection	She had no clue about glide reflection.	A glide reflection is the transformation in which you first reflect the figure over a given mirror and translate it by the given glide vector like by the same length and same direction. This [mirror and glide vector] is going to parallel.

Concept map

The following concept maps made by Abbey show a difference between the first and the second one (Figures 4.8, 4.9, & 4.10). The connections she made in her first concept map were very limited and basic or maybe separate. For example, each transformation - translation, rotation, reflection, glide reflection, and dilation - was connected to the title 'transformations.' The connection between dilation and rotation using a fixed point was not clear and further shows that she did not quite understand the concept of fixed point. In addition, the two separate connections in the bottom were isolated from the main body of her concept map.

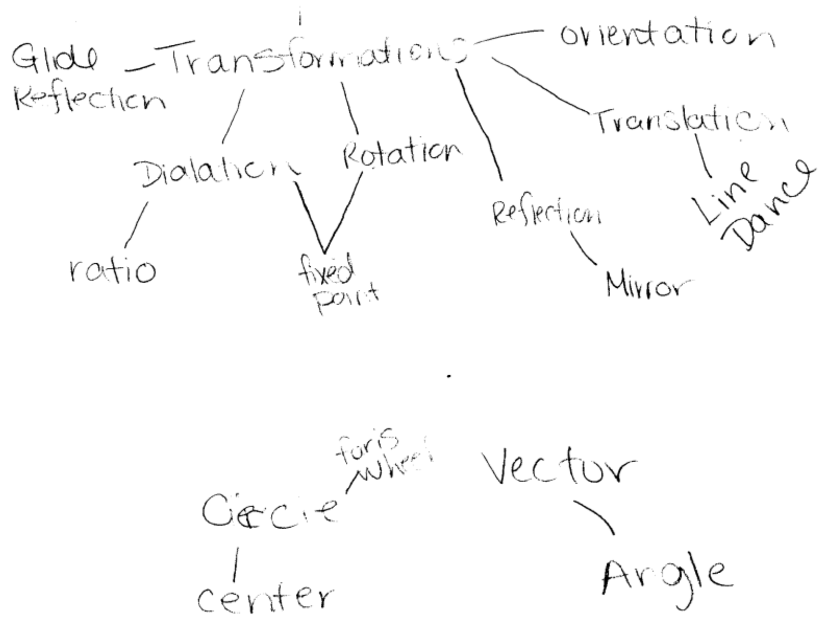


Figure 4.8 Abbey's first concept map - transformations

The Abbey's second concept map of transformation (Figure 4.9) has no isolated part. It is a single unit from the main part. We can recognize the connection between the title 'transformation' and each transformation - translation, rotation, reflection, glide reflection, and dilation. In addition, it shows that glide reflection is the combination of translation and reflection along with the connection among the defining data of translation, reflection, and glide reflection, which shows that she recognized the connection among translation, reflection, and glide reflection. She also made a separate concept map under the title 'symmetry' (Figure 4.10), which is an extended concept based on transformations. Abbey was able to organize concepts of transformations clearly so that the reader can understand transformations (Figure 4.9). The organization of this concept map seems to be affected by the GSP menu. The concept map of symmetry was separately made and the organization was also simple and clear. Any particular connection between transformations and symmetry was not made.

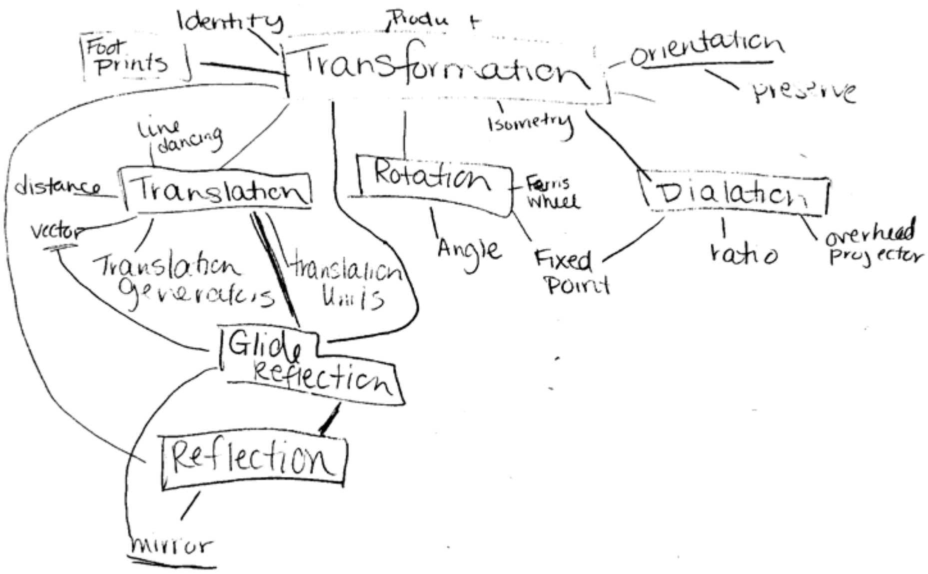


Figure 4.9 Abbey's second concept map - Transformations

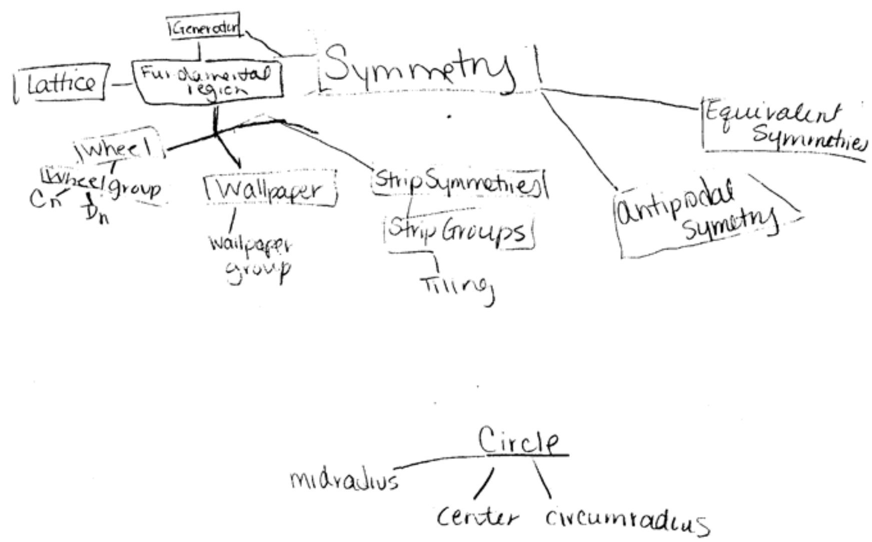


Figure 4.10 Abbey's second concept map - Symmetry

Emily's representations and a concept map in a diagnostic test

Written representation

Emily also had experiences with transformation for only two class periods from the previous semester. In addition, it had been about three months since those two class periods before she took the first diagnostic test. When she answered the first diagnostic test, she did not depend on the specific pictures in order to express her understanding of each transformation (Table 4.6). Instead, she described each transformation based on her own memory or previous understanding of each transformation from her head. Although her description included 'moved' meaning 'translated' and 'flipping' meaning 'reflecting', she also used the words such as 'reflects' and 'rotates,' in her expression of reflection and rotation respectively. She possessed a misconception about 'orientation'. However, Emily had no knowledge about how to express transformations using mathematical symbols at all in the first diagnostic test.

We can tell easily that Emily's responses to the second diagnostic test about translation, rotation, and reflection are different from those of the first diagnostic test. Her expressions in the second were simpler but much clearer than that of the first. There was no redundancy in her expression and she included the main essence in order to present each transformation in the second. She did not recall how to express a glide reflection using mathematical symbols.

Table 4.6 Emily's first & second diagnostic test - Written representation

Transformation\Test	First diagnostic test	Second diagnostic test
Translation	A translation is a construction where an original figure is translated or moved, and its original size, shape, and orientation is preserved.	Vector = (a , b) T (x , y) = (x+a , y+b)
Rotation	In a rotation, an object is rotated, or moved about a fixed point. Here its size and shape are preserved, but its orientation is changed. You can either rotate through a fixed point or by a marked angle (as in the 2nd drawing).	Matrix of rotation : $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ Point on figure (x , y) & center (0, 0) (xcosθ - ysinθ, xsinθ + ycosθ) = (x' , y')
Reflection	A reflection is performed by taking the original figure and flipping over a mirror that you designate. A mirror can be any straight object in your sketch. The original figure is then reflected across that mirror. Its size and shape are preserved, but not its orientation.	An isometry such that ∃ point p such that xp = x'p and the line L through p is perpendicular to xx'.
Glide Reflection	She had no clue about Glide reflection.	She knew what it is, but could not recall.

Pictorial representation

Emily did not use any label for figures and did not present any defining data for the translation in the first diagnostic test (Figure 4.11). So, her pictorial representation of translation was not informative enough to demonstrate an understanding of it. It just seemed to tell us that there are two figures of the same size and shape on the paper and then moving from one to another is a translation. On the other hand, although she did not use many labels in the second diagnostic test (Figure 4.12), she did present the defining data for translation. She did not mention anything about the relation between the translation vector and segments connecting corresponding points.

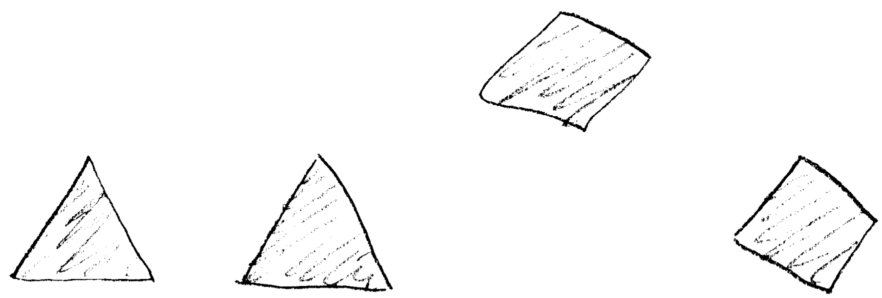


Figure 4.11 Emily's pictorial representation of translation in the first diagnostic test

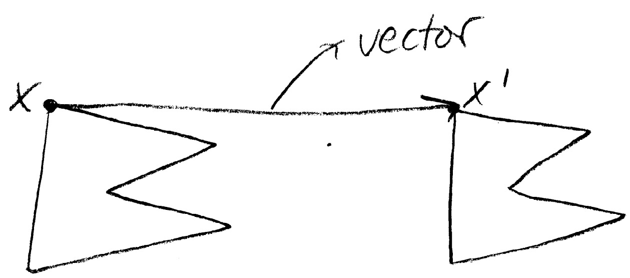


Figure 4.12 Emily's pictorial representation of translation in the second diagnostic test

In both pictorial representations of rotation in the first and second diagnostic test (Figures 4.13 & 4.14), Emily presented two defining data for rotation. In her first pictorial representation about rotation, we can see figures, center point (A), and even angle (∠CAB). There is no direction of angle in Figure 4.13, but in the second diagnostic test she gave the appropriate labels, X and X', for representing the direction of an angle for rotation. She also made the concept of rotation clear by connecting the center and two corresponding points although she did not specifically state that two angles must be the same, ∠XOX' = ∠CAB.

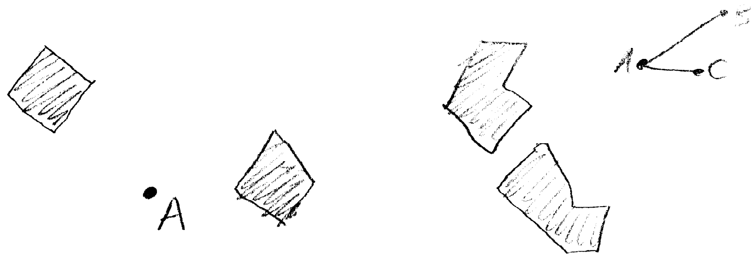


Figure 4.13 Emily's pictorial representation of rotation in the first diagnostic test

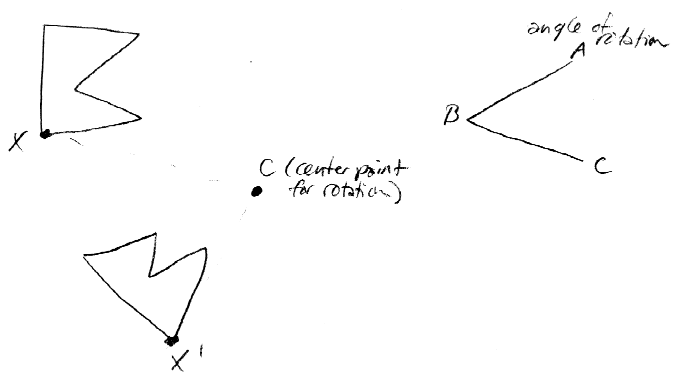


Figure 4.14 Emily's pictorial representation of rotation in the second diagnostic test

The pictorial representation of a reflection in the first diagnostic test is very simple, but at least it shows something about reflection (Figure 4.15). There were no labels for the corresponding points. On the second diagnostic test, Emily gave the appropriate labels on the figures (Figure 4.16). She did not present the relation between the mirror and the segments connecting corresponding points, but she knew something about the relation in a reflection although she did not specifically state it. Emily did not know anything about a glide reflection in the first diagnostic test, but she presented the concept of a glide reflection using appropriate labels, defining data, and figures including the intermediate transformation, translation, in the second diagnostic test (Figure 4.17).

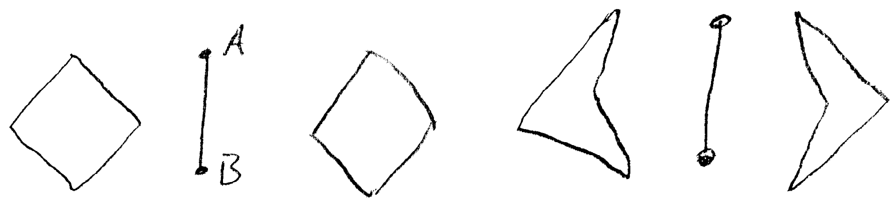


Figure 4.15 Emily's pictorial representation of reflection in the first diagnostic test

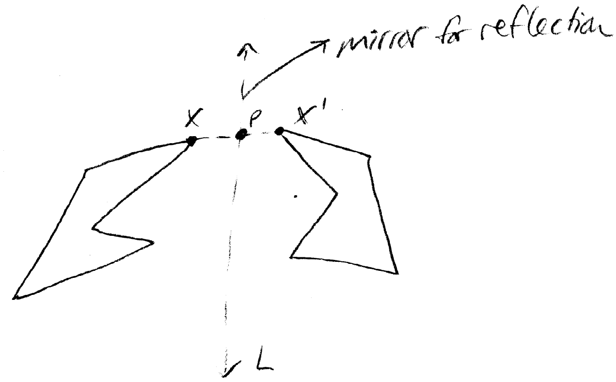


Figure 4.16 Emily's pictorial representation of reflection in the second diagnostic test

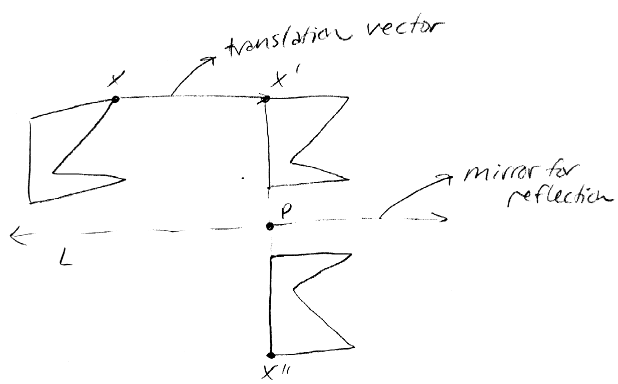


Figure 4.17 Emily's pictorial representation of glide reflection in the second diagnostic test

Verbal representation

When Emily verbally presented each transformation in her first diagnostic test (Table 4.7), she tried to explain the concept of each transformation depending on GSP. For example, as for a translation vector, she mentioned either a rectangular or a polar vector for the translation vector, which we can choose as an option in GSP, rather than simply a segment from one end to the other end point. She also presented the angle in rotation using GSP, which we can type in the number to designate the angle for rotation. Emily knew the concept of each transformation except glide reflection. But, her expressions were also not clear and informative. Especially, the omissions of expressions about the relation between the defining data and the segments connecting corresponding points made her description ambiguous.

In the second diagnostic test (Table 4.7), Emily was independent of GSP when she presented a translation. She was clear about the concept of translation vector and the result of the original image after translation. Although she did not specifically state the relation between translation vector and those segments connecting corresponding points, she had that concept in her mind: "the length of your vector is moved from that same amount." As for rotation, she was also independent from GSP in her expression. But, after she explained the concept of rotation without GSP, she introduced GSP for more powerful exploration of rotation, which is 'marking an angle' on GSP to have students vary the size of an angle by just dragging a point of one side of the marked angle. This shows that she knew the characteristics of GSP as well as understood the concept of rotation. As for a glide reflection, she presented it very well step by step without any hesitation and confusion. Her understanding about the relation between defining data, translation vector and mirror, was exact and so was the movement of the original figure depending on the two defining data. Emily also seemed to lean on mathematical symbols used during the course for her verbal representation.

Table 4.7 Emily's first & second diagnostic test - Verbal representation

Transformation\Test	First diagnostic test	Second diagnostic test
Translation	If you are a given original figure and you can translate it usually by or at least in GSP by rectangular or polar vector anything and normally on a coordinate axes by a rectangular. In coordinates, you can move translate the vector which is keeping its	To do a translation, you're given a figure and you're given a vector and ... that figure is trans.. [translated] or moved by that vector,… by the whatever direction that vector is, move to that direction and then the length of your vector is moved from

	original size and shape and orientation and you can translate it like units in the horizontal or vertical but it preserves all its size and shape and orientation.	that same amount and then so it preserves orientation.
Rotation	Rotation is reformed if you are a given original figure and if you're given a fixed point and then you can rotate it by either a fixed angle, which is an angle that you designate, and then will rotate that original figure by that angle. So its orientation is changed but size and shape remain. Or you can do it, rotate it, through a fixed point and you can designate angle, you know, in GSP.	Rotation preserves orientation and it's an isometry that you're given a figure and you're given a center point and you're given an angle to either you can draw a fixed.... you can draw an angle on your picture or especially in GSP, you can mark it, a certain angle, or you can (inaudible) [give] angle you'd like for your arbitrary [choice]... rotated by an object is rotated about that center point of that angle, positive directions and counter clock wise and orientation is preserved.
Reflection	Reflection is you can designate a mirror like a straight object to use and you're gonna, the given original object, you wannna reflect it or flip it over and so the orientation will be opposite. Orientation won't be the same but the size and shape are preserved.	A reflection is an isometry that is .... orientation is reversing. And for reflection, you're given a figure and you're given a mirror for reflection which can be a line or a segment and you're reflecting so you are flipping the object over the mirror wherever the mirror is placed.
Glide Reflection	Uh, Similar to reflection. I don't know. [She did not have basic knowledge about glide reflection at the moment when the first test was given.]	Glide reflection is orientation reversing as an isometry and you're given a figure and you're given a mirror and you're a vector that's parallel to that mirror. And the figure can either be translated first and then reflected or reflected and then translated. And you translate by the length of that vector in that same direction that the vector is pointing and then the mirror is parallel to the vector so the figure is then reflected.

Concept map

Two concept maps made by Emily also show a big difference between the first and the second one (Figure 4.18 and 4.19). Comparing those two concept maps, the first thing that we can recognize is the number of lines connecting concepts. Obviously, the second is much more complicated and multi-connected. She categorized a glide reflection as one type of reflection ignoring the role of translation. The fixed point was used differently from what it is supposed to be used in transformation. Here is Emily's concept map made when she was given the first diagnostic test.

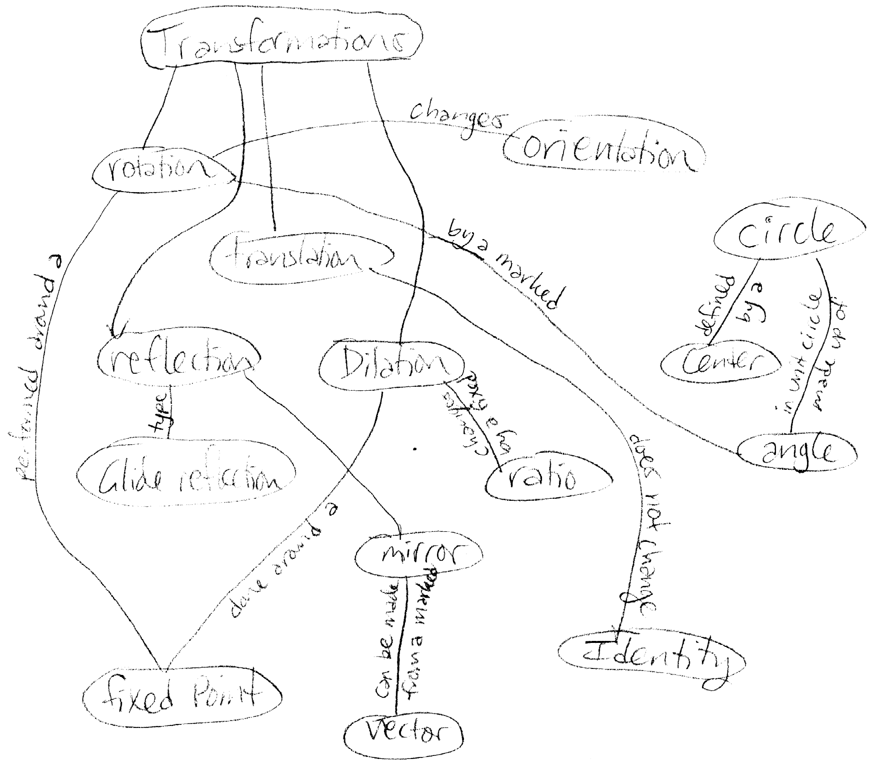


Figure 4.18 Emily's first concept map

The first thing that stands out in Emily's second concept map (Figure 4.9) is the connection between two main parts, isometry and symmetry. In addition, she added a connection between isometry and platonic solids. The platonic solids was the main topic after they finished transformations. The second concept map was made after she learned about platonic solids. She made a connection between the title, 'defining data' and all defining data of each transformation, which is the most important information for doing any transformation. Another thing we can recognize from this concept map is the connection between each transformation and an example that represents a specific transformation. Emily's second concept map shows that her understanding and knowledge about transformations extends to connected topics such as symmetry and platonic solids.

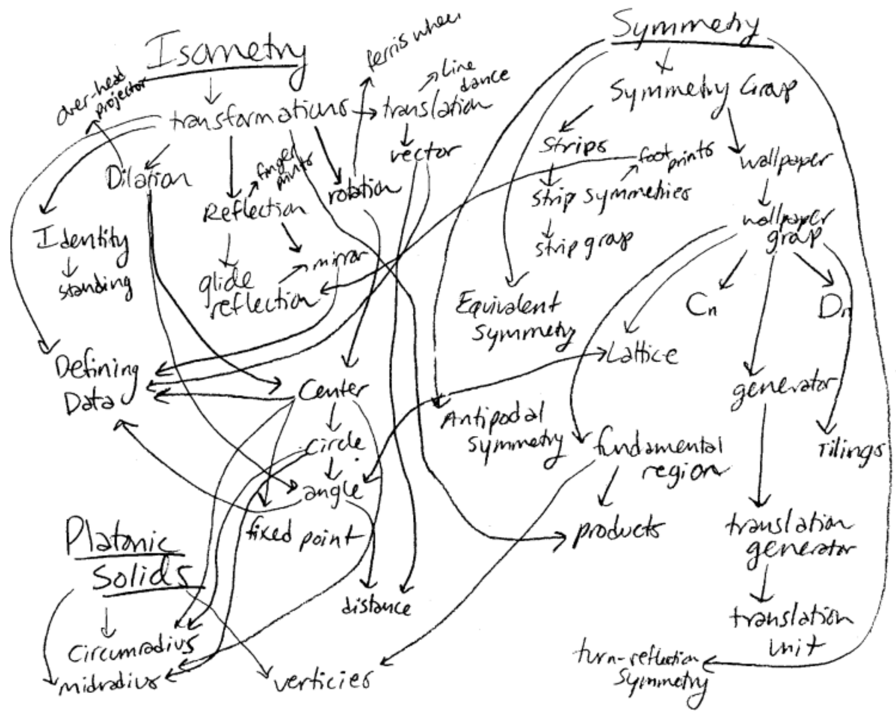


Figure 4.19 Emily's second concept map - Transformation & Symmetry

<u>Summary</u>

Throughout the three types of representations, the second diagnostic test shows that the concepts of each transformation to both Abbey and Emily became more precise and clear than the first one. Although they had some concepts of each transformation except a glide reflection in the beginning of this study with two class periods of experiences about transformation, their three representations in the first diagnostic test were not clear and informative enough to deliver the full concept of each transformation. They were simply able to show some aspects of each transformation in the first test. It seemed they had a general idea of transformation. On the other hand, they were able to present three representations a lot better in the second diagnostic test. The words they chose were more appropriate and refined, and they seemed to have more correlation among three representations in presenting a specific transformation. Sometimes, they did not specifically state the relation between the defining data and those segments connecting corresponding points. But, I could read that not only they had that relation in their mind, but also they knew the importance of that relation in doing transformations. Especially, both of them were able to present the concept of glide reflection, which they had no idea about in the beginning of the study, in three ways without any hesitation and difficulty. In addition, their concept maps show their improvement in understanding of transformations. However, they still had some difficulty in presenting each transformation using mathematical symbols, i.e., written representation.

Students' representations: Written, Pictorial, and Verbal

Abbey's written, pictorial, and verbal representation in a context

Written Representation

As mentioned earlier, written representation specifically means presenting concepts using mathematical symbols in this study. Although Abbey had some knowledge and understanding of transformation prior to this study, she could not present concepts of transformation using mathematical symbols in the first diagnostic test.

R: I remember that you had a hard time in presenting concepts of transformations using mathematical symbols. Why do you think it is difficult for you to do so?

A: I don't know why ..... It was just .... [time passes about 5 seconds]

R: You were very professional in presenting those with figures using your own words.

A: Right. I guess because I just haven't had as much practice with writing things out ... like as explanation. Maybe, it's a little bit harder.

Second of all, she could not make any connection between her understanding of transformations and mathematical symbols that she possessed at the time of the first diagnostic test. The first diagnostic test was given on January 9, 2001 and the first assignment (Appendix J), 'Writing a laboratory report on the Transform menu and the arrow tool in The Geometer's Sketchpad', was due on January 11, 2001. So, the first diagnostic test was given while she was preparing for the laboratory report. She did a wonderful job with the laboratory report throughout each transformation. For example, she knew many things about rotation with GSP. She felt comfortable about the location of a center, that is, a center for rotation can be either on the figure or off the figure. In addition, she knew that angles for rotation can be assigned in different ways: one is to assign a fixed angle by typing a number into and the other is to use a marked angle directly on GSP so that she could dynamically move that angle around as she wanted. But, she left the spaces for written representation for each transformation blank. Here is Abbey's laboratory report about rotation.

For both figures 1 and 2 below, I rotated using the fixed angle 45 degrees. In figure 1, I used point I as my marked center and the blue figure [thick] is the result of the rotation. In figure 2, I used point D on the figure as the marked center. The pink figure [thick] is the result of the 45 degree rotation. In teaching this, I would stress to my students the importance of where you choose your center, because even though both of these figures were rotated 45 degrees, students might think that figure 1 was rotated more than figure 2.

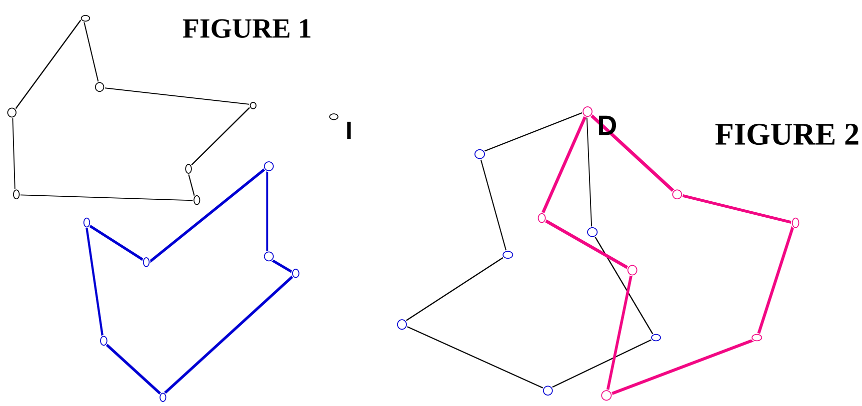


Figure 4.20 Abbey's first laboratory report of rotation - A fixed angle

Now, I rotated by a marked angle ABC. You should let the students know that it does matter which way you select the points of the angle when you are marking it. If you select A then B then C, the rotation will go from A to C, and that is what I chose in this case. Then I had to mark the point F as the center that I was rotating it around.

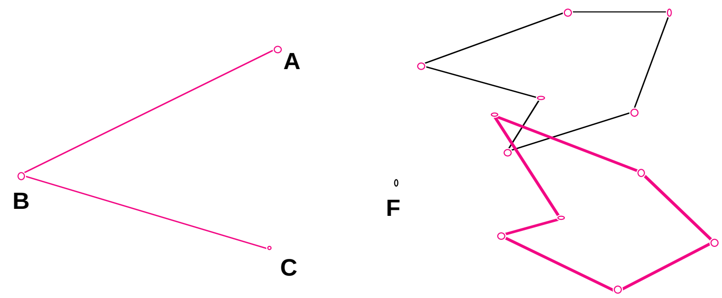


Figure 4.21 Abbey's first lab report of rotation - A marked angle

Abbey's verbal description about rotation was quite adequate. She also reported the other transformations in a similar manner. However, she could not express any transformation using mathematical symbols until the instructor showed how to do it in the class and a new assignment [Homework 2] was assigned 'to give precise mathematical definitions using objects (points, lines, circles, distance, and angle measure) and functions of Euclidean geometry.'

Abbey could fully explain what the rotation is using her own words along with GSP. After the instructor's lecture about mathematical notation and the second assignment asking a precise definition, she could quickly develop the idea of mathematical symbols in presenting transformations. We can see the difference about Abbey's description of rotation between the laboratory report of the first assignment and that of the second assignment.

> ROTATION:
> The defining data for a rotation is a point C which we call the center and an angle. In GSP, there are two ways to specify an angle. You can create an angle and mark that, or you can specify an angle like 45 degrees.

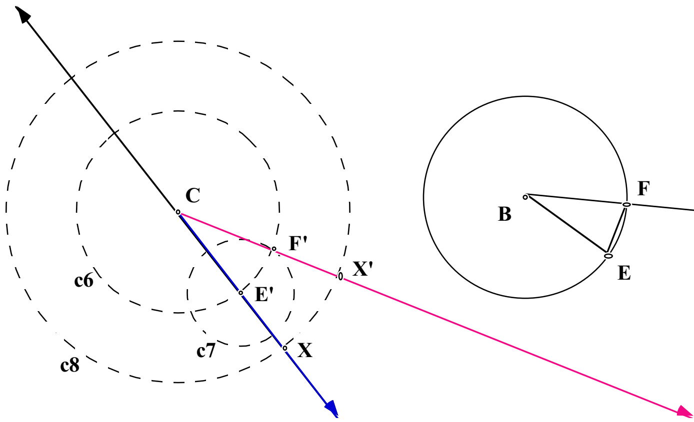


Figure 4.22 Abbey's second report of rotation - Geometric definition

To construct a rotation by only using the GSP menu, I was given the center point C and point X to rotate and we were also given the angle EBF. I used my knowledge from last semester to copy an angle. I constructed a circle c6 using C as my center and $\overline{BE}$ as my radius. The intersection of this circle and the line $\overleftrightarrow{CX}$ is point E'. Then I constructed circle c7 with the center E' and radius $\overline{EF}$. Then, I found point F' which is the intersection of c6 and c7. Then I constructed the line $\overleftrightarrow{CF'}$. Then to find the point X' on the line $\overleftrightarrow{CF'}$, I constructed the circle c8 using the center C and radius $\overline{CX}$. I'm not sure exactly how to tell about the direction of the angle, besides saying I measured the angle EBF, so it is in the counter clockwise direction.

<u>Definition</u> : The reflected point X' is the point such that the measure of angle XCX' = the measure of angle EBA and segment $\overline{CX} = \overline{CX'}$.

This second report of rotation helped Abbey approach the concept of rotation more mathematically using geometric definitions. This experience also helped Abbey to present her understanding of each transformation using mathematical notation in a geometric sense. After this, the instructor developed the formula for each transformation. For example, the formula of the rotation [T] with a center the origin and an angle θ is $T(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. But, she could not recall either the geometric definition or this formula for the second diagnostic test. The followings are the geometric definitions of translation and reflection that Abbey presented.

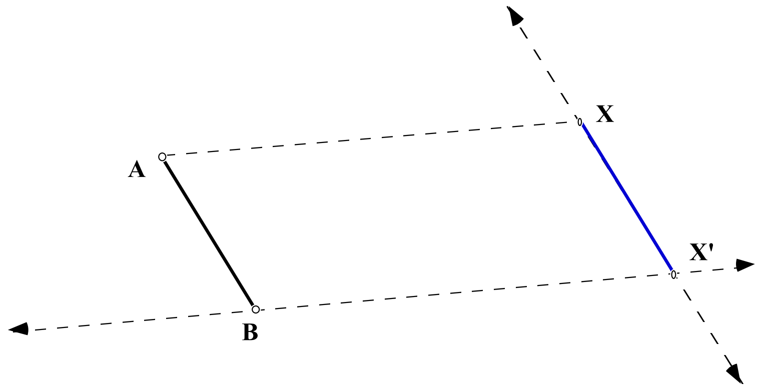


Figure 4.23 Abbey's second report of translation - Geometric definition

Definition: Given a ordered pair of points A and B and point X, the unique point X' has the property that ABX'X is a parallelogram.

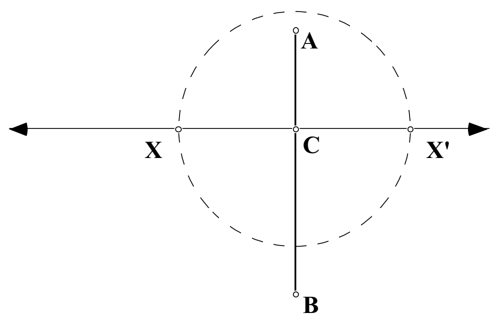


Figure 4.24 Abbey's second report of reflection - Geometric definition

Definition: X' is the point of intersection of the circle C, which is a circle with center C and radius $\overline{XC}$, and the perpendicular line through X which is perpendicular to $\overline{AB}$.

Pictorial Representation

Abbey's responses to pictorial representation of the first diagnostic test (Figures 4.1, 4.3, & 4.5) were rather simple and ambiguous with missing core elements such as the defining data and relations between the original image and the transformed image although she presented outline of each transformation. After lectures from the instructor on the rotation and completing homework 1, 2, and 3, Abbey began to better form the idea of each transformation. For the given mathematical problems, she always drew the situations based on her understanding of the problem and that of each transformation either on paper or on GSP. I did not ask Abbey to present her pictorial representation of each transformation after the first diagnostic test until the second diagnostic test. However, the following figure shows that her pictorial representation is much more exact than that of the first diagnostic test. If she used her first pictorial representation (Figure 4.5) without developing understanding of reflection, she could not complete the proof of two reflections is a rotation because the relation between defining data and figures is very important source for proof.

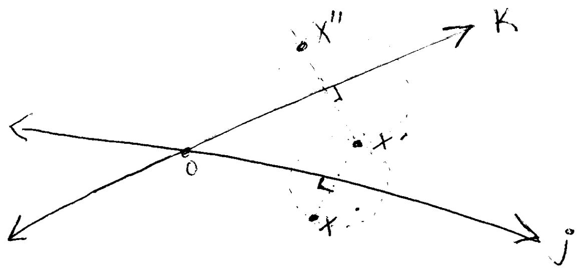


Figure 4.25 Abbey's first pictorial representation of Reflection * Reflection

She also possessed the concept of rotation. So, her goal was to show that two segments, $\overline{OX}$ and $\overline{OX''}$, are congruent and to make it sure that the source of angles for the newly established rotation is from the given defining data. Due to the above drawing based on the mathematical situation and exact pictorial representation of reflection, she could draw another figure below that gave a better glimpse of the accomplishment of proof (Figure 4.26) and finally was able to finish the proof successfully. Here is a proof that Abbey did.

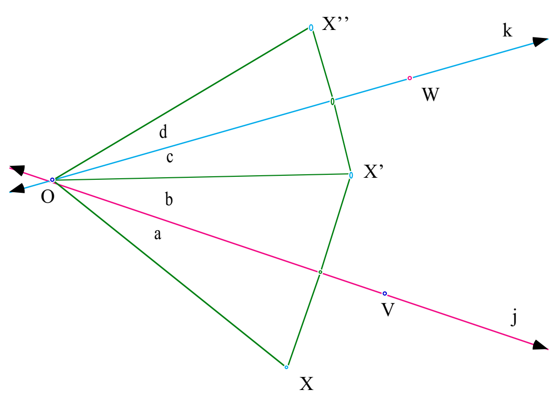


Figure 4.26 Abbey's updated pictorial representation of Reflection * Reflection

First I want to prove that $\overline{OX} = \overline{OX''}$, which means they have the same center of rotation. I know that $\overline{OX} = \overline{OX'}$ because O lies on j which is the perpendicular bisector of line $\overline{XX'}$. The perpendicular bisector theorem says that any point on the perpendicular bisector is equidistant from the endpoints of the segment. For the same reason, $\overline{OX'} = \overline{OX''}$. Then by transitivity, $\overline{OX} = \overline{OX''}$!

Now I want to prove that m(∠XOX") is twice m(∠VOW) [V is the point on j and W on k]. We know m(∠XOX")=a+b+c+d and we know m(∠VOW)=b+c, so we want to show that a+b+c+d=2(b+c).

We know that triangle XOX' and triangle X'OX" are both isosceles triangles (stated above by def. of isosceles triangles). We know that lines j and k are perpendicular bisectors, but in the case of isosceles triangles, we know that the perp. bisectors also bisect the opposite angle. Therefore, angle a=angle b, and angle c=angle d.

Therefore, a+b+c+d=b+b+c+c (substituting b for a and c for d)

2b+2c=2(b+c)!!!! So, a+b+c+d=2(b+c).

After some discussions in the classroom, she realized this is not the only case. That is there are two different cases: one is when two mirrors are not parallel and the other is when two mirrors are parallel. After she finished the first case, the second case was not a big problem for Abbey in proving it. Her drawing was carefully drawn and informative to finish the proof (Figure 4.27) and the method she used to prove that the result of the second case is a translation was similar to the first case. Abbey's exact pictorial representation of reflection made it possible to prove the above two cases.

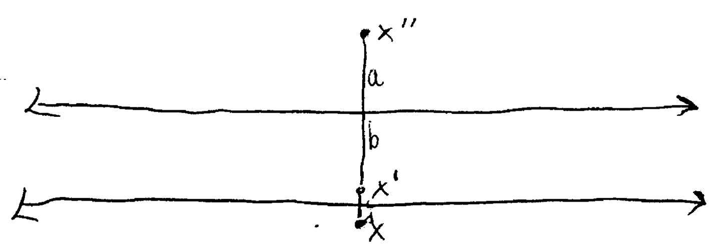


Figure 4.27 Abbey's pictorial representation of Reflection * Reflection - when parallel

The following example is interesting because her pictorial representation was exactly right, but her answer to the question of Homework 5.2 (Appendix J), which is to make a multiplication table for the symmetry group of D5, was not correct (Figure 4.28). She knew that there are five rotations and five reflections in D5 by definition of D5 from the instructor's lecture. Thus, she came up with five angles (72°, 144°, 216°, 288°, and 360°) for rotation and five mirrors which divide a circle into ten equal parts. After she constructed one of shaded quadrilaterals, she reflected that figure through the mirror 'a'. Then she rotated those two figures by four different angles, 72°, 144°, 216°, and 288°. Because 360° of rotation is the same as the identity, it was not necessary to do 360° of rotation. As we can see from the figure below, there are five rotations and five reflections (Figure 4.28).

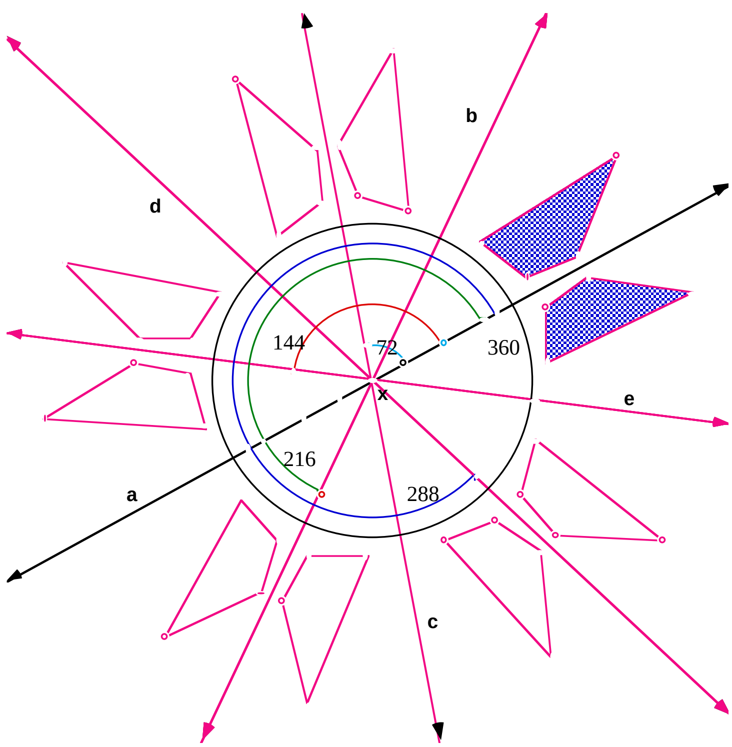


Figure 4.28 Abbey's pictorial representation of D5

However, there is something wrong with the table she made based on her pictorial representation for the given mathematical situation (Table 4.8). The first thing that stands out is that there is no reflection represented in her answers to any set of multiplication. She could not even tell if the table was correct or not. I knew that she possessed the concept of orientation and she also knew that translation and rotation preserve the orientation, but reflection and glide reflection do not. Further, she knew the results of multiplication of two transformations in terms of preserving or reversing orientation (Table 4.9).

[3]Table 4.8 Abbey's multiplication table of the symmetry group, D5

*	R-72	R-144	R-216	R-288	R-360	F-a	F-b	F-c	F-d	F-e
R-72	R-144	R-216	R-288	R-360	R-72	R-72	R-144	R-216	R-288	R-360
R-144	R-216	R-288	R-360	R-72	R-144	R-144	R-216	R-288	R-360	R-72
R-216	R-288	R-360	R-72	R-144	R-216	R-216	R-288	R-360	R-72	R-144
R-288	R-360	R-72	R-144	R-216	R-288	R-288	R-360	R-72	R-144	R-216
R-360	R-72	R-144	R-216	R-288	R-360	R-360	R-72	R-144	R-216	R-288
F-a	R-288	R-216	R-144	R-72	R-360	R-360	R-288	R-216	R-144	R-72
F-b	R-360	R-288	R-216	R-144	R-72	R-72	R-360	R-288	R-216	R-144
F-c	R-72	R-360	R-288	R-216	R-144	R-144	R-72	R-360	R-288	R-216
F-d	R-144	R-72	R-360	R-288	R-216	R-216	R-144	R-72	R-360	R-288
F-e	R-216	R-144	R-72	R-360	R-288	R-288	R-216	R-144	R-72	R-360

Abbey had all information that she needed in order to make a correct multiplication table of the symmetry group, D5, such as correct figures of D5 and knowledge about preserving or reversing orientation when two transformations are multiplied. As we can see from the figure above, Abbey presented the perfect figure for D5. Then, why did she make the wrong table?

[3] This table was slightly modified by the researcher based on what Abbey originally made. R stands for rotation and F stands for reflection. Abbey used 'Rotation' and 'Reflection' instead of R and F respectively in her original table. For example, R-144 means rotation by 144° and F-c means reflection through c. Further, the multiplication table was made by the following rule, i.e. (R-72)(F-b) means do F-b first and R-72 second , where R-72 is from the far left column and F-b from the very top row.

Table 4.9 Multiplication of transformations in terms of orientation

*	Translation	Rotation	Reflection	Glide Reflection
Translation	Preserved	Preserved	Reversed	Reversed
Rotation	Preserved	Preserved	Reversed	Reversed
Reflection	Reversed	Reversed	Preserved	Preserved
Glide Reflection	Reversed	Reversed	Preserved	Preserved

Before she was asked to make a multiplication table of D5, she learned each information as the class was going along and understood each information without connection to other information. There were some problems with her constructions however, although the final result of the figure itself tells us the right information of D5. This is the process that she took while constructing a figure representing D5. She constructed one of two shaded quadrilaterals, reflected it through the mirror 'a', and then rotated two shaded figures by 72° four times. She hid the interiors of the other four sets of quadrilaterals except the original so that she could tell which one is the original set. After all these were done, she rotated the mirror, 'a', by 72° four times to get all the mirrors that she needed in the figure representing D5. So, the figure that Abbey presented was eventually constructed using four rotations without using five distinct reflections. She just used a reflection once for the first set of quadrilaterals. During this process, the original figure for D5 was one set of two shaded quadrilaterals, not the one quadrilateral. From this, she just could see only five rotations with one set of two shaded quadrilaterals regardless of five mirrors in her construction. She knew that reflection is there. But she was not sure how that works in D5. Maybe, she could not recognize reflections because of the way she constructed it. Therefore, reflection did not appear in her multiplication table. The process that she took for construction of D5 using GSP did not allow her to see all the elements of the symmetry group, D5.

For example, her table says (R-144)(F-b) is R-216. This is obviously wrong because it contradicts the table of multiplication of transformations in terms of orientation (Table 4.9). But, her answer was correct according to her construction process because her original figure for D5 was a set of two shaded quadrilaterals instead of only one shaded quadrilateral. After she realized that the original figure is one shaded quadrilateral instead of a set of quadrilaterals, she recognized why she was confused in her table and could easily correct her table. The right answer for (R-144)(F-b) is F-d if the original figure for D5 is only one of two shaded quadrilaterals.

Verbal Representation

I did not specifically ask Abbey to express her understanding of concepts verbally in any interview except the first and the second diagnostic tests. Because each interview would be led over discussing contents covered during the class and regularly given assignments, instead I wanted to see how she verbally presented her understanding while each interview was going on. Her verbal representations were in most cases based on the figures that she constructed.

I was able to trace how her verbal representation of the concept of 'fixed points in a transformation' was changing as the course was going on. The following conversation, which happened right before the concept of fixed point was discussed in the class, is the first time she mentioned the concept of fixed point in a transformation.

A: O.K. Well, these are transformations. Then, orientation has to do with all the transformations. So, I just attached it to the original one. And then, fixed point. I knew how to do it with dilatation and rotation.

R: What is the fixed point in these two pictures?

A: This [The point A in Figure 4.29] and this [The point C in Figure 4.30].

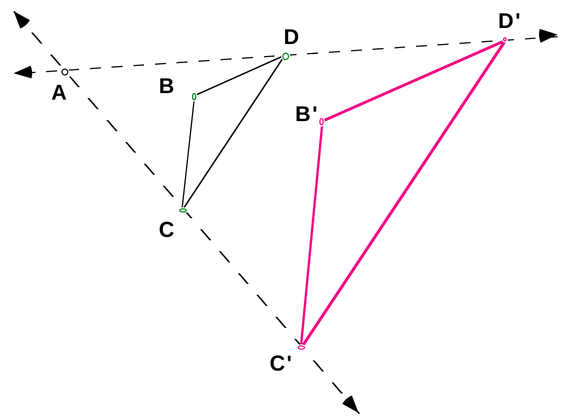


Figure 4.29 Abbey's dilation in connection with fixed point

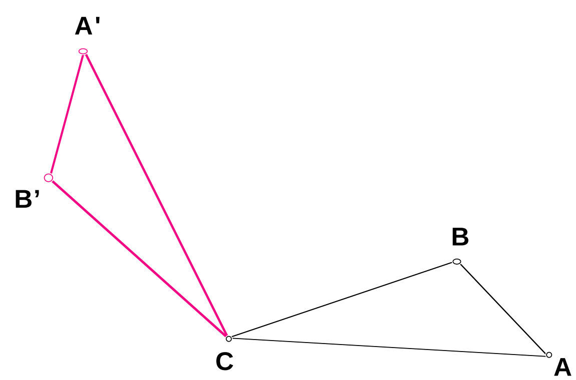


Figure 4.30 Abbey's rotation in connection with fixed point

From the conversation above and the two figures, her understanding about fixed point in transformation was not stable. Her understanding seemed to depend on the meaning of the word itself, fixed point. Two points she thought as fixed points were fixed points in the literal sense but, the point A is not a fixed point, whereas the point C is. If she drew the figure of rotation where the center is off the figure, she might say that that center point off the figure is the fixed point of rotation, which is not true if we look at the figure as the whole set. She was not clear yet with the concept of fixed point although she mentioned the concept. I decided not to talk about the concept of fixed point until it was discussed in the class. The following conversation was just before the concept of the fixed point was discussed in the class.

R: I have one question. What is a fixed point?

A: A fixed point?

R: Yea.

A: Fixed point is a point that doesn't move. It can't be changed.

R: What do you mean by that? Could you say some more?

A: Uhhh like .....

Her verbal representation was quite simple and correct, which is "fixed point is a point that doesn't move". However, it was questionable if she possessed the concept of fixed point because she could not explain it using her own words. Her verbal representation was just recalling of her memory and repeating the definition from the class. Abbey and I talked about the concept of a fixed point specifically with two different cases of rotations: one is when the center point is on the figure and the other is not. After some discussion, Abbey seemed to catch the idea of a fixed point. After she was clear with rotation, we moved to the case of reflection.

A: Oh, if you were to rotate, this would be a fixed point if you rotated it like that.

R: Yes. How about in a reflection? Is there any fixed point in reflection?

A: No.

R: O.K. But what if looks like that [Figure 4.31]?

A: This segment [$\overline{AB}$] would be the fixed point.

R: Segment. That's not the part of your figure. Only this point [A] and this point [B]. But, if we want to find fixed points in the plane, the line, *l*, will be the fixed point in this case.

A: Oh, O.K. O.K.

R: After reflection, these points stay at the same place.

A: Right. So, these two points [A and B] are fixed points or a line *l*.

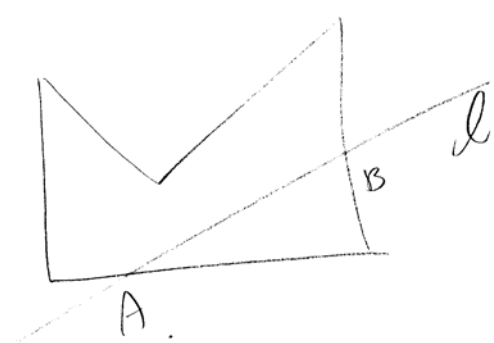


Figure 4.31 Fixed points in reflection when a mirror crosses the figure

After the second interview, there was a week of space until the third interview. During that week, the instructor put the definition of a fixed point on the web, which is 'Definition: A fixed point of a transformation F is a point X such that F(X) = X, and students were given a chance to use the concept of a fixed point in the class. There were some trials and errors for her to get to possess the stable concept of a fixed point. Here is the conversation that shows us that her verbal representation about a fixed point is much more stable than the first conversation. Further, the conversation was much more substantial and interactive, and touched many different cases for each transformation while it was proceeding.

R: I am pretty sure that you understand the concept of fixed points.

A: Umm heum [meaning Yes.].

R: A fixed point of transformation is not going to move at all.

A: Yup. A translation has no fixed point unless it is the identity.

R: That's pretty clear. How about rotation?

A: A rotation has only one fixed point unless it is the identity.

R: What is the fixed point in this case?

A: The center of rotation.

R: O.K. You can't say the center of rotation is always the fixed point.

A: Right.

R: But, you can say when center is on the figure, that's the fixed point. Is that clear?

A: Right.

R: Do you see the difference?

A: Yes, it's because …The figure is rotating but that one fixed point is staying .

R: What if I have a center here [off the figure] and there is a figure over here, …. Is this fixed point?

A: No.

R: Good.

R: What's next? How about reflection?

A: The set of fixed points of reflection is a line.

R: Is that clear to you?

A: Yea. It's a mirror, isn't it?

R: I have a mirror here [No intersection between the mirror and figure.]. Is there any fixed point in this figure?

A: No.

R: If I have it here [The mirror crosses the figure], ...

A: This line is on the top of fixed points. It's the fixed point.

R: That's perfect.

A: It's still going to be that line.

R: That's right.

A: O.K.

R: If you have a figure which is empty inside, then your fixed points will be two.

A: Right.

Emily's written, pictorial, and verbal representation in a context

Written Representation

Emily also, just like Abbey as we saw in the previous section, had a hard time in presenting her understanding or knowledge about transformations using mathematical symbols. Rather, she felt more comfortable in presenting them without using mathematical symbols, actually she was good at presenting them using pictures and her own words in the first diagnostic test although those presentations in pictures and words were not perfect. Maybe, she was not used to describing mathematical situations using symbols. The connection that she constructed between mathematical situations and mathematical symbols was limited somehow or she was not asked to present it with symbols often in the past. In distributing her time in doing mathematics at least in this technology-based environment, creating files to fit to the given mathematical situation is the first priority rather than trying to approach it using mathematical symbols.

R: Did you ever try to figure it out using symbols or proof things like that?

E: I did not because it was so time consuming just to do the sketches and to finish that. I was always lucky I finished as much as I did cause I wrote up so much about them. So, if I had more time, I would have worked on the proofs.

There are two versions of Emily's written presentations of translation. One was made during the class, which was done under the direction of the instructor as an example for guidance of the assignment, and the other was what she turned in as part of assignment, which Emily herself created for the assignment.

> Definition: Given ordered pair of points, (A, B) and a point X, the point X' has the property that it is unique & ABX'X is a parallelogram!
> Special case: What if X is on $\overline{AB}$? (Straight line!) When X is on $\overleftrightarrow{AB}$, you have to draw a circle with a radius $\overline{AB}$ to find X'. (Which point do you choose? - X and X' are on the same side of the line - (Ask!!)

She could follow the instructor's lecture and almost understood the example explained during the class. Further, she knew what she understood and at the same time also recognized which one she needed to ask for clear understanding. The above simple note tells us something about translation, but the following refined version of translation prepared for being turned in for the assignment more clearly unfolds the concept of translation.

> TRANSLATION:
> The defining data for a translation is a vector, and in GSP, there are three ways to determine the vector, either by choosing two points, defining rectangular coordinates (cartesian coordinates, using horizontal and vertical distance), or by defining polar coordinates using the functions of angle and distance. In GSP you would utilize the "marking" command under the transform menu in order to choose the vector.

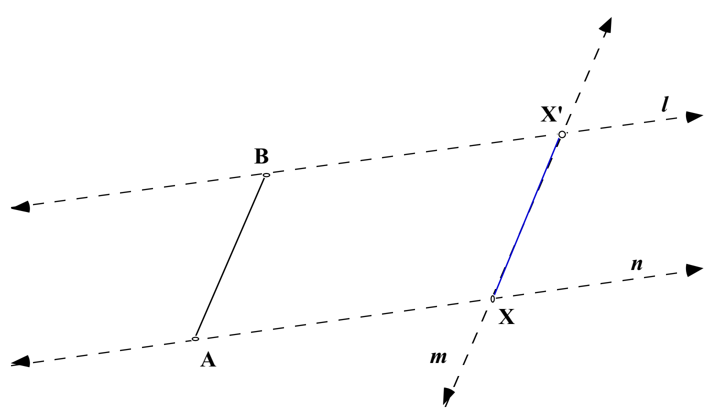


Figure 4.32 Emily's definition of translation - when X is not on $\overleftrightarrow{AB}$,

> Now I am going to attempt a precise mathematical definition for a translation. Given an ordered pair of points (A and B) and a point X, the point X' has the

property that it is unique and that ABX'X is a parallelogram. This definition is illustrated in the diagram above. So, to construct the parallelogram from points A, B, and X that are given, you construct a line '*n*' through the points A and X and then construct a line (*l*) through point B, parallel to 'n', and then another line through point X, parallel to the segment is on $\overline{AB}$.

Here, there is a special case when the point X is on the line AB, as displayed in the diagram below.

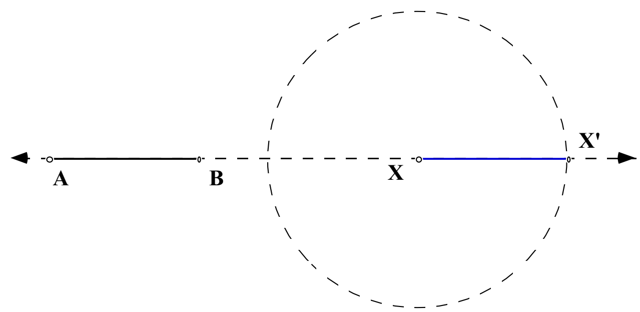


Figure 4.33 Emily's definition of translation - when X is on $\overleftrightarrow{AB}$

For this special case, there would only be a degenerate (collapsed) parallelogram formed. To find point X', you would need to construct a circle with point X as the center, with a radius equal to the distance between the points A and B. To discover which point created on line $\overleftrightarrow{AB}$ is the point X', you need to remember that X' needs to be placed in the direction such that the intersection of the ray $\overrightarrow{AB}$ and ray $\overrightarrow{XX'}$ is also a ray as well. Therefore, if the point X is on the line $\overleftrightarrow{AB}$, the parallelogram ABXX' does not make sense.

Along with figures constructed in GSP, Emily could more clearly present the concept of translation using symbols and written description for the two different cases after she spent hours in investigating the concept of translation and creating a GSP file representing it. Her presentation of translation at this stage was a geometric definition. This idea has been developed through the course and finally she was able to present the concept of translation using algebraic notations more simply and precisely in the second diagnostic test (Table 4.6). Through this challenging experience to present the concept of translation using 'a more abstract definition in terms of X and the defining data', Emily could see the whole picture of translation better. Translation, to Emily, was not just simply moving figures from one place to the different location without changing the size and shape any more.

Before this semester began, Emily did not have any opportunity to explore the relation among the original figure, the transformed figure, and vector (defining data). In addition, she did not hear anything about defining data, which is the critically important element in performing transformations and generalizing the concept of transformations. She might know there must be a vector to do a translation but she could not recognize how it was related to the original and the translated figures. As she began to recognize the importance of defining data, which she officially heard of them for the first time in the class, she never forgot to explain the relation between the original and the transformed figures with relation to the defining data.

Here is one more example, the case of rotation. We can tell the difference very easily between the response that she gave in the first diagnostic test and that in the second diagnostic test (Table 4.6). The first response was basically a simple description about rotation. Although she seemed to know what the rotation is, it was not clear and informative enough to deliver all the messages that should be mentioned about a rotation. Also, it sounded like she just directly interpreted the meaning of rotation in a literal sense instead of giving a full description of rotation in a mathematical sense. But, the following presentation shows us her improvement in understanding the concept of rotation by giving proper elements for rotation, such as defining data (center and angle) and the point on the figures. In addition, she explained the relation between the original and the rotated figure with relation to the defining data.

> ROTATION:
> The defining data for a rotation is a center point C used to perform the rotation, a marked angle, and a point X, which is the object used in the transformation. In GSP, you are given the option of performing a rotation either with a marked angle or with a given angle, where you specify the number of degrees that you want to be able to rotate the given object. Here, I am going to illustrate a

rotation by a marked angle, which is an angle that is already created in my sketch.

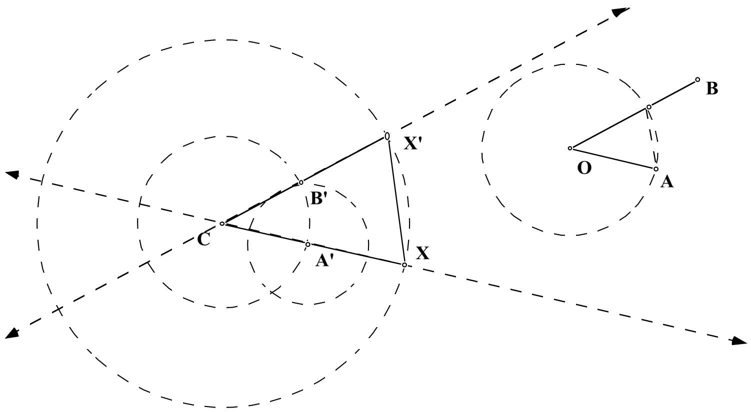


Figure 4.34 Emily's definition of rotation

In the above construction, C is the center point for the rotation, and ∠AOB is the marked angle in my construction. I copied the angle AOB to find the line $\overleftrightarrow{CB}$☾, and then did a circle (C1) with center point C and radius $\overline{CX}$ to find point X', which was the intersection between C1 and the line $\overleftrightarrow{CB}$☾. So, given a center point C, a marked angle AOB, and the point X, the point X' has the property that it is unique and that m(∠XCX') = m(∠AOB) and $\overline{CX}$ = $\overline{CX}$☾. To find the correct direction to place X', you need to measure the angle AOB counterclockwise in the positive direction, so ∠XCX' is also measured in the positive direction.

Pictorial Representation

The pictorial representations of transformations that Emily presented in the first diagnostic test deliver the understandable message of each transformation except a glide reflection (Figures 4.11, 4.13, & 4.15). But, there were some missing important elements such as defining data and relations between defining data and figures, which should be mentioned for the precise concept, and was not enough information with which a reader can grasp the concept of each transformation. After the first diagnostic test, Emily had opportunities to discuss each transformation in the classroom and to think more deeply through successfully accomplishing assignments. As mentioned earlier, she considered herself skillful with the use of GSP and actually she did almost all her work in GSP.

> R: How do the pictures constructed in GSP help you to see the result of combination like ... you were given a translation * reflection without anything else?
>
> E: Umm heum. Because I think because GSP is so precise, and constructing everything and it's a lot different from doing on paper that normally once I do the correct construction, I can see right away what the result's going to be. Umm ... as long as you make the figure. … umm ... different on all the points so that when it either reflects or rotates that it doesn't look the same way you can tell where the corresponding end points are. So, I think that pictures help a whole lot just to be. It would be visualized because I can see what's going on and I can tell umm just right away what the orientation is and narrow it down and then from there I can tell what type of isometries are.

In the first diagnostic test, Emily's pictorial representation of translation (Figure 4.11) simply showed what the translation is, maybe in a literal sense, i.e., moving a figure. There was nothing other than a figure had been moved to a different location in that presentation. We could not tell first, which one is original and which one is translated figure and second, how it had been moved. Especially because the figures that she used looked like regular polygons, it was harder to tell what has been done with those figures from simply looking at figures. But, I do believe that she knew something about translation although her presentation was somewhat ambiguous.

Since the first class of the semester, she worked hard along with good discussion and reasonable amount of assignments in the classroom. It became very easy and clear for Emily to present each transformation using pictures. After she built strong foundations of each transformation, she was challenged to make a multiplication table whose elements are transformations: translation, reflection, rotation, and glide reflection. I would like to pick one of those in the table to show how pictorial representations appear during the process of making a table. She was not simply asked to make a multiplication table. In addition to making a table, she was asked to prove the result of each product of two transformations. Here is the first case, Rotation * Reflection, i.e., R*F [do Reflection first and then Rotation].

> (Rotation) * (Reflection) = ?
>
> To first look at this product, I am going to experiment with GSP to try and figure out the result. I am going to take an object, reflect it over a mirror ($\overline{AB}$) and then rotate it about a center point (O) with a fixed angle (∠CDE).

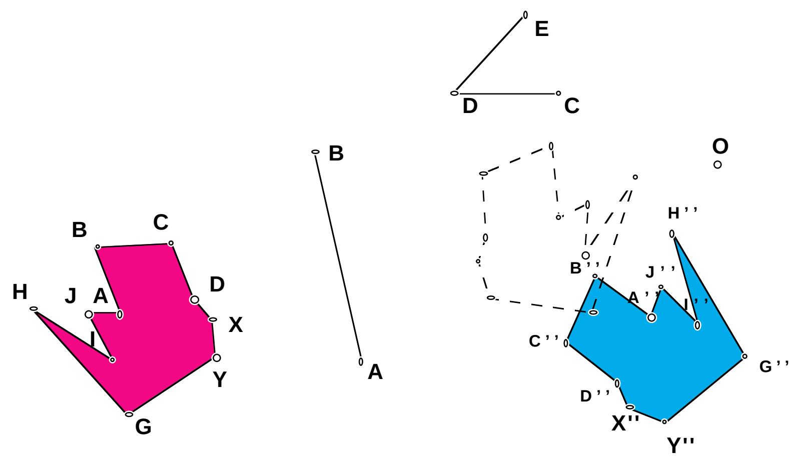


Figure 4.35 Emily's case i) of Rotation * Reflection

> <u>Case #1</u>: The center for the rotation is not on the mirror of reflection.
>
> From above, the pink figure on the left [ABCDXYGHIJ] is the original figure and the light blue figure on the right is the result [A"B"C"D"X"Y"G"H"I"J"]. I began by reflecting the pink figure over the mirror $\overline{AB}$ and then rotating the intermediate figure [figure with dashed segments] about the center point O by the fixed angle ∠CDE. I noticed that when the center is not on the mirror, the resulting figure appears to be a glide reflection of the original figure, because the blue figure is oppositely oriented from the original figure and it is shifted as well from its original reflection position. Therefore, I must find the vector and mirror for the new defining data.

The first thing that she did was to secure all defining data for both reflection and rotation in order to find the result of Rotation * Reflection, which are the most important elements for correct construction. After she had all elements, defining data and a figure, she simply transformed the figure one after another on GSP. She knew how important defining data are in transformations. Further, the figure was well selected for the investigation. The figure above might look too complicated but it was perfect for further investigation because it is irregular enough so that we can tell easily if the orientation was preserved or not. In the first diagnostic test (Figures 4.11, 4.13, & 4.15), the two figures that she drew for translation looked like regular polygons without any labels: one looked like a regular triangle and the other a square or a rhombus. In addition, she drew two figures for both a rotation and a reflection, but one of two figures respectively almost looked like a square without labels, too. Thus, the process of constructing the given mathematical situation, Rotation * Reflection, shows that she knew what to do with this task and she possessed understanding and knowledge of transformations.

She was logically[4] thorough and accurate when she decided the result of Rotation * Reflection. She knew that the result must be either reflection or glide reflection because the orientation of the transformed figure was opposite to that of the original. Two shaded figures above clearly tell us about the change of orientation. Moreover, she was able to recognize that there are two distinct cases thanks to the accurate construction by GSP, the dynamic function of GSP, and her intuition coming from looking at pictures and based on her solid understanding of transformations. Here is another case.

[4] I used the term 'logical' along with approach or thinking. Especially 'logical' means the students' attempt to solve the given situation was based on a mathematical definition.

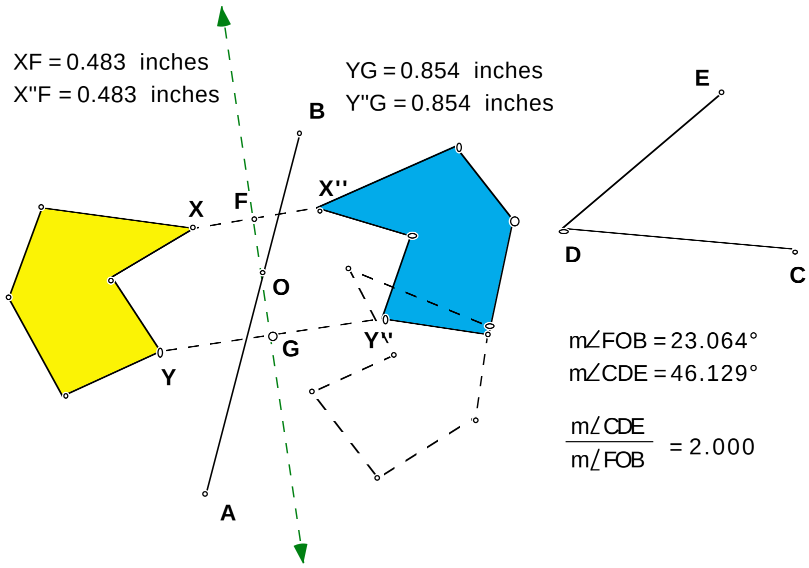


Figure 4.36 Emily's case ii) of Rotation * Reflection

Case #2: The center for rotation is on the mirror ($\overline{AB}$) of reflection.

When the center (O) for rotation is on the mirror of reflection, the result appears to be a reflection. Therefore, I would need to find a new mirror in order to have my new defining data. I noticed that the angle of rotation, ∠CDE, was equal to twice the angle ∠BOF (the angle that the new mirror ($\overleftrightarrow{FG}$), shown here as a green dashed line ($\overleftrightarrow{FG}$), makes with the original mirror ($\overline{AB}$). Therefore, if we were to move half the angle measure of the original angle of rotation from the original mirror, we would find where our new mirror would need to be placed. Also, we know that $\overline{XF} = \overline{X''F}$ and $\overline{YG} = \overline{Y''G}$ would have to hold for our mirror to be in the correct place, by the definition of a reflection.

In analyzing the case ii) of Rotation * Reflection, she reached the result so quickly, maybe this is due to the experience with the first case. Then she quickly moved to find the mirror of the reflection. There was no reason provided why the result is a reflection. But, she seemed to be so confident of the result. I recognized that she used the word "appears" in her answer. Since she gave the logical background in the first case and maybe she thought only the location of a center is different in this second case, she jumped to the next step. For the second case, the figures that she was seeing and her insight generated the result.

She seemed to strongly believe the power of technology and depend on what she saw rather than tried to figure it out based on a logical sense. Because of so strong belief of what she saw in the picture with the help of technology, she concluded so quickly without any doubt. She simply connected X and X" to find a mid point, which is F, and Y and Y" to find a mid point, which is G. She claimed that the line passing through F and G is the new mirror. She also claimed that the new mirror could be found by turning the old mirror by half angle of the given angle for the rotation. All the claims that she made were correct. All these claims came from the fact the result is a reflection. But, there were no strong evidences provided that the result is a reflection in her answer other than the pictures and the expression, "the result appears to be a reflection". If the result is not supported by the logical thoroughness, all these claims are not fully ready yet so that they can be applied. Thus, her strong dependency on figures and technology made her claims relatively weak in a logical sense for this case.

The following figure (Figure 4.37) was created by Emily during the first examination (Appendix K) on February 22, 2001. The question was: "If the center of the half-turn R is on the mirror of the reflection F, what is the product FR? Prove your answer". The half-turn was newly developed after the concepts of transformations were covered, which is restriction of rotation.

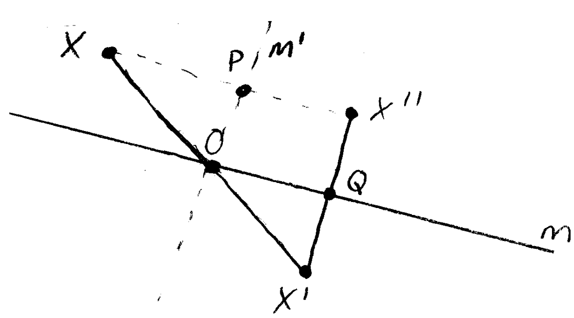


Figure 4.37 Emily's pictorial representation of F*R

Given a center 'O' and a mirror of reflection m, my conjecture is that F*R = F when 'R' is a half-turn and 'O' is a point on mirror m. Therefore, we need to prove that the product is a reflection as well as find the new defining data, a

> mirror m', such that m' is the perpendicular bisector of $\overline{XX''}$, and $m \perp m'$ as well, so, 'O' lies at the intersection of m and m'.

The figure above shows that she understood first, the concept of half-turn and of course reflection, second, what the question asks. Thus, she could pictorially present the problem using half-turn and reflection on GSP. After she finished the construction, she concluded that it is a reflection. It would be better if she explained why it is a reflection. Although her basic pictorial representation tells that she understood everything relating to the problem, her logical process of this problem still seemed to be a little weak. For example, the instructor took off some points for the improperly constructed auxiliary line, $\overleftrightarrow{OP}$, or m'. She just constructed a line m', which could not give any stronger evidence to claim that m' is a new mirror. She should construct the line m', which is a new mirror, so that m is perpendicular to m'. Based on Euclidean construction rule, she could construct a perpendicular line through the point, 'O', to the line m. She seemed to just focus on that the line m' must be perpendicular to the segment, $\overline{XX''}$, which is what she had to show eventually. The pictorial representation correctly presented gave her confidence about a conjecture but did not yet provide justification of it.

As a final example, here is what Emily turned in for the homework 5.2 (Appendix J), which is to make the multiplication table for the symmetry group, D5. She was not confused at all when she made a table of this because of the way she constructed D5 (Figure 4.38). She basically knew that D5 has 10 elements, which are five rotations and five reflections. After she picked the center point, X, and constructed the quadrilateral, ABCD, then she rotated it five times using equally divided angles of a circle, i.e., 72°, 144°, 216°, 288°, and 360°. In the same manner, she constructed five mirrors, i.e., m1, m2, m3, m4, and m5. Then she reflected ABCD to those five mirrors five times one by one. The process she chose to construct D5 was correct and so she was able to present the right concept of D5 through figures. Finally, she could successfully make the table without confusion. The original figure of this construction was ABCD, not a set of two quadrilaterals, which was the key point not to get confused with D5.

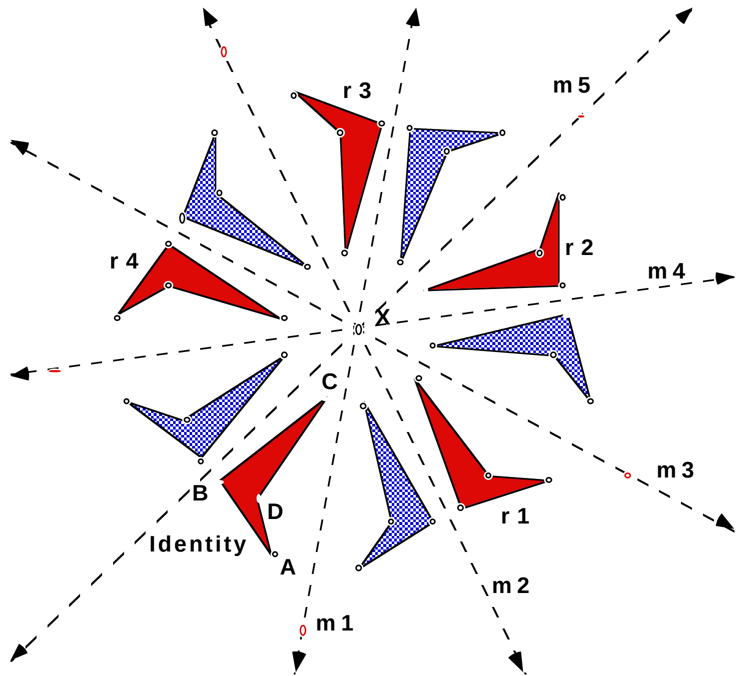


Figure 4.38 Emily's pictorial representation of D5

[5]Table 4.10 Emily's multiplication table of the symmetry group, D5

*	R-72	R-144	R-216	R-288	R-360	F-1	F-2	F-3	F-4	F-5
R-72	R-144	R-216	R-288	R-360	R-72	F-2	F-3	F-4	F-5	F-1
R-144	R-216	R-288	R-360	R-72	R-144	F-3	F-4	F-5	F-1	F-2
R-216	R-288	R-360	R-72	R-144	R-216	F-4	F-5	F-1	F-2	F-3
R-288	R-360	R-72	R-144	R-216	R-288	F-5	F-1	F-2	F-3	F-4
R-360	R-72	R-144	R-216	R-288	R-360	F-1	F-2	F-3	F-4	F-5
F-1	F-5	F-4	F-3	F-2	F-1	R-360	R-288	R-216	R-144	R-72

[5] This table was slightly modified by the researcher based on what Emily originally made. R stands for rotation and F stands for reflection. Abbey used 'Rotation' and 'Reflection' instead of R and F respectively in her original table. For example, R-144 means rotation by 144° and F-3 means reflection through 3. Further, the multiplication table was made by the following rule, i.e. (R-72)(F-2) means do F-2 first and R-72 second , where R-72 is from the far left column and F-2 from the very top row.

F-2	F-4	F-5	F-4	F-3	F-2	R-72	R-360	R-288	R-216	R-144
F-3	F-3	F-1	F-5	F-4	F-3	R-144	R-72	R-360	R-288	R-216
F-4	F-2	F-2	F-1	F-5	F-4	R-216	R-144	R-72	R-360	R-288
F-5	F-1	F-3	F-2	F-1	F-5	R-288	R-216	R-144	R-72	R-360

Verbal Representation

Emily was not asked to express verbally her understanding or knowledge relating to concepts of transformation except for the first and the second diagnostic tests, either. However, while the interview was going on with assignments or contents covered in the class, Emily often verbally presented her understanding or knowledge.

One of the most important concepts in studying transformation is the concept of defining data, which are the main sources in doing each transformation. The understanding of defining data made it possible for students to approach the concept of each transformation more systematically using mathematical symbols as well as to understand it more clearly. Emily never had a chance to talk about specifically the concept of defining data before. Throughout representations of each transformation in the first diagnostic test, the importance of defining data did not stand out in her responses. She only learned for each transformation how a figure simply moves in the plane for two class periods. Maybe that is why her first responses to the first diagnostic test were somewhat vague and did not include the relation between two figures relating to defining data. Here is the first of her verbal expression about defining data.

R: Can you tell me about defining data?

E: Umm. [silence lasts about 5 seconds]

R: What is defining data in transformation to you?

E: Defining data is what you are given to start ... like when you perform a transformation what is needed umm ... in order to do a form of transformation.

R: Great great. So, in order to do some type of transformation, we need some source to do with.

E: Right.

R: The figure itself is just it can be given randomly. But, you have to have something. For example, what is the defining data for translation?

E: A vector (in a low voice). …… (murmuring …inaudible)

R: All right.

R: How about in the rotation?

E: A center point and then ….

R: Is that it?

E: Oh, angle.

R: Right.

The above conversation shows that she has something in her mind about the concept of defining data. But, it does not seem that she quite understood the concept of defining data and the importance of it. After the first homework, she had to use the defining data for constructing the given mathematical situation or investigating it further in the following three assignments, especially the third homework, which is about making a multiplication table for transformations: translation, rotation, reflection, and glide reflection. For example, when she investigated the result of the product, Translation * Reflection, she set up so that the vector [$\overrightarrow{AB}$], which is the defining data of translation and the mirror [$\overline{CD}$], which is the defining data of reflection, are perpendicular each other by accident (Figure 4.39). Here is the conversation at that moment.

R: Then, what is it?

E: Umm (Hesitant)

R: What does that look like?

E: Just looks like a reflection.

R: That's good. Reflection.

E: Right. Because you can draw a mirror here [drawing a line passing through the middle of the original and transformed figure.]

R: Right. My question is …. Is it always reflection then?

E: Umm. [lasts about 7 seconds] Look at the other cases.

R: How do you find other cases?

E: Umm …

R: What controls the figure?

E: The defining (a bit hesitant) data.

R: Right. The location of defining data.

E: Right.

R: So, what is this case then? [Researcher moves the vector around.]

E: This case is ... From the defining data …..

R: When mirror and vector ...

E: Uh huh..

R: are 90 degrees, I have reflection.

E: O.K.

R: If not, 90 degrees? Do you still see the reflection?

E: Glide reflection? [unsure]

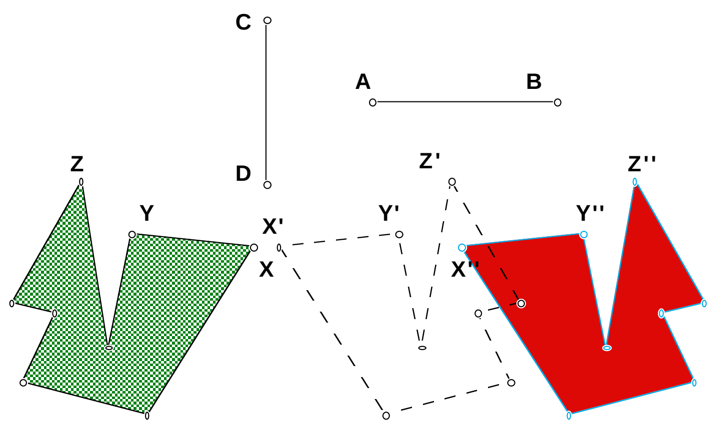


Figure 4.39 Emily's T*F for investigation - when defining data are perpendicular

She finally knew that she could get different results by moving the defining data of each product. Although she was not sure that defining data control the transformed figure, she was able to make a good guess that defining data do control a figure with GSP. After this experience, she not only used the concept of defining data in her investigation, but also could finish almost all the different cases for each product. She did not specifically state the concept of defining data, but she seemed to quite understand the concept of defining data. Emily's verbal presentations had been developed with her better understanding of the concept of defining data. She used the word, defining data, here and there in a right place without any hesitance. She was more confident when she used the concept of defining data than before. It seemed that the concept of defining data is there in investigating the product of transformations, maybe in any type of situation relating to transformation.

> R: What was the most difficult thing when you do this assignment [Homework 2]?
>
> E: The most difficult case? Or ...
>
> R: Most difficult situation or what was bugging you when you do this one?
>
> E: Umm it just took a very long time to try and get through everything and once I got started and was able to see kind of how they work on somewhat kind of similar to be able to find new defining data. So, it helped me to be able to look for things. Once I got started after a while, I was able to do the last few a lot faster than I was the first few. But it still took a long time. It was just kind of stressful for some of the defining data. Because they were funny. Just funny.
>
> R: How much percentage do you think you finished?
>
> E: Umm ...
>
> R: Just roughly.
>
> E: I think I finished about maybe 85 or 90% because I didn't find a lot other special cases.
>
> R: Wow. 90%. That's great.

Students' growth of understanding

Abbey's growth of understanding

Primitive Knowing up to Property Noticing

The following episode is what Abbey tried in order to find the result of combination of rotation and reflection. This is a part of homework 3 (Appendix J), which is to make the multiplication table for isometries. Because there are four elements in this table: translation, rotation, reflection, and glide reflection, she had to fill sixteen blank spaces all total. But, since there are several cases depending on the location of defining data for certain combinations such as the combination of translation and reflection or the following case that Abbey was doing, there are technically more than sixteen cases.

(Abbey creates a GSP file (Figure 4.40) representing the situation of rotation * reflection, where ABCDE is the original figure, A"B"C"D"E" is the final figure, O is the center for rotation, and $\overline{IJ}$ is the mirror for reflection.)

R: O.K. what do you think it is?

A: Looks like a glide reflection. (unsure of her answer)

R: Glide reflection.

A: It looks like … (long pause).

R: Is that it? Can you think of other case?

A: (Long pause)

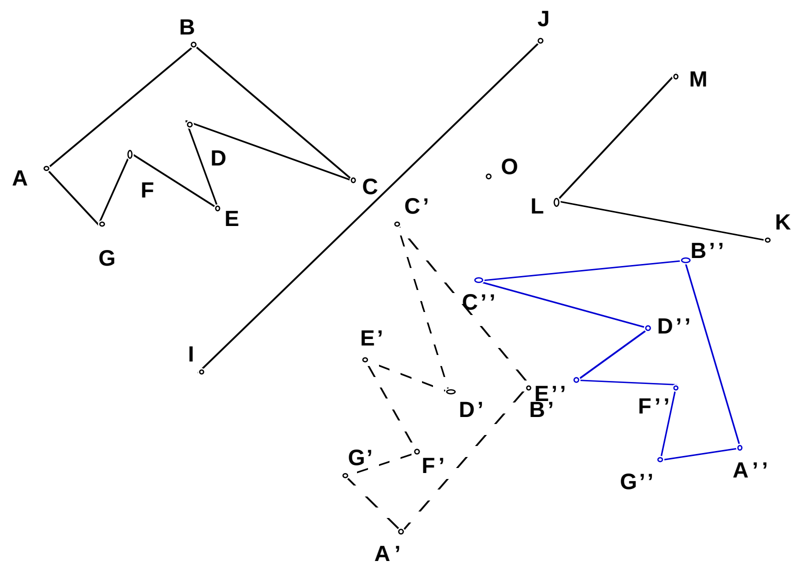


Figure 4.40 Abbey's Rotation * Reflection (R * F) when center is not on mirror

At the time she had this homework 3, she possessed more advanced understanding about the concept of each transformation and defining data than in the very beginning of this semester [Primitive Knowing]. She also knew that the latter transformation is followed by the former in the combination of two transformations, i.e., for rotation * reflection, do reflection first and then rotation. Since she was skillful with GSP, she could easily create the situation of combination of rotation and reflection [Image Making & Image Having - because of the complexity of the mathematical situation, it was hard for her to have the final images directly without constructing them on GSP although she had solid conceptual understanding]. Once she set that up like Figure 4.40, she could tell what it visually looked like using her understanding of a reflection and a glide reflection, especially using the knowledge of orientation change after product of two transformations. GSP allowed her to move around the defining data of rotation and reflection so that she possibly could find a different result, but she did not think of doing that after she found one solution, which is a glide reflection. So, I asked her to try some other cases.

R: Why don't you move around this [center point, O] or that [mirror, $\overline{IJ}$] or both?

A: O.K.

(She moves them around. But she can not find anything new other than glide reflection.)

R: You know that defining data control this figure [the transformed figure]. Let's try to see some kind of relation between the center point and mirror.

R: This is my center, O, right?

A: Right.

R: I am gonna put ….. O.K. when the center is on the mirror, (Figure 4.41)

A: They [two defining data, center point and mirror] overlap …

R: O.K. What do you think it is now?

A: Looks like ... [pause for about 3 seconds] reflection.

R: Good. Then, how would you find defining data for a reflection? That is a big task. That's pretty challenging.

A: (quiet for about 7 seconds.) Oh, no. Construct a line [she meant segment connecting C and C"] and find a midpoint, X [of the segment, $\overline{CC''}$].

R: Is that it?

A: Then, do one up here [making another segment, $\overline{BB''}$ and find a midpoint, Y, of this segment]. Connect those [two mid points, X and Y]. (inaudible) That could be here, mirror.

R: Do you want to test it out?

A: Umm heum.

R: How can you test it?

(She marks the line, $\overleftrightarrow{XY}$, as a mirror, select the original figure, ABCDEFG, and reflect it through $\overleftrightarrow{XY}$.)

A: Yup. It works.

R: Good.

(After this, she reconstructed the second case, which is the case when the center is on the mirror, because if she assigns the center point on the original mirror using the GSP instead of dragging the center point on the mirror, she could construct the situation more exactly. When she tested the second case in conversation, it was approximately exact.)

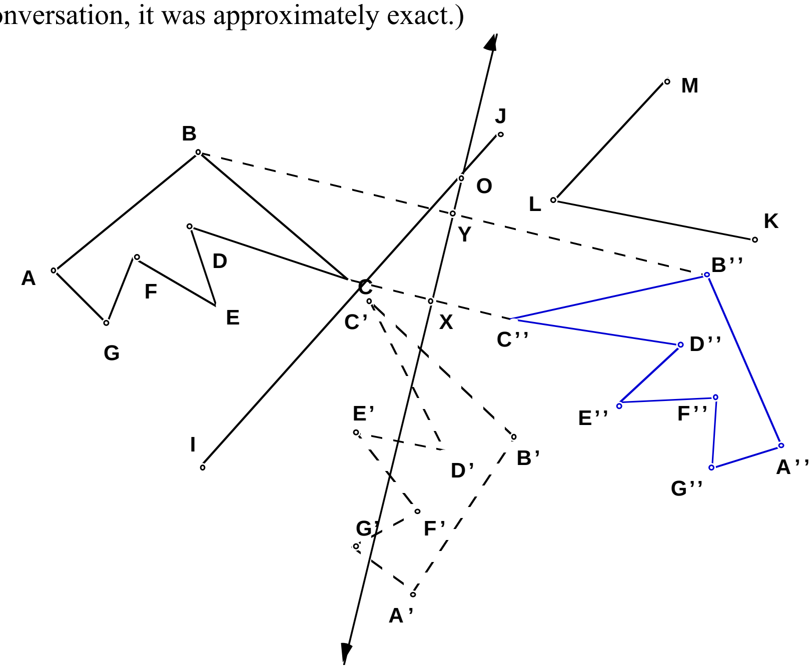
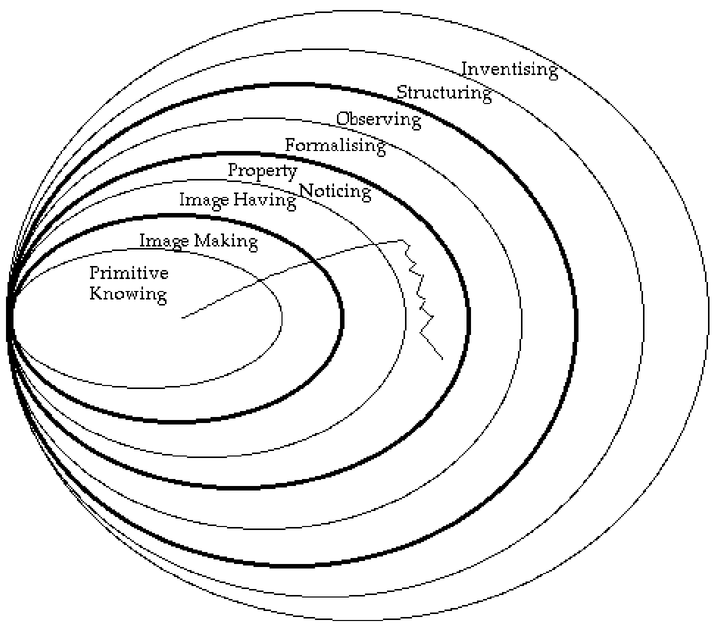


Figure 4.41 Abbey's Rotation * Reflection (R * F) when center is on mirror

The dynamic function of GSP, that is, moving around objects with maintaining the relations as constructed in the beginning, made it possible for Abbey to investigate the other possible cases without additional reconstruction. She had no difficulty in constructing the given situation, but she could not investigate all the possible cases on her own with her construction. She was just satisfied with her initial construction and her sense in deciding the result by looking at her construction [Property Noticing]. She did not know what to do after she finished her construction. She knew not only defining data control the transformed figure from the classroom lecture and discussion, but also she could visually see that the location of final transformed object changes as the angle for rotation varies. But, she could not find any different case by herself. In other words, if she constructed the center for rotation on the mirror in her initial construction, her initial response must be a reflection, not a glide reflection. With the researcher's help, she finally could find the other case which was different from the case she did for herself (Figure 4.41).



[6]Figure 4.42 Abbey's growth of understanding of Rotation * Reflection

As mentioned in the beginning of this section, Abbey had to go through 'Image Making' stage before she reached the 'Image Having' stage because of the complexity of the mathematical situation, not because of her weak understanding of each transformation. Thus, the 'Don't Need Boundary' which lies between 'Image Making' and 'Image Having' was meaningless in this investigation. Abbey quickly moved from 'Primitive Knowing' up to 'Image Having' without any delay. She even could find the

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6 The zigzagged line in 'property noticing' means that Abbey's staying in that level and trying to move forward. In other words that shows that Abbey is using trial and error and thinking hard to move to next level.

result of combination of rotation and reflection based on her initial construction, which is the stage of 'Property Noticing'.

She just moved the angle for rotation, but she could not find anything new other than a glide reflection, which is her first response to her construction. She could not think of any other manipulation or combination of defining data. It was a good try to manipulate the angle for rotation, but if she had combined the other two defining data, center and mirror, she could find the other case, which gave her a different result. Basically, she felt very comfortable with construction itself, but she did not understand what she could do specifically with her construction in order to find all the possible answers. Her 'Property Noticing' was not stable enough in this case, but she finished one case and could completely answer the product of combination of rotation and reflection with a little bit of researcher's help. Although she possessed reasonably solid concept of each transformation, she did not quite understand how each transformation is connected, especially the role of defining data is still somewhat ambiguous. She never did 'Folding Back', but she could not move forward, either.

Primitive Knowing up to Formalising with Folding Back (1)

The following episode is again about finding the product of two transformations: translation and translation. This is also one piece of homework 3 (Appendix J), which is to make the multiplication table for isometries.

R: You wanna try something else here [meaning multiplication table for isometries]?

A: Sure.

(Abbey is doing her work on GSP.)

R: Which one are you trying?

A: I mean translation and translation.

R: O.K.

A: And so, for that, do I have two different vectors like .... I will translate it definitely and create another .... (Figure 4.43)

R: What do you think it is?

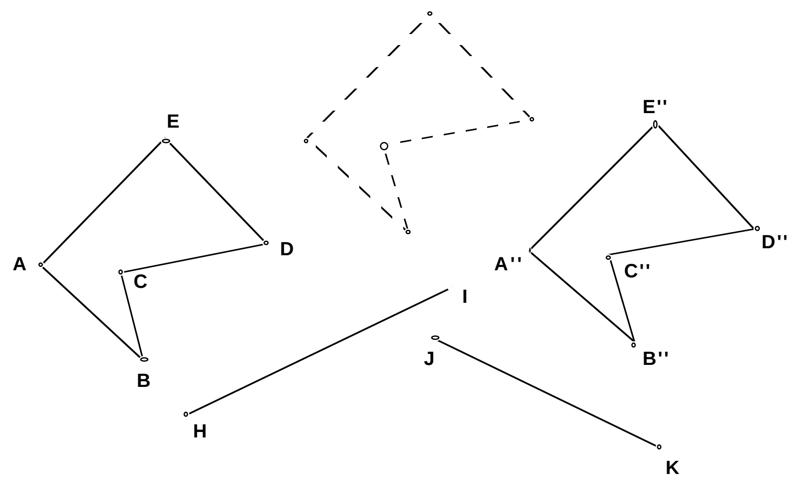


Figure 4.43 Abbey's initial construction of Translation * Translation (T * T)

She basically knew what she was supposed to do with investigating the product of two independent translations [Primitive Knowing]. Her response to the situation shows she knew what the translation is and what the multiplication of two transformations is. She first created a figure (ABCDE) and defining data, a vector ($\overrightarrow{HI}$), then did first translation, whose result is the figure with dashed sides in the middle (Figure 4.43) [Image Making]. Then, she constructed another defining data, a vector ($\overrightarrow{JK}$), for the second translation and did translation, whose result is the figure (A"B"C"D"E") far right above [Image Having]. She was not bothered at all with construction of this situation. Her understanding of translation and skills with GSP made it possible to reach the stage of 'Image Having' so quickly. It was possible for her to have images in her mind representing one translation, but she could not have a clear picture of the combination of two translations. So, she always had to draw the combination on GSP. After she finished her construction, she could not tell what the result was right away. The picture appeared to her as two separate translations, i.e., she could recognize the first and second translation, but could not read the result from the original figure to the final figure right away.

A: Umm. I guess this has to be fairly parallel to the first [$\overrightarrow{HI}$].

R: That [ABCDE] would be a special case. I will start this with a random case and move (two defining data) around freely.

A: O.K.

(Time goes.)

A: (Murmuring. Inaudible)

R: This is your original figure. You translated this [ABCDE] using this vector [$\overrightarrow{HI}$] and then you translated that using that vector [$\overrightarrow{JK}$]. This is what you did.

A: Right. So, if I move the original figure, let's see what it does.

R: You are gonna move the original figure.

A: Right.

R: Any conjecture?

A: About what these two produce?

R: Yea. After you translate and translate, what does that look like?

A: It's like another translation.

R: Another translation. Would you move around these two vectors? Then look at the first and third figure.

A: Yes.

R: Do you still see it as a translation?

A: Yup.

R: So, we are checking experimentally.

She, maybe from the previous experiences with exploration of combination of two transformations, focused on the location of defining data rather than trying to decide what the result is by looking at the first and the final figures. She knew that the relation between two defining vectors would give a clue about the result of product of translation and translation. At last, she could make a conjecture for this combination, which is "another translation" [Property Noticing]. She heavily depended on her intuition based on her understanding of translation when she made this conjecture. Her voice did not show strong confidence of the conjecture. However, she became confident with her technical use of the function of GSP. She moved around especially the second vector, $\overrightarrow{JK}$, to check whether her conjecture is correct using GSP. When the final figure is a bit far away from the original figure, the confidence was not very strong, but when she moved the vector, $\overrightarrow{JK}$, so that the final figure could be close to the original figure, she was confident about her conjecture (Figure 4.44). She did not use the relation between the original figure and the final figure, which is that segments connecting corresponding points are parallel each other and the lengths are all same. Rather, she mainly leaned on the visual confidence from looking at images on computer screen.

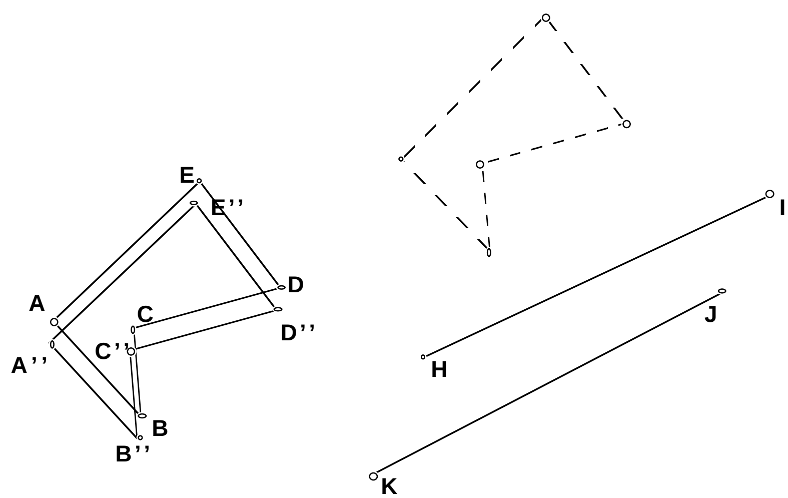


Figure 4.44 Abbey's experimental checking her conjecture

She reached the stage of 'Property Noticing' with the help of her understanding of translation, her intuition, and dynamic function of GSP. Based on these three pieces, she came up with a conjecture although it was still not logically proved yet. Since she made a stable conjecture and experimentally proved it, she needed to give a logical approach, that is, how she could find or construct the defining data for the result and why it must be the way to construct the new vector.

R: Good. So, what do you need for translation?

A: Translation, all you need is a vector.

R: How would you find it? The vector.

A: How would I find a vector of between these two?

R: You are saying that the combination of two translations is a translation.

A: Right.

R: So, how would you find that vector?

A: Umm….. I don't know but…. I would think that it would be something like the hypothenuse of a triangle.

R: Which one are you talking about?

A: (No response)

She knew that she had to construct a vector for the result. But, she did not know how to construct the vector and where she could start to do so or which information she could use. Thus, she had to go back to the stage of 'Image Having' to back up her conjecture [Folding Back to Image Having]. Her construction of the situation was correct, so she did not have to redo the whole construction. Thanks to dynamic function of GSP, she strongly believed that her conjecture is correct. She was also able to express her understanding about translation in three different ways: written, pictorial, and verbal representation. She knew how important it is to mention the relation between the defining data, the vector in this case and the segments connecting corresponding points. However, unfortunately, she could not do anything else other than just moving around two old vectors and figures. She seemed to be somewhat lost what she should do. Maybe, this is from the lack of experience in applying her understanding of translation to the mathematical situation. Or, technology in front of her hindered her from approaching this situation in a different way such as a logical approach.

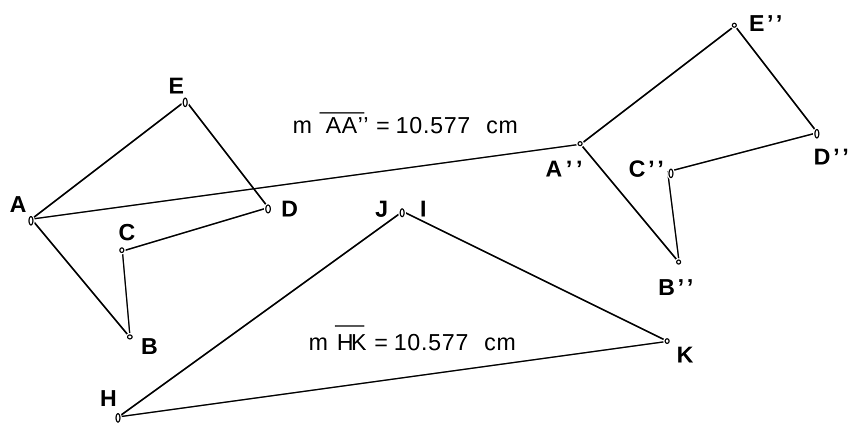


Figure 4.45 Abbey's finding new defining data of Translation * Translation

Abbey moved the second vector, $\overrightarrow{JK}$, so that she could set up to get sum of two vectors like the above (Figure 4.45). Now she came back to the stage of 'Property Noticing', but this time is more stable than the previous time. After she finished this construction, she moved the vector, $\overrightarrow{JK}$, so that two vectors, $\overrightarrow{HI}$ and $\overrightarrow{JK}$, are parallel. Then, she checked if the result was the same. Further, she moved the vector, $\overrightarrow{JK}$, so that two vectors, $\overrightarrow{HI}$ and $\overrightarrow{JK}$, are perpendicular and then checked again [Formalising]. Thus, she basically checked all possible relations that two defining data could make.

R: Are you done? Is this the only case?

A: No. when the two vectors are parallel,

R: That's right. Is this still translation or not?

(Checking what it is when parallel)

A: Translation.

R: And how about the vector?

A: This is still sum of them.

R: When these are parallel, still translation and still sum of two vectors.

A: Umm heum.

R: Is that it?

A: Perpendicular?

R: O.K. Let's try it out.

(She moves around the vector and chekcs.)

A: Yup. Translation.

R: How about the vector? Is it still sum of two vectors?

A: Umm heum. I think that's all.

R: Great. Now we are safe. It must be it.

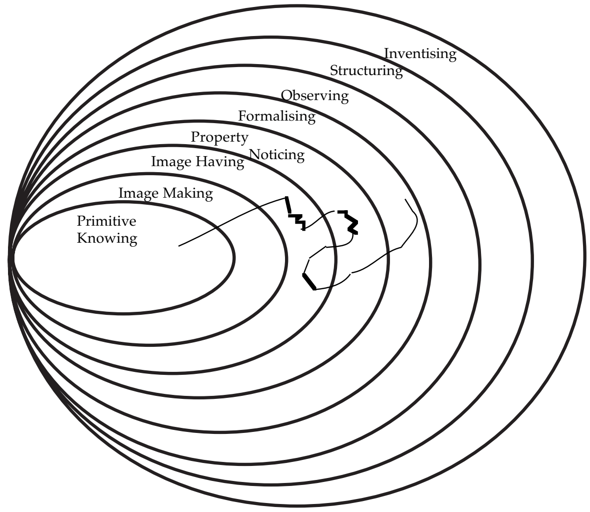


Figure 4.46 Abbey's growth of understanding of Translation * Translation

The concept of translation is maybe the easiest to understand in studying transformations. Abbey definitely felt comfortable with translation [Primitive Knowing], but the combining two transformations was challenging to her, at least for a while in the beginning, even though she knew each transformation fairly well. She was not sure how to explore and what to do with combination of two transformations. Further, since the situation was complex, she had to construct the situation in GSP so that she could look at before she reached the result of it [Image Making and Image Having]. She was also able to make a conjecture based on the figures and the dynamic function of GSP [Property Noticing]. But she struggled with backing up her conjecture. It was challenging for her to find the new defining data from the given situation. She was staring at two figures on computer screen, but it was not very helpful in finding the source of new defining data, which is the task in this investigation. So, she had to heavily lean on the figures just to find the new defining data [Folding Back to Image Having].

Finally, after she could get the idea of the new defining data with help from the researcher, she could move on to next combination very fast from that moment. She seemed to catch how to handle this type of situation. Her skills in GSP coupled with her knowledge of vector sum, which she learned before, made it possible for her to be through with what she was looking for [Property Noticing]. She now had the new defining data to back up her conjecture. Then, she checked all possible cases that two old vectors could make by moving around them, that is, when two old vectors are parallel, perpendicular, and meet each other not perpendicularly [Formalising]. She did everything with figures on GSP. She visually checked everything first. Although she never wrote anything down, she tried to recall her understanding about translation and had a strong impression that she had to find the new defining data. After she checked everything that she could think of in this investigation, she finally proved her conjecture using symbols and definitions of transformation, which is more stable and applicable than making a conjecture from looking at figures.

<u>Primitive Knowing up to Formalising with Folding Back (2)</u>

The following episode is more complex case than the previous, which includes a glide reflection in the combination: Glide reflection * Rotation. Maybe, this was the hardest case (Figure 4.47) in the entire multiplication table of isometries. There are two defining data in each transformation. That is, a center and an angle are the defining data of a rotation, and a mirror and a vector are those of a glide reflection. Further, the vector must be parallel to the mirror in glide reflection in order for them to be the correct defining data, whereas there is no specific relation between the angle and the center in a rotation. The figure right below shows the complexity of the situation. Because the whole picture is so complex, I rearranged and slightly modified what Abbey did appropriately so that we can see it better.

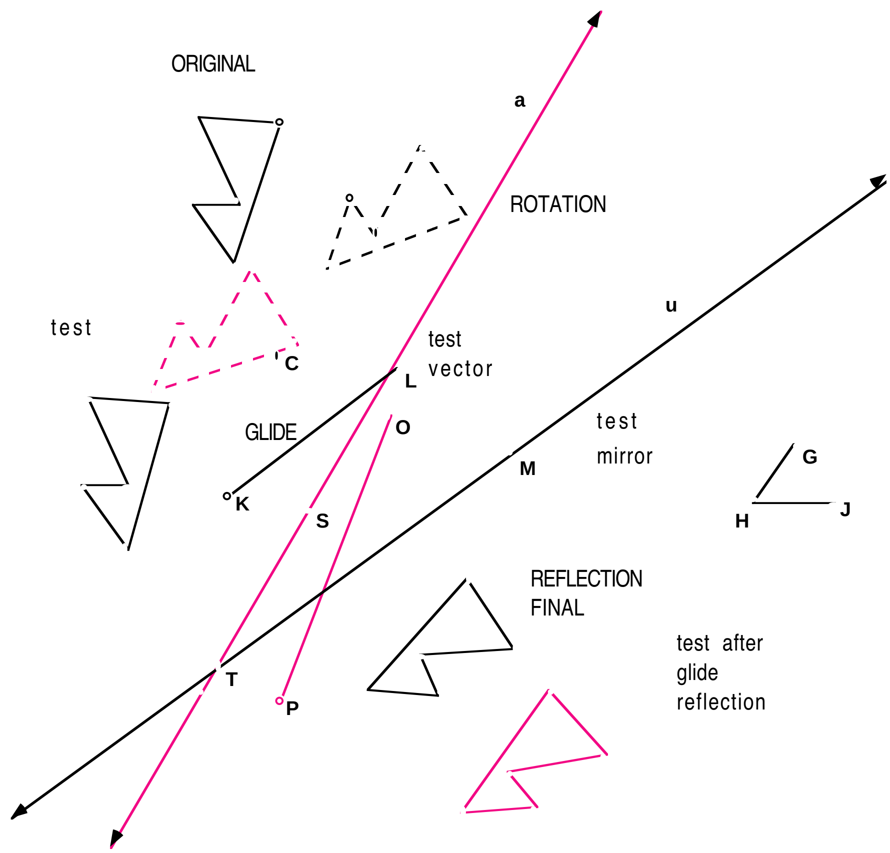


Figure 4.47 Abbey's final construction for Glide Reflection * Rotation

A: O.K. Yes, this is hard. What I did.... This is my original figure [ABCDE] - Figure 4.48. And then I rotated it around C [center point] using this angle [∠GHJ]. And so, I got this one [A'B'C'D'E']. This is the rotation. Then I slid it. I mean ... yea I translated it using L, K [$\overrightarrow{LK}$]. And I got the pink one [A"B"C"D"E"]. And then I reflected it across this big black line [$\overleftrightarrow{MN}$] and I got this pink one [A'"B'"C'"D'"E'"].

R: O.K. Which is which?

A: This [A'"B'"C'"D'"E'"] is my final and that's [ABCDE] my original.

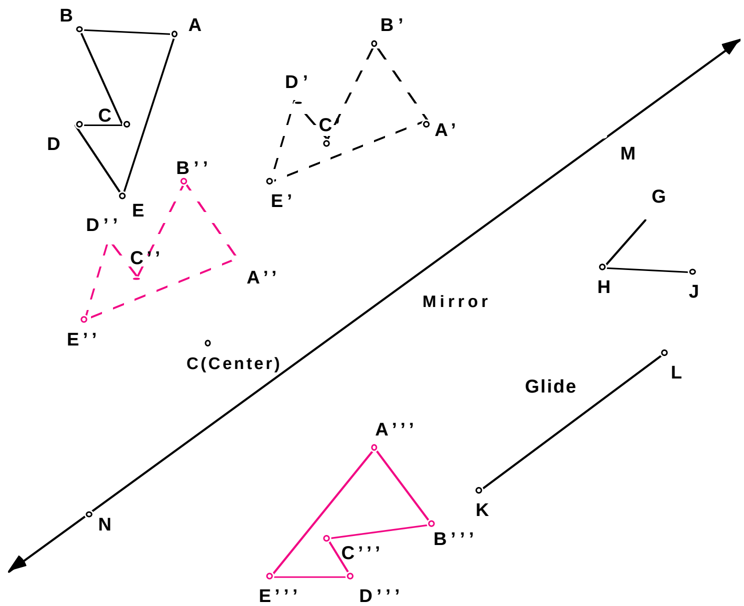


Figure 4.48 Abbey's initial construction for Glide Reflection * Rotation

She possessed solid understanding about a glide reflection and a rotation and also knew how to construct the product of two transformations [Primitive Knowing] on GSP. There was no mistake in setting up two defining data respectively so that she could have the right construction for investigation, Glide reflection and Rotation. Even though the situation was very complex and exact understanding was required for defining data for each transformation, she did not have any difficulty in constructing the figures exactly presenting the product of a glide reflection and a rotation [Image Making and Image Having].

A: Then, I was trying to test it out. So, what I did.. as I … Umm, this is my test object [P1] and I guess this is my test vector [$\overrightarrow{OP}$] (Figure 4.49).

R: What do you mean test object?

A: Like just took … see if I can get one that ended up top one over here. To see what I had to do. You know, just exploring. And cause I figured it was just looking from I thought it would be a glide reflection (a bit hesitant). Cause it's a different orientation and so that either is reflection or glide reflection. And so looking at it, I didn't think it could be a just reflection.

R: Good.

A: So, I figured that it had to be a glide reflection. And so, to test that out, I think I used this vector [$\overrightarrow{OP}$] and slid that one to there.

R: Yea.

A: And then, I umm used …. I guess I used pink line [$\overleftrightarrow{SR}$], I am not sure…. (pause about 5 seconds) This is my test mirror [$\overleftrightarrow{SR}$]. Yea. Pink is my, what I thought, might be the mirror and I can never get this to be exactly lined up. So, I didn't know how I was supposed to do a glide reflection.

R: This is your original (pointing to figure ABCDE).

A: Umm heum.

R: This is your final (pointing to figure A'''B'''C'''D'''E'''). And your decision turned out to be a glide reflection with the process of eliminating transformations.

A: Right.

R: Finally, you decided that that must be a glide reflection. In order to be a glide reflection, you need ...

A: A vector.

R: And what else?

A: A mirror.

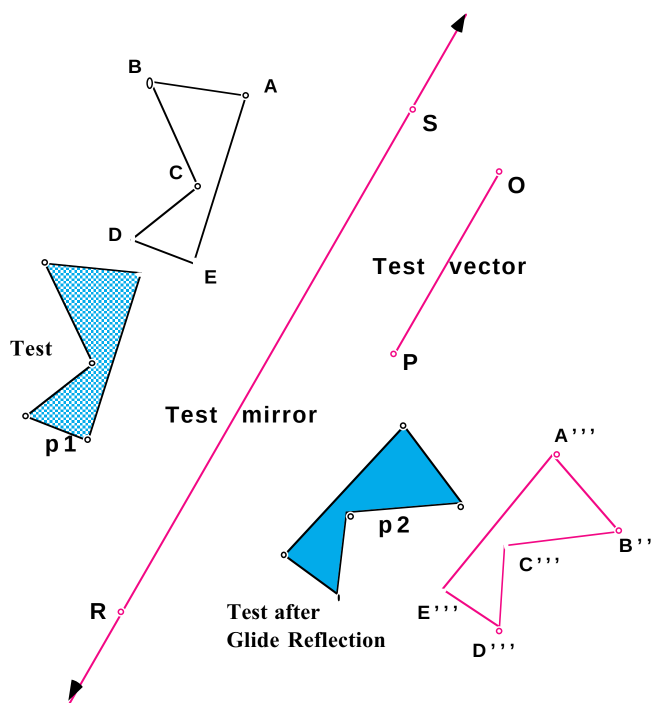


Figure 4.49 Abbey's experimental checking her conjecture

As we can tell from the conversation above, it shows us that she possessed strong and solid understanding about a glide reflection and the product of two transformations having two different orientations. She knew that the result would be either a reflection or a glide reflection just from looking at the combination of a glide reflection and a rotation. Further, because the original figure (ABCDE) could not be directly reflected to the final figure (A'''B'''C''D'''E''') from looking at figures, she was able to make a conjecture that the result is a glide reflection [Property Noticing] by eliminating a reflection using the principle of orientation change for the product of two transformations.

Next task was to locate the defining data for the glide reflection: the mirror and the vector which is parallel to this mirror. Because the construction Abbey created was so complicated, I hid all other elements except main sources, i.e. the original figure (ABCDE) and the final figure (A'''B'''C''D'''E''') for the combination of a glide reflection and a rotation, and the test vector ($\overrightarrow{OP}$), the test mirror ($\overleftrightarrow{SR}$), the test intermediate figure (P1) after translation using test vector, and the test final figure (P2) after reflection using test mirror in the Figure 4.49.

She introduced an experimental test vector and a test mirror which she made up to approximately locate new defining data for the glide reflection. Then she did translation and reflection using the test vector and the test mirror respectively. After she had the final (P2) with the combination using the test defining data, since she knew that those two defining data must be parallel to each other in order to be defining data of the glide reflection. She moved around two test vectors so that those two testing defining data remained parallel. Using this dynamic aspect of GSP she was able to make the figure (A'''B'''C''D'''E''') of initial combination and the figure (P2) of testing combination exactly overlap. Finally, she was able to locate the possible defining data based on what she saw using technology.

However, although she was able to pin down approximate location of new defining data using testing vector and mirror, she could not think of the way of constructing the new defining data from the original defining data of the glide reflection and the rotation. All statements that she made were all based on experimental facts. Thus, she had to go back to the images that she created at first [Folding Back to Image Having].

R: You translated and reflected it. This is your figure. That's good. So,…. One thing for sure that must be a glide reflection. But, you don't know how to find the new vector and mirror.

A: I don't …. I …. Was I …. ? (Silent)

R: The method that you used … What did you do?

A: I just connected two corresponding points. On the mid-point, the perpendicular bisector.

R: What is that for?

A: (No, response. Silent)

R: That's only for reflection.

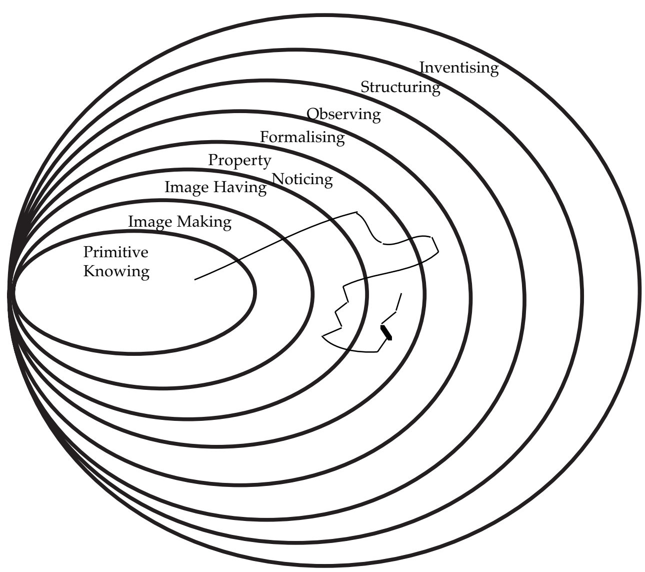
A: Right.

R: That's not for glide reflection.

A: (No response)

She was doing well in her initial construction and came up with a strong conjecture based on the construction and her understanding of each transformation along with well developed skills with the use of GSP. Further, the experimental check of her conjecture was successful in locating the new defining data. But, for the more fundamental question, i.e., how to construct two new defining data of new glide reflection only using those original defining data, her understanding of a glide reflection became shaky. She simply connected two corresponding points and found the midpoint to locate the mirror. Maybe, this came from her understanding of reflection, especially in terms of written representation. She knew that the mirror could be found by doing so in a reflection. It is good to have an idea of defining data for the reflection. But, the original task was to construct new defining data from the original defining data. There was no doubt that the result is a glide reflection and the location of new defining data was approximately located [Property Noticing]. But, the complexity of the problem hindered her from moving forward to the next stage (Figure 4.50). In her final report, she investigated further for the different relations of original defining data of the original glide reflection and rotation.

> I found that a Glide Reflection * Rotation = Glide Reflection.
> In the different cases, such as the center C being on the given mirror u [$\overleftrightarrow{MN}$] or point C [center] being on the given vector $\overrightarrow{KL}$, the result is always a glide reflection.
> I experimented with this for a really long time and couldn't find out how to get the new defining data which is the vector and the mirror. The bold pink line [$\overleftrightarrow{SR}$] is the new mirror and the pink vector $\overrightarrow{PO}$ are the new defining data that I found to make the test figure end up on the original pink final reflection. I did discover that the angle formed between the given mirror u [$\overleftrightarrow{MN}$] and the new pink mirror a [$\overleftrightarrow{SR}$] is half the given angle GHJ. I also know that the new vector $\overrightarrow{PO}$ is parallel to the mirror [new mirror, $\overrightarrow{OP}$], but I couldn't figure out how the length of the new vector related to the given vector $\overrightarrow{KL}$ and the center of rotation C. (Abbey's final report)



[7]Figure 4.50 Abbey's growth of understanding of Glide Reflection * Rotation

[7] The straight line in 'property noticing' means the long period of Abbey's thinking, whereas the straight line in 'formalising' means the short period of her thinking.

Primitive Knowing up to Formalising with Folding Back (3)

The following episode is about the concept of wheel symmetry[8]. This was discussed in the previous section for Abbey's pictorial representation. I will focus here on Abbey's growth of understanding. After an intensive exploration of each transformation or combination of two transformations, the instructor introduced a related concept to transformation, plane symmetry whose elements are wheel symmetry, strip symmetry, and wallpaper symmetry. Especially, wheel symmetry is somewhat related with the concept of 'group' in abstract algebra. But, they did not go that way, instead students were asked to explore the concept of wheel symmetry and also to make the multiplication for the symmetry group.

> R: Did you have any homework with wheel symmetry?
>
> A: Homework 5. I got number 2 wrong. O.K. I did it all, I did my whole chart all in rotation, and he said there should be rotations and reflections. And I see how it could be reflections also, but it's still a rotation also.
>
> R: All right.
>
> A: Like ….
>
> R: Can you show me?
>
> A: Yea, let me show you (Figure 4.28 & Table 4.8). Like I've got ... all of them can be rotation. But I understand how this like say, this to that can be a reflection across D. It can be a reflection across B. But also, it was a rotation of 72, which should be true. It's true also.

Abbey's argument shows she knew the concept of wheel symmetry [Primitive Knowing]. But, she seemed to be confused with the instructor's comments on her work that she turned in. It was not hard for her to construct the situation representing D5 with

[8] A pattern has a wheel symmetry if there is a point of the pattern which is fixed by all the symmetries of the pattern. This includes the very special case when the pattern is asymmetric, i.e. when the only symmetry is the identity transformation (the trivial symmetry). Another special case is when the pattern has bilateral symmetry, i.e. when the only non-trivial symmetry is a reflection, and so there is a line of fixed points. All the other wheel patterns have exactly one fixed point, the center point of the pattern. A wheel pattern has no translation symmetries. There are two types of wheel symmetry, cyclic (Cn)and dihedral (Dn). (Notes from the instructor of MATH 5210)

her knowledge about rotation, reflection, and D5 [Image Making and Image Having]. As discussed before, instead using all five rotations and five reflections, Abbey just constructed the figure that looks like representing D5 using only one reflection and five rotations. However, since there are many different ways to construct D5, as long as the figure represents D5, it should not matter. Actually, Abbey’s final construction did represent D5. But, Abbey could not see the other reflections that are in her figure after she was done with construction. Her initial responses for combinations were all rotations. I believe this is from the way she constructed. The other four reflections were ignored although there were five reflections indicated in her construction.

The following is Abbey's argument about the instructor's comments on her works, which indicates folding back: "So, how would I ... how am I supposed to know that I am supposed to write reflection instead of rotation? So, in class today, he said it was wrong if we had all rotations. And I just don't understand why it can't be all rotations. Like I understand if we were supposed to look for reflections first, and then if there is no reflection, then go to rotations". At the time when she studied a wheel symmetry, she possessed solid understanding of each transformation and especially had no confusion of the combination of two transformations in terms of preserving or reversing orientation. But, she could not see what was wrong with her table.

Abbey and the researcher briefly talked about the combinations of two transformations in terms of preserving or reversing orientation without figures and a table created by Abbey, and then went back to the table again.

R: O.K. Let's look at that one. Look at that. Rotation * Rotation should be rotation. I mean either rotation or translation, isn't it?

A: Right.

R: Positive and positive, so we should have positive, which is either rotation or translation. But, translation is not a question.

A: (at the same time) not a question.

R: So, in the same manner, reflection and reflection will have….

A: Rotation.

R: Rotation. That's clear, right?

A: Right. Cause reflection is negative and negative, so we need positive.

R: O.K. that's clear. Let's look at the rotation, reflection. Can you have a rotation?

A: I guess not. Because reflection is negative and the rotation is positive. So, we need negative.

Now, she began to recognize what was wrong with her table and also knew that half of the table must be reflections instead of rotations. This experience helped her to jump back to 'image having' [Don't Need Boundaries and Image Having]. Because of her construction of the original figure for D5, she still had a hard time in answering each blank for the combination of reflection and rotation.

R: So, let's look at the specific one. Rotation 144° and reflection 'a'. And your answer is rotation 216°. Let's go back to GSP.

A: O.K. take the element and rotate it 144° and then go cross 'a'. So, this [pointing at one set of two figures] is the resulting figure [rotated by 216°].

R: No.

A: No.

R: That's the problem. I see.

A: O.K. what?

R: What is your fundamental figure?

A: This one [pointing two shaded figures].

R: No.

A: No?

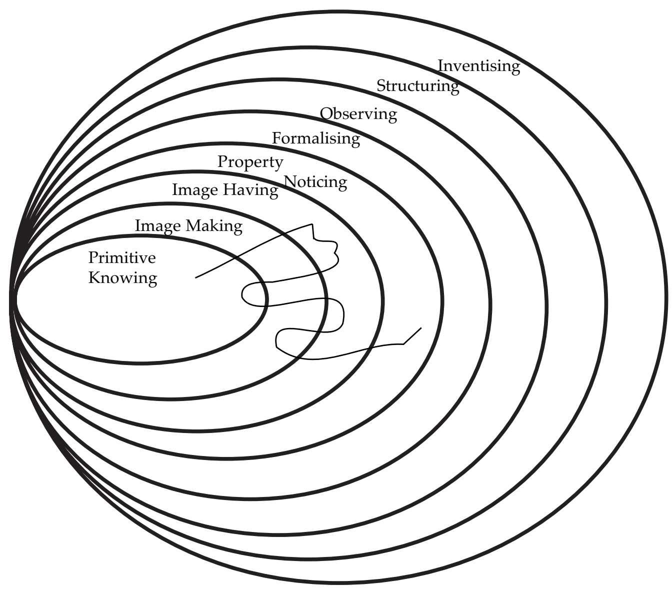
R: One.

A: Oh….

R: One is your fundamental figure.

A: O.K.

Although she started her construction of D5 with one shaded figure, the fundamental figure in constructing D5 became a set of two shaded figures because of the method she took for the final figure. She finally realized that the fundamental figure was not two shaded figures, instead, it must be one shaded figure. After this, we again went through the process of constructing D5. She said that "I see. I just couldn't tell the orientation has changed. O.K. That makes sense." The new construction led Abbey to the next stage, 'Image Having'. After she finished the new construction using five rotations and five reflections (she used the same shape but did it in a different way from what she did before), she could fix the shaded part in the table (Table 4.8) easily. She did not fix all of them at the moment, but she fixed some of them without any mistake [Property Noticing]. Now, she could tell surely the difference between Cn and Dn in a wheel symmetry.



[9]Figure 4.51 Abbey's growth of understanding on the dihedral group, D5

[9] The zigzagged line in 'image having' means Abbey's trial and error with hard thinking. The curve in 'primitive knowing' means almost an immediate moving to 'image having' without staying there for a long period of time, so does each curve in 'image having' and 'image making'.

Emily's growth of understanding

Primitive Knowing up to Property Noticing

The following episode is what Emily did in order to find the result of combination of a reflection and a translation. This is a part of homework 3 (Appendix J), which is to make the multiplication table for isometries.

(Emily creates a GSP file (Figure 4.52) representing the situation of translation * reflection, where ABCDEFGH is the original figure, A"B"C"D"E"F"G"H" is the figure after rotation and reflection, and $\overline{IJ}$ is the mirror for reflection.)

R: O.K. let's just try any one of these [pointing blank spots in the multiplication table].

E: Do translation times reflection.

R: That's a good one.

E: Translation, and then ….

E: Uh [meaning no], reflection and translation. O.K. Reflection first. So we will draw mirror for reflection.

R: Yeah.

E: And mark the mirror and then highlight. And reflect it. And then do a translation. So, highlight … (Emily highlights the segment $\overline{IJ}$ for translation.)

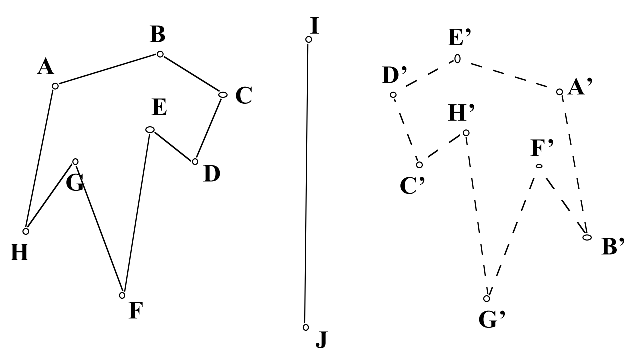


Figure 4.52 Emily's initial stage in construction of Translation * Reflection (T * F)

Emily right away started construction representing translation * reflection on GSP. She finished the first and the second assignments very successfully, which were mainly about exploring and understanding the basic concepts of each transformation. Students were asked to report what they discovered through exploration with GSP and challenged to express the more abstract definitions of each transformation. These two assignments helped her to possess decent knowledge and understanding of transformation. Her reaction, after she specifically decided to do translation * reflection, was very quick in doing a reflection of this combination.

However, she had a trouble with translation right after she finished reflection [Unstable Image Making]. She selected the whole segment, $\overline{IJ}$, as for the translation by clicking the middle of $\overline{IJ}$. After this, she went to 'transform' menu, but the translation menu on GSP was inactive. So, she could not translate the reflected figure. This shows that although she seemed to have decent knowledge and understanding of transformations, she could not properly use those concepts memorized or constructed through experiments or her own conceptualizing. I believe that this is maybe mainly from either the lack of experience with what she discovered about each transformation, especially transformation in this case, or the lack of recognizing a big picture of transformations, that is, lack of connection in transformations. The following conversation and Figure 4.53 show another weak experience or lack of connection in transformations.

E: Umm, I just use this vector and highlight [clicking the middle of $\overline{IJ}$] … and then translate .. and …. [She could not translate it because translation was inactive.]

R: You selected the line segment as a vector, right?

E: Uh huh.

R: What is wrong with that?

E: (Silence)

R: There is no direction. That's why it doesn't work.

E: O.K.

R: So, that one first and select the other one (pointing to the two endpoints of the segment $\overline{IJ}$).

E: Oh, yeah. We just look at this one [point I and point J] and this one.

R: That's right.

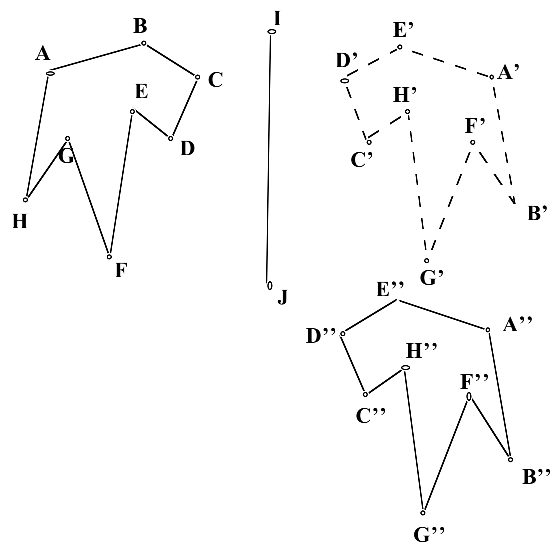


Figure 4.53 Emily's following construction of Translation * Reflection (T * F)

As we can see from Figure 4.53, she used the mirror, $\overline{IJ}$, as a vector for translation, that is, one object was used as the mirror and as the vector at the same time. So, she lost the independent relation of two transformations combined, which is the most important factor in investigating the product of two transformations. Emily's construction was not totally wrong, but it is not the best construction for further exploration. Her construction simply represented the special case when the defining data of the translation, a vector, is on the defining data of the reflection, a mirror.

Eventually she sensed that two defining data, one for each transformation, needed to be set up independently. She recalled how to assign each defining data of each transformation. Now, she finally had a construction done for further exploration [Stable Image Having] (Figure 4.54).

R: What does that look like?

E: Umm (Hesitant) Just looks like a reflection.

R: Reflection?

E: Right. Because you can draw a mirror here [pointing the middle of area between the original and final figures].

R: Right. My question is …. Is it always reflection, then?

E: Umm. (Long pause)

E: Look at the other cases.

R: How do you find other possible cases?

E: Umm …

R: What controls the figure?

E: The defining (a bit hesitant) data.

R: Right. The location of defining data.

E: Right.

R: So, ...

E: This case is ... from the defining data ….. Mirror and vector ...

R: are 90 degrees, I have a reflection.

E: O.K. If not 90 degrees, Glide Reflection?

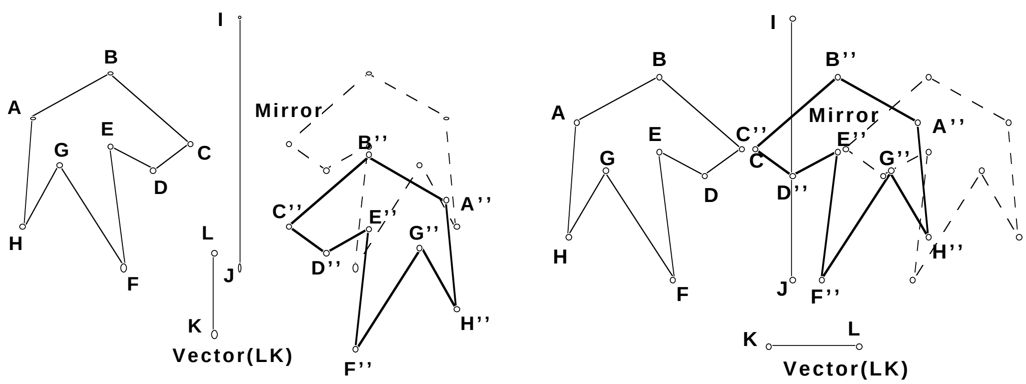


Figure 4.54 Two different cases of Translation * Reflection (T * F)

When Emily had constructed the correct images for further exploration, she moved the vector around on the screen and quickly discovered one case, which is the reflection when two defining data are perpendicular to each other. She then tried to find the other case, that is, when the two defining data are not perpendicular. She seemed to need some more experiences with this type of investigations in general. Although she knew each transformation to some decent extent, she did not know what to do with her knowledge and understanding of each transformation. She picked up very quickly, however how to investigate the product of two transformations and what to do in order to find the result of combination. Once she finished the first case and began the manipulations for a different case, it was not hard for her to find the other case, which is when two defining data are not perpendicular to each other.

R: You are saying that combination of reflection and translation is reflection when vector and mirror is 90 degrees. How do you find your new mirror?

E: Umm [long pause]. Maybe connect the corresponding points and find a bisector.

R: That's right. But, you have to find it from defining data [meaning original defining data for translation and reflection]. That's backward.

E: Right.

R: You have to find it from these [two original defining data]. O.K. we know that….. This [pointing the mid of the original and the final figures] is your mirror.

E: Right.

R: How do you find this mirror?

E: Umm.

R: This is a big task really big task. From this and this (pointing to the mirror line and vector), we have to find it out.

E: Right. [Long pause].

Finally, Emily found all possible cases with her GSP explorations and her conjectures were correct. The formal task was to find the new defining data for the product using only the givens. This is, she needs to use the defining data of the original two transformations to generate new defining data. She experimentally discovered the possible results of the product and was able to locate the new vector [Folding Back to Image Having]. She did not, however, present a method to locate new defining data from original defining data. She simply recalled what she did in the classroom. That is, she constructed new defining data from the old defining data after locating the possible new defining data using the original and final figures, which is going backwards [Property Noticing].

In the process of finding new defining data, Emily needed to go back to 'image having' stage because she did not know of a way to construct new defining data from only original defining data. When she decided to examine constructed images, she knew that the result is either a reflection or a glide reflection. She was attempting to find a way of constructing new defining data by manipulating the figures she constructed. She was unsuccessful. The following is her final report to the assignment where the result is a glide reflection.

Case i) The vector [$\overrightarrow{MN}$] for the translation and the mirror [$\overline{XY}$] for reflection are perpendicular to each other.

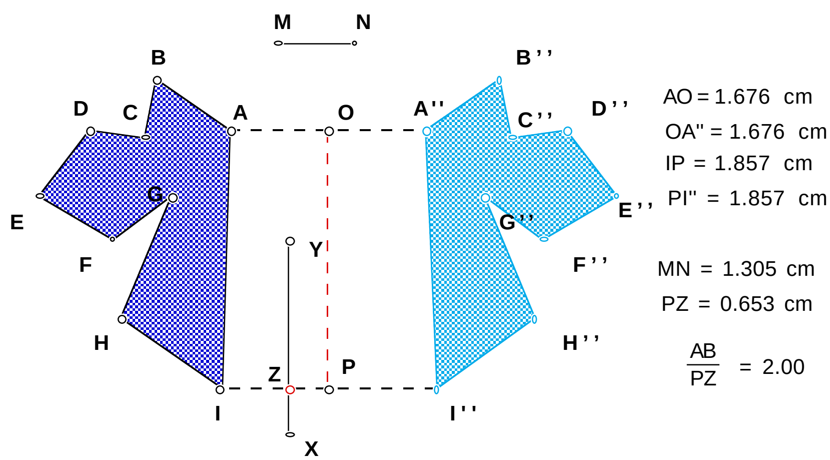


Figure 4.55 Emily's final report (case i) of Translation * Reflection (T*F)

In the sketch above [Figure 4.55], the original figure on the left [ABCDEFGHI] was reflected over the mirror (segment $\overline{XY}$) and then translated by the marked vector [$\overrightarrow{MN}$], to receive the light blue figure [the final figure- A"B"C"D"E"F"G"H"I"] on the right. I believed that the light blue figure [the final figure] was a reflection of the dark blue one [the original figure], and so I connected two pairs of corresponding points on these figures, A and A" and I and I", and constructed the perpendicular bisector of segment AA" (segment $\overline{OP}$]. Next, I checked to see if $\overline{OP}$ was also the perpendicular bisector of segment II" by measuring IP and I"P. Sure enough, they were the same length, so my conjecture was that the product in this case was a reflection. [She did not measure the angle at this point]

Also, through my experiment, I have discovered that the length of my vector used in my translation (segment MN) is twice the length of the distance from my original mirror (segment $\overline{XY}$) to the new mirror, segment $\overline{OP}$.

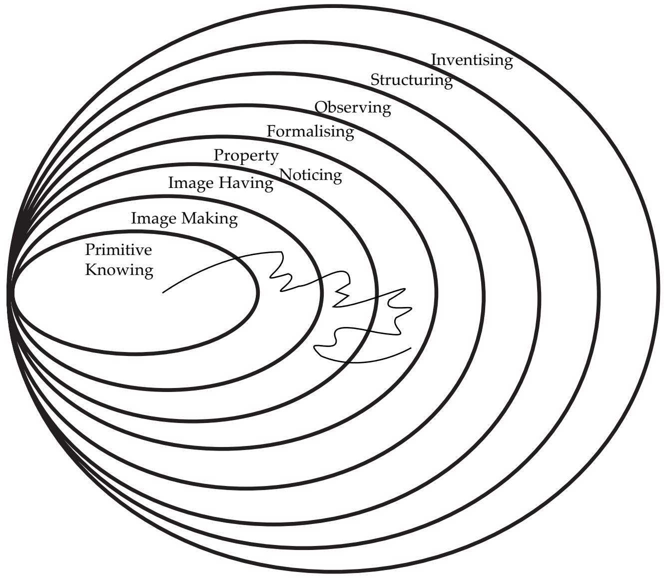


Figure 4.56 Emily's growth of understanding of Translation * Reflection

Primitive Knowing up to Property Noticing with Folding Back (1)

The following episode is what Emily did right after the product of a translation and a reflection. Emily showed more advanced skills with manipulating defining data in finding various possible cases for the result of the combination of two transformations.

(Emily creates a GSP file (Figure 4.57) representing the situation of reflection * rotation, where ABCDEFG is the original figure, A"B"C"D"E"F"G" is the final figure after rotation and reflection, the point 'O' and∠KLM are defining data for rotation, and $\overline{IJ}$ is the mirror for reflection.)

E: This first ... rotation ... and then pick a point [point O] to use this as the center of rotation. And rotate it by angle. Do you recommend [rotation by an arbitrary angle] or by a fixed angle [angle given by typing a specific number]?

R: If you use a fixed angle, ...

E: That's true. [It's hard to explore.]

R: We'd better have the free angle so that we can manipulate more freely.

E: O.K. Then, this is point for the center and … Mark the angle [∠KLM]. And then select object and rotate it by the angle. And this point and then (this angle, ∠KLM). O.K. And then this (mirror, the segment $\overline{IJ}$) is and then reflect this [the rotated object, A'B'C'D'E'F'G']. Umm. then highlight the mirror and then reflect.

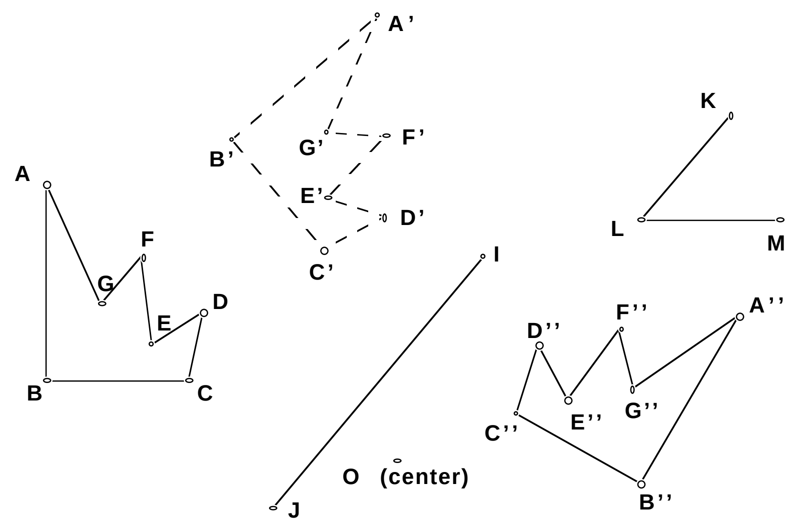


Figure 4.57 Emily's construction of Reflection * Rotation (F * R)

Emily was able to finish the construction representing reflection * rotation along with independent defining data for each transformation so that she could investigate the result dynamically on GSP [Primitive Knowing, Image Making, and Image Having]. She did the construction quickly. Her solid knowledge and understanding of a reflection and a rotation helped her to set up the given defining data for each transformation. Further, she knew the relation between those defining data, i.e., the defining data of a reflection and those of a rotation must be independent each other from the past experience with the product of translation and reflection. At this time, her understanding and knowledge of each transformation were applied to the different situation and that necessarily required the basic understanding or knowledge of each transformation.

After Emily finished the construction, she explored by dragging the defining data such as a center and an angle for rotation, and a mirror for reflection. In a moment, she was able to make a conjecture that the result is a glide reflection [Property Noticing]. While she was dragging around the objects, she also thought of the product of two transformations in terms of an orientation of the figure. She knew that orientation is reversed if a reflection and a rotation are multiplied. In the process of making her initial conjecture, she used this fact along with the dynamic function of GSP based on her construction.

> E: O.K. That's the third [the final figure]. So, I hide this [the intermediate figure, i.e. rotated figure]. All right and then I should draw on the figure maybe …. Umm …. Let me see. It looks like .. maybe a definitely ... it's reflected first ... (it's) flipped over. It's a glide reflection?

Emily's comment shows that she had reached the stage 'property noticing' and also made some connections between the basic understanding of each transformation previously constructed and the newly introduced situation requiring those basic concepts of transformations. She quickly picked what to do for the better investigation and how to manipulate her construction as well as how to construct the given situation.

She needed to find all possible cases that the product of a reflection and a rotation could make and then for each case, she needed to construct defining data using the original defining data.

> R: O.K. If you just look at these two figures [the original and the final figure], the orientation is not the same as that.
>
> E: Right.
>
> R: So, ... (what is it? Either reflection or glide reflection?)
>
> E: Reflection or Glide reflection.
>
> R: It can't be translation or it can't be rotation.
>
> E: Right.
>
> R: When does that look like a reflection (or when is this a glide reflection)?
>
> E: When the defining data ... [long pause]. When the center for rotation is on the mirror of reflection (Figure 4.58), (she answers with a bit of hesitance but at the same time shows confidence at her answer.)
>
> R: Sure, that's good. That can be one case. It is reflection when the center is on the mirror. Is that all?
>
> E: When the center is not on the mirror (Figure 4.59)?
>
> R: What is that?
>
> E: Glide reflection (she carefully checks this case by moving around the center on GSP before she says glide reflection.).

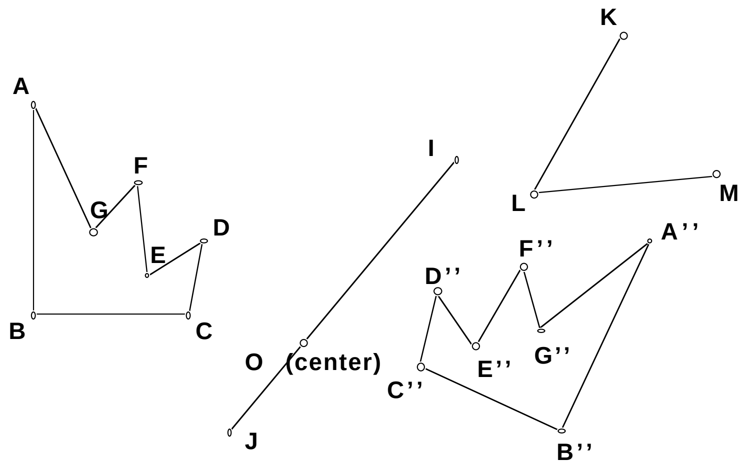


Figure 4.58 Case i) Center is on the mirror

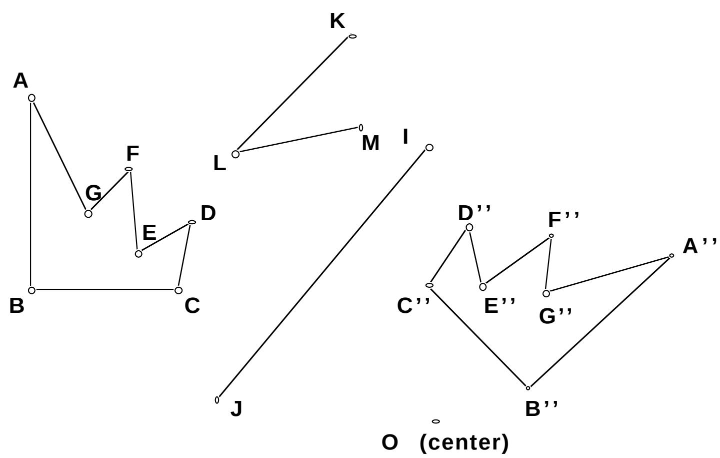


Figure 4.59 Case ii) Center is not on the mirror

Emily basically looked at the relation between the location of the center for a rotation and the mirror for a reflection. She did not mention the angle at all. Since she had not used the assigned angle for further exploration, I asked her, "Then, what is this angle for? You never used this angle for exploration." There was no response and she just kept thinking for a while. She seemed to realize why I was asking this question, but did not know how to include that angle for further investigation. She had the precious experience of dealing with two defining data and she felt comfortable with manipulating two defining data. But, she did not seem to know how to relate three defining data for checking all possible cases. The question about the angle caused her to re-examine her strategies.

R: But, we just cared about these two defining data (i.e. center and mirror). How about the angle?

E: Ah.

R: We didn't use it at all. Is this still reflection (I changed the size of angle leaving the center on the mirror)?

E: Nop.

R: Then, we're wrong. When the center is on the mirror, it is reflection. That's what you said.

E: It looks like it (correcting her answer hesitantly).

R: Are you sure? This is experimental.

E: Right. Oh Yea. I guess it's always reflection.

R: So, reflection is not affected by the size of angle.

E: O.K. Right.

R: Then, I am going to move this point out of the mirror. Is this glide reflection?

E: Uh huh (meaning yes).

R: Is this still glide reflection (the researcher changes the angle of rotation)?

E: Uh huh (meaning yes).

R: So, the size of angle does not affect at all?

E: O.K.

R: Then, we just need to care about the location of the center, if it's on the mirror or not. We have two different cases.

E: Right.

Although she had to reconstruct her explorations, the dynamic function of GSP and the researcher's tips helped her to recognize and overcome her dilemma. After this extended exploration, she exhibited ‘property noticing’ with three defining data quickly. Emily could not, however, figure out how to construct the new defining data from the original defining data. Her initial response to the question was that "It looks tough. How would you find new defining data from these (the original defining data)?" She was not able to figure that out during the interview. She could find the way experimentally for the new defining data to be constructed using the original and the final figures. Thus, she could not complete the intended assignment. The following is her report of the assignment:

Case #2: The center for the rotation is on the mirror.

A special case I found for this product of transformations is when the center for the rotation (in this case point O) is located on the mirror $\overline{IJ}$. I connected points D and D" and constructed a perpendicular bisector to this segment, $\overline{PQ}$. To check on GSP if the product truly was a reflection in this case, then I needed to check if the line $\overleftrightarrow{PQ}$ was also the perpendicular bisector of the line segment $\overline{BB"}$. My measurements show that in fact $\overline{PE}$ is equal to $\overline{PE"}$, so $\overleftrightarrow{PQ}$ would be the mirror of reflection for the original and final figures!

Here, I also discovered that the measure of the fixed angle ∠KLM, is equal to twice the angle that the original mirror used in the construction ($\overline{IJ}$) makes with the "imaginary" mirror $\overleftrightarrow{PQ}$. So, m(∠KLM) = 2(m(∠QOJ))!

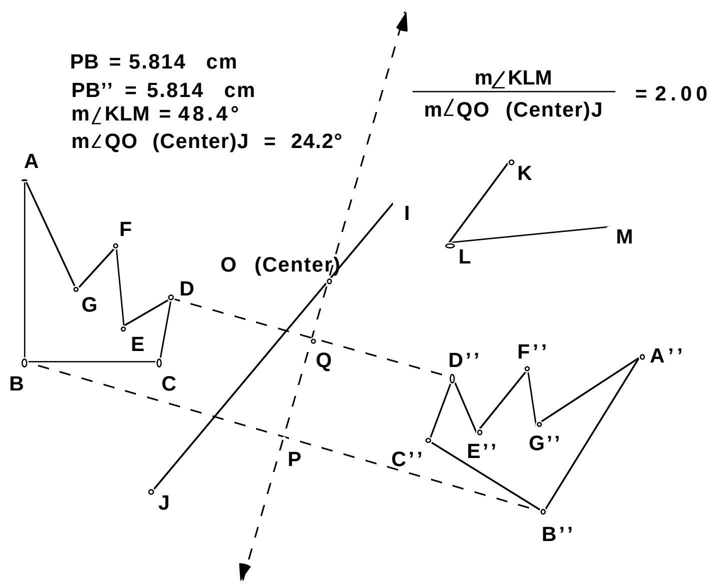


Figure 4.60 Emily's final report when center is on the mirror (F * R)

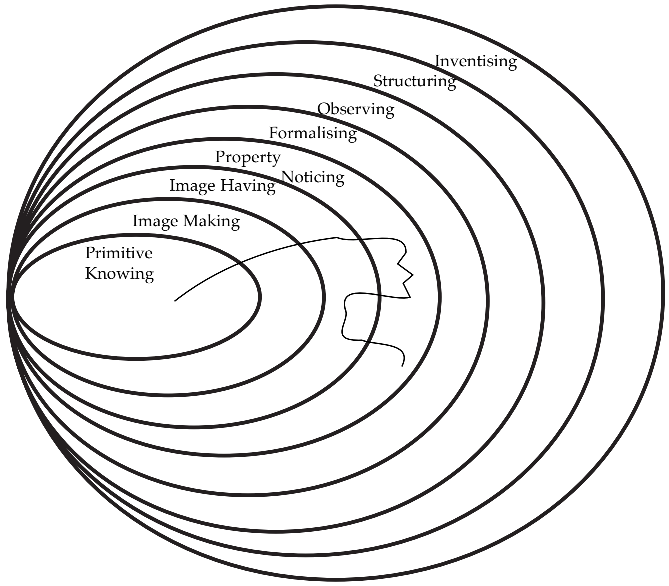


Figure 4.61 Emily's growth of understanding of reflection * rotation

Primitive Knowing up to Property Noticing with Folding Back (2)

The following episode is about the product of two transformations in a strip pattern based on the number 3. (b)[10] of assignment 5 (Appendix J). The strip pattern is an application of transformations. By restricting some conditions of each transformation for the specific situation, students were asked to analyze and do the similar investigation that they had done with transformations in a general sense. For this pattern, there are only two types of reflection: either horizontal (the mirror is the horizontal line in middle of strip pattern) or vertical reflection, the angle of rotation, so called half-turn, is always 180° where the center is on the horizontal mid line of a strip pattern; the vector of translation has only two directions; the mirror of glide reflection is the horizontal mid line of a pattern. The first five weeks were spent mostly for learning the basic concepts of transformations, theory of isometries, and affine transformations, and symmetry group including generator, etc.

[10] 3. (b) If V is a reflection with mirror **n**, and R is a half-turn with center on the line **m** perpendicular to **n**, what is the product VR? Give the defining data for VR in terms of the defining data for V and R. Prove your answer using the geometric definitions of V and R.

Students were very skillful with GSP at the time they took MATH 5210 because they spent a lot of time with GSP in MATH 5200 in the previous semester. Therefore they used GSP extensively in dealing with the further investigations of transformations. At this time, Emily felt very comfortable with transformation and was ready to move on to new topics.

R: O.K. what do want to talk about today?

E: 3 (b).

R: O.K. Was it pretty much similar to what he did in the class?

E: Yes. How it started.

R: Good.

E: Umm … this is …. number 3. (b).

R: Do you want to construct that?

E: Sure. O.K. I am going to do .... And this [center for rotation] is C.

R: Correct.

E: And then this is my original (ABCDEFGHI). Then, do 180 degree rotation and mark the center [the point, C], and click on ... (the original figure). And rotate 180. And this is [the intermediate figure, A'B'C'D'E'F'G'H'I']. Then, mark this [the mirror, *n*, for reflection, *n*] and reflect (the intermediate figure).

R: You are so professional. (Figure 4.62) You are so good.

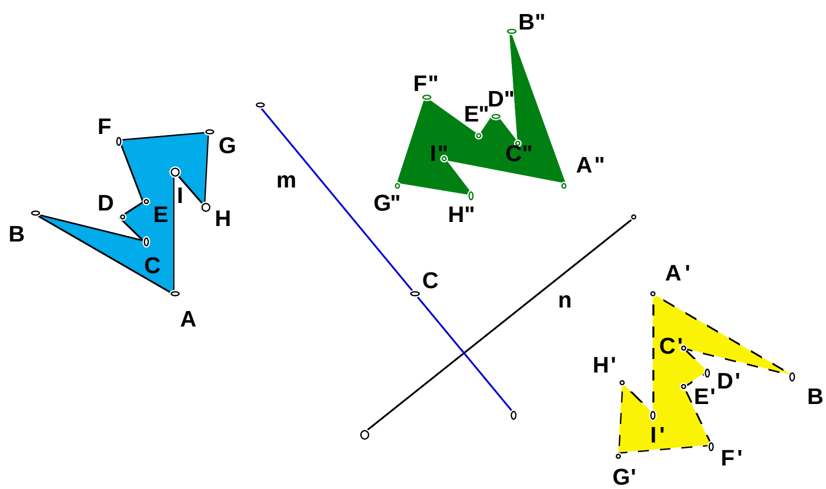


Figure 4.62 Emily's initial construction of Vertical reflection * Rotation

The construction itself of the problem was not a big task to Emily any more. Because she possessed solid understanding about each transformation and had lots of experiences with various products in homework 2, she was able to set up the given situation very quickly [Primitive Knowing, Image Making, and Image Having]. In addition, her tremendously improved skills with the use of GSP accelerated her understanding. At this time, she incorporated many short keys of GSP and this saved a lot of time for construction. Her correct construction quickly led to a conjecture that the result of 'vertical reflection * rotation' is a glide reflection [Initial Property Noticing].

In fact, as long as Emily did the right construction for the combination of two transformations, it was easy to make a conjecture by just looking at the original and the final figures. Although Emily had very solid understanding of each transformation and had enough experience of what to do for further exploration, she still did not find a way to construct new defining data from original defining data. The good news is that she discovered many facts about new defining data with her exploration. She used the GSP based on the confidence of her conjectures from looking at the first and the final figures. In this process, the dynamic function of GSP was the most important driving force for her to discover information. She did not finish constructing new defining data at the time of interview, but here is her written report:

> In the sketch (Figure 4.63), the light blue figure [ABCDEFGHI] on the left is my original figure, the yellow figure [A'B'C'D'E'F'G'H'I'] is my intermediate figure (found after rotating the light blue figure 180º about the center C) and the green figure [A"B"C"D"E"F"G"H"I"] on the right is my resulting figure. My conjecture is that the product of the two transformations is a glide reflection because the orientation is reversed and the resulting figure has been translated as well. Therefore, I would need to find a glide mirror and a glide vector as my new defining data, which are parallel to each other.
>
> I have found that the new glide mirror is the line m, perpendicular to the original mirror n, and that the length of the new glide vector ($\overrightarrow{GG''}$) is twice the distance from C to the original mirror *n*, which is m(CO). Therefore, m($\overrightarrow{GG''}$) =

2*m($\overline{CO}$). The new defining data are also parallel each other, which is required by the definition of a glide reflection.

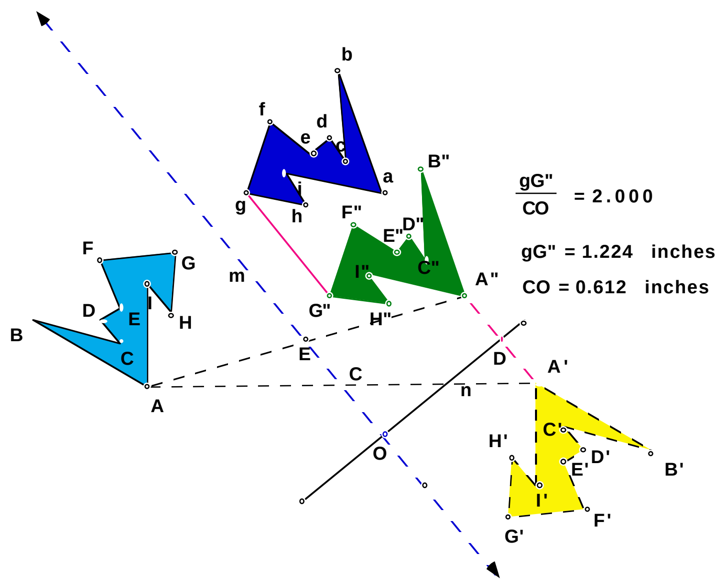


Figure 4.63 Emily's final report of Vertical Reflection * Rotation

The process of making a conjecture shows not only her stable understanding of each transformation, but also her logical sense in making a conjecture. She appropriately used each concept constructed from her efforts, experiences, and her visual ability to read the figures for the right conjecture. After she made a conjecture with confidence, she could pursue constructing new defining data. She could very easily see that her conjecture was correct.

Emily struggled with the construction of defining data from original defining data. In fact, she was unable to implement this goal of the construction. She could, however, discover the new defining data by using her skills of GSP and her knowledge of transformations. Although she could not provide the way to construct new defining data, she was very confident of the conjecture made in the very beginning and it was correct [Property Noticing].

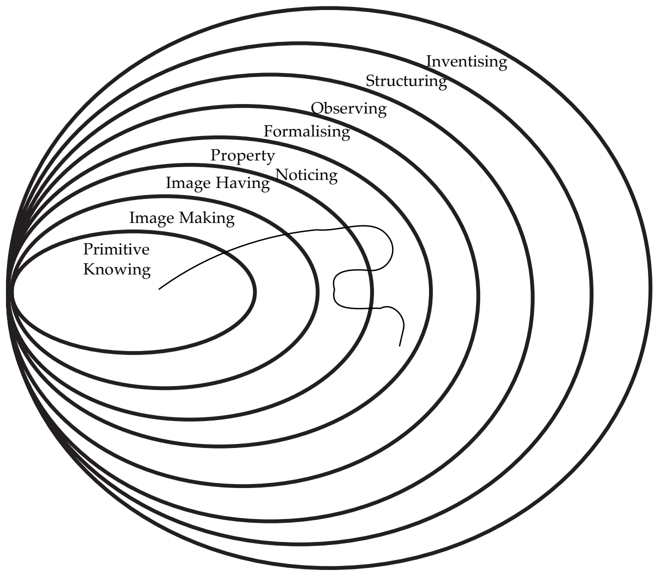


Figure 4.64 Emily's growth of understanding of Vertical Reflection * Rotation

Primitive Knowing up to Property Noticing with Folding Back (3)

The following episode is about the concept of lattice[11] of a wallpaper group. After students learned the basic concept of transformations, they studied other symmetries: a plane symmetry (wheel, strip, & wallpaper symmetry); a wheel symmetry (cyclic & dihedral symmetry), which are applications of the basic concepts of transformations. Seven different types of strip patterns were introduced and students made a multiplication table of strip symmetries. After the intensive study of strip

[11] The **lattice** of a wallpaper group is an array of points obtained from a single point by applying all of the translation symmetries in the group. More formally, let **G** be a wallpaper group and let X be any point in the plane. Then LX is the set of points X' such that there exists a translation symmetry T of the group G with X' = T(X). The lattice L_X doesn't really depend on X, but only on the symmetry group. More precisely, if Y is another starting point, then translation by the vector XY takes the lattice L_X to the lattice L_Y. So we can just talk about "the" lattice L of a wallpaper group. The two vectors V and W **generate** the lattice L if, for every pair of points A and B in L, the vector AB equals mV + nW for some integers m and n. (This is the same thing as saying that, if S is translation by V and T is translation by W, then every translation in the symmetry group of the wallpaper pattern equals T^nS^m.) There are only five types of lattices of wallpaper groups: parallelogram, rectangular, rhombic, square, and hexagonal.

patterns, wallpaper patterns was the main topic. Before the concept of lattice was introduced to students, generators[12] and fundamental regions[13] were discussed first.

E: Now for the lattice, …

R: Yeah, would you tell me about lattice that you know?

E: I know if you take any point in the pattern (Figure 4.65), then you're supposed to be able to find the combination of vector to take this to all the (inaudible) other (vectors) ... going to the same point in the pattern. So, if you picked like this yellow dot thing (the point that the arrow is point in the figure below) right here, and you need to find, the shortest (translation vector) .... I guess to get from ... like to this yellow point [the next yellow point, A, to the left], and say unto this point [the yellow center, B, to the bottom], right there.

R: Yeah. You're going to decide the lattice type using those two vectors, right?

E: Right. You only look at one point in the pattern at a time. And then you're trying to find how to get to the other same point.

R: What do you mean how to get to the other point?

E: Umm.. the translation.

R: O.K.

E: So, only it has to do with translation. And you're looking at just two vectors.

---

[12] A **generating set** for the symmetry group of a pattern is a set **S** of symmetries of the pattern such that every symmetry of the pattern is a product of transformations which are either elements of S or inverses of elements of **S**. A generating set **S** for the symmetry group of a pattern is **minimal** if no proper subset of S is a generating set.

[13] A **fundamental region** (or **unit**) for a pattern P is a region (or piece of the pattern) R such that:

(1) The whole pattern can be obtained from the fundamental region R by moving R around using the symmetries of the pattern. If R' is obtained from R by a symmetry transformation of the pattern, then R' is called an **image** of the fundamental region R.

(2) No region contained in R (except R itself) has property (1).

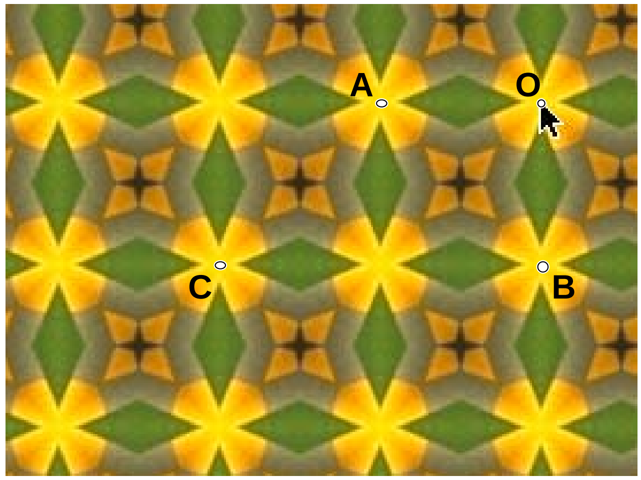


Figure 4.65 Sample image from Kaleidomania for lattice

Emily could recall a lot about lattice from the lectures by the instructor in the classroom [Primitive Knowing]. The images were easily created by the new software, Kaleidomania, which was designed for exploring symmetry. All she had to do with Kaleidomania was to select the picture provided by the Kaleidomania and type of wallpapers, then the software automatically creates the wallpaper pattern using the pictures that she chose. So, 'image making' and 'image having' were easily and quickly achieved with the help of newly introduced software. The task was to make sure two translations she chose were really appropriate for deciding lattice type and further if she could decide a lattice type based on two translations that she chose.

R: So, you're saying this vector ($\overrightarrow{OA}$), this vector ($\overrightarrow{OB}$) will take you anywhere from this point (O).

E: Right.

R: How would you go from here (O) to here (C) using those two vectors ($\overrightarrow{OA}$ and $\overrightarrow{OB}$)?

E: You use the ... (long pause).

R: I want to go from here (O) to there (C).

E: (Long pause) ... when you (are) looking at lattice, you only look at two vectors. Is that correct?

R: That's right.

E: Umm … well if you ….

R: Did you understand my question?

E: Yeah. O.K. Umm…. Well this distance is the same as this distance I think. I don't know. I think that ($\overrightarrow{OA}$)'s the same as that distance ($\overrightarrow{OB}$).

R: What makes you sure about that?

E: I don't know.

E: That's the thing that what I have the question on.

R: O.K. That's the really good part.

E: We hadn't talked about that a lot in the class. I'm starting to understand that better I think, but I still (don't exactly understand it.)

The above short conversation tells us that she simply memorized the concept of lattice from the instructor's lecture and notes. Although she possessed something about lattice, that was just acquired without understanding meaning of lattice. Maybe this might be from the lack of experience to apply this concept to the given situation. So, she could not respond to the simple question that I asked in the conversation. I believe that she knew the vector sum because she studied linear algebra about a year ago and actually we did some vector sum in the beginning of this semester. I slightly changed the figure (Figure 4.66) so that we could easily discuss the lattice in the following dialogue.

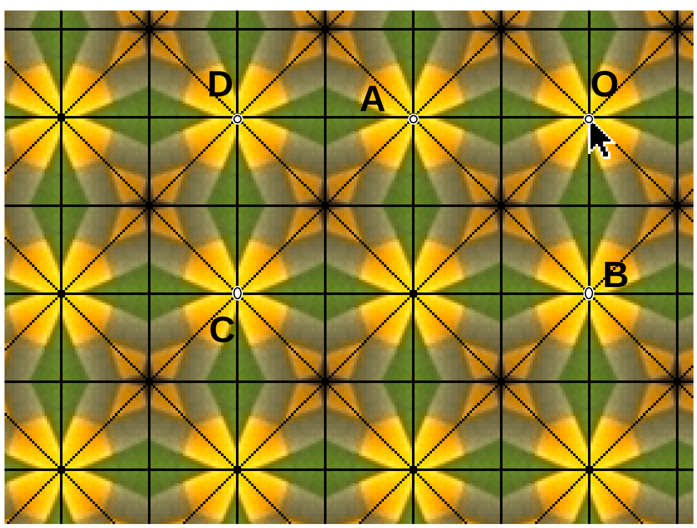


Figure 4.66 Sample image with grid from Kaleidomania for lattice

This figure helped Emily to see better how she could go from O to C using two vectors ($\overrightarrow{OA}$ & $\overrightarrow{OB}$) that she chose. All I did was to extend the vector $\overrightarrow{OA}$ to $\overrightarrow{OD}$. Then, she quickly recognized the way that the point O could move to the point C using vector sum: $2\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OD} + \overrightarrow{OB} = \overrightarrow{OC}$ [Property Noticing]. After this experience, I pointed several more points how she could move the point O to that specific point. Sometimes she quickly found the integers for combination but sometimes it took a while. Finally, she could find the pair of integers for each point that I pointed. We also tried different types of wallpaper patterns in the Kaleidomania. While she was trying some other different patterns, sometimes she had to do 'folding back' to the 'image having' because it was very hard to keep doing without having images. She now was able to show me how she could move the point on the wallpaper to the other point as well as how to select two vectors for lattice in a wallpaper pattern.

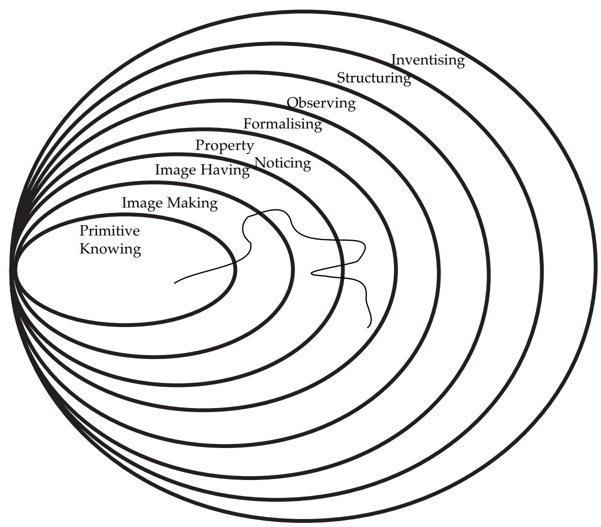


Figure 4.67 Emily's growth of understanding of the concept of lattice

CHAPTER 5

SUMMARY AND CONCLUSIONS

The primary purpose of this study was to investigate how and to what extent 'representations' affect the students' understanding and the growth of understanding in a technology [GSP]-based collegiate mathematics classroom. There are three themes in the framework of the study in support of this purpose mentioned in the first chapter and extended in the second chapter: 1) technology in mathematics education; 2) images on computer screen - visualization and representation; 3) understanding and growth of understanding.

The first theme, which is getting more and more attention in the educational field, triggered this study. Although the form of technology might be different, technology shows up in almost all areas of including mathematics education. Technology can be easily heard and reached in various situations. Especially, people have paid high attention to how they take advantage of technology. Here, technology specifically means The Geometer's Sketchpad (GSP) in this study. Because of the specific software such as GSP, students worked on images they created or those provided by the instructor for most of the time inside and outside the class. Since students spent most of their time in learning and understanding new concepts with images on computer screen, we can easily expect new environments that technology brings might affect students' understanding and a growth of understanding. I believe that if the handy and powerful tool is used wisely and efficiently, we can be more successful in what we are doing, i.e., teaching and learning mathematics with better understanding.

Technology in mathematics education

Over the past decades much investigation has examined the use of technology in mathematics education. Whereas the technology in the past was basically for simple and repeated computation, a different concept about the use of technology now dominates. For example, calculators are highly developed so that we can really explore the mathematical situation rather than simply find an answer by punching in appropriate numbers. We can graph functions, do statistics, use logarithmic and exponential functions, examine trigonometric functions, and so on, which are sometimes tedious without calculators. In addition, we can transfer the data collected from the given mathematical situation to either another calculator or computer. Thus, once students build a strong concept, they do not have to worry about the complex process of handling those data and about accuracy of their hands on calculation. This provides new environments that make it possible for students to spend their energy and time for more substantial growth of their understanding of mathematical concepts.

Along with calculators, the growth of computers and tool software both in quantity and in quality is phenomenal. There has been dramatic increase with the capacity of recent computers. According to the report, *Computers and classrooms: The status of technology in U.S. schools* (Coley, Cradler, & Engel, 1997), although there are differences among schools in their access to different kinds of educational technology, 98% of all schools own computers. The report indicates the student-to-computer ratio ranges from about 6 to 1 to 16 to 1. Eighty five percent of U.S. schools have multimedia computers, three-quarters have access to cable TV, and about one-third have video disc technology. These changes have prompted a transformation of traditional mathematics education by introducing technology into mathematics classrooms. The *Curriculum and evaluation standards for school mathematics* (NCTM, 1989) and *Professional standards for teaching mathematics* (NCTM, 1991) put high values on the appropriate use of technology in mathematics classroom. In addition, recently published documents, *Principles and standards for school mathematics* (NCTM, 2000), intensified the importance of the use of technology in mathematics classroom although we still do not see clearly how we can and why we should use technology for student's better understanding, yet. Although this recent document presented the 'technology principle' and claimed that "technology is essential in teaching and learning mathematics" (NCTM, 2000, p. 24), it is still questionable if people use technology appropriately and take advantage of technology by using it for a reasonable amount of time.

We have seen and experienced the evolution of the concept of technology in mathematics education as well as the growth of technology in quantity and in quality. Mathematics educators have adopted technologies for their researches with various perspectives: CIA (Heid & Zbiek, 1995; Lynch, Fischer, & Green, 1989), Influences (Ruthven, 1990; Thompson, 1992), Problem solving (Blume & Schoen, 1988; Hatfield & Kieren, 1972), Teacher education (Borba, 1995; Bornas, Servera, & Llabrés, 1997), Teaching and learning (Hannafin, Hill, & Land, 1997; Noss, 1988; Wright, 1997), Computer games or World wide web (Bright & Harvey, 1984; Flake, 1996; Starr, 1997). As we can see, technology shows up in many areas of research. But, at the same time, we should consider warnings that Zheng (1998) and Johnson (1997) raised for the detrimental effects to students coming from misuse of technology and our unpreparedness for the use of technology respectively.

Students in this study learned mathematics, specifically transformations, explored it, and completed assignments in environments where technology, GSP, was the main tool without a textbook that students could use whenever they felt the need. These environments required students to use technology almost all the time. They even took a test with technology in the format of take-home test in the previous semester, which is the prerequisite for this semester. Further, since they communicated through e-mails and websites, it was indispensable to them to use technology according to the design of MATH 5210 regardless if technology is essential in learning or teaching mathematics.

Since this semester was the second semester of geometry class in exactly same setting, students were comfortable with the learning environments. Especially, they felt comfortable with GSP and were satisfied with the instructor's teaching style. The only difference from the previous semester was the mathematics content.

Since students were skillful with the use of GSP at this time, the instructor posed more open questions such as 'writing a laboratory report about transformation' or 'investigations of wallpaper patterns.' He sometimes spent class time teaching students how to use GSP so that they could explore the topics with the tool. There were several assignments (Appendix J) that were very hard to finish without GSP. The instructor's well-organized preparedness and guidance to this class based on his experiences with technology and knowledge about contents and the tool brought good effects on students. Both Abbey and Emily loved geometry after this semester and they are deeply interested in the use of technology in their classroom (Tables 4.2 & 4.3).

But, both Abbey and Emily felt that GSP was not very helpful when they did not understand the mathematics behind the given mathematics tasks. Especially, Emily recognized technology as a supplementary tool: "It's good to supplement the mathematics that learned, but not used as substitution. Specially, you can't use GSP to prove things. You can only explore your conjectures." (The first interview) Students could explore more complicated mathematical situations such as the combination of a rotation and a glide reflection or wallpaper patterns, which might be very challenging to students without technology. Concepts of transformations have been taught without technology, but we are able to include topics that might frustrate students if they were asked to do them without technology. All they needed were the correct understanding of basic concepts and maybe their passion or desire to explore those related concepts in order to be ready for extension. One of the reasons the course was successful may be that students reached a decent level of skills with GSP. Thus, the instructor could spend the class time for practical activity or well organized and guided class design targeting mathematical contents, not spending time just for teaching them how to use GSP. Because there was no textbook throughout the course, the instructor seemed to have a lot of responsibility in leading the classroom.

However, as the semester went along, students became more and more dependent on those facts they saw on the computer screen without giving good reasons to make better sense of them with logical approaches. Students' overconfidence may block their creative thoughts and fool them into reaching a conclusion without mathematical support just simply using their insights. Benefits and obstacles of the use of technology will be discussed in detail later.

Images on computer screen - visualization and representation

There can be many different perspectives about the use of technology. Among all those various perspectives, I became very much interested in the effects of images on computer screen to students' understanding and the growth of understanding from three pilot studies. Students spent, maybe had to spend, most of their time with GSP in a unique situation so that they could successfully complete class activities, projects, assignments, and even tests. In this environment, they spent the majority of their time working when they were involved in learning transformations. These unique learning environments forced students to lean on technology and complexity of contents let students rely on figures on a computer screen.

The visualization of mathematical ideas has a long history (Bishop, 1989). Bishop also pointed out that visual presentation could be a powerful way to introduce the complex abstractions of mathematics. Meanwhile, several researchers (Jung, 2000b; Zazkis, Dubinsky, & Dautermann, 1996; Zimmerman & Cunningham, 1991) made their efforts to define the meaning of visualization. There are two elements in their definitions of visualization: one is the physical construction and the other is mental construction. The former means geometrical or graphical representations of mathematical concepts

(Zimmerman & Cunningham, 1991) and latter means connecting those externally presented representations to an internal construct (Zazkis, Dubinsky, & Dautermann, 1996). I viewed visualization as the external construction and the internal construction along with the connection between them. In addition, many researchers (Ben-Chaim & Lappan, 1989; Bishop, 1989, etc.) put their efforts on figuring out the role of visualization in mathematics education. Also, many researchers (Bennett, 1997; Cuoco & Goldenberg, 1997, etc.) found that the use of technology is getting more and more powerful with the dynamic function of computer software.

On the other hand, there is another notion that is similar to visualization, but that contains visualization as the main body in it, which is representation. There are three elements in it: something written on paper (symbols), physical objects (real objects), and arrangement of idea in one's mind (mental object) (Davis, Young, & McLoughlin, 1982; Janvier, 1987) (Figure 1.1). I rephrased these three terms into written representation, pictorial representation, and oral representation respectively for this study. Visualization generally focuses on external or internal constructions, whereas representation has one more element in addition to these, which is 'symbols'. Here, 'symbols' means specifically mathematical notations for definitions, concepts, or the situations having mathematical notions in it. This study shows that students could create physical objects with the tool they had and develop their mental ideas through spending time with tasks and exploring those dynamically. But, they had a hard time in presenting concepts and logical proofs using symbols. Actually the environments that they were given may have hindered them from developing their skills to present their physical and mental constructions in symbols. This shows an unbalance of the three elements in representation in a technology-based classroom.

Students were asked to create specific GSP files as part of assignments and sometimes they had to create them in order to solve the given mathematics tasks. Whatever they did throughout the course, the images they created stayed with them. Not only their images, but also their internal construction process went along with them. Their initial understanding about each transformation gradually developed since their initial laboratory reports, which was the first activity in the first class period and the first homework. The repeated application of each transformation reinforced their understanding and enhanced their abilities to apply each to appropriate tasks. But, in general, they presented their findings or proofs using figures or their own words. On the other hand, they avoided using written symbols for any situation. As a result, their pictorial and oral representations were quite advanced, and further I could tell that students made good progress with these two areas of representation. They did not use symbols unless they had to and thus they did not have enough opportunities to develop skills to use symbols as much as physical or mental constructions.

In a traditional and a typical classroom where there is not any type of technology, there would not be as many varied figures as there are in the classroom that is equipped with computers that run dynamic software such as GSP. This does not mean students in a traditional classroom are behind those who are in a technology-based classroom. Rather, it might be the opposite in the perspective of written representation. Students in a traditional classroom might have more power to imagine or construct physical or mental models based on their written representations. These students in a traditional classroom might develop more logically stronger mathematical concepts than those who are in a technology-based classroom.

Then, do we have to accept as a natural by-product that students in a technology-based classroom will develop their written representations less than those in a traditional classroom? Our goal should be for a balance of verbal, pictorial, and written expression, whether with or without technology. This is not my desire, either. In addition to what we could achieve without technology in the past, I would like to add some more which could not be achieved without it. If we are missing something just because of new environments, it must be time to reflect on what has been done so far with technology and where we should go for students' better understanding. As long as technology is being used as the tool for learning and teaching mathematics, students will have to deal with objects such as tables, graphs, or figures on a screen. The software equipped with dynamic function gave us more freedom and efficiency for investigation, i.e., we could explore the mathematics task with one construction from various perspectives and for diverse cases. Also, we can not ignore accuracy, clearness, quickness, and easiness that technology brought us. On the contrary, despite all these benefits that we can take advantage of from using technology, we should pay attention to obstacles that students might face. As this study shows, the students' insights from looking at images made substantial progress, whereas their use of written expression (symbols) to express their thoughts did not make much progress. Rather their written expression improvement to present proof may have been hindered to some extent by images they created or those provided. Further, we should recognize that incorrect choice of figures could be an another obstacle.

Understanding and growth of understanding

There are many researchers (Ausubel, 1968; Eisenhart et al., 1993; Fuson, Smith, & Lo Cicero, 1997; Hiebert & Carpenter, 1992; Janvier, 1987; Kiser, 1990; Lehrer & Franke, 1992; Stallings & Tascione, 1996; Baranes, Perry, & Stigler, 1989) who put high value on understanding in mathematics education. Understanding is one of the main themes, which appears in many areas of researches. Eisenhart et al. (1993) claimed that "teaching mathematics for understanding is one of the hallmarks of current reform efforts in mathematics teacher education" (p. 8). Further, Hiebert and Carpenter (1992) claimed that the goal of research had been to promote learning with understanding (p. 65). Although we do not have the definition of understanding agreed upon and it is much more complex to understand to understand mathematics than might appear (Byers & Herscovics, 1977, p. 24), there have been efforts to figure out the meaning of understanding. Since Richard Skemp's (1978) 'relational understanding and instrumental understanding', there were researchers who attempted to make the meaning of understanding precise (Backhouse, 1978; Buxton, 1977; Byers & Herscovics, 1977).

In 1946 Brownell and Sims addressed the nature of understanding and in 1987 Janvier suggested features of understanding. Considering the nature and the features of understanding, I was able to recognize three components of understanding which are compatible with those of representations: symbolizing, visualizing, and verbalizing (Figure 1.2). Since symbols (writing) enhance concepts of students (Hiebert and Lefevre, 1986) and visualization (external and mental images) provides a powerful role to the abstract mathematical concepts (Bishop, 1989), these three components of understanding I believe can contribute students' understanding as a group or alone.

There had been many efforts made for developing the notion of understanding, but there were few attempts made for the growth of students' understanding (Pirie, 1988; Pirie & Kieren, 1989, etc.). Pirie and Kieren (1994) created their own model, 'growth in mathematical understanding' (Figure 1.3) and I adopted this model to notice students' growth of understanding. There are eight levels in this model: primitive knowing, image making, image having, property noticing, formalising, observing, structuring, and inventising. Especially, there are two crucial features, 'don't need boundaries' and 'folding back', that make this model more powerful in the sense that this is more flexible and can be applied to any level of mathematics. Growth of understanding does not mean that learners go through every single level as they learn new concepts. Sometimes they skip a certain level, i.e. learners can operate mathematical tasks without reference to previous level (don't need boundary). Or sometimes they had to refer to the inner level in order to extend one's current or inadequate understanding (folding back). Pirie and Kieren suggested that a learner can fold back at any level and maybe folding back is necessary for an advanced, deep, and refined understanding.

Although students often say that 'now, I understand' or 'I got it', the stability of their understanding is still questionable. For example, the two students in this study were able to create figures, present each transformation using their own words, and write what they understood although representation with symbols came last. This was even possible after the first class period and the first assignment about each transformation. However, their understanding of each transformation became shaky as they applied what they thought they understood. That is, the concept of translation was the easiest concept and actually this was one of the concepts they already knew to some extent even before this semester. But, when they were asked to find the result of combination of two translations, both Abbey and Emily had no idea what they should do to solve this problem. Even their pictorial representations for the combination could not be accomplished for a while, which was mostly successful even with short amount of time for each transformation. Here, they had to fold back to basic understanding, which is one element of the given problem.

By defining understanding as to 'make connections', Haylock (1982) claimed that a learner's understanding becomes deeper as he makes more connections. The new experience which failed to make any connection with the previous experience or another new experience, would float around and be lost at some point. Wiggins (1993) viewed understanding as the ability to use knowledge in various ways for diverse contents. Also, Hiebert and Carpenter (1992) defined understanding in terms of a network of concepts and viewed the degree of understanding by the number and the strength of connections within the network. Although it was not clear how to count the number of connections and measure the strength of connections, Hiebert and Carpenter's definition of understanding is persuasive for constructing a global organization for mathematical concepts. It is desirable for continual efforts to be made for understanding of understanding.

I believe that understanding is one of the most important topics in mathematics teaching and learning, and also in the research community. Further I believe that understanding is embedded behind many areas of research in mathematics education. Although it is still questionable if technology plays a crucial or a supplementary role for students' better understanding as we have expected in mathematics classroom, at least we have new tools that did not exist before. This new appearance brought us the possibility to appropriately combine these tools with what we have implemented in the past. I believe that continual efforts like this will make a difference as long as understanding and the growth of understanding can be closely examined.

We have a new tool, in this case GSP, which was not used for teaching and learning mathematics at least a decade ago. With this new tool, we can expect different effects on students' understanding because students will learn mathematics in different environments. Then, what can make a difference? To what extent, does the difference occur? Is it always beneficial to introduce new technology into our classroom? The following three research questions guided this study: 1) How do students present each component of representations when they study 'transformations' in a technology [GSP]-based classroom? If there is any difference between the first and second presentation for each component, how are they different?; 2) How and to what extent do representations affect the students' understanding and the growth of understanding in a technology [GSP]-based classroom?; 3) What types of benefits and obstacles are there when students study 'transformations' in a technology [GSP]-based classroom?

Methodology

Two college students, Abbey and Emily, were purposefully chosen and voluntarily participated for this study. They had successfully finished the first college geometry course, were able to express what they were doing, and were typical age as college juniors. Both of them liked mathematics and their attitudes were positive toward mathematics. At the time of this study, they were very skillful with the use of GSP. Qualitative methodology was used, especially the case study, which is the most powerful and appropriate method "for intensive, in depth examination" (Goetz & LeCompte, 1984, p. 46). I observed the course, MATH 5210, for 12 weeks in Spring, 2001 and each participant was interviewed nine times. All interviews were audiotaped and videtaped, further all interviews were transcribed for an intensive data analysis.

As for data collection, there were five types of methods: descriptive notes, reflective notes, archival data, interviews, and concept maps. Descriptive notes, which mainly focus on what was happening in the class, were taken during the classroom observation. Reflective notes, which contain analysis and interpretation, were taken during or after the classroom observation and sometimes after interviews. Archival data consisted of participant's notes, mid-terms, e-mails, and assignments they turned in. Each interview was conducted based on assignments and class materials. Finally, two concept maps made in the beginning and at the end of the study were compared for each participant. The collected data were analyzed using 'constant comparison method' along with analytic induction.

Findings and conclusions

Two diagnostic tests (one in the beginning and the other at the end of this study) were given to each participant in order to compare their representations presented in three different ways: written, pictorial, and verbal. Abbey and Emily did improve in presenting their understanding of transformations: Translation, Rotation, Reflection, and Glide Reflection. They had only two class periods of experiences with transformations in the previous semester. They were only introduced to the basic idea of transformations, but had never had a chance to explore them in depth. Thus, at the time of first interview, they could barely recall what they learned about transformations. I would like to briefly review their overall representations (written, pictorial, and verbal) found in the first and second diagnostic tests before we look at each research question, and concept maps. The overall goal of these two courses that the instructor was thinking was for the student to develop geometric understanding that is both visual (experimental) and logical (based on definitions and proofs). Further, an important aspect of such understanding is the ability to explain geometric concepts using diagrams and words. Students were able to present their understanding and knowledge about transformations, symmetry, wallpaper patterns, and platonic solids with the help of technology. Especially, they used visual objects such as images on computer screen constructed by GSP or Kaleidomania to explain what they understood.

As for written representation, Abbey and Emily could not present concepts of transformations using any symbol in the first diagnostic test even though they knew something about transformations. They could not find a way to present what they knew except in their own words. Further, they had never heard of a glide reflection before, so they could not do anything with a glide reflection. However, in the second diagnostic test, they presented each transformation using appropriate symbols except a glide reflection. The written representation of a rotation by Abbey was not exactly right, but she knew the systems of a rotation. She just could not recall exactly the symbolic expression. One thing that really stood out and was a very important feature in their second diagnostic test was that they included defining data for each transformation and tried to show the relations between the defining data and two figures (original and transformed figures). In addition, their representations in the second test were more simple, clear, and precise than those in the first. Written representation, however, was the last priority and rarely used unless they were asked to use it.

There were also phenomenal differences in their pictorial representations. In general, it was hard to catch the idea of each transformation by just looking at their pictorial representations in the first diagnostic test if a reader had no knowledge about transformations. The absence of labels and defining data or unclear relations between defining data and figures added ambiguity to the pictorial representations. However, both Abbey and Emily in the second test never failed to include appropriate labels and defining data for each transformation. They did not specifically present the relations between defining data and figures, but sometimes they did indicate the relations in their pictorial representation and did verbally while drawing the figures on paper. They just did not record the relations on paper.

Finally, as for verbal representations, those in the second test were more informative and clear about each transformation, whereas those in the first barely contained the outline of each transformation. Both Abbey and Emily did use terms in an appropriate place and proper time for their verbal representations in the second test. Thus, these final representations were more readable and understandable. In addition, the simple explanations relating to defining data made their final representations more clear. In the case of Emily, she highly depended on GSP during her verbal representations. In other words she did virtual transformations on GSP in her head while she was doing her verbal representation doing the first test, whereas she was very much independent from GSP when she did verbal representation in the second diagnostic test. That is, she was free from GSP and went back to the fundamental concept of each transformation in doing verbal representation.

Along with the two diagnostic tests, Abbey and Emily made concept maps in the beginning and at the end of study. Concepts first selected by the researcher and then revised by the instructor were provided so that the two participants could refer to them as they wished. They also could add any concept that they wanted to include in their concept maps. The first concept map (Figure 4.8) that Abbey made was fairly simple and there were no cross connections among each of the transformation except for the simple case between dilation and rotation. Again, there was nothing about defining data in this concept map. However, the number of lines was highly increased in the second concept maps (Figures 4.9 & 4.10) of Emily's. We can easily see that more cross connections between transformations, and defining data were attached to each transformation. In addition, Emily separated the given concepts into two big categories, transformations and symmetry, so that she could make more effective concept maps.

The first concept map (Figure 4.18) that Emily made was similar to that of Abbey's. It simply classified each transformation under transformations. Although we could not see cross connections except that between a reflection and a glide reflection, she attached proper defining data to each transformation and added appropriate labels linking concepts. The final concept map (Figure 4.19) of Emily's is extremely complicated in terms of the number of concepts used and the number of connections. Not only were there many concepts connected in each category, i.e., isometry and symmetry, but also there were two big categories connected. In general, students were able to perform what they were asked during the course with the help of technology. Now, here are the three research questions that guided this study.

1. How do students present each component of representations when they study 'transformations' in a technology [GSP]-based classroom? If there is any difference between the first and second presentation for each component, how are they different?

Written representation Abbey in general had a hard time in writing the mathematical concepts using symbols. This is not the case just for transformations. According to her, she did not have much practice to do so. In addition, the lack of logically thorough understanding of each transformation made it harder for Abbey to present each transformation in mathematical symbols. Thus, Abbey's first version of laboratory report about rotation was completely done without any symbols. She was asked to give a more abstract definition of each transformation later. Abbey liked to use GSP and felt comfortable with it. Her skills with GSP and basic understanding of each transformation led her to see each transformation systematically. She recognized the importance of defining data for each transformation, how the original figure is transformed depending on defining data, and finally how critically important it is to state clearly the relation between defining data and two figures (original and transformed). Along with her discovery, the instructor's lecture brought confidence to her about the discovery as well as helped her organize conjectures that she made. Especially, the dynamic function, measuring, and coloring of GSP were very useful.

Emily was very similar to Abbey. Her basic understanding of each transformation was not very helpful in presenting it using symbols. However, classroom discussion and the instructor's well organized and guiding lecture helped her see the big picture of transformation. She was skillful enough to do everything that she wanted to try in GSP. Creating a GSP file according to the given mathematical situation was the first priority in exploring each transformation. Then, after she explored it enough, she tried to write things out using words and symbols. In this process, she was finally able to give a more precise and abstract definition of each transformation. Her basic understanding was refined and updated with the correctly constructed mathematical situation coupled with the powerful dynamic functions of GSP.

Although both of them improved their written representations in the second diagnostic test, they did not use written representations much throughout the course. Unless they were asked to use written representations, their reports were in general explanations in words about their discoveries or figures that they constructed. In addition, whenever they had the assignments requiring logical rather than experimental approaches, they experienced difficulty in moving forward.

Pictorial representation Considering the design of the course observed for this study, i.e., there was no textbook used and computer with GSP software was the main tool for studying transformation, we can easily expect the pictorial representation to play an significantly important role throughout the course. Although Abbey and Emily's pictorial representations in the first diagnostic test were not stable, they quickly and fully grasped concepts of each transformation after a couple of classes. They could completely understand transformations through real constructions. Their skills with GSP improved their pictorial representations up to a decent level. However, the difficult task began with the applications of each transformation.

For example, students were asked to make an isometry multiplication table containing translation, rotation, reflection, and glide reflection as elements. Because the product of two transformations is a complicated assignment in almost all the cases, it is really hard to visualize the situations without looking at something such as pictures presenting the given mathematical situation. Further, the instructor asked to find all possible cases for each combination. Both Abbey and Emily heavily relied on their step-by-step constructions for each transformation and the dynamic function of GSP. It still took a while for Abbey and Emily to finish the first combination. They felt very comfortable with each transformation, but did not know how to use each transformation for figuring out the result of their combination, not to mention all the possible cases for one combination. Even with a powerful tool, GSP, and strong basic understanding of each transformation, they could not move forward because they did not understand the role of defining data in each transformation. They were not successful until they actually recognized the role of defining data in the transformations.

Once they determined a method of finding all possible cases for each combination, it took a fairly short time to find all possible results for each combination. In this process, their strong basic concepts of each transformation were very useful in supporting why the result should be that way. As they met more various cases, they came up with different strategies, especially with the help of powerful functions of GSP. They could test their conjectures on GSP easily. The well-constructed pictorial representations were dominant in their finding the combination of two transformations afterward without further additional construction for different cases.

Despite of all these positive roles of pictorial representations, Abbey had an interesting experience that confused her concepts due to her pictorial representation. Although her final construction (Figure 4.28) was correct, the process that she took for construction confused her later on. It is good to take advantage of powerful function of GSP, but students should develop their abilities to interpret the constructed situation correctly.

Verbal representation In general, verbal representations came along with pictorial representations. I found that there were two types of verbal representations. One was the direct memorization of a written version of a definition and the other was simply a verbal explanation of pictorial representations or virtual doings in their head. When Abbey and Emily simply recited the definitions that the instructor provided, their concepts were comparatively unstable. For example, Abbey simply unfolded the definition of fixed points in terms of the literal sense. She recalled the concept of a fixed point, i.e., "Fixed point is a point that doesn't move. It can't be changed," but there was weak understanding behind it. Thus, she could not answer the follow-up questions, i.e., "What do you mean by that? Could you say some more?" as well as paraphrase it using her own words. Verbal representation was kind of by-product of the other two representations: written and pictorial. But, it also works as confirmation of a concept. The newly introduced concepts were understood through written and pictorial representations and reinforced through verbal representation. After students understood a concept in their heads or/and eyes (written or/and pictorial), they confirmed their understanding by speaking out to themselves or others.

2. How and to what extent do representations affect the students' understanding and the growth of understanding in a technology [GSP]-based classroom?

Except understanding basic concepts of each transformation, it was very hard, due to the complexity of contents, for students to visualize almost all those assignments and class activities in their head. But, because of the technology, GSP, they were using at the moment, they leaned on it whenever they needed. Even for those basics, they depended on GSP for a closer look to some extent. Following the well-organized guidance of the instructor, students could build strong written and pictorial representations at least for each transformation and then the verbal representation came along afterwards. Both Abbey and Emily were able to see each transformation as a unique mathematical being which has a specific mathematical property. Especially each transformation's written representation was often used for extension or for proving their conjectures in the following assignments. In the beginning of the study, Abbey and Emily seldom used written representations. As the course progressed, the pictorial representation was dominant in every single assignment and class activity. Although they constructed fairly stable written representations of each transformation, they simply used the facts or properties of each transformation, which they memorized, for exploring more complicated cases rather than used mathematical symbols. Their memorized basic properties and visual capacities were the main tools in exploring the mathematical situations. Written representation was often avoided unless they were asked to use it.

There are two crucial features in Pirie and Kieren's model (1994, see Figure 1.3): 'don't need boundary' and 'folding back.' 'Don't need boundary' occurs when a learner can operate the given situation without reference to the previous stage, whereas 'folding back' occurs when a learner is faced with a question at any stage, that is, when one needs to *fold back* to any previous stage. This folding back can happen to anybody and anytime when it is necessary regardless of the depth of any subject. Because of the complexity of each assignment, Abbey and Emily had to always go through 'image making.' Thus, the 'don't need boundary' lying between 'image making' and 'image having' was not relevant for this study overall.

Students' growth of understanding in transformations was very much based on a) their understanding of basic concepts, b) the applicability of these basics, and c) their mathematical understanding of the given situations. In addition, skillfulness with GSP accelerated their achieving the expected level of understanding. The concepts of translation, rotation, reflection, and glide reflection were the very basic and fundamental concepts of this study. In other words, extension was made based on these basic concepts. Abbey and Emily could understand these basics well and their laboratory reports showed that they touched every important aspect of each transformation, which became a strong stepping stone for the ensuring tasks and extended topics. However, it was questionable how flexibly or appropriately those basic understandings could be applied to new situations. Although their basic understanding strongly stood alone, they had to revisit each concept in the process of applying it to new situations. Further, the lack of interpreting the new situation sometimes resulted in their setting up inappropriately, which led them nowhere after hard work. The journey of students' growth began with the harmony of these three factors.

'Don't need boundary' was not significantly featured in this study. I believe this is because of the complexity of the mathematical content. Students always had to go through 'image making' in order to reach 'image having.' In addition, they always touched 'property noticing' before they reached 'formalising' and they did not reach any stage after 'formalising.' However, they often did 'folding back' in various situations. 'Folding back' generally occurred after both Abbey and Emily made their own conjectures. They never did 'folding back' up to stage of 'image having,' which showed their strong understanding of basic concepts of each transformation and correct interpretations of mathematical situations in conjunction with their skills with GSP. With the help of the dynamic function of GSP, students made a correct conjecture for the given task, which is the stage of 'property noticing.' But, this stage became shaky when the researcher asked them why the conjecture would be correct or how they might prove it. They came up with a conjecture by simply looking at those images appearing on the computer screen, i.e., the main sources in reaching a conjecture were the basic understanding of each concept and their insight coming from pictorial representations. Even after they did 'folding back,' they did not use formal written representations (meaning systemic symbols representing a specific concept), but they just used facts or properties by talking to themselves. Whereas, pictorial representations along with pseudo-verbal representations (meaning simple verbal expressions interpreting images on the computer screen) played a major role for them to progress from the stage after folding back. Especially, the dynamic function of GSP played a critical role in helping this to happen, which allowed them to move freely any object as they wanted. Maybe, the powerful function of GSP encouraged them to rely on pictorial representations throughout the course. If paper and pencil were the main tools for this course, written representations might play a larger role in the whole process.

3. What types of benefits and obstacles are there when students study 'transformations' in a technology [GSP]-based classroom?

Obstacles First, images on computer screen restrict learners' logical thinking. Although students constructed basic concepts of each transformation in three different representations, their explorations were dominated by pictorial representations. Their conjectures were mostly based on the dynamic images on computer screen, which is fine, rather than logical approaches. But, when they were asked to give a reason to support their conjecture, they could not give any explanation and simply went back to what they did with images that they constructed as well as their voice tone became very shaky. Thus, their conjectures were very much dependent on their feelings or insights based on previous understanding of basic concepts by looking at images on screen. Their logical thinking power was tied to the dominant images and was not extended to written representation.

Second, images on computer screen might lead learners nowhere or to misconception. Stable understanding of basic concepts did not guarantee the success in exploring the mathematical situation. When students were exploring the results of combinations of two transformations, the change of orientation was the big clue in narrowing down possible answers. Especially, if they did not label vertices of the image, the only way to tell the change of orientation was to look at shapes of images selected for investigation. But, if the selected image was the symmetric figure, this made it harder for them to make a conjecture or conclusion because the change of orientation could not be seen. Further, because students' eyes and minds were so strongly stuck to images, the inappropriate choices of images easily hinder them from building stable relations between what they understood in their heads and what they saw. Rather, the inappropriate choices weakened their understanding in their heads and possibly might lead to the wrong concepts.

Benefits First, learners could make an accurate, quick, and solid conjecture with technology and images. The accuracy and quickness that technology brought are the phenomenal features that facilitated students' explorations. The issue was whether students took the right steps for construction, not whether the line is straight or square is really a square. Further, GSP allows students to delete or add any additional construction at any time as they want, and even after deletion or addition, the final result always displays as if it was newly constructed without any skid mark or something. Although it might take a while to learn how to use GSP and to be skillful with GSP, once students reached some level, they could save lots of time using appropriate function of GSP such as script or transformations menu. Because of these two features of GSP, students could quickly make a conjecture and this conjecture was almost always correct.

Second, the dynamic function of technology plays a significant role. There are many useful functions that GSP provides. In the process of exploring transformations, the dynamic function of GSP was most useful. For example, when students were exploring the result of combinations of transformations, there could be several cases for one combination. But, they could explore all possible cases with just the initial construction thanks to the dynamic function of GSP. They could clearly see what was happening as they move figures or defining data. This made it possible for them to come up with strategies in finding results and all possible cases. Even after they made a conjecture for a specific case, they could easily try it in a different location in order to check whether they still have the same phenomenon and finally could test it on GSP. All students had to do was to do a correct initial construction, then the dynamic function took the lead until they reached their goals. This dynamic function of GSP seemed to be more powerful when the situation was more complicated. But, GSP did not matter very much about the complexity of any situation.

Third, well-constructed figures lead learners to efficient problem solving and psychological relief. GSP provides students with a strong belief and comfort about accuracy as long as it was constructed according to the right process. This means that once they did a correct initial construction, students can spend substantial time for finding what they are looking for. They do not have to worry about the possibility that they might reach the wrong conjecture coming from not correctly constructed figures. This gives psychological relief about their finding. Further, they could easily test the conjecture right away that they made on GSP at least in the experimental level. Thus, this strong experimental belief could be a guide to an abstract level of proof based on thorough logical approaches.

Limitations of the study

I would like to share some limitations of this study. First of all, it is the lack of clear definition of 'understanding.' Although I agreed that understanding is one of the most important issues in mathematics education and assumed that technology should be used to enhance students' understanding, we do not yet have a clean-cut definition of understanding. Several researchers (e.g. Brownell & Sims, 1946; Janvier, 1987) suggested the nature or features of understanding. Some researchers put their efforts to make the definition of understanding precise (Hiebert & Carpenter, 1992; Wiggins, 1993). Since Skemp (1978) suggested two levels of understanding, relational and instrumental understanding, the classifications of the definition of understanding were more updated and refined (Buxton, 1977; Byers & Herscovics, 1977; Haylock, 1982; Skemp, 1987).

Second, it is the complexity of mathematical content covered during the period of observation, the scope of the mathematics content, and the adequate theory to conceptualize and operate a study on mathematics learning with technology.

The big topic in the course for this study was transformation. After students learned the basic concepts of each transformation, the instructor extended to several related topics such as isometry, the multiplication table for isometries, affine transformations, symmetry (plane, wheel, and strip), wallpaper patterns, fundamental regions, lattices, etc. The understanding of basic concepts of each transformation seemed to be fairly easy for students. Except for the first few class periods, however, it was almost always challenging to them. The scope of the mathematics content was not clearly stated for this study. It was simply termed 'transformation'. Because of the complexity of the content, the researcher had to interrupt too often so that they could move forward, and sometimes the students did not know what to do in order to solve problems. Thus, not only 'don't need boundary', one of two critical features in Pirie and Kieren's (1994) model, could not be recognized, but also students could not reach the levels after 'formalising.' Although Pirie and Kieren's model was adopted for this study and contributed to knowing students' growth of understanding, we still do not have an adequate theory to operate a study on mathematics learning with technology.

Third, there was an unbalanced domination of one of the three components of representations. I believe this phenomenon is from the design of this course, which was mainly taught using GSP without any textbook. I intentionally selected this course, MATH 5210, because I wanted to see students' use of three components of representation and how and to what extent images on computer screen affect students' growth of understanding. But, because pictorial representation was overly dominant due to the learning environment, the other two representations were relatively ignored in the process of students' investigations. The technology seemed to encourage them to rely heavily on pictorial representations in any topics covered. Especially, they almost never used written representation unless they were asked either by the researcher or the instructor. Verbal representation was embedded in the process of their exploring the mathematical situation. In my opinion, it would be better, with respect to balance of three components of representation, if this course was taught with a textbook along with the GSP technology.

As a final limitation, I should include my own bias in collecting and analyzing data. It is obvious that I conducted this study using my personal background and understanding of technology. So, for the same reason, it is quite possible for each person to interpret the situation differently and to have different perspectives. I admit my interpretations were based on my experiences and knowledge with learning mathematics with technology. I also admit that it is quite possible to miss valuable data because of my narrow focus.

Implications for future research

The perspectives may vary as we examine current mathematics classrooms relating to technology. Indeed, we need multiple perspectives. This study focused mainly on geometrical figures on computer screens constructed by the specific software, The Geometer's Sketchpad (GSP). Technology is a part of our living environment at this time; society asks students to be skillful with the use of technology; the speed of improvement of technology in quality and quantity is dramatic; and schools are more and more equipped with technology whether students appropriately use it or not. But it is still questionable if technology has contributed to enhancing students' understanding in mathematics education as much as it has to the current society, although NCTM (2002) claimed that "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24). We do not have enough information to confirm the above statement. We have hope, however, that technology can be efficiently used for enhancing students' understanding mathematics.

Teachers, parents, students, administrators, policy makers, and researchers can work together in developing new curriculum, creating new content structures, and using technology appropriately to help each student reach his or her mathematical potential.

Many researchers (Eisenhart et al., 1993; Fuson, Smith, & Lo Cicero, 1997; Hiebert & Carpenter, 1992; Janvier, 1987) put their first priority on achieving understanding in mathematics education. Some researchers (Buxton, 1977; Byers & Herscovics, 1977; Haylock, 1982; Skemp, 1978; Skemp, 1987) put their efforts to make the meaning of understanding clear. We still need to know more about understanding of understanding. Further, we need a theory or a tool to recognize students' understanding or growth of understanding. Although this study focused on some topics in geometry, technology can be used for other topics in geometry or in other subjects in mathematics. Understanding of mathematics in all of these contexts deserves study.

I believe education is also a part of the social system. Technology impacts all educational fields. As society has achieved many things for a better life from human being, technology could contribute to students' better understanding in all of their learning. The issue is not what we have in our hands, rather how we use what we have. I believe the value of technology in education will be heightened by how it is used to enhance students' understanding of concepts rather than to assist them with routine procedures. Our search for the appropriate uses of technology to enhance student learning and understanding will continue. It is learning and understanding that is our focus, not technology per se.

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APPENDICES

Appendix A. The outline of the research

Ttile: Impacts of representations on college students' understanding transformational geometry in a technology-based classroom

Project director : Inchul Jung (Mathematics Education Department)

Please read the following very carefully.

Period of the Project	1/1/2001 – 9/30/2001
Purpose of the Project	The purpose of this study is to examine the impact of representations on college students' understanding of transformational geometry in a technology-based classroom.
Procedure	1. The students participating in this study will answer the questionnaire relating to technology in a mathematics classroom and be interviewed once a week for class materials and assignment with the participant's pseudonym on it. 2. Because the course will be taught mainly with individual computer exploration, I will observe what participants are doing in the class. The interview will be based on what the participant is doing during the class and on class materials. 3. Also, the regular interviews will help participants to better understand class materials and give a reasonable number of data for Inchul Jung's study. Because there are reasonable amount of assignments, there will be no additional task activities and Inchul Jung will help participants to complete their assignments. 4. Each interview outside of class will be videotaped and audiotaped with the permission of participants.
Benefits	1. Participants do their individual computer explorations with proper help from the researcher (Inchul Jung) during the period of the research study. 2. Participants can get extra help outside the class to complete their assignments and to better understand class materials. So, the researcher will play the role of a free tutor. With the proper help of technology and the researcher, participants are expected to take advantage of the use of technology in a mathematics classroom as of now and in the future. 3. Participants are welcome to ask anything about MATH 5210 or other courses if they want to after the interview is over.
Risks	No discomforts , no risks, or stresses are foreseen.
Eligibility	Undergraduate Students of typical age and average or Above average in grade of MATH 5200.

Researcher: Inchul Jung
ijung@coe.uga.edu , 542-4045 (Office), 543-0679 (Home).

Appendix B. Human subjects consent form

Student representations and understanding of geometric transformations with technology experience

I agree to participate in the research "Impacts of representations on college students' understanding transformational geometry in a technology-based classroom" which is being conducted by Inchul Jung, a graduate student in the Mathematics Education Department of the University of Georgia, under the direction of Dr. James Wilson ((706) 542-4194). I understand that I can withdraw my consent at any time without any penalty and have the results of the participation, to the extent that they can be identified as mine, returned to me, removed from the research records, or destroyed.

I understand the following points that have been explained to me:

1) The reason for the research is to examine the impacts of representations on college students' understanding of transformational geometry in a technology-based classroom. The benefits that I may expect from this include learning how to use technology more effectively in the study of mathematics gaining, better understanding of contents, and accomplishing assignments successfully.

2) The procedures are as follows:

I will answer the questionnaire with my pseudonym on it relating to technology in a mathematics classroom and be interviewed once a week throughout the semester. Because the course will be taught mainly with individual computer exploration, the researcher (Inchul Jung) will observe what I will be doing in the class. The interview will be based on what I do in the class and class materials. Also, the regular interview will help me to understand better the class materials as well as offer a reasonable amount of data for Inchul Jung's study. Because there are a reasonable number of assignments in the course, MATH 5210, already, there will be no extra tasks for the project.

3) No discomforts or stresses are expected.

4) No risks are expected. Participation or non-participation will not effect my course grades at all.

5) Without my written permission, no one except researcher can access the video-tapes and audio-tapes containing my appearance and voice. Any publication or discussion will use pseudonyms (unless otherwise requested by me). Documents such as questionnaires will be confidential, and will not be released in any individually identifiable form without my prior consent, unless otherwise required by me. Formal interviews and tasks with technology after class will be videotaped and audiotaped. Videotapes and audiotapes will be used for this research only and will be destroyed by 2015.

6) Inchul Jung will answer any further questions about the research, now or during this study, and can be reached by telephone at (706) 542-4045 or by e-mail (ijung@coe.uga.edu).

____________________________________ ____________________________________

Signature of Researcher Date Signature of Participant Date

Please sign both copies of this form. Keep one and return the other to the investigator.

Research at the University of Georgia that involves human participants is overseen by the Institutional Review Board. Questions or problems regarding your rights as a participant should be addressed to Julia D. Alexander, M.A., Institutional Review Board, Office of the Vice President for Research, University of Georgia, 606A Boyd Graduate Studies Research Center, Athens, Georgia 30602-7411; Telephone (706) 542-6514; E-Mail Address JDA@ovpr.uga.edu.z

Appendix C. Interview protocol for the first interview

Goal: The goal of this interview protocol is to know general background of participants.

1. Would you tell me about yourself in general?
2. How long have you used The Goemeter's Sketchpad? [Follow up with - do you think you are skillful enough in using GSP?
3. Would you tell me about your experience with mathematics? [Follow up with - can you think of the moment that you like mathematics and that you do not like mathematics?]
4. Do you like mathematics? [Follow up with - If yes, why? If not, why not?]
5. What do you like best about mathematics? [Follow up with - which course in mathematics do you like best?]
6. What do you like least about mathematics? [Follow up with - which course in mathematics do you like lest?]
7. Have you ever learned 'Transformation in Geometry' before? [Follow up with - If yes, when and how long?]
8. What does it mean to know mathematics?
9. What does it mean to understand mathematics?
10. Is mathematics a creative subject? [What does it mean to be creative?]

Appendix D. Technology questionnaire[1]

Goal: The goal of this questionnaire is to know participants' general thoughts and attitudes toward technology.

<u>For each statement, choose the phrase that most closely reflects your point of view.</u>

Name: **Date:** / /

1. High school mathematics students should be allowed to use calculators

a) at all times. b) occasionally. c) only after learning procedures by hands.
d) never.

2. High school mathematics teachers should be proficient in the use of calculators

a) I strongly agree b) I agree c) I am not sure
d) I disagree e) I strongly disagree

and computers

a) I strongly agree b) I agree c) I am not sure
d) I disagree e) I strongly disagree

3. If I were teaching a high school mathematics class, I would feel comfortable using a spreadsheet

a) I strongly agree, b) I agree, c) I am not sure,
d) I disagree, e) I strongly disagree,

Geometer's Sketchpad

a) I strongly agree, b) I agree, c) I am not sure,
d) I disagree, e) I strongly disagree,

or the Internet.

a) I strongly agree. b) I agree. c) I am not sure.
d) I disagree. e) I strongly disagree.

---

[1] This questionnaire was made by Keith Leatham and modified to fit to this study by Inchul Jung.

4. I would feel comfortable taking a high school mathematics class to work in the computer lab

a) I strongly agree. b) I agree. c) I am not sure.
d) I disagree. e) I strongly disagree.

5. I can foresee taking my high school mathematics students to the computer lab

a) everyday. b) several times a week. c) every other week.
d) once a month. e) once a term. f) once in a blue moon.

6. Thinking back to my high school mathematics classes, I wish I had had access to Geometer's Sketchpad

a) I strongly agree, b) I agree, c) I am not sure,
d) I disagree, e) I strongly disagree,

spreadsheets

a) I strongly agree, b) I agree, c) I am not sure,
d) I disagree, e) I strongly disagree,

and the Internet.

a) I strongly agree. b) I agree. c) I am not sure.
d) I disagree. e) I strongly disagree.

7. I

a) always b) often c) seldom d) never

get frustrated when I am using computers.

8. I

a) always b) often c) seldom d) never

find myself helping others with technology-related questions or problems.

9. I feel like I understand mathematics better when I explore it with computer technology

a) I strongly agree, b) I agree, c) I am not sure,
d) I disagree, e) I strongly disagree,

or with graphing calculators

a) I strongly agree. b) I agree. c) I am not sure.
d) I disagree. e) I strongly disagree.

Please write your responses in the space provided.

10. Describe the ideal environment under which you would like to learn mathematics.

11. Describe the ideal environment under which you would like to teach mathematics.

12. In your opinion, what does the phrase " learning with technology" mean?

13. In your opinion, what does the phrase " teaching with technology" mean?

14. In your opinion, what does the phrase "understanding with technology" mean?

Appendix E. Diagnostic test

For the following concepts, present those using mathematical symbols and objects (pictures, tables, figures, etc.) Then, present those using your words to the examiner. You may use any type of technology other than pencil and paper if you want to.

1. Translation

2. Rotation

3. Reflection

4. Glide reflection

5. Dilation

Appendix F. Basic concepts for concept map (Transformation)

Basic Concepts

Angle
Center
Circle
Defining Data
Dilation
Fixed point
Glide Reflection
Identity
Mirror
Orientation
Ratio
Reflection
Rotation
Transformation
Translation
Vector

Examples

Standing
Line Dance
Fairy Wheelchair
Viking
Foot Prints
Finger Prints
Over Head Projector

Appendix G. Updated concepts for concept map (Transformation and Extension)

Concepts

Angle
Antipodal Symmetry
Center
Circle
Circumradius
C_n
Defining Data
Dilation
Discrete
Distance
D_n
Equivalent Symmetries
Fixed point
Fundamental Region
Generator
Glide Reflection
Identity
Isometry
Lattice
Midradius
Mirror
Orientation
Preserve
Product
Ratio
Reflection
Rotation
Strips Group
Strip Symmetries
Strips
Symmetry
Symmetry Group

Tilings
Transformation
Translation
Translation Generators
Translation Units
Turn-Reflection Symmetry
Vector
Vertices
Wallpaper
Wallpaper group
Wheel
Wheel Group

Examples

Standing
Line Dance
Fairy Wheelchair
Viking
Foot Prints
Finger Prints
Over Head Project

Appendix H. Sample transcription for each interview

20010123 **Interview 3**

Participant	Abbey	
Date	Tuesday, January 23, 2001 (2:00 p.m. - 3:00 p.m.)	
Software	The Geometer's Sketchpad	Inchul Jung

R: Researcher A: Abbey

	Counter	Protocol	Comments
1	A 002	R: Good morning? No. Good afternoon? A: Good afternoon? R: What do you wanna start today? Did you bring something today? A: Yes, I am working on Dr. McCrory's. I'm having a trouble with glide reflection. Then Doris said she talked to him after class. And he said O.K. She was like "I can get I can find a mirror like we found a mirror yesterday in class. We couldn't find exact vector. Umm, he was like "Oh, just" Then, connect these points. She was like that. That's not the vector that you can use the two figures. Oh, he said he might miss that we don't have to do ones in glide reflection because we can't find the vector. R: Ah, it makes sense. A: Yea, so we might need to …..	Abbey was having a hard time with 'Glide Reflection.' She knew what she had to find (good basic understanding about the intention of the problem) , but she could not get it. [OR] & [PR] [PK]
2	A 017	R: When I took this course, we didn't have glide reflection. I mean in the table. So, I just had no idea. That was new to me. A: Right. But, I don't know definitely… The one I was having question was the glide reflection * rotation. But I still haven't figured out which the other one .. That's also glide reflection, too. The other one I need to start like cause I just tried to get through and find each of'em. R: O.K. Open any thing that you wanna talk about.	As intended, Abbey brought a question that she could not understand. [PK]
3	A 027	A: O.K. Should we look at the ones with glide reflection or not? R: Doesn't matter. Would you start with the ones you	She seemed to know what she had to find in the

	Counter	Protocol	Comments
		successfully figured out first? A: O.K. These two. I guess. R: I think that's what we did in the class. A: Right. That is. R: Rather than that. That one. A: I don't exactly remember what it was. We worked on this a while and here. And I had get your help on that the other day too. But, this is translation and translation. And uhh you just see the result can be a translation from the first to the third, also. And you do that. You find the defining data by getting a triangle, forming a triangle. And the sum of the original two vectors is the new vector, the new defining vector.	beginning, but now she is a bit confused. [OR] , [PR] Abbey seemed to memorize what has been done in the class rather than building the conceptual understanding of finding new defining data from old defining data. [IM] , [IH]
4	A 039	R: O.K. where is the original? This blue one? A: This yellows, the blue and the green. So, yellow and green. Yea, yellow and blue are the two original ones. R: Ah, this green one is new … A: New defining data.	She knew about what she did with GSP and could explain to me what has been done by her. [IM] . [IH]
5	A 042	R: How did you find that data? A: What I did is I moved this blue vector up to here. R: O.K. A: And then I added the two vectors. And then I connected the two segments and that vector is the new vector. R: Did you try to prove this using mathematical notations or just made a conjecture that that must be the new defining data and then you came with the idea how you can construct that new defining vector? A: Right. R: You didn't try to prove it using mathematical symbols? A: No. R: That's O.K. A: We do have to prove one of these. So, I can work on that. So, I can work on proving it.	Abbey simply depended on pictures, no place for symbols to be used. [OT], [BE] ? [PR], but no [WR] Pictorial presentation is so dominant for figuring out the logical thoroughness for this case.

	Counter	Protocol	Comments
6	A 051	R: It looks very good. And let's go back to question. A: The glide one? R: Yea. A: O.K. Yes, this is hard. What I did.. this is my original figure. And then I rotated it around C using this angle. And so I got this one. This is the rotation. Then I slid it. I mean yea I translated it using L, K. And I got the pink one. And then I reflected it across this big black line and I got this one. R: Why did you reflect this? A: Cause it's glide reflection so I did reflection first and then glide. R: Oh, I see it. So, this is your final. A: Yea, this is my final and that's my original.	Abbey shows her skills with GSP, very good. [IM]. [IH] Also, could explain what she did without any problem.
7	A 060	A: Then, I was trying to test it out. So, what I did.. as I … Umm, this is my test object and I guess this is my test vector. R: What do you mean test object? A: Like just took … see if I can get one that ended up top one over here. To see what I had to do. You know, just exploring. And cause I figured it was just looking from I thought it would be a glide reflection (a bit hesitant). Cause it's a different orientation and so that either is reflection or glide reflection. And so looking at it, I didn't think it could be a just reflection. R: Good. A: So, I figured that it had to be a glide reflection. And so, to test that out, I think I used this vector and slid that one to there. R: Yea. A: And then, I umm used …. I guess I used pink line, I am not sure…. (About 5 seconds lasted.) This is my test mirror. Yea. Pink is my, what I thought, might be the mirror and I can never get this to be exactly lined up. So, I didn't know how I was supposed to do a glide reflection. That's why… that's been **Keeping on going in this manner.**	She used test object and test defining data in order to temporarily check whether her conjecture is correct or not. This is the strategy that I had never thought of. [BE] This is possible thanks to GSP and her skills with GSP. <u>See video</u> [PN] Abbey's logical approach was great and she seemed to be confident with what she did.

Appendix I. The instructor's notes on his personal website

(as of June 9, 2001:

MATH 5210: http://www.math.uga.edu/~clint/2001/5210/home.html#Notes from class.

MATH 5200: http://www.math.uga.edu/~clint/2000/5200/home.html)

Contents covered during the semester (Go to websites above for more information)

1. Transformations:
    1. Translation
    2. Rotation
    3. Reflection
    4. Glide reflection
    5. The isometry game
    6. Theory of isometries
    7. Affine transformations

2. Symmetry:
    1. Plane symmetry - an overview
    2. Wheel symmetry
    3. Strip symmetry products
    4. The midpoint theorem

3. Wallpaper patterns:
    1. Generators and fundamental regions
    2. Lattices of Wallpaper Groups
    3. Translations in wallpaper groups
    4. Classification of wallpaper patterns
    5. More lattices

4. Platonic solids:
    1. Regular polyhedra
    2. Buckyball
    3. Symmetries of the cube
    4. Circumradius of the dodecahedron
    5. Symmetries of the Platonic solids

Appendix J. Assignments of MATH 5210 - Foundations of Geometry II

(as of June 9, 2001: http://www.math.uga.edu/~clint/2001/5210/home.html#F)

Homework 1, due Thursday 1/11

Write a "lab report" on the **Transform** menu and the arrow tool in Geometer's Sketchpad. Using words and pictures, explain how to use these GSP features. What are the properties of Translations, Rotations, Reflections, and Dilations? How can you use GSP to illustrate properties of these transformations? (For example, you may want to use the commands Trace, Locus, Animate, as well as the **Construct** menu.)

Homework 2, due Tuesday 1/16

Explain how to give precise mathematical definitions of the four transformations translation, rotation, reflection, and dilation.

Follow the pattern discussed in class for translation (and include a write-up of what we did in class on translation): For each type of transformation, first explain what are the "defining data," and then explain how the defining data, together with a point X, determine the transformed point X'.

To be very specific, you should first explain how to construct X' starting from X and the defining data, using only the GSP **Construct** menu. You may want to carry out your constructions on GSP to check that they work.

Then give a more abstract definition of X' in terms of X and the defining data, as we did in class for translation. (This definition may involve special cases.) The definition should use only the objects and functions of Euclidean plane geometry: points, lines, circles, distance, angle measure. (See the axioms for plane geometry from MATH 5200.) Don't use coordinates. We'll discuss the coordinate descriptions of these transformations in a couple of weeks.

Homework 3, due Thursday 1/25

A transformation of the plane is an **isometry** if, for all points X and Y, the distance between the image points X' and Y' equals the distance between X and Y. In other words, an isometry is a transformation which preserves distance.

It's easy to see that the product (composition) of two isometries is an isometry. Soon we'll prove that every isometry is either a translation, a rotation, a reflection, or a glide reflection.

The homework assignment is to determine the multiplication table for isometries, following the pattern in class for the product of two reflections.

This homework has two levels: experimental and theoretical. The experimental level is to figure out the answers using GSP. The theoretical level is to prove the answers using the geometric definitions of the different types of isometries.

The homework has several steps (and each of the steps has an experimental and a theoretical level):
(1) Given two types of isometries, what type is their product? For example, in class we figured out that if R_1 and R_2 are reflections, then the product R_2R_1 is a rotation.

Note that GF means do F first and G second. (The order is backwards because that's the way composition of functions is written.)

(2) Are there any special cases? For example, in class we saw that if the mirrors of R_1 and R_2 are parallel, then R_2R_1 is a translation.

(3) If F and G are isometries, how is the defining data of the product GF determined by the defining data of F and the defining data of G? For example, we saw in class that if R_1 and R_2 are reflections, and the mirrors of R_1 and R_2 intersect, then the product R_2R1 is a rotation with center the intersection point of the two mirrors and angle equal to twice the angle from the mirror of R_1 to the mirror of R_2.

<u>Notes added Friday 1/19 and Tuesday 1/23:</u>
As David pointed out Thursday, this is a very big homework assignment! Here's a more manageable goal which I expect everyone in the class to complete. (But I expect the graduate students to do more.)

(a) Do the entire assignment at the experimental level, except that when the product is a glide reflection, you only have to do the case (translation)(reflection).

(b) At the theoretical level, write up the proofs for (reflection)(reflection) discussed in class this week, and then write up proofs for one of the following five additional cases: (translation)(translation), (translation)(rotation), (translation)(reflection), (rotation)(rotation), (rotation)(reflection). Although we'll discuss various indirect methods for doing these proofs Tuesday, the best way is to approach them directly, using the definitions of the different types of transformations.

Don't forget Niki's point that it's useful to look at special cases first, where there are some special relations between the defining data for the two transformations you're multiplying.

Homework 4, due Thursday 2/8.

Part I: Proof of the Classification Theorem.

The Classification Theorem: Every isometry of the plane is one of the following: the identity, a translation, a rotation, a reflection, or a glide reflection.

Write a complete proof of this theorem. You may use any facts (except the Classification Theorem itself) from the theory of isometries.

Follow the outline given in class:

Let T be an isometry of the plane, and let **F** be the set of fixed points of T.

1. If **F** is the whole plane, then T is the identity transformation.
2. If **F** is a line, then T is a reflection.
3. If **F** is a single point, then T is a rotation.
4. If **F** is empty, then T is a translation or a glide reflection.

First show that these are the only possible cases, using the uniqueness part of the Triangle Theorem. Then prove the cases in order. We discussed cases 1, 2, 3 in class.

Here is a small observation about cases 2 and 3:

As a consequence of the Triangle Theorem, to prove that T is the reflection R with mirror M the line of fixed points of T, you just have to find a **single triangle** ABC so that T(A) = R(A), T(B) = R(B), and T(C) = R(C). Let A and B be points on M, and let C (= X in class) be any point not on M. (You could even let C be a point on the perpendicular bisector of AB.) You do not have to check directly that for all points X, T(X) = R(X). This observation doesn't simplify the proof of case 2 much at all, but the same principle applies to cases 3 and 4, where it does help to simplify the proofs.

In case 3, for the same reason, to prove that T is the rotation R with angle a and center O the fixed point of T, you just have to find a single triangle ABC so that T(A) = R(A), T(B) = R(B), and T(C) = R(C). Let A be the center O, let B (= X in class) be any other point, and let C (= Y in class) be a point such that OC = OB. (You could even let C = T(B).) Let a be the angle BOB', where B' = T(B). You do not have to check directly that for all points X, T(X) = R(X).

Here's a strategy for case 4. As usual, let X' = T(X).

4a. Suppose that the isometry T has no fixed points, and suppose that for all points X and Y with X not equal to Y, the line XY is parallel to the line X'Y'. Show that T must be a translation. (Just use the geometric definition of translation.)

4b. Suppose that the isometry T has no fixed points, and suppose that there exists a pair of points X and Y with X not equal to Y, so that the line XY intersects the line X'Y'. Show that T must be a glide reflection.

Strategy for 4b: Suppose that the midpoint M of the segment XX' and the midpoint N of the segment YY' are not the same point. Let L be the line through M and N. Show that T is a glide reflection with mirror L. (If A is the foot of the perpendicular from X to L and B is the foot of the perpendicular from X' to L, then AB is the translation vector of the glide reflection.)

Part II: Coordinate formulas for isometries.

1. Using the GSP Calculate function, check the following coordinate formulas for isometries. In each case apply the given isometry to a dynamic point P and check that, as you move the point P around, the x and y coordinates of the transformed point P' are related to the x and y coordinates of P by the given formula. I did this exercise at the end of class on Tuesday1/30 for the translation with vector (a,b), for which the formula is T(x,y) = (x+a, y+b).

a. Rotation with center the origin and angle t. (t stands for theta.)

T(x,y) = (x cos t - y sin t, x sin t + y cos t)

This says that T is a linear transformation with matrix

cos t	- sin t
sin t	cos t

b. Reflection with mirror the line L through the origin such that the angle from the x-axis to L is t.

T(x,y) = (x cos 2t + y sin 2t, x sin 2t - y cos 2t)

This says that T is a linear transformation with matrix

cos 2t	sin 2t
sin 2t	- cos 2t

c. Rotation with center (a,b) and angle t.

T(x,y) = ((x-a)cos t - (y-b)sin t + a, (x-a)sin t + (y-b)cos t + b)

d. Reflection with mirror the line L through the point (a,b) such that the angle from the x-axis to L is t.

T(x,y) = ((x-a)cos 2t + (y-b)sin 2t + a, (x-a)sin 2t - (y-b)cos 2t + b)

2. Find a coordinate formula for the glide reflection with mirror a line through the point (a,b) and glide vector (c,d). (Remember that the glide vector is parallel to the mirror.)

3. Prove using trigonometry and the geometric definition of reflection: Let T be the reflection whose mirror is the line L through the origin such that the angle from the x-axis to L is t. Then T(1,0) = (cos 2t, sin 2t), and T(0,1) = (sin 2t, -cos 2t).

Homework 5, due Thursday 2/15

1. Let F(X) = MX + B be an affine linear transformation which is an isometry, i.e. M is an orthogonal matrix.

(a) Suppose that F is a rotation, i.e. det(M) = 1. Find the coordinates of the center of this rotation, in terms of the entries of the matrix M and the coordinates of the vector B. *Strategy*: X is the center of the rotation if and only if X is a fixed point of F, i.e. F(X) = X. So you have to solve the equation

$$X = MX + B$$

for X. This is a system of two linear equations in two unknowns. Check your answer using GSP.

(b) Suppose that F is a reflection, i.e. det(M) = -1 and -B is in the column space of the matrix (M - I). Find an equation for the mirror of this reflection, in terms of the entries of the matrix M and the coordinates of the vector B. In analogy with part (a), use the fact that the mirror line is the set of fixed points of F. (In solving this problem you will see where the condition that -B is in the column space of (M - I) comes from.) Check your answer using GSP.

2. The symmetry group D5 (or (*5) in Conway notation) has ten elements. Write down the multiplication table for this symmetry group.

Convention for multiplication tables: If the ith row of the table represents the symmetry F and the jth column represents the symmetry G, then the entry in the ith row and jth column represents the product FG (do G first and then F).

3. (a) If G is a glide reflection with mirror **m** and glide vector **v**, and R is a half-turn (180 degree rotation) with center **c** on **m**, what is the product GR? Give the defining data for GR in terms of the defining data for G and R. Prove your answer using the geometric definitions of G and R.

(b) If V is a reflection with mirror **n**, and R is a half-turn with center on the line **m** perpendicular to **n**, what is the product VR? Give the defining data for VR in terms of the defining data for V and R. Prove your answer using the geometric definitions of V and R.

Homework 6, due Thursday 3/15

Investigations of wallpaper patterns using Kaleidomania

Use Conway's notation for these problems. (In the Kaleidomania menu **Extra**, check **Use Conway's Notation**.)

These are open-ended discussion questions! If you can't answer a question completely, then say as much as you can. If you discover other things, or if you find questions you can't answer, you should report on them.

Class lab time will be provided on Thursday, March 1, and Tuesday, March 13, to work on these problems. As usual, group work is encouraged!

A. Report on the following aspects of each of the 17 wallpaper patterns:

1. Describe the generating set and the fundamental region used by Kaleidomania to make the pattern. Describe step-by-step how the pattern is generated by Kaleidomania.

2. List the wheel patterns which occur in the wallpaper pattern.

3. Determine which type of lattice occurs in the wallpaper pattern.

4. Discuss the relations between the answers to questions 1, 2, and 3. For example, do certain wheel patterns occur only with certain lattices?

B. Explain Conway's notation! How does it relate to the symmetries which occur in the pattern? Hint: Look at the symbols used by Kaleidomania to decorate the boundary of the fundamental region. How do these symbols relate to the symmetries of the pattern?

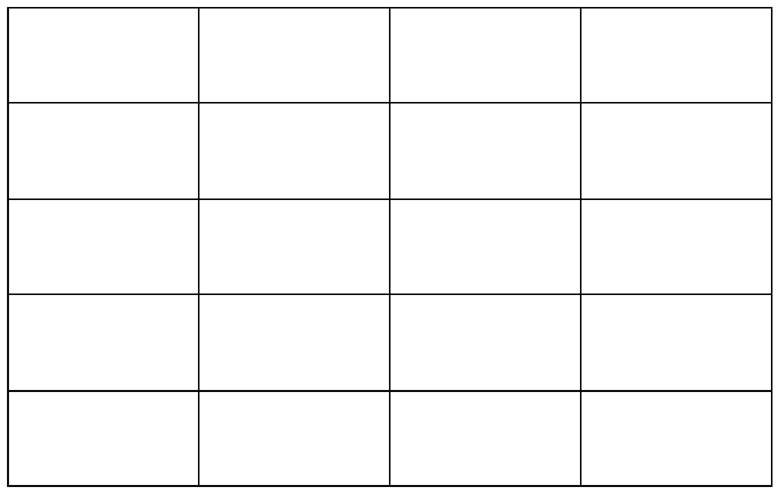
Homework 7, due Tuesday 3/27

Classification of wallpaper patterns

For each of the following 15 wallpaper patterns, determine the Conway symbol and the lattice type of the pattern. Draw a fundamental region on the pattern, and show (using Kaleidomania's symbols) all of the rotation centers and reflection lines which occur on the boundary of the fundamental region.

In the first six "brick patterns," each rectangular "brick" has base 2 and height 1 (or base 1 and height 2).

1.



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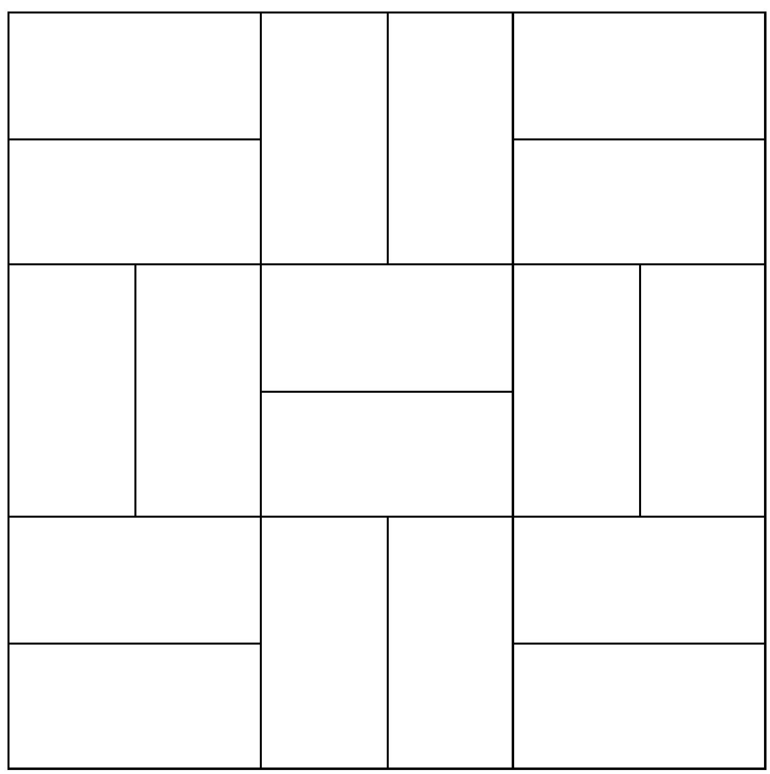

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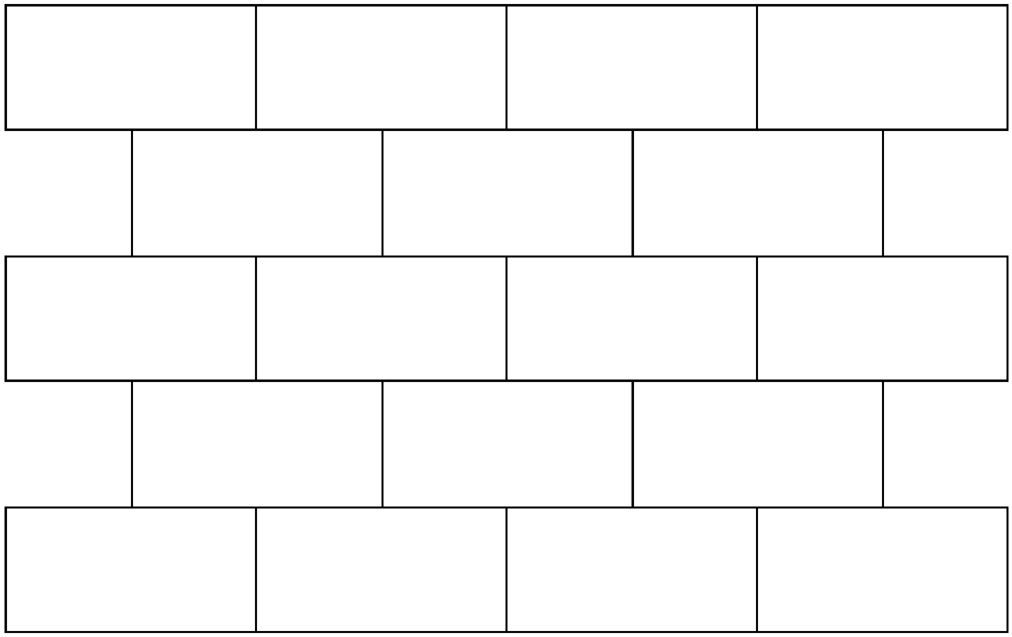

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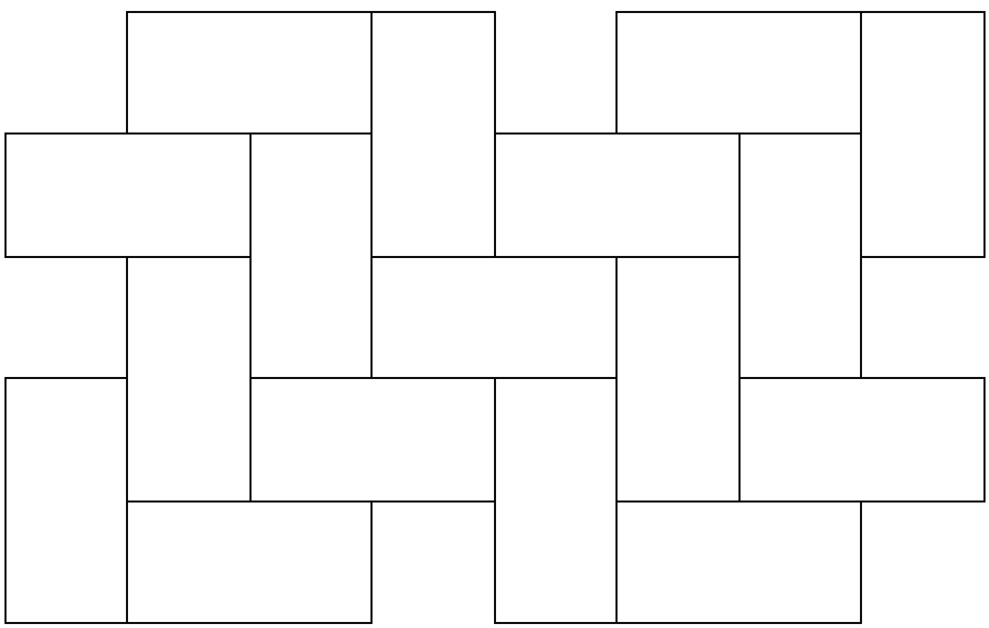

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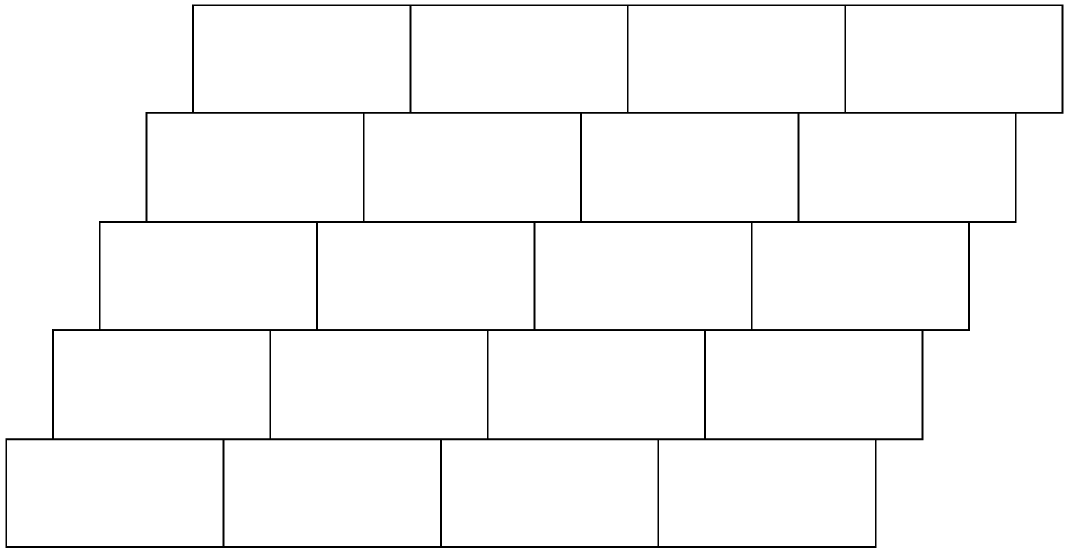

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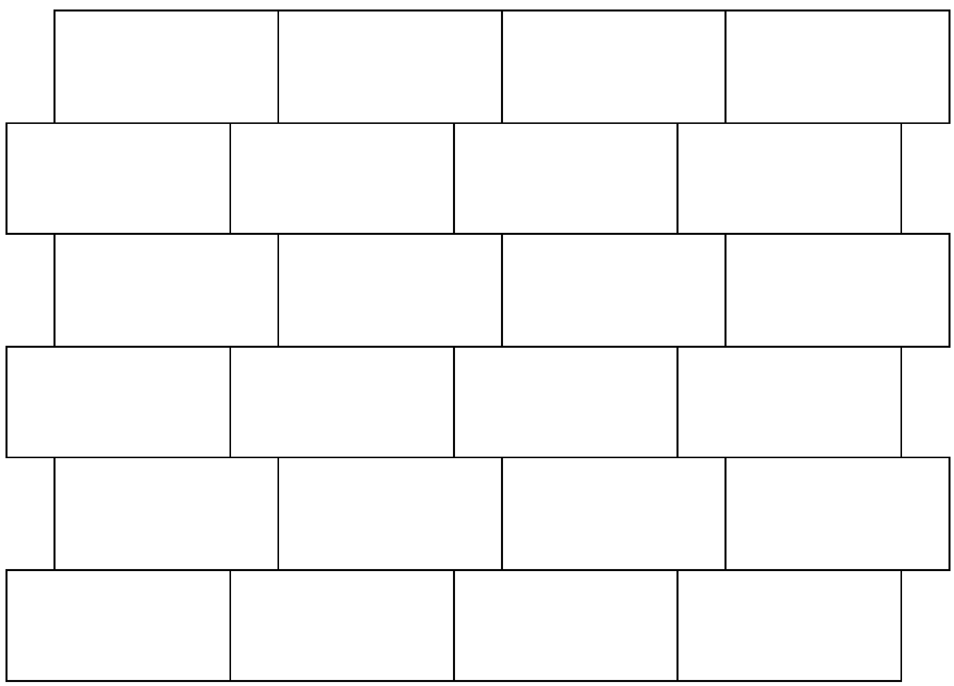

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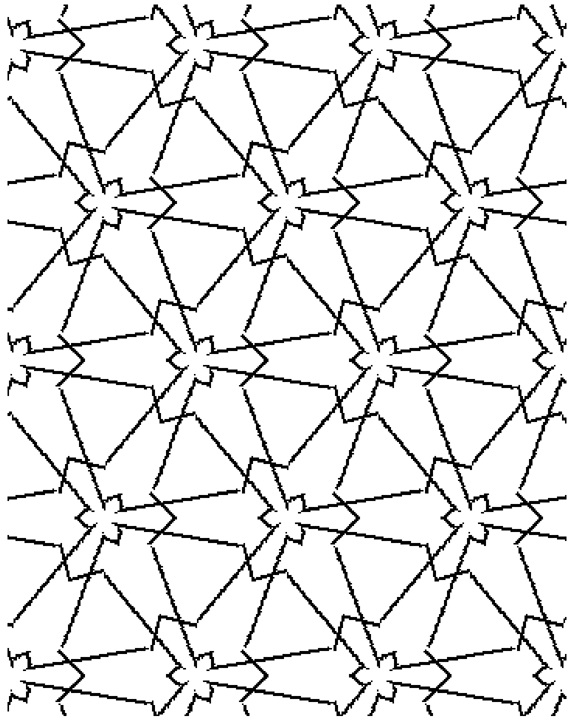

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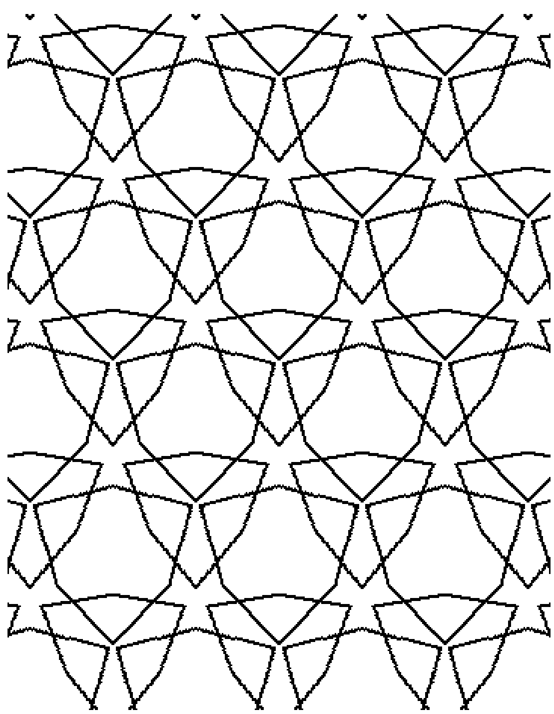

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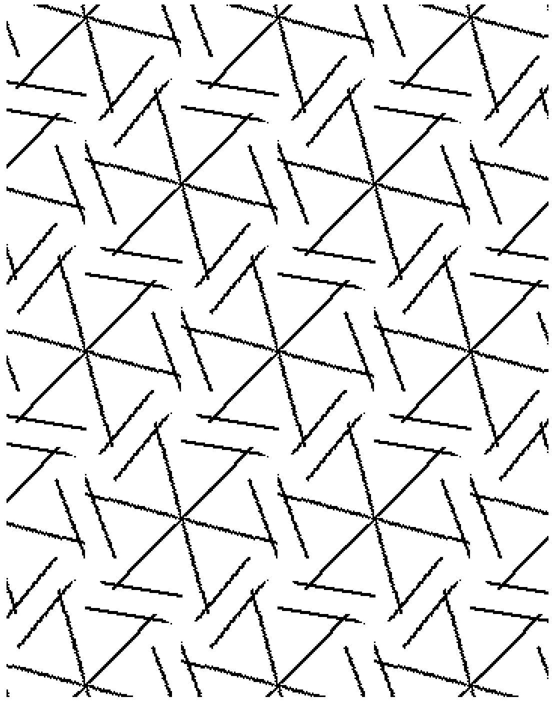

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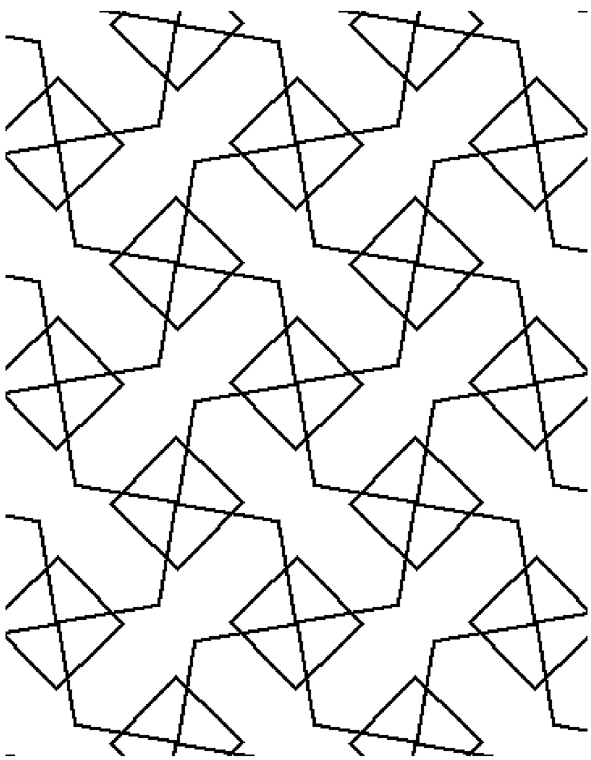

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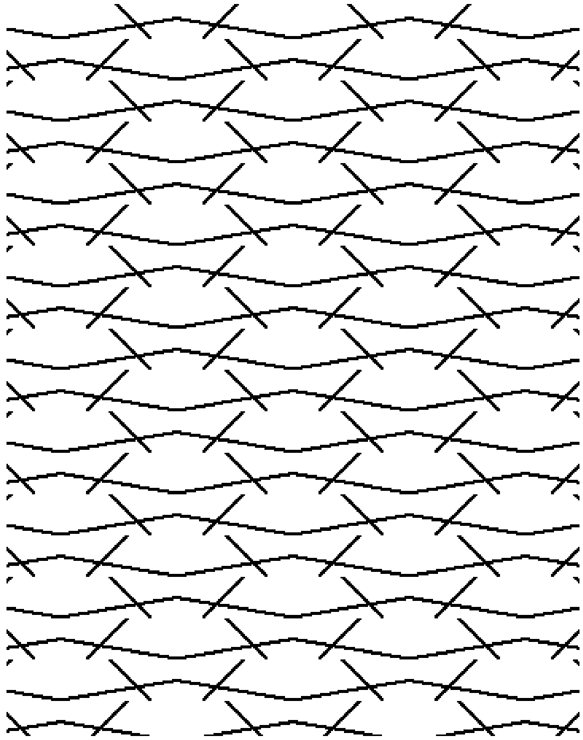

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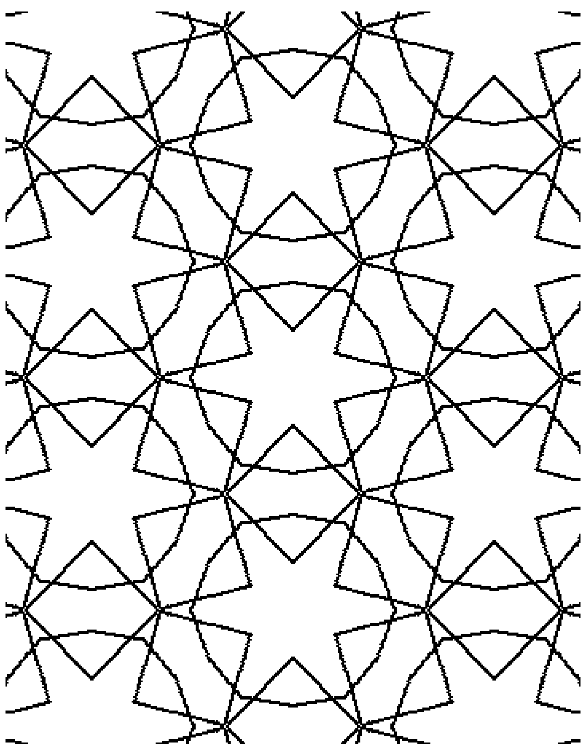

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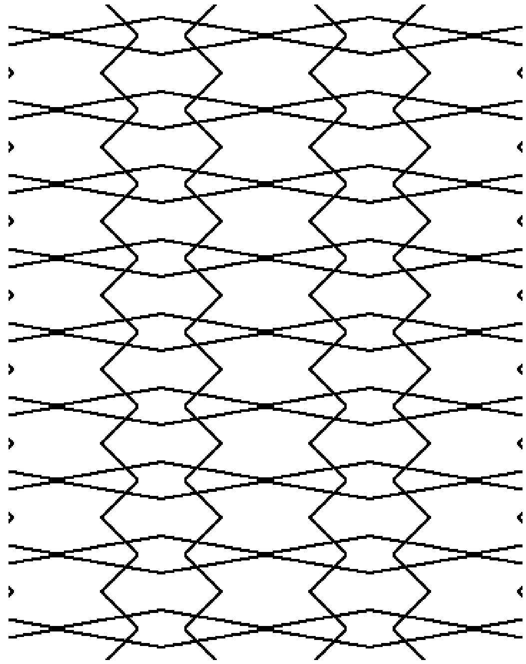

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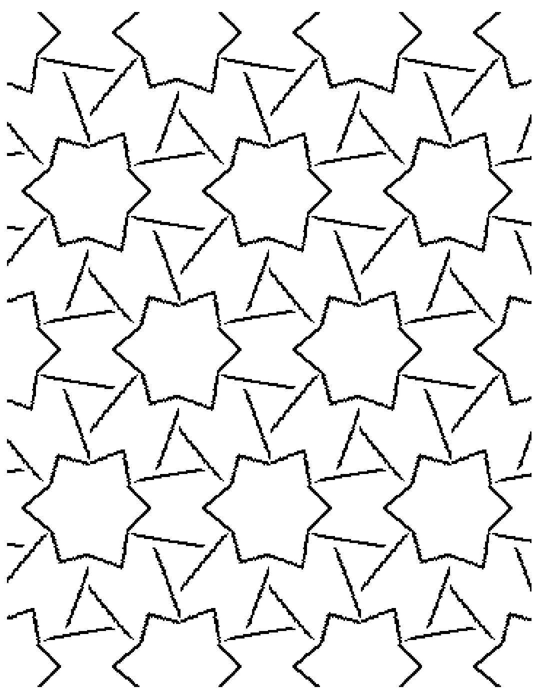

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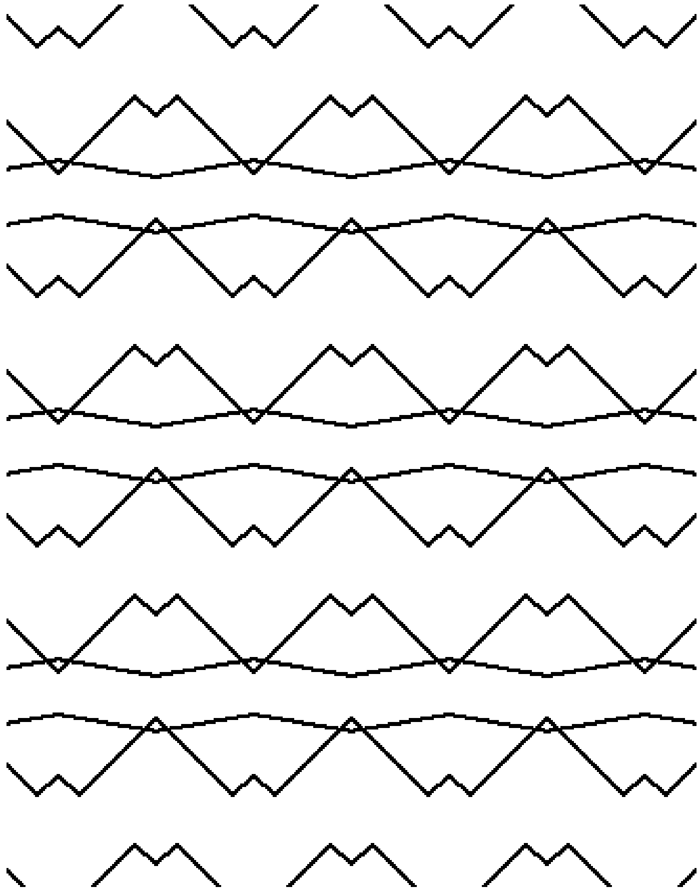


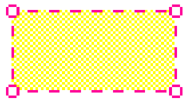
Homework 7, problem 1 solution and explanation

First find all of the symmetries of the pattern. You cannot find a fundamental region until you have found all of the symmetries, or at least a set of generators of the symmetry group.

The reflection mirrors are all of the red dotted lines. There are half-turn centers (wheel type *2) at each of the intersection points of the horizontal and vertical red dotted lines. A simple choice of translation generators is drawn in blue. For every reflection mirror there are also glide reflections with the same mirror, with glide vectors all multiples of the blue translation vector which is parallel to the mirror.

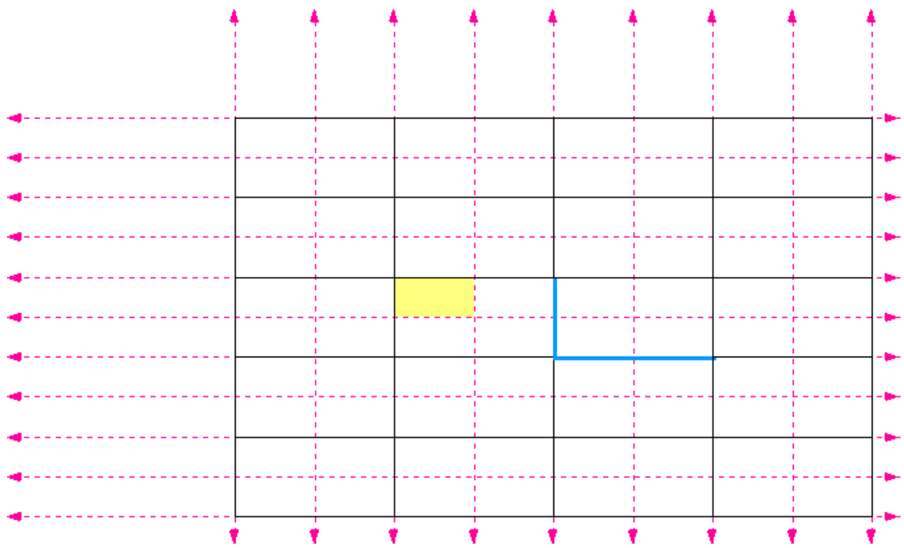
The fundamental region is, by definition, the smallest region which can be used to cover the whole plane by moving it around with the symmetry transformations of the pattern. For this pattern the simplest choice of fundamental region is the small yellow rectangle.

If dotted lines denote reflection mirrors and little circles denote half-turn centers, then the Kaleidomania drawing of the fundamental region is this:



The symmetry group has Conway symbol *2222.

The lattice is rectangular but not square (type B).



Appendix K. MATH 5210 Examination 1

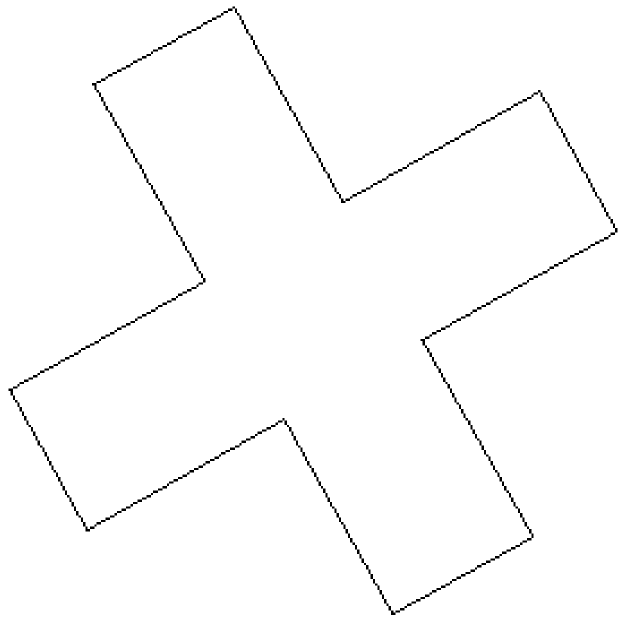
Thursday, February 22, 2001

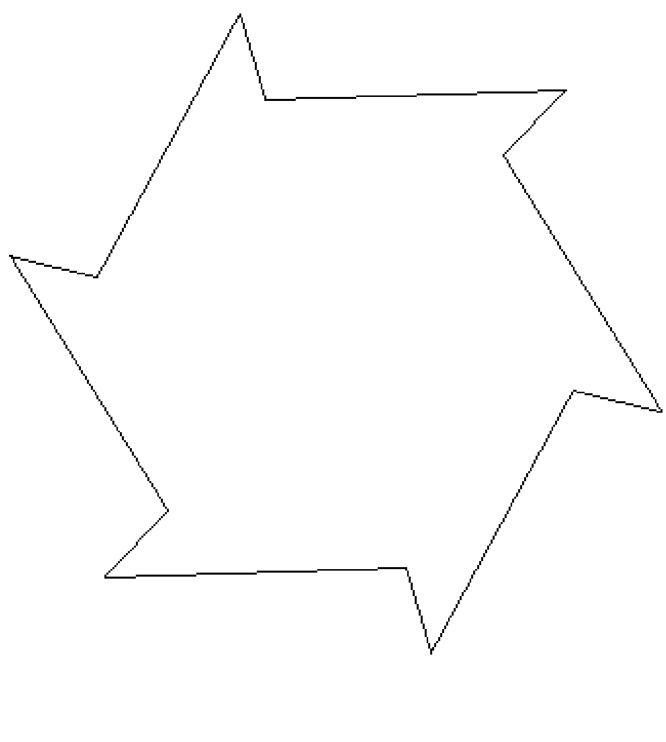
1. (a) Give the geometric definitions of half-turn (rotation by 180 degrees) and reflection.
   (b) If the center of the half-turn R is on the mirror of the reflection F, what is the product FR? Prove your answer.

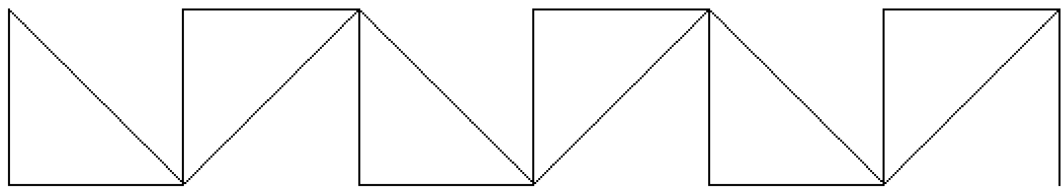
2. State the Triangle Theorem for isometries.

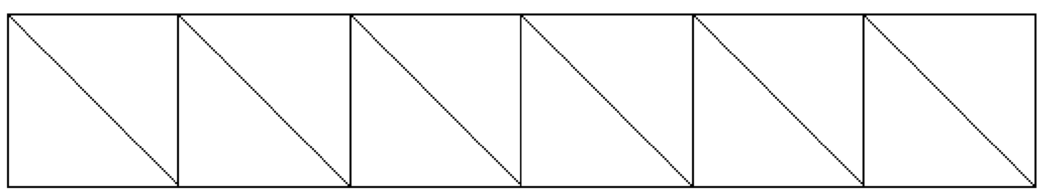
3. Suppose that the isometry F is defined as the product of the following three isometries:
   First, a rotation with center the origin and angle 30 degrees.
   Second, a translation with vector (3, -2).
   Third, a reflection with mirror through the origin and such that the angle from the x-axis to the mirror is 30 degrees.
   Give a coordinate formula for the isometry F. In other words, if X = (x, y) and F(X) = X' = (x', y'), write x' and y' in terms of x and y.

4. Describe all of the symmetries of the wheel patterns and strip patterns on pages 2 and 3.









Appendix L. MATH 5210 Examination 2

Thursday, April 12, 2001

1. Define the following terms:
   (a) a generating set for a wallpaper pattern.
   (b) a fundamental region for a wallpaper pattern.
   (c) the lattice of a wallpaper pattern.

2. For each of the four wallpaper patterns on pages 2, 3, and 4:
   (a) describe all of the symmetries of the pattern.
   (b) draw a fundamental region on the pattern.
   (c) determine the Conway symbol of the pattern.

3. State the definition of a regular polyhedron.

4. Describe all the symmetries of a tetrahedron.