

NILPOTENTS OF REPRESENTATION RINGS OF FINITE p -GROUPS

by

PETER BLAKE HINDMAN

(Under the direction of David J. Benson)

ABSTRACT

The representation ring $a(G)$ of a finite group G provides a context in which to study the behavior of the module category under tensor product. Much work has been devoted to the semisimplicity question for representation rings, by studying the existence and degree of nilpotents in $a(G)$. In this paper, we construct a nilpotent of degree 3 in the representation ring $a(\mathbb{Z}/3 \times \mathbb{Z}/3)$, apparently the first explicit construction of such an element in odd characteristic. We make a number of observations on the general nilpotence question, and discuss applications of techniques developed in this paper to related questions.

INDEX WORDS: Modular Representation Theory, Relative Cohomology, Group Cohomology

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DEDICATION

This dissertation is dedicated to the memory of my father, William M. Hindman.

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CHAPTER 1

INTRODUCTION

D. J. Benson and J. F. Carlson pointed out in 1986 ([3]) that while the tensor product of modules is one of the most frequently used constructions in integral and modular representation theory, few techniques were known at the time for attacking the problem of decomposing a tensor product into a direct sum of indecomposables. The general question remains poorly understood today.

The representation ring $a(RG)$ or $a(G)$ of a finite group G provides one context for studying the decomposition question. Representation rings were first studied systematically by J. A. Green ([6]), who showed that $a(G)$ is semisimple when G is a cyclic p -group and the base field k is of characteristic p . To quote Benson and Carlson,

Much subsequent work has centered on the semisimplicity question in the form: “Does the Green ring have (nonzero) nilpotent elements?”

At the time, it was known that when the base ring is a field k of characteristic p , the representation ring $a(G)$ contains nilpotent elements unless its p -Sylow subgroup is cyclic (in which case Green’s result applies) or an elementary abelian 2-group (where the question appears to be open when the rank is at least 3).

The current paper is largely inspired by the work of Benson and Carlson in [3], which puts earlier investigations (e.g. Zemanek [9], [10]) into a general context. They present a uniform method of construction of nilpotent elements of $a(G)$ for many groups G , using the modules of type L_ζ associated to cohomology elements

$\zeta \in H^n(G, k)$. The following theorem is one of the main results used in [3], and shows how the modules of type L_ζ link group cohomology to modular representation theory:

Theorem 1.1 (Carlson) *Suppose M is a finitely generated kG -module. Let $\hat{\zeta}: \Omega^n k \rightarrow k$ be a representative of $\zeta \in H^n(G, k)$, and let $L_\zeta = \text{Ker}(\hat{\zeta})$. Then the element ζ is sent to zero under the map*

$$H^*(G, k) = \text{Ext}_{kG}^*(k, k) \xrightarrow{\otimes M} \text{Ext}_{kG}^*(M, M)$$

if and only if:

$$\Omega^{-1}L_\zeta \otimes M \cong M \oplus \Omega^{n-1}M \oplus (\text{projective})$$

Proof: The version here is stated and proven in [2, proposition 5.9.5]. A more general version is stated and proven in [3, theorem 3.3]. \square

The first of the two equivalent conditions in the theorem holds if and only if cup product with ζ is identically zero on $\text{Ext}_{kG}^*(M, M)$; we say that the cohomology element ζ *annihilates* $\text{Ext}_{kG}^*(M, M)$ in this situation. Developing the notion further, Carlson and Peng [5] call a homogeneous cohomology element ζ *productive* if it annihilates $\text{Ext}_{kG}^*(L_\zeta, L_\zeta)$; it is known that if p is odd, all cohomology elements of even degree are productive.

Theorem 1.1 allows us to construct nilpotents of $a(G)$ in many cases. Given a productive cohomology element $\zeta \in H^n(G, k)$ of even degree, if it so happens that L_ζ is of period two, then by theorem 1.1 its tensor square decomposes as

$$L_\zeta \otimes L_\zeta \cong L_\zeta \oplus \Omega L_\zeta \oplus (\text{projective})$$

so its core $L_\zeta \oplus \Omega L_\zeta$ is of period one. In other words, modulo the ideal of $a(G)$ generated by projectives, we have congruences

$$[L_\zeta]^2 \equiv [L_\zeta] \cdot [\Omega L_\zeta] \equiv [\Omega L_\zeta]^2$$

and it follows that the element $\mu = [L_\zeta] - [\Omega L_\zeta]$ is nilpotent of degree 2 modulo projectives. If $\dim_k(L_\zeta) = \dim_k(\Omega L_\zeta)$, μ is nilpotent of degree two in $a(G)$, and otherwise some linear combination of μ and classes of projective modules is nilpotent of degree two. One can find productive cohomology elements ζ associated to modules L_ζ of period two when $G = \mathbb{Z}/(p^r) \times \mathbb{Z}/(p^s)$ and p is odd, as well as in certain cases when $p = 2$; a similar approach works when G is a dihedral 2-group. All the examples discussed in [3] are constructed in this way.

As far as the author is aware, the only progress on the nilpotence question published after [3] appears in the thesis of J. Heldner [7]. Working in the representation ring $a(D_8)$ of the dihedral group of order 8, he constructs several classes of nilpotent elements, including some of nilpotence degree three, whose constituent modules are not of type L_ζ , but which are otherwise similar to previous examples. Modulo projectives, these nilpotent elements can be expressed in the form $\mu = [M] - [\Omega M]$ where M is an indecomposable module of period two. In the cases where μ is of nilpotence degree two, there is a decomposition

$$M \otimes M \cong N \oplus \Omega N \oplus (\text{projective})$$

for some indecomposable module N ; in the cases where μ is of nilpotence degree three, $M \otimes M$ has no such decomposition, but we have

$$M^{\otimes 3} \cong N \oplus \Omega N \oplus (\text{projective})$$

for some indecomposable N .

The constructions of Zemanek, Benson and Carlson, and Heldner can be seen as relying on a phenomenon of degeneration of periodicity of periodic modules under tensor powers. We will make this notion precise later, but give the sense of it here. Suppose an indecomposable kG -module M is of period two, but that the core of $M \otimes M$ is of period one. If we define $\gamma = [k] - [\Omega k] \in a(G)$, then modulo the ideal

generated by the isomorphism classes of projective kG -modules, $\gamma.[M] \neq 0$, but $\gamma.[M]^2 \equiv 0$; it follows that $\mu = \gamma.[M]$ is nilpotent of degree two modulo projectives, and some linear combination of μ and isomorphism classes of projectives is nilpotent of degree 2 in $a(G)$. More generally, but for essentially the same reason, if M and the cores of the tensor powers $M \otimes M$ up to $M^{\otimes n-1}$ are all of period two, while the core of $M^{\otimes n}$ is of period one, then a linear combination of $\gamma.[M]$ and isomorphism classes of projectives is nilpotent of degree n .

We take the point of view that the phenomenon of degeneration of periodicity and its implications for decompositions of tensor products are the real subjects of interest; constructing nilpotent elements and understanding them is an approach to this goal. Most known examples of nilpotents in $a(G)$ are consequences of theorem 1.1, which has important applications in other areas of modular representation theory and group cohomology. The only previously published constructions which do not rely on this theorem in some guise are Heldner's [7], where the proofs are computer-assisted and full non-computational proofs do not appear to be known. Some of our methods appear to hold out the possibility of providing full proofs for Heldner's results, and we discuss this prospect in chapter 14.

Considering representation rings and the nilpotence question from this perspective, one may ask whether similar phenomena occur for other notions of periodicity, such as periodicity in cohomology relative to a subgroup H of G . The answer is 'yes'. One example of this is described in [3]: given a short exact sequence of groups

$$0 \rightarrow G'' \rightarrow G \rightarrow G' \rightarrow 0$$

and a nilpotent $\mu = [L_\zeta] - [\Omega L_\zeta]$ in $a(G')$, the inflation

$$\inf_{G,G'}(\mu) = [\inf_{G,G'}(L_\zeta)] - [\inf_{G,G'}(\Omega L_\zeta)]$$

of μ to G is not of the form $[M] - [\Omega M]$ but rather of the form $[M] - [\Omega_{G''} M]$. We will introduce another class of nilpotents of this sort; in our case, translation in relative projectivity will prove to be intrinsic to the construction.

We construct a family of elements of nilpotence degree three in the representation ring of the group $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ over a field k of characteristic 3. These elements are of the form $\mu = [M] - [\Omega_H M]$, where H is a cyclic subgroup of G and where M exhibits degeneration of periodicity relative to H under tensor powers. The elements μ appear to be the first known nilpotents of degree three for odd p , although as we point out in chapter 8, one can construct nilpotents of $a(G)$ of arbitrarily high degree if one takes G of sufficiently high rank, and thus the question of nilpotence degree should be seen as meaningful only relative to the rank of the group. The modules used in this construction are kernels of surjective homomorphisms from modules of type L_ζ to H -projective modules, and exhibit properties similar to those of modules of type L_ζ ; in the process of proving nilpotence of μ we establish some analogues of theorem 1.1 for these modules. Stronger results in this direction seem possible, as we also discuss in chapter 14.

We follow the example of Benson and Carlson and present our results in a context compatible with their work. Our method uses periodicity in relative cohomology rather than ordinary cohomology; to fully develop the analogy with past results, we begin with results on nilpotents in arbitrary commutative rings, and later situate the various constructions for $a(G)$ in that context. By itself, our construction may not seem warrant this level of generality; however, we believe that the methods developed in this paper will be applicable to a number of other problems, and the generality will be useful to this work in the long run. Again, we will say more on the matter in chapter 14.

CHAPTER 2

GENERALITIES

By k , we will always denote a field of characteristic p ; we make no other assumptions about k . By k^\times we mean $k - \{0\}$, the group of multiplicative units of k . By G , we will always mean a finite group.

If H is a subgroup of G , we will denote the restriction of a kG -module M to H by $M \downarrow_H$; more generally, if we wish to stress that a particular module N is a kH -module, we will call it N_H . Thus, k_G denotes the trivial kG -module whereas k_H is the trivial kH -module. Similarly, given an element $x \in M$ of a kG -module M , we will write $\langle x \rangle$ or $\langle x \rangle_G$ for the kG -submodule of M generated by x and $\langle x \rangle_H$ for the kH -submodule of $M \downarrow_H$ generated by x .

We use the notation $\oplus(\text{projective})$ to indicate direct sum with some otherwise unspecified projective kG -module.

We refer to Benson ([1], [2]) for standard background material. The following results will be used frequently.

Lemma 2.1 *Suppose $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence of kG -modules of finite k -dimension. If $M_2 \cong M_1 \oplus M_3$, the sequence splits; i.e., it represents the zero element of $\text{Ext}_{kG}^1(M_3, M_1)$.*

Proof: See [1, lemma 2.6.2]. \square

Lemma 2.2 *If G is a p -group and M is a kG -module with minimal generating set $\{m_1, m_2, \dots, m_n\}$, then any choice of values $\{f(m_1), f(m_2), \dots, f(m_n)\}$ in k defines a kG -homomorphism $f: M \rightarrow k$.*

Proof: Any map $f: M \rightarrow k$ factors through $M/\text{Rad}(M)$, and the images of the generating set span $M/\text{Rad}(M)$. \square

If M and N are kG -modules, the tensor product $M \otimes_k N$ can be given a kG -module structure by letting the elements g of G act diagonally:

$$g(m \otimes_k n) = gm \otimes_k gn$$

Since we will rarely use any other type of tensor product, we will usually drop the subscript and write $M \otimes N$ when we mean $M \otimes_k N$.

Proposition 2.3 *Let k be a field, G any group, $g \in G$, and $\gamma = g - 1 \in kG$. If M and N are kG -modules and $m \in M$ and $n \in N$, then we can describe the action of γ on the tensor product $M \otimes N$ by the identity $\gamma(m \otimes n) = \gamma m \otimes n + m \otimes \gamma n + \gamma m \otimes \gamma n$. In particular, if $\gamma m = 0$, then $\gamma(m \otimes n) = m \otimes \gamma n$.*

Proof: Elements of G act diagonally on tensor products, so

$$\begin{aligned} \gamma(m \otimes n) &= g(m \otimes n) - m \otimes n \\ &= (\gamma + 1)m \otimes (\gamma + 1)n - m \otimes n \\ &= \gamma m \otimes \gamma n + m \otimes \gamma n + \gamma m \otimes n \end{aligned}$$

which proves the first assertion. The second assertion follows immediately. \square

Working with p -groups in characteristic p involves several key simplifications in the general theory:

Lemma 2.4 *Suppose G is a p -group and k is a field of characteristic p .*

i) There is only one simple kG -module, the one-dimensional trivial representation k_G .

ii) The regular representation ${}_k kG$ is the only projective indecomposable kG -module up to isomorphism, and every projective kG -module is free.

iii) *The projective indecomposable module is the projective cover of the simple module k_G , and has one-dimensional head and socle.*

Proof: See [1], section 3.14. \square

CHAPTER 3

DUALITY, RESTRICTION, AND INDUCTION

All material in this chapter is standard and included for reference. Again, for full discussion and proofs, we refer to Benson [1].

Definition 3.1 *If M is a kG -module with underlying k -vector space V on which the action of G is given by $\phi: G \rightarrow GL(V)$, we define the dual module M^* to be:*

$$M^* = \text{Hom}_k(M, k)$$

The module M^ may be viewed as a left kG -module via the following action:*

$$\begin{aligned}\phi^*: G &\rightarrow GL(V^*), \\ \phi^*(g) &= (\phi(g^{-1}))^t\end{aligned}$$

In other words, $g \in G$ acts on $f \in \text{Hom}_k(M, k)$ via $(gf)(m) = f(g^{-1}m)$.

Proposition 3.2 *Let M and N be kG -modules.*

- i) $\text{Hom}_k(M, N) \cong M^* \otimes N$.*
- ii) If G is a p -group, then $\text{Hom}_{kG}(M, N) \cong \text{Soc}(M^* \otimes N)$.*
- iii) $(M \otimes N)^* \cong M^* \otimes N^*$.*

Proof: i) See [1], section 3.1.

- ii) If we denote by M^G the space of G -fixed points of a module M , then by part i) we have:

$$\text{Hom}_{kG}(M, N) = \text{Hom}_k(M, N)^G$$

$$\begin{aligned}
&= (M^* \otimes N)^G \\
&= \text{Soc}(M^* \otimes N)
\end{aligned}$$

iii) The isomorphism holds for the underlying vector spaces, and commutes with the action of G since G acts diagonally on tensor products. \square

If H is a subgroup of G and M is a kG -module, the *restriction* $M \downarrow_H$ of M to H is just the module M viewed as a kH -module via the embedding of kH in kG . Equally important is the induced module $N \uparrow^G$ of a kH -module to G :

Definition 3.3 *Let H be a subgroup of G and let N be a kH -module. We define the induced module $N \uparrow^G$ to be:*

$$N \uparrow^G = kG \otimes_{kH} N$$

An alternate and equivalent way to view $N \uparrow^G$ is to write

$$N \uparrow^G = \bigoplus_{g' \in G/H} g' \otimes N$$

where the sum runs over a set of coset representatives of H in G . The action of G can then be given explicitly: $g(g' \otimes n) = g'' \otimes hn$, where g'' is the representative of the coset of gg' and $g''h = gg'$.

The next proposition gathers a list of standard results relating tensor products, induction, restriction, and homomorphism groups for kH - and kG -modules.

Proposition 3.4 *Let H be a subgroup of G . Let N be a kH -module and let M be a kG -module.*

- i) $\text{Hom}_{kH}(N, M \downarrow_H) \cong \text{Hom}_{kG}(N \uparrow^G, M)$ as vector spaces.
- ii) $\text{Hom}_{kH}(M \downarrow_H, N) \cong \text{Hom}_{kG}(M, N \uparrow^G)$ as vector spaces.
- iii) $(N \otimes M \downarrow_H) \uparrow^G \cong N \uparrow^G \otimes M$ as kG -modules.

Proof: See [1], section 3.3. \square

Remark 3.5 *Induction and restriction are both functorial, and these functors are adjoint to one another. This relationship, which is given explicitly in parts i) and ii) of the previous proposition, is often referred to as Frobenius reciprocity.*

The isomorphisms of Frobenius reciprocity can easily be made explicit:

Definition 3.6 *Let H be a subgroup of G , let N be a kH -module, and let M be a kG -module. Define $\theta_N \in \text{Hom}_{kH_b}(N \uparrow^G \downarrow_H, N)$ to be the following map:*

$$\begin{aligned} \theta_N: N \uparrow^G \downarrow_H &\rightarrow N \\ \theta_N: 1_G \otimes n &\mapsto n \\ g \otimes n &\mapsto 0 \quad \forall g \in G/H, g \neq 1_G \end{aligned}$$

Define $\hat{T}r_{H,G}$ to be the map:

$$\begin{aligned} \hat{T}r_{H,G}: \text{Hom}_{kH}(M \downarrow_H, N) &\rightarrow \text{Hom}_{kG}(M, N \uparrow^G) \\ \hat{T}r_{H,G}(f)(m) &= \sum_{g \in G/H} g \otimes f(g^{-1}m) \end{aligned}$$

Lemma 3.7 *Let M be a kG -module, let N be a kH -module, and let θ_N and $\hat{T}r_{H,G}$ be defined as above. Composing homomorphisms in $\text{Hom}_{kG}(M, N \uparrow^G)$ with θ_N induces an isomorphism:*

$$\begin{array}{ccc} \text{Hom}_{kG}(M, N \uparrow^G) & \xrightarrow{\theta_N \circ} & \text{Hom}_{kH}(M \downarrow_H, N) \\ f & \longrightarrow & \theta_N \circ f \end{array}$$

and applying $\hat{T}r_{H,G}$ to homomorphisms in $\text{Hom}_{kH}(M \downarrow_H, N)$ induces an isomorphism:

$$\begin{array}{ccc} \text{Hom}_{kH_b}(M \downarrow_H, N) & \xrightarrow{\hat{T}r_{H,G}} & \text{Hom}_{kG}(M, N \uparrow^G) \\ f' & \longrightarrow & \hat{T}r_{H,G}(f') \end{array}$$

Proof: Given $f \in \text{Hom}_{kG}(M, N \uparrow^G)$, one may compute directly that that $\hat{T}r_{H,G}(\theta_N \circ f)(m) = f(m)$ for all $m \in M$, and likewise given $f' \in \text{Hom}_{kH}(M \downarrow_H, N)$, that $(\theta_N \circ \hat{T}r_{H,G}(f'))(m) = f'(m)$ for all $m \in M$. \square

Remark 3.8 *This proves part ii) of proposition 3.4. The proof of part i) is similar.*

Theorem 3.9 (Green's indecomposability theorem) *Suppose H is a normal subgroup of G such that G/H is a p -group, and M is an absolutely indecomposable kH -module (i.e. $k' \otimes_k M$ is indecomposable for all extension fields k' of k). Then $M \uparrow^G$ is absolutely indecomposable.*

Proof: See [1], section 3.13. \square

CHAPTER 4

COHOMOLOGY, RELATIVE COHOMOLOGY, AND TRANSLATION

We assume standard results on ordinary cohomology, and only give a few results and definitions which we will use heavily.

Definition 4.1 *Let M be an arbitrary kG -module. We define the core of M to be the smallest direct summand M' of M such that $M = M' \oplus P$ where P is projective. By Krull-Schmidt, the core of a module is well-defined up to isomorphism.*

Definition 4.2 *Let M be any kG -module, let P be a projective kG -module, and let $\rho_M: P \rightarrow M$ be a surjective kG -homomorphism. We define the translate ΩM to be the core of $\ker(\rho)$, so there is a short exact sequence*

$$0 \longrightarrow \Omega M \oplus P' \longrightarrow P \xrightarrow{\rho} M \longrightarrow 0$$

where P' is projective. By Krull-Schmidt, the translate ΩM is unique up to isomorphism, and we may as well assume that we have a short exact sequence

$$0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$$

since any projective (therefore injective) summand P' in the first term can be split off.

We define higher-order translates inductively by setting $\Omega^n M = \Omega(\Omega^{n-1} M)$.

We define translates of negative degree by defining $\Omega^{-1} M$ to be the core of the cokernel of a monomorphism $\iota: M \rightarrow I$ where I is an injective kG -module.

Since group algebras are symmetric algebras, injective modules are the same as projective modules, and it follows that

Proposition 4.3 *Let M and N be non-projective indecomposable kG -modules.*

i) The translations Ω and Ω^{-1} act as inverses up to isomorphism: $\Omega^{-1}(\Omega M) \cong \Omega(\Omega^{-1}M) \cong M$.

ii) The translation $\Omega^n M$ is isomorphic to the core of $\Omega^n k \otimes M$.

iii) Translation respects tensor products up to stable isomorphism: The core of $M \otimes \Omega^n N$ and the core of $\Omega^n M \otimes N$ are both isomorphic to $\Omega^n(M \otimes N)$.

iv) Translation respects direct sums: $\Omega^n(M \oplus N) \cong \Omega^n M \oplus \Omega^n N$.

Proof: i) Consider the following short exact sequence:

$$0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$$

Since projective modules are injective and vice versa, $M = \Omega^{-1}(\Omega M)$ by definition, and a similar argument shows that $\Omega(\Omega^{-1}M) \cong M$.

ii) Given a short exact sequence

$$0 \rightarrow \Omega k \rightarrow P \rightarrow k \rightarrow 0$$

tensoring with M yields a short exact sequence:

$$0 \rightarrow \Omega k \otimes M \rightarrow P \otimes M \rightarrow k \otimes M \rightarrow 0$$

Since $P \otimes M$ is projective and $k \otimes M \cong M$, we have shown that the core of $\Omega k \otimes M$ is isomorphic to ΩM . The result follows by induction.

iii) The proof of ii) generalizes immediately; the tensor product of M with a sequence

$$0 \rightarrow \Omega N \rightarrow P \rightarrow N \rightarrow 0$$

produces a sequence

$$0 \rightarrow M \otimes \Omega N \rightarrow M \otimes P \rightarrow M \otimes N \rightarrow 0$$

with projective middle term, so the core of $M \otimes \Omega N$ is $\Omega(M \otimes N)$, and the same argument applies to $\Omega M \otimes N$. Again, the result follows by induction.

iv) The direct sum of the complexes

$$0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$$

$$0 \rightarrow \Omega N \rightarrow P \rightarrow N \rightarrow 0$$

yields the sequence:

$$0 \rightarrow \Omega M \oplus \Omega N \rightarrow P \oplus P' \rightarrow M \oplus N \rightarrow 0$$

Induction finishes the proof. \square

Lemma 4.4 *Let H be a subgroup of G , let M be a kG -module and let N be a kH -module. Translation respects restriction and induction up to stable isomorphism:*

$$(\Omega M) \downarrow_H = \Omega(M \downarrow_H) \oplus (\text{projective})$$

$$(\Omega N) \uparrow^G = \Omega(N \uparrow^G) \oplus (\text{projective})$$

Proof: Let P be a projective kG -module and let $\rho_M: P \rightarrow M$ be surjective. Since P is projective, the restriction $P \downarrow_H$ is a projective kH -module, and the restriction of ρ_M to H is still surjective, so

$$\Omega(M) \downarrow_H = \ker(\rho_M) \downarrow_H \cong \Omega(M \downarrow_H) \oplus (\text{projective})$$

which proves the first isomorphism.

Let P' be a projective kH -module and let $\rho_N: P' \rightarrow N$ be surjective. Define a map ρ_N :

$$\begin{aligned} \hat{\rho}_N: P' \uparrow^G &\rightarrow N \uparrow^G \\ \hat{\rho}_N &= \sum_{g \in G/H} g \otimes \rho_N \end{aligned}$$

Then $P'\uparrow^G$ is a projective kG -module, ρ_N is surjective, and one may verify directly that $\ker(\rho_N) \cong \ker(\rho_N)\uparrow^G$. This proves the second isomorphism. \square

Definition 4.5 *If it so happens that $\Omega^n M \cong M$ and n is the smallest positive exponent for which this isomorphism holds, we say that M is periodic of period n , or simply n -periodic.*

Definition 4.6 *A kG -homomorphism $f: M \rightarrow N$ is said to factor through a projective module if there exists a projective kG -module P and kG -homomorphisms $f': M \rightarrow P$ and $f'': P \rightarrow N$ such that $f = f'' \circ f'$. By $\text{PHom}_{kG}(M, N) \subset \text{Hom}_{kG}(M, N)$ we denote the subgroup of kG -homomorphisms factoring through a projective module. By $\text{StHom}_{kG}(M, N)$, the group of stable homomorphisms, we mean the quotient $\text{Hom}_{kG}(M, N)/\text{PHom}_{kG}(M, N)$.*

Lemma 4.7 *Let M and N be kG -modules and let n be any integer. We get the following isomorphism:*

$$\text{StHom}_{kG}(M, N) \cong \text{StHom}_{kG}(\Omega^n M, \Omega^n N)$$

Proof: Let f be a kG -homomorphism $f: M \rightarrow N$, and choose sequences

$$0 \longrightarrow \Omega M \longrightarrow P \xrightarrow{\rho_M} M \longrightarrow 0$$

$$0 \longrightarrow \Omega N \longrightarrow P' \xrightarrow{\rho_N} N \longrightarrow 0$$

with projective middle terms. We can lift f to a map $\tilde{\Omega}f$ by chasing elements around the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & P & \xrightarrow{\rho_M} & M \longrightarrow 0 \\ & & \tilde{\Omega}f \downarrow & & \downarrow & & f \downarrow \\ 0 & \longrightarrow & \Omega N & \longrightarrow & P' & \xrightarrow{\rho_N} & N \longrightarrow 0 \end{array}$$

The map $\tilde{\Omega}f$ is not unique, but we claim that two such liftings will differ by a map factoring through a projective. Since P is projective, the map $f \circ \rho_M: P \rightarrow N$

lifts to a map $\hat{f}: P \rightarrow P'$ which induces the map $\tilde{\Omega}f: \Omega M \rightarrow \Omega N$. If we choose a different lifting $\hat{f}': P \rightarrow P'$ of f , inducing a different map $\tilde{\Omega}f': \Omega M \rightarrow \Omega N$, then since $\rho_N \circ \hat{f} = \rho_N \circ \hat{f}'$, the difference $\hat{f} - \hat{f}'$ is killed by composition with ρ_N . It follows that $\hat{f} - \hat{f}'$ sends P into ΩN , and so $\tilde{\Omega}f - \tilde{\Omega}f'$, which is just the restriction of $\hat{f} - \hat{f}'$ to ΩM , factors through P . The lift $\tilde{\Omega}f$ thus corresponds to a well-defined element $\Omega f \in \text{StHom}_{kG}(\Omega M, \Omega N)$. The lemma then follows from 4.3 and induction on n . \square

Remark 4.8 *The previous lemma amounts to the observation that while the translation operation $\Omega(-)$ is not a functor on the module category, it defines one on the stable module category, the category whose objects are kG -modules and whose arrows are stable homomorphisms.*

We now give a very short sketch of the important definitions and results we will use concerning the groups $\text{Ext}_{kG}^n(M, N)$.

Definition 4.9 *Let M be a kG -module. A projective resolution P_M of M is a long exact sequence of the form*

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0$$

where the modules P_n are all projective and such that $P_0/\text{Im}(\delta_1) \cong M$.

Definition 4.10 *Let M and N be kG -modules, and let P_M be a projective resolution of M . Applying the functor $\text{Hom}_{kG}(-, N)$ to P_M yields a cochain complex:*

$$\text{Hom}_{kG}(P_0, N) \xrightarrow{\epsilon^0} \text{Hom}_{kG}(P_1, N) \xrightarrow{\epsilon^1} \text{Hom}_{kG}(P_2, N) \longrightarrow \cdots$$

We define the groups $\text{Ext}_{kG}^n(M, N)$ to be the cohomology groups of this sequence:

$$\text{Ext}_{kG}^n(M, N) = H^n(\text{Hom}_{kG}(P_M, N), \epsilon^*)$$

There are two other interpretations of the groups $\text{Ext}_{kG}^n(M, N)$ that we will use. First, if $n > 0$ and ζ is an element of $\text{Ext}_{kG}^n(M, N)$, we may associate ζ with an equivalence class of *n-fold extensions* of N by M , that is, equivalence classes of exact sequences of the following form:

$$0 \rightarrow N \rightarrow M_{n-1} \rightarrow M_{n-2} \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$$

In particular, an element ζ of $\text{Ext}_{kG}^1(M, N)$ defines a short exact sequence $0 \rightarrow N \rightarrow M' \rightarrow M \rightarrow 0$, and the middle term M' is entirely determined by the element ζ .

Finally, since truncating a projective resolution P_M of a kG -module M gives rise to an n -fold extension of $\Omega^n M$ by M for every $n > 0$, elements of Ext-groups may be identified with stable homomorphisms via the so-called *dimension shifting* isomorphisms:

Proposition 4.11 *Let M and N be kG -modules, and let $n > 0$. We have the following isomorphisms:*

$$\text{Ext}_{kG}^n(M, N) \cong \text{StHom}_{kG}(\Omega^n M, N) \cong \text{StHom}_{kG}(M, \Omega^{-n} N)$$

Proof: Given an element $\zeta \in \text{Ext}_{kG}^n(M, N)$, choose a representative n -fold extension:

$$C_\zeta: 0 \rightarrow N \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$$

We can then lift the identity homomorphism on M to construct a diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Omega^n M & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \xrightarrow{\rho_M} & M & \longrightarrow & 0 \\ & & \hat{\zeta} \downarrow & & \downarrow & & & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & N & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the existence of $\hat{\zeta}: \Omega^n M \rightarrow N$ is guaranteed since the modules P_i are projective. Proving that this procedure is well-defined and yields an isomorphism

$\text{Ext}_{kG}^n(M, N) \cong \text{StHom}_{kG}(\Omega^n M, N)$ is an extended diagram chase which we omit.

Then by 4.3 and 4.7,

$$\text{StHom}_{kG}(\Omega^n M, N) \cong \text{StHom}_{kG}(M, \Omega^{-n} N)$$

and we are done. \square

A version of Frobenius reciprocity holds for $\text{Ext}_{kG}^n(-, -)$:

Proposition 4.12 (Eckmann-Shapiro) *Let H be a subgroup of G . Let N be a kH -module and let M be a kG -module.*

- i) $\text{Ext}_{kH}^n(N, M \downarrow_H) \cong \text{Ext}_{kG}^n(N \uparrow^G, M)$ as vector spaces.*
- ii) $\text{Ext}_{kH}^n(M \downarrow_H, N) \cong \text{Ext}_{kG}^n(M, N \uparrow^G)$ as vector spaces.*

Proof: See [1], sections 2.8 and 3.3. \square

Corollary 4.13 *Let H be a subgroup of G . Let N be a kH -module and let M be a kG -module. We have the following isomorphisms of vector spaces:*

$$\begin{aligned} \text{StHom}_{kH}(M \downarrow_H, N) &\cong \text{StHom}_{kG}(M, N \uparrow^G) \\ \text{PHom}_{kH}(M \downarrow_H, N) &\cong \text{PHom}_{kG}(M, N \uparrow^G) \\ \text{StHom}_{kH}(N, M \downarrow_H) &\cong \text{StHom}_{kG}(N \uparrow^G, M) \\ \text{PHom}_{kH}(N, M \downarrow_H) &\cong \text{PHom}_{kG}(N \uparrow^G, M) \end{aligned}$$

Proof: The isomorphisms for StHom follow by applying 4.4 and 4.7 to the previous result. The same relations hold for Hom , and so hold for the kernel PHom by a dimension count. More concretely, given

$$f \in \text{PHom}_{kG}(M, N \uparrow^G)$$

it is clear that the Frobenius correspondent

$$\theta_N \circ f \in \text{PHom}_{kH}(M \downarrow_H, N)$$

factors through a projective, so the isomorphisms of 3.7 induce the first two isomorphisms here. \square

Lemma 4.14 *Let H be a cyclic group and let M be an indecomposable kH -module. A kH -homomorphism $f: \Omega k_H \rightarrow M$ factors through a projective if and only if it is not surjective.*

Proof: Observe that any indecomposable kH -module is a uniserial quotient of the indecomposable projective kH -module, so

$$\text{StHom}_{kH}(\Omega k, M) = \text{StHom}_{kH}(k, \Omega^{-1}M)$$

is one-dimensional. Since k is simple,

$$\text{StHom}_{kH}(k, \Omega^{-1}M) = \text{Hom}_{kH}(k, \Omega^{-1}M)$$

and a map in $\text{Hom}_{kH}(k, \Omega^{-1}M)$ is clearly nonzero if and only if it is injective; the result follows from dimension shifting. \square

Suppose M and M' are kG -modules, P and P' are projective kG -modules, $\iota_M: M \rightarrow P$ is a monomorphism and $\rho_{M'}: P' \rightarrow M'$ is an epimorphism. Given any kG -homomorphism $f: M \rightarrow M'$, we can use $\rho_{M'}$ to construct an epimorphism $f': P' \oplus M \rightarrow M'$ which is stably equivalent to f ; likewise, we can use ι_M to construct a monomorphism $f'': M \rightarrow M' \oplus P$ with f'' stably equivalent to f . We thus have short exact sequences

$$0 \rightarrow M'' \rightarrow P' \oplus M \rightarrow M' \rightarrow 0$$

$$0 \rightarrow M \rightarrow M' \oplus P \rightarrow X \rightarrow 0$$

and a diagram chase shows that $X \cong \Omega^{-1}M''$. Expanding this idea, we may observe that, given any short exact sequence

$$0 \longrightarrow M'' \xrightarrow{g} M \oplus (\text{projective}) \xrightarrow{f} M' \longrightarrow 0$$

where the modules M'' , M and M' have no projective direct summands, we have short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & M' \oplus (\text{projective}) & \longrightarrow & \Omega^{-1}M'' \longrightarrow 0 \\ 0 & \longrightarrow & \Omega M' & \longrightarrow & M'' \oplus (\text{projective}) & \xrightarrow{g} & M \longrightarrow 0 \end{array}$$

and so on, where we identify f and g with their stable equivalence classes. This kind of transformation of a short exact sequence is often called *moving around the triangle*, where the ‘triangle’ refers to the sequence

$$M'' \xrightarrow{g} M \xrightarrow{f} M' \longrightarrow \Omega^{-1}M''$$

in the stable module category.

Most of the concepts and constructions we have discussed above fit into the context of the theory of ordinary cohomology of the module category of kG . Next, we discuss the theory of cohomology relative to a subgroup H , where we have analogues for much of this material.

Definition 4.15 *If a kG -module M is a direct summand of $N\uparrow^G$ for some kH -module N , we say that M is relatively projective with respect to H , projective relative to H , or just relatively H -projective. In keeping with the notation $\oplus(\text{projective})$ set previously, we will write $\oplus(H\text{-projective})$ to indicate direct sum with a module relatively projective with respect to a subgroup H of G .*

Definition 4.16 *A kG -homomorphism $f: M \rightarrow N$ is H -split if its restriction to H can be written as a composition $f\downarrow_H = f'' \circ f'$ where f' is a split kH -epimorphism and f'' is a split kH -monomorphism.*

Lemma 4.17 *Let H be a subgroup of G . Let M and M' be kG -modules and let M' be projective relative to H . Then $M \otimes M'$ is projective relative to H .*

Proof: There is some kH -module N such that M' is a direct summand of $N\uparrow^G$. Then $M \otimes M'$ is a direct summand of:

$$M \otimes N\uparrow^G = (M\downarrow_H \otimes N)\uparrow^G$$

□

Definition 4.18 *A relatively H -projective resolution or H -projective resolution X_M or (X_M, δ) of a kG -module M is a long exact sequence of the form*

$$\cdots \longrightarrow X_2 \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0$$

where the modules X_n are all projective relative to H and all the maps δ_n are H -split.

Proposition 4.19 *Let H be a subgroup of G , and let M be an indecomposable kG -module. Define a map ρ by:*

$$\begin{aligned} \rho: M\downarrow_H\uparrow^G &\rightarrow M, \\ \rho: g \otimes m &\mapsto gm \end{aligned}$$

Then $\rho: M\downarrow_H\uparrow^G \rightarrow M$ is the beginning of an H -projective resolution of M , and thus every kG -module has an H -projective resolution.

Proof: The module $M\downarrow_H\uparrow^G$ is H -projective by definition, the map ρ is surjective, and the map $M\downarrow_H \rightarrow M\downarrow_H\uparrow^G\downarrow_H$ sending $m \in M$ to $1_G \otimes m$ gives an explicit splitting of the restriction of ρ to H , so ρ is H -split. □

This construction provides the basis of a cohomology theory in which most standard notions have H -relative analogues. In particular, by the core of a module M relative to H , we mean the smallest direct summand M' of M for which there exists a direct sum decomposition $M = M' \oplus M''$ where M'' is H -projective; by Krull-Schmidt-Azumaya, the core relative to H is well-defined up to isomorphism. By the

translate $\Omega_H M$ of M relative to H , we mean the core relative to H of the kernel of δ_1 for some relatively H -projective resolution (X_M, δ) ; the translate $\Omega_H M$ is well-defined up to isomorphism independent of the choice of resolution, and we can make the following definition:

Definition 4.20 *If it so happens that $\Omega_H^n M \cong M$ and n is the smallest positive exponent for which this isomorphism holds, we say that M is periodic of period n with respect to H or simply n -periodic with respect to H .*

Proposition 4.21 *Suppose M is a kG -module, and H is a normal subgroup of G with G/H cyclic. If $M \downarrow_H$ is projective, then we have:*

$$M \oplus (\text{projective}) \cong \Omega^2 M \oplus (\text{projective})$$

Proof: See [1], corollary 3.5.3. \square

Remark 4.22 *For group algebras kG , projectivity relative to the trivial subgroup $\{1_G\}$ is equivalent to projectivity, and ordinary cohomology is the same thing as cohomology relative to $\{1_G\}$. Every result about relative cohomology thus applies equally to ordinary cohomology.*

CHAPTER 5

INFLATION

Definition 5.1 *Let*

$$0 \longrightarrow G' \longrightarrow G \xrightarrow{\phi} G'' \rightarrow 0$$

be a short exact sequence of finite groups, and let M be a kG'' -module. The surjection

$$G \xrightarrow{\phi} G''$$

makes M into a kG -module via $g.m = \phi(g)m$ for $m \in M$. When we regard M as a kG -module in this manner, we call it the inflation of M to G and denote it $\inf_{G,G''}(M)$.

Lemma 5.2 *Let $G = H \times H'$ be a p -group such that H is cyclic.*

i) Let M be an indecomposable kH -module. We have the following isomorphisms:

$$M \uparrow^G \cong \inf_{G,H}(M) \otimes k_H \uparrow^G \cong \inf_{G,H}(M) \otimes \inf_{G,H'}(kH')$$

$$\text{Soc}(M \uparrow^G) \cong \text{Soc}(\inf_{G,H}(M))$$

ii) Let M and N be kH -modules. Then there is an isomorphism:

$$\text{Hom}_{kH}(M, N) \cong \text{Hom}_{kG}(\inf_{G,H}(M), \inf_{G,H}(N))$$

iii) Every indecomposable relatively H -projective module is of the form $M \uparrow^G$ for some indecomposable kH -module M .

Proof: i) By 3.4, $\inf_{G,H}(M) \otimes k_H \uparrow^G \cong (\inf_{G,H}(M) \downarrow_H \otimes k_H) \uparrow^G$. Then by 6.1, $\inf_{G,H}(M) \downarrow_H \cong M$, giving the first isomorphism. Since H is normal, $k_H \uparrow^G \cong \inf_{G,H'}(kH')$, providing the second isomorphism. Finally, since

$$\mathrm{Hom}_{kG}(k_G, M \uparrow^G) \cong \mathrm{Hom}_{kH}(k_H, M)$$

by Frobenius reciprocity, and

$$\mathrm{Hom}_{kH}(k_H, M) \cong \mathrm{Hom}_{kG}(k_G, \inf_{G,H}(M))$$

we get the isomorphism of socles as stated.

ii) Since H' acts trivially on $\inf_{G,H}(M)$ and $\inf_{G,H}(N)$, every kH -homomorphism in $\mathrm{Hom}_{kH}(M, N)$ may be regarded as a kG -homomorphism between the inflations.

iii) Since H is cyclic, every indecomposable kH -module is absolutely indecomposable, and Green's indecomposability criterion applies. \square

Proposition 5.3 *Let $G = H \times H'$, let M be a kH -module and let N be a kH' -module. Then we have the following algebra isomorphisms:*

$$\begin{aligned} & \mathrm{End}_{kG}(\inf_{G,H}(M) \otimes \inf_{G,H'}(N)) \\ & \cong \mathrm{End}_{kG}(\inf_{G,H}(M)) \otimes \mathrm{End}_{kG}(\inf_{G,H'}(N)) \\ & \cong \mathrm{End}_{kH}(M) \otimes \mathrm{End}_{kH'}(N) \end{aligned}$$

Proof: For the sake of readability, we define \hat{M} and \hat{N} by:

$$\hat{M} = \inf_{G,H}(M)$$

$$\hat{N} = \inf_{G,H'}(N)$$

It is clear that there is an inclusion of algebras of the form:

$$\mathrm{End}_{kG}(\hat{M}) \otimes \mathrm{End}_{kG}(\hat{N}) \subseteq \mathrm{End}_{kG}(\hat{M} \otimes \hat{N})$$

We will show that these algebras are of the same dimension over k , and thus isomorphic. Recall that by M^G we mean the set of elements of M fixed by the action of G , and that $\text{Hom}_{kG}(M_1, M_2) = (\text{Hom}_k(M_1, M_2))^G = (M_1^* \otimes M_2)^G$ for any two kG -modules M_1, M_2 . We then have the following isomorphisms of vector spaces:

$$\begin{aligned}
\text{End}_{kG}(\hat{M} \otimes \hat{N}) &= \text{Hom}_{kG}(\hat{M} \otimes \hat{N}, \hat{M} \otimes \hat{N}) \\
&= (\text{Hom}_k(\hat{M} \otimes \hat{N}, \hat{M} \otimes \hat{N}))^G \\
&= ((\hat{M} \otimes \hat{N})^* \otimes (\hat{M} \otimes \hat{N}))^G \\
&= (\hat{M}^* \otimes \hat{M} \otimes \hat{N}^* \otimes \hat{N})^G \\
&= (\text{Hom}_k(\hat{M}, \hat{M}) \otimes \text{Hom}_k(\hat{N}, \hat{N}))^G \\
&= (\text{End}_k(\hat{M}) \otimes \text{End}_k(\hat{N}))^G
\end{aligned}$$

An element of a module M is stabilized by G if and only if it is stabilized by H and by H' , so $M^G = M^H \cap M^{H'}$. We may thus write

$$\begin{aligned}
(\text{End}_k(\hat{M}) \otimes \text{End}_k(\hat{N}))^G &= (\text{End}_k(\hat{M}) \otimes \text{End}_k(\hat{N}))^H \\
&\quad \cap (\text{End}_k(\hat{M}) \otimes \text{End}_k(\hat{N}))^{H'}
\end{aligned}$$

as subsets of $\text{End}_k(\hat{M}) \otimes \text{End}_k(\hat{N})$.

Since H' acts trivially on \hat{M} , all k -endomorphisms of \hat{M} are kH' -endomorphisms, and similarly every kH -endomorphism of \hat{M} is a kG -endomorphism. Likewise, k -endomorphisms of \hat{N} are kH -endomorphisms and kH' -endomorphisms of \hat{N} are kG -endomorphisms. We thus have:

$$\begin{aligned}
(\text{End}_k(\hat{M}) \otimes \text{End}_k(\hat{N}))^H &\cap (\text{End}_k(\hat{M}) \otimes \text{End}_k(\hat{N}))^{H'} \\
&= \text{End}_{kG}(\hat{M}) \otimes \text{End}_k(\hat{N}) \cap \text{End}_k(\hat{M}) \otimes \text{End}_{kG}(\hat{N}) \\
&= \text{End}_{kG}(\hat{M}) \otimes \text{End}_{kG}(\hat{N})
\end{aligned}$$

Therefore, $\text{End}_{kG}(\hat{M} \otimes \hat{N}) \cong \text{End}_{kG}(\hat{M}) \otimes \text{End}_{kG}(\hat{N})$ as vector spaces, which proves the first isomorphism. The second isomorphism is a consequence of 5.2. \square

Corollary 5.4 *Let k be algebraically closed, let $G = H \times H'$, let M be a kH -module and let N be a kH' -module. If M and N are indecomposable, then $\inf_{G,H}(M) \otimes \inf_{G,H'}(N)$ is an indecomposable kG -module.*

Proof: Since the base field k is algebraically closed, a kG -module is indecomposable if and only if its endomorphism algebra is local, so $\text{End}_{kH}(M)$ and $\text{End}_{kH'}(N)$ are local. By the previous proposition, $\text{End}_{kG}(\inf_{G,H}(M) \otimes \inf_{G,H'}(N)) \cong \text{End}_{kH}(M) \otimes \text{End}_{kH'}(N)$ is a tensor product of local rings, thus local, and therefore $\inf_{G,H}(M) \otimes \inf_{G,H'}(N)$ is indecomposable. \square

CHAPTER 6

REPRESENTATION RINGS

Given a group G and base field k , the representation ring $a(G)$ or $a(kG)$ is the ring whose generators are isomorphism classes $[M]$ of kG -modules M , with relations:

$$[M] + [N] = [M \oplus N]$$

$$[M] \cdot [N] = [M \otimes N]$$

The usual properties of direct sum and tensor product show that $a(G)$ is a commutative ring with identity $1_{a(G)} = [k_G]$ and zero $0_{a(G)} = [0_G]$. As an additive group, $a(G)$ is the free abelian group generated by the isomorphism classes of indecomposable kG -modules, since the Krull-Schmidt-Azumaya theorem holds for kG ; every element can be written uniquely as a finite sum

$$\sum n_i [M_i]$$

with coefficients $n_i \in \mathbb{Z}$ and all M_i indecomposable. We will need to consider other coefficient rings for the representation ring, and in accordance with standard notation we define the following extended representation rings:

$$A(G) = \mathbb{C} \otimes_{\mathbb{Z}} a(G)$$

$$a(G)_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} a(G)$$

$$a(G)_p = \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} a(G)$$

One may also introduce various ideals and subrings of $a(G)$. Given a subgroup H of G , we denote by $a(G, H)$ the ideal of $a(G)$ generated by the relatively H -projective modules; by $a_0(G, H)$, we mean the ideal generated by elements of the

form $[M_2] - [M_1] - [M_3]$ for all H -split short exact sequences $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$. In particular, if G is a p -group and H is the trivial subgroup, then $a(G, H)$ is the ideal generated by isomorphism classes of projective modules, and $a_0(G, H)$ is the ideal of elements for which $\sum n_i \dim_k(M_i) = 0$. In keeping with the notation for extended representation rings, we will write $a(G, H)_p$ for $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} a(G, H)$, $A(G, H)$ for $\mathbb{C} \otimes_{\mathbb{Z}} a(G, H)$, and so on.

If H is a subgroup of G , the induction and restriction functors induce maps $\text{ind}_H^G: a(H) \rightarrow a(G)$ and $\text{res}_H^G: a(G) \rightarrow a(H)$. The restriction map res_H^G is a ring map, while the induction map ind_H^G is a map of abelian groups whose image is an ideal.

If G' is a quotient of a finite group G , any kG' -module may be regarded as a kG -module by inflation; this induces an inclusion of rings $\text{inf}_{G,G'}: a(G') \rightarrow a(G)$. Given a direct product of groups $G = H \times H'$, we have the following result:

Proposition 6.1 *Let $G = H \times H'$ be a group. The map*

$$\text{res}_H^G \circ \text{inf}_{G,H}: a(H) \rightarrow a(H)$$

is an isomorphism of rings.

Proof: Let $\iota_H: H \rightarrow G$ be the natural inclusion and $\rho_H: G \rightarrow H$ be the natural projection associated to the given product; the composition $\rho_H \circ \iota_H: H \rightarrow H$ is the identity on H by definition. If M is any $k(H)$ -module, $\text{res}_H^G \circ \text{inf}_{G,H}(M)$ is isomorphic to M , and the result follows. \square

Abusing notation slightly, the previous proposition lets us view $a(H)$ and $a(H')$ as subrings of $a(G)$. We may think of $a(H)$ as generated by those kG -modules on which H' acts trivially, and likewise we may think of $a(H')$ as generated by the modules on which H acts trivially. In this view, the intersection $a(H) \cap a(H')$ is thus just the additive subgroup of $a(G)$ generated by $[k_G]$, and the next result follows:

Proposition 6.2 *Let k be algebraically closed, and let $G = H \times H'$. The map*

$$\begin{aligned}\phi: a(H) \otimes_{\mathbb{Z}} a(H') &\rightarrow a(G), \\ \phi([M] \otimes_{\mathbb{Z}} [N]) &\mapsto [\inf_{G,H}(M)][\inf_{G,H'}(N)]\end{aligned}$$

of abelian groups is a monomorphism of rings.

Proof: The underlying additive group of a representation ring is the free abelian group generated by isomorphism classes of indecomposable modules. Let M be an indecomposable kH -module and let N be an indecomposable kH' -module, so $[M]$ is an element of the generating set of the additive group of $a(H)$ and $[N]$ is an element of the generating set of the additive group of $a(H')$. Then by 5.4, $\inf_{G,H}(M) \otimes \inf_{G,H'}(N)$ is indecomposable, and so $[\inf_{G,H}(M)][\inf_{G,H'}(N)]$ is an element of the generating set of the additive group of $a(G)$. Thus ϕ is a monomorphism of abelian groups. On the other hand, ϕ is clearly a homomorphism of rings, and we are done. \square

Remark 6.3 *In fact, the previous result is true for arbitrary base fields by a standard result that states that if k' is an extension field of k and M and N are kG -modules, that $M \cong N$ if and only if $M \otimes_k k' \cong N \otimes_k k'$.*

Corollary 6.4 *Let $G = H \times H'$ and let $x \in a(H)$ and $y \in a(H')$ be nonzero. Then the product $(\inf_{G,H}(x))(\inf_{G,H'}(y)) \in a(G)$ is nonzero.*

Proof: The element $x \otimes_{\mathbb{Z}} y \in a(H) \otimes_{\mathbb{Z}} a(H')$ is zero if and only if $x = 0$ or $y = 0$. \square

There exist a variety of results on direct sum decompositions of representation rings; we will need the following version of a theorem of Dress:

Proposition 6.5 (Dress) *The representation ring $a(G)_p$ has a direct sum decomposition:*

$$a(G)_p = a(G, H)_p \oplus a_0(G, H)_p$$

Proof: See [1, theorem 5.7.1]. \square

Benson and Parker prove a similar result:

Proposition 6.6 *The representation ring $A(G)$ has a direct sum decomposition:*

$$A(G) = \text{Im}(\text{ind}_H^G) \oplus \text{Ker}(\text{res}_H^G)$$

Proof: See [1, corollary 5.4.11]. \square

Note that $\text{Im}(\text{ind}_H^G)$ is generated by isomorphism classes of relatively H -free modules, so $\text{Im}(\text{ind}_H^G) \subseteq a(G, H)$ and therefore $a_0(G, H) \subseteq \text{Ker}(\text{res}_H^G)$. In the cases of interest to us, these inclusions are equalities, and so the results give identical decompositions:

Proposition 6.7 *Let $G = H \times H'$ be a p -group such that H is cyclic. The representation ring $a(G)_p$ has a direct sum decomposition*

$$a(G)_p = e_H a(G)_p \oplus e'_H a(G)_p$$

where $e_H = \frac{1}{|G:H|} [k_H \uparrow^G]$ and $e'_H = [k] - e_H$.

Furthermore, we have the following equalities:

$$e_H a(G)_p = a(G, H)_p = \text{Im}(\text{ind}_H^G)$$

$$e'_H a(G)_p = a_0(G, H)_p = \text{Ker}(\text{res}_H^G)$$

Proof: Since $k_H \uparrow^G \downarrow_H$ is a direct sum of $|G : H|$ copies of the trivial representation k_H , the isomorphism $k_H \uparrow^G \otimes k_H \uparrow^G \cong (k_H \uparrow^G \downarrow_H \otimes k_H) \uparrow^G$ shows that the product $k_H \uparrow^G \otimes k_H \uparrow^G$ is isomorphic to a direct sum of $|G : H|$ copies of $k_H \uparrow^G$; it follows that e_H is idempotent. It is clear that $e_H \in a(G, H)_p$, so $e_H a(G)_p \subseteq a(G, H)_p$; on the other hand, by 5.2, if M is a relatively H -projective kG -module, it is of the form $M = N \uparrow^G$ for some kH -module N . Then by 5.2, $M \cong \inf_{G,H}(N) \otimes k_H \uparrow^G$, so

$[M] = |G : H|e_H[\inf_{G,H}(N)]$. Thus $a(G, H)_p \subseteq e_H a(G)_p$, so $a(G, H)_p = e_H a(G)_p$.

The rest follows immediately. \square

In the general case, where these ideals do not coincide, we will use e_H and e'_H to refer to the orthogonal idempotents associated to the ideals $a(G, H)_p$ and $a_0(G, H)_p$.

Many of the arguments we will make would be much cleaner if it were true that $\Omega_H k \otimes M \cong \Omega_H M$, but in general we only have equivalence modulo relatively H -projective direct summands. One way around this problem is to work modulo the ideal $a(G, H)$ and lift; proposition 6.7 implies that we may as well work inside the ideal $a_0(G, H)$ instead, and the next lemma makes this explicit:

Lemma 6.8 *Let H be a subgroup of G , let M be an indecomposable kG -module not projective relative to H , and let e'_H be the idempotent associated to the ideal $a_0(G, H)_p$. Then $[\Omega_H k]e'_H[M] = e'_H[\Omega_H M]$.*

Proof: Since $\Omega_H k \otimes M \cong \Omega_H M \oplus (H\text{-projective})$, we have

$$\begin{aligned} [\Omega_H k]e'_H[M] &= e'_H[\Omega_H k \otimes M] \\ &= e'_H[\Omega_H M \oplus (H\text{-projective})] \\ &= e'_H[\Omega_H M] \end{aligned}$$

and we are done. \square

CHAPTER 7

SYMMETRIC AND ALTERNATING POWERS

Let $T = \langle \sigma \mid \sigma^2 = 1 \rangle$ be a cyclic group. Given a kG -module M , the group T acts on the product $M \otimes M$ by:

$$\sigma(x \otimes y) = y \otimes x$$

If the characteristic of the base field k is odd or zero, the group algebra kT has primitive orthogonal idempotents $e = \frac{1}{2}(1 + \sigma)$, $e' = \frac{1}{2}(1 - \sigma)$. These idempotents define a splitting

$$M \otimes M = S^2(M) \oplus \Lambda^2(M)$$

where the *symmetric square* $S^2(M) = e(M \otimes M)$ is the $+1$ eigenspace of the action of T and spanned by elements of the form $\frac{1}{2}(x \otimes y + y \otimes x)$ and the *alternating square* $\Lambda^2(M) = e'(M \otimes M)$ is the -1 eigenspace of the action, spanned by elements $\frac{1}{2}(x \otimes y - y \otimes x)$.

More generally, if C is a complex of kG -modules, T acts on $C \otimes C$ via

$$\sigma(x \otimes y) = (-1)^{\deg(x)\deg(y)} y \otimes x$$

and provides a splitting $C \otimes C = S^2(C) \oplus \Lambda^2(C)$. Furthermore, the Künneth formula $H(C \otimes C) \cong H(C) \otimes H(C)$ implies that:

$$H(S^2(C)) = S^2(H(C \otimes C))$$

$$H(\Lambda^2(C)) = \Lambda^2(H(C \otimes C))$$

Lemma 7.1 *Let G be a p -group and let k be a field of characteristic $p \neq 2$. If n is an even integer, then $S^2(\Omega^n k) = \Omega^{2n} k \oplus (\text{projective})$ and $\Lambda^2(\Omega^n k)$ is projective; if n is odd, then $\Lambda^2(\Omega^n k) = \Omega^{2n} k \oplus (\text{projective})$ and $S^2(\Omega^n k)$ is projective.*

Proof: Given any kG -module M of dimension m , we have $\dim_k(S^2(M)) = m(m+1)/2$ and $\dim_k(\Lambda^2(M)) = m(m-1)/2$. We also have an isomorphism $\Omega^n k \otimes \Omega^n k \cong \Omega^{2n} \oplus (\text{projective})$, where Ω^{2n} is indecomposable and $\dim_k(\Omega^{2n} k) \equiv 1 \pmod{p}$; it follows that any direct summand of $\Omega^n k \otimes \Omega^n k$ is projective if and only if its dimension is divisible by p . If n is even, $\dim_k(\Omega^n k) \equiv 1 \pmod{p}$, so $\dim_k(\Lambda^2(\Omega^n k)) \equiv 0 \pmod{p}$ and $\Lambda^2(\Omega^n k)$ is projective. If n is odd, $\dim_k(\Omega^n k) \equiv -1 \pmod{p}$, so $\dim_k(S^2(\Omega^n k)) \equiv 0 \pmod{p}$ and $S^2(\Omega^n k)$ is projective. The rest follows immediately. \square

If H is a normal subgroup of G , the module $\Omega_H k$ is just the inflation from G/H to G of $\Omega k_{G/H}$, and we get the following corollary:

Corollary 7.2 *Let G be a p -group with normal subgroup H and let k be a field of characteristic $p \neq 2$. If n is an even integer, then $S^2(\Omega_H^n k) = \Omega_H^{2n} k \oplus (H\text{-projective})$ and $\Lambda^2(\Omega_H^n k)$ is projective relative to H ; if n is odd, then $\Lambda^2(\Omega_H^n k) = \Omega_H^{2n} k \oplus (H\text{-projective})$ and $S^2(\Omega_H^n k)$ is projective relative to H . In these decompositions, all H -projective indecomposable summands are of the form $k_H \uparrow^G$.*

Proof: Since $\Omega_H k = \inf_{G,H}(\Omega k_{G/H})$, applying the previous lemma to $\Omega^n k_{G/H}$ and inflating the results to G proves the corollary. \square

Proposition 7.3 *Let M and N be kG -modules. The symmetric and alternating squares of the tensor product $M \otimes N$ may be decomposed as:*

$$S^2(M \otimes N) \cong S^2(M) \otimes S^2(N) \oplus \Lambda^2(M) \otimes \Lambda^2(N)$$

$$\Lambda^2(M \otimes N) \cong S^2(M) \otimes \Lambda^2(N) \oplus \Lambda^2(M) \otimes S^2(N)$$

Proof: It is clear that we have the following decomposition:

$$(M \otimes N) \otimes (M \otimes N) \cong (M \otimes M) \otimes (N \otimes N)$$

We have decompositions

$$(M \otimes N) \otimes (M \otimes N) = S^2(M \otimes N) \oplus \Lambda^2(M \otimes N)$$

$$(M \otimes M) \otimes (N \otimes N) = (S^2(M) \oplus \Lambda^2(M)) \otimes (S^2(N) \oplus \Lambda^2(N))$$

Matching the $+1$ and -1 eigenspaces of the action of T to these decompositions yields the formulas of the proposition. \square

Corollary 7.4 *Let M be a kG -module and let H be a normal subgroup of G . If n is an even integer, we have isomorphisms*

$$S^2(\Omega^n M) \cong \Omega^{2n}(S^2(M)) \oplus (\text{projective})$$

$$\Lambda^2(\Omega^n M) \cong \Omega^{2n}(\Lambda^2(M)) \oplus (\text{projective})$$

$$S^2(\Omega_H^n M) \cong \Omega_H^{2n}(S^2(M)) \oplus (H\text{-projective})$$

$$\Lambda^2(\Omega_H^n M) \cong \Omega_H^{2n}(\Lambda^2(M)) \oplus (H\text{-projective})$$

and if n is odd, we have isomorphisms

$$S^2(\Omega^n M) \cong \Omega^{2n}(\Lambda^2(M)) \oplus (\text{projective})$$

$$\Lambda^2(\Omega^n M) \cong \Omega^{2n}(S^2(M)) \oplus (\text{projective})$$

$$S^2(\Omega_H^n M) \cong \Omega_H^{2n}(\Lambda^2(M)) \oplus (H\text{-projective})$$

$$\Lambda^2(\Omega_H^n M) \cong \Omega_H^{2n}(S^2(M)) \oplus (H\text{-projective})$$

Proof: Since $\Omega^n k \otimes M \cong \Omega^n M \oplus (\text{projective})$ and $\Omega_H^n k \otimes M \cong \Omega_H^n M \oplus (H\text{-projective})$, the result follows directly from 7.1, 7.2, and 7.3. \square

Remark 7.5 *Generalizing this approach, one may observe that if M is a module for any Hopf k -algebra, the symmetric group S_n acts naturally on the tensor power module $M^{\otimes n}$ by permuting the tensor basis, and any decomposition*

$$1 = \sum e_j \in kS_n$$

of the identity element of kS_n into orthogonal idempotents yields a direct sum decomposition

$$M^{\otimes n} = \bigoplus e_j(M^{\otimes n})$$

of the tensor power. This approach provides many of the strongest general results known on tensor decomposition.

CHAPTER 8

NILPOTENT ELEMENTS OF COMMUTATIVE RINGS

In this chapter, we develop a context in which we will be able to present our results and those of earlier papers in a uniform manner. Recall that lemma 6.8 shows that the action of the translation operation $\Omega_H(-)$ on elements of $a_0(G, H)_p$ is equivalent to multiplication by $[\Omega_H k]$. The following definition abstracts this approach to periodicity.

Definition 8.1 *Let R be a commutative ring and let $r_1, r_2 \in R$. We will say that r_2 is of period n with respect to r_1 , or n -periodic relative to r_1 , if n is the smallest positive integer such that $r_1^n r_2 = r_2$.*

The next lemma makes explicit the connection between periodicity in the sense of the preceding definition, periodicity of a module, and periodicity of a module relative to a subgroup.

Lemma 8.2 *Let M be a kG -module, let H be a subgroup of G , and let e be the idempotent in $a(G)$ such that $ea(G) = a_0(G, H)$. Then M is periodic of period n relative to H if and only if $e[M] \in a(G)$ is periodic of period n with respect to $[\Omega_H k]$.*

Proof: This is a direct consequence of 6.8 and the preceding definition. \square

Note that if r_2 is n -periodic relative to r_1 , then any positive power r_2^i is of period at most n with respect to r_1 .

The following result gives us a general construction and method of proof for nilpotent elements.

Proposition 8.3 *Let R be a commutative ring and let $r_1, r_2 \in R$ be elements such that r_2 is of period two with respect to r_1 and n is the smallest positive integer such that r_2^n is of period one with respect to r_1 . Then the element $\nu = r_2 - r_1 r_2$ is nilpotent of degree at most n . Provided that $2 \in R$ is not a zero divisor, ν is of nilpotence degree n .*

Proof: First, note that $r_2 - r_1 r_2 \neq 0$, since r_2 is not of period one relative to r_1 . On the other hand, we can factor $\nu = (1 - r_1)r_2$ and write

$$\begin{aligned}
 \nu^n &= ((1 - r_1)r_2)^n \\
 &= (1 - r_1)^n r_2^n \\
 &= (1 - r_1)^{n-1} (1 - r_1) r_2^n \\
 &= (1 - r_1)^{n-1} 0 \\
 &= 0
 \end{aligned}$$

since $r_2^n = r_1 r_2^n$ by assumption. Finally, if 2 is not a zero divisor, consider the element $s = \frac{1}{2}(1 - r_1)$, adjoining $\frac{1}{2}$ to R if necessary. This element acts as an idempotent on any element $r \in R$ of period two with respect to r_1 , in the sense that $s^2 r = sr$:

$$\begin{aligned}
 s^2 r &= \left(\frac{1}{2}(1 - r_1)\right)^2 r \\
 &= \frac{1}{4}(1 - 2r_1 + r_1^2)r \\
 &= \frac{1}{4}(r - 2r_1 r + r_1^2 r) \\
 &= \frac{1}{4}(2r - 2r_1 r) \\
 &= \frac{1}{2}(1 - r_1)r \\
 &= sr
 \end{aligned}$$

By induction, if r is of period two with respect to r_1 , then $s^i r = sr$ for all positive integers i . Furthermore, for any $r \in R$, it is clear that $sr = 0$ if and only if r is of

period 1 with respect to r_1 . It follows that $(sr_2)^j = sr_2^j \neq 0$ for $j < n$, and likewise $\nu^j \neq 0$ for $j < n$, since $\nu = 2sr_2$. Thus ν is of nilpotence degree n . \square

Remark 8.4 *Every nilpotent described in [3] and [7], which is to say every published example known to the author, may be viewed as an example of this construction. In [3], let $r_1 = [\Omega k]$ or, in a few cases, $r_1 = [\tilde{\Omega}R]$; then $r_2 = e_{1_G}[L_\zeta]$ is of period two with respect to r_1 and r_2^2 is of period one with respect to r_1 . In [7], take $r_1 = [\Omega k]$ and $r_2 = e_{1_G}[M]$, where M is any one of the modules Heldner uses.*

The next propositions describe a way to use nilpotents of low degree to construct nilpotents of higher degree.

Proposition 8.5 *Let R be a commutative ring, let $r_1, r_2 \in R$ be elements of nilpotence degrees n_1 and n_2 respectively such that $r_1^{n_1-1}r_2^{n_2-1} \neq 0$. Then the element $\nu = r_1 + r_2$ is of nilpotence degree at most $n_1 + n_2 - 1$. If $(n_1 + n_2 - 2)!$ is not a zero divisor in R , then ν is of nilpotence degree $n_1 + n_2 - 1$.*

Proof: First, it is clear that $\nu^{n_1+n_2-1} = 0$, since every term in the binomial expansion of $(r_1 + r_2)^{n_1+n_2-1}$ is a multiple of either $r_1^{n_1}$ or $r_2^{n_2}$. Next, assume that $(n_1 + n_2)!$ is not a zero divisor in R , and consider $\nu^{n_1+n_2-2} = (r_1 + r_2)^{n_1+n_2-2}$. Expanding the right hand side of the equality, we note that only one term in the resulting sum is not a multiple of either $r_1^{n_1} = 0$ or of $r_2^{n_2} = 0$; we thus have $\nu^{n_1+n_2-2} = \binom{n_1+n_2-2}{n_1-1} r_1^{n_1-1} r_2^{n_2-1}$, which is nonzero by assumption. \square

The previous result may be extended to arbitrary finite sums of nilpotents; the proof is straightforward and omitted:

Proposition 8.6 *Let R be a commutative ring, and let $r_1, r_2, \dots, r_m \in R$ be of nilpotence degrees n_1, n_2, \dots, n_m respectively. Suppose that $\prod_{i=1}^m r_i^{n_i-1} \neq 0$, and that $(\sum_{i=1}^m n_i - 1)!$ is not a zero divisor in R . Then the element $\nu = \sum_{i=1}^m r_i$ is of nilpotence degree $(\sum_{i=1}^m n_i) - m + 1$.*

Some attention has been given to the question of how high the nilpotence degree of a nilpotent element of $a(G)$ can be; the next result implies that one may construct a nilpotent element of arbitrarily high degree, provided that one is willing to use an elementary abelian group of arbitrarily high rank in the process.

Proposition 8.7 *Let k be a field, and let G_1 and G_2 be arbitrary finite groups. Let $x_1 \in a(G_1)$ be of nilpotence degree n_1 and $x_2 \in a(G_2)$ be of nilpotence degree n_2 . Then, identifying x_1 and x_2 with their inflations to $a(G)$, the element $\nu = x_1 + x_2 \in a(G_1 \times G_2)$ is of nilpotence degree $n_1 + n_2 - 1$. More generally, given $x_1 \in a(G_1), x_2 \in a(G_2), \dots, x_m \in a(G_m)$ of nilpotence degrees n_1, n_2, \dots, n_m respectively, and $G = \prod_{i=1}^m G_i$, the element $\nu = \sum_{i=1}^m x_i \in a(G)$ is nilpotent of nilpotence degree $(\sum_{i=1}^m n_i) - m + 1$.*

Proof: In the first case, the product $x_1^{n_1-1}x_2^{n_2-1}$ is nonzero by corollary 6.4, and the result follows immediately from proposition 8.5. Extending the result to the second case is straightforward. \square

CHAPTER 9

MODULES FOR $\mathbb{Z}/p \times \mathbb{Z}/p$

Our arguments will use modules constructed by generators and relations. We start constructing them here.

Let p be an odd prime.

Definition 9.1 *Let $G = \langle a, b | a^p = b^p = [a, b] = 1 \rangle$ be the elementary abelian group of order p^2 , and let $H_a = \langle a \rangle$ and $H_b = \langle b \rangle$. Let $\alpha = a - 1$ and $\beta = b - 1 \in kG$, so that $\alpha^p = \beta^p = 0$, and α and β generate $\text{Rad}(kG)$. By V_i we will mean the indecomposable kH_b -module of dimension i , so that $V_1 = k_{H_b}$, $V_{p-1} = \Omega k_{H_b}$, and $V_p = kH_b$ as a kH_b -module. By D_i we will mean the induced module $V_i \uparrow^G$.*

Lemma 9.2 *i) The module D_i is a self-dual, indecomposable, relatively H_b -projective module of dimension ip , and its head $D_i/\text{Rad}(D_i)$ and socle $\text{Soc}(D_i)$ are one-dimensional.*

ii) There are isomorphisms of the form:

$$\begin{aligned} D_i &\cong \inf_{G, H_b}(V_i) \otimes k_{H_b} \uparrow^G, \\ D_i \downarrow_{H_b} &\cong \bigoplus_{j=1}^p V_i, \\ D_i \downarrow_{H_a} &\cong \bigoplus_{j=1}^i kH_a \end{aligned}$$

iii) If x is any generator of D_i , then $\text{Soc}(D_i)$ is generated by $\beta^{i-1} \alpha^{p-1} x$.

Proof: i) Since kH is cyclic, every kH -module is self-dual and every indecomposable kH -module is absolutely indecomposable and uniserial. Now apply 5.2, and observe that

$$\dim_k(\operatorname{Hom}_{kG}(k, D_i)) = \dim_k(\operatorname{Hom}_{kG}(D_i, k)) = 1$$

by 3.4.

ii) The isomorphisms follow directly from 5.2 and 3.4.

iii) Since the head $D_i/\operatorname{Rad}(D_i)$ is one-dimensional, the module D_i is generated by a single element. If x is such an element, then $\beta^{i-1}\alpha^{p-1}x$ is annihilated by $\operatorname{Rad}(kG)$, so it must lie in $\operatorname{Soc}(D_i)$, which is one-dimensional, and so $\beta^{i-1}\alpha^{p-1}x$ spans it. \square

Since G is a p -group, the free (left) kG -module ${}_G kG$ is indecomposable. Since $\operatorname{Rad}(kG) = \Omega k$ is generated by α and β , we get an easy explicit presentation:

Lemma 9.3 *There is an isomorphism of the form:*

$$\Omega k \cong \langle u_1, u_2 | \beta^{p-1}u_1, \alpha^{p-1}u_2, \alpha u_1 - \beta u_2 \rangle$$

Proof: The elements u_1 and u_2 encode the relations on α and β respectively, and

$$\alpha \mapsto u_1$$

$$\beta \mapsto u_2$$

gives the isomorphism. \square

We will also need a presentation for $\Omega^{-1}k$:

Lemma 9.4 *There is an isomorphism of the form:*

$$\Omega^{-1}k \cong \langle v | \alpha^{p-1}\beta^{p-1}v \rangle$$

Proof: This follows directly from the fact that $\Omega^{-1}k \cong kG/\operatorname{Soc}(kG)$. \square

We will deal with the operations Ω of translation and Ω_{H_b} of translation relative to H_b in what follows. In some respects, translation relative to H_b is easier

to work with. Since H_b is normal, the module $\Omega_{H_b}k$ can be expressed as an inflation $\inf_{G, G/H_b}(\Omega k_{G/H_b})$. Because the quotient group $G/H_b \cong H_a$ is cyclic, we know that $\Omega k_{G/H_b} \otimes \Omega k_{G/H_b} = k_{G/H_b} \oplus (\text{projective})$ and $\Omega^2 k_{G/H_b} = k_{G/H_b}$; it follows that $\Omega_{H_b}^2 k_G = k_G$. As an important consequence, we have the following result:

Lemma 9.5 *If M is an indecomposable kG -module and not projective relative to H_b , then M is periodic relative to H_b of period one or two.*

Proof: By proposition 7.2, $\Omega_{H_b}^2 M$ is a direct summand of $\Omega_{H_b}^2 k \otimes M = k \otimes M \cong M$, and similarly $\Omega_{H_b}^{-2}(\Omega_{H_b}^2 M) \cong M$ is a summand of $\Omega_{H_b}^{-2} k \otimes \Omega_{H_b}^2 M = k \otimes \Omega_{H_b}^2 M \cong \Omega_{H_b}^2 M$, so $M \cong \Omega_{H_b}^2 M$. \square

Using the presentation for Ωk as the beginning of a projective resolution of k , we can construct a presentation for $\Omega^2 k$:

Lemma 9.6 *There is an isomorphism of the form:*

$$\Omega^2 k \cong \langle w_1, w_2, w_3 | \beta w_1, \alpha w_3, \alpha w_1 - \beta^{p-1} w_2, \beta w_3 - \alpha^{p-1} w_2 \rangle$$

Proof: Let $\langle \hat{u}_1 \rangle \oplus \langle \hat{u}_2 \rangle$ be a projective cover of Ωk so that, using the presentation of the previous lemma, $\hat{u}_1 \mapsto u_1$ and $\hat{u}_2 \mapsto u_2$. Then the elements w_1, w_2 and w_3 correspond to the generators $\beta^{p-1} u_1, -\alpha^{p-1} u_2$, and $\alpha u_1 - \beta u_2$ of the kernel of the cover. \square

If $\zeta \in H^n(G, k)$, then ζ can be represented by a unique homomorphism $\hat{\zeta} : \Omega^n k \rightarrow k$, and such a map allows us to associate a unique module $L_\zeta = \text{Ker}(\hat{\zeta})$ to ζ .

We wish to consider a particular subset of the modules of type L_ζ associated to cohomology elements of degree 2. We use the presentation of $\Omega^2 k$ given in 9.1.

Definition 9.7 *Given $\lambda \in k$, define $\hat{\zeta}(\lambda) \in \text{Hom}_{kG}(\Omega^2 k, k)$ by:*

$$\hat{\zeta}(\lambda)(w_1) = 0$$

$$\hat{\zeta}(\lambda)(w_2) = -\lambda$$

$$\hat{\zeta}(\lambda)(w_3) = 1$$

By 2.2, $\hat{\zeta}(\lambda)$ extends to a kG -homomorphism. We then define $L(\lambda) = \text{Ker}(\hat{\zeta})$.

It is clear that $L(\lambda)$ is the module of type L_ζ associated to the cohomology element $cls(\hat{\zeta}(\lambda)) \in H^2(G, k)$.

Lemma 9.8 *There is an isomorphism of the form:*

$$L(\lambda) \cong \langle l_1, l_2 | \beta l_1, \beta^{p-1} l_2 - \alpha l_1 - \lambda \beta^{p-2} \alpha^{p-1} l_2 \rangle$$

Proof: Since it is the kernel of the map $\hat{\zeta}(\lambda)$, the module $L(\lambda)$ is generated by the elements w_1 and $w_2 + \lambda w_3$ in $\Omega^2(k)$. Setting $l_1 = w_1$ and $l_2 = w_2 + \lambda w_3$ gives the presentation. \square

Proposition 9.9 *Let $\lambda \in k^\times$ and let $L = L(\lambda)$ using the presentation in lemma 9.8. The restrictions $L \downarrow_{H_a}$ and $L \downarrow_{H_b}$ have decompositions*

$$L \downarrow_{H_a} = \langle l_1 \rangle_{H_a} \oplus \bigoplus_{i=0}^{p-2} \langle \beta^i l_2 \rangle_{H_a}$$

$$L \downarrow_{H_b} = \langle l_1 \rangle_{H_b} \oplus \bigoplus_{i=0}^{p-1} \langle \alpha^i l_2 \rangle_{H_b}$$

where

$$\langle l_1 \rangle_{H_a} \cong \langle \beta^i l_2 \rangle_{H_a} \cong kH_a \text{ for } 1 \leq i \leq p-1$$

$$\langle l_1 \rangle_{H_b} \cong kH_b$$

$$\langle \alpha^i l_2 \rangle_{H_b} \cong kH_b \text{ for } 1 \leq i < p-1$$

$$\langle \alpha^{p-1} l_2 \rangle_{H_b} \cong \Omega kH_b$$

Proof: Follows directly from the presentation. \square

Lemma 9.10 *Let $\lambda \in k^\times$ and let $L = L(\lambda)$ as in 9.8. The map*

$$\gamma: L \rightarrow k$$

$$\gamma: l_1 \mapsto 1_k$$

$$\gamma: l_2 \mapsto 0$$

defines a short exact sequence:

$$0 \longrightarrow \Omega^{-1}k \longrightarrow L \xrightarrow{\gamma} k \longrightarrow 0$$

Proof: The kernel of γ is generated by the element l_2 . By the relations, the annihilator of l_2 is the socle of kG , so $\langle l_2 \rangle \cong kG/\text{Soc}(kG) \cong \Omega^{-1}k$; the rest is clear. \square

CHAPTER 10

THE CLASSES $M(-, -; i)$

As in the previous chapter, $G = \mathbb{Z}/p \times \mathbb{Z}/p$ and k is of characteristic $p > 2$. By L we will mean any module of the form $L(\lambda)$ for $\lambda \in k^\times$ with the presentation given in lemma 9.8. Recall that if $1 \leq i \leq p$, we defined V_i to be the indecomposable kH_b -module of dimension i , and D_i to be the induced module $V_i \uparrow^G$. We will also need to use the kH_b -maps $\theta_{V_i}: D_i \rightarrow V_i$ as defined in 3.6.

In this chapter, we define classes of modules $M(-, -; i)$ which arise as kernels of maps $L(\lambda) \rightarrow D_i$.

Let us consider $\text{Hom}_{kG}(L, D_i)$. Since

$$\text{Hom}_{kG}(L, D_i) \cong \text{Hom}_{kH_b}(L \downarrow_{H_b}, V_i)$$

by 3.4, and all indecomposable kH_b -modules are uniserial, applying proposition 9.9 shows that $\text{Hom}_{kG}(L, D_i)$ is $(p+1)$ -dimensional and the stable homomorphism group $\text{StHom}_{kG}(L, D_i)$ is two-dimensional.

Given any homomorphism $f \in \text{Hom}_{kG}(L, D_i)$, the values $f(l_1)$ and $f(\beta^{i-1}\alpha^{p-1}l_2)$ determine the stable class of f , as the next set of results demonstrate.

Lemma 10.1 *Let $f \in \text{Hom}_{kG}(L, D_i)$, let $1 \leq i < p-1$ and let x generate D_i . Then $f(l_1) = \mu\beta^{i-1}\alpha^{p-1}x$ and $f(\beta^{i-1}\alpha^{p-1}l_2) = \nu\beta^{i-1}\alpha^{p-1}x$ for some $\mu, \nu \in k$. The map f is surjective if and only if $\nu \neq 0$.*

Proof: By 9.2, $\beta^{i-1}\alpha^{p-1}x$ generates $\text{Soc}(D_i)$. Since

$$\alpha l_1 = \beta^{p-1}l_2 - \lambda\beta^{p-2}\alpha^{p-1}l_2$$

is a multiple of β^{p-2} , and β^{p-2} annihilates D_i , it follows that $f(\alpha l_1) = 0$, and so $\alpha f(l_1) = 0$. Furthermore, $\beta f(l_1) = f(\beta l_1) = f(0) = 0$. The image $f(l_1)$ is thus annihilated by $\text{Rad}(kG)$, and must therefore lie in $\text{Soc}(D_i)$, and so be a multiple of $\beta^{i-1}\alpha^{p-1}x$.

Similarly, $f(\beta^{i-1}\alpha^{p-1}l_2)$ is annihilated by $\text{Rad}(kG)$, so it too lies in $\text{Soc}(D_i)$ and is a multiple of $\beta^{i-1}\alpha^{p-1}x$.

Finally, if $\nu \neq 0$, then

$$\begin{aligned} \beta^{i-1}\alpha^{p-1}f(l_2) &= f(\beta^{i-1}\alpha^{p-1}l_2) \\ &= \nu\beta^{i-1}\alpha^{p-1}x \end{aligned}$$

which implies that $f(l_2) - \nu x$ lies in $\text{Rad}(D_i)$, and so $f(l_2)$ generates D_i . If $\nu = 0$, then $f(l_2)$ itself lies in $\text{Rad}(D_i)$; since $f(l_1)$ lies in $\text{Soc}(D_i) \subseteq \text{Rad}(D_i)$, the image of f lies in the radical and so h cannot be surjective. \square

Remark 10.2 *The proof of the previous proposition shows that, for computations that only depend on the stable class of f , we might as well assume that $f(l_2) = \nu x$ if $f(\beta^{i-1}\alpha^{p-1}l_2) = \nu\beta^{i-1}\alpha^{p-1}x$.*

Proposition 10.3 *Any two maps $f, f' \in \text{Hom}_{kG}(L, D_i)$ are stably equivalent if and only if $f(l_1) = f'(l_1)$ and $f(\beta^{i-1}\alpha^{p-1}l_2) = f'(\beta^{i-1}\alpha^{p-1}l_2)$.*

Proof: Suppose $(f - f')(l_1) = (f - f')(\beta^{i-1}\alpha^{p-1}l_2) = 0$. Then $\theta_{V_i} \circ (f - f') \in \text{Hom}_{kH_b}(M \downarrow_{H_b}, V_i)$ also kills l_1 and $\beta^{i-1}\alpha^{p-1}l_2$. But by proposition 9.9, $\theta_{V_i} \circ (f - f')$ then kills $\langle l_1 \rangle_{H_b} \cong k_{H_b}$, and its restriction to the summand $\langle \alpha^{p-1}l_2 \rangle_{H_b} \cong \Omega k_{H_b}$ is not surjective and thus factors through a projective by 4.14; since its restriction to the non-projective part of $L \downarrow_{H_b}$ factors through a projective, the whole map factors through a projective. By 3.4, $f - f'$ factors through a projective as well, and so f is stably equivalent to f' . Conversely, if $f - f'$ does not kill l_1 or $\beta^{i-1}\alpha^{p-1}l_2$, then $\theta_{V_i} \circ (f - f')$ does not factor through a projective, and so neither does $f - f'$. \square

Corollary 10.4 *For any choice of values $\mu, \nu \in k$, there is a map $f \in \text{Hom}_{kG}(L, D_i)$ for which $f(l_1) = \mu\beta^{i-1}\alpha^{p-1}x$ and $f(\beta^{i-1}\alpha^{p-1}l_2) = \nu\beta^{i-1}\alpha^{p-1}x$.*

Proof: Given any map $f \in \text{Hom}_{kG}(L, D_i)$, its stable equivalence class is entirely determined by the values of $f(l_1)$ and $f(\beta^{i-1}\alpha^{p-1}l_2)$, and the stable homomorphism group $\text{StHom}_{kG}(L, D_i)$ is two-dimensional. \square

Definition 10.5 *Let $\lambda_1, \lambda_2 \in k$ be nonzero, let $1 \leq i < p-1$, let $L = L(\lambda_2)$ and let x be a generator of D_i . Choose $f \in \text{Hom}_{kG}(L, D_i)$ so that $f(l_1) = -\lambda_1\beta^{i-1}\alpha^{p-1}x$ and $f(\beta^{i-1}\alpha^{p-1}l_2) = \beta^{i-1}\alpha^{p-1}x$. By $M(\lambda_1, \lambda_2; i)$, we shall mean $\text{Ker}(f)$. By proposition 10.3, $M(\lambda_1, \lambda_2; i)$ is well-defined, and by lemma 10.1 f is surjective, so we have a short exact sequence:*

$$0 \rightarrow M(\lambda_1, \lambda_2; i) \rightarrow L \rightarrow D_i \rightarrow 0$$

We will refer to the set of modules $\{M(\lambda_1, \lambda_2; i)\}_{\lambda_1, \lambda_2 \in k^\times}$ as $M(-, -; i)$. The case where $i = p-2$ will be of special interest. By $M(\lambda_1, \lambda_2)$ we will mean $M(\lambda_1, \lambda_2; p-2)$, and by $M(-, -)$ we will mean the set $\{M(\lambda_1, \lambda_2)\}_{\lambda_1, \lambda_2 \in k^\times}$.

As a submodule of $L(\lambda_2)$, $M(\lambda_1, \lambda_2; i)$ is generated by elements $m_1 = l_1 + \lambda_1\beta^{i-1}\alpha^{p-1}l_2$ and $m_2 = \beta^i l_2$. Since $L(\lambda_2)\downarrow_{H_a}$ is a free kH_a -module of rank p , and $D_i\downarrow_{H_a}$ is a free kH_a -module of rank i , $M(\lambda_1, \lambda_2; i)\downarrow_{H_a}$ is a free kH_a -module of rank $p-i$. It is straightforward to verify that β acts on $M(\lambda_1, \lambda_2; i)$ by $\beta m_1 = \lambda_1\alpha^{p-1}m_2$, $\beta^{p-i-1}m_2 = \alpha m_1 + \lambda_2\beta^{p-i-2}\alpha^{p-1}m_2$.

Proposition 10.6 *Let $\lambda_1, \lambda_2 \in k^\times$, let $1 \leq i < p-1$ and let $M = M(\lambda_1, \lambda_2; i)$. We have decompositions*

$$M\downarrow_{H_b} = \langle m_1 \rangle_{H_b} \oplus \bigoplus_{j=0}^{p-2} \langle \alpha^j m_2 \rangle_{H_b}$$

$$M\downarrow_{H_a} = \langle m_1 \rangle_{H_a} \oplus \bigoplus_{j=0}^{p-i-1} \langle \beta^j m_2 \rangle_{H_a}$$

where we have:

$$\langle m_1 \rangle_{H_b} \cong V_{p-i}$$

$$\langle \alpha^j m_2 \rangle_{H_b} \cong V_{p-i} \text{ for } 0 \leq j < p-2$$

$$\langle m_1 \rangle_{H_a} \cong \langle \beta^j m_2 \rangle_{H_a} \cong kH_a \text{ for } 0 \leq j \leq p-i-2$$

Proof: First, $\beta^{p-i-1}m_1 = \lambda_1\beta^{p-2}\alpha^{p-1}l_2 \neq 0$ but $\beta^{p-i}m_1 = 0$, so $\langle m_1 \rangle_{H_b} \cong V_{p-i}$. We know from the decomposition of $L \downarrow_{H_b}$ that $\langle \alpha^j l_2 \rangle_{H_b} \cong kH_b$ for $0 \leq j < p-1$, so $\langle \alpha^j m_2 \rangle_{H_b} = \langle \alpha^j \beta^i l_2 \rangle_{H_b} \cong V_{p-i}$ for $0 \leq j < p-1$. Since $\beta^{p-i-1}m_1 = \lambda_1\beta^{p-2}\alpha^{p-1}l_2$ does not lie in $\bigoplus_{j=0}^{p-2} \langle \alpha^j m_2 \rangle_{H_b}$, we get the direct sum decomposition as claimed.

The decomposition of $M \downarrow_{H_a}$ is easier; we need only observe that h is a surjective map from the free kH_a -module $L \downarrow_{H_a}$ of rank p to the free kH_a -module $D_i \downarrow_{H_a}$ of rank i ; then $M \downarrow_{H_a}$ is a free kH_a -module of rank $p-i$ and the given elements constitute a minimal generating set. \square

The particular case where $i = p-2$ motivates the following definition:

Definition 10.7 Let λ_1 and λ_2 be arbitrary elements of k^\times . We define $\beta(\lambda_1, \lambda_2)$ to be the following 2×2 matrix with entries in kH_a :

$$\beta(\lambda_1, \lambda_2) = \begin{pmatrix} 0 & \alpha \\ \lambda_1 \alpha^{p-1} & \lambda_2 \alpha^{p-1} \end{pmatrix}$$

Remark 10.8 This definition gives a presentation of the module $M(\lambda_1, \lambda_2)$ in terms of how β acts on the underlying free kH_a -module on the generators m_1 and m_2 . We will find this presentation convenient for some computations.

Proposition 10.9 Let λ_1 and λ_2 be arbitrary elements of k^\times , and let $1 \leq i < p-1$. The module $M = M(\lambda_1, \lambda_2; i)$ is indecomposable.

Proof: By 10.6, we know that $M(\lambda_1, \lambda_2) \downarrow_{H_b}$ decomposes as the direct sum of p copies of V_{p-i} , and $M \downarrow_{H_a}$ is a free kH_a -module of rank $p-i$. The dimension of

any direct summand of M must therefore be a multiple of both p and $p - i$. Since $\dim_k(M) = p(p - i)$, it follows that M has no proper direct summands. \square

Proposition 10.10 *Let λ_1 and λ_2 be arbitrary elements of k^\times , and let $1 \leq i < p - 1$. The module $M = M(\lambda_1, \lambda_2; i)$ is periodic of period 2.*

Proof: Since M_{H_a} is free, and $G/H_a \cong H_b$ is cyclic, proposition 4.21 states that M has period one or two. If M were of period one, there would exist a short exact sequence of the form

$$0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$$

with projective middle term, and thus $2 \dim_k(M)$ would be a multiple of $p^2 = \dim_k(kG)$. Since $0 < \dim_k(M) = p(p - i) < p^2$, and p is odd, this is impossible, thus M is not of period one. \square

Proposition 10.11 *Let $\lambda_1, \lambda'_1, \lambda_2$, and λ'_2 be nonzero elements of k . Let $M = M(\lambda_1, \lambda_2)$ and $M' = M(\lambda'_1, \lambda'_2)$. Then*

$$\dim_k(\text{Hom}_{kG}(M, M')) = \begin{cases} 2p & \text{if } \lambda_1 \neq \lambda'_1, \\ 2p + 1 & \text{if } \lambda_1 = \lambda'_1 \text{ and } \lambda_2 \neq \lambda'_2, \\ 2p + 2 & \text{if } \lambda_1 = \lambda'_1 \text{ and } \lambda_2 = \lambda'_2. \end{cases}$$

Proof: By 10.6, $M \downarrow_{H_a}$ and $M' \downarrow_{H_a}$ are free kH_a -modules of rank 2. Let us pick generators m_1, m_2 and m'_1, m'_2 of M and M' be pairs of kH_a -generators on which β acts via the matrices $\beta(\lambda_1, \lambda_2)$ and $\beta(\lambda'_1, \lambda'_2)$ respectively. We may then realize $\text{Hom}_{kH_a}(M, M')$ as the group of 2×2 matrices with elements in $\text{End}(kH_a)$. Since kH_a is a commutative ring, $kH_a \cong \text{End}(kH_a)$, with the isomorphism sending $x \in kH_a$ to multiplication by x . Using this identification, we may write any kH_a -homomorphism $f \in \text{Hom}_{kH_a}(M, M')$ as a matrix

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

with entries f_{ij} in kH_a , and any such matrix defines a kH_a -homomorphism. Since $\dim_k(kH_a) = p$, we see that $\dim_k \operatorname{Hom}_{kH_a}(M, M') = 4p$.

Any given kH_a -homomorphism f is a kG -homomorphism if and only if it commutes with the action of β ; in terms of the corresponding matrices, f is a kG -homomorphism if and only if we have:

$$\beta(\lambda'_1, \lambda'_2)f - f\beta(\lambda_1, \lambda_2) = 0$$

If we set

$$\begin{aligned} R_1 &= \alpha f_{21} - \lambda_1 \alpha^{p-1} f_{12} \\ R_2 &= \alpha f_{22} - \alpha f_{11} - \lambda_2 \alpha^{p-1} f_{12} \\ R_3 &= \lambda'_1 \alpha^{p-1} f_{11} + \lambda'_2 \alpha^{p-1} f_{21} - \lambda_1 \alpha^{p-1} f_{22} \\ R_4 &= \lambda'_1 \alpha^{p-1} f_{12} + \lambda'_2 \alpha^{p-1} f_{22} - \alpha f_{21} - \lambda_2 \alpha^{p-1} f_{22} \end{aligned}$$

then the set of kG -homomorphisms in $\operatorname{Hom}_{kH_a}(M, M')$ can be seen as the set of solutions to the equations $R_1 = R_2 = R_3 = R_4 = 0$.

If we write

$$\begin{aligned} f_{11} &= \sum_{j=0}^{p-1} f_{11,j} \alpha^j, \\ f_{12} &= \sum_{j=0}^{p-1} f_{12,j} \alpha^j, \\ f_{21} &= \sum_{j=0}^{p-1} f_{21,j} \alpha^j, \\ f_{22} &= \sum_{j=0}^{p-1} f_{22,j} \alpha^j \end{aligned}$$

then it is easy to see that the condition $R_1 = 0$ is equivalent to the conditions

$$\begin{aligned} f_{21,j} &= 0, & 0 \leq j < p-2, \\ f_{21,p-2} + \lambda_1 f_{12,1} &= 0 \end{aligned}$$

on the coefficients of f_{21} and f_{12} , and thus is equivalent to $p - 1$ independent linear relations. By a similar argument, the condition $R_2 = 0$ imposes $p - 1$ linear relations on the coefficients of a kG -homomorphism f ; these relations are independent of one another and of those imposed by $R_1 = 0$, since f_{21} does not appear in R_2 and f_{22} does not appear in R_1 .

Next, since $\alpha^{p-2}R_1 = \alpha^{p-1}f_{21} = 0$, the relation R_3 simplifies to:

$$R_3 = \lambda'_1 \alpha^{p-1} f_{11} - \lambda_1 \alpha^{p-1} f_{22} = 0$$

If $\lambda_1 = \lambda'_1$, R_3 is a multiple of $\alpha^{p-2}R_2$, and thus redundant, and otherwise it imposes a single additional linear relation.

Finally, adding R_1 to R_4 yields a relation:

$$\begin{aligned} R'_4 &= \lambda'_1 \alpha^{p-1} f_{12} + \lambda'_2 \alpha^{p-1} f_{22} - \lambda_1 \alpha^{p-1} f_{12} - \lambda_2 \alpha^{p-1} f_{22} \\ &= (\lambda'_1 - \lambda_1) \alpha^{p-1} f_{12} + (\lambda'_2 - \lambda_2) \alpha^{p-1} f_{22} \end{aligned}$$

If $\lambda'_1 = \lambda_1$ and $\lambda'_2 = \lambda_2$, then R'_4 is zero and imposes no relations; otherwise it imposes a single additional linear relation.

We conclude that $\text{Hom}_{kG}(M, M')$ is the subspace of solutions to a set of $2p$, $2p - 1$ or $2p - 2$ linearly independent linear equations on the $4p$ -dimensional space $\text{Hom}_{kH_a}(M, M')$, and the result follows. \square

Corollary 10.12 *If $M = M(\lambda_1, \lambda_2)$ and $M' = M(\lambda'_1, \lambda'_2)$ are modules in $M(-, -)$, then $M \cong M'$ if and only if $\lambda_1 = \lambda'_1$ and $\lambda_2 = \lambda'_2$.*

Proof: Clear. \square

We now show that the class $M(-, -)$ is closed under translation relative to H_b .

Proposition 10.13 *Let $M = M(\lambda_1, \lambda_2)$. Then $\Omega_{H_b}(M) \cong M(\lambda_1, -\lambda_2)$.*

Proof: We prove the result by constructing the natural H_b -projective cover

$$\rho: M \downarrow_{H_b} \uparrow^G \twoheadrightarrow M$$

$$\rho: g \otimes m \mapsto gm$$

of M , restricting ρ to a direct summand of $M \downarrow_{H_b} \uparrow^G$, and analyzing the kernel of the restriction.

By proposition 10.6, we have the following:

$$\begin{aligned} M \downarrow_{H_b} &= \langle m_1 \rangle_{H_b} \oplus \bigoplus_{j=0}^{p-2} \langle \alpha^j m_2 \rangle_{H_b} \\ &\cong \bigoplus_{j=0}^{p-1} V_2 \end{aligned}$$

It follows that there is a corresponding decomposition of the induced module:

$$M \downarrow_{H_b} \uparrow^G = \langle 1_G \otimes m_1 \rangle_G \oplus \bigoplus_{j=0}^{p-2} \langle 1_G \otimes \alpha^j m_2 \rangle_G$$

Let ρ' denote the restriction of ρ to $\langle 1_G \otimes m_1 \rangle_G \oplus \langle 1_G \otimes m_2 \rangle_G$.

By definition, $\rho(1_G \otimes m_1) = m_1$ and $\rho(1_G \otimes m_2) = m_2$ generate M , so ρ' is surjective, thus H_b -split. The kernel of ρ' has generators corresponding to the relations on m_1 and m_2 ; that is, $\text{Ker}(\rho')$ is generated by the following elements:

$$\begin{aligned} m'_1 &= \beta(1_G \otimes m_1) - \lambda_1 \alpha^{p-1}(1_G \otimes m_2), \\ m'_2 &= -\beta(1_G \otimes m_2) + \lambda_2 \alpha^{p-1}(1_G \otimes m_2) + \alpha(1_G \otimes m_1) \end{aligned}$$

(The motivation for our choice of signs will become clear in the next step.)

Since $M \downarrow_{H_a}$ is a free kH_a -module on two generators and

$$\langle 1_G \otimes m_1 \rangle_G \downarrow_{H_a} \oplus \langle 1_G \otimes m_2 \rangle_G \downarrow_{H_a}$$

is a free kH_a -module with four generators, $\text{Ker}(\rho') \downarrow_{H_a}$ is a free kH_a -module on two generators. It is straightforward to then check that β acts on m'_1 and m'_2 as the

matrix $\beta(\lambda_1, -\lambda_2)$, so $\text{Ker}(\rho') \cong M(\lambda_1, -\lambda_2)$. On the other hand, ρ' is an H_b -split surjection from an H_b -projective module onto M , so the H_b -core of $\text{Ker}(\rho)$ is $\Omega_H(M)$. Since $M(\lambda_1, -\lambda_2)$ is indecomposable, we conclude that $\Omega_H(M) = M(\lambda_1, -\lambda_2)$. \square

Proposition 10.14 *Let $M = M(\lambda_1, \lambda_2)$. Then $M^* \cong M(-\lambda_1, -\lambda_2)$.*

Proof: Let $\alpha' = a^{-1} - 1$. Given $f \in M^* = \text{Hom}_k(M, k)$ and $m \in M$, we have:

$$\begin{aligned} (\alpha' f)(m) &= (a^{-1} f)(m) - f(m) \\ &= f(am) - f(m) \\ &= f(\alpha m) \end{aligned}$$

Likewise, if we let $\beta' = b^{-1} - 1$, then $(\beta' f)(m) = f(\beta m)$.

The elements $\alpha^i m_1$ and $\alpha^i m_2$ for $0 \leq i < p$ form a k -basis of M . We will write $(\alpha^i m_1)^*$ and $(\alpha^i m_2)^*$ for the elements of the dual basis of M^* . By an easy computation, we have that $\alpha'(\alpha^i m_j)^* = (\alpha^{i-1} m_j)^*$ for $1 \leq i < p$ and $j = 1, 2$. It follows that the elements $(\alpha^{p-1} m_1)^*$ and $(\alpha^{p-1} m_2)^*$ generate M^* as a kH_a -module, and the relations on m_1 and m_2 imply the following relations on $(\alpha^{p-1} m_1)^*$ and $(\alpha^{p-1} m_2)^*$:

$$\begin{aligned} \beta'(\alpha^{p-1} m_1)^* &= (\alpha^{p-2} m_2)^* \\ &= \alpha'(\alpha^{p-1} m_2)^*, \\ \beta'(\alpha^{p-1} m_2)^* &= \lambda_1(m_1)^* + \lambda_2(m_1)^* \\ &= \lambda_1 \alpha'^{p-1}(\alpha^{p-1} m_1)^* + \lambda_2 \alpha'^{p-1}(\alpha^{p-1} m_2)^* \end{aligned}$$

We would like to present M^* using generators m'_1 and m'_2 with relations expressed in terms of α and β rather than α' and β' . To this end, observe that $\alpha' = -a^{-1}\alpha$ and $\alpha'^{p-1} = \alpha^{p-1}$; similar relations hold for β' . Furthermore, since $M \downarrow_{H_b}$ is a direct sum of two-dimensional uniserial modules, so is $M^* \downarrow_{H_b}$, and it follows that $\beta' = -b^{-1}\beta$

acts identically to $-\beta$ on M^* . Taking these facts into account, we may rewrite the relations as follows:

$$\begin{aligned} -\beta(\alpha^{p-1}m_1)^* &= -a^{-1}\alpha(\alpha^{p-1}m_2)^*, \\ -\beta(\alpha^{p-1}m_2)^* &= \lambda_1\alpha^{p-1}(\alpha^{p-1}m_1)^* + \lambda_2\alpha^{p-1}(\alpha^{p-1}m_2)^* \end{aligned}$$

Now set $m'_1 = a^{-1}(\alpha^{p-1}m_2)^* + \lambda_2\alpha^{p-2}(\alpha^{p-1}m_1)^*$ and $m'_2 = (\alpha^{p-1}m_1)^*$. The elements m'_1 and m'_2 also generate M^* as a kH_a -module, and the previous relations are equivalent to:

$$\begin{aligned} \beta m'_1 &= -\lambda_1\alpha^{p-1}m'_2, \\ \beta m'_2 &= \alpha m'_1 - \lambda_2\alpha^{p-1}m'_2 \end{aligned}$$

These are precisely the relations for $M(-\lambda_1, -\lambda_2)$, proving the result. \square

Proposition 10.15 *Let $\lambda_1, \lambda_2 \in k$ be nonzero, and let $M = M(\lambda_1, \lambda_2)$. Then the modules $M, \Omega_{H_b}(M)$, and M^* are pairwise nonisomorphic.*

Proof: This is an immediate consequence of corollary 10.12 and propositions 10.13 and 10.14. \square

CHAPTER 11

TENSOR POWERS OF COMPLEXES

We continue to use all notation as in the previous chapter.

Proposition 11.1 *There is a split short exact sequence of the form:*

$$0 \longrightarrow S^2(\Omega^{-1}(k)) \longrightarrow S^2(L) \xrightarrow{\gamma'} L \longrightarrow 0$$

We may choose a split map $\phi: L \rightarrow S^2(L)$ for this sequence so that $\gamma' \circ \phi = id_L$, and so that $\phi(l_1) = l_1 \otimes l_1 + \beta^{p-1}y$ and $\phi(\alpha^{p-1}l_2) = \alpha^{p-1}l_2 \otimes l_1 + l_1 \otimes \alpha^{p-1}l_2 + \beta y'$ for $y, y' \in S^2(\Omega^{-1}k)$.

Proof: Let $\gamma: L \rightarrow k, \gamma(l_1) = 1, \gamma(l_2) = 0$ be as in lemma 9.10, so we have a short exact sequence:

$$0 \longrightarrow \Omega^{-1}k \longrightarrow L \xrightarrow{\gamma} k \longrightarrow 0$$

We truncate this sequence to obtain a complex

$$C: 0 \longrightarrow L \xrightarrow{\gamma} k \longrightarrow 0$$

with homology $\Omega^{-1}k$ in degree 1. Taking the antisymmetric square and simplifying yields a complex

$$\Lambda^2(C): 0 \longrightarrow S^2(L) \xrightarrow{\gamma'} L \longrightarrow \Lambda^2(k) \longrightarrow 0$$

with homology $S^2(\Omega^{-1}k)$ in degree 2, where γ' is the restriction of the map $\gamma'(x \otimes y) = \gamma(x)y$ to $S^2(L)$. Since $\Lambda^2(k) = 0$, we have an exact sequence

$$0 \longrightarrow S^2(\Omega^{-1}k) \longrightarrow S^2(L) \xrightarrow{\gamma'} L \longrightarrow 0$$

as desired.

The module $S^2(\Omega^{-1}k)$ is projective by lemma 7.1, so the sequence splits and there exists a map $\phi: L \rightarrow S^2(L)$ such that $\gamma' \circ \phi = id_L$. Since $\gamma'(l_1 \otimes l_1) = \gamma(l_1)l_1 = l_1$, we have $\gamma'(l_1 \otimes l_1 - \phi(l_1)) = 0$, so $\phi(l_1) = l_1 \otimes l_1 + z$ for some $z \in S^2(\Omega^{-1}k)$. Furthermore, since β kills l_1 , it kills $l_1 \otimes l_1$ by 2.3 and $\phi(l_1)$, so $\beta z = 0$. Since $S^2(\Omega^{-1}k)$ is projective, its restriction to H_b is projective, and thus there must exist some $y \in S^2(\Omega^{-1}k)$ such that $\beta^{p-1}y = z$. Similarly,

$$\gamma'(\alpha^{p-1}l_2 \otimes l_1 + l_1 \otimes \alpha^{p-1}l_2) = \alpha^{p-1}l_2$$

so

$$\phi(\alpha^{p-1}l_2) = \alpha^{p-1}l_2 \otimes l_1 + l_1 \otimes \alpha^{p-1}l_2 + z'$$

for some $z' \in S^2(\Omega^{-1}k)$. Then since $\beta l_1 = 0$ and $\beta^{p-1}\alpha^{p-1}l_2 = 0$, applying 2.3 gives us the following:

$$\begin{aligned} 0 &= \phi(\beta^{p-1}\alpha^{p-1}l_2) \\ &= \beta^{p-1}\phi(\alpha^{p-1}l_2) \\ &= \beta^{p-1}(\alpha^{p-1}l_2 \otimes l_1 + l_1 \otimes \alpha^{p-1}l_2 + z') \\ &= \beta^{p-1}\alpha^{p-1}l_2 \otimes l_1 + l_1 \otimes \beta^{p-1}\alpha^{p-1}l_2 + \beta^{p-1}z' \\ &= \beta^{p-1}z' \end{aligned}$$

Thus there exists $y' \in S^2(\Omega^{-1}k)$ such that $\beta y' = z'$. This finishes the proof. \square

CHAPTER 12

THE SYMMETRIC SQUARE OF $M(-, -; 1)$

We retain the notation of earlier chapter.

For the rest of this chapter, fix $\lambda_1, \lambda_2 \in k$ be nonzero, let x be a generator of $D_1 = k_{H_b} \uparrow^G$, let $L = L(\lambda_2)$, and let $M = M(\lambda_1, \lambda_2; 1)$. By construction of M , we have a sequence

$$0 \longrightarrow M \longrightarrow L \xrightarrow{f} D_1 \longrightarrow 0$$

where $f(l_1) = -\lambda_1 \alpha^{p-1} x$ and $f(\alpha^{p-1} l_2) = \alpha^{p-1} x$.

The map $f \otimes f: L \otimes L \rightarrow D_1 \otimes D_1$ sends symmetric tensors to symmetric tensors and alternating tensors to alternating tensors, so $f \otimes f$ restricts to maps $S^2(L) \rightarrow S^2(D_1)$ and $\Lambda^2(L) \rightarrow \Lambda^2(D_1)$, which by abuse of notation we will also call $f \otimes f$ when it is clear from context which map we mean.

Proposition 12.1 *The kernel of $f \otimes f: S^2(L) \rightarrow S^2(D_1)$ is isomorphic to the following module:*

$$M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus \bigoplus_{j=1}^{(p-1)/2} \Omega(D_1) \oplus (\text{projective})$$

Proof: The map $f \otimes f$ is surjective, so the kernel is entirely determined by the stable class of $f \otimes f$. By 11.1 we have a splitting $S^2(L) = \phi(L) \oplus (\text{projective})$, so it suffices to determine the stable class of the restriction of $f \otimes f$ to $\phi(L)$. In particular, observe that

$$(f \otimes f)(\phi(l_1)) = (f \otimes f)(l_1 \otimes l_1 + \beta^{p-1} y)$$

$$\begin{aligned}
&= (-\lambda_1 \alpha^{p-1} x) \otimes (-\lambda_1 \alpha^{p-1} x) + \beta^{p-1} (f \otimes f)(y) \\
&= \lambda_1^2 (\alpha^{p-1} x \otimes \alpha^{p-1} x)
\end{aligned}$$

where the last equality holds because β kills $S^2(D_1)$, and also the following:

$$\begin{aligned}
(f \otimes f)(\phi(\alpha^{p-1} l_2)) &= (f \otimes f)(\alpha^{p-1} l_2 \otimes l_1 + l_1 \otimes \alpha^{p-1} l_2 + \beta y') \\
&= (\alpha^{p-1} x) \otimes (-\lambda_1 \alpha^{p-1} x) \\
&\quad + (-\lambda_1 \alpha^{p-1} x) \otimes (\alpha^{p-1} x) + \beta (f \otimes f)(y') \\
&= -2\lambda_1 (\alpha^{p-1} x \otimes \alpha^{p-1} x)
\end{aligned}$$

Next, set $x' = \frac{1}{2}(x \otimes \alpha^{p-1} x + \alpha^{p-1} x \otimes x)$, and note that:

$$\begin{aligned}
\alpha^{p-1} x' &= \alpha^{p-1} \left(\frac{1}{2} (x \otimes \alpha^{p-1} x + \alpha^{p-1} x \otimes x) \right) \\
&= \alpha^{p-1} x \otimes \alpha^{p-1} x
\end{aligned}$$

Since x' is not annihilated by α^{p-1} and $S^2(D_1)$ is a direct sum of copies of D_1 , it follows that $\langle x' \rangle \cong D_1$ is a direct summand of $S^2(D_1)$; a dimension count shows the following: $S^2(D_1) = \langle x' \rangle \oplus N$ for some

$$N \cong \bigoplus_{j=1}^{(p-1)/2} D_1$$

Given this splitting, let $\rho: S^2(D_1) \rightarrow \langle x' \rangle$ be the projection with kernel N and let $1 - \rho: S^2(D_1) \rightarrow N$ denote the orthogonal projection $1_{S^2(D_1)} - \rho$.

We can then write $f \otimes f = (\rho \circ (f \otimes f), (1 - \rho) \circ (f \otimes f))$ where the maps

$$\begin{aligned}
\rho \circ (f \otimes f): S^2(L) &\rightarrow \langle x' \rangle, \\
(1 - \rho) \circ (f \otimes f): S^2(L) &\rightarrow N
\end{aligned}$$

reflect the splitting. Examining these homomorphisms, we see that:

$$(\rho \circ (f \otimes f))(l_1) = \lambda_1^2 x'$$

and

$$(\rho \circ (f \otimes f))(\alpha^{p-1}l_2) = -2\lambda_1 x'$$

$$(1 - \rho) \circ (f \otimes f)(l_1) = (1 - \rho) \circ (f \otimes f)(\alpha^{p-1}l_2) = 0$$

Thus $f \otimes f$ is stably equivalent to $\rho \circ (f \otimes f)$, while $(1 - \rho) \circ (f \otimes f)$ factors through a projective. Let $F: \phi(L) \rightarrow \langle x' \rangle$ be the restriction of $\rho \circ (f \otimes f)$ to $\phi(L)$; by proposition 10.3 and definition 10.5, $\ker(F) = \ker((-2\lambda_1)^{-1}F) \cong M(\frac{1}{2}\lambda_1, \lambda_2; 1)$; the rest is straightforward. \square

By taking the exact sequence

$$0 \longrightarrow M \longrightarrow L \xrightarrow{f} D_1 \longrightarrow 0$$

and moving around the triangle, we can construct an exact sequence

$$0 \longrightarrow L \xrightarrow{f'} D_1 \oplus (\text{projective}) \longrightarrow \Omega^{-1}(M) \longrightarrow 0$$

where f' is stably equivalent to f .

Corollary 12.2 *The cokernel of $f' \otimes f'$ is isomorphic to the module:*

$$\Omega^{-1}M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus \bigoplus_{j=1}^{(p-1)/2} D_1 \oplus (\text{projective})$$

Proof: Since $f \otimes f$ is an epimorphism and $f' \otimes f'$ is a monomorphism, the core of the cokernel of $f' \otimes f'$ is just $\Omega^{-1} \ker(f \otimes f)$. Now apply the previous proposition. \square

We can now prove the following result:

Proposition 12.3 *There is a short exact sequence of the following form:*

$$\begin{aligned} C_{\Lambda^2(\Omega^{-1}M)}: 0 &\rightarrow \Omega^{-1}M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus \bigoplus_{j=1}^{(p-1)/2} D_1 \oplus (\text{projective}) \\ &\rightarrow D_1 \otimes \Omega^{-1}(M) \rightarrow \Lambda^2(\Omega^{-1}(M)) \rightarrow 0 \end{aligned}$$

Proof: Truncate this sequence to obtain the complex

$$C: 0 \rightarrow D_1 \oplus (\text{projective}) \rightarrow \Omega^{-1}(M) \rightarrow 0$$

with homology L in degree 1. The alternating square of this complex,

$$\begin{aligned} \Lambda^2(C): 0 \rightarrow S^2(D_1 \oplus (\text{projective})) &\rightarrow D_1 \otimes \Omega^{-1}M \\ &\rightarrow \Lambda^2(\Omega^{-1}(M)) \rightarrow 0 \end{aligned}$$

has homology $S^2(L)$ in degree 2. We thus have an exact sequence:

$$\begin{aligned} 0 \rightarrow S^2(L) \rightarrow f' \otimes f' S^2(D_1 \oplus (\text{projective})) &\rightarrow D_1 \otimes \Omega^{-1}M \\ &\rightarrow \Lambda^2(\Omega^{-1}(M)) \rightarrow 0 \end{aligned}$$

By the previous result, $S^2(D_1 \oplus (\text{projective}))/S^2(L)$ is isomorphic to

$$\Omega^{-1}M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus \bigoplus_{j=1}^{(p-1)/2} D_1 \oplus (\text{projective})$$

and we get the sequence $C_{\Lambda^2(\Omega^{-1}M)}$ as claimed. \square

Theorem 12.4 *The symmetric square $S^2(M)$ is isomorphic to the module:*

$$\Omega\Omega_{H_b}M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus \bigoplus_{j=1}^{(p-3)/2} D_1 \oplus (\text{projective})$$

Proof: Consider the short exact sequence $C_{\Lambda^2(\Omega^{-1}M)}$ of proposition 12.3. If we restrict this sequence to H_b , we get:

$$\begin{aligned} C_{\Lambda^2(\Omega^{-1}M)} \downarrow_{H_b}: 0 &\rightarrow \bigoplus_{j=1}^{p(p+1)/2} k_{H_b} \oplus (\text{projective}) \\ &\rightarrow \bigoplus_{j=1}^{p^2} k_{H_b} \oplus (\text{projective}) \\ &\rightarrow \bigoplus_{j=1}^{p(p-1)/2} k_{H_b} \oplus (\text{projective}) \rightarrow 0 \end{aligned}$$

By lemma 2.1, this sequence splits, so $C_{\Lambda^2(\Omega^{-1}M)}$ is H_b -split. On the other hand, the middle term of $C_{\Lambda^2(\Omega^{-1}M)}$ is projective relative to H_b . It follows that:

$$\Omega_H \Lambda^2(\Omega^{-1}M) \cong \Omega^{-1}M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus (H_b\text{-projective})$$

By 7.2 and 7.3, we have:

$$\Lambda^2(\Omega^{-1}M) \oplus (\text{projective}) \cong \Omega^{-2}(S^2(M)) \oplus (\text{projective})$$

By 4.21, all kG -modules involved are of period one or two, so in particular:

$$\Omega^{-2}(S^2(M)) \oplus (\text{projective}) \cong S^2(M) \oplus (\text{projective})$$

Putting everything together, it follows that

$$\begin{aligned} S^2(M) &\cong \Omega_{H_b}^{-1} \Omega^{-1}M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus (H_b\text{-projective}) \\ &\cong \Omega \Omega_{H_b} M(\tfrac{1}{2}\lambda_1, \lambda_2; 1) \oplus (H_b\text{-projective}) \end{aligned}$$

the last line by 4.21 and 9.5, and the fact that Ω and Ω_H commute. Comparing the restrictions $S^2(M) \downarrow_{H_b}$ and $(\Omega M^* \oplus (H_b\text{-projective})) \downarrow_{H_b}$ shows that the relatively H_b -projective summand is of the form

$$\bigoplus_{j=1}^{(p-3)/2} D_1 \oplus (\text{projective})$$

which completes the proof. \square

CHAPTER 13

A NILPOTENT OF ORDER 3

We continue to use the same definitions and notation as in the previous chapter, and impose one additional condition: we require that $p = 3$. Since $3 - 2 = 1$, the classes $M(-, -; 1)$ and $M(-, -)$ are identical in this case. This allows us to prove a number of results which either do not hold more generally, or which do appear to hold in general but for which the proofs at $p = 3$ are substantially less complicated. We will return to a discussion of the situation at other primes in chapter 14.

Proposition 13.1 *Let $\lambda_1, \lambda_2 \in k^\times$ and let $M = M(\lambda_1, \lambda_2)$. The symmetric square $S^2(M)$ is isomorphic to $\Omega M^* \oplus (\text{projective})$.*

Proof: If $p = 3$, then $\frac{1}{2} = -1$. By 10.13, 10.14, and 12.4, we then have:

$$\begin{aligned} S^2(M) &\cong \Omega \Omega_{H_b} M\left(\frac{1}{2}\lambda_1, \lambda_2\right) \oplus (\text{projective}) \\ &\cong \Omega \Omega_{H_b} M(-\lambda_1, \lambda_2) \oplus (\text{projective}) \\ &\cong \Omega M(-\lambda_1, -\lambda_2) \oplus (\text{projective}) \\ &\cong \Omega M^* \oplus (\text{projective}) \end{aligned}$$

□

Proposition 13.2 *Let $\lambda_1, \lambda_2 \in k^\times$ and let $M = M(\lambda_1, \lambda_2)$. The alternating square $\Lambda^2(M)$ is isomorphic to $\Omega \Omega_{H_b}(M^*) \oplus (H_b\text{-projective})$, and the relative core of $M \otimes M$ with respect to H_b is periodic relative to H_b of period 1.*

Proof: Recall that $\Omega_H M \cong M(\lambda_1, -\lambda_2)$ by 10.13. By 7.4, we have isomorphisms as follows:

$$\begin{aligned}
 S^2(\Omega_{H_b} M) \oplus (H_b\text{-projective}) &\cong S^2(\Omega_{H_b} k \otimes M) \\
 &\cong \Omega_{H_b}^2 \Lambda^2(M) \oplus (H_b\text{-projective}) \\
 &\cong \Lambda^2(M) \oplus (H_b\text{-projective})
 \end{aligned}$$

But by 10.13 and 13.1, we have:

$$\begin{aligned}
 S^2(\Omega_{H_b} M) &\cong \Omega(\Omega_{H_b} M)^* \oplus (\text{projective}) \\
 &\cong \Omega(M(\lambda_1, -\lambda_2)^*) \oplus (\text{projective}) \\
 &\cong \Omega M(-\lambda_1, \lambda_2) \oplus (\text{projective}) \\
 &\cong \Omega \Omega_{H_b} M^* \oplus (\text{projective})
 \end{aligned}$$

Thus the relative core of $\Lambda^2(M)$ with respect to H_b is isomorphic to $\Omega \Omega_{H_b} M^*$. It follows immediately that

$$M \otimes M = \Omega M^* \oplus \Omega \Omega_{H_b} M^* \oplus (H_b\text{-projective})$$

and thus the relative core of $M \otimes M$ with respect to H_b is of period one relative to H_b . \square

Proposition 13.3 *Let $\lambda_1, \lambda_2 \in k^\times$, let $M = M(\lambda_1, \lambda_2)$, and let $N = M \oplus M^*$. Then N and the relative core of $N \otimes N$ with respect to H_b are of period two relative to H_b , and the relative core of $N^{\otimes 3}$ with respect to H_b is of period one relative to H_b .*

Proof: It is clear that N is of period two relative to H_b , since $M^* \not\cong \Omega_H M$ by proposition 10.15. Now consider the following tensor product:

$$N \otimes N = M \otimes M \oplus M \otimes M^* \oplus M^* \otimes M \oplus M^* \otimes M^*$$

By corollary 13.2, we know that the relative cores of $M \otimes M$ and $M^* \otimes M^*$ with respect to H_b are of period one relative to H_b . However,

$$\dim_k(\text{Soc}(M \otimes M^*)) = \dim_k \text{Hom}_{kG}(M, M) = 8$$

while

$$\dim_k(\text{Soc}(\Omega_{H_b} M \otimes M^*)) = \dim_k \text{Hom}_{kG}(\Omega_{H_b} M, M) = 7$$

by 10.11. It follows that the relative core of $M \otimes M^* \cong M^* \otimes M$ with respect to H_b is not of period one with respect to H_b , and so the relative core of $N \otimes N$ is not. On the other hand, the expansion of $N^{\otimes 3}$ may be expressed as a direct sum of modules isomorphic to $M^{\otimes 3}$, $M \otimes M \otimes M^*$, $M \otimes M^* \otimes M^*$, and $(M^*)^{\otimes 3}$, all of which are of period one relative to H_b . \square

Theorem 13.4 *Let $M = M(\lambda_1, \lambda_2)$. Then the element*

$$\mu = [M] - [\Omega_{H_b}(M)] + [M^*] - [\Omega_{H_b}(M^*)] \in a(kG)$$

is nilpotent of degree 3.

Proof: First, note that element μ may be expressed as a product:

$$\mu = ([k] - [\Omega_{H_b} k])e'_{H_b}([M \oplus M^*])$$

By 13.3 and 8.2, the element $m = e'_{H_b}[M \oplus M^*] \in a(G)$ is periodic with respect to $[\Omega_{H_b} k]$ of period two, as is m^2 , while m^3 is of period one with respect to $[\Omega_{H_b} k]$. The theorem follows immediately from 8.3. \square

Remark 13.5 *i) The theorem depends on the fact that the elements $e'_{H_b}[M]^2$ and $e'_{H_b}[M^*]^2$ are of period one relative to $[\Omega_{H_b} k]$, but that $e'_{H_b}[M][M^*]$ is of period two relative to $[\Omega_{H_b} k]$. In fact, the same argument shows that*

$$([k] - [\Omega_{H_b} k])e'_H(\omega_1[M] + \omega_2[M^*])$$

is a nilpotent of nilpotence degree three for any nonzero constants $\omega_1, \omega_2 \in k^\times$.

ii) Benson and Carlson [3] say that a kG -module M is absolutely p -divisible if for every extension field k_1 of k and every direct summand M_1 of $k_1 \otimes_k M$ as a k_1G -module, $p \mid \dim_{k_1}(M_1)$. They show that the linear span $a(G; p)$ in $a(G)$ of the absolutely p -divisible kG -modules is an ideal, and define $A(G; p) = a(G; p) \otimes_{\mathbb{Z}} \mathbb{C} \subset A(G)$. They then prove the following theorem:

Theorem 13.6 (Benson-Carlson) *For an arbitrary finite group G , the ring $A(G)/A(G; p)$ has no nonzero nilpotent elements.*

The constituent indecomposable modules of a nilpotent must therefore have dimensions divisible by p over all extension fields. The construction of theorem 13.4 shows that this bound is sharp, in that $p \mid \dim_k(M)$ but $p^2 \nmid \dim_k(M)$ for $M \in M(-, -)$. In all previous constructions of nilpotents in $a(G)$ known to this author, $p^2 \mid \dim_k(M)$ for the constituent modules M of the nilpotent.

Corollary 13.7 *Let G be an elementary abelian 3-group of rank $2n$. The representation ring $a(G)$ contains a nilpotent of degree $2n + 1$.*

Proof: This follows directly from theorem 13.4 and proposition 8.7. \square

Remark 13.8 *It should be possible to reduce the rank of G to $n + 1$ by taking*

$$G = H_a \times H_{b_1} \times H_{b_2} \times \cdots H_{b_n}$$

and inflating nilpotent elements from the subgroups $H_a \times H_{b_1}, H_a \times H_{b_2}, \cdots H_a \times H_{b_n}$.

CHAPTER 14

FUTURE DEVELOPMENT

14.1 GENERALIZATIONS: $M(H, \zeta, \lambda)$

Let G be an arbitrary p -group, let H be a cyclic subgroup of G , and let $\zeta \in H^n(G, k)$ be a cohomology element which is annihilated on restriction to H . Then $L_\zeta \downarrow_H = k_H \oplus \Omega k_H \oplus (proj)$, the stable homomorphism group $\text{StHom}_{kG}(L_\zeta, k_H \uparrow^G)$ is two-dimensional, and the kernels of maps $f \in \text{StHom}_{kG}(L_\zeta, k_H \uparrow^G)$ whose Frobenius correspondents in $\text{StHom}_{kH}(L_\zeta \downarrow_H, k_H)$ are split maps form a one-parameter family. We might thus define a class $M(H, \zeta, \lambda)$ composed of such modules, and much of the framework of our analysis would extend immediately to such a class.

14.2 PRIMES $p > 3$

Note that most of the ‘module-theoretic’ results in chapter 10 hold for the class $M(-, -)$, while the more homological constructions in chapter 11 and afterward are proven for $M(-, -; 1)$. When $p = 3$, these classes are identical, which simplifies matters significantly. At other primes, the situation is less clear; for example, neither $\Omega_{H_b} M(\lambda_1, \lambda_2; 1)$ nor $M(\lambda_1, \lambda_2; 1)^*$ lie in the class $M(-, -; 1)$ (although $\Omega_{H_b} M(\lambda_1, \lambda_2; 1)^*$ *does*, reflecting a similar result for modules of type L_ζ .) However, note that if $M \in M(-, -)$ and

$$\mu = ([k] - [\Omega_{H_b} k])e_{H_b}([M] + [M^*]) \in a(G)$$

then μ is a nilpotent of order three as long as $M \otimes M = N \oplus \Omega_{H_b} N \oplus (H\text{-projective})$ for some module N , a much weaker condition than the one proved in chapter 11; we have verified, using MAGMA, that this construction does indeed yield a nilpotent of degree 3 when $p = 5$ and $p = 7$. We believe it very likely that the construction works at arbitrary odd primes.

14.3 THE CASE $p = 2$: DIHEDRAL 2-GROUPS

The representation theory of the dihedral 2-groups D_{2^n} in characteristic 2 is of interest in part because groups with dihedral Sylow subgroups are among those of ‘tame’ representation type, and we thus know much more about their module theory than we can ever hope to for most groups. In fact, we have a complete classification, due to Ringel, of the indecomposable kD_{2^n} -modules, and thus we have a complete description of $a(D_{2^n})$ as an additive group. Despite this, very little is known about the tensor product structure. In his thesis [7], Heldner constructed the first known example of a nilpotent element of degree 3 of a representation ring, working with the dihedral group D_8 of order 8. Heldner produced his example using custom software written in Cayley. Heldner’s example is somewhat unsatisfactory in that he provides no proof beyond the output of the software itself, giving us little insight into the situation.

Let H be the center $Z(D_8)$ of the dihedral group D_8 of order 8. We have been able to show that Heldner’s construction is of the form $x = [M] - [\Omega_H M]$ for modules M of type $M(H, \zeta, \lambda)$ in the sense we give above. Using MAGMA, we have generated dozens of other examples of nilpotents over D_8 and D_{16} which represent similar constructions. We believe that developing the theory of modules of type $M(H, \zeta, \lambda)$ in this context has a strong chance of not only giving the proper setting for Heldner’s

constructions, but of making substantial progress on the problem of describing the multiplicative structure of $a(D_{2^n})$.

14.4 THE CASE $p = 2$: ELEMENTARY ABELIAN 2-GROUPS

As already noted, the only groups for which the nilpotence question is unresolved are the elementary abelian 2-groups E_{2^n} of rank $n \geq 3$. The usual constructions are of no use here; a standard result states that all periodic modules for elementary abelian 2-groups are of period 1. No such result holds for periodicity relative to a subgroup H , however, and we have constructed modules of type $M(H, \zeta, \lambda)$ over E_8 which are of period $n > 1$ relative to H . It thus may be possible to find a nilpotent of the form $x = [M] - [\Omega_H M]$ using modules of this type.

14.5 COHOMOLOGY RELATIVE TO A kG -MODULE W

Okuyama [8] has introduced a new type of cohomology theory to modular representation theory, that of cohomology relative to a kG -module W . It subsumes the theory of cohomology relative to a subgroup in the following sense: if $H < G$, then cohomology relative to H in the standard sense is equivalent to cohomology relative to the module $k_H \uparrow^G$ under Okuyama's definition. This theory is of interest in the context of our program for two reasons.

First, if G is an elementary abelian p -group, there is an important formal analogy between the cyclic subgroups $H < G$ and certain points of the rank variety V_G , an analogy which extends to other points of V_G in the form of so-called 'shifted' cyclic subgroups $\langle x \rangle$ of G , which are cyclic subgroups of the multiplicative group of kG . This analogy generalizes to arbitrary groups G via the Quillen stratification. If $\langle x \rangle$ is a cyclic shifted subgroup corresponding to a point of the variety of L_ζ , then we

can define modules of type $M(\langle x \rangle, \zeta, \lambda)$ in terms of kernels of maps $L_\zeta \rightarrow k_{\langle x \rangle} \uparrow^G$. Such modules may have properties similar to those modules of type $M(H, \zeta, \lambda)$.

Second, we have computed examples of modules in whose behavior this new type of cohomology emerges naturally. For instance, we have examples of D_8 -modules M and $W = W'_{V_4} \uparrow^{D_8}$ for which

$$M \otimes M \otimes M = M \oplus \Omega_W(M) \oplus \Omega_W(M) \oplus \Omega_W^2(M) \oplus (W\text{-projective})$$

and which cannot be similarly described by other cohomology theories. This type of relative cohomology thus appears to be of practical interest in understanding the behavior of these modules.

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