

QUANTITIES AND COVARIATION: AN INQUIRY INTO THE REASONING OF EXPERTS
ENGAGED IN GRAPHICALLY REPRESENTING DYNAMIC SITUATIONS

by

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(Under the Direction of Kevin C. Moore)

ABSTRACT

Past researchers have illustrated students' and teachers' difficulties reasoning covariationally and have also claimed certain ways of thinking that are propitious for such reasoning abilities. In this thesis, I investigate two experts' reasoning when tasked with drawing graphs that relate two varying quantities. By comparing each experts' activities I argue that constructing and coordinating amounts of change and accumulations thereof afforded these experts the ability to interpret and graphically represent a dynamic situation covariationally. I present evidence that in some (but not all) cases these experts engaged in these ways of reasoning. I end by describing other strategies used by the experts while attempting to reason covariationally that I hypothesize may present barriers for one to successfully engage in such reasoning.

INDEX WORDS: Graphing, Quantitative Reasoning, Covariational Reasoning,
Multiplicative Object

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BS, University of Washington, 2012

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A Thesis Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment
of the Requirements for the Degree

MASTERS OF ARTS

ATHENS, GEORGIA

2017

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August 2017

ACKNOWLEDGEMENTS

This thesis would not have been possible without the support of many people. I am thankful to my adviser, Kevin C. Moore, who has patiently and thoughtfully provided me feedback and engaged with me in rich discussions. I am thankful to my research team in the math education department at UGA, especially Irma Stevens and Biyao Liang. I am also thankful to Amy Ellis and Andrew Izsák for their feedback on this work. Many other mathematics education researchers have influenced and inspired me during this project, and I would particularly like to thank Elise Lockwood, Marilyn Carlson, Pat Thompson, and Kristin Frank.

I would also like to thank Rob Kulow for editing this work. I am beyond grateful for his detailed attention to this thesis all while seeming happy to read it.

Finally, I would like to acknowledge my wonderful family and support system. Mom, Dad, Manda, Bro, Jac, Jakers, Emma, Torrey, Yasin, and Dilara, your love and acceptance provide me with an invisible shield against frustrations I hold from the world and myself. You provide me with a space to be loved, accepted, and happy. And for the record, a *buttload* involves making a quantitative comparison and is used to infer a large number *or* amount, as in “You ordered a *buttload* of nuggets!”

This material is based upon work supported by the National Science Foundation under Grant No. DRL-1350342. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

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CHAPTER 1

INTRODUCTION

Constructing, representing, and interpreting relationships between quantities are prominent mathematical activities one engages in from the early grades to post-collegiate studies. These activities require covariational reasoning. Covariational reasoning describes how someone might conceive a relationship between quantities that change in tandem. Many researchers have illustrated the importance of covariational reasoning to students' and teachers' engagement in and development of many concepts in mathematics. Such concepts include understanding families of relationships (e.g., linear, quadratic, trigonometric, exponential), modeling and reasoning about dynamic situations, and developing mathematical ideas in calculus (e.g., limits, rates of change, differential equations) (Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Moore, 2010; Moore & Carlson, 2012; Moore & Thompson, 2015; Strom, 2008).

While researchers have repeatedly illustrated the importance of covariational reasoning in the study of mathematics, these researchers have also argued that students and teachers have not been given enough opportunities to develop sophisticated covariational reasoning abilities (Carlson, Larsen, & Lesh, 2003; Carlson & Oehrtman, 2004; Oehrtman et al., 2008; P. W. Thompson & Carlson, 2017). In regards to (co)variation and graphs, researchers have discussed how students have a tendency to think about graphs as a finite collection of discrete points (regardless if the displayed graph entails a continuous trace) (Bell & Janvier, 1981; Goldenberg, Lewis, & O'Keefe, 1992; Stein, Baxter, & Leinhardt, 1990), calling into question students' propensities and capacities to conceive a trace of a graph in terms of continuous covariation.

Others have found that students encounter difficulty attending to variations in multiple attributes or quantities in a situation or graph (Moore, 2012; Saldanha & Thompson, 1998; Stalvey & Vidakovic, 2015). Such difficulties signal to researchers the need for further exploration into how students think about quantities and how students might come to construct and reason productively with two simultaneously varying quantities.

Researchers have been developing covariational reasoning as a theoretical construct for the past thirty years (P. W. Thompson & Carlson, 2017). During this time, these researchers have proposed and refined definitions and theories in order to communicate ideas of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1994; Saldanha & Thompson, 1998; A. G. Thompson & Thompson, 1996; P. W. Thompson, 1990a, 1994a, 1994b, 2011). The results of such work include frameworks of mental actions (Carlson et al., 2002; P. W. Thompson & Carlson, 2017), refined characterizations of observed behavior (Castillo-Garsow, 2010, 2012), and examples of students' and children's reasoning about dynamic quantities (Ellis et al., 2015; Johnson, 2012a, 2012b; Lobato, Rhodehamel, & Hohensee, 2012; Stalvey & Vidakovic, 2015). Researchers have also shown the importance and necessity of certain cognitive constructs for an individual to reason covariationally. Particularly, Thompson and colleagues proposed that constructing a multiplicative object is critical to an individual constructing and representing covariational relationships (P. W. Thompson, Hatfield, Joshua, Yoon, & Byerley, 2016).

In this study, I investigate the mental actions and ways of reasoning of two mathematics education graduate students with teaching experience while working on tasks that were designed to investigate covariational reasoning. In doing so, I extend the aforementioned body of research to include examples of experts' reasoning in a dynamic situation. I use "expert" to refer to an

individual who engaged in a significant length of study of mathematics (e.g., beyond an undergraduate degree and with teaching experience), not necessarily one known for his sophisticated thinking. By comparing the two experts' reasoning and activities, I identify strategies that are productive or problematic in their reasoning covariationally. I specifically highlight the importance of joining two quantities' amounts of change and accumulations into a multiplicative object for interpreting and graphing relationships involving varying quantities. Although this study is not intended to be a comparison study across populations, it does provide examples of ways of thinking within the population of experts and thus lays some groundwork for future research in this direction.

My motivation in studying an expert population is not to seek, in his strategies directly, a prototype of ways of thinking or activities that educators should strive to engender in their own students. Rather, my motivation comes from a desire to explore what covariational reasoning might look like from an individual who has continued to engage in mathematics education beyond a formal college experience focused on mathematics. Mathematics instructors are represented in such a population. Identifying problematic and productive features of their reasoning might, in turn, become useful in supporting their development of sophisticated covariational reasoning abilities.

The following paper is organized into four parts. First, I provide background on theoretical constructs related to covariational reasoning and discuss motivating research and empirical findings relevant to my study. In this chapter, I frame the discussion around ways of thinking about graphs in order to classify associations one might make while engaged in a graphing activity. This provides a lens through which I relate and compare my observations of the participants' activities. Then, following a description of the tasks, methods, and participants

of the study, in Chapter 3, I present the results in Chapter 4, which includes transcript excerpts, images of participant work, and my analysis of the data. In the final chapter, I discuss main themes from my analysis of the experts' activities. My major findings corroborate previous researchers' findings that the construction and coordination of corresponding amounts of change and accumulations are essential for one to represent and interpret dynamic situations covariationally and that such activity is difficult even for experts. I illustrate this claim by comparing instances in the experts' activities when I inferred such structures to be present or absent in their covariationally reason while performing the task. In this final chapter, I also highlight alternative strategies my participants had tendencies to enact that did not have a basis in constructing amounts of change, further justifying the importance of constructing and coordinating amounts of change and providing evidence that such activity was difficult even for the experts.

CHAPTER 2

BACKGROUND

“A quantity is in a mind, it is not in the world.”

(Thompson, 2011, p. 33)

In this chapter, I describe the main concepts and structures underlying my investigation. In the descriptions of these terms, I attempt to clarify those words and concepts constituting my analysis of the participants’ activities and the descriptions of the participants’ mathematical thought processes. By illustrating my perspective, I hope to craft for the reader the lens through which I came to develop and understand the results of this study.

Quantity, Quantitative Reasoning, and Covariational Reasoning

This study is focused on students’ covariational reasoning; such reasoning rests on students having constructed objects to reason covariationally about. Thus, I first describe elements and features foundational to such reasoning.

Quantity

The concept of covariation involves how two *quantities* change in tandem. I follow Thompson’s definition of a *quantity* to mean one’s conceptualization of an attribute of an object such that one understands the attribute to have a *measurable magnitude* (2011). I use *magnitude* to mean an “amountness” of an attribute independent of a determined unit by which one might measure the attribute. For example, a magnitude could include someone’s height, the straight-line distance from your nose to your computer, or the horizontal length from a point on a

Cartesian coordinate system (CCS) to an axis perpendicular to that horizontal length. A magnitude of a quantity is then one's conceptual image of a measurable attribute of an object. Moreover, one's image of a magnitude is developmental, having several levels of sophistication (Thompson, 2011).

A quantity's *measure* (or value) then describes a numerical value one might assign to a quantity's magnitude by means of constructing a certain unit magnitude and a comparison of the quantity's magnitude with that unit (Steffe, 1991a, 1991b). To continue an example from above, a measure of a person's height could be the number of inch magnitudes one must accrue to span the length from that person's head to toes. A measurement of this person's height could also be obtained by using centimeters or inches—each resulting in a different height measure. Invariant across all such measure-unit magnitude pairs associated with one's height (i.e., a pair consisting of a value of the height and the unit value used to measure the height) is the magnitude amount of one's height. In this sense, a person attending to a quantity's value is shaded by that person's conception of the quantity's magnitude (i.e., amount) and how to measure it (unit). My focus throughout this work is on quantities' magnitudes, rather than values. I use quantity and magnitude interchangeably and specify if necessary when I am attending to a numerical value or measure of a quantity's magnitude.

Several researchers have found focusing on how students reason about quantities' magnitudes (as opposed to measures) to be productive. For example, P. W. Thompson, Carlson, Byrley, and Hatfield (2014) conjectured that thinking with magnitudes is highly related to students' abilities to reason with and form algebraic relationships. As such, attending to students' abilities to reason with magnitudes could support their development of algebraic reasoning. With regards to graphing, Saldanha and Thompson (1998) focused on magnitudes in order to explore

the conceptual operations involved in envisioning and reasoning with two quantities that continuously change magnitude. Similarly and more specifically, (Moore, Paoletti, Stevens, & Hobson, 2016; Moore, Stevens, Paoletti, & Hobson, under review) described how reasoning about quantities' magnitudes is related to how one might imagine, coordinate, and represent quantities' magnitudes in flux without the necessity of specific measurements of the quantities' values.

Quantitative reasoning

Within a situation there are multiple quantities that one might imagine. Quantitative reasoning describes an individual conceptualizing a situation as being composed of multiple magnitudes and relationships between these magnitudes (Smith III & Thompson, 2008; P. W. Thompson, 1990b, 1993, 1994b, 2011). As an example, consider the situation illustrated in Figure 1 of a rider traversing around the path of a fixed square wheel (this is also the situation represented in the Square Ride Task in Figure 15). In this situation, one might conceptualize the quantities of distance a rider has traveled around the square wheel, TD (as illustrated by the blue magnitude bar), and the height of a rider from the starting height of the ride above the ground, H , (as illustrated by the green magnitude bar), and recognize that these two magnitudes have a relationship. With my focus on magnitudes then, by a relationship between quantities, I do not mean algebraic formulas or numerical comparisons. For example, a relationship between magnitudes in the square ride could include a conceptualization that at any particular position of the rider, the magnitude of height of the rider from the ground and the distance the rider has traveled exist simultaneously, and that such a simultaneous existence is entailed over an interval (or range) of magnitudes as the position of the rider changes and the magnitudes vary. I do not mean to deemphasize the important role that formulas and reasoning about numerical values can

play in quantitative reasoning. Particularly, formulas can result from one propagating a calculation she has deduced from a relationship between quantities (P. W. Thompson, 1990b; P. W. Thompson & Carlson, 2017); one acquiring an intention to construct a formula can foster algebraic reasoning. Students generating values and formulas is also productive in their developing concepts of function, rate of change, fractional reasoning, etc. (Confrey & Smith, 1994; Ellis, 2011; Izsák, 2003; Johnson, 2015; Steffe, 1988, 1991b; Steffe & Olive, 2010). However, algebraic formulas can sometimes introduce an emphasis on calculations and cloud a student's understanding of imagining continuous variation happening with the quantities or magnitudes constituting the relationship (Moore & Carlson, 2012).

One's conception of a relationship between quantities may result in one conceptualizing a new quantity produced from operating with two original quantities. Such an operation is referred to as a *quantitative operation* (Thompson, 1990). For example, in the square wheel situation described above, one may compare how much more (or less) distance the rider is located above the ground for two instances of the rider's location on the square wheel. In this sense, one makes an additive comparison between two states of the rider's height from the ground and the result is a new quantity representing the change between the rider's height from the ground. Such a quantitative operation is distinct from an arithmetic (or numerical) operation (i.e., a calculation with numerical values) because it involves one's conceptualized image, not numbers. For example, one may calculate a *measure* of the change associated with a quantitative operation by subtracting the measures of two instances of the quantities or by tracking a growth (or decay) from the reference of one quantity to the result of the other. This example also illustrates one calculating the measure of an *amount of change* of a quantity; I elaborate on an amount of change further in the next sections.

Covariational reasoning

Quantitative reasoning about a dynamic situation involving two (or more) quantities involves covariational reasoning. *Covariational reasoning* describes, “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (Carlson et al., 2002, p. 354). My focus on quantities’ magnitudes aligns with the foundations of an idea of covariation as an image of sustained magnitudes in flux. A person conceiving of a single magnitude in flux can involve one imagining the quantity undergoing continuous change throughout an interval of time. In this sense, one’s image of covariation could entail “understanding time as a continuous quantity, so that in one’s image, the two quantities’ values persist” (Saldanha & Thompson, 1998). I compare that concept of covariation—focusing on variations over a continuum of quantities’ states of values—to a notion of covariation that involves variations across corresponding sequences of values. In particular, (Confrey & Smith, 1994, 1995) provided work on developing a concept of covariation that involves attending to and identifying patterns across successive paired values. They described this notion as productive in helping students generate a rule to describe the correspondence between two sequences of values. This notion of covariation requires the availability to manipulate numerical values and is well suited for situations that allow one to construct an analytic rule describing a numerical correspondence. However, this notion takes for granted how an individual might be thinking about the values in the correspondence in terms of quantitative imagery and more specifically what is happening quantitatively from one number to the next down a column. Within such a setting, students are often provided with or instructed to generate tables of numbers without regard to developing explicit images of measurable attributes that vary in amountness. As such, it would be quite unclear as to what sense students make of

such a table of numbers and their constructed relationship between the values. In an attempt to overcome such constraints of numerical values and maintain a focus on explicit quantitative images, my view of covariation focuses on quantities' magnitudes and their sustained correspondence under variation.

To further explain the notion of covariational reasoning that I focus on in this work, let me leverage Thompson's (1994) description of accumulations and accruals as foundations to one's image of rate. By *accrual* I mean one's conception of a determined unit magnitude of which a quantity's total magnitude can be composed. By an *accumulation* I mean a total amount of a quantity that is constructed by iterating accruals (see Figure 1). While I describe an accrual as a "unit magnitude," I do not necessarily mean this as a unit for the purpose of measuring, but rather the conception is that the accumulation can be thought of as composed of and occurring through accruals without measurement. One can compare one's conception of an accrual with that of an *amount of change* of a quantity. In some situations, I classify one's construct of a quantitative magnitude as both an accrual and an amount of change. In other situations, I classify these constructs distinctly. In particular, an amount of change requires one to have conceptualized two states of a quantity, but while an accrual could be considered an amount of change, it could also consist of one's constructed unit by which a single state of the quantity is constructed (thereby, resulting in the construct of an accumulation). Thus, these objects are distinguished by the way in which they are constructed and acted on—with regards to two (not necessarily fixed) states of a quantity (i.e., an amount of change) or with regards to a single (not necessarily fixed) state of a quantity (i.e., an accrual). I make the distinction here between a conceptualized 'difference magnitude' and a conceptualized unit magnitude (by state of a quantity is conceptualized). Thompson (1994) maintained that a sophisticated conception of

covariation and rate involves the construction and coordination between two quantities' accumulations and those accruals that can be thought of as resulting in the accumulations. For example, in the square wheel situation, such a coordination could include one imagining that the rider's total distance traveled in relation to his height from the ground accrued (or accumulated) through accruals of distance traveled and accruals of height so that at any moment during the first half of his trip around the wheel, the total distance traveled in relation to his height from the ground is the same as the accruals of total distance around in relation to the accruals of height (see Thompson, 1994, p. 5).

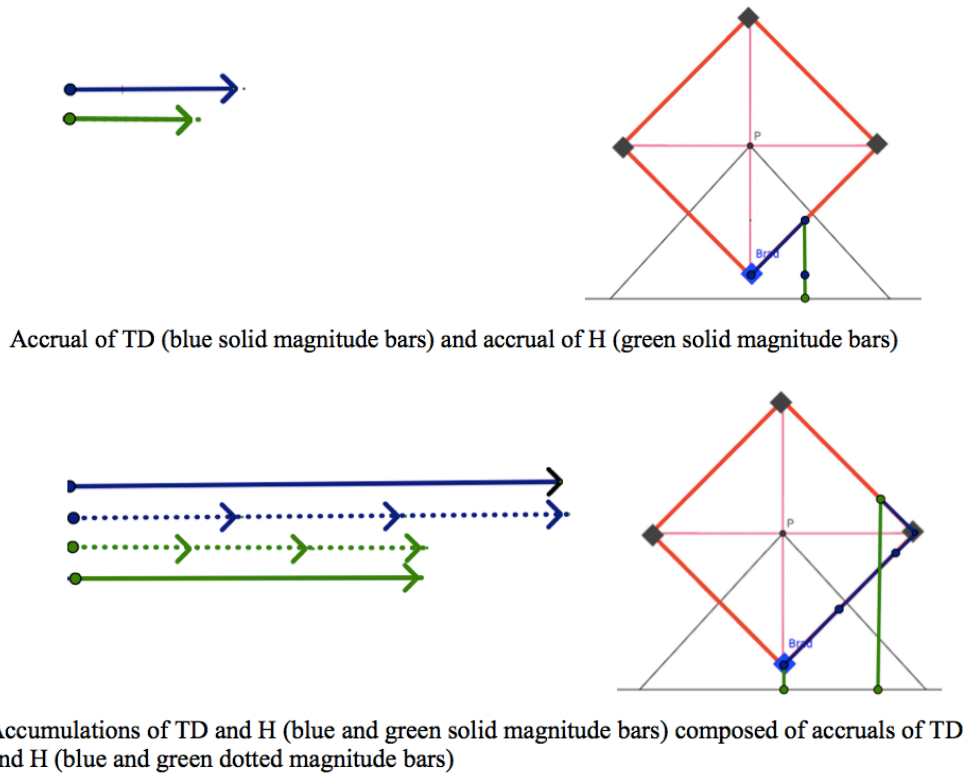


Figure 1. Accrual and accumulation (composed of accruals) of total distance (TD) traveled by a rider and height (H) of that rider from the start of a ride traversing along a square wheel representing a constant rate of change of TD and H (Thompson, 1994, p. 5).

As identified in Thompson and Carlson (2017), an individual conceptualizing rate of change involves him or her reasoning covariationally *and* constructing ratios and reasoning proportionally about accruals and accumulations. Thus, rather than focus on rate of change in this paper, I instead focus on the quantitative constructs of accumulations and accruals involved in Thompson's discussion about rate. As such, I further describe these constructs and their role in covariational reasoning.

Building on the previous discussion, one's image of an accrual and one's image of an amount of change are similar in that each is a quantity related to, but distinct from, the accumulation of a quantity. It seems then that investigating one's images and reasoning with accruals and change may provide researchers insight into understanding how students conceptualize variation of the quantity held in mind. Drawing on students' mental operations involved in variational reasoning, Castillo-Garsow, Johnson, and Moore (2013) classified two forms of students' images of change— *chunky* and *smooth*. Through the use of a metaphor, the authors compared chunky and smooth images of change to one's perception of an animated movie created from still frames: one perception may include the characters in the movie as still figures in one frame to the next, another perception may include the characters moving in continuous motion as the frames traverse. The authors claimed that conceptualizing continuous motion in progress is necessary for one to think about continuous variation of a quantity. However, as was also pointed out by Thompson and Carlson (2017), one's conception of motion alone cannot support one's conception of smooth variation. One may imagine a quantity's magnitude increasing (or decreasing) without engaging in operations that constitute thinking about the continuous variation in the quantity's value. For example, one may imagine the height of the rider in the square wheel ride (as in the situation related to Figure 1) as continuously

moving upward as the rider travels along the first part of the wheel while at the same time holding in mind an image of the rider's height as increasing in completed interval segments as opposed to imagining the height of the rider passing through all intermediate measures within the interval segments.

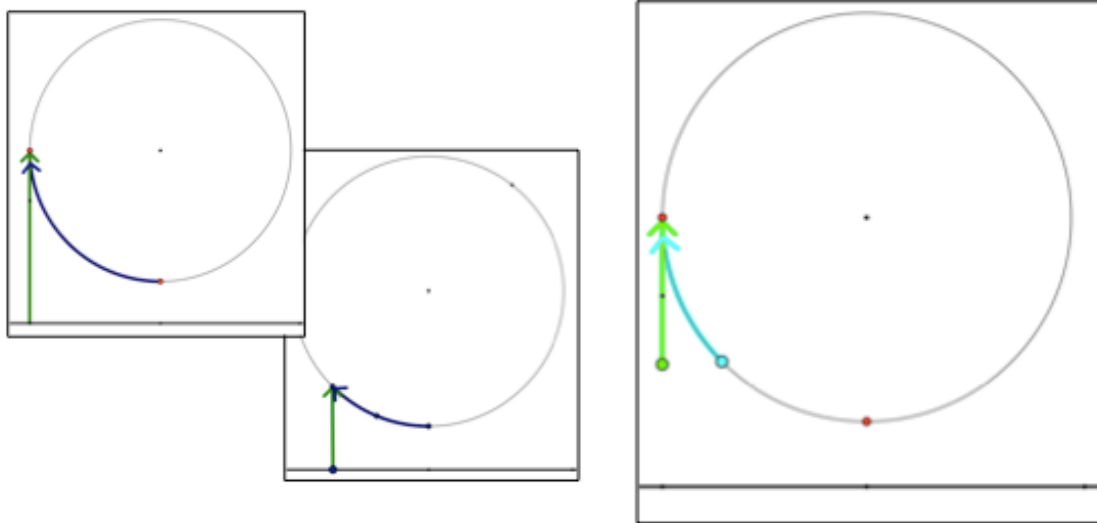
Images of variation are involved in one reasoning covariationally, but also involved is the simultaneity or joining of these varying quantities. In particular covariational reasoning involves two quantities united into a “highly schematized mathematical concept that a person can operate upon mentally” (Thompson and Carlson, 2017, p. 433). This construct is what Saldanha and Thompson (1998) called a *multiplicative object*—a conceptual object formed by uniting two elements so that the resulting gestalt simultaneously represents each element. For example, one could form a multiplicative object from her position on the earth's surface as being simultaneously a latitude and a longitude coordinate. I interpret Thompson's (1994) attention to the relationship between covarying quantities' accumulations and accruals to involve forming multiplicative objects both between corresponding accumulated amounts of quantities and the accruals making up these quantities.

To describe what I mean by the relationships and multiplicative objects involving accruals (or amounts of change) and accumulations of quantities, I illustrate (in Figures 2a-c) some possible elements of these constructs relative to one's thinking about a graph and the situation while constructing the quantities of a rider's total distance and height during a ride on a circle wheel (this is the situation represented in the task in Figure 15). I do not mean these illustrations to be interpreted as three independent examples or, on the opposite extreme, to encompass the entirety of one's construction of a multiplicative object in this setting. Rather, I hope to present possible illustrations that might be helpful for communicating or framing how

one might reason with amounts of change to construct multiplicative objects within a situation and a graph.

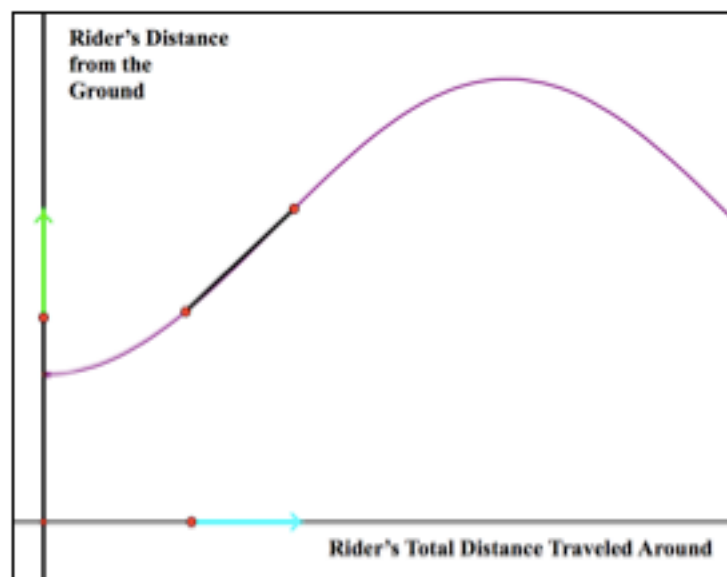
The relationships and multiplicative objects involving amounts of change and accumulations of quantities that I discussed above is a helpful backdrop for framing my own understanding of the notion of a covariational relationship. The framework by Carlson and colleagues (Carlson et al., 2002) of one's mental actions involved in reasoning about covariational relationships aligns with such a perspective. In particular, the structure of this framework provides a language to describe what quantitative objects one may construct, the ways in which one might coordinate these objects, and features one may attend to as she acts on these objects while engaging in a task involving covarying quantities.

Recently, Thompson and Carlson (2017) suggested a revised framework to identify characteristics of an individual's covariational activity. They designed this revised framework based on developments by Castillo-Garsow and colleagues on images of change (Castillo-Garsow, Johnson, & Moore, 2013), Thompson's developed notion of recursive image of change (P. W. Thompson, 2008, 2011), and multiple studies that have emphasized the importance of one conceiving multiplicative objects (Frank, 2016; Stalvey & Vidakovic, 2015; P. W. Thompson et al., 2016; P. W. Thompson & Saldanha, 2003). Particularly, Thompson and Carlson intended their framework to focus on one's conceptualization of *variation* of quantities and one's coordination of the *images* of quantities' values varying. Hence, this updated framework seeks to incorporate an individual's way of reasoning variationally and an individual's construction of a multiplicative object of the quantities' values.



(a)

(b)



(c)

Figure 2a-c. Multiplicative object consisting of (a) accumulated total distance traveled (TD) and height (H), (b) changes of TD and changes of H, and (c) changes of TD and changes of H represented orthogonally.

I find merit in the earlier Carlson et al. (2002) framework, however, to the extent in which it characterizes an individual's constructions of quantities (e.g., amounts of change) and coordination of quantities' magnitudes or values (e.g., directional). Specifically, several of the mental actions described in this framework involve an individual's construction of an *amount of change* and *rate of change*. The participants I observed seemed to describe and identify such constructs in their graphs or situations while engaging in the tasks without any direct prompting. Because of this, I find the language of the framework presented by Carlson et al. (2002) beneficial in identifying and describing some of the participants' behaviors I observed in this study. In the next section I elaborate on the study by Carlson et al. (2002) and their framework, as well as other guiding empirical studies.

Guiding Studies on Students' Thinking in Dynamic Situations

In order to understand an individual's reasoning about covarying quantities, various researchers have investigated students' and teachers' thinking while engaging in activities with dynamic situations (Carlson et al., 2002; Clement, 2000; Ellis et al., 2015; Johnson, 2012a, 2012b; Kaput, 1994; Monk, 1992a, 1992b; Moore & Carlson, 2012; Rasmussen, 2001; Saldanha & Thompson, 1998). Collectively, these studies have reported complexities involved in covariational reasoning and have identified aspects of covariational reasoning that are productive or problematic for such populations. I highlight a subset of these studies here, primarily focusing on those studies and findings that have informed my work.

Dynamic activities and thinking about graphs

With the basis that settings requiring students to reason with changing quantities is productive in order for them to think about a graph as tracking (or providing a record of) two

varying quantities simultaneously, Saldanha and Thompson (1998) explored a student's activity in a task from which the Going Around Gainesville task was developed (see Figures 3 and 14). In their study, they led a three-phase teaching experiment with an 8th grade student, Shawn. Their intent was to investigate what conceptual operations are involved in students coming to envision and reason about continuous covariation of quantities. The activity of each phase provided Shawn with an opportunity to engage with a configuration of a road and two cities and describe the behavior of the distance between the car and each city while the car traveled along the road. I focus on the first two phases of their study due to the similarities to the tasks in these phases and the Going Around Gainesville task. In the first phase, using Geometer's Sketchpad, a diagram of a path and two cities (labeled A and B) and a car (labeled C) along the path were presented to Shawn. The magnitudes representing distances AC and BC from the car to each city respectively were dotted in the diagram (see Figure 3). These distances were then displayed orthogonally on a set of axes. In this situation, Shawn had control of moving the car along the road and the components of the sketch moved correspondingly. He was asked questions related to the behavior of each quantity, the meaning of point P produced on the graph, and the trace (locus) left by P. In the second phase of the teaching experiment, Shawn was given new city-road configurations and was asked to produce the trace left by a car traveling along the road.

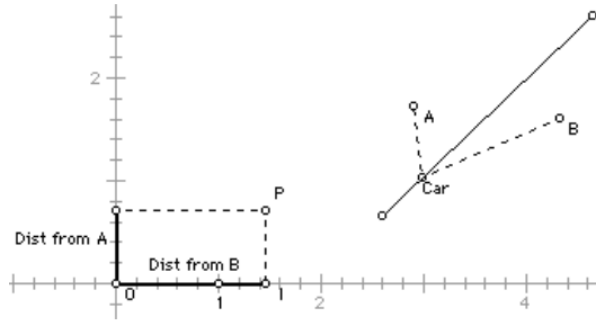


Figure 3. Diagram in GSP used by Saldanha and Thompson (1998, p. 300).

In this study, the authors invited Shawn to construct and coordinate images, each varying quantity AC and BC individually. The authors, however, questioned the extent to which his conception of the quantities in the situation entailed a tight coupling (i.e., so that one quantity's variation is necessarily envisioned with variation in the other quantity). They reported multiple instances of his referring to variation in one quantity or the other. Of particular note in their findings is the role that the car's *position* played in Shawn's explanations. For example, when Shawn was asked about the speed at which AC changed (when AC and BC were represented as magnitude bars which he could manipulate), he focused on describing and coordinating changes in the height of the bar representing the distance AC with changes in the car's *position* as opposed to a change of distance the car traveled or change in BC. The distinction between changes in the car's position and changes in the car's distance that I infer the authors to have made is the extent to which Shawn was imagining explicit images of quantities' magnitudes (and further, changes in such magnitude) when engaged in coordinating varying aspects of the car. In my study, I observed participants acting in ways that suggested their coordinating variations in an object's proximity to other points of interest in the situation. I relate the distinctions I make to

one coordinating an object's proximity, as opposed to an object's distance, to that distinction made by Saldanha and Thompson. I further elaborate on what I mean by proximity and this difference in the Methods Chapter of this paper. With regards to the graph, Shawn explained point P represented the distances AC and BC for a certain position on the road. He at first associated the trace of point P with the trace of the road itself, asserting "the road must be the graph" and that the road must really be curved and not the straight line that was represented in the diagram.

Thus, despite Shawn constructing and coordinating the varying quantities AC and BC, there were other features (i.e., the car's position and shape of road) that Shawn reasoned with in his response. This suggested various non-quantitative elements of the situation and experience of the car that Shawn may have brought to mind when thinking about the graph. This finding motivated me to look deeper into what aspects of the situations my participants attended to, the ways in which they reasoned with these aspects, and the extent to which these aspects afforded them images of quantities. Additionally, the setup in this study provided Shawn with illustrations of the magnitudes AC and BC in the diagram and orthogonally in the graph. Thus, also of interest to me was the extent to which someone might construct and coordinate such images on a set of axes spontaneously or without such perceptually available material.

Constructing and Coordinating Quantities

In the context of exploring students' concepts of functions, Carlson and colleagues classified mental activities involved in covariational reasoning into a five-part framework (Carlson et al., 2002). They then observed the behaviors of high-performing undergraduate students engaged in a covariational reasoning task and identified their mental actions using this

framework. These students were “high-performing” in that they had recently completed a 2nd semester calculus course with a grade of A or better.

The framework describes mental actions inferred from one’s behavior while graphing or reasoning about changing quantities and quantities represented in a function relationship. The authors suggested that the levels outlined in the framework are developmental, the sophistication of one’s image of covariation progressing to include an image of covariation involving (1) the coordination of quantities’ values, (2) the direction of change of values, (3) the construction and coordination of amounts of change of each quantity, (4) coordinating the average rate-of-change between the two quantities across increments of the input quantities, and finally (5) coordinating an instantaneous rate of change between the two quantities.

One task which Carlson and colleagues asked of their high-performing calculus students was the bottle problem or flask task (Bell & Janvier, 1981; Janvier, 1987). The task provided students with a profile outline of a bottle and asked students to graph, as the bottle filled with water, the height of water as a function of the volume of water in the bottle (see Figure 4). The first part of the interview included written responses of 20 students. Of these 20 responses only five were determined as acceptable graphs of the relationship. Six of the 20 students, chosen based on selecting students representing a diverse collection of responses, were then interviewed by the research team.

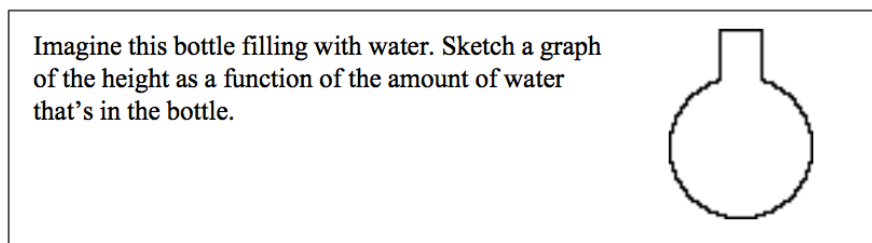


Figure 4. The Bottle Problem from Carlson et al. (2002).

In the interviews with these students, the authors observed that most of the students (five of the six) engaged in behaviors suggestive of their constructing and coordinating amounts of change of the height and volume of water in the bottle as the bottle filled with water. However, only two of these five students drew an “accurate” graph of the situation, suggesting subtleties in how one might attend to amounts of change over intervals of the quantities’ values and might reason with amounts of change to graphically represent a relationship between covarying quantities. With these findings in mind, I was particularly interested in understanding the role that amounts of change played in my participants’ reasoning during the tasks, particularly in their constructing graphs to represent (covariational) relationships they understood to constitute some situation.

Reasoning with Quantities in Rate

In order to classify and distinguish students’ reasoning with amounts of change and rates of change when making sense of a covariational relationship, Johnson (2012a, 2012b) described two forms of reasoning with quantities involved in rate: simultaneous-independent and change-dependent. She identified these two forms of reasoning while investigating secondary students’ work in the bottle problem (see Figure 4) and the reverse problem of asking students to draw a bottle from a given graph representing the height and volume of the bottle. In particular, these two forms of reasoning differ in the ways in which the quantities of change are constructed and the ways in which one compares quantities while making inferences about the covariational relationship in question.

Simultaneous-independent reasoning involves one making comparisons across the amounts of change of two corresponding quantities (Johnson, 2012a). Someone engaged in this

type of reasoning could attend to a change in one quantity, then the other, and then compare these amounts of change by forming a multiplicative relationship between them. The changes in the covarying quantities are simultaneous (i.e., amounts of change are both being constructed) and independent (i.e., the amount of change of each quantity is constructed without need of the other quantity's variation). *Change-dependent reasoning* involves systematically varying one quantity and simultaneously attending to the variation of the corresponding quantity and the intensity of this variation (Johnson, 2012b). For example, I interpret such reasoning to include someone constructing a collection of corresponding changes of two quantities and then comparing the changes constructed from one of the quantity's variations.

Johnson inferred that during the reverse bottle task students made associations about the width of the bottle based on comparing the amount of change of volume and amount of change of height in the graph. Johnson characterized such reasoning as simultaneous-independent. These students inferred, for example, that the bottle would be wider if the amount of change of volume was more than the amount of change of height and would be narrower if the amount of change of volume was less than the amount of change of height (see Figure 5 illustrating the bottle and graph). These students were making comparisons between quantities of volume and height to infer properties of the bottle they conceived their graph was representing. I speculate that Johnson classified such reasoning as simultaneous-independent in that the students' thinking she described progressed from the construction of each quantity (independently) and then formed a multiplicative relationship between these quantities (inferring a property of the bottle based on this relationship).

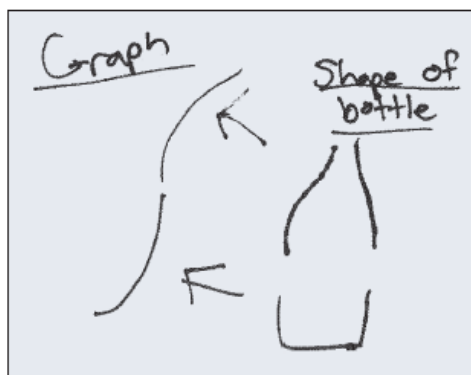


Fig. 3 Jacob sketched this bottle.

Figure 5. Student work from Johnson (2013, p. 706).

In contrast, Johnson inferred another student in this task attend to the variation of volume of water in the bottle in relation to the continually increasing height of water in the bottle (Johnson, 2012b). This student, Hannah, claimed, “Towards the bottom of it, like, it doesn’t increase as much, but as you go along, it definitely increases more. The volume and the height... it increases more as you go. The volume of it increases a lot, like from here to here [*puts two fingers on graph*], like compared to down here [*points to fingers near origin*].” (Johnson, 2013). From Hannah’s continued response, Johnson identified her to be systematically varying the height of water in the bottle (continuously with a constant speed) while simultaneously attending to the variation in the volume of water in the bottle. To draw the bottle from the graph, Hannah then made associations between the variation in the increase in the volume of water and the curved shape of the bottle as the height of the water in the bottle increased continuously and constantly. For example, she inferred that the bottle would be wider if the volume was getting larger. Johnson characterized this way of thinking as change-dependent reasoning.

From the interviews with these students, Johnson suggested that in order for one to consider the variation in the intensity of change (i.e., a rate of change as increasing, decreasing,

or remaining constant), one must “systemically [vary] one quantity and simultaneously [attend] to variation in the intensity of change in [another] quantity” (Johnson, 2012b, p. 327). Johnson’s studies and insights suggest there are a multitude of features researchers and teachers should attend to in students’ reasoning with quantities involved in rate beyond that of just what quantities students are constructing and how they are coordinating them.

Associations Within One’s Meanings for Graphs

In an attempt to characterize individuals’ ways of thinking about graphs, Moore and Thompson (under review) described, three forms of shape thinking: *static shape thinking*, *emergent shape thinking*, and *holistic shape thinking*. Each form is characterized by the aspects one attends to in regards to some conceived graph and the nature of one’s activities while engaging with a graph. To distinguish one form from another, I find it productive to identify the types of associations one may make while making sense of a graph and consider the basis from which such associations result. In Table 1, I describe such associations one may make within each form of shape thinking. In this table, I list distinguishing features and provide examples of each. In this section, I elaborate on the associations listed in the table in order to describe (and distinguish) features specific to each form of shape thinking.

Table 1. Classification of Associations

Static	Holistic	Emergent
Types of Associations		
Graph shape or perceptual feature associated with visual property of event (iconic) or physical feature of situation (thematic). (Contextualized)	Graph shape or perceptual feature associated with abstracted schemes of covariation (that can be enacted if necessary).	Associating images of quantities to construct covariational relationship between quantities.
Graph shape associated to mathematical property, analytic rule, or function class. (Decontextualized)	Emergent properties of two quantities' covariation (within a certain coordinate system) associated to analytic rule.	Images of quantities' covariation in displayed graph associated to images of covariation of quantities in situation (and covariation is conceived as invariant across these representations).
Features of Associations		
Associations constrained to particular coordinate system or graphing conventions.		
Learned associations based on previous experience with shapes (and possibly abstracted schemes of covariation).		
Associations that are pointers to learned facts (such as function class terminology, analytic rules, number patterns, shapes).	Associations abstracted from acting on quantities in a particular coordinate system (or systems).	
Involving quantities and how they co-vary.		
	Properties of covariation are implications of assimilation and can be enacted if necessary.	Associations are based on activity with quantities and result from one's images of quantities. Associations are a means for assimilation.
Associations from perceived perceptual features without regards to quantities.	Associations from abstracting the coordination of quantities.	Associations made while one acts with and on quantities in regards to both of the objects being associated.

Static Shape Thinking

Static shape thinking involves thinking of a graph as an object in and of itself and attending to visual features or perceptual cues of the graph (Moore & Thompson, under review). The figurative shape of a graph is the main feature one attends to while making associations descriptive of static shape thinking. One may associate a perceived graph's shape with a mathematical property, analytic rule, or a physical feature in a situation. Although someone engaged in static shape thinking might make inferences about quantities represented by a graph from the graph's shape, static shape thinking does not involve constructing and coordinating quantities in the moment of assimilation. For example, someone engaged in static shape thinking may infer that a graph that rises left-to-right implies a quantity increases without unpacking this claim by attending to the quantities represented on the axes. Across past studies and reports on students' thinking, there are two main categories of associations that I identify as static shape thinking relevant to this thesis: *contextualized static shape associations* and *decontextualized static shape associations*. My use of "context" in this classification refers to the extent to which the association lies within an experiential situation. Broadly, I use contextualize to refer to an association occurring with regards to an experienced phenomenon (i.e., a "real world" situation, as in a ride around a wheel) and decontextualized to refer to an association with regards to what some might describe as a canonical mathematical representation (i.e., a graph or table). In this section, I describe each category and expand on more specific types of associations within each. I find making such distinctions within static shape thinking relevant because this way of thinking often involves multiple perceptual influences that I find productive for framing associations I have observed students and the participants in this study make. I consider previous researchers to have made similar distinctions in their observations of students' behaviors, and so

I leverage these classifications developed in past studies for clarifying students' associations that I identify as containing elements of static shape thinking.

Contextualized static shape associations. In regards to modeling and interpreting dynamic contextualized situations, many authors have reported on students' and teachers' tendencies to identify and associate visual features, perceptual characteristics, and physical sensations between a situation and a graph intended to represent a relationship between quantities in the situation (Clement, 1989; Kaput, 1987a, 1994; McDermott, Rosenquist, Popp, & van Zee, 1983; Monk, 1992a, 1992b; Schultz, Clement, & Mokros, 1986). In this regard, researchers have identified two types of associations consistent with static shape thinking: *iconic* and *thematic*.

Iconic translations. Monk (1992a) used the notion of *iconic translation* to describe students making global correspondences between the shape of a graph and shapes perceived to constitute some situation. In his study, Monk provided college-level students with a hands-on physical model of a situation representing a ladder against a wall (see Figure 6 for a representation of the model). His intention was to investigate students' conceptions of quantities in a dynamic situation without signaling visual cues to students through pictures of graphs or diagrams. The model Monk provided students was a one-foot to one-inch replica of a ladder against a wall. The students were then asked questions about various behaviors of the ladder's height against the wall and the distance of the base of the ladder from the wall. A final phase of the interview required students to draw a graph representing these two quantities as the ladder slides down the wall. Monk found that five of the 17 students interviewed drew graphs of the situation that he attributed to the students making iconic translations. The graph in Figure 7 is what Monk referred to as a "Sagging" graph. A student who drew such a graph stated, "I've

drawn this as a curve that slopes in [sags] because that's intuitively what I feel it is" (Monk, 1992a, p. 189). Monk described this student constructing this graph "with quite distinctive movements of his hand, wrist, and forearm in a manner that suggests a strong kinesthetic aspect to the construction of graphs" (Monk, 1992a, p. 189). Another student to draw such a graph justified the trace by saying, "[I]f I were to watch the middle of the ladder—this is what it would look like..." (Monk, 1992a, p. 189).

The findings of Monk's study illustrate students' tendencies to spontaneously generate and impose visual features on their constructed graphs from their own internal conceptualization (image) of the situation independent from the way in which the situation was communicated to them. Specifically, students in Monk's study associated iconic kinesthetic features of their movement with iconic shapes in their graph.

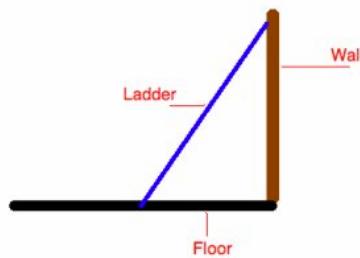


Figure 6. Diagram to represent model of ladder and wall in Monk (1992a).

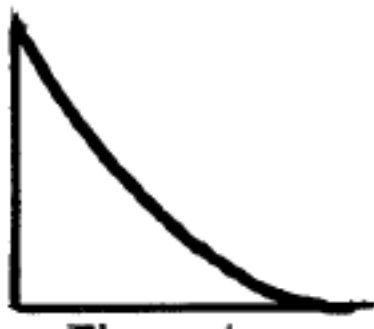


Figure 7. Saggy graph from Monk (1992a, p. 188).

Monk's intention when providing students with a physical model of the ladder, as opposed to a visual diagram, was to investigate whether students' iconic translations reported on in other studies (Clement, 1989; Janvier, 1978; Kaput, 1987a, 1987b; McDermott et al., 1983; Schultz et al., 1986) were primarily artifacts of the way in which the information was being communicated to the students (e.g., visual diagram or graph), rather than the students' own conceptualizations of the situation. These previous studies illustrated students' tendencies to make iconic translations when they were either given a graph (and asked to draw a diagram from interpreting quantities in the graph) or were given a visual diagram of a situation (and asked to draw a graph representing certain quantities in the diagram). In the diagram tasks, Schultz et al. (1986) provided students with a diagram of a hill and asked students to draw a graph of speed versus position of a bicycle along the path (see diagram in Figure 8). Half of the students in their study drew a graph of the situation with visual features similar to that in the given diagram. In regards to the second type of task, Kaput (1987a) gave students a speed versus time graph of a racecar and asked students to draw a diagram of a racetrack for a car whose speed and time could be represented by the graph. He found that the students had a tendency to draw a diagram of a racetrack that resembled the trace of the graph given to students.

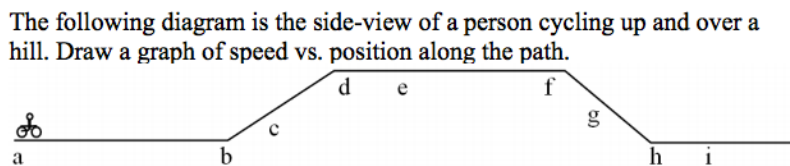


Figure 1. A problem in which students must distinguish between visual features of a situation and representational features of a graph. (From Monk, 1992).

Figure 8. Reproduction from Oehrtman et al. (2008).

Thematic associations. In addition to students making iconic translations of global shapes of graphs and situations, researchers have also observed students associating various features or qualities of a graph to situations or mathematical attributes. I identify such associations as static shape thinking in that these associations have their basis in perceptual features and not in properties of covarying quantities. More precisely, *thematic associations* refer to one associating particular phenomenon (that could be quantitative) with a graph or relationship between quantities (i.e., the objects being associated may be quantitative) such that the association made between the objects are without regard to quantities. For example, a student graphing the relationship between a rider's height and distance around a square wheel might infer that a rider moving around the wheel at a constant speed would imply his graph would be a straight line. Although the notion of "constant speed" has a quantitative basis, the association with a straight line is not made with regards to the quantities to be represented graphically.

As another example, Clement (1989) observed students infer a conclusion about a situation based on visual characteristics of a provided graph; I attribute such activity to students making associations while engaging in static shape thinking about a graph. Clement referred to such student behavior as a *local correspondence error*. In his study, he observed students to infer from a graph of two cars' (A and B) velocity versus time graph (see Figure 9) that the two cars (A and B) would cross each other at the time corresponding to the intersection of the two traces in the given graph. Clement claimed that students associated the feature of two traces intersecting with the two cars crossing paths.

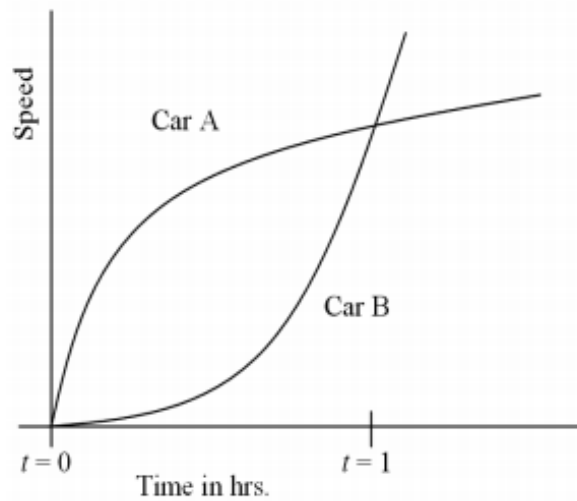


Figure 9. Graph of Car A's and Car B's velocities versus time (Monk, 1992, p. 175).

Clement identified another type of “error” in student work that was reported in a study by McDermott et al. (1983) that I also classify as students making an association while engaged in static shape thinking about a graph. Clement called the error a *height-for-slope error*. In their study, McDermott et al. (1983) gave college students focused in science a graph of two objects' positions versus time (see Figure 10). They asked students which object was moving faster when the time was at two seconds. From the graph in Figure 10, when the time is two seconds object B is at a larger position than object A; however, each object is moving at a constant speed with object A moving faster throughout the ten seconds represented on the graph in Figure 10. Clement (1989) suggested that such an inference (i.e., that object B moved faster) is a result of students associating the graphical feature of height with the speed of the car, thus associating the vertical magnitude of the trace of a graph with the speed of the car corresponding to the given trace. He described this association as a *height-for-slope error* (Clement, 1989, p. 6) and cites other studies in which students associate the height of a graph with the slope or derivative of the graph (Janvier, 1978).

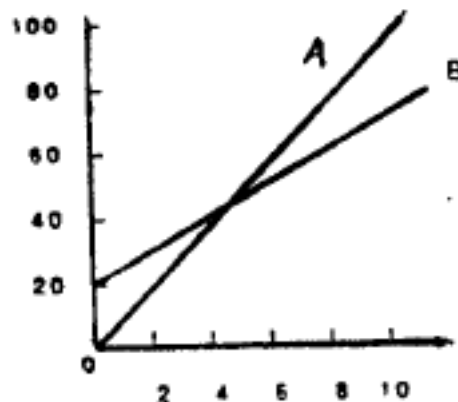


Figure 10. Position of object A and B position (vertical axis) versus time in seconds (horizontal axis) reproduced in Monk (1992, p. 175).

In attempting to assess teachers' meanings for variation and covariation, P. W. Thompson (2016) identified other thematic associations that individuals might make between a feature of a graph and a situation the graph is modeling. Thompson and his team gave teachers a graph representing the relationship between the displacement of a ball from its resting point and the total distance a ball attached to a rubber cord and given a push downward travels. The trace in the graph is given by a continuous trace of upward and downward (left-to-right) line segments (Figure 11). The research team found that teachers had a tendency to respond in ways of thinking about the graph that could be characterized as static shape thinking. Thompson provides examples of responses not implying variation, but which instead are based on the individual attending to the feature of the straight lines in the graph (e.g., inferring the ball moves in straight lines as a result of measuring distance or justifying the straight lines that appear in the graph as a result of measuring distance) or attending to the motion of the ball to justify the trace of the graph (e.g., the ball moving up and down implies the trace of the graph should move up and

down). The associations the teachers made were thematic in that the attributes they associated were tying together some feature of the phenomenon and the perceived shape of the graph—the motion of the ball and the perceptual feature of the graph.

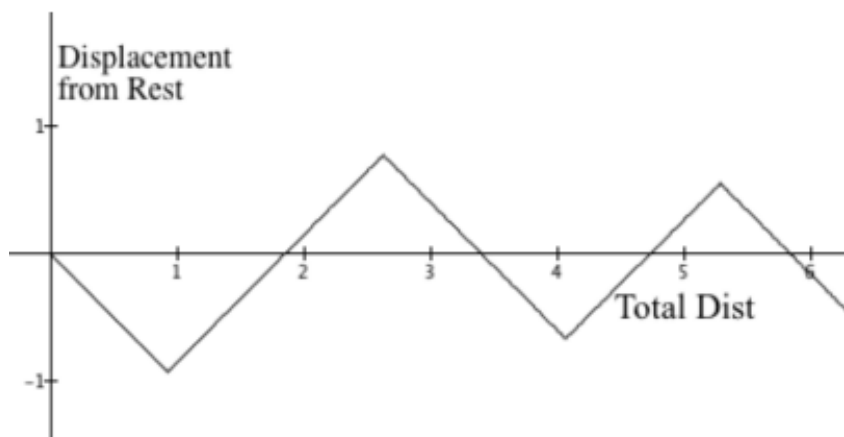


Figure 11. Graph representing relationship between the displacement of a ball from its resting point and the total distance of that ball on a string (Thompson, 2016, p. 449).

Decontextualized static shape associations. The above types of associations (i.e., iconic and thematic) that I classified as examples of static shape thinking were each situated in contextualized problems and focused on associations regarding the context of a situation. In addition to such associations, researchers have also observed students making associations, again without constructing or acting on images of quantities, with regards to mathematical properties (e.g., slope, rate of change) from figurative and perceptual properties of a graph (Moore, Silverman, Paoletti, & LaForest, 2014; Planinic, Milin-Sipus, Katic, Susac, & Ivanjek, 2012; Stump, 1999; Zaslavsky, Sela, & Leron, 2002). I identify such associations as *decontextualized static shape associations*. Although the associations I speak of as decontextualized are within the context of what might typically be regarded as a mathematical representation, they are not within

the context of an experiential event or phenomenon (i.e., racing cars or carnival rides on wheels), as in the case of contextualized static associations. Similar to iconic and thematic associations within contextualized problems, I identify such associations as static shape thinking in that these associations have their basis in perceptual features and not in explicit images of covarying quantities. For an example of such an association, a student may associate the slope of a graph with the visual feature of the incline of the trace of a graph without regards to the quantities being represented on the axes. Expanding on work by Moore and Thompson in regards to preservice teachers' responses to a graph of $y=3x$, a student might claim that the graph in Figure 12 has a negative slope despite the fact that the increase in any amount of the y quantity is 3 times the corresponding increase in the x quantity. In this sense, the student is associating the visual feature of the line positioned downward left-to-right with a negative slope, thus inferring a property about the relationship of the quantities (i.e., slope—a mathematical property) with a perceptual feature of the graph.

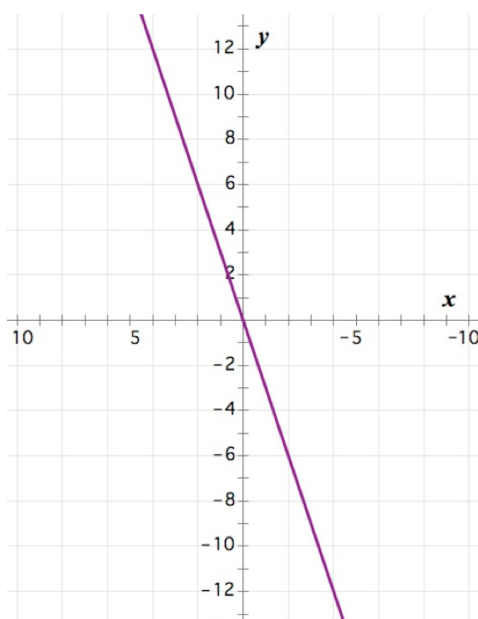


Figure 12. Graph of $y=3x$ (Moore & Thompson, 2015, under review p. 7).

Another example of students making decontextualized static shape associations is illustrated in Moore and Thompson's discussion of a participant's activity in a study by Aspinwall, Shaw, and Presmeg (1997). In (Aspinwall et al., 1997), the authors gave the participant a graph (illustrated in Figure 13) and asked the participant to draw the graph of the derivative function. The participant's activity involved a "common treatment" of such a task that entailed "imagining a sliding tangent line and using visual judgments and comparisons of shape to sketch the derivative function" (Moore and Thompson, 2015, under review p. 51). In such an approach, one may be associating steepness of the graph with the shape of the derivative graph. For example, the participant said that, as the slope of the tangent lines became more positive (as a sliding tangent line traversed the graph left to right), the graph of the derivative function will move up. In the excerpt of the participant reported by Moore and Thompson, it is not clear to what extent the participant could infer an emergent trace of a derivative from constructing and attending to quantities and their covariation represented in the original graph; the associations made in a sliding tangent line strategy— perceptual changing steepness of a tangent line associated to movement up or down of a derivative function—could be exclusively based in a domain of reasoning with properties of shapes. Such an association may be viable only when one is graphing the derivative function on a set of orthogonal axes labeled with conventional axes labels. If one were to draw the derivative graph of a coordinate system without orthogonal axes, the same associations would not suffice to draw a trace of the covariation.

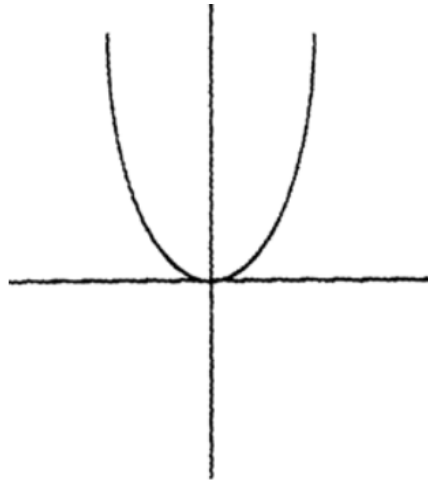


Figure 2.

Figure 13. Quadratic from Aspinwall et al. (1997).

Emergent Shape Thinking

Emergent shape thinking involves thinking of a trace of a graph simultaneously with how the trace is made. In this sense, emergent shape thinking describes thinking that involves the trace of a graph emerging hand-in-hand with one's image of two coordinated simultaneously varying quantities. As an example of this way of thinking, Moore and Thompson describe another preservice teacher's response to the graph of $y=3x$ illustrated in Figure 12. This participant conceived the equation $y=3x$ as representing a rate of change of three and explained that a student who constructed the graph in Figure 12 might have been thinking that "whatever he went up or increased by in x , he went over or increased by three times that in y " (2015, p. 5). Moore and Thompson claim that this preservice teacher understood the graph in regards to representing images of quantities' covariation along the axes and making associations between changes of those quantities.

As another example, Moore, Stevens, Paoletti, and Hobson (in preparation), illustrate via still snapshots and written explanation how one might think emergently about a graph as representing a car's distances to Atlanta and Tampa as it travels along a road in an animation. The Going Around Gainesville Part II task I used with the participants in this study came from this same task (see Figure 14). In the animation, the car travels along a path connected from Atlanta to Tampa at a constant speed. One's thinking emergently to represent the relationship between the car's distance to Atlanta and Gainesville for the first part of the path could involve her conceiving that at the start of the trip (when the car is in Atlanta) the car starts at a magnitude of zero from Atlanta and some positive magnitude from Gainesville. She represents this instance of both quantities by plotting a point on the set of axes given in the task. Next, she conceives that as the car approaches Gainesville towards the semicircle, for any amount of distance that the car increases from Atlanta, the car's distance to Gainesville simultaneously decreases by that same amount of distance, and such a varying relationship is maintained within those amounts as well. She would represent this relationship on the axes by producing a trace originating from the previously plotted point by projecting, in equal amounts of variation, an increase in the car's distance to Atlanta and a decrease in the car's distance to Gainesville, coordinating her pen in the appropriate directions she sees as depicting the quantities' directions along the axes.

Some associations one makes while engaged in *emergent shape thinking* are between different representations of quantities (i.e., representations of like quantities) and between one's constructed images of quantities in conceiving a covariational relationship (i.e., between quantities constituting a relationship). As emergent shape thinking entails one's in the moment actions with quantities, I determine someone to make an emergent association if the basis of her association results from these actions on images of quantities. In an emergent shape association,

one's images of quantities could be constructed from a real world situation (i.e., from a diagram or animation of a situation) or from the context of a graph (e.g., as magnitudes on axes of a graph).

I attribute one's associations while engaged in emergent shape thinking to provide the grounding for me to describe how one might communicate a covariational relationship she conceives. By way of illustration, consider again the circle wheel ride described in the Background section. Figure 2b illustrates a multiplicative object consisting of an amount of change of total distance and an amount of change of height of a rider along the circle wheel. Such a multiplicative object is constructed via one associating an amount of change of each quantity making up the relationship. By comparing various associated amounts of change, one may describe a covariational relationship between the quantities. In this case, for the first quarter of the trip around the wheel, the amount of change of the rider's height increases in relation to equal changes of the rider's distance around the wheel.

Emergent associations can also provide a language to describe one connecting her conceptualized relationship of quantities between representations. Such representations could include, for example, the context of a displayed graph or real world situation and could involve associations between contexts of her conceived relationship in different representations. A student in Johnson's studies (2012a, 2012b) (with the reversed bottle problem that I described in the Background Section) provides a good example of someone making an emergent shape association between representations of her conceived relationship. Recall, in her work with secondary students in the reversed bottle task, Johnson observed one of her students attend to the variation of volume of water in the bottle in relation to the continually increasing height of water in the bottle. This student constructed a bottle that she conceived to constitute the relationship

represented in the given graph by associating variations in the intensity of the volume with the shape of the bottle. I infer then that this student made associations of the trace of the given graph with images of (the intensity of) variation of volume and (continuous) variation of the height (thus associating the trace of a graph with images of quantities). This student also made associations between images of (the intensity of) variation of volume and (continuous) variation of the height of water in the bottle with a bottle's shape (thus associating image of quantities' variations and attributes in the situation). Notice that this students' association is different than her transferring an image of one type of quantity between a graph and a context (e.g., associating the height of the water in the bottle with a magnitude represented horizontally along the axis in her graph). This student is making associations between her conceived relationship within the representation of a graph and a bottle. In this way, those objects the student is associating are her constructed relationships between quantities, and hence, her images of these quantities form the basis of the association.

Holistic Shape Thinking

Holistic shape thinking describes a way of thinking about graphs entailing both static and emergent schemes (Moore & Thompson, 2015, under review). Such a way of thinking involves one attending to the visual shape of a graph and anticipating emergent relationships between the represented quantities. For example, a student engaged in holistic shape thinking might first assimilate her graph as if it were an object in and of itself and then unpack the static shape quantitatively (i.e., by constructing quantities on the axes from the trace). The inferences someone might make about quantities while engaging in holistic shape thinking involve one also constructing those quantities on the graph; this construction of quantities distinguishes such inferences from those one might make while engaged in static shape thinking. I identify a *holistic*

shape association as an association one might make between a physical or perceptual feature of a graph and a feature of two quantities' covariation (in which case the perceptual feature of the graph spurs the construction of quantities).

As holistic shape thinking is a hybrid of emergent and shape thinking, the associations one might make descriptive of holistic shape thinking have similarities and differences with associations I attributed to those other two ways of thinking. First, similar to emergent shape thinking, the associations one might make while engaged in holistic shape thinking can entail the construction of inferences about quantities. However, such associations are implications of assimilations in that the construction of or attention to quantities takes place *after* one has first made an assimilation based on a figurative element of the graph or situation. In this sense, constructing and acting with quantities does not provide the basis for forming the associations. Thus, one might verbally associate quantities while engaged in holistic shape thinking, these quantities and associations are constructed from one assimilating a displayed graph as shape rather than from the shape emerging from one's constructed images. For example, a student might describe a covariational relationship associated with the trace of a graph in the Cartesian coordinate system (CCS) without engaging in activity involving coordination of the quantities represented on the axes.

Similar to the associations descriptive of shape thinking, associations descriptive of holistic shape thinking are learned and constrained to a particular coordinate system. The associations one makes in holistic shape thinking, however, are from abstractions one has formed while previously acting emergently with quantities. In this sense, the associations are constrained to the particular coordinate system in which the activity was abstracted from. For example, a student engaged in holistic shape thinking might associate a straight line graph in a CCS (i.e.,

with x represented on a horizontal and y represented on a vertical axis) with the equation $y=mx+b$. This student might then associate the equation and graph with representing the covariational relationship entailing the concept that for any change of the x quantity, the corresponding change of the y quantity will be m times as large. In this sense, the association is (presumably) resulting from one's previous activity constructing and comparing changes on a set of axes, if necessary, such a student could engage in activity to represent this covariational relationship.

CHAPTER 3

METHODS

The focus of this thesis is to explore the covariational reasoning of individuals who have continued to engage in mathematics education beyond experience in a formal undergraduate mathematics degree program and who are interested in continuing on in mathematics. In an attempt to develop models of the study participants' cognitive structures related to covariational reasoning, I have used semi-structured, clinical, task-based interviews (Clement, 2000; Goldin, 2000) to collect data. Given that one's knowledge structures and reasoning processes are extremely complex and fundamentally impossible for an observer to uncover, I explain and justify this data collection technique and my method of analysis. In this section, I begin by describing the subjects of this study and illustrate the setting and tasks I used. I end this section by describing the data collection and analysis.

Subjects

The two participants of this study, Jake and Dan, were first year students in a mathematics education PhD program at a southeastern U.S. university. Both participants had previously completed a Bachelor's degree in mathematics and had taught secondary school mathematics for at least two years. These participants' interviews were part of a larger study aimed at investigating teachers' and college students' quantitative and covariational reasoning (see goo.gl/dAA7Re). Particular to the present thesis, the research team of this larger study (of which I am included) conducted individual interviews with a population of 10 experts from the

same university. By “expert,” I mean an individual who was enrolled in a doctoral program in mathematics or mathematics education with teaching experience in secondary or undergraduate mathematics.¹ Due to their backgrounds, I also expected these individuals to have had opportunities to represent and reason about quantities both in schooling and in teaching. Thus, I use “expert” in the sense that these participants probably had many experiences engaging in this type of task, not necessarily in the sense that they had been identified as sophisticated in their thinking while doing so. Recruiting from the mathematics and mathematics education PhD populations, we chose the subjects on a volunteer basis based on their agreement to participate and the compatibility of their schedules with the research team.

Data Collection and Methods of Analysis

With the 10 experts, we used semi-structured, clinical, task-based interviews individually with each participant (Carlson & Bloom, 2005; Clement, 2000; Ginsburg, 1997; Goldin, 2000) to collect data on their mathematical activity. Compatible with the goals of this study, semi-structured, clinical, task-based interviews are designed to provide a researcher with a look into an individual’s thinking in ways that mitigate shifts in the individual’s thinking due to researcher questioning. Such an interview broadly includes the subject, interviewer(s), task sequence, and interview script.

The subjects were encouraged to talk aloud about their thinking as much as possible, this offered important insights into their thinking. Consistent with clinical interviews, each interview entailed an interview script that included a sequence of tasks with potential questions and interventions. Reflecting the semi-structured nature of the interviews, however, the interview

¹ I note that, the 10 experts also include mathematics and mathematics education faculty.

scripts were not designed to dictate the entire course of the interview (e.g., the interview scripts did not contain a sequence of questions to be asked in order). Rather, the interviewers had the flexibility to formulate questions during an interview based on the subject's actions in order to gain additional insights into a subject's thinking while minimally influencing the subject's thinking. This type of interview format is important in that it allows the interviewer to react responsively to the subject as she collects data and thus attempt to collect deeper insight into the participant's thinking.

One or two members of our research team led each interview. Each interview lasted about 60 minutes in length. Based on availability, each participant was interviewed once or twice (Jake and Dan were interviewed twice). The interviews were video- and audio-recorded using two cameras and were transcribed post-session. I annotated and reproduced parts of these transcripts for the excerpts in the Results chapter of this thesis; I use "I1" or "I2" in the excerpt to refer to the primary or secondary interviewers (other members of the research team or myself). We used two cameras to video record the interviews. One camera was positioned to capture the participant's written work and was set up behind the participant with a view of the participant's hands and arms as they sketched their work. This camera also captured the animation on the computer in front of the participant and his interactions (such as hand gestures) with the display. The second camera was positioned in front of the participant and captured the participant and researcher interactions (including facial expressions and gestures). The participants' work was digitized. The data for this study includes the artifacts collected from the two interviews each with Jake and Dan. These artifacts include written and drawn work (i.e., graphs and diagrams) that are additionally explained or clarified by the participant in a recorded (audio and video) format.

Each of the interviews with the 10 experts was watched by at least one member of the research team and summaries of the interviews were reported to the entire group. Summaries included descriptions of the participants' main techniques and activities during the interview and engagement with the tasks. Jake's and Dan's activities seemed to provide the most enlightening comparison for me, as both participants' behavior seemed to focus on quantities and did not resort to memorized formulas, yet there were subtle difference in their thinking.

Focusing on the two experts for this thesis, I performed a conceptual analysis (P. W. Thompson, 2000, 2008) of their thinking by recording themes I observed in their behavior patterns and problem solving techniques across and within their activity in each task. These themes emerged and were refined through my repeated reflection on and engagement with the participants' interview videos and transcripts. In order to generate and test models of their thinking so that these models provided viable explanations of their behaviors, I reflected on my observations in conjunction with current theoretical concepts, particularly focusing on those that I described in the Background chapter of this paper. I used parts of the structure and language of the covariational reasoning framework developed by Carlson and colleagues (2002) to describe some of the quantitative objects I inferred Jake and Dan to construct and to delineate the ways in which they coordinated and attended to the variations of these quantitative objects (e.g., as varying directionally or with regards to an amount of change). Using the framework in this way is compared to using the framework to determine the experts' covariational reasoning ability across all tasks (as was done in the study by Carlson et al.). In my analysis, I focused on creating models of the experts' thinking in order to understand the ways in which they imagined quantities and their covariation rather than focusing on determining the level of development of

their image of covariation. I illustrate and justify the models I develop in the Results chapter of this thesis. I summarize main themes that I draw from the analysis in the Discussion chapter.

As an illustration of how themes emerged from my data analysis upon several iterations of working with the study data, I identified an important difference in how Jake and Dan described the quantities represented and visual features of their graphs with regards to objects in the contextualized situation. Identifying these differences resulted in me going back through the data to be more attentive to the experts' thinking quantitatively versus perceptually. I relate one's attention of a relevant subject's (e.g., car's or rider's, see the Tasks section below) proximity to a relevant reference point (e.g., Gainesville or ground) to Saldanha and Thompson's (1998) description of their study participant's attention to the position of a car in the City Travels task. Both instances suggest that one may maintain an awareness of variation within a situation (i.e., a car or rider moving in time and space) without explicitly acting on quantities' magnitudes, in this sense, engaging in *pseudo-quantitative reasoning*. I do not claim that behaviors indicating one's attention to proximity or perceptual features of a situation necessarily imply one to have not constructed or to not be coordinating images of quantities and their variation; however, I claim that observing this type of behavior requires further observation of one's behaviors in order to determine to what extent the individual is conceiving and imagining quantities in the situation or graph. Across both tasks in this study, Jake and Dan often used language that referred to the car's proximity to a certain city. Specifically in this case, by *proximity* I mean a sense of closeness or farness, suggesting a gross comparison of an anticipated distance quantity. I attend to Jake's and Dan's description of the quantities in this way because various instances of their actions constructing and reasoning with their graphs and their description of the car's proximity (from my perspective) did not necessarily imply an explicit image of a distance quantity or variations

of a constructed image of a distance quantity. Thus, I distinguish in this thesis between when a participant's activity suggested his thinking was about the car's proximity (and variations of such) to a given city (e.g., the car is *close* to Gainesville or *getting closer* to Gainesville) and when a participant's activity suggested his thinking was about the car's distance (and variations of such) to a city (e.g., the distance of the car to Gainesville is at a minimum *value* and the distance of the car to Gainesville is *decreasing*).

Tasks

In this section, I describe the tasks used in the two interviews with both Jake and Dan. Each interview centered around one animated situation. We began each interview task by having the participant watch a short (about 20 second, looped) animated clip that depicted the situation and then asking the participant what he noticed about the animation. The tasks in the first interview were *Going Around Gainesville (GAG) Part I and II* (see Figure 14) and the animation played a "car" moving back and forth along the path indicated in the figure. The situations in the second interview were *Taking a Circle Ride* and *Taking a Square Ride* (see Figure 15). These used two animations to depict different variations of a "rider" taking a ride around a wheel. In *Taking a Circle Ride*, the animation played a circle wheel spinning around an axle and in *Taking a Square Ride*, the animation played a fixed square "wheel" in which the carts traversed along the frame of the "wheel." All three of these animations played so that the motion was mostly at a constant speed. After playing the participant the animation and asking him what he noticed in the situation, we then gave him the corresponding prompt (Figures 14 and 15), which asked him to graph the relationship between certain quantities in the animated situation. We decided to move

on to the next part of a task or a new task when we felt sufficiently satisfied in probing the participant in ways we could use to later analyze his behavior.

Going Around Gainesville Part I

You've decided to road trip to Tampa Bay for Spring Break. Of course, this means traveling around Gainesville on your way down and back, because who would want to go through Gainesville? The animation represents a simplification of your trip there and back. Create a graph that relates your total distance traveled and your distance from Gainesville during your trip.

Going Around Gainesville Part II

Create a graph that relates your distance from Gainesville and your distance from Athens during your trip.

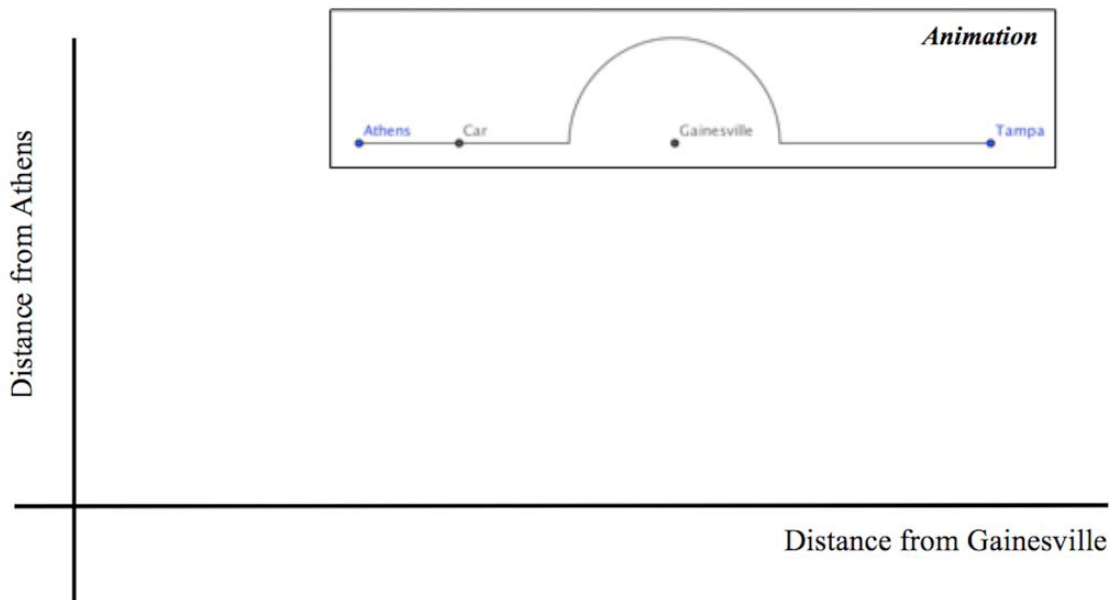
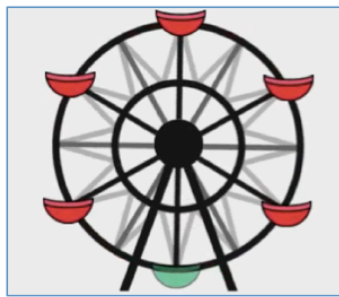


Figure 14. The *Going Around Gainesville* (GAG) tasks Part I and II.²

² This task is a modification of the task provided by Saldanha and Thompson (1998).



Taking a Circle Ride

Graph the relationship between the total distance the rider has traveled *around* the Ferris wheel and the rider's distance from the ground.

Taking a Square Ride

Graph the relationship between the total distance the rider has traveled *around* the Square Wheel and the rider's distance from the ground.

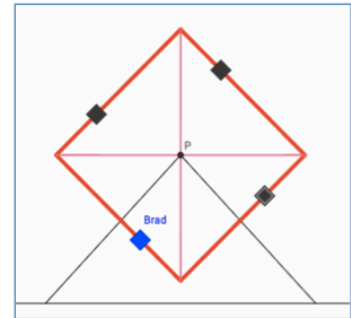


Figure 15. Taking a Circle and Square Ride tasks.

Our interest was to explore the extent to which participants conceived and coordinated *magnitudes* and *images* of covariation. As such, we designed the situations in the animation to provide the participants an opportunity to construct multiple measurable attributes and prompts to invite them to relate and graphically represent various relationships between quantities' magnitudes that they conceived constituting the situation. We intended that the participants not rely on learned facts or formulas to relate the quantities' magnitudes (Saldanha & Thompson, 1998). We attended to these goals by: (a) creating situations with relationships (e.g., distance of car to Gainesville in GAG task or height of rider on a square wheel in the Taking a Ride tasks) that we anticipated students did not have memorized formulas or graphs for; (b) not providing the subjects with or asking the subjects about numerical values; (c) prompting the subjects to graph multiple orientations of their conceived relationships (e.g., GAG Part II with fixed orientation of axes with "distance from Gainesville" along the horizontal axis, which we anticipated would be a different orientation from what the participants generated in GAG Part I); and (d) avoiding time as a quantity for participants to represent. The focus on novel attributes and multiple orientations of graphs was intended to perturb students' ways of thinking about graphs (Moore et al., 2014). Multiple researchers have reported on the role time plays in

students' understating of covariation (Ellis et al., 2015; Lobato et al., 2012; Stalvey & Vidakovic, 2015; P. W. Thompson, 1994a, 1994b). A general theme in these studies is the extent to which researchers observed time as an implicitly conceived quantity within a student's image of an explicitly conceived quantity. Rather than explore students' disembedding of time and the role of time in one's image of covariation, we avoided time as one of the explicit quantities for students to relate and represent. We also hoped that prompting students to graph a relationship between two length quantities would allow us to investigate students' images of covariation, as students can often reason univariationally when time is one of the quantities under consideration (Leinhardt, Zaslavsky, & Stein, 1990).

The GAG task is a modification of the City Travels tasks used by Saldanha and Thompson (1998). However, there are noticeable differences in our modifications with regards to the level of technological support provided to our participants. In the diagram of the cities and road, we did not provide the participants with bars representing the magnitudes corresponding to the distance of the car to each city either in the diagram itself or positioned orthogonally on a set of axes. Furthermore, before asking students the prompt to graph a relationship between quantities, we played an animation of the car moving along the road. Although our participants could choose to operate on the video itself (i.e., pause, play, and rewind the video) while creating their graph, participants were not explicitly directed to do this. This is in contrast to the instructions that Saldanha and Thompson gave the student in their study, that is, to move and interact with the diagram and sketch. We did not explicitly instruct Jake and Dan to interact with the video, as we were interested in investigating the extent to which they generatively constructed and coordinated images of quantities.

CHAPTER 4

RESULTS

In this chapter, I present the results of this study by describing, in succession, Jake's and Dan's activities in each task. I present their work in the tasks in the order in which we asked Jake and Dan to engage with them. I use a combination of excerpts, participants' drawings, and my own descriptions of participants' behaviors as evidence to justify the claims I make regarding their thinking in the tasks. My claims focus on the ways in which Jake and Dan conceptualized the situations in terms of images of quantities' covariation and how they thought of their graphs as representing these relationships.

Going Around Gainesville Part I

Jake: Going Around Gainesville Part I

In this section, I describe Jake's activity with the GAG Part I Task (Figure 14) and conclusions I draw from his behaviors about his thinking. The ways in which Jake first described and related features of his graph and features of the animation suggest that Jake did not initially conceptualize and coordinate images of quantities' magnitudes while thinking about or drawing his graph. Rather, Jake initially relied on thematic associations with the speed of the car in the animation and the approach of the drawing of his trace to the horizontal axis. Additionally, the ways in which Jake described certain visual or perceptual properties of line segments in his graph (e.g., that an upward left-to-right linear segment has an "increasing *slope*") illustrates a theme that emerges across Jake's interviews of his tendency to attribute visual features of line

segments (e.g., as increasing) with notions he has associated with lines (e.g., slope). As Jake's work continued in this task, his attempts to interpret his proposed trace for a faster speeding car resulted in his constructing amounts of change in his graph of the distance quantities he conceived in the situation. It was in constructing and relating these amounts of change between his graph and situation that allowed Jake to ultimately think about his graph as representing a relationship between two covarying distance quantities consistent with his image of the situation and independent of the speed of the car.

To begin the task, we played the animation for Jake and asked him, "So what's going on here [*in the animation*]?" He responded by describing the car's speed and Gainesville's location between Atlanta and Tampa. He explained, "I'm assuming we're driving at a constant rate for the most part...Gainesville is halfway between Atlanta and Tampa? Yes?" We then gave Jake the task prompt and asked him to read the prompt aloud. After doing so, Jake drew a perpendicular set of axes, labeled *Total Distance* on the horizontal axes and *Distance from Gainesville* on the vertical axis, and whispered to himself, "Let's see, they're gonna get closer..." as he slowly swept his hand in the air mirroring the motion of the car in the animation. He then watched the animation play through once more and drew the black trace in Figure 16 all at once without verbally responding.

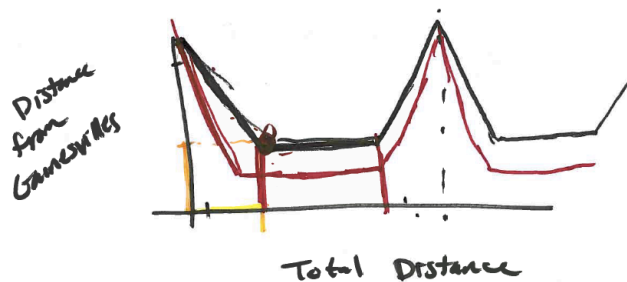


Figure 16. Jake's graph for GAG Part I.

We asked Jake to talk about how he drew his graph (Excerpt 1).

Excerpt 1. *Jake's initial explanation of his graph.*

Jake: Um, okay, so your, let's see, total distance traveled. You didn't travel anywhere, and so you're going to be a certain set distance from Gainesville, [*pointing to the vertical intercept on his graph*] I don't know, like 100 miles away, and as you drive towards it, and I'm assuming we're going at a constant rate, so you'll go down a steady slope [*traces over first linear segment*], and then as you reach the I guess semicircle, your distance isn't changing. You're just going around [*motions a semicircle above screen*] because you've got the same radius, so that's why you have the flat slope, and then you're moving further away again, so that's why you have a positive constant change [*traces over third linear segment*], and then it's just kind of the reverse going back, I would assume.

In the above excerpt, Jake described a physical phenomenon of the rider (“going at a constant rate”), the rider’s proximity to Gainesville (“drive towards it”), and perceptual features of his drawn graph (e.g., “going down a steady slope... flat slope”). Based on these comments and actions, the extent that Jake was imagining *two* distance quantities while thinking about the trace of his graph was not clear.

To gain additional insights into Jake’s thinking, we returned his focus to the return trip from Tampa to Atlanta. Jake continued to describe the proximity of the car to Gainesville (Excerpt 2).

Excerpt 2. *Jake describing the return trip of the car and “steady slope” in his graph.*

Jake: So I guess this would be the first part of the trip [*pointing at first half of trace*], going

to Tampa, and then kind of the same thing, you're approaching Gainesville [*traces fourth linear segment on graph*], so the distance is getting closer and levels out [*traces horizontal line in graph*] because you're still the same distance away going back around the semicircle, and then you get further away as you head towards Atlanta.

I1: Cool. You also talked a little bit about steady rate. Could you say a little bit more about that? You were just talking about steady rate and somehow that related to the graph.

Jake: Um, so I'm assuming the car is traveling at a set constant speed, so when you're leaving Atlanta the distance is going to get closer at a constant rate because your speed is constant, and then as you go around the semicircle it just kind of flattens out because you're not getting further or closer away, you're just going around the radius of the circle, so along the circumference, I guess.

I1: Cool. And so how does that show a steady rate, the graph? How do we see that in the graph and what does it mean to go at a steady rate?

Jake: The flatness of the graph is not changing at all, it's just [*holding hand in horizontal position*].

I2: What do you mean by flatness?

Jake: Slope of the arrow along this line.

In the above excerpt, Jake associated the “flatness” feature he described of his graph with the car “not getting further or closer away.” From this response, I infer that Jake had associated the car’s approach to Gainesville (“towards it”) or a fixed distance around Gainesville with how an “arrow” tracing along his graph (left-to-right) would project towards or parallel to the horizontal

axis. Additionally, Jake responded to our question about a “steady rate” by saying, “I’m just assuming the car is going at a set speed, so the distance would be changing at a constant rate, so that’s why you would see a straight line with a decreasing slope.” This suggests that Jake was associating the curvature of his graph (i.e., straight line segments) with the constant speed of the car in the animation. These associations Jake made calls into question the extent to which Jake was explicitly imagining covarying magnitudes when thinking about his graph.

In order to further understand Jake’s reasoning about the speed of the car in relation to his graph, we asked Jake what would happen if the car were to move a set speed but twice as fast as the speed in the first animation (Excerpt 3).

Excerpt 3. *Jake drawing graph for car moving twice as fast as original speed.*

Jake: Okay. So let’s see. [pause] So it would just be a steeper slope. [pause] I imagine it would be something similar to that. [drawing red trace in Figure 16] [pause]

I1: So what are you thinking?

Jake: [pause] Trying to think. [pause] Slope would definitely have to be steeper. I’m thinking it would still be flat or slope of zero in the same instance [tracing over first red horizontal segment in graph] because we’re talking about the total distance traveled [drawing vertical red segments in graph], so it still would be the same, I guess, spot where you traveled where the distance wasn’t changing from Gainesville anymore [motioning over a semicircle in the air]. And then let’s see. [pause] So I’m thinking that intervals for total distance traveled, that slope wouldn’t change, but the slopes outside of these intervals [tracing over red vertical segments in Figure 16] would, right? Yeah.

Jake's response in Excerpt 3 provides evidence that Jake associated the intensity of speed with the intensity of "steepness" of the graph (i.e., a faster car would indicate the slope of a line in his graph has a larger absolute value). After drawing the red trace to correspond to the car traveling twice as fast (Figure 16), Jake constructed an interval on the horizontal axis, which he described as "interval(s) for total distance traveled" while the car traversed along the semicircle. This suggests Jake was thinking about an interval along the horizontal axis as representing a change in the car's total distance traveled. He then inferred that the "slope" of his graph associated to this interval would not change if the car moved twice as fast, but the slope of his graph outside of that interval would change. He seemed to make this claim about the slope of his graph "not changing" along the indicated interval by thinking about the variation of the car's distance to Gainesville along the semicircle ("same... where the distance wasn't changing from Gainesville anymore"). However, due to his attention to the "slope" of this horizontal segment (as opposed to a fixed vertical magnitude of the horizontal segment), it was unclear if Jake explicitly imagined a magnitude of the car's distance to Gainesville while thinking about his graph or rather continued to associate movement toward or away from Gainesville in the situation with the trace of his graph moving towards or away from the horizontal axis. Thus, although Jake's attention to intervals on the *horizontal axis* of his graph suggested he was imagining a quantity of total distance while thinking about the horizontal displacement of the trace of his graph, it was not clear if Jake was imagining a distance quantity while thinking about the vertical displacement of his graph.

To further investigate what Jake was thinking about when considering the horizontal segment, we asked him to say more about why the "slopes" for the horizontal segments in his graph wouldn't change (Excerpt 4).

Excerpt 4. *Jake describing horizontal segment in graph.*

Jake: [pause] Alright. [pause] Because you're still going the same distance no matter if you're like changing speed, so it would still be at this point [*tracing over first vertical red segment in graph in Figure 16*] along your travel where you would meet, I guess where you're starting to go around the circumference, so it would still be the same I guess width of the interval where it's still at a constant rate.

I1: So you're talking about this width here? [*pointing to segment along horizontal axis on graph in Figure 16*]

Jake: Mmm. And the only thing I think what would change would be the slopes outside of those intervals. They would be steeper.

I1: And why is that the case?

Jake: Because you've increased the speed, so you're going to get there faster. [pause]

Jake's response of the claim that for either car, as the car traveled along the semicircle, the "width of the interval [*along the horizontal axis*] where it's still at a constant rate" would be the same further confirms that he was interpreting the horizontal axis on his graph as representing a distance quantity, that distance being a car's total distance traveled along the road. Jake also continued to associate the intensity of the car's speed with the steepness of the trace of his graph. This is suggested by his claim that for the faster car the non-horizontal line segments would be "steeper" than the previous graph ("They would be steeper... Because you've increased the speed, so you're going to get there faster"). Moreover, I emphasize that, while Jake maintained a horizontal segment in his graph (corresponding to the car's travel on the semicircle), there is still no evidence that Jake gave explicit attention to the magnitude of the vertical displacement of his

horizontal segments. Thus, it was still unclear if Jake was imagining the car's distance to Gainesville and variations of this distance while thinking about his graph.

As the interview continued, Jake began to question whether the trace of his graph would be different for the faster car (Excerpt 5).

Excerpt 5. *Jake questioning the trace of his graph for a car moving twice as fast as the original speed.*

Jake: Um, I'm trying to think whether that changed the slope now. Um, I'm pretty sure that it would though, because we're talking about total distance, distance from Gainesville. [*pause*]

I1: What are you thinking about?

Jake: I'm thinking I don't think it changes the slope at all. Uh [*pause*] yeah, I don't think it's going to change at all.

I1: So what's making you change your mind now?

Jake: I guess I was thinking more in terms of speed, but I guess speed really wouldn't matter in this case, because the distances are still staying constant, right? No matter where you're along your total distance traveled, you're still going to be the same distance from Gainesville, no matter your rate of speed.

Jake's question as to whether the speed of the car would affect his graph ("speed really wouldn't matter") seemed to be motivated by his thinking about *two* quantities of distance ("the distances are still staying constant"). In his response, however, it was unclear what Jake was comparing when he described the distances as "staying constant." As the interaction continued, he added, "no matter what speed, I'm still going to travel the same amount and my distance from Gainesville is not going to change even though I increased my speed. Still going to be the same

distance no matter if I'm going 45 and you're going 65." I interpret his response to suggest that he was comparing a car's total distance and distance from Gainesville across two different situations: a car traveling at the fixed original speed or a car traveling at a fixed speed faster than that of the original speed. However, it was still unclear what Jake imagined when he said the "distance from Gainesville is not going to change," as this statement might suggest that he was imagining variation within both cars' distances to Gainesville, exclusively relating both cars' distances to Gainesville across speed situations.

We asked Jake to say more about why he thought the speed of the car would no longer matter (Excerpt 6).

Excerpt 6. *Jake explaining why his graph would be the same for faster car.*

Jake: Um, [*pause*] no matter what speed we travel, we're still going to travel the same distance, total distance, right? Um, so no matter where I am along this graph, if at some point in time when you take a snapshot and just pause it, like if I was driving 45, somebody else was driving 65, we would still be that same distance from Gainesville at that moment in time. So I don't think this graph would change at all because the distances are not changing.

I1: So when you're saying at the same point in time, what do you imagine – Are you – Yeah, when you're saying at the same point in time you're meaning they're at the same what?

Jake: Like driving down the road, somebody comes and passes me. If we can just like stop that moment in time where we're at the exact same point in our travel, we're the exact same distance from Gainesville.

I1: So we're at like the same location on the path?

Jake: Yeah. Same location and same distance.

From the comparison of two cars' distances across two situations (original fixed speed and faster fixed speed), Jake inferred that the speed of the car would not affect his graph. Jake's response suggested he was imagining two cars at the same fixed location along the path and comparing the total distance traveled of each car and the distance to Gainesville of each car at this fixed location. This suggests that Jake was comparing static images of like quantities across the two situations of cars traveling at constant but different speeds.

We next asked Jake some general questions about the quantities in the situation. First, we asked Jake to interpret a point on his graph. In his initial response, Jake associated a point on his graph with a description of the car moving near a certain fixed location on the road. He explained that the marked point on the graph is "the point where I begin to make my transition along the circumference around Gainesville." When we asked how he determined this, he explained, "Because my distance from Gainesville is no longer changing. It's remained constant because you're traveling along the circumference and you're along like... it's following the radius, so it's... the distance is held constant."

We then pushed Jake to specifically identify what a point on the graph told us with respect to distances. The intention of our questioning was to understand how Jake was imagining and isolating quantities represented on his graph (e.g., did a point represent the uniting of two projected magnitudes?). After several minutes of our attempting to clarify our question to Jake, he continued to provide responses describing the local behavior of a car moving along the road ("still driving") with respect to the point. His persistent focus on describing a point on his graph in terms of the car's movement along the path suggested he thought about his graph in a way that was tied to movement of the car traveling along the path.

Recall, that at the beginning of the task Jake originally claimed that the shape of his graph was a result of the constant speed of the car. Since Jake's thinking seemed to shift after we introduced a car that was traveling at a fixed greater speed, we returned to the question regarding why the trace of his graph was the shape he originally drew. He responded by saying, "I think I initially said it had to do with the speed of the car and [pause]... So as I move towards Tampa in the beginning, my distance from Gainesville is decreasing, so I just drew a downward slope." This seemed to suggest that Jake was thinking about the directional change of the car's distance to Gainesville.

We further questioned Jake as to how steep he knew to draw his graph and questioned why his graph was a straight line. He stated, "Because it's still like a, I guess, one-to-one relationship as I... for every mile I move closer to Gainesville or towards Tampa I'm still moving that much closer to Gainesville." His response suggested that he was constructing successive and corresponding amounts of change in the situation, but it was unclear how Jake thought about these amounts of change when creating the trace of his graph. We asked him how he drew the graph from this explanation of the situation (Excerpt 7).

Excerpt 7. Jake using amounts of change to reason about shape of graph.

Jake: How to draw it? So I mean, you would have that constant slope, one-to-one. I mean, for every one mile you move towards the total distance towards Tampa you're still getting that one mile closer to Gainesville at least for this part of the graph.

I1: Do you think that's true for this part of the graph [*pointing to upward left-to-right line segment in graph in Figure 16*]?

Jake: Let's see, this is when we're leaving, so for every one mile which moves, distance from Gainesville would still be constant, yeah.

I1: Distance from Gainesville will be what?

Jake: Would still be... I mean, that kind of... for every one increase in mile after you leave the other end of the circle, it's still increasing another mile away from Gainesville as you move towards Tampa, so I would definitely think these would be similar, just one is negative, one is positive.

To justify the first part of the trace of his graph, Jake attended to the car getting “closer” to Gainesville and moving “towards Tampa.” In his explanation, he was associating changes in the car’s proximity to Gainesville with the changes in the car’s distance to Gainesville and changes in the car’s proximity to Tampa with the changes in the car’s total distance. He made this association by describing values for each change in distance quantity (“for every one mile you move towards... Tampa you’re still getting that one mile closer to Gainesville”). This suggests that Jake conceived images of quantities implied by his thinking about the *movement* of the car’s location relative to each city. Further evidence of this is provided by Jake’s response to justify the upward left-to-right linear segment of his graph (i.e., the segment in his graph corresponding to the car traveling on the final segment of the road— see Figure 16). To justify this linear segment, Jake similarly described paired changes of the car’s total distance (“for every one increase in mile”) and distance from Gainesville (“increasing another one mile away”) in a way that suggested these changes arose from the car’s movement. These instances of Jake’s attention to and measurement of changes of the car’s proximity to Gainesville and Tampa to explain the trace of his graph illustrated that Jake came to reason covariationally about the trace of his graph by coordinating amounts of change of distance in the situation but that such amounts of change were tacit in his thinking about perceptual features of the situation.

In summary, Jake's thinking about the trace of his graph progressed from his thematic associations with speed and perceptual features of the trace of his graph to reasoning covariationally with implicit images of distance quantities. Jake came to think about his graph in this way by first constructing an amount of change of the car's total distance in his graph and relating this quantity to those he conceived in the situations of two cars traveling at different fixed speeds. Furthermore, it was Jake reflecting on these changes of total distances across the situations with two cars that Jake came to imagine invariance in the distance relationships regardless of the cars' speed. It was in this iterative and reflective process of constructing and comparing quantities in the situations of two different cars that Jake came to think of his graph as representing two distance quantities and a relationship between them. Throughout the entire interview, his thinking about his graph (with regards to the curvature or a marked point) involved his thinking about the car's proximity to either city in a way that entailed the car in motion. In this sense, Jake's thinking about the quantities required his thinking about the car in motion, suggesting that Jake's image of the quantities were imbedded in his thinking about the car's movement.

Dan: Going Around Gainesville Part I

In this section, I describe Dan's activities during the first interview session with the GAG Task Part I (see Figure 14). I make claims about his thinking based on these descriptions. Dan's interview began similarly to Jake's in that Dan began his work by drawing the trace of his graph from a thematic association with the speed of the car in the animation. The interview continued with Dan justifying his trace. Dan's work diverged from Jake's in that Dan conceived his alternative trace (as representing a car speeding up—see Figure 17 on next page) in terms of constructed amounts of change of both distance quantities he conceived his alternative trace to

represent (cf., Jake constructed amounts of change in *two* situations rather than in his two traces). Similar to Jake, Dan justified the trace of his original graph as representing the covarying relationship between distances he conceived in the situation by reasoning with amounts of change. However, in contrast to Jake, Dan’s activities provide evidence that he held in mind explicit images of distance quantities while thinking about his graph that were not necessitated by the car’s movement along the path.

The interview began with us playing the animation for Dan and prompting him by saying, “So just take a look at it, watch it, and just let me know what you notice.” Dan claimed that the car was moving at a “constant rate.” We then gave Dan the prompt. After reading the prompt aloud he reread and underlined the words *total distance traveled* and their *distance from Gainesville* on the provided written task. Dan then drew a set of perpendicular axes, labeled the horizontal axis as *Total Distance Traveled* and the vertical axis as *Distance from Gainesville* (Figure 17).

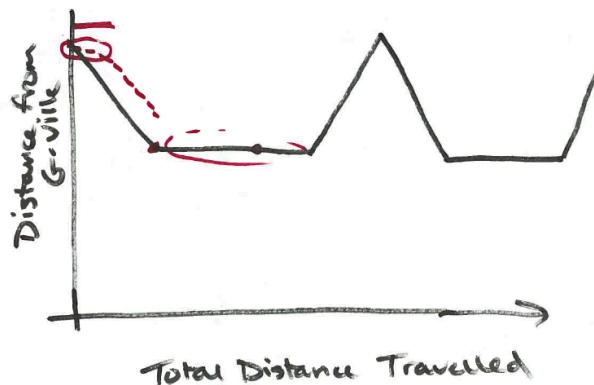


Figure 17. Dan’s graph for GAG Part I.

Dan then looked back and forth between the animation and his paper as he described the situation and drew the black trace in Figure 17 (Excerpt 8).

Excerpt 8: *Dan's response while drawing black trace in Figure 17.*

Dan: Alright, so total distance traveled and distance from Gainesville. So, it starts at its furthest point from Gainesville, then becomes 0, and then it gets to a constant and it never gets close to, so it's gonna be something like, along the lines of, it's getting closer [*places pen on vertical axis and draws the downward linear segment from left-to-right of Figure 17*], and then it gets to a constant away from it [*traces horizontal segment of graph in Figure 17*], and then it's gonna come back [*continues trace by drawing third linear segment of graph in Figure 17*]. And then I guess this furthest point kinda goes around this space [*draws downward left-to-right line segment and second horizontal segment of graph in Figure 17*], and then it's gonna be further away again [*draws final linear segment of graph in Figure 17*]. Oops, sorry.

Dan seemed to be focusing on the proximity of the car to Gainesville throughout this excerpt, leaving the question open as to whether or not he was coordinating two quantities to draw his graph. Additionally, Dan did not discuss why he drew each segment of his graph as linear. As he drew his trace, he used the phrase "it" in multiple instances. This also left us unclear as to how Dan was imagining the situation and his graph, if at all, and whether he was describing features of his graph or features of the situation.

In order to gain additional insights into Dan's thinking, we asked him to further explain his graph. He began his explanation by first attending to the direction of change of total distance, stating that this quantity will always be increasing. He said, "Sure. So, the total distance traveled is, no matter which direction the car is moving, is going to continue to increase [*motioning along horizontal axis*]. So, it's kinda like the odometer reading. Verses the distance from Gainesville."

Dan then seemed to be thinking about a maximum quantity represented by the trace of his graph in relation to the car's location from Gainesville; he stated, "they actually start furthest from Gainesville," while he pointed to the vertical intercept of his graph. He continued, "So, you're at your furthest point from it, and it's getting closer to Gainesville. So this would obviously represent zero [*pointing to origin of graph in Figure 17*], and it never gets to Gainesville." This seemed to suggest that Dan was thinking about the vertical axis of his graph as representing the car's distance from Gainesville and was making gross comparisons of this quantity by attending to the car's proximity to Gainesville. However, Dan's continued use of the word "it" while seeming to refer to different objects made us unsure whether Dan thought about an explicit image of the car's distance to Gainesville in his graph or was associating perceptual features of the car's "closeness" to Gainesville (in the animation) with the trace of his graph's "closeness" to the horizontal axis.

Dan continued by describing the horizontal segment in his graph while the car traveled along the semicircle (Excerpt 9).

Excerpt 9: *Dan describing the car traveling around semicircle and the horizontal segment of the black trace in Figure 17.*

Dan: So, it gets down to its closest distance where it seems to hit what looks to be a bypass of some kind or just something along those lines, and it's in a circle so it's never getting any closer, or a semicircle, any closer to Gainesville, so it's remaining a constant distance from Gainesville [*motioning along first horizontal segment of graph in Figure 17*], but the distance the car is traveling is still going up. So, this is continuing [*pointing along horizontal segment in graph in Figure 17*].

In this response, Dan made explicit associations between features of the car's proximity to Gainesville ("never getting any closer... to Gainesville") and the distance of the car to Gainesville ("remaining a constant distance from Gainesville"). In addition, Dan described the direction of change of the car's total distance (the "distance the car is traveling is still going up") with the trace of his graph ("this is continuing"). This suggests that Dan's version of the car's closeness to Gainesville did include an explicit image of distance and that Dan was coordinating the directional variation of two quantities to draw the horizontal segment in his graph.

After Excerpt 9, Dan continued describing the remaining trace of his graph by describing the proximity of the car to Gainesville after the car traveled along the semicircle (i.e., "bypass") until the car reached Tampa. He stated, "Then it gets to the other side of the, uh, bypass.... and it's getting further away from it until it gets to Tampa." Dan then used the symmetry of the situation to explain the rest of his trace, "So almost this should be a mirror [*spans fingers across first half of the graph and translates it over the second half of the graph*]." In his final response, Dan's language continued to suggest his attention to the directional change of the car's distance to Gainesville. It was still unclear to what extent Dan was coordinating the total distance traveled by the car to draw the entire trace of his graph, as this quantity seemed to become implicit while he thought about the entire trace of his graph. That is, although his previous explanation suggested he was coordinating two explicit images of quantities to draw the horizontal segment of his graph, the way in which he was thinking about the linear segments in his graph was unclear from continued explanation of the situation that emphasized the car's proximity from Gainesville.

To follow up on Dan's activity, we asked Dan why he drew his graph with straight lines. Initially, Dan responded by attending to the speed of the car, inferring that his graph may not be

“really exact” if he took into account the car “leaving” (i.e., “going slower and then... faster”), in which case the trace of the graph would be “more flat.” He stated that he was assuming the car was “moving at a constant speed” which meant the trace of the graph would be “a straight line.” We asked Dan to say more about why this would be the case. As he explained, he concluded that his graph should indeed consist of linear segments as he had drawn (Excerpt 10).

Excerpt 10. *Dan’s initial response concluding graph is linear.*

Dan: Yeah, I’m trying to, maybe I should think before I speak sometimes. Um, distance from Gainesville is travel. Yea, that doesn’t, yea, that actually, the distance traveled. I think I keep trying to relate it to time. Um, that wouldn’t actually make sense for it to be flatter at that point because that would mean that it’s not getting closer to Gainesville but is going further. So yea, there wouldn’t, it would have to be completely linear. There wouldn’t be a flattening like I was thinking of.

Dan’s response in Excerpt 10 suggested he was interpreting a “flatter” graph by constructing changes in both distance quantities (“getting closer to Gainesville” and “going further”). To further clarify, we asked him if he could illustrate this “flattening out” he described on his graph and repeat his argument. Dan then illustrated by drawing the red dotted trace in Figure 18.



Figure 18. Dan’s illustration of “flatter” graph.

Excerpt 11. *Dan describing the red trace in Figure 18.*

Dan: Um, where it's going slower and then it's getting faster. But if this was what I did, then that would imply that I've traveled further but I'm not getting closer to Gainesville, which makes no sense because as I'm traveling from the very beginning, I'm getting closer to Gainesville.

In his response, Dan made inferences about a situation by interpreting a “flatter graph.” He interpreted a “flatter graph” to represent the car “traveled further” (i.e., total distance has a larger increase) while the car is not “getting closer to Gainesville” (i.e., distance from Gainesville has a smaller decrease). This provided evidence that Dan conceived the trace of a “flatter” graph in terms of changes in distance quantities, and that he associated those changes to the behavior of a car in the situation in regards to the car's travel (both in total distance and distance to Gainesville).

Dan's response in Excerpt 11 also suggested that Dan was comparing changes in similar quantities (i.e., total distance traveled or distance from Gainesville) across his graph and a “flatter” graph. We were curious if this was the case, so we asked Dan to explain how he imagined the car “traveled further” and was “getting closer to Gainesville” (Excerpt 12).

Excerpt 12. *Dan's response describing a flatter graph and constant rate of change.*

Dan: If it's flat that implies that you're getting no closer to Gainesville but that the car is moving, so that it would have to be going in a circular pattern around Gainesville at that point. So this line [*pointing to the first segment*] wouldn't make sense to be flatter and then coming down because Gainesville. It's getting, as the car is traveling from the very beginning, from the moment that it starts in Atlanta until it gets to the bypass

around Gainesville, it's getting both closer to Gainesville and traveling, so it has to be a constant rate.

I2: Could you maybe say a little bit more about that, why it has to be a constant rate?

Dan: Uh, because every. There's a relationship between as the car's moving, it's consistently getting closer to Gainesville and it's also consistently traveling a distance. So, whatever it is doing distance-wise is also happening to its distance to Gainesville [*holds fingers around an interval on road above screen*].

In Excerpt 12, Dan described a different path of a car that he inferred as being represented by a flatter graph. He then described the car traveling along the path in the animation in terms of variation in both quantities, saying, "it's getting both closer to Gainesville and traveling," which suggested he was attending to the directional change of both distance quantities. After saying this, he concluded, "it has to be a constant rate." We were unsure how Dan was inferring a constant rate from what we interpreted as a directional change of the quantities, so we asked him to say more about this after he stated the claim. Dan's hand gestures as he explained "whatever it is doing distance-wise" indicated that he was constructing an amount of change of the quantities of total distance and distance to Gainesville, suggesting these two amounts of change were equal. After the dialogue in Excerpt 12, we asked Dan about a specific amount of miles the car traveled. He stated that "(for) 10 miles total distance traveled and it's also 10 miles closer to Gainesville."

In order to gain further insights into Dan's thinking, we then asked Dan how a stop of the car would affect his graph. He stated that his initial reaction would be that a stop of the car would create a "flat" piece of his graph, which he claimed stemmed from his prior experience with calculus problems involving graphs with time. As he continued, he reasoned that a stop of the car would not affect the graph, arguing that a "flat" piece of his graph would represent the car's total

distance increasing (i.e., a car “still moving”) while the car’s proximity to Gainesville remained fixed (i.e., “it’s getting no closer or no further away from Gainesville”) (Excerpt 13).

Excerpt 13. *Dan’s response to a stop of the car.*

Dan: I keep trying to put time into the graph. So, I thought of the old-school calculus problem about when you have time [*positions arm horizontally, creating the “horizontal axis”*] versus distance [*positions other arm vertically, creating the “vertical axis”*], and when they stop it would go a flat line. But then I realized that we’re graphing distance against distance so even if they were to stop, it wouldn’t affect this graph. So, that’s why I realize that the, I was gonna add in an extra flat piece but that didn’t make any sense ‘cause the, a flat line, or a horizontal line should I say in this, in this graph represents the car is still moving, but it’s getting no closer or no further away from Gainesville.

Dan’s comments in Excerpt 13 provided confirming evidence that Dan was thinking about the trace in his graph as representing two distance quantities (“graphing distance against distance”). His continued interpretation of the trace in his graph to represent variations in the car’s movement (“still moving”) and proximity to Gainesville (“getting no closer or no further away from Gainesville”) continued to suggest that Dan’s version of the car’s proximity included an explicit image of variation of distance and that he thought about his graph to represent such a variation.

Next, we shifted Dan’s focus to a point on his graph. We asked him what a specific point on the graph itself represented (Excerpt 14).

Excerpt 14. *Dan’s response to what a point on graph represents.*

Dan: Umm... the point itself... So, this is just a certain distance that the car is traveled, and at the same time, how far that car is from Gainesville. So, if we were thinking of a coordinate pair, a coordinate, a coordinate pair at this moment. This, the x coordinate if this was an x - y graph, if this was x [*pointing at the end of the horizontal axis on graph in Figure 17*] versus y [*pointing at the top of the vertical axis*], the x coordinate would be how far this car has traveled total, and the y coordinate would represent how far the car is from Gainesville [*pointing to vertical axis corresponding to point*].

Dan described the point in terms of the quantities “distance the car has traveled” and “how far that car is from Gainesville.” This suggests that Dan thought about a point on his graph as representing a multiplicative object composed of the quantities of the car’s total distance traveled and distance from Gainesville.

At this point in the interview, Dan had not yet described the third linear segment of his graph (i.e., left-to-right upward line segment). We asked Dan if he could explain this linear segment (Excerpt 15).

Excerpt 15. *Dan describing third segment in graph.*

Dan: So from this point [*pointing to the lowest point of third segment on graph*] up to this point [*pointing to the highest point of third segment on graph*] the car is traveling from the bypass to Tampa. So its distance, straight-line distance from Gainesville, is increasing [*motions along third segment upward*], and at the same time, the distance the car is traveling is also increasing [*motions left to right on graph*].

Dan responded by describing the location of the car along the road corresponding to the third segment of his graph and the direction of change of the quantities of distance from Gainesville (“distance from Gainesville is increasing”) and total distance the car traveled (“the distance the

car is traveling is also increasing”). At this point in the interview, although I infer Dan thought about his graph as representing explicit distance quantities constructed from the situation, his image of the variations of both quantities was directional.

After Dan’s response in Excerpt 15, he made the comment that the trace of his graph was not affected by time. So we asked Dan what *did* affect the trace of his graph (Excerpt 16).

Excerpt 16. *Dan’s response to what does affect the trace of his graph.*

Dan: So it would be, I guess what would influence it would be if, if this was a curvature that wasn’t circular. So if you just, let’s say instead of having this [*tracing along road above animation on screen*], we had from Atlanta to Tampa in like a uh elliptical ovular kind of shape....Well then the distance traveling is still the same, but now it’s getting closer to Gainesville at a slower rate per se than if it were just coming at a straight line. So, then that would, that would cause this initial line segment to be a little less steep because it’s not getting as close to Gainesville at the exact same amount of uh distance.

Dan’s response suggests that he was constructing and then comparing the change of two cars’ distances from Gainesville (one traveling along the original road and one traveling along an oval-shaped road) corresponding to an equal amount of each car’s total distance traveled along either road. Dan’s reference to a graph representing a car traveling along an oval path as being “less steep” suggested that Dan was imagining the trace of a graph to emerge from the relationship between a car’s amounts of change of distances on a road. Dan’s constructions and operations with amounts of change of quantities in the situation and in his graph illustrate a theme that emerges throughout Dan’s work in later tasks of his consistent reasoning with amounts of change.

In summary, Dan concluded the task thinking about the trace of his graph covariationally. After originally making a thematic association with the car's speed and implicitly conflating the car's total distance with time, it was Dan's thinking about an alternative trace as representing amounts of change of distance quantities that led Dan to reason his original trace represented the covariational relationship between distances he conceived in the situation. Dan's image of the quantities he imagined represented in his graph developed from conceptualizing directional variation to conceptualizing amounts of change of each quantity. His construction of amounts of change of distances in alternative situations (i.e., with an oval road) and his imagining a trace of a graph as emerging from representing such amounts suggests that Dan's images of quantities of distance were not necessitated by the movement of the car in the animation (this is as compared to Jake's image of the quantities with this task including movement). Specifically, Dan conceived relationships between two distance quantities in both situations that did not involve his thinking about the speed or movement of the car in the situation. Further evidence of this is also suggested by Dan's thinking about a point on his graph as representing the car's total distance traveled and distance from Gainesville without regards to the car's movement along the road. This illustrates a feature of Dan's thinking that becomes more salient in his work in the GAG Part II task (see below), that Dan imagined explicit images of distance quantities that were disembedded from his thinking about the car's movement.

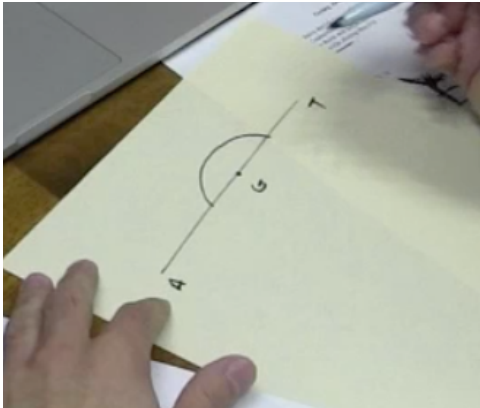
Going Around Gainesville Part II

Jake: Going Around Gainesville Part II

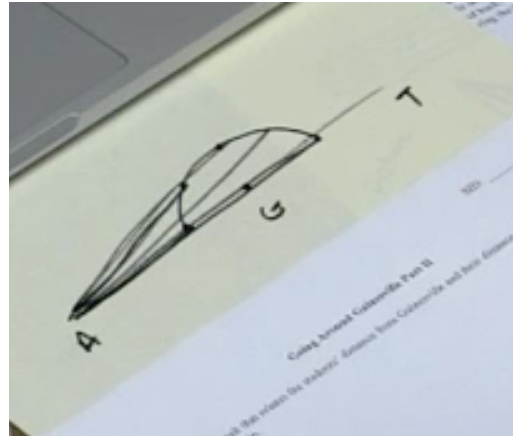
In this section, I provide my analysis of Jake's activity in the second part of the first interview with the GAG Part II task (Figure 14). After attending to the symmetry in the path of

the road (in regards to the car's distance to Gainesville along either the left or right horizontal strip of road), Jake primarily focused on features of the car's varying distances to both Atlanta and Gainesville. Jake drew the initial trace of his graph by associating the variation in distances he conceived represented in the situation with movement along one or the other axis of his graph. His reasoning while drawing and justifying his initial graph suggested that he did not think about his trace as representing a relationship between the car's *accumulated* distances to either city, but rather he thought about his graph as representing the variations of the car's distances to either city. Upon continued reflection and reasoning while making modifications to his graph, his eventual comparisons of accumulated distances of the car to Atlanta at different locations on the path were a marked shift in his thinking and provided him satisfaction that his final trace represented the relationship he conceived in the situation.

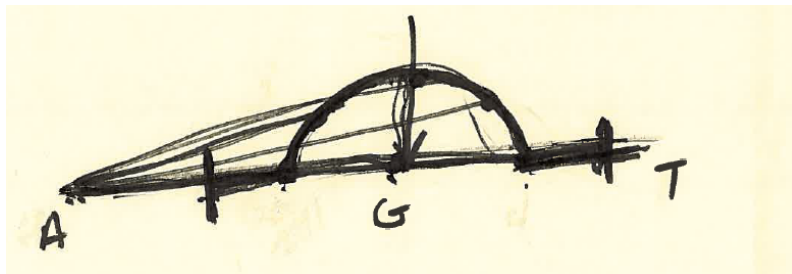
After finishing our inquiries with the GAG Part I task, we informed Jake that we were going to consider the same situation but now with a different prompt. We reset the animation and gave Jake the GAG Part II prompt (Figure 14). After reading the task aloud he expressed that he was unsure how to "start" his graph. He asked for paper and drew the diagram in Figure 19a.



(a)



(b)



(c)

Figure 19a-c. Progression of Jake drawing diagram of road in GAG Part II.

After Jake drew the path in Figure 19a, he marked several points on the semicircle in his diagram of the road and drew line segments on his diagram from ‘A’ to the points he marked on the semicircle (see Figure 19b and Excerpt 17).

Excerpt 17. *Jake’s description while drawing Figure 19b.*

Jake: Like these are all like the same distances from Gainesville [*drawing in lines in semicircle*], but then you have a couple like different distances [*drawing in line segments on diagram from point ‘A’ to marked points on the semicircle—see Figure 19b*].

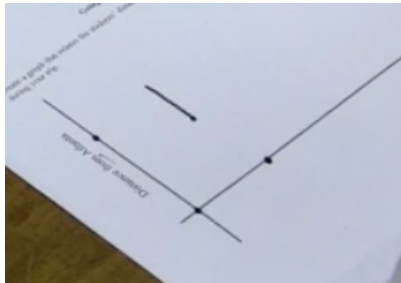
I2: So what are you drawing there?

Jake: The how far I am from Gainesville and then like drawing the distance from this point to Atlanta [*tracing line segment on diagram in Figure 19b from 'A' to left-most point on semicircle*], so you're going to actually have several different distances [*drawing line segments of diagram in Figure 19b from 'A' to remaining marked points on semicircle*], the biggest being from here to here [*drawing line segments of diagram in Figure 19b from 'A' to right-most point on semicircle*], smallest being from there to the other side of Gainesville [*tracing line segment on diagram*]. So the distance from Gainesville, distance [*whispering to self*] [*pause*] Okay, so –

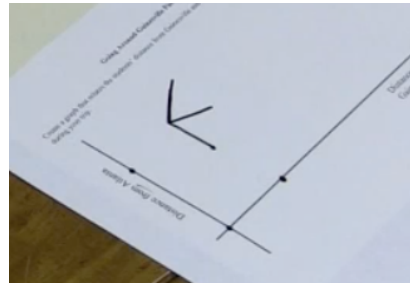
Jake's response suggested that he conceived of the line segments he drew in Figure 19b as representing the distance of the car to Atlanta at the marked positions. He described that the marked positions of the car on the semicircle are all the "same distances from Gainesville" but the corresponding distances to Atlanta are different. This suggests that Jake thought about two distance quantities as represented in the diagram for various positions of the car along the path and that Jake conceptualized the directional variation of these quantities at the various *fixed* locations indicated by his marked points.

Jake then began to draw his graph by plotting a point (the lowest point on the vertical segment of his graph in Figure 20a) and then drawing a vertical line segment (see Figure 20a) as he said, "Actually have a couple of them." Jake continued, "then as you move away [*tracing blue highlighted line segment on diagram in Figure 21*], distance from Atlanta increases, which [*draws in upward line from left-to-right of graph in Figure 20a*]." Jake's activity drawing the remaining trace of his graph (see Excerpts 18 to 21 below) makes me question to what extent at

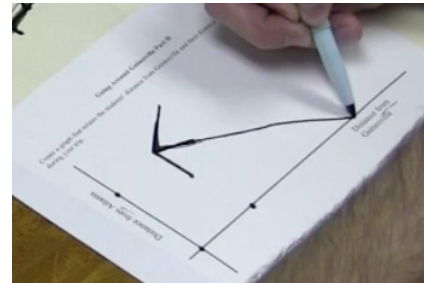
this point of the interview Jake held in mind an image of the car's accumulated distance to Atlanta.



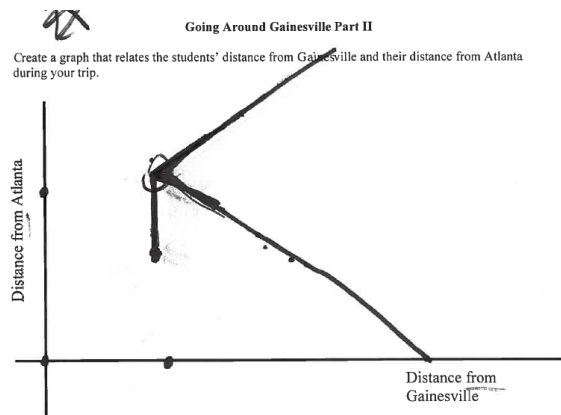
(a)



(b)



(c)



(d)

Figure 20a-d. Progression of Jake drawing initial graph for GAG Part II.

Jake then looked back at his drawing of the path (see Figure 19b). He traced over the two straight-line segments on the road in his diagram (highlighted in yellow and blue in Figure 21). He then drew in two vertical tick marks (highlighted in orange in Figure 21) on his diagram of the road and identified these two locations on his diagram as “equidistant.” He described the car traveling to each of these locations around the semicircle before continuing to draw the trace of his graph in Figure 20b (Excerpt 18).

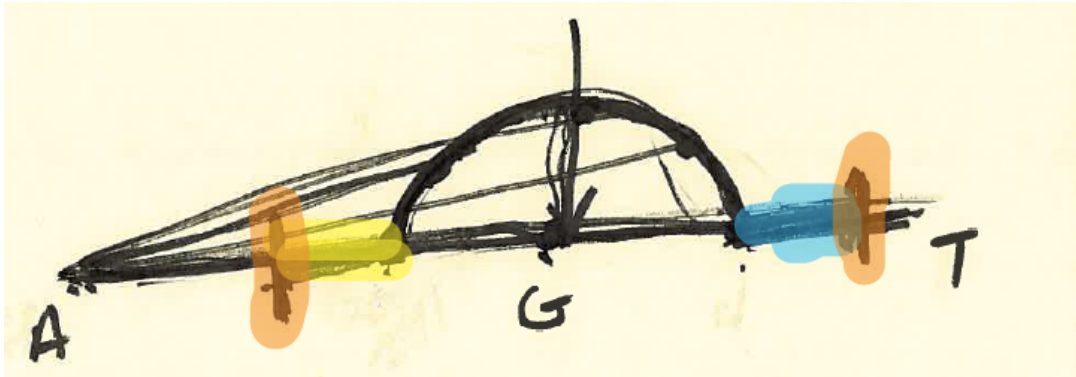


Figure 21. Jake's diagram of road in GAG with highlights.

Excerpt 18. *Jake's response while drawing downward left-to-right line segment in Figure 20b.*

Jake: Okay. [pause] So now I'm thinking about like these two points right here [draws vertical tick marks on diagram highlighted in orange in Figure 21], if these were equidistant. Here your distance from Atlanta obviously is getting longer [tracing line segment in diagram of road from 'A' to right-most tick mark highlighted in orange in Figure 21]. This one not so much [tracing line segment in diagram highlighted in yellow in Figure 21], so one case you have the distance from Gainesville increasing in Atlanta and it's increasing, and then in one point it's also decreasing [draws downward left-to-right linear segment of graph in Figure 20b].

In Excerpt 18, Jake attended to the movement of the car outward from the semicircle around Gainesville in the diagram. This was motivated by Jake identifying two positions along the road in which he conceived the distance of the car to Gainesville as the same (i.e., "equidistant"). In this sense, Jake was orienting himself to the situation and task of representing the car's distance to Gainesville by thinking about the symmetry of the car's distance to Gainesville along the path as the car moved outward along the traces highlighted in yellow and blue in Figure 21. While

reasoning about the car's movement in the diagram, Jake's language and interactions with the diagram provides evidence that Jake conceived of the situation as representing both the car's distances from Atlanta and Gainesville. His behavior also suggests that he conceived of these quantities as varying in direction as the car moved away from the semicircle and towards his two marked points highlighted in orange in Figure 21 ("...your distance from Atlanta obviously is getting longer. This one not so much..."). However, the placement of Jake's non-horizontal line segments suggests he did not hold in mind an explicit image of the car's distance from Atlanta while he drew his graph. Confirming evidence of this is later provided in Jake's continued explanation of the situation and justification of the trace of his graph (see Excerpts 19 to 21).

Next we asked Jake to further explain his graph. He expressed discomfort with what he had drawn and again described the directional variation of the car's distance to Atlanta along each horizontal segment (highlighted in orange in Figure 21) *on the road* (Excerpt 19).

Excerpt 19. *Jake associating directional variation of car's distance to Atlanta with two non-horizontal segments in graph.*

Jake: I don't know, it looks weird, but it somewhat makes sense in my head, I guess. Alright. Distance is still decreasing. But at one point the distance is increasing [*traces blue segment in Figure 21 and then points to upward left-to-right line segment in graph*] from Atlanta and it's also decreasing [*traces yellow line segment in Figure 21 and then points to downward left-to-right line segment on graph*], so you have like two separate parts.

This activity provides evidence that Jake drew his graph by associating the variation of the car's distance to Atlanta he conceived in the situation with the trace of his graph drawn upward or downward (left to right). This also further illustrates that Jake did not conceive his graph as

representing a relationship between the car's *accumulated* distance to Atlanta or Gainesville.

We asked Jake to say more about how he was using his diagram and particularly the vertical tick marks highlighted in orange in Figure 21. He began by justifying the vertical segment of his graph, saying “So that’s why I kind of drew this vertical line [*tracing vertical line segment on graph in Figure 20d*] because, even though the distance isn’t changing, your distance from Atlanta is depending on where you are along this semicircle.” Jake then described the two vertical ticks he drew on his diagram highlighted in orange in Figure 21 as “equal distance from either side of Gainesville.” He then again described the situation as the car moved away from the semicircle to either orange highlighted tick mark while explaining the remaining trace of his graph (Excerpt 20).

Excerpt 20. *Jake describing variations in the car’s distance to Atlanta while the car drives away from semicircle.*

Jake: And again kind of similar to a semicircle depending on what side of the circle you were on. They’re the same distance from Gainesville [*points to each tick mark in diagram highlighted in orange in Figure 21*] but if you go out this path [*traces line segment in diagram highlighted in blue from left-to-right in Figure 21*], it’s increasing the distance from Atlanta [*traces upward left-to-right segment on graph in Figure 20d*], whereas if you’re on the left side of Gainesville [*traces line segment in diagram highlighted in yellow from right-to-left in Figure 21*] it’s decreasing the distance from Atlanta to Gainesville.

I1: So what’s decreasing there?

Jake: Um, decreasing the distance from Atlanta.

Jake persistently attended to both quantities, car's distance to Gainesville and distance to Atlanta, while thinking about the road and situation. After identifying that the car's distance to Gainesville would be the same for either segment he identified on his diagram, he continued to focus on the directional change of the car's distance to Atlanta for those two parts of the road he conceived as the "same distance to Gainesville." In this sense, Jake was talking about the car's distance to Gainesville as the car drove outward along the horizontal segments of his road.

At this point in the interview, it was unclear if Jake was imagining his graph to represent the full trip as he had been attending only to the highlighted path represented in his diagram in Figure 21. We asked him if his graph represented the full trip. In response, he drew the remaining trace of his graph in Figure 20d as he explained (Excerpt 21).

Excerpt 21. *Jake's response representing the entire road trip.*

Jake: Let's see, distance from Atlanta, so it would kind of keep going – [*tracing over downward left-to-right line segment of graph in Figure 20b and continuing this line segment drawing trace in Figure 20c*]. Until you arrived in Atlanta and then [*pause*] this would keep going, and I guess you would stop when you reached Tampa [*tracing over and continuing the upward left-to-right line segment in graph*]. So let's see [*pause*], so I guess if you were thinking like a time series it would just like... you're starting your trip in Atlanta [*pointing to horizontal axis intercept of graph in Figure 20d*], and then you're a certain set distance away from Gainesville, drive a long [*traces finger upward along trace of graph*] distance from Gainesville, decreases, then you reach Gainesville and then you kind of... Well, you never reach Gainesville, but you reach the point at which you have to divert around Gainesville [*tracing finger along semicircle on road in diagram*], and so that would be kind of like this part

going back and forth [*tracing down and up along vertical segment in graph*] and then drive away, which [*pause*].

In Excerpt 21, Jake explained his graph while “thinking like a time series,” from which I interpret Jake to describe his graph as being traced while the car traveled along the road from the beginning of the road trip to Tampa. Throughout this explanation, Jake described the car’s proximity to Gainesville and the direction of variation of the car’s distance to Gainesville. He did not describe an accumulated distance of the car from Atlanta or variations of such in regards to the situation or graph. Jake indicated that movement along the trace of his graph would be “back and forth” as he traced up and down over the vertical segment while the car drove along the semicircle, suggesting he was *not* holding in mind an image of the car’s distance to Atlanta while thinking about this movement on his graph. This provides confirming that Jake was not imagining an explicit image of the car’s distance to Atlanta while thinking about his graph.

After Jake’s explanation, he paused and seemed perturbed by something in his graph. We asked him to share his thinking. He seemed dissatisfied with the initial vertical segment he drew, saying “I think these two parts I’m okay with [*tracing the two (non-vertical) line segments in his graph*].” He then drew a “K” Figure in the margin of his paper as he said, “I’m thinking [*pause*] [*groans*]... I’m thinking maybe it should hit more like [*pause*] this [*draws “K” in margin of paper*].” Below his original graph, he drew a new set of axes and the trace in Figure 22a.

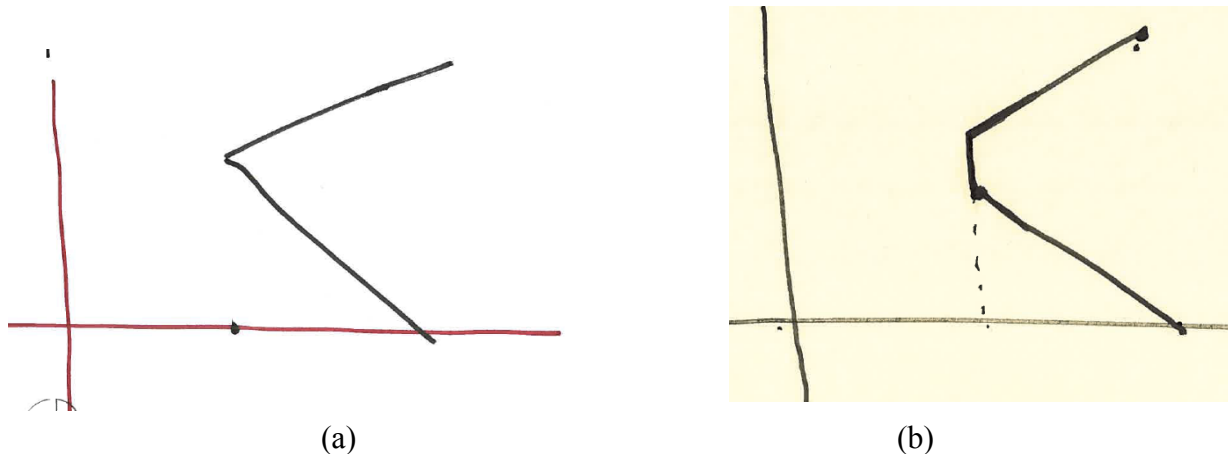


Figure 22a-b. Jake's Second and Third Graphs

We asked Jake to explain his new graph. He responded, “Um, you’re the same distance from Gainesville at this point [*pointing to leftmost point of semicircle in Figure 19c*], it’s not going to change on this point. That starts changing [*slowly traces pen around semicircle arc*]. [*pause*] So....” Jake looked back at his graph in Figure 20d and expressed dissatisfaction again. He asked for a new piece of paper and then drew a new set of axes and immediately drew the graph in Figure 22b. He drew the trace of this graph by starting at the horizontal axis and drawing the trace upward.

Excerpt 22. *Jake explaining trace of graph in Figure 22b.*

I1: So what made you think about that, rather than this kind of ‘K’ or whatever thing?

Jake: Um, yeah. I think I just started thinking about the midpoint, starting at Gainesville, and then I knew that the distance from Atlanta along this kind of arc was changing, and then I kind of realized, well, the maximum is here [*pointing to right end of semicircle of diagram in Figure 19c*], and then it just keeps getting further from that

point, and the smallest distance before you reach the arc was like right here [*pointing to left side of semicircle*]. So [*pause*] Yeah.

Jake's response involved both variation of the car's distance to Atlanta and a comparison of the car's *accumulated* amount of distance to Atlanta at each endpoint of the semicircle. Jake identified that the car's position on the right-most or left-most location of the semicircle corresponded to an extreme value ("maximum" and "smallest," respectively) of the car's distance to Atlanta along the semicircle.

We asked Jake to say more about this, specifically with regards to the placement of the two upward and downward left-to-right linear segments that were no longer connecting at a common point in relation to the vertical segment in his graph in Figure 22b (Excerpt 23).

Excerpt 23. *Jake explaining connection of non-vertical linear segments in graph in Figure 22b.*

Jake: So don't have them connecting anymore because you've reached this point in your trip [*pointing to bottom of vertical segment*] diverting around Gainesville [*traces clockwise along semicircle in diagram*], and then as you kind of divert around – Yeah. As you divert [*pause*] the distance from your point to Atlanta increases [*pause*] until you reach this point [*pointing to right end of semicircle on animation above screen*], and then as you drive away from Gainesville also your distance from Atlanta [*pause*]—Probably not vocalizing that like I should, but –

Jake's explanation in Excerpt 23 of his new graph (in Figure 22b) involved movement along the vertical segment of his graph exclusively upward and associations of this upward movement with the distance of the car to Atlanta increasing (e.g., "distance from your point to Atlanta increases"). Jake's attention to the directional change of the car's distance to Atlanta while he traced upward along the vertical segment of his graph in Figure 22b was a feature that Jake had

previously not attended to while tracing the vertical segment of his graph in Figure 20d. Recall how, in Jake’s explanation of his graph in Figure 20d (see Excerpt 23), he described movement along the vertical segment in his graph “back and forth” as he traced the segment up and down and did not describe what the up-and-down movement would indicate about the variation in the car’s distance to Atlanta. New to Jake’s thinking about his graph in Figure 22b was his imagining the graph as representing the car’s *accumulated* distance from Atlanta. This provided him satisfaction that the trace of his graph represented the relationship between the quantities he conceived in the situation and provides confirming evidence that previously (Excerpts 19-21) Jake was not holding in mind an image of such a quantity while thinking about his graph in Figure 20d.

In this task Jake eventually thought about his graph as representing a relationship between two quantities he conceived in the situation. Within the interactions during the interview, his image of covariation of these quantities (in the situation and graph) obtained a level that involved directional variation; the interview did not involve discussions about the curvature of his graph. Jake’s thinking about his graph progressed from him imagining his graph as representing the relationship between the directional variation of two quantities that did not involve the accumulations of these quantities, to that of him thinking about his graph as representing a relationship between two varying quantities with attention to their accumulated amounts.

Dan: Going Around Gainesville Part II

In this section, I describe Dan’s work and thinking with the GAG Part II task (Figure 14) in the second part of the first interview. Similar to Dan’s activity in GAG Part I, his activity throughout this task provided evidence that he was flexible in constructing and relating quantities

across the context of the situation and his graph. Furthermore, the way in which Dan described characteristics of the car's proximity to both Atlanta and Gainesville while explaining the trace of his graph illustrates that his conception of the relationship between varying distances he imagined his graph to represent entailed an image of the car's proximity to both cities. This suggests that Dan also conceptualized the car's proximity to either city in a way that entailed explicit images of the car's distances to both cities. As the interview continued, the data progressively uncovered the extent to which Dan's actions in perceptual space entailed anticipated operations with distance as a quantity.

To begin this task, we reset the animation and gave Dan the GAG Part II task (Figure 14). After reading the prompt aloud, Dan compared the labels on the axes given in the task with those on his graph in GAG Part I. He described, "now it's flipped over to the distance from Gainesville is now here [*pointing to horizontal axis on GAG Part II task*]. Um, and now it's distance from Atlanta."

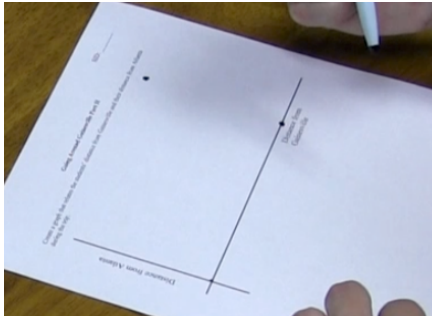
Before drawing a trace on his paper, Dan first described the car's proximity to Atlanta as the car traveled along the semicircle. Dan's language suggested he was describing the variation of the speed at which the car's distance to Atlanta changed along the semicircle in terms of the car's proximity to Atlanta ("it's really not getting further away from Atlanta for a split second, and then it's slowly.... going to hit a quicker rate"). During his explanation, he held his fingers above the screen at the car's location on the path and Atlanta, suggesting an interval of length which I interpret he imagined to represent the quantity of distance to Atlanta.

Dan began to create his graph by first attending to the car's position relative to Atlanta and Gainesville at the beginning of the road trip and when the car was in Tampa (Excerpt 24).

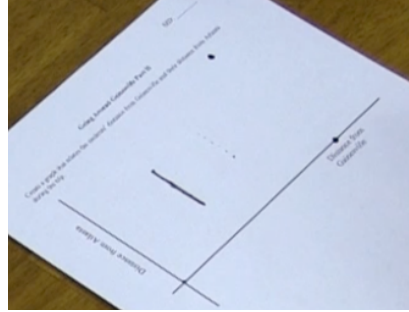
Excerpt 24. *Dan plotting two points on graph in Figure 23a.*

Dan: So, it's in Athens [Atlanta] at the furthest from Gainesville, but it's at Atlanta....So, distance from Gainesville is its furthest here [*marks point on horizontal axis of graph in Figure 2a*] and it's gonna be at the same distance from Gainesville when it gets to Tampa, but now it's gonna be its absolute furthest away from Atlanta. So there's going to be another point somewhere in here [*marks point on his the set of axes directly above initial point in Figure 2a*].

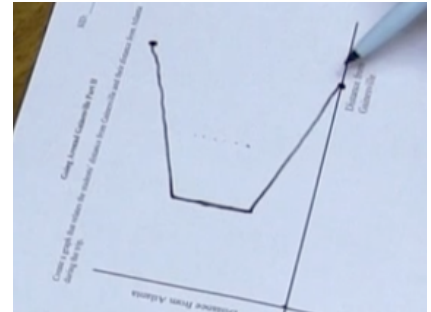
Dan's attention to the maximum qualities of distances at these positions of the car suggests he might have been anticipating gross variations in the car's distance to Gainesville (i.e., inferring the distance was maximum implies his anticipating that distances at other positions were smaller).



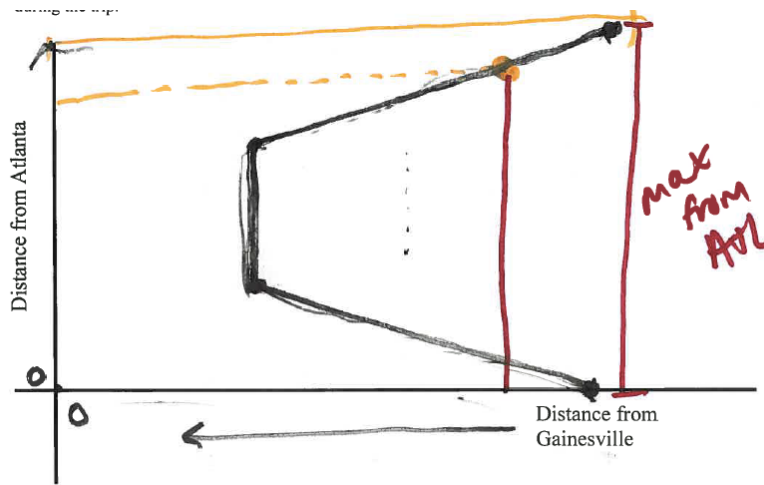
(a)



(b)



(c)



(d)

Figure 23a-d. Progression of Dan drawing graph for GAG Part II.

After Dan plotted the two points in Figure 23a (see Excerpt 24), he described the points again. In his description, Dan associated each point with a car's location on the road. He then compared the car's distances to Atlanta at each location and compared the car's distances to Gainesville at each location. He explained the point he marked on the horizontal axis as "this is when the car is furthest from Gainesville...but is closest to Atlanta. So this would represent when the car is leaving the trip." He then described the car's location in Tampa as "when the car gets to Tampa, it's the same distance to Gainesville as it was in Atlanta... But now it's, its absolute

deviation as far away from Atlanta is as far away as it can be [*pointing to Atlanta and Tampa on video*].”

Dan’s flexibility in describing and comparing the car’s distances at the car’s *fixed location* on the road by attending to the car’s proximity to either city (i.e., how close or far the car was from Atlanta or Gainesville) suggests that his conception of the relationship he intended his plotted point to represent entailed an explicit image of distance quantities from which he could make inferences of the car’s location on the path in regards to the car’s fixed proximity to Atlanta or Gainesville.

Next in the interview, Dan described the variation of both the car’s distance to Gainesville and distance to Atlanta as the car traveled from the start of the trip until it reached the semicircle. He described the direction of the variation of each quantity in terms of the car’s proximity to both cities as the car began its trip saying, “Now the distance from Gainesville closes in but the car is always getting further from Atlanta.” He continued by describing the situation as the car traveled along the semicircle (Excerpt 25).

Excerpt 25. *Dan describing the car traveling along the semicircle.*

Dan: So, if that’s the case, I, it’s closest to Gainesville for a while. So as it gets further from Atlanta, it’s staying the same distance from Gainesville. So there’s gonna be section somewhere in here [*draws dotted vertical line segment*]. It’s ... still getting further away from Atlanta but it’s not getting any closer to Gainesville.

Dan’s response suggests that he drew the dotted vertical line segment by attending to the directional variation of the car’s location relative to Atlanta (“it gets further from Atlanta”) and the car’s distance and location relative to Gainesville (“same distance from Gainesville” and “not getting any closer to Gainesville”).

Dan's action of drawing the vertical line dotted suggested his questioning if his graph could entail vertical line segments. He commented, "The question I have... is [this trace] going to be a vertical line, but it's distance, so again it's not time. So yea, so that makes sense." He then traced over his dotted vertical line with a solid trace (see Figure 23b) while saying "that's gonna be as the car is going around Gainesville." Dan continued to justify his vertical trace by thinking about the quantities he imagined his graph to represent ("but it's distance"). To Dan, this vertical trace represented the relationship he conceived between the car's distances to both cities from the car "getting further away from Atlanta" and "not getting any closer to Gainesville." This provides an instance that illustrates Dan's image of the car's varying proximity (in terms of closeness) to both cities entailed fragments of his anticipated operations with his image of the car's varying distance to both cities as a quantity. However, for this particular part of the path, Dan was not required to attend to variation in *both* of the quantities (in that the distance to Gainesville was fixed as the car traversed along the semicircle). Later excerpts (e.g., Excerpt 28) provide evidence of Dan operating on two varying (non-constant) quantities.

Next Dan connected the end points of his drawn vertical segment with the two points he plotted on his axes (Figure 23d). After completing this trace, he then immediately began to explain the entire trace of his graph (Excerpt 26).

Excerpt 26. *Dan describing upward right-to-left linear segment of graph in Figure 23d.*

Dan: So, this would be the car leaving Atlanta [*pointing to horizontal intercept of graph in Figure 23d*], where it's at its max distance from Gainesville, and it's-it's in Atlanta. So, if this is the distance from Atlanta [*points to vertical axis*], then it's on, it's in Atlanta. And now it's going, it's getting closer to Gainesville, so it, this would be

implying that we're getting closer to Gainesville [*draws arrow pointing left along horizontal axis*]. . . . the distance from Gainesville gets smaller, it's getting closer to Gainesville [*traces along first segment of graph from right to left*] until it gets to the bypass. So, this would represent when the car enters the bypass [*pointing to lowest point on vertical segment of graph*].

In his explanation, Dan frequently associated features of the car's proximity to Gainesville with the car's distance to Gainesville. He made this association with fixed positions of the car ("max distance from Gainesville") and as the car traveled along the path ("getting closer to Gainesville") (i.e., with fixed and varying states of quantities). He represented on his graph the car's movement from Gainesville that he conceived in the situation by drawing a left-pointing, horizontal arrow underneath the horizontal axis of his graph. This suggested that Dan thought about his graph as representing the car's decreasing distance to Gainesville while operating with perceptual features of the car's varying proximity to Gainesville. That is, Dan's thinking of the car "getting closer to Gainesville" entailed an image of the car's distance to Gainesville as a decreasing quantity that he imagined his graph to represent (Excerpt 26).

While Dan described, in Excerpt 26, the upward right-to-left linear segment of his graph, he did not explicitly describe variation in the car's distance to Atlanta. Dan did describe the car "getting closer to Gainesville" but it is not clear from this if Dan interpreted this behavior as also representing the car's distance to Atlanta increasing; after making this claim that the car got closer to Gainesville, he represented this behavior in his graph by exclusively attending to variation along the horizontal axis. He continued by describing the trace of his graph (Excerpt 27).

Excerpt 27. *Dan describing vertical line segment in graph in Figure 23d.*

Dan: From the time that we enter the bypass until the time we leave the bypass, the car is no closer to Gainesville, but it is getting further from Atlanta the entire time. So that would be the distance is strictly going away from Atlanta [*traces along vertical segment*] and no influence on what's going to happen in Gainesville.

Similar to Dan's activity in Excerpt 25, Dan's activity in Excerpt 27 illustrates that his thinking about the vertical segment in his graph involved perceptual features of the car, in this case that the car was "no closer to Gainesville" while also "strictly going away from Atlanta." This suggests his thinking about these features of the car's proximity encompassed a relationship between the distance quantities the vertical segment was to represent. And although Dan attended to variations of two distance quantities in Excerpt 27 while explaining the vertical trace of his graph, he identified that there was no variation in one of these distances (the distance of the car to Gainesville) as the car traveled along the semicircle. At this point in the interview we had not observed Dan engage in actions that suggested he was attending to variations of *both* quantities on his graph in the case when neither quantity remained fixed. Recall that in Excerpt 26 he explicitly described variations only in regards to the car's distance to Gainesville (representing this variation along the horizontal axis).

As Dan continued, he described the upward left-to-right linear segment of his graph and the car traveling along the final segment of the road (Excerpt 28).

Excerpt 28. *Dan describing the upward left-to-right linear segment in graph in Figure 21b*

Dan: Once we leave the bypass, we're still getting further from Atlanta, so we're still going vertically [*draws arrow upwards along vertical axis*], but we're also getting further from Gainesville, which is that way [*pointing to the right*]. So that's why, again,

we're coming back here until we get to Tampa. When we're at Tampa, we're our absolute furthest we can be from Atlanta. So the graph would never go past here [*pointing to the endpoint of the graph*] unless we drive somewhere else.

In Excerpt 28, Dan coordinated the (directional) variation of *two* quantities while thinking about the trace of his graph. This is illustrated by his identifying a directional change along both axes, indicated by his drawing an arrow along each axis. Additionally, his attention to the car's varying proximity to both cities as he carried out this action illustrates that he also operated perceptually with the feature of the car's proximity to both cities while thinking about his graph. This provides confirming evidence of what I inferred from previous excerpts (see Excerpts 26 and 27) that Dan's conception of the car's varying proximity to each city entailed an explicit image of the car's varying distance to each city.

Dan then considered the return trip of the car. He first described the car as "coming back towards Atlanta" and "getting closer to Gainesville." From this, Dan concluded that his graph represented the full trip "because... it would go back the way that it came." We were unsure what Dan was imagining when he made this comment and how he came to this conclusion about his graph. We asked him if he could explain further (Excerpt 29).

Excerpt 29. *Dan explaining why his graph represented the full trip.*

Dan: So, so yea. So this [*pointing to upper end point of trace of graph*] would be when the car is leaving Tampa, this would represent that point, 'cause that point hasn't moved. It's still the same distance from Tampa and Atlanta. Or, excuse me, Gainesville and Atlanta. So when we're in Tampa, we're the same distances. And then as the car starts traveling back towards Gainesville [*motioning along road in video from Tampa to Gainesville*], it's getting closer to Gainesville and it's also getting closer to Atlanta,

so, and at the same, the same amount of distance as it was on the way there. So, it would just backtrack over itself [*tracing graph from upper right point downward*]. So there would, that would be the entire graph.

Similar to his previous activity, Dan conceptualized the trace of his graph (now being traced out downward from right to left) to represent a covariational relationship entailed by acting perceptually in regards to the car's proximity to the cities in the situation. His comment that "it's getting closer to Gainesville and it's also getting closer to Atlanta... and at the same, the same amount of distance as it was on the way there" suggested that Dan was making comparisons of the amount of variations within each quantity across the forward and return trip.

At this point, Dan had identified the direction of change of the quantities of distance to Gainesville and Atlanta and made comparisons in the amounts of change within these quantities. However, at this point in the interview, he had not explained the curvature of his graph (i.e., straight lines). Thus, we asked Dan about how he determined the steepness of his graph. His response is given in Excerpt 30.

Excerpt 30. *Dan describing the curvature of his graph in Figure 23d.*

Dan: Well, the, I actually didn't put much thought into the uh, steepness of that line segment, I'll be honest. So, I just strictly was thinking about distance in relation to distance. So, this is when the car is in Atlanta [*pointing at starting point*]. It's as far as it ever will be from Gainesville. And then as the car travels towards the bypass, so again, this would represent the bypass [*pointing to the point at the top end of the lowest linear segment of graph in Figure 23d*], it's getting closer to uh Gainesville, and it's also getting further from Atlanta at the same distances....But for every mile I'm traveling from Atlanta, I'm getting one mile to Gainesville. So it's almost a one

to one relationship, if I was to draw this to scale. Um, so that's where that comes from. And then once I get to the bypass, I'm still gonna get further from Atlanta, but I'm not going to get any closer to Gainesville.

II: Mm, okay. So when you're saying "one to one relationship," you're thinking of one mile of distance from Gainesville and one mile from Atlanta?

Dan: One mile toward, or one mile from Atlanta and one mile towards Gainesville, yes.

Dan's response suggests that he conceived the curvature of his graph to represent amounts of change of the car's distances to Atlanta and Gainesville as the car traveled along the path. He seemed to imagine these changes with respect to features of the car's proximity to either city both directionally *and* with regards to an amount of distance ("it's getting closer to uh Gainesville, and it's also getting further from Atlanta at the same distances"). This illustrates that Dan's images of the covarying quantities (entailed from his perception of the car's proximity to either city) also encompassed conceptions of relationships between amounts of change of distance as a quantity.

The interview continued with the interviewer marking a point on the graph (yellow marked point in Figure 23d) and asking Dan what this point represented on the graph and relative to the diagram. Dan explained, "if we're talking about time... this will be a point when this is on the trip. It will also be a point on the way back. But, if we're strictly just talking about distance, it's a location on the road." He then proceeded by describing the location on the road that he associated with this point on the graph. It was unclear how Dan determined this location; we asked him to explain how he did so (Excerpt 31).

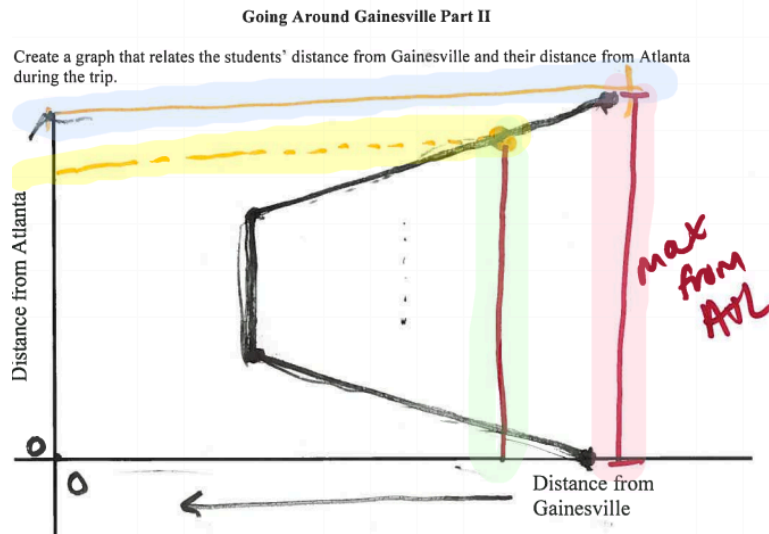


Figure 24. Dan's graph with highlights corresponding to actions in Excerpt 31.

Excerpt 31. *Dan determining location on path from quantities represented by point on graph.*

Dan: So, this is the distance the car is from Atlanta [*draws dotted yellow horizontal segment in Figure 24 from yellow marked point highlighted in yellow*], so it's almost its max. 'Cause again, this is the maximum distance the car will be from Atlanta [*draws yellow horizontal segment in Figure 24 from upper end point of graph trace highlighted in blue*]. So, it's almost as far away as possible. So, max from ATL [*labels "Max from ATL" on graph in Figure 24*], so, it's almost as far as possible from there, and then this is also representing, this is the distance from Gainesville [*draws in red vertical segment in Figure 24 highlighted in green*], which is almost again its max distance that it can be from Gainesville [*draws in red vertical segment in Figure 24 highlighted in red*]. So, that must mean it's extremely far away from both Gainesville and Atlanta.

When thinking about a point on his graph, Dan held in mind explicit images of distance quantities. He determined the car's position on the road by acting with these images. In particular, he first attended to the car's proximity to both cities ("it's almost as far away as possible" and "far away from both Gainesville and Atlanta") by making a quantitative comparison between these quantities he imagined the marked point to represent with the largest distances he conceived his graph to represent. From his inferences about the car's proximity to the cities, he determined the car's location on the path. This illustrates that Dan interpreted and compared images of distance (as represented by a point on graph) in terms of the car's proximity to the cities in the situation and from this inferred a location of the car on the path. Before this instance in the interview, Dan made inferences about quantities from his operations with perceptual structures (i.e., thinking about features of the car's location while drawing the trace of his graph). In this instance, however, Dan inferred perceptual features of the car's location from his conception of the quantities he imagined his graph to represent.

In summary, Dan's activities in this task showed that he thought about the trace of his graph as representing a covariational relationship between quantities he conceptualized in the situation. His conceptualization of this relationship involved his thinking about perceptual features of the car (i.e., the car's proximity to the cities), particularly relating how far (or close) the car approached each city as it traveled along the path. When we asked Dan overtly to show how he inferred such perceptual features in his graph (i.e., the location of the car from a point on his graph), his work suggested that he was thinking about these perceptual features with regards to imagining explicit distance quantities in his graph and the animation.

Taking a Circle Ride

Jake: Taking a Circle Ride.

In this section, I describe and analyze Jake's activity in the second interview with the Taking a Circle Ride Task (Figure 15). Jake began his work by first drawing the trace of his graph. His interview continued with him justifying the trace he drew. Recall, in the first interview in the GAG Part I and II tasks, Jake came to think about his graph as representing a covariational relationship. In the GAG Part I task his conception involved a relationship between amounts of change of two quantities and in the GAG Part II task his conception involved a directional relationship between two quantities. However, in this task I did not observe evidence to suggest Jake imagined a covariational relationship while thinking about his graph. Rather, in this task Jake's thinking primarily involved indexical associations. He associated the curvature of his graph as representing a "varying rate." After experiencing difficulty illustrating why this was so, he eventually reasoned with tangent lines that he constructed at points along his curve. Jake associated visual features of his drawn tangent lines to the "rate of change." His activities suggested that his thinking about "rate of change" did not involve a relationship between changes of two quantities.



Figure 25. Jake's graph for Taking a Circle Ride.

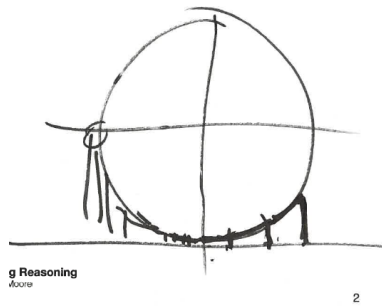


Figure 26. Jake's diagram for Taking a Circle Ride.

After Jake watched the video and read the prompt aloud, he drew a set of axes (without labels). He then drew a dotted horizontal line (see Figure 25). After watching the animation play out several times, he drew the trace of his graph (black trace of graph in Figure 25). We asked Jake to talk about his graph and why he drew it the way he did (Excerpt 32).

Excerpt 32. *Jake's initial explanation for the trace of his graph.*

Jake: Um, starting here at the ground, so this is the total distance, uh, traveled [*labeling horizontal axis "total traveled"*]. This is the height respective to the ground [*labeling vertical axis "ht"*], so start off the ground, uh, they start going around the Ferris wheel. Um, so eventually they'll reach maximum height halfway through and then they'll come back down.

After this initial response, we focused Jake's attention on the shape of his graph and asked if he could explain why he drew a curved graph. He responded with, "Just because it's not a constant rate of change," and claimed the situation instead involved "increasing rates of change."

To explain why the situation was not a constant rate of change, Jake drew the diagram of a wheel (Figure 26). He then partitioned an arc around the circle wheel into equal parts and explained (Excerpt 33).

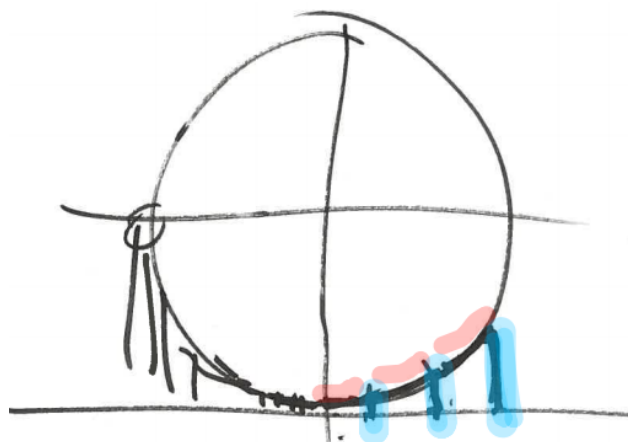


Figure 27. Jake's diagram with highlights corresponding to actions in Excerpt 33

Excerpt 33. *Jake illustrating and describing quantities in diagram in Figure 26.*

Jake: Um, I just see it as I move— a certain, I guess arc along the Ferris wheel [*retracing arcs on wheel in Figure 27 highlighted in red*], um, so all of those are congruent, I guess, arcs. You see that the height [*drawing vertical segments in Figure 27 highlighted in blue*] is a lot different in comparison to these arcs being the same, um, so it's not linear. If it was linear, we would expect these heights to all be the same for each arc along the circle that you move, those heights would all have to be the same.

Jake's activity of identifying equal arc lengths and full height segments suggests that he was comparing changes in arc length around the circle with total height of a rider at the end point of each partitioned arc. Further evidence of this comparison is provided in Jake's response when we asked him what a situation would look like "if the heights were the same?" Jake drew another diagram of the wheel (Figure 28) and then explained such a situation (Excerpt 34).

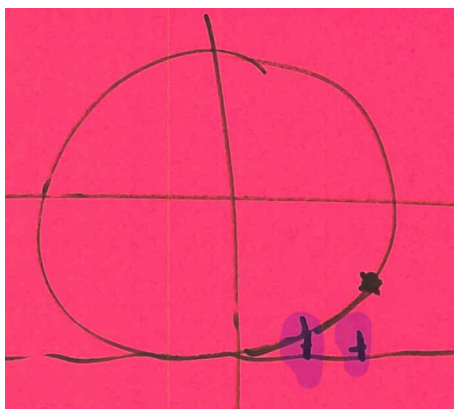


Figure 28. Jake’s second diagram for Taking a Circle Ride Task with highlights.

Excerpt 34. Jake’s response to what a situation would look like “if the heights were the same”

Jake: [pause] [sighs] So carry this thing around. Um, so if these two arcs are congruent [tracing two consecutive arcs on lower right of circle in Figure 28], here’s the height here [drawing left-most highlighted vertical segment in Figure 28], and so if I move this same amount along the arc length [tracing over second arc in partition of wheel in Figure 28], I would expect these heights [drawing right-most highlighted vertical segment in Figure 28] to be the same, um, but it’s not because it’s much larger.

I1: Okay. So because these heights aren’t the same [pointing to two vertical segments highlighted in diagram in Figure 28], we had kind of changing rates of change there.

Jake: Yeah.

Similar to his previous work with the diagram in Figure 27, on Jake’s diagram in Figure 28 he marked two “congruent” arcs along the bottom of the wheel and then drew two vertical segments on his diagram (see highlighted vertical tick mark segments in Figure 28). He drew in the vertical segments to indicate heights being “the same.” Referring back to the original situation, Jake stated that the heights are “much larger.” This provides evidence that Jake’s claim relative to “changing rates of change” stemmed from

him conceiving the accumulated heights associated with the end of each partitioned arc as varying.

Drawing his attention to the trace in his graph, we asked Jake how his *graph* represented this phenomenon of “changing rates of change” (Excerpt 35).

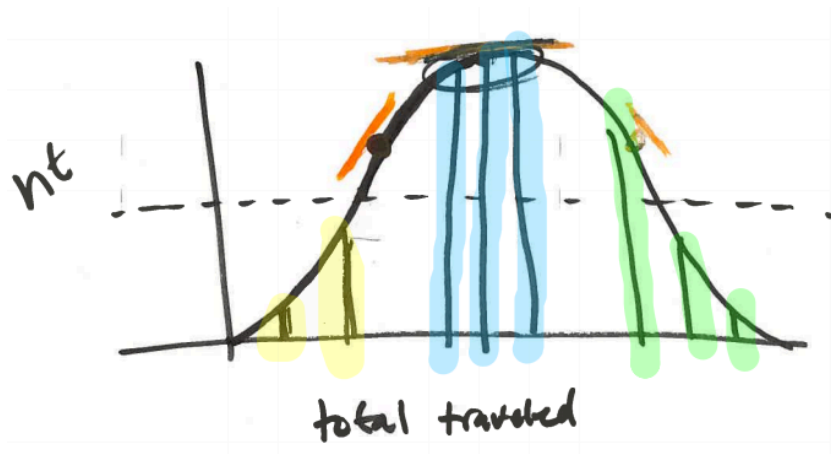


Figure 29. Jake’s graph with highlights corresponding to actions in Excerpt 35.

Excerpt 35. *Jake describing his graph as representing changing rates of change.*

Jake: So [*sighs*] you can just see it here, so initially as you start [*pause*] the one point, the height changed very little [*drawing vertical segments on graph in Figure 29 highlighted in yellow*], but as you move towards the top, the height starts changing much more erectly and much more quickly, and then it starts falling back down as you go towards the ground.

II: Cool. And could you say a little bit more about what it means for like the height to change more rapidly? Like what does that imply and how do we know that graph represents that?

Jake: Um, [*sighs*] you can see like here that’s a big change, but as you start moving towards the top [*drawing vertical segments on graph in Figure 29 highlighted in blue*] these changes [*circling peak of graph above paper*] are not as, uh, different from these, but as you start

falling back down [*drawing vertical segments on graph in Figure 29 highlighted in green*], you can see those [*circling vertical segments in graph in Figure 29*] are much more, I guess, varied or different in heights.

It was unclear what Jake was imagining when he referred to these “changes in height” and how he was making the comparison that “(the) changes are not as... different” or “much more... varied” from the vertical segments he drew. This activity and his circling of the vertical height segments on his graph suggested he might have been making comparisons between the segments he conceived as representing height magnitudes and in doing so he might have been imagining perceptual changes of these segments. Jake’s initial description of the situation representing “changing rates” involved his thinking about the variation of the rider’s accumulated height (see Excerpt 33); he now seemed to be thinking, at least tacitly, about changes in the rider’s height. It was still unclear to what extent Jake was imagining and comparing the “changes” he referred to.

We asked Jake explicitly how he saw “changes” in his graph. Jake responded, “Oh, well, I guess I mean, you can just look at tangents to the curve if you want. So here [*pointing to trace in Figure 29 near yellow highlighted segments*] it’s much more steeper, so I would expect to see a steeper change, but then here [*pointing to trace in Figure 29 near blue highlighted segments*] tangent is pretty much horizontal, so not as much change.” Although I speculate that Jake was imagining a conventional tangent line on his graph at the two points he referred to, it was unclear *what* Jake was imagining to be “more steeper” or how such steepness related to his claims of “changing rates.” Additionally, at this point in the interview, Jake had not engaged in behaviors we interpret to indicate his coordinating amounts of change in the context of the graph or situation, making it unclear how he was thinking about rate.

In order to gain additional insights into Jake’s thinking, we asked if he could say more about tangents and slope. He responded, “So tangents and slope, we’re just talking about a rate of change, so how is, uh, your height changing at a specific place on the wheel or where you are on the Ferris wheel?” From this response, it seemed that Jake associated “rate of change” as “how... your height is changing at a specific place” on the circle wheel, but we remained unsure as to what Jake had in mind when he claimed “how... your height” changed.

We asked Jake if he could provide us an example of what he was describing (Excerpt 36).



Figure 30. Jake’s graph with highlights corresponding to actions in Excerpt 36.

Excerpt 36. *Jake drawing and describing tangent lines on his graph.*

Jake: Okay. So if you take the tangent of the curve at this point to be that [draws yellow highlighted line segment in Figure 30], the rate of change is increasing, whereas maybe as you take it through the maximum point [draws blue highlighted line segment in Figure 30] and your rate of change is zero at that instant.

The first line Jake drew was upward left to right; after drawing such, he claimed, “the rate of change is increasing.” The second line Jake drew was horizontal; Jake claimed here that the “rate of change is zero.” From this activity, it appears that Jake was thinking about the rate of change as a perceptual

feature of a tangent line and as some value (determined by a tangent line). The first conception implied that a line tilting upward left to right (conventionally referred to as an *increasing* line) represented an increasing rate of change; the latter conception implied a horizontal line (conventionally referred to as having a slope of *zero*) representing a rate of change of zero. These ways of thinking about the rate of change may be the same for Jake, in that, in the latter case, an assignment of such a value may be determined perceptually with horizontal indicating no increase or decrease, thus representing “zero.”

We then asked Jake specifically what it meant for the rate of change to be increasing. Jake responded, “It means that for like— as you’re moving along the wheel, as your total distance is changing, your height is also changing as well in the same direction but positive.” He then stated that the rate of change decreasing meant that, “As your total distance traveled was increasing, then your height would have to be decreasing.” Upon further questioning about an increasing or decreasing rate of change in reference to the situation and graph, he persistently repeated these two claims in response to our questions. This provides evidence that Jake’s thinking about varying rates of change involved the *directional* change of two quantities, particularly the increasing (or decreasing) of both quantities.

Across Jake’s activity in this task, we never obtained evidence of him reasoning about rate of change quantitatively by constructing or comparing corresponding changes in *two* quantities. In his work with his diagram, he constructed changes in the rider’s distance around the wheel (see Excerpt 33); in his work with his graph, he referred to “changes” in the rider’s height (see Excerpt 35). However, Jake did not construct or relate changes of *two* quantities in either context of the situation or graph. Jake’s thinking about rate of change led to him drawing tangent lines and primarily attending to perceptual features of such lines. This suggested that Jake’s thinking about varying rate of change and the curvature of his graph were dominated by perceptual features of the shape of a graph.

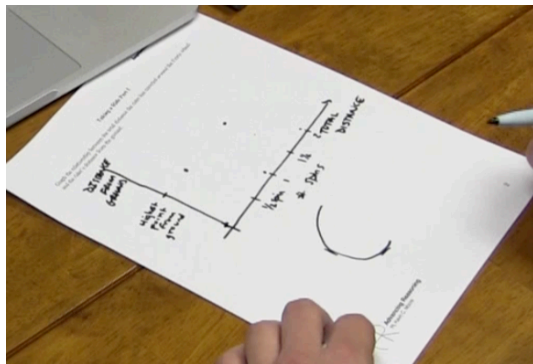
Dan: Taking a Circle Ride.

In this section, I describe and analyze Dan's activity in the second interview with the Taking a Circle Ride Task (Figure 15). Dan's primary activity with this task was to first construct and compare changes in two quantities he conceived represented in the situation. His strategy to do so was systematic in that he constructed equal changes in the rider's distance around the wheel and then constructed and compared the corresponding changes in the rider's height from the ground. To draw his graph, Dan represented these coupled changes of distance and height by accumulating them on his axes. To Dan, the curvature of his graph emerged from his activity representing the coupled changes. Similar to Dan's thinking in the first interview, his conceiving and representing a covariational relationship between quantities involved him reasoning with amounts of change of two quantities.

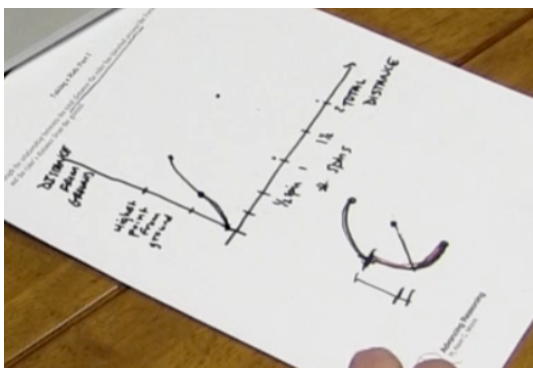
To begin, Dan watched the video and then drew and labeled a set of axes with *total distance* along the horizontal axis and *distance from ground* on the vertical axis. He drew two tick marks on the vertical axis of his graph to represent the maximum and minimum heights of the rider, indicated by him saying, "that's the closest he's gonna be to the ground. And then he will reach a maximum when he's at the very top of the Ferris wheel." Dan then partitioned the horizontal axis of his graph into half-rotations of the wheel (see Figure 31a) and plotted three more points on his graph corresponding to the rider's positions at one half-rotation of the wheel, one full rotation of the wheel, and $1\frac{1}{2}$ rotations around the wheel. As Dan marked the first of such points on his graph, he explained, "Half a spin would imply that he gets to the top of the Ferris wheel, so I know he has to be here [*marking point at first peak in trace, see Figure 31a*]."

Next, Dan questioned how to draw the trace of his graph between these points. "The question I'm having is, will this be a straight line [*indicating a straight line between the first two*

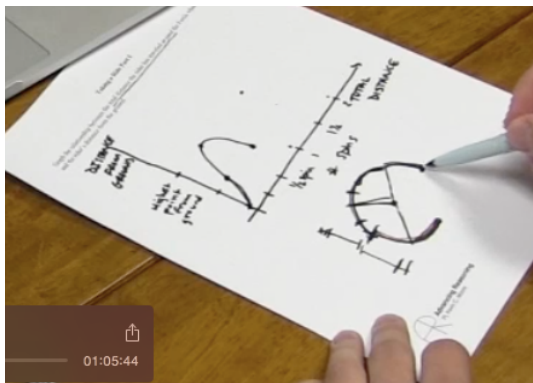
points he marked on graph in Figure 31a] or will there be any curve in it?” To determine the trace, Dan drew a diagram of the wheel (Figure 33) and marked a point at a quarter-spin along the wheel. He then drew a vertical segment corresponding to the rider’s height at this location, saying that this segment is “halfway as high as he can get.” He next referred to a “quarter-spin” being “halfway” along both axes and plotted a point that he marked to represent this. Focusing on the first quarter-arc of the wheel in his diagram, he partitioned this arc into two equal arcs and drew vertical segments that he identified as corresponding to the change of height of the rider along each arc in his partition (Figure 31b). He explicitly compared the amounts of arc lengths and vertical segments he drew in his diagram, identifying the two arcs as equal and stating that the first vertical segment he drew was “less change” than the second vertical segment he drew. He subsequently drew the first part of his graph, corresponding to the rider’s trip along the first quarter of the wheel.



(a)



(b)



(c)

Figure 31a-c. Progression of Dan's work on Circle Wheel Task.

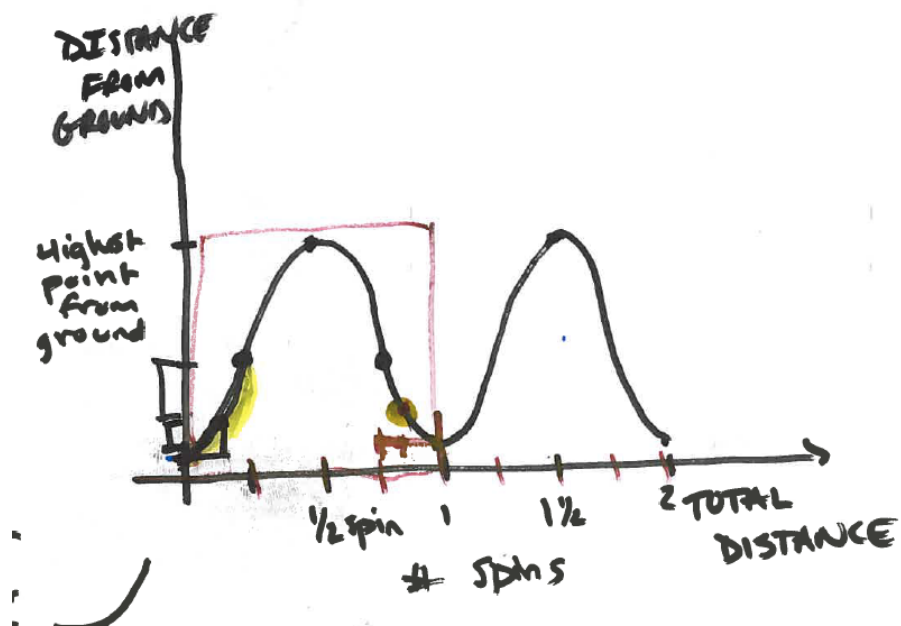


Figure 32. Dan's graph for Circle Wheel Task.

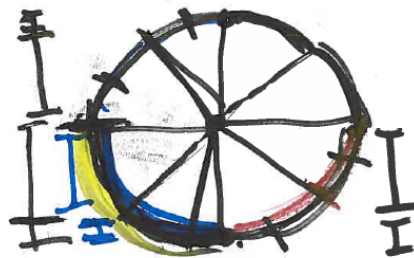


Figure 33. Dan's diagram for Circle Wheel Task.

Dan continued by performing similar activities with the remaining three quarters of the circle in his diagram. For each quarter of an arc, he reasoned in this similar way. He partitioned the arc into two equal arc lengths and constructed the change of the rider's height (which he

represented with vertical segments in his diagram) corresponding to each equal arc in his partition (Figure 31c). He then compared these corresponding changes of the rider's height.

Dan's reasoning with the final quarter of a ride around the wheel suggested Dan drew the trace of his graph to represent perceptual features he attributed to the comparisons he made between the height segments (Excerpt 37).

Excerpt 37. *Dan completing the trace of his graph from the final quarter of the first cycle in Figure 31c.*

Dan: Same thing, one more [*draws final quarter of circle in diagram in Figure 31c*]. That's gotta get us to the bottom again, but then if we draw in our, these should be equal again [*partitions the final quarter circle into two equal arc lengths*], but there's a much bigger change in the height from the ground [*draws vertical segment corresponding to change in height of rider associated to first arc in partition*] versus this one [*draws segment in diagram corresponding to the change of height of rider associated to second arc in partition*]. So it should be steep and then hit [*draws trace of graph connecting plotted points*], and then we would repeat because he goes around the Ferris wheel again, so this is one full revolution and this is just a replica of that [*draws the entire trace of graph in Figure 31c*], but since he's already been around it once, the total distance will continue to increase when he goes around it again. That's why he moves to the right, and it's not just repeating on top of itself.

Dan's activity suggested that he was constructing amounts of change of both quantities he conceived in the diagram and that he thought about his graph as representing these corresponding changes. Dan's reference to the steepness of his graph indicated that Dan conceptualized the perceptual feature of his graph (in terms of "steepness") as representing the relationship he

conceived across his constructed changes of the rider's height in the diagram. For this final quarter of the circle, the "much bigger change" he identified in the second vertical segment was represented with a "steep[er]" trace in his graph.

Although Dan had drawn segments to indicate amounts of change with respect to the situation (see diagram in Figure 33), he did not identify such segments with respect to his graph. Thus, we were interested in better understanding the way in which Dan drew the trace of his graph from the comparisons of the vertical segments in his diagram. We asked Dan to say more about why he drew his graph the way he did from his diagram, particularly from the changes of height and arc lengths that he had constructed and compared in his diagram. Dan responded by explaining why his graph was not a straight line (Excerpt 38).

Excerpt 38. *Dan describing why trace of graph in Figure 32 is not a straight line by using amounts of change.*

Dan: So, this is. So the change in the height from the ground is our y value, what we're calling the y [*holding fingers around vertical axis on graph in Figure 32*] versus this is our total distance x [*holding fingers around horizontal axis on graph in Figure 32*]. So for the same amount of x [*pointing to arcs in circle diagram in Figure 33*], we're having different amounts of change in y [*pointing to vertical segments in diagram in Figure 33*].

II: Okay.

Dan: So the change in y [*pointing to vertical axis of graph in Figure 32*] has to be, so to get from... yea, so to, so the change in y is not staying the same. Or the change in our distance from the ground is not a constant [*holding fingers around vertical axis of graph in Figure 32*].

I1: Okay.

Dan: But the change in x is the way that I'm drawing it are constant, so each one of these sections are the same total distance [*tracing along arcs of circle in diagram in Figure 33*], but the amount that the y value, or the amount that the distance from the ground is changing is not also constant [*holding fingers around vertical segments in diagram in Figure 33*].

I1: Okay.

Dan: So it can't be a straight line.

In his response, he described amounts of change on the axes of his graph both with his words and with his hand gestures suggesting an interval of length. This provides evidence that Dan held in mind the quantities in the situation when thinking about his graph and that he imagined intervals along the axes of his graph as representing changes of the quantities. Dan's justification for concluding why his graph was not a straight line was to systematically construct constant changes represented along the horizontal axis, coupled with changes represented along the vertical axis, and to compare the vertical changes across this collection of coupled amounts of change.

Dan reasoned in Excerpt 38 why his graph was not a straight line, so we asked him why his graph was curved the particular way it was (Excerpt 39).

Excerpt 39. *Dan describing why trace of his graph in Figure 32 is curved.*

Dan: So, I looked at this distance. These, these are all the same amounts on the x -axis [*tracing over arc segments in circle diagram in Figure 33*]. So those are uniform tick marks going pop, pop, pop, pop [*marking horizontal axis along tick marks in graph in Figure 32*]. Like the change in x is a constant.

I1: Okay.

Dan: But then, what I looked at is how much is the y . And if you notice in the first x interval [*holding fingers to indicate first segment on horizontal axis*], there's very little change in y [*marks a segment on diagram corresponding to change of height of rider during first $1/8^{\text{th}}$ of ride in Figure 33*]. It only goes up so much. But then if you look at the next same amount of x [*holding fingers to indicate second segment on horizontal axis*], you get a much bigger change in y [*marks segment on diagram corresponding to change of height of rider during second $1/8^{\text{th}}$ of ride on diagram in Figure 33*]. So it went from only having a little bit of change in y to a big change.

In his explanation, Dan associated the eight equal arc lengths constituting his partition of the circle wheel in Figure 33 with a partition of the horizontal axis of his graph in Figure 32 by accumulating amounts of change on his axis. He then compared changes of the rider's height in the diagram, now attending to the directional variation between two consecutive changes of the rider's height ("But then if you look at the next same amount of x ... you get a much bigger change in y ").

Dan's activity in Excerpt 39, after referring to his partition of the horizontal axis, was to identify and compare changes in the rider's height in the *diagram*. Hence, it was unclear how Dan imagined the changes of height he drew on his diagram in the *graph*. We asked Dan about this explicitly (Excerpt 40).

Excerpt 40. *Dan accumulating changes in the rider's height along vertical axis in graph in Figure 32.*

I1: So then these, these changes [*pointing to vertical segments in wheel diagram in Figure 33*], where are those in the graph?

Dan: Where are the changes? ... The first piece [*pointing to smaller height segment in diagram in Figure 33 near first quarter-spin of wheel*] would represent how much it changed getting from here to here [*marking interval on vertical axis in Figure 33*], and then the second piece would be that larger piece.

In the discussion in Excerpt 40, Dan associated a vertical segment he drew to represent the change in the rider's height in the diagram with an interval segment on his vertical axis. He drew these vertical segments along his vertical axis by accumulating sequential vertical segments from his diagram. This provides evidence that Dan thought about the trace of his graph as representing accumulations of coupled amounts of change on the axes and that these coupled changes represented the coupled amounts of change in the rider's distance around the wheel and amounts of change in the rider's height in the situation.

Although Dan constructed changes in the rider's distance around the wheel into arcs measuring an eighth of the distance around the full wheel and constructed the corresponding changes in the rider's height, he conceived of covariation happening within these completed chunks as well and understood his strategy to include constructing equal changes of total distance around the wheel into recursively smaller-sized arcs. This is shown by him describing that, as "the changes in total distance become smaller then the, these [*pointing to height segments in diagram of wheel in Figure 33*] are gonna kinda shrink down and make more of a curve versus segments." This suggests that Dan's way of thinking about covariation was recursive, in that he anticipated refining his partition around the wheel to entail constructing corresponding changes in the rider's height and that such a refinement would justify the smooth trace of his graph. Such a way of thinking is compatible with Thompson's description of variation as recursively happening within intervals (Thompson, 2011 p. 47).

During this task, Dan reasoned in such a way that the trace of his graph emerged from his representing coupled changes of distance and height. His explicit colored markings of magnitude segments in the diagram suggest that Dan's conception of the covariational relationship in the situation entailed images of amounts of change of the quantities. His thinking about these images of change involved his attending to directional variation between consecutive amounts of change in the rider's height corresponding to a partition of the wheel into equal arc lengths. Features of Dan's reasoning during this task were similar to his reasoning during both of the GAG tasks in the first interview, in that across all of these tasks Dan thought about the trace of his graph as representing amounts of change.

Taking a Square Ride

Jake: Taking a Square Ride

In this section, I describe Jake's activity in the second part of the second interview with the Taking a Square Ride Task (Figure 15). Jake's work with this task began similarly to that of his work in the Taking a Circle Ride task; he began by drawing the trace of a graph and continued by justifying the curvature of his trace and some feature of the "rate of change" that he identified his graph and situation represented. Jake's activity with the Square Ride, however, differed from his activity with the Circle Ride earlier in the interview. Jake justified the straight lines in his graph in Figure 34a by claiming the rider was "increasing at a constant rate." His activity suggests that he thought about constant rate with this task as two quantities that could be related by a scale factor of some fixed unit magnitude of each quantity. Jake illustrated this in the situation and graph by first identifying a "unit magnitude" of the rider's total distance and a "unit

magnitude” of the rider’s height and then constructing other corresponding total distances and height pairs in terms of multiples of the unit magnitudes.

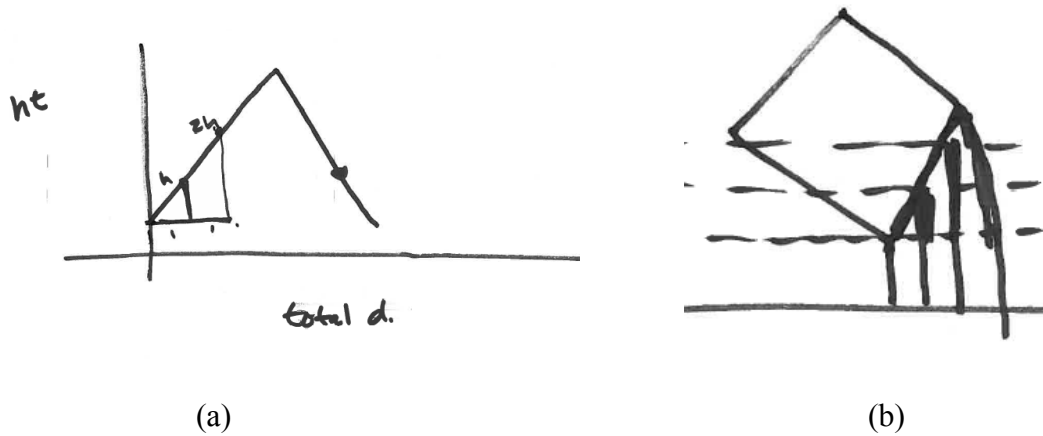


Figure 34a-b. Jake’s graph (a) and diagram (b) in Taking a Square Ride.

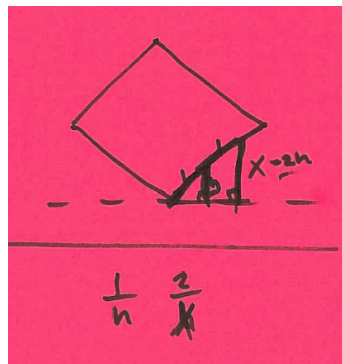


Figure 35. Jake’s second diagram for Taking a Square Ride.

After watching the animation and reading the prompt aloud, Jake drew a set of axes. He labeled the horizontal axis *total d* and the vertical axis *ht* (see Figure 34a), from which I interpret Jake to identify “total distance” and “height.” After pausing to watch the animation play out in several iterations of the carts traversing the wheel, he drew the trace of his graph in Figure 34a. We then asked Jake to explain his thinking. He responded, “He’s increasing at a constant rate in the beginning...still pretty much that same rate along this path [*tracing upper right side of square on animation above computer*],

and then it's the reverse. He starts decreasing at the constant rate [*tracing along upper left side of square on animation above computer*].”

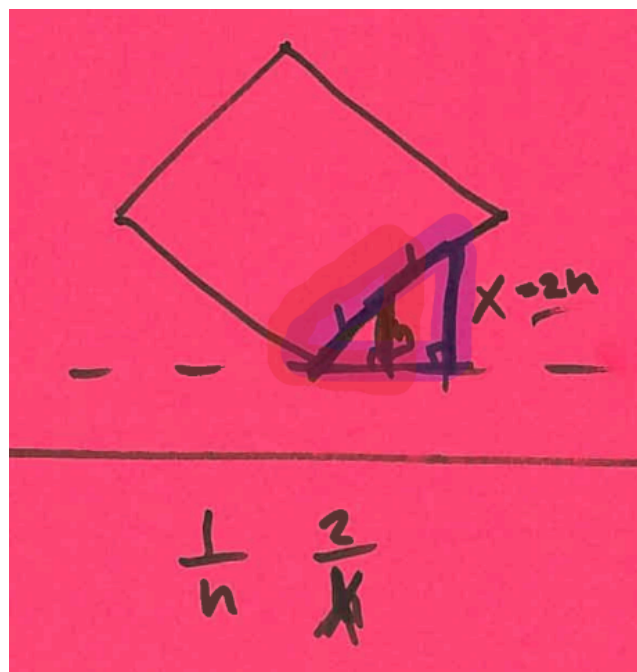


Figure 36. Jake's second diagram for Taking a Square Ride with highlights.

We asked Jake how he saw the rider “increasing at a constant rate.” He seemed unsure how to illustrate this relationship. After various responses, including, “It looks like as he’s moving around the square, the height is increasing by the same amount each time” and “It’s just increasing by the same amount each time,” he drew the square wheel (see Figure 34b). In his diagram, he identified similar triangles (as indicated by the bold segments in Figure 34b and highlighted segments in Figure 36) and explained that the quantities’ magnitudes (indicated by the hypotenuse and vertical leg of the triangle) were each in a fixed multiple of the original corresponding unit magnitudes. Jake explained, “For each unit that I move up, along the square, whatever this height is [*motioning to the total amount of height corresponding to the first “unit up” from original height, i.e., the vertical leg of larger highlighted*

triangle in Figure 36] is going to be twice the original height, or if you want to call it a unit height.”

From my perspective, Jake had not identified a quantity that represented a “change” or “increase” in the height, but rather identified the total height magnitude of the rider as a scaled unit magnitude. We asked Jake where he saw this “increase by a constant amount” that he described. In response, Jake used his diagram in Figure 34b and stated, “As you move here [pointing to top of second vertical segment in Figure 34b], increase by this much [tracing over second vertical segment], if you move maybe twice as long it should be twice as much from here to here [tracing over third vertical segment], three times as much [tracing over third vertical segment].” From my perspective, Jake was identifying each accumulated quantity as a multiple amount of the corresponding unit length that he originally identified (i.e., “twice” the total distance and “twice” the height). I note, that, while subtle, this is different than reasoning about accumulated accruals of amounts of change (i.e., a new quantity obtained from two states of some quantity). Jake’s image of a quantity as a multiple amount of a unit length involved imagining iterating (or perhaps scaling) this *unit*; thus, the image of any two states of the quantity necessarily entailed this unit magnitude rather than a relationship between the states of the quantity representing a change from one state to the next. That is, it was unclear to me how Dan might have been seeing a change between “twice as much” and “three times as much.”

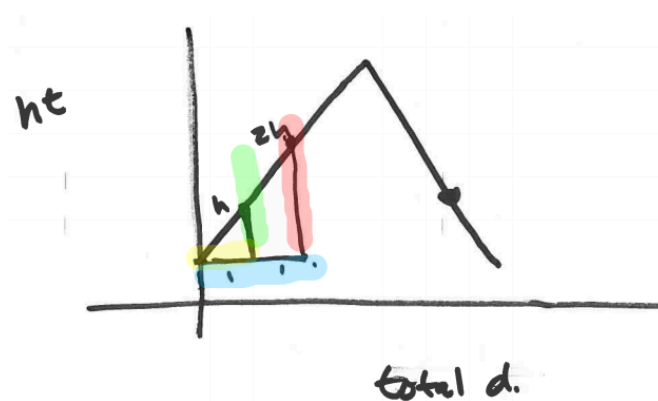


Figure 37. Jake’s graph for Taking a Square Ride with highlights.

Although Jake seemed confident that this “twice as much... three times as much” justified his linear graph, he subsequently expressed skepticism in using this to claim: “height is changing by a constant rate.” We asked him, “How do we see that on the graph? That this idea of constant or rate or whatever?” Jake responded by saying, “You can actually really see it the same here, every one distance you travel [*pointing to yellow then blue horizontal segments in graph in Figure 37*] you should have a given height [*tracing along green highlighted segment in Figure 37*], should be twice the height [*tracing along red highlighted segment in Figure 37*].” Similar to Jake’s activity showing the “constant changes” in his diagram, from my perspective Jake was identifying accumulations of the quantities in his graph, rather than changes.

With this task, we obtained evidence to suggest that Jake came to thinking quantitatively about a covariational relationship he imagined in the situation. However, there was not evidence of him reasoning about rate of change quantitatively by constructing or comparing corresponding *changes* in two quantities. The covariational relationship he conceived was with regards to multiples of unit magnitude of each quantity. His work suggests his image of covariation while engaging in this task did not support his constructing changes in the quantities.

Dan: Taking a Square Ride

In this final section of the Results chapter, I discuss Dan’s work in the second part of the second interview with the Taking a Square Ride task (Figure 15). At the start of this task, Dan acknowledged that he could construct his graph using the same “logic” as in the Taking a Circle Ride task. His activity included him partitioning each quarter of the square (i.e., side of the square) into two equal parts and then constructing the corresponding change in height of the rider along each segment of the wheel. He compared the changes of the rider’s height along each

segment in his partition to draw the trace of his graph. Dan’s work across both tasks in this interview provides evidence that his ways of thinking about covariation and of graphically representing such a relationship involved his constructing, comparing, and accumulating amounts of change.

After watching the animation and reading the task aloud, Dan drew and labeled a set of perpendicular axes (Figure 38). He labeled the horizontal axis as *Total Distance* and the vertical axis as *Distance from Ground*. He then marked a tick on the horizontal axis corresponding to one full trip around the wheel, which he labeled as “one full... squar-ation,” indicating that he imagined the horizontal axis as measuring rotations around the square wheel.

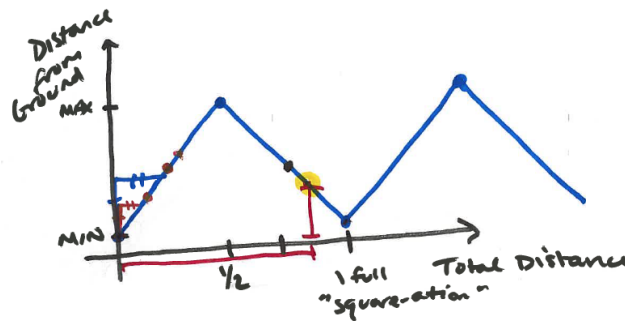


Figure 38. Dan’s graph for Taking a Square Ride.



Figure 39. Dan’s diagram for Taking a Square Ride.

Dan then created tick marks on the vertical axis of his graph in Figure 38 and labeled these marks *max* and *min*. He identified these points as “the highest Brad [the rider] can be from

the ground” and “the lowest Brad can be from the ground.” This indicated that Dan conceived the vertical axis as measuring the Brad’s height from the ground and anticipated an interval measured on this axis within which the rider’s height could be represented.

He then plotted a point on the vertical axis of his graph at the *min* labeled tick mark as he said, “Alright, so Brad has to be at that point, because ... when I start the motion, he hasn’t moved at all total distance, but he’s not on the ground because the Ferris wheel square wheel is slightly off the ground.” This suggests that Dan was imagining points on his graph as representing two quantities—the total distance traveled by the rider and the rider’s height.

After briefly describing certain geometric features of the shape of the square wheel, Dan recalled his work with the Taking a Circle Ride task and acknowledged that he wanted to carry out similar “logic” with this task. He explained, “So I kinda want to go back, though, to my logic which is I know if this is a half a spin [*marks point halfway between origin and “1 full” on horizontal axis in Figure 38*], so he has to be here... here [*marks point on graph corresponding to 1/2 spin and max height*] and he has to be here... here [*marks point on graph corresponding to 1 spin and 0 height*].” Similar to Dan’s work in Taking a Circle Ride, he began by partitioning the interval on the horizontal axis he had previously identified as one full “squar-ation” into two equal parts, identifying the midpoint in his partition as a “1/2” spin (Figure 38). He then plotted two points on his set of axes that he associated with the rider’s location at the top of the square wheel and at the bottom of the square wheel after one trip around (see point at first peak and point at trough of graph in Figure 38). Dan then carried out similar activity as during the Taking a Circle Ride task to draw the trace of his graph (Excerpt 41).

Excerpt 41. *Dan constructing and comparing amounts of change pointing to animation.*

Dan: And then if I was to think about, you guys nicely drew in this halfway point for me [*tracing with finger above the screen over diagonal from center of square to lower right side of square above screen*]. So we know this total distance traveled is the same as that total distance traveled [*motioning with finger above each segment of lower right side of square*] and we also know that the, um, change [*motioning along the corresponding heights associated to each segment of side of square*] in distance from the ground is the same, so that means that Brad's graph would look like this [*draws blue trace of graph in Figure 38*]. And then it would repeat if I was to do it again, so if two full rotations, it would hit back down.

To draw the trace of his graph between his original three plotted points, Dan identified (in the situation) the “halfway point” on the lower right side of the square and compared the total distance traveled along each segment of this side of the wheel (“this total distance traveled is the same as that total distance traveled”). He then claimed that the associated change of distance from the ground for each segment was the same (Excerpt 41). After this explanation, Dan drew the trace of his graph in Figure 38. Dan drawing the entire trace of his graph from this activity with the first side of the wheel suggested that Dan anticipated his activity could be used to reason about the rider's total distance and height along other sides of the wheel as well.

While Dan drew his graph in Excerpt 41, however, he referred to the situation of the wheel and pointed at the animation; he had not used a visual aid or diagram. We were interested in how Dan might illustrate the changes using a visual aid. We asked Dan if he could show the changes he referred to on the screen in a diagram. To do so, Dan drew the diagram representing the first two sides of the square wheel (see Figure 39). He then marked a point on the lower right side of this diagram that he identified as “the halfway point.” After doing so, he identified that

the rider's "total distance traveled is the same" along the segment from the bottom of the wheel to the halfway point and the segment from the halfway point to the end of the lower right side of the wheel. Dan then expressed interest in comparing the corresponding changes in height of the rider associated with each segment constituting his partition of the lower right side of the wheel, saying, "but my question was does that mean that the height from the ground has changed the same amount as well?" He then drew arrows on his diagram to point to these changes of height that he described and said, "So what I did was, what my brain was really trying to get at was is this right here [*drawing arrow pointing to leg of triangle forming change of height along first change of distance*] the same as this [*drawing arrow pointing to leg of triangle forming change of height along second change of distance*] right here? And the answer is yes." This provides evidence to suggest that Dan imagined changes in the rider's height (as indicated by the vertical segments he pointed to in his diagram) and then drew his graph by identifying directional variation between these changes.

Dan then claimed that his strategy could be carried out for any part of the square (Excerpt 42).

Excerpt 42. *Dan using his strategy for "any arbitrary piece" of the square.*

Dan: So, and this is, I said for an eighth but you could do this for any arbitrary piece of the square. So, the total distance, the hypotenuse of these little right triangles [*pointing to triangles in diagram in Figure 39*] are the same and these legs, because they're isosceles right triangles are the same, so that means that the total distance and the distance from ground are the same, so which means that change in distance from the ground over the total distance are the same, which would imply that this should have a straight line.

He summarized by identifying the two triangles he drew in his diagram and claiming that the hypotenuses of the two triangles are the same and the legs (corresponding to the change in height of the rider) are also the same as each other (relating like quantities). He seemed to infer this claim by using the geometry of the triangles in his construction (“isosceles right triangles”). This suggests that Dan’s comment in Excerpt 42 that “the total distance and the distance from the ground are the same” is a comparison with like quantities (i.e., comparison between the total distances and then the distance from the ground) that he associated as being represented in each triangle. From this Dan inferred that the graph would be a straight line.

Dan’s similar activity across both tasks in the second interview (and his acknowledgement that his behavior was consistent), in addition to Dan’s thinking with the tasks in the first interview, provides evidence to suggest that Dan maintained, across all of these tasks, a way of thinking about covariation and graphically representing such a relationship that involved his constructing and operating with amounts of change. In both interviews, for Dan, the trace of his curve emerged from his representing the relationship he conceived between amounts of change he imagined in the animated situations. This is evident across all of Dan’s discussions of curvature. Across all tasks, he thought about the curvature of his graph as either representing the directional variation between consecutive changes of a similar quantity in the situation (e.g., in the Take a Ride tasks he compared consecutive changes of height) or comparisons between corresponding amounts of change across quantities in the situation (e.g., in the GAG tasks he compared change of the car’s distance from Atlanta and distance from Gainesville). More data is necessary to further analyze Dan’s reasoning and images of these amounts of change.

CHAPTER 5

DISCUSSION

In this chapter, I discuss themes that I have drawn from the results of this study. In this study, the participants' activities and tendencies to reason covariationally with respect to their drawn graphs varied. Looking within each participant's work across the interviews, I first present preliminary models of Jake's and Dan's mental actions that surfaced while they engaged in the tasks. I draw these models from those activities that I observed dominated their thinking during the tasks. From these models I attend to the propitious and inauspicious features of the participants' actions. I infer that the construction and coordination of corresponding amounts of change and accumulations thereof afford individuals the ability to consistently interpret and graphically represent the animations covariationally. I continue this chapter by elaborating on this. I then describe other features in the participants' thinking, particularly focusing on the associations Jake made throughout his activity. I illustrate ways in which these features may present a barrier for one to successfully represent or conceive a covariational relationship.

Preliminary Models of Participants' Reasoning

In this section, I provide brief models of the experts' thinking by summarizing the main ways of reasoning that I observed each expert to enact across the interviews. The models I generate of their thinking are an attempt to explain the mental actions that possibly contributed to their behaviors. These models are hypothetical, implying how the expert may have been thinking. More data and interactions with each expert would be necessary to refine and test the

models that I describe. From these preliminary models the two initial inferences of interest are the extent to which each expert maintained a consistent way of reasoning across the tasks and the extent to which the role of quantitative reasoning played a part in characterizing each expert's mental images. In the case that I inferred an expert to maintain a consistent way of reasoning across tasks, I conclude that he had "developed a pattern for utilizing specific meanings... in reasoning about [the] particular ideas" (Thompson et al., 2015, p. 12) during the tasks. These repeatedly constructed meanings constitute his *way of thinking* with regards to the particular ideas. With these models and initial inferences there are several implications for future work towards understanding and supporting students' covariational reasoning.

Jake's: Summary of Actions

I first describe my interpretation of Jake's thinking during the interview sessions. In the first interview, with the GAG Part I and II tasks (Figure 14), Jake eventually conceived that his graph represented a covariational relationship between the quantities he conceived in the situation. Although the progress of his work was different in each task (in GAG Part I Jake first made thematic associations and then holistically thought about his graph in terms of quantities, while in GAG Part II Jake attended to the variation of the quantities in the animation in order to draw the trace of his graph), throughout each task he eventually resorted to thinking about his graph and the situation covariationally. Jake conceived of the relationship represented in the situation and graph in a way that entailed constructing and relating amounts of change of quantities. The manner in which Jake came to think about his graph in this way (particularly those aspects in his reasoning with amounts of change that marked a shift in his thinking) provides insight into particular ways of reasoning with amounts of change that are productive for one to think about when considering graphs as representing a covariational relationship. I expand

on these ways of reasoning in the next section (see the Accumulating Amounts of Change section below for a discussion of one such shift).

Jake's initial activities in the second interview primarily involved making static shape associations between his animation and his graph. Jake associated perceptual features between the animation and the drawn trace of his graph, as well as between "rate of change" and a tangent line. These associations seemed based on perception, in that I did not observe Jake reasoning quantitatively about those objects constituting his associations (i.e., tangent lines or rate of change) to an extent that justified and illustrated the inferences he made. These perceptual associations led to ways of reasoning that (in my view) were not coherent and included contradictory results across the tasks. For example, in the Circle Ride task Jake claimed that the "rate of change" was increasing for the first part of his graph and that an increasing "rate of change" means that "as you're moving along the wheel, as your total distance is changing, your height is also changing as well in the same direction but positive" (see discussion following Excerpt 36). From my perspective, the directional change he described is also represented in the relationship he identified between the quantities in the Square Wheel task. For this latter situation, however, Jake claimed the "rate of change" was constant, rather than increasing. This provides evidence to suggest that Jake did not draw his conclusions from a connected meaning of covariation. I elaborate on the associations Jake made in the final section of this chapter. I note that I use quotation marks around rate of change in order to communicate Jake's use of the phrase, as opposed to the normative definition describing a covariational relationship and multiplicative comparison of how one quantity changes in relation to the other. In what follows, I similarly use quotation marks around words that may hold normative quantitative definitions and meanings when I am not referring to those words' normative definitions.

Jake engaged in different activity across the four tasks and two interviews from my perspective. For Jake, these tasks may have seemed to be addressing different problems in that they involved different forms of reasoning in order to produce a graph. I infer from this that Jake did not hold in mind a way of thinking such that he conceived these tasks to entail the same mental activity. Moreover, Jake's work in the first interview involved his reasoning with amounts of change, but only after interview probing. During the second interview I did not observe Jake to reason with amounts of change in either task. Taken together, Jake's activities also reveal that his way of reasoning about covariation did not involve reasoning with amounts of change consistently.

Dan: Summary of Actions

Next I summarize Dan's thinking during the interview sessions. Throughout each of the four tasks, Dan's conception of the situation and graph entailed reasoning with amounts of change. Across the two interviews there were subtle ways in which Dan's reasoning with amounts of change differed, particularly with regards to Dan making comparisons between amounts of change of different quantities (e.g., distance from Atlanta and distance from Gainesville in the first interview) or between amounts of change of similar quantities (e.g., the rider's height in the second interview). More data would be necessary to further analyze Dan's thinking about these amounts of changes, but the illustrations in the results provide some insight into ways in which his reasoning about amounts of change supported his conceiving his graph as an emergent trace of covariation.

Dan's work in the first interview illustrates the flexibility of his thinking about amounts of change between the context of his graph and the situation. In the GAG tasks in the first interview Dan constructed road and city arrangements from the trace of a graph and then

imagined the reverse, the trace of his graph representing a road and city situation. Both his resulting images (a road and graph) emerged from his reasoning constructing amounts of change in the original context. In GAG Part I, in exploring the trace of a “flatter” graph, Dan inferred that for such a car, the car would travel no closer to Gainesville for the same amount of total distance traveled. Dan also described a road that was “more ovular” and made conclusions about the trace of a graph produced by the relationship between the quantities in such a path. When comparing the relationship between quantities he conceived in a “more ovular” path with that relationship between quantities in the original path, he inferred that “the distance traveling is still the same, but now it’s getting closer to Gainesville...that would cause this initial line segment [in graph] to be a little less steep” (Excerpt 16). This provides evidence that Dan’s thinking about the car’s visual features (i.e., movement and proximity of “getting closer” to Gainesville) entailed his thinking quantitatively with distances.

In the second interview involving the Taking a Ride tasks, Dan’s reasoning with amounts of change was more explicit and systematic in that he drew segments representing amounts of change in his diagram and then represented and coordinated these same segments along his axes in order to draw the trace of his graph. His activity in constructing these changes involved partitioning one quantity (distance around the wheel) into equal parts and then drawing segments representing the change in the rider’s height corresponding to each segment in his partition of the wheel. His coordination of these changes involved his accumulating each pair along his axes. His graph emerged from his representing the relationship between the amounts of variation that he conceived in the diagram. I expand on his thinking about these objects more in the following sections.

Dan’s reasoning with amounts of change helped him in conceiving consistency across the tasks. Dan himself acknowledged the similarity in his reasoning during the Taking a Circle and Square Ride tasks; when working on Taking a Square Ride, Dan explained that he could use the “same logic as before” (referring to the “logic” he used during the Taking a Circle Ride task). Due to the consistency of Dan’s activities, I infer that he held a way of thinking of covariational reasoning that enabled him to conceive and compare situations and graphs in terms of amounts of change.

Reasoning with Amounts of Change: Constructing, Comparing, and Accumulating

The results of this study and the models of the experts’ thinking highlight the central role of reasoning with amounts of change in one conceptualizing and representing a covariational relationship. It was only through joining two quantities’ amounts of change *and* accumulations into a multiplicative object that enabled these experts to reason covariationally about the trace of their graphs. By analyzing and comparing their actions, I have identified important aspects involved in this way of reasoning (e.g., the construction, coordination, and accumulation of amounts of change). To illustrate such aspects, I highlight instances in the participants’ work that demonstrate such ways of reasoning.

Coupled Amounts of Change

First, I illustrate the importance in the experts’ thinking about the trace of their graph as representing amounts of change. Focusing on the GAG Part I Task (Figure 14), each participant initially justified the straight segments in his graph by associating those segments with the constant speed as conceived in the situation; each expert made a thematic association with the speed of the car in the animation. In considering alternative speeds—a faster car (Jake) or the car

starting from a stop (Dan)—each drew a new trace based on associations with speed, again suggesting thematic associations. In attempts to interpret or justify their proposed traces representing a new situation (see Figure 16 for Jake and Figure 17 for Dan), however, each came to discard his new trace and return to his original trace. It was in reasoning with these amounts of change that each participant conceived that the represented relationship (in his alternative trace) was no longer consistent with his image of the situation. Furthermore, by comparing amounts of change, each participant came to understand that his initially constructed trace represented the relationship he understood as constituting the situation. In another example, the progress of Dan's work in the second interview during the Taking a Circle and Square Ride tasks (Figure 15) was to first construct changes in the rider's distance around the wheel and couple these with changes he constructed in the rider's height from the ground (see Excerpts 37 and 41). He then associated such coupled magnitudes with intervals on the axes of his graph. This activity provided Dan a way of drawing his graph such that the trace that emerged represented the quantities in the situation with intervals on his graph. In this sense, for Dan, the relationship between the coupled amounts of change, as constructed and represented in his graph and situation, was invariant. These examples illustrate how reasoning about coupled amounts of change enabled Jake and Dan to conceive the situation and graph in compatible ways entailing accumulated distance quantities and as representing an invariant relationship across contexts.

Second, the above examples of their work emphasize the iterative process of constructing a graph and reflecting on those constructs. Returning to the example of their work in the GAG Part I task, it was by each of them comparing amounts of change that he conceived as represented by his alternative trace with those he constructed in the situation that led to his conceptualization of his original trace as indeed representing the intended relationship. Both Jake

and Dan mentally operated upon their images of amounts of change (constructed in the graph *and* situation) by making comparisons between these objects and across context. Dan's work with the Taking a Ride tasks illustrates a different way of operating with coupled quantities. Dan made comparisons of the changes in the rider's height (across equal changes in the rider's distance around the wheel). To carry out such operations on these coupled objects, I speculate that for both Jake and Dan these amounts of change (from their graphs and situation) were mentally united to "make a new conceptual object that is, simultaneously, one and the other" (Thompson & Carlson, 2017, p. 433). This conceptual object produced is what Thompson and Carlson referred to as a *multiplicative object*.

Accumulating Amounts of Change

Another important aspect of the participants' conceptualizing and representing a covariational relationship was their attention to accumulations inferred by variations of the quantities. Below I discuss the importance of this aspect by illustrating an example in which an expert conceived his graph covariationally in a way entailing *accumulated* amounts of change. I then provide an example in which an expert thinking about accumulated amounts of change made a marked shift in his thinking about his graph as representing the covariational relationship he conceived in the situation.

Dan's work, particularly during the Taking a Ride tasks (Figure 15), provides an illustration of reasoning involving accumulating amounts of change that were productive for his graphically representing a covariational relationship he conceived in the situation. As I summarized above, Dan's reasoning in this interview involved constructing amounts of change in the situation and associating such changes with intervals on his axes. Dan identified these intervals in a way that involved iterating their placement, drawing one after the other by

connecting the segments at their endpoints; thus, he conceived the result of this process to be accumulated magnitudes represented on his set of axes (see Excerpts 39 and 42 and Figures 32a-c and 38). His accumulating the amounts of change along his axes to draw the trace suggests that his thinking about the trace of his graph also entailed a multiplicative object of these accumulated quantities that simultaneously entailed their accruals (e.g., amounts of change).

Evidence of the importance of this way of reasoning, with accumulated amounts of change, is also suggested by Jake's work on the GAG Part II (Figure 15) task. Jake's activity provides an instance in which his reasoning with accumulated amounts of change marks a shift towards him thinking about his graph as representing the covariational relationship he conceived in the situation. In this task, Jake described variations in each of the quantities he conceived within the situation (e.g., "as you move away, distance from Atlanta increases..." see discussion following Excerpt 17). Jake drew his first graph in a way that he understood to represent these (directional) variations (without accumulations). After describing both quantities in the situation as increasing, he drew an upward left-to-right line. After describing the car's distance from Atlanta as decreasing as the car's distance to Gainesville increased (recall, these are the quantities that Jake understood to be represented on the vertical and horizontal axes of his graph, respectively), he drew a downward left-to-right line. In drawing these segments, Jake positioned them on his graph such that their endpoints touched at a common point (see Figure 20d). Jake became unsatisfied that this trace represented the relationship he conceived in the situation, discarding this graph and drawing a new trace (see Figure 22b). Jake's explanation of the trace in his new graph involved both variations of the car's distances *and* a comparison of the car's *accumulated* amounts of distances across two locations of the car on the path (those two locations which Jake conceived as corresponding to those endpoints of his linear segments

previously touching in his original graph) (Excerpt 22). Jake identified that the car's position on the right-most or left-most location of the semicircle corresponded to extreme values ("maximum" and "smallest," respectively) of the car's distance to Atlanta along the semicircle and that the original placement of the line segments touching in his graph did not represent these extreme values. This provides evidence that while Jake was initially thinking about his graph as representing variations in the quantities he conceived in the situation, he was not holding in mind a sustained image of an accumulated quantity that resulted from this variation. This emphasizes the importance of attending to *both* the products of one's actions obtained by covarying quantities (i.e., the changes in the quantities) and the resulting magnitude (i.e., the accumulated amounts) when graphically representing a covariational relationship.

Reasoning with Associations: Lines and "Rates of Change"

As I discussed in the summary models of the experts' thinking, Jake's thinking during the Taking a Ride tasks did not involve his reasoning with amounts of change; rather, much of Jake's work suggested associations involving visual or experiential features which he did not unpack quantitatively. This provides no evidence that the associations Jake made stemmed from his abstractions while operating with quantities. Additionally, such associations led to Jake's inconsistent reasoning. This suggests that such activity may not be productive for one developing a way of thinking about graphically modeling covarying quantities so that quantitative invariance and relationships are maintained. Since similar associations are especially pervasive in student activity as reported in previous studies (see Background chapter) and as I have observed in my own students' work, I find it important to elaborate on these associations Jake made in his work

on the Taking a Circle and Square Ride tasks (Figure 15). I find such a discussion productive for exploring the nature of these associations.

Jake held different ways of understanding “rate of change” across the two tasks in the second interview, neither of which entailed his thinking quantitatively with change. During the Circle Ride task, for Jake *varying* “rates of change” signaled the use of “tangent” lines (which I elaborate on in the next paragraph). His way of thinking about *constant* “rate of change” implied identifying a “constant” between two quantities he conceived in the situation and inferred his graph to represent. We did not obtain evidence throughout either task of Jake constructing or reasoning with changes (in his graph nor in the situation). This suggests Jake was making associations based on thinking about his graph and “rate of change” consistent with static shape thinking.

Jake’s further reasoning with “tangent” lines on the Taking a Circle Ride task illustrates additional evidence that his reasoning was based in static shape thinking. Jake’s work on this task primarily relied on an indexical association between the tangent line at a point (see Figure 25) on his drawn trace and “rate of change.” For Jake, the perceptual feature of a “tangent” line at a point on his graph “increasing” (i.e., upward left to right) meant the “rate of change” was increasing (see Excerpt 36). Jake maintained this association throughout his activity with this task, suggesting that such an association might prevent one’s further construction and exploration of quantities.

This finding is troubling in that in a typical calculus course associations between the tangent lines of a graph and a rate of change of a function are explicitly stated or communicated to students. For example, consider Figure 40 of a snapshot of a summary passage in a college

calculus textbook currently used at the institution where Jake and Dan were students at the time of the interviews.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

Figure 40. Interpretations of “difference quotient” from Weir, Hass, and Thomas (2010).

List items 2 and 3 in Figure 40 illustrate that one can associate the “slope of the tangent to the curve” with “rate of change.” Nowhere within or near this passage is covariation mentioned. Illustrations and summary figures such as those in Figure 40, which emphasize interpretations of quantitative objects (such as the rate of change) without regards to quantities and their covariation, may introduce an environment in which students develop associations grounded in static shape thinking. Without the opportunity to develop these concepts quantitatively, students may attribute various features across these associated objects (such as an “increasing” tangent line with an “increasing rate of change”). For example, there are multiple studies in which researchers have illustrated the complexities of students’ and teachers’ understandings of slope (Coe, 2007; Lobato & Thanheiser, 2002; Nagle, Moore-Russo, Viglietti, & Martin, 2013; Stump, 1999). Thus, encouraging students to interpret “the rate of change of a function” and the “slope of the tangent to the curve” (Weir et al., 2010) provides minimal, if not no, opportunity for students to develop conceptual understandings rooted in covariational reasoning. This also presents potential for students to come away from calculus courses with

certain associations that prevent their later development of such concepts. Listing items such as those represented in Figure 40 does not illustrate a coherent quantitative structure between quantities in which “slope of tangent” and “rate of change of function” both refer to a relationship between covarying quantities.

Conclusion

This report extends previous observations of Carlson et al. (2002) and Johnson (2012a, 2012b) on the difficulties college-level and younger students have in constructing and coordinating amounts of change of quantities. Indeed, I illustrate that maintaining an image of simultaneous amounts of change is also not trivial for experts. The evidence suggests that it was the construction of coupled amounts of change and accumulations of such that afforded Jake and Dan the ability to represent and interpret the animations covariationally. Additionally, this study shows that such ways of reasoning may be productive in one conceiving consistency across these types of task.

These findings are informative for designing productive experiences for both students and mathematics instructors to develop covariational reasoning abilities. Insights into those reasoning processes continued mathematics users engage in provides a picture of what calculus students *are* coming away from mathematics courses with. Understanding this could allow us to make changes in the types of activities developed for students and teachers in order for them to come away from these experiences with more productive reasoning abilities.

Limitations

In this study, I analyzed the activities of two experts' work during two interviews sessions each. As such, the data that I analyzed are limited to two individuals' behaviors over the course of only a few activities. These behaviors are not necessarily those that would be performed by other experts or even by these two particular experts in a different setting. However, what these results do provide are possibilities for actions that did surface in these instances. Such work is meaningful in establishing a foundation from which to further build models of expert thinking in tasks investigating covariational reasoning.

Additionally, the data from both interviews was collected before analysis of the first interview session. As such, the models that I attempt to build of the participants' thinking could not be tested in subsequent interviews in order to refine their viability. The amount of data provides evidence to make general claims about how the experts were thinking about each task, but may not be sufficient to establish the stability and depth of their meanings and actions.

As a final limitation, my access to each expert's thinking was limited to observing his recorded actions, behaviors, and written work. Such behaviors and actions may signal certain ways of thinking but might not depict one's *actual* ways of thinking. Another person's thinking is unknowable to an observer. Additionally, as a researcher I may also misinterpret such recorded interactions. It is through my own observations of the data that I draw results. Thus, the claims I make are from my own interpretations of the videos and work I witnessed and may be different than another observer's analysis of the same data.

Directions for Future Study

This study provides a glimpse into what covariational reasoning might look like in experts and other mathematicians and thus provides new directions to explore in seeking a more developed portrait of this population's mathematics. In this final section I briefly summarize a few main themes that I have observed from the data that seem to crave further exploration and development.

One result of this study describes the role of reasoning with amounts of change in one conceiving and representing a covariational relationship and describes certain aspects that may be involved in such reasoning. This result should be investigated further. One program of research could explore and characterize behaviors of reasoning with amounts of change. The descriptions and aspects of reasoning I provide in the above sections are rather coarse, mostly describing the context of one's constructed amounts of change (i.e., in a situation, graph, or accumulation thereof). This leaves the path of further classifying one's reasoning with these objects wide open. I would be particularly interested in exploring the relationships one might form between amounts of change and across coupled amounts of change. A research program could also investigate the role of this type of reasoning in one developing sophisticated covariational reasoning abilities.

Reflecting on their drawn graph and relating their graph to the situation played important roles in both experts' conceiving a covariational relationship represented in the animation. In particular, I described such activities in Jake's and Dan's work in the GAG Part I task in the section "Coupled Amounts of Change." It was in reflecting on his drawn graph as representing quantities and how they change together that each expert came to conceive a covariational relationship between distance quantities in the situation. It is interesting to consider how one's

process of reflecting on a graph and relating such activity to a phenomenon might result in one refining her or his image of a covariational relationship constituting that phenomenon. A direction of research could entail investigating the nature of one's reflections on a drawn or given graph, the role of such activities in one's developing a sophisticated image of covariation, and those experiences that might perturb one to make such reflections.

Finally, another interesting feature of the experts' activities is the fact they drew and annotated diagrams of the situation in the process of drawing the trace of their graph. This suggests that such diagrams supported them developing images of and relationships between the quantities. Furthermore, it seems that acting with their constructed images of the quantities in their drawn diagram was also productive in their conceptualizing the relationship between the quantities. Another program of research could work towards developing experiences for individuals to construct and engage with images of quantities.

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