TREATMENTS OF TRIGONOMETRIC FUNCTIONS DURING REFORMS IN THE UNITED STATES

by

CELIL EKICI

(Under the Direction of Jeremy Kilpatrick)

ABSTRACT

Textbooks and reform documents provide a multiple source of descriptions of what it is to treat trigonometric functions as a school subject during three periods. Trigonometric functions in schools are introduced and developed in three major mathematical frames over the course of three reform periods in the United States – unified mathematics, new math, and standards-based instruction. Those frames are triangle, circle, and vector. They are used to explain the variations in textbooks' treatments of trigonometry as a phenomenon during and across reforms. Using phenomenology, I focused on ideas of trigonometric functions as a school subject manifested in textbooks and reform documents during reform periods. I used Schubring's methodology of historical textbook analysis to develop a history of interpretations of trigonometric functions as a school subject. I described the appearances and changes in the treatments of the trigonometric functions for each frame along the course of three reform periods from selected textbooks and reform documents.

INDEX WORDS: History of trigonometry, trigonometric functions, textbook analysis, circular functions, vector trigonometry

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DEDICATION

This dissertation is dedicated to my wife Çiğdem. I would be lost without her.

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CHAPTER 1

RATIONALE AND RESEARCH QUESTIONS

> Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. – Bertrand Russell (1917, p. 75)

In the 20th century, major attempts were made to reform school mathematics in the United States: unified mathematics in the first quarter of the century, new math from the 1950s to 1970s, and standards-based mathematics during the 1980s and still in effect in 2010. Trigonometry is considered a school subject that exemplifies unified mathematics and presents "a true correlation of arithmetic, algebra and intuitive geometry" (Breslich & Stone, 1945, p. iii).

In the early phases of a reform movement, advisory committees are formed to develop a set of reform documents specifying their recommendations on the content, organization, and treatment of school mathematics (National Education Association [NEA], 1894, 1899, 1918, 1920; National Committee on Mathematical Requirements [NCMR], 1923; College Entrance Examination Board [CEEB], 1959, 1985; Educational Services Inc., 1963, National Council of Teachers of Mathematics [NCTM], 1940, 1989, 2000). The CEEB (1959) report, for example, was developed over 4 years, and the NCMR (1923) report was developed over 7 years. Each reform attempts to develop a vision of school mathematics with an accompanying manifestation of trigonometry. Each reform document provides the reformers' recommendations for school mathematics and develops an image of school trigonometry.

Reform textbooks are artifacts designed to interpret and translate curriculum reform perspectives into materials for daily classroom practice in schools by teachers and students

(Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002, p. 2). Textbooks play an important part in the mathematics curriculum and are a major influence on daily instructional practices for many teachers (Reys, Reys, & Chavez, 2004; Schmidt, McKnight, & Raizen, 1997).

School mathematics cannot be treated solely as a structural discipline logically constructed or as a plethora of subject matters psychologically interpreted for children. What is defined as school mathematics is shaped and fashioned by social and historical conditions. This definition of school mathematics does not share the meaning of mathematics as an academic discipline (Popkewitz, 1988; Stengel, 1997). School algebra exemplifies the differences in the contents and meanings of associated mathematical objects between school mathematics and academic mathematics. Although trigonometry was associated with and integrated into school algebra throughout the 20th-century, the same association would be alien for an academic algebra especially after its modernization in the 1930s.

The content of the school mathematics curriculum is dynamic and goes through revisions to provide better foundations for evolving and growing practices in mathematics and its applications. There is no permanent definition of school mathematics or for any of its components, neither school algebra, nor geometry, nor trigonometric functions. Klein (1906/2009) made a supporting argument decades ago, describing the dynamic character of elementary mathematics as follows:

> Mathematics has grown like a tree, which does not start at its tiniest rootlets and grow merely upward, but rather sends its roots deeper and deeper at the same time and rate that its branches and leaves are spreading upward. . . . *We see that as regards the fundamental investigations in mathematics, there is no final ending, and therefore, on the other hand, no first beginning, which could offer an absolute basis for instruction*. (Italics in original, p. 15)

In this study, I approached critically the mathematical content of school trigonometry. The basic premise of this work was to question the ontological status of knowledge objects that define trigonometric functions in the curriculum as a formalized body of knowledge. I questioned first the ontological status of trigonometry knowledge considered for schools, determining its conceptual content. This issue precedes other questions of epistemological and axiological aspects of school mathematics, determining how and why it is practiced in schools. This approach is unlike perspectives from the Theory of Didactical Situations (Brousseau,, 1997) or the Didactical Transposition (Chevallard, 1991). Both approaches suffer from an uncritical view of school mathematics and take a disciplinary knowledge of mathematics as given.

The main thesis of this study challenged the status of knowledge objects of trigonometric functions in school mathematics practice using a historical perspective. Knowledge objects of school trigonometry were examined as entities that appear in school mathematics in different forms with alternative conceptualizations and configurations. Thom (1972) said, “The real problem which confronts mathematics teaching is not that of rigor, but the problem of the development of ‘meaning’, of the ‘existence’ of mathematical objects” (p. 202). What kind of mathematical objects are trigonometric functions? How are they developed? In the present study, those objects that define trigonometric functions were treated as dynamic, historical, and cultural objects. They cannot be taken as an a priori to be used as a well-established and unchanging model for testing knowledge in school practice. The content of school mathematics is influenced not only by structural developments and reorganizations of domain of mathematics and mathematical sciences, but also, and more importantly, by foundational developments that call for reorganizations of elementary mathematical objects in school mathematics with their renewed senses. The content of school mathematics responds to developments in the elementary foundations of the subject as mathematics develops.

As school mathematics content, trigonometric functions have always found a place in high school mathematics and appeared as a theme of school mathematics over the course of the three reform periods. The status of trigonometric functions and their conception as a school subject changed during those reforms. Each reform developed its own treatments of trigonometric functions from the reform perspective. During each reform, there were reactions to the proposed changes in the treatments of trigonometric functions in schools (Bressoud, 2010; Hirsch, Weinhold, & Nichols, 1991; Markel, 1982; Moritz, 1908; Rosenberg, 1958). What Bressoud suggested in 2010 parallels what Moritz critically promoted in 1908 by endorsing a shift of emphasis from right triangles to circles as the primary objects of school trigonometry, which makes trigonometry the "science of angular magnitudes" (Moritz, 1908, p. 397).

Textbooks As Historical Documents

A textbook's treatment of school mathematics content is based on implicit assumptions about the contents and objects of school mathematics. Reform textbooks provide multiple sources of descriptions of what it is to treat trigonometric functions as a school subject. During the reform periods, multiple groups responded to the calls for reform and developed textbooks reflecting their interpretations of a reformed school trigonometry. There is hardly ever a unique way of treating a school subject such as trigonometric functions. There is a lack of consensus and apparent plurality among experts and textbooks in their recommended treatments of school subjects. Stanic and Kilpatrick (1992) maintain that

> competing visions—that is, competing answers to the questions of what we should teach, why we should teach one thing rather than another, and who should have access to what knowledge— can be healthy, but only if they are recognized and dealt with. (p. 416)

This plurality of reform treatments of a school subject presents a context to study their variations during and across reforms. Having alternative approaches to treating trigonometric functions in textbooks is something one expects during a reform. An examination of multiple sets of reform textbooks from different reform periods can provide information to determine the status and alternative configurations of learning objects in school mathematics. A history of trigonometric functions as a school subject can be developed by attending to variations in textbook treatments of the subject across the three reforms and within each reform.

Where to place and integrate trigonometry is one of the essential aspects influencing practices of trigonometry in school mathematics. During the past century, there was a trend to treat trigonometry not as a separate but as an integrated subject in Grades 9 to 11. Considering, for the sake of simplicity of argument, the traditional sequence of algebra, geometry, and advanced algebra, there are examples of practices that integrate trigonometry into each of these courses. A trigonometry that can be introduced to a student as a part of the last several units of advanced algebra would be different from a trigonometry introduced as a part of geometry, or earlier as a part of elementary algebra. Trigonometric content and practice would then have a totally different set of constraints and affordances. This variation would yield practices of trigonometry with different sets of assumptions regarding its content and connections, and its structure and sense. For example, when students take their third mathematics course in high school, by the middle of 11th grade, they have already been exposed to the ideas of series, sequences, and sums. It is a rare practice to see a series approach used to further development of trigonometric ideas at that grade level. Therefore, the problem is not about finding one exemplary practice of trigonometry that would work for all grade levels from 9 to 12. Rather, it is about seeing the variations of the practices that make use of the constraints and affordances of a trigonometry integrated at that grade level.

The Problem

The mere existence of textbooks is based on a premise that they model school mathematics for students and teachers. Each series of high school mathematics textbooks is developed by experts in the domain of school mathematics and represents a case for studying a conceptualization of trigonometric functions. A textbook can be the result of a long process of development and field tryouts to ensure that it represents a workable model. In a textbook, a set of mathematical ideas are introduced, developed, organized, and extended. Instructional materials developed by content experts and reformers of different periods represent a source for observing changes in trigonometric functions as a school subject. Although "the role of experts in textbook development is certainly important, the reality of textbook production as an industry brings marketplace factor and editorial influence into play and can diminish the importance of experts. Editors influence also what gets in the textbook" (J. W. Wilson, personal communication, December 6, 2010).

There is a lack of research on the historical status of knowledge objects for trigonometric functions in schools. An emerging knowledge base with this research is developed for trigonometric functions as a school subject from its historical appearances as a school mathematics phenomenon across reforms. Starting with the unified mathematics reform, I examined the textbooks and reform documents in terms of their treatments of trigonometric functions during each reform.

There are examples of research in mathematics education performing historical analyses of school mathematics content. For example, Quast (1968) made an historical analysis of geometry in high schools between 1890 and 1966 in the United States. There is still a lack of study on historical analysis of trigonometric functions as a high school mathematics subject. The major objective of this study was to fill this gap of research in mathematics education.

Significance and Benefits

There is a need to attend to trigonometry as a problematic high school subject that has been under transformation during the last 200 years. Despite the historical roles trigonometry played in the development of calculus and analysis and the unification of algebra and geometry, the reforms during the last 30 years have placed trigonometry in a rather controversial place in school mathematics (Markel, 1982; NCTM, 1989, 2000). The treatment of trigonometry is inconsistent between the two main standards documents from the NCTM (1989, 2000). Whereas the former document placed trigonometry as its 9th among 14 content standards, the latter document decreased the total number of content standards to 5, dropped trigonometry as a separate content standard, and mentioned right-triangle trigonometry under the geometry content standard. The content standards for NCTM (2000) were number and operations, algebra, geometry, measurement, and data analysis and probability.

The mathematical concepts of trigonometric functions are subject to historical revisions and expansions as reflected in the textbooks and reform documents of different periods. The transformation and the changing images of trigonometry have led many to question its place in school mathematics and treat it as an outdated subject. A history of trigonometric functions as a school subject across reforms can benefit not only researchers in mathematics education but also preservice teachers, inservice teachers, and teacher educators. Expository articles and opinion papers have brought forward the diversity of sources of elementary conceptualizations of trigonometric functions and problems regarding their teaching. Usiskin (2001) raised the problem of treating school mathematics from an advanced perspective for the professional development of mathematics teachers. The present research study can fill a critical gap by presenting alternative treatments of trigonometric functions as a high school subject including advanced perspectives.

This work with its historical approach shows how alternative conceptualizations of the same content are possible, even in the same reform period, shaping subject matter knowledge in defining and organizing the content and its connections. This study returns attention to mathematical content and its variations across reforms. This history of trigonometric functions across reforms can benefit mathematics education by providing a deep analysis of the makings of a school mathematics subject across reforms.

Frames for Trigonometric Functions in Schools

Trigonometric functions in the plane are introduced and developed in school mathematics in number of different mathematical frames. I characterized them as triangle, circle, and vector. All three frames have been used in high school mathematics with different degrees of emphasis and integration during the reforms in secondary school mathematics in the United States in the last century.

Each frame develops the reformers' ideas of trigonometric functions and organizes their own set of references to trigonometric functions in high schools. The multiplicity of frames brings a plurality to the school subject. Textbooks series do not have to agree on their use of frames in their treatments of trigonometric functions. With this sense of frames, each textbook has to develop and coordinate multiple references to trigonometric functions. Those frames are used to explain the variations in textbook treatments of trigonometry during and across reforms.

The first frame for trigonometric functions, the triangle frame, is essentially a geometric one using right triangles. It is distance based and demands only the use of nonnegative rational and irrational numbers without requiring a negative sense associated with the notion of direction of lengths; that is, it does not require negative numbers. One of the main features of a trigonometry frame is that number sense is extended so that it affords a change in the geometric meaning of numbers. The second frame, the circle frame, is analytic and calls for a coordinate system with a built-in direction and a point of origin. It relies on a coordinate-based geometry that associates the mathematical objects in the plane such as triangles and circles with their defining points on the assumed coordinate system and uses transcendental numbers, such as π and e. The third frame is vector and provides a further abstraction of coordinate geometry by making it origin free. This vector frame introduces and uses trigonometric functions to express rotations of coordinate-free objects. The vector representations of trigonometric functions such as sine and cosine are unified by complex numbers. This frame mainly works toward extending from real to complex numbers, which unifies trigonometric functions and the operations of rotation and dilation. In vector-based trigonometry, each number has an operator sense, where i is a counterclockwise rotation about a right angle, and -1 is a clockwise rotation through two right angles. For example, the recommendations given by reform documents (e.g., CEEB, 1959) during the new mathematics reform placed emphases on the vector-based frame and complex numbers for reforming trigonometry. Several high school textbooks were written during the

1960s that used the vector idea. The manifestations of this vector-based trend were examined in the present study.

Although vectors and complex numbers are introduced in high school mathematics, there are differences in textbook treatments of trigonometry in this vector framework. The treatment of trigonometric functions in the vector frame in elementary mathematics was a project that started in the early 19th century. De Morgan (1849) and later Hayward (1892) made this frame for trigonometric functions a part of college mathematics textbooks. Vector trigonometry was not well-developed in school mathematics during the unified mathematics reform period. During the new mathematics period, vector-based trigonometry was developed by the University of Illinois Committee on School Mathematics (Szabo, 1969, 1971, 1973; Vaughan, 1971; Vaughan & Szabo, 1971, 1973).

Other than three main frames manifested in high school textbook treatments of trigonometric functions during the 20th century, other frames were used to introduce and study trigonometric functions as mathematical objects. But they are not explicit in high school textbooks or in reform documents, although they can be found in journal articles not intended for high school mathematics (Lunn, 1908; Vaughan, 1955). Therefore those frames are not included here.

Using the three frames, I characterize the treatments of trigonometric functions by examining the set of tasks that introduce and extend trigonometric functions in textbooks during three reform periods.

Research Questions

Focusing on trigonometric functions in the school mathematical content as a dynamic object that can be studied as a phenomenon, I ask how it appeared and changed during three reform periods in the 20th century. I pose two main descriptive questions, one to address textbooks, and the other to address the contemporary reform context.

Question 1. How do textbooks treat trigonometry during each reform period? Three subquestions are the following: How do textbooks treat right-triangle trigonometry during each reform period? How do textbooks treat circular trigonometry during each reform period? How do textbooks treat vector trigonometry during each reform period?

Question 2. What recommendations were made by major reform documents on the content of school trigonometry during each reform period? What reactions and recommendations were made by contemporary experts on teaching trigonometry? How do reform textbooks' treatments of trigonometric functions align with the recommendations of reform documents of their period?

The outline of the thesis is as follows. The second and the third chapter, respectively, give a literature review and discuss methods. The subject of the fourth chapter is right-triangle trigonometry during the reforms. Circle trigonometry is addressed in the fifth chapter, and vector trigonometry in the sixth chapter, which contains a discussion of complex trigonometry as an extension of vector trigonometry. These three chapters address the subquestions of Question 1 corresponding to each of the three frames for trigonometry in schools. I address Question 2 for each trigonometry frame and present contemporary reform context as found in the leading reform documents in secondary school mathematics over the course of a century. The last chapter summarizes the results and the changes in the status of trigonometry as a school subject over a century.

CHAPTER 2

REVIEW OF LITERATURE

The treatment of trigonometry, one of the oldest subjects in the history of mathematics, is full of historical lessons. The applicability of the subject generated stereotypical mathematical applications such as solving triangles that became associated with trigonometry and confusingly understood as the main reason for existence of the subject. Trigonometry had been practiced before in solving quadratics, cubics—as in Cardano's famous formula dating from the1500s—and higher order polynomials. When its stereotypical applications, such as solving triangles and solving equations, became outdated, that resulted in changed images of the subject. This change of object generated a countermovement against the whole subject without regard to other conceptualizations and applications of trigonometry.

The practice of right-triangle trigonometry in schools precedes that of circle trigonometry (Bressoud, 2010). Lardner (1828), Peacock (1830), and De Morgan (1837) all revived and used the ratio definition of trigonometric functions. Approaching sines and cosines as ratios and not lines liberated them from the circle and its radius. Circles were merely introduced and used to furnish measures of angles. Tangent was no longer defined as another geometrical line segment but as the ratio of the sine to the cosine of an angle. Peacock (1845, p. 181) still used the term *species* for the right triangle as Rheticus had done.

About six centuries ago, Regiomontanus, also known as Johann Muller and then Rheticus, made major contributions to the practice of plane trigonometry. Rheticus developed a textbook that examined triangles and circular arcs and chords as a mathematical introduction to astronomy (Maor, 2002). Rheticus's textbook, *De triangulis omnimodis* [On triangles of every kind], was modeled after Euclid's *Elements*, organizing a historical body of knowledge on trigonometry by Greek, Hindu, and Arab scholars. Copernicus was working at the same time and was making discoveries of a trigonometric nature. He developed mathematical descriptions of different periodic motions followed by the earth such as his discovery of earth's motion around its axis. He also noticed that oscillations can be generated by superposition of two circular movements at right angles (Zielinska, 2007).

The plane trigonometry used in schools follows Rheticus with the merging of the ideas of ratio and similarity of right triangles in the development of trigonometry. He developed this method during his study under Copernicus (De Morgan, 1845). Before Rheticus, right-triangle trigonometry on a plane was not developed in Western mathematics. Until Rheticus, tangent as a trigonometric object was not listed along with sine in trigonometric canons. He brought together three species of trigonometric objects as ratios and associated with a circle as shown in Figure 1 and Figure 2 (van Brummellen, 2009, p. 275).

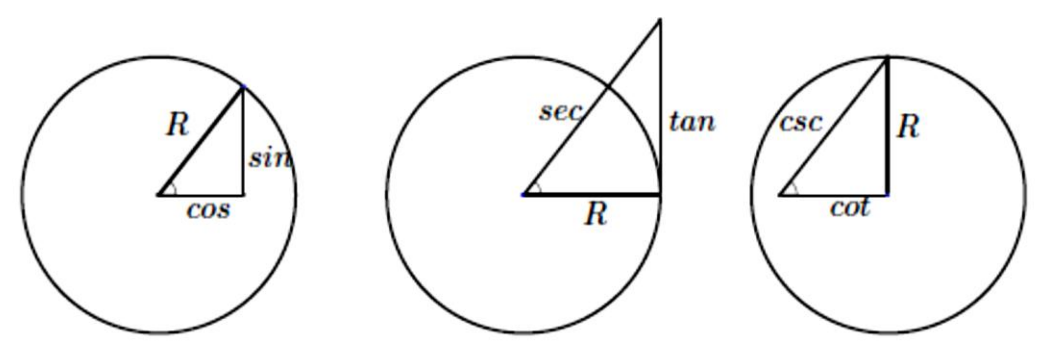


Figure 1. Rheticus and three species

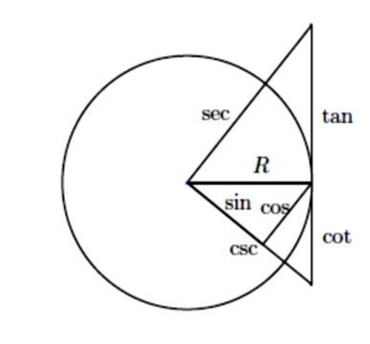


Figure 2. Reconfiguration of trigonometric species.

Trigonometric functions were associated by their line segments in and about a circle with a given radius. The maxim of Rheticus was "*Triquetrum in planicie cum angulo recto, est magister Matheseos* [Right angled triangles in plane guides the mathematical canon]. Magister matheseos was also the nickname for the theorem of Pythagoras. He called his method the doctrine of triangles (De Morgan, 1845). He suggested that there are three kinds of trigonometric ratios for a right-triangle trigonometry and those triangles are similar to each other. What Rheticus did was to suggest a method that used the ratios of the sides of right-angled triangles in developing his trigonometric canon. Rheticus offered the practice of the ratio approach by giving more emphasis to right triangles rather than to the circle in defining the trigonometric sine as a ratio (van Brummelen, 2009). This approach did not become common in school mathematics until the 1800s.

Trigonometric line definitions were used for centuries. As Figure 3 shows, the "trigonometric line" AT was the tangent, OM the cosine, PM the sine, and OT the secant. The circular arc length AP dominated trigonometric conceptions with no restriction on the radius until the 19th century.

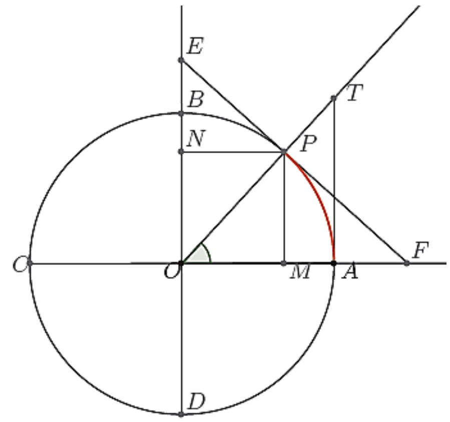


Figure 3. Trigonometric line functions

Trigonometry underwent another major change with the work of Viète (1540-1603). The analytic character of trigonometry began to emerge. Repeated applications of half-angle formulas led to study of infinite processes as mathematical objects in trigonometry. Viète expressed π as a result of an infinite product of nested radicals. This study of infinite products in trigonometry was continued by Euler. Newton (1642–1726) found the relations between an arc (z) and its corresponding sine (x) in a circle of radius 1, through his method of fluxions.

In setting the stage for modern times, Euler wrote his textbook *Introductio in analysin infinitorum* in 1748. Boyer (1951) called that textbook the "foremost textbook of modern times," with a major impact for the development of analysis that can be compared to what Euclid's *Elements* and Al-Khowarizmi's *Al jabr wa'l muquabala* did for synthetic geometry and for elementary algebra, respectively. Euler did not define trigonometric functions as lines in a circle as had been common until then. He defined the trigonometric functions as transcendent quantities arising from the circle: π is an associated transcendental quantity, half the circumference of a circle of radius 1. Transcendental functions are nonalgebraic functions, such as log x, sin x, cos x, e^x, and any functions containing them. Such functions are expressible in algebraic terms only as infinite series. The problem of the quadrature of the circle is historically equivalent to the determination of π. This is a problem of antiquity deeply connected to trigonometric functions, especially in series representations. Euler series were used in approximating unit arc lengths or π which is total arc length for a circle with diameter 1.

Euler defined the trigonometric functions as numerical ratios, discussed their various properties such as periodicity, provided addition formulas, and gave their power series expansion. The function concept and the study of functions with infinite processes were brought to center stage for analysis. If the primary object of algebra study is the variable, the primary object of analysis study is the function. Euler used the Pythagorean identity and sum formulas for sine and cosine for an analytical study of the quantities emerging from the circle. With this approach, he brought a sense of functions to sine and cosine, and he developed a series of powers for sine and cosine and made the trigonometric functions a part of differential and integral calculus (Katz, 1987). Euler provided a systematic graphical study of functions, both transcendental and algebraic functions. *Analytic* gained a new sense, and textbooks emerged on "Analytical Trigonometry," "Analytic Geometry," and other traditional topics with an analytical sense. Euler's *Introductio* became a prototype of modern textbooks. His textbook went through a dozen of editions, was translated into several languages, and became a major source of inspiration for modernizing the treatment of school mathematics.

The trigonometric functions were not always represented as the sinusoid that graphically depicts their periodic behavior. It is interesting to note that the sinusoid did not first appear as defined by an equation $y = \sin x$, but was first given as "an auxiliary curve whose definition is derived from that of the cycloid" (Bourbaki, 2004, p. 153). It was first discussed as a mechanical means of squaring the circle. The cycloid is the locus of points on a rotating circle with radius R. The parametric formula for the Cartesian point can be given as:

$y = R(1 - \cos t)$, $x = R(t - \sin t)$, where t denotes the measure of rotation.

After Leibniz and until the work of Fourier, a convergent sequence of continuous functions was always assumed to converge to a continuous function. Fourier's work on heat with trigonometric functions produced a family of counterexamples that refuted the Leibniz principle. An example is given in Figure 4. The functions plotted in this figure are given in the equation

$$f_k(x) = \sum_{i=0}^{k} (-1)^i i^{-1} \sin ix.$$

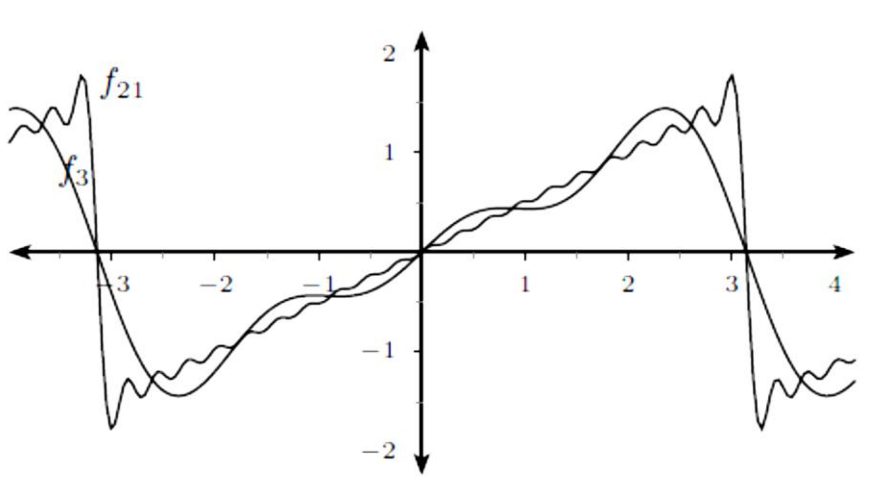


Figure 4. Fourier's discovery with the series of trigonometric functions.

In this series, the signs do not make a difference. Partial sums of trigonometric functions without alternating signs are still continuous, and their sequence converges to noncontinuous functions.

> Fourier's theorem marks one of the great achievements of nineteenth-century analysis. Fourier shows that the sine and cosine functions are essential to the study of all periodic phenomena, simple or complex; they are, in fact, the building blocks of all such phenomena, in much the same way that the prime numbers are

> the building blocks of all integers. Fourier's theorem was later generalized to nonperiodic functions (in which case the infinite series becomes an integral), as well as to series involving nontrigonometric functions. (Maor, 2002, p. 54)

The sinusoid with its periodic wave form used in developing trigonometric functions in school mathematics is a 20th-century phenomenon. It gained its popularity with the availability of graph paper as an instructional tool, and it was promoted in textbooks at the turn of the 20th century. The first use of graph paper was to study of a periodic behavior and trace the variation of a barometer (Howard, 1800). Palmer (1912) noted that no textbook in elementary algebra printed before 1902 introduces any of the theory graphically. In 1903, college entrance requirements in mathematics were revised with a mutual collaboration of committees from the American Mathematical Society, the Society for the Promotion of Engineering Education, and the National Education Association. A new requirement to be included in elementary algebra was "the use of graphic methods and illustrations, particularly with the solution of equations" (Palmer, 1912, p. 693). This experimental tool was then called by several names such as squared paper, graph paper, and coordinate paper. Graphical representations and methods were used for an experimental and numerical study of functions and their behavior.

E. H. Moore (1906), one of the major figures behind the unified mathematics movement, called this tool "cross-section paper." Besides square-ruled paper, he advised the use of various styles of ruling: rectangles, parallelograms, and triangles, and with concentric circles and diverging radii. He recommended that such paper be used systematically as a mathematical instrument by teachers in elementary and secondary schools to unify algebra and geometry. Cross-section paper was one of the main tools in a laboratory method to study mathematics, and it was to be used in conjunction and connection with "double-entry tables" and "graphical computation." Further on the unifying role of cross section paper:

> I know of no medium serving to bring together so closely and so easily the three phases or dialects of pure mathematics – number, form, formula – and to lead so directly to the concept of functionality. (Moore, 1906, p. 318)
>
> By maximizing the function of the cross-section paper we secure, to speak only of pure mathematics, intense reaction between geometry and algebra. Geometry and algebra may certainly be developed independently, each with its relations to arithmetic, and no one doubts their high educational and scientific value as so developed. But this value is indeed small compared with the value to be obtained by developing them together in continuous reaction, thus releasing, as it were, abundant stores of sub-atomic energy. (Moore, 1906, p. 319)

Emphasis gradually shifted from making graphs to interpreting their meaning. The study of graphs became a major trend in algebra. Along with the formula, it helps to clarify the idea of functionality. Its popularity grew in the decades to follow (Reeve, 1929).

The trigonometric functions are defined in high school with reference to a system of rectangular coordinates. When the idea of hyperbolic functions is introduced and explored, students become familiar with their analogies with circular functions. Strand and Stein (1962) pursued the founding idea of an oblique coordinate system generalizing the fixed angle $\pi/2$ to an arbitrary angle lambda and developing quasi-elliptic, quasi-parabolic, and quasi-hyperbolic trigonometries. Then the circular functions of conventional trigonometry become a special case of quasi-elliptic and quasi-parabolic functions.

Argument of a Trigonometric Function and Order

There are discrepancies in elementary and higher conceptualizations of trigonometric functions in the way primitive mathematical objects are used as arguments to develop higher mathematical objects. An algorithmic approach to the trigonometric functions allows a constructive local pointwise definition of the value of the function without requiring an expression of a closed form that describes its global behavior. Either a circular point is defined by a pair of coordinates; or the other way around, a pair of given coordinates is used to define a circular point. The main dilemma is to choose between a circle as a given primary construct and a circle as a construct achieved from its constructible and knowable points.

The choice is whether to limit trigonometry to circular algebraic points or to embrace cyclic nonalgebraic points. This choice is associated with historical debates on the use of numbers. For example, there was a debate between Weierstrass and Kronecker suggesting a conceptual versus a structural approach, respectively. Kronecker argued against relying on real numbers. A construction of circular arc length of a given size has been known to be problematic since ancient timess (Maor, 2002). These are also known as historical problems of the rectification of circle or measuring π. The use of radians in schools calls for using a directed circular arc length as a descriptive position of a point on a circle. Historically, the problem of constructing the rectification of the circle was proven to be impossible during the late 19th century. A circular arc length is defined by school textbooks as the product of the radius and the central angle in radians. This method of defining the angle measure is circular; it starts with the radian, which is already measured by the circular arc. This practice assumes that one can know or locate a circular point with certainty given an angle at the center measured in radians.

The historical approach of trigonometry in dealing with angles helps one to develop a formal approach to an approximate science. Radian measure is based on the concept of the circumference of the circle, which is the limit of the perimeters of inscribed and circumscribed polygons. Radian measure can be made precise through the limit of an infinite process, and it can be measured with precision up to a bounded error. Radian measure as an ideal concept presupposes a nonalgebraic measure designed to bypass the problem encountered in defining angles and their functions. By adopting an imprecise measure, a trigonometric function becomes a constructive object structurally defined with an algorithmic construction process of measuring an angle. "Whenever we consider tan *x* where *x* is measured in radians, we really presuppose a process of the same logical order as the formation of e^x" (Menger, 1945, p. 31). Some mathematicians such as Menger (1945, p. 31) even objected to the use of radian measure in developing the differentiation of trigonometric functions. Whenever π is used as a constant with radians, it is not a concrete but an abstract constant, and it ingeniously bypasses historical dilemmas surrounding the practice with circular arc lengths.

Klein's Erlanger program suggested treating geometry as a set of transformations (Glass, 1993). The geometry of the plane is developed intuitively by studying rotations, reflections, and translations as transformations. Klein offered this approach to provide a foundation that is extensible and helps to better distinguish alternative geometries. The practice of radians is based on standardization of circular arc length. The angle as a circular arc length was not always considered essential to develop the trigonometric functions. An alternative to arc length is the circular sector. A unit circle has an area π and a circumference measure of 2π; trigonometric functions can be defined by using a circular sector area as arguments. In this case, trigonometric sine does not need to use half but full sectors, compared to the use of half chord in the practice of circular arc length as argument in trigonometric functions. Klein (1906/2009) decided not to use angle and avoided arc length; he chose sector area in developing his goniometric functions, as seen in Figure 5. *Goniometry* was the popular name for analytic trigonometry, which studied trigonometric functions for generalized angular quantities and their relationships. It was suggested because Klein wanted to deemphasize triangular connections, which limited the functional study of the subject. Therefore, Klein did not use the term *trigonometric functions* but rather *goniometric functions*. For Klein, sector area was better suited than circular arc to be used as an argument in developing trigonometric functions. Sector area as the argument can still work when one extends it from circular to hyperbolic trigonometric functions, as shown in Figure 5. Arc length, on the other hand, does not work as an argument after this extension. The circular sector area φ can be used to develop sine and cosine functions rather than using the half the arc length P_1P_0 on the unit circle. The sector area in Figure 5 works as the argument for the hyperbolic case of trigonometric functions.

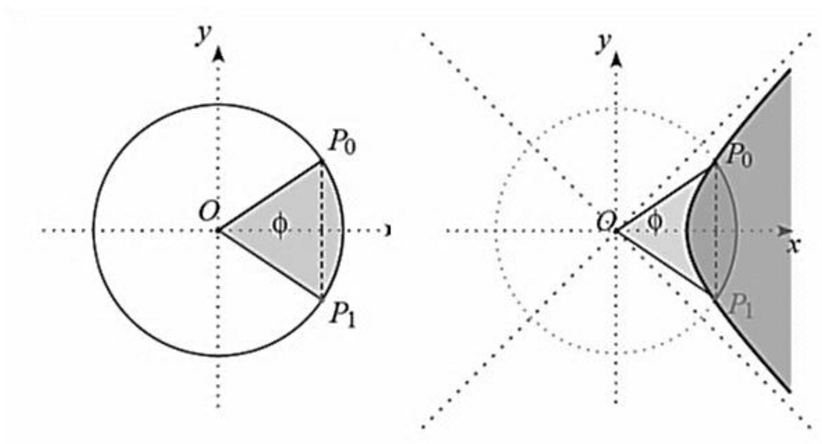


Figure 5. Sector area as the argument for circular and hyperbolic trigonometric functions.

This argument demonstrates that angle is not a simple school mathematics object when approached formally. A measuring tape is often used to suggest a winding function. This tool is acceptable in measuring the circumference of a circle and finding radians for physical models. But it is not acceptable for finding the perimeter of other geometric objects such as the square, where an algebraic equation can readily be deduced. The complexity of the angle suggests that it can be considered as a proper object rather than as an argument in the study of trigonometry. More formal approaches often use inverse trigonometric functions in defining trigonometric functions when trigonometric function ideas are extended from circular to hyperbolic or elliptic forms. High school trigonometry makes possible a comparison of the trigonometric functions for the hyperbolic and circular cases and draws a connection between them.

It is debatable whether the mathematical object of sine or cosine is more primitive than that of the angle. When sine and cosine are defined by projections of circular points rotated through a given directed circular arc length, the point to be projected is not known with certainty but is an abstract point. When students are asked to find the tangent of an angle of 1 degree, they do not question what 1 degree means and how it can be known.

Determining an angle from its trigonometric value is the other category of problem that approaches trigonometry as the study of angles. Such problems led to the development of goniometry. Comte (1851) suggested that trigonometric line segments about the circle could be used to determine angles by their measures. These observations coincide with the remarks by Moritz (1908) that suggested the study of angles and angularity as the object of trigonometry. Viète (1540–1603) suggested that “the analysis of angular sections involves geometric and arithmetic secrets which hitherto have been penetrated by no one” (Maor, 2002, p. 50).

Alternative Trigonometries

Each geometry offers an extension of idea of right-triangle trigonometry depending on a number of fundamental reconsiderations such as its angles, lines, and distance measure. The case of spherical trigonometry represents a historical school subject that was traditionally examined in colleges before 1900s; and it was included in the recommendations by the Committee on College Entrance Requirements Committee of the NEA (1899). It was a subject treated after plane trigonometry in college trigonometry textbooks (e.g., Wentworth, 1891). It kept reappearing in school trigonometry textbooks as an optional unit throughout the century.

Macfarlane (1894) examined the definitions to conclude that the one term *trigonometric ratios* comprised two species: the geometric, or *triangular,* and the *circular*. The triangular ratios were defined independently of the circle, and they included some of the circular ratios as special cases. Macfarlane showed that there are several geometrical generalizations of the circular functions, and that the algebraic series for the simple functions generalize in ways that would never be deduced by taking the elementary series as the general definitions. He presented definitions of several species of trigonometric functions—the triangular, circular, and excircular—which could be harmoniously defined with one another. The ex-circular trigonometric functions refer to the hyperbolic case.

Hayward (1892) introduced three kinds of trigonometry and trigonometric functions on the plane. They were triangular functions, circular functions, and excircular function. Similar to the circular functions' study of the unit circle, excircular functions study the unit rectangular hyperbola. Hayward gave the name *excircular* to hyperbolic functions because of the sense that hyperbolic function can be interpreted geometrically as "circular functions turned inside out" (p. xi).

The generalization of trigonometry with functional equations was suggested by several researchers after the beginning of the century (Lunn, 1908; Thielman, 1937). For instance, Cauchy's method of solving functional equations such as $f(x + y) = f(x) + f(y)$ and showing that $f(x) = cx$ is the solution, for c an arbitrary constant, provides a characterization of linear functions. Addition formulas, such as $\sin(x + y) = \sin x \cos y + \sin y \cos x$, are used as defining characteristics for sine and cosine functions. The sine function from a functional equation approach is defined as the solution of the equation

$$f(x + y)\, f(x - y) = f(x)^2 - f(y)^2,\text{ where } x \text{ and } y \text{ are real numbers.}$$

The cosine function is the solution of the equation

$$f(x + y)\, f(x - y) = 2\, f(x)\, f(y).$$

In the same vein, the tangent function is defined as the function that satisfies the functional equation

$$f(x + y)\,(1 - f(x)\, f(y)) = f(x) - f(y).$$

The idea of a functional equation approach to trigonometric functions provides one further extension of the trigonometric function idea. Trigonometric functions are realized as dynamic objects by the use of a functional approach. Functions were promoted as a unifying theme after the first quarter of the 20th century, for example, by the National Committee on Mathematics Requirements (NCMR, 1923) and the National Council of Teachers of Mathematics (NCTM, 1989). But the functional equation approach did not make it into recommendations in any major mathematics curriculum reform documents for secondary schools during the 20th century.

Circular trigonometry can be extended by studying the more general form, $x^2 - n\, y^2 = 1$; one obtains ordinary circular trigonometry for $n = 1$. For positive n values, an analytic treatment yields elliptical trigonometry, and for negative values of n, it yields hyperbolic trigonometry. The pair cosine and sine similarly defines the point of generalized quadratics. For $n = 0$, a parabolic trigonometry is also suggested and applied to the study of catenaries (Booth, 1857, p. 445). The hyperbolic sine and cosine represent a generalization that can be made in high school. Although found in some textbooks, hyperbolic trigonometric functions are usually elective content and seldom included in a regular mathematics curriculum.

It is often the case that secondary school trigonometry attends to triangles and circles in two major curriculum units to generate associated conceptualizations of the trigonometric functions. In both cases, the trigonometric functions are based on the measurement of distance and angle. Both conceptualizations rely on Euclidean distance, which uses the square root of the sums of squares as its definition of measure. Other types of measures generate alternative unit disks instead of the circle. Taxicab trigonometry is one simple trigonometry that uses what might be called Manhattan distance instead of Euclidean distance. Different trigonometries emerge from alternative conceptualizations of distance and angle as primitives. The conceptualization of a trigonometric function depends on its contextualization.

Euclidean distance is the most common measure used in school trigonometry to define trigonometric functions. An alternative geometry can be defined by using a different metric to measure distances instead of the traditional Euclidean distance. The taxicab metric is an option suggested during the new math years to develop an elementary non-Euclidean geometry and its associated trigonometric functions (Menger, 1971). The square circle is an idea used to introduce non-Euclidean geometries to a general audience. The unit circle is based on Euclidean distance measure. Taxicab geometry uses measures directed lengths along a rectangular grid. The Euclidean distance between two points on a plane is the square root of the sums of the squares of the length of the projections of the connecting segment onto axes. A city block distance between two points on a plane is defined as the sum of the lengths of the projections of the connecting line segment, that is, $d(x, y) = |x_2 - x_1| + |y_2 - y_1|$. For this metric, the set of points that satisfies $d(x, (0, 0)) = 1$ is a square, compared with a unit circle for the Euclidean metric. The corresponding trigonometry for this metric can be defined on its unusual unit circle shaped like a diamond (Menger, 1971; Thompson, & Dray, 2000; University of Chicago School Mathematics Project [UCSMP], 1998, p. 299).

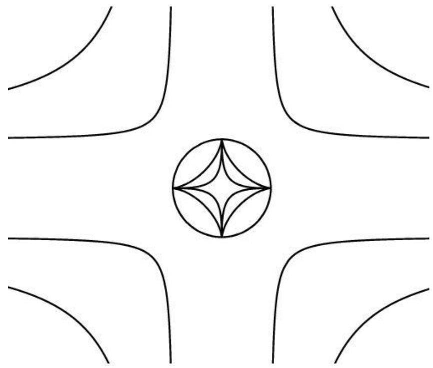


Figure 6. A variety of unit circles depending on a metric.

Other than summing the absolute values as a distance measure, there are alternatives that extend the Euclidean metric to a more general metric. Instead of taking the root of sums of squares of distances, it takes the *p*th root of sums of *p*th powers of distances. Their unit circles can be parameterized by ($\cos^p\varphi$, $\sin^p\varphi$) with unit circles as given in Figure 6.

Staying with plane geometries for school mathematics, some high school textbooks series after the 1980s extended the idea of trigonometric functions further by constructing alternative trigonometric functions by changing the distance norms, one of which is taxicab geometry and its trigonometry (Menger, 1971, p. 17; UCSMP, 1998). It is rare to see a school mathematics textbook that provides excursions into alternative metrics and their corresponding trigonometries. Among the contemporary textbook series of the Standards-based reform era, the University of Chicago School Mathematics Project (UCSMP, 1998) textbook extended its treatment of trigonometric functions and developed alternative trigonometric functions on “noncircular functions” in an end-of-unit project (p. 299). The UCSMP’s extension allowed students to see trigonometric functions not as static but as dynamic mathematical objects that can be reconceptualized by changing the assumptions. Several mathematicians suggested a rational alternative to the development of the trigonometric sine and cosine functions (Klein, 1906/2009; Wildberger, 2005). Instead of using real numbers, they suggested starting with the tangent and building sine and cosine from rational numbers as algebraic numbers. Six standard trigonometric functions have algebraic values for angles measured rationally in degrees (Hamming, 1945). The extent of the claim of rationality is not on sine and cosine, but only on rational angle values. The implications of this misunderstanding were explained further by Olmsted (1945).

As seen in Figure 7, starting with a given *t*, the coordinates of C express points on the unit circle in terms of *t*. For a given point C along a line, one finds a precise circular point D. In contrast, transcendental parameterization (cos *t*, sin *t*) or (sin *t*, cos *t*) points are imprecise and approximate. This approach provides an alternative parameterization of the circle by the transcendental pairs (Gelfand & Saul, 2001). It presents an algebraic expression of circularity by the equations

$$\tan\theta = t;\ x = \frac{t^2-1}{t^2+1};\ y = \frac{2t}{t^2+1}$$

This practice is known in calculus as half tangent substitution to transform and solve complex trigonometric equations using their algebraic correspondents.

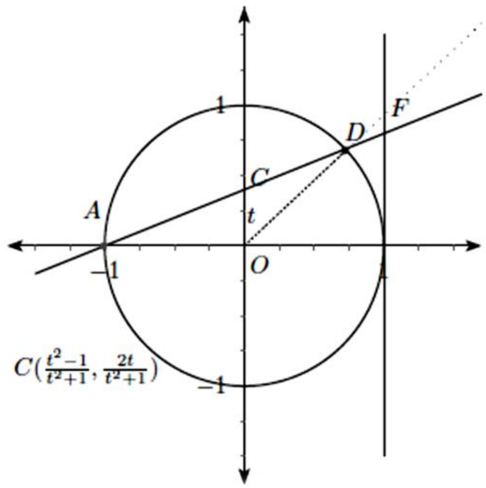


Figure 7. Rational approach for building trigonometric points with circular algebraic points using tangent parameterization.

Therefore, studying circles in school mathematics does not have to be done through circular nonalgebraic points as approached by radians and winding functions based on the use of real numbers. Wildberger (2005) recently presented and extended this rational approach. He proposed a complete rewriting of the elementary foundations of trigonometry by replacing the concepts of angle and distance with new notions of *spread* and *quadrance*. Wildberger developed a trigonometry on the idea of rational numbers without relying on real numbers. By doing so, this trigonometry avoids the dilemmas of continuity and infinite processes. Wildberger works with areas with the notion of quadrance, rather than working with lengths as in Euclidean distance. His choice of area removes the square root associations coming from the use of Euclidean distance and avoids having to develop the real domain in the construction of trigonometry. Rational trigonometry replaces sines, cosines, and tangents with elementary arithmetic. Rational trigonometry is a redefinition and extension of the sense of trigonometry as solving triangles on a plane in a radical way. This new approach exemplifies ongoing debates on the elementary foundations of trigonometry and the search for alternative foundations for trigonometry. This approach suggests an alternative to elementary trigonometry in schools. It has not yet, however, been tried out or incorporated into any school mathematics textbook.

Major Developments Concerning Trigonometric Functions

***De Morgan's modernizing trigonometric lines to circular lines*.** De Morgan (1849) attempted to modernize the trigonometric functions. He defined trigonometry as a branch of algebra where trigonometric functions are studied. In his terms, trigonometry is the science of "continually undulating magnitude" (p. 1), the most simple one being periodic.

The ancient method of defining the trigonometric terms was described by De Morgan (1849) in his *Trigonometry and Double Algebra:*

> A given straight line, called the *radius,* revolves from a starting line *OA,* as in our definitions; but it must be of the same length for all angles, which need not be- the case in ours. The *arc* described by the revolving extremity generally takes the place of the angle. Then [referring to Figure 8] *BM* was called the *sine* of the *arc AB (sinus,* bosom, the literal translation of an Arabic word: if *BAB "*represents a bow *(arcus),* half of the string *BB"* comes against the breast of the archer). And *OM* is the *cosine* of *AB*: this word is an abbreviation of sine of the complement, or *complemental sine;* it was long before *OM* was considered as anything but the sine of another arc, *BA'.* (p. 18)

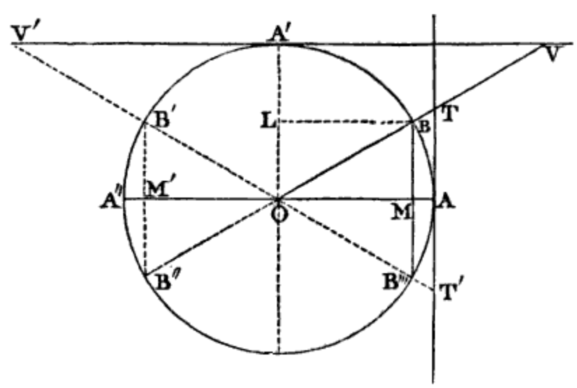


Figure 8. De Morgan's trigonometric lines and circular functions.

Note: Reprinted from *Trigonometry and Double Algebra*, by A. De Morgan, 1849, p. 19. (Public domain)

Here the terms *sine* and *cosine* are applied to certain lines drawn in and about a sector of a circle. These lines are commonly called the *trigonometric* lines. All these definitions are connected; a trigonometric line function divided by the radius gives the numerical function, which is a circular form. Because of their reference to a circular sector and not to a triangle in general, these functions are more properly called *circular* lines. The trigonometric lines may be defined independently of the circle or any other curve. A trigonometric line of a hyperbola or ellipse can be objectified as such. De Morgan also focused on the circular sector as an argument. He did not refer to central angle or arc length but to lines and the sector.

De Morgan (1849) observed that some writers had difficulty understanding *angle* as a *magnitude* because of their constant attention to the arc of a circle. He suggested a shift of focus from circular arc to central angle, θ, suspending the arc $\widehat{AB}$ in defining trigonometric functions.

$$\sin\theta = \frac{AB}{OB} = \frac{\mathrm{Sin}\widehat{\mathrm{AB}}}{r}$$

A variation of this problematic issue is to keep on seeing the angle as a geometric object, rather than shifting one's attention to its measure. He made some pedagogical notes and explained:

> The student may ask, "how can any thing but *an angle* have a sine? I answer, that θ is not an *angle*, but the *number* of arcual units in angle. Every *number* has a *sine*. (p. 37)

De Morgan modernized sine and cosine as projections of a revolving line. When introducing trigonometric functions for an angle, he used the idea of coordinates and projections. In this case, a right triangle represented an angle. The convention of naming the axes of coordinates as *x* and *y* was yet not in place then. A point was used as a starting object. Its position on a coordinate system was set by its projection on the axes chosen (see Figure 9). He then gave the definition of trigonometric functions of an angle as the ratios of projections to the revolving line.

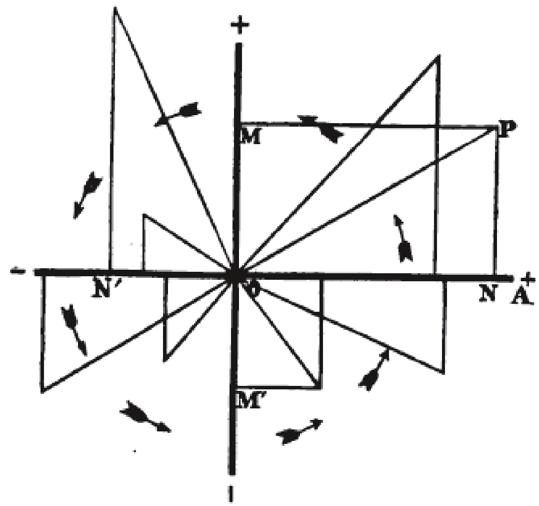


Figure 9. De Morgan’s rotated angles and sine and cosine as projections.
Note: In De Morgan’s *Trigonometry and Double Algebra*, 1849. Not in copyright.

One of the major developments in the history of mathematicians’ treatment of trigonometry is the rationalization of the arcsine function by the substitution of the tangent of the half angle. This thread can be traced from Euler to Peacock, De Morgan, Hardy, Klein, Eberlein, and Wildberger. The choice of circular area allows one to define *arcsine* as a rational mathematical object. The use of area was also a choice the Greeks made. The rationalization project’s impact on the development of trigonometric forms can be observed in discussions on the nature of the proper fundamental object in building and extending trigonometry. In defining trigonometric functions, an integral approach was used by Klein (1906/2009). This approach basically defines *angle* in terms of its sine with an integral area.

Regarding the historical development of the vector idea, analysis of a motion by its parts played a role. Galileo’s discovery of the representation of each motion by its horizontal and vertical components allowed the study of a motion by its components independently. The problem of finding the range of cannon ball led to linking trigonometry to the analysis of quadratic motion (Maor, 2002).

Wessel (1797/1999) generated the idea of vectors as directed line segments. He considered all parallel line segments that had the same length and the same orientation to be equivalent. Vectors were expressions of the form $\cos x + i\sin x$. He developed the idea of addition and subtraction of directed lines. He noticed that -*ba* = *ab* and developed the parallelogram rule for the addition of such expressions. In the multiplication of such forms, the idea was that the direction of product of two directed lines should be the sum of the factor's directions. Generalizing this idea, Wessel deduced $(\cos\frac{\varphi}{m} + i\sin\frac{\varphi}{m})^m = \cos\varphi + i\sin\varphi$. He developed the complex form directly from geometric problems resulting from his surveying practice, using intuitive geometric reasoning to get an algebraic formula. In contrast, Argand began with algebraic quantities and sought a geometric representation for them (Andersen, 1999; Schubring, 2001). Although Wessel's initial formulation was clear and direct, it was ignored for nearly a century.

Miller (1925) discussed the role of the arithmetization of mathematics in transforming trigonometric functions of two geometrical objects to functions of a single abstract number. The term *abstract number* refers to a number that has no geometric associations. The definitions of the elementary trigonometric functions as abstract numbers entailed the use of negative numbers when angles larger than 90 degrees were considered. The imperfect knowledge as regards negative numbers even in the 18th century retarded the arithmetization of the definitions of the elementary trigonometric functions. The usefulness of these numbers for this arithmetization helped them secure a permanent and critical position among the fundamental elements of mathematics. On the other hand, before the place and the use of negative numbers in elementary trigonometry were firmly established, the complex numbers began to enter into the picture. Epitomized by Euler's elegant formula, $e^{\pi i} + 1 = 0$, a bridge was formed between negative numbers and complex numbers. The elementary operations with negative numbers then found solid ground on the more general subject of operating with the ordinary complex numbers. A new horizon opened for the number sense, each number gained an operator sense. With this sense, multiplication (-1)(-1) = 1 meant in polar form that $(1, 180^{o})(1, 180^{o}) = (1, 360^{o})$. Each pair is represented by its radius and angle, $z = x + iy = re^{i\theta}$. With this new sense, multiplication became a transformation providing a conceptualization that unified dilation and rotation: $i^2 = -1$ gained a geometric sense, $(1, 90^{o})(1, 90^{o}) = (1,180^{o})$, and i^i yielded a real number, $e^{-\pi/2}$.

De Morgan (1837), Hardy (1908), Klein (1906/2009), and Eberlein (1966) all resorted to an early introduction of complex numbers in introducing a proper treatment of trigonometric functions. Eberlein (1966) also observed the problem of the proper treatment of trigonometric functions in a calculus course and suggested a series approach:

> No matter how pedagogically justified an intuitive first approach might be, educators have an obligation to put matters eventually on the sound arithmetic basis necessary for real and complex variable theory. The classical geometric procedure translates so clumsily into arithmetic language, however, that the obligation, when recognized, is usually met by an ad hoc definition of the sine and cosine as power series. Contact with geometry is then lost. The smallest positive zero of the cosine defines π/2, and the periodicity becomes an analytic tour de force. (p. 197)

To exemplify one of the higher conceptualizations of trigonometric functions, Fleury, Detraubenberg, and Yamaleev (1993) developed generalized circular functions based on an *n*-dimensional commutative algebra generated by the *n* vectors $<e_1, e_2, ..., e_{n-1}>$. The idea behind this higher conception was to explore what happens if the domain of trigonometric functions is extended from 2-tuples as complex numbers to higher ordered spaces, so that the arguments of the trigonometric functions are matrices. Since the 1960s, this idea has been developed, mainly by Gustafson (2006), as operator trigonometry. The argument becomes a matrix, and trigonometry is reconceptualized for matrix spaces in the theory of linear operators. For every operator, Gustafson defined the angle of an operator with maximum real turning effect the matrix produces, which is connected with the first antieigenvalue. He noted that antieigenvectors, including the higher ones, always occur in pairs. He developed a theory of antieigenvalues and antieigenvectors and operator turning angles during the emergence of noncommutative trigonometry. This theory found applications in various fields that use linear operators and their iterations such as statistics and economics (Gustafson, 2002). Operator trigonometry exemplifies another frame in which trigonometry was redefined, extended, and applied. This approach has not become common in undergraduate mathematics, and it is still a graduate specialization in college. Applications of noncommutative trigonometry include: perturbation theory, operator theory, convexity theory, wavelets, domain decomposition and multilevel methods, control theory, scattering theory, preconditioning and condition number theory, statistical estimation and efficiency, canonical correlations, Bell's inequalities, quantum spin systems, and quantum computing.

Historical Accounts of Teaching Trigonometry

Teaching trigonometry has received quite a lot attention in the field of mathematics education. One of the leading researchers in the field of education who dealt with trigonometry was Johann Friedrich Herbart, who lived between 1776 and 1841, and his views had a strong influence among the early mathematics educators in the United States around the turn of the 20th century. W. J. Eckoff, in the preface of his translation of *Herbart's ABC of Sense-Perception*, explained the impact of Herbart on the American education community: "American educators have begun to live, move, and have their being in an atmosphere of Herbartianism. It is coming to be the pedagogic spirit of the times" (Herbart, 1802/1903, p. xiv). This influence was not surprising since mathematics education reached the status of a university subject in part because of Herbart's works such as *the Science of Education* (Herbart, 1895). The Herbartian influence was brought to the States by Americans such as C. de Garmo, Charles A McMurry, Frank McMurry, and C. C. van Liew, all of whom had their training in Germany. The majority of Herbart's works on education were translated into English between 1890 and 1910. Those were the years when mathematics education emerged as a discipline in the United States. There was a trend to receive scholarship in education from Europe because of its unavailability in the States at that time. Almost every pedagogue of note in this period adopted Herbart's teachings (Röhrs, 1997). He had a major influence on the educational reform movement early in the century (Huemer & Landerer, 2010).

The influence of the Herbartian view in the education community can be observed in the modernization of school mathematics in the United States. The correlation of school subjects is one of the practical recommendations of the school of Herbart. The NEA's 1893 Committee of Ten report called for more correlation in the school curriculum. Herbartian educators in the United States formed the National Society for the Study of Education (NSSE), which was initially called the National Herbart Society for the Scientific Study of Education and was established in 1895 with founding members including J. Dewey, N. M. Butler, and C. A. McMurry.

Herbart placed an emphasis on the relevance of information to the learner. Personal experience and experimentation were essential in his pedagogical approach. Herbartian methods of teaching were based on his psychological theory of interest. Interest is formed when the individual learner deals in depth with many different objects and relates recollections of these many in-depth examinations to each other in an all-embracing unity. Psychologizing the subject matter for student's interest was one of the tenets during progressive reform as advanced by John Dewey and others (Judd, 1915; Kilpatrick, 2009).

Herbart gave trigonometry a central role in his famous *ABC of Sense Perception* (1802/1903). The original 1802 work was entitled *Pestalozzis Idee eines ABC der Anschauung* [Pestalozzi's Idea of an A B C of Sense Perception]. Herbart, extending Pestalozzi's ideas, developed an alphabet of sense perception and affirmed that the development of sense perception fell in the sphere of mathematics. For Herbart (1802/1903), "there is nothing that, so to speak, seems to lie so nearly at the center of mathematics as trigonometry" (p. 162). Spatial forms and measurements are studied to best advantage through trigonometry. In this work, he accordingly developed a system of instruction that analyzed all forms into triangles and discovered the ratios of the sides of the triangle, one to another, as depending upon the size of the angles. The students form for themselves a table of natural tangents and secants in terms of radius by simply measuring, doing exercises with a variety of problems, and discovering the terms required. On the integrating role of trigonometry, Herbart added:

> The consideration of triangles is fundamental to all geometry; and pure analysis, which, strictly speaking, has nothing in common with concepts of space, would frequently not know where to turn in integrating if it did not borrow sequences of proportions from trigonometry. This consideration alone would make it desirable, provided always that we found it compatible with the other purposes of *the A B C of Sense Perception*, that triangles should become the first subject for mathematical exercises. (p. 162)

Herbart provided a series of object lessons in trigonometry. The trigonometry lessons involved a continued occupation in estimating and then actually measuring angles and sides of triangles. Afterward, the student becomes able to make sense and predict the dimensions of the object that he or she sees. The object is not merely to sharpen the observation of material objects, but even more to develop the power of pure geometrical abstraction, and to combine it with arithmetical modes of thought.

Euclidean synthetic geometry is congruency based, and it compares triangles to see whether they are equally determined by some of the angles and sides. By that means, students determine that the triangles are equal in form and magnitude. For Herbart (1802/1903), geometry was not adequate for questioning what those forms are, how the pieces come together, and what the determining elements and relationships are. For him, a real scientific statement of the form of a triangle belongs not to geometry but to trigonometry. Although trigonometry gives a course for analyzing and understanding general rules of triangular forms, its objects are not concrete without a direct corresponding image as in geometry.

> The objectification of the teachings of trigonometry is left, then, for our exercises. Thus we have determined more clearly where in mathematics the means for cultivating sense-perception are to be found, and also the relation in which our preliminary exercises stand to the science. (p. 177)

For Herbart, triangle is the fundamental form for sense-perception. Trigonometry employs in particular the right-angled triangles which offer the fundamental concepts for determining all the other triangles. Herbart (1802/1903) readily acknowledged that "true foundation of trigonometry is higher analysis" (p. 178), but he put a constraint in schools on borrowing foundations from experience with an inductive approach. What he advocated was first to detect certain relations from empirical measurement and then to develop propositions from imperfect inductions experimentally made. He considered that science is the demonstration of the necessity of relationships observed and taking them as propositions to prove their universality theoretically.

In Herbart's object lessons on trigonometry, students were expected to experience a multitude of triangles in a continuous transition and even "anticipate sensuously" the meaning of the different formulas of trigonometry. The study of right-angled and isosceles triangles with trigonometry would orient student toward using angle to determine the form of the triangle. From Herbart's perspective, an angle could be "closed" by its sine or by its tangent. In this sense, an angle was a circular arc "closed" by sine and tangent as geometric line segments about a circle. When a right-angled triangle was formed by a sine and a cosine, they were all included in one circle. At the time, the standardization did not take place yet in divorcing the definition of sine and cosine from the circle radius. Then it was to be understood that the measure of a sine depends on the size of the radius. The smaller sized sine was better measured by the greater sized circles; thus indicating that a smaller angle with its sine was more perceptible to the senses when a greater circle is drawn.

Following Rheticus, every right triangle with sides *a*, *b*, and *c* can be canonically expressed by three species depending on which side is taken as the radius: (cos, sin, *r*), (*r*, tan, sec), and (cot, *r*, cosec). Herbart thought that the child needs unchanging, easily applicable expressions. He suggested conventions for the practice of the trigonometry of right triangles. Instead of sine and cosine, he preferred to use tangents and secants for sense perception. If one keeps the practice of calling the smallest side *radius* and the intermediate side *tangent*, the angles increase from 45° in every right triangle. This way, it becomes unnecessary to work with the angles under 45° (Herbart, 1802/1903, pp.187–188).

Herbart's scientific approach to education suggested an emphasis on sense perception, experimentation, the development of rules, and conventions. During the unified mathematics reform, more important than the integration of applications and science content with school mathematics, pedagogical challenges were made to integrate scientific process with the practice of school mathematics. This effort to reform the process was supported by an emphasis on experimentation and laboratory work for mathematics, with the integration of graph paper.

The principal functional thinking for geometry resided in the mensuration formulas (Hedrick, 1938). Functional ideas in geometry were not addressed adequately in schools. For Hedrick, "the idea of trigonometric calculation is intrinsic in the congruence theorems" (p. 451). The "congruence theorems" were a surprising choice, but the discussion he offered implied that he did not also mean "similarity theorems" or similarity of right triangles. His argument was based on the determinability of a congruent triangle by two sides and the included angle, placing an emphasis on the idea that the other sides must be functions of the three given parts.

Considering the content of geometry and algebra textbooks in the 1930s, Hedrick (1938) thought it a measure of quality for both algebra and geometry textbooks to "give elements of trigonometric calculation for right triangles" (p. 451). The emphasis should be given throughout geometry on the dependence of parts of a figure upon other parts, angles, arcs and sides. Although the relationships between arcs and angles are vital, they are often made secondary to some traditional topics such as incommensurables. Hedrick supported the idea of training the student to think functionally to see the relationships between the various parts of each figure. He called this "the *real geometry*." Hedrick stressed the functional aspect of trigonometry. He argued further that "the trigonometry is functional on its face. It is the first place where we traditionally use the word "*function*" (p. 451)." He objected to the tendency to emphasize formalisms in the practice of trigonometric identities. He called for their de-emphasis except for a very few simple ones, and more emphasis on the way in which the trigonometric functions vary. He called for more emphasis on graphical representation to help students understand trigonometric variations as observed in physics and other fields.

Evanovich (1975) introduced trigonometry in secondary schools through finite mathematical structures. In his dissertation study from Temple University, he proposed a finite mathematical structure for students of secondary school. The study also examined the effect of the finite structure method on students' achievement when used as an introduction to trigonometry.

Kendal and Stacey (1996, 1997) provided a critical review of the unit circle definition of trigonometric functions and compared it with the ratio definition. The unit circle method defines *cosine* and *sine* as the *x* and *y* coordinates of a point on a unit circle. The ratio method, in contrast, defines trigonometric functions as the ratios of pairs of sides in a right triangle. Kendal and Stacey indicated some positive aspects of the unit circle definition. For example, the unit circle provides concrete meanings for the trigonometric functions as lengths of intervals that can be directly measured. Also, it is an ideal definition to be extended beyond the first quadrant. They observed, however, that none of the textbooks that used the unit circle definition went beyond the first quadrant. They found this situation particularly odd, adding the following:

> The problem with the unit circle method is not the unit circle. Rather it is the complex procedure that is required to use the resulting definitions to solve triangles. In recommending the unit circle approach, curriculum developers opted for the more difficult path, with promise but no evidence of any long term benefit. Moreover, the scale factor version of this approach which is designed to eliminate one major difficulty has only replaced it by another of equal or greater magnitude. Since almost all of the curriculum goals that lead to the introduction of the unit circle method are no longer regarded as important at years 9 and 10, the unit circle

> method has been left stranded high and dry when the curriculum tide has gone out. (Kendal & Stacey, 1997, p. 7)

Although Kendal and Stacey's arguments were made in the context of changes in the Australian mathematics curriculum during the 1990s, they may correlate with changes in the practice of trigonometry in the United States during the standards-based reform. The argument above also suggests a complicated practice of solving triangles with the unit circle. Kendal and Stacey claimed that there was a change in practice towards using a circle with any radius and factoring accordingly to develop trigonometric functions. I address these two claims in chapters 4 and 5.

Katz (1998) has been another proponent of more emphasis on circular trigonometry than right-triangle trigonometry for treating trigonometry in schools. Most modem trigonometry texts, according to Katz, begin the subject by defining the basic trigonometric ratios, calculating those ratios using some elementary geometry for angles of 30, 45, and 60 degrees, and then assuming that students can use calculators to find the trigonometric ratios for any other value. In addition, without using a circular approach, students would not see the purpose of deriving the half angle, sum, and difference formulas. It is much more natural to adapt the original order of the treatment by Ptolemy, or even Copernicus, and to develop the subject in a manner inspired by history.

An emphasis on circular trigonometry does not imply the use of chords instead of sines. Katz (1998) suggests a course plan for trigonometry. Given the definition of a trigonometric ratio, one can use geometry to determine those values, not only for angles of 30, 45, and 60 degrees, but also for angles of 18, 36, 54, and 72 degrees. He suggests starting to build a trigonometric function table by calculating the values of sine, cosine, and tangent for the angles from 1 to 90 degrees. During this process, one derives the half-angle and difference formulas to calculate other angles. This way, students come to another historical realization: that it is not possible to calculate the sine of 1 degree exactly. A mathematical connection can be built here to the problem of trisecting an angle. If students notice that the sine function is essentially linear for small values, one can then approximate the sine of 1 degree to a reasonable level of accuracy and then use the sum formulas to calculate in principle the sine, cosine, and tangent of any angle of an integral number of degrees. The linearity of the sine function for small values is an important idea for later use. With the trigonometric tables calculated, students can use them to solve triangles of various types. In particular, another goal of the trigonometry course, again one based on history, should be to solve spherical triangles as well as plane triangles. Katz argued that the major use of trigonometry was to solve spherical triangles related to astronomy.

Gluchoff (1994) presented a historical tour through theories of integration to establish the need for refining and developing the idea of the integral in the context of trigonometric series. He claimed that the focus on trigonometric series can provide a way of grounding the development of the idea of integrability for students. The historical development of integration in analysis can be traced by the treatment of the problem of “uniqueness of representation of functions by trigonometric series” (Ash, 1989, p. 873) and the development of continuity theory. Lebesgue extended the concept of integrability to the discontinuous functions of a real variable to resolve the problem of representation of such functions by trigonometric series. Trigonometric functions were a critical impetus in the development in calculus and analysis.

Cavey (2002) studied the growth in a teacher’s mathematical understanding while learning to teach right-triangle trigonometry. She employed a method of lesson plan study adapted from the Japanese lesson study method of professional development and explored patterns of growth and connection building through lesson plan study. Cavey and Berenson

(2005) described one preservice teacher's growth in understanding right-triangle trigonometry as she participated in lesson plan study. The study was not particularly about right-triangle trigonometry per se, but it was about process of growth in understanding that is known as the Pirie and Kieren (1994) model. Trigonometry was used to exemplify this process for growth in understanding of subject matter by student teachers. Right-triangle trigonometry was studied through student-teacher-developed lesson plans that would introduce right-triangle trigonometry through connections to ratio and similarity. Cavey and Berenson claimed to have observed substantial growth in one preservice teacher's understanding of right-triangle trigonometry. They explained what changed in understanding and noted that the teacher's understanding was extended by noticing properties and formalizing mathematically precise information in relation to solving for missing parts of right triangles where noticing and formalizing are parts of the growth process.

Fulling (2005) noted the problem of the inverse secant function with negative arguments and suggested that after a decade of calculus reform the secant function and its inverse had been deemphasized to the vanishing point. The justification for teaching the secant and tangent functions is that they arise when algebraic functions are integrated by trigonometric substitutions. Fulling suggested using hyperbolic substitution rather than secant and inverse secant.

Thompson's (2007) dissertation study focused on the development of understanding of trigonometric functions in context by conducting action research from the theoretical framework of realistic mathematics education (Treffers, 1993). A contextual realistic problem-solving scenario was used as the instructional starting point for a trigonometry unit of instruction. The Ferris wheel context was used to support the concepts of amplitude, period, and the general behavior of the graph increasing or decreasing through various angles. Context became an advantage for many students, but not for all, to support their thinking concerning the unit circle and the graphing of trigonometric functions. In the development of students' mathematical understanding in context, a shift was observed from a *model of* to a *model for*. This transition is further described by realistic mathematics educators (van den Heuvel-Panhuizen, 2003). The Ferris wheel context as a starting point supported students in developing an understanding in context by allowing them to shift from considering the animation as a model of trigonometric functions to a model for trigonometric functions. This foundation provided students with a rich initial experience to further build upon to generate a connected and flexible understanding of graphing trigonometric functions with the unit circle.

Realistic mathematics education called for a dual functioning of context problems: not only as an application as before but also as a source of mathematical ideas. In realistic mathematics education, there were attempts to develop a theory of levels of understanding as a framework for mathematical development in context. Mathematizing everyday subject matter and mathematizing the mathematical subject matter yields horizontal and vertical mathematization. The shift from *model of* to *model for*, which is parallel to Sfard's (1991) process of reification in which processes are reinterpreted as objects. Operational matter at one level becomes a subject matter at the next level (Freudenthal, 1991). The models emerge as context-specific models that refer to concrete situations and that are experientially real for students. Gradually, the contextual meaning may shift to the background, but the roots of the model are preserved. Emergent models mediate a shift from an informal situated mathematics to more formal mathematics. *Horizontal mathematization* refers to the process of describing a context problem and solving it in mathematical terms. *Vertical mathematization* refers to mathematizing the mathematical activity, where the focus of attention is decontextualized.

After writing a history of early trigonometry, Van Brummelen (2010) posed some fundamental historiographic questions on trigonometry as a school subject. How is one to determine what belongs to trigonometry and what does not? To what extent can one legitimately talk about knowledge crossing cultural boundaries intact? Although these questions do not have clear answers, their introduction in a classroom setting could enrich and deepen students' perceptions of what mathematics is, and how culture interacts with it.

Historical Treatments of Trigonometry in Schools

I present below some exemplary approaches to trigonometry taken from textbooks published before 1893 to give the reader a sense of how trigonometry was approached prior to 20th-century reforms in school mathematics in the United States.

One of the first American trigonometry textbooks by Flint (1938) defined *sine* as follows "The sine of an arc is a line drawn from one end of the arc, perpendicular to the radius or diameter drawn through the other end: or, it is half the chord of double the arc" (p. 11). It was a geometrical definition based on the circle and its chords. Trigonometric functions were defined as functions of an arc, determined by line segments in and about a circle. There was no standardization of the radius. For example, "tangent of an arc of 45 degrees is equal in length to the radius of the circle of which the arc is a part" (p. 11).

The case of William George Spencer (1886) from the 19th century is interesting in the sense that it exemplifies textbooks with a problem-solving and discovery method. Spencer's case shows that textbooks began to appear during the 18th century that followed a discovery approach to mathematics by employing a programmed sequence of problems. The father of Herbert

Spencer, Spencer wrote an introductory level geometry textbook entitled *Inventional Geometry* in the 1830s to "introduce the beginner to geometry by putting him at work on problem which will not only thoroughly familiarize his mind with geometrical ideas" (p. 3) and to exercise "his inventive and constructive faculties"(p. 3). This textbook became well known and popular not only in England but also in America (Elliot, 2004). The 1860 edition of the book comes with a preface in which the author explains more about the method:

> Instead of dictating to the pupil how to construct a geometrical figure—say a square—and letting him rest satisfied with being able to construct one from that dictation, the author has so organized these questions that by doing justice to each in its turn, the pupil finds that when he comes to it, he can construct a square without aid. (Spencer, 1860, p. 2)

The language of trigonometry that Spencer used in the problems he posed suggests how he wanted to clarify the contemporary geometric conception of trigonometry and its language of practice. Spencer introduces the "line of sines," "line of secants," and the double meaning of *secant* and *tangent*: One meaning refers to an arc, and the other to a circle. In his exposition of trigonometric concepts, he asks a series of questions and then stops to give a definition, as demonstrated in the following excerpt. It also clarifies the argument used for trigonometric functions in Spencer's practice. In his approach, the line of sines is used to measure the angle.

> If to one extremity of an arc, not greater than that of a quadrant, there be drawn a radius, and if from the other extremity there be let fall a perpendicular to that radius, such perpendicular is called a sine of that arc. (p. 69)

> Which of the sines is equal in length to the radius of the line of sines? … Measure by the line of sines a few acute angles. Can you make an angle of 70° by the line of sines? The sine of the complement of an arc is called the co-sine of that arc. (p. 70)

In Spencer's (1860) practice of trigonometry, a secant was defined as a construct associated with a circle. It is a line that begins outside a circle, and "on being produced enters it, and traverses it until stopped by the other side of it" (p. 90). It indicates a practice of making a line of secants, fitting a secant to a circle, and making a secant of an arc measured by degrees. His approach used a trigonometric function as a geometric line segment rather than conceptualizing it as an algebraic ratio with a geometric meaning. The ratio method of introducing trigonometry was the new method of introducing the subject in textbooks after the early 1830s. Spencer's method and its prevalence suggest that the transition to the ratio method was not yet complete before the early reforms of the 20th century.

Orton (1987) points out that an elementary study of right-triangle trigonometry may contribute to the formation of the difficult ideas of similarity and ratio. Also, without forming the ideas of similarity and ratio, sines and cosines would have to be learned by rote. This dilemma is tied to the problem of integrating right-triangle trigonometry into algebra at Grade 9, or into geometry at Grade 10. Although the idea of ratio is contained in the traditional content of Grade 9, the idea of similarity was better formed in Grade 10 than Grade 9 because of the traditional geometry content of Grade 10.

Greene (1935) studied the evolution of mathematics for the secondary schools between 1834 and 1934 with respect to content and method of teaching. She found that the introduction of trigonometry was one of the "three major changes" in the teaching of algebra in the last century (p. 227). Two other major changes were the introduction of the function concept and the elimination of progressions.

Brown (2005) studied students' understanding of the portion of trigonometry that moves from right triangles to the coordinate plane and then establishes sine and cosine as functions. Its theoretical framework was based on the model of Schoenfeld, Smith, and Arcavi (1993), which gave a four-tier structure for a microgenetic analysis of a particular content topic. Two of the tiers involved aspects that are important for building a coherent understanding: (1) fundamental background concepts, and (2) issues related to context. Brown developed a content framework for a portion of coordinate trigonometry, and then he applied it in a case study of a group of students at the end of their work with this topic. He used the results to create a model of students' understanding of sine and cosine and also to refine the content framework. As part of the framework, he set forth a set of foundation concepts that underlie the subject of coordinate trigonometry. He called these ideas the *trigonometric connection*. Brown's research revealed that many students had an incomplete or fragmented understanding of three major ways to view sine and cosine: as coordinates of a point on the unit circle, as the horizontal and vertical distances that are the graphical entailments of those coordinates, and as ratios of the sides of a reference triangle.

In addition, Brown (2005) identified several cognitive obstacles. Some involved issues specific to the study of trigonometry. These included a fragile conception of rotation angle and unit, and a failure to connect a rotation on the unit circle to a point on the graph of the cosine or sine function. Other obstacles involved more fundamental topics, such as inability to relate the coordinates of a point on a graph to the horizontal and vertical segments that connect it to the axes. A difficulty related to context was the nature of sine and cosine as both ratios and numbers.

The trigonometric functions have always found a place in school mathematics during all efforts to reform it. Their treatment, however, has not been a subject on which experts could readily agree.

CHAPTER 3

METHOD

I used a historical and phenomenological approach to study trigonometric functions as a school subject during three reform periods. These two approaches complement each other. One guides the structure and the other guides the method of investigation. A historical approach as promoted by Schubring (1987) provides a general methodology in studying trigonometric functions as a school subject during three periods. A phenomenological approach helps one to focus on each textbook's perspective on the trigonometric functions. For the analysis of curriculum and instructional materials in trigonometry, I used Alkin's (1973) theoretical framework, which helps one to organize levels of information on a mathematical curriculum from the level of courses to units to lesson activities.

Three Mathematical Content Frames for the Practices of Trigonometry

In a preliminary study of textbooks during three periods, I observed the practice of three trigonometries– triangle, circle and vector/complex. It has been observed that students are exposed to two distinct trigonometries in schools: triangle trigonometry and circle trigonometry (Thompson, 2008, Bressoud, 2010). Each of these frames in practice is used to develop a set of ideas for trigonometric functions and organizes a set of references for treating trigonometric functions in high schools. The multiplicity of reference frames brings a plurality to the school subject. The sine as a mathematical object has a different sense depending on its reference frame. Textbooks do not always agree on their use of frames in their treatments of trigonometric functions. Depending on the grade level at which trigonometry is treated, each frame has a variety of practices for trigonometry among textbooks. It is also the case that the practices for each of those frames are not disjoint. For example, a circle trigonometry is blended with the practices of a triangle trigonometry when an emphasis given to reference right triangles. The blended practices of the frames require students to deal with the associations of the trigonometric objects in the former frame in the blend. Although in one frame the variations were on the sizes of right triangles, the circle frame demands that students shift their focus on variations on the measure of the angle, or rotation. A previous association with a practice of reference right triangle may become a hindrance. In this sense, each textbook has to develop and coordinate multiple references to trigonometric functions. Reform textbooks provide a multiple source of descriptions of what it is to treat trigonometric functions as a school subject. I used those frames to explain the variations in textbook treatments of trigonometry during and across reforms.

Methodological Framework

Historical Approach

The changes in ideas of school mathematics are traced by employing a historical method. Each textbook represents an expert's or group of experts' historical position on the content of trigonometric functions. Schubring's historical approach to school textbooks was used to study the development of trigonometric function ideas in school mathematics during three major periods. A textbook analysis with a historical perspective requires an elaboration of the textbook's contemporary context by representing other perspectives existing during its time. These perspectives include the context of debates on trigonometry and the trigonometric conceptions of contemporary textbook authors. On the historical study of mathematics textbooks, Schubring (1987, 2006) suggested contemporary context as an essential dimension of a textbook analysis.

Schubring (1987) offers a methodological approach to the study of history of mathematics education by proposing a “three-dimensional scheme for the methodology of textbook analysis to account for their social context” (p. 45). My interpretation of this methodology is as follows: The first dimension consists of analyzing the changes within the various editions of one textbook series chosen as a starting focus. This dimension of analysis traces the changes within the same series across time. The second dimension requires finding and analyzing corresponding changes in trigonometry in other school mathematics textbooks of the period. This dimension traces what other series are doing with the same content during the same period. The third and last dimension relates the changes in the textbooks to changes in the context. It concerns demands created by contemporary school mathematics reform documents; federal, state, and local constraints on school mathematics; debates and research on teaching and learning the specific subject matter; the evolution of mathematics; and so on. Each of the three dimensions was incorporated into my historical analysis of textbooks. This approach has been adopted before by other mathematics education researchers in the study of historical textbooks. A first example of a three-dimensional investigation is a study of the history of negative numbers in France and Germany between 1750 and 1850 (Schubring, 1986). Another example is Fujita and Jones’ (2003) examination of the place of experimental tasks in geometry textbooks from the early 20th century.

Schubring (1987) further elaborates this point for an analysis of Lacroix’s texts:

> It is necessary to enlarge the interpretation of a text in order to reconstruct its meaning: a first basic rule for such an endeavor is that a text can only be interpreted adequately together with its context. And, as an approximation to a

> reconstruction in its proper conceptual field, one should analyze its contemporary context. . . . One has to reconstitute the whole context of the debates and the conceptions of the contemporary authors together with their embeddings in the cultural structures of the time. (p. 44)

The context of reform is critical in interpreting textbooks. Schubring (1987) emphasizes the importance of the contemporary context to reconstruct the conceptual field as a method of textbook analysis. A historical text needs to be enlarged to interpret and reconstruct its meaning. A basic rule is that a text can be interpreted adequately only together with its context. As an approximation to a reconstruction in its proper conceptual field, one should analyze its contemporary context. One has to reconstitute the context of the debates and the conceptions of contemporary authors together with the "embeddings of those debates" and conceptions in the cultural structures of the time (p. 44). I provide the context for each reform by laying out the critical events and documents that shape the dynamics of reform. Then each reform perspective is given to indicate its position and its interpretation of a reform perspective as it relates to the practice of trigonometry. I present sketches of conceptual orientations towards trigonometry by contemporary authors during the three reform eras. I describe the textbooks' practices on the placement of trigonometric functions in the high school mathematics program, observing the patterns and their change during reforms.

Because of the lack of direct access to previous reforms, sources and artifacts on the reforms are limited. There is no library of school reform records of the artifacts and the manifestations of reform mathematics curricula in the form of audio-recordings, video-recordings, and photographs. As a consequence of these limitations, I develop a history of treatments of trigonometric functions during reforms based on sources that I have direct access for each reform period. For all three reform periods, I have direct access only to written historical artifacts such as textbooks as primary source. I also have resources such as books and journal articles on the reforms as secondary sources. Each source describes what the period was about from the authors' perspective, written from an internal or external perspective depending on the affiliation of the authors with the curricula under discussion. A historical approach combines multiple perspectives and helps one to bring together descriptions of the past as related to each reform. It is used to generate a multilayered description of what the reform was about as it pertains to trigonometric functions as a school subject.

Phenomenological Method

I adopted phenomenology as a research method. The phenomenological method was developed by Edmund Husserl between the 1890s and 1940 and is still active in both the social and the physical sciences (Creswell, 2007; Moustakas, 1994; Reeder, 1986). The application of the method of phenomenological description to textbook analysis was suggested and used by Kang (1990) in his description of mathematical knowledge in didactical transpositions in textbooks (see also Chevallard, 1991; Kang & Kilpatrick, 1991).

Husserl's phenomenological analysis can be interpreted as a project of conceptual clarification. In the present study, it was implemented to clarify the conceptual field of school trigonometry. If there is a core for trigonometry or for any other subject of school mathematics, phenomenology provides a methodological approach to those core mathematical ideas and developing their essences. An essence is a fundamental idea that anchors concepts in intuition. For Husserl, following Kant and Descartes, knowledge based on intuition and essence precedes empirical knowledge.

Phenomenology is a philosophy, and as a philosophy it develops methods for a rigorous science in uncovering universal elements of knowledge and providing grounds for its further exploration. By design, phenomenology keeps a keen eye on mathematical ideas. It suggests a balanced approach in addressing subjective and objective aspects of mathematical ideas to present school mathematics. Husserl's transcendental phenomenology provides a rigorous method for moving from subjective to objective (Lauer, 1979).

As an explanation of the intent of phenomenological description, Lauer (1979) added:

> Husserl was opposed to what he called the "dualism" of Kant, the "constructionism" of Hegel, and the "naturalism" or "psychologism" of the positivists. He agrees with them in asserting that only phenomena are given, but he would claim that in them is given the very essence of that which is. Here there is no concern with reality as existing, since existence is at best contingent and as such can add to reality nothing which would be the object of scientific knowledge. If one has described phenomena, one has described all that can be described, but in the very constant elements of that description is revealed the essence of what is described. Such a description can say nothing regarding the existence of what is described, but the phenomenological "intuition" in which the description terminates tells us what its object necessarily is. To know this is to have an "essential" hence a "scientific" knowledge of being. (pp. 3–4)

A phenomenological method was used in the present study to find the essence of the ideas a textbook series developed in treating trigonometric functions. Phenomenology suggests a progressive description of the essence from the layers of ideas of treating trigonometric functions in textbooks from different reform periods. A textbook is an artifact that manifests its authors' ideas of school mathematics in a way that models and supports classroom practice. It is a result of a knowledge claim of experts consensually responding to the question of what essentially is there for learners to experience about trigonometric functions. *Phenomenon* refers to the collected totality that is made manifest. The phenomenon in the sense of mathematics is best explained by Cassirer (1978):

> Mathematics does not move in the circle of sensory existence, yet its truth too, is firmly anchored in original being; in the reality or essence of things. This essential being can be apprehended only through concepts, yet abstract thinking in no way creates but only discovers it. In this sense, even the pure form mathematical

> reality is always related to a definite “matter,” though this is not a sensible but a wholly “intelligible matter.” (p. 62)

Husserl’s phenomenology has been relevant to mathematics education since its emergence. From the 17th to the 19th century, practice moved toward an algebraic approach to geometry. Descartes showed the possibility of approaching geometry through numbers, which gave rise to analytic geometry (Otte, 1997). By 1850, Weierstrass was arguing that geometry must be approached through numbers to satisfy the burden placed on geometrical intuition. Husserl studied mathematics under Weierstrass, who had a project to expose the original roots of analysis at the same time as Kronecker and Cantor (Hill, 1997). Husserl turned to philosophy to examine the nature of elementary mathematical objects and relationships. During the 19th century, geometrical intuition broke down as a way of characterizing mathematical relationships as the new geometries emerged. As the method became the object of study, the existence of counterintuitive mathematical objects generated debates. The subjects of continuity, the real numbers, and the mathematical object of infinitesmals were some of the topics debated in modernizing the elementary foundations of mathematics. Method as the object attempted to be used to organize the mathematical content to be presented to the learner (e.g., Pólya & Szegö). Husserl provided a methodological approach that balanced the subjective and objective nature of human sciences. The quest for Husserlian scientific knowledge for mathematics education is marked by a special emphasis on starting with subjective intentional acts. Performing a series of reductions from the subjective experiences of a knowledge object**,** the researcher progressively reaches higher states of awareness that culminates in intersubjective experience that together intuits a universal subjectivity of human experience. In a sense, the essence of a mathematical experience can be experience in a pure enough state that merges a transcendental merging of subjectivity and objectivity.

Husserl believed in the unity of formal and intuitive aspects of mathematics. Following Leibniz, he emphasized the objectivity of mathematical relationships. He developed a method that would not obscure that essential objectivity. His motto was, “Back to the things themselves!” While he was pushing toward objectivity, he was also adopting his approach to the subjective aspects of mathematics. His method was unique in that sense by pushing and balancing two opposite directions at the same time. Husserl was concerned about the adequacy of a psychological approach in describing those objective, normative aspects of mathematical relationships. Illustrating mathematical relationships through their volitional and arbitrary individual psychological constructions creates a problem of addressing the objectivity of those relationships. Toward developing a balanced methodology, he accepted an emphasis on subjectivity far beyond other empiricists such as Kant. Husserl’s phenomenological approach to mathematics suggests an account for the dependence of mathematical relationships on subjective constructions and for the normative functions of those constructions in mathematical practice.

At the beginning of his career, Husserl studied the concept of number. He developed a rigorous method to study subjective experiences; and some of his works were on geometry. This approach was supported and adopted by some mathematicians, among whom were Gödel (Tieszen, 1992), with his questioning of the limits; Rota (Rota, Sharp, & Sokolowski, 1988), with his practice of phenomenology in mathematics; and Tiezsen (2005a, 2005b), with not only his expositions of Husserl’s descriptive method for clarifying the meaning of mathematical objects but also his demonstration of the application of method of free variation in the study of essences in geometry. Some mathematics educators have begun to take phenomenological stances, such as Brown (2008), with a hermeneutic orientation; Campbell (2010), with a cognitive neuroscience orientation; Duval (2000, 2006), with a semiotic orientation; and Roth (Roth & Thom, 2009), with a dialectical orientation.

Phenomenology in the study of curriculum and textbooks is not new. A phenomenological approach can be used towards understanding of curriculum as a "phenomenological text" (Pinar, Reynolds, Slattery, & Taubman, 1995, pp. 404–449). Phenomenology is useful in studying the history of those mathematical ideas taken as school mathematics. Among other researchers, Tieszen (2005) interpreted the method of ideation in the Husserlian phenomenology and applied it to the study of essences of mathematical ideas observed in invariants across different kinds of free variations. Tieszen argued, and later Hartimo (2008) substantiated, that the group-theoretical approach to modern geometry can be seen as a realization of Husserl's view of eidetic intuition.

A phenomenological approach can help one focus on the content given in a reform textbook series. The practices of bracketing and retention are important for a researcher to see and faithfully describe how an ordinary object like sine is given in school mathematics textbooks written in different reform periods. The practice of bracketing what is previously known about trigonometric functions helps the researcher see how trigonometric functions are presented in a reform textbook series. Bracketing makes a familiar practice unfamiliar, which is an attitude needed to see what is made available by a reform textbook. Phenomenology focuses on providing fair and multilayered descriptions of the appearances of the phenomena and refrains from providing explanations for why they appear the way they appear. Practicing phenomenology helps a researcher to focus attention to describe what is given.

The phenomenological approach to a textbook presentation of content keeps the researcher's attention on the intended mathematical object rather than confusing it with the content of presentation. I explain next what essential content is and how it is different from the content of presentation. Trigonometric functions as objects of study are ordinary objects that require a proper distancing to attend to their meanings manifested in textbook treatments across reforms. As a researcher, I have associations with angle and sine as objects of study under trigonometric functions. Some of these associations may not be essential but were established during their practice, such as cosine as a horizontal projection and trigonometric functions as a study of angles. I explain what I meant by nonessential associations related to the content and constraints of presentation. As my approximation of essential content, I start with an exemplary intuition about the core idea of circle trigonometry. A circle trigonometry can frame an analytic study of circle that traditionally posits point $(1, 0)$ as starting point and counterclockwise direction as positive direction for a rotation. Then the analytical expression of a circular point is manifested by the pair $(\cos\vartheta,\ \sin\vartheta)$. The pair $(\sin\vartheta\ ,\ \cos\vartheta)$ can still be used in analytic treatment of the circle. The association, "cosine" as horizontal projection, does not hold, and circularity is analytically defined in the clockwise direction for increasing ϑ values with a North Pole $(0, 1)$ position as the starting point. From these observations, I can suggest that some ideas used with circle trigonometry are contextual and conventionally became part of the treatment of circle trigonometry. The essence of the idea of circle trigonometry does not necessarily include the content of presentation. Upon free variations of founding intuition about the core idea, it further refines the idea of circularity to the idea of periodicity. This phenomenological analysis is based on an assumed circle trigonometry for the sake of giving an example.

In a phenomenological analysis of a circle trigonometry manifested in a textbook, I am also bracketing what I have just described. I needed to keep an open mind and suspend my own preconceptions and biases during the process of attending to a textbook treatment of trigonometric functions. I described the textbooks' treatment of trigonometric functions as faithfully as possible. *Bracketing* is a technique for suspending background presuppositions about what there is to know about trigonometric functions in schools. It allowed me to access what textbooks offer about trigonometric functions while suspending my judgment about trigonometric functions as everyday objects of school mathematics. By doing so, I attempted to bring forth the intrinsic nature of the trigonometric functions as they are manifested in different frames of high school mathematics textbooks during reform eras. By using this method, I shifted my attention from a particular instance of a treatment of trigonometric functions as manifested in a reform textbook to an abstracted essence across its many instances as manifested in textbooks. The essence of a textbook's treatment of trigonometric functions then offered itself for analysis and phenomenological description. During this process, I was an integral part of the research process and involved in the research design as a part of the study phenomenon with my sustained effort to bracket and keep proper distancing while attending to each textbook's treatment of trigonometric functions.

Phenomenology is useful to capture what is different about a new textbook. It provides a method for assessing whether and how an ontological innovation, if any, has been designed, tried out, and embodied in a school mathematics textbook.

I present, in this report, phenomenological descriptions of three trigonometry frames from their treatments in textbooks to clarify their idea and their manifestations. Phenomenological description involves a process of eidetic reduction. I progressively reach at the essence of right-triangle trigonometry and its treatment from its practices manifested across textbooks following the process of eidetic reduction. This process has three parts: exemplary intuition, its manifested variations, and synthesis. Exemplary intuitions are first located in the textbook. This exemplary intuition serves as a generic idea of the trigonometry whether it is triangle, circle, or vector. It is used to explain the basic mathematical idea intended by the textbook's treatment of right-triangle trigonometry. This is a process of conceptual analysis in the Husserlian sense and based on a founding intuition on a frame of trigonometry whether it is concrete or abstract.

Textbook authors often include some passages to pinpoint the generic idea about a trigonometry either in the preface or in the introduction while treating the trigonometry. This founding intuition serves as an exemplary model for the ideational abstraction towards the essence of right-triangle trigonometry. I trace how its aspects are varied and anchored in the subsequent parts of the textbooks on the trigonometry frame while bracketing. A textbook treatment of a trigonometry represents realizations of this exemplary intuition. The imaginative repetition is how this exemplary intuition repeats and holds across its variations. When I implement this process for different series of textbooks, I have a variation in generic ideas of trigonometry and a multiplicity of its variation within a textbook series. The last part is the synthesis where I intuit overarching themes observing the invariants. Along this progressive reduction procedure, I employed bracketing and distinguished the core idea from their sedimented associations. The notion of "sedimentation" is suggested in Husserl's historical constitutions of geometry. It was also interpreted for the constitution of the mathematical concept of symmetry (Kolen, 2005). Derrida (1989) also has an appendix with the Husserl's text on *Origin of Geometry* translated by D. C. Simms that further adds to that notion:

> Husserl calls this same phenomenon the historical “sedimentation of meaning”: the techniques and symbolisms of mathematics are applied and taken for granted, but its original meaning is forgotten and eventually distorted. Scientific inquiry becomes technologized and sediments into unquestioned cultural traditions and assumptions. (Simms, 2005, p. 166)

This process shows parallels with Menger’s (1961) use of Occam’s Razor in the study of the concept of variable by dissociating it from its semantics. It is a maxim to ensure that entities must not be multiplied beyond necessity. Menger applied this razor or prism to resolving “conceptual medleys into the spectra of their meanings” and “comb disentangling and straightening out the various threads of thought” (p. 332).

Progressively constructing and validating a core idea across its uses, I reach at an essence of right-triangle trigonometry. That is how I studied the fundamental ideas of a trigonometry frame along with its textbook treatments as manifestations of essences of school mathematics.

Textbook and Document Selection Process

There are three choices to make in a historical study of trigonometry in school textbooks. Each choice concerns the selection of textbooks and materials. First, one should select a focal textbook series with multiple editions representing a leading reform perspective of the period. Second, one should select some other popular textbook series of the period to contrast their treatments of trigonometry with that of the first series. Third, one should select several categories of documents to establish the contemporary context.

Selected Textbooks

In this study, four textbooks were selected from each period. One series from each period was used as the focus, and the other three were used to describe other reform perspectives and the contemporary context. The series were chosen from among the reform textbooks that either received major support or were widely disseminated during their time. The focal series are Breslich's Integrated Mathematics series from the unified mathematics period, the School Mathematics Study Group (SMSG) series from the new math period, and the Core-Plus Mathematics Project (CPMP) series from the standards-based reform period. I also used documents and articles that were developed by leading writers and project group members to give their perspectives on the material development and revision process while designing and revising their series (e.g., Hirsch, 2007; Baber, Stilwell, Benignus, Ashleman, Myers, Atwood, et al., 1901; Myers, Wickes, Breslich, & Wreidt, 1907; SMSG, 1965d; Wooton, 1965).

Unified mathematics reform period textbooks. The selection of textbooks representing this period was partly informed by Donoghue's (2003) review of algebra and geometry textbooks in 20th-century America. I also consulted Sigurdson's (1962) and Quast's (1968) dissertations on the leading textbook series as well as reviews of contemporary perspectives during the first reform period such as Betz (1950) and Stanic (1987).

Among the reasons I chose the Breslich series as focal series were the following: the amount of support it received from the mathematics education community during its time as a successor of University of Chicago school mathematics reform experiments; the emphasis placed on this collective effort by the 1923 Committee of the NEA as to model their recommendations for reform in high school mathematics; Sigurdson's (1962) identification of the series as a successful effort; and the reactions of conservative figures, such as D. E. Smith (1915), who was against the unification of geometry and algebra.

The Breslich series was not the result of the efforts of a single author; instead, it was a result of the collaboration of the teachers from the University of Chicago Laboratory High School and educators from the University of Chicago (Breslich, Schorling, Wright, & Irwin, 1916). The series represented the continuation of Breslich's involvement in the program development efforts started by G. Myers and E. H. Moore. The books were developed as a part of the University of Chicago Mathematical Series, School of Education Texts and Manuals edited and approved by Myers and Moore. Depending on the edition, right-triangle trigonometry was incorporated into either the first year or the second year. The trigonometric functions were mainly generalized in the third year, *Third Year Mathematics* (Breslich, 1917). The Breslich series was revised during the late 1920s as *Senior Mathematics*, integrating trigonometry into Book II on geometry and as *Purposeful Mathematics* in 1938 and 1943 with a change of publisher. This series was translated into Spanish and adopted internationally, which further indicates its popularity. For instance, the Breslich series played a major role in mathematics curriculum reforms of 1931 and 1941 in Brazil (de Carvalho, 2006).

Having chosen the Breslich series as focal series to address to the first dimension of analysis, I located copies of all four editions. I list all the copies in Table 1. I counted the first two-year version as the first edition of the series. It was developed under the leadership of Myers with an authoring team from the University High School that included Breslich. Therefore, Breslich was clearly involved throughout the life of series until the 1950s. After the 1950s, I found no evidence to suggest the series was in continued use. I located no new editions or impressions of the textbook series, nor additional reviews or discussions in mathematics education journals and magazines.

In Table 1, I present all the editions I located for the Breslich series. They were published for several decades as a part of The University of Chicago Mathematical Series under the editorship of E. H. Moore. They were edited by G. W. Myers as part of The School of Education Text and Manuals. From the early 1900s to the mid 1930s, the University of Chicago affiliation of the series continued. At the beginning, the books were used only in the University High School. The 1909 edition had eight impressions by 1914, which indicated that there was a demand for this edition. After 1915, another major edition of the series was prepared under the leadership of Breslich. This edition, unlike the earlier one, provided a four-book series. This series was made available by publishing houses in New York, Los Angeles, London, Shanghai, Tokyo, and Leipzig. This is also an indication of an international demand for the series. The next major edition came in 1928 as the Senior Mathematics Series, and it was distributed from Chicago, New York, Toronto, London, Tokyo, and Shanghai. There is no international distribution information for the Laidlaw Brothers edition of the series, unlike the University of Chicago Press editions.

Table 1
The Breslich Series

Year	Title and publisher	Authors
1907	*First Year Mathematics for Secondary Schools*. University of Chicago Press.	G. E. Myers, R. Wickes, E. R. Breslich, & E. A. Wreidt
1907	*Geometric Exercises for Algebraic Solution–Second Year Mathematics for Secondary Schools.* University of Chicago Press.	G. E. Myers, R. Wickes, E. A. Wreidt, & E. R. Breslich
1909	*First Year Mathematics for Secondary Schools*. University of Chicago Press.	G. E. Myers, R. Wickes, H. F. MacNeish, E. R. Breslich, E. A. Wreidt, E. L. Caldwell, & A. Dresden
1910	*Second Year Mathematics for Secondary Schools.* University of Chicago Press.	G. E. Myers, R. Wickes, E. A. Wreidt, E. R. Breslich, A. Dresden, E. L. Caldwell, and R. M. Mathews
1915, 1916	*First-Year Mathematics for Secondary Schools*. University of Chicago Press.	E. R. Breslich
1916	*Second Year Mathematics for Secondary Schools.* University of Chicago Press.	E. R. Breslich
1917	*Third Year Mathematics for Secondary Schools.* University of Chicago Press.	E. R. Breslich

1919	*Correlated Mathematics for Junior Colleges.* University of Chicago Press.	E. R. Breslich
1928, 1936, 1940, 1943, 1945	*Trigonometry-Plane and Spherical with Tables* (Rev. ed.). Laidlaw Brothers	E. R. Breslich & C. A. Stone
1928	*Senior Mathematics–Book I*. University of Chicago Press.	E. R. Breslich
1927	*Senior Mathematics–Book II*. University of Chicago Press.	E. R. Breslich
1929, 1934	*Senior Mathematics–Book III*. University of Chicago Press.	E. R. Breslich
1939	*Purposeful Mathematics: Algebra – First Course*. Laidlaw Brothers	E. R. Breslich
1938	*Purposeful Mathematics- Plane Geometry*. Laidlaw Brothers.	E. R. Breslich
1929, 1943	*Purposeful Mathematics- Algebra–Second Course*. Laidlaw Brothers.	E. R. Breslich
1940	*Trigonometry with Tables for Use in Senior High Schools and Colleges.* Laidlaw Brothers.	E. R. Breslich & C. A. Stone

Donoghue's (2003) review emphasizes the importance of the Wentworth textbook series during this period. The series represented "the most popular" high school textbooks widely circulated in the States during the early years of the 20th century (p. 331). The Wentworth textbooks were "used in over two-thirds of the schools in the country" (Middleton, 1911, pp. 146–147). Donoghue (2003) consulted the lists of textbooks used in three master's theses from the University of Chicago on the most widely circulated geometry and algebra textbooks during the first two decades of the century. The Wentworth series were included in almost all textbook studies covering the period. The Wentworth-Smith series were developed by the collaboration of A. G. Wentworth with D. E. Smith to continue the popular series authored earlier by the father, G. Wentworth. The Breslich series and the Wentworth-Smith series were both selected in John

Elbert Stout's (1921, p. 230) historical study, *The Development of High School Curricula in the North-Central States from 1860 to 1918*, to represent the reform and contemporary perspective. The Wentworth series books were in circulation for almost 70 years. The last Wentworth-Smith trigonometry was published in 1951. The length of the period between 1883 and 1951 further indicates its popularity as a high school mathematics textbook series. The series was embraced internationally, including Australia, China, and Latin America. So they made a remarkable history especially in the first half of the 20th century. Finkel (1907) wrote a short biography on G. A. Wentworth. Cajori (1890) added that "if we were called upon to name the writer whose books have met with more wide-spread circulation during the last decennium than those of any other author we should answer, Wentworth" (p. 294). I concluded that the Wentworth series was popular before, and remained popular during, the unified mathematics reform.

I used different editions of Wentworth's *Plane Trigonometry* to describe the changes in a popular textbook series' treatment of trigonometric functions before and after the unified mathematics reform (Table 2). I located and obtained copies of Wentworth's (1887, 1903) *Plane Trigonometry* and the subsequent Wentworth-Smith (1914) *Plane Trigonometry* as different editions of the same textbook series.

Table 2
The Wentworth Series

Year	Title/Publisher	Authors
1882, 1885, 1902, 1903	*Plane Trigonometry and Tables* (2nd ed.). Boston, MA: Ginn.	G. A. Wentworth
1914,1915, 1938	*Plane trigonometry and tables.* Boston, MA: Ginn.	G. Wentworth & D. E. Smith
1918	*Junior High School Mathematics, Book 3*. Boston, MA: Ginn.	G. Wentworth, D. E. Smith, & J. C. Brown
1922	*Fundamentals of Practical Mathematics*. Boston, MA: Ginn.	G. Wentworth, D. E. Smith, & H. D. Harper

There are other examples of reform-oriented textbook series from the period. William Charles Brenke was another prolific reform-oriented textbook author of the period with a complete series. His textbook series represents the Lincoln, Nebraska, brand of the reform. He collaborated with Edith Long to develop *Algebra* (1913) and *Plane Geometry* (1916). He also collaborated with other mathematicians and authors during this period. Some of his textbooks led the field, such as his textbook on *Calculus* with Hedrick. Brenke's textbooks on trigonometry were in demand for three decades. His *Advanced Algebra and Trigonometry* (1910) went into a second edition as *Elements of Trigonometry* (1917), and a third edition as *Plane and Spherical Trigonometry* (1943).

The other series I chose to examine was John A. Swenson's (1934) *Integrated Mathematics 1-2-3*. Donoghue's (2003) review pointed out that Swenson's integrated mathematics series was a major successor committed to following Perry's (1900, 1902) program of school mathematics. The Swenson books were unusual with their integration of trigonometric functions into second-year high school mathematics, which was commonly reserved for geometry. The textbook series of the later reform periods often treated trigonometric functions in the third course in high school mathematics. This third course was where the second algebra course was located in a traditional Algebra—Geometry—Algebra/Trigonometry course sequence. Stout (1921) traced this order of mathematics subjects in the curricula. According to Stout, when a second year of algebra followed geometry, it was designated as a review subject.

I included *Plane Geometry* by William Betz and Harrison E. Webb (1912) for two reasons. One was the authorship. William Betz was one of the most prolific mathematics educators in the first half of the 20th century. The second reason was that the series represented a case of early plane geometry textbooks that introduced trigonometric functions. I also included the *Plane Trigonometry and Applications* textbook by Wilczynski and Slaught (1914) to establish a context for the Breslich Series among other contemporary textbook authors of the period. This was a peculiar textbook with its use of the unit circle and inclusion of a chapter on wave functions. Ernest H. Wilczynski was the author, and H.E. Slaught was the editor of that textbook. The author received his doctorate in Germany and followed a heuristic approach. This textbook introduced trigonometric functions with an emphasis on their practical utility.

New math reform period textbooks. The textbook series I chose to represent the second reform period, new math, were the Mathematics for High School series (SMSG, 1960, 1961a, 1961b, 1961c,1961d, 1964, 1965a, 1965b, 1965c), the University of Illinois Committee on School Mathematics (UICSM) series (Vaughan & Szabo, 1973), the *Unified Modern Mathematics* series of the Secondary School Mathematics Curriculum Improvement Study (SSMCIS, 1970), and the Comprehensive School Mathematics Program's (CSMP, 1973, 1974; Kaufman & Steiner, 1969) *Elements of Mathematics* series. The UICSM and SMSG were both large-scale programs during the period. The new math period lasted about two decades (Fey, 1978; Hayden, 1982; National Advisory Committee on Mathematics Education [NACOME], 1975), leaving behind the newly developed commercial textbook series, some of which are still adopted and in use, such as the *Structure and Method* Series (Brown, Dolciani, Sorgenfrey, Kane, 2000; Brown, Dolciani, Sorgenfrey, Kane, Dawson, & Nunn, 2010; Dolciani, Berman, Wooton, & Meder, 1963; Wooton, Beckenbach, & Dolciani, 1966). Although the textbook series started out with SMSG origins, the new editions no longer reflect an image of the new math because of the editorial process in the reality of commercial textbook publishing. The length of the period between 1963 and 2010 indicates the popularity of the Dolciani series as a high school mathematics series. For the present study, different editions of Dolciani's *Algebra and*

Trigonometry: Structure and Method Book 2 represented a case to observe the changes in a commercial textbook's treatment of trigonometric functions before and after the new math reform. I used the 1963 and 1999 editions, the latter being used recently and adopted widely. For example, in Ohio, there is a report for Grade 12 about textbook use in the state. The Ohio teachers reported using 43 texts. No single text was dominant. The leading textbook for the 12th grade was the McDougal Littell *Algebra & Trigonometry: Structure & Method*, *Book 2* by Brown et al. (2000). Nine percent of the teachers used this series. The UCSMP *Advanced Algebra* was among the second most popular series with six percent. The Dolciani series descendents from the new math are still popular around the United States. Regarding the context of new math reform, I consulted available histories (Gibb, Karnes, & Wren, 1970; Hayden, 1982; Roberts & Walmsley, 2003; Walmsley, 2003).

Standards-based mathematics reform period textbooks. For the last reform period, I selected the Core-Plus Mathematics Project (CPMP; Coxford, Fey, Hirsch, Schoen, et al., 1997, 2008, 2009) as the focal series. The rationale for choosing this series as the focal series was mainly their leading authors' involvement in the development of the NCTM (1989) Standards. Hirsch and Schoen were two of six members of the working group who developed NCTM's Standards for School Mathematics at Grades 9–12. I located both editions of the CPMP series that addressed the trigonometric functions as they became available.

I analyzed three other textbook series to address the second dimension of analysis: the *Mathematics: Modeling of Our World* (MMOW) series by the Consortium for Mathematics and Its Applications (COMAP, 1998, 1999), the *Interactive Mathematics Program* (IMP; Fendel, Resek, Alper, & Fraser, 2000), and the University of Chicago School Mathematics Project (UCSMP, 1998) series.

Selected Reform Documents

As a critical dimension for the theoretical framework, I examined leading reform documents and research articles, dissertations, and commentaries on school mathematics and trigonometry during the period. I used them to present the contemporary context for selected textbooks of the period. To provide the context for the unified mathematics period, I examined the content of school trigonometry in the following reform documents: National Education Association (NEA, 1894, 1899, 1912, 1918, 1920), the National Committee on Mathematics Requirements (NCMR, 1923), Progressive Education Association (PEA, 1940), and National Council of Teachers of Mathematics (NCTM, 1940). The College Entrance Examination Board (CEEB, 1923) responded to the recommendations of the 1923 NCMR report and revised the college entrance requirements. Those became the official requirements until 1935 to inform both secondary schools and colleges as to how to organize their mathematics programs for college intending students. The report of the National Committee had a profound influence on the teaching of secondary school mathematics and particularly on the preparation of textbooks in algebra (Longley, 1927). The committees that prepared the 1923 CEEB mathematics requirements and the 1923 NCMR report had common members. It is not surprising to observe alignment in their recommendations.

For the new math period, I examined the content of school trigonometry in the reform documents of the CEEB Commission on Mathematics (1959) and Educational Services Inc. (1963). For the standards-based mathematics period, I examined the recommendation on the content of school trigonometry in *Academic Preparation* (CEEB, 1985) and NCTM's *Standards* (1989, 2000). These recommendations were included as a third dimension of analysis to establish the context for reform.

CHAPTER 4

TRIANGLE TRIGONOMETRY DURING REFORMS

A history of treatments of triangle trigonometry is here developed from textbook treatments during reforms. In this chapter, triangle trigonometry is presented as it was treated during reform periods in school mathematics. I present a phenomenological description of triangle trigonometry from its treatments in textbooks to clarify the idea and its manifestations. Textbook authors often include some passages to pinpoint the generic idea of triangle trigonometry either in the preface or in the introduction of a unit or course on trigonometry.

There are three sections in this chapter. In each, I address a reform period's treatments of triangle trigonometry: unified, new math, and standards. Triangle trigonometry during a period is further described in three subsections: contemporary reform context, editions of focal textbook series, and other textbook series of the period.

Unified Mathematics Period Practices of Triangle Trigonometry

The idea of unified mathematics first started with more general sense of unifying mathematics and other school subjects, especially physics. Then, in less than decade, it evolved into the idea of correlating mathematical school subjects such as geometry, algebra, and trigonometry. In the beginning, among schools giving unified mathematics were a high school in Lincoln, Nebraska, and the University High School in Chicago, Illinois. The idea of unified mathematics was "in accordance with the dominant educational tendencies" of the times (Myers, 1916, p. 82). Those tendencies were the following:

> Resolving the problem of lack of contiguity in mathematics work in the high school; the point of view of general mathematics is psychological; general mathematics is practical in content and method; and the algebra and geometry was kept apart due to the separation of subjects by the Ancient Greeks. (p. 82)

The laboratory method suggested an inductive method of teaching by making students more active and integrating the use of different apparatus in the laboratory such as algebraic balances, plane tables, modeling frames, composition of motion apparatus and parallelogram of forces apparatus, and slide rules (Collins, 1907). While there was a push in the Chicago area toward making practical experiments a part of school mathematics by the laboratory method, there was strong opposition from the East expressed primarily by D. E. Smith. He argued that mathematics and science should be kept separate in schools (Smith, 1907). After such rebuttals, the majority of subsequent efforts by the Chicago movement shifted their focus to unifying different branches of school mathematics—algebra, geometry, and trigonometry—rather than focusing on unifying science and mathematics as reflected in the course of studies of Myers' efforts in the laboratory school. The leadership of the curriculum experiments passed to Breslich after 1910 for the next three decades (Members of the Department of Mathematics of the University High School of the University of Chicago, 1940, p. viii).

The opposition the laboratory method in mathematics classes was fed by "a fear" that mathematics would be lost as separate branch of education (Sigurdson, 1962, p. 157). Developments in the following decades confirmed the aptness of this fear. A greater emphasis was placed on functional mathematics based on the social utility of its objectives showing more parallels with the recommendations of W. H. Kilpatrick's (NEA, 1918) committee than with the recommendations of the NCMR (NEA, 1923). Kilpatrick's report on the problem of mathematics in secondary education advocated a functional mathematics based on values. The decades following the NCMR report were characterized by a “nationwide battle of objectives” (Betz, 1950, p. 380), which led to “the creation of thousands of curricula in mathematics alone” (p. 381; see also Bruner, Evans, Hutchcroft, Wieting, & Wood, 1941)

Progressive influences were prominent among educators during the unified mathematics era. To meet the needs of increasing demands of a mass education at higher levels, educators in response developed high school mathematics programs for “all the children of all the people” (Betz, 1950, p. 80) with a social efficiency emphasis. Starting in the early 1920s, educators worked fervently all around the country to develop justifiable and flexible mathematical content to fit the needs of society. Betz reflected back on this period and said: “Someday the fantastic story of the era of curriculum revision, beginning in the early 1920s, will have to be written” (p. 381). He added that “many thousands of mathematical courses of the study are now cluttering the shelves of our curriculum morgues” (p. 381). Sixty years after Betz’s remark, the mathematics education story of this period is still to be written. Calling this period the period of depression in mathematics education does not do justice to the efforts by educators at that time to provide a flexible and socially efficient mathematics programs for all. It is clear that the course of study movement did not last “because there is nothing to hold it together” (p. 382). Subsequently, a period of depression started in school mathematics with a drop in mathematics requirements by a majority of states from the 1920s to the 1940s (Betz, 1950; Stanic, 1986b). Betz observed that around 1950,

> it became current doctrine that only a minority, “the few,” had any use for the customary offerings in algebra and geometry. All the rest, “the other 85%”needed a totally different diet, the “mathematics of daily life,” or the “mathematics of life situations.” (Betz, 1950, p. 383)

The suggested mathematics programs for all placed more emphasis on the intuitive and informal aspects of mathematics, and did not provide a proper balance for intuitive and formal aspects of mathematical practice.

Contemporary Reform Context on Triangle Trigonometry

Secondary education before unified mathematics (Doty & Harris, 1874) involved the study of the following courses: (a) algebra, geometry, calculus, some forms of engineering, surveying, and navigation; (b) natural philosophy or physics; and (c) physical geography or natural history. In secondary education, surveying and navigation involved not only plane trigonometry but also spherical trigonometry. At that time, algebra and geometry were taught separately and in parallel in Grades 9 and 10. Educators became dissatisfied with this arrangement, however, and some attempted to unify algebra and geometry in the secondary school (Sigurdson, 1962).

The image of mathematics during this time was that “mathematics is the abstract form of the natural sciences; and that it is valuable as a training of the reasoning powers, not because it is abstract, but because it is a representative of actual things” (Safford, 1896, p. 9). The suggested modernization of the method of teaching mathematics before the unified mathematics was toward “teaching the concrete first and thoroughly [leading] up to the abstract by easy stages” (p. 46). Safford supported only numerical trigonometry for all students:

> It is more proper to teach the practical side of trigonometry first; the theoretical part, analytical trigonometry or “goniometry,” should be made a strong mathematical discipline, if possible; otherwise not meddled with. The capacity to measure and solve triangles is of practical value; an imperfect knowledge of goniometry is not useful, but rather tends to confuse and dishearten the pupils. (p. 46)

To replace the static view of mathematical quantities, the function concept was introduced into all areas of school mathematics following Felix Klein’s notion of functional reasoning. The colloquium lectures Klein delivered in 1893 in Evanston, Illinois, were considered a landmark event in the history of American mathematics (Gray & Rowe, 1993). His fields of interest were geometry and the theory of functions. He questioned the degree of emphasis placed on algebra in the name of rigor and called for more emphasis on applications tied to physics.

Reform documents during unified mathematics. The trigonometry in textbooks and their changes across its editions were sensitive to the contemporary context reflected by the reform documents of the period.

NEA (1894). The release of the report of the Committee of Ten appointed by the NEA (1894) was one of the first events in shaping U.S. high school mathematics reform during the early 20th century. The general report did not address trigonometry separately, recommending only that ``boys going to a scientific school might profitably spend a year on trigonometry and some of the higher parts of algebra, after completing the regular course in algebra and geometry" (p. 107). Trigonometry was first introduced in the first half of the fourth year of secondary school for two periods a week. Higher algebra was to replace trigonometry in the second half year and was to be continued in the allocated two periods a week.

Concerning the reception of the Committee of Ten report, Dexter (1906) examined its impact and studied how fully the changes that had taken place during 10 years in the high-school curriculum coincided with the specific recommendations of the committee. Mathematics programs for secondary schools were compared between 1894 and 1904. Regarding the impact of recommendations for trigonometry, Dexter reported that trigonometry was in 1894 offered in a quarter of the schools studied; in 1904, almost half. It was almost without exception that trigonometry was given in the fourth year. Both these facts were in accordance with the committee's report. When offered, trigonometry was offered with advanced algebra as recommended.

NEA (1899). The 1899 report of the NEA Committee on College Entrance Requirements (CCER) did not emphasize the subject of trigonometry. The Chicago Section of the American Mathematical Society (AMS) appointed a committee in 1898 to cooperate with the NEA to help prepare the CCER report "on the scope, aim, and place of mathematics in the secondary schools and in preparation for colleges with model courses in algebra, geometry, and trigonometry" (NEA, 1899, p. 135). The CCER's recommendations on mathematics in 1899 aligned with those of the Committee of Ten (NEA, 1894). In secondary schools, work in mathematics should be required from all students during each of the 4 years of the course. The committee suggested a divided mathematics practice during the 9th and 10th grades. The committee recommended that "the time allotted to mathematics in the ninth and tenth grades be divided equally between algebra and plane geometry" (p. 22). Solid geometry and plane trigonometry was recommended for Grade11 for 4 periods a week. The report recommended a half year of plane trigonometry.

The CCER committee report (NEA, 1899) recommended that some elements of plane trigonometry should be an integral part of the school course, in continuation of algebra and plane geometry and a "fitting sequel to them" (p. 143). When trigonometry was treated, it should be "restricted to that needed for the solution of plane triangles—numerous, but simple applications to the determination of heights and distances should be made" (p. 143). The committee suggested the trigonometric functions should be defined as ratios, and the whole treatment should be based exclusively on the ratio definitions. The CCER (NEA, 1899) further recommended that the study of geometry should proceed the study of algebra: "Geometry is less abstract, less artificial, lends itself less readily to mere mechanical manipulations, and is more easily illustrated by concrete and familiar examples from algebra" (p. 145). "The unity of work in mathematics is emphasized and correlation and inter-application of its different parts recommended" (p. 148).

The report (NEA, 1899) suggested the following:

> There is no intrinsic reason why the elements of plane trigonometry should not be an integral part of the school course in mathematics; it can be developed well in continuation of algebra and plane geometry and is a fitting sequel to them. (p. 154)

When offered as a high school subject, the report remarks that "the matter should be restricted to that needed for the solution of plane triangles" (p. 154). The committee takes a stance on the conceptualization of trigonometric functions. Although trigonometry was pushed down to high school, the NEA report excluded the treatment of spherical trigonometry. This subject had been paired with the content of plane trigonometry in college.

NCMR (1923). The National Committee on Mathematical Requirements (NCMR) was formed in 1916, worked until 1923, and as a result provided a national expression of the movement for reform in the teaching of mathematics (NEA, 1923). Reflecting back on the reform, NCMR (1922) stated that:

> A convenient starting point for the history of modern movement in this country may be found in E. H. Moore's presidential address before the American Mathematical Society in 1902. The movement here is only one manifestation of a movement that is world-wide and in which very many individuals and organizations have played a prominent part. (p.vi)

As suggested by the NCMR (1923) report, the reform had already made considerable progress in various parts of the country. The reform efforts, however, were not coordinated or united, as the report pointed out. The NCMR report included numerical trigonometry to the content of algebra. Algebra was defined as a composite of six topics: (1) the formula, (2) graphs and graphic representation, (3) positive and negative numbers, (4) the equation and its use in solving problems, (5) algebraic technique, and (6) numerical trigonometry. Compared with the recommendations by the NEA (1899), algebra was reconstructed to include graphs and graphical representation and numerical trigonometry in solving right triangles.

Inclusion of a general mathematics class for all students can be considered the main impact of the unified mathematics movement. One of the main critics of the movement was D. E. Smith. He was on the NCMR, and so was the main proponent of unified mathematics, E. H. Moore.

The NCMR (1923) recommended the treatment of the following topics for numerical trigonometry in Grades 7 to 9:

> (a) Definition of sine, cosine, and tangent;
> (b) Their elementary properties as functions;
> (c) Their use in solving problems involving right triangles,
> (d) The use of tables of these functions. (p. 25)

The report justified the early introduction to numerical trigonometry by emphasizing more of the practical usefulness of the subject as in the application of its methods concerned with indirect measurement. Intuitive geometry was suggested as a basis for the work on trigonometry. It was indicated that the treatment of the subject should be confined to the material needed for the numerical treatment of the problems indicated. No further relations between the trigonometric functions needed to be considered for Grades 7 to 9.

On the subject of problems, emphasis was shifted from formal exercises to “concrete” problems that were practical for the students and suited to their level of maturity and experience. The committee recommended the use of problems “real *to the pupil*” (NCMR, 1923, p. 28). A conscious effort was to be made to select problems that would connect mathematics with other subjects, especially the sciences. Problems became an essential part of the junior high school mathematics program demanded of all students. Other than numerical trigonometry as a separate content strand, trigonometry was also included in the recommendations of “the numerical computation, use of tables” (p. 28) to further suggest development of students’ understanding and using tables for trigonometric functions.

The committee suggested four plans for arranging courses for Grades 10, 11, and 12. No preference was given to any one of the plans. Grade 12 was common to all four plans. The courses for Grades 10 and 11 in the plans are shown in Table 3.

Table 3
1923 NCMR Recommendations for Arranging Courses

Plan A.	Tenth year: Plane demonstrative geometry, algebra.
	Eleventh year: Statistics, trigonometry, solid geometry.
Plan B.	Tenth year: Plane demonstrative geometry, solid geometry.
	Eleventh year: Algebra, trigonometry, statistics.
Plan C.	Tenth year: Plane demonstrative geometry, trigonometry.
	Eleventh year: Solid geometry, algebra, statistics.
Plan D.	Tenth year: Algebra, statistics, trigonometry.
	Eleventh year: Plane and solid geometry.

The NCMR (1923) provided detailed content recommendations for the senior high school mathematics classes. The trigonometry recommended for Grades 10 to 12 included the following topics:

> The work in elementary trigonometry begun in the earlier years should be completed by including the logarithmic solution of right and oblique triangles, radian measure, graphs of trigonometric functions, the derivation of the fundamental relations between the functions and their use in proving identities and in solving easy trigonometric equations. (p. 38)

The other recommendations made by the committee concerned the instruments to be used in teaching trigonometry. The use of the transit was recommended for conducting simple surveying. The report proposed the use of the sextant in conducting simple astronomical observations, such as finding local time. Improvisation of a simple apparatus for measuring angles was recommended when no transit or sextant was available. Graphing was continued to form an essential part of the numerical work of trigonometry by drawing to scale. The slide rule was proposed as an instrument to facilitate and check computations requiring only three-place accuracy.

The NCMR (1923) conducted a survey of college teachers on the selection of topics to be included in high school according to values attached to them for "securing an adequate mathematical preparation" for the study of the physical sciences and social sciences in college (p. 43). Numerical trigonometry was ranked as one of the highest valued subjects, next to the following topics: simple formulas, their meaning and use, linear and quadratic functions, the use of logarithms and numerical computation, and statistics. These results indicated that some modifications were needed in mathematics requirements from the perspective of the fields of study other than mathematics. The main recommendations were that work on numerical trigonometry and statistics should be included and that "functional relationships" must be made underlying principle of the course (p. 44).

The CEEB Mathematics Requirements (1923). The 1923 CEEB report gave a definition of requirements in Elementary Algebra, Advanced Algebra, and Trigonometry. Four of the authors had served on the NCMR, which prepared the 1923 report for the reorganization of secondary school mathematics.

The CEEB (1923) requirements for trigonometry were as follows:

1. Definition of the six trigonometric functions of angles of any magnitude as ratios. The computation of five of these ratios from any given one. Functions of 0°, 30°, 45°, 60°, 90° and of angle different from these of multiples of 90°.
2. Determination, by means of a diagram of such functions as $\sin(A+90^\circ)$ in terms of the trigonometric functions of A.

3. Circular measure of angles; length of an arc in terms of the central angle in radians.
4. Proofs of the following fundamental formulas and of simple identities derived from them; ratio formulas and the Pythagorean formulas as below

   $$\tan x = \frac{\sin x}{\cos x},\ \sin^2 x + \cos^2 x = 1$$

   In addition, the Board requires knowledge of the derivation and proofs of the addition theorems and the double-angle formulas.
5. Solution of simple trigonometric equations such as 6 sin x+cos x = 2.
6. Theory and use of logarithms without the introduction of work involving infinite series. Use of trigonometry tables with interpolation.
7. Derivation of the Law of Sines and the Law of Cosines
8. Solution of right and oblique triangles (both with and without logarithms) with special reference to the applications. (p. 9)

Unified mathematics period discussions on right-triangle trigonometry. As for the status of triangle trigonometry before and during the unified reform era, right-triangle trigonometry was seen by many educators as a critical part of school mathematics. Making trigonometry the capstone of elementary mathematics was a major change during the unified reform period between 1893 and 1923. Several presidents of the American Mathematical Society addressed the critical role of trigonometry in blending arithmetic and geometry (Moore, 1903; Hedrick, 1917). This higher status for trigonometry was shared by mathematicians such as W. E. Story, who had a German educational background. In line with his educational upbringing, Story (1903) recommended the unification of mathematics and gave a capstone role to trigonometry in the school curriculum.

During this period, Rugg also gave trigonometry a central place in school mathematics in his scientific construction of the content of that mathematics. He joined the University of Chicago and collaborated with Clark on the development of a ninth-grade textbook with a 5-year experimental study (Rugg & Clark, 1918). This was experimental work conducted under a project commissioned by the Illinois Committee on Standardization of Ninth-Grade Mathematics. The work was about “investigations of the socially worthwhile material of algebra, geometry, and arithmetic courses, of measured experiments in learning and of results of utilizing standardized tests" (Rugg, 1975, p. 297). Rugg and Clark's (1919) ninth-grade textbook came with an ambitious claim that it exemplified a scientific method for the construction of school textbooks. A progressively improved method was suggested for reconstructing a school subject. It provided a detailed description of a scientific study of the effectiveness and the organization of ninth-grade mathematics. Rugg's efficiency-based reconstruction effort was clearly reflective of his engineering orientation.

Geometry textbooks began to integrate trigonometry as an integral part during early years of the unified mathematics period. Trigonometric ratios were introduced in the context of surveying in observational geometry. Students at work while studying the trigonometry portions of observational geometry are shown in Figure 10 and 11 (Campbell, 1899, pp. 233–234).

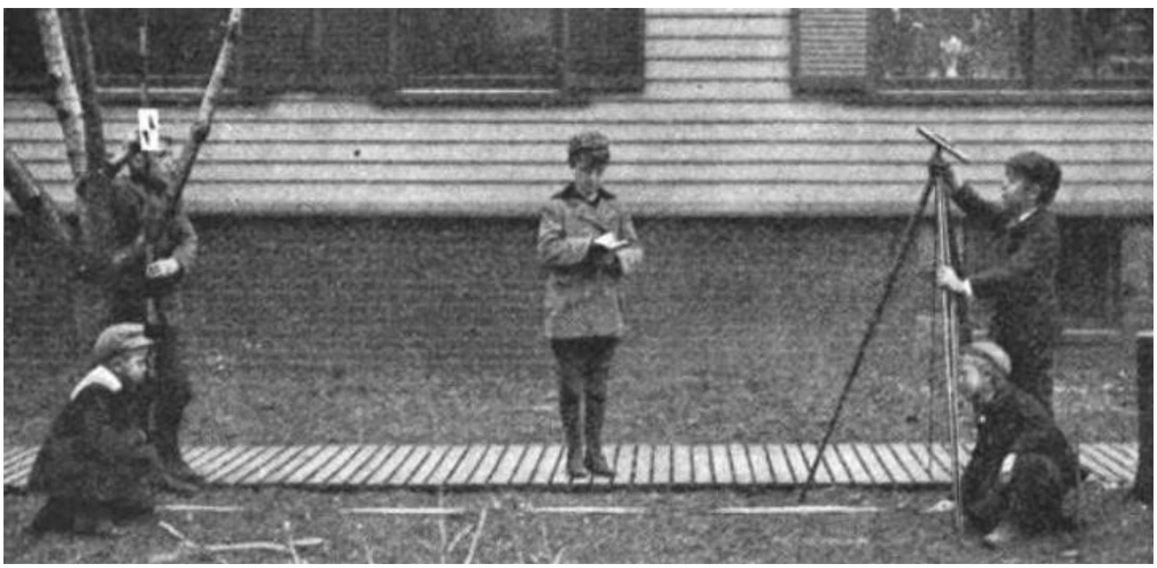


Figure 10. Students measuring the height of a tree during the unified mathematics period in Boston
Note: Reprinted from *Observational Geometry*, by W. T. Campbell, 1899, p. 231, public domain.

Figure 10 shows students at work taking measurements and calculating the height of a tree; they were practicing right-triangle trigonometry in the context of an experimental geometry class.

As one can observe in this picture, the lesson was conducted outside the classroom, and the students worked as surveyors taking measurements and gathering data. The students assumed complementary roles and made use of available instruments and tools in this practice of trigonometry. Angles were measured with a transit. It was used for determining the angle of sight for indirect measurements of distant objects by right-triangle trigonometry. In this context, the use of negative numbers and directed lines was not needed. Observation is essential in Figure 11.

Figure 11. Shadows as a triangle trigonometry context in geometry textbook.
Note: Reprinted from *Observational Geometry*, by W. T. Campbell, 1899, p. 233 (public domain).

Campbell's work was not unique. It followed the tradition of inventive geometry after Pestalozzi, Herbart, and Spencer. On the subject of trigonometry, this tradition placed an emphasis on student experiments that involved taking measurements and practicing numerical trigonometry.

Moritz (1908) and Young (1906). At the turn of the 20th century, Moritz (1908, 1913) observed that trigonometry had received little attention from the experts compared with all the subjects that made up the high school curriculum. As a professor of mathematics, he not only wrote trigonometry textbooks for colleges and technical schools but also expressed clearly the contemporary changes in trigonometry as a school subject amid the reform. He added that the lack of recorded discussion cannot be attributed to a lack of importance of the subject:

> Few subjects combine in so high a degree, as does plane trigonometry, both practical and disciplinary elements. If algebra and geometry are the pillars on which all exact science rests, trigonometry is the lintel which bridges these pillars and supports the superstructure. At this point algebra and geometry are merged into one in the investigation of the circular functions. This, together with the constant reference to arithmetic in its abundance of numerical processes, makes trigonometry the ideal mathematical discipline for the senior year in high school, a most fitting capstone for the mathematical curriculum in the secondary schools. (Moritz, 1908, p. 303)

Moritz (1908) made a critique of the lack of discussion on the treatment of trigonometry in schools. He observed that "trigonometry today is probably the least organized of the mathematical disciplines from arithmetic to and through the infinitesimal calculus" (p. 393). Moritz observed that there was no recognized order of precedence in the treatment of different topics of trigonometry by various authors. He attributed this lack of order to an absence of unity in the conception of the subject. He examined 20 textbooks published during the decade and generated a list of 10 different definitions, which he noted was a nonexhaustive list. He listed alternative ways of defining trigonometry as follows: solution and calculation of triangles, numerical methods for finding angles and triangles, calculations concerning lines and angles, a branch of geometry where algebraic methods are used to treat relations of lines and angles, properties and measurement of angles and triangles, a branch of algebra treating periodic functions, and the mathematical doctrine of angles, sides, and areas of plane and spherical triangles. For Moritz, trigonometry embraces goniometry for its study of singly periodic functions. If trigonometry is seen as a method of solving triangles, then there is no logical reason to include, understand, or use radian measure, inverse function, extension of trigonometric functions for general values of angles, or the study of graphs. Within this limited conception, it is hard to extend the treatment of the subject to develop an understanding and appreciation of Euler's or De Moivre's identities.

Young (1906) attributed narrow and different conceptions of trigonometry to a lack of a central idea in defining trigonometry compared with the other subjects of school mathematics. He argued that a clear-cut central idea can give a subject its individuality and associated arithmetic with the number concept as the central idea, algebra with the generalized number concept and the equation, and geometry with the concept of space. Young questioned trigonometry in terms of its distinctive characteristic. He asked; "Is it the study of trigonometric ratios?" (p. 288). If it is, that would make it the subject of geometry. "Is it the solution of triangles, as name implies?"(p. 288). Then, that would still make the subject as fitting continuation as a part of geometry. "Is it the manipulation of formulas and solution of equations involving trigonometric ratios?" (p. 288). That would make the subject algebraic in essence. He developed a list of eight separable parts for trigonometry: ratios, use of logarithms, solution of triangles, ratio relations and identities, extended definitions of angle and associated ratios, extended identities, trigonometric equations, and the addition theorem. He did not explain how he came up with this list. He suggested that these parts are different and have no close connection. He further suggested that their order can be varied and that they can be more directly related to parts of algebra and geometry than to each other. Young's stance on trigonometry was clearly to distribute and merge its contents with algebra and geometry instead of treating it separately.

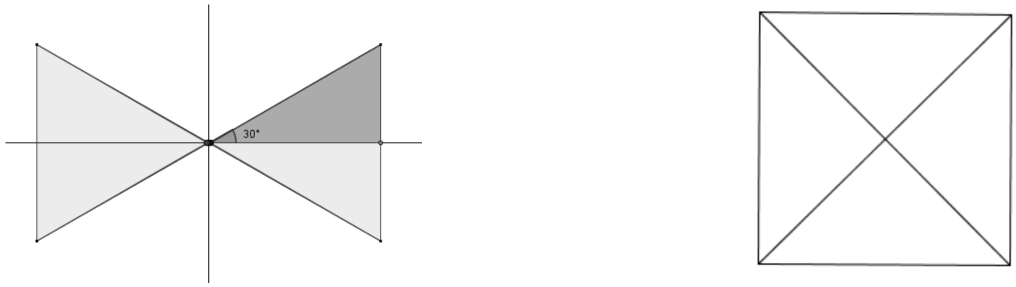
For Young (1906), "the fundamental proposition of trigonometry is the similarity of right triangles having one acute angle of each of the same" (p. 290). In other words, the same acute angle specifies similar right triangles. "The consensus of modern opinion" (p. 290), said Young, was that the trigonometric function should be defined as ratios exclusively. An algebraic phase of the work in trigonometry is the recognition of the fundamental relations between the ratios.

Moritz (1908), in contrast, objected to Young's view of the lack of a central idea in trigonometry and suggested the angle as the central idea. The usual treatment confined the subject to the solution of triangles and put an end to the science. Defining trigonometry as the science of angular magnitudes broadened the subject and allowed one to cover the theory of elliptic functions and of Fourier series. Hyperbolic functions were linked through this conception by applying it to complex numbers, and reconceiving an imaginary angle. Moritz's main recommendation was that the trigonometry should be divorced from its etymology as the study of measures of triangles and be reconceptualized as the science of angular magnitudes.

Karpinski (1939) and O'Toole (1939). Karpinski (1939) addressed the problem of presentation of trigonometry and suggested that the number of definitions and formulas should be reduced and limited to the following:

$$\sin\theta = \frac{y}{r};\ \cos\theta = \frac{x}{r};\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x};\ \sin^2\theta + \cos^2\theta = 1$$

To introduce trigonometric function values for 30, 45, and 60 degrees, he recommended using two equilateral triangles and a square as given in Figure 12. The *x*-axis split the equilateral triangle on the right into two equal right triangles that can be used to read the values for 30, –30, or 330 degrees. Rotating the equilateral triangle about the origin 180 degrees gives the equilateral triangle on the left and similarly yields the values for 150, –150, or 210 degrees. Karpinski recommended a square with its diagonal as an easier connection than the isosceles right triangle to arrive at the function values for 45 degrees and related angles.



(a) (b)

Figure 12. Equilateral triangle and diagonals of the square for special trigonometric values

Responding to Karpinski’s (1939) remarks on the problem of presenting trigonometry, O’Toole (1939) made further recommendations. His primary recommendation was on setting up definitions of trigonometric functions. Instead of placing too much emphasis on certain right triangles, O’Toole suggested that the teacher should put major emphasis on a point *P* on the terminal side of the angle, its coordinates *x* and *y*, and the distance *r* from the origin to *P* instead of on the right triangle *OMP* as shown in Figure 13. O’Toole further noted that no emphasis was given to the triangle *ONP*.

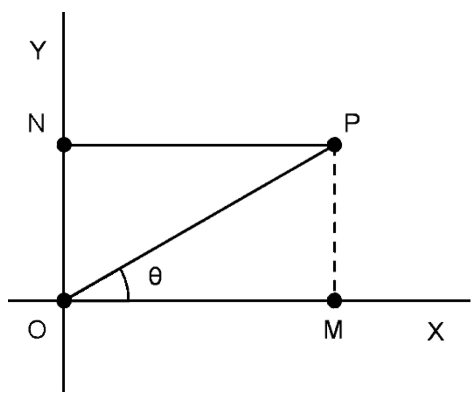


Figure 13. Angle and method of coordinates.

He further raised some issues with this emphasis on the right triangle *OMP*. This emphasis generates problems when the angle is extended further to the second quadrant. The idea of trigonometric functions in coordinates demands that attention must be shifted from the straight line segment *MP* to the number *y* using rectangular coordinates of the point P. O'Toole (1939) further suggested that the line *MP* be dotted to avoid reliance on the triangle. He remarked that "the general formulas for the trigonometric functions are algebraic rather than geometric" (p. 374). This approach to trigonometry requires similar triangles and the Pythagorean Theorem. Taking several positions of the point *P* on the terminal side of the angle, similar right triangles are used to show that the ratios *y/r*, *x/r*, and *y/x* are independent of the position of *P* on the terminal side.

Trigonometric formulas are algebraic, however, only in the sense that they are a ratio of coordinates and a distance. However, the exact nature of point P is not clear. A given angle is assumed to have a terminal line where the point P is selected. Therefore, the transcendental nature of trigonometric function is shifted now to the definability of terminal line of an angle. The nature of point P because of the approximate nature of the terminal line as a mathematical object must be noted.

Variation Across Editions of Breslich Series as Focal

The editions of the focal reform textbook series, that of Breslich, were analyzed to yield the first dimension of the analysis of right-triangle trigonometry during the unified mathematics period.

Breslich (1934) reflected on the image and status of trigonometry in the preface of the fourth edition of his unified mathematics textbook series. This edition was in use between 1927 and 1934. For Breslich, trigonometry was an excellent example of unified mathematics. He objected to the way trigonometry was frequently presented as giving the appearance of a difficult course. He favored a more intuitive approach. More explicitly, he objected to starting with abstract definitions of the trigonometric functions without developing meanings for them through concrete experiences. He observed that this practice invited difficulties for the students and led to their "manipulating symbols without understanding the principles" (Breslich, 1940, p. iii).

Starting with a laboratory method, Breslich (1940) thought that the meanings of the trigonometric ratios and of other fundamental concepts could be developed through the activities of drawing, observing, and measuring. He noted that the method was "superior to the customary procedure of beginning trigonometry with abstract definitions. According to the method used in this text, understanding comes first and the definitions are last in the procedure" (p. iii).

The ratio idea helped the learner to connect the angle and the similarity of right triangles. The focus was not on varying the angle but rather on conceptualizing angle so as to characterize varying sizes of similar right triangles. I applied the method of free variation of this founding intuition across the tasks to see how this idea becomes manifest.

The first editions of the Breslich series did not include trigonometry. They were *First Year Mathematics* by Myers et al. (1909) and, in the second year, supplemental titles such as *Geometric Exercises for Algebraic Solution* (Myers et al., 1907). Although the second edition of *First Year Mathematics* by Myers et al. (1909) did not have trigonometry, Breslich's (1910) *Second Year Mathematics* integrated the subject.

Breslich's Second Year Mathematics (Myers, Wickes, Breslich, & Dresden, 1910). The major emphasis in the second year textbook was on geometry. Here, the introduction of trigonometric ratios was preceded by chapters on proportion, similar triangles, and right triangles, Pythagoras theorem, radicals, and similarity of right triangles. The trigonometry chapter included the trigonometric ratios in right triangles; radicals and exact values of special 30, 45, and 60 degrees; relations of trigonometric functions; and graphical solutions of quadratic equations.

In developing triangle trigonometry, Myers et al. (1910) started with an acute angle represented by AOD as shown in Figure 14. Right triangles OMP and OM_1P_1 are constructed. Then, the ratio of MP to OP is called the sine of angle O. The idea emphasized by the exercises is that the sine of O and the other trigonometric ratios do not depend on the size of the right triangle that contains the angle O (p. 103). They are constant numbers for a given angle. As the ratios or quotients of certain lengths, they are positive numbers.

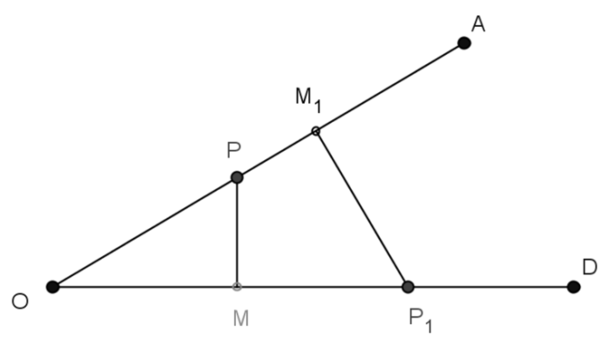


Figure 14. Similar right triangles.
Note: As manifested in *Second Year Mathematics* in Myers et al. (1910), p. 103 (public domain).

A study of circle was presented without a connection to circle trigonometry. The presented trigonometry was not a circle trigonometry, but a triangle trigonometry. Therefore, the synthetic study of connections between circle and its chords was not manifested in this textbook. Toward discovering and measuring π, regular polygons in and about a circle were discussed. The intuitive connection was made that by increasing the number of sides of a polygon “perimeters of the inscribed and circumscribed polygons *approach the circumference as a limit"* (Breslich, 1910, p. 216, italics in original). The idea of limit was used only informally.

Breslich's (1915) First Year Mathematics. The 1915 edition included right-triangle trigonometry. Moore, Myers, and Judd wrote a preface for this textbook. Their review of this edition observed that the textbook placed greater emphasis on experimental and intuitive geometry. They also indicated that the book presented a pedagogical rather than a logical organization of general and fundamental mathematical notions. Rigor in the pure mathematical sense was not attempted in definitions, axioms, or principles.

Breslich, as a pedagogical rule, determined that the practical use of every new feature should be clearly set forth before the abstract theory was developed. Ideas of graphing and variation were included as with all the other reform textbooks. Although the chief emphasis was on algebra, the new edition placed more emphasis on geometry than the earlier edition did. The chapter on trigonometric ratios followed ideas of measuring lines in space and similar figures, which were introduced in the previous chapter. Right-triangle trigonometry treated the following topics: drawing to scale, ratio, similar figures, and problems in similar figures. In the preface of this edition, Breslich (1915) stated that this arrangement serves to help students see and appreciate "the superiority of algebraic methods over geometric methods and of trigonometric methods over both" (p. xvii). In the earlier editions, this position was not taken. This statement clarified the newfound place of trigonometry in first-year mathematics. It was a Herbartian leap taken from experimental geometry to algebra. Trigonometry was introduced with the similarity of right triangles.

The same position was reiterated in the preface of *Second Year Mathematics* (Breslich, 1916)*:* Trigonometry's replacing geometric and algebraic methods was a defining idea:

> Trigonometric methods here often replace algebraic and geometric methods, giving the student the opportunity to see some of the advantages of trigonometry over algebra and geometry. In addition to the foregoing aims the following are included: (a) the application of three trigonometric functions (sine, cosine, and tangent) to the solution of the right triangle and to a number of practical problems; (6) the development of some of the fundamental relations between these important functions. (p. xi)

Right-triangle trigonometry provides a static depiction of trigonometric functions based on the similarity of right triangles. In right-triangle trigonometry, the focus is not on variations of the angle, but on the sides. Angle can be abstracted as a similarity class. This ratio sense is limited because it does not ascribe negative values without introducing the method of coordinates or directed line segments. Also the unit circle is used as a conceptual anchor to trace the variations of an angle and to reframe the right-triangle sense on a fixed coordinate plane.

The Breslich series provided an overall assessment of the processes of measuring and finding ratios in developing the ideas of trigonometry. The authors found that this approach was highly successful in developing the ratio idea, which is an important aspect of trigonometry. However, they added that students following this method did not have a clear understanding of the function idea, which is equally or even more important. This understanding "is attained by the use of graphical methods. The graph of a trigonometric function is not an end in itself, but [it is] a means of securing understanding of the real character of the trigonometric ratios" (Breslich, 1940, pp. iii–iv).

The Breslich series did not have a standard placement of triangle trigonometry in the curriculum. In one edition, trigonometry was placed in the first course and reviewed later. In another edition, it was integrated into the second course, which was traditionally a geometry course. It became a critical issue to determine for each year of high school which trigonometry to develop and to what extent.

Horizontal Variation Comparing Focal with Other Series Including Wentworth

In this section, I contrast the right-triangle trigonometry in other contemporary textbooks with that of the Breslich series. I mainly focus on the Wentworth and Wentworth-Smith series among other textbooks during the reform period. I analyzed four books by Wentworth and three other reform textbooks from the period: Long and Brenke's *Plane Geometry*, Betz and Harrison's *Plane Geometry,* and Palmer and Leigh's *Plane and Spherical Trigonometry*.

Wentworth's triangle trigonometry. I compared how Wentworth introduced the idea of right-triangle trigonometry comparing the various editions of his textbook (Wentworth, 1887, 1897, 1903; Wentworth-Smith, 1915). I have two observations. One is that authors provided the same definitions for about 35 years with only slight variations. The definition of trigonometric functions and associated figures were invariably the same. The other observation is that angular measure was introduced before discussing trigonometric functions for right triangles. The angular measure section was not included in the first edition of the book (Wentworth, 1887). Wentworth (1897) wrote a revised edition and introduced angular measure before presenting trigonometric functions for right triangles.

Wentworth (1895, 1903) defined *radian* as an angle at the center of a circle subtended by an arc equal in length to the radius. He states on the first page that "it is proved in geometry" that the number of times the length of the radius is contained in the length of the circumference is the same for all circles, and it is 2π. Angle is measured by circular arc length and connected to π. "Introduce circular measure first" was the first idea manifested in Wentworth's treatment of triangle trigonometry.

Wentworth and Smith's (1914) Plane Trigonometry. In the Wentworth-Smith textbook of 1914, trigonometric functions for general angle were introduced as ratios involving the coordinates of a point on the rotating line or terminal side of the angle. Line definitions were given at once to visualize trigonometric functions. Fundamental ideas were given in italics, which facilitated the extracting of important mathematical ideas emphasized by the authors.

The textbook made a connection between side ratios of a series of similar right triangles and the angle itself. As Figure 15a shows, the ratios of the sides of each right triangle were the same as the corresponding ratio of sides of each similar triangle. The similarity ratios here were not haphazard but standardized. The ratios were not of corresponding sides between similar triangles, but the comparison was based on internal ratios. When the angle is fixed, these ratios remain unchanged without depending on the choice of points on a line *P* and its perpendiculars on *X*. Six possible ratios within a right triangle were introduced as six trigonometric functions of an acute angle. Therefore the essential idea was to shift the focus from the geometric associations of similar sides to comparing ratios as numbers.

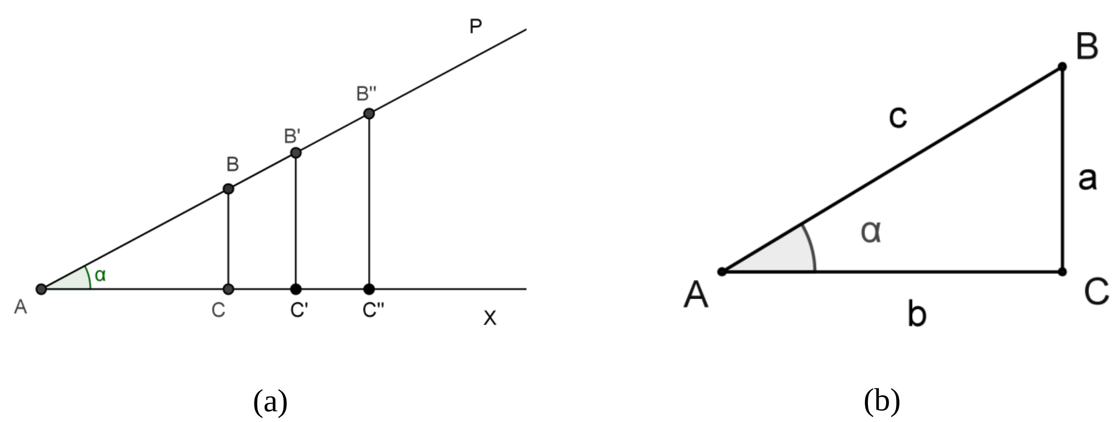


Figure 15. Series of similar right triangles whose corresponding sides are proportional
Note: Figures adapted from G. Wentworth, D. E. Smith, *Plane Trigonometry*, 1915, p. 14; W. Betz and E. W. Webb, *Plane Geometry*, 1912, p. 59. Public Domain.

During this period, there was a practice of integrating right-triangle trigonometry into ninth-grade mathematics, traditionally in a chapter on ratio and proportion. As shown in Figure

15a and Figure 15b, Wentworth did not use a coordinate method and therefore did not account for negative ratios when trigonometric functions were first introduced.

Although Long and Brenke's (1913) *Algebra: First Course* did not integrate trigonometry, Betz's (1912) *Plane Geometry* and Long and Brenke's (1916) *Plane Geometry* introduced trigonometric functions. There is one feature that distinguishes Long and Brenke's (1913, 1916) exposition of ideas from the rest. They attempted to develop a strong dialogue with the student and reader by posing finely graded tasks and questions.

In contrast, Palmer and Leigh's *Plane Trigonometry* (1916, 1934) was another textbook of the period. Palmer was one of the early proponents of graphs and their incorporation in teaching school mathematics topics. This textbook incorporated the method of coordinates in a ratio definition of trigonometric functions. Their exemplary idea for triangle trigonometry was stated as follows: *"To each and every angle there corresponds but one value of each trigonometric ratio"* (p. 14). This definition started with an angle. As shown in Figure 16, first an angle was drawn and three arbitrary points were chosen on its terminal line OP.

The arguments used to generalize trigonometric ratios were still synthetic, by taking the quotients of the distances rather than the ordinates of the points P_1, P_2, and P_3. Internal ratios of a pair of sides are the same as the ratios of the corresponding sides of the similar right triangles. The sine ratios of the given triangles are $\frac{M_1P_1}{OP_1}, \frac{M_2P_2}{OP_2}, \frac{M_3P_3}{OP_3}$. They remain the same as long as the angle does not change. The same is true for the other ratios. Therefore trigonometric functions are still based on right triangles. Although coordinates were used, as shown in Figure 16, it does not account for coordinates when defining ratios. Signs of trigonometric functions were treated separately for each quadrant.

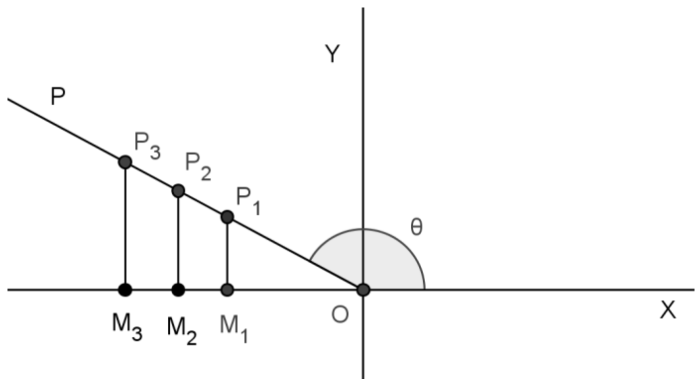


Figure 16. Ratio approach with the method of coordinates.

Note: Adapted from Palmer and Leigh's *Plane Trigonometry*, 1916, p. 14. Public Domain.

The core idea emerging for triangle trigonometry was progressively challenged and reassessed by the included variation of each textbook. The essence of triangle trigonometry remained the same by including the variation of Palmer and Leigh's manifestation of triangle trigonometry. The search for the fundamental meaning of coordinate trigonometry remained unfulfilled with this particular experience of presenting trigonometric functions. A transition from right-triangle trigonometry to coordinates did not occur. The founding idea of coordinate trigonometry as an extension of right-triangle trigonometry also remained unfulfilled. The authors' development of trigonometry did not present a shift in perspective that would bring to the fore the Cartesian points and a trigonometric expression of their positions as an intuition that remained to be challenged. The triangle manifested in this coordinate frame is no longer an Euclidean triangle. But it is a standardized one, with one of its vertices at the origin and another as the *x*-coordinate. The standardization starts by measuring angle from the positive *x*-axis with a counterclockwise sense. This is the exemplary intuition that is subjected to the test of variation experiments with the other cases represent this passage from the right-triangle trigonometry to a trigonometric analysis of points in rectangular coordinates.

The coordinate aspect of this extension of right-triangle trigonometry allows one to incorporate negative values. Since to every value of the angle, there is a corresponding value for each ratio, the ratios are called trigonometric ratios. Rather than distances, the coordinate values are used in developing trigonometric ratios. How was this idea manifested in the practice of right-triangle trigonometry?

This ratio approach on a coordinate system allowed the development of trigonometric ratios even when the triangle was undefined, such as at angles 0, 90, or 180. Having defined trigonometric functions for general angles in a coordinate system, the authors next apply them to right triangles. Trigonometric ratios were not first introduced from right triangles. Without putting restrictions on domain and sense, the trigonometric function was developed first. Then it was applied to right triangles:

> Attacking the triangle, trigonometry, in many ways, is a more powerful tool than geometry, which makes little use of the angles while trigonometry makes use of the angles, as well as the sides, of a triangle.(Palmer & Leigh, 1916, p. 37)

Although right-triangle trigonometry was introduced in coordinates, it was not practiced in its generalized form with negative value and generalized angles. The authors went back to positive valued and a restricted domain definition in the applications they provided for right-triangle trigonometry.

Integrated Mathematics with a Special Application to Geometry (Swenson, 1934). Trigonometric functions were integrated by Swenson (1934) into a geometry course. Before introducing trigonometric functions, he discussed the conditions that determine the ratio of two sides of a triangle: “If two triangles agree in two angles, they agree also in the ratio of any two sides similarly situated in two triangles” (p. 364). Therefore, in a triangle with sides *a, b, c*, if the triangles agree in two angles, the value *a/b* is the same in both triangles and is constant. Swenson went on to state the ratio of any two sides of a triangle is a function of any two angles, or

$$\frac{a}{c} = f(A, B); \;\; \frac{a}{b} = f(A, B)$$

For a right triangle,

$$f(\frac{a}{c}) = A; \; f(\frac{a}{b}) = A$$

Swenson continued by presenting ratios as a function of two other ratios and each element of a triangle as a function of the remaining elements. Neither sine nor cosine nor tangent was introduced as a trigonometric ratio; each was called a function *f* without specifying it further. This part was only a brief introduction to circle trigonometry. Right-triangle trigonometry was not emphasized. Only its connection with the function idea was put forward. Swenson presented a rare manifestation of a functional approach in geometry. The way he presented trigonometry aligned with his use of the function idea as a unifying theme for school mathematics. He adopted this functional idea as a unifying perspective from the National Committee on Mathematics Requirements (1923) proposal to modify the “static geometry of Euclid” (Swenson, 1934, p. i).

In Swenson’s geometry course, right-triangle trigonometry was applied in formally expressing the mathematical idea of measuring a circle by regular polygons with n-sides. Circles and related lines were discussed towards measuring circumference and area of a circle by polygons. They were regular polygons with *n* sides inscribed in and circumscribed about a circle. The inscribed polygon utilized the mathematical object of sine, and the other used the tangent. This practice aligned with the historical origins of trigonometric functions in the study of a circle and its chords. This problem of measuring the circle generated a discussion of the tangent and sine function value for small degrees. The notion of limit was intuitively used as "approaching" (Swenson, 1934, p. 444), and π was then measured by tabulating its approximations for large *n* values.

The integration of trigonometry into plane geometry was one of the trends emerged during the unified mathematics period. It was often a synthetic triangle trigonometry, treating trigonometric ratios without using the method of coordinates. Comparing the textbook trends in plane geometry before 1930, Freeman (1932) observed that more trigonometric exercises were incorporated into contemporary textbooks than in the earlier ones. She further noted:

> Attempts to effect some correlation with other subjects is seen in the increased numbers and percentages of algebraic exercises, especially in recent textbooks; in the introduction of trigonometric exercises and topics in many current and recent books. (p. 293)

During this period, the practice of triangle trigonometry as a part of second-year algebra incorporated more of the idea of coordinates generalizing the trigonometric functions for any angle using ratios. This practice can be called triangle trigonometry in coordinates.

New Math Period Practices of Triangle Trigonometry

Contemporary Reform Context of Right-Triangle Trigonometry

The reform curriculum for the new math period was also called modern mathematics, but the reform proposals and activities were diverse (Kilpatrick, 1997, 2008). The modernization of school mathematics by making it more similar and closer to collegiate mathematics was one of the main motivations for the new math reform (Begle, 1971). Begle explained this motivation in his retrospective account of the School Mathematics Study Group (SMSG) in a conference held at Stanford University in 1969.

Recommendations of the Commission on Mathematics of the CEEB (1959) were realized by high school programs such as that of the SMSG. Taking a structuralist approach toward mathematics, SMSG placed emphasis on the formal development of mathematics and the discovery of mathematical structures. The new math programs included more demanding content that was intended to be more appropriate for college-intending students. Later efforts attempted to make the content more accessible to students and more teachable for teachers because the programs required a high level of mathematical competence on the part of the teacher. Critics claimed that the curricula were weak in psychological content and treated students as miniature mathematicians and scientists. The new math movement's stance toward modernization was similar to that of the Bourbaki movement.

The Bourbaki movement after 1939 followed a belief that the mathematical universe is unique. Their attempt was to reorganize all mathematical knowledge hierarchically on a singular foundation from simple to complex and general to specific. The Bourbaki group was a dynamic collective of mathematicians in France who worked under the pseudonym Nicholas Bourbaki. They were propounding Van der Waerden's (1931) approach to mathematics as used in his textbook *Modern Algebra*. They began the task of expanding Van der Waerden's process of reconstruction from algebra to the entire field of mathematics after the 1940s. Bourbakians assumed a project of major unification and modernization of mathematics by reworking its elementary foundations and developing the knowledge tree of mathematics from the ground up based on axiomatic foundations and a structural approach. The Bourbakians' unitary structural view of mathematics had defined the spirit of its times in mathematics circles. The new math reform had a major involvement of mathematicians in developing their mathematics programs. The Bourbakian influence and structures of discipline approach were new ways of looking at mathematics and organizing the curriculum as prevalent during the new math era in the United States (Begle, 1971; Corry, 1998).

CEEB (1959). The College Entrance Examination Board (1959) Commission on Mathematics placed an emphasis on trigonometry and suggested a modernization of the subject:

> Trigonometry is the part of school mathematics related most clearly to technical applications. In the past, these had mainly to do with surveying and navigation. Therefore, much attention was paid to the method of solving plane and spherical triangles by logarithmic computation. But that era has passed. Special tables, computing machines, and other equipment have made the logarithmic solution of triangles an almost obsolete tool. Instead, there are more substantial and challenging applications of trigonometry now evident in many areas of science and technology—especially, in statics and dynamics, electromagnetic waves, and vibration problems of all sorts. (p. 28)

The commission proposed a program sequence that contained four trigonometric units. The first was the basic numerical trigonometry of right triangles. The Commission recommended that it be included in Grade 9 as an optional unit. The second was the trigonometry of coordinates, vectors, and complex numbers. The third was the cosine and sine laws, addition theorems, and identities. The last unit was circular measure, circular functions, and their wave nature. The second and third units were suggested for the end of Grade 11 for about a third of a semester. The first unit might be anticipated in Grade 8 or combined with the second unit in Grade 11. The circular trigonometry unit was recommended to be treated toward the end of Grade 12.

Under right-triangle trigonometry, the CEEB (1959) commission listed five subtopics:

> 1. Ratio and proportion; similar triangles;
> 2. Angles; right triangle; trigonometric ratios of acute angles (sine, cosine, tangent);
> 3. Tables of ratios; Pythagorean rule; review of square root;
> 4. Solution of right triangles; computation with numbers that are approximations; and
> 5. Problems. (p. 37)

Right-triangle trigonometry in the *Cambridge proposal* (Educational Services Inc., 1963). The case of triangle trigonometry represents an interesting feature of the Cambridge proposal, *Goals for School Mathematics* (ESI, 1963). I examined how triangle trigonometry was manifested in the reform documents. Inspecting all the content items in the recommended curriculum topics from Grades K to 12, I found that triangle trigonometry was not included. The earliest introduction of a trigonometry-related subject in the recommendations was at Grade 9. It was the treatment of complex numbers and rotation in the plane, as well as trigonometry.

National Advisory Committee on Mathematical Education (NACOME, 1975). At the end of the new math period, I checked the status of right-triangle-related recommendations that NACOME (1975) provided. However, no curriculum content recommendations were made, with the following justification:

> Curriculum content subject to the flux of accelerating change in all areas of our society, cannot be viewed a fixed set of goals or ideas; it must be allowed to emerge, ever changing, responsive to the human and technological lessons of the past , concerns of the present, and hopes for the future. With this is mind, no definite curriculum can ever be recommended. (p. 138)

New mathematics period discussions of right-triangle trigonometry.

Rosenberg (1958). Rosenberg remarked that trigonometry put an excessive emphasis on the solution of triangles and neglected the graphic aspects of trigonometry. This contributed to the irrelevance to secondary school mathematics of modern mathematics and its applications to modern science. He suggested transforming trigonometry from static to dynamic by not giving too much emphasis on the routine solution of triangles and instead using logic. First, he suggested defining such terms as the *tangent function, inverse sine function,* and *radian,* and then applying deductive reasoning to these definitions and to previously accepted assumptions and proved theorems of geometry and algebra to establish the major properties of the trigonometric functions.

Bowie (1967). Bowie (1967) discussed the "modern" definition of trigonometric functions using a coordinate system with an angle in standard position. On using coordinates, he added:

> This relates the subject to analytic geometry. Special definitions which agree with these are made for an acute angle as an angle of a right triangle. It is important to show that the position of the end-point of the radius vector on the terminal side of the angle does not affect the values of the functions. This involves similar triangles and establishes the dependence of trigonometry on plane Euclidean geometry. (p. 80)

Bowie's remarks corroborated Hart's idea of using *r* as a vector and not a scalar. This special attention of two figures to the vector idea in regards to *r* suggests that the idea of *r* as a vector required further examination. As a variation of the coordinate method, I followed its other manifestations to get the essence of the idea. A candidate for a founding intuition was suggested here as the idea of using *r* as a vector. Free variation of this intuition generates a mathematical construct that demands taking the ratio of two vectors, *x* and *r*, as a number that corresponds the value of sine. Vector division, however, is a problematic construct that is beyond the intent of the method proposed at this level. It would imply a conception of quarternions and requires a definition of 1/ *r* for a vector *r* that would demand a notion of conjugate as in the case of complex numbers.

Another idea Bowie suggested was that the practice of similarity of triangles in developing trigonometry generates a dependence of trigonometry on Euclidean geometry. The idea of similarity of triangles is not extensible to other frames such as spherical geometry where trigonometry can be defined. Although similarity is not an essential idea, it is a frame-dependent construct that is used to build triangle trigonometry.

Variation Across Editions of a Focal Series

In setting up the school mathematics program for the new math, the ideas of J. Bruner and Z. P. Dienes played important parts with their structuralism in developing mathematics instruction. This idea referred to teaching mathematical structures by using concrete manipulative materials to help developing deeper understandings of mathematical ideas than traditional approaches. The structure of the disciplines approach was at the forefront in the curriculum reform movement during 1950 to 1975 with their continuing commercial availability.

School Mathematics Study Group (SMSG). Following the CEEB's recommendations, the School Mathematics Study Group (SMSG) series of school mathematics textbooks were *First Course in Algebra, Geometry, Intermediate Mathematics, Elementary Functions,* and *Introduction to Matrix Algebra.* In their revised editions some there were minor changes in the name of the courses. *Geometry* was revised and called *Geometry with Coordinates.* SMSG's (1961a, 1961b) *First Course of Algebra* did not have trigonometry.

I observed that there is a lack of emphasis on triangle trigonometry in SMSG. *Geometry* (SMSG, 1961c) included right-triangle trigonometry only in an appendix as an elementary study. The authors first introduced one theorem that placed the focus on the angle in determining the similarity of right triangles. The theorem stated: "If an acute angle of one right triangle is congruent to an acute angle of another right triangle, then two triangles are similar" (p. 355).

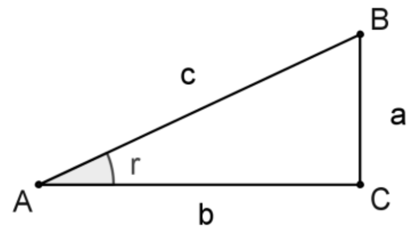


Figure 17. A right triangle to introduce trigonometric ratio as manifested by the School Mathematics Study Group.

SMSG's *Geometry* makes a further point that, as in Figure 17, "the ratio *a/c* does not depend on a particular triangle we use, but only the measure *r* of the acute angle. The value of this ratio is called the sine of r^o." (SMSG, 1961c, p. A-58). The authors provide an explanation for specifying degree measure and suggest that a different measure of angle, radian measure, is used commonly in more advanced aspects of trigonometry. Here it appears that degree measure is associated with right-triangle trigonometry. Other trigonometric ratios defined are cos r^o and tan r^o. The revised version, *Analytic Geometry*, repeats the same treatment of right-triangle trigonometry and adds it as an appendix (SMSG 1965c, p. 963).

Intermediate Mathematics (SMSG, 1961d) does not contain right-triangle trigonometry. It focuses on circle trigonometry introducing the idea of circular functions. *Elementary Functions* (SMSG, 1961e) has only an appendix on trigonometric identities. Triangle trigonometry was part of the traditional mathematics content added to appendices of the SMSG series.

Horizontal Variation Comparing Focal With Other Series

Right Triangle Trigonometries by Hart and Hart (1942) and Hart (1961). A trigonometry textbook selected from the new math era was popular before and during that period. This series was *Plane Trigonometry, Solid Geometry and Spherical Trigonometry* (Hart & Hart, 1942) *and Modern Plane Trigonometry* (Hart, 1961). Both editions start defining trigonometric functions of angles by first describing what angle is accounting for and how it is measured. A static concept of angle by a ready-made figure is changed to a dynamic concept of rotation to trace and measure the direction and magnitude of an angle. The notion of sense is introduced, and standardizations are made to operationalize the dynamic conception of angle as rotation such as terminal side, initial side, and vertex. Sense is clockwise or counterclockwise. This idea allowed trigonometric functions to be defined in an extended domain. In defining trigonometric functions, "reference right triangle" as a mathematical construct began to be used in a coordinate system. For a given angle, it is first placed in a standard position on a coordinate system, with its initial line coinciding with the x-axis. Then a point $P(x, y)$ is selected on the terminal line of the angle. This idea assumes that a terminal line can be determined for a given angle. However, for an angle such as $\pi/3$, there is not a real terminal line on which to select a point P. Hart starts with the angle as given, then selects a free point depending on the angle, and bases on that the rest of the ratio definitions for trigonometric functions. The vertices of the reference right triangle is determined by the origin O, P, and A as the projection of P on the x-axis. OP and AP are now directed segments. All the arguments are based on the determinability of the point P. This argument further indicates the problems with starting with an angle. Ratios of directed segments to the radius vector are said to be used to define the trigonometric ratios. OP is assumed to be a "radius vector" taken as positive for all angles (Hart, 1961, p. 32). The definition of *radius vector* further implies that to define the sine function y as a directed line segment is divided to $\vec{r}$. The vector division is beyond the intent of introduction of trigonometric functions. Otherwise, it would imply the notion of quaternion, which extends the idea of complex numbers as an algebra of pairs to an algebra of fours. So *r* being a vector does not hold as a founding idea and is an unfulfilled intuition. The fraction $y/\vec{r}$ is a representation with an empty content. An assumption of angle as an approximate object allows the terminal line and any choice of *P* on that line to appear indefinitely. This way, *P* can further be projected onto the *x*-axis and connected to the origin to form the reference right triangle.

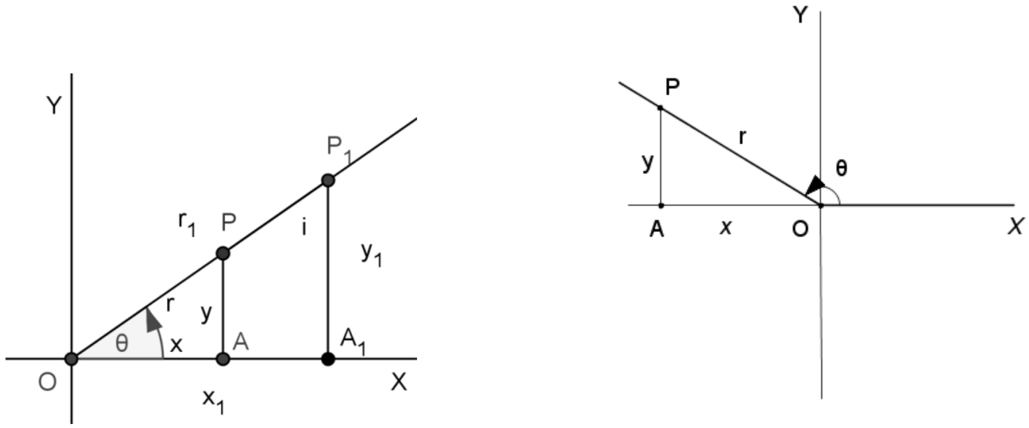


Figure 18. A practice of reference right triangle starting with an angle.
Note: Adapted from Hart's *Modern Trigonometry*, 1961, pp. 32, 33.

The fundamental idea that Hart (1961) establishes is that the values of the ratios depend only on the position of the terminal side of the angle and not on the particular point *P* used to find *x*, y and *r*, as seen in Figure 18. He later develops a section where "construction of an angle if the value of one of its functions is known" (p. 40), which indicates there is a change of perspective towards integrating a view of trigonometry as the science of angular magnitudes (Moritz, 1908). Hart's use of coordinate methods to introduce trigonometric functions paralleled O'Toole's (1939) with its shift of attention from the measure of an angle to a coordinate point.

Another important feature of Hart's (1961) approach was the inclusion of two more notes in the appendices on developing trigonometric functions. One described a novel method, and the other was an old method modernized. The first method used a winding process that the author described. It suggested a distinction to be made between trigonometric functions of angles and circular functions of real numbers. This process was also introduced by new math reform curricula such as SMSG s. The other method used the line values of trigonometric functions on a unit circle placed at the center of a coordinate system. The modernized trigonometric line functions were conceptualized as directed line segments in and about a unit circle. Neither method was present in Hart and Hart (1942). Hart (1961) did not incorporate either perspective from the appendices to his treatment of trigonometry in the main text.

***Elementary Mathematics*, Comprehensive School Mathematics Project (*CSMP*).** The CSMP was an exemplary program developed with an international collaboration. The project developed the EM, *Elementary Mathematics (EM),* which is a seventh- through twelfth-grade mathematics program. Although the program did not associate itself with the new math movement, their ideas were not totally different. The authors also believed that mathematics should be taught as a unified whole (Kaufman & Steiner, 1969). Professor Exner from Syracuse University served as senior author for 10 years and played a major role in the development of the trigonometry portions of the program. The program mainly aimed at introducing good students to contemporary mathematics as early as possible. It was designed for well-motivated secondary school students to familiarize them with important mathematical problems, ideas, theories, and techniques that were contemporary to the practice of mathematicians. Only two books from the series had trigonometry content. Like the other mathematical content developed in the program, the first exposition of trigonometry was more intuitive than the second, more formal, one.

The CSMP's first exposition of trigonometry was in *Elements of Mathematics, Book O* (CSMP, 1973). When triangle measurement is chosen as the primary object of study in trigonometry, sine as trigonometric ratio reifies the idea of similarity as an extension and condensation, as in Figure 19. Such an idea of sine represents an invariant association between the angle of a right triangle and the class of similar triangles on the same plane. Attention is directed towards the internal ratio of sides of right triangles. Similarity is captured and encapsulated in the idea of sine. An angle in this sense is not a priori but an abstraction from similarity ratios. The CSMP's conceptual orientation for trigonometric ratios associates the angle and the internal ratios of pair of sides of right triangles. The trigonometric ratios are defined for a $\alpha^{\circ}-$similarity class.

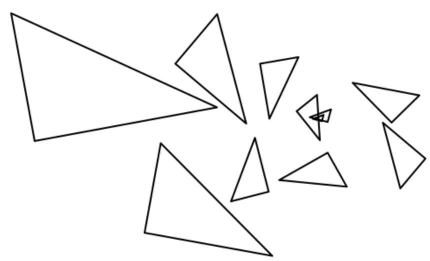


Figure 19. CSMP's trigonometric ratio as an extension and condensation of the idea of similarity.

Standards-Based Mathematics Period Practices of Triangle Trigonometry

The mathematics in context approach is a common perspective in standards-based reform textbooks. The tasks in standards based curricula are typically given in context. My first observation is that more than the textbooks of other reform period textbooks, standards era textbooks tend to offer more chances for students to experiment with mathematical ideas, and to construct, order, and symbolize the mathematics by using those ideas.

Contemporary Reform Context on Triangle Trigonometry

CEEB (1983, 1985). Among the seven members of the Mathematical Sciences Advisory Committee for the CEEB, there were five mathematicians, one mathematics educator, and one mathematics teacher. *Academic Preparation for College* (CEEB, 1983) and *Academic Preparation in Mathematics* (CEEB, 1985) did not make specific recommendations about right-triangle trigonometry. The books commented that general mathematics was a course created during the 1920s for those who were deemed unprepared for the study of algebra. It was designed as a course for Grades 7 to 12 that would integrate algebra, geometry, trigonometry, and elementary statistics, stressing the function concept as the unifying idea. The course was widely accepted for Grades 7 and 8, and at higher grades, it was used as an alternative to algebra. Therefore, the report made recommendations for a 3-year sequence of high school mathematics that they called Mathematical Topics 1-2-3. This sequence included basic topics in computing, statistics, algebra, geometry and functions. Lack of inclusion of right-triangle trigonometry may have been due to the assumption that this topic belonged to Grade 7 or 8.

Both CEEB reports included mathematical proficiencies expected from students. Algebra proficiencies included skill in solving trigonometric, exponential, and logarithmic equations, and skill in operations with complex numbers for all college-intending students. For those who aimed to study engineering and sciences, proficiency expected for college entrants included familiarity with arithmetic and geometric series and with proofs by mathematical induction. On the subject of geometry, the expectation was for students to gain familiarity with vectors and with the use of polar coordinates. About functions, students were expected to develop knowledge of various types of functions, including polynomial, exponential, logarithmic, and circular functions. Students were also expected to graph such functions and to use them in the solution of problems. This competency suggests that a graphical solution was promoted for problem solving as opposed to an algebraic solution. The ability to gather and interpret data and to represent them graphically became a proficiency of statistics.

Standards period discussions on triangle trigonometry. Hirsch, Weinhold, and Nichols (1991) examined the place and teaching of trigonometry in a standards-based curriculum. They considered that the study of similarity can serve as a natural introduction to trigonometric ratios in right triangles. Their arguments further corroborates to the fundamental idea for right-triangle trigonometry as the idea of using ratios to express the similarity of right triangles.

Variation Across Editions of the Core-Plus Mathematics Series as Focal

The Core-Plus Mathematics Project (CPMP) ws a 4-year program emphasizing mathematical modeling, featuring the full use of graphing calculators, building on the theme of mathematics as sense-making, and involving investigations of real-life contexts that lead to reinventing important mathematics (Hirsch, 1995).

Right-triangle trigonometry of CPMP (1997). Core-Plus (1997) presented right-triangle trigonometry in Unit 6 of second-year mathematics. Right-triangle trigonometry was given with strong utilitarian themes using concrete objects. The unit studied "geometric form and its function" from everyday objects. It represented a unique study of functional mathematics, questioning how a shape allows a tool to function, or in other words "form follows function." The geometric forms studied were quadrilaterals, triangles, and circles. The triangles and quadrilaterals were designed and manufactured objects with flexible and rigid sides, and with hinged or pivoting vertices. Mechanical context terms, such as *crank*, *frame*, and *coupler*, were used to study how these tools function and make objects work, such as wheels, with complete rotation, or windshield wipers, with partial rotation. Trigonometric methods were introduced by mathematical modeling while designing and analyzing mechanisms whose function is based on either triangles or circles and their properties.

Among the objectives for the unit were

> to investigate characteristics of quadrilaterals and the mechanical uses of quadrilateral linkages; to investigate ways that triangles are used to maintain rigidity in structures with one side of variable length; to explore properties and applications of the sine, cosine, and tangent ratios for the length of sides of right triangles; to explore characteristics of circles and relate circles to rotating objects, angular velocity, and the graphs of trigonometric functions. (p. T367)

Fundamental ideas were embedded in context. The first lesson used the mathematical model of flexible quadrilaterals, which involved activities demanding the description and determination of similarity of plane shapes. The idea was that quadrilaterals became rigid when triangulated. A truck crane was used as model to represent a right triangle with flexible and adjustable side length. Trigonometric ratios were introduced in the context of a triangle with variable side. Students discovered the similarity conditions for two triangles.

Angles and sides relationships were investigated using physical models of triangles with a variable length side. Contextual variation was presented using the same idea of a flexible triangle. Those were reclining lawn chairs, two different kinds of automobile jacks, drawing tables with a variable tilted top, a cold frame, and ironing boards. In studying these situations, students were asked to gather, plot, and interpret data to examine the effect of the variations of the flexible side. In the extension, students were exposed to a hoisting derrick with a long boom raised by a cable that could be shortened using a small winch at different rates. With this new context, students were expected to discover the pattern of changes in the angle of elevation, and the height of the tip of the book from the ground as the boom moved to a vertical position. This model is a contextual representation of the rotation of a point on circle in the first quadrant. An intuitive development of the concept of angle and a cosine pattern preceded their treatment in the sections that followed. The title of the next investigation was “What’s the Angle?” The teacher explanation provided the essence of the mathematical idea treated in this lesson:

> In this investigation, students begin exploring the conditions that guarantee that two triangles are similar. Now they explore special cases of similar right triangles, thus developing an understanding of trigonometric ratios. (p. T395)

As a part of this investigation, students were asked to draw right triangles for a given angle of 35^o and calculate three triangle side ratios. Those ratios are *a/b*, *a/c*, and *b/c* for a right triangle with its length of its sides *a, b,* and *c.* Students were to make conjectures as to the invariant pattern across a variety of triangles they drew for 35^o. They were asked to interpret “what seems to cause the differences in the results” (p. 397), and to deduce that an “important variable is the measure of one acute angle of the right triangle” (p. T397).

Horizontal Variation Comparing Focal with Other Series

COMAP’s *(1999) Mathematics Modeling Our World Course 3.* COMAP (1999) introduced triangle trigonometry in the first unit,en titled “Geometry of Art,” where students were introduced to basic ideas of projective geometry. Key ideas introduced in the unit were “size, and similarity, parallel lines, corresponding angles, projections, proportions, and trigonometric ratios” (p. 2).

The foreshortening phenomenon is used as a founding intuition to build the idea of triangle trigonometry. “It is manifested when lines or surface perpendicular to the line of sight appear increasingly shorter as they are rotated away from an observer” (COMAP, 1999, p. 86). Foreshortening is used to make sense of the shortened appearance of an object such as a pencil when it is tilted away from the plane of view. The mathematical idea of representing accurately the length of such objects was used to develop trigonometry. “You will solve this problem by building on what you already know about similarity. The new mathematics dealing with angles is called trigonometry" (p. 87).

Tilting was further associated with an angle to indicate its degree. Students were asked to find the apparent length for special angles of tilt: 0, 30, 45, 60, and six more angles of their choosing. The metaphor called for the measurement of tilt from the vertical as the object rotated at special angles. After determining apparent lengths, students were to tabulate and graph their results. The trigonometric ratio idea was explored in an activity. Six ratios were defined as ratios of the sides of a right triangle. Students were asked to use a ruler and protractor or a geometric drawing utility to draw several right triangles. They were asked to vary the sizes of the triangles, and to include some triangles that were similar to each other. They were asked to tabulate angle measure and side length and to examine sets of similar triangles. A scientific calculator or a graphing calculator was to be used to check the results. Although students were learning about the trigonometric ratios in triangles, they were to use calculators to plot the graphs for sine and cosine and to observe similarities and the odd behavior of tangent near 90^o. Students also were to graph $\sin^2 x$ and $\cos^2 x$, and observe how these graphs are related. They were asked further to "predict what the graph of $\sin^2 x + \cos^2 x$ will look like" (p. 108).

The triangle trigonometry practice of MMOW is blended with a function approach by the exploratory use of technological tools such as graphing calculators. All triangles are synthetic, and the coordinate system is not blended into discussion of trigonometric ratios. Indirect measurement was practiced by using the ideas of similarity of triangles and of scale.

COMAP's Precalculus Modeling Our World (2000, 2002). The COMAP precalculus book was part of a series that utilized an activity and problem-based approach, presenting concepts and skills with applications and a modeling point of view. Triangle trigonometry was introduced after trigonometric functions, meaning that the general definitions of trigonometric functions were given first. The treatment started with an application that emphasized the indirect measurement of the height of inaccessible objects from their shadows as an intuitive origin of the ideas of right-triangle trigonometry. An activity was used to adapt the proportions of similar right triangles to determine the indirect measurement of heights. The activity presented a well-posed task with scaffolding to help learners deal with the parts of a problem situation presented at the beginning of the unit. Each scaffold was phrased in question form to provide learners with a guided discovery of ideas and student modeling of the mathematical problem situation that requires right-triangle trigonometry.

I focused on how the authors used the notion of angle in defining trigonometric ratios for right triangles. The first step was to place a unit circle on a rectangular coordinate system so that the vertex of one acute angle was at the origin, and one side of the angle lay on the positive *x*-axis (as shown in Figure 20). First the authors defined, with an angle measured in radians, sine as the vertical displacement of the arc *DC*, cosine as the horizontal displacement of the arc *AC*, and their ratio as the tangent. Although it is not indicated, displacement in this coordinate setting can take negative values. So the book starts with a general definition of directed length *DC*, not its length but more referring to its projection onto the *y*-axis, which is implied.

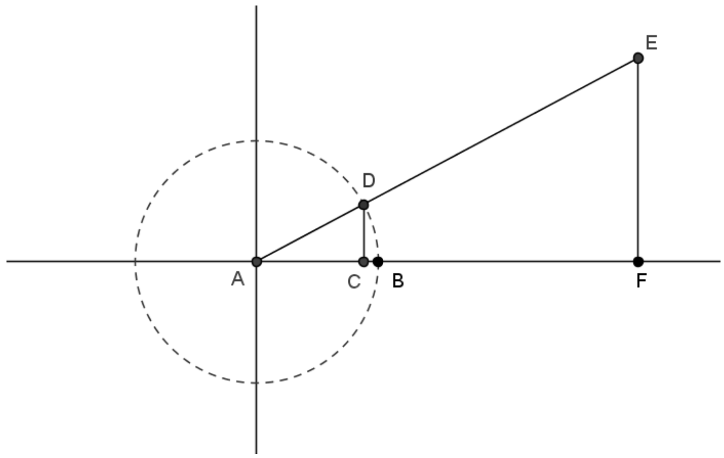


Figure 20. Right-triangle trigonometry defined after trigonometric functions.

The COMAP (2002) authors further suggest that “however, triangles *ACD* and AFE are similar, and it is convenient to describe angle *A*’s trigonometric ratios in terms of the triangle *AFE*’s sides” (p. 307). So, they suggest that the definitions shift from the triangle *ACD* determined by radian measure to any other similar right triangle. And then the authors make a statement indicating the intuitive idea of using angle as a reference: “Since not all triangles are labeled with letters it is convenient to describe the trigonometric ratios in terms of sides relative to the chosen angle” (p. 307). So the practice of right-triangle trigonometry is based only on side ratios relative to the angle without a circle construct imposed. The authors continue with the degree measure without using radians. Other than its definition, the use of right-triangle trigonometry does not reflect anything about trigonometric functions. A right-triangle trigonometry is placed in precalculus without a clear integration into the grade level, the other content, and most importantly to the earlier treatment of trigonometric functions in a general sense. I checked the earlier edition of *Precalculus* (COMAP, 2000), and it contains neither a triangle trigonometry chapter nor a trigonometric functions chapter. Both chapters were included in the 2002 edition as new additions to the book with almost the same content as that used in the third year book. The COMAP 2002 *Precalculus* edition has chapters in common with the third-year book in the series. This commonality raised questions about the nature of the integration of the right trigonometry content to the grade level and to the other content. The way the textbook was revised with the inclusion of trigonometry suggests that the revision allows adopters more content options for different levels of students.

***Algebra and Trigonometry Structure and Method* (Brown, Dolciani, Sorgenfrey, and Kane, 2000).** In Brown et al. (2000), trigonometric functions of an acute angle, θ, are defined by the ratio method (De Morgan, 1849; O'Toole, 1939). A point P on a rectangular coordinate system is chosen on the terminal side of the angle θ placed in its standard position. The distance of the point P to the origin is defined as r. This definition is Euclidean-distance-based and assumes that a point can be chosen on the terminal line. The core idea emphasized was that the trigonometric ratios for sine, cosine and tangent depend only on θ and not on the choice of the point P. The choice of point P depends on the assumption that terminal line is known for the angle. It can only be approximated in general, however, because π is transcendenta, and the measure cannot be exact. This nature of trigonometric functions demands their experimental and numerical treatments. In generalizing this idea for any angle, one finds that the sign of θ is assumed to depend on the signs of x and y. For any angle, a reference angle is used as a basis for the definitions of trigonometric functions, and a reference angle always maps any angle to an acute angle.

Triangle Trigonometry of the *Interactive Mathematics Program* (IMP, 1999). IMP (1999) introduces the extended definitions of the trigonometric sine and cosine functions in the fourth book of their high school mathematics series. Although the trigonometric content was handled mainly in the first chapter, the treatment was continued throughout the series. For example, the radian concept was introduced in the third chapter. The ratio definition was used to obtain the extended definitions. Points and a rectangular coordinate system were assumed to be primary objects to frame the conceptualization of trigonometric functions. Trigonometric functions were to be considered as tools to express the coordinates of the points on a coordinate plane by projection. The sine and cosine functions were defined for any point on an analytical plane by using the projections on the *y*- or *x*-axis and their ratio to the distance to the origin. The function is not the projected line segment but the position of the projected line. This definition of trigonometric function employs a Cartesian point for an angle and essentially maps that point (x, y) to its sine and cosine values. The mapping is defined from $\mathbf{R}^2$ to $\mathbf{R}$.

IMP (1999) defines the cosine function for all angles by an equation. For a given angle, this cosine definition requires first finding an associated point. That point is chosen along the terminal line for this angle. The founding idea is that the choice of point along the terminal line does not affect the sine or cosine ratios using the similarity of triangles. The issue of the sign of the trigonometric functions is resolved by following the Cartesian coordinate values for the point. This definition gives the same values for acute angles as the right-triangle definition. The conception of trigonometric functions is not dynamic, and attention is not focused on the variation of the argument for the trigonometric function. The fundamental mathematical idea of right-triangle trigonometry for IMP is based on the similarity of right triangles and the equivalence of the internal ratios of similar right triangles in a coordinate setting.

Triangle Trigonometry Across Reforms

The common theme across reforms was that right-triangle trigonometry is reframed within coordinates connecting algebraic tools in studying geometric objects of trigonometry. The fundamental idea in school practice since Rheticus has been to pair right triangles with an angle. For each acute angle, there is a class of similar right triangles. The horizon of the states of affairs of mathematical objects in a right-triangle trigonometry expands from similarity ratios to a similarity class represented by angles. This expansion means that a right triangle specifies an angle. And vice versa, an angle is specified by a right triangle. De Morgan used this exemplary idea by rotating triangles to show the rotation of angles. This angle-associated right triangle became a reference right triangle when used with the analytical sense of coordinates. The essential idea is that any angle is associated with an acute angle. Then, any angle is associated with a right triangle.

The practices of trigonometry across reforms suggest that right-triangle trigonometry is a frame for trigonometry in which a right triangle and the measures of its sides are primary constructs. In this frame, attention is averted from measuring angles and carefully placed on the ratios of pairs of sides of right triangles. Trigonometric ratios are introduced as six possible ratios among three measures of sides. Those ratios are associated with two acute angles for a right triangle. Studying right triangles with flexible sides suggests an extension of the static view of the ratio approach in practice. The limitation of this conception is that it is length based, and thus, does not have its primary focus on the angle and its measure. Right-triangle trigonometry assumes the triangle as an a priori construct. The definability of triangles challenges the threshold trigonometric function values for this frame of 0 and 90 degrees. This triangle assumption generates the fringe of the states of affairs of trigonometric functions with this sense.

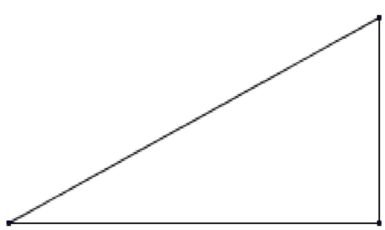
Trigonometric ratios in this sense are limited to positive values. This frame with integration of coordinate or vector methods is extended to account for negative trigonometric ratios with the incorporation of relative coordinates with respect to the axes and origin or directed distances on an oriented plane.

This framing inherently represents a static perspective because of its lack of focus on variation. The fundamental shift required for a higher level conceptualization is to make a transition from variation in the sides to variations in the angle. The act of variations in the sides of the right triangle shapes the horizon for the state of affairs of trigonometric objects activated in the first frame. The similarity idea is used to form trigonometric objects as ratios of side lengths. The ratio idea is associated with the trigonometric objects. On the fringe of the state-of-affairs of trigonometric objects of the first frame, a shift needs to occur from variation in the sides to abstracting angle first as a static and then as a dynamic object. This angle object condenses the idea of similarity of right triangles. Shifting from the idea of similarity of right triangles to the idea of angle requires a special coordination that makes it possible to use angle as a dynamic object.

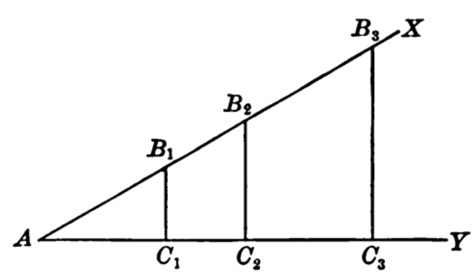
Trigonometric sine as a mathematical object in this perspective is not associated with angle but with the notion of constant side length ratios of similar right triangles. This perspective perceives angle as a static object to be achieved as an abstraction of similarity. It does not perceive the angle as a dynamic object. A flexible triangle approach can be seen as an extension of this approach. It generates a limited dynamic perspective by exploring the variations in the sides and its effect on the ratios. This approach can provide a link for building circular functions, but the focus of attention is on variation not on the angle as argument, but on the sides.

Triangle trigonometry is extended by the method of coordinates. The extension is made to incorporate negative values for the triangular side lengths by adding the notion of sense with directed lengths. It keeps the ratio idea. This conceptualization began to emerge during the 15th century and became a part of the discourse in textbooks in the early 1800s. This conceptualization is based on a point and its position in a coordinate system. The idea of right triangle is reconceptualized by point, origin, and the projection on axis. This frame presumes a choice of origin and orientation for setting up a Cartesian coordinate system. Trigonometric functions are used to denote the positions in the fixed coordinate system.

Sediments of cultural traditions appear in the form of conventions and assumptions regarding the practices of right-triangle trigonometry. When the trigonometric ratios are defined, the orientation of the right triangle is one of the conventional practices. A right triangle is conventionally oriented as in Figure 21a. Even similar right triangles are defined in this manner, as shown in Figure 21b. The Core-Plus series, on the other hand, used a left-oriented right triangle in defining trigonometric functions as the ratios of the sides of a right triangle (CPMP, 1997, p. 398).



(a)



(b)

Figure 21. Sediment of practices with the assumed orientation of a right triangle when ratios defined. As manifested by Betz and Webb's (1912) *Plane Geometry* (p. 259). Not-in-copyright.

Another convention has been to use the sine-as-a height or a vertical projection. This is a conceptual metaphor that can work only under standardization. After being introduced, the standardization becomes implicit and is hidden within this convention. The alternative standardization breaks the mold and sets a case for making the familiar unfamiliar. It helps to provide a deeper sense of connections between sine and cosine and the quadratic form for a circle. It represents a case of the complacency of conventions in the everyday practice of trigonometry in schools. A parallel development is also possible for the pair (sec φ, tan φ) to represent hyperbola parametrically as another quadratic form. This order of parametric representation traces a hyperbola counterclockwise starting at (1, 0) for varying φ.

Overall, the horizon of state of affairs of mathematical objects in a right-triangle trigonometry expands from similarity ratios to a similarity class represented by angles. Synthetic geometry suffices for right trigonometric objects to be manifested. Angle as a construct associated with the right-triangle frame manifests itself as acute angles. States of affairs of an angle object have the horizon on null and right angles. Sine and cosine are objectified with a limited sense. Negative quantities do not become manifest for trigonometric functions unless triangle trigonometry is reframed in a rectangular coordinate frame. Direction of segments and angle measures are not required for defining and applying right-triangle trigonometry. The founding idea of approaching angle as a class for similar right triangles is challenged by the lack of similarity in other frames. Angles and distances on triangles on a sphere would give spherical triangle trigonometry without the similarity condition. The similarity triangles in a hyperbolic or spherical geometry are congruent. And the presence of the similarity construct of Euclidean geometry makes it hard to disassociate and determine the necessity of this frame-based construct in defining a right-triangle trigonometry.

Therefore, the status of the mathematical construct of similarity as an essential idea for right-triangle trigonometry must be reassessed and further investigated to see whether it is a cultural sediment that became a cultural, if not essential, part of the practice of triangle trigonometry.

CHAPTER 5

CIRCLE TRIGONOMETRY DURING REFORMS

In this chapter, I present a phenomenological description of circle trigonometry and its treatments. There are three sections in this chapter. In each, I address a reform period's treatment of circle trigonometry: unified, new math, and standards. In three further subsections, I describe circle trigonometry for the period, focusing on, respectively, the contemporary reform context, editions of the focal textbook series, and other textbook series of the period.

Unified Mathematics Period Practices of Circle Trigonometry

Contemporary Reform Context on Circle Trigonometry

Reform documents. The Report of the Mathematics Commission of the Committee of Ten does not make a reference to circle trigonometry (NEA, 1894). Neither does the NEA (1899) committee on college entrance requirements. The NCMR's (1923) recommendation was that the work in elementary trigonometry begun in earlier years should be completed by including the logarithmic solution of right and oblique triangles, radian measure, graphs of trigonometric functions, the derivation of the fundamental relations between the functions, and their use in proving identities and in solving easy trigonometric equations. The report emphasized the mathematical idea of the functional relation to unify the courses in high school mathematics. The function idea was suggested to be used to integrate the mathematical courses. With these recommendations, I observed that there was a transition toward making circle trigonometry more of a central object in school mathematics.

The CEEB (1923) board on mathematics said that the instruction should include the treatment of inverse trigonometric functions, with notations, y = arc sin x or $y = \sin^{-1} x$, if $x=\sin y$ where y is measured in radians. The board did not use the term *anti-trigonometric functions*. Graphical solutions were encouraged to be used by means of scale and protractor as checks. Although the board endorsed the use of slide rules in instruction, they did not permit their use in the examinations.

Discussions of the Practice of Circle Trigonometry During Unified Mathematics

The contemporary reform context is depicted from the discussions on circle trigonometry during the unified mathematics reform period. As part of one of three analytical dimensions, it serves to purpose of putting in perspective the treatment of circle trigonometry by leading textbooks of the period.

Directed lines began to be seen at the turn of the century as the essential idea that connected algebra and geometry. Introducing trigonometry by means of directed lines was the novel method used by Ashton and Marsh (1908) in their *Plane and Spherical Trigonometry*. This 1908 reference is to the 1902 edition of their trigonometry. It is a part of a full series of high school mathematics textbooks that was popular just after the turn of the century. The directed line was the first mathematical object discussed and was used to introduce trigonometric functions for any angle. In the preface of this textbook, Ashton and Marsh discussed and contrasted two methods of introducing trigonometry. One was purely geometrical without paying attention to the direction of lines. The results do not hold in general but are specific to a figure. When this method is followed, an algebraic process must be used to generalize the results. If the second method is used, all lines have direction and magnitude, and the results hold for all figures. The authors objected to “the usual mistake of mixing the two methods” (p. v) of introducing trigonometry; they chose the second method. The rotation idea was used to define measure of angles between directed lines. The notion of directed angles was also introduced to account for angles with negative measures. The introduction of the addition and subtraction of directed angles established that it is not necessary to distinguish further between an angle and its measure. Line values of trigonometric functions were again used with a difference to account for direction. The trigonometric functions were defined as the ratios of lines to each other, and they are abstract numbers. The coordinate method was introduced to standardize the treatment of an angle as a mathematical object. When an angle was drawn in a coordinate setting, the notion of directed line also became straightforward in defining the ratios. This method thereby accounted for negative values, which expanded both the domain and the range for trigonometric functions.

Moritz's* (1908) *science of angular magnitudes. As a unifying element, Moritz (1908) suggested defining trigonometry as the science of angular magnitudes. The study of circular functions merges algebra and geometry, and by taking the angular magnitudes for its domain, trigonometry could be enriched by the introduction of several topics. Moritz added:

> The angle as the central idea, not only introduces unity into the science, but it emphasizes that element of the subject which is of greatest importance. Knowledge of the relations between angles and their functions, that is fundamental. Without this knowledge progress in higher mathematics and in a dozen applied sciences is impossible, for most branches of science, knowledge of the solution of oblique triangles is of minor importance. The solution of triangles, far from being the aim of the science, is only one of its many applications, though an important one. (p. 397)

A graphical approach to study trigonometric functions is suggested by Moritz (1908). This approach includes the behaviors of a wave curve corresponding to a change in its amplitude or wave length, the curve of damped vibrations, the angular velocity and periodicity, simple and compound harmonic motion, and the composition of harmonics. Each of these conceptions is of fundamental importance to a number of sciences. The study of graphs provides students with skills needed not only in mathematics but also in all scientific activities in school.

The extension of the domain of trigonometric functions to imaginary and complex numbers comes along with a new conception of addition and multiplication that translates, dilates, and rotates. As the science of angular magnitudes, trigonometry provides rich, stimulating, and practical mathematical connections and content.

Griffin (1915) and Nyberg (1916) discussed the unification of trigonometric functions in freshman mathematics. On the one hand, Griffin cautioned against the unification of a freshman mathematics movement that called for inclusion of more advanced and abstract parts of trigonometric analysis, such as the general treatment of imaginaries and vectors, progressive waves, the solution of cubics, and so forth, or the study of higher plane curves, hyperbolic functions, and so forth., Nyberg, however, favored an integration of trigonometric functions into calculus, which justified their content. This treatment also included polar graphs. Vertical integration was addressed to target their content. Unity was considered critical for a nonspecialist student.

In contrast, Miller (1925) presented some historical connections and observed that the subject of trigonometry is arithmetized, as seen in the change in the definitions of the elementary trigonometric functions. Trigonometric functions were generally regarded as line segments up to the time of Euler (1707-1783). Until then, trigonometric functions were not considered functions of a single variable, such as the measure of angle, but as functions of two variables, composed of this magnitude and the length of the radius of the circle or the hypotenuse of the triangle with respect to which the angle was considered. The arithmetization of the definitions of the elementary trigonometric functions led to their conceptualization as functions of a single variable. The elementary trigonometric functions were modernized in a way to break from their geometrical origins and started to be used as abstract numbers. Here an essential tension appears in the use of trigonometric functions. The mathematical is here to arithmetize—to break away from geometric associations of knowledge objects of trigonometry and study them by algebraic means.

Variation Across Editions of the Breslich Series as Focal

The 1906-1914 Breslich series. The first edition of the series was written under the leadership of Dr. Myers and was only for the first two years of high school. The content of the second year was mainly geometry, and the authors published a supplementary exercise book to further establish a connection between geometry and algebra. Myers described the prevailing procedure in developing the content as “isolation in details, but correlation in major matters” (Breslich, 1917, p. vii). Although the first book features algebra with associated arithmetic and geometry, the second develops geometry with associated algebra and trigonometry. The trigonometry content is based on right triangles and follows ratio, proportion, and similarity of triangles. More specifically, it provides the following sequence of topics: ratio of segments, commensurable and incommensurable segments and magnitudes, proportion, problems of construction, method of analysis, algebraic exercises, similar triangles, similar polygons, similar right triangles and trigonometry, problems of right triangles, relations of trigonometric ratios, and problems and exercises. The next chapter treats the study of the circle by measurement of angles of arcs of the circle.

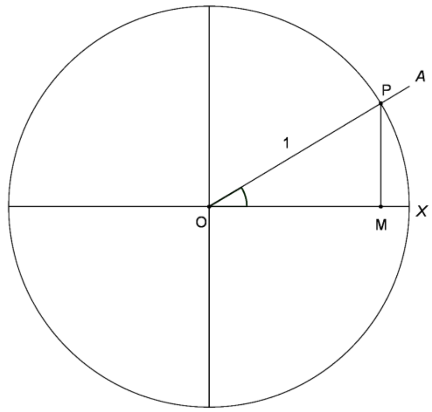


Figure 22. Representation by Breslich of unit circle and trigonometric ratio.

Breslich’s (1917) *Third Year Mathematics for Secondary Schools* introduces the ratio approach to develop general trigonometric functions for any angle. As in Figure 22, first, an angle is defined as the amount of rotation of a line from *OX* to *OA*, where the lines *OX* and *OA* are the initial and terminal sides of the angle. This definition is mixed because it defines angle as a measure but also as a geometric object mechanically defined with initial and terminal sides. Standardizing *OX* as the initial side for the rotation makes it easy to use the method of Cartesian coordinates. Notice that *OX* is not defined as the *x*-axis. The standard of using axes *x* and *y* for Cartesian coordinates is manifested here. It indicates that method of coordinates was not yet established with its conventions. Rather, quadrants are defined. The sign is used in standardizing the direction of rotation in defining an angle. The resulting angle is assumed to be negative when the rotation is clockwise. Any given angle is considered to be in one of four quadrants. The angle *XOA* is drawn for each of four quadrants. *OX* is not defined as an axis but only as an initial line, and the *y*-axis is drawn but not labeled. A line is drawn at *O* perpendicular to *OX* to demonstrate the four quadrants.

The ratio method treats each quadrant by first drawing an angle *XOA* for that quadrant. For example, the terminal line *OA* for the angle *XOA* is called a line but it is drawn as a ray. There is no differentiation between ray and line.

Extending the definition to the second quadrant demonstrates Breslich's use of trigonometric ratios with a limited incorporation of the method of coordinates. As shown in Figure 23, Brelich did not associate the coordinate of point *P* as (-b, a), nor the the point O as the origin or (0, 0). The emerging form of directed line segment refers to *OM* as "-*b*" and *PM* as "*a*." The signs are treated without a reference to an *x*- or *y*-axis. The notion of axes is not manifested. The coordinate method is in pieces of its emergent form but has not yet established its norms as a reference system in school mathematics. Signs are defined by the direction of the extension with respect to the initial line. Vertex refers to the point *O* around which the line *OX* rotates to define the angle and its sign. The sign is positive when the side opposite the vertex extends upward from the initial line *OX*. It is negative when it extends downward. The signs of functions are treated for each quadrant.

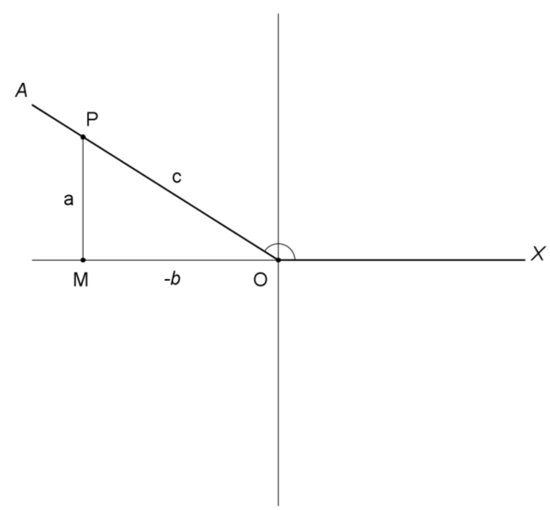


Figure 23. Ratio definitions for trigonometric functions of angles in the second quadrant.

Breslich (1917) states that “since the trigonometric functions are ratios it is possible to represent them graphically by means of line segments” (p. 56). Each trigonometric function can be associated with a trigonometric line segment for a unit circle. These representations are used to study the change in the behavior of trigonometric functions for a varying angle. The trigonometric line values are studied in pairs for sine and cosine, tangent and secant, cotangent and cosecant functions as the angle changes. The degree measure is first used to observe the behavior of trigonometric functions in four quadrants of the circle with the six associated trigonometric line-segments. The radian measure is not limited to four quadrants. It is used to introduce graphical representations of the change in trigonometric function. Radian measure allowed Breslich to incorporate rotations to present graphs in larger domains so as to better reflect the global periodic behavior of trigonometric functions.

The unit circle can be powerful device to show the repeating behavior of trigonometric functions for angles larger than 360^o. The behavior of trigonometric functions can be observed better when a new representation makes it easier to see the periodicity. It allows one to observe point-wise behavior, global behavior, and recurrence. The graphical representation of a trigonometric function addresses questions of the interpretation of their local and global behavior. Some of the questions Breslich (1917) provides concern variations in the trigonometric function as θ varies. Then Breslich introduces informally the notions of domain, range, rate of change, and minimum and maximum values of the trigonometric functions. For example, students are asked to discuss “the changes of the trigonometric functions as the angle changes from 0 to 360, using the straight-line representation” (p. 51). Periodicity and recurrence of trigonometric functions are examined by their graphical representations as the function repeats its values at intervals.

Breslich's (1917) examination of the range of the tangent function leads to the discussion of bounds, limits, and infinity. When θ lies in the second quadrant and decreases approaching 90^o, tan θ increases without bound, always being negative. This behavior is expressed in symbols by means of the statement "tan $90^o = -\infty$" (p. 35). This would exemplify the problem of expressing the limit values during this period. Breslich committed to a treatment of infinity as a number.

Breslich (1929, 1943) were the third-year books of the subsequent major editions of the *Senior Mathematics* and *Purposeful Mathematics* series. Neither included a treatment of trigonometric functions as a part of third-year mathematics. This absence indicated that the treatment of trigonometric functions was taken out in response to lowering the mathematics requirements during this period.

Horizontal Variation Comparing Focal with Other Series

Circle trigonometry in the Wentworth series. My first observation is on the status of circle trigonometry in the Wentworth series. The Wentworth (1903) series manifested circle trigonometry only as a part of *Plane Trigonometry*. The Wentworth series did not provide an integrated treatment of circle trigonometry in other school mathematics courses, that is, Algebra, Geometry, or Advanced Algebra. More specifically, circle trigonometry is not found in his *1898 New School Algebra*, *1891 Higher Algebra*, *1913 School Algebra Book 2*, or *1913 Plane Geometry* (Wentworth, 1891, 1898; Wentworth-Smith, 1913a, 1913b). Some other plane geometry textbooks of the period integrated trigonometry but only in the frame of right triangles (Betz & Webb, 1912; Long & Brenke, 1916).

For example, Long and Brenke's (1916) *Plane Geometry* incorporated only a right-triangle trigonometry after studying the circle and its chords without making a connection to a circle trigonometry. Long and Brenke's treatment of plane geometry focused on ratio, proportion, and similar figures and built trigonometry on them. It followed a rigorous approach, with theorems selected from Euclid's Geometry Book IV. The order of presentation is that the book first gives the definitions, states the theorems, and then works on corollaries. The authors define trigonometric functions as the ratio of line segments, which leads to a treatment of the commensurable and incommensurable ratios of line segments.

Betz and Webb's (1912) *Plane Geometry* makes the study of the circle a major theme of the whole treatment but only introduces triangle trigonometry. The authors claimed that their geometry course was built up "not only in a topical but also in a psychological order" (p. vi). The book's last chapter is about the mensuration of the circle, which was set up earlier by the introduction of inscribed regular polygons. It was built toward the discovery of the transcendental number π without a formal introduction of limits. Trigonometric connections between the circle and the chord or half chord are explicitly treated in this textbook.

The circle trigonometry is historically built from the idea of connecting the circular arc *EA* and the half-chord *EC*, as in Figure 24. For the history of this ancient connection, see van Brummelen (2009). *O* is the center. The trigonometric tangent line *TA* and its connection to *EA* suggest the circumscribing option for approximating circular arcs.

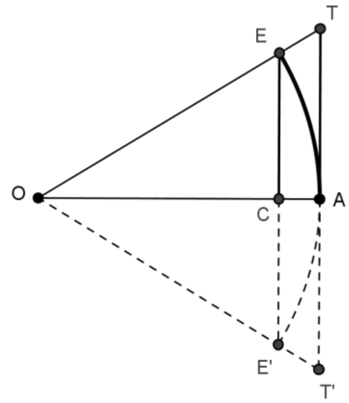


Figure 24. Founding idea as connecting arcs and chords.

Tracing this idea, I located the following exercise as a part of the use of squared paper for implementing the method of coordinates for an intuitive study of curves, their graphs, and their relationships.

> The graphic method may be used to illustrate geometric relations even when the equation showing the relation is not given. For example, draw a circle of radius 2 in. By means of the protractor mark off on this circle from some point A arcs of 10°, 20°, 30°, 40°, etc., to 180°. Draw from A the chords of these arcs. Using the number of degrees in the arcs as abscissas, and the measured lengths of corresponding chords as ordinates, plot the points thus determined. Is the graph a straight or a curved line? Continue the graph by increasing the number of the arcs at intervals of 10° up to 360°. What change is there in the corresponding chords? (Betz & Webb, 1912, p.180)

This intuitive founding idea is left unfulfilled in both series from the unified mathematics reform period.

Wentworth's *Plane Trigonometry* textbooks introduce trigonometric functions for generalized angles in the goniometry chapter. The explicit use of the term *goniometry* also indicates a shift of focus from triangles to angles and the circle. The Wentworth (1884, 1896, 1903) trigonometries treated generalization of trigonometric functions for any angle without incorporating the method of coordinates. Wentworth-Smith (1914) also used trigonometric lines.

When it comes to generalizing trigonometric functions for any angle, Wentworth uses trigonometric line functions instead of ratio. Wentworth (1884) provides the following justification:

> Functions of an angle being ratios are numbers; but we may represent them by lines if we first choose a unit of length and then construct right triangles, such that denominators of the ratios shall be equal to this unit. (p. 7)

Wentworth's presentation of trigonometric functions for general angle is shown in Figure 25. It demonstrates that the line values of the trigonometric functions become equal numerically to the ratio values for varying angles for a circle with unit radius. *AT* is tangent; *PM* is sine.

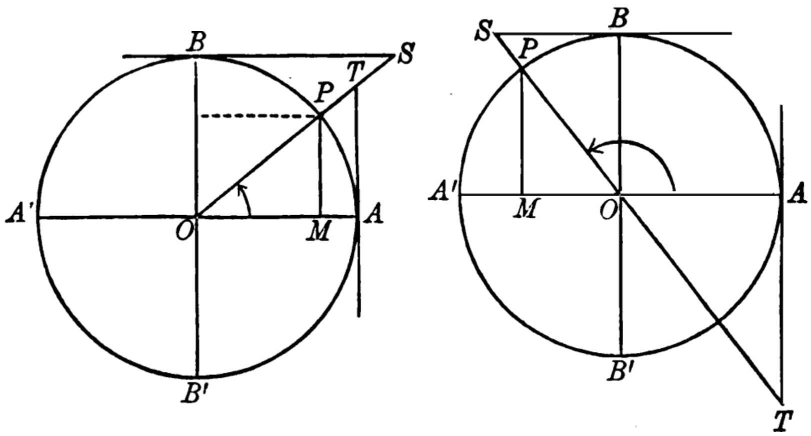


Figure 25. Wentworth's trigonometric line values for general angles.
Note: Trigonometric lines are given for an angle for each quadrant of the circle without a reference to coordinates (Wentworth, 1884, p. 2, p. 32). Not in copyright.

As shown in Figure 26, Wentworth (1903, p. 10) presented the use of trigonometric line values to demonstrate the changes in trigonometric functions for a variable angle. As shown in the figure, no reference was made to coordinates. What was manifested in Wentworth's trigonometry was a geometrical perspective towards variation without graphs. Goniometry follows the chapter on right-triangle trigonometry and its application to the solution of right triangles. Wentworth introduced positive and negative quantities.

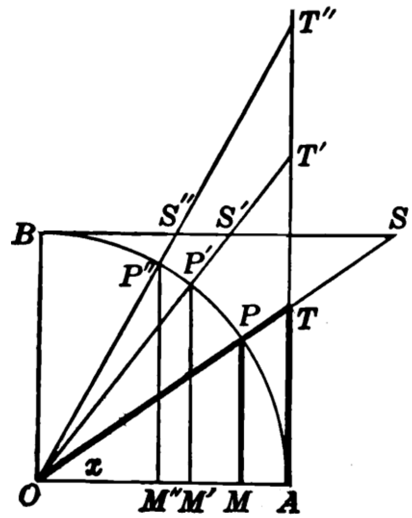


Figure 26. Variation in trigonometric line values for a variable angle.
Note: From Wentworth (1882, 1903, 1895). Not in copyright.

Circle trigonometry was included by the use of trigonometric lines. Trigonometric functions for a general angle were introduced as ratios involving the coordinates of a point on the rotating line or terminal side of the angle. Line representations were displayed to visualize the changes of trigonometric functions as the angle changes.

Wentworth-Smith's (1914) Plane Trigonometry and Tables. The first edition of Wentworth's *Plane Trigonometry* appeared in 1882. Wentworth (1903) started by introducing the idea of angular measure and radian as a unit measure. Wentworth's (1895) edition presented two systems based on circle measurement. Both measures are directly dependent upon the circle. One system is given as the sexagesimal system, where the circumference of a circle is divided into 360 equal parts. The other is a circular system, where "an arc is laid off equal in length to the radius" (p. 1). "The angle at the center subtended by this arc is taken as the unit angle and is called a *radian*" (p. 1). Wentworth associates radian with the central angle. He calls the chapter "Trigonometric Functions of Acute Angles" and then introduced the idea of general angle measure. I found this "introducing the angular measure first" to be an intuition to develop the idea of right-triangle trigonometry and applied it to the process of variation across the editions and its further parts to see how the idea of angle measure manifested itself. Then I determined whether the intuition was fulfilled or not as an essential aspect of right-triangle trigonometry in Wentworth's treatment. The manifestation of the radian idea is traced in the later sections of the book. The author does not use this idea in developing and applying the ratio definitions of trigonometric functions. He only employs degree measure while developing right-triangle trigonometry; he presented its applications of solving right triangles and indirect measurement in chapters 1 and 2. In the third chapter, he introduced trigonometric functions for any angle. He called this study *goniometry* and presented relationships among trigonometric functions for general angles. His use of goniometry also did not employ radian measure. I located only one manifestation of radian measure in the treatment of "anti-trigonometric" functions, or inverse trigonometric functions, where the author mixed the practice of degrees and π, for example, $\tan^{1}1 = 45^{o} \pm 2n\pi$ (Wentworth, 1903, p. 58). The exercises all indicated that the author did not use π as a radian but to refer to 180^{o}. In Wentworth-Smith (1914), the authors rectified this situation and removed the mixed uses. The number of terms used to refer to the same mathematical object indicates that consensus was not reached on the use of this mathematical object. In one usage, arc sin *y* is introduced to read "the arc whose sine is y," or "the angle whose sine is y." The arc is the reference used as the argument for the sine function. The other usage is $\sin^{-1}y$. They are called inverse trigonometric functions or "the antisine of *y*" (p. 156).

***Chauvenet* (1850-1908).** I briefly present here another popular textbook to provide an idea about the practice of right-triangle trigonometry on the wake of unified mathematics reform.

Chauvenet's (1908) *Treatise in Plane and Spherical Trigonometry* was in its 10th edition after its first in 1850. He provided line definitions for trigonometric functions (see Figure 27).

> Trigonometric sine of an arc or of the angle at the center measured by that arc, is the perpendicular let fall from one extremity of the arc upon the diameter passing through the other extremity. (p. 16)

Chauvenet used a circle with a radius *r*. Sine of an arc was considered to be sine of the central angle, which is half chord over radius. It depended on the ratio of the arc to the whole circumference and not on the absolute value of the arc. However, with a choice of unit radius, all the trigonometric functions could then be represented by the straight lines in and about the circle. As seen in the figure, the full chord was still displayed. This conception is length based, and the reference to negative quantities is complex.

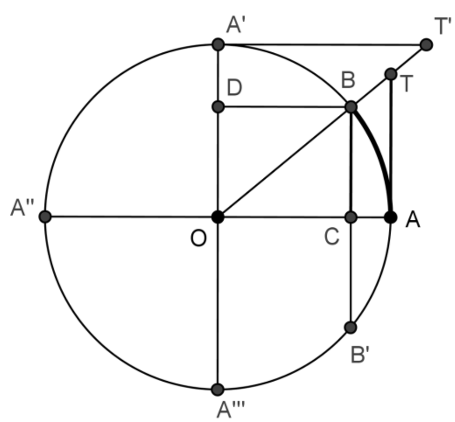


Figure 27. Trigonometric line functions of an arc.
Note: As manifested in Chauvenet's trigonometry between 1850 and 1908. Not in copyright.

In Chauvenet's practice, degrees were used in extending trigonometric functions for angular magnitude in general. The case of 0˚ and 90˚ was explored to discuss the corresponding values of trigonometric functions that lead to the use and discussion of infinite and infinitesimal quantities.

***Unit circle and graphical method* (Wilczynski & Slaught, 1914; Ettlinger, 1920).** The rotation definition of angle is introduced in books by Wilczynski and Slaught (1914) and Ettlinger (1920). In that way, radian measure is simplified by the arc subtended on a unit circle: θ is arc sin *y*. The method is implemented in the solution of triangles. Graphical methods meant the use of ruler, compasses, and protractor. Angle was measured by a protractor, which is a mechanical method. Tables of sines and cosines are prepared from an angle drawn by a protractor, measuring the lengths of trigonometric lines in and about a unit circle on squared paper. An application of the graphical method was the construction of an angle when one of the ratios was given. The unit circle method was superimposed on existing practices without challenging fundamentally how trigonometry was practiced. The main object of study was still the solution of triangles. There was no manifestation of a notion of directed angle or a sense of arc length. Although the authors mentioned waves, periodicity was not brought forward as an invariant characteristic of trigonometric functions, sine, cosine, or tangent. Although graphical connections were made between sine and cosine, periodicity was not made an issue, and graphical behavior beyond the measure of 2π was not discussed.

Integrated Mathematics with a Special Application to Geometry (Swenson, 1934). The generalized definitions of trigonometric functions were defined by Swenson (1934) in an analytical plane. The order of presentation is important for discussing how the mathematical idea was built. It started with a point $P(x, y)$ rather than starting with the angle. Connected with every point in the *XY*-plane, Swenson introduced four geometrical magnitudes *x*, *y*, *r*, and α. He started with a given point *P* in the *XY*-plane; *x* and *y* were the coordinates, the distance of *P* to the origin was represented by *r*. The angle α was then defined by the angle that *OP* makes with the positive *x*-axis. Therefore, a point and a coordinate system are first assumed, then the angle is dependent on the point and the coordinate system. The trigonometric idea was built toward expressing a given position $P(x, y)$ by its distance and the angle it makes with the positive x-axis. This idea anticipates the complex representation of the point P. I traced how this idea progressed toward vector trigonometry with polar and complex forms in their unified expression of the coordinates.

As displayed in Figure 28, α is generated by the ray *OP* revolving counterclockwise about *O*. Angles are measured in degrees. In this geometry course, a theorem was presented on the ratio definition of trigonometric function on a coordinate plane: "To a given α corresponds unique values of each of the ratios, $\frac{x}{r}, \frac{y}{r}, \frac{y}{x}$" (Swenson, 1934, p. 366). Similar right triangles with different r values demonstrated that the values of the ratios had nothing to do with the separate values of x, y, and r. They depended only on the value of α. The value of an angle was measured in degrees in Swenson's practice. In other words, trigonometric ratios were emphasized to be functions of the angle α. *Angle* is defined to be both the rotation by implication and the angle that *OP* makes with the positive x-axis at the beginning. The initial sense of the angle is not kept, and these two ways of defining the angle α are not necessarily compatible.

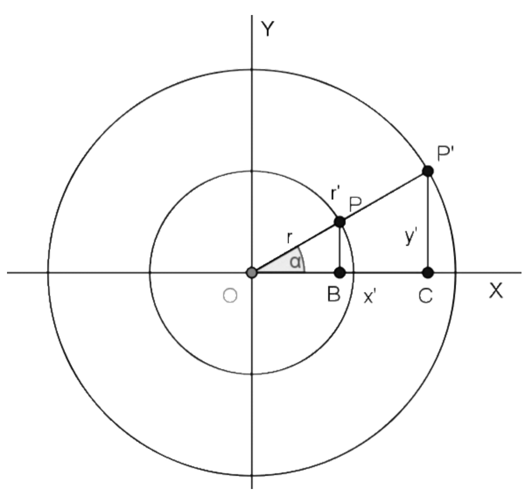


Figure 28. Trigonometric ratios, circles, and coordinate plane.

Swenson (1934) again defines circular function, first, not by their usual names, *sine* and *cosine*. The angle α determines uniquely each of the six ratios $y : r$, $x : r$, $y : x$ and their reciprocals. Each of these ratios is a one-valued function of α, such as $\frac{y}{r} = f(\alpha)$. And after placing his emphasis on the function aspect, Swanson starts specifying each one by its appropriate name, so that $\frac{y}{r}$ is called the sine-function of α. Having now defined trigonometric functions in an analytic frame, he starts varying the angles and discussing the specific trigonometric function values in different quadrants. In the subsequent sections, the author treats variation, graphing, tabulation, and interpolation of trigonometric function values.

New Math Period Practices of Circle Trigonometry

Contemporary Reform Context of Circle Trigonometry

I present below some of the leading perspectives on teaching circle trigonometry as reflected by the reform committees and leading journal articles during the new math reform period.

CEEB (1959). The College Entrance Examination Board (CEEB, 1959) Commission on Mathematics placed an emphasis on trigonometry and suggested a modernization of the subject. Instead of trigonometry approaching trigonometric functions in solving plane and spherical triangle, they found "more substantial and challenging applications of trigonometry now evident in many areas of science and technology– especially, in statics and dynamics, electromagnetic waves, and vibration problems of all sorts" (p. 28).

The commission proposed a program sequence that contained four trigonometric units. The first was the basic numerical trigonometry of right triangles. The commission recommended that it be included in Grade 9 as an optional unit. The second was the trigonometry of coordinates, vectors, and complex numbers. The third was the cosine and sine laws, addition theorems, and identities. The last unit was circular measure, circular functions, and their wave nature. The second and third units were suggested for the end of Grade 11 for about a third of a semester. The first unit might be anticipated in Grade 8, or combined with the second unit in Grade 11. The circular trigonometry unit was recommended to be treated toward the end of Grade 12.

The Commission on Mathematics noted that stress should be placed on the definition of the circular functions in terms of real numbers. To develop that idea, the Commission suggested using a winding line around a unit circle for Grade 12. The treatment of functions of angles measured in degrees was recommended for Grade 10. The radian measure of an angle was to be considered independently of the circle since it is a number defined by the ratio of two lengths. The Commission further underlined the importance of radian measure as a mean of transition from the functions of angles to the functions of real numbers at the beginning of Grade 12:

> It is possible to develop the essentials of trigonometry (included in units two through four) by starting with the circular functions in terms of real numbers and ending with angles, radian measure, and degree measure. Such passage from "pure'" to "applied" trigonometry may have greater brevity and mathematical elegance, but it presents a higher level of abstraction to the learner. The Commission's proposed treatment of trigonometry should not be construed as a dogmatic judgment in favor of angles over numbers. Rather, it reflects the Commission's decisions to stress coordinates, vectors, and complex numbers in grade 11, and functional properties in grade 12. (p. 29)

Circle Trigonometry in the Cambridge Conference Report (Educational Services Inc., 1963). ESI's *Goals* document suggested a mathematics curriculum for Grades 7–12. It was written by members of the Cambridge Conference on School Mathematics. Their main observation was that most curriculum reports were practical, had chosen to limit their goals, and tended to create new courses so that existing teachers with brief training could competently handle them.

Members of the conference (ESI, 1963) developed a program for Grades 7–12 school mathematics to set forth goals for the future. The report stated the view of 25 mathematicians and natural scientists as to the direction school mathematics should be going:

> These are the curricula toward which the school should be aiming. If teachers cannot achieve them today, they must set their course so that they may begin to achieve them in ten years, or twenty years, or thirty. If this is what the teacher of the future must know, the schools of education of the present must begin at once to think how to prepare those teachers. There must still be short-term curriculum reforms, they must look upon themselves as constituting a stage toward the larger goals, and they must at all costs be consistent with the larger goals. (p. viii)

> The topics proposed for the high school have become the foundation upon which applications to the sciences, engineering, and mathematics itself are built. … Moreover, the concrete treatment that we propose and the avoidance of a loose use of symbolism ought to ease the transition to applied mathematics in general and to computing in particular. For example, loose calculus deals with “variables” (in a Leibnizian sense) rather than functions, while both rigorous analysis and computing deal with functions rather than variables: you cannot explain to transistors the meaning of the symbol *dy/y*. We believe that this principle applies rather broadly; significant applications of mathematics require, at least, that clear intuitive grasp of the concepts which is best gained from precise formulations. (p. 43)

The Goals conference members recommended an informal and experimental approach to trigonometric function (ESI, 1963, p. 39). Trigonometric functions were to be defined intuitively on the whole line by using a point moving along uniformly on the unit circle. Functional characteristics were then recommended to be studied, such as maxima, minima, and periods. Applications suggested for trigonometric functions were harmonic motion, oscillations and pendulums, and waves. Approximate values of the functions for acute angles were to be found by measurement. Students were asked to build their own trigonometric tables with the expectation that a student who did the job conscientiously would acquire a good intuitive grip on the qualitative properties. Symmetry, periodicity, and interpolation were to be used in tasks of experimental measurements and tabulations. Applications to indirect measurement were expected. The notion of mathematical "model" was emphasized to capture the real situation in part or approximately (p. 40).

In their ambitious goals for 2000, the conference members recommended a Grade 9 geometry course that included rotations in the plane and in space and complex numbers and rotations in the plane with trigonometry. Linear algebra was recommended for Grade 10. It included the geometry of complex numbers, inner products, and orthogonal transformations. They recommended the study of analysis in Grades 11 and 12. Analysis included the topic of trigonometric functions as one of the transcendental functions.

Discussions on the Treatment of Circle Trigonometry.

Newsom and Randolph (1946) and the unit circle method. During postwar reform attempts on school mathematics, Newsom and Randolph (1946) urged a reform of school trigonometry. In their clear suggestions, they proposed a change of perspective in approaching the practice of school trigonometry. Their main recommendation was to place heavy emphasis on trigonometric functions as functions of numbers and not as functions of angles. Their objection to the reliance on angles as arguments in defining trigonometric functions was that such reliance provided a misplaced foundation for the subject. For Newsom and Randolph, functions of angles should be introduced only as an application of trigonometric functions. Trigonometry as an important mathematical system would be based on one of its applications if angle were used to define trigonometric functions:

> Either by definition or by implication, the great majority of textbooks on trigonometry regard the subject as fundamentally concerned with angles. In fact, the idea is presented with such insistence and emphasis that advanced instruction

> in science and mathematics is definitely handicapped. (Newsom, & Randolph, 1946, p. 66)

According to Newsom and Randolph (1946), a focus on angle in trigonometry creates a problem in the higher courses that students take:

> In calculus the trigonometric transformations are a perpetual headache for students who have been disciplined to always think of cos *x* as meaning the cosine of angle *x*. Very good scientists have confided to the authors of this article that the graphical representation of $y = \sin x$ upon a Cartesian axis-system always bothered them because they had difficulty accepting the device of marking *x* upon a straight axis; such a situation undoubtedly resulted from the false emphasis found in our textbooks on trigonometry. (p. 66)

The authors noted that the trigonometric functions do not require the restriction that the arguments be angles. They revised and provided a general definition of *trigonometry* as the mathematical science concerned with the trigonometric functions. This shift in perspective divorced trigonometry from its argument and turned attention to the functional characteristics of trigonometric functions. The new demand was to understand sin *x* and cos *x* as numbers whenever *x* is a number.

Newsom and Randolph (1946) described the unit circle method as an alternative way to define sin *x* and cos *x* for *x* a real number. A unit circle was defined to be a circle with its center located at the origin of a rectangular coordinate system with a unit radius. To make the proposed unit circle approach compatible with the former trigonometry of angular quantities, Newsom and Randolph decided to use the radian system. Radian measure provides a one-to-one correspondence between an arc on a unit circle and its angular measure. In this system, the angle at the center of a unit circle has the same measure as its intercepted arc. A trigonometric function of a central angle is identical to the function of its arc.

Rosenberg's dynamic trigonometry (1958). Rosenberg (1958) discussed the changing concept of trigonometry as a school subject during the first half of the century. He suggested transforming trigonometry from static to dynamic by not putting too much emphasis on the routine solution of triangles and instead using logic. First, he suggested defining such terms as *the tangent function, inverse sine function,* and *radian* and then applying deductive reasoning to these definitions and to previously accepted assumptions and proved theorems of geometry and algebra to establish the major properties of the trigonometric functions.

Rosenberg's analysis of the nature of trigonometry revealed that trigonometry has three basic components: analytic, geometric, and algebraic. His suggestions for a dynamic trigonometry included that modern trigonometry should be a general area of mathematics in which the geometric point of view is only one of many possible interpretations. With this approach, the x variable in $\sin x$ should represent more than an angle; it could be an abstract number that is real or complex, or it could be time or some other magnitude.

Variation Across Editions of a Focal Series

The School Mathematics Study Group (SMSG) texts were initially written in accordance with the recommendations of the CEEB Commission on Mathematics for high school programs. These recommendations included integrating algebra and trigonometry, and integrated plane and solid geometry with coordinate methods for Grade 11.

***Intermediate Mathematics* (SMSG, 1961d).** SMSG (1961d) defines trigonometric functions and circular functions separately. *Intermediate Mathematics* provides an introduction to trigonometry in chapter 10 of 15 chapters intended for Year 11 Mathematics. The textbook first generalizes the idea of arc length by introducing path length, the basic difference being not limited to physical circumference. It operationalized the process of winding around a circle. A path assumes a starting point and a measure of directed distance, which is a real number. If *P* is a starting point of the path, then (*P*, 0) describes a stationary point on a circle. A signed angle is defined as the triple (*AP*, *AQ*, (*P*, θ)) where *P* and *Q* are initial and terminal points of the path; and *A* is the center. A signed angle is briefly denoted by (*A*, *P*, θ). The size of an angle (*A*, *P*, θ) is determined by the length of the path (*P*, θ). SMSG's definition of signed angles extends the absolute valued arc length so as to describe directed arcs on a circle. A distinction is made between equivalent and equal parts.

Trigonometric functions are defined as functions of angles in standard position and then, they are further defined for arbitrary angles. The definition was as follows:

> Let (O, X, θ) be any angle in standard position, and let (x_0, y_0) be the intersection of its terminal side with the standard unit circle. Then sine of (O, X, θ) = y_0, sin θ = y_0; cosine of (O, X, θ) = x_0, cos θ = x_0; tangent of (O, X, θ) = y_0/x_0, tan θ = y_0/x_0. (p. 561)

SMSG's (1961d) definition is clear in terms of the assumptions made: an origin *O*, a line *X*, and θ. Exercises provided variations of the exemplary idea: (*O*, *X*, 120^{o}) specifies an angle precisely. And the authors further note:

> These definitions do not enable us to calculate these six functions except in a few special cases since it is usually not possible to find the coordinates of a point on the terminal side of the angle (O, X, θ). (p. 562)

The generalization of trigonometric functions for any angle is defined as follows. First any angle is defined as (*A*, *P*, θ) for any vertex *A*, initial line segment *P*, and angle θ. Then, any angle (*A*, *P*, θ) is mapped to a unique angle in standard position (*O*, *X*, θ). Then sin (*A*, *P*, θ) = sin (*O*, *X*, θ). This maps any angle to a signed angle. When the signed angle (*A*, *P*, θ) is paired with a real number sin (*A*, *P*, θ), then we have a function whose domain is set of all signed angles. In this case, we can use the first definition. It is denoted by sine θ. If the same process is repeated for cos (*A*, *P*, θ), it means that as paired with angle (*A*, *P*, θ), it defines a function from the domain of all signed angles to [-1, 1].

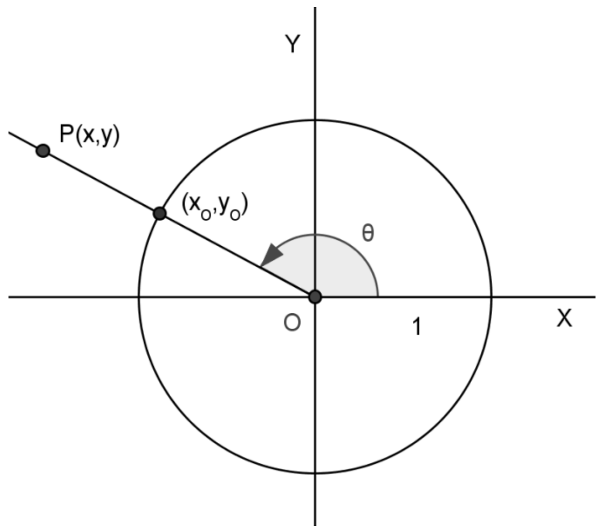


Figure 29. SMSG's trigonometric functions for any angle.
Note: Adapted from SMSG (1964d) *Intermediate Algebra*, p. 564.

The SMSG's (1964d) approach to trigonometric functions is displayed in Figure 29. SMSG first gave the definition of trigonometric functions for any angle, rather than giving it first for the acute angles on a right triangle. In the definition, the origin *O* and the *X* axis are assumed to be given. A line *OP* also is assumed to be given as the terminal line for a given angle θ. The point *P*(x, y) is chosen as any point on the terminal line. Here there is no mention of the measure of the angle. Therefore, angle is assumed to be an entity whose measure is not needed to be known to start with. The angle θ is the only variable, and its measure is unknown.

The definition finds the intersection of the line *OP* and the unit circle and calls the coordinates of this point (x_o, y_o). Then the pair of points on the unit circular were found to be as follows:

$$x_o = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}};\ y_o = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}.$$

Finally, SMSG (1961c) introduces the ratio definition for the sine and cosine functions for any angle (p. 564). They are as follows:

$$\cos\theta = x_o = \frac{x}{r};\ \sin\theta = y_o = \frac{y}{r}$$

The SMSG (1961c) authors at this point remark that corroborated the former essence of right-triangle trigonometry in the coordinate plane: If a point (x_0, y_o) is a point on a terminal side of the angle, then all points have coordinates of the form "(rx_0, ry_o) for $r > 0$" (p. 565).

SMSG's study of circular functions is a course on an integration of algebra, trigonometry, and elementary functions but "not in the solution-of-triangles sense" as the study of angles and triangles. The rotation and circular symmetries are founding ideas in an analytical study of circular functions. SMSG's treatment of circular functions puts an emphasis on the periodic property of sine and cosine. It makes extensive use of the idea of rotating the plane about a perpendicular through the origin. The claim is that this approach gives a certain unity to the discussion. Using this orientation of the plane, the authors derive the addition formulas for sine and cosine. This derivation further expresses a belief that this approach is a natural one, "which does not involve a trick."

Sine and cosine are defined using the unit circle in a *uv*-plane. Later, the authors use the *xy*-plane to plot the graphs for $y = \sin x$, $y = \cos x$. The argument they use is that since x is used for arc length, it might be confusing for students to visualize x as both the horizontal axis in the plane of the unit circle and at the same time a length of the circular arc. SMSG suggested that a more exact way of defining *sine* and *cosine* is by the composition of two functions. One maps the set of real numbers **R** to the set of geometric points on the unit circle, and the other maps the points on the circle to the real numbers. Therefore, if x is in **R**, and if P is a point on the unit circle, we have a function $f: x \rightarrow P$, another function $g: P \rightarrow \cos x$, from which $gf: x \rightarrow \cos x$. Similarly, the sine function is defined as a mapping from **R** to **R** using a composite function. SMSG further emphasizes that cosine and sine are functions from real numbers to real numbers.

> You might point out to the student that nowhere in this section have we used an angle, although we have used the concept of arc length, sine and cosine are completely divorced from any geometric considerations. They are functions on the set of real numbers in the same sense as polynomials. Too often when we speak of sin A the students feel that A must be an angle. Sometimes they think of A as being the degree measure or radian measure of angle, but the idea that A need have no connection with an angle is usually very strange. (SMSG, 1961c, p. 183)

The ideas of functions and their analysis were explored throughout the course. Students were asked to study functional characteristics of periodic functions. Algebraic properties of periodic functions were investigated. For example, students were asked to analyze the periodicity of the addition of periodic functions. In another exercise, the periodicity of multiplication of a periodic function and a nonperiodic function is to be shown.

***Elementary Functions* (SMSG, 1961e, 1965a).** *Elementary Functions- Teachers' Commentary* (SMSG, 1965a) indicated that "the text follows generally the outline recommended by the Commission on Mathematics for the first semester of the 12th grade" (p. i). Sine and cosine are defined with the unit circle on a *uv*-plane. The argument for using the *uv*-plane was to avoid confusion. Toward defining sin x, the contention is that x is already reserved for arc length. Then it might be confusing for students to visualize x as both the horizontal axis on the plane of the unit circle and at the same time a length of the circular arc. Then SMSG(1961e) presents the following:

> Consider a moving point *P* starting with (1, 0) on the *u*-axis and proceeding in a counterclockwise direction around the circle. We can locate *P* exactly by knowing the distance x which it has traveled along the circle from (1, 0). The distance *x* is the length of an arc of the circle. Since every point on the circle $u^2 + v^2 = 1$ has associated with an ordered pair of real numbers (*u, v*) as coordinates. (SMSG, 1961e, p. 226)

The motion of the point *P* defines a function

$$\wp : x \rightarrow (u, v)$$

that associates every nonnegative arc length *x* with an ordered pair of real numbers (*u*, *v*). By doing so, SMSG (1961e) defined a function from real numbers to circular points. Notice that this mapping is only part of the step toward defining trigonometric functions. The student textbook further states, "It is inconvenient to work with a function whose range is a set of ordered pairs rather than single numbers" (p. 226). Two functions are defined as follows: cos: $x \rightarrow u$, and sin: $x \rightarrow v$. They are called circular functions, and a remark was made that "these circular functions are related by not identical with the familiar functions studied in elementary trigonometry" (p. 227).

Horizontal Variation Comparing Focal With Other Series

University of Illinois Committee on School Mathematics (UICSM). Standard Euclidean geometry usually treats the angle addition in a restrictive way. The transition from high school geometry to trigonometry requires extensions of angle concepts. Szabo (1971) developed a trigonometry of sensed angles as an analogue to circular functions. "Sensed angles" were defined as ordered pairs of rays in a way to yield the familiar angle properties of plane geometry. The transition from high school geometry to trigonometry required extensions of angle concepts. They suggested analogues to the cosine and sine functions whose domain is the set of sensed angles. They were the cosine and "sin-perp" functions whose domain is the set of sensed angles of an oriented plane (Szabo, 1969, p. 37; Vaughan & Szabo, 1971). Perp was indicated by $\perp$.

Then trigonometric sine and cosine functions are defined as

$$\cos(a, b) = \vec{u}_a \cdot \vec{u}_b, \ \sin^{\perp}(a, b) = \vec{u}_a^{\perp} \cdot \vec{u}_b.$$

where the rays u_a and u_b are the unit vectors in the senses of the rays u and v, respectively; the perp of the vector u_a is the unit vector in the sense of a (Szabo, 1969, p. 37). This work led to UICSM's experimental geometry and trigonometry course by "A Vector Approach to Euclidean Geometry" (Vaughan & Szabo, 1971).

This frame is based on extension of the number concept to the doubles, to the concept of vectors that extends analytic geometry by making the analysis coordinate free. Trigonometric ideas are fundamental in the foundations of the elementary algebra of vectors. UICSM defined two products of vectors. One was inner product: $a.b = ab_{||} = |a|\ |b|\cos \alpha$.

The outer product is defined as the product of one vector with the orthogonal component of the other: $a \wedge b = a\ b_{\perp}$. The values of sine and cosine determine a unique angle in $(0, 2\pi)$. The angle between two vectors is a sensed magnitude, positive if counterclockwise. The inverse vector, u^{-1} has the same direction as u and the inverse modulus $1/u$. An angle is defined by two vectors, $\alpha(u, v)$. Trigonometric function values are defined for the angle.

$$\cos \alpha(u, v) = \frac{u_1 v_1 + u_2 v_2}{|u|\,|v|}, \ \sin \alpha(u, v) = \frac{u_1 v_1 - u_2 v_2}{|u|\,|v|}$$

Circular functions in UICSM's inner products, Euclidean geometry, and trigonometry, Vol. 2 (Vaughan & Szabo, 1973). Vaughan and Szabo (1973) defined a number of cosine and sine functions with different senses and arguments. For example, three cosine functions were introduced. First, the authors defined cosine and sine functions whose arguments were angles rather than real numbers. Next, cosine and sine functions were introduced using arguments as sensed angles, and for each orientation of the plane a sine perp function was defined using sensed angles as arguments; $\sin^{\perp}$ denoted the sine perp function. They were closely related in the sense that

> the cosine of an angle is the cosine of each of the corresponding sensed angles and the sine of an angle is the absolute value of the $\sin^{\perp}$ of each of the corresponding sensed angles. (Vaughan & Szabo, 1973, p. 413)

The circular functions were introduced by a winding function, **W**, that maps a real line onto the unit circle "in such a way that if *a* and *b* are any real two numbers such that $0<b-a<2\pi$, then the image of the interval $\{x: a < x < b\}$ of **R** is an arc whose measure is $b - a$—that is, is the same as the measure of the interval"(p. 409). After defining this mapping, the authors introduced the definitions of a pair of cosine and sine functions. For any real number *t*, **W**(*t*) is a point of the unit circle, which is a pair of real numbers, and cosine and sine are defined as the coordinates of W(*t*):

$$(\cos(t), \sin(t)) = W(t).$$

Degree-sine and degree-cosine functions are defined from circular functions as follows:

$$^{o}\cos \alpha = \cos (\pi a/180);\ ^{o}\sin \alpha = \sin (\pi a/180).$$

There is clearly an inflation of trigonometric functions defined by UICSM. There were four cosine defined, while the authors developed precise definitions for trigonometric functions. With each new sense and formalization of the argument for trigonometric functions, they considered a different function. It is clear that during the new math, the authors provided further clarification and more precise definitions for trigonometric functions in school. Although their differentiation of trigonometric functions was informative, the number of trigonometric functions generated is a clear indication that overspecification can make the subject more incoherent. Changing the name does not necessarily change the essence of trigonometric functions. What I observed was the authors' imposition of the contextual constraints as essential content and their presenting them as rules to define trigonometric functions. These four kinds of trigonometric functions are only instances of trigonometric functions that can be defined given a mathematical context. They should not necessarily merit a conceptual category other than trigonometric functions sine and cosine.

Standards-Based Mathematics Period Practices of Circle Trigonometry

Contemporary Reform Context of Circle Trigonometry

During the third reform period, standards-based textbooks placed more importance on mathematical experimentation and graphical methods. Kilpatrick (1997) observed that "a large part of the standards-based reform is built on the view that mathematics itself has become more computational and less formal" (p. 957). Most textbooks during this period incorporated an intuitive development of trigonometric functions by a numerical approach and investigation of data patterns that could be modeled by trigonometric functions, usually sinusoidals.

The Conference Board of the Mathematical Sciences (CBMS, 1983). The 1983 report of the CBMS, entitled *What is Still Fundamental and What is Not,* suggested that topics from discrete mathematics and from data analysis and statistics are "more important than … what is now taught in trigonometry beyond the definition of the trigonometric functions themselves" (CBMS, 1983, p. 13).

Circle trigonometry in NCTM (1989, 2000) documents. According to the NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics,* "knowing" mathematics is "doing" mathematics (p. 7). In this standards-based reform document, NCTM endorsed a process-oriented-approach to school mathematics. Mathematics was defined by its methods and the processes involved rather than by the structural and logical aspects of learning objects and logical aspects of elementary mathematics.

NCTM (1989) proposed standards for Grades 9–12, recommending that the mathematics curriculum should include the study of trigonometry so that all students could: apply trigonometry to problem situations involving triangles; and explore periodic real-world phenomena using the sine and cosine functions. In addition, college-intending students should understand the connection between trigonometric and circular functions; use circular functions to model periodic real-world phenomena; apply general graphing techniques to trigonometric functions; solve trigonometric equations and verify trigonometric identities; and understand the connections between trigonometric functions and polar coordinates, complex numbers, and series. Scientific calculators could and should significantly facilitate the teaching of trigonometry, providing more class time and computational power to develop conceptual understanding and address realistic applications. Graphing utilities would provide dynamic tools that would permit students to model many realistic problem situations using trigonometric equations or inequalities. Graphing utilities should play an important role in students' development of an understanding of the properties of trigonometric functions and their inverses. In addition, college-intending students should solve trigonometric equations and inequalities by computer-based methods, such as those described in the NCTM standard on algebra.

In connection with the 1989 NCTM Standards, an overview of the standards document (NCTM, 1988) provided a summary of the changes in content and emphases in Grade 9–12 mathematics. Trigonometry topics to receive increased attention were as follows: "The use of appropriate scientific calculators; realistic applications and modeling; connections among the right triangle ratios, trigonometric functions, and circular functions; and the use of graphing utilities for solving equations and inequalities" (p. 27). In contrast, trigonometry topics to receive decreased attentions were as follows: "The verification of complex identities; numerical applications of sum, difference, double-angle, and half-angle identities; calculations using tables and interpolations; and paper-and-pencil solutions of trigonometric equations" (p. 27).

Decreased attention was recommended (NCTM, 1988, p. 27) for some geometry topics traditionally related to trigonometry. These topics were the study of inscribed and circumscribed polygons and the study of theorems for circles involving segment ratios. Meanwhile, other geometry topics were recommended for more attention. Among those were computer-based explorations of 2-D and 3-D figures; and coordinate and transformation approaches. For algebra, increased attention was to be given to computer-based methods such as successive approximations, to graphing utilities for solving equations and inequalities, and to matrices and their applications.

The content standards developed by NCTM (1989) were algebra, geometry, trigonometry, functions, statistics, probability, discrete mathematics, conceptual underpinnings of the calculus, and mathematical structure. These standards were not to be treated separately. In general, an integrated treatment was recommended for all mathematics content, challenging the practices associated with traditional boundaries and sequences among the topics in the high school mathematics program. Geometry was not to be treated as a separate subject that should be given at Grade 10 in high school. Geometry topics were to be integrated at all grade levels. Analytical geometry and functions were not recommended to be given as separate courses.

The 2000 NCTM *Principles and Standards* document, in contrast, had five content strands: number and operations, algebra, geometry, measurement, and data analysis and probability. No separate content strand was allocated to trigonometry.

Standards period discussions on circle trigonometry. Hirsch, Weinhold, and Nichols (1991) illustrated how the standards' recommendations for trigonometry can be implemented in a technology-rich environment to support reasoning, communication, and problem solving. The authors discussed three fundamental mathematical themes for trigonometry: functions, graphical sense, and modeling. Graphing tools and scientific calculator recommended for reducing time to find and calculate the values of trigonometric functions, and their graphs. The authors further presented the use of graphical methods in showing identities. The authors suggested that by using these tools, students can sense the identities and then develop a proof; if not, they suggest that students can provide a counterexample. Their approach to mathematical practice was in line with the standards' view of mathematics as a science of patterns.

I observe that there is a tendency to make the mathematics class a science class. Hirsch et al. (1991) describe a laboratory approach as a fitting analogy to investigate and make sense of mathematical objects such as polar equations:

> Investigation of graphs of polar equations challenges students' ability to set up a controlled experiment; making a formal laboratory report gives them practice in communicating in precise language and furnishing supporting data for their conclusions. (p. 105)

The mathematics class as a laboratory experiment is an old theme from the early unified mathematics period.

Variation Across Editions of a Focal Series— Core Plus

***Contemporary Mathematics in Context, Course 2* (CPMP, 2008).** The Core-Plus project (CPMP, 2008) introduced right-triangle trigonometry and coordinate methods before introducing the circular functions. The content of the coordinate methods included transformation matrices with translations and rotation of images. The earlier practice of treating rotation matrices before introducing circular functions was used in the second course. Special rotations of a pair of coordinates were used for angles such as 0, 45, 90, and 180 degrees.

***Contemporary Mathematics in Context, Course 2B* (CPMP, 1997).** In the first edition of CPMP (1997), trigonometry was studied in the second year in Unit 6, "Geometric Form and Its Function." First triangle and then circle trigonometry were introduced in the same unit.

Introducing circular functions, the first edition of CPMP placed more emphasis on studying machines that use a system of rotating circular objects connected by pulleys or drive systems, such as automobile engines or sprockets. Complete relational symmetry was discussed as the essential characteristics of the circle and central to its use. The authors discussed measuring rotation by revolution and degrees, the connection between radii, and transmission factors for rotating circular objects that are connected.

Radian measure was introduced by modeling angular velocity with two strips attached to a cupboard disk. One was fixed at the 9 o'clock position, and the other was a movable strip. The idea behind radian measure was manifested by students' discovery that "the size of the central angle of a circle, determined by an arc equal in length to the radius of the circle, remains the same for all circles" (CPMP, 1997, p. T419).

***Contemporary Mathematics in Context, Course 3* (CPMP, 2009).** A first observation was that the new edition (CPMP, 2009) shifted the content of circle trigonometry to the third year. CPMP studied circular motion and developed trigonometric function ideas. The authors developed key ideas in two major lessons. The first lesson was a new lesson that was not included in the first edition prior to circle trigonometry. The properties of the circle studied in the first section were used in the second lesson to model circular motion. In the first lesson, the circle was studied by focusing on its chords, arcs, and angles. The properties of the circle were investigated intuitively and confirmed deductively. Students were asked to explore properties of special lines and angles in circles by using use compass and straightedge, paper folding, measuring, or geometry software. And then, students were asked to use trigonometric functions and coordinate methods "to reason about the properties" (p. 397). This lesson provided a balanced analytic, synthetic, and deductive study of the circle in which students could develop and justify mathematical reasoning using proof and conjectures. One weakness of this section is a lack of accounting for the law of sines in connecting inscribed angles and radius and the chords. The law of sines provides a link between circle and triangle trigonometry. This section prior to circle trigonometry provided an exemplary context to introduce this historical connection.

In the second lesson, systems of circular rotating objects are studied, and their motion is modeled using sine and cosine functions as functions of angles and radians. To model circular motion, the authors used a Ferris wheel with the following justification:

> Ferris wheels are circular and rotate about the center. The spokes of the wheel are radii, and the seats are like points on the circle. The wheel has horizontal and vertical lines of symmetry through the center of rotation. This suggests a natural coordinate system for describing the circular motion. (CPMP, 2009, p. 425)

Although the first edition introduced the radian measure earlier for studying coordinates of points on a rotating wheel, the second edition did not. It used degree measure for the angle. The Ferris wheel model is placed on a coordinate system with its center at the origin. Then "you can use what you know about geometry and trigonometry to find *x*- and *y*- coordinates of any point on the circle" (CPMP, 2009, p. 425).

The CPMP authors link angular and linear motion by devising radian measure. Measuring the angle in radians was defined as wrapping a number line around a circle (CPMP, 2009, p. 430). This is a mechanical approach to measuring an angle that assumes the circle as a given physical object that one can wrap a measuring tape around. The resulting effect is a correspondence between points on the number line and points on the circle. The authors discuss that when angles are defined in radians then trigonometric functions can be defined as functions with the domain all real numbers. After extending the domain, "wrapping a number line around a circle shows how the radius of the circle can be used to measure angles, arcs, and rotations in radians and how the cosine and sine can be defined as functions of real numbers" (p. 432).

The materials emphasized in the first edition were circular rotating object systems connected by pulleys or belts or sprockets and machines. They could also be found in this edition: not at the beginning but toward the end as applications, connections, and extensions.

A cursory reading of the circular functions chapter and associated activities suggested that the integration of instructional technology tools was still weak. Most activities were the same as the first edition, and they did not directly call for the integrated CPMP software toolkit. For example, an investigation activity was given on the concept of shapes with constant width. The activity asked students to make a cardboard model of this shape. However, a digital manipulative would have facilitated and directed attention more to the mathematical object than setting up the manipulatives. Very little direct integration of learning technology was observed in this unit on circular functions.

Horizontal Variation Comparing Focal With Other Series

COMAP's (1999) *Mathematical Modeling: Our World-Course 3*. In developing ideas of trigonometric functions, COMAP followed a modeling orientation. Without formally introducing any trigonometric functions, first, the idea of periodicity was introduced. COMAP (1999) presented multiple data sets that built the idea of period as a characteristic feature of oscillating data patterns. Students were expected to describe data patterns, graph data, and develop a mathematical model. COMAP gave students a description of the modeling process that they are expected to follow:

> In seeking the best possible predictions for periodic phenomena, the usual modeling advice still applies. Start simple. Examine the context carefully to understand its properties: Those properties need to be inherited by your model. Work with familiar descriptions before trying to invent new ones. Again, match properties of the context to properties of your mathematical descriptions. (p. 394)

Following this practice, periodic data patterns were first approximated by piecewise parabolic equations that students had studied earlier. A goodness-of-fit measure is used for the piecewise parabolic model to explain the model's appropriateness. Upper loops and lower loops were described by different parabolic equations. Activities were purposefully designed to lead students realize that "parabolic models are fairly close over a limited domain, but consistently wrong models even there" (p. 406) to explain periodic data patterns.

After introducing the idea of periodicity, a circle model then was introduced. Students were exposed to activities to discover the pattern of the change of vertical position of a fixed point on rotating circular objects. Those objects varied from one activity to next; they were Ferris wheel, bicycle wheel, and others.

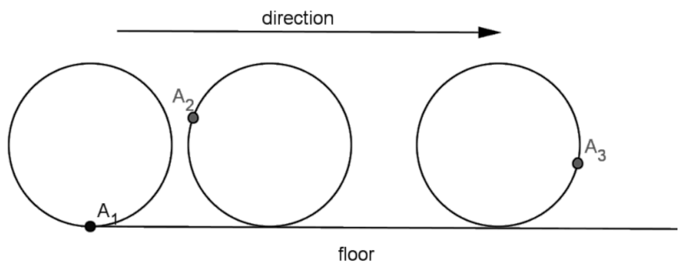


Figure 30. Measuring the height of a dot on a rolling circular object.
Note: Adapted from COMAP (1999) *Mathematics: Modeling Our World,* p. 404.

The first model was the height of the points on rolling circular objects. Students were asked to collect data to measure the height of a fixed point on rolling wheel: "Measure and record the dot above the floor and the horizontal distance the wheel has traveled" (p. 404, see Figure 30). The position of the point of A_1 changes to A_2 and A_3 after rolling on the floor towards the direction shown. The advantage of this rolling model is that instead of measuring angle as the amount of turn and finding circular arc length, it mechanically maps the points on a circle to a line. While A_1 changes to A_2, it travels a distance that unwinds a circle. Then how much the circle turned is an arc length, and it is directly manifested by the distance of the point A_1 to the point where the second circle touches the floor. The travelled distances are plotted against the height of the points A_2 and A_3. Notice that so far students did not have to measure and discuss angles but were able to observe the connection between circularity and periodic data patterns.

This rolling task is conceptually rich and requires carefully planned activity. It is not the study of changes in the coordinates of the point on a rolling circle. Otherwise, tracing the path of the point *A* would lead to a study of cycloids. But it is only an activity on changes in height. Students are expected to note the periodicity of the graph.

One of the features of the COMAP text is that it demands interpretation and contextual reasoning to develop intuitive meanings. After collecting data with a rolling wheel, students are asked to interpret the meaning of negative values for the variable in context.

The next shift of perspective is to study a rotating circular object such as a Ferris wheel to model circular motion by only focusing on the changes in the height of a point on a rotating Ferris wheel from the ground. COMAP (1999) stated, "You can determine the position of the rider on a Ferris wheel when you know the distance the rider has traveled since boarding the ride" (p. 406). This assumes an a priori knowledge of a circular arc length to measure a height of an object. COMAP clarified the fundamental idea further: "In riding a Ferris wheel, progress can be measured by how far around the circumference of the wheel you have moved" (p. 407). In the rolling activity,

> progress was recorded as how far forward the wheel had rolled. Since the amount of forward progress for a rolling wheel is exactly equal to the amount of its circumference that has rolled past the contact point, these two methods of measuring the turning of a circle are identical. (p.407)

A further standardization was given by a rotating point on a static unit circle centered at the origin. COMAP (1999) uses an analogy to describe the radian measure. It is "an ant walking around the circumference of a unit circle" (p. 407). According to COMAP, the ant's position on the unit circle can be determined by the directed distance of its walk. It is positive if the ant's walk is counterclockwise, negative if clockwise. "If the ant walks 2π radians then it has walked one around the circle" (p. 407). Radian is defined as the directed arc length of an arc that begins at (1, 0) on the unit circle.

Then after introducing radian, the COMAP (1999) authors discussed the implication of the radian measure on the behavior of periodic functions revisiting the previous activities. It is a transformation of the periodic pattern so that “changing the radian measure standardizes the period” (p. 409). Having introduced the radian and studied the oscillating graphs, the authors define the sine function as follows

> As the ant (or rotating dot, or Ferris wheel rider) moves from its initial position, the point (1, 0) on the unit circle, a graph of its height versus the directed distance (radian measurement) of its walk produces the oscillating graph. [This new function is called the sine function:] “the vertical displacement from the horizontal axis of a point on the unit circle. That is, the value of the sine is the *y*-coordinate of a point on the unit circle. (p. 410)

The COMAP (1999) authors do not introduce the cosine function right away. Instead, they do something unusual: They present a treatment of the cosine function as intuitively connected to sine. They first ask students to find the times at which the Ferris wheel run is the most exciting. A contextual connection is introduced between an exciting moment and the moment at which the height above the ground changes most rapidly or most slowly. Then the authors introduce a rate of change construct contextually. Rate of change is applied to the height of the point as measured by the slope, assuming the Ferris wheel revolves at a constant rate. The difference quotient (DQ) can be used as a measure of rate of change. This method is an intuitive derivative-finding process that was first proposed by Leibnitz. In COMAP, the cosine pattern was then developed from sine function data by symmetric DQs. The similarity of $y = \sin(t)$ and its rate of change was observed before cosine was introduced as a separate trigonometric function.

Cosine was also generated as a horizontal displacement. COMAP explained the relationship between the *x*-coordinate and *t*-value of a dot rotating around a circle. The similarity of patterns of cosine is presented as a shifted sine by observing similarity of patterns and explaining it by a phase shift of $\pi/2$.

Finally, formal discussions were given by COMAP to show how the right triangle sense and the circle sense of trigonometric functions are consistent. The point was made by using the following metaphor:

> Imagine you are walking a unit circle starting at (1, 0). Every now and then you stop and record in radians the directed length of your arc. These recordings, based on the definitions of sine and cosine, can be used to find the coordinates of your location. (p. 443)

By definition, a circular point is already assumed to be given as the position of the walker. It is also clear from this definition that one can also assume that given the point, one can measure the length of a directed circular arc. My clarification of that argument is this measure can be only an estimation. This estimated circular position is then project onto the *x*- and *y*-axes to yield the cosine and sine function values for the position. Then, COMAP (1999) connects this definition with the ratio definition by using the following argument:

> Continue to imagine that you are walking around a unit circle. Add a spotlight at the center of the circle. Its operator tracks you in its beam during your walk. Your starting location is (1, 0). When you stop, the ray made by the spotlight beam has turned a certain number of degrees as it followed you along your arc. Your location on the unit circle can also be determined by this angle θ, called the central angle—an angle whose vertex is the center of a circle. (p. 443)

At this point the text connects angle and arc length. From an arc length, one can assume that one knows the position of the point. Toward that point, a ray is defined that begins at the origin, and as the point rotates, a ray traces it and defines the corresponding central angle. The above unification of definitions lies in the certainty of knowing the point. Without assuming exactness, COMAP's argument would hold for the unification of definitions of trigonometric functions in the triangle and the circle sense.

Interactive Mathematics Program, Year 4 (IMP; Fendel, Resek, Alper, & Fraser, 2000). In a large problem context based on a Ferris wheel, IMP (Fendel et al., 2000) provided an integrated treatment of linear, quadratic, and circular motion. The problem concerned the coordination of a circus act that involves a diver jumping from a rotating Ferris wheel into a tub of water on a cart moving at a linear rate. The cart's motion is linear, the diver's motion is quadratic, and the Ferris wheel's is circular. The students had previously been exposed to mathematical models that explained linear and quadratic but not circular motion. The students were expected to learn about trigonometric functions that could model circular motion in the context of this large problem. The whole unit was to take about 32 class days. In a carefully planned series of small problems, the authors incrementally introduced and treated the problem by dealing with different combinations of its smaller parts. These small problems were presented in context and focused on different parts of the larger problem with different degrees of complexity. For example, the circular part of the problem was treated along with the quadratic part by reducing the problem to a diver jumping from the 1 o'clock position on a stationary Ferris wheel. During the activities, the authors revisited the larger problem to help the students gauge their progress, having finished some of the smaller parts of the big problem in a variety of related contexts. The mathematical concepts were introduced as tools to resolve parts of a big problem situation.

IMP (Fendel et al., 2000) introduced the extended definitions of trigonometric sine and cosine functions in the fourth year course. Although the trigonometric functions had been introduced in the first chapter of the first year, the treatment of trigonometry was continued throughout the series. For example, the radian concept was introduced in the third chapter of the fourth year. The ratio definition was used to obtain the extended definitions in a coordinate setting. The angle θ and a Cartesian coordinate system were assumed to be primary objects to frame the conceptualization of trigonometric functions.

For a given angle θ,

> the angle is placed within coordinate system, with its vertex at the origin, and is measured clockwise from the positive direction of the *x*-axis. The goal is to express sin θ in terms of the *x*- and *y*- coordinates of a point on the ray defining the angle (IMP, 2000, p. 17).

From this setup, the angle and the terminal ray were ontological commitments of the trigonometric object to be defined. Therefore, given angle θ, IMP (2000) assumed that there is a definite terminal line. The point *A*(*x*, *y*) was to be chosen along that ray to further build the definition. As shown in Figure 31, IMP used a directed angle with a point *A* on its terminal line *r* that is introduced to represent the distance from the origin.

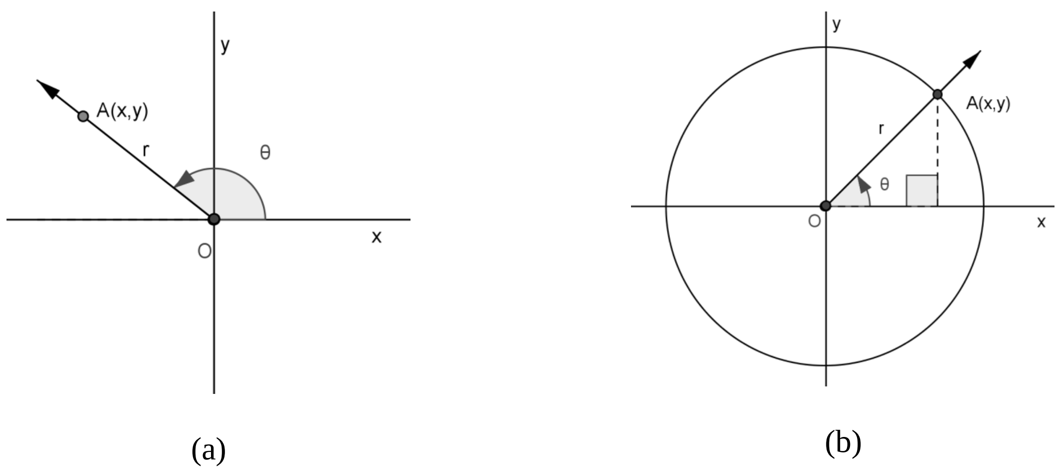


Figure 31. IMP extended definitions for trigonometric functions.

Trigonometric functions were considered tools to express the coordinates of the points that can be chosen for a given angle on a coordinate plane by projections. The sine and cosine functions were defined for any point in the analytic plane by using its projections on the *y*- or *x*-axis and their ratio to the distance to the origin. The cosine value was not that of the projected line segment but referred to the position of the projected point on the *x*-axis. This definition of the trigonometric functions employed a Cartesian point for an angle and essentially mapped that point (*x*, *y*) to its sine and cosine values. The mapping was defined from $\boldsymbol{R}^2$ to $\boldsymbol{R}$.

Fendel et al. (2000) defined the cosine function for all angles. For a given angle, that cosine definition required first finding an associated point. That point was to be chosen along the terminal line for the angle. It can be proven using the similarity of triangles that the choice of point along the terminal line does not affect the sine and cosine ratios. The issue of sign for trigonometric functions is resolved by following the Cartesian coordinate values for the point. This definition gives the same values for acute angles as the right-triangle definition. The conception of trigonometric functions is not dynamic, and attention is not focused on the variation of the argument for the trigonometric function. The conception is based on similarity of right triangles and equivalence of the internal ratios of similar right triangles. This is signified by the use of any point on the terminal line for the angle θ.

In the IMP program, the circle was not used as a conceptual tool. IMP used both the Cartesian and the polar coordinate systems. The variation in angle was not the primary concern in conceptualizing trigonometric ratio. The unit circle definition was not used. The circle is superimposed onto the definition as in Figure 31b. The definitions were applied to any circle through the Ferris wheel problem.

The development of trigonometric functions in IMP was based on degree measure, which was used to convert the argument from the speed of rotation of a circular object investigated throughout the unit. The degree-based and time-focused development, however, meant that an introduction of radian measure was postponed, which created some potential confusion toward the end of the chapter. At that point, problems were posed that called for finding the simultaneous rate of change of position horizontally and vertically. The choice not to use radian measure created a situation such as the following:

> For a circular platform with its center 65 feet above ground and a 50-foot radius, rotating at constant rate of 9 degrees per second, the equation is $h(t) = 65 + 50 \sin (9t)$. Taking the derivative of the height function, which is constantly changing, the vertical component of the velocity of circular rotating platform is $2.5\pi \cos(9t)$, and the horizontal component is $-2.5\pi \sin(9t)$. (Adapted from Fendel et al., 2000, p. 84, p. 110)

Before the above problem was posed, the chapter on the Ferris Wheel problem had not created a major obstacle for learners using degree measure. At that point, however, the IMP authors could not avoid using π. The derivative for a degree-based function is not straightforward, and the use of radian measure simplifies the calculation of the derivative. Without resorting to derivatives, one can assume that the tangential speed of a rotating object is constant because the object is a circle. Finding the tangential velocity vectors for points rotating around a circle does not require using derivatives. One could define an elliptic path with parametrization, $x=a \cos t$, $y=b \sin t$, where the nonzero a and b are not equal. Then the rate of change would not stay constant, as with planetary motion around the sun.

IMP uses rotation matrices after introducing circular functions to animate a cube as a long-term project in the fourth textbook of the series. Using trigonometric forms of rotation matrices, the authors introduced general rotations of pairs of coordinates without a restriction to special angles such as 0, 45, 90, and 180 degrees.

***Functions, Statistics, and Trigonometry* (FST) by the University of Chicago School Mathematics Project (UCSMP).** Among the series I examined, only FST by UCSMP addressed the connection to trigonometry using generalized quadratic equations in closed form, that is, $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$ (UCSMP, 1998, p. 788). Coordinate transformations and rotations are used to simplify and generate the generic quadratic forms such as circle, ellipse, and parabola. Through translation and rotation, any given quadratic form can be transformed into a generic form. Rotation serves the purpose of fitting a Cartesian plane rotating its pairs of coordinates by using trigonometric functions.

***Algebra and Trigonometry: Structure and Method* (Brown, Dolciani, Sorgenfrey, & Kane, 2000).** The book by Brown et al. (2000) started by defining circular functions to respond to the need to define the trigonometric functions of a number. The radian measure of an angle was defined by the ratio of arc length to the radius. The unit circle was not used to define radian measure. Circular functions were defined on a unit circle $x^2 + y^2 = 1$.

> Given any real number s, start at A and measure $|s|$ units around O in a counterclockwise direction if $s \geq 0$, and in a clockwise direction if $s < 0$, arriving at a point $P(x, y)$. The sine of s and cosine of s are then defined by the coordinates of point P. The tangent, cotangent, secant, and cosecant functions are defined in terms of sine and cosine. (p. 613)

The authors further remarked on the close relationship between the circular and the corresponding trigonometric functions. This remark suggests that they differentiated circular functions from trigonometric functions.

Brown et al. (2000) presented and discussed functional characteristics of circular functions such as periodicity and symmetry as well as their graphs. They presented the effects of constants in changing the graphs of the circular functions. In discussing the graphing of circular functions such as tangent, the authors presented trigonometric line representations of tangent. This approach was similar to that of the old trigonometric line functions presentation that was practiced during the early 1800s before the introduction of the ratio approach in school trigonometry. The only difference was that circle was the unit, which was De Morgan’s (1849) suggestion.

The Frame of Circle Trigonometry

The frame for circle trigonometry develops a function approach for the treatment of trigonometric objects such as sine and cosine. It is based mainly on the mathematical idea of connecting circular arc lengths and associated chords. The unit circle provides a frame for conceptualizing angle as a dynamic object. An angle is associated with a real number by creating a mapping from the real line onto the points on a unit circle with a standardized circular directed arc length. Each point of the unit circle is mapped by a rotation, $\boldsymbol{R}$, of the point (1, 0), and the circle is represented as $\boldsymbol{R}_{\theta}(1, 0) = (\cos \theta, \sin \theta)$. This approach suggests a transition from an angle-based analytic geometry perspective to a real-value based conception of trigonometric functions. An arc-length-based approach to defining trigonometric functions is ancient, but its unitizing and standardization of circular arc length has found a place in textbooks during the last 200 years. The sinusoid as the graph of the sine function was not originally associated with (θ, sin θ). Sinusoids became associated only after the practice was established of plotting Cartesian graphs of changes in signed trigonometric line values in and about the unit circle. This practice was a relatively new phenomenon during the early 20th century.

Circle trigonometry varied in textbooks because it was introduced through either a unit-circle approach or a function approach. Research on secondary school mathematics has often underlined the conceptual difficulty of the transition from arithmetic to algebra. The first extension of the trigonometric function idea is from arithmetic to algebra by shifting the character of the mathematical object from numeric to algebraic. This extension produces trigonometric functions as functions of numbers. Another shift represents a change of perspective to seeing the trigonometric function as a variable object, which is called a functional approach. This extension shifts trigonometric functions to a functional approach. A function approach in trigonometry extends the algebra of numbers to the algebra of functions and explores the patterns of composite trigonometric functions such as $\sin x + \cos x$, or $\sin^2 x + \cos^2 x$, and multiplicative and additive structures with basic trigonometric functions. A function approach also builds on patterns emerging from the additive and multiplicative structures of trigonometric functions. In the function approach to trigonometry, trigonometric functions are considered as solutions of functional equations. As arithmetic seeks a numerical solution to equations, algebra seeks algebraic variable solutions to functions and then the analysis of variable function solutions to functional equations.

CHAPTER 6

VECTOR TRIGONOMETRY DURING REFORMS

In this chapter, I present a phenomenological description of vector trigonometry and its treatments. I address in three sections the reform periods' treatments of circle trigonometry. In three subsections, I describe circle trigonometry for the period, focusing, respectively, on the contemporary reform context, editions of the focal textbook series, and other textbook series of the period.

Vector Trigonometry during the Unified Mathematics Reform

Contemporary Reform Context on Vector Trigonometry

The NEA (1894, 1899, 1923) mathematic commission reports during unified mathematics did not make references to vector trigonometry. The mathematics committee for CEEB (1923) specified one requirement related to vector trigonometry. In connection with problems in physics, students were expected to know the principle of parallelogram of forces (CEEB, 1923, p. 11).

Vertical Variation Across Editions of Series as Focal

I examined the place and treatment of vector trigonometry in different editions of the Breslich series.

Breslich (1919) *Correlated Mathematics.* The idea of a coordinate system is to represent the position of a point by assuming two reference lines. Breslich (1919) presented this idea after reviewing rectangular coordinate systems and possible oblique coordinate systems as alternative systems. When the lines are perpendicular, then the author noted that one has a rectilinear system of coordinates.

By using a radius vector and vectorial angle, Breslich (1919) defined polar coordinates of a point *P*. A point *P* in a plane may be located by its direction and distance from a fixed reference point *O*. This frame assumes one initial line *OX* as the polar axis. The author also inserted small historical notes related to the subject such as “polar coordinates were first used by James Bernoulli in 1691” (p. 6). In the polar setting, a point on the plane can be determined definitely by letting the vectorial angle and the radius vectors have positive and negative values. The sense of radius vector was then assumed to be positive when the author developed a relationship between Cartesian and polar coordinates. The polar form of the complex number directly relates to this representation. But then the radius vector was restricted to the positive sense and defined as the modulus, and the vectorial angle was the amplitude or argument of the complex number. Algebraic operations of addition and subtraction and multiplication were then defined. Division was not discussed without introducing the conjugate first. De Moivre’s theorem was introduced by observing the multiplication pattern for the *n*th power of a complex number. The polar form was considered to suggest a geometric construction of multiplication and the quotient of two complex numbers. Then Breslich applied the polar idea to study a straight line. Trigonometric forms were used to study the line as a mathematical object. One was in the normal form, $x \cos\omega + y \sin\omega - p = 0$. The other was in the polar form expressed by the equation $\rho\cos(\theta - \omega) = p$, where ω is the angle of the perpendicular to the origin. The line was the author’s choice of mathematical objects as an application of polar and trigonometric forms. The author did not represent the circle or ellipse in polar form as a simple quadratic form.

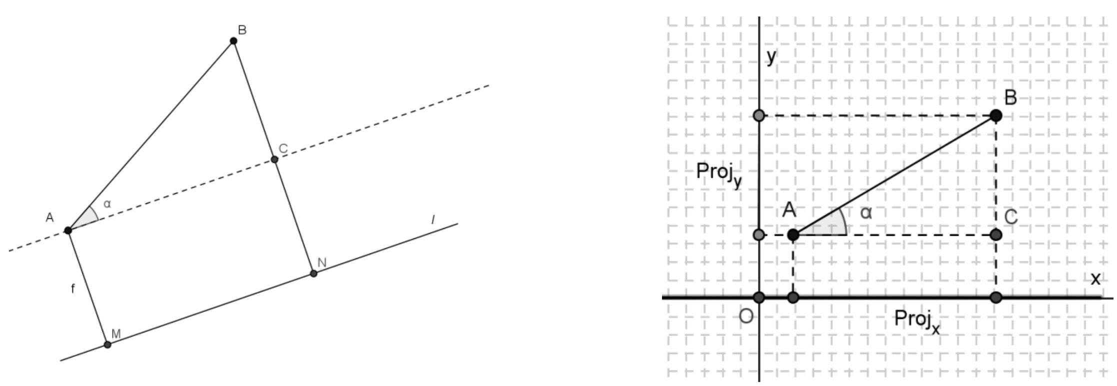
Horizontal Variation Comparing Focal With Other Series

Palmer and Leigh (1916). Palmer and Leigh (1916) started with a point *P* and a straight line. The orthogonal projection was used to define the projection of the point on any straight line. The definition of the projection depended on drawing a perpendicular, which could be described further by a shortest distance. Hence, the vector approach, if progressively described, is distance based, without a loss in the chain of descriptions. With a projection approach, the line segment does not have to be defined or placed at the origin. The projections are usually made onto a horizontal line OX and vertical line *OY*; if the length of the segment of line projected is *l*, then $x = l \cos \theta$, $y = l \sin \theta$, where θ is the angle of inclination of the line to be projected to the assumed horizontal line. Angle is not predetermined; it is determined by the horizontal line and the line to be projected. Based on the idea of projection, the authors developed the vector idea. A vector was defined to be "a line representing a directed quantity" (p. 54) and applied to the concept of force and its composition from physics.

Swenson's *Integrated Mathematics* (1935a). Swenson (1931, 1935b) wrote a calculus textbook for use in Grade 12. Analytic geometry and calculus were treated as a high school mathematics course and were introduced "as the mortar or unifying element which is capable of binding together the various parts of algebra, geometry, and trigonometry" (Swenson, 1935a, preface). The textbook started with the function concept. The chapter on the number system of algebra treated first the real numbers and then gave complex numbers by using vectors. Complex numbers were used to give the proof of addition formulas of trigonometry. The fundamental theorem of algebra and De Moivre's theorem were given.

Kenyon and Ingold's *Trigonometry* (1914, 1921). During the unified mathematics period, the vector approach was first utilized by Kenyon and Ingold (1914). They saw the first fundamental idea about trigonometric functions as introducing trigonometric ratio. That idea was made explicit; stating that "the ratio of two sides of a triangle does not depend upon the size of the triangle, but only upon the angles" (p. 7). Then a progression was made from right triangles to their similarity and then to an angle measure. The idea of triangle similarity was associated with trigonometric ratio for an angle measure. Radian measure or any circle-based definition of angle was given.

After introducing a right-triangle definition, Kenyon and Ingold (1914) reframed the ratio idea as emerging from the right triangle by the method of projections. The projection of a segment *AB* on a given line *l* was equal to the product of the length of the segment and the cosine of the angle the segment makes with the given line, that is, $MN = AB \cos \alpha$. In Figure 32b, a given line segment *AB* on a coordinate frame is a projection on the *x*-axis and *y*-axis instead of the line *l*.



(a) (b)

Figure 32. Trigonometric function as projection of line segments in a manifestation of the vector approach. Adapted from Kenyon and Ingold, *Trigonometry*, 1914, p. 20.

This frame shift allowed a new state of affairs for trigonometric objects, giving a broader horizon. When *A* is chosen to be *O*, then the coordinate *B* can be defined in terms of its projections onto the *x*- and *y*-axes. In the projection method, the use of a right triangle is replaced by the use of a line segment and a reference line. Starting with two directed lines, the angle determines the direction of the cosine.

One intuition in the vector frame is the idea of using only two segments—a given segment and a reference segment—and reducing the direction to an angle. The right triangle is reconceptualized as a segment and its projections onto reference axes.

Kenyon and Ingold's (1914, 1921) treatment of trigonometric functions for any angle incorporated projections. It refined and clarified the meanings of trigonometric objects. No auxiliary circle was drawn. The circular definition of angles was not used. A point (*x, y*) was fixed on a coordinate plane (Figure 33). Its coordinates were expressed by the projections of the point onto axes, $x = \text{Proj}_x r = r \cos \alpha$; $y = \text{Proj}_y r = r \sin \alpha$. This practice directly called for the use of polar coordinates.

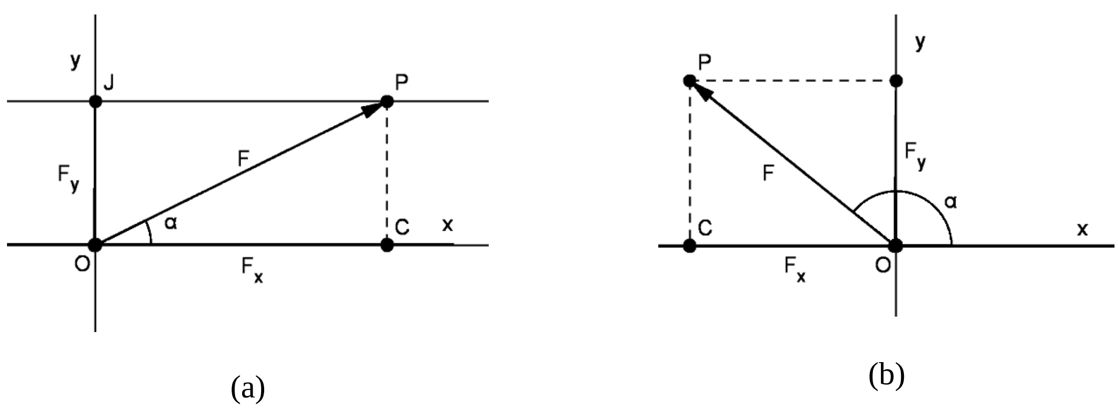


Figure 33. Manifested vector definition of trigonometric functions using projections. Adapted from Kenyon and Ingold, *Trigonometry*, 1914, p. 60; *Elem. of Plane Trigonometry*, 1921, p. 89.

Sine and cosine were used by Kenyon and Ingold (1914, 1921) as tools to indicate that a point *P* depended on the angle and its projection. The authors provided a distance-based definition of trigonometric functions where *r* was a positive number representing the measure of distance from the point *P* to the origin. Based on the manifestation of the basic intuition, the variations suggested that the distance *r* and the *x*- and *y*-axes were invariants in this frame. Despite the misleading geometric representation of unequal *r*s appearing in the third and fourth quadrant, the idea focused on the variations of the angle. The distance was measured by the Euclidean metric $d^2(x, y) = x^2 + y^2$.

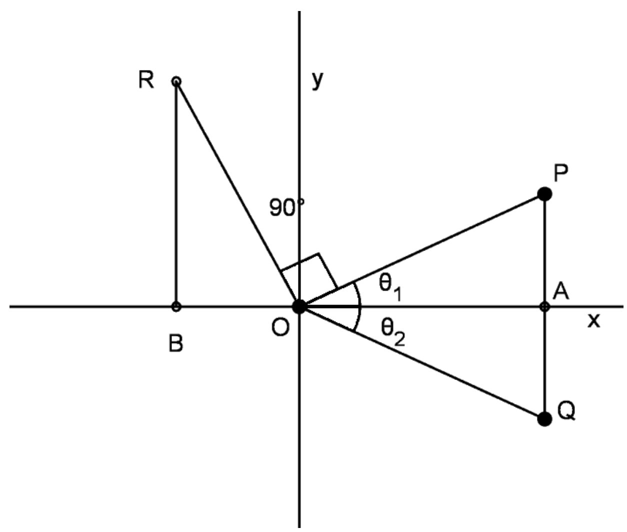


Figure 34. Graphical representation of trigonometric identities after projections. Adapted from *Trigonometry* by Kenyon and Ingold, 1914, p. 64.

Kenyon and Ingold started with a point *P*(*a, b*) in the coordinate system as given in Figure 34. The angle θ_2 was assumed to be $-\theta_1$. The authors treated trigonometric identities with the variations of the angle accompanied by graphical representations to show symmetries and repeating patterns between the trigonometric values. The transformation of the point *P*(*a, b*) to *R*(-*b, a*) and *P*(*a, b*) to *Q*(*a, -b*) gave students the possibility of having a rotational framework for understanding numbers. It further established a founding intuition toward a unification of the pair *x* and *y* for the point *P* by a complex number. If the pair (1, 0) were chosen to represent a unit vector for *x*, then rotation by a right angle would represent the pair (0, 1) expressing the unit vector that generates the *y*-axis. This argument suggests that genetic constitution of the *y*-axis is achieved from the *x*-axis within the vector trigonometry framework.

Swenson's (1934) Integrated Mathematics-Geometry. Swenson's (1934) *Integrated Mathematics* provides a good example of a textbook series developed in line with progressive education principles. In the preface, the author indicates that the 1923 report of the National Committee on Mathematical Requirements emphasized using the mathematical idea of the functional relation to unify mathematics courses in high school.

Swenson's second book of his series was written for the 10th-grade geometry course. In the preface, he noted a pattern emerging in U.S. mathematics education during the 1920s and 1930s toward tracking and provision of courses for more specialized instruction to satisfy the needs of higher ability students. He observed that the practice of making every student, irrespective of ability, cover the same amount of work was fast disappearing from better schools. For that reason, while Swenson developed a series for all students, he also included considerable material that could be used as supplementary to keep the brighter student occupied and interested.

Swenson (1934) included two additional chapters on Cartesian geometry for experimentation purposes. Trigonometric functions were also integrated into the course not as an extra but as an essential part of the integrated course. To put this choice of topic sequencing in context, the trigonometric function in its general form was started early compared with its counterparts in subsequent generations of integrated mathematics textbook series. Trigonometric functions were commonly introduced with right triangles as ratios of the sides in a geometry course when it was a second high school mathematics course. It was unusual to integrate the extended definitions of trigonometric functions into the second course, which was mainly focused on geometry. The common trend after the Committee of Ten recommendations in 1893 was to integrate those definitions into the end of an Algebra II course, which was usually the third course in high school mathematics. Later, some groups such as the Interactive Mathematics Program (IMP) tried to delay the topic until the beginning of the fourth year course, but in their 2010 revision, Fendel et al. put the trigonometric functions back into the later parts of a third course. This shift put the trigonometric functions topic at the end of Grade 11, which is a more traditional place for trigonometric functions since the 1890s. It represents another case of how the high school mathematics curriculum is resistant to change.

Swenson's (1934) geometry treated the trigonometric functions as a function of angle. Angles were defined in degrees, and the radian measure was not defined. The unit circle was not used in the book. "Adding Line Segments," for example, was a section in the trigonometric functions chapter. The earlier development of the Cartesian connection in two chapters between line segments and numbers allowed Swenson to introduce an algebra of directed line segments. The introduction to algebra was primitive. The direction could be either positive or negative depending on the sign of the numbers on a number line. "In using directed segments to represent numbers we may take them from any part of the number scale provide the segments selected have the correct length and direction" (p. 388). Any of the segments [3 0], [0, -3] and [-2, -5] on a number line could be used to represent -3. Two directed line segments were considered equal when their directions were the same and their lengths were equal. So *AB* was not equal to *BA* as a directed segment; the initial and terminal points mattered. The addition operation for the algebra of directed line segments was introduced. In the addition of line segments, the direction of a segment was used: (+2) + (-3). In this context, direction was used only to define the sense or orientation in the measurement of the length of the segment. Swenson then introduced an application that presented vectors as directed line segments and showed their addition. He did not go into multiplication or division for directed line segments. The tangent, on the other hand, was defined as an identity; that is, as the quotient of sine and cosine.

The subsequent chapter in Swenson (1934) is entitled "Circles and Related Lines." It started by examining the relation between a chord of a circle and its distance from the center. There was no reference to the sine function or to its relationship to the chords of a circle. Although the previous chapter concerned the trigonometric functions and although the sine function is connected historically to the chords of a circle, this lack of accounting for the historical and logical connection decreased the coherence of the approach for this integrated mathematics textbook.

Vector Trigonometry During the New Math Reform

Contemporary Reform Context on Vector Trigonometry

CEEB(1959). The CEEB (1959) report presented clear recommendations regarding the place of trigonometry.

> Trigonometry must be reorganized to meet these contemporary needs. Computational emphasis should shift from triangles to vectors, and analytic emphasis from identities to functional properties. … The vital material of the reorganized trigonometry lies in the rectangular and polar description of points, vectors, and complex numbers, and in the addition theorems and periodic character of the circular functions. In particular, the one-to-one correspondence between the ordered set of the pairs of real numbers, and the set of points in plane, or the set of vectors drawn from the origin, or the set of complex numbers provides a superlative example of the manner in which important mathematical ideas coalesce. (p. 28)

The CEEB recommended vector trigonometry for Grade 11 mathematics. Coordinate trigonometry and vectors were suggested to be treated as a topic. Sine, cosine and tangent of a general and directed angle were to be defined in terms of *x*, *y*, and *r*. Rectangular and polar coordinates were to be included in vector trigonometry. A vector was connected to a directed angle. A fundamental connection was to be made between complex numbers and vectors.

The CEEB (1959) further emphasized the importance of the treatment of complex numbers as vectors:

> [This treatment] is a most important mathematical by-product of the reorganized trigonometry. Reinforcing the student's earlier contact with complex numbers in the study of quadratic equations, this trigonometric treatment should round out a good basic knowledge of complex numbers in grade 11. At the end of unit four, an informal discussion of Euler's formula … and series expansion serves to relate the exponential and circular functions in striking fashion, and to free the latter from angles without a shadow of doubt. (p. 29)

The Committee noted the fundamental unity of coordinates and vectors and complex numbers that was presented decisively by Whitehead (1958).

Discussions on Vector Trigonometry. Picken (1946) provided a vector trigonometry perspective prior to the new math reform efforts and discussed the ideas of sign and vector in plane analytic geometry and trigonometry.

Wooton (1965) criticized the high school trigonometry courses during the 1950s: "Instead of focusing on those parts of trigonometry of use and value in more advanced mathematics, an inordinate amount of time was being spent on computational trigonometry of a sort no longer useful either theoretically or practically" (p. 34).

Willoughby (1967) noted the change of emphasis in the study of trigonometry in high school from the study of measurement to the study of functions. He also argued that there was a place for algebraic geometry in high schools (Willoughby, 1966). He presented the results of an experimental class for 10th graders taught in Connecticut during the 1958–1959 academic year. The course introduced an affine geometry, which does not place Euclidean restrictions early on, and it provided a substantial introduction to trigonometry. The materials used for the course were highly algebraic and started with the geometry and algebra of the line. The materials were written by Howard Levi and directed by Howard Fehr of Teachers College. They had used the materials prepared by the UICSM for the ninth-grade course. Willoughby found limited evidence that the students benefited from a more algebraic course.

Another experiment was the Wesleyan experiment, which also used the materials developed by Levi (Sitomer, 1964). The course treated Euclidean geometry as a special case of affine geometry preserved by parallel projections, not imposing some of Euclid's postulates, and working on a more general case of Euclidean geometry that did not change by scaling or slanting. It did not require one-to-one correspondence with points but with vectors; therefore, the origin was free.

The University of Illinois Committee on School Mathematics (UICSM) developed a two-year course based on vector spaces. They sought to develop high school geometry course on the principles of projective geometry. Following intuitive ideas on projection, perpendicularity and distance were defined by the dot product of translation (Szabo, 1969, 1971, 1973).

Amir-Moez (1958) discussed the teaching of trigonometry through vectors and gave a course plan. Trigonometric functions for any angle were defined for vectors. The author noted that "we consider only vectors with beginning at the origin 0 of a rectangular coordinate system and denote $\overrightarrow{0A}$ by $\vec{A}$"(p. 19). This approach reduced the conception of vector to a directed line segment. Vectors are quantities with direction and size but not a starting value. The author further introduced the Pythagorean identity, addition, inner product, and projections of the vectors.

Rosenberg (1958). Rosenberg (1958) presented a perspective of dynamic trigonometry. He noted that "a genuinely modern trigonometry course destroys the fallacy that the only 'true' trigonometry is the trigonometry based on the assumptions of Euclidean geometry. The truly modern trigonometry course explodes the myth that the only type of trigonometry worthy of study is the trigonometry of circular functions" (p. 251).

Toward transforming trigonometry into a dynamic subject, Rosenberg pointed out that it is not enough only to integrate trigonometry into traditional courses or topics in secondary school mathematics. Concepts of modern mathematics should be utilized such as the polar, exponential forms of complex variables to derive the law of cosines. Further, he noted that alternative procedures might be discovered in higher geometry and higher algebra.

Rosenberg added that a modern trigonometry course could be an introductory exploration of the theory of the elementary transcendental functions, trigonometric, inverse trigonometric, logarithmic, exponential, hyperbolic, and so on. It is a course in which students might discover alternative sequences are available from mathematical logic for proving trigonometric identities and realize that there exist other worlds where the sine of an angle might be greater than 1 or in which negative numbers might have logarithms.

Vertical Variation Across Editions of Focal Series

SMSG. Begle reported that in April, 1962, the SMSG steering committee met in San Francisco and discussed whether to accept a recommendation from the Steering Committee that a sample textbook be prepared stressing affine geometry based on coordinates (Wooton, 1965, p. 82). At that time, he reported that the advisory board declined to pursue this recommendation but "found the project and interesting one and would be happy to provide assistance to any group working in this area" (p. 82). SMSG later produced supplemental materials on vector geometry.

SMSG's (1961d) Intermediate Mathematics. SMSG (1961d) provided a study of vectors as directed line segments. The idea of rotation was applied to a vector. Rotation was treated as a function. Here, rotation *x* means the geometric rotation of a vector through an arc *x*. The rotation function of a vector was represented by its components on unit *u* and *v* vectors. Placing emphasis on the idea of functions, the rotation function was studied as a function to determine its characteristics. When the series introduced a new function, it was formally treated. The composite of rotation functions was introduced. Complex numbers were then used to develop addition formulas.

Vectors have both geometric and algebraic aspects. The vector trigonometry frame suggests a transition to be made from coordinates of points to components of vectors. Algebraic aspects of vectors were examined by showing their equivalence, their addition and scalar multiplication by real numbers. The concept of inner product of vectors was introduced. With these algebraic and geometric aspects, vector algebra was presented as a substitute for Euclidean geometry. The inner product of two vectors requires cosine by definition. This practice suggests that angle be defined from its vectors, avoiding pitfalls of the idea of rotating a vector by a measure of an angle and coming up with an indeterminate vector as a result of an transcendental measure of a circular arc.

Analytic Geometry (SMSG, 1965c). The SMSG (1965c) treatment of vectors represented an ontological shift; it reoriented the study of familiar tools that had been experienced by students before in new mathematical context. The elementary study of vector calculus allowed working with a new set of powerful algebraic tools associated with addition, scalar multiplication, inner product, and external product of vectors. It allowed regenerating many familiar theorems with a new sense, such as finding areas using the inner product of vectors, study of diagonals, medians of triangles, and quadrilaterals.

SMSG (1965c) showed the simplicity and directness of analytical proofs for some theorems from plane geometry and trigonometry:

> It is time we paused to survey the variety of problem-solving tools which are not at our disposal. We have a choice of three basic systems, rectangular coordinates, polar coordinates, and vectors, within each system we have a different representations to suit different purposes. But the question uppermost in your mind at the moment probably is "How do I decide which method is the best one to use?" Questions regarding distances between points, slopes of lines and midpoints of segments are easily handled in rectangular coordinates. (p. 155)

Therefore, when these ideas were present, SMSG suggested that one should try to fit rectangular coordinate axes to the problem. If the problem involved angular motion or circular functions, polar forms were suggested first. Vectors are quite versatile and work under a wide range of conditions. SMSG (1965c) added that "concurrence, parallelism, and perpendicularity of lines as well as problems of physical forces are situations which might lead you to choose a vector approach" (p. 156).

SMSG's (1965c) *Analytic Geometry* introduced vectors through directed line segments. With a line represented analytically, a sense of direction is assumed for a line. If there are two points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ then the pair (l, m) are called direction numbers, where $l = x_1 - x_0$ and $m = y_1 - y_0$. They are associated with the slope of the line, m/l. The cosines of the direction angles of a line l are called direction cosines. Direction angles and direction cosines are defined only for a line with a specified sense of direction. Such a line is called a directed line. The origin-principle emphasized the freedom of choosing any point in the space as an origin. This principle allowed relating a vector to any point in space. The simplest representative vector is called the origin vector. Introducing inner product of vectors using cosines, SMSG showed its uses: in finding the area of the triangles, showing perpendicularity of the diagonals of a rhombus, and concurrency of the altitudes of a triangle. The Ceva and Menelaus theorems were proven through the use of vectors.

In introducing vectors, SMSG (1965c) started with directed line segments, where the end points were differentiated qualitatively: one as starting or initial, and the other as the ending or terminal point. Then the vectors were defined as an infinite set of directed line segments equivalent to any given directed line segment. This concept was also explained through an analogy from arithmetic. The analogical concept is rational number, where an infinite set of equivalent fractions represent the same quantity; that is 1/2, 2/4, 3/6, 4/8, … . This set is called a rational number: “It is common in many texts to use the vector to mean not the whole set of equivalent directed line segments, but any single member of that set (p. 93).

Horizontal Variation Comparing Focal with Other Series

During the new math period, the main series that implemented vector trigonometry was the UICSM series. UICSM’s efforts were directed toward the development of textbook materials for Grades 9 through 12 supported by funds from the Carnegie Foundations of New York. After 1962, new materials were developed to extend the curriculum to seventh and eighth grade. The position of the UICSM vector-based approach followed the trend of the algebraization of geometry and an advanced method of coordinates for an analytic treatment of geometry. The methods of coordinates were advanced by incorporating vector methods. This UICSM textbook claimed that the algebra of points and translations using a vector approach was more efficient than the algebra of analytic geometry. UICSM developed a two-volume geometry textbook on a vector approach to Euclidean geometry that dealt with inner product spaces, Euclidean geometry, and trigonometry. The UICSM textbooks were later published commercially (Vaughan & Szabo, 1973).

The UICSM used the notion of vector concepts in unifying high school mathematics. Szabo (1969) had written his dissertation on vector trigonometry for secondary schools at the University of Illinois-Urbana Champaign under Max Beberman. He observed that “the notion of vector was discussed as a part of trigonometry courses on the sixties just as logarithms were in the fifties” (Szabo, 1969, p. 24).

UICSM’s Vector Based Trigonometry by Vaughan and Szabo (1973). Trigonometry in Vaughan and Szabo (1973) was developed on a vector geometry. The founding intuition for this geometry was that translations of a Euclidean space constitute a vector space with an inner product. The conviction of the authors were that “the algebra of analytic geometry is in many instances is less efficient than is the algebra of points and translations which is developed in this course” (p. 2). The chapters in this textbook were the following: Inner Product Spaces, Perpendicularity, Distance, Angles, Triangles and Quadrilaterals, Circles, Oriented Planes and Sensed Angles, and the Circular Functions. In three chapters trigonometric functions are introduced.

In the Angles chapter (Vaughan & Szabo, 1973), the notion of sense was incorporated into the treatment of the mathematical object of angle. The definition of cosine and sine of an angle was first based on sensed angles. To develop the founding idea for cosine, the authors made several metacognitive statements about its genetic procedure: “We can link up the notion of the cosine of an angle with the notions concerning orthogonal projections” (p. 215). Another statement was that “the ratio in which an interval BC is foreshortened when it is projected orthogonally onto a line *l* is the absolute value of the cosine of “the angle between the lines BC and *l*” (p. 216). Then, the cosine of an angle was defined as the dot product of the unit vectors of the senses of the sides of the angle. Given a triangle ABC, the vector *a* is A – C and *b* from B – C. Then cos ABC= $a \cdot b/(\|a\|\,\|b\|)$ (p. 215). The authors next presented a theorem suggesting that angles are congruent if and only if they have the same cosine (p. 219). This approach suggests “an intuitive notion of the meaning of congruence” (p. T219(2)). The sine of an angle, C, is defined by sin C=$\sqrt{1-(\cos C)^2}$. Then the sine of an angle is a positive number that can not be greater than 1. The phenomenon of “foreshortening” was also the founding intuition of COMAP’s (1999) right-triangle trigonometry.

Vector Trigonometry During Standards Based Math Reform

Contemporary Reform Context on Vector Trigonometry

CEEB (1985). The endorsement of using polar coordinates in CEEB (1985) moved textbooks toward preparing students for complex trigonometry. One of the exemplary intuitions I had was that the polar form as a mathematical object is a quasi-complex object. The textbooks had drastic variations in how they treated polar forms. Trigonometric connections of polar and complex instances suggested a critical role for polar representation toward yielding higher knowledge objects in complex form. This role was not clear from the observations I made. The nature of intensions and extension of polar forms as knowledge objects in school mathematics requires further investigation. This study of vector/complex trigonometry makes it clear that further research is needed to study manifestations of polar connections.

Vertical Variation Across Editions of Series as Focal

Coxford, Fey, Hirsch, Schoen's Core Plus Mathematics Project, Course 3 (2009). Among the goals of the geometry and trigonometry strand for the Core-Plus curriculum was "representing patterns with drawings, coordinates, or vectors" (Coxford et al., 2009, p. xiii). The textbook does not mention vectors or trigonometric connections of complex numbers in polar form. A trigonometry problem is posed at one point, as the very last problem at the end of a treatment of quadratic polynomials. The problem included a ray as a terminal side of an angle θ passing through a point P in the third quadrant. Angle θ was drawn without a direction. Students were supposed to find sin θ, cos θ, and tan θ (p. 363). Directed angles were not used in the second edition of Course 3. Instead, "the positive number line is wrapped in a counterclockwise direction around the unit circle starting at (1, 0)" (p. 430).

Coxford, Fey, Hirsch, Schoen, Hart, Keller, Watkins et al.'s (2001a). Core Plus Mathematics Project Course 4 Part A. Coxford et al. (2001a) introduced vectors as a tool for modeling motions because they could simultaneously represent distance and direction. First, linear motion was discussed by using navigation context to introduce vectors and vectors operations. The geometry of addition of vectors was discussed and connected to right-triangle trigonometry. Navigation provided a context in which angles were consensually measured by degrees from due North in a clockwise orientation, as opposed to the *x*-axis and counterclockwise orientation of a rectangular coordinate system. The algebra of vectors was discussed. The fundamental connection of coordinates and vectors was presented by the method of decomposition of a vector into its horizontal and vertical components. The standardization of a vector by a coordinates with a set of assumptions was introduced. Although vectors are free objects, this standardization picked a vector from the origin of a coordinate system as its initial point. Thereby, the authors conclude that there are several ways to represent vectors; among which are *(r, θ)* and (*x, y*)as its endpoint. The authors described further this process of coordinate representation of vectors in a manner much like the following:

> Suppose OB is a vector whose initial point is at the origin. If B has coordinates (5 cos 135°, 5 sin 135°), what is the length of OB? What lines or segments determine the sides of the angle that has measure sin 135°. Describe the relationships among a vector, its component vectors, and the coordinates of the terminal point when the initial point of the vector is at the origin. (Adapted from Coxford et al. 2001, p. 97)

The Core-Plus authors represented circular motion by the rotation of position vector OP in standard position with its initial point at the origin. Based on the earlier setup, a notion of directed angles was already given in this parameterization of circular motion by the components of a standard vector. The parametric representations of linear and circular motions were exemplified as "$x = 3t \cos 42^\circ$; $y = 3t \sin 42^\circ$; x = 3 cos t, y = 3 sin t" (p. 122). Graphing calculators were used to dynamically represent the rotational motion by its "trace" feature (p. 124). Its variations were examined to extend the parameterization of a motion along an elliptical path. The series did not further develop connections to complex numbers and exponentials.

Coxford, Fey, Hirsch, Schoen, Hart, Keller, Watkins et al.'s (2001b). Core Plus Mathematics Project Course 4 Part B. In year 4, Coxford et al. (2001b) provide a problem context at the beginning that calls for modeling of projectile motion of a golf ball, by developing horizontal and vertical components of its initial speed vector with the golfer's strike.

Core-Plus extended the family of trigonometric functions by incorporating the notion of vector: θ denotes both an angle and its measure. A coordinate system with O and axes are defined. A point A(*x, y*) other than the origin is chosen on the coordinate plane. Then θ is assumed to be "the measure of the angle determined by rotating a position vector of length r from a position along the positive *x*-axis to its position $\overrightarrow{OA}$"(p. 459), as shown in Figure 35. The measure θ is positive if counterclockwise, depending on the rotation. Then this setup assumes that point A is the first object and θ is known as a measure from the *x*-axis to $\overrightarrow{OA}$.

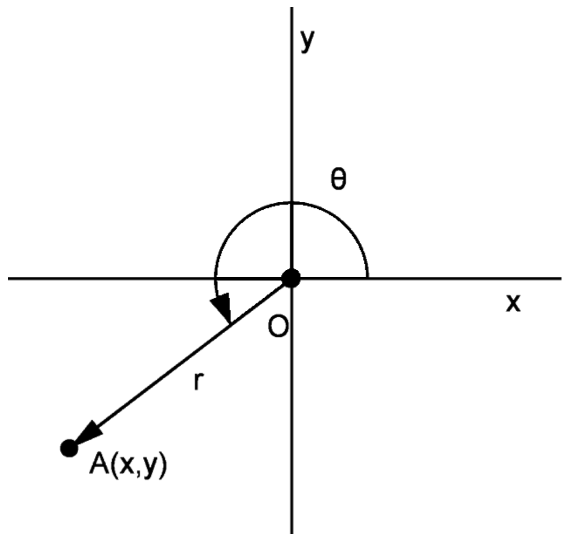


Figure 35. Extending trigonometric functions manifested by Core Plus Mathematics Project.

Coxford et al. (2001b) next provide standard ratio definitions of the trigonometric functions as functions of θ in the coordinate setting, sin θ = *y/r*, cos θ = *x/r*, tan θ = *y/x*, *x* is nonzero. They next provide an activity where "A(x, y) is defined as a point on the terminal side of the angle" (p. 459). Then A became secondary to the angle and became dependent on the measurability and determinability of its terminal line. The authors were inconsistent. In the majority of tasks, they defined A as point on terminal side of the angle. The authors further defined position vectors with length *r*, whose terminal point lies in a circle. Its coordinates are "functions of cosine and sine of the direction angle of the vector" (p. 461).

Polar coordinates were defined at this point (Coxford et al., 2001b), and sum and difference identities were developed. Notice that this polar form assumes that the position vector

$\overrightarrow{OP}$ is always positive, see Figure 36. This form was manifested during the transition to the study of geometry of complex numbers. Adding the variation in the Core Plus treatment of the polar form, the earlier intuition on the status of polar form as a “protention” (see Husserl) of complex form is fulfilled. The geometry of complex numbers is introduced next; complex numbers and vectors are associated.

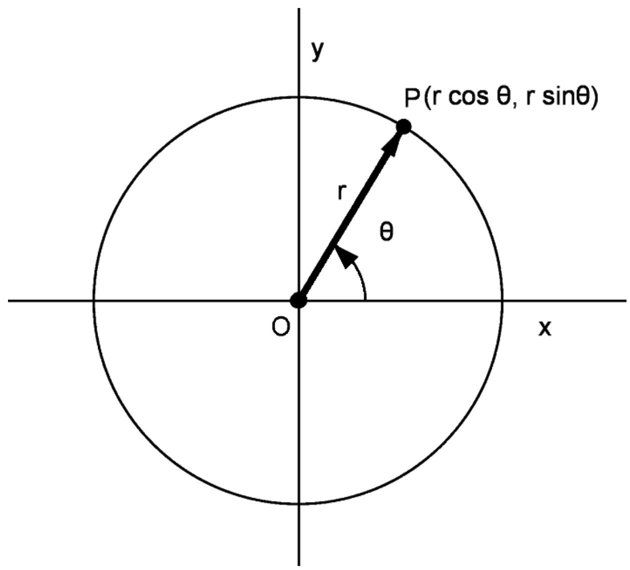


Figure 36. From position vectors to polar form.

Coxford et al. (2001b) say, “You can use the relationship between the rectangular and polar coordinates of a complex number to express a complex number in trigonometric form” (p. 496). Students were posed a series of mathematical tasks to study the geometry of addition and multiplication of complex numbers and their corresponding trigonometric forms. The connections between rotations of the plane and the multiplication of complex numbers were observed. This observed pattern is further generalized into De Moivre’s theorem.

Horizontal Variation Comparing Focal with Other Series

Brown, Dolciani, Sorgenfrey, and Kane (2000). In Brown et al. (2000), vectors in the plane were defined and basic operations with vectors were presented, such as their addition and their multiplication by a scalar. Vectors were defined as directed magnitudes without requiring a starting point. The directed nature of a vector was expressed by its bearing, which represented an angle measured clockwise from due north, and the direction of heading was counterclockwise from due north. Triangle trigonometry and the measurement of triangles were handled by vector addition without fixing them to a coordinate system. Two vectors and their sum represented a free triangle. A vector was expressed in a coordinate system by its components. The dot product of two vectors was defined and connected to the cosine function. This practice of dot product suggests that given two vectors, one can define the angle between them by its cosine.

Vector-Based Frame for Trigonometry

Vector trigonometry presents a third frame of conceptualization extending the methods of coordinates using vectors. It essentially presents an analytic trigonometry in an origin-free manner and bases it on only one reference line, not two traditional *x*- and *y*-axes. The algebra of vectors is developed from vectors as directed numbers. Vector spaces are defined from vector products. Espoused by the analytic study of coordinates, this approach began to develop during the middle 1800s while mathematicians were extending the algebra of numbers to develop a unified algebra of number pairs, a failed algebra of triplets, and an achieved algebra of quadruples (Crowe, 1967/1994). Complex numbers became a part of the algebra of doubles presenting a unified algebra of Cartesian pairs of numbers. Complex numbers and vector trigonometry entered the college textbooks during the late 1800s (Hayward, 1892).

Vector trigonometry builds toward developing a dynamic view of numbers that express a rotational symmetry. A multiplication operationally defines a unified transformation that rotates and dilates an object simultaneously. A multiplication by 1 + *i* rotates an object counterclockwise by π/4 and dilates it by $\sqrt{2}$. This way, vector trigonometry describes a spiral similarity of objects. A spiral similarity can be given by a simple formula:

$$f(z) = r(z - k)(\cos\varphi + i\sin\varphi) + k$$

where *k* is the center, *r* the real coefficient of dilation, and φ the angle of rotation.

During the new math era, technology was limited to embodying a dynamic view of rotation as a function for vectors as mathematical objects.

> You should do a lot of blackboard work here, giving a variety of simple illustrations. By using chalk of different colors, you can improve on some of the figures in the book. Show vectors in both directions, illustrate rotations followed by rotations; show the rotations of the components of the vector as the vector rotates. (SMSG, 1961e, p. 196)

Although the suggested variations in the mathematical objects were clear, the instructional technology to achieve those variations was limited, and demonstrating dynamic representations was time consuming. During the unified mathematics era, squared paper was to be used in mathematics laboratories to gather data from the physical manipulatives that could be used to derive mathematical ideas. With the emergence of digital manipulative tools during the standards period, textbooks increasingly incorporated more digital manipulatives for student experimentations with dynamic mathematical object configurations that can lead to the emergence and grounding of trigonometric mathematical objects. Another observation was that standards-based mathematics textbooks incorporated chapters on graphical animations that called for rotation matrices. The many textbooks of the standards era incorporating learning technologies included Core-Plus, IMP, and UCSMP (e.g., the *As the Cube Turns* unit on animation of a 3d rotation by Fendel et al., 2000).

Textbooks increasingly had more of an instructional technology dimension that called for the presence of graphing calculators or computer-based instructional tools during everyday classroom practice on algebra, geometry, statistics and discrete mathematics. The second editions of Core-Plus did not just call for supportive instructional technologies but provided their own software toolkit. For each activity, Core-Plus provided a corresponding digital manipulative for classroom use and investigation of the mathematical objects. The given digital sketches were to facilitate the intensive process of setting up digital manipulatives that would carefully coordinate didactical objects to accompany the textbook activity (Hirsch, Fey, Hart, Schoen, & Atkins, 2009, pp. 404, 406, 410).

The vector-based approaches to trigonometry were little incorporated during the first wave of reform of high school mathematics. Although complex numbers were present in all three reform periods, the treatment of complex numbers often failed to develop a unified understanding of complex form as an extension of a trigonometric conception developed in Cartesian form. The algebraic meaning of complex numbers as roots of higher degree equations often dominated the treatments of complex numbers at the expense of the trigonometric and geometric meaning of simple operations of complex numbers as pairs. The emphasis given to this subject during the new math period is to be applauded. During the standards-based reform era, new content such as linear algebra and transformation geometry were introduced without accounting for its deeper connections with vector trigonometry and a complex number sense.

The unified look into circle frame was presented as $\boldsymbol{R}_{\theta}(1, 0) = (\cos\theta, \sin\theta)$. A vector and complex representation refined and empowered this representation so that $\boldsymbol{R}_{\theta}(1, 0) = \cos\theta + i\sin\theta = e^{i\theta}$. This representation gave a new sense to exponential functions through De Moivre's theorem as a topic integrated often into mathematics for Grade 11.

Complex numbers with trigonometric connection should be emphasized more, and students should be exposed not only to the elegance but also to the power of vector trigonometry in expressing transformative geometry with rotational elements.

CHAPTER 7

DISCUSSION AND CONCLUSION

Trigonometry was a controversial school topic during three reforms. At the end of the first reform period, trigonometry found itself a solid place in secondary school mathematics as a triangle trigonometry. Before and during the unified reform era, triangle trigonometry was seen by many educators as a critical part of school mathematics. Making trigonometry the capstone of elementary mathematics was a major change during the unified mathematics reform period between 1893 and 1923.

Parallel debates about trigonometry can be observed in the standards-based reform period and the earlier unified mathematics and new math periods. A diversity of approaches can be observed in each reform period. It is clear from earlier reform efforts that an integrated treatment of mathematical subjects such as geometry and algebra is not a new idea. Starting with the idea of correlation, a numerous terms came to be used over the century: *unified, correlated, fusion, connected,* and *integrated*.

During the third reform period, standards-based textbooks placed more importance to mathematical experimentation and graphical methods. Most textbooks during this period incorporated an intuitive development of trigonometric functions through a numerical approach and investigation of data patterns that can be modeled by trigonometric functions, usually sinusoids. This graphical approach with analysis of data patterns also originated in the unified mathematics period with an introduction of a graphical approach in studying mathematics. The practice of using graph paper as a tool is the precursor to the use of spreadsheets and the graphing calculators in the mathematics classroom in the latter part of the century.

Those who believed in the unity of mathematical thought had always reacted against creating artificial boundaries between mathematical subjects. Trigonometry was often considered an inhibitor of the integration of many areas of mathematics such as geometry, algebra, and arithmetic. Integration of arithmetic and geometry was what Descartes set out to do 500 years ago. He made geometry a more mechanical and metrical activity (Bos, 2001). Right-triangle trigonometry also made geometry numerical. Using the structure of arithmetic and its four types of calculations, Descartes set out to classify and present geometry problems consistently. He did so by developing geometrical equivalents of arithmetical operations and "he redefined algebraic operations so as to be applicable outside the domain of numbers" (Bos, 2001, p. 154). Otte (1997) also confirms an arithmetic approach to geometry by Descartes. In a sense, Descartes developed a geometrical algebra. Ever since then, many mathematicians have developed alternative paths toward the integration of different areas of mathematics. Since early school mathematics reform efforts at the turn of the 20th century, a large body of mathematics educators have been committed to developing an integrated treatment of mathematics in school, not only integrating mathematics within itself but integrating it with other subjects.

Integrating a school mathematics subject with other subjects requires clarifying what that subject is and has been. The method of teaching is an epistemological aspect of school mathematics. The conceptual field of school trigonometry is dynamic. The body of school mathematics is subject to refinement and redefinition in different frames and practiced at different grade levels. This study has shown that there is not one school trigonometry. It has been manifested in various forms during reform periods.

During the standards-based reform period, contextual approaches were used by the reform-oriented textbooks in representing a school mathematics subject area such as trigonometry. I observed that the reform textbooks carried an inherent tension between contextual diversity and conceptual depth and complexity. This tension affected the choices made by the textbook designers when they developed variations in context and concepts across their sets of textbook problems. Contextual diversity refers to the richness of meaning with different contexts, and conceptual depth to the depth and complexity of the instructional material. Contextual diversity reflects variations in the surface structure of mathematical problems and tasks in context, whereas conceptual depth dwells on the deep structure of problems. Contextual diversity is a motivation to provide connections of the subject matter to science, other subject areas, and other contexts from occupations and real life. Mathematics curriculum developers make an ontological commitment to what mathematical knowledge and connections are to be developed in context. Conceptual depth is an inherent motivation behind providing further connections and abstractions within mathematics subjects. The crowning mathematical object for high school trigonometry is De Moivre's theorem, which requires a conceptual complexity that integrates different mathematical concepts such as complex numbers, exponents, and basic trigonometric functions. It allows further connections between e, π, and i. Developing such mathematical objects in context requires a careful coordination of tasks and their variations. Connections within and outside mathematics are developed by purposeful variations in problem contexts, facilitating manifestations of mathematical concepts in their living context. The nature of integration and connections is made explicit through such variations in reform textbooks.

The instructional materials for mathematics are designed to develop a frame and a set of tasks that leads to developing mathematical concepts. Teachers make assumptions about the logical order of concepts and the ordinal relationship of one learning object to the next in the construction of a mathematical object. If the tangent function is defined just as the ratio of sine and cosine, then the tangent can be divorced from its geometric intuitive associations as a separate entity. This preference inserts a logical order, not justified by history. The development and emergence of sine and tangent took historically different paths. Therefore, there is a split in preferences made by school trigonometry experts on introducing trigonometric tangent. Tangent is either defined as an identity or as a separate entity.

In textbook practice it is common to start with angles to define the trigonometric functions. There is no real consensus about defining trigonometric sine and treating a measure of central angle as the first object to build from. There is no right definition of trigonometric functions. The reference frame determines the forms of trigonometric objects to study. So it is impossible to get the definitions right to begin with. Although the definitions are not fixed, they are aligned across frames and reconceptualized depending on the frame.

The definition of sine as a mathematical abstraction depends on reference frame that affords and calls for its emergence. When the Euclidean form is used as a reference frame then sine is an abstraction of similarity ratio, becoming of a rational trigonometry. The notion of sense and the direction of segment are not integral to this framing. When the sense is fixed to a coordinate system, the analytic trigonometry provides the next referential frame for conceptualization of sine as a ratio.

The sine as a mathematical object is a mathematical abstraction invariant of its representations. Although learners do not have a direct access to mathematical objects, knowledge of those objects can be communicated only through their representations. As mathematical objects, sine, cosine, and sinusoidal, A sin (B*x* + C) + D, are the same. A choice of coordinate system and orientation generate an instance of the sine in its representation as a mathematical object. These can be perceived different in terms of their representations whether they are graphical or algebraic. But these representational differences are not a sufficient ground to claim that they call for different mathematical objects. The same sine wave can be perceived differently under the frame of representation. Representations can only intuit a mathematical object, they do not define it. A representation inherits a process of transformation on the abstract object so as to make it accessible. For example, the Cartesian coordinate system as a frame is a sign system that assumes an origin and an orientation for axes. Any graphical object in the Cartesian system must presume and inherit a choice, frames and fixes a mathematical object into a concrete entity. Therefore, a graphical representation of sine is not a mathematical object as a mapping of coordinates. The same sine form can have different algebraic representations. The passage and extensibility of one instance of mathematical object to the next defines the conceptual field. A mathematical object by its nature is dynamic, but its representations may be concrete and static. Invariant form across representations as an abstract structure objectifies the mathematical object such as sin *t*. Sine as an abstract object indicates a form, just as a circle indicates a form.

Constructivist approach calls for proof of existence of a mathematical object through its constructibility. The circular approach of conceptualization is a way of abstracting the mathematical objects of sine or cosine. They can be abstracted from a rotated circle or from a rotating point on a circle. But a mathematical sine can be constructed mechanically utilizing the frame of coordinates as a locus of points on rotating circular object. Then the amount of rotation carries the burden of being a measure of a transcendental construct in generating trigonometric object. Therefore practice with radian and π is an ingenious solution to envelop the conceptual complexity of trigonometric objects in bridging linear and circular and facilitating their practice in school mathematics.

The practice of effects of constants of sinusoids on their graphs provides a backward approach to trigonometric functions. The trigonometric sine as a knowledge object remains in the form that is invariant under transformations and translations. A practice with an emphasis on certain type of algebraic or graphical representations brings along their states of affairs accompanied by certain conventions and assumptions to reflect knowledge object of trigonometry. Those associated practices when unchallenged develop extra sediments to veil the essence of trigonometric ideas.

A mathematical object becomes manifest as a reification of given species of structures suggested in graphical and algebraic forms in one form or other. The mathematical object is a result of interconnection providing a special coordination of a system of a structure to form a higher structure. Trigonometric forms when they come together express congruent mathematical species. Trigonometric pairs of sine-cosine and tangent-secant, for instance, express a congruent right triangle.

Circularity and periodicity do not need to be examined through an assumption of knowledge of a center and radius. Vectors as functions with their direction cosines and sines suggest alternative ways to generate them simply. The Comprehensive School Mathematics Program (CSMP) used the turtle to generate such curves.

Reflecting on the Present—The Common Core Curriculum and Trigonometry

The Common Core State Standards Initiative (CCSSI) generated a new rigorous set of shared standards for school mathematics that states could voluntarily adopt. The CCSSI has been maintaining the progress of states' adoption on their Web site: http://corestandards.org. As of November 15, 2010, a large majority of the U.S. states had officially adopted the CCS as a school mathematics framework.

The CCS extended the process standards of NCTM (1989, 2000) by incorporating strands for mathematical proficiency identified by the National Research Council's (2001) *Adding It Up* and set eight standards for mathematical practice. Those were as follows: make sense of problems and persevere in solving them; reason abstractly and quantitatively; model with mathematics; use appropriate tools strategically; attend to precision; look for and make use of structure; look for and express regularity in repeated reasoning. The proficiency strands in *Adding It Up* were developed and exemplified especially for Grades Pre-K to 8. The CCS further expanded this approach for the practice of mathematics in high school.

The CCS proposed six content strands for high school mathematics: number and quantity, algebra, functions, modeling, geometry, and statistics and probability. The modeling strand was the most complicated. The modeling strand was treated unlike the other content strands. Rather than taking mathematical modeling as a collection of isolated topics, the writers considered it in relation to other standards. Modeling was also taken as a standard for mathematical practice. The CCS integrated the modeling standard throughout the other high school standards. Trigonometry appeared in two places in the recommendations of the CCS. One was under geometry, and the other was under functions.

Under the geometry strand, the CCS set a standard on circles for finding arc lengths and areas of sectors of circles. Although it is not listed as such in the latest form of the CCS, this topic is a highly relevant subject for trigonometry. With this standard, students are expected to derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and they are to define the radian measure of the angle as the constant of proportionality; they should also derive the formula for the area of a sector.

The treatment of trigonometry starts with the similarity of triangles. It asks that students use the definition of similarity in terms of similarity transformations of given two figures to decide if they are similar and explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

> *Define trigonometric ratios and solve problems involving right triangles.* Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. Explain and use the relationship between the sine and cosine of complementary angles. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems employing mathematical modeling.
>
> *Apply trigonometry to general triangles.* Derive the formula $A = \frac{1}{2}ab \sin C$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side. Prove the Laws of Sines and Cosines and use them to solve problems. Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles. They are applied to surveying problems, resultant forces.

Triangle trigonometry is here described mostly synthetically. Its application to resultant forces would be better if vector methods were incorporated. CCS does not account for vectors as an essential part towards building higher conceptions of trigonometric functions.

The CCS made further recommendations on trigonometric functions as a part of the functions strand. It makes core content demands on the topic of trigonometric functions. The first four core content demands use the idea of unit circle to extend the domain of trigonometric functions.

> 1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. 2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. 3. Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$ degrees, and use the unit circle to express the values of sine, cosines, and tangent for x, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number. 4. Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
>
> *Model periodic phenomena with trigonometric functions:*
>
> 5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. 6. Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. 7. Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.
>
> *Prove and apply trigonometric identities*. Prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta)=1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle. The second part of the same standard was revised from the earlier version "use it to calculate trigonometric ratios". 9. Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Although the use of trigonometric functions instead of circular functions reduces number of terms used, the recommendation on the use of unit circle in their development is limiting. During standards based reform, it is well manifested that any sized circle can be used to develop trigonometric functions especially when it is introduced in context. Another concern is that this new set of standards is weak in terms of vector trigonometry with its connections to the geometry of complex numbers.

Reflecting on the Study

If I repeated this study, I would narrow its scope. Although Schubring's (1987) methodology provided a way to approach historical analysis of textbooks, its implementation requires one to address three dimensions that proved to be very difficult for each reform period. It demanded collection of resources, various editions of series, and materials by contemporary authors. Moreover, during the analysis of documents, I found that there were alternative practices that integrated different trigonometries into 9th, 10th, and 11th and even a separate 12th grade. There were many examples of trigonometries practiced at every grade level as a part of algebra, geometry, precalculus, and statistics. I also observed that same series might present the same trigonometry content in two different grades to meet the needs of diverse groups of adopters with different grade level expectations. During the standards era, commercial textbooks presented state specific editions of the same textbook series to fit the need of textbook adoption states in the south. A popular commercial textbook series may change its content from one state to the next. This brings another level of complexity to study treatments of trigonometric functions in popular textbooks series around the states. At a given time, one series have multiple editions in practice. With the Common Core Standards, this practice may change.

I also observed that authors usually do not necessarily have a particular approach in developing trigonometry. A popular trigonometry textbook author, Michael Sullivan, developed alternative versions of his textbook-one employing right triangle approach and another employing a unit-circle approach (Sullivan, 2008; Sullivan & Sullivan III, 2009). This practice suggests a viability of case studies to explore intended and enacted practices of two different trigonometries by the same author. All these present challenges to frame a study of this kind.

Trigonometry is a challenging subject to study with its algebraic, geometric, functional connections that allow a variety of practices. This situation made it compelling to coordinate the study especially if one pursued to attend connections. Rather than defining and studying trigonometric objects by their connections, I made a choice and I kept the focus on the essential ideas of trigonometric functions for right triangle, circle, and vector frames.

Reflections and Recommendations

As established during this study, trigonometric objects do not have a single frame of reference. There is a conceptual manifold emerging that needs to account for transitions and couplings of frames of references for trigonometric objects. For future research, the context in developing mathematical concepts should be examined. I recommend a constitutive phenomenology of mathematical objects of trigonometry, where contexts are variations of frames that allows a conceptual space for a mathematical object to emerge.

The choice and the variation of the context are important determinants in developing mathematical concepts. A mathematical object is emergent from what is present as invariant across its variations in context. There is a contextual binding for intuition and manifestation of mathematical objects. Mathematical objects are categorical entities, requires an intuition of a category from individuals. An individual representation cannot be sufficient to generate an abstraction. A representation should not be confused with a mathematical concept. Multiplicities of representations lead students to experience the nature of the invariant need for the mathematical abstraction of such objects. School trigonometry frames trigonometric objects in contexts in which they can exist. The context determines how intended objects such as sine and cosine behave.

When a textbook presents a development of trigonometric functions, it lays out a path for development of a conceptual network on trigonometric functions, making implicit assumptions regarding its conceptual foundations. Textbooks as resources can set a direction for teachers to present, develop, and connect mathematical content. A lesson in a textbook can represent multiplicities of a mathematical object across a series of mathematical tasks for its meaning and application. The variations become intentional in a categorical sense when the mathematical object is intuited. Some textbooks from the standards-based era contain reflecting back portions in each lesson that focus students' attention on a categorical sense of the intended mathematical objects.

There is a strong coupling and interplay between the context and the conceptualization it affords; context influences the conception, and the conception effects how the context is viewed. When a textbook provides a problem context, that context intends to achieve the functionality of certain elements in the conceptual structure. A problem context prioritizes a conceptual need and calls for a particular conceptualization. The viable connections between conceptual elements in the conceptual structure are reconfigured to fit the need of the context. Study of sound with trigonometric functions suggests not only visual but also auditory experiences of mathematical structures emergent from the context. Explorations of additive structures of trigonometric functions were manifested by several textbooks among which COMAP's (1999) *Mathematics: Modeling Our World*. One popular example of a phenomenon emerging from additive structures of trigonometric functions is the beat phenomenon, which is manifested by a combination of slightly phased sine waves. Another additive structure is the mathematical modeling of the concept of a chord in the context of sound. The chord phenomenon becomes manifest as one of the emergent patterns from the combination of simple waves following with a mathematical intuition of the addition of basic sine forms. Additive structures of trigonometric functions lead to patterns emerging as a part of their genetic constitution in context. The connection and integration of exponential forms to sine forms is emergent when a shift is made in the context to expose the mathematical structure of dissipating simple sounds. Among other manifestations, COMAP's (1999) *Mathematics: Modeling Our World* series provided intuitive functional analysis of trigonometric objects and exemplified how contextual reasoning and mathematical reasoning can be unified by a mathematical modeling approach toward school mathematics.

The representational system for trigonometric functions can be geometric and algebraic. A concept comes with a context. A concept does not become manifest without a context. The representational framing generates affordances and constraints for mathematical connections in the conceptual space of trigonometric functions. Although a coordinate frame can afford a conceptual organization for sine and cosine as trigonometric objects, their unification is afforded by extension of the frame to the complex frame to set it free of constraints of the coordinate frame. The trigonometric functions and their intensions and extensions are affected by the mathematical context that frames the nature of conceptual field as a school mathematics domain. In developing mathematical concepts in context, a careful design of singular and sequential mathematical tasks is needed for the emergence, condensation, and extension of mathematical structures.

Triangular and circular objects are different types of objects, both of which are associated with trigonometric functions. Whether circular or triangular, objects under study generate a certain class of abstract objects primarily associated with trigonometric functions. The choice of context frames the types of abstraction that one generates. Context limits conceptualizations of trigonometric objects. Context sets a horizon for the states of affairs of trigonometric objects. A trigonometric abstraction comes with a contextual binding resulting from the assumed utility of trigonometric functions. The objects of study inherit some constraints on the level of conceptualization abstracted as trigonometric functions. For example, a problem of surveying land can be reduced to the problem of measuring triangles through triangulation. In this case, a triangle can be chosen as a natural object of study. Then triangle trigonometry as a level of conceptualization may be sufficient to solve all such surveying problems through triangulation. Trigonometric functions become manifest as tools for measuring triangles and surfaces. If trigonometric functions are needed only to study the measurement of triangles, then the level of conceptualization for trigonometric function is constrained by the concept of triangle and thereby limited by abstractions resulting from obtuse triangle cases. Length-based ratios as trigonometric functions can be sufficient in many contexts. Variations on the sides of a right triangle dominate the conception; the variation on the angle is naturally put aside in this conceptualization. The measures of angles are bounded in the analytic study of coordinates in a Cartesian setting. The objective setting for the study and implementation of trigonometric functions influences the required level of conceptualization, determines what is necessary, and sets a level for sufficiency.

The required level of conceptualization and generalization is dependent upon the context. As an example of a constraint on the conceptualization because of its framing, a typical student question might be “Why do sine and cosine have values at zero and ninety degrees?” In that case, the precondition is challenged that sine and cosine are defined in terms of a right triangle, and ratio has to deal with the notion of infinitesimal. The conceptualization of sine and cosine as ratios in right triangles goes through a stretching of the constraints of its definition, and this conceptual breaking point leads to a need for reconceptualization.

When the object of study changes to the positions of rotating circular objects, that context reprioritizes the elements of the conceptual structure formed as trigonometric functions. In a triangular conception, a measure of an angle is bounded by π. Unlike the triangular case, angle can become an a priori construct without a bound on its size if the object is rotation of or on a circular form. An analytical study of curves, such as the circle, in a polar setting makes prominent the centrality of angle, and radial distance in the conceptualization emerges at the outset. The parameterization of the circle by sine and cosine as directed half chords is usually defined in a Cartesian setting. The complexity of mathematical structures in describing circularity is a representational problem. The polar setting can simplify the representations of the functions a great deal. Two conceptions of trigonometric functions can be unified using the circle metaphor and reference right triangles to obtain a conceptual blend that can work in both cases.

Depending on the frame that puts the mathematics in a context, the trigonometric functions can be conceptualized differently. When one extends the method of trigonometric functions as employed in the analytic study of circular forms, one can generate the hyperbolic trigonometric functions by choosing a hyperbola as the object of study rather than a circle. Hyperbolic trigonometry is often incorporated into high school trigonometry as a close analogy to circular trigonometry. The trigonometric functions are essential mathematical constructs for an analytical study of basic nonlinear forms such as quadratics where the circle can be seen as their most basic form. With the circle as the simplest quadratic form, the circular and hyperbolic trigonometric functions can be generated from the quadratic forms. Elliptic trigonometric functions and hyperbolic trigonometry functions are only two cases of trigonometries that are possible depending on the analysis of the objects of study and the assumptions made during the study.

It is often that case that secondary school trigonometry attends to triangles and circles in two major curriculum units to generate associated conceptualizations of the trigonometric functions. In both cases, the trigonometric functions are based on the measurement of distance and angle. Both conceptualizations rely on Euclidean distance, which uses the square root of the sums of squares as its definition of measure. Other types of measures generate alternative unit disks instead of the circle. Taxicab trigonometry is one simple trigonometry that uses what might be called Manhattan distance instead of Euclidean distance. The University of Chicago School Mathematics Project's trigonometry also incorporates the idea of a development of alternative trigonometries as one of its project topics at the end of the trigonometric functions chapter. Based on distance and angle as primitives, their alternative conceptualizations yield different trigonometries. The conceptualization of a trigonometric function depends on its contextualization.

Another area of needed research is focusing on different grade levels and conducting a horizontal examination of other knowledge objects of study as context. This examination should reassess the objects and methods of study of trigonometry at those grade levels to see affordances and constraints. Further research needs to address how a circle trigonometry in a 10th-grade course on geometry should be considered differently than an 11th-grade one by accounting for other mathematical and relevant subjects studied. A unified framework needs to address to the treatment of complex numbers in high school as well. The case of transformation and rotation matrices is a critical content in this sense. Rotation matrices can be practiced early before introducing circular functions in the second-year high school course as in the case of *Core-Plus Mathematics* (Coxford et al., 2008) or postponed to the fourth year as in the case of *Interactive Mathematics Program* (IMP, Fendel et al., 2000). There is a variation of this practice in different levels that requires further attention. The mathematical ideas in rotation matrices can be organized by trigonometric sine and first as a parametric connected pair or coordinated and then reorganized again with the basic algebraic operations of the complex numbers.

In depth case studies are needed on lived experiences of mathematical objects of trigonometry from one frame to the next. This study revealed that sine as a mathematical object in a triangle frame is not the same sine when it was presented in circle frame or in vector frame. Extensibility of mathematical objects and their required changes of perspectives should be a subject of a special investigation. These transitions can be informative about dynamics behind the changes of perspectives and coordination of mathematical objects as shifting from triangle to circle and to vector trigonometry. Further work is needed locating and overcoming epistemological ruptures during frame changes and transfer of knowledge objects.

Douek (1999) research provides a suggestive methodology to study students' experiences with alternative trigonometries at different grade levels. Further case studies are needed to understand students' experiences with alternative trigonometries for school mathematics such as functional trigonometry and finite trigonometry (Evanovich, 1975). Elementary trigonometric connections used in school textbooks require an update to adjust to changes in connections between trigonometry and other fields. It is not sufficient to conceptualize trigonometry as a connection between geometry and arithmetic. Some linear algebra connections and statistics are already being put in standards-based mathematics textbooks. Their practices needed to be explored in the enactments and classroom practices. Other trigonometric connections to analysis, statistics, and combinatorics are weak and need to be explored further.

Reflecting back on a century of reforms on school mathematics, Kilpatrick (1996) observed that:

> Americans have tended to approach the reforming of school mathematics in a pragmatic fashion: find out what is wrong and replace it with something that works. They address changing school practice as though it were a technological problem rather than a human problem. (p. 256)

Following this thread, any reform on school trigonometry should also account for perspectives of ordinary students, ordinary teachers, as well as public. Mathematics educators and mathematicians as experts are only parts of a culture that contribute to shaping of the practice of school trigonometry. Further phenomenological studies are needed to understand the translation of the intended mathematical ideas embedded into textbooks and curriculum materials in the lived ordinary experiences of students, teachers, and parents. Otherwise, only illusions of trigonometric ideas will be reflected on the textbooks, not the ideas that can be realized in ordinary practices.

The power of a mathematical idea, the beauty of a mathematical idea, and the structure of a mathematical idea, they all become sensible with a multileveled study of trigonometry in schools. Trigonometry provides a cascaded and integrated set of ideas that can build across triangle, circle, vector frames. It gives a critical school subject that can benefit from a vertical organization of a major school theme with huge potential to organize, integrate and develop connections in mathematics and its applications.

REFERENCES

Alkin, M. C. (1973). Theoretical framework for the analysis of curriculum and instructional reform. *International Review of Education, 19*, 195–207.

Amir-Moez, A. R. (1958). Teaching trigonometry through vectors. *Mathematics Magazine, 32*(1), 19–23.

Andersen, K. (1999). Wessel's work on complex numbers and its place in history. In B Banner, &J Lützen (eds.), *Caspar Wessel: On the analytic representation of direction* (pp. 65–100). Copenhagen, Denmark: Royal Danish Academy of Sciences and Letters.

Ash (1989). Uniqueness of representation by trigonometric series. *The American Mathematical Monthly, 96*, 873–885. doi:10.2307/2324582

Ashton, C. H., & Marsh, W. R. (1908). *Plane and spherical trigonometry.* New York, NY: Scribner.

Baber, Z., Stilwell, K. M., Benignus, S., Ashléman, L. A., Myers, G. W. Atwood, W. W., Rice, E. J., & Fallersleben, H. (1901). *Course of Study, 1*, 929–939

Betz, W. (1950). Five decades of mathematics reform. *Mathematics Teacher, 43*, 377–387.

Begle, E. G. (1971). SMSG: Where we are today. In E. W. Eisner (Ed.), *Confronting curriculum reform* (pp. 68–90). Boston, MA: Little, Brown.

Betz, W., & Webb, E.W. (1912). *Plane geometry* (P. F. Smith, Ed.). Boston, MA: Ginn.

Booth, J. (1857). On the application of parabolic trigonometry to the investigation of the properties of the common catenary. *Proceedings of the Royal Society of London, 8*, 443–447.

Bos, H. J. M. (2001). *Refining geometrical exactness: Descartes' transformation of the early modern concept of construction.* New York, NY: Springer

Bourbaki, N. (2004). *Elements of mathematics: Functions of a real variable*. New York, NY: Springer.

Bowie, H. E. (1967). Teaching trigonometry. *Improving College and University Teaching, 15*, 80–83.

Boyer, C. B. (1951). The foremost textbook of modern times. *American Mathematical Monthly, 58*, 223–226. doi:10.2307/2306956

Brenke, W. C. (1910). *Advanced algebra and trigonometry.* New York, NY: Century.

Brenke, W. C. (1917). *Elements of trigonometry.* New York, NY: Century.

Brenke, W. C. (1943). *Plane and spherical trigonometry* (E. R. Smith, Ed.). New York, NY: Dreyden Press.

Breslich, E. R. (1915, 1916). *First year mathematics for secondary schools.* Chicago, IL: University of Chicago Press.

Breslich, E. R. (1917). *Third year mathematics for secondary schools.* Chicago, IL: University of Chicago Press.

Breslich, E. R. (1919). *Correlated mathematics for junior colleges.* Chicago, IL: University of Chicago Press.

Breslich, E. R. (1929, 1934). *Senior mathematics, Vol. 3*. Chicago, IL: University of Chicago Press.

Breslich, E. R. (1929, 1943). *Purposeful mathematics- Algebra – second course*. New York, NY: Laidlaw Brothers.

Breslich, E. R., Schorling, R., Wright, H. C., & Irwin, H. N. (1916). Course of study in secondary mathematics in the University High School. *School Review, 24*, 648–674. doi: 10.1086/436678

Breslich, E. R., & Stone, C. A. (1928, 1940, 1945). *Trigonometry: Plane and spherical with tables* (Rev. ed.). New York, NY: Laidlaw Brothers.

Breslich, E. R., & Stone, C. A. (1940, 1945). *Trigonometry with tables for use in senior high schools and colleges*. New York, NY: Laidlaw Brothers.

Bressoud, D. M. (2010). Historical reflections on teaching trigonometry. *Mathematics Teacher, 104*, 107–112.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des mathématiques 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Dordrecht, Netherlands: Kluwer.

Brown, S. A. (2005). The trigonometric connection: Students' understanding of sine and cosine. (Doctoral dissertation, Illinois State University). *Dissertation Abstracts International, 67*(9), 3336A. (UMI No. 3233908)

Brown, R. G., Dolciani, M. P., Sorgenfrey, R. H., & Kane, R. B. (2000). *Algebra and trigonometry: Structure and method*. Boston, MA: Houghton-Mifflin.

Brown, M. W. (2008). The place of description in phenomenology's naturalization. *Phenomenology and the Cognitive Sciences, 7*, 563–583. doi:10.1007/s11097-007-9085-8

Brown, R. G., Dolciani, M. P., Sorgenfrey, R. H., Kane, R. B., Dawson, S. K., & Nunn, B. (2010). *Algebra and trigonometry: Structure and method*. Boston, MA: Houghton-Mifflin.

Bruner, H. B., Evans, H. M., Hutchcroft, C. R., Wieting, C. M., & Wood, H. B. (1941). *What our schools are teaching*. New York, NY: Teachers College, Columbia University, Bureau of Publications.

Cajori, F. (1890). *The teaching and history of mathematics in the United States.* Washington, DC: Government Printing Office, Bureau of Education

Campbell, S. R. (2010). Embodied minds and dancing brains. In B. Sriraman, & I. English, (Eds.), *Theories of mathematics education* (pp. 309–331). New York, NY: Springer-Verlag. doi:10.1007/978-3-642-00742-2_31

Campbell, W. T. (1899). *Observational geometry.* New York, NY: American Book Company.

Cassirer, E. (1978). *The problem of knowledge.* New Haven, CT: Yale University Press.

Cavey, L. O. (2002). Growth in mathematical understanding while learning to teach right triangle trigonometry: Patterns of growth and connection building through lesson plan study (Doctoral dissertation, North Carolina State University). *Dissertations Abstracts International, 63*(5), 1754A. (UMI No. 3052748)

Cavey, L. O., & Berenson, B. (2005). Learning to teach high school mathematics: Patterns of growth in understanding right triangle trigonometry during lesson plan study. *Journal of Mathematical Behavior, 24*, 171–190. doi:10.1016/j.jmathb.2005.03.001

Chauvenet, W. (1908). *A treatise in plane and spherical trigonometry* (10th ed.). Philadelphia: J. B. Lippincott.

Chevallard, Y. (1991). La transposition didactique: Du savoir savant au savoir enseigné [The didactical transposition: From the expert knowledge to the taught knowledge] (2nd ed.). Grenoble, France: Editions La Pensée Sauvage.

College Entrance Examination Board, Commission on Mathematics. (1923). *Definition of the requirements in elementary algebra, advanced algebra, trigonometry* (Document No. 107). New York, NY: Author.

College Entrance Examination Board, Commission on Mathematics. (1959). *Program for college preparatory mathematics*. New York, NY: Author.

College Entrance Examination Board. (1983). *Academic preparation for college: What students Need to know and be able to do.* New York, NY: Author.

College Entrance Examination Board. (1985). *Academic preparation in mathematics: Teaching for transition from high school to college.* New York, NY: Author.

Collins, J. V. (1907). Home-made or inexpensive mathematical apparatus. *School Science and Mathematics, 7,* 524–528. doi:10.1111/j.1949-8594.1907.tb17563.x

Common Core State Standards Initiative. (2010). *The standards- Mathematics.* Retrieved from http://www.corestandards.org

Comprehensive School Mathematics Program. (1973). *Geometry: Similitudes, coordinates and trigonometry: Elements of mathematics, Book 0: Intuitive background* (R. Exner, Ed.). St. Louis, MO: CEMREL.

Comprehensive School Mathematics Program. (1974). *Linear algebra and geometry with trigonometry: Elements of mathematics, Book 9*. St. Louis, MO: CEMREL.

Comte, A. (1851). *The philosophy of mathematics.* New York, NY: Harper. Retrieved from http://books.google.com

Conference Board of the Mathematical Sciences (CBMS) (1983).*The mathematical sciences curriculum K-12: What is still fundamental and what is not.* Report to the NSB commission on precollege education in mathematics, science, and technology. Washington, D.C.: Author. (ERIC Document Reproduction Service No. ED225806)

Consortium for Mathematics and Its Applications (1999). *Mathematics: Modeling our world. Course 3* (S. Garfunkel, L. Godbold, & H. Pollak, Eds.). New York: W. H. Freeman.

Consortium for Mathematics and Its Applications (2000). *Mathematics: Modeling our world, Pre-Calculus.* (S. Garfunkel, L. Godbold, & H. Pollak, Eds.). New York: W. H. Freeman.

Consortium for Mathematics and Its Applications (2002). *Mathematics: Modeling our world, Pre-Calculus.* (S. Garfunkel, L. Godbold, & H. Pollak, Eds.). New York: W. H. Freeman.

Corry, L. (1998). The origins of eternal truth in modern mathematics: Hilbert to Bourbaki and beyond. *Science in Context, 12,* 137 –183.

Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Burrill, G., Hart, E. W., et al. (1997). *Contemporary mathematics in context: A unified approach. Course 3a*. Chicago, IL: Everyday Learning.

Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Hart, E. W., Keller, B. A., et al. (2001a). *Contemporary mathematics in context: A unified approach. Course 4a*. Chicago, IL: Everyday Learning.

Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Hart, E. W., Keller, B. A., et al. (2001b). *Contemporary mathematics in context: A unified approach. Course 4b*. Chicago, IL: Everyday Learning.

Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Burrill, G., Hart, E. W., et al. (2008). *Contemporary mathematics in context: A unified approach. Course 2* (2nd ed.). New York, NY: Glencoe/McGraw-Hill.

Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Burrill, G., Hart, E. W., et al. (2009). *Contemporary mathematics in context: A unified approach. Course 3* (2nd ed.). New York: Glencoe/McGraw-Hill.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage.

Crowe, M. J. (1967/1994). *A history of vector analysis: the evolution of the idea of a vectorial system.* Mineola, NY: Dover.

De Carvalho, J. P. (2006). A turning point in secondary school mathematics in Brazil: Euclides Roxo and the mathematics curricular reforms of 1931 and 1942. *The International Journal for the History of Mathematics Education, 1*, 69–86.

De Morgan, A. (1837). *Elements of trigonometry and trigonometrical analysis.* London: Taylor, & Walton.

De Morgan, A. (1845). On the almost total disappearance of the earliest trigonometrical canon. *Monthly Notices of the Royal Astronomical Society, 6*, 221–228.

De Morgan, A. (1849). *Trigonometry and double algebra.* London, UK: Taylor, Walton, & Maberly. Retrieved from http://books.google.com

Dexter, E. G. (1906). Ten years' influence of the report of the Committee of Ten. *The School Review, 14*, 254–269. doi:10.1086/434806

Derrida, J. (1989). *Edmund Husserl's Origin of Geometry* (J. Leavey, trans.). Lincoln, NE: University of Nebraska Press.

Dolciani, M. P., Berman, S. L., Wooton, W., & Meder, A. E. Jr. (1963). *Modern algebra and trigonometry: Structure and method*. Boston, MA: Houghton-Mifflin.

Donoghue, E. F. (2003). Algebra and geometry textbooks in twentieth-century America. In G. M. A. Stanic & J. Kilpatrick (Eds.), *A history of school mathematics* (Vol. 1, pp. 329–398). Reston, VA: National Council of Teachers of Mathematics.

Doty, D., & Harris, W. T. (1874). *A statement of the theory of education in the United States of America as approved by many leading educators.* Washington, DC: Government Printing Office. Retrieved from http://books.google.com/books

Douek, N. (1999). Argumentation and conceptualization in context: A case study on sun shadows in primary school. *Educational Studies in Mathematics, 39*, 89–110. doi:10.1023/A:1003800814251

Duval, R. (2000 July). Basic issues for research in mathematics education. In T. Nakahara, & M. Koyama, (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (July 23–27). Hiroshima, Japan: Hiroshima University (ERIC Document Reproduction Service No. ED466737)

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131. doi: 10.1007/s10649-006-0400-z

Eberlein, W. F. (1966). Circular function(s). *Mathematics Magazine, 39,* 197–201. doi:10.2307/2688079

Educational Services Inc. (ESI) (1963). *Goals for school mathematics: The report of the Cambridge Conference on School Mathematics.* Boston, MA: Houghton Mifflin.

Elliot, P. (2004). "Improvement, always and everywhere": William George Spencer (1790-1866) and mathematical, geographical and scientific education in 19th-century England. *History of Education, 33,* 391–417.

Ettlinger, H. J. (1920). An introduction to plane trigonometry by graphical methods. *The American Mathematical Monthly, 27,* 63–65. doi:10.2307/2973167

Evanovich, G. P. (1975). A finite mathematical structure for students of secondary trigonometry: A study of the effect upon achievement of students utilizing the finite structure as an introduction to trigonometry (Doctoral dissertation, Temple University). *Dissertations Abstracts International*, *36*(6), 3475A. (UMI No. 7528108)

Fehr, H. F. (1976). Toward a unified mathematics curriculum for the secondary school. A report of the origin, work and development of Unified Mathematics (Report No. NSF-PES-69-0167). National Science Foundation, Washington, DC: Bureau of Research. (ERIC Document Reproduction Service No. ED129630).

Fendel, D., Resek, D., Alper, L., & Fraser, S. (2000). *Interactive mathematics program: Integrated high school mathematics, Year 4.* Berkeley, CA: Key Curriculum Press.

Fendel, D., Resek, D., Alper, L., & Fraser, S. (2010). *Interactive mathematics program: Integrated high school mathematics, Year 3* (2nd ed.). Berkeley, CA: Key Curriculum Press.

Fey, J. T. (1978). Change in mathematics education since the late 1950s – Ideas and realisation: U.S.A. An ICMI report. Part 2. *Educational Studies in Mathematics*, *9*(3), 339–353. doi:10.1007/BF00241036

Finkel, B. F. (1907). Biography of George Albert Wentworth. *School Science and Mathematics, 7*, 485–488. doi: 10.1111/j.1949-8594.1907.tb17553.x

Flint, A. (1838). *A system of geometry and trigonometry.* Hartford, CT: Belknap & Hamersley. Retrieved from http://books.google.com

Freeman, E. M. (1932). Textbook trends in plane geometry. *The School Review, 40,* 282–294.

Freudenthal, H. (1991). *Revisiting mathematics education.* Boston, MA: Kluwer.

Fujita, T., & Jones, K. (2003). The place of experimental tasks in geometry teaching: learning from the textbook designs of the early 20th century. *Research in Mathematics Education*, 5(1&2), 47–62.

Gelfand, I. M., & Saul, M. (2001). *Trigonometry*. Boston, MA: Birkhäuser.

Gibb, E. G., Karnes, H. T., & Wren, F. L. (1970). The modern era: 1945–present. In P. S. Jones & A. Coxford (Eds.), *A history of mathematics education in the United States and Canada* (1970 Yearbook of the NCTM, pp. 327–350). Reston, VA: NCTM.

Glass, E. (1993). From form to function: A reassessment of Felix Klein's unified programme of mathematical research, education and development. *Studies in History and Philosophy of Science, 24,* 611–631.

Gluchoff, A. D. (1994, February). Trigonometric series and theories of integration. *Mathematics Magazine, 67*(1), 3–20.

Greene, L. E. (1935). A century of progress in secondary school mathematics 1834–1934. *Peabody Journal of Education, 12,* 220–232. doi:10.1080/01619563509535268

Griffin, F. L. (1915). An experiment in correlating freshman mathematics. *American Mathematical Monthly, 22,* 325–330. doi:10.2307/2974264

Gustafson, K. (2002). Operator trigonometry of statistics and economics. *Linear Algebra and Its Applications, 354,* 141–158. doi:10.1016/S0024-3795(01)00315-9

Gustafson, K. (2006). Non commutative trigonometry. *Operator theory: Advances and Applications, 167*, 127–155.

Gray, J. J., & Rowe, D. E. (1993). Felix Klein at Evanston: Learning, teaching, and doing mathematics in 1893. *For the Learning of Mathematics, 13*(3), 31–36.

Fulling, S. A. (2005). How to avoid the inverse secant (and even the secant itself). *College Mathematics Journal, 36,* 381 –387. doi:10.2307/30044889

Fleury, N., Detraubenberg, M.R., & Yamaleev, R.M. (1993). Commutative extended complex numbers and connected trigonometry. *Journal of Mathematical Analysis and Applications, 180,* 431–457. doi:10.1006/jmaa.1993.1410

Hardy, G. H. (1908). *A course of pure mathematics*. Cambridge, UK: Cambridge University Press.

Hart, W. L. (1961). *Modern plane trigonometry.* Boston, MA: Heath.

Hart, W. L. (1942). *Plane trigonometry, solid geometry, and spherical trigonometry.* Boston, MA: Heath.

Hartimo, M. H. (2008). From geometry to phenomenology. *Synthese, 162,* 225–233. doi:10.1007/s11229-007-9177-6

Hamming, R. W. (1945). The transcendental character of cos *x*. *American Mathematical Monthly 52,* 336-337. doi:10.2307/2305295

Hayden, R. W. (1982). A history of the "new math" movement in the United States. (Doctoral dissertation, Iowa State University). *Dissertation Abstracts International, 42*(11), 4753A. (UMI No. 8209127)

Hayward, R. B. (1892). *The algebra of coplanar vectors and trigonometry*. New York, NY: Macmillan. Retrieved from http://archive.org/

Hedrick, E. R. (1917). The significance of mathematics. *American Mathematical Monthly, 24,* 401–406. doi:10.2307/2972761

Hedrick, E. R. (1938). The function concept in elementary teaching and in advanced mathematics. *American Mathematical Monthly, 45,* 448–455. doi:10.2307/2304151

Herbart, J. F. (1895). *The science of education.* (H. M. Felkin, & E. Felkin, Trans.). Boston, MA: Heath.

Herbart, J. F. (1903). Herbart's ABC of sense-perception and minor pedagogical works. In W.T. Harris (Series Ed.) & W. J. Eckoff (Trans.), *International Education Series: Vol. 36.* New York, NY: D. Appleton. (Original work published 1802)

Hill, C. O. (1997). Did George Cantor influence Edmund Husserl? *Synthese*, *113*, 145–170. doi: 10.1023/A:1005099615326

Hirsch, C. R., Weinhold, M., & Nichols, C. (1991). Trigonometry today. *Mathematics Teacher, 84*(2), 98–106.

Hirsch, C. R. (1991). Teaching sensible mathematics in sense-making ways with the CPMP. *Mathematics Teacher*, *88,* 694–700.

Hirsch, C. R. (Ed.) (2007). *Perspectives on the design and development of school mathematics curricula.* Reston, VA: National Council of Teachers of Mathematics.

Howard, L. (1800). On a periodical variation of the barometer, apparently due to the influence of the sun and moon on the atmosphere. *Philosophical Magazine Series 1*(7), 355–363. doi: 10.1080/14786440008562596

Huemer, W., & Landerer, C. (2010). Mathematics, experience and laboratories: Herbart's and Brentano's role in the rise of scientific psychology. *History of the Human Sciences,* 23(3), 72–94. doi: 10.1177/0952695110363639)

Husserl, E. (2003). *Philosophy of arithmetic* (Trans. by Dallas Willard). Dordrecht: Kluwer.

Judd, C. H. (1915). *Psychology of high school subjects.* Boston, MA: Ginn.

Kang, W. (1990). Didactical transposition of mathematical knowledge in textbooks. (The University of Georgia, Athens, GA). *Dissertation Abstracts International, 52*(1), 104A, 1991. Retrieved from ProQuest Digital Dissertations (AAT 9117305)

Kang, W., & Kilpatrick, J. (1992, February). Didactical transposition in mathematics textbooks. *For the Learning of Mathematics, 12*(1), 2–7.

Karpinski, L. C. (1939). A problem of presentation in trigonometry. *National Mathematics Magazine, 13,* 240–241. doi:10.2307/3028655

Katz, V. J. (1987). The calculus of the trigonometric functions. *Historia Mathematica, 14,* 311–324. doi:10.1016/0315-0860(87)90064-4

Katz, V. J. (1998). A history of mathematics (2nd ed.). New York, NY: HarperCollins.

Kaufman, B. A., & Steiner, H. G. (1969). The CSMP approach to a content-oriented, highly individualized mathematics education. *Educational Studies in Mathematics, 1,* 312–326. doi:10.1007/BF00558316

Kendal, M. & Stacey, K. (1996). Trigonometry: Comparing ratio and unit circle methods. In P. Clarkson, (Ed.), *Technology in mathematics education* (Proceedings of the 19th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 322–329). Melbourne, Australia: MERGA.

Kendal, M., & Stacey, K. (1997). Teaching trigonometry. *Vinculum, 34*(1), 4–8.

Kenyon, A. M., & Ingold, L. (1914). *Trigonometry* (E. R. Hedrick, Ed.). New York, NY: Macmillan.

Kenyon, A. M., & Ingold, L. (1921). *Elements of Plane Trigonometry* (E. R. Hedrick, Ed.). New York, NY: Macmillan.

Kilpatrick, J. (1996). Réformer les programmes de mathématiques aux U.S.A. depuis 1900: realité et imaginaire [The reform of the U.S. School mathematics programs since 1900: Reality and imagination]. In B. Belhoste, H. Gispert & N. Hulin (Eds.), *Les sciences au lycée: Un siècle de réformes des mathématiques et de la physique en France et à l'étranger* (pp. 247–258). Paris: Vuibert.

Kilpatrick, J. (1997). Confronting reform. *American Mathematical Monthly, 104*, 955–962. doi:10.2307/2974478

Kilpatrick, J. (2008). Five lessons from the new math era. *New York State Mathematics Teachers' Journal, 58*(3), 87–90.

Kilpatrick, J. (2009). The social efficiency movement in the United States and its effects on school mathematics. In K. Bjarnadóttir, F. Furinghetti, & G. Schubring (Eds.), "*Dig where you stand": Proceedings of the conference On-going Research in the History of Mathematics Education* (pp. 113–122). Reykjavík, Iceland: University of Iceland, School of Education.

Klein, F. (2009). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis.* New York, NY: Cosimo. (Original work published 1906, translated 1932) Retrieved from http://books.google.com

Kolen, F. (2005). An interpretation of Husserl's concept of constitution in terms of symmetry. In A. T. Tymieniecka (Ed.), *Analecta Husserliana, 88*, 307–316. doi:10.1007/1-4020-3680-9_15

Lardner, D. (1828). *An analytical treatise on plane and spherical trigonometry and the analysis of angular sections* (2nd ed.). London: John Taylor.

Lauer, J.Q. (1979). *Triumph of subjectivity: An introduction to transcendental phenomenology* (2nd ed.). New York, NY: Fordham University Press.

Long, E., & Brenke, W. C. (1913). *Correlated mathematics for secondary schools: Algebra – First course.* New York, NY: Century.

Long, E., & Brenke, W. C. (1916). *Plane geometry.* New York, NY: Century.

Longley, W. R. (1927 April). [Review of the book *The Work of the College Entrance Examination Board 1901–1925*]. *American Mathematical Monthly, 34*, 206–208. doi:10.2307/2299869

Lunn, A. C. (1908). The foundations of trigonometry. *Annals of Mathematics, Series 2, 10*(1), 37–45. doi:10.2307/1967303

Macfarlane, A. (1894). *On the definitions of the trigonometric functions.* Boston, MA: Norwood.

Maor, E. (2002). *Trigonometric delights.* Princeton, NJ: Princeton University Press.

Markel, W. D. (1982). Trigonometry–forgotten and abused? *School Science and Mathematics, 82*, 548–551. doi: 10.1111/j.1949-8594.1982.tb10055.x

Members of the Department of Mathematics of the University High School of the University of Chicago. (1940). *Mathematics instruction in the University High School.* Chicago, IL: University of Chicago.

Menger, K. (1945). Methods of presenting *e* and *π*. *American Mathematical Monthly, 52,* 28–33. doi:10.2307/2304832

Menger, K. (1961). A counter of Occam's Razor in pure and applied mathematics: Semantic uses. *Synthese, 13,* 331–349. doi:10.1007/BF00486631

Menger, K. (1971).The geometry relevant to modern education. *Educational Studies in Mathematics, 4,* 1–17. doi: 10.1007/BF00305793

Middleton, G. (1911). The text-book game and its quarry. *The Bookman, 33,* 146–147.

Miller, G. A. (1925). Arithmetization in the history of mathematics. *Proceedings of the National Academy of Sciences of the United States of America, 11,* 546–548.

Moore, H. E. (1903). On the foundations of mathematics. *Bulletin of American Mathematical Society, 9,* 402–424. doi:10.1090/S0002-9904-1903-01007-6

Moore, E. H. (1906). The cross-section paper as a mathematical instrument. *School Review, 14,* 317–338. doi:10.1086/434821

Moritz, R. E. (1908). On the definition and scope of plane trigonometry. *School Science and Mathematics, 8,* 392–399. doi:10.1111/j.1949-8594.1908.tb01200.x

Moustakas, C. E. (1994). *Phenomenological research methods*. Thousand Oaks, CA: Sage.

Myers, G. E., Wickes, R., Wreidt, E. A., Breslich, E. R., Dresden, A., Caldwell, E. L., & Mathews, R. M. (1910). *Second year mathematics for secondary schools.* Chicago, IL: University of Chicago Press.

National Advisory Committee on Mathematical Education (NACOME). (1975). *Overview and analysis of school mathematics: Grades K–12.* Washington, DC: Conference Board of the Mathematical Sciences.

National Education Association. (1920). The problem of mathematics in secondary education – A report of the Commission on the Reorganization of Secondary Education. *Department of Interior Bureau of Education Bulletin, 1* (pp.1–24). Washington, DC: Government Printing Office. Retrieved from http://books.google.com

National Committee on Mathematical Requirements (NCMR). (1922). The reorganization of mathematics in secondary education- A summary of the report by the National Committee on Mathematical Requirements. *Department of Interior Bureau of Education Bulletin, 32* (pp. 1–70).Washington, DC: Government Printing Office. Available from http://archive.org.

National Committee on Mathematical Requirements (NCMR). (1923). *The reorganization of mathematics in secondary education*. Oberlin, OH: Mathematical Association of America.

National Council of Teachers of Mathematics. (1940). *The place of mathematics in secondary education* (15thYearbook of the NCTM). Reston, VA: Author.

National Council of Teachers of Mathematics. (1988). *Curriculum and evaluation standards for school mathematics: Overview*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Education Association (NEA). (1894). *Report of the Committee of Ten on secondary school studies* (pp. 104–116). New York, NY: American Book Company.

National Education Association. (1899). *Report of the committee on college entrance requirements*. Chicago: Author. Retrieved from http://books.google.com

National Education Association. (1912). *Final report of the national committee of fifteen on geometry syllabus*. Chicago, IL: University of Chicago Press.

National Education Association. (1918). *Cardinal principles of secondary education: A report of the Commission on the Reorganization of Secondary Education.* Washington, DC: U.S. Government Printing Office.

National Education Association. (1920). The problem of mathematics in secondary education – A report of the Commission on the Reorganization of Secondary Education. *Department of Interior Bureau of Education Bulletin, 1* (pp.1–24). Washington, DC: Government Printing Office. Retrieved from http://books.google.com

Newsom, C. V., & Randolf, J. F. (1946, February). Trigonometry without angles. *Mathematics Teacher, 39,* 66–68.

Norton, A. (1987). *Learning mathematics.* Philadelphia, PA: Cassel Educational.

Nyberg, J. A. (1916). The unification of freshman mathematics. *American Mathematical Monthly, 23,* 101–106.

Olmsted, J. M. H. (1945). Rational values of trigonometric functions. *American Mathematical Monthly,* 52, 507 –508.

O'Toole, A. L. (1939). An approach to trigonometry. *National Mathematics Magazine, 13,* 373–375. doi:10.2307/3028490

Otte, M. (1997, February). Mathematics, semiotics, and the growth of social knowledge. *For the Learning of Mathematics, 17*(1), 47–54.

Palmer, E. G. (1912). History of the graph in elementary algebra in the United States. *School Science and Mathematics, 12*, 692–693. doi: 10.1111/j.1949-8594.1912.tb04744.x

Palmer, C. I., & Leigh, C. W. (1916). *Plane and spherical trigonometry* (2nd ed.). New York, NY: McGraw-Hill.

Palmer, C. I., & Leigh, C. W. (1934*). Plane and spherical trigonometry* (4th ed.). New York, NY: McGraw-Hill.

Peacock, G. (1830). *A treatise of algebra* (Vol. 1). London, UK: F. J. Rivington &Whittaker.

Peacock, G. (1845). *A treatise of algebra – Symbolic algebra applied to geometry of position* (Vol. 2). London, UK: F. J. Rivington &Whittaker.

Perry, J. (1900). The teaching of mathematics. In J. Perry (Ed.), *England's neglect of science* (pp. 44–57). London, UK: T. Fisher Unwin. Retrieved from http://books.google.com

Perry, J. (1902). *Discussion on the teaching of mathematics*. London, UK: Macmillan. Retrieved from http://books.google.com

Picken, D. K. (1946). Sign, and elementary vector Ideas, in plane analytical geometry and trigonometry. *Mathematical Gazette, 30,* 200–208. doi:10.2307/3611218

Pinar, W. F., Reynolds, W. M., Slattery, P., & Taubman, P. M. (1995). *Understanding curriculum – An introduction to the study of historical and contemporary curriculum discourses*. New York, NY: Peter Lang.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, *26*, 165–190. doi:10.1007/BF01273662

Popkewitz, T. S. (1988). Institutional issues in the study of school mathematics: Curriculum research. *Educational Studies in Mathematics, 19*, 221–249. doi:10.1007/BF00751234

Progressive Education Association. (1940). *Mathematics in general education.* New York, NY: D. Appleton-Century.

Quast, W. G. (1968). Geometry in the high schools of the United States: An historical analysis from 1890 to 1966. (Doctoral dissertation, Rutgers University). *Dissertation Abstracts International, 28*(12), 4888B. (UMI No. 689162)

Reeder, H. P. (1986). *Theory and practice of phenomenology.* New York, NY: University Press of America.

Reeve, W. D. (1929). United States. In W. D. Reeve (Ed.), *Significant changes and trends in the teaching of mathematics throughout the world since 1910* (4th Yearbook of the National

Council of Teachers of Mathematics, pp. 131–186) Reston, VA: NCTM. (ERIC Document Reproduction Service No. ED096165)

Reys, B., Reys, R., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership*, *61*(5), 61–66.

Roberts, D. L., & Walmsley, A. L. E. (2003). The original new math: Storytelling versus history. *Mathematics Teacher*, *96*, 468–473.

Rota, G. C., Sharp, D. H., & Sokolowski, R. (1988). Syntax, semantics, and the problem of the identity of mathematical objects. *Philosophy of Science*, *55*(3), 376–386. doi:10.1086/289442

Rosenberg, H. (1958). The changing concept of trigonometry as a school subject. *Mathematics Teacher, 51*, 240–245.

Roth, W-M., & Thom, J. S. (2009). Bodily experience and mathematical conceptions: From classical views to a phenomenological reconceptualization. *Educational Studies in Mathematics*, *70*, 175–189. doi: 10.1007/s10649-008-9138-0

Röhrs, H. (1997). Progressive education in the United States and its influence on related educational developments in Germany. *Paedagogica Historica*, *33*, 45–68. doi: 10.1080/0030923970330103

Rugg, H. O., & Clark, J. R. (1918). *Scientific method in the reconstruction of ninth grade mathematics* (Supplementary Educational Monographs, Vol.2, No.1). Chicago, IL: University of Chicago Press.

Rugg, H. O., & Clark, J. R. (1919). *Fundamentals of high school mathematics.* Chicago, IL: World Book Company.

Rugg, H. O. (1975). Curriculum-making and the scientific study of education since 1910. *Curriculum Theory Network, 4*, 295–308. doi:10.2307/1179267

Russell, B. (1917). *Mysticism and logic*. London, UK: George Allen & Unwin.

Safford, T. (1896). *Mathematics teaching and its modern methods.* Boston, MA: Heath.

Schmidt,W. H., McKnight, C. C., & Raizen, S. A. (1997). *A splintered vision: An investigation of U.S science and mathematics education*. Norwell, MA: Kluwer.

Schoenfeld, A.H., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp.55-176). Hillsdale, NJ: Erlbaum.

School Mathematics Study Group. (1961a). *A first course in algebra. Teacher's commentary, Part 1, Unit 11*. New Haven, CT: Yale University Press. (ERIC Document Reproduction Service No. ED135619)

School Mathematics Study Group. (1961a). *A first course in algebra. Teacher's commentary, Part 1, Unit 12*. New Haven, CT: Yale University Press. (ERIC Document Reproduction Service No. ED135620)

School Mathematics Study Group. (1961c). *Geometry. Student's Text, Part 2, Unit 14*. New Haven, CT: Yale University Press. (ERIC Document Reproduction Service No. ED135622)

School Mathematics Study Group. (1961d). *Intermediate mathematics. Teacher's commentary, Part 2, Unit 20*. New Haven, CT: Yale University Press. (ERIC Document Reproduction Service No. ED135628)

School Mathematics Study Group. (1961e). *Elementary functions, Student's text, Unit 21*. New Haven, CT: Yale University Press. (ERIC Document Reproduction Service No. ED135629)

School Mathematics Study Group. (1964). *Supplementary and enrichment series: Functions and circular function* (Ed. R. Dubinsky). Stanford, CA: Author.

School Mathematics Study Group. (1965a). *Elementary functions*, *Teacher's commentary*, *Unit 22* (Rev. ed.). Stanford, CA: Author. (ERIC Document Reproduction Service No. ED 135630)

School Mathematics Study Group. (1965b). *Geometry. Student's Text, Part 2, Unit 48* (Rev. ed.). Stanford, CA: Author.

School Mathematics Study Group. (1965c). *Analytic geometry. Student's Text, Part 2, Unit 64* (Rev. ed.). Stanford, CA: Author.

School Mathematics Study Group. (1965d). *Philosophies and procedures of SMSG writing teams*. Stanford, CA: Author. (ERIC Document Reproduction Service No. ED130879)

Schorling, R. (1917). Significant movements in secondary school mathematics. *Teachers College Record*, *18*, 438–457.

Schorling, R., & Reeve, W. D. (1919). *General mathematics 1*. Boston, MA: Ginn. Retrieved from http://books.google.com

Schubring, G. (1986). Ruptures dans le statut mathématique des nombres négatifs [Ruptures in the mathematical status of negative numbers]. *Petit*, *10*(12), 5–32.

Schubring, G. (1987). On the methodology of analysing historical textbooks: Lacroix as textbook author. *For the Learning of Mathematics*, *7*(3), 41–51.

Schubring, G. (2001). Argand and the early work on graphical representation. In J. Lutzen (Ed.) *Around Caspar Wessel and the geometric representation of complex numbers* (pp. 125–146). Copenhagen, Denmark: Royal Danish Academy of Sciences and Letters.

Schubring, G. (2006). Researching into the history of teaching and learning mathematics: The state of the art. *Paedagogica Historica*, *42*, 665–677. doi:10.1080/00309230600806757

Secondary School Mathematics Curriculum Improvement Study. (1970). *Unified modern mathematics, Course 3, Part 2.* New York, NY: Teachers College Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36. doi:10.1007/BF00302715

Sigurdson, S. E. (1962). The development of the idea of unified mathematics in the secondary school curriculum: 1890–1930 (Doctoral dissertation, University of Wisconsin—Madison). *Dissertation Abstracts International*, *23*(6), 1997. (UMI No. 6300616)

Simms, E. M. (2005). Goethe, Husserl, and the crisis of the European sciences. *Janus Head, 8,* 160–172

Smith, D. E. (1907). The preparation of the teacher of mathematics in secondary schools. *School Science and Mathematics, 7,* 247–253. doi:10.1111/j.1949-8594.1907.tb01013.x

Smith, D. E. (1915, December). [Review of the book *First-Year Mathematics for Secondary Schools*, by E. R. Breslich]. *Bulletin of the American Mathematical Society*, *22*(3), 136–139. doi:10.1090/S0002-9904-1915-02739-4

Sitomer, H. (1964). Coordinate geometry with an affine approach. *Mathematics Teacher*, *57,* 404–405.

Spencer, W. G. (1860). *Inventional geometry.* London, UK: J. & C. Mozley, Paternoster Row.

Spencer, W. G. (1886). *Inventional geometry.* New York, NY: D. Appleton.

Stanic, G. M. A. (1986). The growing crisis in mathematics education in the early twentieth century. *Journal for Research in Mathematics Education, 17,* 190–205. doi:10.2307/749301

Stanic, G. M. A. (1987). Mathematics education in the United States at the beginning of the twentieth-century. In T. S. Popkewitz (Ed.) *The formation of school subjects: The struggle for creating an American institution* (pp. 145–175). New York, NY: Falmer.

Stanic, G. M. A., & Kilpatrick, J. (1992). Mathematics curriculum reform in the United States: A historical perspective. *International Journal of Educational Research, 17*, 407–417. doi:10.1016/S0883-0355(05)80002-3

Stengel, B. S. (1997). "Academic discipline" and "school subject": Contestable curricular concepts. *Journal of Curriculum Studies, 29,* 585–602. doi:10.1080/002202797183928

Story, W. E. (1903). The unification of mathematics in school curriculum. *School Review, 11,* 832–855. doi:10.1086/434542

Stout, J. E. (1921). *The development of high-school curricula in the North-Central states from 1860 to 1918* (Supplementary Educational Monographs, Vol. 3, No.3). Chicago, IL: University of Chicago Press.

Strand, A., & Stein, F. M. (1962). Quasi-trigonometry. *American Mathematical Monthly, 69,* 143–147. doi:10.2307/2312548

Sullivan, M. (2008). Trigonometry: A unit circle approach (8th ed.). Upper Saddle River, NJ: Prentice Hall.

Sullivan, M., & Sullivan III, M. (2009). *Trigonometry: A right triangle approach* (5th ed.). Upper Saddle River, NJ: Prentice Hall.

Swenson, J. A. (1931). Calculus in the high school. *Junior-Senior High School Clearing House, 5*, 347–349.

Swenson, J. A. (1934). *Integrated mathematics with special application to geometry* (Vol. 2). Ann Arbor, MI: Edwards Brothers.

Swenson, J. A. (1935a). *Integrated mathematics with special application to analysis* (Vol. 3). Ann Arbor, MI: Edwards Brothers.

Swenson, J. A. (1935b). *Integrated mathematics with special application to calculus* (Vol.4). Ann Arbor, MI: Edwards Brothers.

Szabo, S. (1969). *Vector trigonometry for secondary schools* (Doctoral dissertation, University of Illinois at Urbana–Champaign). *Dissertation Abstracts International, 30*(7), 2733A. (UMI No. 7000998)

Szabo, S. (1971). *The trigonometry of sensed angles: An analogue to the circular functions*. Retrieved from ERIC database. (ED047982)

Szabo, S. (1973). A vector approach to Euclidean geometry. In K. B. Henderson (Ed), *Geometry in the mathematics curriculum* (NCTM Thirty-Sixth Yearbook, pp. 232–302). Reston, VA: National Council of Teachers of Mathematics.

Swenson, J. A. (1935). *Integrated mathematics with special application to analysis.* Ann Arbor, MI: Edwards Brothers.

Thielman, H. P. (1937). A generalization of trigonometry. *National Mathematics Magazine, 11*, 349–351. doi:10.2307/3028764

Thom, R. (1972). Modern mathematics: Does it exist? In A. G. Howson (Ed.), *Developments in mathematical education*, Proceedings of ICMI 2 (pp. 194–209). Cambridge, UK: Cambridge University Press.

Thielman, H. P. (1937). A generalization of trigonometry. *National Mathematics Magazine, 11*, 349–351. doi:10.2307/3028764

Thompson, K., & Dray, T. (2000). Taxicab angles and trigonometry. *Pi Mu Epsilon Journal, 11*, 87–97.

Thompson, K. A. (2007). Students' understanding of trigonometry enhanced through the use of a real world problem: Improving the instructional sequence. (Doctoral dissertation, Illinois State University). *Dissertation Abstracts International, 68*(9) A. (UMI No. 3280913)

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Joint 32nd Conference of the International Group for the Psychology of Mathematics Education and the 30th Meeting of North American Chapter* (Vol. 1, pp. 45–64). Morélia, México: Cinvestav-UMSNH.

Tieszen, R. (1992). Kurt Gödel and phenomenology. *Philosophy of Science, 59*, 176–194. doi:10.1086/289661

Tieszen, R. (2005a). Free variation and the intuition of geometric essences. *Philosophy and Phenomenological Research*, *70*, 153–173. doi: 10.1111/j.1933-1592.2005.tb00509.x

Tieszen, R. (2005b). *Phenomenology, logic, and the philosophy of mathematics.* Cambridge, UK: Cambridge University Press.

Treffers, A. (1993). Wiskobas and Freudenthal: Realistic Mathematics Education. *Educational Studies in Mathematics, 25*, 89-108.

University of Chicago School Mathematics Project. (UCSMP) (1998). *Functions, statistics and trigonometry Teacher's edition* (2nd ed.). Glenview, IL: Scott Foresman Addison Wesley.

Usiskin, Z. (2001). Teachers' mathematics: A collection of content deserving to be a field. *The Mathematics Educator*, *6*(1), 86–98. (Published by Association of Mathematics Educators in Singapore)

Valverde, G.A., Bianchi, L.J., Wolfe, R.G., Schmidt, W.H. & Houang, R.T. (2002). *According to the Book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Dordrecht, the Netherlands: Kluwer Academic Publishers.

Van Brummelen, G. (2009). *The mathematics of the heavens and the earth: The early history of trigonometry.* Princeton, NJ: Princeton University Press.

Van Brummelen, G. (2010). Filling in the short blanks: musings on bringing the historiography of mathematics to the classroom. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 25(1), 2–9. doi: 10.1080/17498430903321125

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in Realistic Mathematics Education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*, 9–35. doi:10.1023/B:EDUC.0000005212.03219.dc

Van der Ploeg, A. J. (2001). *K-12 mathematics education in Ohio: What districts intend to teach, what teachers teach.* Cleveland, OH: Ohio Mathematics and Science Coalition.

Van der Waerden, B. L. (1931). *Modern algebra.* Berlin, Germany: Verlag-Springer.

Vaughan, H. E. (1955). Characterization of the sine and cosine. *American Mathematical Monthly, 62,* 707–713. doi: 10.2307/2307075

Vaughan, H. E. (1971). Development of Euclidean geometry in terms of translations. *Educational Studies in Mathematics, 4*, 104–110. doi:10.1007/BF00305801

Vaughan, H. E., & Szabo, S. (1971). *Vector spaces and affine geometry, Vol. 1: A vector approach to Euclidean geometry*. New York, NY: Macmillan. (ERIC Document Reproduction Service No. ED186276)

Vaughan, H. E., & Szabo, S. (1973). *Inner products, Euclidean geometry, and trigonometry, Vol. 2: A vector approach to Euclidean geometry*. New York, NY: Macmillan. (ERIC Document Reproduction Service No. ED186277)

Walmsley, A. L. E. (2003). *A history of the"new mathematics" movement and its relationship with current mathematical reform.* Lanham, MA: University Press of America.

Wentworth, G. A. (1882). *Plane trigonometry and tables.* Boston, MA: Ginn.

Wentworth, G. A. (1884). *Plane and spherical trigonometry.* Boston, MA: Ginn.

Wentworth, G. A. (1887). *Plane trigonometry*. Boston, MA: Ginn.

Wentworth, G. A. (1891). *Higher algebra.* Boston, MA: Ginn.

Wentworth, G. A. (1895). *Trigonometry surveying and navigation* (Rev. ed.). Boston, MA: Ginn.

Wentworth, G. A. (1897). *Plane and spherical trigonometry* (Rev. ed.). Boston, MA: Ginn.

Wentworth, G. A. (1898). *New school algebra.* Boston, MA: Ginn.

Wentworth, G. A. (1903). *Plane trigonometry* (2nd ed.). Boston, MA: Ginn.

Wentworth, G., & Smith, D. E. (1913a). *Plane geometry.* Boston, MA: Ginn.

Wentworth, G., & Smith, D. E. (1913b). *School Algebra Book 2.* Boston, MA: Ginn.

Wentworth, A. G., & Smith, D. E. (1914). *Plane trigonometry and tables*. Boston, MA: Ginn.

Wentworth, A. G., & Smith, D. E. (1915). *Plane and spherical trigonometry*. Boston, MA: Ginn.

Wentworth, G., Smith, D. E., & Harper, H. D. (1922). *Fundamentals of practical mathematics.* Boston, MA: Ginn.

Wessel, C. (1797/1999). On the analytical representation of direction (Tran. F. Damhus). In B Banner, &J Lützen (eds.), *Caspar Wessel: On the analytic representation of direction* (pp. 101–122). Copenhagen, Denmark: Royal Danish Academy of Sciences and Letters.

Whitehead, A. N. (1958). *An introduction to mathematics*. New York, NY: Oxford University Press.

Wilczynski, E. J., & Slaught, H. E. (1914). *Plane trigonometry and applications*. Chicago, IL: Allyn & Bacon. Retrieved from http://books.google.com

Wildberger, N. J. (2005). *Divine proportions: Rational trigonometry to universal geometry.* Sidney, Australia: Wild Egg Books.

Willoughby, S. S. (1966). Algebraic geometry for high school pupils. *American Mathematical Monthly, 73,* 650–654.

Willoughby, S. S. (1967). *Contemporary teaching of secondary school mathematics*. New York, NY: Wiley.

Wooton, W., Beckenbach, E. F., & Dolciani, M. P. (1966). *Modern trigonometry*. Boston, MA: Houghton-Mifflin.

Wooton, W. (1965). *SMSG:The making of a curriculum*. New Haven, CT: Yale University Press.

Young, J. W. A. (1906). *Teaching of mathematics in the elementary and the secondary school.* New York, NY: Longman, Green.

Zielinska, T. (2007). Nicholas Copernicus. In M. Ceccarelli (ed.), *Distinguished figures in mechanism and machine science* (pp. 117–134). Dordrecht, the Netherlands: Springer. doi: 10.1007/978-1-4020-6366-4_5