Block-Intersection Graphs of Steiner Triple Systems

by

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(Under the Direction of E. Rodney Canfield)

Abstract

A Steiner triple system of order n is a collection of subsets of size three, taken from the n-element set $\{0, 1, ..., n-1\}$, such that every pair is contained in exactly one of the subsets. The subsets are called triples, and a block-intersection graph is constructed by having each triple correspond to a vertex. If two triples have a non-empty intersection, an edge is inserted between their vertices. It is known that there are eighty Steiner triple systems of order 15 up to isomorphism. In this paper, we attempt to distinguish the eighty systems using their block-intersection graphs, as well as discuss general properties of block-intersection graphs of Steiner triple systems.

Index words: Steiner triple system, Block-intersection graph, Isomorphism, Spectra

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Chapter 1

Introduction

In 1847, Reverend Thomas P. Kirkman published "On a Problem in Combinations" [11] that first answered questions on the existence of what would later be called Steiner triple systems. The problem he considered was to find a collection of subsets of size three, taken from the n-element set $\{0, 1, ..., n-1\}$, with the property that every pair is contained in exactly one of the subsets. This problem was a specific case of a more general question on combinations posed by W.S.B. Woodhouse in 1844. Several years later, Jakob Steiner, apparently unaware of Kirkman's article, similarly asked about the existence of various designs, including the one solved by Kirkman. It was Steiner's name that was eventually used for the systems, although Steiner himself did not offer any solutions. Such systems were an early example of what are now known as balanced incomplete block designs in the field of combinatorial design.

Kirkman's name did become associated with a related topic that originated with "Kirkman's schoolgirl problem," in which he wanted to arrange fifteen girls into sets of three such that each girl was paired with every other girl once. This describes a Steiner triple system, but Kirkman added the condition that the thirty-five sets of three be expressible as seven partitions of $\{0, 1, ..., 14\}$. The triple system would then describe a week-long schedule for daily walks where the fifteen girls were in different sets each day. In the same year his question was published, Kirkman offered a solution, as did Arthur Cayley [5]. Kirkman also asked about orders other than 15 for which an analogous problem could be solved. These designs are a subtype of Steiner triple systems often called Kirkman triple systems, and we will revisit this idea below.

1.1 Combinatorial Definitions and Background

A *Steiner triple system of order* n is a pair $(V, \mathcal{B})$ comprising the n-set $V = \{0, 1, ..., n-1\}$ and a set $\mathcal{B}$ of sets of size three, each a subset of V, such that any pair of elements of V appears in exactly one set of $\mathcal{B}$. The sets in $\mathcal{B}$ are called *triples* or *blocks*. We will abbreviate "Steiner triple system of order n" as STS(n). A necessary and sufficient condition for the existence of a Steiner triple system of order n was proved by Kirkman in his 1847 paper, that $n \equiv 1, 3 \bmod 6$ [11].

As a balanced incomplete block design, an STS(n) is a $(n, 3, 1)$-design, where 3 is the size of a block and 1 is the number of blocks in which a pair of elements appears. In any STS(n), there are $\frac{n(n-1)}{6}$ triples. We obtain this total by computing $\binom{n}{2}/\binom{3}{2}$, as there are $\binom{n}{2}$ ways to select unordered pairs from the n-set and each triple contains $\binom{3}{2}$ pairs. The replication number of an STS(n) is defined as the number r of triples in which each element of V appears. A theorem on block designs states that $(n-1) = r(3-1)$ for an STS(n), yielding the replication number equal to $\frac{n-1}{2}$.

The subtype of STS(n)s known as Kirkman triple systems, as described above with Kirkman's schoolgirl problem, have the additional property of *resolvability.* First, we define a *parallel class* in an STS(n) as a set of pairwise disjoint triples that partitions the n-set; the union of the triples in a parallel class will contain each element once. An STS(n) is resolvable if the triples of $\mathcal{B}$ can be partitioned into parallel classes. In Kirkman's problem, each day represents a parallel class. A necessary and sufficient condition on n for the existence of a resolvable STS(n) was not proved until 1965 [5], but it is quite simple: $n \equiv 3 \bmod 6$.

Up to isomorphism, there is one STS each for orders 3, 7, and 9. There are two STS(13)s, eighty STS(15)s, over 11 billion STS(19)s, and the numbers grow exponentially from there (and are not completely determined for orders over 19) [10]. Richard Wilson showed that the number N of Steiner triple systems on a permissible order n is such that $(e^{-5}n)^{n^2/6} \leq N \leq (e^{-1}n)^{n^2/6}$ [18]. In this paper we will focus on the Steiner triple systems of order 13 and 15, in particular on distinguishing among isomorphism types. The STS(15)s have been known for over half a century and have a standard numbering in the literature, which can be found in [5]. In addition, the two STS(13)s are included in Appendix A in the order by which they will be referenced.

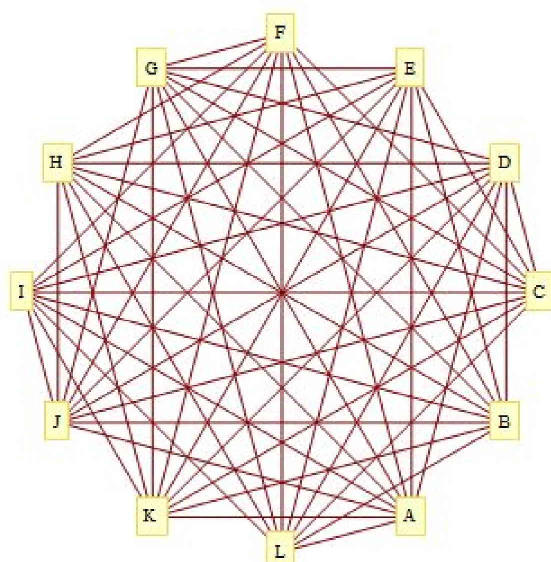
1.2 Graph Theory Definitions

A *graph* is an object made up of *vertices* and *edges*, where edges are links between vertices. If two vertices have an edge between them, they are *adjacent*. The number of edges leaving a vertex is called the *degree* of that vertex, and a *simple graph* contains no loops and at most one edge per pair of vertices; all graphs considered here will be simple. A graph with every vertex having equal degree is called *regular*. If for each pair of vertices, there is a sequence of edges starting from one vertex and ending at the other, the graph is *connected*. The *complement* $\overline{G}$ of a graph G is formed using the same vertex set as G with edges inserted between vertices that were not adjacent in G.

For the following definitions, consider a graph on n vertices. If every pair of vertices is joined by an edge, the graph is *complete* and denoted by K_n. A graph can also be represented in the form of an *adjacency matrix*, which is an $n \times n$ matrix with entry a_{ij} equal to the number of edges between the i^{th} and j^{th} vertices. In a simple graph, this matrix is composed entirely of 0's and 1's.

1.3 Block-Intersection Graphs

From an STS(n), different graphs can be formed. For example, if the vertex set is $\{0, 1, ..., n-1\}$ and n is an admissible order, then defining an STS(n) is equivalent to decomposing K_n into triangles (copies of K_3). But a different sort of graph can be constructed by taking the vertex set as $\mathcal{B}$, that is, every triple in the STS(n) corresponds to a vertex. If two triples have a non-empty intersection then an edge is inserted between their vertices. The resulting graph is known as the *block-intersection graph* or the BIG. It is this graph that will be considered as a possible way to distinguish Steiner triple systems of the same order. By construction, the block-intersection graphs are finite and simple.



Vertex	Triple
A	$(0, 1, 2)$
B	$(3, 4, 5)$
C	$(6, 7, 8)$
D	$(0, 3, 6)$
E	$(1, 4, 7)$
F	$(2, 5, 8)$
G	$(0, 4, 8)$
H	$(1, 5, 6)$
I	$(2, 3, 7)$
J	$(0, 5, 7)$
K	$(1, 3, 8)$
L	$(2, 4, 6)$

Figure 1.1: The BIG for STS(9)

In order to study the properties of the BIGs, we construct them using SAGE, the open-source mathematics software package [14]. Given a Steiner triple system, a matrix is built to serve as an adjacency matrix for the BIG. The code cycles through the list of triples, entering a "1" in position ij if triple i shared a common element with triple j and "0" otherwise (for $i \neq j$). A function in SAGE to create a graph on such an adjacency matrix is then applied to each matrix corresponding to an STS(13) and STS(15). For the relevant program code, please see Appendix B.

One can use BIGs of Steiner triple systems to attempt to distinguish the systems because two STS(n)s are isomorphic if and only if their block-intersection graphs are isomorphic. The reverse direction was proved by David Pike in 1996 (although the article was not printed until 1999) and the forward direction was stated without a formal proof by Charles Colbourn and Alexander Rosa in 1999 [5, 13]. Hence if two BIGs can be shown to be non-isomorphic via a graph invariant, we can conclude their systems are also non-isomorphic.[1]

1.4 Properties of the Block-Intersection Graphs

Some of the very basic graph invariants will be identical for BIGs of the same order. The number of vertices is equal to the number of triples, which as mentioned above is $\frac{n(n-1)}{6}$. To calculate the degree of a vertex, note that any element of V is in $\frac{n-1}{2}$ triples, so for an arbitrary element a in a triple, a is in $\frac{n-1}{2} - 1$ other triples. Now multiply by three for the three elements in each triple, and we obtain the degree of each vertex: $3(\frac{n-1}{2} - 1) = \frac{3(n-3)}{2}$. Thus each BIG is a regular graph. Because the BIGs are regular, their total number of edges is straightforward to compute using the Handshaking Lemma: (number of vertices) $\times$ (degree) / 2.

For lower orders of n, the block-intersection graphs are fairly small in terms of both number of vertices and of edges. The BIG for STS(3) consists of a single vertex representing

[1] Although the results cited for STS(n)s specifically are from the late 1990s, block-intersection graphs had been used to distinguish non-isomorphic balanced incomplete block designs since at least 1985. For example, see M.J. Colbourn (1985), Algorithmic aspects of combinatorial designs: a survey, *Annals of Discrete Mathematics* **26**, 67-136.

the only triple $(0, 1, 2)$, and for STS(7) the graph is K_7. If $n = 9$, its BIG has twelve vertices and fifty-four edges as shown above in Figure 1.1. But these totals grow quickly: each BIG of an STS(13) contains twenty-six vertices and 195 edges and is regular of degree 15; the corresponding totals for $n = 15$ are thirty-five, 315, and 18.

In fact, not only is a BIG regular, it is also strongly regular. A *strongly regular graph* with parameters (v, k, e, f) is a k-regular graph on v vertices such that each pair of adjacent vertices has e neighbors in common and each pair of non-adjacent vertices has f neighbors in common. We prove strong regularity in the following proposition.

Proposition: The block-intersection graph of an STS(n) for $n \geq 9$ is strongly regular with parameters $(\frac{n(n-1)}{6}, \frac{3(n-3)}{2}, \frac{n+3}{2}, 9)$.

Proof: We know the number of vertices in a BIG is $\frac{n(n-1)}{6}$ and the graph is regular of degree $\frac{3(n-3)}{2}$. Now we determine the parameters e and f.

Consider two adjacent vertices in a BIG. They represent triples B_1 and B_2 with one common element a; suppose $B_1 = (a, b, c)$ and $B_2 = (a, d, e)$. Then their common neighbors must intersect both B_1 and B_2, so four of these neighbors will have the form $(b, d, \_)$, $(b, e, \_)$, $(c, d, \_)$, and $(c, e, \_)$, where $\_$ denotes any other permissible element of V. The other neighbors in common will be the triples also containing a, which appears in a total of $\frac{n-1}{2}$ triples. Because B_1 and B_2 both already contain a, the remaining number is $\frac{n-1}{2} - 2 = \frac{n-5}{2}$. Hence the number of common neighbors is $\frac{n-5}{2} + 4 = \frac{n+3}{2}$.

Let B_1 and B_2 now correspond to non-adjacent vertices. They must be of the form $B_1 = (a, b, c)$ and $B_2 = (d, e, f)$. A common neighbor of both of these triples must contain an element from each. For instance, we know that the pair $\{a, d\}$ must occur in exactly one triple, so one of the neighbors in common has the form $(a, d, \_)$. Similarly, two other common neighbors are $(a, e, \_)$ and $(a, f, \_)$. There are nine possible ways to pair one element from B_1 and one element from B_2, so any non-adjacent vertices will have nine common neighbors (for $n \geq 9$). Thus the BIG of an STS(n) is a strongly regular graph with parameters $(\frac{n(n-1)}{6}, \frac{3(n-3)}{2}, \frac{n+3}{2}, 9)$. □

Clearly every block-intersection graph G is connected. In a strongly regular graph, the vertex connectivity $\kappa(G)$, which is the smallest number of vertices whose removal results in a disconnected graph, is equal to the degree of each vertex [4]. Edge connectivity is defined analogously and denoted by $\kappa'(G)$. A well-established result in graph theory is the inequality $1 \leq \kappa(G) \leq \kappa'(G) \leq \delta$ for connected graphs, where δ is the minimum degree. In a strongly regular graph, $\kappa(G) = \delta$, implying that the edge connectivity $\kappa'(G)$ is also equal to the degree. Thus both measurements of connectivity are $\frac{3(n-3)}{2}$ in each BIG.

Another measure related to graph connectivity is the *diameter* of a graph, the supremum of the set of distances between any two vertices u and v, where such a distance is defined as the smallest number of edges required to go from u to v. In a BIG, two vertices correspond to two triples B_1 and B_2 of an STS(n), and they either have a common element or they do not. If they share an element, there is an edge between the vertices and so the distance is 1. If not, since each element of B_1 must occur in a triple with each element of B_2, the non-adjacent vertices for B_1 and B_2 have a common neighbor and the distance is 2. Hence the possible non-trivial distances in a BIG are 1 and 2, so the diameter is equal to 2.

The block-intersection graphs are also *distance-regular*. Let G be regular and connected with diameter d, and consider two vertices u and v of distance i from each other. If non-negative integers b_0, $b_1, ..., b_{d-1}$, $c_1, ..., c_d$ exist such that for $i = 0, 1, ..., d$, v has b_i neighbors a distance $i+1$ from u and c_i neighbors a distance $i-1$ from u, then G is distance-regular. These values form an array $\{b_0, ..., b_{d-1}; c_1, ..., c_d\}$ associated to the graph called the *intersection array.*

Proposition: The block-intersection graph of an STS(n) is distance-regular.

Proof: We determine the entries of the intersection array, which shows the existence of the required integers b_i and c_j for appropriate i and j.

- $i = 0$: If u and v are at distance 0, then $u = v$ and the number of neighbors distance 1 from u is the degree k, so $b_0 = \frac{3(n-3)}{2}$.
- $i = 1$: Let u and v be at distance 1. Then they correspond to intersecting triples B_1 and B_2 of the form (a, b, c) and (a, d, e). Consider the number of adjacent vertices to v distance 2 from u: these are triples that have a common element with B_2 but not B_1, so any triple containing d or e but not b or c. Each of d, e appears in $\frac{n-1}{2}$ triples, but we subtract 2 to not double count B_2 and subtract 4 for the triples containing $\{d, b\}, \{d, c\}, \{e, b\}$, or $\{e, c\}$, yielding $b_1 = 2(\frac{n-1}{2}) - 6 = n - 7$. Also, the number of neighbors of v distance 0 from u is only u itself, so $c_1 = 1$.
- $i = 2$: If u and v are 2 edges apart, then they are non-adjacent vertices. The BIG is strongly regular and the corresponding parameter applicable here is 9, thus $c_2 = 9$.

Hence the intersection array for a BIG is $\{\frac{3(n-3)}{2}, n - 7; 1, 9\}$. From these quantities we can also define integers a_i as the number of neighbors of v a distance i from u, yielding $a_1 = \frac{3(n-3)}{2} - b_1 - c_1 = \frac{n+3}{2}$ and $a_2 = \frac{3(n-3)}{2} - c_2 = \frac{3(n-9)}{2}$. □

Note that a connected graph is distance-regular of diameter 2 if and only if it is strongly regular [3].

Yet another regularity property satisfied by the block-intersection graphs is *walk-regularity*. A graph G is walk-regular if the number of closed walks of length s that start and end at the same vertex is equal for every vertex. A walk of length s is a sequence of adjacent vertices v_1, v_2, ..., v_s, v_{s+1}, not necessarily distinct, and the walk is closed if $v_1 = v_{s+1}$. A standard theorem is that given the adjacency matrix A of G, the entry $a_{ij}^{(s)}$ in A^s is the number of s-length walks starting at v_i and ending at v_j. If $i = j$, this is the number of closed walks.

For example, in a simple graph, $a_{ii}^{(1)} = 0$ because there are no loops, and $a_{ii}^{(2)} = d_i$, the degree of v_i. Then the total number of closed walks of length s in G is equal to $\text{tr}(A^s)$. The fact that the BIGs are walk-regular is implied by their distance-regularity.

The basic properties of the block-intersection graphs offer a good overview of the highly structured nature of these graphs. Because the invariants we have covered in this section are the same for BIGs corresponding to STS(n)s on the same n, we will consider other graph invariants in further chapters. First, though, we discuss how to obtain the Steiner triple systems from which the BIGs are formed.

Chapter 2

Methods of Construction

In this chapter we describe several ways of producing Steiner triple systems of a given order, starting with either a permissible n-set or a smaller system from which a larger system can be created.

2.1 The Doubling Construction

There is a "doubling construction" for Steiner triple systems that is valid for all n, which provides a relatively quick and easy way of obtaining Steiner triple systems of larger orders.

Proposition: If there is an STS(n), then there is an explicit method for producing an STS(2n+1) for any $n \geq 3$.

Proof: Because we necessarily have $n \equiv 1, 3 \bmod 6$, in either case it is true that $2n+1 \equiv 1, 3 \bmod 6$; hence there is an STS(2n+1). The triples of the larger system are those from the original system, plus triples containing elements from $\{n, n+1, ..., 2n-1, 2n\}$. For each (a, b, c) of the STS(n), we create $(a+n, b+n, c)$, $(a+n, b, c+n)$, and $(a, b+n, c+n)$. Finally, add the triple $(a, a+n, a+2n)$ for all $a \in \{0, ..., n-1\}$. This process yields three types of triples that total to $\frac{n(n-1)}{6} + 3(\frac{n(n-1)}{6}) + n = \frac{(2n)(2n+1)}{6}$, the right number for an STS(2n+1).

Now we verify that each pair of elements of $\{0, ..., 2n\}$ appears in exactly one triple. For any pair in $\{0, ..., n-1\}$, there is a unique triple of the STS(n) containing it. A pair with both elements in $\{n, ..., 2n-1\}$ or one element in $\{0, ..., n-1\}$, one element in $\{n, ..., 2n-1\}$ will occur in a triple of the second type. Finally, any pair including the element $2n$ occurs in the third type of triple. □

2.2 Lexicographically Least STS(n)

We consider another method for the creation of an STS(2n+1) from an STS(n), this time resulting in a system that is lexicographically least among Steiner triple systems of order $2n+1$. With the usual ordering $<$ on the integers, we define a lexicographic ordering of blocks in a Steiner triple system. Given triples (a, b, c) and (d, e, f) such that $a < b < c$ and $d < e < f$, we say that $(a, b, c) \prec (d, e, f)$ if $a < d$ or $a = d, b < e$ or $a = d, b = e, c < f$. It follows that there is a lexicographically least STS(n) for any n, but such a Steiner triple system can actually be generated with a backtrack-free algorithm for certain n.

The Algorithm: One method of trying to produce an STS(n) is to create triples in a lexicographic order. Beginning with the triple $(0, 1, 2)$, one then takes the next-lowest triple possible while maintaining the conditions necessary for an STS(n), i.e., making sure not to have any pair of elements represented in more than one triple. Note that the $\frac{n-1}{2}$ triples containing 0 are always the first generated; because n is necessarily odd, we obtain $(0, 1, 2), (0, 3, 4), ..., (0, n-2, n-1)$. If an STS(n) can be produced this way without ever having to backtrack, it will certainly be lexicographically least among all STS(n)s.

Example: For $n = 7$, the element set is $V = \{0, ..., 6\}$ and the first three triples produced using the algorithm are $(0, 1, 2)$, $(0, 3, 4)$, and $(0, 5, 6)$. We start the next block with 1; since 1 and 2 have appeared together, the next element added is 3. The last element in this triple is 5, as 3 and 4 were already paired, yielding $(1, 3, 5)$. Similar reasoning produces $(1, 4, 6)$, and now 1 has been with all other elements. Finally, $(2, 3, 6)$ and $(2, 4, 5)$ complete the system. At no point did we have to stop and backtrack; there was always a greater appropriate element in V.

Example: For $n = 9$, $V = \{0, ..., 8\}$ and the triples produced are $(0, 1, 2)$, $(0, 3, 4)$, $(0, 5, 6)$, $(0, 7, 8)$, $(1, 3, 5)$, and $(1, 4, 6)$. At this point 1 needs to appear with 7 and 8, but $(1, 7, 8)$ is not a valid triple because 7 and 8 have already been paired. This is problematic because we are going in increasing lexicographic order, so the next triple must begin with 2, implying there will not be a triple containing 1 and 7 or 1 and 8. If we tried to continue, we obtain $(2, 3, 6)$ and $(2, 4, 5)$, and here we run into the same problem. Clearly there are pairs of elements not appearing in any triple, in addition to the fact that an STS(9) should comprise twelve triples. Thus there is no STS(9) that can be produced without backtracking in the lexicographically-least method.

Now we characterize lexicographically-least, backtrack-free STS(n)s by a property of n.

Proposition: If $n = 2^k - 1$ for $k \geq 2$, then there exists an STS(n) produced by the lexicographically-least, backtrack-free method.

Proof by induction on k: Base case $k = 2$: There is one STS(3), consisting of the triple $(0, 1, 2)$.

Base case $k = 3$: As listed above, the STS(7) has triples $(0, 1, 2)$, $(0, 3, 4)$, $(0, 5, 6)$, $(1, 3, 5)$, $(1, 4, 6)$, $(2, 3, 6)$, and $(2, 4, 5)$.

Induction step: Assume we have a lexicographically-least, backtrack-free STS(n), where $n = 2^k - 1$ and $k \geq 3$. We must show there exists an STS($2^{k+1} - 1$) = STS(2n+1) that can be created in the same manner. First we note that if there is an STS(n), $n \equiv 1, 3 \bmod 6$. As mentioned in the previous proof, this implies $2n + 1 \equiv 1, 3 \bmod 6$, so there exists an STS(2n+1). We complete this proof by outlining a method of construction.

First, we keep the triples of the STS(n), of which there are $\frac{n(n-1)}{6}$, and let $m = 2^{k-1}$. Add the triples $(0, n, n+1)$, $(0, n+2, n+3)$, ..., $(0, 2n-1, 2n)$. In each of these triples beginning with 0, there is a pair of consecutive elements from $\{n, ..., 2n\}$; label the pairs $p_1, p_2, ..., p_m$. For example, $p_1 = \{n, n+1\}$, $p_2 = \{n+2, n+3\}$, and $p_m = \{2n-1, 2n\}$. Because we are adding $n + 1 = 2^k$ new elements, there are $\frac{2^k}{2} = 2^{k-1} = m$ such pairs. These pairs are to be put into sets of size four as follows (note that m is divisible by 4):

stage 1: $\{p_1, p_2\}, \{p_3, p_4\}, ..., \{p_{m-3}, p_{m-2}\}, \{p_{m-1}, p_m\}$

stage 2: $\{p_1, p_3\}, \{p_2, p_4\}, ..., \{p_{m-3}, p_{m-1}\}, \{p_{m-2}, p_m\}$

stage 3: $\{p_1, p_4\}, \{p_2, p_3\}, ..., \{p_{m-3}, p_m\}, \{p_{m-2}, p_{m-1}\}$

...

stage $m-1$: $\{p_1, p_m\}, \{p_2, p_{m-1}\}, ..., \{p_{m/2-1}, p_{m/2+2}\}, \{p_{m/2}, p_{m/2+1}\}$

At each stage any pair p_i is put into a 4-set with a new pair, the next available pair it has not yet been joined with.

Now the 4-sets are to be put into triples: At each stage, we do the following for each set of size four. Starting with pairs $p_i = \{a, b\}$ and $p_j = \{c, d\}$ from the 4-set, create the triples $(base_1, a, c)$, $(base_1, b, d)$, $(base_2, a, d)$, and $(base_2, b, c)$, where the bases are

stage 1: $base_1 = 1, base_2 = 2$

stage 2: $base_1 = 3, base_2 = 4$

...

stage $m-1$: $base_1 = n-2, base_2 = n-1$

There will be m new bases $m-1, m, ..., n-1$, in addition to the original bases of $0, 1, ..., m-2$ (we know these are the original bases since we began with the STS(n)). By construction, after stage i, the bases $2i-1$ and $2i$ will have been in a triple with every element of the (2n+1)-set, thereby creating an STS(2n+1).

As a check, we calculate the number of triples in the new system. We have added $n \times m$ new triples, as there will be n total bases and we added m triples per base. The STS(n) had $\frac{n(n-1)}{6}$ triples, so the new system will have $\frac{n(n-1)}{6} + nm = \frac{(2^k-1)(2^k-2)}{6} + (2^k-1)(2^{k-1}) = \frac{(2^{k+1}-1)(2^{k+1}-2)}{6} = \frac{(2n+1)(2n)}{6}$ triples, which is the correct number in an STS(2n+1). □

Remark: This creates a sequence of nested Steiner triple systems, each of order $2^k - 1$ for $k \geq 2$.

2.3 Other Methods

While the previous two sections detailed the creation of an STS(2n+1) from an STS(n), we now consider direct constructions of a Steiner triple system for any permissible value of n. This is in contrast to Kirkman's proof that $n \equiv 1, 3 \bmod 6$ is necessary and sufficient for the existence of an STS(n), which relied on using smaller systems to obtain systems of a larger order. The following are due to Thoralf Skolem as discussed in [1].

Skolem's method for the construction of an STS(n) where $n = 6m+3$. We work with elements of the set $\{0, 1, ..., 6m+2\}$. First create the following array:

0	1	2	3	...	$2m-1$	$2m$
$2m+1$	$2m+2$	$2m+3$	$2m+4$	...	$4m$	$4m+1$
$4m+2$	$4m+3$	$4m+4$	$4m+5$	...	$6m+1$	$6m+2$

Form triples $(i, i+2m+1, i+4m+2)$ from the columns of this array for $i = 0, 1, ..., 2m$. Now for each pair $\{a, b\}$ in any row, the third element c of the triple (a, b, c) is the entry in the next row such that $2c \equiv a + b \bmod (2m+1)$ (the next row of the bottom is the top row). There are $\binom{2m+1}{2}$ pairs of elements in each row, so because there are 3 rows, we obtain $3\binom{2m+1}{2} = 3m(2m+1)$ triples. Adding this number to the $2m+1$ previously created triples yields a total of $(2m+1)(3m+1) = \frac{(6m+3)(6m+2)}{6}$ triples, as required. It is straightforward to verify that each pair of elements from $\{0, 1, ..., 6m+2\}$ occurs in exactly one triple.

Skolem's method for the construction of an STS(n) where $n = 6m+1$. Here the set of elements is $\{0, 1, ..., 6m\}$, and we begin in a similar manner with the array:

$$\begin{array}{cccc|cccc} 0 & 1 & ... & m-1 & m & m+1 & ... & 2m-1 \\ 2m & 2m+1 & ... & 3m-1 & 3m & 3m+1 & ... & 4m-1 \\ 4m & 4m+1 & ... & 5m-1 & 5m & 5m+1 & ... & 6m-1 \end{array}$$

Now we create three types of triples.

1. Triples from the first m columns (those to the left of the bar): $(i, 2m+i, 4m+i)$ for $i = 0, 1, ..., m-1$.

2. Triples of the following forms: $(m+i, 2m+i, 6m)$, $(3m+i, 4m+i, 6m)$, and $(i, 5m+i, 6m)$, each of these for $i = 0, 1, ..., m-1$.

3. Triples (a, b, c), where $\{a, b\}$ is a pair of elements from the same row and c is in the next row such that if $a+b$ is even, $2c \equiv a+b \bmod 2m$ and c is in the left half, and if $a+b$ is odd, $2c \equiv a+b-1 \bmod 2m$ and c is in the right half.

The number of triples is then $4m + 3\binom{2m+1}{2}$, where the first summand corresponds to types (1) and (2) and the latter summand is for type (3). This adds to the correct total $m(6m+1) = \frac{(6m+1)(6m)}{6}$. Again one can confirm that each pair of elements is part of exactly one triple.

Now that we have reviewed algorithms yielding Steiner triple systems on permissible n-sets, we turn to the primary issue at hand, using block-intersection graphs to distinguish among isomorphism types of systems on the same order.

Chapter 3

Cliques

One graph invariant is the number of cliques of a given size in a graph. A *clique* is a complete graph on m vertices, often called an m-clique, where we consider $m \leq \frac{n(n-1)}{6}$ (recall that $\frac{n(n-1)}{6}$ is the total number of vertices in a BIG of an STS(n)). A *maximum clique* is the largest possible clique in a graph, whose number of vertices is called the *clique number* and denoted by $\omega(G)$. A *maximal clique* is a clique for which it is not possible to add another vertex; alternatively, a maximal clique is a clique not contained in a larger clique. In a BIG, the vertices correspond to triples of an STS(n), so a clique represents a collection of triples with pairwise non-empty intersections.

Because each element of an STS(n) is in $\frac{n-1}{2}$ triples, the BIG will have at least n cliques of size $\frac{n-1}{2}$ induced by each of $0, 1, ..., n-1$, namely all the triples that contain a given point a. As one element $a \in V$ generates the clique and hence will be contained in each triple, these are always maximum cliques. If the size were increased, an additional triple would have to contain a in order to intersect every triple currently present; however, all triples containing a are already vertices of the clique, showing maximality. In this paper we call this type of clique, of size $\frac{n-1}{2}$ with one common element in each triple, a "canonical clique."

3.1 Maximal Cliques in a BIG

In order to consider maximal cliques of larger sizes, we begin with triangles. There are two types of 3-cliques possible in a BIG:

1. Triples containing a common element, such as $(a,b,c),(a,d,e)$, and (a,f,g).

2. Triples with no element in common, such as $(a,b,c),(a,d,e)$, and (c,e,f).

Neither of these are maximal cliques: each can be expanded to create a clique of size four. For the first type, a 4-clique can be obtained by adding the triple (a,h,i). Because a must appear with every other element of V and we are considering $n = 13$ or 15, such a triple will always exist and allow the creation of a 4-clique. It is also possible to add any of the triples (b,d,f), (b,d,g), (b,e,f), (b,e,g), (c,d,f), (c,d,g), (c,e,f), or (c,e,g), if they are part of the STS(n).

For the second type, if we start with the triangle $(a,b,c),(a,d,e),(c,e,f)$, there are four pairs of the elements $\{a,...,f\}$ that have not yet appeared in a triple in this triangle: $\{a,f\},\{b,d\},\{b,e\}$ and $\{c,d\}$. Because the pairs $\{a,f\},\{b,e\}$, and $\{c,d\}$ already intersect all triples in this 3-clique, it does not matter what the third element is in the triple containing any of these pairs. By definition of a Steiner triple system, the pairs must occur in some triple, so any of these triples can be included to increase from a 3-clique to a 4-clique. In addition, if the triple (b,d,f) is part of the system, it offers another option in expanding to a 4-clique. Hence a triple of the form $(a,f,\_)$, $(b,e,\_)$, $(c,d,\_)$, or (b,d,f) can always be added.

Unlike cliques of size three, some cliques of size four are maximal for both $n = 13$ and 15. Again we consider two cases:

1. Triples with a common element, that is, $(a,b,c),(a,d,e),(a,f,g)$, and (a,h,i). Clearly this can be expanded to a 5- or 6-clique for orders 13 and 15, and a 7-clique for order 15 by choosing any other triple containing a.

2. Triples with no one element in common. Theoretically, each of these 4-cliques can be expanded to a maximum clique of size six for order 13 and a maximum clique of size seven for order 15. For example, (a, b, c), (a, d, e), (a, f, g), (b, d, f) becomes (a, b, c), (a, d, e), (a, f, g), (b, d, f), (b, e, g), which then yields (a, b, c), (a, d, e), (a, f, g), (b, d, f), (b, e, g), (c, d, g), which in turn is enlarged to (a, b, c), (a, d, e), (a, f, g), (b, d, f), (b, e, g), (c, d, g), (c, e, f). However, maximality occurs when the desired next triple is not in a particular system. In this example, the 6-clique could be stopped from increasing to a 7-clique if the pair $\{c, e\}$ exists in a triple that does not contain the required element f.

Example: In system 2 of order 15, the triples $(1, 3, 5)$, $(1, 7, 9)$, $(3, 7, 11)$, and $(5, 7, 14)$ form a clique of size four. This clique is maximal because the triples that could possibly be added that intersect with these four are $(1, 11, 14)$, $(3, 9, 14)$, and $(5, 9, 11)$, none of which is part of system 2.

3.2 Maximum Cliques for $n = 15$

In the BIG for an STS(15), there is a special property of maximum cliques without a common element in each triple, which we now state and prove.

Proposition: If the BIG for an STS(15) contains a non-canonical maximum clique of size 7, it is the BIG of an STS(7), i.e., it represents a subsystem of size 7.

Proof: As above, we consider the possible ways to obtain a 7-clique from a 4-clique. The first possible 7-clique contains triples with a common element; this type is a canonical clique. The second type has the form (a, b, c), (a, d, e), (a, f, g), (b, d, f), (b, e, g), (c, d, g), (c, e, f), up to labeling, in which each element is represented in three triples. In fact, that must be the case in a clique of size seven for $n = 15$. Consider if we had four, five, or six triples with a common element. For example, take $(a,b,c),(a,d,e),(a,f,g)$, and (a,h,i) of size four. At this point there is no possible triple to add that will intersect these four other than one containing a (which would make a canonical clique), and the same clearly holds for size five or six. Finally, if one element is in only one or two triples, we would not have enough triples to make a 7-clique. Hence an element occurs in exactly three triples.

Now we confirm that the clique is an STS(7). By inspection of (a,b,c), (a,d,e), (a,f,g), (b,d,f), (b,e,g), (c,d,g), (c,e,f), we have seven distinct elements $\{a,...,g\}$, each in three triples, which equals the replication number $\frac{7-1}{2}$. Also, there are seven triples, and the required number is $\frac{7(7-1)}{6}=7$. Finally, each possible pair of $\{a,...,g\}$ appears in exactly one triple. Thus we do have a Steiner triple system of size 7. □

Example: System 1 of the STS(15)s corresponds to the lexicographically-least, backtrack-free system guaranteed by $15=2^4-1$. Because it can be generated starting with the STS(7) and using the proof in Chapter 2, system 1 contains a non-canonical maximum clique corresponding to the triples from the STS(7): $(0,1,2)$, $(0,3,4)$, $(0,5,6)$, $(1,3,5)$, $(1,4,6)$, $(2,3,6)$, and $(2,4,5)$.

Example: Another non-canonical maximum clique in STS(15) #1 is formed by $(1,3,5)$, $(1,8,10)$, $(1,12,14)$, $(3,8,12)$, $(3,10,14)$, $(5,8,14)$, and $(5,10,12)$, which is isomorphic to the STS(7) in the previous example.

Remark: Maximum cliques in BIGs for order $n=13$ will not correspond to subsystems, as they are of size six, not an admissible order for a Steiner triple system (clearly $6\not\equiv 1,3 \mod 6$).

3.3 Clique Census

As part of the effort to distinguish Steiner triple systems of a given order, we conducted a maximal clique census. SAGE provides a function to identify all maximal cliques of a graph, so we ran the function on each BIG and tallied the number of maximal cliques for each possible clique size.

For $n = 13$, both BIGs have clique number $\omega(G) = 6$, as expected because $\frac{n-1}{2} = \frac{12}{2} = 6$, and both graphs have thirteen of these maximum cliques. The other possible clique sizes were four and five; the total number of maximal cliques was 201 for system 1 and 156 for system 2. For the complete census please see Table 3.1. Note that the different number of maximal cliques immediately distinguishes the two STS(13)s, as does the differing numbers of 4- and 5-cliques.

Table 3.1: Clique Census for $n = 13$

System	Total Maximal Cliques	6-Cliques	5-Cliques	4-Cliques
1	201	13	24	164
2	156	13	39	104

Now for order $n = 15$, all BIGs have $\omega(G) = 7$, and all have at least fifteen such maximum cliques corresponding to the canonical cliques. There is one system each that has thirty and twenty-two maximum cliques, respectively. Five systems have eighteen and sixteen have sixteen; all of the rest have fifteen. The difference between the number of maximum cliques and fifteen corresponds to the number of subgraphs that represent STS(7)s. The other possible clique sizes were four, five, or six and the number of maximal cliques ranged from 30 to 435.

We initially ranked and compared systems by total number of maximal cliques. There are only four systems with a total under 150, one of which will be discussed further below, and another four systems between 150 and 200. Of the 90% with totals over 200, sixteen were between 200 and 300 (20%), fifty-four were between 300 and 400 (67.5%), and just two had totals over 400.

Out of the eighty STS(15)s, seventeen have a unique number of maximal cliques. There are nine pairs of systems with the same total, three sets of size 3 that share the same total, two sets of size 4, three sets of size 5, and one each of size 6 and size 7. All of these systems that have the same number of maximal cliques also have identical clique censuses (the same number of 4-cliques, 5-cliques, 6-cliques, and 7-cliques), except in one case. There are four systems with 345 maximal cliques, three of which have the same census and one which does not. As a result of the clique census, the STS(15)s were partitioned into thirty-seven classes.

Table 3.2: Partial Clique Census for $n = 15$

System(s)	Total Maximal Cliques	7-Cliques	6-Cliques	5-Cliques	4-Cliques
1	30	30	0	0	0
2	86	22	8	24	32
3	114	18	12	36	48
4, 5	158	18	8	36	96
35, 40, 59	325	15	1	33	276
46, 49, 60, 65, 75	372	15	0	21	336
77	417	15	0	6	396
80	435	15	0	0	420

There are a few other interesting points that arose but which we will not elaborate on at this time. In general, as the total number of maximal cliques increased, so did the number of systems with identical censuses. All of the systems with a total greater than 350 had only the fifteen canonical maximum 7-cliques; almost half of the systems fell into this category without subsystems of size 7 (36 out of 80 for 45%). Just over half (41) had no 6-cliques, and there was a rough correlation between number of maximal cliques and absence of 6-cliques. Only two had no 5-cliques and these were the systems with the lowest and highest total of maximal cliques. Every STS(15) except one had maximal 4-cliques; the exception, system 1, had only 7-cliques in its BIG.

The maximal clique census did provide a way to distinguish many of the STS(15)s from each other based on their BIGs. However, there were still many small groups of systems that had identical BIGs in this regard and thus could not be differentiated using cliques alone.

3.4 System 1 for $n = 15$

The lexicographically-least, backtrack-free STS(15) labeled as system 1 has a very curious maximal clique census: it has thirty maximal cliques, which is by far the lowest number, and all are maximum cliques of size 7. STS(15) #1 has the usual fifteen canonical cliques, meaning that it has another fifteen cliques whose triples do not all contain a common element, and it is also the only system without maximal cliques of lower size.

A simple and appealing explanation for the large number of maximum cliques in system 1 for $n = 15$ is provided using properties of finite projective space. The three-dimensional projective space $\mathrm{PG}(3, 2)$ over GF(2), the field containing two elements, can be defined as a set P of fifteen points and a set L of subsets (called "lines") of P satisfying the following axioms for projective spaces:

1. Any line contains at least three points.
2. Any two points lie on a unique line.
3. A transversal to two sides of a triangle also meets the third side.

Subspaces of dimension 0, 1, and 2 are points, lines, and planes, respectively. In a finite projective space, an equal number of points lie on every line [17].

In $\mathrm{PG}(3, 2)$, there are thirty-five lines and fifteen planes to go with the fifteen points (see Figure 3.1). Every line contains three points, each point lies on seven lines, and any pair of points appears on exactly one line. As a result, the properties of $\mathrm{PG}(3, 2)$ coincide with those of a Steiner triple system of order 15, with points and lines instead of elements and triples. Thus PG$(3, 2)$ can be thought of as an STS(15). There is only one PG$(3, 2)$ up to isomorphism, meaning that it represents one of the eighty STS(15)s, which corresponds to system 1.

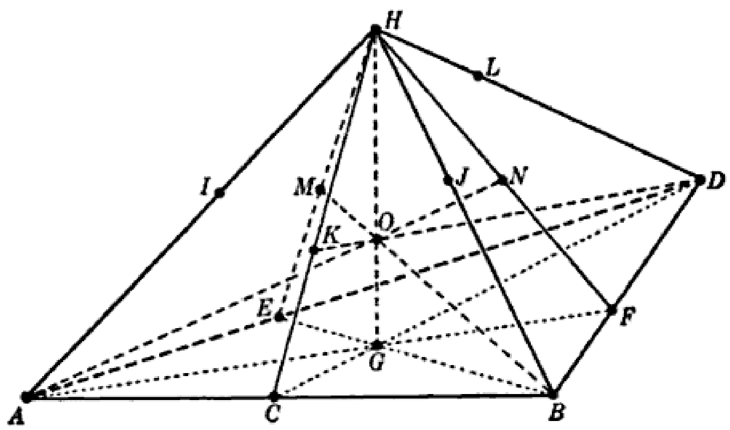


Figure 3.1: PG(3,2)

Now, the planes of PG$(3, 2)$ have seven points and seven lines and are each isomorphic to the Fano plane PG$(2, 2)$, the projective plane of order two. The Fano plane is also a visual representation of the STS(7), so each plane of PG$(3, 2)$ corresponds to a subsystem STS(7) of STS(15) #1. In Figure 3.2 of the Fano plane, the triples of the STS(7) are formed by the six lines and one circle. Recall from above that a subsystem of size 7 in an STS(15) corresponds to a non-canonical maximum clique. Then the fact that system 1 has fifteen subsystems isomorphic to STS(7) implies that its block-intersection graph has another fifteen maximum cliques in addition to the fifteen canonical maximum cliques.

Remark: The relationship between PG$(3, 2)$ and an STS(15) holds for projective spaces of higher dimension. In fact, Colbourn and Rosa state the theorem:

> Theorem: The points and lines of PG$(n, 2)$ form the elements and triples of an STS$(2^{n+1} - 1)$.

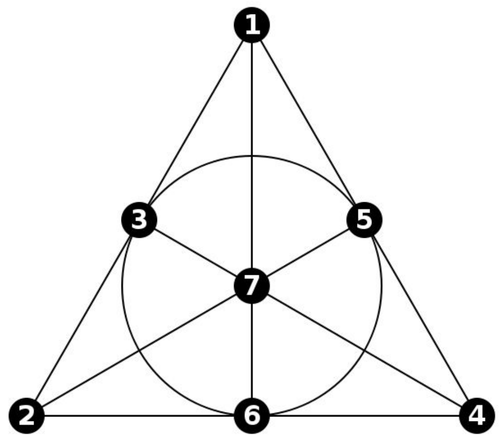


Figure 3.2: The Fano Plane

The argument of their proof goes as follows: Let P_n be an $(n+1)$-dimensional vector space over GF(2) with vector addition represented by $\oplus$. For nonzero $\vec{x}, \vec{y} \in P_n$, $\{0, \vec{x}, \vec{y}, \vec{x} \oplus \vec{y}\}$, is a subspace of dimension two. Hence each pair of nonzero vectors defines one punctured subspace (obtained by removing 0 from the set in the previous sentence) containing three vectors. Defining the elements to be nonzero vectors and punctured subspaces of dimension two to be triples yields an STS$(2^{n+1} - 1)$. A Steiner triple system resulting from PG$(n, 2)$ is called *projective* [5].

In summary, cliques were helpful in distinguishing some STS(15)s but were not enough to completely identify all eighty up to isomorphism, although they did differentiate the two STS(13)s. In the next chapter we consider using independent sets and related properties.

Chapter 4

Independent Sets

In a graph, an *independent set* is a set of mutually non-adjacent vertices; a *maximal independent set* is an independent set such that adding another vertex would force the induced subgraph to contain an edge, that is, at least two vertices in the set would be adjacent, violating independence. Finally, a *maximum independent set* is the largest possible independent set, whose size is denoted by $\alpha(G)$. It is well-established that a set of vertices is independent if and only if the set is a clique in the graph's complement. With that in mind, we produced the graph complement of each BIG using SAGE, then identified independent sets in BIGs by finding cliques in these complements.

4.1 Independent Sets in a BIG

As the vertices of a BIG represent triples of an STS(n), an independent set in such a graph is a set of triples that are mutually disjoint. In the following sections, we consider different characteristics of independent sets in the block-intersection graphs.

4.1.1 Cardinality

In the BIGs of STS(13)s and STS(15)s, the cardinality of a maximal independent set is at least three: given an independent set $\{(a, b, c), (d, e, f)\}$ of size two, there is always a third triple (g, h, i) that does not intersect the first two triples.

For $n = 13$, an independent set is at most size four, for suppose we have an independent set of triples $\{(a, b, c),\ (d, e, f),\ (g, h, i),\ (j, k, l)\}$. Increasing this set would require an additional three distinct elements to form a mutually non-intersecting triple. But $n = 13$, so there is only one more element and thus no other triple can be added. In fact, this shows that any maximum independent set will have one element of V not represented, and in system 2, each of the thirteen maximum independent sets corresponds to missing a different element of V.

Now if $n = 15$, it is possible to have an independent set of size five, as such a set would contain five triples, each made up of distinct elements. Then clearly this set would be maximal because 5 triples $\times$ 3 elements is 15 total elements; every element of $\{0, ..., 14\}$ would be represented in the independent set. Of course, just as in the case $n = 13$, maximality can occur with sets of lower sizes (three and four) if unused elements are only paired with elements already included in the independent set.

Although in theory an independent set of size three can be expanded to include one more triple for either $n = 13$ or 15, maximality occurs when the desired elements do not appear in a triple together. Such a set contains nine distinct elements $a, b, ..., i$, leaving four or six others not yet represented for $n = 13$ or 15, respectively. But if no 3-subset of the remaining elements forms a triple in the system, the independent set of size three is maximal.

Example: In the second STS(13), the triples $(0, 1, 2), (3, 6, 11)$, and $(5, 9, 12)$ form a maximal independent set. There are four unused elements: 4, 7, 8, and 10, but each triple that contains at least one of these elements also contains at least one of 0, 1, 2, 3, 6, 5, 9, 11, and 12. Adding such a triple would violate independence, so we have maximality.

4.1.2 Resolvability

Recall from the Introduction that a resolvable STS(n) is one whose set of triples can be partitioned into parallel classes. A related idea is that of a *partial parallel class*, which is a set of mutually disjoint triples that may not partition V. Because a parallel class is a set of mutually disjoint triples whose union is the n-set, a parallel class in an STS(15) is exactly a maximum independent set of size five in its BIG; maximal independent sets of sizes three or four are partial parallel classes. However, a maximum independent set in the BIG of an STS(13) is of size four, and hence cannot form a parallel class due to the absence of one element. For $n = 13$, then, any maximal or maximum independent set forms just a partial parallel class.

For an STS(15), note that if a block-intersection graph contains at least seven maximum independent sets, or equivalently, the system has at least seven parallel classes, this does not necessarily imply the system is resolvable. Not only do the triples within a parallel class have to be pairwise disjoint, but the set of parallel classes must also be pairwise disjoint so as to partition the set $\mathcal{B}$ of triples. If the system is resolvable and the BIG does contain seven mutually disjoint independent sets, these sets must be maximum of size five to obtain the total of thirty-five vertices.

Example: System 61 of order 15 is resolvable, so it contains seven parallel classes that partition its set of triples. Each parallel class is a maximum independent set in the block-intersection graph; these are listed in Table 4.1. The table also represents one solution to Kirkman's schoolgirl problem if each element corresponds to a girl and the columns are taken as the days of the week.

Remark: Because an STS(n) comprises blocks of size three, it is straightforward to see that a necessary condition for resolvability is $n \equiv 0 \bmod 3$. As Steiner triple systems exist for $n \equiv 1, 3 \bmod 6$, a system is potentially resolvable only if $n \equiv 3 \bmod 6$. It was not until 1965, however, that this condition was also shown to be sufficient [5].

Table 4.1: Maximum Independent Sets in the BIG of STS(15) #61

Parallel Class						
1	2	3	4	5	6	7
$(0,1,2)$	$(0,3,4)$	$(0,5,6)$	$(0,7,8)$	$(0,9,10)$	$(0,11,12)$	$(0,13,14)$
$(3,8,10)$	$(1,8,11)$	$(1,7,9)$	$(1,4,6)$	$(1,12,14)$	$(1,10,13)$	$(1,3,5)$
$(4,9,13)$	$(2,9,12)$	$(2,8,14)$	$(2,11,13)$	$(2,4,5)$	$(2,3,6)$	$(2,7,10)$
$(5,11,14)$	$(5,7,13)$	$(3,12,13)$	$(3,9,14)$	$(3,7,11)$	$(4,7,14)$	$(4,8,12)$
$(6,7,12)$	$(6,10,14)$	$(4,10,11)$	$(5,10,12)$	$(6,8,13)$	$(5,8,9)$	$(6,9,11)$

4.1.3 Graph Coloring

A classic problem in graph theory is that of *coloring* a graph G: assigning each vertex a color such that adjacent vertices are of different colors. The lowest possible number of colors is the *chromatic number* $\chi(G)$. Every set of vertices of the same color, called a *color class*, clearly forms an independent set. Because each STS(n) block-intersection graph G contains cliques, in which every pair of vertices is connected, $\chi(G)$ will be at least the clique number $\omega(G)$. Note that the size of any color class is at most $\alpha(G)$, the size of the maximum independent set. We obtain a connection between colorings, independent sets, and resolvability as stated in the following:

Proposition: Let G be the block-intersection graph of an STS(15). Then $\chi(G) = 7$ if and only if the system is resolvable.

Proof: $\chi(G) = 7 \iff G$ contains seven color classes on thirty-five vertices $\iff G$ has seven maximum independent sets of size five $\iff \mathcal{B}$ can be partitioned into seven mutually disjoint sets of triples, i.e., parallel classes $\iff$ the STS(15) is resolvable. □

We know there exists at least one resolvable STS(15) since $15 \equiv 3 \bmod 6$ (and because we were able to identify the parallel classes in system 61), so the chromatic number of at least one BIG for an STS(15) attains the lowest possible value $\omega(G) = 7$. The other possible values for the chromatic number for $n = 15$ were eight and nine; both systems of order 13 have chromatic number equal to eight [12].

4.2 Independent Set Census

Like with maximal cliques, we compiled a census of maximal independent sets. For $n = 13$, the size of the largest independent set is four for both systems, as expected. System 1 has eight maximum and 106 maximal independent sets, and system 2 has thirteen maximum and 91 maximal independent sets. As mentioned above, non-maximum maximal independent sets are of size three. The data are shown in Table 4.2. Any of the unequal numbers of such sets is enough to distinguish the two STS(13)s, similar to the differing totals of maximal cliques.

Table 4.2: Independent Set Census for $n = 13$

System	Total Maximal Independent Sets	Size 4	Size 3
1	106	8	98
2	91	13	78

For each BIG of order $n = 15$, the maximum independent set is either of size four, for ten out of eighty systems, or size five for the remaining systems. The number of maximum independent sets ranges from 1 to 224. Although this is a broad range, 22.5% of systems contain only one such set, and three-fourths of the systems have under ten (61 of 80), with nine more falling between 11 and 56. The total then increases considerably: the remaining ten STS(15)s have between 182 and 224 maximum independent sets, and these ten coincide with those having maximum independent sets of size four. There are only two systems such that every maximal independent set is also maximum; their respective totals are 56 and 224.

The number of maximum independent sets of the BIGs yields information on parallel classes in the STS(15)s. Because ten systems have such sets of size four, not five, in their graphs, these systems contain no parallel classes. To consider those that do have at least one parallel class, we look at the BIGs with maximum independent sets of size five. Hence the number of parallel classes in an STS(15) with such a set falls between one and fifty-six.

The number of maximal independent sets goes from 56 to 261, with 76 of the systems having 200 or higher and only one system with under 152. The four systems that have under 200 maximal independent sets all have between sixteen and fifty-six maximum independent sets, which is on the large end of the totals of maximum sets under 182. All except four STS(15)s have maximal independent sets of size three, and those that have none fall in the lowest 20% in total number of maximal sets.

Considering the independent set census as a way of differentiating systems, there are sixteen with a unique number of maximal independent sets, and twelve pairs of systems with the same total. Of these pairs, only one has systems with identical censuses; all the others can be distinguished. In fact, several of the pairs differ in size of maximum independent set, making the full census unnecessary in these cases. Six sets of size three match in number of maximal independent sets, but five of these have different censuses. The sixth has two with identical censuses and one without. There are only three sets of size four with the same total of maximal sets, and all of these systems within each total have different censuses. Finally, there are ten STS(15)s that have 251 maximal independent sets; of these only two have the same census.

The independent set census was more successful in distinguishing the eighty STS(15)s than the clique census. Using maximal independent sets in the BIGs, only six systems could not be declared completely different from every other graph on this invariant. In fact, a majority of the systems that had the same total of maximal independent sets had an unequal number of maximum independent sets, in which case calculating the numbers for each possible set size is not required. The remaining six systems were divided into three pairs, with the two BIGs in each pair sharing the same census, resulting in seventy-seven classes based on independent sets.

Compared to cliques and independent sets, a more recently explored subfield of graph theory is algebraic graph theory, whose focus is using the spectra of graphs as an invariant to show non-isomorphism. We will cover this subject in the next chapter.

Chapter 5

Spectra

Spectral graph theory uses techniques of linear algebra to study structural properties of graphs and can be used in the graph isomorphism problem. We consider such methods applied to the block-intersection graphs.

5.1 Definitions and Background

Let A denote the adjacency matrix of a block-intersection graph G on v vertices. The characteristic polynomial of A is $det(xI - A)$, where I is the $v \times v$ identity matrix. Because similar matrices have the same characteristic polynomial, any vertex labeling of G used to create A results in the same polynomial. Thus there is exactly one characteristic polynomial for every graph, and the roots $\lambda_1, \lambda_2, ..., \lambda_v$ of this polynomial are the *eigenvalues of G*. The eigenvalues are typically listed in non-increasing order, and the multi-set of eigenvalues is the *spectrum of G*, with the number of times λ_i appears called its *multiplicity*. If two graphs have the same spectrum, they are *cospectral*, but they may not be isomorphic. However, if two graphs are isomorphic then they are cospectral, meaning that a graph's spectrum is an invariant.

Because the spectrum of a graph constitutes an invariant and is straightforward to compute, we turned to graph spectra as a possible method of distinguishing BIGs. SAGE has a function to find the spectra of a graph; these eigenvalues were determined for the BIG of each system. We initially considered eigenvalues of each graph's *Laplacian matrix* as well. The Laplacian matrix is $L = D - A$, where D is the degree matrix, a $v \times v$ diagonal matrix such that the non-zero entry d_i is equal to the degree of vertex i. In general, there is no easy relationship between the usual spectra and the Laplacian spectra. However, with a k-regular graph, each entry in D is the same and results in the eigenvalues of L being $k - \lambda_1, k - \lambda_2, ..., k - \lambda_v$. Since the BIGs are regular and the Laplacian spectra can be characterized in terms of the usual spectra, we focused on eigenvalues of the adjacency matrix.

Before discussing the spectra of the BIGs, we make some general remarks. The first eigenvalue λ_1 is equal to k for a k-regular graph G: each row of A has k entries equal to 1, so $A\vec{\mathbf{j}} = k\vec{\mathbf{j}}$, where $\vec{\mathbf{j}}$ denotes the all-1 vector. This also implies that $\vec{\mathbf{j}}$ is an eigenvector of G. In addition, k must be the first eigenvalue of a k-regular graph because $|\lambda_i| \leq \Delta(G)$, the maximum degree of G, for all i. Finally, in a connected graph, the multiplicity of k is 1.

5.2 Spectra of the BIGs

We found the spectra for each BIG for orders $n = 13$ and 15; within the respective orders, all graphs were cospectral, that is, had the same eigenvalues with the same multiplicities. Each order had three distinct eigenvalues. The spectrum for $n = 13$ is $15, 2,$ and -3 with respective multiplicities $1, 12,$ and 13, and the same figures for $n = 15$ are $18, 3,$ and -3 with multiplicities $1, 14,$ and 20. In hopes of a general characterization, we also considered the spectra of BIGs of one STS(n) each for permissible orders up to $n = 25$, using SAGE to generate the Steiner triple systems. The results are in Table 5.1; $n = 7$ was excluded because its BIG is K_7, and complete graphs have their own classification of spectra.

Table 5.1: Spectra of the Block-Intersection Graphs

STS Order	Number of Vertices	Degree k	Eigenvalues	Respective Multiplicities
9	12	9	9, 0, -3	1, 8, 3
13	26	15	15, 2, -3	1, 12, 13
15	35	18	18, 3, -3	1, 14, 20
19	57	24	24, 5, -3	1, 18, 38
21	70	27	27, 6, -3	1, 20, 49
25	100	33	33, 8, -3	1, 24, 75
...	...	...	...	...
n	$\frac{n(n-1)}{6}$	$\frac{3(n-3)}{2}$	k, $k-n$, -3	1, $n-1$, $\frac{n(n-7)}{6}$

As is evident from the table, the BIGs all had three distinct eigenvalues, with -3 appearing in the spectrum of each graph. Using the output seen in the table, the eigenvalues and multiplicities could both be predicted based on the order n. Because the spectra were highly structured, we researched further in algebraic graph theory. It can be shown that strongly regular graphs with parameters (v, k, e, f) have eigenvalues k, s, and t, where $s, t = \frac{1}{2}(e-f \pm \sqrt{\Delta})$ and $\Delta = (e-f)^2 + 4(k-f)$. The respective multiplicities are 1, ℓ, and m, where $\ell, m = \frac{1}{2}(v - 1 \mp \frac{2k+(v-1)(e-f)}{\sqrt{\Delta}})$ [8]. Using the parameters $(\frac{n(n-1)}{6}, \frac{3(n-3)}{2}, \frac{n+3}{2}, 9)$ of the BIGs, the spectra calculated based on [8] matches the values predicted by our investigation.

5.3 Graph Complements and Line Graphs

Two cospectral graphs do not necessarily have cospectral graph complements, so one might consider determining the spectra of the complements of the BIGs in an effort to distinguish among systems. However, it is straightforward to show that the complement $\overline{G}$ of a strongly regular graph G is itself strongly regular. In fact, the parameters of $\overline{G}$ can be computed in terms of the original parameters (v, k, e, f): $\overline{G}$ has parameters $(v, v-k-1, v-2-2k+f, v-2k+e)$. Since the BIG complements are all strongly regular on the same parameters, they are also all cospectral for each n and hence cannot be used to differentiate STS(n)s of the same order.

Another simple graph operation used to obtain a new graph whose spectrum can be informative is the creation of the *line graph L(G)* of a graph G. The vertex set of $L(G)$ is the edge set of G, and two vertices are joined in $L(G)$ if their respective edges in G have a vertex in common. The number of vertices for a line graph of a BIG is then 195 for $n = 13$ and 315 for $n = 15$. Now, each BIG is regular of degree k. It follows that its line graph is $(2k-2)$-regular: consider an edge e in the BIG incident to two vertices v_1 and v_2, which each have k edges leaving them. Two of those, one each from v_1 and v_2, correspond to e, implying that e shares a vertex with $2k-2$ edges. Hence any k-regular graph will have a line graph that is $(2k-2)$-regular.

We produced the line graphs for the BIGs and found their spectra; again all BIG line graphs for the same order STS(n) were cospectral. In this case, strong regularity was unnecessary in determining the spectra of each $L(\text{BIG})$, as regularity itself implies cospectral line graphs. Because each BIG has a $(2k-2)$-regular line graph, a result of Sachs states that if G has eigenvalues k, s and t with multiplicities 1, ℓ, and m, then the respective spectrum for $L(G)$ is $2k-2$, $k-2+s$, $k-2+t$, and -2 with multiplicities 1, ℓ, m, and $|E(G)| - v$, where $|E(G)|$ is the number of edges of the original graph G [3]. Because the BIGs are of the same degree, their line graphs have identical spectra, so this spectra cannot be used to differentiate the STS(n)s.

Remark: Although the line graph of a regular graph is regular, the same relationship does not necessarily hold for strong regularity. This follows either from the fact that a graph is strongly regular if and only if it has three distinct eigenvalues or by a counting argument. First, the line graphs of the BIGs have four eigenvalues and so are not strongly regular. Second, consider two adjacent vertices u_1 and u_2 in $L(G)$, where G is k-regular. Adjacency implies they share a vertex v_1 in G. A common neighbor u_3 in the line graph would have to be adjacent to both u_1 and u_2, meaning it shares a vertex with both of the corresponding edges in G. This vertex could be v_1, in which case there are $k-2$ possible edges. If not v_1, then the edge u_3 in G must have one incident vertex v_i in common with u_1 and the other incident vertex v_j in common with u_2. However, this edge u_3 may not be present in G if v_i and v_j are non-adjacent. Because two adjacent vertices in $L(G)$ have a variable number of common neighbors (either $k-2$ or $k-1$), $L(G)$ is not strongly regular.

5.4 Eigenspaces

For both orders $n = 13$ and 15, although the BIGs are cospectral, each graph's adjacency matrix is distinct up to permutation by vertex relabeling and so has a distinct set of eigenvectors, as tested using SAGE. Because the eigenvalues and multiplicities are identical, every graph has for each eigenvalue a vector space of the same degree (equal to the number of vertices) and same dimension (equal to the eigenvalue's multiplicity). However, the basis matrices, composed of the eigenvectors, are different. It is true that the first eigenspace always corresponds to the eigenvalue k, the degree of the BIG, and the all-1 eigenvector $\vec{\mathbf{j}}$.

In looking for algebraic invariants associated to graphs, Dragoš Cvetković defined the *angle matrix* of a graph G. Given a graph on v vertices with t distinct eigenvalues $\mu_1 > ... > \mu_t$, define β_{ij} to be the angle between the eigenspace of μ_i and $\mathbf{e}_j$, where $\{\mathbf{e}_1, ..., \mathbf{e}_v\}$ is the standard basis of $\mathbb{R}^v$. The angles of G are then $\alpha_{ij} = \cos(\beta_{ij})$, which form the $t \times v$ angle matrix (α_{ij}). This matrix is an invariant if the vertices of G are labeled such that the columns of the angle matrix are in lexicographic order (all references in this section are to [7]).

The advantage to considering angle matrices is that they can distinguish many graphs with identical spectra. Using results from Cvetković, Rowlinson, and Simić, we calculate the angle matrices of the BIGs. They have $v = \frac{n(n-1)}{6}$ vertices and $t = 3$ distinct eigenvalues. Because each BIG is walk-regular, $\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{iv}$ for $i = 1, 2, 3$.[1] The dimension of the eigenspace corresponding to μ_i is the multiplicity of μ_i, and by Proposition 4.2.1 in [7], equals $\sum_{j=1}^{v} \alpha_{ij}^2$ where $i = 1, 2, 3$. For ease of notation here, denote the multiplicity of the i^{th} eigenvalue as c_i. Then we have $c_i = \sum_{j=1}^{v} \alpha_{ij}^2 = v\alpha_{ij}^2$, yielding angles $\alpha_{ij} = \sqrt{\frac{c_i}{v}}$. As each BIG of an STS(n) on the same n has the same eigenvalues of the same multiplicities, this implies they have identical angle matrices. Thus these matrices unfortunately cannot differentiate the block-intersection graphs.

Remark 1: The logic of the preceding paragraph shows that any two strongly regular graphs with the same parameters have the same angle matrices, as they are walk-regular with equal spectra. In fact, any two walk-regular graphs with equal spectra have identical angle matrices. Hence although the angle matrix is an algebraic graph invariant, it does not determine a graph up to isomorphism.

Remark 2: If graphs have four distinct eigenvalues, they also have the same angles. In particular, this means the line graphs of the BIGs cannot be distinguished via angle matrices.

[1]This follows from the fact that $a_{kk}^{(s)} = \sum_{j=1}^{v} \mu_j^s \alpha_{jk}^2$, and $a_{kk}^{(s)}$ is the number of closed walks of length s that begin and end at vertex k.

5.5 Distance-Regularity

Recall that the block-intersection graphs of STS(n)s are distance-regular. This property yields information on graph spectra just as strong regularity did. First we define the *adjacency algebra* $\mathcal{A}(G)$ of a graph G with adjacency matrix A as the set of all polynomials in A with real coefficients. Now denote the diameter of G as d. A basis for $\mathcal{A}(G)$ is then given by the set $\{A_0, A_1, ..., A_d\}$, where the A_h are *distance matrices* defined as follows. Entry a_{ij} of A_h is 1 if the distance between the i^{th} and j^{th} vertices equals h and 0 otherwise. In a BIG, $d = 2$, and note that $A_0 = I$, $A_1 = A$, and $A_0 + A_1 + A_2 = J$, J being the all-1 matrix. Because the distance matrices are a basis for $\mathcal{A}(G)$, the degree of the minimal polynomial of A is $d + 1 = 3$ and so G has exactly three distinct eigenvalues. This of course matches the total dictated by strong regularity.

Furthermore, the parameters of a distance-regular graph, represented in the intersection array, can be used to calculate the three eigenvalues and their multiplicities. In the Introduction, we computed the intersection array of a BIG on an STS(n) to be $\{b_0, b_1; c_1, c_2\}$ $= \{\frac{3(n-3)}{2}, n - 7; 1, 9\}$, with related values $a_1 = \frac{n+3}{2}$ and $a_2 = \frac{3(n-9)}{2}$. From the distance matrices, Biggs develops a different set of $(d + 1) \times (d + 1)$ matrices, in particular resulting in the *intersection matrix* B based on the a_i, b_i, and c_i, which we list here for the BIGs:

$$B = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3(n-3)}{2} & \frac{n+3}{2} & 9 \\ 0 & n-7 & \frac{3(n-9)}{2} \end{pmatrix}$$

whose eigenvalues coincide with those of the graph G [3]. This method can be advantageous for reducing the size of the matrix of interest from $\frac{n(n-1)}{6} \times \frac{n(n-1)}{6}$ to 3×3 for the BIGs. The eigenvalues we calculated from the intersection matrices for orders $n = 13$ and 15 did indeed match those listed above. Biggs also includes a formula to determine the multiplicity of each eigenvalue, involving the number of vertices and the inner product of left and right eigenvectors.[2] Thus the highly-structured nature of the BIGs offers multiple ways to compute their spectra.

Remark: The adjacency algebra $\mathcal{A}(G)$ of any strongly regular graph G has as a basis $\{I, A, J - I - A\}$ [8]. Because the block-intersection graphs are of diameter 2, this basis is the same as the set of distance matrices $\{A_0, A_1, A_2\}$.

Although spectral graph theory yields many interesting results on the block-intersection graphs, the spectrum and eigenspaces of the BIGs unfortunately were not able to aid in distinguishing the underlying Steiner triple systems of orders 13 and 15 (or any STS(n)s on the same order). In the final chapter, we address the success of the methods considered thus far in differentiating systems.

[2]To determine the multiplicity of eigenvalue λ_i, compute $m(\lambda_i) = \frac{|V(G)|}{(\mathbf{u}_i, \mathbf{v}_i)}$, where $\mathbf{u}_i$ and $\mathbf{v}_i$ are left and right eigenvectors, respectively, corresponding to λ_i and with first entry 1. $|V(G)|$ is the number of vertices and the denominator represents the inner product.

Chapter 6

Differentiating Systems

As discussed in the Introduction, there is only one Steiner triple system up to isomorphism for orders $n = 3, 7$, and 9, but there are two isomorphism types for STS(13)s and eighty for STS(15)s. We turned to block-intersection graphs and their invariants in an attempt to differentiate these systems.

6.1 Block-Intersection Graphs

The primary reason we have studied block-intersection graphs of Steiner triple systems, other than to determine some of their basic characteristics and properties, was to find a way to distinguish non-isomorphic systems of the same order. Clearly this is not a problem for $n = 3, 7$, or 9, which have only one STS(n) up to isomorphism. The two systems of order 13 were easily distinguished by either the maximal clique census or the maximal independent set census, as their respective totals differed on these graph invariants. However, Steiner triple systems of order 15 proved to be more of a challenge. Because each BIG had identical graph spectra, we considered cliques and independent sets.

6.1.1 Cliques and Independent Sets

As stated above, only eighteen STS(15)s had distinct clique censuses, so sixty-two had the same census as at least one other system. We also had seventy-four with distinct independent set censuses and six without. We then used both censuses simultaneously, which resulted in seventy-six systems having a unique breakdown of cliques and independent sets (Table 6.1 shows the data for the additional two systems). But there were still two pairs of systems that could not be distinguished by this method, as each pair had an identical number of cliques and independent sets of the various sizes. Now, because there are eighty STS(15)s and systems of the same order are isomorphic if and only if their BIGs are isomorphic, we know there are eighty distinct BIGs up to isomorphism. Yet we still had four graphs that could not be differentiated; the two pairs were systems 20 and 21 and systems 46 and 65.

Table 6.1: Systems Differentiated Using Both Censuses

System	Total Maximal Independent Sets	Size 5	Size 4	Size 3	Total Maximal Cliques
58	251	3	168	80	370
74	251	3	168	80	363

Remark: Because the amount of time required for a computer to find cliques and independent sets is small, these censuses of the block-intersection graphs do represent an efficient way to distinguish almost all of the STS(15)s.

6.1.2 Line Graphs

Two line graphs $L(G_1)$ and $L(G_2)$ are isomorphic if and only if their original graphs G_1 and G_2 are isomorphic, except in the case where G_1 and G_2 are K_3 and $K_{1,3}$[1], a result of Whitney's from 1932 [19]. The line graphs of the two BIGs for order $n = 13$ were tested

[1] $K_{1,3}$ consists of four vertices, one of which is adjacent to each of the other three vertices.

using SAGE and found to be non-isomorphic, and all BIG line graphs for $n = 15$ were mutually non-isomorphic as well. Because these line graphs were clearly not defined from K_3 or $K_{1,3}$, their corresponding graphs are also non-isomorphic. While this constitutes a method of distinguishing systems of a given order, the method in all likelihood would not be preferred, as it still relies on testing for graph isomorphisms and on much larger graphs. The number of vertices jumps from 26 in the BIG to 195 in its line graph just for $n = 13$. It would almost certainly be quicker to simply test the BIGs themselves for isomorphism.

6.2 Cycle Structure

Another graph that can be formed from a Steiner triple system is defined as follows. For any pair of elements $\{a, b\}$ of V, the graph G_{ab} has vertex set $V - \{a, b, c\}$, where c is the other element from the triple containing a and b. Vertices i and j are connected by an edge if (a, i, j) or (b, i, j) is a triple in $\mathcal{B}$. It follows that the resulting graph is regular of degree 2, but G_{ab} may not be connected. Each component, a maximal connected subgraph, is an even cycle, which is a 2-regular graph on an even number of vertices. This graph is described but not given a name by Mathon, Phelps, and Rosa in [12], in which the results discussed below appear. Colbourn and Rosa later call it the *double neighborhood* of a and b, although they allow c to be a vertex of degree 0 [5].

Each cycle in G_{ab} has length at least four and at most $n - 3$, due to the removal of three elements in creating the vertex set. The list of the cycle lengths in lexicographic order is the *cycle list* and partitions $n - 3$ into even parts not less than 4. The *cycle structure* is the multiset of cycle lists in lexicographic order for all possible $\binom{n}{2}$ pairs from V. Now let $\pi(n)$ be the number of distinct cycle lists for an STS(n). A useful compressed form of the cycle structure data is the *cycle vector*, which comprises $\pi(n)$ components such that position

i equals the number of occurrences of the i^{th} cycle list. The sum of the coordinates of the cycle vector will be $\binom{n}{2}$.

Cycle structure completely distinguishes Steiner triple systems of orders 13 and 15. In general, isomorphic systems yield identical cycle structures, but the converse may not hold. For $n = 13$, there are $\binom{13}{2} = 78$ cycle lists, each of the form $\ell_1 = 4 + 6$ or $\ell_2 = 10$. For order 15, the number of cycle lists is $\binom{15}{2} = 105$, and there are four distinct lists: $\ell_1 = 4 + 4 + 4$, $\ell_2 = 4 + 8$, $\ell_3 = 6 + 6$, and $\ell_4 = 12$. The cycle vectors for an STS(13) and STS(15) are then (a_1, a_2) and (a_1, a_2, a_3, a_4), respectively, where a_i is the number of occurrences of ℓ_i among the different cycle lists.

It is possible for two systems to have identical cycle vectors but different cycle structures, so cycle vectors alone are not always enough information. However, the two pairs of STS(15)s not differentiated by the block-intersection graphs were distinguished by their cycle vectors, shown in Table 6.2, so the entire cycle structure was unnecessary.

Table 6.2: Cycle Vectors for Select STS(15)s

System	Cycle Vector
20	(4, 48, 4, 49)
21	(5, 45, 7, 48)
46	(1, 18, 14, 72)
65	(0, 21, 14, 70)

6.3 Non-Graphical Methods

Because the BIGs were not enough to completely distinguish the STS(15)s, we researched alternate methods in the literature to evaluate whether other approaches were successful.

6.3.1 Convex Hulls of the Triples

In 2005, Olivier Anglada and Jean-François Maurras published their results on differentiating the eighty STS(15)s by the convex hulls of the characteristic vectors of the triples. For each block $B \in \mathcal{B}$ we define its *characteristic vector* x_B in $\mathbb{R}^{15}$ by $x(i) = 1$ if $i - 1 \in B$ and $x(i) = 0$ otherwise, for $i = 1, 2, ..., 15$. The *convex hull* associated with a particular STS(15) is then the set of all convex combinations of the characteristic vectors of its triples, where a convex combination is a linear combination of a subset of the characteristic vectors such that the real-valued coefficients are all non-negative and sum to 1.

Anglada and Maurras used computer code to count the number of facets of the convex hull for each STS(15) and found that the totals were all distinct. They were thus able to conclude that these systems are all non-isomorphic. The number of facets varied greatly, from 150 for the first STS(15) to 32,699 for STS(15) #77. Although the range is large, only two systems have facets totaling under 2,316. Note the number of facets provides an alternate ordering for the systems.

6.3.2 Binary Linear Codes

Vladimir D. Tonchev and Robert S. Weishaar detailed their success in differentiating the STS(15)s using binary linear codes generated by the incidence matrix of each system. Given an STS(n), its *incidence matrix* M is the $n \times \frac{n(n-1)}{6}$ element-by-triple matrix where entry $a_{ij} = 1$ if the element i is in the j^{th} triple and 0 otherwise. If the rows and columns of the incidence matrix for a system can be permuted into the incidence matrix of another system, the two STS(n)s are isomorphic. Tonchev and Weishaar note that a k-length code is a linear subspace of the vector space of dimension k over some finite field F. Because the article uses binary codes, F is GF(2). Finally, codes are isomorphic if a permutation of the k coordinate positions results in one code being obtained from the other.

The *code* C of an STS(n) is defined to be the row space of its incidence matrix M; C has length $\frac{n(n-1)}{6}$ and dimension equal to the rank of M over GF(2). Since the rows of M correspond to the elements or points of the 15-set, C is more specifically called the *code of the points*. Similarly, one can consider the *code of the blocks*, of length n, as the column space of M. For the STS(15)s, Tonchev and Weishaar list several computations: size of the automorphism groups of the system and of its code, rank of the incidence matrix of a system over GF(2), and the weight distribution (number of codewords with a particular weight, using the Hamming weight of a vector).

The codes of the blocks were not able to distinguish the systems of order 15, as they were partitioned into five isomorphism classes according to the rank of the incidence matrices. But the length-35 codes of the points were all non-isomorphic, thereby showing another way to distinguish the STS(15)s. The weight distributions and column sums of matrices formed by codewords of weight 7 associated to each code of the points was enough to show non-isomorphism. The authors remark on an interesting result, that the system with the largest code automorphism group was not the system with the largest automorphism group. System 16 has a code group of order 225,792 but its automorphism group has order 168. On the other hand, STS(15) #1 has the largest automorphism group, which actually has the same size as its code automorphism group: order 20,160.

6.3.3 Lattices

A *k-dimensional lattice* L is a free abelian group of rank k or a discrete group of vectors in $\mathbb{R}^k$, and Michel Deza and Viatcheslav Grishukhin use lattices as a way to categorize STS(15)s. The set $\mathcal{B}$ of triples of an STS(n) can be used to form the set $\mathcal{V}(\mathcal{B})$ of vectors corresponding to $\mathcal{B}$, which generates affinely a lattice $L(\mathcal{B})$ contained in the lattice $\sqrt{2}A_{14}$. Vectors of norm 4 in $L(\mathcal{B})$ are themselves a set $R(\mathcal{B})$ that form a *root system*, all vectors of norm 2 in a lattice composed of vectors with even norms. The authors study the lattices generated by the STS(15)s, and they find that the lattices have a total of five different root systems, all of them sub-lattices of A_{14}. The eighty STS(15)s are partitioned into the root systems $\emptyset$, A_1^7, $A_2A_3^3$, A_6A_7, and A_{14}. Systems 1 and 2 are the only systems that fall into $\emptyset$ and A_1^7, respectively. Numbers 3 through 7 are in the third type, 8 through 22 and 67 are in the fourth type, and the remaining systems all fall into the final type. The two pairs of STS(15)s not distinguished by the BIGs, systems $\{20, 21\}$ and $\{46, 65\}$, are of the same type of lattice root system so cannot be distinguished by this method either.

Remark: The partition of STS(15)s resulting from classifying the systems into the five types of root systems coincides with the partition resulting from the isomorphism classes of the binary linear codes of the blocks discussed in the previous section. The equivalence of the two methods was shown by Patrick Solé in 1997 [15].

6.3.4 Miscellaneous

We survey several other properties of triple systems not derived from the block-intersection graphs.

Automorphism Groups: As briefly mentioned above, the first STS(15) has an automorphism group of order 20,160. This is by far the largest size; the next highest is 288, and after that there are two systems with orders 192 and 168. Seven systems have automorphism groups of order between 21 and 96, and there are another seven between 5 and 12. The majority of the systems are of smaller order: there are eight, twelve, and six each of orders 4, 3, and 2, respectively. Finally, nearly half of the STS(15)s, thirty-six of eighty, have automorphism groups of order 1. In the literature, triple systems with only the identity automorphism are called *automorphism-free*. Note that systems with automorphism groups of unequal orders are certainly non-isomorphic.

Chromatic Index: The *chromatic index CI* of an STS(n) is the smallest number of colors assigned to the triples such that no intersecting triples have the same color. In a BIG, each vertex corresponds to a triple, and so the chromatic index of the system coincides with the chromatic number $\chi(G)$ of the block-intersection graph G. Every STS(15) has chromatic index $7, 8$, or 9. Because only four are of $CI = 7$ and thirteen of $CI = 8$, with over three-fourths of the systems having $CI = 9$, the chromatic index can be informative but is not the best way to distinguish systems. Systems of order 15 with $CI = 7 = \chi(G)$ are resolvable, as shown in Chapter 4.

Parallel Classes: We defined parallel classes in the introduction in the context of Kirkman triple systems, and because the number of such classes varies per system, this total can be used to differentiate STS(15)s. The smallest number is 0 and the largest is 56, but just six have 16 or more. In fact, 12.5% of the systems do not contain any parallel classes, 22.5% have only 1, and 15% have 2, so fully half of the systems fall between 0 and 2. The number of systems with a count of parallel classes equal to 3 or 4 is sixteen, with eight systems having 5 or 6, and ten with somewhere between 7 and 12.

Since a parallel class in an STS(15) is an independent set in its BIG, these figures on parallel classes match the totals we obtained on maximum independent sets in the BIGs. For example, we found that ten systems had $\alpha(G) = 4$, which implied they contained no parallel classes, and ten of eighty is 12.5%. Of those with $\alpha(G) = 5$, the number of maximum independent sets did range from 1 to 56.

Pasch Configurations: A *Pasch configuration* is any collection of four triples on six elements of the form (a, b, c), (a, d, e), (b, e, f), and (c, d, f). Note that each element will not be paired with all of the five other elements. Within an STS(n), many other configurations can be defined by changing the number of triples or number of elements involved, but Pasch configurations have been frequently studied in conjunction with Steiner triple systems because they are the smallest configuration (in terms of number of triples) that occurs a variable number of times per STS(15). There are from 0 to 105 Pasch configurations in the systems of order 15, although only six fall above 37.

Resolvability: Resolvability of a Steiner triple system has been discussed in other contexts above, but it is another property that shows non-isomorphism of systems on the same order. Recall that a resolvable STS(n) is one whose set of triples can be partitioned into parallel classes. There are four resolvable STS(15)s, which cannot be isomorphic to a non-resolvable system. One of these has only a single resolution, but the other three each have two different resolutions [6]. This is an interesting point: there are seven possible resolutions of an STS(15) up to isomorphism, but they only occur on four underlying systems. Resolvability is clearly not a very efficient method of differentiating systems, but it does constitute an invariant. The resolvable STS(15)s are systems 1, 7, 19, and 61.

Remark: The two pairs of STS(15)s that could not be distinguished by graph invariants of their block-intersection graphs were systems $\{20, 21\}$ and $\{46, 65\}$. The properties associated with Steiner triple systems discussed above that are identical for each of these pairs are the size of automorphism group, chromatic index, number of parallel classes, number of Pasch configurations, and resolvability.

6.4 Summary

Of the properties discussed in this chapter, the invariants of Steiner triple systems that could distinguish each of the eighty systems of order 15, up to isomorphism, were cycle structure, number of facets of the convex hulls of the triples' characteristic vectors, and binary linear codes of the points (elements in V). Using a test for graph isomorphism, the STS(15)s were also distinguished by their block-intersection graphs and by the corresponding line graphs. The other methods studied in relation to the BIGs could not fully differentiate the systems, although we were successful for seventy-six out of eighty. The most useful BIG invariant was maximal independent sets; the number and size of such sets were distinct for seventy-four systems.

Appendix A

Steiner Triple Systems of Order 13

1. { (0,1,2), (0,3,4), (0,5,6), (0,7,8), (0,9,10), (0,11,12), (1,3,5), (1,4,7), (1,6,8), (1,9,11), (1,10,12), (2,3,9), (2,4,5), (2,6,10), (2,7,12), (2,8,11), (3,6,11), (3,7,10), (3,8,12), (4,6,12), (4,8,9), (4,10,11), (5,7,11), (5,8,10), (5,9,12), (6,7,9) }

2. { (0,1,2), (0,3,4), (0,5,6), (0,7,8), (0,9,10), (0,11,12), (1,3,5), (1,4,7), (1,6,8), (1,9,11), (1,10,12), (2,3,9), (2,4,5), (2,6,10), (2,7,11), (2,8,12), (3,6,11), (3,7,12), (3,8,10), (4,6,12), (4,8,9), (4,10,11), (5,7,10), (5,8,11), (5,9,12), (6,7,9) }

Appendix B

Code Used to Generate Block-Intersection Graphs

```
# Create a list with each STS(15) as an entry #
# The STS(15)s are labeled x0 through x79 #
List_15 = [x0,x1,x2,x3,x4,x5,x6,x7,x8,x9,
           x10,x11,x12,x13,x14,x15,x16,x17,x18,x19,
           x20,x21,x22,x23,x24,x25,x26,x27,x28,x29,
           x30,x31,x32,x33,x34,x35,x36,x37,x38,x39,
           x40,x41,x42,x43,x44,x45,x46,x47,x48,x49,
           x50,x51,x52,x53,x54,x55,x56,x57,x58,x59,
           x60,x61,x62,x63,x64,x65,x66,x67,x68,x69,
           x70,x71,x72,x73,x74,x75,x76,x77,x78,x79]
n = len(x0)  # number of triples

# Initialize lists that will contain adjacency matrices and BIGs #
M_15 = [a for a in range(80)]
```

```
G_15 = [a for a in range(80)]


# Create adjacency matrices and BIGs for each STS(15) #
for a in range(80):
    M_15[a] = matrix(n,n)  # this creates a matrix filled with 0's


    for i in range(n):
        for j in range(n):
            for k in range(3):
                for l in range(3):
                    if j!=i and List_15[a][i][k] == List_15[a][j][l]:
                        M_15[a][i,j] = 1


    G_15[a] = Graph(M_15[a])
```

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