Vector bundles of conformal blocks in types A and C from a combinatorial approach

by

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(Under the Direction of Angela Gibney)

Abstract

Vector bundles of conformal blocks on $\overline{M}_{0,n}$ provide a collection of base point free divisors on $\overline{M}_{0,n}$ defined using representation theory. Specifically, from the data of a simple Lie algebra $\mathfrak{g}$, a nonnegative integer ℓ (called the *level*), and an n-tuple of dominant integral weights $\vec{\lambda}$, one can construct the bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$. The first Chern classes of nontrivial such bundles are base point free and so give rise to morphisms from $\overline{M}_{0,n}$ to other projective varieties. By studying the divisor classes, we can begin to classify the images of the induced maps.

The main results of this dissertation concern combinatorial aspects of bundles defined using $\mathfrak{sl}_r$ and $\mathfrak{sp}_{2r}$. We give identities between the divisors, study the cones they generate, and the associated morphisms. Our main tool involves a theorem known as Witten's Dictionary, which allows us to deploy methods of representation theory and combinatorics to analyze the behavior of vector bundles of conformal blocks.

Index words: Moduli space of curves, vector bundles of conformal blocks, quantum cohomology, first Chern classes

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Dedication

To my students, past, current, and future.

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"...it is by standing on the shoulders of giants."

- Isaac Newton

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Chapter 1

Introduction

1.1 The moduli space of stable n-pointed rational curves

The moduli space $\overline{M}_{0,n}$ parameterizes stable rational curves with n marked points. A point in the interior $M_{0,n}$ consists of an equivalence class $[(C; p_1, ..., p_n)]$ where C is a smooth genus zero curve (i.e., a rational curve, isomorphic to $\mathbb{P}_1$) and $p_i \in C$ for $i = 1$ to n are n distinct marked points of C. We say that two such collections $(C; p_1, ..., p_n)$ and $(C'; p'_1, ..., p'_n)$ are isomorphic if there is an isomorphism $\phi : C \to C'$ of the curves such that $\phi(p_i) = p'_i$ for $i = 1$ to n. We sometimes drop the square brackets and write $(C; p_1, \ldots p_n)$ for the class $[(C; p_1, \ldots p_n)]$ when it is clear from context.

A point $[(C; p_1, ..., p_n)] \in \overline{M}_{0,n}$ can be represented by a curve C consisting of a tree of projective lines with $p_i \in C$ for $i = 1$ to n distinct marked smooth points and every irreducible component has at least three nodes or marked points (see Figure 1.1). By a tree of projective lines, we mean a connected curve with irreducible components isomorphic to the projective line such that the intersection points are ordinary double points and there are no closed circuits.

The space $\overline{M}_{0,n}$ has many beautiful properties. For instance, it is a smooth projective variety and a fine moduli space. The boundary points $\overline{M}_{0,n} \backslash M_{0,n}$ correspond to reducible curves. The boundary has a stratification in which the components are determined by the number of nodes on a curve. The numerical equivalence classes of the one-dimensional strata are called *F-curves* (see Definition 1.1) . A representative of each such class can be obtained by defining a map $\overline{M}_{0,4} \to \overline{M}_{0,n}$ [25, Thm. 2.2]. We give a description of such a map in the following definition.

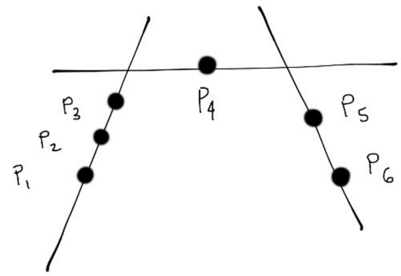


Figure 1.1: Representative of a point in boundary of $\overline{M}_{0,6}$

Definition 1.1. [25, Thm. 2.2] Let $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 = [n]$ be a partition of $[n] = \{1, ..., n\}$ into four nonempty, disjoint sets. Consider the morphism, $\phi : \overline{M}_{0,4} \to \overline{M}_{0,n}$ sending

$$(C; p_1, p_2, p_3, p_4) \mapsto (X; q_1, ..., q_n)$$

where X is built from $(C; p_1, p_2, p_3, p_4)$ in the following sense: If $|I_i| \geq 2$, then attach a point $(\mathbb{P}_1; a_i, \{q_j : j \in I_i\}) \in \overline{M}_{0,|I_i|+1}$ at the marked point $a_i \in \mathbb{P}_1$ to the marked point $p_i \in C$. If $|I_j| = 1$, then label the point p_j of $(C; p_1, p_2, p_3, p_4)$ as q_j. We refer to an *F-curve* as any element in the numerical equivalence class of the image of ϕ and denote any such element by F_{I_1,I_2,I_3,I_4}. The points (nodes or marked points) on the irreducible component of X associated to the original marked points $p_i \in C$ are referred to as the *attaching points*.

1.2 Vector bundles of conformal blocks

A vector bundle of conformal blocks on $\overline{M}_{0,n}$ is defined with three ingredients: a simple Lie algebra $\mathfrak{g}$, a nonnegative integer ℓ called the *level*, and an n-tuple $\vec{\lambda} = \{\lambda_1, ..., \lambda_n\}$ of dominant integral weights of $\mathfrak{g}$ at level ℓ (see Definition 2.5). We denote such a bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$. We denote the first Chern class, the conformal blocks divisor, of such a bundle $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$. In this section, we briefly define a fiber of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ over a point in $M_{0,n}$. For more details on the construction of the bundle

$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ see [46, 5, 17]. This construction was extended to be defined for general vertex algebras in [19].

Let [,] denote the usual bracket on $\mathfrak{g}$ and (,) the scaled Killing form on $\mathfrak{g}$ (see Definition 2.1).

Definition 1.2. For a simple Lie algebra $\mathfrak{g}$ and each $i = 1$ to n, one can define an affine Lie algebra

$$\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((z_i)) \oplus \ \mathbb{C} \cdot c,$$

where $\mathbb{C}((z_i))$ is the field of Laurent power series over $\mathbb{C}$ in the variable z_i and c belongs to the center of $\hat{\mathfrak{g}}_i$. The bracket defined on simple tensors is

$$\Big[(X \otimes f, \alpha c), (Y \otimes \ g, \beta c)\Big] = \Big([X, Y] \otimes fg,\ c(X, Y) \cdot \mathrm{Res}_{z_i=0}(g(z_i) df(z_i))\Big),$$

for $X, Y \in \mathfrak{g}$ and $f, g \in \mathbb{C}((z_i))$. This bracket is extended linearly to be defined on all of $\hat{\mathfrak{g}}_i$. We also define the Lie subalgebra of $\hat{\mathfrak{g}}_i$

$$\hat{\mathfrak{g}}_i^+ = \mathfrak{g} \otimes z_i \mathbb{C}[[z_i]].$$

For $\lambda_i \in \vec{\lambda}$ and nonnegative integer ℓ, there is a $\hat{\mathfrak{g}}_i$-module H_{λ_i} at level ℓ and highest weight λ_i. H_{λ_i} is characterized up to isomorphism by the property that the subspace of H_{λ_i} annihilated by $\hat{\mathfrak{g}}_i^+$ is isomorphic as a $\mathfrak{g}$-module to V_{λ_i} (see [28, Chapt. 12] or [5, Sect. 1.6] for more details on the construction of H_{λ_i}).

Let $(C; p_1, ..., p_n) \in M_{0,n}$. The ring of regular functions on the curve C outside the marked points $\bar{p} = (p_1, ..., p_n)$ is given by $\mathcal{O}(C - \bar{p})$. For each i, a ring homomorphism

$$\mathcal{O}(C - \bar{p}) \to \mathbb{C}((z_i))$$

can be defined by expanding a function $f \in \mathcal{O}(C - \bar{p})$ about the point p_i; denote this expansion f_{p_i}. Let $\mathfrak{g}(C - \bar{p}) := \mathfrak{g} \otimes \mathcal{O}(C - \bar{p})$. The ring homomorphisms above induce maps

$$\mathfrak{g}(C - \bar{p}) \to \hat{\mathfrak{g}}_i. \tag{1.1}$$

Let $\mathcal{H}_{\vec{\lambda}} := H_{\lambda_1} \otimes ... \otimes H_{\lambda_n}$. Define the map

$$\mathfrak{g}(C - \bar{p}) \times \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\vec{\lambda}}$$

by

$$(X \otimes f) \cdot (v_1 \otimes ... \otimes v_n) \mapsto \sum_{i=1}^{n} v_1 \otimes ... \otimes v_{i-1} \otimes (X \otimes f)v_i \otimes v_{i+1} \otimes ... \otimes v_n,$$

where $(X \otimes f)v_i$ is given in (1.1). By the Residue Theorem $\sum_{i=1}^{n} \operatorname{Res}_{p_i} f_{p_i} dg_{p_i} = 0$ for nodal curves. This can be used to show the map just defined is an action of $\mathfrak{g}(C - \bar{p})$ on $\mathcal{H}_{\vec{\lambda}}$.

Definition 1.3. The fiber of the bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ over the point $(C; \bar{p})$, is

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C;\bar{p})} = [\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(C-\bar{p})},$$

where $[\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(C-\bar{p})}$ is the space of coinvariants of $\mathcal{H}_{\vec{\lambda}}$. This space is the largest quotient of $\mathcal{H}_{\vec{\lambda}}$ on which $\mathfrak{g}(C - \bar{p})$ acts trivially. Its dual is the *vector space of conformal blocks.*

A dimension formula of the vector space in Definition 1.3 (i.e., the rank of the vector bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$) is given by the *Verlinde formula* and was conjecturally given by Verlinde in 1988 [48]. The formula has since been proven in various cases by a number of authors [12, 45, 6, 18, 32, 40, 33].

1.3 Motivation and main results

Central to representation theory is the study of symmetries, and often in representation theory one sees important and sometimes unexpected identities. Since vector bundles of conformal blocks are defined using representation theory, it may be no surprise, then, that conformal blocks are also frequently subject to symmetries and unexpected identities [44, 3, 17, 22, 38, 2, 39, 11, 8, 30].

For instance, we know that there are additive identities between first Chern classes, the conformal blocks divisors, which allow one to decompose conformal blocks divisors into sums of simpler divisors, as long as certain rank conditions are satisfied [8, Prop. 19]. In this decomposition, one of the summands is the first Chern class of a rank one bundle. One important class of rank one bundles is the set of bundles in type A at level one. The associated morphisms have been described

[21, 22], and moreover, while there are an infinite number of these bundles, using the fact that the images are constructed as GIT quotients, it has been shown that the cone of divisors they generate is polyhedral [22, Thm.1.1].

With the importance of rank one bundles and the desire to determine the structure of the cone generated by conformal blocks divisors, one may naturally ask the following questions.

Problem 1.4. 1. Describe sets $\mathcal{S} := \{\mathbb{V} = \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell) \mid \operatorname{rank}(\mathbb{V}) = 1\}$.

2. Show that $\mathcal{C}(\mathcal{S}) := \operatorname{ConvHull}\{c_1(\mathbb{V}) \mid \mathbb{V} \in \mathcal{S}\}$ is finitely generated.

Our first main result completely determines a solution for this problem for bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$, $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$, and $\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}, \ell)$ when the weight data is rectangular (see Definition 4.1). More specifically, in the above cases, we give necessary and sufficient conditions on $\vec{\lambda}$ and ℓ which determine when the rank of such a bundle is zero, one, and larger than one. Explicitly, our first main result is the following classification theorem.

Theorem 1.5. Let $\mathbb{V}_m = \mathbb{V}(\mathfrak{sl}_{2m}, (a_1\omega_m, ..., a_n\omega_m), \ell)$ be a vector bundle of conformal blocks such that $a_1 \geq a_2 \geq ... \geq a_n$ and $\sum_{i=1}^{n} a_i = 2(k\ell + p)$, for some integers p and k such that $1 \leq p \leq \ell$ and $k \geq 0$. Define $\Lambda := \sum_{i=2k+2}^{n} a_i$ where $\Lambda := 0$ if $2k + 2 > n$. Then

1. $\operatorname{rk}(\mathbb{V}_m) = 0$ if and only if $\Lambda < p$;
2. $\operatorname{rk}(\mathbb{V}_m) = 1$ if and only if either
   (a) $\Lambda = p$, or
   (b) $\Lambda > p$ and weight content is maximal (see Definitions 4.2 and 4.4); and
3. $\operatorname{rk}(\mathbb{V}_m) > 1$ if and only if $\Lambda > p$, and the weight content is not maximal.

Our approach to proving Theorem 1.5 is combinatorial and relies on work by Belkale and Witten (see [9, 49]) that translates the rank of a vector bundle of conformal blocks defined with $\mathfrak{g} = \mathfrak{sl}_r$ to a coefficient appearing as a structure constant in a product of classes in the (quantum) cohomology of the Grassmannian variety. This translation is known as *Witten's Dictionary* and is stated explicitly in Proposition 3.15. Using Witten's Dictionary as a tool, we first establish the statement for $\mathfrak{sl}_2$

bundles in Section 4.2. We then apply a rank scaling statement for rank one bundles defined using $\mathfrak{sl}_{2m}$ and rectangular weights, which relies partially on work of the authors of [10]. Further details and proof of their results are given in Proposition 4.44.

The set $\mathcal{S}$ of rank one bundles described in Theorem 1.5 contains an infinite number of elements, including all $\mathfrak{sl}_2$ level one bundles. Following work of Fakhruddin [17, Thm. 4.3], $\mathcal{C}(\mathcal{S})$ forms a full dimensional subcone of the nef cone $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. We prove this cone $\mathcal{C}(\mathcal{S})$ is finitely generated, providing an answer to the second part of Question 1.4.

Theorem 1.6. Let $\mathcal{S} := \{\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{2m}, (a_1\omega_m, ..., a_n\omega_m), \ell) \mid \mathrm{rk}(\mathbb{V}) = 1\}$. Then

$\mathcal{C}(\mathcal{S}) := \mathrm{ConvHull}\{c_1(\mathbb{V}) \mid \mathbb{V} \in \mathcal{S}\}$ is finitely generated.

To prove this, we show the following result.[1]

Theorem 1.7. Each element of $\mathcal{C}(\mathcal{S})$ can be expressed explicitly as an effective linear combination of level one divisors.

Theorem 1.5, also gives information about vector bundles of conformal blocks for the Lie algebra $\mathfrak{sp}_{2\ell}$ at level one, using the relationship

$$\mathrm{rank}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rank}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$$

from [17, Sect. 5.2.3], where $\vec{\lambda} = (a_1\omega_1, ..., a_n\omega_1)$ and $\vec{\lambda}^T = (\omega_{a_1}, ..., \omega_{a_n})$. The same conditions on $(a_1, \ldots, a_n)$ given in Theorem 1.5 also determine when $\mathbb{V} = \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ has rank zero, one, and greater than one.

Our next main result shows the first Chern classes of bundles for $\mathfrak{sp}_{2\ell}$ at level one are equal to divisors of bundles for $\mathfrak{sl}_2$ at level ℓ if and only if the rank of the bundles is zero or one (Theorem 1.8). As an application, we show the cone generated by the divisors from such rank one bundles is polyhedral: All extremal rays are spanned by first Chern classes of rank one bundles for $\mathfrak{sp}_2$ at level one. Using this result, we are able to explicitly describe the associated morphisms of these divisors (see Section 5.4 and Proposition 5.14).

[1]See Theorem 6.11 for a precise statement.

Theorem 1.8. Let $\vec{\lambda}$ be an n-tuple of nonnegative integers of weights for $\mathfrak{sl}_2$ at level ℓ or $\mathfrak{sp}_{2\ell}$ at level one. Then for bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ the identity

$$c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$$

holds if and only if $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 1$ or 0.

The final main result of this work shows that the divisors from bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ are all linearly equivalent when the Lie algebra rank ℓ is taken large enough.

Theorem 1.9. For a fixed n-tuple of nonnegative integers $\vec{\lambda}$, there is an integer $r(\vec{\lambda})$ such that for any $\ell \geq r(\vec{\lambda})$, the following identity holds,

$$c_1(\mathbb{V}(\mathfrak{sp}_{2r(\vec{\lambda})}, \vec{\lambda}, 1)) = c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)).$$

In Section 5.4 we explicitly compute the integer $r(\vec{\lambda})$ and determine when the bundle $\mathbb{V}(\mathfrak{sp}_{2r(\vec{\lambda})}, \vec{\lambda}, 1)$ is nontrivial.

We now give an outline of this dissertation. In Chapter 2, we provide background information regarding the Lie algebras $\mathfrak{sl}_r$ and $\mathfrak{sp}_{2r}$ important to our work for vector bundles of conformal blocks. In Chapter 3, we describe our main tools and methods of investigation. Particularly, we focus on Witten's Dictionary, quantum Kostka computations, and known formulas for computing ranks and first Chern classes of vector bundles of conformal blocks. In Chapter 4 we prove Theorem 1.5 and show the cone generated by divisors from $\mathfrak{sl}_2$ rank one bundles is polyhedral (Theorem 1.6). In Chapter 5 we investigate bundles with $\mathfrak{sp}_{2\ell}$ at level one and prove Theorem 1.8. We end with Chapter 6 in which we establish rank and divisor identities between bundles with $\mathfrak{sp}_{2\ell}$ at level one.

Chapter 2

The Lie Algebras $\mathfrak{sl}_{r+1}$ and $\mathfrak{sp}_{2r}$

We begin by explicitly describing the two main Lie algebras of our study: $\mathfrak{sl}_{r+1}$ and $\mathfrak{sp}_{2r}$. These are the Lie algebras associated to the special linear group and symplectic group, respectively. We state background information of both Lie algebras relevant to the study of vector bundles of conformal blocks. For further information see [46], [5], and [17].

2.1 Notation and general background

For a simple Lie algebra $\mathfrak{g}$ let $\mathfrak{h}$ denote a Cartan subalgebra and $\mathfrak{h}^*$ denote the dual (Euclidean) vector space. We use R to denote the root system,

$$R := \{0 \neq \alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\},$$

where $\mathfrak{g}_\alpha$ is the root space,

$$\mathfrak{g}_\alpha := \{v \in \mathfrak{g} : \forall h \in \mathfrak{h}, ad(h)(v) := [h, v] = \alpha(h)v\}.$$

Let Δ denote a fixed base for the root system R. This means Δ is a basis of $\mathfrak{h}^*$ and each root $\beta \in R$ can be written as

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with c_α either all nonnegative or all nonpositive integers. The elements in Δ are called *simple* roots. The *height* of a root β is then the sum $\sum c_\alpha$ from the expression stated above. Since any $\beta \in R$ is either a nonnegative sum or nonpositive sum of simple roots, we can partition the root system R into positive and negative parts (denoted R^+ and R^- respectively). The notion of height of a root also provides an ordering on the root system and allows us to define a *highest root*. As is conventional, the highest root is denoted θ.

The Killing form $\mathcal{K}(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $\mathcal{K}(X, Y) := \operatorname{trace}(\operatorname{ad} X \circ \operatorname{ad} Y)$ is nondegenerate on $\mathfrak{h}$ and so provides an isomorphism $\phi : \mathfrak{h} \to \mathfrak{h}^*$, given by

$$h \mapsto \phi_h : \mathfrak{h} \to \mathbb{C},$$

where, for $H \in \mathfrak{h}$,

$$\phi_h(H) = \mathcal{K}(h, H).$$

Using the map ϕ, we can define an inner product on $\mathfrak{h}^*$.

Definition 2.1. Let $T_\lambda := \phi^{-1}(\lambda) \in \mathfrak{h}$. For any $\mu, \nu \in \mathfrak{h}^*$, an inner product on $\mathfrak{h}^*$ is given by

$$(\mu, \nu) := \mathcal{K}(T_\mu, T_\nu).$$

In this document, we normalize this inner product so that $(\theta, \theta) = 2$, where θ is the highest root of $\mathfrak{g}$.

For any positive root $\alpha \in R^+$, we have a notion of an $\mathfrak{sl}_2(\alpha)$-triple of $\mathfrak{g}$, that is, a triple of elements $\{E, H, F\}$ of elements in $\mathfrak{g}$ with

$$H \in \mathfrak{h}, \quad E \in \mathfrak{g}_\alpha, \quad F \in \mathfrak{g}_{-\alpha}$$

and the following relationship with the commutator is satisfied,

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H.$$

The $\mathfrak{sl}_2(\theta)$-triple associated to the highest root θ, can be used to describe the $\hat{\mathfrak{g}}$-modules H_λ used in the construction of the vector space of conformal blocks Definition 1.3 (see [28, Exer. 12.12] and [5, Section 1.2]).

Definition 2.2. The *weight lattice* of $\mathfrak{g}$, denoted Λ, is a set of elements in $\mathfrak{h}^*$ given by

$$\Lambda := \{\lambda \in \mathfrak{h}^* : \forall \alpha \in R, (\lambda, \alpha) \in \mathbb{Z}\}.$$

Definition 2.3. For a fixed base Δ a weight $\lambda \in \Lambda$ is *dominant* if the pairing (λ, α) as in Definition 2.1 is nonnegative for all $\alpha \in \Delta$. The set of all dominant weights is denoted Λ^+.

Definition 2.4. Given a fixed ordering on a base $\Delta = \{\alpha_1, ..., \alpha_\ell\}$, the *fundamental weights*, denoted $\Omega = \{\omega_1, ..., \omega_\ell\}$, are those weights forming a dual basis to the coroots of Δ with inner product as in Definition 2.1. That is, for any $\alpha_j \in \Delta$ and $\omega_i \in \Omega$ the following holds,

$$2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if and only if $i = j$ and $\delta_{ij} = 0$ otherwise. The *coroot* of a root $\alpha \in R$ is given by $\check{\alpha} := \frac{2}{(\alpha,\alpha)}\alpha$.

Definition 2.5. For a fixed nonnegative integer $\ell \in \mathbb{Z}^{\geq 0}$, the set of *dominant weights for* $\mathfrak{g}$ *at level* ℓ, denoted $P_\ell(\mathfrak{g})$, is the set of dominant weights which pair with the longest root θ to a value less than or equal to ℓ. This set is given by

$$P_\ell(\mathfrak{g}) := \{\lambda \in \Lambda^+ : (\lambda, \theta) \leq \ell\}.$$

2.2 The Lie algebra $\mathfrak{sl}_{r+1}$

The Lie algebra $\mathfrak{sl}_{r+1}$ is the complex simple Lie algebra of the special linear group $SL(r+1)$. We first define the Lie algebra $\mathfrak{sl}_{r+1}$ and explicitly give components important to our study.

Definition 2.6. For $r \geq 1$ the Lie algebra $\mathfrak{sl}_{r+1}$ consists of the following elements,

$$\mathfrak{sl}_{r+1} := \{(r+1)\times(r+1) \text{ matrices with entries in } \mathbb{C} \text{ and trace } 0\}.$$

The Lie bracket in $\mathfrak{sl}_{r+1}$ is given by the commutator $[A, B] = AB - BA$.

2.2.1 Basis and dimension

A *basis* for $\mathfrak{sl}_{r+1}$ is given by the following set of matrices

$$\{E_{i,j} : 1 \leq i \leq r+1, 1 \leq j \leq r+1, i \neq j\} \cup \{E_{i,i} - E_{1+i,1+i} : 1 \leq i \leq r\}, \tag{2.1}$$

where $E_{i,j}$ denotes an $(r+1)\times(r+1)$ matrix with a one in the (i,j) entry and zeros everywhere else (these are often called the *elementary matrices*). This gives the dimension

$$\dim(\mathfrak{sl}_{r+1}) = (r+1)r + r = r^2 + 2r.$$

A Cartan subalgebra of $\mathfrak{sl}_{r+1}$ is given by the following span of matrices,

$$\mathfrak{h} = \text{Span}\{E_{i,i} - E_{1+i,1+i} : 1 \leq i \leq r\}.$$

These are exactly the diagonal matrices in the above basis for $\mathfrak{sl}_{r+1}$.

2.2.2 Roots and Weights

With the above choice of Cartan subalgebra, the roots for $\mathfrak{sl}_{r+1}$ are,

$$R = \left\{\epsilon_i - \epsilon_j : 1 \leq i \leq r+1, 1 \leq j \leq r+1, i \neq j\right\},$$

where ϵ_i denotes the operator on a matrix M which reads off the (i,i) entry of M.

Remark 2.7. We can show that the basis element $E_{i,j}$ $(i \neq j)$ is in the root space $\epsilon_i - \epsilon_j$. That is, for an element $H_{\ell,m} := E_{\ell,\ell} - E_{m,m}$ in the Cartan subalgebra, we have

$$[H_{\ell,m}, E_{i,j}] = H_{\ell,m}E_{i,j} - E_{i,j}H_{\ell,m} = (\epsilon_i - \epsilon_j)(H_{\ell,m})E_{i,j}.$$

A base for the root system is given by,

$$\Delta = \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq r\}. \tag{2.2}$$

With this base, the positive roots are

$$R^+ = \Big\{\epsilon_i - \epsilon_j = \sum_{k=i}^{j-1} \alpha_k \mid 1 \leq i < j \leq r+1\Big\}. \tag{2.3}$$

The highest root is,

$$\theta = \sum_{i=1}^{r} \alpha_i = \epsilon_1 - \epsilon_{r+1}. \tag{2.4}$$

2.2.3 Fundamental and dominant integral weights

With the base in (2.2), the fundamental weights are given by the following set,

$$\Omega = \left\{\omega_i := \sum_{j=1}^{i} \epsilon_j - \frac{i}{r+1}\sum_{k=1}^{r+1} \epsilon_k \mid 1 \leq i \leq r\right\}. \tag{2.5}$$

The dimension of the weight lattice Λ of $\mathfrak{sl}_{r+1}$ is,

$$\dim(\Lambda) = |\Omega| = \dim(\mathfrak{h}) = r.$$

The dominant integral weights for $\mathfrak{sl}_{r+1}$ are positive integer linear combinations of the fundamental weights,

$$\Omega^+ = \left\{ \sum_{i=1}^{r} c_i \omega_i \mid c_i \in \mathbb{Z}^{\geq 0} \right\}. \tag{2.6}$$

Definition 2.8. For a fixed integer $\ell \in \mathbb{Z}^{\geq 0}$ the *dominant integral weights for* $\mathfrak{sl}_{r+1}$ *at level* ℓ are given by the following set,

$$P_\ell(\mathfrak{sl}_{r+1}) = \left\{ \sum_{i=1}^{r} c_i \omega_i \mid c_i \in \mathbb{Z}^{\geq 0} \text{ and } \sum_{i=1}^{r} c_i \leq \ell \right\}. \tag{2.7}$$

Remark 2.9. In the ϵ basis, a dominant integral weight for $\mathfrak{sl}_{r+1}$ can be written as

$$\lambda = \sum_{i=1}^{r+1} (\lambda^{(i)} - k)\epsilon_i$$

for some constant integer k and integers $\lambda^{(i)}$. Since the highest root of $\mathfrak{sl}_{r+1}$ is $\theta = \epsilon_1 - \epsilon_{r+1}$, if $\lambda \in P_\ell(\mathfrak{sl}_{r+1})$, then the integers $\lambda^{(i)}$ will necessarily be weakly decreasing $\lambda^{(1)} \geq \cdots \geq \lambda^{(r+1)} \geq 0$ with $\ell \geq \lambda^{(1)}$ and $\lambda^{(r+1)} = 0$ [47, Sect. 6.5.1]. We often denote such a weight as the r-tuple $(\lambda^{(1)}, \ldots, \lambda^{(r)})$. The weight $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ corresponds to a unique Young diagram that fits inside of an $r \times \ell$ rectangle. Particularly, such a *Young diagram* is a collection of rows of boxes with $\lambda^{(i)}$ boxes in the i^{th} row (see Definition 3.7).

2.2.4 The $\mathfrak{sl}_2(\theta)$-triple in $\mathfrak{sl}_{r+1}$

With the highest root $\theta = \epsilon_1 - \epsilon_{r+1}$, from (2.4), the corresponding $\mathfrak{sl}_2(\theta)$-triple in $\mathfrak{sl}_{r+1}$ is given by

$$H_\theta := E_{1,1} - E_{r+1,r+1}, \quad X_\theta := E_{1,r+1}, \quad Y_\theta := E_{r+1,1}. \tag{2.8}$$

Recall the set $\{H_\theta, X_\theta, Y_\theta\}$ being an $\mathfrak{sl}_2(\theta)$-triple means the following relationships with the commutator hold,

$$[H_\theta, X_\theta] = 2X_\theta, \quad [H_\theta, Y_\theta] = -2Y_\theta, \quad [X_\theta, Y_\theta] = -H_\theta. \tag{2.9}$$

2.2.5 Example: $\mathfrak{sl}_3$

We next study $\mathfrak{sl}_3$. The basis of elements from (2.1) is the following set,

$$E_{1,1} - E_{2,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_{0,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_{1,0} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{2,2} - E_{3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; E_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; E_{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$E_{1,1} - E_{3,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; E_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have written the basis in this order to indicate the $\mathfrak{sl}_2(\alpha)$-triples. In order, these rows correspond to the $\mathfrak{sl}_2(\epsilon_1 - \epsilon_2)$, $\mathfrak{sl}_2(\epsilon_2 - \epsilon_3)$, and $\mathfrak{sl}_2(\epsilon_1 - \epsilon_3)$ triples of $\mathfrak{sl}_3$. A Cartan subalgebra corresponding to this basis is given by the (complex) span of the matrices,

$$E_{1,1} - E_{2,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_{2,2} - E_{3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A base for the root system is given by $\Delta = \Big\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\Big\}$. We denote $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3$ as is convention. We also have $|\Delta| = 2$, showing that the weight lattice is two dimensional. The highest root is the sum,

$$\theta = \alpha_1 + \alpha_2 = \epsilon_1 - \epsilon_3.$$

The fundamental weights of $\mathfrak{sl}_3$ are

$$\Omega = \Big\{\omega_1 = \epsilon_1 - \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3),\ \omega_2 = \epsilon_1 + \epsilon_2 - \frac{2}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)\Big\}.$$

The normalized Killing form on the fundamental weights is,

$$\begin{aligned}(\omega_1, \omega_1) &= ||\omega_1||^2 = 2/3,\\ (\omega_1, \omega_2) &= 1/3,\\ (\omega_2, \omega_2) &= ||\omega_2||^2 = 2/3.\end{aligned} \tag{2.10}$$

For elements v, w in a Euclidean space V, we can compute the angle γ formed by $v, w \in V$ using the inner product $(\ ,\ )$ on V since

$$(v, w) = ||v||||w|| \cos(\gamma).$$

This allows us to visually represent the 2-dimensional weight lattice associated to $\mathfrak{sl}_3$ generated by ω_1 and ω_2 as in Figure 2.1 (see also [20, p. 331]).

2.3 The Lie algebra $\mathfrak{sp}_{2r}$

The Lie algebra $\mathfrak{sp}_{2r}$ is the complex simple Lie algebra of the symplectic group $Sp(2r)$. We define $\mathfrak{sp}_{2r}$ and explicitly give relevant information as we did for $\mathfrak{sl}_{r+1}$.

2.3.1 Definition and basis of $\mathfrak{sp}_{2r}$

To define $\mathfrak{sp}_{2r}$ we first define the following $2r \times 2r$ matrix S.

$$S := \begin{pmatrix} \bar{0} & I_r \\ -I_r & \bar{0} \end{pmatrix}$$

where I_r denotes the $r \times r$ identity matrix and $\bar{0}$ denotes the $r \times r$ zero matrix.

Definition 2.10. For $r \geq 2$ the Lie algebra $\mathfrak{sp}_{2r}$ consists of the following elements,

$$\mathfrak{sp}_{2r} := \Big\{ 2r \times 2r \text{ matrices } M \text{ with entries in } \mathbb{C} \text{ such that } SM = -M^t S \Big\},$$

where $-M^t$ denotes the negative transpose of the matrix M. The Lie bracket in $\mathfrak{sp}_{2r}$ is given by the commutator, $[A, B] = AB - BA$.

2.3.2 Basis and Dimension

Let $E_{i,j}$ denote the $2r \times 2r$ matrix with a one in the (i, j) entry and zeros everywhere else. A basis for $\mathfrak{sp}_{2r}$ is given by the following set of matrices,

$$\begin{gathered} \{E_{i,i} - E_{r+i,r+i} : 1 \leq i \leq r\} \cup \{E_{i,j} - E_{r+j,r+i} : 1 \leq i \leq r, 1 \leq j \leq r, i \neq j\} \cup \\ \{E_{i,r+i} : 1 \leq i \leq r\} \cup \{E_{i,r+j} + E_{j,r+i} : 1 \leq i \leq r, 1 \leq j \leq r, i < j\} \cup \{E_{r+i,i} : 1 \leq i \leq r\} \\ \cup \{E_{r+i,j} + E_{r+j,i} : 1 \leq i \leq r, 1 \leq j \leq r, i < j\}. \end{gathered} \tag{2.11}$$

We have

$$\dim(\mathfrak{sp}_{2r}) = 2r^2 + r.$$

A Cartan subalgebra of $\mathfrak{sp}_{2r}$ is given by the (complex) span,

$$\mathfrak{h} = \mathrm{Span}\{E_{i,i} - E_{r+i,r+i} : 1 \leq i \leq r\}.$$

These elements generating this set are exactly the diagonal matrices in the basis for $\mathfrak{sp}_{2r}$ from (2.3.2).

2.3.3 Roots and weights

With our choice of Cartan subalgebra, the roots for $\mathfrak{sp}_{2r}$ are,

$$R = \{\epsilon_i - \epsilon_j : 1 \leq i \leq r, 1 \leq j \leq r, i \neq j\} \cup \{2\epsilon_i : 1 \leq i \leq r\} \cup$$
$$\{\epsilon_i + \epsilon_j : 1 \leq i \leq r, 1 \leq j \leq r, i < j\} \cup$$
$$\{-2\epsilon_i : 1 \leq i \leq r\} \cup \{-\epsilon_i - \epsilon_j : 1 \leq i \leq r, 1 \leq j \leq r, i < j\}. \tag{2.12}$$

Here again ϵ_i denotes the operator on a diagonal matrix M which gives the (i, i) entry of M.

A base for the root system associated to $\mathfrak{sp}_{2r}$ is given by,

$$\Delta = \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < r\} \cup \{\alpha_r := 2\epsilon_r\}. \tag{2.13}$$

With the base in (2.13) the positive roots are given by,

$$R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq r\} \cup \{2\epsilon_i \mid 1 \leq i \leq r\} \cup \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq r\}. \tag{2.14}$$

The highest root is

$$\theta = \sum_{i=1}^{r-1} 2\alpha_i + \alpha_r = 2\epsilon_1. \tag{2.15}$$

2.3.4 Fundamental and dominant integral weights

With the base from (2.13), the fundamental weights are given by the following set,

$$\Omega = \left\{ \omega_i = \sum_{j=1}^{i} \epsilon_j \mid 1 \leq i \leq r \right\}. \tag{2.16}$$

The dimension of the weight lattice Λ of $\mathfrak{sp}_{2r}$ is

$$\dim(\Lambda) = |\Omega| = \dim(\mathfrak{h}) = r.$$

The dominant weights for $\mathfrak{sp}_{2r}$ are positive integer linear combinations of the fundamental weights,

$$\Lambda^+ = \left\{ \sum_{i=1}^{r} c_i \omega_i \mid c_i \in \mathbb{Z}^{\geq 0} \right\}.$$

Remark 2.11. For $\lambda = \sum_{i=1}^{r} c_i \omega_i \in \Lambda$, a weight of $\mathfrak{sp}_{2r}$, we have the following pairing of the normalized Killing form of λ and the highest root θ,

$$(\lambda, \theta) = \sum_{i=1}^{r} c_i.$$

Definition 2.12. For a fixed integer $\ell \in \mathbb{Z}^{\geq 0}$ the *dominant integral weights for* $\mathfrak{sp}_{2r}$ *at level* ℓ are given by the following set,

$$P_\ell(\mathfrak{sp}_{2r}) = \left\{ \sum_{i=1}^{r} c_i \omega_i : c_i \in \mathbb{Z}^{\geq 0} \text{ and } \sum_{i=1}^{r} c_i \leq \ell \right\}. \tag{2.17}$$

2.3.5 The $\mathfrak{sl}_2(\theta)$-triple in $\mathfrak{sp}_{2r}$

With the highest root, $\theta = 2\epsilon_1$, the corresponding $\mathfrak{sl}_2(\theta)$-triple is given by

$$H_\theta := E_{1,1} - E_{r+1,r+1}, \quad X_\theta := E_{1,r+1}, \quad Y_\theta := E_{r+1,1}. \tag{2.18}$$

2.3.6 Example: $\mathfrak{sp}_4$

We concretely state the above components for $\mathfrak{sp}_4$. A basis for this Lie algebra is given by,

$$E_{1,1} - E_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_{2,2} - E_{4,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ E_{1,2} - E_{4,3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$E_{2,1}-E_{3,4}=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&-1\\0&0&0&0\end{pmatrix},\ E_{1,3}=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ E_{2,4}=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$E_{1,4}+E_{2,3}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ E_{4,1}+E_{3,2}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

$$E_{3,1}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\ E_{2,4}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}.$$

The Cartan subalgebra $\mathfrak{h}$ is the (complex) span of the first two basis vectors in the above list. A base for the root system is given by

$$\Delta=\{\alpha_1:=\epsilon_1-\epsilon_2,\alpha_2:=2\epsilon_2\},$$

where α_i is the conventional notation. The highest root is

$$\theta=2\alpha_1+\alpha_2=2\epsilon_1,$$

and the set of positive roots is given by the set,

$$R^+=\Big\{\alpha_1=\epsilon_1-\epsilon_2,\alpha_2=2\epsilon_2,\alpha_1+\alpha_2=\epsilon_1+\epsilon_2,2\alpha_1+\alpha_2=2\epsilon_1\Big\}.$$

The fundamental weights of $\mathfrak{sp}_4$ are,

$$\Omega=\{\omega_1=\epsilon_1=\alpha_1+\alpha_2/2,\ \omega_2=\epsilon_1+\epsilon_2=\alpha_1+\alpha_2\}.$$

The normalized Killing form on the fundamental weights is as follows,

$$(\omega_1, \omega_1) = ||\omega_1||^2 = 1/2,$$
$$(\omega_1, \omega_2) = 1/2, \quad \text{and}$$
$$(\omega_2, \omega_2) = ||\omega_2||^2 = 1. \tag{2.19}$$

Recall, in this case, the highest root is $\theta = 2\alpha_1 + \alpha_2 = 2\epsilon_1 = 2\omega_1$. The length squared of this weight is two in the normalization. The two dimensional weight system for $\mathfrak{sp}_4$ generated by ω_1 and ω_2 is represented in Figure 2.2 (see also [20, p. 331]).

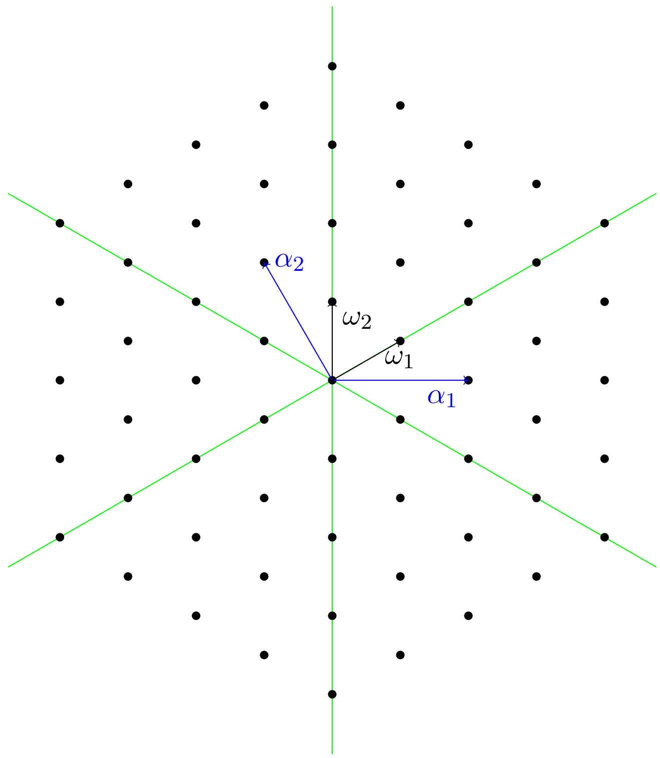


Figure 2.1: Weight system for $\mathfrak{sl}_3$

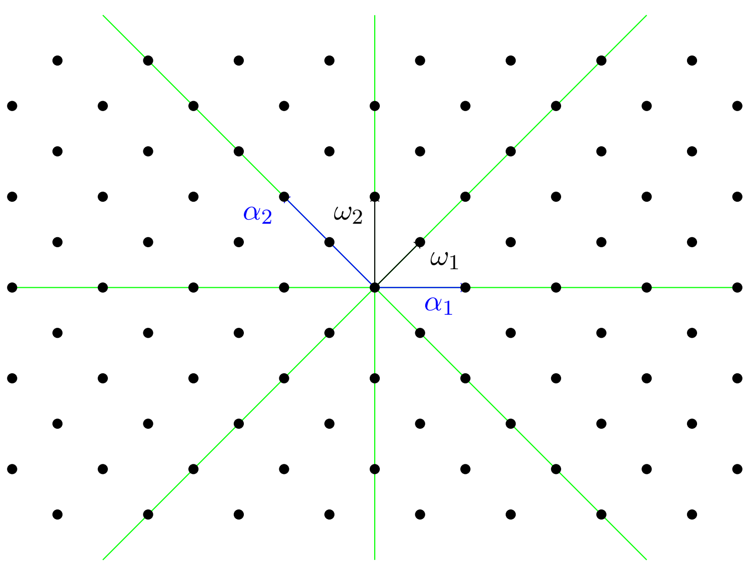


Figure 2.2: Weight system for $\mathfrak{sp}_4$

Chapter 3

Methods for Investigating Vector Bundles of Conformal Blocks

In this chapter we describe the main tools we use to compute rank and determine the first Chern class of bundles $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$. We begin by describing *factorization* which allows one to compute the rank of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ with any $\mathfrak{g}$ and level ℓ in terms of ranks of bundles with fewer weights. We then relate bundles with $\mathfrak{sl}_{r+1}$ at level ℓ with those defined with $\mathfrak{sp}_{2\ell}$ at level one. We describe several rank computation methods specific to bundles defined with $\mathfrak{sl}_{r+1}$. We conclude with stating several formulas appearing in [17] that we use to determine first Chern classes in the chapters that follow.

3.1 Factorization

In [46, Prop. 2.2.6] the authors described an isomorphism of the fiber of a bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ over a point associated to a degenerate curve. This isomorphism is known as *factorization* and is proved in [42, 2.4.2], [17, Prop. 2.4(2)], and [37, Thm.13]. The result of the isomorphism is that the fiber of a bundle on $\overline{M}_{0,n}$ over a point represented with a nodal curve can be understood as a sum of products of bundles defined over irreducible components of the curve. The sum is taken over all possible weights in $P_\ell(\mathfrak{g})$; such weights are associated to the "gluing" or nodal point on the irreducible components of the curve. We state the isomorphism specifically for vector bundles of conformal blocks defined over $\overline{M}_{0,n}$. Recall, for a point $(C_0; p_1, ..., p_n) \in \overline{M}_{0,n}$, the marked points p_i are smooth points on C_0.

Proposition 3.1. *(Factorization). Let $(C_0; p_1, ..., p_n)$ be a point in $\overline{M}_{0,n}$ with a node x_0. Let $\nu : C_1 \cup C_2 \to C_0$ be the normalization of C_0 at x_0, with $\nu^{-1}(x_0) = \{x_1, x_2\}$ and $x_i \in C_i$. Let $\nu^{-1}(\vec{p})_i \in C_i$ denote the tuple of points corresponding to the preimage of those marked points $\vec{p}$ on C_0 landing on the component C_i (see Figure 3.1). Then*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})} \cong \bigoplus_{\mu \in P_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \lambda(C_1) \cup \mu, \ell)|_{(C_1;\nu^{-1}(\vec{p})_1)} \otimes \mathbb{V}(\mathfrak{g}, \lambda(C_2) \cup \mu^*, \ell)|_{(C_2;\nu^{-1}(\vec{p})_i)}, \tag{3.1}$$

where $\lambda(C_i) = \{\lambda_j | \nu^{-1}(p_j) \in \nu^{-1}(\vec{p})_i\}$ and particularly, x_0 is not one of the marked points p_i.

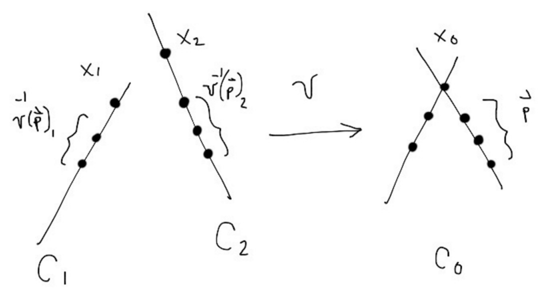


Figure 3.1: Normalization of a nodal genus 0 curve, C_0

In this proposition, for a weight $\mu \in P_\ell(\mathfrak{g})$, the weight $\mu^* \in P_\ell(\mathfrak{g})$ is the lowest weight of the irreducible highest weight $\mathfrak{g}$-module V_μ with highest weight μ. Equivalently, the symbol * represents the involution on the weight lattice of $\mathfrak{g}$ defined by $\mu^* = \sigma(\mu)$ where $\sigma \in W(\mathfrak{g})$ is the longest word in the Weyl group. In the case with $\mathfrak{g} = \mathfrak{sl}_2$ the involution is the identity. Since this is one of our main cases of interest, when we apply the factorization formula to $\mathfrak{sl}_2$ we use $\mu^* = \mu$.

The factorization formula is useful in computing ranks and first Chern classes of vector bundles of conformal blocks. For example, one can compute the rank of a bundle by computing instead the rank of the factorized sum which amounts to computing ranks of bundles on $\overline{M}_{0,n'}$ with $n' < n$. By applying factorization repeatedly (i.e., using the isomorphism to decompose a bundle over a curve with multiple nodes) one can compute the rank of a bundle in terms of dimensions of fibers of bundles on $\overline{M}_{0,3}$. As another application, in [17, Prop. 2.7] Fakhruddin applies factorization to compute the degree of any vector bundle of conformal blocks intersected with an F-curve (see Proposition 3.17 and [17, Prop. 2.7]). As the set of F-curves spans the vector space of 1-cycles on $\overline{M}_{0,n}$ (see Observation 4.43(3)), by computing all such degrees determines the first Chern class.

3.2 Preliminaries relating $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$

For the Lie algebra $\mathfrak{sl}_2$ the dominant integral weights at level ℓ are given by $a\omega_1$ where $a \leq \ell$ is some nonnegative integer and ω_1 is the fundamental weight for $\mathfrak{sl}_2$ (see Definition 2.8). For the Lie algebra $\mathfrak{sp}_{2\ell}$ the dominant integral weights at level one are given by the fundamental weights ω_a where $a \leq \ell$ is some nonnegative integer (see Definition 2.12). In this way, the set of dominant integral weights at level ℓ for $\mathfrak{sl}_2$ and at level one for $\mathfrak{sp}_{2\ell}$ are given by the following sets respectively,

$$\begin{aligned} P_\ell(\mathfrak{sl}_2) &= \{a\omega_1\}_{a=0}^{\ell} \text{ and} \\ P_1(\mathfrak{sp}_{2\ell}) &= \{0\} \cup \{\omega_a\}_{a=1}^{\ell}. \end{aligned} \tag{3.2}$$

For a fixed positive integer ℓ, an n-tuple of integers $\vec{\lambda} = (a_1, ..., a_n)$ such that $0 < a_i \leq \ell$ determines an n-tuple of weights $(a_1\omega_1, ..., a_n\omega_1) \in P_\ell(\mathfrak{sl}_2)^n$ or an n-tuple of weights $(\omega_{a_1}, ..., \omega_{a_n}) \in P_1(\mathfrak{sp}_{2\ell})^n$. In this document, we write $\vec{\lambda}$ to denote an n-tuple of nonnegative integers and the set of weights that this n-tuple of integers refers to will be determined by context, where if an integer $a_i = 0$ then the corresponding weight is the zero weight. When we want to be precise, we use $\vec{\lambda}$ to denote the n-tuple in $P_\ell(\mathfrak{sl}_2)^n$ and $\vec{\lambda}^T$ to denote the n-tuple in $P_1(\mathfrak{sp}_{2\ell})^n$. In order for the bundle to not be necessarily trivial, we always assume the sum $|\vec{\lambda}| = \sum_{i=1}^{n} a_i$ is even. When starting with a fixed integer $\ell \in \mathbb{Z}^{>0}$ and an n-tuple of positive integers $\vec{\lambda} = (a_1, ..., a_n)$, we will always assume $\ell \geq a_1 \geq ... \geq a_n > 0$ (see Remark 3.16).

In [34, p. 42], Littelmann constructs a generalized Littlewood-Richardson rule which relates the ranks of $\mathfrak{sl}_2$ level ℓ bundles and $\mathfrak{sp}_{2\ell}$ level one bundles. We state this consequence of Littelmann's work in the following fact.

Fact 3.2. For a fixed ℓ and n-tuple $\vec{\lambda}$ the vector bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ have the same rank. That is,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)).$$

Such an equality on the ranks also follows from rank-level duality (equivalently, in this case, "strange duality" for parabolic symplectic bundles, see the discussion in [1, Section 1.2]). Specifically, Abe shows in [1, 1.4] that over any smooth point of $\overline{M}_{0,n}$, the fiber of $\mathbb{V}(\mathfrak{sp}_{2r}, \vec{\lambda}, \ell)$ and the fiber of $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, r)$ are isomorphic where weights of $\vec{\lambda}^T$ are the transposed weights of $\vec{\lambda}$. That is, for a weight $\lambda \in \vec{\lambda}$ such that $\lambda = \sum_{i=1}^{r} c_i\omega_i$, the transposed weight $\lambda^T = \sum_{i=1}^{r} i\omega_{c_i}$. In the case with $r = 1$, the bundles $\mathbb{V}(\mathfrak{sp}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ are equivalent, as all one dimensional Lie algebras are isomorphic. This provides the following isomorphism of fibers over interior points $(C; \vec{p}) \in M_{0,n}$,

$$\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)|_{(C;\vec{p})} \cong \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, r)|_{(C;\vec{p})}. \tag{3.3}$$

The proof of the isomorphism (3.3) given in [1, 1.4] relies on a geometric interpretation of the fibers. Such a geometric interpretation does not necessarily extend to boundary points of $\overline{M}_{0,n}$ (see [10, Sect. 4.1]) and so the divisors, or first Chern classes, of the bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ are not necessarily linearly equivalent. Indeed, a main result of this dissertation (Theorem 1.8) gives necessary and sufficient conditions for when the divisors $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ *are* linearly equivalent.

With the isomorphism of fibers in (3.3), equality of ranks of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ (i.e., Fact 3.2) also follows. Because of this rank result, all rank computations and methods which we discuss for bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ can be applied to bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$.

To reduce confusion when referring to the rank of the bundle $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ or the rank of the Lie algebra $\mathfrak{sp}_{2\ell}$ we distinguish these two ranks. We refer to the rank ℓ, of the Lie algebra $\mathfrak{sp}_{2\ell}$, as the *Lie rank* and the rank of the vector bundle $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ as the *vector bundle rank*.

3.3 Fusion rules for $\mathfrak{sl}_2$

As mentioned above, using factorization (Proposition 3.1), the rank of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ can be determined by computing ranks of bundles on $\overline{M}_{0,3}$ (for example, by applying factorization to decompose a fiber of the bundle over a completely degenerate curve in $\overline{M}_{0,n}$). The conditions determining the ranks of such bundles are known as the *fusion rules* for $\mathfrak{g}$. An algorithm for computing fusion rules for any Lie algebra $\mathfrak{g}$ was determined by Kac and Walton and is known as the *Kac-Walton algorithm*. It is straightforward to compute a closed formula for such rules for the Lie algebra $\mathfrak{sl}_2$ [5, Lemma 4.2, Cor. 4.4]. However, for other Lie algebras, such closed formulas are not as easily determined. A closed formula for $\mathfrak{sl}_3$ was written in [7] and later verified in [4, Prop. 2.2]. We state the fusion rules for $\mathfrak{sl}_2$ of a bundle on $\overline{M}_{0,3}$ from [5, Cor. 4.4] as this is used extensively in Section 3.5 and Chapters 4 and 5

Proposition 3.3. *(Fusion rules for $\mathfrak{sl}_2$) Let $\ell, a, b,$ and c be fixed nonnegative integers with $a, b, c < \ell$. Then*

$$\operatorname{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) \;=\; 1$$

if and only if $a+b+c$ is even and the following four inequalities are satisfied,

$$\begin{aligned} a &\leq b+c, \\ b &\leq a+c, \\ c &\leq a+b, \text{ and} \\ a+b+c &\leq 2\ell. \end{aligned} \tag{3.4}$$

If the above are not satisfied, then $\operatorname{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) \;=\; 0$.

3.4 Rank tools for $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$

To compute the rank of a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ on $\overline{M}_{0,n}$, one could use factorization and the fusion rules from Proposition 3.3. However, in the case of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ with four weights defined on $\overline{M}_{0,4}$ there is a known formula for determining the rank. This formula was written in [44, Lemma 3.3].

Lemma 3.4. Let ℓ, a, b, c and d be fixed integers with $\ell \geq a \geq b \geq c \geq d \geq 0$ and $a+b+c+d = 2(\ell+s)$ for some integer s. The rank of $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2,(a,b,c,d),\ell))$ is given by the formula below, unless the value is negative.

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2,(a,b,c,d),\ell)) = \begin{cases} 1+\ell-a & \text{if } a+d \geq b+c \text{ and } s \geq 0 \\ 1+\ell-\frac{1}{2}(a+b+c-d) & \text{if } a+d \leq b+c \text{ and } s \geq 0 \\ \frac{1}{2}(b+c+d-a)+1 & \text{if } a+d \geq b+c \text{ and } s \leq 0 \\ 1+d & \text{if } a+d \leq b+c \text{ and } s \leq 0 \end{cases}$$

If the value of the formula is negative, then $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2,(a,b,c,d),\ell)) = 0$.

The result in Lemma 3.5 is known as the Generalized Triangle Inequality and can be shown using factorization and induction [2, Lemma 3.8].

Lemma 3.5 (Generalized Triangle Inequality)**.** Let ℓ be a fixed nonnegative integer and $\vec{\lambda} = (a_1,...,a_n)$ be a fixed n-tuple with $a_i \leq \ell$. If $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},\ell)) > 0$, then for all $i \in \{1,\ldots,n\}$ the inequality holds,

$$a_i \leq \sum_{j\neq i} a_j.$$

3.5 Plussing for $\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},\ell)$ bundles

To determine the rank of an $\mathfrak{sl}_{r+1}$ bundle a method called "plussing" on the weights can be used [10, Def. 8.2]. We state and prove the result for bundles $\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},\ell)$ using factorization.

Proposition 3.6. *Let $\vec{\lambda} = (a_1,...,a_n)$ be a fixed n-tuple of weakly decreasing positive integers such that the sum $|\vec{\lambda}| = \sum_{i=1}^n a_i$ is even. Let ℓ be a fixed integer such that $\ell \geq a_1$. Let $I \sqcup J = [n]$ be a partition into two disjoint subsets such that the size $|I|$ is even. Let $\vec{\lambda}^{\ell-I}$ denote the n-tuple of weights given by $\{\ell-a_i\}_{i\in I} \cup \{a_j\}_{j\in J}$. Then*

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},\ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda}^{\ell-I},\ell)).$$

Proof. First, suppose $\vec{\lambda} = (a,b,c)$, then the rank of $\mathbb{V}(\mathfrak{sl}_2,\vec{\lambda},\ell)$ can be determined by the fusion rules in Proposition 3.3.

If the inequalities in (3.4) are satisfied, then the set of inequalities in (3.5) follow,

$$\begin{aligned}(\ell - b) &\leq (\ell - a) + c \\ (\ell - a) &\leq (\ell - b) + c \\ (\ell - a) + (\ell - b) + c &\leq 2\ell \\ c &\leq (\ell - a) + (\ell - b).\end{aligned} \tag{3.5}$$

By Proposition 3.3, these relationships imply $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (\ell - a, \ell - b, c), \ell)) = 1$. Hence, the result is true for $n = 3$ weights. We now consider two cases determined by the parity of n.

First suppose n is some fixed odd integer. For our base case, we have shown the result holds for $n = 3$ weights. For the inductive step, consider a second partition $A \sqcup B = [n]$ such that $|B| = \{i, j\}$ with $i \in I$ and $j \notin I$. Let $\vec{\lambda}_{\hat{i},\hat{j}}$ be the $(n-2)$-tuple of weights from $\vec{\lambda}$ with weights in the i^{th} and j^{th} spot removed (that is, the weights associated to those from set A). Using factorization (Proposition 3.1) along the partition $A \sqcup B$, the rank of the original bundle becomes the following (where all bundles are $\mathfrak{sl}_2$ bundles at level ℓ),

$$\begin{aligned}\mathrm{rk}(\mathbb{V}(\vec{\lambda})) &= \sum_{\mu=0}^{\ell} \mathrm{rk}(\mathbb{V}(\vec{\lambda}_{\hat{i},\hat{j}}, \mu))\, \mathrm{rk}(\mathbb{V}(\lambda_i, \lambda_j, \mu)) \\ &= \sum_{\mu=0}^{\ell} \mathrm{rk}(\mathbb{V}(\vec{\lambda}_{\hat{i},\hat{j}}^{\ell-(I-i)}, \ell - \mu))\, \mathrm{rk}(\mathbb{V}(\ell - \lambda_i, \lambda_j, \ell - \mu)) \\ &= \mathrm{rk}(\mathbb{V}(\vec{\lambda}^{\ell-I})),\end{aligned} \tag{3.6}$$

The second equality in (3.6) follows from the induction hypothesis and the base case; the first and third equalities follow from factorization.

Now suppose n is even. This case follows from the odd result. Similar to the above, one can compute the rank by factorizing along a partition $A \sqcup B = [n]$ where $B = \{i, j\}$ with now $i, j \in I$.

Thus,

$$\begin{aligned}
\operatorname{rk}(\mathbb{V}(\vec{\lambda})) &= \sum_{\mu=0}^{\ell} \operatorname{rk}(\mathbb{V}(\vec{\lambda}_{\hat{i},\hat{j}}, \mu)) \operatorname{rk}(\mathbb{V}(\lambda_i, \lambda_j, \mu)) \qquad (3.7)\\
&= \sum_{\mu=0}^{\ell} \operatorname{rk}(\mathbb{V}(\vec{\lambda}_{\hat{i},\hat{j}}^{\ell-(I-B)}, \mu)) \operatorname{rk}(\mathbb{V}(\ell-\lambda_i, \ell-\lambda_j, \mu))\\
&= \operatorname{rk}(\mathbb{V}(\vec{\lambda}^{\ell-I})).
\end{aligned}$$

The second equality in (3.7) follows from the odd case (notice, the bundles in this sum each have an odd number of weights); the first and third equalities follow from factorization. □

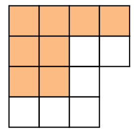
3.6 Young diagrams and tableaux

As stated in Remark 2.9, dominant integral weights at level ℓ for $\mathfrak{sl}_{r+1}$ are parametrized by Young diagrams in an $r \times \ell$ rectangle. In Section 3.7, we see that such objects also parametrize certain cohomology classes relevant to our study. In this section we define Young diagrams and related objects.

Definition 3.7. A *Young diagram* (also referred to as a partition) is a collection of rows of boxes, left-justified, with a weakly decreasing number of boxes in each row. A Young diagram can be denoted by a weakly decreasing sequence of integers $\lambda = (\lambda^{(1)} \geq \cdots \geq \lambda^{(r)} \geq 0)$ in which $\lambda^{(i)}$ denotes the number of boxes in the i^{th} row of the Young diagram λ. If $\lambda^{(1)} \leq \ell$, then we say that the Young diagram fits inside of an $r\times\ell$ rectangle and we write $\lambda \subset r\times\ell$. We denote $|\lambda| = \sum_{i=1}^{r} \lambda^{(i)}$ for the area or number of boxes in λ.

Definition 3.8. We use the word *shape* to refer to a collection of rows of boxes that are not necessarily left-justified or weakly decreasing in number of boxes in each row. For Young diagrams λ and ν such that $\nu \subset \lambda$ (i.e., each row of ν is less than or equal to the size of the corresponding row of λ), the shape λ/ν is the complement of the diagram of ν in λ. See Example 3.9 for an example. In the case that the shape ν has rows that are left-justified, we denote it as we would a Young diagram, that is $\nu = (\nu_1, \ldots, \nu_r)$ where ν_i is the number of boxes in row i of ν.

Example 3.9. With the Young diagrams $\lambda = (4,4,3,3)$ and $\nu = (4,2,2,0)$ the shape λ/ν is given by the complement of ν in λ. The figure below is the Young diagram λ with the boxes in ν highlighted. The white boxes in the figure illustrate the shape λ/ν.



Definition 3.10. Let $\mu = (a_1, \ldots, a_n)$ be a collection on nonnegative integers. We refer to such an n-tuple as *content* when referring to a collection of a_1 1's, a_2 2's, ..., and a_n n's. To help eliminate confusion between the values of a_i and the values of i in this collection, we refer to the number a_i as the *amount* of content and the number i as the *flavor* of the content.

Definition 3.11. Let $\mu = (a_1, \ldots, a_n)$ be a collection of nonnegative integers and λ a shape such that $|\mu| = |\lambda|$. A *semistandard tableau* on λ with content μ is a filling (or labeling) of the boxes of λ with content μ such that flavors are weakly increasing (left to right) across rows and strictly increasing (top to bottom) down columns. A semistandard tableau is *proper* for some fixed positive integer r, if for each flavor i in the first column and row $r+q$ (for $q \geq 1$), the box in the last column and row q is either not in λ or else contains flavor j with $j \leq i$. See Example 3.12 for an example. Sometime we omit the adjective *semistandard* and *proper*.

See Remark 4.9 for a discussion on the proper condition specific to our focus with $r = 2$.

Example 3.12. For the Young diagram $\lambda = (4,4,3,3)$ and collection of integers $\mu = (3,3,3,2,2,1)$ the figures below are semistandard tableaux with shape λ and content μ. For the integer $r = 2$, the first semistandard tableau is proper and the second semistandard tableau is not. The boxes in the second figure whose content flavor does not satisfy the proper condition are highlighted.

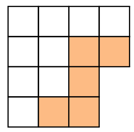
1	1	1	2
2	2	3	3
3	4	4	
5	5	6	

1	1	1	3
2	2	2	5
3	3	5	
4	4	6	

Definition 3.13. Let λ be a Young diagram. A *rim hook* of λ of size k is a collection of k contiguous boxes in the bottom and right border of λ. See Example 3.14 for an example of a rim hook. A

rim hook of size k *added* to λ is a collection of k contiguous boxes placed adjacent to the bottom and right border of λ such that the resulting collection of boxes is a Young diagram. In the case that k is longer than the number of boxes in the largest row of λ, extra boxes of the rim hook are accumulated below the first column of λ. See Example 4.6 for an example of a rim hook added to a Young diagram.

Example 3.14. For the Young diagram $\lambda = (4, 4, 3, 3)$, a rim hook of size five starting in the second column and ending in the fourth column of λ is shaded in the figure.



3.7 Witten's dictionary and computing ranks

Let $Gr(r, \mathbb{C}^m)$ denote the Grassmannian of r-dimensional linear subspaces of the vector space $\mathbb{C}^m$. A basis for the integral cohomology $H^*(Gr(r, \mathbb{C}^m); \mathbb{Z})$ is given by Schubert classes. These classes are parametrized by Young diagrams $\lambda \subset r \times (m - r)$ and denoted σ_λ. As in Definition 3.7 such a Young diagram is given by a collection of r weakly decreasing integers $\lambda = (\lambda^{(1)} \geq \cdots \geq \lambda^{(r)} \geq 0)$ such that $\lambda^{(1)} \leq m - r$.

The (small) quantum cohomology ring $QH^*(Gr(r, \mathbb{C}^m); \mathbb{Z})$ is an associative ring with underlying abelian group $H^*(Gr(r, \mathbb{C}^m); \mathbb{Z}) \otimes \mathbb{Z}[q]$. This quantum ring has an additive basis given by $q^d \omega_\lambda$ where d varies over all nonnegative integers and λ varies over all partitions in $r \times (m - r)$. A relationship between fusion rings and quantum cohomology of the Grassmannian is established in [49, Sect. 4.7] and stated in [9, Thm. 3.6]. Using this result and [9, Prop. 3.4, Eq. 3.10] an explicit translation can be made from the rank of $\mathbb{V}(\mathfrak{sl}_{r+1}\vec{\lambda}, \ell)$ to a coefficient appearing in a (quantum) cohomology computation. This translation and the following result is often referred to as *Witten's Dictionary* [10, Thm. 2.4]. We state this translation below.

Proposition 3.15. ***Witten's Dictionary.***
For a fixed positive integer $\ell \in \mathbb{Z}^{>0}$ and $\vec{\lambda} \in P_\ell(\mathfrak{sl}_{r+1})^n$, suppose there is some integer s such that $|\vec{\lambda}| = (r + 1)(\ell + s)$. Then to compute $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell))$ we consider the following two cases:

(1) If $s \leq 0$, then $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell))$ *is equal to the coefficient of the class of a point* $\sigma_{(\ell+s)^{r+1}} = \sigma_{(\ell+s,\ldots,\ell+s)}$ *in the classical product:*

$$\sigma_{\lambda_1} \cdot \ldots \cdot \sigma_{\lambda_n} \in H^*(Gr(r+1, \mathbb{C}^{r+1+\ell+s})).$$

(2) If $s > 0$, then $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell))$ *is equal to the coefficient of* $q^s[pt] = q^s\sigma_{(\ell,\ldots,\ell)}$ *in the quantum product:*

$$\sigma_{\lambda_1} * \ldots * \sigma_{\lambda_n} * \sigma^s_{\ell\omega_1} \in QH^*(Gr(r+1, \mathbb{C}^{r+1+\ell})),$$

where $\sigma^s_{\ell\omega_1}$ *is the s-fold quantum product* $\sigma^s_{(\ell,0,\ldots,0)}$.

Remark 3.16. The above products are commutative (see also [13, Sect. 3]) and the rank of $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ is invariant under the ordering of the weights in $\vec{\lambda}$. For a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ or $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ with weight $\vec{\lambda} = (a_1, \ldots, a_n)$, we usually assume $a_1 \geq \ldots \geq a_n$.

3.8 Kostka numbers and tableaux

According to [13, Eq. 10], to compute the classical or quantum product of a collection of n simple classes in cohomology (i.e., classes of the form $\sigma_{(a_i,0,\ldots,0)}$ where a_i is a positive integer) with a class σ_λ, we use the following rule, where $\mu = (a_1, \ldots, a_n)$. These calculations require enumerating certain tableaux; see Section 3.6 for the definitions of these objects.

1. Classical: $\sigma_{(a_1,0,\ldots,0)} \cdot \ldots \cdot \sigma_{(a_n,0,\ldots,0)} \cdot \sigma_\lambda = \sum K^\nu_{\lambda,\mu}\sigma_\nu \in H^*(Gr(r+1, \mathbb{C}^{r+1+\ell+s}); \mathbb{Z})$, where we sum over partitions $\nu \subset (r+1) \times (\ell+s)$ such that $|\nu| = |\lambda| + \sum_i^n a_i$.

   The coefficient $K^\nu_{\lambda,\mu}$ is called the *classical Kostka number*. This number is equal to the number of tableaux on the shape ν/λ with content μ.

2. Quantum: $\sigma_{(a_1,0,\ldots,0)} * \ldots * \sigma_{(a_n,0,\ldots,0)} * \sigma_\lambda = \sum K^\nu_{\lambda,\mu,m}(r+1,\ell) q^m \sigma_\nu \in QH^*(Gr(r+1, \mathbb{C}^{r+1+\ell}); \mathbb{Z})$, where we sum over partitions $\nu \subset (r+1) \times \ell$ and $m \geq 0$ such that $|\nu| + m(r+1+\ell) = |\lambda| + \sum_i^n a_i$ and $\mu = (a_1, \ldots, a_n)$.

   The coefficient $K^\nu_{\lambda,\mu,m}(r+1,\ell)$ is called the *quantum Kostka number*. This number is equal to the number of proper tableaux with shape $\nu[m]/\lambda$ and content μ. The Young diagram $\nu[m]$

is obtained from ν by adding m rim hooks of size $\ell+2$ to the Young diagram ν each starting in the first column and ending in the ℓ^{th} column.

In Section 4.1 we use this result to determine an explicit computation of the rank of a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ by enumerating tableaux.

3.9 Degree formula for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$

For any simple Lie algebra $\mathfrak{g}$, one can compute the intersection number of $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$ with an F-curve (Definition 1.1) by using [17, Prop. 2.7]. We state the formula here for later reference.

Recall from Definition 1.1, an F-curve, F_{I_1,I_2,I_3,I_4} is determined by a partition $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 = [n]$. Let $\vec{\lambda}_{I_k}\mu_k$ be the $|I_k|+1$-tuple of weights $(\lambda_{i_1}, ..., \lambda_{i_{\tilde{k}}}, \mu_k^*)$ where $I_k = \{i_1, ..., i_{\tilde{k}}\}$ and μ_k^* is the image of a weight $\mu_k \in P_\ell(\mathfrak{g})$ under the involution of the weight system of $\mathfrak{g}$ determined by the symmetry in the Dynkin diagram (see the discussion following Proposition 3.1).

Proposition 3.17. *[17, Prop. 2.7] The following formula determines the degree of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ restricted to the curve F_{I_1,I_2,I_3,I_4},*

$$\deg(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{F_{I_1,I_2,I_3,I_4}}) = \sum_{\vec{\mu} \in P_\ell(\mathfrak{g})^4} \deg(\mathbb{V}(\mathfrak{g}, \vec{\mu}, \ell)) \prod_{k=1}^{4} \mathrm{rk}(\mathbb{V}(\mathfrak{g}, \vec{\lambda}_{I_k}\mu_k^*, \ell)),$$

where $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$.

In order to make the computation in Proposition 3.17, one must first compute degrees of bundles with four weights. Bundles $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ with four weights live over $\overline{M}_{0,4} \cong \mathbb{P}^1$, and hence, the degree of such a bundle determines its divisor class. Fakhruddin has a formula in [17, Cor. 3.5] for computing the degree of any bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell) \to \overline{M}_{0,4}$. This formula was specialized to bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ in [17, Prop. 4.2] and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ in [17, Prop. 5.4]. We state these previous results and deduce our own formula to relate the degree and rank of $\mathfrak{sp}_{2\ell}$ bundles at level one in Chapter 5. We write these formulas explicitly in the following lemmas. Let ℓ, a, b, c, and d be fixed integers such that $\ell \geq a \geq b \geq c \geq d \geq 0$ and $a+b+c+d = 2(\ell+s)$ for some integer s. Set $\vec{\lambda} = (a, b, c, d)$.

Lemma 3.18. [17, Prop. 4.2] The degree of the bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) \to \overline{M}_{0,4}$, is given by

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \begin{cases} \max\{0, (\ell+1-a)s\} & \text{if } a+d \geq b+c \\ \max\{0, (1+d-s)s\} & \text{if } a+d \leq b+c \end{cases}$$

Note that these two formulas are the same for $a+d = b+c$. For in this case, we have $2(\ell+s) = a+b+c+d = 2(a+d)$ which implies $a = \ell+s-d$. Using this substitution in the first formula, it follows that the two rules in the formulas are equal.

Lemma 3.19. [44, Lemma 3.3] Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ be a bundle over $\overline{M}_{0,4}$.
If $s \geq 0$, then

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) \cdot s$$

If $s \leq 0$, then $\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 0$.

Lemma 3.20. [17, Prop. 5.4] Let $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be a bundle over $\overline{M}_{0,4}$.

If $a \leq \ell + s$, then

$$\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = \begin{cases} \max\{0, (\ell+1-a)(\ell+2s-a)/2\} & \text{if } a+d \geq b+c \text{ and } 0 < s \\ (\ell+s+1-a)(\ell+s-a)/2 & \text{if } a+d \geq b+c \text{ and } 0 \geq s \\ \max\{0, (1+d-s)(d+s)/2\} & \text{if } a+d \leq b+c \text{ and } 0 < s \\ d(d+1)/2 & \text{if } a+d \leq b+c \text{ and } 0 \geq s \end{cases}$$

If $a > \ell + s$, then $\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 0$.

Similar to the reasoning of equality of the formulas in Lemma 3.18 when $a+d = b+c$, we can show equality of the formulas for each case of s by using the substitutions $a = \ell + s - d$.

Remark 3.21. Variations of these formulas have appeared in previous work. The original formula, from which we obtain Lemma 3.19, first appeared in [17, Prop. 4.2]. Further simplifications were made by B. Alexeev and stated in [44, Lemma 3.3]. The formula in Lemma 3.20 first appeared in [17, Prop. 5.4] with certain conditions of the weights implicitly assumed. We briefly justify the additional condition in our formula in Lemma 3.20.

Claim 3.22. If $a > \ell + s$, then $\text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ is zero and thus,

$$\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 0.$$

Proof. We use the Generalized Triangle Inequality in (Proposition 3.5 [2, Lemma 3.8]) to show $\text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 0$, from this the degree result follows. The Generalized Triangle Inequality is for ranks of $\mathfrak{sl}_2$ bundles, so by Fact 3.2, this inequality provides a condition for when $\text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ is necessarily zero. The Generalized Triangle Inequality gives that if $a > b+c+d$ then $\text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = \text{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 0$. If we assume $a > \ell+s$ then since $a+b+c+d = 2(\ell+s)$, it follows immediately that $a > b + c + d$ and so $\text{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 0$. □

By comparing the degree formula in Lemma 3.20 with the rank formula for a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ (and hence $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$) in [44, Lemma 3.3], we can rewrite the degree formula for a bundle $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1) \to \overline{M}_{0,4}$ in terms of rank. We deduce the analogous degree formula for that of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) \to \overline{M}_{0,4}$ in Lemma 3.19 but for bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1) \to \overline{M}_{0,4}$.

Lemma 3.23. Let $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be a bundle over $\overline{M}_{0,4}$.

If $a \leq \ell + s$, then

$$\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = \begin{cases} \text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))(\ell + 2s - a)/2 & \text{if } a + d \geq b + c \text{ and } 0 < s \\ \text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))(\ell + s - a)/2 & \text{if } a + d \geq b + c \text{ and } 0 \geq s \\ \text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))(d + s)/2 & \text{if } a + d \leq b + c \text{ and } 0 < s \\ \text{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))d/2 & \text{if } a + d \leq b + c \text{ and } 0 \geq s \end{cases}$$

Otherwise, $\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 0$.

Comparing degree terms in Lemma 3.19 and Lemma 3.23, we obtain the following result.

Corollary 3.24. For bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ over $\overline{M}_{0,4}$, the following inequality holds:

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) \leq \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)).$$

Chapter 4

Rank One and Finite Generation for Bundles $\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}, \ell)$

In this chapter we prove Theorem 1.5 and Theorem 1.6 regarding the rank classification of bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and the finite generation of first Chern classes $c_1(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}, \ell))$ with rectangular weights. This work has also been written in [26].

We begin in Section 4.1 by recalling and summarizing a description of the data defining bundles $\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}, \ell)$. We then combine Witten's Dictionary (see Section 3.7) with the Kostka formulas (see Section 3.8) to describe a method to determine the rank of a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ as an enumeration of explicitly defined tableaux (Lemma 4.8). We end Section 4.1 by stating several definitions relevant to our approach and describe an algorithm for constructing tableaux. In Section 4.2 we prove our main result, Theorem 1.5, by applying the Reverse Fill Algorithm from Section 4.1. In Section 4.3, we describe the decomposition of the rank one bundles described in Theorem 1.5 and thus prove Theorem 1.6. In the final section, we give a proof of the scaling statement, Proposition 4.44, communicated to us by the authors of [10].

4.1 Definitions and lemmas

Recall, we defined the dominant integral weights at level ℓ for the simple Lie algebra $\mathfrak{sl}_{r+1}$ in Definition 2.8. By Remark 2.9, the dominant integral weights at level ℓ for $\mathfrak{sl}_{r+1}$ are parametrized by Young diagrams in a $r \times \ell$ rectangle (see Definition 3.7). Thus, such a weight is given by an r-tuple of integers $\lambda = (\lambda^{(1)}, ..., \lambda^{(r)})$ such that $\ell \geq \lambda^{(1)} \geq ... \geq \lambda^{(r)} \geq 0$. The Young diagram associated to this weight contains r rows with $\lambda^{(i)}$ boxes in the i^{th} row. Recall that $|\lambda| := \sum_{i=1}^{r} \lambda^{(i)}$ is the *area* of the weight λ. With this notation, a basis of the fundamental dominant weights ω_i for $\mathfrak{sl}_{r+1}$ is written $\omega_j = (1, \ldots, 1, 0, \ldots, 0)$ where $|\omega_j| = j$. If $\vec{\lambda} = (\lambda_1, ..., \lambda_n)$ is a collection of n dominant integral weights, then the total area of the weights is the sum of the areas of all weights in this collection, $|\vec{\lambda}| = \sum_{i=1}^{n} |\lambda_i|$. In order to define a vector bundle of conformal blocks with $\mathfrak{sl}_{r+1}$, weight $\vec{\lambda}$, and level ℓ it is necessary that $|\vec{\lambda}| = (r+1)(\ell+s)$ for some integer s. The Young diagrams corresponding to a collection of weights for bundles we analyze in Theorem 1.5 all look rectangular. We refer to these weights as such.

Definition 4.1. A collection of n weights $\vec{\lambda}$ for the Lie algebra $\mathfrak{sl}_{2m}$ is called *rectangular* if $\vec{\lambda} = (a_1\omega_m, ..., a_n\omega_m)$. In this case, the Young diagrams corresponding the weights in $\vec{\lambda}$ are all rectangles of height m. We often write $\vec{\lambda}$ as $(a_1, ..., a_n)$ when the fundamental dominant weight ω_m is clear from context.

4.1.1 Maximal weights

In order to distinguish vector bundles of $\mathfrak{sl}_2$ which have rank one for maximal reasons, we state two definitions. Given the data of a vector bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ with $\vec{\lambda} = (a_1\omega_1, \ldots, a_n\omega_1)$ we will always assume $a_1 \geq ... \geq a_n$. We sometimes call the n-tuple of integers $\mu := (a_1, ..., a_n)$ the content (as in Definition 3.11) associated to the weight $\vec{\lambda}$ but refer to these objects interchangeably. We will see the reason for this in Section 4.1.2.

Definition 4.2. A collection $\mu = (a_1, ..., a_n)$, of content containing n flavors, is *ℓ-maximal* if $n-3$ or more flavors have amounts of size ℓ.

Remark 4.3. Suppose there are integers $k \geq 0$ and $1 \leq p \leq \ell$ such that $\sum_{i=1}^{n} a_i = 2(k\ell + p)$, content $\mu = (a_1, ..., a_n)$ being ℓ-maximal implies one of two situations depending on the parity of n. If n is odd then $n = 2k+1$ or $n = 2k+3$ and if n is even then $n = 2k+2$ or $n = 2k+4$.

Definition 4.4. A collection $\mu = (a_1, ..., a_n)$, of content containing n flavors is *sum-maximal* if $\sum_{i=1}^{n} a_i = 2m\ell$ where $n = 2m$ or $n = 2m+1$.

Remark 4.5. Suppose there are integers $k \geq 0$ and $1 \leq p \leq \ell$ such that $\sum_{i=1}^{n} a_i = 2(k\ell + p)$, content $\mu = (a_1, ..., a_n)$ being sum-maximal implies $k = m - 1$ and $p = \ell$. When n is odd, this also implies the sum Λ from Theorem 1.5 contains only two terms.

We now apply Witten's Dictionary (see Section 3.7) to a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$.

4.1.2 Witten's Dictionary and classical Kostka applied to $\mathfrak{sl}_2$

For a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ with weight $\vec{\lambda} = (a_1, ..., a_n)$ such that $|\vec{\lambda}| = \sum_{i=1}^{n} a_i = 2(\ell+s)$ for an integer $s \leq 0$, Witten's Dictionary (Proposition 3.15) and the classical Kostka equation (Section 3.8) give

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell)) = K^{(\ell+s,\ell+s)}_{(a_1,...,a_n)}, \tag{4.1}$$

where $K^{(\ell+s,\ell+s)}_{(a_1,...,a_n)}$ is the number of tableaux with Young diagram $(\ell+s, \ell+s)$ given by a rectangular box with two rows, each with $\ell + s$ boxes, and filled with content $(a_1, ..., a_n)$.

4.1.3 Witten's Dictionary and quantum Kostka applied to $\mathfrak{sl}_2$

For a bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ with weight $\vec{\lambda} = (a_1, ..., a_n)$ such that $|\vec{\lambda}| = \sum_{i=1}^{n} a_i = 2(\ell+s)$ for an integer $s > 0$, Witten's Dictionary (Proposition 3.15) and the quantum Kostka equation (Section 3.8) give

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell)) = K^{(\ell,\ell)}_{\ell\omega_1,(a_1,...,a_n,\ell^{s-1}),s}(2, \ell), \tag{4.2}$$

where $K^{(\ell,\ell)}_{\ell\omega_1,(a_1,...,a_n,\ell^{s-1}),s}(2, \ell)$ is the number of proper tableaux with shape $\nu[s]/\lambda$ and content $(a_1, ..., a_n, \ell^{s-1})$ (the superscript denotes the number of content of amount ℓ). Here $\nu = (\ell, \ell), \lambda = (\ell)$, and $\nu[s]$ is obtained from ν by adding s rim hooks to ν of size $\ell + 2$ starting in the first column of ν and ending in the ℓ^{th} column (see Figure 4.1 and Definitions 3.13 and 3.8).

To carry out the quantum computation, we must first consider the shape $\nu[s]/\lambda$. We give an example to motivate the general shape of $\nu[s]/\lambda$. We refer the reader to Section 3.6 and [13] for more information on the description of these objects.

Example 4.6. Let $\ell = 4$, we construct the tableau $\nu[s]/\lambda$, $s = 5$, $\nu = (4, 4)$, and $\lambda = (4)$. The $s = 5$ rim hooks of size $\ell + 2 = 6$ that have been added to ν are distinguished from each other with

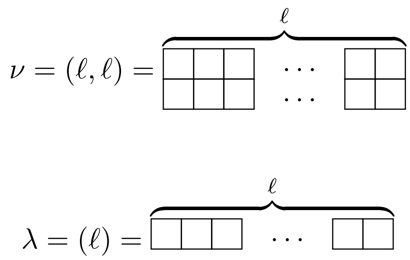
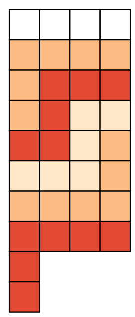


Figure 4.1: Young diagrams in Equation 4.2

a different shade. The Young diagram λ has been deleted from $\nu[5]$.



We generalize this example to describe the shape of $\nu[s]/\lambda$ for any integer $s > 0$ and $\ell > 0$.

Lemma 4.7. Let $\nu = (\ell, \ell)$ and $\lambda = (\ell)$ be Young diagrams and $s > 0$. Let p, k be integers such that $1 \leq p \leq \ell$ and $s = (k-1)\ell + p$ (the relevance of such integers will become apparent in Lemma 4.8 when analyzing the area of a weight vector), then $\nu[s]/\lambda = (\ell^{s+2k-1}, p^2)$.

Proof. Following the construction of the tableau in Example 4.6, we see that if we add $0 < s < \ell$ rim hooks of size $\ell + 2$ to ν, we add s new rows with ℓ boxes in each row (full rows) and two rows, each with s boxes in each row. When we add $s = \ell$ rim hooks, we add $s = \ell$ full rows of size ℓ and an additional two rows of size ℓ. When $p = 0$, the resulting shape $\nu[s]/\lambda$ will be a rectangle with $(k-1)(\ell+2)+1$ rows. $\square$

4.1.4 Tableaux to count ranks of $\mathfrak{sl}_2$ bundles

Lemma 4.8. For $\mathbb{V}_1$ as in Theorem 1.5, the rank of $\mathbb{V}_1$ is equal to the number of proper (semistandard) tableaux with content $\mu = (a_1, ..., a_n)$ given by $\vec{\lambda}$ on Young diagram $\varrho := (\ell^{2k}, p^2)$ (see Figure 4.2). The shape ϱ is a vertical concatenation of two rectangular shapes, one of dimension $2k \times \ell$ and one of dimension $2 \times p$.

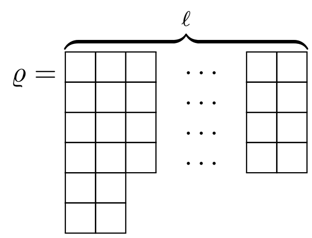


Figure 4.2: Young diagram $\varrho = (\ell^{2k}, p^2)$ for rank computations

Proof. First, for $s \leq 0$, we have $k = 0$ and $p = \ell - s$. Comparing the lemma statement in this case to the description of the coefficient in Equation 4.1 yields equivalent descriptions.

When $s > 0$, we consider the description of the coefficient in Equation 4.2. We must analyze the number of proper (semistandard) tableaux on the diagram $\nu[s]/\lambda$ from Lemma 4.7 with content $(\ell^{s-1}, a_1, ..., a_n)$. The smallest content flavors (from 1 to $s-1$) each have size ℓ. Since columns must be strictly increasing, we must fill rows 1 to $s-1$ in $\nu[s]/\lambda$ with the first $s-1$ content flavors of size ℓ, each such flavor fills the entire row. The remaining boxes must now be filled with the remaining content $(a_1, ..., a_n)$. These remaining boxes consist of the shape ϱ described in the lemma statement. In this case, the shape is a Young diagram.

This concludes that the rank computation in Equation 4.1 and 4.2 is equivalent to counting the number of proper tableaux on ϱ as claimed in the lemma. □

Remark 4.9. For the Young diagram ϱ, described in this lemma, the proper condition is equivalent to the condition that the flavor in the first column and p row be greater than or equal to the flavor in the final column and $p-2$ row. This means that for a given tableau, if any flavor i is contained in three (or fewer) consecutive rows, the tableau will be proper. Due to this remark, the tableaux we create in Section 4.2 are all proper.

4.1.5 The Reverse Fill Method for filling Young diagrams with content

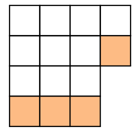
We now describe a method of placing content in a Young diagram to produce a tableau. We utilize this method to construction tableaux in the proof of Theorem 1.5.

Let λ denote a Young diagram in a $r \times \ell$ box. We denote $B_{(a,b)}$ the box of λ in row a and column b. We consider boxes in a diagram to have *lexicographical* ordering with row and column. That is, $B_{(a,b)} \leq B_{(a',b')}$ if and only if $a < a'$ or else ($a = a'$ and $b \leq b'$). We refer to the boxes of λ as being *larger* or *smaller* if we are referring to this ordering and *higher* or *lower* if they are visually displayed in that manner (i.e., box $B_{(a,b)}$ is *higher* than box $B_{(a',b')}$ if $a < a'$).

Definition 4.10. Define the *low-row* of a diagram λ to be all of the boxes of λ, $B_{(a,b)}$, such that $B_{(a+1,b)}$ is *not* a box of λ. The first row of the low-row of λ is the highest row containing such a box, note that this row will necessarily be the i^{th} row of λ where i is the number of boxes in the last column of λ. We use $l_\lambda = (l_\lambda^{(1)}, ..., l_\lambda^{(k)})$ to denote the sizes of the low-row of λ. If r is the total number of rows of λ, then for t such that $1 \leq t \leq k = r - i + 1$, the value $l_\lambda^{(t)}$ is the number of low-row boxes in the $(i+t-1)^{th}$ row of λ. In this case, since the boxes of the low-row are contained in the i to r rows of λ, we say the low-row *contains* $r - i + 1$ rows.

Remark 4.11. With this definition we require $l_\lambda^{(1)} \neq 0$; this convention considers the highest (vertical) row of a low-row as the first row. However, if i denotes the row of λ that contains the first row of the low-row of λ, then we could have $l_\lambda^{(t)} = 0$ if the $i+t-1$ and $i+t$ rows of λ contain the same number of boxes (i.e., there are no boxes $B_{i+t-1,b}$ in row $i+t-1$ such that $B_{i+t,b}$ is not a box in λ). See Example 4.12 for an example of this. We omit the subscript λ if the diagram we are referring to is clear. Note that the number of boxes in the low-row of λ will be equal to the length of the first row of λ and the number of rows of λ containing the low-row cannot be larger than the total number of rows of λ.

Example 4.12. The boxes in the low-row of the Young diagram $\lambda = (4, 4, 3, 3)$ are shaded in the figure. The low-row lengths are given by $(1, 0, 3)$.



With a (partial) filling of a Young diagram λ, we define $\lambda(i)$ to be the Young diagram obtained from λ by removing all boxes containing content of flavors i or larger[1]. For λ in $r \times \ell$ and content $\mu = (a_1, ..., a_n)$ such that $\sum_{i=1}^{n} a_i = |\lambda|$ we describe an algorithm for placing content μ in the boxes of λ. This algorithm has been programed in `Macaulay2` and is available on the author's website [27]. The code includes programs to compute the low-row of a tableau and create the semistandard tableau produced by the Reverse Fill Method of a given set of content with a given Young diagram.

Algorithm 1 Reverse Fill Method

For $i = n$ to 1 (in decreasing order) place all content a_i of flavor i in the largest boxes of the low-row of $\lambda(i + 1)$ (the ordering on boxes in a low-row is inherited by the ordering of the boxes of λ). Continue until either all content is placed, resulting in a tableau, or content a_i does not fit in the low-row of $\varrho(i + 1)$.

Example 4.13. Let $\lambda = (7, 7, 7, 7, 5, 5)$ and $\mu = (7, 6, 6, 6, 6, 6, 1)$ the following is the result of the Reverse Fill Method for placing μ in λ.

1	1	1	1	1	1	1
2	2	2	2	2	2	3
3	3	3	3	3	4	4
4	4	4	4	5	5	6
5	5	5	5	6		
6	6	6	6	7		

4.1.6 Notation and definitions for describing the decomposition of $\mathfrak{sl}_{2m}$ divisors

The following weight vector and vector bundle are used in Theorem 6.11(1).

[1] We again refer the reader to [13] for more background definitions related to Young diagrams and tableaux

Definition 4.14. For $\mathbb{V}_m = \mathbb{V}(\mathfrak{sl}_{2m}, (a_1\omega_m, ..., a_n\omega_m), \ell)$ and k, p integers as in Theorem 1.5, with $A, B \subset \{0, ..., n\}$ define the weight vector and vector bundle,

$$\vec{v}_{A,B} := (v_1\omega_m, ..., v_n\omega_m) \text{ and}$$

$$V_{A,B} := \mathbb{V}(\mathfrak{sl}_{2m}, \vec{v}_{A,B}, 1),$$

where $v_i = 1$ if $i \in (\{1, ..., 2k+1\} - A) \cup B$ and $v_i = 0$ otherwise. In the case that A or B is a singleton set (which is typically the case in our work, see Theorem 6.11) we denote the set by its one element. In the case that A or B is empty, we denote the set with a zero, 0. This convention is used to simplify our notation when decomposing a divisor of an $\mathfrak{sl}_2$ bundle (see Section 4.3).

Example 4.15. As an example of Definition 4.14, for $n = 9$, $k = 2$, and $m = 2$, the weight vector with 9 weights $(\omega_2, \omega_2, 0, \omega_2, \omega_2, 0, 0, \omega_2, 0)$ can be written as $\vec{v}_{3,8}$. In this case, $A = \{3\}$ and $B = \{8\}$.

The following weight vector and vector bundle are used in Theorem 6.11(2) and (3).

Definition 4.16. For $j \in \{0, 1, ..., n\}$, let

$$\vec{v}_{\hat{j}} := (v_1\omega_m, ..., v_n\omega_m) \text{ and}$$

$$V_{\hat{j}} := \mathbb{V}(\mathfrak{sl}_{2m}, \vec{v}_{\hat{j}}, 1),$$

where $v_i = 1$ if $i \neq j$ and $v_j = 0$.

4.2 Rank classification of the vector bundles $\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}, \ell)$

In this section we prove Theorem 1.5. As in the statement of Theorem 1.5, let $\mathbb{V}_1 = \mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ be the vector bundle of conformal blocks with weakly decreasing weights $\vec{\lambda} = (a_1\omega_1, ..., a_n\omega_1)$, where $|\vec{\lambda}| = 2(k\ell+p)$ for some integers k and p with $k \geq 0$ and $1 \leq p \leq \ell$ and $\Lambda := \sum_{i=2k+2}^{n} a_i$. We denote the content produced by the weight vector $\vec{\lambda}$ as $\mu = (a_1, ..., a_n)$ and use notation ϱ from Lemma 4.8

to refer to the diagram in Figure 4.2. The main lemmas and propositions in this section have been programed into `Macaulay2` and are available at the author's website [27].

4.2.1 Rank greater than zero

In this section, we show the construction of proper tableaux to conclude when ranks are necessarily positive.

Proposition 4.17. *If $\Lambda \geq p$ then a proper tableau can be constructed on the Young diagram $\varrho = (\ell^{2k}, p^2)$ with content μ using the Reverse Fill Method (see Algorithm 1).*

The following lemma is the main technical result used in the proof of the above proposition. In this lemma, we analyze the low-row of the diagrams $\varrho(i)$ that are produced after carrying out the Reverse Fill Method with content $(a_i, ..., a_n)$ on ϱ. We use the result to justify the construction of a proper tableau with content μ on shape ϱ using the Reverse Fill Method in the proof of Proposition 4.17.

Lemma 4.18. For a collection of weakly decreasing content $(a_1, ..., a_n)$, the following statements hold for the low-row of shape $\varrho(i)$ obtained after placing content $(a_i, ..., a_n)$ in shape ϱ using the Reverse Fill Method.

1. The low-row of $\varrho(i)$ will contain at most three rows. We denote the low-row of $\varrho(i)$ as $(l_i^{(1)}, l_i^{(2)}, l_i^{(3)})$.

2. After placing content a_{i-1} into $\varrho(i)$ using the Reverse Fill Method, the low-row of $\varrho(i-1)$ will satisfy $0 \leq l_{i-1}^{(3)} \leq l_i^{(3)} \leq p$.

3. If the amount of content a_{i-1} of flavor $i-1$ is such that $l_i^{(3)} \leq a_{i-1}$, then $l_j^{(3)} \leq a_{j-1}$ for all $j \leq i$.

4. If $l_i^{(3)} \neq 0$ for some i, then the amount of content of flavor $t-1 \geq i$, must be either $0 < a_{t-1} < l_t^{(3)}$ or $l_t^{(2)} + l_t^{(3)} \leq a_{t-1} \leq l_t^{(1)} + l_t^{(2)} + l_t^{(3)}$

Proof. We observe that the initial low-row of ϱ is $(l_\varrho^{(1)}, l_\varrho^{(2)}, l_\varrho^{(3)}) = (\ell - p, 0, p)$ contains three rows. Inductively, we consider what happens when placing content a_{i-1} into $\varrho(i)$ in the Reverse Fill

Method. There are three cases we must consider, determined by the amount a_{i-1} relative to the low-row of $\varrho(i)$. Consequences of each case follow immediately; we compute the resulting low-row of $\varrho(i-1)$ explicitly below. For an example of each case, see Example 4.20.

Case 1: $0 < a_{i-1} < l_i^{(3)}$, content of flavor $i-1$ does not fill the third row of the low-row of $\varrho(i)$. After placing a_{i-1} content of flavor $i-1$, the low-row of $\varrho(i-1)$ will be $(l_{i-1}^{(1)}, l_{i-1}^{(2)}, l_{i-1}^{(3)}) = (l_i^{(1)}, l_i^{(2)} + a_{i-1}, l_i^{(3)} - a_{i-1})$.

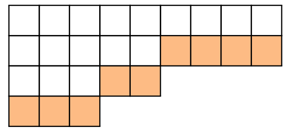
Case 2: $l_i^{(3)} \leq a_{i-1} < l_i^{(2)} + l_i^{(3)}$, content of flavor $i-1$ fills the third row of $\varrho(i)$ but does not fill the second row of the low-row of $\varrho(i)$. After placing a_{i-1} content of flavor $i-1$, the low-row of $\varrho(i-1)$ will be $(l_{i-1}^{(1)}, l_{i-1}^{(2)}, l_{i-1}^{(3)}) = (l_i^{(1)} + a_{i-1} - l_i^{(3)}, l_i^{(2)} + 2(l_i^{(3)}) - a_{i-1}, 0)$.

Case 3: $l_i^{(2)} + l_i^{(3)} \leq a_{i-1} \leq l_i^{(1)} + l_i^{(2)} + l_i^{(3)}$, content of flavor $i-1$ fills the third and second row of the low-row of $\varrho(i)$. After placing a_{i-1} content of flavor $i-1$, the low-row of $\varrho(i-1)$ will be $(l_{i-1}^{(1)}, l_{i-1}^{(2)}, l_{i-1}^{(3)}) = (a_{i-1} - l_i^{(2)} - l_i^{(3)}, (l_i^{(1)} + l_i^{(2)} + l_i^{(3)}) - (a_{i-1} - l_i^{(2)} - l_i^{(3)}) - (l_i^{(3)}), l_i^{(3)})$. Additionally, the sum of the second and third rows in the low-row increase, $l_{i-1}^{(2)} + l_{i-1}^{(3)} = l_i^{(2)} + l_i^{(3)} + (l_i^{(1)} + l_i^{(2)} + l_i^{(3)}) - a_{i-1}$.

Analyzing the result of each case outlined above, we see that placing content a_{i-1} will have the following effect in all cases, $0 \leq l_{i-1}^{(3)} \leq l_i^{(3)}$. Inductively, it follows that $l_i^{(3)} \leq l_\varrho^{(3)} = p$ and $l_i^{(1)} \leq l_\varrho^{(1)} \leq \ell - p$ for any shape $\varrho(i)$, showing the first two statements of the Lemma. The last two statements in the Lemma follow from the content $(a_1, ..., a_n)$ being weakly decreasing and analyzing the inequalities in each case. □

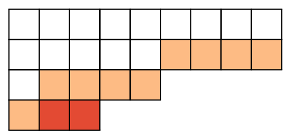
Remark 4.19. The statement of Lemma 4.18 (3) implies that if a_i is in Case 2 or 3 (in the proof of the Lemma), then all remaining content to place will also be in Case 2 or 3. The statement of Lemma 4.18 (4) implies that if the third row of the low-row of $\varrho(i)$ is nonzero, then all previously placed content must be in Case 1 or 3 (in the proof of the Lemma). In the remaining discussion, when referring to cases in the lemma, we mean that to refer to the three cases appearing in the proof of the lemma.

Example 4.20. We begin with shape $\varrho(i) = (9, 9, 5, 3)$ with low-row $(4, 2, 3)$ shaded in the image below. We demonstrate the effect on the low-row after filling with content a_{i-1} with amount relative to each case above.

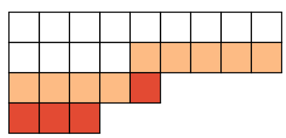


In the following cases, the boxes in the low-row of $\varrho(i)$ which are filled with content $i-1$ are darkly shaded and the new low-row for $\varrho(i-1)$ is lightly shaded.

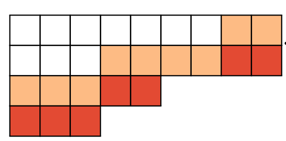
Case 1: $0 < a_{i-1} = 2 < 3$. After filling the low-row $(4,2,3)$ with 2 content values the low-row becomes $(4,4,1)$



Case 2: $3 \leq a_{i-1} = 4 < 5$. After filling the low-row $(4,2,3)$ with 4 content values the low-row becomes $(5,4,0)$



Case 3: $5 \leq a_{i-1} = 7 \leq 9$. After filling the low-row $(4,2,3)$ with 7 content values the low-row becomes $(2,4,3)$


.

Proof of Proposition 4.17. We show that all flavors from content μ can be placed in ϱ using the Reverse Fill Method.

As noted in Algorithm 1, if the Reverse Fill Method finishes by using all content from μ, the result will be a tableau and by Remark 4.9, this tableau will be proper. We now show that indeed, the Reverse Fill Method with content μ in shape ϱ and $\Lambda \geq p$ will always complete.

By the description of the Reverse Fill Method, content flavor i can always be placed using this method if $a_i \le l_{i+1}^{(1)} + l_{i+1}^{(2)} + l_{i+1}^{(3)}$ (i.e., content of flavor i fits in the low-row of $\varrho(i+1)$). When this sum, $l_{i+1}^{(1)} + l_{i+1}^{(2)} + l_{i+1}^{(3)}$, is equal to ℓ, then as $a_i \le \ell$ for all content in μ, the algorithm continues.

Let m denote the smallest content flavor such that $l_{m+1}^{(1)} + l_{m+1}^{(2)} + l_{m+1}^{(3)} = \ell$. We analyze the shape of $\varrho(m)$ after placing content a_m.

If $m = 1$, then $\varrho(1)$ is empty (as $|\mu| = |\varrho|$). A proper tableau has thus been created.

Thus, assume $m > 1$ and after filling with content of flavor m and larger, the low-row of $\varrho(m)$ is such that $0 < l_m^{(1)} + l_m^{(2)} + l_m^{(3)} < \ell$. We show $\varrho(m) = (r_1, r_2)$ for some nonegative integers $0 \le r_2 < r_1 < \ell$.

Recall, the sum of the rows in the low-row of $\varrho(i+1)$ is equal to the number of boxes in the largest row of $\varrho(i+1)$. It must be that $\varrho(m+1)$ contains a row with ℓ boxes and the largest row of $\varrho(m)$ contains less than ℓ boxes. Since content a_m is placed in at most one column of each row in $\varrho(m+1)$, the shape $\varrho(m+1)$ must contain exactly one row with ℓ boxes, and such a row must also contain boxes in the low-row of $\varrho(m+1)$ in order for content a_m to be placed in it. Additionally, this row must be the first row of ϱ as content is placed (using the Reverse Fill Method) in the lowest rows of $\varrho(m+1)$ first. From Lemma 4.18 (1), the low-row of $\varrho(m+1)$ is contained in at most three consecutive rows of ϱ; since one such row is the first row of ϱ, the shape $\varrho(m+1)$ has at most three nonzero rows. From this, we infer $\varrho(m+1) = (\ell, l_{m+1}^{(2)} + l_{m+1}^{(3)}, l_{m+1}^{(3)})$ (with possibly $l_{m+1}^{(2)} = 0$ or $l_{m+1}^{(3)} = 0$).

Our assumptions imply a_m content of flavor m is placed in $\varrho(m+1) = (\ell, l_{m+1}^{(2)} + l_{m+1}^{(3)}, l_{m+1}^{(3)})$ and the largest row of $\varrho(m)$ is less than ℓ. Considering the cases of Lemma 4.18 for which this happens, it must be that a_m is in Case 3 of Lemma 4.18 and $l_{m+1}^{(1)} + l_{m+1}^{(2)} + l_{m+1}^{(3)} = \ell$. From this case, it follows that after placing content a_m, we have $\varrho(m) = (\ell - (a_m - l_{m+1}^{(3)} - l_{m+1}^{(2)}), l_{m+1}^{(3)}, 0)$. Letting $r_1 = \ell - (a_m - l_{m+1}^{(3)} - l_{m+1}^{(2)})$ and $r_2 = l_{m+1}^{(3)}$, we can write the shape $\varrho(m)$ after filling with content a_m as $\varrho(m) = (r_1, r_2)$ with $0 \le r_2 < r_1 < \ell$.

We now show the remaining content can be placed in $\varrho(m)$ using the Reverse Fill Method. We consider two possible cases of $\varrho(m)$.

First, if $r_2 = 0$, then the shape $\varrho(m) = (r_1)$ is a single row. As $|\mu| = |\varrho|$, the remaining content must be less than r_1 (the size of the low-row of $\varrho(m)$). The Reverse Fill Method can thus continue with the remaining content. This completes the proof in this case.

Now suppose $r_2 \neq 0$. Let j be the maximum content such that $a_j \geq l_{j+1}^{(3)}$ (i.e., a_j is in Case 2 or 3 of Lemma 4.18). We can show that $j \geq 2k+2$ and all content $a_j, ..., a_{j-q}$ (for some q such that $m = j - q$) is in Case 3 of Lemma 4.18.

First, as the sum $\sum_{i=2k+2}^{n} a_i = \Lambda \geq p$ and the low-row of ϱ is $(\ell, 0, p)$, some content from the sum Λ satisfies $a_j \geq l_{j+1}^{(3)}$ (and so, in particular, a_j is in Case 2 or 3 of Lemma 4.18). By Lemma 4.18 (3) all subsequent content placed after a_j in the Reverse Fill Method will be in Case 2 or 3 of Lemma 4.18.

Additionally, our assumption $r_2 \neq 0$ implies $\varrho(m+1)$ has three nonzero rows and a_m is in Case 3 of Lemma 4.18. By Lemma 4.18 (4) all content placed before a_m (i.e., flavor larger than m) is in Case 1 or 3 of Lemma 4.18.

Together, these two results imply content $a_j, ..., a_{j-q}$ is in Case 3 of Lemma 4.18 and content $a_n, ..., a_{j+1}$ is in Case 1 of Lemma 4.18. By Lemma 4.18, we have $l_{j+1}^{(2)} + l_{j+1}^{(3)} = p$ and the low-row of $\varrho(j)$ after placing content a_j is:

$$l_j^{(2)} + l_j^{(3)} = p + \ell - a_j.$$

And similarly, for any i from j to $j - q$ we have $l_i^{(2)} + l_i^{(3)} = l_{i+1}^{(2)} + l_{i+1}^{(3)} + (\ell - a_i)$ (this follows from Case 3 of Lemma 4.18). We can conclude the sum of the low-row of $\varrho(j-q)$ after placing content $a_{j-q}, ..., a_n$ in the Reverse Fill Method satisfies:

$$l_{j-q}^{(2)} + l_{j-q}^{(3)} = r_2 + (r_1 - r_2) = p + \sum_{i=j-q}^{j} (\ell - a_i).$$

Furthermore, using this equality with the remaining sum, we obtain

$$\sum_{i=1}^{j-q-1} a_i = r_1 + r_2 > r_1 = r_2 + (r_1 - r_2) = p + \sum_{i=j-q}^{j} (\ell - a_i)$$

$$\sum_{i=1}^{j-q-1} a_i > p + (q+1)\ell - \sum_{i=j-q}^{j} a_i.$$

Solving for q gives,

$$\sum_{i=1}^{j-q-1} a_i + \sum_{i=j-q}^{j} a_i > p + (q+1)\ell,$$

$$\sum_{i=1}^{n} a_i > p + (q+1)\ell,$$

$$2k\ell + 2p = \sum_{i=1}^{n} a_i > p + (q+1)\ell,$$

$$2k\ell + p > (q+1)\ell,$$

$$2k\ell + \ell > (q+1)\ell,$$

$$2k > q.$$

Using this last inequality and that $j \geq 2k+2$, we obtain

$$m = j - q > 2k + 2 - 2k = 2.$$

It remains to fill $\varrho(m)$ with content $(a_1, ..., a_{m-1})$ for some $m-1 \geq 2$. Since $m-1 \geq 2$ there are at least two flavors in the remaining content to fill in the two rows $\varrho(m) = (r_1, r_2)$.

Since a_m with shape $\varrho(m+1)$ is in Case 3 of Lemma 4.18 we have that $a_m > l_{m+1}^{(3)} = r_2$. Since content amounts are weakly decreasing, we have that $a_{m-1} \geq a_m$ and so it follows, $a_{m-1} \geq r_2$. Thus there is enough content of flavor $m-1 \geq 2$ to fill the second row of r_2 boxes in $\varrho(m)$.

Additionally, since content is weakly decreasing, we also have $\sum_{i=1}^{m-2} a_i \geq a_{m-1}$. The remaining $a_{m-1} - r_2$ amount of content of flavor $m-1$ will be $a_{m-1} - r_2 < r_1 - r_2$ and so we can continue the Reverse Fill Method with content flavor $m-1$ and $\varrho(m-1)$ will be $\varrho(m-1) = (r_1 + r_2 - a_m)$, that is, one row. The Reverse Fill Method will then continue with the remaining content. This completes the proof in this case. □

4.2.2 Rank one and zero

We now justify the sufficient conditions of Λ in Theorem 1.5 for rank to be zero and one. As in the previous section, let $\mathbb{V}_1$ and Λ be defined as in Theorem 1.5 and ϱ and μ the Young diagram and content as in Lemma 4.8 from the data of $\mathbb{V}_1$.

Lemma 4.21. If $\Lambda < p$, then $\text{rk}(\mathbb{V}_1) = 0$.

Proof. Applying Lemma 4.8, we show there are no possible tableaux with shape ϱ and content μ.

In order to produce a tableau, the largest content flavors must be placed in rows with larger boxes than smaller flavors. Since $a_1 \geq ... \geq a_n$ and $\Lambda < p$, we must fill the final $2k+2$ row (which contains p boxes) with the largest flavors $(a_{2k+2}, ..., a_n)$. However, since the amount of content of flavor $2k+2$ to n (i.e., $\sum_{i=2k+2}^{n} a_i$) is less than p, we will not entirely fill this last row with this content. After placing the content $(a_{2k+2}, ..., a_n)$ the remaining collection of empty boxes creates a shape with a column of length $2k+2$. Such a column cannot be filled with the remaining content, $(a_1, ..., a_{2k+1})$ with flavors all less than $2k+2$ in strictly decreasing order. Thus, no such tableau exists. □

Lemma 4.22. If $\Lambda = p$, then $\text{rk}(\mathbb{V}_1) = 1$.

Proof. By Proposition 4.17 we know a proper tableau $T_{\vec{\lambda}}$ exists with shape ϱ and content μ given by $\vec{\lambda}$. We show this tableau cannot be modified.

The construction of $T_{\vec{\lambda}}$ places content flavors from $2k+2$ to n entirely in the $2k+2$ row of ϱ with p boxes. Any modification of a such content would move a content flavor from $(a_1, ..., a_{2k+1})$ (the content not in Λ) into row $2k+2$. This would contradict column content flavors strictly increasing.

Arguing by induction, we can show that content $(a_1, ..., a_{2k+1})$ must be placed in ϱ as described in the Reverse Fill Method. As $\varrho(2k+2)$ contains columns with $2k+1$ rows, content a_{2k+1} must be placed in the lowest rows of $\varrho(2k+2)$ in such columns and the remaining content must be placed in the next largest boxes so as to maintain weakly increasing rows. And so, content a_{2k+1} must be placed in the largest boxes in the low-row of $\varrho(2k+2)$. Since the Reverse Fill Method does produce a tableau (Proposition 4.17), after placing content a_{2k+1} in this way, the shape $\varrho(2k+1)$ now contains a column with exactly $2k$ rows and the same reasoning can be applied to content a_{2k} to argue that content a_{2k} must be placed in the $\varrho(2k+1)$ using the Reverse Filled Method. Continuing with this same reasoning, it follows that all content $(a_1, ..., a_{2k-1})$ must also be placed using the Reverse Fill Method. We can thus conclude that the Reverse Fill Method is the only way to create a tableau with shape ϱ and content μ.

By applying Lemma 4.8, we obtain that $\mathrm{rk}(\mathbb{V}_1) = 1$. See Example 4.23 for the tableau in a situation that $\Lambda = p$. □

Example 4.23. For the vector bundle $\mathbb{V}_1 = \mathbb{V}(\mathfrak{sl}_2, (6,6,5,5,5,2,1), 6)$, we have $k = 2$, $p = 3$, and $\Lambda = \sum_{i=6}^{7} a_i = 3 = p$. The following is the only tableau which can be produced on shape $\varrho = (6,6,6,6,3,3)$ with content $\mu = (6,6,5,5,5,2,1)$. This shows $\mathrm{rk}(\mathbb{V}_1) = 1$. The content in the sum Λ is shaded.

1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	4
4	4	4	4	5	5
5	5	5			
6	6	7			

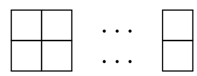
Lemma 4.24. If $\Lambda > p$ and weight content $\mu = (a_1, ..., a_n)$ is ℓ-maximal (see Definition 4.2), then $\mathrm{rk}(\mathbb{V}_1) = 1$.

Proof. In this argument we use the smallest content amounts to show there is only one possible tableau on ϱ with content μ from $\vec{\lambda}$ in Lemma 4.8. The tableau constructed in this argument will be the same as $T_{\vec{\lambda}}$ constructed using the Reverse Fill Method in Proposition 4.17. We further analyze the possible construction of a tableau in this specific case to conclude there is only one such possible tableau.

From Remark 4.3 and $\Lambda > p$, either $n = 2k + 2$ if n is even or $n = 2k + 3$ if n is odd. We examine each case of the parity of n.

First, if n is odd, then we have $n = 2k + 3$. We must have the highest $n - 3$ rows of the Young diagram ϱ from Lemma 4.8 filled with the first $n - 3$ content flavors of size ℓ. The boxes of $T_{\vec{\lambda}}$ containing content (a_{n-2}, a_{n-1}, a_n) has shape $\tilde{\varrho} = (p, p)$.

We seek to fill the remaining shape with content (a_{n-2}, a_{n-1}, a_n), the remaining empty boxes form the shape,



two rows of length p. Since $n - 2k - 1 = 2$, we have assumed $a_{n-1} + a_n > p$.

If n is even then $n = 2k+2$. Again, the highest $n-3$ rows of the Young diagram ϱ from Lemma 4.8 must be filled with the first $n-3$ content flavors of size ℓ. In this case, the empty boxes create the shape $\tilde{\varrho} = (\ell, p, p)$ and our remaining content is (a_{n-2}, a_{n-1}, a_n).

In either situation, in order to produce a tableau with three content flavors, the largest content of flavor n must be placed in the largest boxes, the smallest content of flavor $n-2$ must be placed in the smallest boxes, and the remaining content of flavor $n-1$ must then be placed in the remaining boxes. Such placement is equivalent to the Reverse Fill Method of (a_{n-2}, a_{n-1}, a_n) in shape $\tilde{\varrho}$ and describes the only possible filling to produce a tableau.

Considering the unique placement of content $(a_1, ..., a_{n-3})$, we confirm there is a unique tableau. By applying Lemma 4.8, we obtain that $\text{rk}(\mathbb{V}_1) = 1$.

See Example 4.25 for a tableau with content that is ℓ-maximal. □

Example 4.25. For the vector bundle $\mathbb{V}(\mathfrak{sl}_2, (6,6,6,6,2,2,2), 6)$, we have $k = 2$, $p = 3$, and $\Lambda = \sum_{i=6}^{7} a_i = a_6 + a_7 = 2 + 2 = 4 > p$. The following is the only tableau which can be produced on ϱ with content $(6,6,6,6,2,2,2)$. This shows $\text{rk}(\mathbb{V}) = 1$. The content from Λ is shaded. This can be compared to Example 4.23.

1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	6			
6	7	7			

Example 4.26. For the vector bundle $\mathbb{V}_1 = \mathbb{V}(\mathfrak{sl}_2, (5,5,5,5,5,3,3,3), 5)$, we have $k = 3$, $p = 2$, and $\Lambda = \sum_{i=8}^{8} a_i = a_8 = 3 > p$. The following is the only tableau which can be produced with content $(5,5,5,5,5,3,3,3)$ with shape $\varrho = (\ell^6, 2^2)$. This shows $\mathrm{rk}(\mathbb{V}_1) = 1$. The content from the sum Λ is shaded.

1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5	5	5	5	5
6	6	6	7	8
7	7			
8	8			

Lemma 4.27. If $\Lambda > p$ and weight content $\mu = (a_1, ..., a_n)$ is sum-maximal (see Definition 4.4), then $\mathrm{rk}(\mathbb{V}_1) = 1$.

Proof. Observe from Definition 4.4, if content is sum-maximal, then the Young diagram used to compute the rank (see Lemma 4.8 and Remark 4.5) is a rectangle of size $2m \times \ell$ where $n = 2m$ or $n = 2m + 1$.

If n is even, then each content must be of size ℓ. It follows immediately that there is only one proper tableau in Lemma 4.8.

If $n = 2m + 1$ and $\Lambda = a_{n-1} + a_n \geq p$, then by Proposition 4.17 the Reverse Fill Algorithm produces a semi standard tableau with content $(a_1, \ldots, a_{2m+1})$ with shape $2m \times \ell$. One can observed that in this case, each row i of the tableau constructed, will contain content of flavor i or $i+1$. Thus, to create a new tableau from this constructed one, we would have to switch a content of flavor $i+1$ in row i with a flavor $i+2$ in row $i+1$. However, in order for the modified column, now containing content flavor $i+2$ in its i^{th} row, to be strictly increasing flavors in rows $i+1$ to

$2m = n-1$ would at least require content from flavors $i+3$ to $i+(2m-i)+2 = 2m+2 = n+1$. Since our content flavors are only between 1 and n, this cannot happen. □

Example 4.28. For the vector bundle $\mathbb{V}_1 = \mathbb{V}(\mathfrak{sl}_2, (4,3,3,3,3), 4)$, we have $k=1$, $p=\ell=4$, and $\Lambda = \sum_{i=4}^{5} a_i = a_4 + a_5 = 6 > p$. The following is the only tableau which can be produced with content $(4,3,3,3,3)$ with shape $\varrho = (\ell^4)$. This shows $\mathrm{rk}(\mathbb{V}_1) = 1$. The content from the sum Λ is shaded.

1	1	1	1
2	2	2	3
3	3	4	4
4	5	5	5

With Lemmas 4.22, 4.24, and 4.27 above, we have shown all necessary conditions in the statement of Theorem 1.5 for $m=1$.

4.2.3 Rank greater than one

We now show the proper tableau as constructed in Proposition 4.17 can be modified when $\vec{\lambda}$ is not ℓ-maximal or sum-maximal to create a new proper tableau with the same shape and content. We use this to determine when the rank is larger than one. To begin, we use the following lemma.

Lemma 4.29. If $\Lambda > p$, the tableau, $T_{\vec{\lambda}}$, constructed in Proposition 4.17 using the Reverse Fill Method with content μ given by weight $\vec{\lambda}$ contains a column in which content increases by an increment larger than one.

Proof. Given $\Lambda > p$, we have that content of flavor $2k+2$ appears in a row smaller than the final $2k+2$ row of ϱ and let r denote this row. Denote by s the smallest column of $T_{\vec{\lambda}}$ containing content flavor $2k+2$ in row r. Consider the content flavors in column s. We can show that some content in column s must increase by an increment larger than one.

If such was not the case, then since content of flavor $2k+2$ is in some row $r < 2k+2$, flavor in row one of column s must necessarily be larger than 1 (by assuming each flavor appearing in column s above row r decreases by one increment from boxes in row r to 1). Since $T_{\vec{\lambda}}$ is a tableau, content of flavor 1 must fit inside the $s-1$ boxes in row one. Thus, we have $a_1 < s$.

However, we can show that for $j \leq 2k+1$, content amounts must be $a_j > s$. Indeed, since s is the smallest column containing content flavor $2k+2$ and flavor $2k+1$ is in row $r-1$ and column s, content of flavor $2k+1$ must also appear in row r and columns 1 to s, this implies $a_{2k+1} > s$. Since content amounts are weakly decreasing, $a_j \geq a_{2k+1} > s$ for $j = 1$ to $2k$. □

Proposition 4.30. *Let $T_{\vec{\lambda}}$ be the tableau constructed in Proposition 4.17 using the Reverse Fill Method with content μ given by weight $\vec{\lambda}$. If $\vec{\lambda}$ is not ℓ-or sum-maximal and $\Lambda > p$, then $T_{\vec{\lambda}}$ can be modified to create a new proper tableau on ϱ with content μ.*

Proof. In this proof, we analyze four cases. We describe procedures to create a proper tableau with shape ϱ and content μ by modifying the placement of content in $T_{\vec{\lambda}}$. In Cases 1-3, boxes of $T_{\vec{\lambda}}$ are identified with the same parameters and content contained in these boxes are switched, however, the justification that this procedure produces a tableau is different in each case. In Case 4, a slightly different procedure must take place. Examples 4.31 4.32, 4.33, and 4.34 illustrate these four cases.

Let s_1 denote the leftmost column of $T_{\vec{\lambda}}$ in which content increases by an increment larger than one. We know such a column exists by Lemma 4.29 for some $s_1 \leq p$ since $\Lambda > p$.

Let $r+1$ denote the first row in column s_1 in which content has increased by an increment larger than one. It follows that $r \leq 2k+1$, content in box B_{r,s_1} is of flavor r, box B_{r+1,s_1} contains flavor $d \geq r+2$, and $a_d < \ell$. Indeed, if $a_1 = \ell$, then by the Reverse Fill Method, we would have content of flavor $a-1$ in B_{r,s_1} with $r+1 \leq d-1$ (recall, the largest rows of ϱ contain ℓ boxes).

Since $T_{\vec{\lambda}}$ was filled using the Reverse Fill Method and box B_{r+1,s_1} has flavor d and B_{r,s_1} has flavor r, all content $a_{d-1}, ..., a_{r+1}$ of flavors $d-1$ to $r+1$ are in at most the r and $r+1$ row of ϱ. Particularly, all content in row $r-1$ must have flavor smaller than $r+1$.

Let b denote the largest flavor in row r smaller than d and larger than r. Let s_2 denote the largest column such that B_{r,s_2} contains flavor b. We show that in Cases 1-3 that flavor b and column s_2 exist. And that switching content of flavor b in B_{r,s_2} with content of flavor d in B_{r+1,s_1} will result in a new proper tableau with shape ϱ with content μ.

Case 1: $s_1 = 1$.

If $s_1 = 1$, then since $B_{r,1}$ contains content r, there are at most $a_d - 1$ other boxes in row r above content d in row $r+1$. As all columns are strictly increasing, we must have flavor $d-1$ with $d > d-1 > r$ in row r. Since $a_{d-1} \geq a_d$, it follows that $a_{d-1} > a_d - 1$ and so the largest column containing content $b = d-1$ must be above a box containing content larger than d (or such a box is not in ϱ).

It follows that switching content of flavor b in B_{r,s_2} with content of flavor a in B_{r+1,s_1} will result in a new proper tableau with shape ϱ and content μ. See Example 4.31 for such a case.

Case 2: $s_1 > 1$ and $r \leq 2k$.

Since $r \leq 2k$, row $r+2$ is a row in ϱ. Since $s_1 > 1$, we have that columns $1, ..., s_1 - 1$ increase by increments of size one, thus, content $r+2$ is in row $r+2$ of ϱ in boxes $B_{r+2,1}, ..., B_{r+2,s_1-1}$. We can further show that $d = r+2$; if this was not the case, then by the Reverse Fill Method, content of flavor d must appear in row $r+3$ below the content of flavor $r+2$. From the choice of s_1, this would force $d = r+3$. And specifically, $\varrho(r+3)$ would have the form $(\ell - (s_1 - 1), 0, s_1 - 1)$ (a jump in the low-row). However, from Lemma 4.18 a low-row of this form only appears for $\varrho(n+1)$ (the initial shape ϱ). However, if $n+1 = r+3$, then $n = r+2$, contradicting a content flavor of d in μ. In summary, columns 1 to s_1 contain s_1 amounts of content of flavor $r+2$ (in boxes $B_{r+2,1}, ..., B_{r+2,s_1-1}$ and B_{r+1,s_1}) and $s_1 - 1$ amounts of content flavor $r+1$ (in boxes $B_{r+1,1}, ..., B_{r+1,s_1-1}$). Since content amounts are weakly increasing $a_{r+1} \geq a_{r+2}$ so after placing $r+1$ in the $s_1 - 1$ boxes in row $r+1$, there must be additional content of flavor $r+1$ to place in row r. The largest box in the low-row of $\varrho(r+2)$ in row r will contain flavor $r+1$ and must be above content of flavor larger than $r+2$ (or such a box may not be in ϱ). Denote the column containing this largest box s_2.

It follows that switching content of flavor $r+1$ in B_{r,s_2} with content of flavor $r+2$ in B_{r+1,s_1} will result in a new proper tableau with shape ϱ and content μ. See Example 4.32 for an illustration of this case.

Case 3: $s_1 > 1$, $r = 2k+1$, and $n \geq j+1$.

Now, suppose $r = 2k+1$. In this case, row $r+1$ with box B_{r+1,s_1} containing content flavor a is in the final $2k+2$ row of ϱ. Since $s_1 > 1$, we have that content of flavor $r+1 = 2k+2$ is in row $r+1 = 2k+2$ in the boxes $B_{r+1,1}, ..., B_{r+1,s_1-1}$ and content a is only contained in row $r+1 = 2k+2$.

Since $\Lambda = \sum_{i=2k+2}^{n} a_i > p$ there must be content of flavor $2k+2 = r+1$ in row $r = 2k+1$. It follows that content $r+1 = 2k+2$ will be placed in column p of row $2k+1$ above a box containing content flavor $n > j$.

It follows that switching content of flavor $2k+2$ in $B_{r,p}$ with content of flavor a in B_{r+1,s_1} will result in a new proper tableau with shape ϱ and content μ. See Example 4.33.

Case 4: $s_1 > 1$, $r = 2k+1$ and $n = j$.

In this case, a different modification of $T_{\vec{\lambda}}$ than in Cases 1-3 must take place.

As in the previous situation with $s_1 > 1$ and $r = 2k+1$, we have that content of flavor $r+1 = 2k+2$ is in row $r+1 = 2k+2$ in boxes $B_{2k+2,1}, ..., B_{2k+2,s_1-1}$, content of flavor $r = 2k+1$ is in boxes $B_{2k+1,1}, ..., B_{2k+1,s_1}$, and content of flavor $a = 2k+3$ is in B_{2k+2,s_1}. Our parameters are thus, $r = 2k+1$ and $n = a = r+2 = 2k+3$.

Now, with $n = 2k+3$, the sum Λ contains only two content flavors. Additionally, since $\Lambda = a_{2k+2} + a_{2k+3} > p$, row $2k+1$ must also contain content of flavor $2k+2$ and so we have a strict inequality, $s_1 < p$. Let s_3 denote the *smallest* column such that B_{2k+1,s_3} contains content $2k+2$ (we will have $s_1 < s_3 \leq p$). See Figure 4.3 for an illustration of the $2k$, $2k+1$, and $2k+2$ rows of $T_{\vec{\lambda}}$ for such a case.

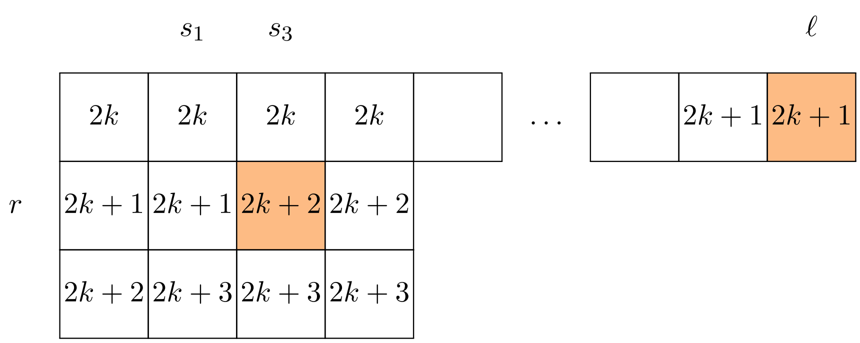


Figure 4.3: Tableau in Case 4 of Proposition 4.30

We have previously shown, all content in row $r-1$ must have flavor smaller than $r+1$. And so, for this case, it follows that all content in row $2k$ has flavor smaller than $2k+2$.

However, since $\vec{\lambda}$ is not ℓ-maximal, and $n = 2k+3$, we have $a_{2k} < \ell$. So box $B_{2k,\ell}$ must contain content of flavor $2k+1$.

In this case, it follows that switching content of flavor $2k+1$ in $B_{2k,\ell}$ with content of flavor $2k+2$ in B_{2k+1,s_3} will result in a new proper tableau with shape ϱ and content μ. These selected boxes are highlighted in Figure 4.3. □

We now illustrate examples for each case in the proof of Proposition 4.30. In each example, the first tableau is constructed using the Reverse Fill Method, the boxes containing content to switch are highlighted. The tableau obtained from the switching the highlighted content is illustrated.

Example 4.31. Consider the vector bundle $V = \mathbb{V}(\mathfrak{sl}_2, (3,3,3,2,2,1), 4)$. In this case, we have $n = 6$, $\ell = 4$, $k = 1$, $p = 3$ and $\Lambda = \sum_{i=4}^{6} a_i = 2+2+1 = 5 > p$. We are in Case 1 of Proposition 4.30 with parameters $r = 3$ and $d = 5$. The tableau $T_{\vec{\lambda}}$, constructed using the Reverse Fill Method, is first illustrated below. Content of flavors 4 and 5 to be switched specified by the description in Proposition 4.30 is shaded. As one can check, switching the flavors in these boxes produces a new proper tableau. Additionally, $\mathrm{rk}(V) = 6$.

1	1	1	2
2	2	3	3
3	4	**4**	
5	5	6	

1	1	1	2
2	2	3	3
3	4	**5**	
4	5	6	

Example 4.32. Consider the vector bundle $V = \mathbb{V}(\mathfrak{sl}_2, (4,3,2,2,1), 5)$. In this case, we have $n = 5$, $\ell = 5$, $k = 1$, $p = 1$ and $\Lambda = \sum_{i=4}^{5} a_i = 2+1 > p$. We are in Case 2 of Proposition 4.30. The tableau $T_{\vec{\lambda}}$, constucted using the Reverse Fill Method, is first illustrated below. Content of flavors 2 and 3 to be switched specified by the description in Proposition 4.30 is shaded. As one can check, switching the flavors in these boxes produces a new proper tableau. Additionally, $\mathrm{rk}(V) = 2$.

1	1	1	1	**2**
2	2	**3**	3	4
4				
5				

1	1	1	1	**3**
2	2	**2**	3	4
4				
5				

Example 4.33. Consider the vector bundle $V = \mathbb{V}(\mathfrak{sl}_2, (5,5,3,3,1,1), 5)$. In this case, we have $n = 6$, $\ell = 5$, $k = 1$, $p = 4$ and $\Lambda = \sum_{i=4}^{6} a_i = 3+1+1 = 5 > p$. We are in Case 3 of

Proposition 4.30 with parameters $r = 3$, $b = 4$, and $s_1 = 3$. The tableau $T_{\vec{\lambda}}$, constucted using the Reverse Fill Method, is first illustrated below. Content of flavors 4 and 5 to be switched specified by the description in Proposition 4.30 is shaded. As one can check, switching the flavors in these boxes produces a new proper tableau. Additionally, $\mathrm{rk}(V) = 2$.

1	1	1	1	1
2	2	2	2	2
3	3	3	**4**	
4	4	**5**	6	

1	1	1	1	1
2	2	2	2	2
3	3	3	**5**	
4	4	**4**	6	

Example 4.34. Consider the vector bundle $V = \mathbb{V}(\mathfrak{sl}_2, (4, 3, 3, 2, 2), 4)$. In this case, we have $n = 5$, $\ell = 4$, $k = 1$, $p = 3$ and $\Lambda = \sum_{i=4}^{5} a_i = 2 + 2 = 4 > p$. We are in Case 4 of Proposition 4.30 since $r = 3$ and $n = d = 5$. The tableau $T_{\vec{\lambda}}$, constucted using the Reverse Fill Method, is illustrated below. Content of flavors 3 and 4 to be switched specified by the description in Proposition 4.30 is shaded. As one can check, switching the flavors in these boxes produces a new proper tableau. Additionally, $\mathrm{rk}(V) = 2$.

1	1	1	1
2	2	2	**3**
3	3	**4**	
4	5	5	

1	1	1	1
2	2	2	**4**
3	3	**3**	
4	5	5	

We are now able to conclude the following result.

Proposition 4.35. *If $\Lambda > p$ and content μ is not ℓ-or sum-maximal then* $\mathrm{rk}(\mathbb{V}_1) > 1$.

Proof. By Proposition 4.17, a proper tableau can be constructed with shape ϱ and content μ. If $\vec{\lambda}$ is not ℓ-or sum-maximal, then the tableau $T_{\vec{\lambda}}$ construction in Proposition 4.17 satisfies Lemma 4.29. Hence by Proposition 4.30, more than one proper tableau can be produced with shape ϱ and content μ. By Lemma 4.8, we can conclude $\mathrm{rk}(\mathbb{V}_1) > 1$. □

4.2.4 Main Theorem

We collect our results together to prove the main theorem.

Proof of Theorem 1.5. By collecting the results from Lemmas 4.21 to 4.27 and Proposition 4.35 we can conclude the statement of the main theorem for the case $m = 1$. By the scaling property in Proposition 4.44, we conclude the full statement of Theorem 1.5. □

4.3 Decomposition of first Chern classes of rank one bundles

divisors

We now prove our final result which describes the decomposition of first Chern classes of rank one bundles appearing in Theorem 1.5 into an effective sum. We write this sum explicitly in Theorem 6.11. To reduce notation, we write the sum using the vector bundles $V_{A,B}$ and $V_{\hat{j}}$ we defined in Definitions 4.14 and 4.16.

Theorem 4.36. In the case $\mathrm{rk}(\mathbb{V}_m) = 1$ the first Chern class $c_1(\mathbb{V}_m)$ decomposes into an effective sum of ℓ first Chern classes of the form $c_1(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}', 1))$ where the weight vector $\vec{\lambda}'$ is determined by the original weight $\vec{\lambda} = (a_1\omega_m, ..., a_n\omega_m)$. We state the explicit decomposition for each possible form of weights. The sum always reduces to ℓ nonzero terms. As in Theorem 1.5, let p and k be integers such that $\sum_{i=1}^{n} a_i = 2(k\ell + p)$ with $1 \leq p \leq \ell$ and $k \geq 0$. Define $\Lambda := \sum_{i=2k+2}^{n} a_i$ where $\Lambda := 0$ if $2k + 2 > n$.

1. When $\Lambda = p$:

$$c_1(\mathbb{V}_m) = \sum_{i=1}^{2k+1} (\ell - a_i) \cdot \mathrm{c}_1(V_{i,0}) + \sum_{j=2k+2}^{n} a_j \cdot \ \mathrm{c}_1(V_{0,j}).$$

2. When $\Lambda > p$, content is ℓ-maximal, and

   (a) n is odd:

$$c_1(\mathbb{V}_m) = (\ell - p) \cdot c_1(V_{2k+1,0}) + \sum_{j=2k+1}^{n} (p - a_j) \cdot \ \mathrm{c}_1(V_{\hat{j}}),$$

   (b) n is even:

$$c_1(\mathbb{V}_m) = p \cdot c_1(V_{\hat{0}}) + (a_{2k} - p) \cdot \mathrm{c}_1(V_{2k+1,0}) + (a_{2k+1} - p) \cdot \mathrm{c}_1(V_{2k,0}) + (a_{2k+2} - p) \cdot \mathrm{c}_1(V_{\{2k,2k+1\},2k+2}).$$

3. When $\Lambda > p$ and content is sum-maximal:

$$c_1(\mathbb{V}_m) = \sum_{i=1}^{n} (\ell - a_i) \cdot \mathrm{c}_1(V_{\hat{i}}).$$

Our method used to decompose a first Chern class involves the *rank one peeling* result from [8, Prop. 19]. We state this result here in the special case of rank one bundles from Theorem 1.5.

Proposition 4.37. *If* $\vec{\mu} \in P_\ell(\mathfrak{sl}_{2m})^n$ *and* $\vec{\nu} \in P_{\tilde{\ell}}(\mathfrak{sl}_{2m})^n$ *such that*

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\mu} + \vec{\nu}, \ell + \tilde{\ell})) = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\mu}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\nu}, \tilde{\ell})) = 1$$

then

$$c_1(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\mu} + \vec{\nu}, \ell + \tilde{\ell})) = c_1(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\mu}, \ell)) + c_1(\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\nu}, \tilde{\ell})).$$

The result of Proposition 4.37 allows us to decompose $c_1(\mathbb{V}_m)$, where $\mathbb{V}_m$ is a bundle as in Theorem 1.5 of rank one, by describing a decomposition of the weight data $\vec{\lambda}$ that can be associated to rank one bundles. As a first step, we show in Lemma 4.38 that a decomposition of $\vec{\lambda}$ can be obtained from separating content in the final column of the unique tableau constructed to compute the rank of $\mathbb{V}_m$. Repeating this decomposition, we can completely decompose $c_1(\mathbb{V}_m)$ into an effective sum of ℓ terms as described in Proposition 6.11.

Lemma 4.38. Let $\mathbb{V}_1 = \mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ be a bundle of rank one from Theorem 1.5 with $\ell > 1$ and T the unique tableau constructed using the Reverse Fill Method with content μ from weight $\vec{\lambda}$. Let $(\tilde{c}_1, ..., \tilde{c}_n)$ denote the content appearing in the final column of T (note that each $\tilde{c}_i = 0$ or 1). Let $\vec{\lambda}^{-c} = ((a_1 - \tilde{c}_1)\omega_1, ..., (a_n - \tilde{c}_n)\omega_1)$ (i.e., the weight data from content not in the final column of T). Then,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}^{-c}, \ell - 1)) = 1.$$

Proof. We show this result using the rank one classification determined in Theorem 1.5.

Using the notation of Theorem 1.5, the final row of the tableau constructed in Proposition 4.17 to compute $\mathrm{rk}(\mathbb{V}_1)$ will contain $2k+2$ or $2k$ boxes, where $|\vec{\lambda}| = \sum_{i=1}^{n} a_i = 2(k\ell + p)$. Thus, we will have one of the following cases

$$|\vec{\lambda}^{-c}| = 2(k(\ell-1)+p-1) \text{ or } |\vec{\lambda}^{-c}| = 2(k(\ell-1)+p),$$

determined by the final column of T containing $2k+2$ or $2k$ boxes respectively. Since $\ell > 1$, if we were to remove a column of size $2k+2$ then it must be that $p > 1$ and so the integers k_c and p_c associated to $\vec{\lambda}^{-c}$ in Theorem 1.5 will be

$$k_c = k$$

and

$$p_c = p - 1 \text{ or } p_c = p.$$

Denote $\Lambda_c = \sum_{i=2k_c+2}^{n}(a_i - \tilde{c}_i)$ for the sum in Theorem1.5 associated to the weight $\vec{\lambda}^{-c}$. Particularly, since $k_c = k$, the sums Λ_c and Λ contain the same flavors.

If $\Lambda = p$, then the content from Λ fits exactly in the $2k+2$ row of T (see Lemma 4.22). If the final column of T contains only $2k$ boxes, then $\Lambda_c = \Lambda$ (as no content from the sum Λ was removed from the last column). If the final column of T contains $2k+2$ boxes, then it contains exactly one content from Λ, so $\Lambda_c = \Lambda - 1$. But also in this case, we have $p_c = p - 1$, and so $\Lambda_c = p_c$.

Now suppose $\Lambda \geq p$ and $\vec{\lambda}$ is $\ell-$maximal. This implies that removing the final column from T removes content from each $a_i = \ell$ in the content μ. The weight λ^{-c} will be $(\ell-1)-$maximal (that is, it will contain the same number of weights of size $\ell - 1$ that $\vec{\lambda}$ contains of size ℓ). As all content from Λ is first filled into the entire final $2k+2$ row of T, removing the final column of T will leave $\Lambda_c \geq p_c$.

Finally, suppose $\Lambda \geq p$ and $\vec{\lambda}$ is sum-maximal. In this case, we have $|\vec{\lambda}| = 2m\ell$ with $n = 2m$ or $n = 2m+1$. In either case, the final column of T contains $2m$ boxes. Thus $|\vec{\lambda}^{-c}| = 2m(\ell-1)$, showing that $\vec{\lambda}^{-c}$ is also sum-maximal at level $\ell - 1$. Similar to the previous cases, since the content from Λ_c is filled into the entire final $2m$ row of T, removing the final column of T will leave $\Lambda_c \geq p_c$.

In each of the above cases, Theorem 1.5 implies $\operatorname{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}^{-c}, \ell - 1)) = 1$. $\square$

4.3.1 Description of column content when $\Lambda = p$

We see from Lemma 4.38, that to explicitly describe the bundles appearing in our method of decomposing $c_1(\mathbb{V}_1)$, we must describe the content in each column of the tableau constructed to compute $\operatorname{rk}(\mathbb{V}_1)$. We describe the content in each column of the unique tableau constructed in Proposition 4.17 using the Reverse Fill Method for the case that $\Lambda = p$. Let j be a column of this tableau.

If j is such that $p < j \leq \ell$, then the j^{th} column contains $2k$ boxes with content from flavors 1 to $2k+1$ which are strictly increasing. There is at most one occurrence between consecutive rows in this column where the content flavors increase by two and all other content flavors increase by exactly one. Denote i_j the flavor missing by this increase of size two from column j. The weight vector obtained from the content of this column is $\vec{v}_{i_j,0}$ (see Definition 4.14).

If j is such that $1 \leq j \leq p$, then the j^{th} column contains $2k+2$ boxes in which the content in the first $2k+1$ rows are precisely the flavors 1 to $2k+1$ and the final box contains a content flavor from $\{2k+2, ..., n\}$. Let i_j denote this flavor. The weight vector given by the flavors in this column is given by $\vec{v}_{0,i_j}$. The transpose of column j has the following form:

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 \end{array} \;\cdots\; \begin{array}{|c|c|c|} \hline 2k & 2k+1 & i_j \\ \hline \end{array}.$$

For example, consider the conformal block, $V = \mathbb{V}(\mathfrak{sl}_2, (9, 8, 8, 8, 8, 8, 8, 2, 1), 9)$. In this case, $\ell = 9$, $k = 3$, $p = 3$. For each column j, if $j = 4, ..., 9$, the content $i_j - 1$ and $i_j + 1$ are lightly shaded (or else not in this column, see column $j = 4$). For columns $j = 1, 2, 3$ the content i_j is darkly shaded. In this example, $i_1 = 8, i_2 = 8, i_3 = 9, i_4 = 7, i_5 = 6, i_6 = 5, i_7 = 4, i_8 = 3$, and $i_9 = 2$.

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	3
3	3	3	3	3	3	3	4	4
4	4	4	4	4	4	5	5	5
5	5	5	5	5	6	6	6	6
6	6	6	6	7	7	7	7	7
7	7	7						
8	8	9						

4.3.2 Proof of divisor decomposition

We now prove our result on the decomposition of $c_1(\mathbb{V}_m)$.

Proof of Theorem 6.11. Let $\mathbb{V}_1$ be a rank one bundle. As in the statement of Lemma 4.38, let $\vec{\lambda}^{-c}$ be the weight vector obtained by removing the content from $\vec{\lambda}$ in the final column of the tableau produced in Proposition 4.17. The results of Lemmas 4.38 and 4.8 imply,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}^{-c}, \ell - 1)) = 1.$$

As nontrivial $\mathfrak{sl}_r$ bundles at level one are also rank one [17, Sect. 5.2.1], we have,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{v}_{i_\ell,0}, 1)) = 1,$$

where $\vec{v}_{i_\ell,0}$ is the weight vector associated to column ℓ.

By Proposition 4.37, the first Chern class decomposes as,

$$c_1(\mathbb{V}_1) = c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}^{-c}, \ell - 1)) + c_1(V_{i_\ell,0}).$$

We repeat this decomposition now with the rank one vector bundle $\mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}^{-c}, \ell - 1)$. This process continues until we have completely decomposed the original divisor into a sum of level one divisors. Observing the behavior of the weight content in column j with $p < j$ and $j \leq p$ of the tableau produced in Proposition 4.17 allows us to conclude the explicit equations in the theorem statement. □

Example 4.39. We demonstrate the decomposition of $c_1(V_1)$ from Theorem 6.11 with $V_1 = \mathbb{V}(\mathfrak{sl}_2, (9, 8, 8, 8, 8, 8, 8, 2, 1), 9)$. We have $|\vec{\lambda}| = 2(3 \cdot 9 + 3)$, and so $k = 3$, $p = 3$, and $\sum_{i=2(3)+2}^{9} a_i = 2 + 1 = 3 = p$. By Theorem 1.5, $\text{rk}(V_1) = 1$.

The unique tableau from Lemma 4.22 is the following:

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	3
3	3	3	3	3	3	3	4	4
4	4	4	5	4	4	5	5	5
5	5	5	5	5	6	6	6	6
6	6	6	6	7	7	7	7	7
7	7	7						
8	8	9						

Each column of this tableau gives the weight content associated to the tableau for the divisors of level one in the decomposition in Theorem 6.11. For columns $j = 4$ to 9, the content flavors $i_j - 1$ and $i_j + 1$ associated to missing content flavor i_j in column j are highlighted (or else not in column j if $i_j - 1$ is the box content in the last box of the column). For columns $j = 1$ to 3 boxes containing content flavors of i_j are highlighted (see Section 4.3.1)

1	1	1	1	1	1	1	1	**1**
2	2	2	2	2	2	2	**2**	**3**
3	3	3	3	3	3	**3**	**4**	4
4	4	4	5	4	**4**	**5**	5	5
5	5	5	5	**5**	**6**	6	6	6
6	6	6	**6**	**7**	7	7	7	7
7	7	7						
8	**8**	**9**						

From this, we see that the divisor decomposes as

$$c_1(V) = c_1(V_{2,0}) + c_1(V_{3,0}) + c_1(V_{4,0}) + c_1(V_{5,0}) + c_1(V_{6,0}) + c_1(V_{7,0}) + 2(c_1(V_{0,8})) + c_1(V_{0,9}).$$

4.4 Rank one vertical scaling

In this section, for the Lie algebra $\mathfrak{sl}_r$ and the set of dominant integral weights $P_\ell(\mathfrak{sl}_r)$, we describe two types of scaling operations on the Lie data referred to as *vertical* and *horizontal* (Definition 4.40 and 4.41). We state results on the vector bundles and first Chern classes of bundles obtained by scaling the Lie data defining a rank one bundle on $\overline{M}_{0,n}$. Below, we let $\lambda = \sum_{j=1}^{r-1} a_j\omega_j \in P_\ell(\mathfrak{sl}_r)$, $m \in \mathbb{N}$, and set $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ with $\vec{\lambda} = (\lambda_1, ..., \lambda_n)$.

Definition 4.40. The vertical scaling of a weight λ is defined to be the weight $\lambda(m) = \sum_{j=1}^{r-1} a_j\omega_{mj} \in P_\ell(\mathfrak{sl}_{mr})$. We denote $\mathbb{V}_m = \mathbb{V}(\mathfrak{sl}_{mr}, \vec{\lambda}(m), \ell)$, where $\vec{\lambda}(m) = (\lambda_1(m), ..., \lambda_n(m))$ to be the bundle obtained by vertically scaling the Lie data.

Definition 4.41 (see Def. 2.5 [10])**.** The horizontal scaling of a weight λ is defined to be the weight $m\lambda = \sum_{j=1}^{r-1} mc_j\omega_j \in P_{m\ell}(\mathfrak{sl}_r)$. We denote $\mathbb{V}[m] = \mathbb{V}(\mathfrak{sl}_r, m\vec{\lambda}, m\ell)$, where $m\vec{\lambda} = (m\lambda_1, ..., m\lambda_n)$ to be the bundle obtained by horizontally scaling the Lie data.

The following result is a special case of *horizontal projective rank scaling* for bundles of rank one (cf. [10, Thm. 3.1]). This was proved initially for bundles with $\mathfrak{g} = \mathfrak{sl}_r$ at level one in [22, Prop. 1.3] and generalized to bundles with projective rank scaling in [10, Thm. 3.1].

Claim 4.42 (Horizontal projective rank scaling)**.** For $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ of rank one, the bundles $\mathbb{V}[m]$ for $m \in \mathbb{N}$ are also rank one and the first Chern classes are related by the identity, $c_1(\mathbb{V}[m]) = m \cdot c_1(\mathbb{V})$.

The analogous vertical scaling result for rank one bundles also holds. The precise statement and proof, communicated by Belkale, Gibney, and Kazanova, are given in Proposition 4.44. The main methods in the argument involve: (1) *rank-level duality* of the fiber of the bundle over a smooth point [38, Cor. 4], (2) intersection theoretic computations using formulas by Fakhruddin [17, Prop. 2.7], and (3) factorization formulas of vector bundles of conformal blocks (as in [17, Prop. 2.2]). In Observation 4.43, we summarize three results of rank one bundles used in the argument. These observations follow from the fact that the set of F-curves spans the vector space of 1-cycles on $\overline{M}_{0,n}$ and the rank of a bundle can be computed using factorization with a partition of the weights determined by an F-curve.

Observation 4.43.

(1) For $\mathbb{V} = \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ of rank one, there is at most one nonzero term in the summation to compute rank by factorization along the partition determined by an F-curve, $\mathcal{F}$.

(2) Particularly, the four weights $\vec{\mu} = (\mu_1, \mu_3, \mu_3, \mu_4)$, indexing this nonzero term determine $\deg(\mathbb{V}|_{\mathcal{F}}) = \deg(\mathbb{V}(\mathfrak{g}, \vec{\mu}, \ell))$ [17, Prop. 2.7].

(3) Two divisors $c_1(\mathbb{V})$ and $c_1(\mathbb{V}')$ on $\overline{\mathrm{M}}_{0,n}$ are linearly equivalent if and only if the divisors intersect all F-curves in the same degree.

Proposition 4.44. *Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ be a bundle on $\overline{M}_{0,n}$ such that $\mathrm{rk}(\mathbb{V}) = 1$, then $\mathrm{rk}(\mathbb{V}_m) = 1$ and $c_1(\mathbb{V}_m) = m \cdot c_1(\mathbb{V})$*

Proof. If $\mathbb{V}$ has rank one, then by taking its rank-level dual (see [38, Cor. 4]), one obtains the rank one bundle $\mathbb{V}(\mathfrak{sl}_\ell, \vec{\lambda}^T, r)$, where $\vec{\lambda}^T = (\tilde{\lambda_1}^T, ..., \tilde{\lambda_n}^T)$ with $\tilde{\lambda_i}$ the modified Young diagram obtained by adding $\ell - \lambda_i^{(1)}$ columns of length r to the Young diagram λ_i (see Section 7.1 in [38]) and $\tilde{\lambda_i}^T$ is the transpose of the Young diagram $\tilde{\lambda_i}$. The weight $\tilde{\lambda_i}$ corresponds to adding trivial representations, ω_0, to the weight λ_i so that $\tilde{\lambda_i}$ has $\tilde{\lambda}_i^{(1)} = \ell$. Now, by Claim 4.42, $\mathbb{V}(\mathfrak{sl}_\ell, m\vec{\lambda}^T, mr)$ also has rank one. By taking the rank-level dual of this bundle, one obtains $\mathbb{V}(\mathfrak{sl}_{mr}, \vec{\lambda}(m), \ell) = \mathbb{V}_m$ and so also has rank one. One can check that indeed, normalizing the weights $(m\tilde{\lambda_i}^T)^T$ (i.e., removing columns of length mr from the Young diagrams corresponding to the weight $(m\tilde{\lambda_i}^T)^T$) appearing in the rank-level dual of the scaled bundle, correspond to the desired scaled weights $\vec{\lambda}(m)$.

For any F-curve, we can apply the rank result to the rank one bundle with four weights, $\mathbb{V}(\mathfrak{sl}_r, \vec{\mu}, \ell)$, in Observation 4.43(2) associated to the bundle $\mathbb{V}$, to deduce $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{mr}, \vec{\mu}(m), \ell)) = 1$. Since we showed $\mathrm{rk}(\mathbb{V}_m) = 1$, the scaled weight $\vec{\mu}(m)$ must be the weight from Observation 4.43(2) for the bundle $\mathbb{V}_m$. In Lemma 4.45 we show $m \cdot \deg(\mathbb{V}(\mathfrak{sl}_r, \vec{\mu}, \ell)) = \deg(\mathbb{V}(\mathfrak{sl}_{mr}, \vec{\mu}(m), \ell))$. From Observation 4.43 (2, 3) the proposition follows. □

Lemma 4.45. Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ be a bundle on $\overline{M}_{0,4}$ with $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\mathbb{V}_m$ be the bundle obtained by vertically scaling the Lie data (Definition 4.40). If $\mathrm{rank}(\mathbb{V}) = 1$, then

$$m \cdot \deg \mathbb{V} = \deg \mathbb{V}_m.$$

Proof. Assuming $\mathrm{rk}(\mathbb{V}) = 1$, it follows from the first argument in Proposition 4.44 that for all $m \geq 1$, $\mathrm{rk}(\mathbb{V}_m) = 1$. By Claim 4.42 we then also have $\mathrm{rk}(\mathbb{V}_m[m]) = 1$.

To compute the degree of a four weighted bundle, we use Fakhruddin's formula [17, Cor. 3.5],

$$\deg \mathbb{V}(\mathfrak{sl}_r, (\lambda_1, \lambda_2, \lambda_3, \lambda_4), \ell) = \frac{1}{2(r+\ell)} \Bigg\{ \sum_{i=1}^{4} \hat{c}_r(\lambda_i) -$$

$$\Bigg\{ \sum_{\lambda \in P_\ell(\mathfrak{sl}_r)} \hat{c}_r(\lambda) \Big\{ \sum_{\{a\} \sqcup \{bc\} = \{234\}} \operatorname{rk} \mathbb{V}(\mathfrak{sl}_r, (\lambda_1, \lambda_a, \mu), \ell) \operatorname{rk} \mathbb{V}(\mathfrak{sl}_r, (\lambda_b, \lambda_c, \mu^*), \ell) \Big\} \Bigg\} \Bigg\}. \quad (4.3)$$

In this formula, $\hat{c}_r(\lambda)$ is the scalar by which the Casimir element of $\mathfrak{sl}_r$ acts on the irreducible $\mathfrak{sl}_r$ representation V_λ (see for example [14, IX.7.6] or [16, p. 511]). Since $\mathbb{V}$ is rank one, there is exactly one weight $\mu \in P_\ell(\mathfrak{sl}_r)$ such that

$$\operatorname{rk} \mathbb{V}(\mathfrak{sl}_r, (\lambda_1, \lambda_a, \mu), \ell) \operatorname{rk} \mathbb{V}(\mathfrak{sl}_r, (\lambda_b, \lambda_c, \mu^*), \ell) = 1 \quad (4.4)$$

for each $a = 2, 3, 4$ in the indexing set of the final sum in Equation 4.3 (and zero otherwise). Thus, this sum indexing over $\mu \in P_\ell(\mathfrak{sl}_r)$ in Equation 4.3 reduces to a sum of three terms. Equation 4.3 can be written as,

$$\deg \mathbb{V} = \frac{1}{2(r+\ell)} \Bigg\{ \sum_{i=1}^{4} \hat{c}_r(\lambda_i) - \sum_{j=1}^{3} \hat{c}_r(\mu_j) \Bigg\}, \quad (4.5)$$

where $\mu_j \in P_\ell(\mathfrak{sl}_r)$ are the three weights making each case of a for Equation 4.4 true. By the rank scaling result of Proposition 4.44 and Claim 4.42, the (vertically and horizontally) scaled weights $m\mu_j(m) \in P_{m\ell}(\mathfrak{sl}_{mr})$ will also be those appearing in the similarly reduced degree computation for $\mathbb{V}_m[m]$. This implies,

$$\deg \mathbb{V}_m[m] = \frac{1}{2(mr+m\ell)} \Bigg\{ \sum_{i=1}^{4} \hat{c}_{mr}(m\lambda_i(m)) - \sum_{j=1}^{3} \hat{c}_{mr}(m\mu_j(m)) \Bigg\}. \quad (4.6)$$

From the definition of the Casimir operator ([14, IX.7.6] or [16, p. 511]) a formula for the Casimir scalar for $\mathfrak{sl}_r$ and weight $\lambda = \sum_{i=1}^{r-1} c_i \omega_i$ can be computed. We write one such standard formula below in Equation 4.7. An equivalent formula, using another choice of basis can be found in [35, Sect. 6].

$$\hat{c}_r(\lambda) = \frac{1}{r} \sum_{i=1}^{r-1} (r-i) i c_i^2 + \frac{2}{r} \sum_{1 \le i < j \le r-1} (r-j) i c_i c_j + \sum_{i=1}^{r-1} (r-i) i c_i. \quad (4.7)$$

We observe that the relationship of the Casimir scalar for $\mathfrak{sl}_r$ and weight λ, and $\mathfrak{sl}_{mr}$ and scaled weight $m\lambda(m)$ is

$$\hat{c}_{mr}(m\lambda(m)) = m^3\hat{c}_r(\lambda).$$

Using this relationship and Claim 4.42 with the rank one bundles $\mathbb{V}_m$ and $\mathbb{V}_m[m]$ leads to

$$\begin{aligned}
\deg \mathbb{V}_m &= \frac{1}{m}\deg \mathbb{V}_m[m] \\
&= \frac{1}{m}\frac{1}{2(mr+m\ell)}\left\{\sum_{i=1}^{4} m^3\hat{c}_r(\lambda_i) - \sum_{j=1}^{3} m^3\hat{c}_r(\mu_j)\right\} \\
&= \frac{1}{(m^2)}\frac{1}{2(r+\ell)}(m^3)\left\{\sum_{i=1}^{4} \hat{c}_r(\lambda_i) - \sum_{j=1}^{3} \hat{c}_r(\mu_j)\right\} \\
&= m\frac{1}{2(r+\ell)}\left\{\sum_{i=1}^{4} \hat{c}_r(\lambda_i) - \sum_{j=1}^{3} \hat{c}_r(\mu_j)\right\} \\
&= m\cdot \deg \mathbb{V}. \qquad \square
\end{aligned}$$

When the weight vector $\vec{\lambda}$ has rectangular weights, the converse of the rank relationship in Proposition 4.44 is also true. The argument in the first paragraph in the proof of Proposition 4.44 can be reversed to obtain the following result.

Corollary 4.46. For $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{2m}, \vec{\lambda}, \ell)$, we have $\mathrm{rk}(\mathbb{V}) = 1$ if and only if $\mathrm{rk}(\mathbb{V}_m) = 1$.

Chapter 5

Divisor Equivalence of Bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$

In this chapter, we investigate the first Chern classes of bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ (see Section 3.2) and prove Theorems 1.8 and 1.9. Our methods involve comparing intersection numbers of $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ with an arbitrary F-curve by applying the intersection formula in Proposition 3.17 (from [17, Prop. 2.7]) and using Fact 3.2 on the equality of the rank terms appearing in this formula for bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$. In Section 5.1 we restate the results of Theorem 1.5 in the case when $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ is a bundle over $\overline{M}_{0,4}$ in order to describe the four pointed bundles appearing in the intersection formulas of rank one bundles. In Section 5.2 we prove Theorem 1.8 by first showing the result for bundles with four weights (see Lemma 5.4). With this result and the observation made in Observation 5.2 for intersection numbers of rank one bundles, it follows that for $n > 4$, the corresponding intersections numbers of $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ are equal if the four pointed bundles appearing in the intersection formulas from Proposition 3.17 are rank one (or zero) bundles. In Proposition 5.7 we show such a condition with all F-curves implies the original bundles have rank one or zero. We end this chapter in Section 5.4 by explicitly describing the maps from rank one (or zero) bundles given by $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$.

5.1 Rank one classification of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ with four weights

In Theorem 1.5 a complete description of $\vec{\lambda}$ and ℓ is given to determine when the rank of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ is one. Due to the equality of ranks of bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ in Fact 3.2 this result also applies to bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$. We state this result explicitly for four pointed bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ by specializing the classification in Theorem 1.5 to this case.

Lemma 5.1. Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be defined with a fixed $\vec{\lambda} = (a, b, c, d)$ at level ℓ such that $a + b + c + d = 2(\ell + s)$, for some integer s. Then $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell, \vec{\lambda}, 1})) = 1$ if and only if one of the following sets of conditions is satisfied:

1) $s \geq 0$,
2) $a, b, c, d \geq s$, and
3) $a = \ell$ or $d = s$,

or

1) $s < 0$ and
2) $a = \ell + s$.

Furthermore, $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell, \vec{\lambda}, 1})) > 1$ if and only if one of the following sets of conditions is satisfied:

1) $s \geq 0$,
2) $a, b, c, d > s$, and
3) $a \neq \ell$,

or

1) $s < 0$ and
2) $a < \ell + s$.

5.2 Proof of divisor equivalence for rank one bundles

In this section we prove Theorem 1.8. Our method is to show the divisor classes $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ intersect all F-curves in the same degree if and only if the bundles have rank one (or zero). As the set of F-curves spans the vector space of 1-cycles on $\overline{M}_{0,n}$ (see Observation 4.43(3)), this will conclude our result.

We first note the following observation for rank one bundles.

Observation 5.2. For $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ of rank one, there is at most one nonzero term in the intersection formula in Proposition 3.17 from [17, Prop. 2.7]. Particularly, this potential nonzero term is the degree of a four pointed bundle with rank one.

From this observation, together with the equality of ranks in Fact 3.2, the following simplification for computing degrees for rank one bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ is deduced.

Corollary 5.3. Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be defined for a fixed ℓ and $\vec{\lambda}$. If $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 1$, then for any F-curve, F_{I_1,I_2,I_3,I_4}, there is a four-tuple of nonnegative integers $\vec{\mu} = (a, b, c, d)$ such that

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)|F_{I_1,I_2,I_3,I_4}) = \deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\mu}, \ell))$$

and

$$\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)|F_{I_1,I_2,I_3,I_4}) = \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\mu}, 1)).$$

From this corollary, it follows that to compare intersection numbers for rank one bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$, we need only compare four pointed degrees of rank one bundles. We now prove a result about such bundles.

Lemma 5.4. Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be defined with a fixed $\vec{\lambda} = (a, b, c, d)$ and integer ℓ such that $a + b + c + d = 2(\ell + s)$, for some integer s. Then

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 1 \text{ or } 0$$

if and only if

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)).$$

Particularly, in the case when degrees are equal, they are equal to $\max\{0, s\}$ and when they are not equal, $\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) < \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$.

Proof. We first prove the forward direction. If the rank of the bundles is zero, then degree formulas will be consistent (both zero, implying the bundles are trivial). Assume then that $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 1$. We show the degree formulas in Lemmas 3.19 and 3.20 are equal. To do this, we compare the corresponding degree formulas in the four cases determined by the relationship of a, b, c, d, ℓ, and s in Lemma 3.20. We go through the first case; the other cases follow from similar calculations.

In the first case of Lemma 3.20, suppose $a + d \geq b + c$ and $s \geq 0$. Since we are assuming $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 1$, Lemmas 3.19 and 3.20 give

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = s \tag{5.1}$$

and

$$\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = \max\{0, (\ell + 1 - a)(\ell + 2s - a)/2\}. \tag{5.2}$$

Now, since $a \leq \ell$ and $s \geq 0$ the value $(\ell+1-a)(\ell+2s-a)/2$ in (5.2) is nonnegative. Furthermore, since $\mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 1$, Lemma 5.1 implies $d \geq s$ and either $a = \ell$ or $d = s$. However, since $a + b + c + d = 2(\ell + s)$ and $a + d \geq b + c$ it follows that $a + d \geq \ell + s$ and so indeed, $a = \ell$. Using this, the right-hand side of (5.2) becomes,

$$(\ell + 1 - \ell)(\ell + 2s - \ell)/2 = s,$$

showing (5.1) and (5.2) are equal and hence $\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = \deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$.

For the reverse implication, assume $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) > 1$. From Lemma 5.1 we can assume $s > 0$ and both $a < \ell$ and $d > s$. We compare the four point degree formula from [17, Prop. 4.2] for $\mathfrak{sl}_2$ bundles with our corresponding formula for $\mathfrak{sp}_{2\ell}$ in Lemma 3.20. We must consider two cases.

Case 1: $a + d \geq b + c$. We compare,

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \max\{0, (\ell + 1 - a)s\}$$

and

$$\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = \max\{0, (\ell + 1 - a)(\ell + 2s - a)/2\}.$$

With our assumptions, it follows that both values are nonzero. Since $a < \ell$, it follows that $\frac{\ell - a}{2} > 0$ and so $(\ell+1-a)s < (\ell+1-a)(\ell+2s-a)/2$. This shows $\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) < \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$.

Case 2: $a + d < b + c$. We compare,

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \max\{0, (1 + d - s)(s)\} \text{ and}$$

$$\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = \max\{0, (\ell + 1 - d)(d + s)/2\}.$$

With our assumptions, both values are nonzero (as $d > s$) and $\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) < \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$.

□

The results of Lemma 5.4 and the method of computing vector bundle rank using factorization (Proposition 3.1) in the discussion of Observation 5.2 allow us to explicitly determine when two intersection numbers for $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ with an arbitrary number n of weights are equal. We summarize this result in the following corollary.

Corollary 5.5. Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be defined for some fixed integer ℓ and n-tuple $\vec{\lambda}$. Given a partition $\{1, ..., n\} = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$ determining an F-curve on $\overline{M}_{0,n}$, the intersection numbers for $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ are equal on that F-curve if and only if the four pointed bundles appearing as the degree terms in Proposition 3.17 (from [17, Prop. 2.7]) are rank one or zero bundles. Furthermore, the following relation always holds

$$\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)|F_{I_1,I_2,I_3,I_4}) \leq \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)|F_{I_1,I_2,I_3,I_4}).$$

As we prove in Proposition 5.7, if a vector bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ or $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ satisfies the conditions in Corollary 5.5 with every possible partition determining an F-curve (see Definition 1.1) then this also implies the bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ have rank one. The argument in the proposition uses the following result. To reduce notation, we carry out the argument for $\mathfrak{sl}_2$ bundles and write $\mathbb{V}_{\vec{\lambda}} = \mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ where we have fixed a level ℓ.

Lemma 5.6. Fix an integer $\ell \geq 0$ and n-tuple of weights $\vec{\lambda} = (\lambda_1, ..., \lambda_n) \in P_\ell(\mathfrak{sl}_2)^n$. Let $\tilde{\mu} < \mu \in P_\ell(\mathfrak{sl}_2)$. If

$$\mathrm{rk}(\mathbb{V}_{(\vec{\lambda},\mu)}), \mathrm{rk}(\mathbb{V}_{(\vec{\lambda},\tilde{\mu})}) > 0,$$

then also

$$\mathrm{rk}(\mathbb{V}_{(\vec{\lambda},\tilde{\mu}+2)}) > 0.$$

Proof. We show this result by induction on the number n of weights in $\vec{\lambda}$.

Let $\tilde{\mu} < \mu \in P_\ell(\mathfrak{sl}_2)$ be as in the lemma statement. As stated in Sections 3.2 and 3.4, the weights $\tilde{\mu}$ and μ will have the same parity. If $\tilde{\mu} + 2 = \mu$, then the lemma is vacuous, thus assume $\tilde{\mu} + 2 < \mu$.

For our base case, we show the result for $n = 2$. In this case, we have:

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\mu}) = 1$$

and

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\tilde{\mu}}) = 1.$$

From the three point fusion rules with weights $\lambda_1, \lambda_2, \mu$ and $\lambda_1, \lambda_2, \tilde{\mu}$ (3.4) the fusion rules with $\lambda_1, \lambda_2, \tilde{\mu} + 2$ follow. Thus,

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\tilde{\mu}+2}) = 1.$$

Now, suppose the statement holds for $n \geq 2$. We show the result for $n + 1$ weights in $\vec{\lambda}$. The rank of $\mathbb{V}_{(\vec{\lambda},\mu)}$ and $\mathbb{V}_{(\vec{\lambda},\tilde{\mu})}$ can be computed using factorization along the partition $\{1, 2, 3, ..., n\} \sqcup \{n + 1, n + 2\}$. This gives the following sums, which, by the assumption of the lemma are nonzero.

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\lambda_{n+1},\mu}) = \sum_{\nu \in P_\ell(\mathfrak{sl}_2)} \mathrm{rk}(\mathbb{V}_{\mu,\lambda_{n+1},\nu}) \, \mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\nu}) > 0 \tag{5.3}$$

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\lambda_{n+1},\tilde{\mu}}) = \sum_{\tilde{\nu} \in P_\ell(\mathfrak{sl}_2)} \mathrm{rk}(\mathbb{V}_{\tilde{\mu},\lambda_{n+1},\tilde{\nu}}) \, \mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\tilde{\nu}}) > 0 \tag{5.4}$$

Using the same partition, we can compute the rank of a bundle with weights $(\vec{\lambda}, \tilde{\mu}+2)$,

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\lambda_{n+1},\tilde{\mu}+2}) = \sum_{\tilde{\tilde{\nu}} \in P_\ell(\mathfrak{sl}_2)} \mathrm{rk}(\mathbb{V}_{\tilde{\mu}+2,\lambda_{n+1},\tilde{\tilde{\nu}}})\,\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\tilde{\tilde{\nu}}}) > 0. \tag{5.5}$$

We show the sum in (5.5) contains a nonzero term.

Let ν and $\tilde{\nu}$ be two fixed weights appearing as nonzero terms in (5.3) and (5.4) respectively. Explicitly, we have

$$\mathrm{rk}(\mathbb{V}_{\mu,\lambda_{n+1},\nu}), \mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\lambda_{n+1},\nu}), \mathrm{rk}(\mathbb{V}_{\tilde{\mu},\lambda_{n+1},\tilde{\nu}}), \mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\tilde{\nu}}) > 0 \tag{5.6}$$

We consider three possible cases for the relationship between these weights.

Case 1: $\tilde{\nu} < \nu$.

We show that $\tilde{\nu}$ or $\tilde{\nu}+2$ is a weight appearing as a nonzero term in (5.5). We consider two subcases.

Case 1(a): $\tilde{\mu}+2 \leq a+\tilde{\nu}$.

In this case, we show ν is a weight appearing as a nonzero term in (5.5). Using the assumption of this subcase and the fusion inequalities (3.4) with weights $\tilde{\mu}, a, \tilde{\nu}$, the fusion inequalities with $\tilde{\mu}+2, a, \tilde{\nu}$ follow. Thus

$$\mathrm{rk}(\mathbb{V}_{\tilde{\mu}+2,\lambda_{n+1},\tilde{\nu}}) > 0.$$

By (5.6) the following term is nonzero in the sum (5.5),

$$\mathrm{rk}(\mathbb{V}_{\tilde{\mu}+2,\lambda_{n+1},\tilde{\tilde{\nu}}})\,\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\tilde{\tilde{\nu}}}) > 0.$$

Case 1(b): $\tilde{\mu}+2 > a+\tilde{\nu}$.

We now show $\tilde{\nu}+2$ is a weight appearing as a nonzero term in (5.5). Since $\tilde{\mu} \leq a+\tilde{\nu}$ (3.4), parity of the three weights and the assumption of this subcase implies $\tilde{\mu} = a+\tilde{\nu}$. We consider the fusion inequalities with weights $\tilde{\mu}+2, a, \tilde{\nu}+2$. The first three fusion inequalities follow directly. We must check $a+(\tilde{\mu}+2)+(\tilde{\nu}+2) \leq 2\ell$.

Suppose this was not the case, we derive a contradiction. Assume $a+(\tilde{\mu}+2)+(\tilde{\nu}+2)>2\ell$. But as we also have $\tilde{\mu}=a+\tilde{\nu}$, it would follow that

$$a+(\tilde{\mu}+2)+(\tilde{\nu}+2)=(a+\tilde{\nu})+\tilde{\mu}+4=2\tilde{\mu}+4>2\ell,$$

and so

$$\tilde{\mu}+2>\ell.$$

This would imply $\tilde{\mu}=\ell$ or $\tilde{\mu}+1=\ell$ which contradicts $\tilde{\mu}+2<\mu\leq\ell$. Thus, the fusion rules must also be satisfied for weights $\tilde{\mu}+2,a,\tilde{\nu}+2$ and so $\mathrm{rk}(\mathbb{V}_{\tilde{\mu}+2,\lambda_{n+1},\tilde{\nu}}+2)>0$.

Since $\tilde{\nu}<\nu$, the inductive assumption implies

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\tilde{\tilde{\nu}}+2})>0,$$

which allows us to conclude the following term is nonzero in the sum (5.5)

$$\mathrm{rk}(\mathbb{V}_{\tilde{\mu}+2,\lambda_{n+1},\tilde{\nu}+2})\,\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\tilde{\nu}+2})>0.$$

Case 2: $\tilde{\nu}=\nu$.

We can assume $\tilde{\nu}=\nu$ are the only weights associated to a nonzero term in the sum of (5.3) and (5.4) in order to avoid repeating a case. More specifically then, the only possible weights ν and $\tilde{\nu}$ for which $\mathrm{rk}(\mathbb{V}_{\nu,a,\mu})>0$ and $\mathrm{rk}(\mathbb{V}_{\tilde{\nu},a,\tilde{\mu}})>0$ are the same weights $\nu=\tilde{\nu}$. For two fixed weights (say, μ and a) fusion inequalities determine such possible weights ν with nonzero rank terms $\mathbb{V}_{\nu,a,\mu}$. Thus, the upper and lower bounds on the weight ν determined be the fusion rules must be the same with weights μ,a and $\tilde{\mu},a$. This implies:

$$\max\{a-\mu,\mu-a\}=\max\{a-\tilde{\mu},\tilde{\mu}-a\}=\min\{2\ell-\mu-a,a+\mu\}=\min\{2\ell,\tilde{\mu}-a,a+\tilde{\mu}\}.$$

Since we have assumed $\tilde{\mu}<\mu$ (and given the above equality) the maximum and minimum can be determined, providing,

$$\mu-a=a-\tilde{\mu}=2\ell-\mu-a=a+\tilde{\mu}.$$

From these equalities, we can deduce the following relations.

$$\tilde{\mu} = 0 \qquad a = \ell/2 \tag{5.7}$$
$$\mu = \ell \qquad \nu = \ell/2$$

We show that ν is a weight appearing as a nonzero term in (5.5). From our deductions, we consider $\mathrm{rk}(\mathbb{V}_{\nu,a,\tilde{\mu}+2}) = \mathrm{rk}(\mathbb{V}_{\ell/2,\ell/2,2})$. That this term is nonzero follows directly from checking the three pointed fusion rules (3.4) with weights $\ell/2, \ell/2, 2$. By the nonzeroness of the term associated to $\nu = \tilde{\nu}$ in (5.6), we can deduce

$$\mathrm{rk}(\mathbb{V}_{\nu,a,2})\,\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\nu}) > 0,$$

and thus contributes a nonzero term to the sum of (5.5).

Case 3: $\tilde{\nu} > \nu$.

We show $\tilde{\nu}$ is a weight appearing as a nonzero term in (5.5). First, we can assume $\nu + 2 = \tilde{\nu}$. If this was not the case, then by (5.6) and our inductive assumption it follows that

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\nu+2})$$

is also nonzero. Repeating this argument, say m times, we can continue until we obtain

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\nu+2m}) > 0.$$

We can now take ν to be the weight $\nu + 2(m-1)$ and continue the case with this weight for ν.

With this additional condition, the fusion inequalities with weights ν, a, μ and the fusion inequalities with weights $\tilde{\nu}, a, \tilde{\mu}$, then the fusion inequalities with weights $\nu, a, \tilde{\mu}+2$ follow. This shows $\mathrm{rk}(\mathbb{V}_{\nu,a,\tilde{\mu}+2}) > 0$. From (5.6), we also have $\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\ldots,\lambda_n,\nu}) > 0$ and so, the product of these nonzero terms provides a nonzero term in (5.5). □

Proposition 5.7. *Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ be defined for some fixed ℓ and n-tuple $\vec{\lambda}$. Let $\{1, ..., n\} = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$ be any partition into nonempty sets. Consider the sum in the rank computation using factorization along any such partition. If the four pointed bundles appearing in the factorization* *sum associated to the weights in the indexing set of the sum (i.e., the 'attaching' or 'gluing' weights) have rank one or zero, then* $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 1$.

To prove Proposition 5.7, we first prove Lemma 5.8 and show the result for a bundle with five weights. We then use induction on the number of weights to prove the general statement of Proposition 5.7.

Lemma 5.8. Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ be defined for some fixed ℓ and weights $\vec{\lambda} = (a, b, c, d, e)$ such that $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) > 0$. If the rank computation along any partition $\{1, 2, 3, 4, 5\} = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$ into nonempty sets has rank one or zero terms for the four pointed bundles appearing in the terms in the rank factorization sum (see (5.8) below for where the bundles with four attaching weights appear explicitly), then $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 1$.

Proof. As in the Lemma statement, let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ be a bundle with weights $\vec{\lambda} = (a, b, c, d, e)$ (we use this notation to match the notation from Lemma 5.1). Using the result of plussing (Proposition 3.6, see [10, Def. 8.2]), we can assume four of our weights are $a, b, c, d \leq \frac{\ell}{2}$. Further, suppose $a \geq b \geq c \geq d$. Consider the partition $[5] = \{1\} \sqcup \{2\} \sqcup \{3\} \sqcup \{4, 5\}$. Using factorization along this partition, we have the following computation (where $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ and we denote $\mathbb{V}_{\vec{\lambda}}$ for the bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$)

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \sum_{\vec{\mu} \in P_\ell(\mathfrak{sl}_2)^4} \mathrm{rk}(\mathbb{V}_{\vec{\mu}}) \, \mathrm{rk}(\mathbb{V}_{a,\mu_1}) \, \mathrm{rk}(\mathbb{V}_{b,\mu_2}) \, \mathrm{rk}(\mathbb{V}_{c,\mu_3}) \, \mathrm{rk}(\mathbb{V}_{d,e,\mu_4}). \tag{5.8}$$

Using the fusion rules for $\mathfrak{sl}_2$ in (3.4), the two pointed rank terms are nonzero (and equal to one) if and only if the two weights are equal. Hence, equation 5.8 reduces to,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \sum_{\mu \in P_\ell(\mathfrak{sl}_2)} \mathrm{rk}(\mathbb{V}_{a,b,c,\mu}) \, \mathrm{rk}(\mathbb{V}_{d,e,\mu}). \tag{5.9}$$

The assumption of the Lemma is that $\mathrm{rk}(\mathbb{V}_{a,b,c,\mu}) = 1$ or 0 for any $\mu \in P_\ell(\mathfrak{sl}_2)$. In order to deduce the original bundle has rank one, we first show there is only one nonzero term appearing in (5.9); we will then show such a term is one.

First, since $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) > 0$, we know there must be at least one nonzero term in (5.9) (otherwise the bundle would have rank zero). For contradiction, suppose there are at least two nonzero terms in (5.9), that is,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) \geq \mathrm{rk}(\mathbb{V}_{a,b,c,\mu})\,\mathrm{rk}(\mathbb{V}_{d,e,\mu}) + \mathrm{rk}(\mathbb{V}_{a,b,c,\tilde{\mu}})\,\mathrm{rk}(\mathbb{V}_{d,e,\tilde{\mu}}) \geq 2 \tag{5.10}$$

and $\mu \neq \tilde{\mu}$; particularly, let's assume $\tilde{\mu} < \mu$. As we previously discussed, by the fusion rules for $\mathfrak{sl}_2$ in (3.4), in order to two nonzero rank terms, the weights d, e must not be zero. With our previous assumption we have $a \geq b \geq c \geq d > 0$. Further, let s and $\tilde{s}$ be integers such that

$$a + b + c + \mu = 2(\ell + s) \tag{5.11}$$

and

$$a + b + c + \tilde{\mu} = 2(\ell + \tilde{s}). \tag{5.12}$$

Since $\tilde{\mu} < \mu$, it follows that $\tilde{s} < s$ and parity provides the further relation $\tilde{\mu} + 1 < \mu$. Solving for s in the above set of equations, we obtain the relationship

$$s = \tilde{s} + \frac{\mu - \tilde{\mu}}{2} \tag{5.13}$$

There are three possible cases between $\tilde{s}$ and s arising from the conditions in the classification of rank one bundles (Lemma 5.1): $\tilde{s} < s < 0$, $\tilde{s} < 0 \leq s$, and $0 \leq \tilde{s} < s$.

Case 1: $\tilde{s} < s < 0$.

In this case, the second collection of conditions in Lemma 5.1 must be satisfied for weights $\{a, b, c, \mu\}$ and $\{a, b, c, \tilde{\mu}\}$. Hence, the largest weight in each set must be equal to $\ell + s$ and $\ell + \tilde{s}$ respectively. Since we assumed the weights are ordered $a \geq b \geq c$, the largest weight in the set $\{a, b, c, \mu\}$ is either a or μ and the largest weight in the set $\{a, b, c, \tilde{\mu}\}$ is either a or $\tilde{\mu}$. If the largest weight in each collection is a, then we would have $a = \ell + \tilde{s} = \ell + s$, implying $\tilde{s} = s$. And similarly, if the largest weights in each collection are μ and $\tilde{\mu}$, then $a + b + c = \ell + s = \ell + \tilde{s}$, implying $\tilde{s} = s$. Hence, we must have μ the largest weight in the set $\{a, b, c, \mu\}$ and a the largest weight in the set $\{a, b, c, \tilde{\mu}\}$. This implies $\mu = \ell + s$ and $a = \ell + \tilde{s}$.

We further consider two subcases:

Case 1(a): $\tilde{\mu} + 2 < \mu$.

From this assumption, (5.11), and (5.12), it follows that $\tilde{s}+1 < s$. Consider the term indexed by the weight $\tilde{\mu}+2 \in P_\ell(\mathfrak{sl}_2)$ in the sum (5.9). Since we are assuming the terms associated to μ and $\tilde{\mu}$ in (5.10) are nonzero, it must be that $\mathrm{rk}(\mathbb{V}_{d,e,\mu}) = \mathrm{rk}(\mathbb{V}_{d,e,\tilde{\mu}}) = 1$, the fusion inequalities with weights d, e, μ and $d, e, \tilde{\mu}$ imply the fusion inequalities with weights $d, e, \tilde{\mu}+2$ are satisfied, and so $\mathrm{rk}(\mathbb{V}_{d,e,\tilde{\mu}+2}) = 1$. Furthermore, consider $\mathrm{rk}(\mathbb{V}_{a,b,c,\tilde{\mu}+2})$, the assumption of the Lemma is that the rank of this bundle must be one. However, we have

$$a + b + c + (\tilde{\mu}+2) = 2(\ell+\tilde{s}) + 2 = 2(\ell+\tilde{s}+1),$$

which by Lemma 5.1, implies the largest weight in the set $\{a, b, c, \tilde{\mu}+2\}$ must be equal to $\ell+\tilde{s}+1$. However, since $\tilde{\mu} \leq a = \ell+\tilde{s}$, the largest weight is $\tilde{\mu}+2$, implying $\tilde{\mu} = \ell+\tilde{s}+1$. From (5.12) $a+b+c = \ell+\tilde{s}+1$ follows and from (5.11) $a+b+c = \ell+s$ follows. Together, these equalities contradict $\tilde{s}+1 < s$. Hence, it must be that $\tilde{\mu}+2 < \ell+\tilde{s}+1$ and so $\mathrm{rk}(\mathbb{V}_{a,b,c,\tilde{\mu}+2}) > 1$, a contradiction to the Lemma.

Case 1(b): $\tilde{\mu}+2 = \mu$.

We derive a contradiction. From $\tilde{\mu}+2 = \mu$, it follows that $s = \tilde{s}+1$ and

$$a + b + c + \mu = 2\ell + 2\tilde{s} + 2.$$

Recall, in Case 1 we also have $\mu = \ell + s$ and $a = \ell + \tilde{s}$. Together, this implies

$$(\ell+\tilde{s}) + b + c + (\ell+\tilde{s}+1) = 2\ell + 2\tilde{s} + 2,$$

forcing $b = 1$ and $c = 0$. This contradicts the weights a, b, c, d being nonzero.

Case 2: $\tilde{s} < 0 \leq s$.

First, from $\mathrm{rk}(\mathbb{V}_{a,b,c,\mu}) = 1$, the rank one classification result in Lemma 5.1 implies $a, b, c, \mu \geq s$ and one of the following is satisfied

(a) $c = s < \mu$ (c is smallest weight),

(b) $\mu = s \leq c$ (μ is smallest weight),

(c) $\mu = \ell$ (μ is largest weight).

If (a), then since $a + b + c + \mu = 2\ell + 2s$ it follows

$$a + b + \mu = 2\ell + s.$$

As $\ell \geq a + b$ (by plussing the weights, see Proposition 3.6), this would give,

$$\ell + \mu \geq 2\ell + s$$

forcing $\mu = \ell$ and $s = 0$ (that is, condition (c) from Lemma 5.1 is implied) and so we can ignore this as a separate case.

If (b), then we obtain a contradiction. Recall, we are assuming

$$a + b + c \leq \frac{\ell}{2} + \frac{\ell}{2} + \frac{\ell}{2} = \frac{3}{2}\ell,$$

but $\mu = s$ forces $a + b + c = 2\ell + s \geq 2\ell$ (as $s \geq 0$).

If (c), it follows that

$$a + b + c = \ell + 2s.$$

By the Generalized Triangle Inequality for $\mathrm{rk}(\mathbb{V}_{a,b,c,\mu})$ (Proposition 3.5) it follows that

$$a + b + c \leq \mu = \ell,$$

forcing $s = 0$ and

$$a + b + c = \ell. \tag{5.14}$$

Now we consider $\mathrm{rk}(\mathbb{V}_{a,b,c,\tilde{\mu}}) = 1$. Again, using the classification of rank one from Lemma 5.1, we must have the largest weight of $\{a, b, c, \tilde{\mu}\}$ equal to $\ell + \tilde{s}$. We consider two cases for this largest weight.

Case 2 (a): $a = \ell + \tilde{s}$ is the largest weight in $\{a, b, c, \tilde{\mu}\}$.

We show such a bundle cannot satisfy the conditions of the lemma. With the assumption in Case 2 and $a \leq \frac{\ell}{2}$, we have $a = \ell + \tilde{s} \leq \frac{\ell}{2}$ forcing $\tilde{s} \leq -\frac{\ell}{2}$. From (5.13) and $s = 0$, it follows that

$$0 = \tilde{s} + \frac{\mu - \tilde{\mu}}{2} \leq -\frac{\ell}{2} + \frac{\mu - \tilde{\mu}}{2}.$$

Now, substituting $\mu = \ell$ into this inequality, it follows,

$$0 \leq -\tilde{\mu},$$

forcing $\tilde{\mu} = 0$. Using $\tilde{\mu} = 0$ and (5.14),

$$2\ell + 2\tilde{s} = a + b + c + \tilde{\mu} = a + b + c = \ell.$$

This shows $\tilde{s} = \frac{-\ell}{2}$ (which requires ℓ to be even) and forces $a = \frac{\ell}{2}$ and $b + c = \frac{\ell}{2}$.

In summary, we have shown $\mu = \ell$ and $\tilde{\mu} = 0$ are the two attaching weights appearing as nonzero terms in (5.9). Since we are assuming these terms are nonzero, it must also be that $\text{rk}(\mathbb{V}_{d,e,\ell}) = \text{rk}(\mathbb{V}_{d,e,0}) = 1$. Thus, the inequalities from the fusion rules in (3.4) with weights d, e, ℓ and $d, e, 0$ are satisfied. Particularly, $\text{rk}(\mathbb{V}_{d,e,0}) = 1$ implies $d = e$ and $\text{rk}(\mathbb{V}_{d,e,\ell}) = 1$ implies $d + e = \ell$. From this, we must have $d = e = \frac{\ell}{2}$. We can conclude then that our five weights are $\vec{\lambda} = (\frac{\ell}{2}, b, c, \frac{\ell}{2}, \frac{\ell}{2})$ with $b + c = \frac{\ell}{2}$.

With such a $\vec{\lambda}$, consider the rank computation using factorization along the partition $\{1\} \sqcup \{2, 3\} \sqcup \{4\} \sqcup \{5\}$. This gives the sum,

$$\text{rk}(\mathbb{V}_{\vec{\lambda}}) = \sum_{\mu \in P_\ell(\mathfrak{sl}_2)} \text{rk}(\mathbb{V}_{\frac{\ell}{2},\frac{\ell}{2},\frac{\ell}{2},\mu}) \, \text{rk}(\mathbb{V}_{b,c,\mu}).$$

By the assumption in the lemma,

$$\text{rk}(\mathbb{V}_{\frac{\ell}{2},\frac{\ell}{2},\frac{\ell}{2},\mu}) \, \text{rk}(\mathbb{V}_{b,c,\mu}) = 1 \text{ or } 0, \tag{5.15}$$

for any $\mu \in P_\ell(\mathfrak{sl}_2)$. We show this is not true for the weight $\mu = \frac{\ell}{2}$.

First, consider $\text{rk}(\mathbb{V}_{b,c,\frac{\ell}{2}})$. With $b + c = \frac{\ell}{2}$, it is straightforward to check the fusion inequalities with $b, c, \frac{\ell}{2}$ in (3.4) are satisfied, so $\text{rk}(\mathbb{V}_{b,c,\mu}) = 1$.

Now consider $\mathrm{rk}(\mathbb{V}_{\frac{\ell}{2},\frac{\ell}{2},\frac{\ell}{2},\frac{\ell}{2}})$. From the rank formula in Proposition 3.4 (see [44, Lemma 3.3]), we compute

$$\mathrm{rk}(\mathbb{V}_{\frac{\ell}{2},\frac{\ell}{2},\frac{\ell}{2},\frac{\ell}{2}}) = 1 + \frac{\ell}{2} > 1.$$

This contradicts the product in (5.15).

Case 2 (ii): $\tilde{\mu} = \ell + \tilde{s}$ is the largest weight in $\{a, b, c, \tilde{\mu}\}$.

If $\tilde{\mu} = \ell + \tilde{s}$, then by (5.14), $\ell = a + b + c = \ell + \tilde{s} < \ell$ (recall, Case 2 assumption is $\tilde{s} < 0 \leq s$), a contradiction.

Case 3: $0 \leq \tilde{s} < s$.

From Case 2, if we assume $a, b, c \leq \frac{\ell}{2}$, $s \geq 0$, and $\mathrm{rk}(\mathbb{V}_{a,b,c,\mu}) = 1$ it follows that $\mu = \ell$. Since we are now also assuming $\tilde{s} \geq 0$ and $\mathrm{rk}(\mathbb{V}_{a,b,c,\tilde{\mu}}) = 1$, it follows that $\tilde{\mu} = \ell$. This contradicts $\tilde{\mu} < \mu$. We conclude the sum in (5.9) consist of one nonzero term. Particularly, since this term is a product of two ranks each equal to one, the sum is one and hence a bundle with five weights that satisfies the assumptions of the lemma has rank one. □

We now use induction on the number of weights of a bundle to prove Proposition 5.7.

Proof of Proposition 5.7. In Lemma 5.8, we show the result of the proposition for $n = 5$ weights, this provides our base case of the induction. Now assume the proposition holds for a bundle with $n \geq 5$ weights, we show the case with $\vec{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n, \lambda_{n+1})$ follows.

Consider the factorization sum using the partition $\{1\} \sqcup \{2\} \sqcup \{3\} \sqcup \{4, \ldots, n+1\}$ with $\frac{\ell}{2} \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$ (by plussing the weights, see Proposition 3.6). The rank computation using factorization with this partition gives the sum,

$$\mathrm{rk}(\mathbb{V}_{\vec{\lambda}}) = \sum_{\mu \in P_\ell(\mathfrak{sl}_2)} \mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\mu})\, \mathrm{rk}(\mathbb{V}_{\lambda_4,\lambda_5,\ldots,\lambda_{n+1},\mu}). \tag{5.16}$$

As in Lemma 5.8, we first show that this sum contains at most one nonzero term. For contradiction, suppose $\tilde{\mu} < \mu \in P_\ell(\mathfrak{sl}_2)$ are associated to nonzero terms in the sum (5.16). Then by Lemma 5.6, since $\mathrm{rk}(\mathbb{V}_{\lambda_4,\lambda_5,\ldots,\lambda_{n+1},\mu}), \mathrm{rk}(\mathbb{V}_{\lambda_4,\lambda_5,\ldots,\lambda_{n+1},\tilde{\mu}}) > 0$ it follows that we also have

$$\mathrm{rk}(\mathbb{V}_{\lambda_4,\lambda_5,\ldots,\lambda_{n+1},\tilde{\mu}+2}) > 0, \tag{5.17}$$

(with the cases that $\tilde{\mu}+2=\mu$ or $\tilde{\mu}+2<\mu$).

By the assumption of the Proposition,

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\mu})=1 \tag{5.18}$$

and

$$\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\tilde{\mu}})=1. \tag{5.19}$$

As in Lemma 5.8 with five weights (cf. (5.11) and (5.12)), let s and $\tilde{s}$ be integers such that

$$\lambda_1+\lambda_2+\lambda_3+\mu=2(\ell+s) \tag{5.20}$$

and

$$\lambda_1+\lambda_2+\lambda_3+\tilde{\mu}=2(\ell+\tilde{s}). \tag{5.21}$$

The same relation between s and $\tilde{s}$ from (5.13) holds and we again have three possible cases between $\tilde{s}$ and s resulting from the conditions in the classification of rank one bundles in Lemma 5.1. The arguments and results with five weights in Lemma 5.8 can be used to reason with the bundles in (5.18) and (5.19); specifically, we can follow the same arguments using $a=\lambda_1, b=\lambda_2, c=\lambda_3$.

Case 1: $\tilde{s}<s<0$.

We consider the two subcases considered in the $n=5$ argument of Lemma 5.8.

Case 1 (a): $\tilde{\mu}+2<\mu$.

Repeating the argument in Case 1(a) with $n=5$, it follows that $\mathrm{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\tilde{\mu}+2})>1$. By (5.17), this contradicts the assumption of the proposition (if we didn't have (5.17), then it might be that the product that appears in the factorization sum is zero because the term (5.17) is zero).

Case 1 (b): $\tilde{\mu}+2=\mu$. The same argument with $n=5$ weights derives a contradiction.

Case 2: $\tilde{s}<0\leq s$.

Following the argument with $n=5$, we must consider when $\mu=\ell$ is the largest weight in $\{\lambda_1,\lambda_2,\lambda_3,\mu\}$. We further consider the two subcases determined by the largest weight; that is we consider when $\lambda_1=\ell+\tilde{s}$ or $\tilde{\mu}=\ell+\tilde{s}$ are the largest weights in $\{\lambda_1,\lambda_2,\lambda_3,\tilde{\mu}\}$. Recall, in the $n=5$ argument, assuming $\tilde{\mu}=\ell+\tilde{s}$ is the largest weight derives a contradiction; this contradiction would similarly be derived in this case. Thus we just consider the case that $\lambda_1 = \ell + \tilde{s}$ is the largest weight in $\{\lambda_1, \lambda_2, \lambda_3, \tilde{\mu}\}$ and from this assumption the following relations are deduced (again, following the argument with $n = 5$ weights),

$$\begin{aligned} &\mu = \ell && s = 0 && \lambda_1 = \ell/2 \\ &\tilde{\mu} = 0 && \tilde{s} = -\ell/2 && \lambda_2 + \lambda_3 = \ell/2. \end{aligned} \tag{5.22}$$

Consider the weight $\tilde{\mu}+2 = 2$ in the sum (5.16). Given the nonzero rank of the bundle in (5.17), the assumption of the proposition is that $\text{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\tilde{\mu}+2}) = 1$. We will derive a contradiction by deducing from the rank one classification results in Lemma 5.1 that such a bundle has rank larger than one. We compute, $\lambda_1 + \lambda_2 + \lambda_3 + 2 = 2(\ell + \tilde{s}) + 2 = 2(\ell - \ell/2 + 1)$. Then $\hat{s} = -\ell/2 + 1$ is the parameter appearing in Lemma 5.1 with the bundle $\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\tilde{\mu}+2}$. Since $\lambda_3 = \frac{\ell}{2}$ is the largest weight in $\{\lambda_1, \lambda_2, \lambda_3, 2\}$, it follows from Lemma 5.1 that $\text{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,2}) > 1$.

Case 3: $0 \leq \tilde{s} < s$.

Here, the same contraction from the $n = 5$ argument is deduced. Specifically, we obtain $\mu = \ell = \tilde{\mu}$ which contradicts $\tilde{\mu} < \mu$.

It follows that the sum in (5.16) reduces to one term,

$$\text{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \text{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\mu})\,\text{rk}(\mathbb{V}_{\lambda_4,\lambda_5,\lambda_6,\ldots,\lambda_{n+1},\mu}). \tag{5.23}$$

We must show that this term is one. Particularly, since we are assuming $\text{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\mu}) = 1$, we must show

$$\text{rk}(\mathbb{V}_{\lambda_4,\lambda_5,\lambda_6,\ldots,\lambda_{n+1},\mu}) = 1.$$

We proceed by showing the bundle $\mathbb{V}_{\lambda_4,\lambda_5,\lambda_6,\ldots,\lambda_{n+1},\mu}$ satisfies the assumptions in the Proposition. Since this bundle now has $n - 1 < n$ weights, by our inductive assumption, we will conclude this bundle has rank one.

Let $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 = \{4, 5, ..., n+1\}$ be any partition. We want to show that the four weight bundles appearing in the rank factorization sum with the bundle $\mathbb{V}_{\lambda_4,\lambda_5,\lambda_6,\ldots,\lambda_{n+1},\mu}$ and partition $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$ have rank one.

We begin by computing the rank of the original bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ using the partition $[n] = \{1,2,3\} \sqcup \{I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4\}$. In the following $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ and $\lambda_{I_i} = \{\lambda_j\}_{j \in I_i}$. This gives,

$$\begin{aligned}
\operatorname{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) &= \sum_{\nu \in P_\ell(\mathfrak{sl}_2)} \operatorname{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\nu}) \operatorname{rk}(\mathbb{V}_{\lambda_4,\lambda_5,\lambda_6,\ldots,\lambda_{n+1},\nu}) \\
&= \sum_{\nu \in P_\ell(\mathfrak{sl}_2)} \operatorname{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\nu}) \Big(\sum_{\vec{\mu} \in P_\ell(\mathfrak{sl}_2)^4} \operatorname{rk}(\mathbb{V}_{\vec{\mu}}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_1},\nu,\mu_1}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_2},\mu_2}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_3},\mu_3}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_4},\mu_4}) \Big) \\
&= \sum_{\nu \in P_\ell(\mathfrak{sl}_2)} \sum_{\vec{\mu} \in P_\ell(\mathfrak{sl}_2)^4} \operatorname{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\nu}) \operatorname{rk}(\mathbb{V}_{\vec{\mu}}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_1},\nu,\mu_1}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_2},\mu_2}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_3},\mu_3}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_4},\mu_4}) \\
&= \sum_{\vec{\mu} \in P_\ell(\mathfrak{sl}_2)^4} \sum_{\nu \in P_\ell(\mathfrak{sl}_2)} \operatorname{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\nu}) \operatorname{rk}(\mathbb{V}_{\vec{\mu}}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_1},\nu,\mu_1}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_2},\mu_2}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_3},\mu_3}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_4},\mu_4}) \\
&= \sum_{\vec{\mu} \in P_\ell(\mathfrak{sl}_2)^4} \operatorname{rk}(\mathbb{V}_{\vec{\mu}}) \Big(\sum_{\nu \in P_\ell(\mathfrak{sl}_2)} \operatorname{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\nu}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_1},\nu,\mu_1}) \Big) \operatorname{rk}(\mathbb{V}_{\lambda_{I_2},\mu_2}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_3},\mu_3}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_4},\mu_4}) \\
&= \sum_{\vec{\mu} \in P_\ell(\mathfrak{sl}_2)^4} \operatorname{rk}(\mathbb{V}_{\vec{\mu}}) \operatorname{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\lambda_{I_1},\mu_1}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_2},\mu_2}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_3},\mu_3}) \operatorname{rk}(\mathbb{V}_{\lambda_{I_4},\mu_4}).
\end{aligned}$$

The last line follows from factorizing $\operatorname{rk}(\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3,\lambda_{I_1},\mu_1})$ with the partition of weights $\{\lambda_1, \lambda_2, \lambda_3\} \sqcup \{\lambda_{I_1}, \mu_1\}$ in reverse.

The assumption of the proposition is that all ranks $\operatorname{rk}(\mathbb{V}_{\vec{\mu}})$ appearing in this sum are one or zero. Since the partition $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$ was an arbitrary partition of $\{4, 5, 6..., n+1\}$, we can conclude that the assumption of the proposition is satisfied for the $n-2$ weight bundle $\mathbb{V}_{\lambda_4,\lambda_5,\lambda_6,\ldots,\lambda_{n+1},\mu}$. By our inductive assumption, the rank of this bundle is one. □

Remark 5.9. In Observation 5.2 we discussed the converse of Proposition 5.7. Specifically, for a vector bundle of rank one, when we compute the rank using factorization along any partition of $\{1, ..., n\}$ determined by an F-curve, the sum in the factorization formula is one term equal to one.

We summarize our results of this section with the proof of our main result.

Proof of Proposition 1.8. Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be defined for a fixed integer ℓ and n-tuple $\vec{\lambda}$. Observation 5.2 and Proposition 5.7 show that such bundles have rank one if and only if the rank calculated by factorizing along the partition of the n weights determined by any F-curve has rank one on the four pointed bundles associated to the four attaching weights. By

Lemma 5.4, the degrees of four pointed bundles $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ are equal if and only if the corresponding bundles have rank one or zero, and otherwise, by Corollary 3.24 the degree term of $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ is larger. Hence, such bundles will have equal intersection on every F-curve if and only if the bundles have rank one or zero. □

5.3 Examples of divisor identities

Here we give examples illustrating Theorem 1.8 and emphasize the necessity of the rank one condition in Theorem 1.8. In the first two examples, the bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ has certain properties described in previous work, however the first Chern classes $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ and $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1))$ are not linearly equivalent. Particularly, Example 5.10 illustrates an $\mathfrak{sl}_2$ bundle that has *projective rank scaling* ([10, Def. 2.16]) and in Example 5.11 the level (or Lie rank) is above the critical level for $\mathfrak{sl}_2$ (see Remark 6.2). In Example 5.13 we illustrate two rank one bundles with linearly equivalent divisors classes (see proof of Proposition 5.16).

Example 5.10. Let $\ell = 5$ and weights be given by $\vec{\lambda} = (4, 4, 4, 4)$. Consider the bundles:

$$\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) = \mathbb{V}(\mathfrak{sl}_2, (4, 4, 4, 4), 5) \quad \text{and} \quad \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1) = \mathbb{V}(\mathfrak{sp}_{2\cdot 5}, (4, 4, 4, 4), 1).$$

We have that $|\vec{\lambda}| = 12 = 2(5+1)$ (showing that $\ell = 5$ is at the critical level for $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$). The ranks of these bundles are, $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = 2$. However, the degree formulas in Section 3.9 give $\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 6$ while $\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = 7$.

Note that in Example 5.10 since $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 2$, the bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ is said to have *projective rank scaling* [10, Def. 2.16]. This example shows that such divisors are not linearly equivalent with the corresponding $\mathfrak{sp}_{2\ell}$ bundle at level one.

Example 5.11. Let $\ell = 5$ and $\vec{\lambda} = (2, 2, 1, 1)$. Consider the bundles:

$$\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) = \mathbb{V}(\mathfrak{sl}_2, (2, 2, 1, 1), 5) \quad \text{and} \quad \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1) = \mathbb{V}(\mathfrak{sp}_{2\cdot 5}, (2, 2, 1, 1), 1).$$

Computing ranks gives (for example, by using the `Macaulay2` package [43]), $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = 2$. Furthermore, we have that $|\vec{\lambda}| = 6 = 2(5-2)$ and so ℓ is above the critical level (or stabilizing Lie rank). Using the formulas in Lemmas 3.19 and 3.20, we obtain that $\deg(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = 0$ and $\deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = 1$.

Example 5.12. Let $\ell = 5$ and $\vec{\lambda} = (4, 4, 4, 4, 3, 3)$. Consider the bundles,

$$\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) = \mathbb{V}(\mathfrak{sl}_2, (4, 4, 4, 4, 3, 3), 5) \quad \text{and} \quad \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1) = \mathbb{V}(\mathfrak{sp}_{2\cdot 5}, (4, 4, 4, 4, 3, 3), 1).$$

Computing ranks (for example, by using the `Macaulay2` package [43]), we have $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = 2$. Using [31], a `Macaulay2` code to compute the divisor class of a vector bundle of conformal blocks on $\overline{M}_{0,6}$, we can explicitly write the divisor of each bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ in the nonadjacent basis of $\mathrm{Pic}(\overline{M}_{0,6})$, the divisor class group of $\overline{M}_{0,6}$, in terms of boundary divisors (see [36, Example 4.4]). For a subset $I \subset [6]$ such that $|I| \geq 2$, the divisor class δ_I is the closure of the collection of points in $\overline{M}_{0,6}$ represented by a curve with one node, separating the marked points corresponding to I on one irreducible component and the marked points corresponding to I^c on the other irreducible component. We give the coordinates of each divisor class with the basis ordered as $\{\delta_{13}, \delta_{14}, \delta_{15}, \delta_{24}, \delta_{25}, \delta_{26}, \delta_{35}, \delta_{36}, \delta_{46}, \delta_{124}, \delta_{125}, \delta_{134}, \delta_{135}, \delta_{136}, \delta_{145}, \delta_{146}\}$. This shows the divisors are not linearly equivalent.

This computation gives,

$$c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = (12, 6, 12, 12, 6, 12, 12, 0, 12, 2, 2, 6, 24, 2, 2, 6)$$

and

$$c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = (14, 8, 14, 14, 8, 14, 14, 3, 14, 4, 4, 8, 28, 4, 4, 8).$$

We end with an example illustrating two bundles of rank one which by Theorem 1.8 do have linearly equivalent first Chern classes.

Example 5.13. Let $\ell = 5$ and $\vec{\lambda} = (5, 5, 5, 3, 1, 1)$. Consider the bundles,

$$\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) = \mathbb{V}(\mathfrak{sl}_2, (5, 5, 5, 3, 1, 1), 5) \quad \text{and} \quad \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1) = \mathbb{V}(\mathfrak{sp}_{2\cdot 5}, (5, 5, 5, 3, 1, 1), 1).$$

Computing ranks, we have $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = 1$. We can explicitly write each divisor in the nonadjacent basis of $\mathrm{Pic}(\overline{M}_{0,6})$ as in Example 5.12. This gives equality

$$c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = (5,0,0,3,1,1,0,0,0,0,0,2,4,4,0,0).$$

5.4 Generalized Veronese quotients and maps given by rank one $\mathfrak{sl}_2$ and $\mathfrak{sp}_{2\ell}$ bundles

There are birational models of $\overline{M}_{0,n}$ given by so called *generalized Veronese quotients*, $V^d_{\gamma,\mathcal{A}}$. These projective varieties parametrize configurations of n weighted points lying on (limits of) weighted Veronese curves of degree d in projective d space. They were first constructed in [21] with S_n-invariant weights $\mathcal{A}$ on the n marked points and weight $\gamma = 0$ on the underlying curve. They were later generalized in [22, 23]. These moduli spaces receive birational morphisms from $\overline{M}_{0,n}$ and are constructed as GIT quotients generalizing Kapranov's birational model of $\overline{M}_{0,n}$ given by $(\mathbb{P}^1)^n//SL(2)$ in [29].

In the case $\gamma = 0$ the birational contractions

$$\varphi_{0,\mathcal{A}} : \overline{M}_{0,n} \to V^d_{0,\mathcal{A}}$$

are known to correspond to conformal blocks divisors in type A at level 1, $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, 1))$ [22, Thm. 3.2]. The higher level divisors $c_1(\mathbb{V}(\mathfrak{sl}_2, (\omega_1)^n, \ell))$ are also know to give contractions with $\varphi_{\frac{\ell-1}{\ell+1},\mathcal{A}}$ [23].

From Theorem 1.7 we have that for $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ of rank one we can explicitly write $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ as a sum of divisors for $\mathfrak{sl}_2$ at level one. Using this decomposition, the description of the maps from divisors at level one in [22, Thm. 3.2], and Theorem 1.8, we obtain the following result about the maps from the divisors $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ when $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 1$.

Proposition 5.14. *Let $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ and $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)$ be defined for some fixed ℓ and n-tuple $\vec{\lambda}$. Let $|\vec{\lambda}| = 2(d\ell + p)$ for some $d \geq 0$ and $\ell > p \geq 0$. If $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = 1$, then the* *contraction* $\phi_{\mathbb{D}}$ *given by* $\mathbb{D} = c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ *maps to a product of* ℓ *generalized Veronese quotients,*

$$\phi_{\mathbb{D}} : \overline{M}_{0,n} \to \prod_{i=1}^{p} V^{d}_{0,(\frac{1}{2})^{2d+2}} \times \prod_{j=p+1}^{\ell} V^{d-1}_{0,(\frac{1}{2})^{2d}}.$$

Remark 5.15. The map $\phi_{\mathbb{D}}$ first factors through a product of ℓ forgetful maps, where in each factor we have forgotten all but either $2d+2$ or $2d$ marked points. By a dimension count, the map $\phi_{\mathbb{D}}$ is not surjective to this product. Furthermore, the following result shows the map $\phi_{\mathbb{D}}$ does not factor through a smaller product of these generalized Veronese quotients when $d > 1$.

Proposition 5.16. *The map* $\phi_{\mathbb{D}}$ *in Proposition 5.14 does not factor through a smaller product of these generalized Veronese quotients when* $d > 1$.

Proof. Write $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) = \sum_{i=1}^{\ell} c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_i, 1))$, then no linear combination with fewer terms of divisor classes from $\mathfrak{sl}_2$ level one bundles is linearly equivalent to $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1)) = c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$.

Indeed, the divisors $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_i, 1))$ are such that $|\vec{\lambda}_i| = 2d+2$ or $2d$ (where d is obtained from $\vec{\lambda}$ as in Proposition 5.14). In [17, Sect. 4.2] Fakhruddin explains that such divisors are nontrivial when $d > 1$ and shows they form a basis of the Picard group $\mathrm{Pic}(\overline{M}_{0,n})$ [17, Thm. 4.3]. Hence, there is only one way to write $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ in this basis. □

Example 5.17. Let $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1) = \mathbb{V}(\mathfrak{sp}_{2\cdot 9}, (\omega_9, \omega_8, \omega_8, \omega_8, \omega_8, \omega_8, \omega_2, \omega_1), 1)$. It was shown in [26, Example 5.3] that the bundle $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) = \mathbb{V}(\mathfrak{sl}_2, (9\omega_1, 8\omega_1, 8\omega_1, 8\omega_1, 8\omega_1, 8\omega_1, 8\omega_1, 2\omega_1, 1\omega_1), 9)$ has rank one and the first Chern class decomposes as a sum of first Chern classes of $\mathfrak{sl}_2$ bundles at level 1. That is, $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ decomposes into a sum

$$c_1(V_{(1,1,1,1,1,1,1,1,0)}) + c_1(V_{(1,1,1,1,1,1,1,1,0}) + c_1(V_{(1,1,1,1,1,1,1,0,1)}) +$$

$$c_1(V_{(1,1,1,1,1,1,0,0,0)}) + c_1(V_{(1,1,1,1,1,0,1,0,0)}) + c_1(V_{(1,1,1,1,0,1,1,0,0)}) +$$

$$c_1(V_{(1,1,1,0,1,1,1,0,0)}) + c_1(V_{(1,1,0,1,1,1,1,0,0)}) + c_1(V_{(1,0,1,1,1,1,1,0,0)}).$$

The subscript of these bundles denotes the weights. For example, $V_{(1,1,1,1,1,0,1,0,0)}$ is the $\mathfrak{sl}_2$ bundle at level one and weights $(\omega_1, \omega_1, \omega_1, \omega_1, \omega_1, 0, \omega_1, 0, 0)$. Such a decomposition is seen from the column data of the unique tableau formed to compute $\mathrm{rank}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ (see Figure 5.1). In this tableau, each column corresponds to a level one first Chern class in the sum. The image of such a map is then into a product of generalized Veronese quotients as in Claim 5.14 where in each component, we have forgotten the zero weights determined by the corresponding level one first Chern class in the decomposition. By our result of Proposition 1.8, the first Chern class $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$ is linearly equivalent to $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}^T, 1))$ and the corresponding maps contract the same F-curves.

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	3
3	3	3	3	3	3	3	4	4
4	4	4	5	4	4	5	5	5
5	5	5	5	5	6	6	6	6
6	6	6	6	7	7	7	7	7
7	7	7						
8	8	9						

Figure 5.1: The unique tableau determining $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell))$

Chapter 6

Rank and Divisor Identities of Bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$

We now investigate bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ with $\mathfrak{sp}_{2\ell}$ at level one. In this chapter we prove Theorem 1.9 which states that the first Chern classes $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1))$ become fixed when ℓ is taken large enough. In Section 6.1 we explicitly define the rank ℓ above which such divisors become fixed. In Section 6.3 we prove Theorem 1.9 by first showing in Section 6.2 that ranks of bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ strictly increase when the Lie algebra rank ℓ increases in a certain range (Proposition 6.5). We provide several examples in Section 6.4 to show linear equivalence of certain divisors $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ for various Lie algebra ranks ℓ. We end this chapter by stating several consequences of our results.

6.1 Stabilizing Lie rank

We now define a term and divisor class that will be used to describe the relationship between divisor classes associated to $\mathfrak{sp}_{2\ell}$ at level one when ℓ is taken large enough.

Definition 6.1. Let $\vec{\lambda} = (a_1, ..., a_n)$ be an n-tuple of weakly decreasing nonnegative integers such that $|\vec{\lambda}| = \sum_{i=1}^n a_i$ is even. We define the *stabilizing Lie rank* associated to $\vec{\lambda}$ to be,

$$r(\vec{\lambda}) = \frac{|\vec{\lambda}|}{2} - 1.$$

If $r(\vec{\lambda}) \geq a_1$ (i.e., $\vec{\lambda} \in P_1(\mathfrak{sp}_{2r(\vec{\lambda})})^n$), then we call $c_1(\mathbb{V}(\mathfrak{sp}_{2r(\vec{\lambda})}, \vec{\lambda}, 1))$ the *stable Lie divisor* for $\vec{\lambda}$.

Remark 6.2. Interpreting $\vec{\lambda} = (a_1, ..., a_n)$ in Definition 6.1 as a set of n dominant integral weights for $\mathfrak{sl}_2$, the value of $r(\vec{\lambda})$ in Definition 6.1 is called the *critical level* of $\vec{\lambda}$ [17, Sect. 4.3] (see also [11, Def. 1.1] for a general definition of critical level associated to bundles of type $\mathfrak{sl}_{r+1}$). Additionally, one can show that if $a_1 > r(\vec{\lambda})$ then $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{2r}, \vec{\lambda}, 1)) = 0$ (and hence the divisor is trivial) for any integer r. Because of this, we will assume the weight vector $\vec{\lambda}$ is such that $a_1 \leq r(\vec{\lambda})$.

Recall from Section 3.2, that we refer to the rank ℓ, of the Lie algebra $\mathfrak{sp}_{2\ell}$, as the *Lie rank* and the rank of the vector bundle $\mathbb{V}_{\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1}$ as the *vector bundle rank*.

6.2 Ranks of $\mathfrak{sl}_2$ bundles below critical level

We now go through a brief interlude to show several results on ranks of conformal blocks bundles for $\mathfrak{sl}_2$ with a fixed set of weights as the level defining the bundle varies. By Fact 3.2 our results in this section are relevant for $\mathfrak{sp}_{2\ell}$ bundles at level one when the Lie algebra rank ℓ increases within a certain range. We use the results of this discussion to prove Theorem 1.9 in Section 6.3.

Throughout this section, we fix an integer $\ell \geq 0$ and a vector of weakly decreasing integers $\vec{\lambda} = (a_1, a_2, ..., a_n)$ with $n \geq 4$. We denote $r(\vec{\lambda})$ the *stabilizing Lie rank* (or the *critical level* for $\mathfrak{sl}_2$) associated to this fixed $\vec{\lambda}$ (see Definition 6.1 and Remark 6.2).

6.2.1 Ranks of bundles at varying levels for $\mathfrak{sl}_2$ and $n = 4$

To compute the rank of $\mathbb{V}(\mathfrak{sl}_2, (a_1, a_2, a_3, a_4), \ell)$ we use the formula in Lemma 3.4.

Lemma 6.3. Fix some $\ell \geq 0$ and $\vec{\lambda} = (a_1, a_2, a_3, a_4)$ and $r(\vec{\lambda})$ as in Definition 6.1. Let t be some integer such that $\ell = r(\vec{\lambda}) + 1 - t$. We have the following rank relationships:

1. If $t > 0$, then

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1 - t)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1)) - t.$$

2. If $t \leq 0$, then the vector bundle rank becomes fixed,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1 - t)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1)).$$

Remark 6.4. In the first case with $t > 0$, we also say that ℓ is at or below the critical level for $\vec{\lambda}$. In the second case with $t \geq 0$ we say that ℓ is above the critical level for $\vec{\lambda}$.

Proof. This result follows immediately from computing the ranks $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1 - t))$ and $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1))$ in the formula in Lemma 3.4. Particularly, when $t > 0$, we have $a_1 + a_2 + a_3 + a_4 = 2(\ell + s)$ with $s > 0$ and when $t \leq 0$ we have $a_1 + a_2 + a_3 + a_4 = 2(\ell + s)$ with $s \leq 0$. These cases of s determine the formula to follow in Lemma 3.4. In either case, the comparison of the rank of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1 + t)$ with the rank of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1)$ follows.

□

6.2.2 Ranks of bundles at varying levels for $\mathfrak{sl}_2$ and $n \geq 4$

We now show that the ranks of $\mathfrak{sl}_2$ bundles with a fixed, arbitrary number of weights strictly increases or becomes fixed when the level is increased. We show explicitly the range of the level for which the rank increases.

Proposition 6.5. *Let t be some integer such that $\ell = r(\vec{\lambda}) + 1 - t$. We have the following rank relationships:*

1. *If $t > 0$, then*
$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) < \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1)).$$

2. *If $t \leq 0$, then*
$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell) = \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, r(\vec{\lambda}) + 1)).$$

Remark 6.6. We can make the same remark Remark 6.4. That is, in the first case with $t > 0$, we also say that ℓ is at or below the critical level for $\vec{\lambda}$. In the second case with $t \leq 0$ we say that ℓ is above the critical level for $\vec{\lambda}$.

Proof. We show the first case by induction on the number of weights. The second case will follow from Witten's Dictionary (Proposition 3.15).

Recall, in this section, we are assuming $n \geq 4$. In Lemma 6.3 we showed for $n = 4$ the conclusion follows. For any integer $4 \leq k \leq n$, define the k-tuple of weights $\vec{\lambda}_k := (a_1, \ldots, a_k)$ and integer $r(\vec{\lambda})(k) := \sum_{i=1}^{k} a_i/2 - 1$. For our inductive assumption, we assume that for any $k < n$ and $\ell = r(\vec{\lambda})(k) + 1 - t$, if $t > 0$ then we have a strict inequality of ranks,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_k, r(\vec{\lambda})(k) + 1 - t)) < \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_k, \ell + 1)). \tag{6.1}$$

We want to show this relationship is true for $k = n$. Let $\ell = r(\vec{\lambda})(n) + 1 - t$ for some $t > 0$. Using factorization with the partition $\{1, ..., n-2\} \sqcup \{n-1, n\}$ we can compute the following ranks,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_n, \ell)) = \sum_{\mu \in P_\ell(\mathfrak{sl}_2)} \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (\vec{\lambda}_{n-2}, \mu), \ell))\, \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_{n-1}, a_n, \mu), \ell)) \text{ and} \tag{6.2}$$

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_n, \ell + 1)) = \sum_{\mu \in P_{\ell+1}(\mathfrak{sl}_2)} \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (\vec{\lambda}_{n-2}, \mu), \ell + 1))\, \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_{n-1}, a_n, \mu), \ell + 1)), \tag{6.3}$$

We now compare each term in the sums (6.2) and (6.3).

First observe that if $\mu \in P_\ell(\mathfrak{sl}_2)$, then $\mu \in P_{\ell+1}(\mathfrak{sl}_2)$ and so a weight μ that appears in the sum (6.2) will also appear in the sum (6.3). Note, that the value of the ranks may be zero. However, with our inductive assumption, we have the following relationship between these terms,

$$\begin{aligned} &\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (\vec{\lambda}_{n-2}, \mu), \ell))\, \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_{n-1}, a_n, \mu), \ell)) \\ &\qquad \leq \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (\vec{\lambda}_{n-2}, \mu), \ell + 1))\, \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_{n-1}, a_n, \mu), \ell + 1)). \end{aligned} \tag{6.4}$$

It follows immediately that,

$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_n, \ell)) \leq \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}_n, \ell + 1)). \tag{6.5}$$

We want to show that such a relationship is strict.

Consider the weight vector $(\vec{\lambda}_{n-2}, \mu)$ from (6.2). The critical level for this weight vector is $r(\vec{\lambda}_{n-2}, \mu)(n-1)$ (i.e., stabilizing Lie rank for $(\vec{\lambda}_{n-2}, \mu)$). From our inductive assumption, we have that the relationship in (6.4) is strict whenever $\ell = r(\vec{\lambda}_{n-2}, \mu)(n-1) + 1 - t'$ for some t' with $t' > 0$.

Making the substitution $r(\vec{\lambda}_{n-2}, \mu)(n-1) = (\sum_{i=1}^{n-2} a_i + \mu)/2 - 1$, we can restate this condition to be that the relationship is strict whenever $\ell \leq (\sum_{i=1}^{n-2} a_i + \mu)/2$. Hence, the inequality in (6.5) is strict and our conclusion follows whenever we have at least one μ in the sum (6.2) such that $\ell \leq (\sum_{i=1}^{n-2} a_i + \mu)/2$.

Suppose then that each μ in the sum (6.2) is such that

$$\ell > (\sum_{i=1}^{n-2} a_i + \mu)/2. \tag{6.6}$$

In this case, just comparing terms appearing in (6.2) and (6.3) from $\mu \in P_\ell(\mathfrak{sl}_2)$ does not guarantee an increase in ranks between each term in the sum. In such a situation, we show that the sum (6.3) includes an additional nonzero term not in the sum (6.2) and can thus conclude a strict inequality.

Suppose μ is the largest weight appearing as a nonzero term in (6.2). Using (6.6) and recalling we have $\ell = r(\vec{\lambda})(n) + 1 - t$ with $t > 0$, we obtain,

$$\sum_{i=1}^{n-2} a_i + \mu < 2\ell \leq 2(\ell + t) = 2(r(\vec{\lambda})(n) + 1) = \sum_{i=1}^{n} a_i. \tag{6.7}$$

From this, two relationships follow,

$$\mu < a_{n-1} + a_n, \tag{6.8}$$

$$a_{n-1} + a_n + \mu < 2\ell. \tag{6.9}$$

Here (6.9) follows from the assumption that the weights are weakly decreasing, $a_1 \geq ... \geq a_{n-2} \geq a_{n-1} \geq a_n$. Combining (6.8) and (6.9), we obtain a strict inequality, $\mu < \ell$, from which the weak relationship follows, $\mu + 2 \leq \ell + 1$. Hence, $\mu + 2 \in P_{\ell+1}(\mathfrak{sl}_2)$ and appears as a possible weight for a term in the sum (6.3). Furthermore, comparing (6.8) and (6.9) with the three point fusion rules (3.4), we can conclude $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_{n-1}, a_n, \mu + 2), \ell + 1)) = 1$ so is nonzero in the sum (6.2). We must now consider $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_1, ..., a_{n-2}, \mu + 2), \ell + 1))$.

By (6.6), assuming μ is the largest weight in a nonzero term in the sum (6.2), we must have $\sum_{i=1}^{n-2} a_i + \mu = 2p$ for some $p < \ell$ (this condition on p follows from (6.7)). And so also $\sum_{i=1}^{n-2} a_i +$

$\mu + 2 = 2(p + 1)$. Now, since $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_1, ..., a_{n-2}, \mu), \ell)) > 0$, it follows from the nonzero rank condition for $\mathfrak{sl}_2$ bundles in Theorem 1.5 that $\sum_{i=2}^{n-2} a_i + \mu \geq p$ and so also $\sum_{i=2}^{n-2} a_i + \mu + 2 \geq p + 1$. Again, using the result of Theorem 1.5, we can conclude $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_{n-1}, a_n, \mu + 2), \ell + 1)) > 0$.

Recall, that all other weights $\mu \in P_\ell(\mathfrak{sl}_2)$ appearing in the sum (6.2) satisfy the weak inequality in (6.4). Since μ was assumed to be the largest weight in (6.2), the weight $\mu + 2$ does not appear in the rank calculation for level ℓ in (6.2) but does contribute a new nonzero term in the sum (6.3). This allows us to conclude the relationship in (6.5) is strict. This concludes the first case of the Proposition with $t < 0$.

In the second case, with $\ell = r(\vec{\lambda}) + 1 - t$ and $t \leq 0$ the rank computation using Witten's Dictionary (Proposition 3.15) is the same calculation for all t in this range. This shows the equality of the ranks. This same method was used in the proof of vanishing above critical level showed in [11, Section 4]. □

Remark 6.7. Work by Alex Yong, Anders Buch and others has resulted in bounds on the structure constants for the product of Schubert classes (e.g., [41, 15, 50]). However, such results compare values within a fixed quantum cohomology ring of the Grassmannian. The result of Proposition 6.5 describes the behavior of structure constants appearing in products of Schubert classes across different rings. That is, the coefficients we analyze appear in products of cohomology classes living in the cohomology rings of Grassmannian varieties with varying parameters.

In Section 6.4, we show examples of ranks of bundles with varying level to demonstrate this rank behavior.

6.3 Proof of Proposition 1.9

To prove Proposition 1.9 we show the bundles in the statement have equal intersection with any F-curve. To make this comparison, we first establish the result for bundles on $\overline{M}_{0,4}$. We use notation as in the degree formula of Lemma 3.20.

Lemma 6.8. For a fixed level ℓ and $\vec{\lambda} = (a, b, c, d)$, let $r(\vec{\lambda})$ be the stabilizing Lie rank and let t be some integer such that $\ell + t = r(\vec{\lambda}) + 1$. If $t \leq 1$ then we have equality,

$$\deg(\mathbb{V}(\mathfrak{sp}_{2r(\vec{\lambda})}, \vec{\lambda}, 1)) = \deg(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)).$$

This result follows immediately from comparing the formulas in Lemma 3.20. Using the language of the stabilizing Lie rank, this lemma says that the degrees of divisors with $\mathfrak{sp}_{2\ell}$ at level one with four weights become equal when the Lie rank ℓ is chosen to be at or above the stabilizing Lie rank for the weight vector $\vec{\lambda}$. We are now ready to prove this result for an arbitrary number of weights.

Proof of Proposition 1.9. Let $\vec{\lambda} = (a_1, ..., a_n)$ be an n-tuple of weakly decreasing integers such that $|\vec{\lambda}| = \sum_{i=1}^{n} a_i$ is even. Using Definition 6.1, the stabilizing Lie rank is the integer $r(\vec{\lambda})$ such that $|\vec{\lambda}| = 2(r(\vec{\lambda}) + 1)$. Now suppose ℓ is some integer such that $\ell \geq a_1$ and $\ell > r(\vec{\lambda})$. Thus, we can write $|\vec{\lambda}| = 2(\ell + t)$ with $t \leq 1$ (i.e., $\ell \geq r(\vec{\lambda})$ as specified by the proposition statement). We denote $\mathbb{V}_{\vec{\lambda},\ell} := \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ to simply notation throughout this section. We want to show linear equivalence,

$$c_1(\mathbb{V}_{\vec{\lambda},r(\vec{\lambda})}) = c_1(\mathbb{V}_{\vec{\lambda},\ell}).$$

We compare the intersection numbers of these two bundles with an arbitrary F-curve, F_{I_1,I_2,I_3,I_4}, determined by a partition $\{1, ..., n\} = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$. The formula in Proposition 3.17 (from [17, Prop. 2.7]) provides the following degree computations,

$$\deg(\mathbb{V}_{\vec{\lambda},r(\vec{\lambda})}|_{F_{I_1,I_2,I_3,I_4}}) = \sum_{\vec{\mu} \in P_1(\mathfrak{sl}_{2r(\vec{\lambda})})^4} \deg(\mathbb{V}_{\vec{\mu},r(\vec{\lambda})}) \prod_{i=1}^{4} \mathrm{rk}(\mathbb{V}_{(\lambda_{I_i},\mu_i),r(\vec{\lambda})}) \tag{6.10}$$

$$\deg(\mathbb{V}_{\vec{\lambda},\ell}|F_{I_1,I_2,I_3,I_4}) = \sum_{\vec{\nu} \in P_1(\mathfrak{sl}_{2\ell})^4} \deg(\mathbb{V}_{\vec{\nu},\ell}) \prod_{i=1}^{4} \mathrm{rk}(\mathbb{V}_{(\lambda_{I_i},\nu_i),\ell}) \tag{6.11}$$

where λ_{I_j} denotes the weight vector with weights a_i for $i \in I_j$ and $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ and $\vec{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4)$ denote the attaching weight vectors.

First, we show that the attaching weights $\vec{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4)$ appearing in the degree term of (6.11) are all such that $\nu_i \leq r(\vec{\lambda})$. This will allow us to take the above sums over the same set of integers.

Consider all possible terms in (6.11) and let $|I_i| := \sum_{j \in I_i} a_j$, the sum of just those weights appearing in a partition determined by I_i. By the the Generalized Triangle Inequality for ranks of $\mathfrak{sl}_2$ bundles (Proposition 3.5) and Fact 3.2, in order for the rank of $\mathbb{V}_{(\lambda_{I_i}, \nu_i), \ell}$ to be nonzero it is necessary that

$$\nu_i \leq |I_i|. \tag{6.12}$$

Thus, in order for a term in (6.11) to be nonzero, it is necessary that this condition holds for $i = 1, 2, 3, 4$. Adding all such inequalities gives,

$$\sum_{i=1}^{4} \nu_i \leq \sum_{i=1}^{4} |I_i| = \sum_{i=1}^{n} a_i = 2(r(\vec{\lambda}) + 1). \tag{6.13}$$

Suppose for some weight $\nu_k \in \vec{\nu}$, we had $r(\vec{\lambda}) + 1 < \nu_k$; it would follow that

$$\sum_{i=1}^{4} \nu_i \leq \sum_{i=1}^{4} |I_i| = 2(r(\vec{\lambda}) + 1) < 2\nu_k.$$

Canceling ν_k from this inequality would imply $\sum_{i \in \{1,2,3,4\} - \{k\}} \nu_i < \nu_k$. So by the Generalized Triangle Inequality with these weights $\text{rk}(\mathbb{V}_{\vec{\nu}, \ell}) = 0$. This implies the degree, $\deg(\mathbb{V}_{\vec{\nu}, \ell})$, is zero. This shows that all nonzero terms in (6.11) have attaching data $\vec{\nu}$ such that each $\nu_i \leq r(\vec{\lambda}) + 1$. We need to check that in fact this inequality is strict so that all attaching data in (6.11) are such that $\nu_i \leq r(\vec{\lambda})$. For contradiction, assume for some k, $\nu_k = r(\vec{\lambda}) + 1$, we compute $\deg(\mathbb{V}_{\vec{\nu}, \ell})$. Using (3.20), it follows immediately that $\deg(\mathbb{V}_{\vec{\nu}, \ell}) = 0$.

We can thus assume that all nonzero terms appearing in the sums (6.10) and (6.11) have attaching weights $\vec{\mu}$ and $\vec{\nu}$ with $\mu_i, \nu_i \leq r(\vec{\lambda})$. Particularly, we can assume corresponding terms in these sums have the same attaching data. To finish the proof, we compare corresponding terms in each sum and show they are equal.

First, we compare rank factors. Consider the following ranks,

$$\mathrm{rk}(\mathbb{V}_{(\lambda_{I_i},\mu_i),r(\vec{\lambda})}) \quad \text{and} \quad \mathrm{rk}(\mathbb{V}_{(\lambda_{I_i},\mu_i),\ell}). \tag{6.14}$$

Define σ_i to be the integer such that $2\sigma_i := |\lambda_{I_i}| + \mu_i$, then from Proposition 6.5, we see that these ranks will be equal whenever we have $\sigma_i \leq r(\vec{\lambda})$. So suppose $r(\vec{\lambda}) < \sigma_k$ for some $k = 1, 2, 3$, or 4. Without loss of generality (and clarity in the following argument) we assume $k = 1$. From this, it follows

$$\sum_{i=1}^{4} |\lambda_{I_i}| = 2(r(\vec{\lambda}) + 1) \leq 2\sigma_1 = |\lambda_{I_1}| + \mu_1,$$

(where the strict inequality of our assumption provides the weak inequality $r(\vec{\lambda})+1 \leq \sigma_1$). Canceling $|\lambda_{I_1}|$ gives the relationship, $\sum_{i=2}^{4} |\lambda_{I_i}| \leq \mu_1$.

Using this relationship, (6.13), and the Generalized Triangle Inequality for $\mathrm{rk}(\mathbb{V}_{\vec{\mu},\ell})$ we have,

$$\mu_2 + \mu_3 + \mu_4 \leq \sum_{i=2}^{4} |\lambda_{I_i}| \leq \mu_1 \leq \mu_2 + \mu_3 + \mu_4.$$

The equality follows,

$$\mu_1 = \mu_2 + \mu_3 + \mu_4. \tag{6.15}$$

Consider the degree factors $\deg(\mathbb{V}_{\vec{\mu},r(\vec{\lambda})})$ and $\deg(\mathbb{V}_{\vec{\mu},\ell})$ appearing in (6.10) and (6.11) for the attaching weight μ as in (6.15). By Lemma 3.20 the degree is zero. Hence, terms in the sums associated to such $\vec{\mu}$ are also zero. Thus, we can always assume $\sigma_i \leq r(\vec{\lambda})$ and thus rank factors are equal.

We now compare the corresponding degree factors in each sum (6.10) and (6.11). In order for the product of rank factors (in either sum) to not necessarily be zero, the relationship in (6.13) must be satisfied (using notation $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ for the attaching weights). From this relationship and Lemma 6.8 it follows that the four pointed degree factors are equal.

We can now conclude that the terms appearing in (6.10) and (6.11) are always equal, concluding the proposition. □

It was shown for $\mathfrak{sl}_2$, that for $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ to be nontrivial is equivalent to $\ell \leq r(\vec{\lambda})$ and $0 < \mathrm{rk}(\mathbb{V})$ (see [11, Prop. 1.3]). Considering the degree formula for $\mathfrak{sp}_{2\ell}$ divisors in Lemma 3.20, the

Table 6.1: Divisors and ranks for $\mathbb{V}_\ell = \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ and varying ℓ

ℓ	$c_1(\mathbb{V}_\ell)$	$\mathrm{rk}(\mathbb{V}_\ell)$
5	$(7, 1, 1, 5, 2, 2, 1, 1, 1, 1, 1, 3, 7, 6, 1, 1)$	3
6	$(11, 4, 2, 9, 4, 4, 3, 3, 2, 4, 2, 6, 12, 10, 3, 3)$	7
7	$(12, 5, 3, 10, 5, 5, 4, 4, 3, 5, 3, 7, 14, 11, 4, 4)$	10
8	$(12, 5, 3, 10, 5, 5, 4, 4, 3, 5, 3, 7, 14, 11, 4, 4)$	11
9	$(12, 5, 3, 10, 5, 5, 4, 4, 3, 5, 3, 7, 14, 11, 4, 4)$	11
10	$(12, 5, 3, 10, 5, 5, 4, 4, 3, 5, 3, 7, 14, 11, 4, 4)$	11

nontrivality of $\mathfrak{sp}_{2r}$ divisors above stabilizing Lie rank follows whenever the rank of the stabilizing Lie bundle is nonzero. We state this nonvanishing result explicitly below.

Corollary 6.9. Let $c_1(\mathbb{V}(\mathfrak{sp}_{2r(\vec{\lambda})}, \vec{\lambda}, 1))$ be the stable Lie divisor for a fixed n-tuple, $\vec{\lambda}$. Then if $\mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2r(\vec{\lambda})}, \vec{\lambda}, 1)) > 0$, the divisor $c_1(\mathbb{V}(\mathfrak{sp}_{2r}, \vec{\lambda}, 1))$ is nontrivial for all $r \geq r(\vec{\lambda})$.

6.4 Examples

Here we give an example to illustrate our results on ranks and first Chern classes of bundles $\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ with varying Lie (algebra) rank, ℓ. Particularly, we use this example to illustrate the rank behavior of Lemma 6.5 and the stable Lie divisor of Theorem 1.9 for bundles with $\mathfrak{sp}_{2\ell}$ and level one.

Example 6.10. Let $\vec{\lambda} = (5, 4, 3, 2, 1, 1)$. We consider the bundles, $\mathbb{V}_\ell := \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)$ for varying ℓ. Using the same ordering on the nonadjacent basis of $\mathrm{Pic}(\overline{M}_{0,6})$ as in Example 5.11, we compute the divisor class and rank of each $\mathbb{V}_\ell$. The stabilizing Lie rank for $\vec{\lambda}$ is $r(\vec{\lambda}) = 7$. By Proposition 1.9 the divisors defined at or above $r(\vec{\lambda}) = 7$ are all equal. By Lemma 6.5 the ranks of the bundles with Lie rank at or above $r(\vec{\lambda}) + 1$ are equal. This is shown from computations displayed in Table 6.1.

6.5 Consequences for conformal blocks of Type C at level one

The main propositions of our results in this chapter have several consequences to the study of understanding conformal blocks divisors in $\mathrm{Nef}(\overline{M}_{0,n})$. Specifically, we combine the results of

Proposition 1.8 with previous results related to vector bundles of conformal blocks with $\mathfrak{sl}_2$ to conclude several consequences for bundles with $\mathfrak{sp}_{2\ell}$ at level one.

The finite generation of the cone of all conformal blocks divisors in $\mathrm{Nef}(\overline{M}_{0,n})$ is an open question. This problem was considered for conformal blocks with $\mathfrak{sl}_n$ at level one in [22] and for bundles with $\mathfrak{sl}_2$ of rank one in [26]. Using these results, we are able to make the following conclusion related to conformal blocks with $\mathfrak{sp}_{2\ell}$ at level one.

Corollary 6.11. Let $\mathcal{S} := \{\mathbb{V} = \mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1) \mid \mathrm{rk}(\mathbb{V}) = 1\}$. Then

$$\mathcal{C} = convHull\{c_1(\mathbb{V}) \mid \mathbb{V} \in \mathcal{S}\},$$

is the same cone as that generated by conformal blocks divisors with $\mathfrak{sl}_2$ and rank one. Particularly, $\mathcal{C}$ is finitely generated.

Another open problem in the study of vector bundles of conformal blocks is to determine necessary and sufficient conditions for when a conformal blocks divisor is nonzero [8, Question 1]. Due to results in [17, Prop. 4.3] and [8, Cor. 3.6] we can conclude the following nonvanishing result for $\mathfrak{sp}_{2\ell}$ conformal blocks at level one.

Corollary 6.12. For a fixed integer ℓ and n-tuple of weights $\vec{\lambda} \in P_1(\mathfrak{sp}_{2\ell})^n$, let $r(\vec{\lambda})$ be the stabilizing Lie rank as in Definition 6.1. Then we have the following nonvanishing result

$$c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) \text{ is nontrivial } \Leftrightarrow \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) > 0 \text{ and } \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2r(\vec{\lambda})}, \vec{\lambda}, 1)) > 0.$$

Additionally, using the decomposition of [8, Prop. 1.2], Proposition 1.8 provides new decomposition and scaling identities for the divisors $c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1))$ with $\mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\lambda}, 1)) = 1$.

Corollary 6.13. Let $\vec{\mu} = (\omega_{a_1}, ..., \omega_{a_n})$ such that $0 < a_i \leq m$ and $\vec{\nu} = (\omega_{b_1}, ..., \omega_{b_n})$ such that $0 < b_i \leq \ell$ (so that $\vec{\mu} \in P_1(\mathfrak{sp}_{2m})^n$ and $\vec{\nu} \in P_1(\mathfrak{sp}_{2\ell})^n$). If $\mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2m}, \vec{\mu}, 1)) = \mathrm{rk}(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\nu}, 1)) = 1$ then

$$c_1(\mathbb{V}(\mathfrak{sp}_{2(m+\ell)}, (\omega_{a_1+b_1}, ..., \omega_{a_n+b_n}), 1)) = c_1(\mathbb{V}(\mathfrak{sp}_{2m}, \vec{\mu}, 1)) + c_1(\mathbb{V}(\mathfrak{sp}_{2\ell}, \vec{\nu}, 1)).$$

Iterating this result leads to the following scaling behavior.

Corollary 6.14. Define $\mathbb{V}_N := \mathbb{V}(\mathfrak{sp}_{2(N\ell)}, (\omega_{Na_1}, ..., \omega_{Na_n}), 1)$ for $(\omega_{a_1}, ..., \omega_{a_n}) \in P_1(\mathfrak{sp}_{2\ell})^n$ and $N \geq 1$. If $\mathbb{V}_1$ has rank one, then we have the following divisor identity:

$$c_1(\mathbb{V}_N) = Nc_1(\mathbb{V}_1).$$

Remark 6.15. Similar scaling behavior appears for $\mathfrak{sl}_{r+1}$ in [8, Cor. 4.6], for $\mathfrak{sl}_2$ and $\vec{\lambda} = (\omega_1, ..., \omega_1)$ in [24, Prop. 5.2], and analogous results for $\mathfrak{so}_{2r+1}$ and $\vec{\lambda} = (\omega_1, ..., \omega_1)$ in [39, Thm. 1.2].

Chapter 7

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Appendix A

Young diagrams in rank computations

In these appendices, we expand on some of the tools used in the document. Some of these examples and results were preliminary to the construction and generalization of results in this document.

A.1 Young diagrams (shapes)

In this section we expand on the construction of the Young diagram shapes in Lemma 4.8 used to compute ranks of $\mathfrak{sl}_2$ bundles.

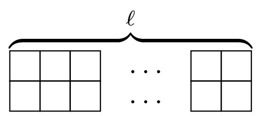
First, Witten's Dictionary (Section 3.15) gives us that $\text{rk}(\mathbb{V}(\mathfrak{sl}_2, (a_1, ..., a_n), \ell))$ is equal to the coefficient, $K^{(\ell,0)}_{(\ell,0),(a_1,...,a_n,\ell,...,\ell),s}(2, \ell)$, of the term $q^s \sigma_{(\ell,0)}$ in the following product:

$$\sigma_{a_1} * ... * \sigma_{a_n} * \sigma_\ell^{s-1} * \sigma_\ell \in QH^*(Gr(r, r + \ell)).$$

From the section on quantum Kostka numbers (Section 3.8), the desired coefficient in this product is the number of tableau with shape $\nu[s]/\lambda$, where $\nu = (\ell, \ell)$ and $\lambda = (\ell, 0)$ as in Figure A.1, and content $(a_1, ..., a_n, \ell, ..., \ell)$. This content contains our original content flavors and amounts with $s - 1$ additional flavors with ℓ amounts of each flavor. That is, we have

$$\mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell) = K^{\ell\omega_2}_{\ell\omega_1,(a_1,...,a_n,\ell,...,\ell),1}.$$

To carry out this computation, we first consider the shape $\nu[s]/\lambda$, where ν and λ are the same as in Figure A.1. We give an example to motivate the general shape.

$\nu = (\ell, \ell) =$ 

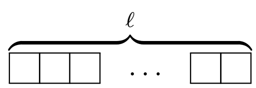
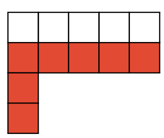
$\lambda = (\ell, 0) =$ 

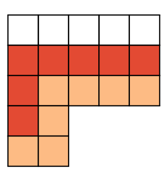
Figure A.1: Young diagrams $\nu = (\ell, \ell)$ and $\lambda = (\ell, 0)$

Example A.1. Let $\ell = 5$, we construct the tableau $\nu[s]/\lambda$ for $s = 1, ..., 6$ and ν and λ as given in Figure A.1 by adding rim hooks to ν each containing $\ell + 2 = 5 + 2 = 7$ boxes.

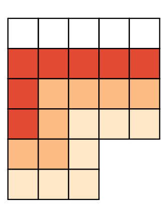
$s = 1, \nu[1]/\lambda$



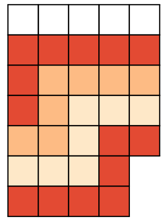
$s = 2, \nu[2]/\lambda$



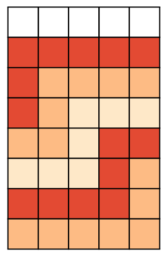
$s = 3, \nu[3]/\lambda$



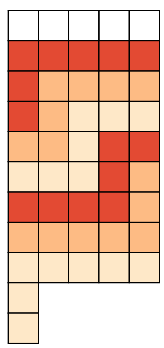
$s = 4, \nu[4]/\lambda$



$s = 5, \nu[5]/\lambda$

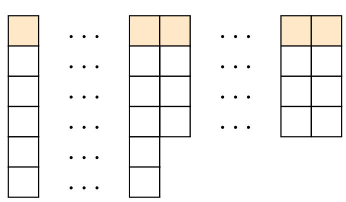


$s = 6, \nu[6]/\lambda$

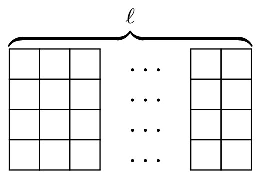


We can generalize this example to describe the shape of $\nu[s]/\lambda$ for any integers $s > 0$ and $\ell > 0$.

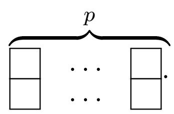
Lemma A.2. Let $\nu = \ell\omega_2$, $\lambda = \ell\omega_1$ and $s > 0$. Let p, m be integers such that $1 \leq p \leq \ell$ and $s = m\ell + p$. Then we obtain the following Young diagram, $(\ell^{(m(\ell+2)+p+1)}, p^2)$,

$\nu[s]/\lambda =$ 

(highlighted boxes indicating λ). After removing λ, this shape is a vertical concatenation of the two rectangular shapes, one of dimension $(m(\ell+2)+p+1)\times\ell$:



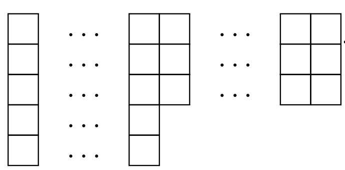
and one of dimension $2\times p$:



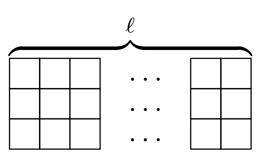
Proof. Following the construction of the tableau in the above example, we see that if we add $s<\ell$ rim hooks, we add s new rows with ℓ boxes in each row (full rows) and two rows, each with s boxes in each row. When we add $s=\ell$ rim hooks, we add $s=\ell$ full rows and two rows of size ℓ, that is 2 full rows. When $p=0$, this will close up the shape giving a rectangle of $m(\ell+2)+1$ full rows. ☐

Using the results of Lemma A.2, we can prove Lemma 4.8. We work out the full proof here.

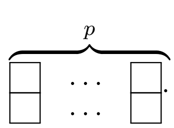
Lemma A.3. For $\mathbb{V}=\mathbb{V}(\mathfrak{sl}_2,(a_1,...,a_n),\ell)$, and integers s,m, and p such that $\sum_{i=1}^n a_i=2(\ell+s)$ with $s=m\ell+p>0$, and $1\leq p\leq\ell$, the rank of $\mathbb{V}$ is equal to the number of proper tableaux with content $(a_1,...,a_n)$ on shape, $(\ell^{2(m+1)},p^2)$,



This is a vertical concatenation of the two rectangular shapes, one of dimension $(2(m+1))\times\ell$,



and one of dimension $2\times p$,



Proof. As in Section 4.1.3, let $\nu = (\ell, \ell)$ and $\lambda = (\ell)$ be Young diagrams. We show the empty boxes in $\nu[s]/\lambda$ after placing the content (ℓ^{s-1}) from $(a_1, \ldots, a_n, \ell^{s-1})$ forms a Young diagram given by $(\ell^{2(m+1)}, p^2)$. The lemma result then follows from Witten's Dictionary.

From Remark 3.16 the number we want to compute is independent of the ordering of the content, $(a_1, \ldots, a_n, \ell^{s-1})$; thus, we can choose to make $A_1 = \ell$, ..., $A_{s-1} = \ell$ and consider creating a tableau on $\nu[s]/\lambda$ with content $(\ell^{s-1}, a_1, \ldots, a_n)$.

Since rows must be strictly decreasing in flavor, we must fill the first row with $A_1 = \ell$ amount of 1's, the second row with $A_2 = \ell$ amount of 2's, and continue filling full rows of ℓ boxes in the shape $\nu[s]/\lambda$ from Lemma A.2 until we have filled the $s-1$ row with $A_{s-1} = \ell$ amount of $(s-1)$'s. This was the only way to place the $s-1$ content flavors of size ℓ into $\nu[s]/\lambda$. The remaining empty boxes of $\nu[s]/\lambda$ creates the Young diagram $(\ell^{2(m+1)}, p^2)$. Thus, the total number of tableaux with shape $\nu[s]/\lambda$ and content $(\ell^{s-1}, a_1, \ldots, a_n)$ is equal to the total number of tableaux with shape $(\ell^{2(m+1)}, p^2)$ and content $(a_1, \ldots, a_n)$. □

For convenience in Lemmas 4.7 and 4.8, we denote $k = m + 1$.

A.2 Rank of $\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)$ at critical level

We now demonstrate computations of quantum Kostka numbers to determine ranks of vector bundles of conformal blocks of type $\mathfrak{sl}_2$ *at the critical level* [11, Def. 1.1]. That is, we compute ranks of the vector bundles, $\mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell)$ when $\sum a_i = 2(\ell + 1)$. Bundles *at the critical level* were the first type of bundles we considered at the beginning of our search for the general result obtained in Theorem 1.5.

Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell)$ for a fixed ℓ and n-tuple of weights $\vec{\lambda} = (a_1\omega_1, ..., a_n\omega_1)$ with $\ell \geq a_1 \geq ... \geq a_n > 0$. By Witten's Dictionary, we have

$$\text{rk}(\mathbb{V}) = K^{(\ell,\ell)}_{\ell\omega_1,(a_1,\ldots,a_n),1}$$

That is, we compute $K^{\nu}_{\lambda,\mu,1}(2,\ell)$ with $\nu = (\ell,\ell)$, $\nu[1] = (\ell,\ell,\ell,1,1)$, $\lambda = (\ell,0)$, and $\nu[1]/\lambda = (\ell,\ell,1,1)$. These shapes are given in Figure A.2 (the boxes in the $(2+\ell)$-rim hook added to ν are highlighted in the figure).

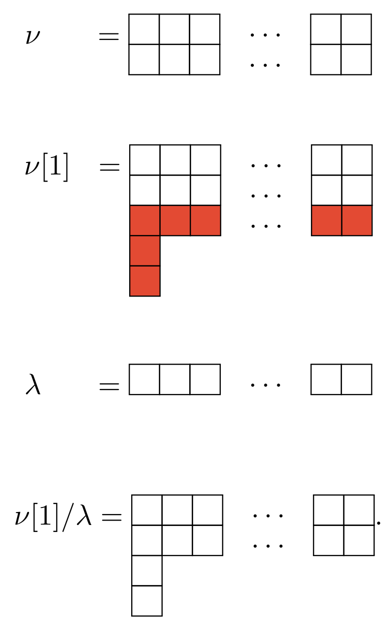


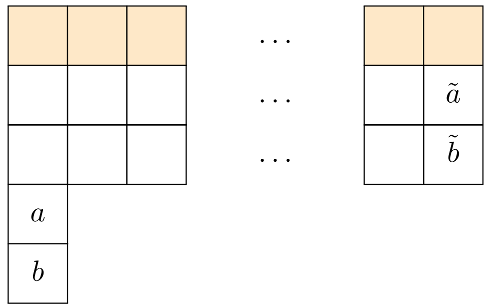
Figure A.2: Shapes for computing rk($\mathbb{V}$) for $\mathbb{V}$ at critical level

To compute the rank we must find the number of proper tableaux with shape $\nu[1]/\lambda$ and content $(a_1, ..., a_n)$. Note that for any n, the shape $\nu[1]/\lambda$ above will be the same for $\mathfrak{sl}_2$ and any fixed ℓ. The n in the calculations changes the number of flavors in content $\nu = (a_1, ..., a_n)$, it does not affect the parameters ν, m, λ, or the shape of the tableaux we are using to determine rank. For different values of ℓ the shape will look similar with the first two rows having ℓ boxes in each row.

The content of $\nu[1]/\lambda$ must be strictly increasing in the columns (top to bottom) and weakly increasing in the rows (left to right). The shape of $\nu[1]/\lambda$ will have *proper* content if and only if

$$a \geq \tilde{a} \text{ and } b \geq \tilde{b},$$

where $a, \tilde{a}, b, \tilde{b}$ are the flavors in shape $\nu[1]/\lambda$, as follows



A.2.1 Ranks of bundles at critical level when $n = 4$

We now further restrict our focus to bundles $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell)$ at critical level and with four weights.

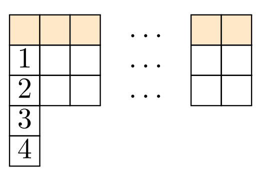
Lemma A.4. If $(a_1\omega_1, a_2\omega_1, a_3\omega_1, a_4\omega_1)$ are four nonzero weights of level $\leq \ell$ such that $a_1 + a_2 + a_3 + a_4 = 2(\ell + 1)$ then

$$\operatorname{rk} \mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, a_2\omega_1, a_3\omega_1, a_4\omega_1), \ell) = 1$$

if and only if

$$a_i = \ell \text{ or } a_i = 1 \text{ for some weight } a_i.$$

Proof. First, with four weights, our content will contain the flavors 1, 2, 3, and 4. As we fill in $\nu[1]/\lambda$ with such content, since columns must be strictly increasing, the first column in $\nu[1]/\lambda$ (see Figure A.2) must be filled with increasing values as follows,

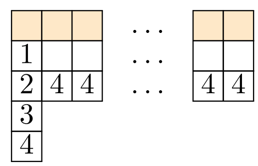


We have reduced our problem to finding the number of proper tableaux with content $(a_1 - 1, ..., a_4 - 1)$ on a Young diagram of shape $2 \times (\ell - 1)$.

Case 1: $a_i = \ell$.

If $a_i = \ell$ we can assume this is true for a_4 (rank is invariant with respect to the order of the weights, see Remark 3.16) and so we have $\ell - 1$ amount of content with flavor 4 remaining. Since the remaining $\ell - 1$ columns (each with two boxes) must be strictly increasing the full second row of the empty boxes must be filled with content flavor 4. This takes up all of the $a_4 - 1$ remaining content. We now have the following shape to fill with the remaining content $(a_1 - 1, a_2 - 1, a_3 - 1)$,

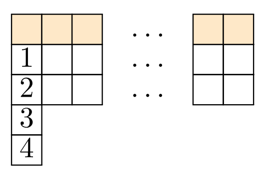
a row of $\ell - 1$ boxes, the empty boxes of,



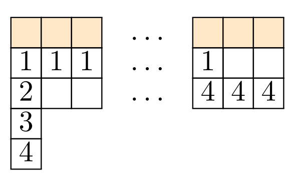
Now, for the remaining content $(a_1 - 1, a_2 - 1, a_3 - 1)$, there will be one and only one way to place this content in a single row in weakly increasing order.

Case 2: $a_i = 1$.

We can assume $a_2 = 1$ (again, rank is invariant on order of weights, Remark 3.16). As discussed above, we will have to fill in the shape below with the content $(a_1 - 1, 0, a_3 - 1, a_4 - 1)$.



Again, to maintain strictly increasing columns and weakly increasing rows, we must have the a_4 content in the last row and the a_1 content in the first row. This gives the following tableau,



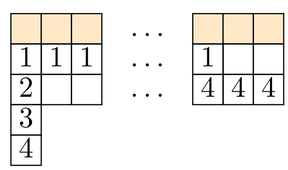
This will give us two disjoint rows to fill in with the remaining $a_3 - 1$ content. Indeed, these rows will be disjoint as we can assume $a_1, a_4 \geq a_3$, from which it follows that $a_1 + a_4 \geq (2(\ell + 1))/2$

since the sum of all weights is $2(\ell+1)$ and we have assumed a_1, a_4 are the largest. There will then be only be one way to fill two disjoint rows with the $a_3 - 1$ amount of content of flavor 3.

In the above two cases, we have determined that there is only one possible way to fill the shape $\nu[1]/\lambda$ with content μ. Thus, the value $K^{\nu}_{\lambda,\mu,1}(2,\ell)$ and hence the $\operatorname{rk} \mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, a_2\omega_1, a_3\omega_1, a_4\omega_1), \ell)$, will be one in either case.

Case 3: $1 < a_i < \ell$ for all $i = 1, 2, 3, 4$.

We can begin this discussion in the same way. We want to fill the shape $\nu[1]/\lambda$ with content (a_1, a_2, a_3, a_4). The first column of $\nu[1]/\lambda$ is already determined as above and we can assume a_1, a_4 are the largest amount of content and so $a_1 + a_4 \geq \ell + 1$. As discussed above, in order for any such content to be strictly decreasing down columns we must have the a_1 1's in the beginning of the first row and a_4 4's in the end of the second row. This again will give us two disjoint rows to fill in with the remaining (nonzero) content $(a_2 - 1, a_3 - 1)$ in the empty boxes below,



We can see that there is more than one way to fill in these empty boxes with the remaining content so that we have a proper tableau. One option is to place $a_2 - 1$ number of 2's in the top row immediately after the 1's, continuing to the second row if necessary, and then fill the remaining boxes with $a_3 - 1$ 3's. A second option is to fill the end of the first row with the $a_3 - 1$ 3's, continuing to the empty boxes at the end of the second row if necessary. These will produce two different proper tableaux. If $a_3 - 1$ is less than or equal to the number of empty boxes in the first row, then the first tableau created will have flavor 3 in the second row; the second tableau created will not. If $a_3 - 1$ is greater than the number of empty boxes in the first row, then the first method will produce a tableau with flavor 2 in the first row and the second method will not.

This covers all of the cases of weights $(a_1\omega_1, a_2\omega_1, a_3\omega_1, a_4\omega_1)$ at the critical level. □

A.2.2 Ranks of bundles at critical level when $n > 4$

We now extend the results of Section A.2.1 to include ranks of bundles $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell)$ at critical level with $n > 4$ weights. We show that any such bundle at critical level will have rank larger than one.

Claim A.5. Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (a_1\omega_1, ..., a_n\omega_1), \ell)$ for some fixed integer $\ell > 3$ and $n > 4$ integers $\ell \geq a_1 \geq ... \geq a_n > 0$ such that $\sum_{i=1}^{n} a_i = 2(\ell + 1)$, then $\mathrm{rk}(\mathbb{V}) > 1$.

Proof. As we discussed in the introduction to this section, we can compute the rank by calculating the Kostka coefficient,

$$K^{(\ell,\ell)}_{(\ell),(a_1,...,a_n,1)}.$$

This number is the number of ways to fill the shape $\nu[1]/\lambda$ (see Figure A.2) with content $(a_1, ..., a_n, 1)$ to obtain a proper semistandard tableau. We show there are two methods to do this.

We make two observations. First, as we reasoned in the $n = 4$ case, the first a_1 boxes of the first row must contain all content of flavor 1 and the last $a_n - 1$ boxes of the second row must contain all content of flavor n. Additionally, the last (fourth) box of the first column must have content n (this is so the tableau is proper). Hence, any such proper semistandard tableau must have the filling as in Figure A.3 with a_1 amount of flavor 1 and a_n amount of flavor n. We now show two methods to fill the remaining content $(a_2, ..., a_{n-1})$. For each method, we reason that the resulting object is a proper semistandard tableau.

1	1	1	1					
						n	n	n
n								

Figure A.3: Filling of content a_1 and a_n

Method 1: Fill the remaining empty two boxes in the first column of Figure A.3 with content of flavor 2 and $n - 1$. Place the remaining content $(a_2 - 1, a_3, ..., a_{n-2}, a_{n-1} - 1)$ in the remaining empty boxes left to right, top to bottom, by placing all content $a_2 - 1$ of flavor 2, followed by content a_3 of flavor 3, etc. See Figure A.4 for an example of a filling using this method.

Method 2: Fill the remaining empty two boxes in the first column of Figure A.3 with content of flavor 2 and $n-2$. Place the remaining content $(a_2-1, a_3, ..., a_{n-3}, a_{n-2}-1, a_{n-1})$ in the remaining empty boxes left to right, top to bottom, by placing all content a_2-1 of flavor 2, followed by content a_3 of flavor 3, etc. See Figure A.5 for an example of a filling using this method.

We now show that Method 1 and 2 each result in a distinct proper semistandard tableau.

First, each tableau has weakly increasing content flavors left to right in the rows. Indeed, the initial content placement, as in Figure A.3, is of content of flavor 1 or n and both methods specified placement of flavor 2 in the second row, first column. The remaining flavors to fill in the first and second rows are between 2 and $n-1$, all of which are either greater than or equal to 1 (initial content in first row) or less than or equal to n (initial content at end of second row). We specified the placement of the remaining content in either case to exhaust content amounts in weakly increasing order. Thus the two rows will have content flavors weakly increase.

Now, each filling will have strictly increasing column content flavors. This is clear in the first column, since $n > 4$ which implies $1 < 2 < n-1 < n$ (for the first method) and $1 < 2 < n-2 < n$ (for the second method). Additionally, each column in the tableau containing a 1 or n (the initial placed content) will be strictly increasing (since no other content of such flavor remains). And since content amounts are weakly decreasing $a_i \geq a_{i+1}$ the placement of content in first row, followed by second row (in each case) will not result in any column (of two boxes) with the same content flavor i.

Finally, in each method, the tableau will be proper. We must check that the content in the final box of the first row is less than or equal to $n-1$ (in the first method) and $n-2$ in the second method. Thus, it is sufficient, for either case to show that such content, denote this content flavor j is such that $j \leq n-2$. For contradiction, assume this wasn't the case, and so $j = n-1$. Then the amount of content of flavor $n-1$ must be $a_{n-1} \geq \ell - 1 - (a_n - 1) + 1 = \ell - a_n + 1$, the sum of the number of empty boxes in the second row and one box from the first row (i.e., those boxes that containing content of flavor $n-1$). This would imply $a_{n-1} + a_n \geq \ell + 1$. Since weights are weakly decreasing, this would further imply

$$a_1 + a_2 + a_{n-2} + a_{n-1} + a_n > 2\ell + 2.$$

However since $2\ell + 2 = \sum_{i=1}^{n} a_i \geq a_1 + a_2 + a_{n-2} + a_{n-1} + a_n$, this would provide a contradiction. Thus, content of flavor j in the last column of the first row must be $j \leq n - 2$.

Since each method of filling involves placing a different flavor in the third box in the first column, the resulting tableaux are distinct. Hence, the rank of the bundle $\mathbb{V}$ is larger than one. $\square$

Example A.6. Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (6, 5, 3, 2, 2, 2), 9)$ be an $\mathfrak{sl}_2$ bundle with weights $(6, 5, 3, 2, 2, 2)$ at level 9. We check that $6 + 5 + 3 + 2 + 2 + 2 = 20 = 2(9 + 1)$ so $\mathbb{V}$ is at critical level. The following are the tableaux produced in Claim A.5 using Method 1 and Method 2.

1	1	1	1	1	1	2	2	2
2	2	3	3	3	4	4	5	6
5								
6								

Figure A.4: Tableau produced from Method 2 of Claim A.5

1	1	1	1	1	1	2	2	2
2	2	3	3	3	4	5	5	6
4								
6								

Figure A.5: Tableau produced from Method 2 of Claim A.5