Quiver Grassmannians and their Quotients by Torus Actions

by

Tyler L. Kelly

(Under the direction of Elham Izadi)

Abstract

We introduce the construction of quiver Grassmannians and the quotients of quiver Grassmannians by torus actions. After defining these objects, we give an approach to describing the space of a quiver Grassmannian. We then compute some examples of quiver Grassmannians and a quotient by a torus action.

Index words: Algebraic Geometry, Quiver Grassmannians, Torus Actions

Quiver Grassmannians and their Quotients by Torus Actions

by

Tyler L. Kelly

A Thesis Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment
of the
Requirements for the Degree

Master of Arts

Athens, Georgia

2009

Quiver Grassmannians and their Quotients by Torus Actions

by

Tyler L. Kelly

Approved:

Major Professor: Elham Izadi

Committee: Theodore Shifrin
Pete L. Clark

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2009

Dedication

I dedicate this thesis to the memory of Steve Sigur. He was an inspiration to all young mathematicians that came into contact with him. With a warm smile and care, he made me understand more about what I loved. Thank you for all you have done for me, Steve.

Acknowledgments

First of all, I would like to thank Elham Izadi for her aid, insight and care. She has been the primary person to help me in my growth as an algebraic geometer. Patient when needed, but challenging me to continue, she has made me the mathematician I am today.

I would be hard-pressed to not also thank Ted Shifrin for his many hours of advice, both mathematically and professionally. He has been a role model when needed and has challenged me to be the best mathematician I can be.

I also want to thank David Swinarski for his help in computational algebraic geometry in regards to this project. During this project, he has helped teach me and coded many calculations presented here. He deserves much credit for some results in this paper.

My mathematical training at UGA has been fostered by the entire department, but Malcolm Adams, Mitch Rothstein, Daniel Nakano, Lenny Chastkofsky, William Graham, Shuzhou Wang, Robert Varley, Brian Boe, Pete Clark, and Ed Azoff deserve special recognition.

I would be nowhere if it were not for the camaraderie in the graduate body. Fellow algebraic geometers such as Lev Konstantinovskiy and Maxim Arap need special mention, while Brandon Samples and Jennifer Belton have given me support in mathematics throughout all four years of my UGA tenure. I thank you all for your support and care.

I also thank my family for their support during my time at UGA, and I would be doing little justice if I omitted Adam Thomas. He has been the one that has cared for me the most and kept me sane during my graduate study at UGA. Thank you, Adam.

Table of Contents

Chapter 1

Introduction and definitions

Let $Gr(2, n)$ be the Grassmannian of 2-dimensional subspaces in n-dimensional space, and $\overline{M_{0,n}}$ denote the Deligne-Mumford compactification of the moduli space of n-pointed genus 0 curves. Kapranov proved that the Chow quotient of $Gr(2, n)$ by the $(n-1)$-dimensional torus T_{n-1} can be identified with $\overline{M_{0,n}}$ [K]. Derksen, Weyman, and Zelevinsky study quiver Grassmannians for their interplay with cluster algebras (page 3 of [DWZ]). It has been shown that their Euler characteristics "define sets of generators of cluster algebras in the acyclic type" (page 2369 of [CR]). We will introduce the concept of a quiver Grassmannian and compute examples, then we will compute the quotient of a quiver Grassmannian by a torus action.

1.1 Quivers

Definition 1.1.1. *A quiver $Q := (Q_0, Q_1)$ is the collection of data consisting of*

1.) a finite set $Q_0 = \{1, \dots, n\}$ of vertices and

2.) a finite collection $Q_1 = \{a_1, \dots, a_m\}$ of arrows.

Let $a \in Q_1$, where a goes from $i \in Q_0$ to $j \in Q_0$. We call j and i the head $h(a)$ and tail $t(a)$ of a, respectively. Note that the head and tail could be the same element of Q_0 and there may be more than one arrow from $h(a)$ to $t(a)$.

Definition 1.1.2. *Let P be a finite sequence of arrows, $a_n, \dots, a_1$ in Q_1. P is a path in Q if $h(a_i) = t(a_{i+1})$ for all $i \in \{1, \dots, n-1\}$. The head $h(P)$ of P is $h(a_n)$ and the tail $t(P)$ of P is $t(a_1)$.*

Note that when following the arrows of a path, we move from right to left. A special type of path is a cycle:

Definition 1.1.3. *A cycle C is a path in Q where $h(C) = t(C)$.*

We will assume that, for any quiver Q in this paper, there does not exist a cycle C in Q (i.e., Q is acyclic). We now define a quiver representation. Let $\mathbb{C}$ be the field of complex numbers. We will work over $\mathbb{C}$ during the course of this thesis.

Definition 1.1.4. *A quiver representation (X, f) associated to the quiver Q is the data of*
1.) a collection X of assignments of finite-dimensional complex vector spaces X_i to every vertex $i \in Q_0$, and
2.) a collection f of assignments of linear maps $f_a : X_{t(a)} \to X_{h(a)}$ to every arrow $a \in Q_1$.

A quiver representation is a generalization of a vector space. Consider the quiver Q with one vertex (i.e., $Q_0 = \{1\}$) and no arrows. A vector space W can be considered as a quiver representation (X, f) associated to Q where X consists of one vector space $X_1 := W$ and f is just the empty collection (since there are no arrows).

Another example shows that quiver representations also generalize linear maps. Consider the quiver $Q' = (Q'_0, Q'_1)$ with two vertices (i.e., $Q'_0 = \{1, 2\}$) and one arrow $a \in Q'_1$ with head 1 and tail 2. Assign the complex vector spaces V_1 and V_2 to the vertices 1 and 2, respectively, and assign the linear map $f_a : V_1 \to V_2$ to the arrow a. This quiver representation contains the same data as the linear map f_a.

Definition 1.1.5. *A quiver subrepresentation (S, g) of the quiver representation (X, f) associated to the quiver Q is the data of*
1.) a collection S of linear subspaces V_i of the vector spaces X_i in X, each associated to a vertex $i \in Q_0$, and
2.) a collection g of linear maps such that for every arrow $a \in Q_0$, there is a linear map $g_a : V_{t(a)} \to V_{h(a)}$ so that $g_a = f_a|_{V_{t(a)}}$.

We remark that, for every arrow $a \in Q_1$, the linear map g_a is determined by the mapsf_a, so, to specify a subrepresentation, only a choice of subspaces of X_i is necessary; however, one cannot choose arbitrary subspaces of every vector space X_i in order to create a subrepresentation. If (S, g) is a subrepresentation of (X, f), then for every arrow a in the quiver, $\operatorname{im} g_a = \operatorname{im} f_a|_{V_{t(a)}} \subset V_{h(a)}$.

Note that for any arrow a, the diagram

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{f_a|_{V_{t(a)}}} & V_{h(a)} \\ \downarrow{\scriptstyle \iota_{V_{t(a)}}} & & \downarrow{\scriptstyle \iota_{V_{h(a)}}} \\ X_{t(a)} & \xrightarrow{f_a} & X_{h(a)} \end{array}$$

commutes, where $\iota_{V_{t(a)}}$ and $\iota_{V_{h(a)}}$ are the natural inclusions.

1.2 Grassmannians

Definition 1.2.1. *The Grassmannian $Gr(d, n)$ is defined to be the set of d-dimensional vector subspaces of the n-dimensional vector space $V = \mathbb{C}^n$.*

We view d-dimensional subspaces as equivalence classes of decomposable elements in $\Lambda^d(V)$. Let W be a d-dimensional subspace of V. We choose a basis $w_1, w_2, \ldots, w_d$ for W and define $\omega = w_1 \wedge w_2 \wedge \ldots \wedge w_d \in \Lambda^d(V)$. Any change of basis for W will only multiply ω by the (nonzero) determinant of the change-of-basis matrix. Consider the equivalence relation $\sim$ where $\omega \sim \eta$, $\omega, \eta \in \Lambda^d(V)$ if and only if there exists a $c \in \mathbb{C}^*$ such that $\omega = c\eta$.

By assigning to W the $\sim$-equivalence class $[\omega]$ we obtain a well-defined map

$$\phi : G(d, n) \to \mathbb{P}\left(\Lambda^d(V)\right), \quad W \mapsto [w_1 \wedge \ldots \wedge w_d],$$

where $\mathbb{P}\left(\Lambda^d(V)\right)$ is the projective space of one-dimensional subspaces of the vector space $\Lambda^d(V)$ (page 64 of [H]). Looking at the image of $W \in G(d, n)$ under ϕ, we can recover all of the vectors in W since $v \wedge (w_1 \wedge \ldots \wedge w_d) = 0$ if and only if $v \in W$. We call ϕ the Plücker embedding (cf. page 64 [H]). Indeed, $Gr(d, n)$ has more structure than that of a set. We first build up some machinery.

Definition 1.2.2. *Let* $T \subset \mathbb{C}[x_1, \dots, x_n]$ *(in the projective case,* T *must be a set of homogeneous polynomials). Let* $\mathbb{A}^n$ $(\mathbb{P}^n)$ *be affine (projective) n-space. The zero set* $Z(T)$ *of* T *is*

$$Z(T) = \{P \in \mathbb{A}^n : f(P) = 0 \textit{ for all } f \in T\}, \quad (Z(T) = \{P \in \mathbb{P}^n : f(P) = 0 \textit{ for all } f \in T\}).$$

We call $Z(T)$ *a complex affine (projective) algebraic variety. An open subset of a complex affine algebraic variety is a quasi-affine (quasi-projective) variety.*

Note that we do not require $Z(T)$ to be irreducible.

Lemma 1.2.3 (Page 64 of [H])**.** *Let* $0 < d < n$*. The Grassmannian* $G(d, n)$ *is a complex projective algebraic variety.*

Consider the standard basis $\epsilon_1, \dots, \epsilon_n$ of V. We take a multivector $\omega = w_1 \wedge \dots \wedge w_k \in \Lambda^d(V)$. We can write $w_i = c_{1i}\epsilon_1 + \dots + c_{ni}\epsilon_n$ for some $c_{ji} \in \mathbb{C}$, so

$$\omega = (c_{11}\epsilon_1 + \dots + c_{n1}\epsilon_n) \wedge \dots \wedge (c_{1d}\epsilon_1 + \dots + c_{nd}\epsilon_n) = \sum_{1 \le i_1 < i_2 < \dots < i_d \le n} d_I \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_d},$$

$$\text{where} \quad d_I = \begin{vmatrix} c_{i_1 1} & c_{i_1 2} & \dots & c_{i_1 d} \\ c_{i_2 1} & c_{i_2 2} & \dots & c_{i_2 d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_d 1} & c_{i_d 2} & \dots & c_{i_d d} \end{vmatrix}.$$

There are $N+1 := \binom{n}{d}$ such multivectors $E_I = \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_d}$. Suppose we order the multivectors E_I in lexicographical order (e.g., $E_0 := \epsilon_1 \wedge \epsilon_2 \wedge \dots \wedge \epsilon_d$). We define a map $\tilde{\phi} : G(d, n) \to \mathbb{P}^N$ so that

$$W \mapsto (d_0, \dots, d_N), \quad \text{where} \quad \omega = \sum_{i=0}^{N} d_i E_i.$$

This map is the same map as ϕ; however, we have used a particular basis for $\Lambda^d(V)$.

Interpreted in a linear algebraic setting, W is spanned by d vectors $v_1, \dots, v_d$. We construct an $n \times d$ matrix whose columns are the vectors v_i and the Plücker embedding is mapping this matrix to the line spanned by the vector of its $d \times d$ minors.

1.3 Quiver Grassmannians

Consider a quiver Q, letting $Q_0 = \{1, \ldots, n\}$ be the set of vertices and Q_1 be the set of arrows of Q.

Definition 1.3.1. *Let (X, f) be a quiver representation associated to the quiver Q. The dimension vector of (X, f) is the n-tuple $\underline{d} = (\dim X_i)_{i \in Q_0}$.*

Definition 1.3.2. *Let Q be a quiver, and let (X, f) be a quiver representation associated to the quiver Q. Let $\underline{e} = (e_i)_{i \in Q_0}$ be an n-tuple where $e_i \leq \dim X_i$ for all $i \in Q_0$. The quiver Grassmannian $Gr_{\underline{e}}(X, f)$ associated to the quiver representation (X, f) is the set of subrepresentations of (X, f) with dimension vector $\underline{e}$.*

Note that $Gr_{\underline{e}}(X, f)$ has more structure than that of a set as it can be endowed with the structure of a complex algebraic variety. Since a subrepresentation is a choice of n vector subspaces, we can view the quiver Grassmannian $Gr_{\underline{e}}(X, f)$ as a subset of the product space $Gr(e_1, \dim X_1) \times \ldots \times Gr(e_n, \dim X_n)$. Moreover, $Gr_{\underline{e}}(X, f)$ is a closed subvariety of $Gr(e_1, \dim X_1) \times \ldots \times Gr(e_n, \dim X_n)$; hence it is also projective (page 2370 of [CR]).

Consider the following example. Let Q be the quiver with one vertex and no arrows. As reasoned above, an n-dimensional vector space W can be viewed as the quiver representation (X, f) associated to Q, where X is the collection of one vector space $X_1 := W$ and f is the empty collection. The dimension vector of (X, f) is $\underline{d} = (\dim W) = (n)$. Let $\underline{e} = (d)$. Consider the quiver Grassmannian $Gr_{\underline{e}}(X, f)$. For any d-dimensional subspace W_0 of W, there is one subrepresentation (S, g) such that $S_1 = W_0$. Therefore $Gr_{\underline{e}}(X, f) = Gr(d, n)$. We conclude that the quiver Grassmannian is a generalization of the Grassmannian.

Chapter 2

Quiver Grassmannians

We now discuss the quiver Grassmannian of a given quiver representation. We will first work out an approach for the quiver Grassmannians in general and then compute some examples.

First, consider the case of a quiver with two points and n arrows. Let $Q = (Q_0, Q_1)$ be a quiver where $Q_0 = \{t, h\}$ are the vertices and $Q_1 = \{a_1, \ldots, a_n\}$ are the arrows. Assume that Q is acyclic, so, without loss of generality, every arrow has tail t and head h. To construct a representation (X, f) associated to Q, we assign vector spaces X_t and X_h to the vertices t and h, respectively, and linear maps $f_{a_i} : X_t \to X_h$ to every arrow a_i. The dimension vector of (X, f) is $(\dim X_t, \dim X_h)$. Given a dimension vector $\underline{e} = (e_1, e_2)$, where $e_1 \leq \dim X_t$ and $e_2 \leq \dim X_h$, we investigate the quiver Grassmannian $Gr_{\underline{e}}(X, f)$. Recall that such a quiver Grassmannian is a subvariety of $Gr(e_1, \dim X_t) \times Gr(e_2, \dim X_h)$.

We now describe what is in the quiver Grassmannian $Gr_{\underline{e}}(X, f)$. Recall that if $V_t \times V_h \in Gr(e_1, \dim X_t) \times Gr(e_2, \dim X_h)$ is an element of the quiver Grassmannian, then, for every i, the diagram

$$\begin{array}{ccc} V_t & \xrightarrow{f_{a_i}|_{V_t}} & V_h \\ \downarrow{\scriptstyle \iota_t} & & \downarrow{\scriptstyle \iota_h} \\ X_t & \xrightarrow{f_{a_i}} & X_h \end{array}$$

commutes, where ι_t and ι_h are the natural inclusions $V_t \hookrightarrow X_t$ and $V_h \hookrightarrow X_h$.

We will outline an approach to investigate the space $Gr_{\underline{e}}(X, f)$. For every $V_t \in Gr(e_1, \dim X_t)$, we ask what, if any, $V_h \in Gr(e_2, \dim X_h)$ fulfill the criteria that $\operatorname{im}(f_{a_i}|_{V_t}) \subseteq V_h$ for all i. We will answer this question using multivectors and the Plücker embedding.

Since V_t is a vector subspace of X_t, the image of V_t under the linear map f_{a_i} is a vector subspace of X_h for all i. Considering $V_t \in Gr(e_1, \dim X_t)$, let $\beta_{V_t} = \{v_{t,1}, \ldots, v_{t,e_1}\}$ be a basis. Then $\operatorname{span}(\cup_{i=0}^n f_{a_i}(\beta_{V_t})) \subseteq V_h$. The key here is that the quiver Grassmannian has the fixed dimension vector $\underline{e} = \underline{\dim}(V)$. Thus, if $\dim(\operatorname{span}(\cup_{i=0}^n f_{a_i}(\beta_{V_t}))) = e_2$, then $V_h = \operatorname{span}(\cup_{i=0}^n f_{a_i}(\beta_{V_t}))$.

If $\dim(\operatorname{span}(\cup_{i=0}^n f_i(\beta_{V_t}))) > e_2$, then there does not exist a $V_h \in Gr(e_2, \dim X_h)$ such that $\operatorname{span}(\cup_{i=0}^n f_i(\beta_{V_t})) \subset V_h$; therefore, there does not exist a $V_h \in Gr(e_2, \dim X_h)$ such that $V_t \times V_h \in Gr_{\underline{e}}(X, f)$.

On the other hand, if $\dim(\operatorname{span}(\cup_{i=0}^n f_i\,(\beta_{V_t}))) < e_2$ then we have different possibilities for V_h. Let $d := \dim(\operatorname{span}(\cup_{i=0}^n f_{a_i}\,(\beta_{V_t})))$ and say $\beta_W = \{w_1, \ldots, w_d\}$ is a basis for $\operatorname{span}(\cup_{i=0}^n f_{a_i}(\beta_{V_t}))$. Since V_h has dimension e_2, we add $e_2 - d$ more basis vectors. A nice way to calculate the space of all possible d-dimensional subspaces V_h is to look at the space of $(e_2 - d)$-dimensional subspaces of $X_h/\operatorname{span}(\beta_W)$. Such spaces are in one-to-one correspondence with the e_2-dimensional subspaces of $\mathbb{C}^n$ that contain $\operatorname{span}(\beta_W)$; therefore, the different possibilities for V_h are parameterized by the space $Gr\,(e_2 - d, \dim X_h - d)$.

We can compute $\dim(\operatorname{span}(\cup_{i=0}^n f_{a_i}(\beta_{V_t})))$ using the Plücker embedding. Consider bases $E = \{\epsilon_1, \ldots, \epsilon_{\dim X_t}\}$ of X_t and $\Gamma = \{\gamma_1, \ldots, \gamma_{\dim X_h}\}$ of X_h. For every j, $1 \leq j \leq \dim X_t$, there exist unique scalars $a_{lj} \in \mathbb{C}$, $1 \leq l \leq m$, such that

$$f_{a_i}\,(\epsilon_j) = \sum_{l=1}^{\dim X_h} a_{lj}\gamma_l.$$

There exists a matrix representation of f_{a_i} with respect to the ordered bases E and Γ, denoted $[f_{a_i}]_{E,\Gamma}$ where the lj entry of $[f_{a_i}]_{E,\Gamma}$ is a_{lj}.

Rewrite the vectors $v_{t,j} \in \beta_{V_t}$ as linear combinations of standard basis vectors, then map every basis vector through every matrix $[f_{a_i}]_{E,\Gamma}$, and make them column vectors of an $\dim X_h \times n \dim V_t$ matrix:

$$M = \begin{pmatrix} | & & | & | & & | \\ [f_{a_1}]_{E,\Gamma}(v_{t,1}) & \cdots & [f_{a_1}]_{E,\Gamma}(v_{t,k}) & [f_{a_2}]_{E,\Gamma}(v_{t,1}) & \cdots & [f_{a_n}]_{E,\Gamma}(v_{t,k}) \\ | & & | & | & & | \end{pmatrix}.$$

The span of the column vectors of M is, in fact, a subspace of a possible V_h, so we check the dimension of the span of the column vectors $[f_{a_i}]_{E,\Gamma}(\nu_{V_h,j})$. The dimension of this span is equal to the rank of M, so to find the dimension we look at the $d \times d$ minors of the matrix M. If all of the $d \times d$ minors vanish, then the dimension of the span is less than d, and if there exists a nonzero $d \times d$ minor, then the dimension of the span is greater than or equal to d. We use this observation to compute the dimension of the span and then use this information to see what freedom, if any, we have when choosing V_h given a fixed V_t.

Consider a quiver $Q = (Q_0, Q_1)$, where $Q_0 = \{1, \ldots, n\}$ with representation (X, f). Suppose we want to describe the quiver Grassmannian $\mathscr{Q} = Gr_{\underline{e}}(X, f)$. Consider a pair of vertices i, j that are connected by an arrow with tail i and head j. Using the approach outlined above, first compute the quiver Grassmannian, $\mathscr{Q}_{i,j} \subset Gr(e_i, \dim X_i) \times Gr(e_j, \dim X_j), i, j \in Q_0$, between these two vertices with n arrows. Note that if there are no arrows with neither tail i and head j nor tail j and head i, then $\mathscr{Q}_{i,j} = Gr(e_i, \dim X_i) \times Gr(e_j, \dim X_j)$. We have a description of the quiver Grassmannian between these two vertices. Then, one may intersect this quiver Grassmannian with all other quiver Grassmannians obtained from such pairings to obtain the subvariety that is the quiver Grassmannian. To do this, consider the projection maps

$$\pi_{ij} : \prod_{k=1}^{n} Gr(e_k, \dim X_k) \to Gr(e_i, \dim X_i) \times Gr(e_j, \dim X_j); \qquad \pi_{ij}(V_1 \times \ldots \times V_n) = V_i \times V_j.$$

Then

$$\mathscr{Q} = \bigcap_{i,j \in Q_0} \pi_{ij}^{-1}(\mathscr{Q}_{i,j}).$$

Chapter 3

Examples of Quiver Grassmannians

3.1 Two points, one arrow with a map of rank two

Consider the quiver $Q = (Q_0, Q_1)$ with $Q_0 = \{1, 2\}$ and $Q_1 = \{a\}$, where a has tail 1 and head 2. We construct a representation (X, f). Let X_1 be the vector space $\mathbb{C}^4$, with a basis of vectors x_1, x_2, x_3, and x_4, associated to the vertex 1, and X_2 be the vector space $\mathbb{C}^4$, with a basis of vectors y_1, y_2, y_3, and y_4, associated to the vertex 2. Let $f_a : X_1 \to X_2$ be the linear map corresponding to the arrow a, and suppose that $f_a(x_1) = y_1$, $f_a(x_2) = y_2$, and $f_a(x_3) = f_a(x_4) = 0$. We will compute the quiver Grassmannian $Gr_{(2,2)}(X, f) \subset Gr(2, 4) \times Gr(2, 4)$. Let $\pi_1 : Gr_{(2,2)}(X, f) \to Gr(2, 4)$ be the projection onto the first factor.

Proposition 3.1.1. $\pi_1(Gr_{(2,2)}(X, f)) = Gr(2, 4)$. *There exists a closed subvariety W in $Gr(2, 4)$ that is a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$ such that for all but one point $p_0 \in W$, the fiber of a point $p \in W \setminus \{p_0\}$ over π_1 is $\pi_1^{-1}(p) \cong \mathbb{P}^2$. At p_0, $\pi_1^{-1}(p_0) = p_0 \times Gr(2, 4)$. For all $p \in Gr(2, 4) \setminus W$, $\pi_1^{-1}(p)$ is a point.*

Proof. We look at an arbitrary element of the Grassmannian, $V_1 \in Gr(2, 4)$. V_1 is the span of two linearly independent vectors in $\mathbb{C}^4$, say v and w. Rewrite, without loss of generality, $v = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4$ and $w = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4$, where $\lambda_i, \mu_i \in \mathbb{C}$, and assume that v and w form a basis. We have the following commutative diagram:

$$\begin{array}{ccc} V_1 & \xrightarrow{f_a|_{V_1}} & V_2 \\ \downarrow \iota & & \downarrow \iota \\ X_1 := \mathbb{C}^4 & \xrightarrow{f_a} & X_2 := \mathbb{C}^4. \end{array}$$

The image of the map f can be computed:

$$V_1 = \operatorname{span}\left(\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix}, \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{pmatrix}\right) \mapsto f_a(V_1) = \operatorname{span}\left(\begin{pmatrix}\lambda_1\\ \lambda_2\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}\mu_1\\ \mu_2\\ 0\\ 0\end{pmatrix}\right) \subseteq V_2.$$

Thus following the approach described in Chapter 2, we consider the matrix

$$M = \begin{pmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\\ 0 & 0\\ 0 & 0\end{pmatrix}.$$

We have three cases based on the dimension of im f_a. We use these three cases to describe what $V_2 \subset X_2$ can be combined with a fixed V_1 so that $V_1 \times V_2 \in Gr_{(2,2)}(X, f)$.

Case 0: Consider the case where $\dim(T(V_1)) = 0$. Here, we have no restriction on V_2, so V_2 can be any element of $Gr(2, 4)$. This case only occurs when V_1 is spanned by x_3 and x_4. This vector space is the point $p_0 \in Gr(2, 4)$ outlined in the proposition.

Case 1: Suppose $\dim(f_a(V_1)) = 1$. Without loss of generality, assume that $(\lambda_1, \lambda_2) \neq (0, 0)$, i.e., that $f_a(V_1)$ is spanned by the vector $f_a(v) = \lambda_1 y_1 + \lambda_2 y_2$ and that $w \in \ker f_a$. Now $f_a(V_1) \subset V_2$, so the basis of V_2 has an extra vector, say η. Choosing such an η is equivalent to choosing a nonzero element (up to scalars) of the quotient $X_2/f_a(V_1)$. Choosing η up to scalars is equivalent to choosing an element of $Gr(1, X_2/\operatorname{span}(f_a(v)))$. Since $\dim X_2/f_a(V_1) = \dim X_2 - \dim f_a(V_1) = 4 - 1 = 3$, we can see that the space of the V_2 that can be paired with V_1 to make an element of $Gr_{(2,2)}(X, f)$ is isomorphic to $Gr(1, 3) \cong \mathbb{P}^2$. Therefore, if $V \in Gr(2, 4)$ and $\dim(f_a(V)) = 1$ then $\pi_1^{-1}(V) \cong \mathbb{P}^2$.

The other question we ask is what is the space of $V_1 \in Gr(2, 4)$ such that $\dim(f_a(V_1)) = 1$. Without loss of generality, we assume that for the basis vectors v and w of V_1, $f_a(v) \neq 0$ and $f_a(w) = 0$. Then $v = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4$ and $w = \mu_3 x_3 + \mu_4 x_4$. The space of lines spanned by such vectors w is $Gr(1, 2) \cong \mathbb{P}^1$. Given a vector w, we choose v so that it is a nonzero element of $X_1/\operatorname{span}(w)$. The space of such vectors v up to scalars is equivalent to $Gr(1, X_1/\operatorname{span}(w)) \cong \mathbb{P}^2$. This shows that the variety of linear subspaces V_1 that have 0- or 1-dimensional images under f_a is a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$.

Case 2: Otherwise, it is a matrix of full rank; therefore $f_a(v)$, $f_a(w)$ span a plane, fully defining V_2.

Now that we know all of these spaces, we take the union and that is the quiver Grassmannian. □

3.2 Three dimensional ambient space

Consider the quiver $Q = (Q_0, Q_1)$, where $Q_0 = \{t, h\}$ and $Q_1 = \{a_1, a_2, a_3\}$ where $t(a_i) = t$ and $h(a_i) = h$ for all i. Construct a quiver representation (X, f) associated to Q as follows. At t and h, we have vector spaces $X_t = \mathbb{C}^3$ and $X_h = \mathbb{C}^3$, respectively. We now define the linear maps $f_{a_i} : X_t \to X_h$. Let these maps be the permutation maps that have the following matrix representations for the standard basis $\beta = \{\epsilon_1, \epsilon_2, \epsilon_3\}$:

$$[f_{a_1}]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad [f_{a_2}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

$$[f_{a_3}]_\beta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

3.2.1 The case $\underline{e} = (1, 1)$

Consider the quiver Grassmannian $Gr_{(1,1)}(X, f)$. Let ζ be a primitive third root of unity.

Proposition 3.2.1. *$Gr_{\underline{e}}(X, f) \subset Gr(1, 3) \times Gr(1, 3) \cong \mathbb{P}^2 \times \mathbb{P}^2$ is the set of three points $\{(1 : \zeta^n : \zeta^{2n}) \times (1 : \zeta^n : \zeta^{2n}) : n \in \{0, 1, 2\}\}$.*

Proof. Let $V_t \in Gr(1,3)$. We ask for what vector spaces V_t there exists a vector space V_h such that the pair $V_t \times V_h$ gives rise to a subrepresentation, hence a point in the quiver Grassmannian $V_t \times V_h \in Gr_{\underline{e}}(X,f)$. V_t is defined by a basis of one nonzero vector; call it $v_t = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3$. Due to the reasoning described in Chapter 2, we know that $f_{a_i}(v_t) \in V_h$ if $V_t \times V_h \in Gr_{\underline{e}}(X,f)$.

Therefore, if $V_t \times V_h \in Gr_{\underline{e}}(X,f)$, then

$$\operatorname{span}\left(\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}, \begin{pmatrix}\lambda_3\\ \lambda_1\\ \lambda_2\end{pmatrix}, \begin{pmatrix}\lambda_2\\ \lambda_3\\ \lambda_1\end{pmatrix}\right) \subset V_h.$$

Using the approach described in Chapter 2, we will consider the matrix

$$M = \begin{pmatrix}\lambda_1 & \lambda_3 & \lambda_2\\ \lambda_2 & \lambda_1 & \lambda_3\\ \lambda_3 & \lambda_2 & \lambda_1\end{pmatrix}.$$

We know that there exists a 1×1 minor that is nonzero, since the λ_i cannot all be zero. Since $e_2 = 1$, any V_h would be defined by the corresponding V_t. Now we see what values of λ_i have only vanishing 2×2 minors. Let the graded polynomial ring $\mathbb{C}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ be the projective coordinate ring of $\mathbb{P}^2$. The set of values of λ_i that give vanishing 2×2 minors is a variety in $\mathbb{P}^2$. The zero set of each 2×2 minor is a curve in $\mathbb{P}^2$. We look at the intersection H of all such curves, where by just looking at the first two columns we obtain

$$Z(\lambda_1^2 - \lambda_2\lambda_3, \lambda_2^2 - \lambda_1\lambda_3, \lambda_3^2 - \lambda_1\lambda_2) \supseteq H.$$

We first note that if $(\lambda_1 : \lambda_2 : \lambda_3) \in H$, then $\lambda_1^3 = \lambda_2^3 = \lambda_3^3 = \lambda_1\lambda_2\lambda_3$ (by multiplying a λ_i to every equation). Therefore $|\lambda_1| = |\lambda_2| = |\lambda_3|$ thus every λ_i is nonzero. Without loss of generality, let $\lambda_1 = 1$. Then $\lambda_2^3 = 1$ and $\lambda_3 = \lambda_2^2$. Only the points $\{(1 : \zeta^n : \zeta^{2n}) : n \in \{0,1,2\}\}$ fulfill these criteria. Furthermore, if $(\lambda_1 : \lambda_2 : \lambda_3) \in \{(1 : \zeta^n : \zeta^{2n}) : n \in \{0,1,2\}\}$ then the rank of M is 1. So $H = \{(1 : \zeta^n : \zeta^{2n}) : n \in \{0,1,2\}\}$.

Recalling the original question we asked, we wanted to know what vector spaces V_h make a subrepresentation when coupled with V_t. We conclude that the only $V_t \in Gr(1,3)$ that can be paired with a one-dimensional subspace of X_h, $V_h \in Gr(1,3)$ such that $V_t \times V_h \in Gr_{\underline{e}}(X, f)$ are the one-dimensional subspaces spanned by the basis vectors $v_n = \epsilon_1 + \zeta^n \epsilon_2 + \zeta^{2n} \epsilon_3$. Moreover, we know that V_h is the same line since f_{a_1} is the identity. □

3.2.2 The case $\underline{e} = (1, 2)$

Consider the quiver Grassmannian $Gr_{\underline{e}}(X, f) =: \mathcal{Q}_2$. In the previous case, we found the set of one-dimensional vector spaces V_t whose images under f_{a_i} span a one-dimensional subspace of X_h. Let $\pi_1 : \mathcal{Q}_2 \to Gr(1,3)$ where $V_t \times V_h \mapsto V_t$ be the projection onto the first factor.

Proposition 3.2.2. $\pi_1(\mathcal{Q}_2) \subset Gr(1,3) = \mathbb{P}^2$ *is the union of the three lines that form the edges of the triangle with vertices* $(\lambda_1 : \lambda_2 : \lambda_3) \in \{(1 : \zeta^n : \zeta^{2n}) : n \in \{0, 1, 2\}\}$. *Moreover, if* $p \in \pi_1(\mathcal{Q}_2)$ *is not in* H, *its fiber* $\pi_1^{-1}(p)$ *is a point; if* $p \in \pi_1(\mathcal{Q}_2) \cap H$, *then* $\pi_1^{-1}(p) \cong \mathbb{P}^1$.

Proof. Look at the 3×3 "minor" or the determinant of M in order to find the restrictions as to what V_t has a V_h so that $V_h \times V_t \in \mathcal{Q}_2$. This zero set of this determinant is the variety

$$\pi_1(\mathcal{Q}_2) = Z(\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3),$$

which is a reducible curve in $\mathbb{P}^2$ since $\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3 = (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \zeta\lambda_2 + \zeta^2\lambda_3)(\lambda_1 + \zeta^2\lambda_2 + \zeta\lambda_3)$ with three irreducible components:

$$\pi_1(\mathcal{Q}_2) = Z(\lambda_1 + \lambda_2 + \lambda_3) \cup Z(\lambda_1 + \zeta\lambda_2 + \zeta^2\lambda_3) \cup Z(\lambda_1 + \zeta^2\lambda_2 + \zeta\lambda_3).$$

Every irreducible component is a line in $\mathbb{P}^2$. They intersect at the points in H, so they create the triangle with the set of vertices H. Consider the fibers of all the elements of $\pi_1(\mathcal{Q}_2)$. Recalling results from the previous subsection, for all but the three points in H, there is a nonzero 2×2 minor, so V_h is fixed (since the span of $\cup_i f_{a_i}(V_t)$ is two-dimensional). However, for the three points in H, the image of V_t by the maps f_{a_i} is just a line, so we have some freedom. The set of two-dimensional subspaces that contain $f_{a_i}(V_t)$ for all i is equal to the set of lines in $X_h/\operatorname{span}(\cup_{i=1}^{3} f_{a_i}(V_t))$. Since $\dim X_h/\operatorname{span}(\cup_{i=1}^{3} f_{a_i}(V_t)) = \dim X_h - \dim\operatorname{span}(\cup_{i=1}^{3} f_{a_i}(V_t)) = 3-1 = 2$, this is the same as choosing a point in $Gr(1,2) \cong \mathbb{P}^1$. □

3.3 Quartic hypersurface as subvariety of a quiver Grassmannian

During a colloquium at UGA, Zelevinsky commented that one can find a quartic hypersurface that is a subvariety of a quiver Grassmannian. In this section, we will build up to an example of such a quiver Grassmannian and relate it to the geometry of a tetrahedron. Consider the quiver $Q = (Q_0, Q_1)$, where $Q_0 = \{t,h\}$ and $Q_1 = \{a_1,a_2,a_3,a_4\}$ where $t(a_i) = t$ and $h(a_i) = h$ for all i.

We construct a quiver representation (X,f) associated to Q. At t and h, we have vector spaces $X_h = \mathbb{C}^4$ and $X_t = \mathbb{C}^4$, respectively. We now define the linear maps $f_{a_i} : X_t \to X_h$. Let these maps be the permutation maps that have the following matrix representations for the standard basis $\beta = \{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\}$:

$$[f_{a_1}]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad [f_{a_2}]_\beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$[f_{a_3}]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \qquad [f_{a_4}]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

3.3.1 The case $\underline{e} = (1,1)$

Consider the quiver Grassmannian $\mathscr{Q}_1 := Gr_{(1,1)}(X,f)$. Let ζ be a primitive fourth root of unity.

Proposition 3.3.1. *$Gr_{\underline{e}}(X,f) \subset Gr(1,4) \times Gr(1,4) \cong \mathbb{P}^3 \times \mathbb{P}^3$ is the set of four points $\{(1:\zeta^n:\zeta^{2n}:\zeta^{3n}) \times (1:\zeta^n:\zeta^{2n}:\zeta^{3n}) : n \in \{0,1,2,3\}\}$.*

Proof. Let $V_t \in Gr(1,4)$. We ask what, if any, vector spaces V_h give us a subrepresentation when coupled with V_t, hence a point in the quiver Grassmannian $V_t \times V_h \in \mathcal{Q}_1$. So, V_t is defined by a basis of one nonzero vector; call it $v_t = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3 + \lambda_4\epsilon_4$. Due to the reasoning described in Chapter 2, we know that $f_{a_i}(v_t) \in V_h$ if $V_t \times V_h \in \mathcal{Q}_1$.

Therefore, if $V_t \times V_h \in \mathcal{Q}_1$, then

$$\operatorname{span}\left(\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix}, \begin{pmatrix}\lambda_4\\ \lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}, \begin{pmatrix}\lambda_3\\ \lambda_4\\ \lambda_1\\ \lambda_2\end{pmatrix}, \begin{pmatrix}\lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_1\end{pmatrix}\right) \subset V_h$$

Using the approach described in Chapter 2, we will consider the matrix

$$M = \begin{pmatrix}\lambda_1 & \lambda_4 & \lambda_3 & \lambda_2\\ \lambda_2 & \lambda_1 & \lambda_4 & \lambda_3\\ \lambda_3 & \lambda_2 & \lambda_1 & \lambda_4\\ \lambda_4 & \lambda_3 & \lambda_2 & \lambda_1\end{pmatrix}.$$

We know that there exists a 1×1 minor that is nonzero, since the λ_i cannot all be zero. Since $e_2 = 1$, any V_h would be defined by the corresponding V_t. Now we see what values of λ_i give us only vanishing 2×2 minors. Let the graded polynomial ring $\mathbb{C}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ be the projective coordinate ring for $\mathbb{P}^3$. The set of values of λ_i that give vanishing 2×2 minors is a variety in $\mathbb{P}^3$: The zero set of each 2×2 minor is a surface in $\mathbb{P}^3$. We look at the intersection H of all such surfaces, where by just looking at the first two columns we obtain

$$Z\left(\lambda_1^2 - \lambda_2\lambda_4, \lambda_2^2 - \lambda_3\lambda_1, \lambda_3^2 - \lambda_4\lambda_2, \lambda_4^2 - \lambda_1\lambda_3, \lambda_1\lambda_2 - \lambda_3\lambda_4, \lambda_2\lambda_3 - \lambda_4\lambda_1\right) \supseteq H.$$

We first note that if $(\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4) \in H$, then $\lambda_1^2 = \lambda_2\lambda_4 = \lambda_3^2$ and $\lambda_2^2 = \lambda_1\lambda_3 = \lambda_4^2$. Since we are working over $\mathbb{C}$, we now know that $\lambda_1 = \pm\lambda_3$ and $\lambda_2 = \pm\lambda_4$. Moreover, since $\lambda_1^2 = \lambda_2\lambda_4$, $\lambda_1^4 = \lambda_2^2\lambda_4^2 = \lambda_2^4$, we know that $\lambda_1 = \zeta^n\lambda_2$ for some $n \in \{0, 1, 2, 3\}$. Hence $|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4|$, so they are all nonzero. Without loss of generality, set $\lambda_1 = 1$. Then $\lambda_2 = \zeta^n$. Since $\lambda_2^2 = \lambda_3\lambda_1$, $\lambda_3 = \zeta^{2n}$, and since $\lambda_2\lambda_3 = \lambda_4\lambda_1$, $\lambda_4 = \zeta^n\lambda_3 = \zeta^{3n}$. So there are four possible points on H by looking at the minors of only the first two columns. We can check that indeed these points have vanishing 2×2 minors. So, $H = \{(1 : \zeta^n : \zeta^{2n} : \zeta^{3n}) : n \in \{0, 1, 2, 3\}\}$.

Recalling the original question we asked, we wanted to know for what vector spaces V_t there exists a V_h such that $V_t \times V_h$ gives a subrepresentation of (X, f). We conclude that the only $V_t \in Gr(1, 4)$ that can be paired with a one-dimensional subspace of X_h, $V_h \in Gr(1, 4)$, such that $V_t \times V_h \in \mathcal{Q}_1$ are the one-dimensional subspaces spanned by the basis vectors $v_n = \epsilon_1 + \zeta^n \epsilon_2 + \zeta^{2n} \epsilon_3 + \zeta^{3n} \epsilon_4$. Moreover, we know that V_h is the same line. □

3.3.2 The case $\underline{e} = (1, 2)$

Consider the quiver Grassmannian $Gr_{(1,2)}(X, f) =: \mathcal{Q}_2$. In the previous case, we found the set of vector spaces V_t whose images under the maps f_{a_i} span a one-dimensional subspace of X_h. Let $\pi_1 : \mathcal{Q}_2 \to Gr(1, 4)$ where $V_t \times V_h \mapsto V_t$ be the projection onto the first factor.

Proposition 3.3.2. $\pi_1(\mathcal{Q}_2) \subset Gr(1, 4) = \mathbb{P}^3$ *is the variety of six lines that form the edges of the tetrahedron with vertices* $H = \{(1 : \zeta^n : \zeta^{2n} : \zeta^{3n}) : n \in \{0, 1, 2, 3\}\}$. *Moreover, if* $p \in \pi_1(\mathcal{Q}_2)$ *is not in* H, *its fiber* $\pi_1^{-1}(p)$ *is a point; if* $p \in \pi_1(\mathcal{Q}_2) \cap H$, *then* $\pi_1^{-1}(p) \cong \mathbb{P}^2$.

Proof. If $V_h \in \pi_1(\mathcal{Q}_2)$, then the dimension of the image of its span under the T_i maps is less than three, so all 3×3 minors of the matrix M vanish. There are 16 such minors of M (choice of deleting one row and one column); however, there are only 4 equations for minors due to the symmetry of the matrix, so we cut out the variety

$$Z(\lambda_1^3 + \lambda_4^2\lambda_3 + \lambda_2^2\lambda_3 - \lambda_3^2\lambda_1 - 2\lambda_1\lambda_2\lambda_4, \lambda_2^3 + \lambda_1^2\lambda_4 + \lambda_3^2\lambda_4 - \lambda_4^2\lambda_2 - 2\lambda_1\lambda_2\lambda_3, \lambda_3^3$$

$$+\lambda_2^2\lambda_1 + \lambda_1^2\lambda_4 - \lambda_1^2\lambda_3 - 2\lambda_2\lambda_3\lambda_4, \lambda_4^3 + \lambda_3^2\lambda_2 + \lambda_1^2\lambda_2 - \lambda_1^2\lambda_3 - 2\lambda_1\lambda_3\lambda_4).$$

This subvariety of $\mathbb{P}^3$ is the space $\pi_1(\mathcal{Q}_2)$. If $V_t \in \pi_1(\mathcal{Q}_2)$, we can look at the fiber $\pi_1^{-1}(V_t)$. By the previous section, for all but the four points in H, there is a nonzero 2×2 minor, so V_h is fixed (since the span of the image of V_t by the maps f_i is two-dimensional). However, for these four points, the image of V_t by the maps f_{a_i} is just a line, so we have a degree of freedom.

The set of two dimensional subspaces that contain $f_{a_i}(V_t)$ for all i is equal to the set of lines in $X_h/\operatorname{span}(\cup_{i=1}^4 f_{a_i})(V_t)$. Since $\dim X_h/\operatorname{span}(\cup_{i=1}^4 f_{a_i}(V_t)) = \dim X_h - \dim\operatorname{span}(\cup_{i=1}^4 f_{a_i}(V_t)) = 4-1=3$, this is the same as choosing a point in $Gr(1,3)\cong\mathbb{P}^2$.

We now use MacCaulay 2 and Magma to investigate what this variety looks like. The code used here is shown in Appendix A, and was written with the aid of David Swinarski. Through the code, we find that $\pi_1(\mathscr{Q}_2)$ is a reducible curve and is the set of six lines that are the edges of a tetrahedron. Moreover, the vertices of the tetrahedron are the four points of $\pi_1(\mathscr{Q}_1)$, H. The singular locus of $\pi_1(\mathscr{Q}_2)$ is H. □

3.3.3 The case $\underline{e} = (1,3)$

We will denote this quiver Grassmannian by $\mathscr{Q}_3$. We redefine $\pi_1 : \mathscr{Q}_3 \to Gr(1,4)$, where π_1 projects $V_t \times V_h$ to V_t.

Proposition 3.3.3. *$\pi_1(\mathscr{Q}_3) \subset Gr(1,4) = \mathbb{P}^3$ is the variety of four planes that are the faces of the tetrahedron with vertices $H = \{(1:\zeta^n:\zeta^{2n}:\zeta^{3n}) : n \in \{0,1,2,3\}\}$. Moreover, if $p \in \pi_1(\mathscr{Q}_3)$ is not in $\pi_1(\mathscr{Q}_2)$, its fiber $\pi_1^{-1}(p)$ is a point; if $p \in \pi_1(\mathscr{Q}_3)\cap\pi_1(\mathscr{Q}_3)$ but not in H, then $\pi_1^{-1}(p)\cong\mathbb{P}^1$. Lastly, if $p\in H$, then $\pi_1^{-1}(p)\cong(\mathbb{P}^2)^*$.*

Proof. We are now looking at the 4×4 "minor" or the determinant of M in order to get some restriction on what V_t will give us a V_h so that $V_t \times V_h \in \mathscr{Q}_3$. The determinant gives us a variety

$$\pi_1(\mathscr{Q}_3) = Z(\lambda_1^4 - \lambda_2^4 + 4\lambda_1\lambda_2^2\lambda_3 - 2\lambda_1^2\lambda_3^2 + \lambda_3^4 - 4\lambda_1^2\lambda_2\lambda_4 - 4\lambda_2\lambda_3^2 + 2\lambda_2^2\lambda_4^2 + 4\lambda_1\lambda_3\lambda_4^2 - \lambda_4^4)$$

which is a reducible hypersurface in $\mathbb{P}^3$. Through computation, we see that this is a reducible surface of degree 4. The Hilbert polynomial of $\pi_1(\mathscr{Q}_3)$ is $h(x) = 2x^2+2$, so its arithmetic genus is $(-1)^{\dim\pi_1(\mathscr{Q}_3)}(h(0)-1) = (-1)^2(2-1) = 1$. By looking at the irreducible components, we see that it is the union of four planes:

$$\pi_1(\mathscr{Q}_3) = Z(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \cup Z(\lambda_1 + \zeta^2\lambda_2 + \lambda_3 + \zeta^2\lambda_4) \cup$$

$$Z(\lambda_1 + \zeta^3\lambda_2 + \zeta^2\lambda_3 + \zeta\lambda_4) \cup Z(\lambda_1 + \zeta\lambda_2 + \zeta^2\lambda_3 + \zeta^3\lambda_4).$$

Through computation we can see that the singular locus of $\pi_1(\mathscr{Q}_3)$ is $\pi_1(\mathscr{Q}_2)$. An irreducible component of $\pi_1(\mathscr{Q}_2)$ corresponds to the intersection of two irreducible components of $\pi_1(\mathscr{Q}_3)$. Moreover, any point is in three irreducible components of $\pi_1(\mathscr{Q}_3)$ if and only if it is in $\pi_1(\mathscr{Q}_1)$. We can also state that the choice we have for V_h varies depending on what V_t we start with. Let i be the minimal number such that $V_t \in \pi_1(\mathscr{Q}_i)$. If $i = 1$ then we have a choice of $Gr(3-1, 4-1) = Gr(2,3) \cong (\mathbb{P}^2)^*$, if $i = 2$ then we have a choice of $Gr(3-2, 4-2) = Gr(1,2) = \mathbb{P}^1$ and if $i = 3$ then V_h is completely fixed. □

Chapter 4

Example of a quiver Grassmannian and its quotient by a torus action

In this chapter, we will construct another example of a quiver Grassmannian, give an example of a complex algebraic torus acting on it, then take the geometric quotient on an open set of the quiver Grassmannian. Let $Q := (Q_0, Q_1)$ be a quiver with vertices $Q_0 = \{1, 2\}$ and one arrow $a \in Q_1$ with tail 1 and head 2. Consider the representation (X, f) of the quiver Q where $X_1 = \mathbb{C}^4$, with basis $\{x_1, x_2, x_3, x_4\}$, and $X_2 = \mathbb{C}^4$, with basis $\{y_1, y_2, y_3, y_4\}$, while $f_a : X_1 \to X_2$ is the linear map associated to the arrow a where $f_a(c_1x_1+c_2x_2+c_4x_3+c_4x_4) = c_1y_1$, $c_i \in \mathbb{C}$.

Before we further investigate $\mathscr{Q}$, let us consider $Gr(2, 4)$ more carefully. Here the Plücker embedding is given by $\tilde{\phi} : Gr(2, 4) \to \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$, where

$$V = \operatorname{span}\left\{\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix}, \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{pmatrix}\right\} \mapsto$$

$$(\lambda_1\mu_2 - \lambda_2\mu_1 : \lambda_1\mu_3 - \lambda_3\mu_1 : \lambda_1\mu_4 - \lambda_4\mu_1 : \lambda_2\mu_3 - \lambda_3\mu_2 : \lambda_2\mu_4 - \lambda_4\mu_2 : \lambda_3\mu_4 - \lambda_4\mu_3)$$

Note that these are all the 2×2 minors of the matrix with the vectors λ and μ as column vectors. We can then see that $Gr(2, 4)$ can be expressed as the zero set $Z(x_0x_5+x_1x_4-x_2x_3) \subset \mathbb{P}^5$ (cf. page 211 of [GH]).

4.1 The Quiver Grassmannian

Recall that $Gr_{(2,2)}(X, f) \subset Gr(2, 4) \times Gr(2, 4)$. Define the embedding $\Phi : Gr(2, 4) \times Gr(2, 4) \to \mathbb{P}^5 \times \mathbb{P}^5$ by $\Phi(V_1 \times V_2) = \tilde{\phi}(V_1) \times \tilde{\phi}(V_2)$. Given coordinate rings $\mathbb{C}[x_0, \ldots, x_5]$

and $\mathbb{C}[y_0,\ldots,y_5]$ of two copies of $\mathbb{P}^5$, we may say that $Gr_{(2,2)}(X,f) \subset Gr(2,4) \times Gr(2,4) = Z(x_0x_5 + x_1x_4 - x_2x_3) \times Z(y_0y_5 + y_1y_4 - y_2y_3) \overset{\Phi}{\hookrightarrow} \mathbb{P}^5 \times \mathbb{P}^5$.

Proposition 4.1.1. *The map* Φ *maps* $Gr_{(2,2)}(X,f)$ *isomorphically onto* $Z(x_0x_5 + x_1x_4 - x_2x_3) \times Z(y_3,y_4,y_5) \cup Z(x_0,x_1,x_2) \times Z(y_0y_5 + y_1y_4 - y_2y_3) \subset \mathbb{P}^5 \times \mathbb{P}^5$.

Proof. For any $V_1 \times V_2 \in \mathscr{Q}$, we have the following commutative diagram:

$$\begin{array}{ccc} V_1 & \xrightarrow{f_a|_{V_1}} & V_2 \\ \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle \iota} \\ X_1 := \mathbb{C}^4 & \xrightarrow{f_a} & X_2 := \mathbb{C}^4 \end{array}$$

We look at an arbitrary element of the Grassmannian, $V_1 \in Gr(2,4)$. V_1 is the span of two linearly independent vectors in $\mathbb{C}^4$, v, w. Write, without loss of generality, $v = \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 + \lambda_4x_4$ and $w = \mu_1x_1 + \mu_2x_2 + \mu_3x_3 + \mu_4x_4$, where $\lambda_i, \mu_i \in \mathbb{C}$.

Since the diagram commutes, we know that, for all $u \in V_1$, $f_a \circ \iota(u) \in \iota(V_2)$, hence $f_a(V_1) \subseteq V_2 \Leftrightarrow f_a|_{V_1}(v), f_a|_{V_1}(w) \in V_2$. In this case, this means that $\{\lambda_1y_1, \mu_1y_1\} \subset V_2$. We now have two cases for V_2, based on the values of λ_1 and μ_1.

Case 1: If $\lambda_1 = \mu_1 = 0$, then $f_a \circ \iota(v) = f_a \circ \iota(w) = 0$, so we have no criteria for V_2, so V_2 is an arbitrary element of $Gr(2,4)$.

We now look for when we have this case. We have that $\lambda_1 = \mu_1 = 0$, so we will look at the image of such a V_1 by the Plücker embedding $\tilde{\phi}$:

$$V_1 = \text{span}\left(\begin{pmatrix} 0 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}\right) \mapsto (0:0:0:\lambda_2\mu_3 - \lambda_3\mu_2 : \lambda_2\mu_4 - \lambda_4\mu_2 : \lambda_3\mu_4 - \lambda_4\mu_3),$$

so $\phi(V_1)$ is in the hyperplane $Z(x_0, x_1, x_2)$ of $\mathbb{P}^5$. So

$$U_1 = \{V_1 \times V_2 : V_1 = \operatorname{span}(\lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4, \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4) \in Gr(2,4),$$

$$V_2 \in Gr(2,4); \lambda_i, \mu_i \in \mathbb{C}\} \subset Gr_{(2,2)}(X, f).$$

Then $\Phi(U_1) = Z(x_0, x_1, x_2) \times Gr(2,4) = Z(x_0, x_1, x_2) \times Z(y_0 y_5 + y_1 y_4 - y_2 y_3)$.

Case 2: We now consider the case where $V_1 \in Gr(2,4) \setminus \tilde{\phi}^{-1}(Z(x_0, x_1, x_2))$. Without loss of generality, we change coordinates so that $\lambda_1 = 1$ and $\mu_1 = 0$. Then we know that $v' = y_1 \in V_2$, but we need a vector linearly independent of y_1, say w'. We can assume $w' = \mu_2 y_2 + \mu_3 y_3 + \mu_4 y_4$, $\mu_i \in \mathbb{C}$, since $\operatorname{span}\{y_1, w'\} = \operatorname{span}\{y_1, w' + c y_1\}$ for all $c \in \mathbb{C}$. We can conclude that this case is the set

$$U_2 = \Big\{V_1 \times V_2 : V_1 \in Gr(2,4), V_2 = \operatorname{span}(y_1, \mu_2 y_2 + \mu_3 y_3 + \mu_4 y_4) \in Gr(2,4)$$

$$\text{for some } \mu_i \in \mathbb{C}\Big\} \subset Gr_{(2,2)}(X, f)$$

The choice of such a w' is the same as choosing an element of $Gr(1,3) \cong \mathbb{P}^2$; however, we can give a description of this space as a subvariety of $Gr(2,4)$. V_2 has the following image by the Plücker embedding:

$$V_2 = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}\right) \mapsto (\mu_2 : \mu_3 : \mu_4 : 0 : 0 : 0),$$

so $\Phi(U_2) = (Z(x_0 x_5 + x_1 x_4 - x_2 x_3) \setminus Z(x_0, x_1, x_2)) \times Z(y_3, y_4, y_5) \subset \mathbb{P}^5 \times \mathbb{P}^5$.

Consider the maps $\iota_i : U_i \hookrightarrow \mathbb{P}^5 \times \mathbb{P}^5$ where ι_i is Φ composed with the inclusion $U_i \hookrightarrow Gr(2,4) \times Gr(2,4)$. We can glue their images together to get the variety induced by the quiver. $\mathscr{Q} = \operatorname{im} \iota_1 \cup \operatorname{im} \iota_2 \subset Gr(2,4) \times Gr(2,4)$ is the quiver Grassmannian associated to the quiver Q. Note that $Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5) \subset \tilde{\phi}(U_1)$, so then $\tilde{\phi}(U_1 \cup U_2) = Z(x_0 x_5 + x_1 x_4 - x_2 x_3) \times Z(y_3, y_4, y_5) \cup Z(x_0, x_1, x_2) \times Z(y_0 y_5 + y_1 y_4 - y_2 y_3)$. □

Note that the intersection of the two irreducible components is $Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5)$.

4.2 The Torus and the Torus Action

We define a one-dimensional complex algebraic torus to be the spectrum $T = \text{Spec}\,(\mathbb{C}[x, x^{-1}])$. Since $\mathbb{C}$ is algebraically closed, prime ideals of T can be written as $\langle x - c\rangle$, so the set of closed points of the spectrum is $\mathbb{C}^*$. Generalizing, the m-dimensional algebraic torus is the spectrum $T^m := \text{Spec}\left(\mathbb{C}[x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}]\right) = (\mathbb{C}^*)^m$.

Lemma 4.2.1. $(\mathbb{C}^*)^m$ *is an affine variety.*

Proof. It is sufficient to prove that $\mathbb{C}^*$ is an affine variety. By definition, $\mathbb{C}^* = \{(x) : x \neq 0\} \subset \mathbb{A}^1$. We see that $\mathbb{C}^*$ is the variety $\{(x, y) : xy = 1\} \subset \mathbb{A}^2$. This is the trick of Rabinowitsch, an approach typically taken to prove Strong Nullstellensatz using Weak Nullstellensatz. (page 59 of [H])

Alternatively, we may view $(\mathbb{C}^*)^m$ as an affine subvariety of GL_m. GL_m can be interpreted as an affine subvariety of $\mathbb{A}^{m^2+1}$. Using the trick of Rabinowitsch, $GL_m = \{M \times x \in M_{m\times m}(\mathbb{C}) \times \mathbb{C} : x \det M = 1\} \subset \mathbb{A}^{m^2+1}$. In this sense, $(\mathbb{C}^*)^m$ is the affine subvariety of GL_m only consisting of all of the diagonal matrices. □

Definition 4.2.2 ([S])**.** *A linear algebraic group G satisfies one of the following (equivalent) conditions:*

1.) the radical of the connected component G^0 of the unit element of G is an algebraic torus;

2.) the unipotent radical of the group G^0 is trivial; or

3.) the group G^0 is a product of closed normal subgroups S and T that are a semisimple algebraic group and an algebraic torus, respectively.

Clearly, any algebraic torus is a linear algebraic group.

Definition 4.2.3 ([S])**.** *A linear algebraic group G is linearly reductive if one of the following (equivalent) statements are true:*

1.) each rational linear representation of G is completely reducible; or

2.) for each rational linear representation $\rho : G \to GL(W)$ and any $\rho(G)$-invariant vector $w \in W \setminus \{0\}$ there is a $\rho(G)$-invariant function f on W such that $f(w) \neq 0$.

Since we work over characteristic 0, a group G is linearly reductive if and only if it is reductive [S]. In the case of the algebraic torus, the first statement holds, so an algebraic torus is a linearly reductive algebraic torus (cf. Proposition 8.4 on page 113 of [B]). Note that an algebraic torus is a linearly reductive algebraic group in any characteristic (cf. page 464 of [E]).

We have a nice symmetry between the two cases. $\operatorname{im}\iota_1$ cuts out a hyperplane in the $Gr(2,4)$ associated to V_1 and, similarly, $\operatorname{im}\iota_2$ cuts out a hyperplane isomorphic to the first one in the $Gr(2,4)$ associated to V_2.

Definition 4.2.4 (cf. page 51 of [F] and pages 2-3 of [MFK])**.** *An action ψ of an algebraic group G on a variety V is a morphism $G \times V \to V$ that satisfies the criteria 1.) $\psi(xy \times v) = \psi(x \times \psi(y \times v))$, and 2.) $\psi(1 \times v) = v$.*

We will describe a map of the 2-torus $(\mathbb{C}^*)^2$ on $\hat{\mathscr{Q}} = \Phi(\mathscr{Q})$ by first defining a torus action on $\mathbb{P}^5 \times \mathbb{P}^5$. Define $\Psi : (\mathbb{C}^*)^2 \times (\mathbb{P}^5 \times \mathbb{P}^5) \to \mathbb{P}^5 \times \mathbb{P}^5$:

$$(g_1, g_2) \times ((x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \times (y_0 : y_1 : y_2 : y_3 : y_4 : y_5)) \mapsto$$

$$(g_1x_0 : g_1x_1 : g_1x_2 : x_3 : x_4 : x_5) \times (y_0 : y_1 : y_2 : g_2y_3 : g_2y_4 : g_2y_5)$$

The goal is to prove that $\Psi|_{\hat{\mathscr{Q}}}$ is an action of $(\mathbb{C}^*)^2$ on $\hat{\mathscr{Q}}$.

Lemma 4.2.5. *Ψ is an action of $(\mathbb{C}^*)^2$ on $\mathbb{P}^5 \times \mathbb{P}^5$.*

Proof. We first prove that Ψ is a morphism and then prove that the other two criteria are fulfilled. We can cover $\mathbb{P}^5 \times \mathbb{P}^5$ by an affine cover of open sets $U_i \times V_j$, where $U_i = \mathbb{P}^5 \setminus Z(x_i) \cong \mathbb{A}^5$ and $V_j = \mathbb{P}^5 \setminus Z(y_j) \cong \mathbb{A}^5$.

Suppose $i = 5$ and $j = 5$. We construct a map $\tilde{\Psi}_{55} : (\mathbb{C}^2)^2 \times (\mathbb{A}^5 \times \mathbb{A}^5) \to \mathbb{A}^5 \times \mathbb{A}^5$ that maps

$$(r \times s) \times (x \times y) = ((r_1, r_2) \times (s_1, s_2)) \times ((x_0, \dots, x_4) \times (y_0, \dots, y_4)) \mapsto$$

$$(r_1x_0, r_1x_1, r_1x_2, x_3, x_4) \times (s_2y_0, s_2y_1, s_2y_2, y_3, y_4).$$

Every component is a polynomial equation, so $\tilde{\Psi}_{55}$ is a morphism. We note that, on the subvariety $Z(r_1r_2-1)\times Z(s_1s_2-1)$, $r_2=\frac{1}{r_1}$ and $s_2=\frac{1}{s_1}$. Consider the natural inclusion $\iota:\mathbb{A}^5\times\mathbb{A}^5\hookrightarrow\mathbb{P}^5\times\mathbb{P}^5$. So if $(r\times s)\times(x\times y)\in(Z(r_1r_2-1)\times Z(s_1s_2-1))\times(\mathbb{A}^5\times\mathbb{A}^5)$, then when we map through $\iota\circ\tilde{\Psi}_{55}:(\mathbb{C}^2)^2\times(\mathbb{A}^5\times\mathbb{A}^5)\to\mathbb{A}^5\times\mathbb{A}^5\hookrightarrow\mathbb{P}^5\times\mathbb{P}^5$, then

$$\iota\circ\tilde{\Psi}_{55}((r\times s)\times(x\times y))=\iota((r_1x_0,r_1x_1,r_1x_2,x_3,x_4)\times(s_2y_0,s_2y_1,s_2y_2,y_3,y_4))=$$

$$(r_1x_0:r_1x_1:r_1x_2:x_3:x_4:1)\times(s_2y_0:s_2y_1:s_2y_2:y_3:y_4:1)=$$

$$(r_1x_0:r_1x_1:r_1x_2:x_3:x_4:1)\times(y_0:y_1:y_2:s_1y_3:s_1y_4:s_1y_5),$$

which agrees with the action Ψ. Using the same techniques shown here, we can make the morphisms $\tilde{\Psi}_{ij}$ for any $i,j\in\{0,\dots,5\}$. We then have proven that Ψ is locally a morphism, which is sufficient.

We prove the rest by brute force.

$$\Psi((g_1h_1,g_2h_2)\times((x_0:x_1:x_2:x_3:x_4:x_5)\times(y_0:y_1:y_2:y_3:y_4:y_5))=$$

$$(g_1h_1x_0:g_1h_1x_1:g_1h_1x_2:x_3:x_4:x_5)\times(y_0:y_1:y_2:g_2h_2y_3:g_2h_2y_4:g_2h_2y_5).$$

And

$$\Psi((g_1,g_2)\times\Psi((h_1,h_2)\times((x_0:x_1:x_2:x_3:x_4:x_5)\times(y_0:y_1:y_2:y_3:y_4:y_5)))=$$

$$\Psi((g_1,g_2)\times((h_1x_0:h_1x_1:h_1x_2:x_3:x_4:x_5)\times(y_0:y_1:y_2:h_2y_3:h_2y_4:h_2y_5)))=$$

$$(g_1h_1x_0:g_1h_1x_1:g_1h_1x_2:x_3:x_4:x_5)\times(y_0:y_1:y_2:g_2h_2y_3:g_2h_2y_4:g_2h_2y_5).$$

The second criterion is trivial. □

Proposition 4.2.6. $\Psi|_{(\mathbb{C}^*)^2\times\hat{\mathcal{Q}}}$ *is an action on* $\hat{\mathcal{Q}}$.

Proof. Let $x\times y\in\hat{\mathcal{Q}}$. First, let us assume that $\Phi^{-1}(x\times y)\in U_1$. This means that $x=(0:0:0:x_3:x_4:x_5)\in Z(x_0,x_1,x_2)$, thus $\tilde{\phi}(V_1)$ is fixed. Since x stayed in $Z(x_0,x_1,x_2)$

we need to check that $(y_0 : y_1 : y_2 : g_2y_3 : g_2y_4 : g_2y_5) \in Z(y_0y_5 + y_1y_4 - y_2y_3)$. Now $y \in Z(y_0y_5 + y_1y_4 - y_2y_3)$, so we just need to plug in to see that

$$y_0(g_2y_5) + y_1(g_2y_4) - y_2(g_2y_3) = g_2(y_0y_5 + y_1y_4 - y_2y_3) = 0$$

so $\Psi((g_1, g_2) \times (x \times y)) \in \hat{\mathscr{Q}}$.

If we now assume that $\Phi^{-1}(x \times y) \in U_2$, $y \in Z(y_3, y_4, y_5)$, so y is fixed by our action of $\mathbb{C}^*$. Also, $x \in Z(x_0x_5 + x_1x_4 - x_2x_3)$ hence, as above,

$$(g_1x_0)x_5 + (g_1x_1)x_4 - (g_1y_2)x_5 = g_1(x_0x_5 + x_1x_4 - x_2x_3) = 0$$

so then we know that $\Psi((g_1, g_2) \times (x \times y)) \in \hat{\mathscr{Q}}$.

Since $\hat{\mathscr{Q}} \subset \mathbb{P}^5 \times \mathbb{P}^5$ is a subvariety, $\Psi|_{(\mathbb{C}^*)^2 \times \hat{\mathscr{Q}}}$ is a morphism. □

Definition 4.2.7. *The orbit of $v \in V$ by an action $\psi : G \times V \to V$ is the set $o_\psi(v) = \psi(G \times v) = \{\psi(x \times v) \in V : x \in G\}$.*

Using the action of ψ, we may define an equivalence relation $\sim$ on V so that $v \sim v'$ if and only if $o_\psi(v) = o_\psi(v')$ (cf. page 55 of [F]). We investigate the orbit of an element $x \times y \in \hat{\mathscr{Q}}$.

Lemma 4.2.8. *If $x \times y \in Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5)$, then $x \times y$ is fixed by Ψ. If $x \times y = ((a_0 : a_1 : a_2 : a_3 : a_4 : a_5) \times (b_0 : b_1 : b_2 : 0 : 0 : 0)) \in (Z(x_0x_5 + x_1x_4 - x_2x_3) \times Z(y_3, y_4, y_5)) \setminus (Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5))$, then the orbit of $x \times y$ is one-dimensional and closed in $(Z(x_0x_5 + x_1x_4 - x_2x_3) \times Z(y_3, y_4, y_5)) \setminus (Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5))$.*

Proof. If $x \times y \in Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5)$, (i.e., $x \times y = (0 : 0 : 0 : a_3 : a_4 : a_5) \times (b_0 : b_1 : b_2 : 0 : 0 : 0) \in \mathbb{P}^5 \times \mathbb{P}^5$) then

$$\Psi((g_1, g_2) \times (x \times y) = \Psi((g_1, g_2) \times ((0 : 0 : 0 : a_3 : a_4 : a_5) \times (b_0 : b_1 : b_2 : 0 : 0 : 0))$$

$$= (0 : 0 : 0 : a_3 : a_4 : a_5) \times (b_0 : b_1 : b_2 : 0 : 0 : 0)$$

So, $x \times y$ is Ψ-invariant.

If $x \times y = ((a_0 : a_1 : a_2 : a_3 : a_4 : a_5) \times (b_0 : b_1 : b_2 : 0 : 0 : 0)) \in Z(x_0x_5 + x_1x_4 - x_2x_3) \times Z(y_3, y_4, y_5) \setminus (Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5))$, then

$$\Psi((g_1, g_2) \times (x \times y)) = (g_1a_0 : g_1a_1 : g_1a_2 : a_3 : a_4 : a_5) \times (b_0 : b_1 : b_2 : 0 : 0 : 0).$$

We know that there exists $i \in \{0, 1, 2\}$ such that $a_i \neq 0$. Suppose $(a_0 : a_1 : a_2 : a_3 : a_4 : a_5) \times (b_0 : b_1 : b_2 : 0 : 0 : 0)$ is in the same orbit as $(a'_0 : a'_1 : a'_2 : a'_3 : a'_4 : a'_5) \times (b'_0 : b'_1 : b'_2 : 0 : 0 : 0)$ then we know that $b'_i = b_i$ for $i \in \{0, 1, 2\}$.

Moreover, if all a_i are nonzero, then we have the equations

$$\frac{a'_0}{a_0} = \frac{a'_1}{a_1} = \frac{a'_2}{a_2}, \quad \frac{a'_3}{a_3} = \frac{a'_4}{a_4} = \frac{a'_5}{a_5}.$$

The values $(a'_0 : a'_1 : a'_2 : a'_3 : a'_4 : a'_5)$ that fulfill all of these equations cuts out a one-dimensional closed variety in $\mathbb{P}^5$. The orbit of $x \times y$ is contained in this variety. If one of the a_i is zero, say $i = 0$, and the others are nonzero, then we have the equations

$$a'_0 = 0, \quad \frac{a'_1}{a_1} = \frac{a'_2}{a_2}, \quad \frac{a'_3}{a_3} = \frac{a'_4}{a_4} = \frac{a'_5}{a_5},$$

that cuts out a one-dimensional variety in $\mathbb{P}^5$. Note for every a_i that is equal to zero, we lose one equation and gain another, so the dimension of the variety is fixed. However, we cut out from this variety the element that is in $Z(x_3, x_4, x_5)$. We then have the orbit of $x \times y$ by the action Ψ. Note that inside the open set $(Z(x_0x_5 + x_1x_4 - x_2x_3) \times Z(y_0, y_1, y_2)) \setminus (Z(x_3, x_4, x_5)) \times Z(y_3, y_4, y_5))$, the orbit of $x \times y$ is closed. □

We now introduce some definitions and a theorem of Mumford that will allow us to take the quotient.

Definition 4.2.9 (Pages 55-6 of [F])**.** *Let G be an affine group and V an affine variety. A quotient of V by G is a pair (V_0, α) where G acts trivially on V_0 and $\alpha : V \to V_0$ is a G-morphism (i.e., $\alpha(gv) = g\alpha(v)$ for all $g \in G$, $v \in V$) such that if $\delta : V \to V'_0$ is a G-morphism and V'_0 is G-trivial, there exists a G-morphism ξ such that $\delta = \xi \circ \alpha$.*

Definition 4.2.10 (Page 56 of [F]). *Let G act on an affine variety V. We say that the quotient (V_0, α) is an orbit space if the following conditions hold:*

1.) for every $w \in V_0$, $\alpha^{-1}(w)$ is an orbit,

2.) $U \subset V_0$ is open if and only if $\alpha^{-1}(U)$ is open in V, and

3.) α^* is a $\mathbb{C}$-isomorphism of $\mathbb{C}[V_0]$ onto $\mathbb{C}[V]^G$.

Theorem 4.2.11 (Page 159 of [F]). *(Mumford's Theorem) Let G be a linearly reductive affine group and let V be an affine algebraic set. Let ψ be an action of G on V over $\mathbb{C}$. Then a quotient of V by G exists. If the orbits of ψ in V are closed, then this quotient is an orbit space.*

Note that we have not defined a quotient for a projective variety. For the purposes of this thesis, we will define the quotient of a projective variety as the set of all positive-dimensional orbits of an action. To investigate such a quotient, we will take the variety, cover it with globally invariant affine open sets (cutting out all points with zero-dimensional orbits) and then find the orbit space of the variety with respect to the action. We then will glue it back together on the intersections of the open sets to form the set of positive-dimensional orbits. As we will see, this quotient is a projective variety.

We first look at the irreducible component $Z(x_0x_5 + x_1x_4 - x_2x_3) \times Z(y_0, y_1, y_2)$. We look at the action Ψ:

$$\Psi((g_1, g_2) \times (x \times y)) = (g_1x_0 : g_1x_1 : g_1x_2 : x_3 : x_4 : x_5) \times y.$$

In this component, y is fixed, so we look at the map $\Psi_1 : \mathbb{C}^* \times Z(x_0x_5 + x_1x_4 - x_2x_3) \rightarrow Z(x_0x_5 + x_1x_4 - x_2x_3)$ where $\Psi_1(g \times (x_0, x_1, x_2, x_3, x_4, x_5)) = (g_1x_0 : g_1x_1 : g_1x_2 : x_3 : x_4 : x_5)$. Note that the Ψ_1-invariant elements of $Z(x_0x_5 + x_1x_4 - x_2x_3)$ make up the subvariety $Z(x_0, x_1, x_2) \cup Z(x_3, x_4, x_5)$.

Proposition 4.2.12. *The quotient $Z(x_0x_5 + x_1x_4 - x_2x_3) \setminus (Z(x_0, x_1, x_2) \cup Z(x_3, x_4, x_5))/\mathbb{C}^*$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^2$.*

Proof. We cover $Z(x_0x_5 + x_1x_4 - x_2x_3) \setminus (Z(x_0, x_1, x_2) \cup Z(x_3, x_4, x_5))$ with affine varieties that do not touch the zero-dimensional orbits. We then take their quotients with respect to the action Ψ_1. Let $i \in \{0, 1, 2\}$. Then $V_i := \mathbb{P}^5 \setminus Z(x_i) \cong \mathbb{A}^5$ is an affine variety. Using the trick of Rabinowitsch, if $j \in \{3, 4, 5\}$, then $V_{ij} = \mathbb{P}^5 \setminus (Z(x_i) \cup Z(x_j))$ is an affine variety. Moreover, $(Z(x_0, x_1, x_2) \cup Z(x_3, x_4, x_5)) \cap (\mathbb{P}^5 \setminus (Z(x_i) \cup Z(x_j))) = \emptyset$, so the Ψ_1-invariant elements are already not in the affine variety. We now intersect V_{ij} with $Z(x_0x_5 + x_1x_4 - x_2x_3)$ to get an affine subvariety of V_{ij}. Call this intersection U_{ij}.

If we look at the affine variety V_0, we can express the action Ψ_1 as a mapping $\mathbb{C}^* \times \mathbb{P}^5 \to \mathbb{P}^5$:

$$g \times (1 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (g : gx_1 : gx_2 : x_3 : x_4 : x_5) = (1 : x_1 : x_2 : hx_3 : hx_4 : hx_5),$$

or, alternatively, as the map $\mathbb{C}^* \times \mathbb{A}^5 \to \mathbb{A}^5$:

$$g \times (x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, hx_3, hx_4, hx_5),$$

where $h = \frac{1}{g} \in H := \mathbb{C}^*$.

Cut V_0 into two proper open subsets, which are affine varieties in their own right, V_{03} and V_{04}. Intersecting with the hypersurface, we obtain $U_{03} = V_{03} \cap Z(x_0x_5 + x_1x_4 - x_2x_3)$ and $U_{04} = V_{04} \cap Z(x_0x_5 + x_1x_4 - x_2x_3)$, as above. Note that these two varieties cover $V_0 \cap Z(x_5 + x_1x_4 - x_2x_3) \setminus Z(x_3, x_4, x_5)$, so in order to understand the quotient on V_0, we just need to know the quotient on all of these affine varieties and then glue. Recall that the orbits of Ψ_1 are closed.

Using Mumford's theorem, we see that the quotient of U_{03} is an orbit space, since algebraic tori are linearly reductive algebraic groups, and the orbits of Ψ_1 are closed. We now want to compute the coordinate ring of the quotient so we look at the ring of invariants of U_{03}. $\mathbb{C}[U_{03}] = \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_3^{-1}]/\langle x_5 + x_1x_4 - x_2x_3\rangle = \mathbb{C}[x_1, x_2, x_3, x_4, x_3^{-1}]$ and $\mathbb{C}[U_{04}] = \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_4^{-1}]/\langle x_5 + x_1x_4 - x_2x_3\rangle = \mathbb{C}[x_1, x_2, x_3, x_4, x_4^{-1}]$. Note that since x_0 is nonzero, x_5 is a linear combination of x_3 and x_4, so we only need these two open subsets to cover U_0. We now find the ring of invariants for all of these rings under the action of Ψ_1.

Let $f \in \mathbb{C}[U_{03}]$. $f = \sum x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$, where $i_1, i_2, i_4 \in \mathbb{N} \cup \{0\}$, and $i_3 \in \mathbb{Z}$. Then $\psi^*(g \times f) = \sum h^{i_3+i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ (since $\Psi_1^*(F) = F \circ \Psi_1$ for a function F). If $f \in \mathbb{C}[U_0]^H$, then $\Psi_1(g \times f) = f$ so then

$$\sum h^{i_3+i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} = \sum x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}, \quad \text{for all } h \in \mathbb{C}^*.$$

This yields that $h^{i_3+i_4} = 1$ for any $h \in \mathbb{C}^*$, therefore $i_3 + i_4 = 0$ and $i_3 = -i_4$. Therefore $f = \sum x_1^{i_1} x_2^{i_2} x_3^{-i_4} x_4^{i_4}$ and $\mathbb{C}[U_{03}]^H = \langle x_1, x_2, x_3^{-1} x_4 \rangle$, thus $\operatorname{Spec}(\mathbb{C}[U_{03}]^H) \cong \mathbb{A}^3$. By symmetry, we have $\mathbb{C}[U_{04}]^H = \langle x_1, x_2, x_4^{-1} x_3 \rangle$ and likewise $\operatorname{Spec}(\mathbb{C}[U_{04}]^H) \cong \mathbb{A}^3$.

Note that the intersection of these two open sets, say U_{034}, is an open set such that $\mathbb{C}[U_{034}]^H = \langle x_1, x_2, x_3 x_4^{-1}, x_4 x_3^{-1} \rangle$, so

$$U_{034}/H = \operatorname{Spec}(\mathbb{C}[U_{034}]^H) = \operatorname{Spec}(\langle x_1, x_2, x_3 x_4^{-1}, x_4 x_3^{-1} \rangle) = \mathbb{A}^3 \setminus P,$$

where the coordinate ring for $\mathbb{A}^3$ is $\mathbb{C}[x_1, x_2, x_3]$ and P is the plane $Z(x_3)$. The gluing of $\operatorname{Spec}(\mathbb{C}[U_{03}]^H)$ with $\operatorname{Spec}(\mathbb{C}[U_{04}]^H)$ over the intersection gives us:

$$\begin{array}{ccc} U_{034}/H = \mathbb{A}^3 \setminus P & \hookrightarrow & U_{03}/H = \mathbb{A}^3; \\ \left(x_1, x_2, \frac{x_3}{x_4}\right) & \mapsto & \left(x_1, x_2, \frac{x_3}{x_4}\right); \\ U_{034}/H = \mathbb{A}^3 \setminus P & \hookrightarrow & U_{04}/H = \mathbb{A}^3; \\ \left(x_1, x_2, \frac{x_3}{x_4}\right) & \mapsto & \left(x_1, x_2, \frac{x_4}{x_3}\right). \end{array}$$

So gluing these two open sets of the cover gives us $U_0/H = U_{03}/H \cup U_{04}/H = \operatorname{Spec}(\mathbb{C}[x_1, x_2]) \times \operatorname{Proj}(\mathbb{C}[x_3, x_4]) \cong \mathbb{A}^2 \times \mathbb{P}^1$.

We now can see from symmetry that $U_1/H = \operatorname{Spec}(\mathbb{C}[x_0, x_2]) \times \operatorname{Proj}(\mathbb{C}[x_3, x_5])$ and $U_2/H = \operatorname{Spec}(\mathbb{C}[x_0, x_1]) \times \operatorname{Proj}(\mathbb{C}[x_4, x_5])$. The affine schemes in all of these product spaces will glue together to form $\operatorname{Proj}(\mathbb{C}[x_0, x_1, x_2]) \cong \mathbb{P}^2$, but the copy of $\mathbb{P}^1$ will change over this space, thus we have a $\mathbb{P}^1$ bundle over $\mathbb{P}^2$. □

We now investigate what bundle it is. We will compute the transition functions between the open sets U_i/H, and look at how the $\mathbb{P}^1$ changes as we vary where we are in $\mathbb{P}^2$. Let us start by computing the transition function from U_0/H to U_1/H over the intersection.

The affine plane $\mathrm{Spec}(\mathbb{C}[x_1, x_2])$ can be thought of as the affine open set $\mathbb{P}^2 \setminus Z(x_0)$ where the homogeneous coordinate ring of $\mathbb{P}^2$ is $A = \mathbb{C}[x_0, x_1, x_2]$. Computing the transition functions we obtain:

$$\begin{array}{c} U_0/H \to U_1/H \\ (1, x_1, x_2) \mapsto \left(\frac{1}{x_1}, 1, \frac{x_2}{x_1}\right), \end{array} \qquad \begin{pmatrix} 1 & 0 \\ x_2 & -x_1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix};$$

Analogously, we can compute transition functions from U_1/H to U_2/H

$$\begin{array}{c} U_1/H \to U_2/H \\ (x_0, 1, x_2) \mapsto \left(\frac{x_0}{x_2}, \frac{1}{x_2}, 1\right), \end{array} \qquad \begin{pmatrix} x_2 & -x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix};$$

and also from U_0/H to U_2/H

$$\begin{array}{c} U_0/H \to U_2/H \\ (1, x_1, x_2) \mapsto \left(\frac{1}{x_2}, \frac{x_1}{x_2}, 1\right), \end{array} \qquad \begin{pmatrix} 0 & 1 \\ x_2 & -x_1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix};$$

We cannot simultaneously diagonalize these transition matrices, so this bundle does not split, and does not have an easy description. Looking back at the original action Ψ, there is a gluing. Let B be the $\mathbb{P}^1$ bundle over $\mathbb{P}^2$. By symmetry, we now have a description of the quotient of an open set of $\hat{\mathscr{Q}}$ by $(\mathbb{C}^*)^2$. Let S be the variety $(Z(x_0, x_1, x_2) \times Z(y_3, y_4, y_5)) \cup (Z(x_3, x_4, x_5) \times Z(y_3, y_4, y_5)) \cup (Z(x_0, x_1, x_2) \times Z(y_0, y_1, y_2))$. Therefore,

$$(\hat{\mathscr{Q}} \setminus S)/(\mathbb{C}^*)^2 = B \times Z(y_3, y_4, y_5) \sqcup Z(x_0, x_1, x_2) \times B.$$

Bibliography

[B] A. Borel. Linear Algebraic Groups, Springer-Verlag, New York, 1991.

[CR] P. Caldero, M. Reineke. On the quiver Grassmannian in the acyclic case. J. Pure Appl. Algebra 212 (2008), 2369-2380.

[DWZ] H. Derksen, J. Weyman, A. Zelevinsky. Quivers with potentials and their representations II: Applications to cluster algebras. arXiv:0904.0676v1 [math.RA], April 3, 2009.

[E] D. Eisenbud. Commutative Algebra, with a view toward algebraic geometry. Springer-Verlag, New York, 1991.

[F] J. Fogarty. Invariant Theory, W. A Benjamin, New York, 1969.

[GH] P. Griffiths, J. Harris. Principles of Algebraic Geometry, Wiley, New York, 1978.

[H] J. Harris. Algebraic Geometry: A First Course, Springer-Verlag, Berlin, New York: 1995.

[K] M. M. Kapranov. Chow quotients of Grassmannians, I in I. M. Gelfand Seminar, Adv. Soviet Math. 16, Part 2, Amer. Math. Soc., Providence, 1993, 29-110.

[MFK] D. Mumford, J. Fogarty, F. Kirwan. Geometric Invariant Theory, Third Enlarged Edition, Springer-Verlag, Berlin, New York, 1994.

[S] M. Hazewinkel ed. Encyclopaedia of Mathematics: Reductive Group. eom.springer.de/R/r080440.htm May 4 2009.

Appendix A

Macaulay 2 and Magma Coding Used

This is the Macaulay 2 coding to give us the geometry of $\pi_1(\mathcal{Q}_2)$ in Chapter 3:

```
i1 : R=QQ[x,y,z,w]
o1 = R
o1 : PolynomialRing
i2 : M = matrix { {x,w,z,y},{y,x,w,z},{z,y,x,w},{w,z,y,x}}
o2 = | x w z y |
     | y x w z |
     | z y x w |
     | w z y x |
             4       4
o2 : Matrix R  <--- R
i3 : I = minors(3,M) --Built in Macaulay 2 for the ideal defined by kxk minors
i4 : hilbertPolynomial(I, Projective=>false)
o4 = 6i - 2
--This tells us we have a curve of degree 6 and arithmetic genus 3
o4 : QQ[i]
i5 : Z = singularLocus(I);
i6 : Z
o6 = Z
i8 : hilbertPolynomial(Z,Projective=>false)
o8 = 16
o8 : QQ[i]
--This tells us the curve is singular
i10 : toString I
o10 = ideal(x^3+y^2*z-x*z^2-2*x*y*w+z*w^2,x^2*y+
      y*z^2-y^2*w-2*x*z*w+w^3,x*y^2-x^2*z+z^3-2*y*z*w+x*w^2,
      y^3-2*x*y*z+x^2*w+z^2*w-y*w^2,y^3-2*x*y*z+x^2*w+z^2*w-
      y*w^2,x^3+y^2*z-x*z^2-2*x*y*w+z*w^2,x^2*y+
      y*z^2-y^2*w-2*x*z*w+w^3,x*y^2-x^2*z+z^3-2*y*z*w+x*w^2,
      x*y^2-x^2*z+z^3-2*y*z*w+x*w^2,y^3-2*x*y*z+
      x^2*w+z^2*w-y*w^2,x^3+y^2*z-x*z^2-2*x*y*w+z*w^2,
      x^2*y+y*z^2-y^2*w-2*x*z*w+w^3,x^2*y+y*z^2-y^2*w-
      2*x*z*w+w^3,x*y^2-x^2*z+z^3-2*y*z*w+x*w^2,y^3-2*x*y*z+
```

```
      x^2*w+z^2*w-y*w^2,x^3+y^2*z-x*z^2-2*x*y*w+z*w^2)
i11 : quit
Process M2 finished
```

Then we move to Magma in order to finish.

```
> Q:=RationalField();
> P3<x,y,z,w>:=ProjectiveSpace(Q,3);
> X:=Scheme(P3,[x^3+y^2*z-x*z^2-2*x*y*w+z*w^2,x^2*y+y*z^2-
y^2*w-2*x*z*w+w^3,x*y^2-x^2*z+z^3-2*y*z*w+x*w^2,y^3-2*x*y*z+x^2*w
+z^2*w-y*w^2,y^3-2*x*y*z+x^2*w+z^2*w-y*w^2,x^3+y^2*z-x*z^2-2*x*y*w
+z*w^2,x^2*y+y*z^2-y^2*w-2*x*z*w+w^3,x*y^2-x^2*z+z^3-2*y*z*w
+x*w^2,x*y^2-x^2*z+z^3-2*y*z*w+x*w^2,y^3-2*x*y*z+x^2*w+z^2*w-
y*w^2,x^3+y^2*z-x*z^2-2*x*y*w+z*w^2,x^2*y+y*z^2-y^2*w-2*x*z*w
+w^3,x^2*y+y*z^2-y^2*w-2*x*z*w+w^3,x*y^2-x^2*z+z^3-2*y*z*w
+x*w^2,y^3-2*x*y*z+x^2*w+z^2*w-y*w^2,x^3+y^2*z-x*z^2-2*x*y*w+ z*w^2]);
> Dimension(X);
1
> C:=Curve(X);
> ArithmeticGenus(C);
3
> Degree(C);
6
> IsReduced(C);
true
> IsIrreducible(C);
false
> IrreducibleComponents(C);
[
    Scheme over Rational Field defined by
    y^2 + 2*y*z + 2*z^2 + 2*z*w + w^2,
    x + y + z + w,
    Scheme over Rational Field defined by
    y^2 - 2*y*z + 2*z^2 - 2*z*w + w^2,
    x - y + z - w,
    Scheme over Rational Field defined by
    x + z,
    y + w,
    Scheme over Rational Field defined by
    x - z,
    y - w
]
> SingularPoints(C);
{@ (-1 : 1 : -1 : 1), (1 : 1 : 1 : 1) @}
```

```
> K<i>:=CyclotomicField(4);
> C:=BaseChange(C,K);
> PK3<x,y,z,w>:=AmbientSpace(C);
> IrreducibleComponents(C);
[
    Scheme over K defined by
    x + i*z + (i + 1)*w,
    y + (-i + 1)*z - i*w,
    Scheme over K defined by
    x + i*z + (-i - 1)*w,
    y + (i - 1)*z - i*w,
    Scheme over K defined by
    x - i*z + (i - 1)*w,
    y + (-i - 1)*z + i*w,
    Scheme over K defined by
    x - i*z + (-i + 1)*w,
    y + (i + 1)*z + i*w,
    Scheme over K defined by
    x + z,
    y + w,
    Scheme over K defined by
    x - z,
    y - w
]
> SingularPoints(C);
{@ (-1 : 1 : -1 : 1), (1 : 1 : 1 : 1), (-i : -1 : i : 1), (i : -1 : -i : 1) @}

--Dimension -1 means the intersection is empty
> L1:=IrreducibleComponents(C)[1];
> L2:=IrreducibleComponents(C)[2];
> L3:=IrreducibleComponents(C)[3];
> L4:=IrreducibleComponents(C)[4];
> L5:=IrreducibleComponents(C)[5];
> L6:=IrreducibleComponents(C)[6];
> Dimension(Intersection(L1,L2));
-1
> Dimension(Intersection(L1,L3));
0
> Dimension(Intersection(L1,L4));
0
> Dimension(Intersection(L1,L5));
0
> Dimension(Intersection(L1,L6));
0
```

```
> Dimension(Intersection(L2,L3));
0
> Dimension(Intersection(L2,L4));
0
> Dimension(Intersection(L2,L5));
0
> Dimension(Intersection(L2,L6));
0
> Dimension(Intersection(L3,L4));
-1
> Dimension(Intersection(L3,L5));
0
> Dimension(Intersection(L3,L6));
0
> Dimension(Intersection(L4,L5));
0
> Dimension(Intersection(L4,L6));
0
> Dimension(Intersection(L5,L6));
-1
```

So we know that it is a tetrahedron since it has six lines where they intersect at four points in such a way to create a tetrahedron. We now go back to MaCaulay 2 to look at the geometry of $\pi_1(\mathscr{Q}_3)$:

```
i1 : R=QQ[x,y,z,w]
o1 = R
o1 : PolynomialRing
i2 : M = matrix { {x,w,z,y},{y,x,w,z},{z,y,x,w},{w,z,y,x}}
o2 = | x w z y |
     | y x w z |
     | z y x w |
     | w z y x |
               4       4
o2 : Matrix R  <--- R
i3 : I = minors(4,M)
o3 : Ideal of R
i4 : hilbertPolynomial(I, Projective=>false)
       2
o4 = 2i  + 2
o4 : QQ [i]
i5 : Z=singularLocus(I);
i6 : hilbertPolynomial(Z, Projective=>false)
```

```
o6 = 6i - 2
o6 : QQ [i]
i7 : toString I
o7 = ideal(x^4-y^4+4*x*y^2*z-2*x^2*z^2+z^4-4*x^2*y*w-4*y*z^2*w
      +2*y^2*w^2+4*x*z*w^2-w^4)
i8 : toString Z
o8 = R/(x^4-y^4+4*x*y^2*z-2*x^2*z^2+z^4-4*x^2*y*w-4*y*z^2*w+2*y^2*w^2+
     4*x*z*w^2-w^4,4*x^3+4*y^2*z-4*x*z^2-8*x*y*w+4*z*w^2,-4*y^3+8*x*y*z
     -4*x^2*w-4*z^2*w+4*y*w^2,4*x*y^2-4*x^2*z+4*z^3-8*y*z*w+4*x*w^2,
     -4*x^2*y-4*y*z^2+4*y^2*w+8*x*z*w-4*w^3)
```

We note that the Hilbert polynomial of the singular locus is equal to that of $\pi_1(\mathscr{Q}_2)$ which makes us curious. We move back to Magma.

```
> Q:=RationalField();
> K<i>:=CyclotomicField(4);
> P3<x,y,z,w>:=ProjectiveSpace(K,3);
> X:=Scheme(P3,[x^4-y^4+4*x*y^2*z-2*x^2*z^2+z^4-4*x^2*y*w-4*y*z^2*w+2*y^2*w^2+
4*x*z*w^2-w^4]);
> Dimension(X);
2
> Degree(X);
4
> IsReduced(X);
true
> IsIrreducible(X);
false
> IrreducibleComponents(X);
[
    Scheme over K defined by
    x - y + z - w,
    Scheme over K defined by
    x + y + z + w,
    Scheme over K defined by
    x - i*y - z + i*w,
    Scheme over K defined by
    x + i*y - z - i*w
]
> SingX:=Scheme(P3,[x^4-y^4+4*x*y^2*z-2*x^2*z^2+z^4-4*x^2*y*w-4*y*z^2*w+2*y^2*
w^2+4*x*z*w^2-w^4,4*x^3+4*y^2*z-4*x*z^2-8*x*y*w+4*z*w^2,-4*y^3+8*x*y*z-4*x^2*w
-4*z^2*w+4*y*w^2,4*x*y^2-4*x^2*z+4*z^3-8*y*z*w+4*x*w^2,-4*x^2*y-4*y*z^2+4*y^2*
w+8*x*z*w-4*w^3]);
> IrreducibleComponents(X);
```

```
[
    Scheme over K defined by
    x - y + z - w,
    Scheme over K defined by
    x + y + z + w,
    Scheme over K defined by
    x - i*y - z + i*w,
    Scheme over K defined by
    x + i*y - z - i*w
]
> IrreducibleComponents(SingX);
[
    Scheme over K defined by
    x + i*z + (i + 1)*w,
    y + (-i + 1)*z - i*w,
    Scheme over K defined by
    x + i*z + (-i - 1)*w,
    y + (i - 1)*z - i*w,
    Scheme over K defined by
    x - i*z + (i - 1)*w,
    y + (-i - 1)*z + i*w,
    Scheme over K defined by
    x - i*z + (-i + 1)*w,
    y + (i + 1)*z + i*w,
    Scheme over K defined by
    x + z,
    y + w,
    Scheme over K defined by
    x - z,
    y - w
]
```

Therefore, the suspicions are confirmed.