

HODGE SPACES OF REAL TORIC VARIETIES

by

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(Under the direction of Clint McCrory)

ABSTRACT

We introduce the notion of a cosheaf on a fan Σ and define the \mathbb{Z}_2 Hodge spaces $H_{pq}(\Sigma)$, which are the homology groups $H_p(\wedge^q \mathcal{E})$ of the q th exterior power of the cosheaf \mathcal{E} on Σ . Geometrically, for $\sigma \in \Sigma$ the stalk E_σ of the cosheaf \mathcal{E} is the compact real torus in the real orbit $O_\sigma(\mathbb{R})$ of the real toric variety $X_\Sigma(\mathbb{R})$. The \mathbb{Z}_2 Hodge spaces $H_{pq}(\Sigma)$ are indexed by pairs p, q with $0 \leq q \leq p \leq d$, where $d = \dim \Sigma$. When Σ is a smooth fan, we have $H_{pq}(\Sigma) = 0$ for $p \neq q$. However, for $p > q$ the spaces $H_{pq}(\Sigma)$ are not generally well understood. If Σ is the normal fan of a reflexive polytope Δ then we use polyhedral duality to compute the \mathbb{Z}_2 Hodge Spaces of Σ . In particular, if the cones of dimension at most k in the face fan Σ^* of Δ are smooth then we compute $H_{pq}(\Sigma)$ for $p \leq k - 2$. Moreover, if Σ^* is a smooth fan then we completely determine the spaces $H_{pq}(\Sigma)$.

The \mathbb{Z}_2 Hodge spaces of Σ are related to the topology of both the real and complex points of the toric variety X_Σ in the following way:

$$H_{pq}(\Sigma) = \overline{E}_{p,q}^1 = E_{p,q}^2,$$

where $(\overline{E}^r, \overline{d}^r)$ is a spectral sequence with

$$\overline{E}_{p,q}^1 \implies H_p(X_\Sigma(\mathbb{R}), \mathbb{Z}_2)$$

and (E^r, d^r) is a spectral sequence with

$$E_{p,q}^2 \implies H_{p+q}(X_\Sigma(\mathbb{C}), \mathbb{Z}_2).$$

When Σ^* is a smooth fan, we show the spectral sequence $(\overline{E}^r, \overline{d}^r)$ for X_Σ collapses at \overline{E}^1 and hence X_Σ is maximal, meaning that

$$\sum_i b_i(X(\mathbb{R})) = \sum_j b_j(X(\mathbb{C})),$$

where $b_i(*) = \text{rank} H_i(*, \mathbb{Z}_2)$.

In 2004, Bihan, Franz, McCrory, and van Hamel conjectured that every toric variety is maximal. All nonsingular projective toric varieties are known to be maximal. Moreover, if X is a projective toric variety such that either X has isolated singularities or the dimension of X is at most three then X is maximal. We present a six dimensional projective toric variety which is not maximal and hence gives a counterexample to the conjecture of Bihan, Franz, McCrory, and van Hamel.

INDEX WORDS: real toric variety, Hodge space, reflexive polytope, Smith-Thom inequality, Chow group

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DEDICATION

To my grandfather:



John Aurie Dean

(1921-2001)

As a chemistry professor at the University of Tennessee–Knoxville from 1950 to 1981, Dr. John Dean supervised 23 Ph.D. students and 50 Masters students. He authored or coauthored over 20 text and reference books as well as over 100 research articles. His life is an inspiration to any aspiring academic.

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CHAPTER 1

TORIC VARIETIES

1.1 FUNDAMENTAL DEFINITIONS AND PROPERTIES

We begin by reviewing background on toric varieties. We refer the reader to [Ful1] for a more detailed discussion.

1.1.1 FANS AND TORIC VARIETIES

Let $N \cong \mathbb{Z}^d$ and $M = \text{Hom}(N, \mathbb{Z})$ be dual lattices with dual pairing denoted $\langle \cdot, \cdot \rangle$. A *cone* σ in $N \otimes \mathbb{R}$ is the positive hull of a finite number of elements of N . We assume that a cone does not contain any lines through the origin. We will use the notation $\sigma \subset N$ in this case. The dual cone $\sigma^\vee \subset M$ determines a semigroup $\sigma^\vee \cap M := \{u \in M \mid \langle u, v \rangle \geq 0 \ \forall v \in \sigma\}$. The affine toric variety U_σ is defined to be $\text{Spec } \mathbb{C}[\sigma^\vee \cap M]$.

A *fan* Σ in $N \otimes \mathbb{R}$ is a collection of cones satisfying the following properties.

- If $\tau \in \Sigma$ and σ is a face of τ (written $\sigma < \tau$) then $\sigma \in \Sigma$.
- If $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face of both σ and τ .

We will write $\Sigma(i)$ for the set of i dimensional cones in Σ and $\sigma(i)$ for the set of i dimensional faces of the cone σ .

Suppose $\sigma < \tau$ in Σ . Then, $\tau^\vee \cap M \subset \sigma^\vee \cap M$ and hence $\mathbb{C}[\tau^\vee \cap M] \subset \mathbb{C}[\sigma^\vee \cap M]$. This gives $U_\sigma \subset U_\tau$ and the affine varieties $(U_\sigma)_{\sigma \in \Sigma}$ piece together to form a complex algebraic

variety $X_\Sigma = X_\Sigma(\mathbb{C})$. The algebraic torus

$$\begin{aligned} T &:= \text{Spec } \mathbb{C}[M] \\ &= \text{Hom}(M, \mathbb{C}^*) \\ &= N \otimes \mathbb{C}^* \end{aligned}$$

is dense in the variety X_Σ .

1.1.2 SUBVARIETIES AND THE CHOW GROUPS

Each cone $\sigma \in \Sigma$ defines a sublattice $\sigma \cap N + (-\sigma \cap N)$ of N which is generated as a subgroup by $\sigma \cap N$. We will abuse notation and write this sublattice as $\sigma \cap N$. We have a corresponding torus

$$O_\sigma := \frac{N}{\sigma \cap N} \otimes \mathbb{C}^*$$

of dimension $d - \dim \sigma$. The closed subvariety $V(\sigma)$ of X_Σ is defined to be the closure of the torus O_σ . This gives a one-to-one inclusion reversing correspondence:

$$\begin{array}{ccc} \text{Cones} & \longleftrightarrow & \text{Orbit closures} \\ \sigma \in \Sigma & & V(\sigma) \subset X_\Sigma \end{array} .$$

Note that $N/\sigma \cap N$ is the “ N ”-space for $V(\sigma)$ and $\sigma^\perp \cap M = \text{Hom}(N/\sigma \cap N, \mathbb{Z})$ is the “ M ”-space for $V(\sigma)$.

For an algebraic variety Y the integral homology Chow groups $A_k(Y)$ of Y are generated by the k dimensional irreducible closed subvarieties of Y . The relations in $A_k(Y)$ are generated by divisors $\text{div } f$ where f is a nonzero rational function on a $(k + 1)$ dimensional subvariety. For toric varieties, the Chow groups are discussed in [Ful1 §5.1] or [Ful2]. Proposition 1.1 in [Ful2] states that the Chow groups of toric varieties are equal to the torus invariant Chow groups, which we review here. The k th torus invariant Chow group is generated by the cycles $[V(\tau)]$ for $\tau \in \Sigma$ of codimension k . The relations are generated by torus invariant divisors of the following form. For $\sigma \in \Sigma$ of codimension $k + 1$, each $\mathbf{u} \in \sigma^\perp \cap M$ gives a

function $\chi^u \in \mathbb{C}[\sigma^\perp \cap M]$. The divisor $\text{div}\chi^u$ gives a relation in the k th torus invariant Chow group for X_Σ . If $\sigma < \tau$ with $\text{codim } \tau = k$ then we have the following

$$\text{ord}_{V(\tau)}(\text{div}\chi^u) = \langle \mathbf{u}, \mathbf{n}_{\sigma,\tau} \rangle$$

where $\mathbf{n}_{\sigma,\tau}$ is a lattice generator of $\tau \cap N / \sigma \cap N$. This gives the following relation.

$$\text{div}\chi^u = \bigoplus_{\sigma < \tau, \tau \in \Sigma(d-k)} \langle \mathbf{u}, \mathbf{n}_{\sigma,\tau} \rangle [V(\tau)]$$

1.1.3 PROJECTIVE TORIC VARIETIES AND THE MOMENT MAP

We will primarily be concerned with projective toric varieties. X_Σ is a projective toric variety provided the fan Σ is *strongly polytopal*. That is, there exists a polytope $\Delta = \text{conv}\{p_1, p_2, \dots, p_n\}$ with $p_1, p_2, \dots, p_n \in M$ such that the cones in Σ are in correspondence with the faces of Δ .

$$\sigma(= \sigma_f) = \{v \in N \otimes \mathbb{R} \mid \langle u, v \rangle \geq \langle w, v \rangle \quad \forall u \in f, w \in \Delta\}$$

In this case we say $\sigma \in \Sigma$ is the cone *dual* to the face f of Δ . When X_Σ is projective, we have a moment map $\mu : X_\Sigma \rightarrow M \otimes \mathbb{R}$ given by the following formula.

$$\mu(x) = \frac{1}{\sum \chi^u(x)} \sum_{u \in \Delta \cap M} |\chi^u(x)| u$$

The image of μ is the convex polytope Δ . We refer the reader to [Ful1 §4.2] for further properties of the moment map μ .

1.2 THE REAL POINTS OF A TORIC VARIETY

A real toric variety is a normal real variety that contains a real algebraic torus $(\mathbb{R}^*)^d$ as a dense subset. When $X_\Sigma = X_\Sigma(\mathbb{C})$ is a projective toric variety embedded in $\mathbb{C}\mathbb{P}^r$, the real variety $X_\Sigma(\mathbb{R})$ is the intersection of X_Σ with the real projective space $\mathbb{R}\mathbb{P}^r$. A discussion of real toric varieties can be found in [Sot]. Suppose X_Σ is projective with moment map μ . Since μ is invariant under complex conjugation, we have maps $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ for the complex

and real toric varieties $X_\Sigma(\mathbb{C})$ and $X_\Sigma(\mathbb{R})$, respectively. We write the complex torus as $T = (\mathbb{R}_{>0})^d \times (N \otimes \mathbb{R})/N$ and the real torus as $T(\mathbb{R}) = (\mathbb{R}_{>0})^d \times \frac{1}{2}N/N$. The map μ is shown below.

$$\begin{array}{ccccc}
 (\mathbb{R}_{>0})^d & \times & (N \otimes \mathbb{R})/N & \subset & X_\Sigma(\mathbb{C}) \\
 \parallel & & \cup & & \cup \\
 (\mathbb{R}_{>0})^d & \times & \frac{1}{2}N/N & \subset & X_\Sigma(\mathbb{R}) \\
 \cong \downarrow \mu & & & & \downarrow \mu_{\mathbb{R}} \\
 \text{int } \Delta & & & \subset & \Delta
 \end{array}
 \begin{array}{l}
 \curvearrowright \\
 \mu_{\mathbb{C}}
 \end{array}$$

It follows that if $p \in \text{int } \Delta$ then we have the following.

$$\begin{array}{ccc}
 \mu_{\mathbb{C}}^{-1}\{p\} & = & (N \otimes \mathbb{R})/N \cong (S^1)^d \\
 \cup & & \cup \\
 \mu_{\mathbb{R}}^{-1}\{p\} & = & \frac{1}{2}N/N \cong (S^0)^d
 \end{array}$$

Suppose $p \in \text{int } f$ with $\sigma \in \Sigma$ the cone dual to f . Then, we have the following.

$$\begin{array}{ccc}
 \mu_{\mathbb{C}}^{-1}\{p\} & = & \frac{(N \otimes \mathbb{R})/\sigma}{N/\sigma \cap N} \cong (S^1)^{\dim f} \\
 \cup & & \cup \\
 \mu_{\mathbb{R}}^{-1}\{p\} & = & \frac{\frac{1}{2}N/N}{\sigma \cap \frac{1}{2}N/\sigma \cap N} \cong (S^0)^{\dim f}
 \end{array} \tag{1.1}$$

In the sequel, we will use $N/2N$ as the compact torus contained in $T(\mathbb{R})$ and $N(\sigma) := \frac{N/2N}{\sigma \cap N/\sigma \cap 2N}$ as the compact torus in $O_\sigma(\mathbb{R})$ for convenience. The group $\sigma \cap N/\sigma \cap 2N$ will be denoted N_σ . The real moment map $\mu_{\mathbb{R}}$ induces a polyhedral cell structure on $X_\Sigma(\mathbb{R})$. Explicitly, we realize $X_\Sigma(\mathbb{R})$ as the quotient space (1.2)

$$X_\Sigma(\mathbb{R}) \cong \frac{\Delta \times N/2N}{\sim} \tag{1.2}$$

$$(x, t) \sim (x', t') \iff x = x' \quad \text{and} \quad t - t' \in N_\gamma$$

where γ is dual to $g < \Delta$ and $x \in \text{int } g$. Thus, the cells in (1.2) are of the form (g, t) with $t \in N(\gamma)$. A discussion of this cell structure can be found in [Gel2 §11.5 B].

1.3 GLUING COPIES OF Δ TO FORM A REAL TORIC VARIETY

In this section, we present an algorithm for identifying the faces of the 2^d copies of Δ . We will begin by gluing the facets and at step j we will glue faces of codimension j . When we identify two faces, we will glue the closed faces. The reason for this is as follows. Let (g, t) and (g, t') be two open faces identified in the relation \sim of (1.2). Then, $t - t' \in N_\gamma$ where γ is dual to g . Suppose $f < g$ in Δ and β is the cone dual to f . We have $\gamma < \beta$ and $N_\gamma \subset N_\beta$ yielding $t - t' \in N_\beta$. Thus, (f, t) and (f, t') are identified in the relation \sim .

Step 1: Let f_1, f_2, \dots, f_p be the facets of Δ and r_i the ray in Σ dual to f_i . If R_i is the image of r_i in $N/2N$ then (f_i, t) is identified with (f_i, t') if and only if $t - t' = R_i$.

Step j : After step $j - 1$ we have identified faces according to the vector spaces N_τ for $\tau \in \Sigma(j - 1)$. Let $\sigma \in \Sigma(j)$ be dual to $f < \Delta$. We have two cases.

$$1. \ 2^{d-j} \text{ copies of } f \text{ remain} \iff \bigoplus_{\gamma \in \sigma(j-1)} N_\gamma \longrightarrow N_\sigma \text{ is surjective}$$

$$2. \ 2^{d-j+1} \text{ copies of } f \text{ remain} \iff \bigoplus_{\gamma \in \sigma(j-1)} N_\gamma \longrightarrow N_\sigma \text{ is not surjective}$$

where the map $\bigoplus_{\gamma \in \sigma(j-1)} N_\gamma \longrightarrow N_\sigma$ is the direct sum of the inclusion maps $N_\gamma \xrightarrow{\subset} N_\sigma$. In

Case 1 we need not make any further identifications among the cells (f, t) since these cells are in correspondence with the \mathbb{Z}_2 vector space $N(\sigma)$ of rank $d - j$. In Case 2 we have exactly

$j - 1$ basis elements of N_σ contained in $\bigoplus_{\tau \in \sigma(j-1)} N_\tau$. Find the j th basis element of N_σ of the

following form. If r_1, r_2, \dots, r_k are the rays of σ then any element of σ is of the form

$$r = a_1 r_1 + a_2 r_2 + \dots + a_k r_k \quad \text{with} \quad a_i \in \mathbb{Z}.$$

To get an element in N_σ we take the first integer point on r and take its image R in $N/2N$.

If R is the element of this form not contained in $\bigoplus_{\tau \in \sigma(j-1)} N_\tau$ then at step j we identify pairs of faces $(f, t), (f, t')$ with $t - t' = R$.

The link of the open cell (f, t) acquires a cell structure from the cell structure (1.2) on $X_\Sigma(\mathbb{R})$. The cells in $Lk(f, t)$ are supported on cells of $X_\Sigma(\mathbb{R})$ of codimension $j - 1$ and less.

Thus, the link $Lk(f, t)$ is glued at step $j - 1$ or earlier. Let $t \in N(\sigma)$. In Case 1 $Lk(f, t)$ is connected while in Case 2 $Lk(f, t)$ has 2 connected components.

Note that the above process stops in at most $d = \dim \Delta$ steps. The next lemma will be useful in subsequent chapters.

Lemma 1. *After step 1 we are left with 2^{d-s} components, where*

$$s = \text{rank}(\text{span}\{R_1, R_2, \dots, R_p\})$$

with R_i the image of r_i in $N/2N$ and $\{r_1, r_2, \dots, r_p\}$ the rays of Σ .

To prove Lemma 1 we fix $t \in N/2N$. The d -cell (Δ, t) will be glued to the d -cells $(\Delta, t + R_1), (\Delta, t + R_2), \dots, (\Delta, t + R_p)$ along the facets f_1, f_2, \dots, f_p respectively. Next, note that the cell $(\Delta, t + R_i + R_j)$ is in the same component as (Δ, t) since the cell $(\Delta, t + R_i)$ is glued to $(\Delta, (t + R_i) + R_j)$ along the facet f_j . Similarly, we can see that $(\Delta, t + \sum_{i \in I} R_i)$ is in the same connected component as (Δ, t) where $\sum_{i \in I} R_i \in \text{span}\{R_1, R_2, \dots, R_p\}$. As the rank of the \mathbb{Z}_2 vector space is s , there are exactly 2^s d -cells in this connected component yielding $2^d/2^s = 2^{d-s}$ connected components in total after step 1.

Remark. *If for each cone σ in Σ , the image in $N/2N$ of the rays of σ contains a basis for N_σ then the above process stops after step 1. This is also equivalent to the link of each cell in $X_\Sigma(\mathbb{R})$ being connected.*

1.4 ALGEBRAICALLY ISOMORPHIC VERSUS T -HOMEOMORPHIC

Suppose $\Sigma_1 \subset N_1$ and $\Sigma_2 \subset N_2$ are two fans with $N_1 \cong \mathbb{Z}^d$ and $N_2 \cong \mathbb{Z}^n$. Following [Oda §1.5], let A be a $d \times n$ integer valued matrix such that for each cone $\sigma_1 \in \Sigma_1$ there is a cone $\sigma_2 \in \Sigma_2$ with $A\sigma_1 \subset \sigma_2$. Then, A gives a *map of fans* and determines a map of toric varieties.

$$\tilde{A} : X_{\Sigma_1} \longrightarrow X_{\Sigma_2}$$

The restriction of \tilde{A} to the dense torus $T_1 \subset X_{\Sigma_1}$ coincides with the homomorphism of tori

$$T_1 = N_1 \otimes \mathbb{C}^* \xrightarrow{A \otimes 1} N_2 \otimes \mathbb{C}^* = T_2.$$

When $d = n$, the map \tilde{A} is an *algebraic isomorphism* of X_{Σ_1} and X_{Σ_2} provided the cones in Σ_2 are all of the form $A\sigma$ for $\sigma \in \Sigma_1$ and $|\det A| = 1$.

Example 1. Let Σ_1 and Σ_2 be the 2 dimensional complete fans in Figure 1.1.

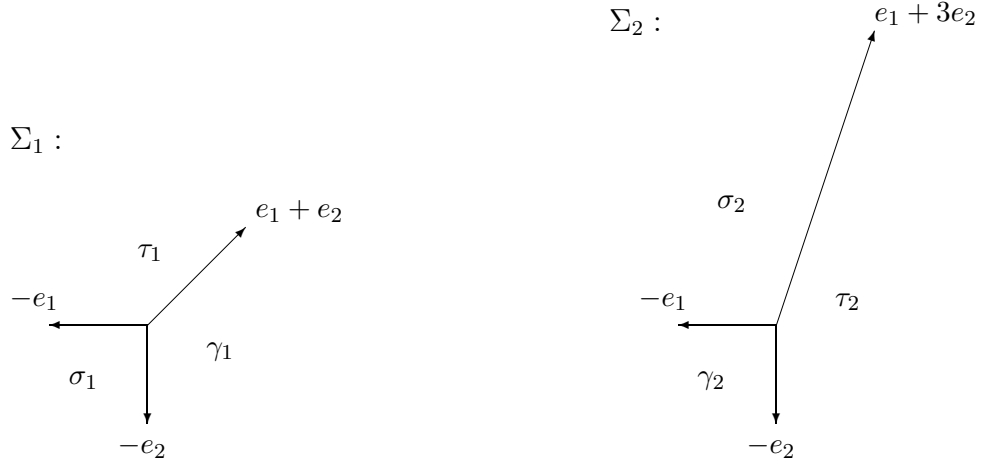


Figure 1.1: Fans Σ_1 and Σ_2

The matrix $A := \begin{bmatrix} -1 & 1 \\ -3 & 0 \end{bmatrix}$ satisfies

$$A(-e_1) = e_1 + 3e_2$$

$$A(-e_2) = e_1$$

$$A(e_1 + e_2) = -3e_2 \subset \text{poshull}(-e_2)$$

$$A\sigma_1 \subset \sigma_2$$

$$A\tau_1 \subset \tau_2$$

$$A\gamma_1 \subset \gamma_2$$

and thus we have a map $\tilde{A} : X_{\Sigma_1} \longrightarrow X_{\Sigma_2}$. The map \tilde{A} cannot be an algebraic isomorphism because the cone $\sigma_2 \in \Sigma_2$ is a singular cone, while the fan Σ_1 is a smooth fan and defines \mathbb{P}^2 . Note that the determinant of A is 3. The fan Σ_2 is the normal fan of a lattice triangle. We label 4 copies of a triangle with the elements $(0, 0), (0, 1), (1, 0), (1, 1)$ of $N_2/2N_2 \cong (\mathbb{Z}_2)^2$

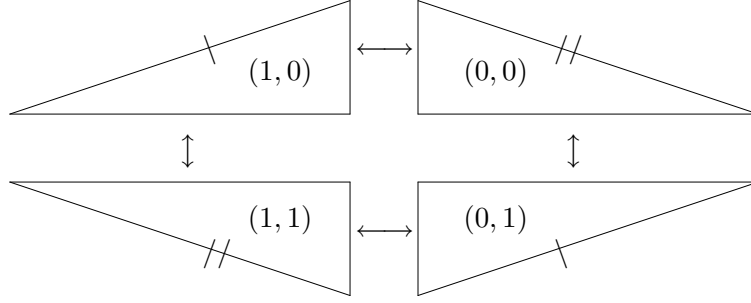


Figure 1.2: The topological space $X_{\Sigma_2}(\mathbb{R})$

and glue as in Section 1.3. Using Figure 1.2 and the classification of surfaces, we see $X_{\Sigma_2}(\mathbb{R})$ is homeomorphic to $\mathbb{R}P^2 = X_{\Sigma_1}(\mathbb{R})$.

The above example illustrates that two real toric varieties may be homeomorphic even though they are not isomorphic as algebraic varieties. This motivates the notion of T -homeomorphic real toric varieties. Let's assume A is a $d \times d$ integer valued matrix with $\det A$ an odd integer. We consider A as a map between the d dimensional lattices N_1 and N_2 . Then we have $A(2N_1) \subset 2N_2$ and hence A determines a map

$$A' : N_1/2N_1 \longrightarrow N_2/2N_2.$$

Since the determinant of A is odd, the preimage $A^{-1}\{2N_2\}$ is contained in $2N_1$. Thus A' is an injective map between \mathbb{Z}_2 vector spaces of the same rank yielding A' is an isomorphism of vector spaces.

Definition 1. *Suppose the $d \times d$ matrix A determines a map of fans and $\det A$ is an odd integer. Then A induces a T -homeomorphism $\tilde{A} : X_{\Sigma_1}(\mathbb{R}) \xrightarrow{\cong} X_{\Sigma_2}(\mathbb{R})$ provided*

1. For each $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ such that the restriction

$$A'|_{N_{\sigma_1}} : N_{\sigma_1} \longrightarrow N_{\sigma_2}$$

is a vector space isomorphism.

2. For each $\sigma_2 \in \Sigma_2$, $N_{\sigma_2} = A'(N_{\sigma_1})$ for some $\sigma_1 \in \Sigma_1$.

We say $X_{\Sigma_1}(\mathbb{R})$ and $X_{\Sigma_2}(\mathbb{R})$ are T -homeomorphic.

When the varieties X_{Σ_1} and X_{Σ_2} are projective, the dual map $A^* : M_2 \rightarrow M_1$ gives a combinatorial equivalence between the polytopes Δ_2 and Δ_1 which define X_{Σ_2} and X_{Σ_1} , respectively. Hence, the T -homeomorphism \tilde{A} is a cellular map

$$\frac{\Delta_1 \times N_1/2N_1}{\sim_1} \xrightarrow[\cong]{\tilde{A}} \frac{\Delta_2 \times N_2/2N_2}{\sim_2}.$$

We say a cone $\sigma \subset N$ is \mathbb{Z}_2 regular provided the image in $N/2N$ of the rays of σ forms a basis for the \mathbb{Z}_2 vector space N_σ . Suppose σ is \mathbb{Z}_2 regular and $\dim \sigma = k$. If r_1, r_2, \dots, r_k are the rays of σ then we extend R_1, R_2, \dots, R_k to a basis $\{R_1, R_2, \dots, R_k, B_{k+1}, \dots, B_d\}$ for $N/2N$. Then, the matrix

$$\begin{bmatrix} | & | & & | & | & & | \\ r_1 & r_2 & \cdots & r_k & b_{k+1} & \cdots & b_d \\ | & | & & | & | & & | \end{bmatrix}$$

determines a T -homeomorphism between the affine variety $U_\sigma(\mathbb{R})$ and \mathbb{R}^{d-k} . We thus have the following lemma.

Lemma 2. *Suppose Σ consists of \mathbb{Z}_2 regular cones. Then, $X_\Sigma(\mathbb{R})$ is a topological manifold of dimension d .*

We conclude this section by summarizing the implications discussed.

$$\begin{array}{ccccc} X_{\Sigma_1} \cong X_{\Sigma_2} & \implies & X_{\Sigma_1}(\mathbb{R}) \cong X_{\Sigma_2}(\mathbb{R}) & \implies & X_{\Sigma_1}(\mathbb{R}) \cong X_{\Sigma_2}(\mathbb{R}) \\ \text{algebraically} & & T\text{-homeomorphically} & & \text{topologically} \end{array}$$

CHAPTER 2

THE SPECTRAL SEQUENCE \overline{E}^r AND THE \mathbb{Z}_2 HODGE SPACES

2.1 THE \mathbb{Z}_2 HODGE SPACES

In this section, we introduce cosheaves on a fan Σ . We also discuss cosheaf homology and some basic properties of cosheaves on fans. We then define the cosheaf \mathcal{E} and the \mathbb{Z}_2 Hodge spaces $H_{pq}(\Sigma)$. The terminology \mathbb{Z}_2 Hodge spaces is inspired by work of Brion [Bri] who considered similar spaces associated to a fan Σ . In later chapters, we will also use sheaves on Σ and sheaf cohomology. The definition and properties of sheaves on fans are similar to those of cosheaves and are developed in [Bri §1.1]. Sheaves on fans are also studied in [Bre] and [Bar]. The main difference between our work with sheaves and that in [Bri] is that our sheaves are sheaves of \mathbb{Z}_2 vector spaces. Throughout this and subsequent chapters, all homology and cohomology groups will be with \mathbb{Z}_2 coefficients, unless otherwise stated.

2.1.1 SHEAVES AND COSHEAVES ON A FAN

A *cosheaf* \mathcal{F} of \mathbb{Z}_2 vector spaces on a fan Σ is a collection of vector spaces $(F_\sigma)_{\sigma \in \Sigma}$ over \mathbb{Z}_2 together with face restriction maps $\rho_{\tau,\sigma} : F_\sigma \rightarrow F_\tau$ for $\sigma < \tau$ satisfying the following two conditions.

- If $\sigma < \tau < \beta$ then $\rho_{\beta,\tau}\rho_{\tau,\sigma} = \rho_{\beta,\sigma}$
- $\rho_{\sigma,\sigma}$ is the identity map

We define *cosheaf homology* groups as follows. The chain groups $C_p(\mathcal{F})$ are the \mathbb{Z}_2 vector spaces defined by

$$C_p(\mathcal{F}) := \bigoplus_{\sigma \in \Sigma(d-p)} F_\sigma$$

and the boundary map $\partial_p : C_p(\mathcal{F}) \longrightarrow C_{p-1}(\mathcal{F})$ is the direct sum of the maps

$$\sum_{\sigma < \tau} \rho_{\tau, \sigma} : F_\sigma \longrightarrow \bigoplus_{\tau \in \Sigma(d-p+1), \sigma < \tau} F_\tau. \quad (2.1)$$

Note that if we were working with cosheaves of k vector spaces with $\text{char } k \neq 2$ then we would need to introduce signs in (2.1) to guarantee $\partial^2 = 0$. We define $H_p(\mathcal{F})$ to be the p th homology group of the complex $(C_*(\mathcal{F}), \partial_*)$.

Suppose \mathcal{F}^1 and \mathcal{F}^2 are two cosheaves on Σ . A *morphism of cosheaves* $\Phi : \mathcal{F}^1 \longrightarrow \mathcal{F}^2$ is a collection of vector space homomorphisms $(\phi_\sigma)_{\sigma \in \Sigma}$ with $\phi_\sigma : F_\sigma^1 \longrightarrow F_\sigma^2$ such that if $\sigma < \tau < \beta$ the following diagram commutes.

$$\begin{array}{ccccc} F_\sigma^1 & \xrightarrow{\rho_{\tau, \sigma}^1} & F_\tau^1 & \xrightarrow{\rho_{\beta, \tau}^1} & F_\beta^1 \\ \downarrow \phi_\sigma & & \downarrow \phi_\tau & & \downarrow \phi_\beta \\ F_\sigma^2 & \xrightarrow{\rho_{\tau, \sigma}^2} & F_\tau^2 & \xrightarrow{\rho_{\beta, \tau}^2} & F_\beta^2 \end{array}$$

Hence, an *exact sequence of cosheaves* $0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \longrightarrow \dots \longrightarrow \mathcal{F}^n \longrightarrow 0$ is a collection of exact sequences of \mathbb{Z}_2 vector spaces for $\sigma \in \Sigma$

$$0 \longrightarrow F_\sigma^1 \xrightarrow{\phi_\sigma^1} F_\sigma^2 \xrightarrow{\phi_\sigma^2} \dots \xrightarrow{\phi_\sigma^{n-1}} F_\sigma^n \longrightarrow 0$$

such that the maps ϕ_σ^i are natural with respect to face restriction for $1 \leq i \leq n-1$. In subsequent chapters, we will frequently use the fact that a short exact sequence of cosheaves

$$0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \longrightarrow \mathcal{F}^3 \longrightarrow 0$$

induces a long exact sequence on homology groups

$$\dots \longrightarrow H_{p+1}(\mathcal{F}^3) \longrightarrow H_p(\mathcal{F}^1) \longrightarrow H_p(\mathcal{F}^2) \longrightarrow H_p(\mathcal{F}^3) \longrightarrow H_{p-1}(\mathcal{F}^1) \longrightarrow \dots$$

2.1.2 THE COSHEAF \mathcal{E} AND THE \mathbb{Z}_2 HODGE SPACES $H_{pq}(\Sigma)$

We define the cosheaf \mathcal{N} on Σ by

$$N_\sigma := \sigma \cap N / \sigma \cap 2N \quad \text{for } \sigma \in \Sigma$$

with the restriction map $\rho_{\tau,\sigma}$ given by inclusion

$$\rho_{\tau,\sigma} : N_\sigma \xrightarrow{\subset} N_\tau \quad \text{for } \sigma < \tau.$$

We encountered the cosheaf \mathcal{N} in Section 1.3 when gluing $X_\Sigma(\mathbb{R})$ from 2^d copies of Δ . If $\dim \sigma = q$ then N_σ is a rank q vector space over \mathbb{Z}_2 . The cosheaf \mathcal{E} is defined so that

$$E_\sigma := N(\sigma) = \frac{N/2N}{N_\sigma}.$$

If $\sigma < \tau$ then the restriction map

$$\varpi_{\tau,\sigma} : E_\sigma = \frac{N/2N}{N_\sigma} \longrightarrow \frac{N/2N}{N_\tau} = E_\tau$$

is induced from the identity on $N/2N$ which takes N_σ to N_τ . Thus, the cosheaf \mathcal{E} is the cokernel of the inclusion $\mathcal{N} \hookrightarrow N/2N$, where $N/2N$ is the constant cosheaf assigning $N/2N$ to each cone in Σ . The \mathbb{Z}_2 Hodge spaces of Σ are defined to be the homology groups

$$H_{pq}(\Sigma) := H_p(\wedge^q \mathcal{E}),$$

where $\wedge^q \mathcal{E}$ is the q th exterior power of the cosheaf \mathcal{E} .

Remark. Since $\wedge^q E_\sigma$ is zero for $q > \text{codim } \sigma$, we have $C_p(\wedge^q \mathcal{E}) = 0$ for $p < q$. Hence the \mathbb{Z}_2 Hodge spaces $H_{pq}(\Sigma)$ are indexed by p, q with $0 \leq q \leq p \leq d$.

2.2 THE SPECTRAL SEQUENCE \overline{E}^r

We recall that for X_Σ projective, the cells in $X_\Sigma(\mathbb{R})$ are of the form (f, t) with $f < \Delta$ and $t \in N(\sigma)$, $\sigma \in \Sigma$ is dual to f . The boundary map ∂ for the chain complex $C_*(X_\Sigma(\mathbb{R}))$ is given by the maps $\varpi_{\tau,\sigma} : N(\sigma) \longrightarrow N(\tau)$ from Section 2.1. It follows that

$$C_j(X_\Sigma(\mathbb{R})) = \bigoplus_{\sigma \in \Sigma(d-j)} H_0(N(\sigma)) \tag{2.2}$$

where the boundary map ∂_j is the direct sum of the maps

$$\sum_{\sigma < \tau} (\varpi_{\tau,\sigma})_* : H_0(N(\sigma)) \longrightarrow \bigoplus_{\tau \in \Sigma(d-j+1), \sigma < \tau} H_0(N(\tau))$$

In [Bih] Bihan et al. want to understand the relationship between the topology of $X_\Sigma(\mathbb{R})$ and that of $X_\Sigma(\mathbb{C})$. The authors show that the chain complex $C_*(X_\Sigma(\mathbb{R}))$ is a filtered differential graded vector space. The associated spectral sequence $(\overline{E}^r, \overline{d}^r)$ converges to $H_*(X(\mathbb{R}))$ and is known to collapse at \overline{E}^1 when X_Σ is complete and has isolated singularities or when the dimension of X_Σ is at most 3 [Bih]. Our notation is slightly different than in [Bih] and hence we briefly review their construction.

Using (2.2) we may specify a filtration on $C_*(X_\Sigma(\mathbb{R}))$ by giving a filtration of $H_0(N(\sigma))$, the \mathbb{Z}_2 group algebra of $N(\sigma)$. We use the *augmentation* homomorphism ϵ_σ .

$$\begin{aligned} \epsilon_\sigma : H_0(N(\sigma)) &\longrightarrow \mathbb{Z}_2 \\ \sum_{n_i \in \mathbb{Z}_2, g_i \in N(\sigma)} n_i g_i &\longmapsto \sum n_i \end{aligned}$$

We define I_σ , an ideal in $H_0(N(\sigma))$, via $I_\sigma := \ker \epsilon_\sigma$. This gives a filtration of $H_0(N(\sigma))$

$$0 = I_\sigma^{j+1} \subset I_\sigma^j \subset \dots \subset I_\sigma^2 \subset I_\sigma \subset I_\sigma^0 = H_0(N(\sigma))$$

where $j = \text{rank } N(\sigma) = \text{codim } \sigma$. We reindex by setting $J_\sigma^p = I_\sigma^{d-p}$ so that

$$0 = J_\sigma^{d-j-1} \subset J_\sigma^{d-j} \subset \dots \subset J_\sigma^{d-2} \subset J_\sigma^{d-1} \subset J_\sigma^d = H_0(N(\sigma)) \quad (2.3)$$

is an increasing filtration of $H_0(N(\sigma))$.

Lemma 3. *The filtrations of $H_0(N(\sigma))$ of the form (2.3) for $\sigma \in \Sigma$ determine an increasing filtration of F of $C_*(X_\Sigma(\mathbb{R}))$ and we have the following.*

$$\partial_j : F_q C_j(X_\Sigma(\mathbb{R})) \longrightarrow F_q C_{j-1}(X_\Sigma(\mathbb{R}))$$

To prove Lemma 3 note that for $\sigma < \tau$ in Σ the map $(\varpi_{\tau,\sigma})_*$ commutes with the augmentation homomorphisms.

$$\begin{array}{ccc} H_0(N(\sigma)) & \xrightarrow{(\varpi_{\tau,\sigma})_*} & H_0(N(\tau)) \\ & \searrow \epsilon_\sigma & \swarrow \epsilon_\tau \\ & \mathbb{Z}_2 & \end{array}$$

Thus we have $(\varpi_{\tau,\sigma})_*(I_\sigma) \subset I_\tau$ and $(\varpi_{\tau,\sigma})_*(I_\sigma^k) \subset I_\tau^k$. Lemma 3 follows. We will denote $\underline{\varpi}_{\tau,\sigma}^k$ for the induced map on the quotient

$$\underline{\varpi}_{\tau,\sigma}^k : \frac{I_\sigma^k}{I_\sigma^{k+1}} \longrightarrow \frac{I_\tau^k}{I_\tau^{k+1}}.$$

We now use the spectral sequence of a filtered module as developed in [Mac §XI.3] to define the spectral sequence $(\tilde{E}^r, \tilde{d}^r)$. From Theorem 3.1 in [Mac] we have

$$\tilde{E}_{p,q}^1 = H_{p+q} \left(\frac{F_p C_*(X_\Sigma(\mathbb{R}))}{F_{p-1} C_*(X_\Sigma(\mathbb{R}))} \right)$$

and

$$\tilde{E}_{p,q}^1 \implies H_{p+q}(X_\Sigma(\mathbb{R})).$$

Lemma 4. *For each p, q we have*

$$\tilde{E}_{p,q}^1 \cong H_{p+q, d-p}(\Sigma)$$

where $H_{p+q, d-p}(\Sigma)$ is the \mathbb{Z}_2 Hodge space of Σ defined in Section 2.1.

We illustrate this lemma by showing the term $\tilde{E}_{p,q}^1$ when the dimension of Σ is 5.

$$\begin{array}{ccccccc}
 & & & & & & H_{55}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{54}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{44}(\Sigma) \quad H_{53}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{43}(\Sigma) \quad H_{52}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{33}(\Sigma) \quad H_{42}(\Sigma) \quad H_{51}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{32}(\Sigma) \quad H_{41}(\Sigma) \quad H_{50}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{22}(\Sigma) \quad H_{31}(\Sigma) \quad H_{40}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{21}(\Sigma) \quad H_{30}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{11}(\Sigma) \quad H_{20}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{10}(\Sigma) \\
 & & & & & & \\
 & & & & & & H_{00}(\Sigma)
 \end{array}$$

p

To prove Lemma 4 consider the following.

$$\begin{aligned} \frac{F_p C_{p+q}(X_\Sigma(\mathbb{R}))}{F_{p-1} C_{p+q}(X_\Sigma(\mathbb{R}))} &= \bigoplus_{\sigma \in \Sigma(d-(p+q))} \frac{J_\sigma^p}{J_\sigma^{p-1}} \\ &= \bigoplus_{\sigma \in \Sigma(d-(p+q))} \frac{I_\sigma^{d-p}}{I_\sigma^{d-(p-1)}} \end{aligned}$$

The following claim is Proposition 6.1 in [Bih].

Claim. *For each $\sigma \in \Sigma$ we have*

$$\frac{I_\sigma^q}{I_\sigma^{q+1}} \cong \wedge^q N(\sigma)$$

canonically as \mathbb{Z}_2 vector spaces.

This yields

$$\begin{aligned} \frac{F_p C_{p+q}(X_\Sigma(\mathbb{R}))}{F_{p-1} C_{p+q}(X_\Sigma(\mathbb{R}))} &= \bigoplus_{\sigma \in \Sigma(d-(p+q))} \wedge^{d-p} N(\sigma) \\ &= C_{p+q}(\wedge^{d-p} \mathcal{E}). \end{aligned}$$

To see the lemma we need only compare boundary maps. The map

$$\tilde{d}_{p,q}^0 : \frac{F_p C_{p+q}(X_\Sigma(\mathbb{R}))}{F_{p-1} C_{p+q}(X_\Sigma(\mathbb{R}))} \longrightarrow \frac{F_p C_{p+q-1}(X_\Sigma(\mathbb{R}))}{F_{p-1} C_{p+q-1}(X_\Sigma(\mathbb{R}))}$$

is given by the collection

$$\varpi_{\tau,\sigma}^{d-p} : \frac{I_\sigma^{d-p}}{I_\sigma^{d-p+1}} \longrightarrow \frac{I_\tau^{d-p}}{I_\tau^{d-p+1}}$$

of maps for $\sigma < \tau$ in Σ with $\dim \sigma = d - (p + q)$. By construction, this is induced from the map $\varpi_{\tau,\sigma}$. The cosheaf differential for $\wedge^{d-p} \mathcal{E}$ is given by the collection of face restriction maps

$$\wedge^{d-p} N(\sigma) \longrightarrow \wedge^{d-p} N(\tau) \quad \text{for } \sigma < \tau$$

which are also induced from the maps $\varpi_{\tau,\sigma}$, and hence the two maps are equal. Taking homology we arrive at Lemma 4.

As $(\tilde{E}^r, \tilde{d}^r)$ is the spectral sequence associated to the filtered complex of cellular chains in $X_\Sigma(\mathbb{R})$, we have

$$\tilde{E}_{p,q}^\infty = \frac{F_p H_{p+q}(X_\Sigma(\mathbb{R}))}{F_{p-1} H_{p+q}(X_\Sigma(\mathbb{R}))}$$

with

$$0 = F_{d-(p+q)-1} H_{p+q}(X_\Sigma(\mathbb{R})) \subset F_{d-(p+q)} H_{p+q}(X_\Sigma(\mathbb{R})) \subset \cdots \subset F_d H_{p+q}(X_\Sigma(\mathbb{R})) \subset H_{p+q}(X_\Sigma(\mathbb{R}))$$

the bounded filtration of $H_{p+q}(X_\Sigma(\mathbb{R}))$. Since $\tilde{d}_{d-(p+q),2(p+q)-d}^r = 0$ for all r , we have a sequence of surjections

$$\tilde{E}_{d-(p+q),2(p+q)-d}^1 \longrightarrow \tilde{E}_{d-(p+q),2(p+q)-d}^2 \longrightarrow \tilde{E}_{d-(p+q),2(p+q)-d}^3 \longrightarrow \cdots \longrightarrow \tilde{E}_{d-(p+q),2(p+q)-d}^\infty$$

which are called edge homomorphisms in §11.1 of [Mac]. Moreover, as

$$\tilde{E}_{d-(p+q),2(p+q)-d}^\infty = \frac{F_{d-(p+q)} H_{p+q}(X_\Sigma(\mathbb{R}))}{0},$$

we can compose the edge homomorphisms with inclusion to obtain a natural homomorphism

$$\tilde{E}_{d-(p+q),2(p+q)-d}^1 \longrightarrow H_{p+q}(X_\Sigma(\mathbb{R})). \quad (2.4)$$

At this point, we introduce reindexing which we will use in the sequel. We rotate the \tilde{E}^1 term clockwise by 90° and then shear so that each diagonal line $p + q = c$ becomes the vertical line $p = c$. We write (\bar{E}^r, \bar{d}^r) for the spectral sequence obtained after reindexing. The total grading of (\bar{E}^r, \bar{d}^r) is p , and Lemma 4 gives the identity $\bar{E}_{p,q}^1 = H_{pq}(\Sigma)$. Figure 2.1 shows the terms $\tilde{E}_{p,q}^1$ and $\bar{E}_{p,q}^1$ for a 4 dimensional fan Σ .

For the spectral sequence (\bar{E}^r, \bar{d}^r) the boundary map \bar{d}^r satisfies

$$\bar{d}_{p,q}^r : \bar{E}_{p,q}^r \longrightarrow \bar{E}_{p-1,q+r}^r.$$

We show the first five boundary maps in Figure 2.2. Next, we point out that after reindexing the natural homomorphism (2.4) becomes a map which we denote $f_*^{\mathbb{R}}$ and will use in future chapters where

$$f_q^{\mathbb{R}} : H_{qq}(\Sigma) = \bar{E}_{q,q}^1 \longrightarrow H_q(X_\Sigma(\mathbb{R})). \quad (2.5)$$

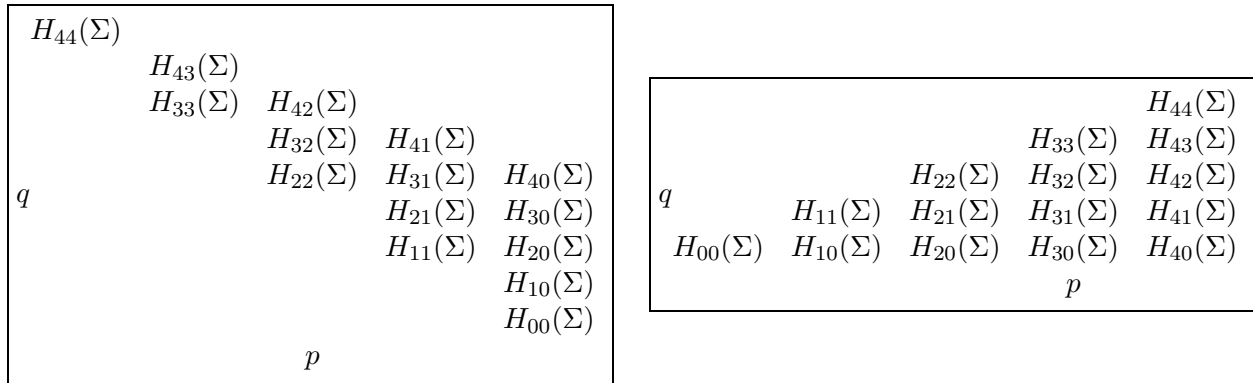


Figure 2.1: The terms $\tilde{E}_{p,q}^1$ (left) and $\bar{E}_{p,q}^1$ (right)

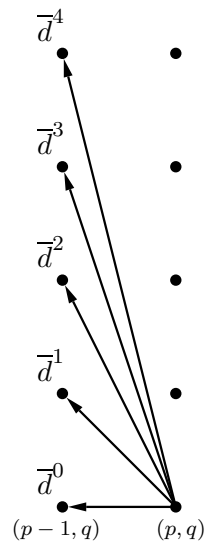


Figure 2.2: The differentials $\bar{d}^0, \bar{d}^1, \bar{d}^2, \bar{d}^3$, and \bar{d}^4

Lemma 5. *The spectral sequence $(\overline{E}^r, \overline{d}^r)$ satisfies*

$$\overline{E}_{p,q}^1 \cong E_{p,q}^2 \quad \text{for } 0 \leq p, q \leq d$$

where (E^r, d^r) is the \mathbb{Z}_2 Leray spectral sequence for the map $\mu_{\mathbb{C}}$.

We refer the reader to [Mac §XI.7] or [McC §5] for construction of the Leray spectral sequence. We will show $E_{p,q}^2 \cong H_{pq}(\Sigma)$ which will prove the lemma. Suppose $\sigma \in \Sigma$ is dual to $f < \Delta$. From (1.1) if $z \in \text{int } f$ then

$$\mu_{\mathbb{C}}^{-1}\{z\} = \frac{(N \otimes \mathbb{R})/\sigma}{N/\sigma \cap N} \cong (S^1)^p \quad p = \dim f$$

Next, we note that $H_q((S^1)^p, \mathbb{Z}_2) = \wedge^q (H_1((S^1)^p, \mathbb{Z}_2))$ and

$$\begin{aligned} H_1 \left(\frac{(N \otimes \mathbb{R})/\sigma}{N/\sigma \cap N}, \mathbb{Z}_2 \right) &= H_1 \left(\frac{(N \otimes \mathbb{R})/\sigma}{N/\sigma \cap N}, \mathbb{Z} \right) \otimes \mathbb{Z}_2 \\ &= (N/\sigma \cap N) \otimes \mathbb{Z}_2 \\ &= \frac{N/2N}{\sigma \cap N / \sigma \cap 2N} \\ &= N(\sigma) \end{aligned}$$

where each equality above is canonical. This gives the following.

$$\begin{aligned} E_{p,q}^1 &= \bigoplus_{\sigma \in \Sigma(d-p)} H_q \left(\frac{(N \otimes \mathbb{R})/\sigma}{N/\sigma \cap N}, \mathbb{Z}_2 \right) \\ &= \bigoplus_{\sigma \in \Sigma(d-p)} \wedge^q N(\sigma) \end{aligned}$$

Using Proposition 5.1 of [Bih], the differential $d_{*,q}^1 : E_{*,q}^1 \longrightarrow E_{*-1,q}^1$ is equivalent to the boundary map for the cosheaf $\wedge^q \mathcal{E}$ and hence we have Lemma 5.

We now have a way to compare the topology of the real and complex points of a toric variety. The Smith-Thom Inequality states

$$\sum_i b_i(X_{\Sigma}(\mathbb{R})) \leq \sum_j b_j(X_{\Sigma}(\mathbb{C}))$$

and we say X_Σ is *maximal* or an *M-variety* if equality is obtained. Using the the Smith-Thom inequality and Lemma 5 we have the following diagram.

$$\begin{array}{ccc} \sum \text{rank}(\overline{E}_{p,q}^1) & = & \sum \text{rank}(E_{p,q}^2) \\ \downarrow & & \downarrow \\ \sum b_i(X(\mathbb{R})) & \leq & \sum b_j(X(\mathbb{C})) \end{array} \quad (2.6)$$

If the spectral sequence \overline{E}^r collapses at \overline{E}^1 then the left vertical inequality in (2.6) is equality. This forces both the right vertical inequality and the lower horizontal inequality to both be equalities. In this case we obtain X_Σ is maximal.

CHAPTER 3

PRELIMINARY RESULTS

3.1 THE DIAGONAL ENTRIES $H_{qq}(\Sigma)$

In this section we show that the diagonal \mathbb{Z}_2 Hodge space $H_{qq}(\Sigma)$ can be identified with the q th \mathbb{Z}_2 torus invariant Chow group $A_q^T(X_\Sigma)$, and we interpret the map $f_q^{\mathbb{R}}$ in (2.5) in this setting. We also determine the groups $A_q^T(X_\Sigma)$ when certain hypotheses are placed on Σ .

Proposition. *For each q with $0 \leq q \leq d$ we have*

$$H_{qq}(\Sigma) \cong A_q^T(X_\Sigma),$$

where $A_q^T(X_\Sigma)$ is the q th \mathbb{Z}_2 torus invariant Chow group of X_Σ .

To prove the proposition, let $\sigma < \tau$ in Σ with $\text{codim } \sigma = q + 1$ and $\text{codim } \tau = q$ and we consider the restriction map

$$\rho_{\tau,\sigma} : \wedge^q N(\sigma) \longrightarrow \wedge^q N(\tau) \tag{3.1}$$

for the cosheaf $\wedge^q \mathcal{E}$. Choose a basis $\mathcal{B} := \{t_1, t_2, \dots, t_q\}$ for $N(\tau)$ and extend \mathcal{B} to a basis $\mathcal{B}' := \mathcal{B} \cup \{n_{\sigma,\tau}\}$ for $N(\sigma)$, where $n_{\sigma,\tau}$ is the nonzero element in N_τ/N_σ . We determine the map $\rho_{\tau,\sigma}$ in (3.1) using the basis elements of $\wedge^q N(\sigma)$:

$$\begin{aligned} \rho_{\tau,\sigma}(t_1 \wedge t_2 \wedge \dots \wedge t_q) &= t_1 \wedge t_2 \wedge \dots \wedge t_q \\ \rho_{\tau,\sigma}(n_{\sigma,\tau} \wedge * \wedge \dots \wedge *) &= 0 \end{aligned}$$

where $t_1 \wedge t_2 \wedge \dots \wedge t_q$ is the generator of $\wedge^q N(\tau)$ and the $*$ -entries can be any elements of \mathcal{B}' . Since $N(\sigma)$ has rank $q + 1$, we have an isomorphism $\wedge^q N(\sigma) \cong N(\sigma)$ which identifies a q -tuple of basis elements $b_1 \wedge b_2 \wedge \dots \wedge b_q$ to the one element of $\mathcal{B}' \setminus \{b_1, b_2, \dots, b_q\}$. Hence

we can interpret the map $\rho_{\tau,\sigma}$ in (3.1) as a map $N(\sigma) \longrightarrow \mathbb{Z}_2[\tau]$ which sends the nonzero element $n_{\sigma,\tau} \in N_\tau/N_\sigma$ to $1 \in \mathbb{Z}_2[\tau]$ and any t_k to zero.

Next, we consider the relations in the \mathbb{Z}_2 torus invariant Chow groups $A_q^T(X_\Sigma)$. The integral torus invariant Chow groups are reviewed in Section 1.1.2. When considering the \mathbb{Z}_2 torus invariant Chow groups, we define $A_q^T(X_\Sigma) := Z_q^T(X_\Sigma)/R_{q+1}^T(X_\Sigma)$, where $Z_q^T(X_\Sigma)$ is the \mathbb{Z}_2 vector space generated by cycles $[V(\tau)]$ for $\tau \in \Sigma$ of codimension q . The subspace $R_{q+1}^T(X_\Sigma)$ is generated by $\text{div}\chi^u$ as in Section 1.1.2 except the coefficients are taken mod 2. That is, the coefficient of $[V(\tau)]$ in the relation $\text{div}\chi^u$ is

$$\langle \mathbf{u}, \mathbf{n}_{\sigma,\tau} \rangle \pmod{2} \quad (3.2)$$

where $\mathbf{n}_{\sigma,\tau}$ is a lattice generator of $\tau \cap N / \sigma \cap N$. The element of \mathbb{Z}_2 in (3.2) is the same as the element when we take u , the image of \mathbf{u} in $M(\sigma) = \sigma^\perp \cap M / \sigma^\perp \cap 2M$ and $n_{\sigma,\tau}$, the image of $\mathbf{n}_{\sigma,\tau}$ in N_τ/N_σ . Next, we have $M(\sigma) \cong \text{Hom}(N(\sigma), \mathbb{Z}_2)$ and hence the subspace $R_{q+1}^T(X_\Sigma)$ is generated by the image of

$$\text{Hom}(N(\sigma), \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2[\tau] \quad (3.3)$$

$$u \longmapsto \langle u, n_{\sigma,\tau} \rangle \pmod{2}. \quad (3.4)$$

We take the dual basis for \mathcal{B}' as a basis for $M(\sigma)$ which consists of elements $\{t_1^*, t_2^*, \dots, t_q^*, n_{\sigma,\tau}^*\}$. We determine the map (3.3) using this basis, and we see the image of (3.3) in $\mathbb{Z}_2[\tau]$ is equal to the image of the differential (3.1) in $\wedge^q N(\tau) \cong \mathbb{Z}_2[\tau]$. Moreover since $C_{q-1}(\wedge^q \mathcal{E}) = 0$, we have

$$\ker(C_q(\wedge^q \mathcal{E}) \longrightarrow C_{q-1}(\wedge^q \mathcal{E})) = C_q(\wedge^q \mathcal{E})$$

and as $C_q(\wedge^q \mathcal{E}) \cong Z_q^T(X_\Sigma)$ we obtain

$$\begin{aligned} \frac{C_q(\wedge^q \mathcal{E})}{\text{im}(C_{q+1}(\wedge^q \mathcal{E}) \longrightarrow C_q(\wedge^q \mathcal{E}))} &\cong \frac{Z_q^T(X_\Sigma)}{R_{q+1}^T(X_\Sigma)} \\ H_{qq}(\Sigma) &\cong A_q^T(X_\Sigma), \end{aligned}$$

which proves the proposition.

Next, we recall the natural map

$$f_q^{\mathbb{R}} : A_q^T(X_\Sigma) \cong H_{qq}(\Sigma) \longrightarrow H_q(X_\Sigma(\mathbb{R}))$$

given in (2.5) which arises from the edge homomorphisms of the spectral sequence $(\tilde{E}^r, \tilde{d}^r)$. Thinking cellularly, an algebraic cycle $[V(\beta)]$ in $Z_q^T(X_\Sigma)$ is represented by the sum of all 2^q cells in the orbit $O_\beta(\mathbb{R})$. Hence, if g is a q dimensional face of Δ with β the cone dual to g then $f_q^{\mathbb{R}}([V(\beta)])$ is the homology class of C_β where $C_\beta := \sum_{t \in N(\beta)}(g, t)$ is the cellular chain in $C_q(X_\Sigma)$ obtained by adding all 2^q copies of g in $X_\Sigma(\mathbb{R})$.

Bihan et al. [Bih] show that if Σ consists of \mathbb{Z}_2 regular cones then the map $f_q^{\mathbb{R}}$ is an isomorphism for $0 \leq q \leq d$. We will see in Chapter 5 that $f_q^{\mathbb{R}}$ is in general neither injective nor surjective. Next, we determine $A_q^T(X_\Sigma)$ in some cases. The following lemma will be useful in Section 4.5.

Lemma 6. *Assume for each $\tau \in \Sigma(d - k - 1)$ we have $V(\tau)(\mathbb{R})$ is T -homeomorphic to $\mathbb{R}\mathbb{P}^{k+1}$. If $q \leq k$ then $A_q^T(X_\Sigma)$ is generated by the orbit closure of any q dimensional torus orbit in X_Σ .*

To prove the lemma, we first look at the relations coming from a single cone σ of codimension $q + 1$. Since σ is a face of a cone in $\Sigma(d - k - 1)$, $V(\sigma)(\mathbb{R})$ is T -homeomorphic to $\mathbb{R}\mathbb{P}^{q+1}$. Thus, $M(\sigma) \cong \text{Hom}(N(\sigma), \mathbb{Z}_2)$ is generated by elements a_1, a_2, \dots, a_{q+1} and σ is contained in the $q + 2$ cones $\omega_1, \omega_2, \dots, \omega_{q+2}$ of codimension q , where

$$n_{\sigma, \omega_i} = \begin{cases} a_i & \text{if } 1 \leq i \leq q + 1 \\ a_1 + a_2 + \dots + a_{q+1} & \text{if } i = q + 2. \end{cases}$$

Hence, the map

$$M(\sigma) \longrightarrow Z_q^T(X_\Sigma) \tag{3.5}$$

is given by basis elements as follows.

$$\begin{aligned}
a_1 &\longmapsto [V(\omega_1)] + [V(\omega_{q+2})] \\
a_2 &\longmapsto [V(\omega_2)] + [V(\omega_{q+2})] \\
&\dots \\
a_{q+1} &\longmapsto [V(\omega_{q+1})] + [V(\omega_{q+2})]
\end{aligned}$$

Extending linearly to all of $M(\sigma)$, we see that in the image of (3.5) we obtain the sum of any even number of $[V(\omega_i)]$'s. For any r with $1 \leq r \leq q+2$, the cycle $[V(\omega_r)]$ is not in the image of (3.5). Moreover, if C is a sum of an odd number of $[V(\omega_i)]$'s then $C + [V(\omega_r)]$ is in the image of (3.5). Hence for any r , the cycle $[V(\omega_r)]$ is the generator of $\text{coker}(M(\sigma) \longrightarrow \bigoplus_{\sigma < \omega_i} [V(\omega_i)])$.

Next, we consider the map

$$\bigoplus_{\sigma \in \Sigma(d-q-1)} M(\sigma) \longrightarrow \bigoplus_{\omega \in \Sigma(d-q)} [V(\omega)], \quad (3.6)$$

where the map on each $M(\sigma)$ is described above.

Claim. *If ω and ω' are codimension q cones in Σ then $[V(\omega)] + [V(\omega')]$ is in the image of (3.6).*

To see the claim, note that if $\omega \cap \omega'$ is a codimension $q+1$ cone then $[V(\omega)] + [V(\omega')]$ is in the image of (3.6), as discussed above. Else, find a sequence of codimension q and codimension $q+1$ cones

$$\begin{array}{ccccccc}
\omega = \omega_0 & & \omega_1 & & \omega_2 & \dots & \omega_{n-1} & & \omega_n = \omega' \\
& \searrow & & \nearrow & \searrow & & \nearrow & & \searrow \\
& & \sigma_{01} & & \sigma_{12} & & \dots & & \sigma_{n-1n}
\end{array}$$

where $\sigma_{i-1i} < \omega_{i-1}$ and $\sigma_{i-1i} < \omega_i$ for $1 \leq i \leq n$. There exists $u_{i-1i} \in M(\sigma_{i-1i})$ with $\text{div}(\chi^{u_{i-1i}}) = [V(\omega_{i-1})] + [V(\omega_i)]$. Adding, we obtain

$$\sum_{i=1}^n u_{i-1i} \in \bigoplus_{\sigma \in \Sigma(d-q-1)} M(\sigma)$$

and

$$\sum_{i=1}^n u_{i-1i} \longmapsto [V(\omega)] + [V(\omega')]$$

which proves the claim. Thus, the sum of any even number of $[V(\omega)]$'s is in the image of (3.6) and the cycle $[V(\omega')]$ is not. Moreover, if C is a chain in $Z_q^T(X_\Sigma)$ with an odd number of $[V(\omega)]$'s then $C + [V(\omega')]$ is in the image of (3.6). Hence, $A_q^T(X_\Sigma)$ is generated by $[V(\omega')]$ where ω' is any cone of codimension q . This proves Lemma 6.

3.2 THE RIGHT-MOST COLUMN $H_{dq}(\Sigma)$

In this section, we show that the higher boundaries $\bar{d}_{d,q}^r$, $r \geq 1$ for the spectral sequence (\bar{E}^r, \bar{d}^r) are zero and we determine the ranks of the \mathbb{Z}_2 Hodge spaces $H_{dq}(\Sigma)$ in the right-most column. Let $\{r_1, r_2, \dots, r_k\}$ be the rays of Σ and R_i the image of r_i in $N/2N \cong (\mathbb{Z}_2)^d$. Suppose the rank of the \mathbb{Z}_2 vector space $V := \text{span}(R_1, R_2, \dots, R_k)$ is $s \leq d$. Choose a basis for V consisting of a subset of the R_i , and reorder if needed so that $\{R_1, R_2, \dots, R_s\}$ is a basis for V . Note that the elements of the form

$$R_i \wedge * \wedge * \wedge \dots \wedge *$$

generate

$$\ker(\wedge^q N/2N \longrightarrow \wedge^q N(r_i)),$$

where the $*$ -entries are any elements of $N/2N$. Hence,

$$\ker(\wedge^q N/2N \longrightarrow \bigoplus_{i=1}^s \wedge^q N(r_i))$$

is generated by elements of the following form.

$$R_1 \wedge R_2 \wedge \dots \wedge R_s \wedge * \wedge * \wedge \dots \wedge *$$

This is a subspace of $\wedge^q N/2N$ of rank $\binom{d-s}{q-s}$, where $\binom{n}{k} = 0$ if $k < 0$. Hence, the kernel of the boundary map

$$\bar{d}_{d,q}^0 : \wedge^q N/2N \longrightarrow \bigoplus_{r \in \Sigma(1)} \wedge^q N(r)$$

has rank at most $\binom{d-s}{q-s}$. This shows

$$b_d(X_\Sigma(\mathbb{R})) \leq \sum_{q=s}^d \binom{d-s}{q-s} = 2^{d-s} \quad (3.7)$$

and equality holds if and only if for each q , $\ker \bar{d}_{d,q}^0$ has rank $\binom{d-s}{q-s}$ and all higher differentials $\bar{d}_{d,q}^r$, $r \geq 1$ with source the rightmost column are zero. Next, we use Lemma 1. After identifying all facets, we are left with 2^{d-s} components which shows

$$b_d(X_\Sigma(\mathbb{R})) \geq 2^{d-s}. \quad (3.8)$$

Combining (3.7) and (3.8) we have equality yielding

$$b_d(X_\Sigma(\mathbb{R})) = 2^{d-s}$$

and the higher boundaries $\bar{d}_{d,q}^r$, $r \geq 1$ are zero.

3.3 EXACTNESS OF “KOSZUL” SEQUENCES

For a \mathbb{Z}_2 vector space E of rank r , we have the Koszul complex as in [McC p. 259] or [Lan p. 861]

$$0 \longrightarrow \wedge^r E \otimes SE \longrightarrow \wedge^{r-1} E \otimes SE \longrightarrow \cdots \longrightarrow \wedge^1 E \otimes SE \longrightarrow \wedge^0 E \otimes SE \longrightarrow 0 \quad (3.9)$$

with boundary map $\partial_p : \wedge^p E \otimes SE \longrightarrow \wedge^{p-1} E \otimes SE$ given by

$$\partial_p(x_i \wedge \cdots \wedge x_p \otimes y) = \sum_{i=1}^p x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_p \otimes x_i y \quad (3.10)$$

The complex (3.9) has $H_0 = \mathbb{Z}_2$ and $H_i = 0$ for $i > 0$, and hence

$$0 \longrightarrow \wedge^r E \otimes SE \longrightarrow \wedge^{r-1} E \otimes SE \longrightarrow \cdots \longrightarrow \wedge^1 E \otimes SE \longrightarrow \wedge^0 E \otimes SE \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \quad (3.11)$$

is exact. Decomposing (3.11) into its graded pieces, we obtain for $j \geq 1$ the following exact sequence of \mathbb{Z}_2 vector spaces.

$$0 \longrightarrow \wedge^j E \otimes S^0 E \longrightarrow \wedge^{j-1} E \otimes S^1 E \longrightarrow \cdots \longrightarrow \wedge^1 E \otimes S^{j-1} E \longrightarrow \wedge^0 E \otimes S^j E \longrightarrow 0 \quad (3.12)$$

Next, let G be a \mathbb{Z}_2 vector space of rank p and we consider the following for $k \geq 1$

$$0 \longrightarrow \left(\bigoplus_{i+j=k} \wedge^i G \otimes \wedge^j E \right) \otimes S^0 E \longrightarrow \left(\bigoplus_{l+t=k-1} \wedge^l G \otimes \wedge^t E \right) \otimes S^1 E \longrightarrow \cdots \longrightarrow \wedge^0 G \otimes \wedge^0 E \otimes S^k E \longrightarrow 0 \quad (3.13)$$

$$0 \longrightarrow \wedge^k E \otimes S^0 E \otimes S^0 G \longrightarrow \wedge^{k-1} E \otimes \left(\bigoplus_{l+t=1} S^l E \otimes S^t G \right) \longrightarrow \cdots \longrightarrow \wedge^0 E \otimes \left(\bigoplus_{i+j=k} S^j E \otimes S^i G \right) \longrightarrow 0 \quad (3.14)$$

where in (3.13) the boundary map is given by

$$\begin{aligned} \wedge^i G \otimes \wedge^j E \otimes S^l E &\longrightarrow \wedge^i G \otimes \wedge^{j-1} E \otimes S^{l+1} E \\ \alpha \otimes \beta \otimes \gamma &\longmapsto \alpha \otimes \partial_j(\beta \otimes \gamma) \end{aligned}$$

and in (3.14) the boundary map is given by

$$\begin{aligned} \wedge^l E \otimes S^j E \otimes S^i G &\longrightarrow \wedge^{l-1} E \otimes S^{j+1} E \otimes S^i G \\ \beta \otimes \gamma \otimes \rho &\longmapsto \partial_l(\beta \otimes \gamma) \otimes \rho. \end{aligned}$$

We use the sequences (3.13) and (3.14) to construct the exact sequences (3.15) and (3.16).

For (3.13) we use the fact that

$$\left(\bigoplus_{i+j=q} \wedge^i G \otimes \wedge^j E \right) \otimes S^{k-q} E \cong \bigoplus_{i+j=q} (\wedge^i G \otimes \wedge^j E \otimes S^{k-q} E)$$

and that the boundary map in (3.13) is a direct sum. On one summand, we obtain

$$\begin{aligned} &\ker(\wedge^i G \otimes \wedge^j E \otimes S^{k-q} E \longrightarrow \wedge^i G \otimes \wedge^{j-1} E \otimes S^{k-q+1} E) \\ &= \wedge^i G \otimes \ker(\wedge^j E \otimes S^{k-q} E \longrightarrow \wedge^{j-1} E \otimes S^{k-q+1} E) \\ &\cong \begin{cases} \wedge^i G \otimes \operatorname{im}(\wedge^{j+1} E \otimes S^{k-q-1} E \longrightarrow \wedge^j E \otimes S^{k-q} E) & \text{if } k-q \neq 0 \\ 0 & \text{if } k-q = 0 \text{ and } j \neq 0 \\ \wedge^k G \otimes \wedge^0 E \otimes S^0 E & \text{if } k-q = 0 \text{ and } j = 0. \end{cases} \end{aligned}$$

Since

$$\begin{aligned} &\wedge^i G \otimes \operatorname{im}(\wedge^{j+1} E \otimes S^{k-q-1} E \longrightarrow \wedge^j E \otimes S^{k-q} E) \\ &\cong \operatorname{im}(\wedge^i G \otimes \wedge^{j+1} E \otimes S^{k-q-1} E \longrightarrow \wedge^i G \otimes \wedge^j E \otimes S^{k-q} E), \end{aligned}$$

we have exactness at each piece except for the far left and

$$\begin{aligned}
& \ker\left(\bigoplus_{i+j=k} \wedge^i G \otimes \wedge^j E \otimes S^0 E \longrightarrow \bigoplus_{i+j-1=k-1} \wedge^i G \otimes \wedge^{j-1} E \otimes S^1 E\right) \\
&= \wedge^k G \otimes \wedge^0 E \otimes S^0 E \\
&\cong \wedge^k G.
\end{aligned}$$

Hence,

$$0 \longrightarrow \wedge^k G \longrightarrow \left(\bigoplus_{i+j=k} \wedge^i G \otimes \wedge^j E\right) \otimes S^0 E \longrightarrow \cdots \longrightarrow \wedge^0 G \otimes \wedge^0 E \otimes S^k E \longrightarrow 0 \quad (3.15)$$

is an exact sequence of \mathbb{Z}_2 vector spaces.

The proof for (3.14) is similar, since the map

$$\bigoplus_{i+j=q} \wedge^{k-q} E \otimes S^i E \otimes S^j G \longrightarrow \bigoplus_{i+1+j=q+1} \wedge^{k-q-1} E \otimes S^{i+1} E \otimes S^j G$$

is a direct sum, and

$$\begin{aligned}
& \ker(\wedge^{k-q} E \otimes S^i E \otimes S^j G \longrightarrow \wedge^{k-q-1} E \otimes S^{i+1} E \otimes S^j G) \\
&= \ker(\wedge^{k-q} E \otimes S^i E \longrightarrow \wedge^{k-q-1} E \otimes S^{i+1} E) \otimes S^j G \\
&\cong \begin{cases} \text{im}(\wedge^{k-q+1} E \otimes S^{i-1} E \longrightarrow \wedge^{k-q} E \otimes S^i E) \otimes S^j G & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \text{ and } k - q \neq 0. \end{cases}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
& \text{coker}\left(\bigoplus_{i-1+j=k-1} \wedge^1 E \otimes S^{i-1} E \otimes S^j G \longrightarrow \bigoplus_{i+j=k} \wedge^0 E \otimes S^i E \otimes S^j G\right) \\
&= \wedge^0 E \otimes S^0 E \otimes S^k G.
\end{aligned}$$

As

$$\begin{aligned}
& \text{im}(\wedge^{k-q+1} E \otimes S^{i-1} E \longrightarrow \wedge^{k-q} E \otimes S^i E) \otimes S^j G \\
&\cong \text{im}(\wedge^{k-q+1} E \otimes S^{i-1} E \otimes S^j G \longrightarrow \wedge^{k-q} E \otimes S^i E \otimes S^j G)
\end{aligned}$$

we have exactness of (3.14) at each term except the far right. This gives the following exact sequence of \mathbb{Z}_2 vector spaces.

$$0 \longrightarrow \wedge^k E \otimes S^0 E \otimes S^0 G \longrightarrow \cdots \longrightarrow \wedge^0 E \otimes \left(\bigoplus_{i+j=k} S^j E \otimes S^i G\right) \longrightarrow S^k G \longrightarrow 0 \quad (3.16)$$

Next, suppose we have a short exact sequence of \mathbb{Z}_2 vector spaces

$$0 \longrightarrow G \xrightarrow{\Phi} F \xrightarrow{\Psi} E \longrightarrow 0. \quad (3.17)$$

We can form

$$0 \longrightarrow \wedge^k G \longrightarrow \wedge^k F \longrightarrow \wedge^{k-1} F \otimes S^1 E \longrightarrow \dots \longrightarrow \wedge^1 F \otimes S^{k-1} E \longrightarrow S^k E \longrightarrow 0 \quad (3.18)$$

with the boundary map

$$\begin{aligned} \wedge^l F \otimes S^{k-l} E &\longrightarrow \wedge^{l-1} F \otimes S^{k-l+1} E \\ x_1 \wedge \dots \wedge x_l \otimes a &\longmapsto \sum_{i=1}^l x_i \wedge \dots \wedge \widehat{x}_i \wedge \dots \wedge x_l \otimes \Psi(x_i) a \end{aligned}$$

where

$$\sum_{i=1}^l x_i \wedge \dots \wedge \widehat{x}_i \wedge \dots \wedge x_l \otimes \Psi(x_i) a = \sum_{x_i \in \text{coker } \Phi} x_i \wedge \dots \wedge \widehat{x}_i \wedge \dots \wedge x_l \otimes \Psi(x_i) a$$

because $x_i \in \Phi(G) \iff \Psi(x_i) = 0$. Note that the boundary map in (3.18) is equivalent to the boundary map in (3.15), where we have used that $\bigoplus_{i+j=l} \wedge^i E \otimes \wedge^j G \cong \wedge^l F$. This shows the sequence (3.18) is exact. Moreover, the boundary map in (3.18) is independent of the choice of splitting of the short exact sequence (3.17).

Similarly, if

$$0 \longrightarrow E \xrightarrow{\Phi} H \xrightarrow{\Psi} G \longrightarrow 0 \quad (3.19)$$

is a short exact sequence of \mathbb{Z}_2 vector spaces then we can form

$$0 \longrightarrow \wedge^k E \longrightarrow \wedge^{k-1} E \otimes S^1 H \longrightarrow \dots \longrightarrow \wedge^1 E \otimes S^{k-1} H \longrightarrow S^k H \longrightarrow S^k G \longrightarrow 0 \quad (3.20)$$

with boundary map

$$\begin{aligned} \wedge^l E \otimes S^{k-l} H &\longrightarrow \wedge^{l-1} E \otimes S^{k-l+1} H \\ x_1 \wedge \dots \wedge x_l \otimes a &\longmapsto \sum_{i=1}^l x_i \wedge \dots \wedge \widehat{x}_i \wedge \dots \wedge x_l \otimes \Phi(x_i) a. \end{aligned}$$

As before, we can see that this map is equivalent to the boundary map in (3.16) and hence (3.20) is an exact sequence of \mathbb{Z}_2 vector spaces. The boundary map is defined independent

of a choice of splitting of (3.19).

Next, we will obtain two more exact sequences. As $0 \longrightarrow E \longrightarrow H \longrightarrow G \longrightarrow 0$ is exact, we have $0 \longrightarrow G^* \longrightarrow H^* \longrightarrow E^* \longrightarrow 0$ is an exact sequence of \mathbb{Z}_2 vector spaces. In this case the exact sequence (3.18) gives

$$0 \longrightarrow \wedge^k G^* \longrightarrow \wedge^k H^* \longrightarrow \wedge^{k-1} H^* \otimes S^1 E^* \longrightarrow \dots \longrightarrow \wedge^1 H^* \otimes S^{k-1} E^* \longrightarrow S^k E^* \longrightarrow 0. \quad (3.21)$$

Applying $\text{Hom}(-, \mathbb{Z}_2)$ to (3.21) we obtain the exact sequence

$$0 \longrightarrow S^k E \longrightarrow \wedge^1 H \otimes S^{k-1} E \longrightarrow \dots \longrightarrow \wedge^{k-1} H \otimes S^1 E \longrightarrow \wedge^k H \longrightarrow \wedge^k G \longrightarrow 0 \quad (3.22)$$

where we have used that

$$\begin{aligned} (\wedge^j H^* \otimes S^{k-j} E^*)^* &\cong (\wedge^j H^*)^* \otimes (S^{k-j} E^*)^* \\ &\cong \wedge^j H \otimes S^{k-j} E \end{aligned}$$

as vector spaces over \mathbb{Z}_2 . Moreover, the boundary maps in (3.22) do not depend on a choice of splitting of the short exact sequence $0 \longrightarrow E \longrightarrow H \longrightarrow G \longrightarrow 0$ because the maps in (3.21) do not depend on a splitting of $0 \longrightarrow G^* \longrightarrow H^* \longrightarrow E^* \longrightarrow 0$.

We also have (3.20) which yields the exact sequence

$$0 \longrightarrow \wedge^k G^* \longrightarrow \wedge^{k-1} G^* \otimes S^1 H^* \longrightarrow \dots \longrightarrow \wedge^1 G^* \otimes S^{k-1} H^* \longrightarrow S^k H^* \longrightarrow S^k E^* \longrightarrow 0.$$

Applying $\text{Hom}(-, \mathbb{Z}_2)$ as before, we obtain

$$0 \longrightarrow S^k E \longrightarrow S^k H \longrightarrow \wedge^1 G \otimes S^{k-1} H \longrightarrow \dots \longrightarrow \wedge^{k-1} G \otimes S^1 H \longrightarrow \wedge^k G \longrightarrow 0 \quad (3.23)$$

and again the boundary maps do not depend on a choice of splitting of $0 \longrightarrow E \longrightarrow H \longrightarrow G \longrightarrow 0$.

We conclude this section with our interest in the exact sequences (3.18), (3.20), (3.22), and (3.23).

Proposition. *Suppose*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0 \quad (3.24)$$

is an exact sequence of sheaves (or cosheaves) on a fan Σ . Then the following sequences are exact for $k \geq 1$.

$$0 \longrightarrow \wedge^k \mathcal{A} \longrightarrow \wedge^k \mathcal{B} \longrightarrow \wedge^{k-1} \mathcal{B} \otimes S^1 \mathcal{C} \longrightarrow \dots \longrightarrow \wedge^1 \mathcal{B} \otimes S^{k-1} \mathcal{C} \longrightarrow S^k \mathcal{C} \longrightarrow 0 \quad (3.25)$$

$$0 \longrightarrow \wedge^k \mathcal{A} \longrightarrow \wedge^{k-1} \mathcal{A} \otimes S^1 \mathcal{B} \longrightarrow \dots \longrightarrow \wedge^1 \mathcal{A} \otimes S^{k-1} \mathcal{B} \longrightarrow S^k \mathcal{B} \longrightarrow S^k \mathcal{C} \longrightarrow 0 \quad (3.26)$$

$$0 \longrightarrow S^k \mathcal{A} \longrightarrow \wedge^1 \mathcal{B} \otimes S^{k-1} \mathcal{A} \longrightarrow \dots \longrightarrow \wedge^{k-1} \mathcal{B} \otimes S^1 \mathcal{A} \longrightarrow \wedge^k \mathcal{B} \longrightarrow \wedge^k \mathcal{C} \longrightarrow 0 \quad (3.27)$$

$$0 \longrightarrow S^k \mathcal{A} \longrightarrow S^k \mathcal{B} \longrightarrow \wedge^1 \mathcal{C} \otimes S^{k-1} \mathcal{B} \longrightarrow \dots \longrightarrow \wedge^{k-1} \mathcal{C} \otimes S^1 \mathcal{B} \longrightarrow \wedge^k \mathcal{C} \longrightarrow 0 \quad (3.28)$$

For each $\sigma \in \Sigma$, we may apply (3.18), (3.20), (3.22), and (3.23) to the short exact sequence of \mathbb{Z}_2 vector spaces given by the stalks $0 \longrightarrow A_\sigma \longrightarrow B_\sigma \longrightarrow C_\sigma \longrightarrow 0$. Since each boundary map in (3.18), (3.20), (3.22), and (3.23) is independent of a choice of splitting of the short exact sequence $0 \longrightarrow A_\sigma \longrightarrow B_\sigma \longrightarrow C_\sigma \longrightarrow 0$ and is induced from one of the the sheaf (or cosheaf) maps in (3.24), the collection of maps on stalks gives rise to a sheaf (or cosheaf) homomorphism. Thus, we obtain the exact sequences of sheaves (or cosheaves) in the proposition.

CHAPTER 4

\mathbb{Z}_2 HODGE SPACES FOR REFLEXIVE POLYTOPES

4.1 A CORRESPONDENCE BETWEEN SHEAVES AND COSHEAVES

A *reflexive* polytope is a lattice polytope Δ with $0 \in \text{int}\Delta$ and such that the polar polytope Δ^* is also a lattice polytope. A discussion of reflexive polytopes can be found in [Bat1 §4.1].

Throughout this chapter, we will use the following notation.

$$\begin{array}{ll}
 \Delta \subset M & \text{a reflexive polytope} \\
 \Delta^* \subset N & \text{the polar polytope of } \Delta \\
 \Sigma \subset N & \text{the normal fan of } \Delta \\
 & (= \text{the face fan of } \Delta^*) \\
 \Sigma^* \subset M & \text{the normal fan of } \Delta^* \\
 & (= \text{the face fan of } \Delta)
 \end{array}$$

If $\tau \in \Sigma$ and $\dim \tau = j > 0$ then $\tau = \text{poshull} f^*$ where $f^* < \Delta^*$ is a face of dimension $j - 1$. We define τ^* to be the cone in Σ^* of dimension $d - j + 1$ which is dual to f . The correspondence

$$\tau \in \Sigma \longleftrightarrow \tau^* \in \Sigma^*$$

gives a one-to-one inclusion reversing correspondence between the cones in Σ of positive dimension and the positive dimensional cones in Σ^* . We show this correspondence by dimension below.

$$\begin{array}{ccccccc}
 \text{Cone in } \Sigma : & 1 & 2 & 3 & \cdots & d-2 & d-1 & d \\
 & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\
 \text{Cone in } \Sigma^* : & d & d-1 & d-2 & \cdots & 3 & 2 & 1
 \end{array}$$

Let \mathcal{H} be a sheaf on the fan Σ^* . We use the correspondence above to create a cosheaf $\widehat{\mathcal{H}}$ on Σ by defining for $\tau \in \Sigma$

$$\widehat{H}_\tau := \begin{cases} H_{\tau^*} & \text{if } \dim \tau > 0 \\ 0 & \text{if } \dim \tau = 0 \end{cases} \quad (4.1)$$

with restriction map for $\sigma < \tau$

$$\widehat{\rho}_{\tau,\sigma} := \begin{cases} \rho_{\tau^*,\sigma^*} & \text{if } \dim \sigma > 0 \\ 0 & \text{if } \dim \sigma = 0 \end{cases} \quad (4.2)$$

where ρ_{τ^*,σ^*} is a restriction map for the sheaf \mathcal{H} . Using (4.1) we have equality of chain groups for the sheaf \mathcal{H} on Σ^* and the cosheaf $\widehat{\mathcal{H}}$ on Σ , as depicted below.

$$\begin{array}{ccccccccc} C_{d-1}(\widehat{\mathcal{H}}) & \longrightarrow & C_{d-2}(\widehat{\mathcal{H}}) & \longrightarrow & \cdots & \longrightarrow & C_2(\widehat{\mathcal{H}}) & \longrightarrow & C_1(\widehat{\mathcal{H}}) & \longrightarrow & C_0(\widehat{\mathcal{H}}) \\ \parallel & & \parallel & & & & \parallel & & \parallel & & \parallel \\ C^0(\mathcal{H}) & \longrightarrow & C^1(\mathcal{H}) & \longrightarrow & \cdots & \longrightarrow & C^{d-3}(\mathcal{H}) & \longrightarrow & C^{d-2}(\mathcal{H}) & \longrightarrow & C^{d-1}(\mathcal{H}) \end{array}$$

From (4.2) the horizontal sheaf and cosheaf boundary maps are equal. Hence we may equate sheaf cohomology groups for \mathcal{H} on Σ^* with cosheaf homology groups for $\widehat{\mathcal{H}}$ on Σ .

$$H^p(\mathcal{H}) \cong H_{d-p-1}(\widehat{\mathcal{H}}) \quad 1 \leq p \leq d-2$$

One more construction which we will use in the sequel is that of the cosheaf \mathcal{A}° . If \mathcal{A} is a cosheaf on Σ then \mathcal{A}° is defined by

$$A_\sigma^\circ := \begin{cases} A_\sigma & \text{if } \dim \sigma > 0 \\ 0 & \text{if } \dim \sigma = 0 \end{cases}$$

with the restriction map for $\sigma < \tau$ defined by

$$\rho_{\tau,\sigma}^\circ := \begin{cases} \rho_{\tau,\sigma} & \text{if } \dim \sigma > 0 \\ 0 & \text{if } \dim \sigma = 0. \end{cases}$$

Note that by definition

$$C_p(\mathcal{A}) = C_p(\mathcal{A}^\circ) \quad \text{for } p \leq d-1,$$

and hence we have

$$H_p(\mathcal{A}) = H_p(\mathcal{A}^\circ) \quad \text{for } p \leq d-2.$$

4.2 THE COSHEAVES $\widehat{\mathcal{F}}$, $\widehat{\mathcal{G}}$, AND \mathcal{C} ON Σ

The sheaf \mathcal{F} on Σ^* is defined as follows. For $\sigma^* \in \Sigma^*$ the stalk is

$$F_{\sigma^*} := \frac{(\sigma^*)^\perp \cap N}{(\sigma^*)^\perp \cap 2N}$$

and the face restriction map

$$\rho_{\sigma^*, \tau^*} : F_{\sigma^*} \xrightarrow{\subset} F_{\tau^*}$$

is given by inclusion for $\tau^* < \sigma^*$ in Σ^* . The sheaf \mathcal{G} on Σ^* is then defined to be the cokernel of the inclusion $\mathcal{F} \hookrightarrow N/2N$ so that

$$0 \longrightarrow \mathcal{F} \longrightarrow N/2N \longrightarrow \mathcal{G} \longrightarrow 0 \quad (4.3)$$

is a short exact sequence of sheaves on Σ^* .

Lemma 7. *For $\tau \in \Sigma, \tau^* \in \Sigma^*$ as in the previous section with $\dim \tau = l > 0$, we have the following containment of \mathbb{Z}_2 vector spaces*

$$\frac{(\tau^*)^\perp \cap N}{(\tau^*)^\perp \cap 2N} \subset \frac{\tau \cap N}{\tau \cap 2N}.$$

Note that the \mathbb{Z}_2 vector space on the left has rank $l - 1$ while the one on the right has rank l . To prove the lemma, suppose

$$\begin{aligned} \tau &= \text{poshull } f^* \\ &= \text{poshull}\{q_1, q_2, \dots, q_s\} \end{aligned}$$

and

$$\begin{aligned} \tau^* &= \text{poshull } f \\ &= \text{poshull}\{p_1, p_2, \dots, p_k\} \end{aligned}$$

where $f < \Delta$ and $f^* < \Delta^*$. Then we have $\langle p_i, q_j \rangle = -1 \quad \forall i, j$. This gives

$$(\tau^*)^\perp = \text{span}\{q_i - q_j \mid 1 \leq i, j \leq s\}$$

yielding

$$\frac{(\tau^*)^\perp \cap N}{(\tau^*)^\perp \cap 2N} \subset \frac{\tau \cap N}{\tau \cap 2N}.$$

Next, we note that this inclusion

$$\frac{\text{span}\{q_i - q_j \mid 1 \leq i, j \leq s\} \cap N}{\text{span}\{q_i - q_j \mid 1 \leq i, j \leq s\} \cap 2N} \subset \frac{\text{poshull}\{q_1, q_2, \dots, q_s\} \cap N}{\text{poshull}\{q_1, q_2, \dots, q_s\} \cap 2N}$$

is compatible with the face restriction maps for cones in Σ . Thus, we obtain an injective homomorphism of cosheaves $\widehat{\mathcal{F}} \hookrightarrow \mathcal{N}$. We define the cosheaf \mathcal{C} to be the cokernel of this homomorphism so that

$$0 \longrightarrow \widehat{\mathcal{F}} \longrightarrow \mathcal{N} \longrightarrow \mathcal{C} \longrightarrow 0 \quad (4.4)$$

is an exact sequence of cosheaves on Σ . Note that if $\dim \sigma = 0$ then the stalks \widehat{F}_σ , N_σ , and C_σ are all zero. Moreover, for each $\sigma \in \Sigma$ of positive dimension the stalk C_σ is a rank one \mathbb{Z}_2 vector space.

Lemma 8. *For $\sigma < \tau$ in Σ with $\dim \sigma > 0$ the map $C_\sigma \longrightarrow C_\tau$ is the identity.*

We prove the lemma by contradiction. Assume we have $\tau_1 < \tau_2$ in Σ and $C_{\tau_1} \longrightarrow C_{\tau_2}$ is the zero map. We have the follow diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{(\tau_1^*)^\perp \cap N}{(\tau_1^*)^\perp \cap 2N} & \longrightarrow & \frac{\tau_1 \cap N}{\tau_1 \cap 2N} & \longrightarrow & C_{\tau_1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \phi \\ 0 & \longrightarrow & \frac{(\tau_2^*)^\perp \cap N}{(\tau_2^*)^\perp \cap 2N} & \longrightarrow & \frac{\tau_2 \cap N}{\tau_2 \cap 2N} & \longrightarrow & C_{\tau_2} \longrightarrow 0 \end{array}$$

where the rows are short exact sequences and vertical maps are cosheaf restriction maps. We have assumed $\phi = 0$ and hence

$$\frac{\tau_1 \cap N}{\tau_1 \cap 2N} \subset \frac{(\tau_2^*)^\perp \cap N}{(\tau_2^*)^\perp \cap 2N}. \quad (4.5)$$

Let r_1 be the first lattice point on a ray of $\tau_1 \subset \tau_2$ and r_2 be the first lattice point on a ray of τ_2^* . Due to the inclusion (4.5) we have $\langle r_1, r_2 \rangle = 0 \pmod{2}$. This is a contradiction because $\langle r_1, r_2 \rangle = -1$. Thus we have Lemma 8.

Next, note that

$$0 \longrightarrow \widehat{\mathcal{F}} \longrightarrow N/\widehat{2N} \longrightarrow \widehat{\mathcal{G}} \longrightarrow 0$$

is a short exact sequence of cosheaves of Σ where $\widehat{N/2N} = N/2N^\circ$ as cosheaves on Σ . We can combine this short exact sequence with the short exact sequence (4.4) into the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & & 0 & & \mathcal{K} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \widehat{\mathcal{F}} & \longrightarrow & \widehat{N/2N} & \longrightarrow & \widehat{\mathcal{G}} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \Phi \\
 0 & \longrightarrow & \mathcal{N} & \longrightarrow & N/2N^\circ & \longrightarrow & \mathcal{E}^\circ \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathcal{C} & & 0 & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array} \tag{4.6}$$

where the rows and columns are exact and $\mathcal{K} := \ker \Phi$.

Next, we use the Snake Lemma on the diagram (4.6) to obtain $\mathcal{K} \cong \mathcal{C}$ as cosheaves yielding

$$0 \longrightarrow \mathcal{C} \longrightarrow \widehat{\mathcal{G}} \longrightarrow \mathcal{E}^\circ \longrightarrow 0 \tag{4.7}$$

is a short exact sequence of cosheaves on Σ . As $(\wedge^q \mathcal{E})^\circ = \wedge^q \mathcal{E}^\circ$ we have

$$H_p(\wedge^q \mathcal{E}) = H_p(\wedge^q \mathcal{E}^\circ) \quad \text{for } p \leq d - 2.$$

4.3 VANISHING OF THE HOMOLOGY GROUPS $H_p(\wedge^k \widehat{\mathcal{G}})$

Our goal in the next two sections is to establish the vanishing of certain \mathbb{Z}_2 Hodge spaces $H_{pq}(\Sigma)$. We use the short exact sequence (4.7) and information about the vanishing of the

homology groups $H_p(\wedge^k \widehat{\mathcal{G}})$. We remind the reader that a cone $\sigma \in \Sigma$ is \mathbb{Z}_2 regular provided the image in $N/2N$ of the rays of σ forms a basis for N_σ .

Lemma 9. *Assume the cones in Σ^* of dimension at most e are \mathbb{Z}_2 regular. Then*

$$H_p(\wedge^k \widehat{\mathcal{G}}) = 0 \quad \text{for } 1 \leq p < e - 1.$$

First, we will assume that $e = d$. Following [Bri §1.2], as the cones in Σ^* are \mathbb{Z}_2 regular the sheaf \mathcal{G} on Σ^* can be written as follows.

$$\mathcal{G} = \bigoplus_{r_i \in \Sigma^*(1)} \mathcal{G}(r_i)$$

is a direct sum of the sheaves $\mathcal{G}(r_i)$ with

$$G(r_i)_{\tau^*} = \begin{cases} \mathbb{Z}_2 & \text{if } r_i \in \tau^*(1) \\ 0 & \text{else.} \end{cases}$$

Moreover, we have

$$\wedge^k \mathcal{G} = \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} \mathcal{G}(r_1, r_2, \dots, r_k)$$

is the direct sum of sheaves $\mathcal{G}(r_1, r_2, \dots, r_k)$ with

$$\mathcal{G}(r_1, r_2, \dots, r_k)_{\tau^*} = \begin{cases} \mathbb{Z}_2 & \text{if } r_1, r_2, \dots, r_k \in \tau^*(1) \\ 0 & \text{else.} \end{cases}$$

Using the proof of the proposition in Section 1.2 of [Bri], the cohomology groups $H^p(\mathcal{G}(r_1, r_2, \dots, r_k))$ vanish for $p > 0$ because Σ^* is a complete fan. Hence we have

$$\begin{aligned} H^p(\wedge^k \mathcal{G}) &= H^p\left(\bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} \mathcal{G}(r_1, r_2, \dots, r_k)\right) \\ &\cong \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} H^p(\mathcal{G}(r_1, r_2, \dots, r_k)) \\ &= 0 \quad \text{for } p > 0. \end{aligned}$$

Hence, as $\wedge^k \widehat{\mathcal{G}} = \widehat{\wedge^k \mathcal{G}}$ we have

$$H_p(\wedge^k \widehat{\mathcal{G}}) = H^{d-p-1}(\wedge^k \mathcal{G}) = 0$$

for p such that $1 \leq p \leq d - 2$ and $d - p - 1 > 0$, which proves Lemma 9 when $e = d$.

Next, assume $e < d$. Let $\Sigma_{\leq e}^* := \bigcup_{i \leq e} \Sigma^*(i)$ be the subfan of Σ^* consisting of the cones of dimension at most e , and let \mathcal{G}' be the restriction of \mathcal{G} to $\Sigma_{\leq e}^*$. Then as $\Sigma_{\leq e}^*$ consists of \mathbb{Z}_2 regular cones, we have

$$\mathcal{G}' = \bigoplus_{r_i \in \Sigma^*(1)} \mathcal{G}(r_i)'$$

where $\mathcal{G}(r_i)'$ is the restriction of $\mathcal{G}(r_i)$ to $\Sigma_{\leq e}^*$. Moreover, we have

$$\wedge^k \mathcal{G}' = \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} \mathcal{G}(r_1, r_2, \dots, r_k)'$$

By definition, for each k we have the equality

$$H^p(\wedge^k \mathcal{G}) = H^p(\wedge^k \mathcal{G}') \quad \text{for } p > d - e$$

which gives $H^p(\wedge^k \mathcal{G}) = 0$ for $p > d - e$. Thus, $H_p(\wedge^k \widehat{\mathcal{G}}) = 0$ for p such that $1 \leq p \leq d - 2$ and $d - p - 1 > d - e$ yielding

$$H_p(\wedge^k \widehat{\mathcal{G}}) = 0 \quad \text{for } 1 \leq p < e - 1.$$

4.4 VANISHING OF THE \mathbb{Z}_2 HODGE SPACES $H_{pq}(\Sigma)$

Theorem 1. *Assume the cones in Σ^* of dimension at most e are \mathbb{Z}_2 regular. Then*

$$H_p(\wedge^q \mathcal{E}) = 0 \quad \text{for } q < p < e - 1.$$

To prove the theorem, we use the short exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow \widehat{\mathcal{G}} \longrightarrow \mathcal{E}^\circ \longrightarrow 0$$

of cosheaves of Σ and the associated degree q sequence from (3.27)

$$0 \longrightarrow S^q \mathcal{C} \longrightarrow \wedge^1 \widehat{\mathcal{G}} \otimes S^{q-1} \mathcal{C} \longrightarrow \dots \longrightarrow \wedge^{q-1} \widehat{\mathcal{G}} \otimes S^1 \mathcal{C} \longrightarrow \wedge^q \widehat{\mathcal{G}} \longrightarrow \wedge^q \mathcal{E}^\circ \longrightarrow 0$$

which we break into short exact sequences below

$$0 \longrightarrow S^q \mathcal{C} \longrightarrow \wedge^1 \widehat{G} \otimes S^{q-1} \mathcal{C} \longrightarrow W_1 \longrightarrow 0 \quad (4.8)$$

$$0 \longrightarrow W_1 \longrightarrow \wedge^2 \widehat{G} \otimes S^{q-2} \mathcal{C} \longrightarrow W_2 \longrightarrow 0 \quad (4.9)$$

$$\dots \quad (4.10)$$

$$0 \longrightarrow W_{q-2} \longrightarrow \wedge^{q-1} \widehat{G} \otimes S^1 \mathcal{C} \longrightarrow W_{q-1} \longrightarrow 0 \quad (4.11)$$

$$0 \longrightarrow W_{q-1} \longrightarrow \wedge^q \widehat{G} \longrightarrow \wedge^q \mathcal{E}^\circ \longrightarrow 0. \quad (4.12)$$

These induce long exact sequences on homology groups. From Lemma 8 we see that

$$\mathcal{C} \cong (\mathbb{Z}_2)^\circ$$

$$S^{q-k} \mathcal{C} \cong (\mathbb{Z}_2)^\circ$$

where \mathbb{Z}_2 is the constant cosheaf on Σ . Thus we have

$$H_p(\wedge^k \widehat{\mathcal{G}} \otimes S^{q-k} \mathcal{C}) \cong H_p(\wedge^k \widehat{\mathcal{G}}) \otimes S^{q-k} \mathcal{C} = 0 \quad \text{for } 1 \leq p < e-1,$$

where we have used Lemma 9. We begin with the long exact sequence induced from (4.8).

$$\dots \longrightarrow H_p(\wedge^1 \widehat{G} \otimes S^{q-1} \mathcal{C}) \longrightarrow H_p(W_1) \longrightarrow H_{p-1}(S^q \mathcal{C}) \longrightarrow \dots$$

We have

$$H_p(\wedge^1 \widehat{G} \otimes S^{q-1} \mathcal{C}) = 0 \quad \text{for } 1 \leq p < e-1$$

$$H_{p-1}(S^q \mathcal{C}) = 0 \quad \text{for } 1 \leq p-1 < e-1$$

and hence $H_p(W_1) = 0$ for $1 < p < e-1$. Next we use the long exact sequence induced from (4.9).

$$\dots \longrightarrow H_p(\wedge^2 \widehat{G} \otimes S^{q-2} \mathcal{C}) \longrightarrow H_p(W_2) \longrightarrow H_{p-1}(W_1) \longrightarrow \dots$$

We have

$$H_p(\wedge^2 \widehat{G} \otimes S^{q-2} \mathcal{C}) = 0 \quad \text{for } 1 \leq p < e-1$$

$$H_{p-1}(W_1) = 0 \quad \text{for } 1 < p-1 < e-1$$

and hence $H_p(W_2) = 0$ for $2 < p < e - 1$. We continue this process to obtain

$$\begin{aligned}
H_p(W_1) &= 0 & \text{for } 1 < p < e - 1 \\
H_p(W_2) &= 0 & \text{for } 2 < p < e - 1 \\
&\dots & \dots \\
H_p(W_{q-2}) &= 0 & \text{for } q - 2 < p < e - 1 \\
H_p(W_{q-1}) &= 0 & \text{for } q - 1 < p < e - 1 \\
H_p(\wedge^q \mathcal{E}^\circ) &= 0 & \text{for } q < p < e - 1.
\end{aligned}$$

Moreover, as $e - 1 \leq d - 1$ we have $H_p(\wedge^q \mathcal{E}) = H_p(\wedge^q \mathcal{E}^\circ)$ for $q < p < e - 1$ and hence Theorem 1 holds.

4.5 THE DIAGONAL ENTRIES $H_{qq}(\Sigma)$

In this section, we are under the assumption that the cones in $\Sigma_{\leq e}^*$ are \mathbb{Z}_2 regular. Let $\tau^* \in \Sigma^*(e)$, $\tau^* = \text{poshull } f$ where $f = \text{conv}\{p_1, p_2, \dots, p_e\}$.

Lemma 10. *The toric subvariety $Y(\mathbb{R})$ of $X_\Sigma(\mathbb{R})$ defined by the face f of Δ is T -homeomorphic to $\mathbb{R}\mathbb{P}^{e-1}$.*

To prove the lemma, let $\Psi = \text{conv}\{0, p_1, p_2, \dots, p_e\} \subset M$. We extend $\{p_1, p_2, \dots, p_e\}$ to a basis $\{p_1, p_2, \dots, p_e, t_1, t_2, \dots, t_{d-e}\}$ for $M \otimes \mathbb{R}$ with $\{t_1, t_2, \dots, t_{d-e}\}$ orthonormal. We have a map $M' \longrightarrow M$ given by the $d \times d$ matrix

$$A := \begin{bmatrix} | & | & & | & | & & | \\ p_1 & p_2 & \cdots & p_e & t_1 & \cdots & t_{d-e} \\ | & | & & | & | & & | \end{bmatrix}$$

which has odd determinant because τ^* is \mathbb{Z}_2 regular. Moreover,

$$\begin{aligned} Ae_1 &= p_1 \\ Ae_2 &= p_2 \\ &\dots \\ Ae_e &= p_e \end{aligned}$$

and A takes the simplex $\Psi' = \text{conv}\{0, e_1, e_2, \dots, e_e\}$ to the simplex Ψ . The matrix A^* gives a map $N \longrightarrow N'$ which induces an isomorphism

$$N/2N \xrightarrow{\cong} N'/2N'. \quad (4.13)$$

Let r_i be the first integer point on the ray dual to the facet $\text{conv}\{0, p_1, p_2, \dots, \widehat{p_i}, \dots, p_e\}$ of Ψ . We have the following for $j \in \{1, 2, \dots, \widehat{i}, \dots, e\}$.

$$0 = \langle Ae_j, r_i \rangle = \langle e_j, A^*r_i \rangle$$

and hence the vector A^*r_i lies on the ray dual to the facet $\text{conv}\{0, e_1, e_2, \dots, \widehat{e_i}, \dots, e_e\}$ of Ψ' . This gives $A^*r_i = ke_i$, where k is an odd integer. (if k were even, then the image of r_i in $N/2N$ would map to $0 \in N'/2N'$ contradicting the isomorphism (4.13)) Similarly, if r is the first integer point along the ray in N dual to the facet $\text{conv}\{p_1, p_2, \dots, p_e\}$ of Ψ then A^*r is an odd multiple of the vector $-e_1 - e_2 - \dots - e_e$. Thus, the isomorphism in (4.13) gives an isomorphism $N_\sigma \xrightarrow{\cong} N_{\sigma'}$ where σ is a cone in the normal fan of Ψ and σ' is in the normal fan of Ψ' . We obtain a T -homeomorphism.

$$\frac{\Psi \times N/2N}{\sim} \cong \frac{\Psi' \times N'/2N'}{\sim'}$$

As $f < \Psi$ the toric variety Y is T -homeomorphic to projective space and we obtain Lemma 10. Next, we use Lemma 6 to arrive at the following proposition.

Proposition. *If the cones in $\Sigma_{\leq e}^*$ are \mathbb{Z}_2 regular then for $q < e - 1$ we have $H_{qq}(\Sigma) (\cong A_q^T(X_\Sigma))$ has rank 1 and is generated by the orbit closure of any q dimensional torus orbit.*

4.6 COLLAPSING OF THE SPECTRAL SEQUENCE \overline{E}^r

In this section, we show that the spectral sequence \overline{E}^r for X_Σ collapses at \overline{E}^1 when the cones in Σ^* are \mathbb{Z}_2 regular. Combining the work in Section 4.4 and Section 4.5, we have that the ranks of the entries in the \overline{E}^1 term are as follows

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & * & * \\
 & & & & & 1 & * & * \\
 & & & & 1 & 0 & * & * \\
 q & & 1 & 0 & 0 & * & * & \\
 & & & 1 & 0 & 0 & 0 & * & * \\
 & & & & 1 & 0 & 0 & 0 & 0 & * & * \\
 & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & p
 \end{array} \tag{4.14}$$

where the $*$ entries are possibly nonzero and occur in the columns $p > d - 2$.

Remark. We can completely determine the $*$ entries. From Section 3.2, if s is the rank of the image of the rays of Σ in $N/2N$ then the rank of $\overline{E}_{d,q}^1$ is the binomial coefficient $\binom{d-s}{q-s}$. The Euler characteristic of the q th row is $(-1)^q h_q$ where $h = (h_0, h_1, h_2, \dots, h_d)$ is the h -vector of the polytope Δ . Thus, we may determine the ranks of the vector spaces $\overline{E}_{d-1,q}^1$, giving us knowledge of the ranks of all the entries in $\overline{E}_{p,q}^1$.

By looking at the \overline{E}^1 term for X_Σ , we see that the only possible nonzero higher differentials have target $\overline{E}_{d-2,d-2}^1$. To show that the spectral sequence \overline{E}^r for X_Σ collapses at \overline{E}^1 , we need only show the following lemma.

Lemma 11. *The map*

$$f_{d-2}^{\mathbb{R}} : \overline{E}_{d-2,d-2}^1 \longrightarrow H_{d-2}(X_\Sigma(\mathbb{R}))$$

from (2.5) is nonzero.

Let g be a $d-2$ dimensional face of Δ with $\beta = \text{poshull}\{q_1, q_2, \dots, q_k\}$ the cone dual to g . As mentioned in Section 3.1, $f_{d-2}^{\mathbb{R}}$ sends $[V(\beta)]$ to the homology class of $C_\beta := \sum_{t \in N(\beta)} (g, t)$. Suppose $C_\beta = \partial C$, $C = (g_1, t_1) + (g_2, t_2) + \dots + (g_r, t_r) \in C_{d-1}(X_\Sigma)$ where $g_i < \Delta$, $\sigma_i \in \Sigma$ is dual to g_i , and $t_i \in N(\sigma_i)$. We include $N \subset \tilde{N}$ where $\tilde{N} \cong \mathbb{Z}^{d+1}$ and $\Delta^* \times [-1, 1] \subset \tilde{N}$. Note that $\Delta^* \times [-1, 1]$ is reflexive and the normal fan of $\Delta^* \times [-1, 1]$ consists of \mathbb{Z}_2 regular cones. Let Ξ be the face fan of $\Delta^* \times [-1, 1]$. Then, Ξ is the normal fan of $B_\Delta := (\Delta^* \times [-1, 1])^*$ the bipyramid with base Δ . We define the cone $\tilde{\beta} \in \Xi$ to be the positive hull of the rays $\{(q_1, 1), (q_1, -1), (q_2, 1), (q_2, -1), \dots, (q_k, 1), (q_k, -1)\}$ in \tilde{N} . Note that $\tilde{\beta}$ is dual to g considered as a face in B_Δ . Moreover, we have

$$\begin{aligned} N(\tilde{\beta}) &= \frac{\tilde{N}/2\tilde{N}}{\tilde{\beta} \cap \tilde{N} / \tilde{\beta} \cap 2\tilde{N}} \\ &\cong \frac{N/2N \oplus \langle e_{d+1} \rangle}{\beta \cap N / \beta \cap 2N \oplus \langle e_{d+1} \rangle} \\ &\cong \frac{N/2N}{\beta \cap N / \beta \cap 2N} \\ &= N(\beta). \end{aligned}$$

This gives that inclusion of $[V(\tilde{\beta})]$ in $H_{d-2}(X_\Xi(\mathbb{R}))$ is also represented cellularly by C_β . Similarly, for each $\sigma_i \in \Sigma$ appearing in the chain C , $\tilde{\sigma}_i := \sigma_i \times [-1, 1] \in \Xi$ satisfies $N(\tilde{\sigma}_i) \cong N(\sigma_i)$. Hence, C can be viewed as a chain in $C_{d-1}(X_\Xi(\mathbb{R}))$. We have

$$\tilde{\partial}C = C_\beta \tag{4.15}$$

where $\tilde{\partial}$ is the cellular boundary map for $X_\Xi(\mathbb{R})$. Equation (4.15) holds because $\tilde{\sigma}_i < \gamma$ in Ξ implies γ must be of the form $\gamma = \sigma \times [-1, 1]$ with $\sigma \in \Sigma$. As $[V(\tilde{\beta})]$ generates $A_{d-2}^T(X_\Xi)$, the map $A_{d-2}^T(X_\Xi) \rightarrow H_{d-2}(X_\Xi(\mathbb{R}))$ must be zero. However, the ranks of the entries \overline{E}^1 for X_Ξ have the form (4.14) and $\overline{E}_{d-2, d-2}^1$ is in the fourth column from the right. There cannot be higher boundaries with target $\overline{E}_{d-2, d-2}^1$ which contradicts the fact that $A_{d-2}^T(X_\Xi) \rightarrow H_{d-2}(X_\Xi(\mathbb{R}))$ is the zero map. Hence, Lemma 11 holds and the spectral sequence \overline{E}^r for X_Σ collapses at \overline{E}^1 .

Corollary. *If Σ^* consists of \mathbb{Z}_2 regular cones then X_Σ is maximal.*

Remark. *We have proved the maximality of toric varieties associated to the Fano polyhedra.*

Definition 2. *Let $\text{vert}\Delta = \{v_1, v_2, \dots, v_n\}$. The d dimensional polytope Δ is a Fano polyhedron provided*

1. $0 \in \text{int}\Delta$
2. *Each face of Δ is a simplex*
3. *If $v_{i_1}, v_{i_2}, \dots, v_{i_d}$ are the vertices of a $(d - 1)$ dimensional face of Δ then*

$$\det[v_{i_1} v_{i_2} \cdots v_{i_d}] = \pm 1.$$

If Δ is a Fano polyhedron then Δ^ defines one of the so called smooth toric Fano manifolds. A classification of the Fano polyhedra is known for dimension at most 4. There are 5 Fano polyhedra of dimension 2. Batyrev classified the 18 Fano polyhedra of dimension 3 in [Bat3] and the 123 Fano polyhedra of dimension 4 in [Bat2].*

4.7 EXAMPLES

In this section, we illustrate the results of this chapter with some examples. In each example, we use `torhom` [Fra1] to compute the \mathbb{Z}_2 Hodge spaces $H_{pq}(\Sigma)$ and the f -vector of the polytope Δ .

Example 2. *The five dimensional cross polytope.*

Let Δ be the five dimensional cross polytope. Then, Δ is the convex hull of the ten vertices

$$\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5\}.$$

The polar polytope Δ^* is the five dimensional cube, which defines the nonsingular toric variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The f -vector for Δ is $(10, 40, 80, 80, 32)$. Below I compute

the h -vector for Δ :

$$\begin{aligned}
 h_5 &= && 1 && = 1 \\
 h_4 &= && f_4 - 5 && = 27 \\
 h_3 &= && f_3 - \binom{4}{3}f_4 + \binom{5}{3} && = -38 \\
 h_2 &= && f_2 - \binom{3}{2}f_3 + \binom{4}{2}f_4 - \binom{5}{2} && = 22 \\
 h_1 &= && f_1 - 2f_2 + 3f_3 - 4f_4 + 5 && = -3 \\
 h_0 &= && 1 && = 1.
 \end{aligned}$$

We compute the ranks of the \mathbb{Z}_2 Hodge spaces for Σ .

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 31 & 4 \\
 & & & & & 1 & 45 & 6 \\
 & & & & & q & 1 & 0 & 25 & 4 \\
 & & & & & & 1 & 0 & 0 & 5 & 1 \\
 & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & p
 \end{array}$$

One can check that $(-1)^q h_q$ is the Euler characteristic of the q th row. We compute the \mathbb{Z}_2 Betti numbers for the real points $X_\Sigma(\mathbb{R})$ by adding along the columns.

$$[1 \ 1 \ 1 \ 1 \ 106 \ 16]$$

We compute the \mathbb{Z}_2 Betti numbers for the complex points $X_\Sigma(\mathbb{C})$ by adding along the diagonals.

$$[1 \ 0 \ 1 \ 0 \ 1 \ 5 \ 27 \ 49 \ 37 \ 4 \ 1]$$

Example 3. *A seven dimensional example.*

We define Δ to be the convex hull of the following nine vertices.

$$\{-e_1, -e_2, -e_3, -e_4, -e_5, -e_6, -e_7, e_1 + e_2 + e_3 + e_4, e_5 + e_6 + e_7\}$$

The f -vector for Δ is $(9, 36, 84, 125, 120, 70, 20)$. The polar polytope Δ^* is the product $P_3 \times P_4$, where P_i is the i dimensional simplex. Moreover, Δ^* defines the nonsingular toric variety $\mathbb{P}^3 \times \mathbb{P}^4$. Below are the ranks of the \mathbb{Z}_2 Hodge spaces $H_{pq}(\Sigma)$.

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & & 15 & 2 \\
 & & & & & 1 & 31 & 1 \\
 & & & & 1 & 0 & 34 & 0 \\
 q & & & 1 & 0 & 0 & 21 & 0 \\
 & & & 1 & 0 & 0 & 0 & 7 & 0 \\
 & & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & p
 \end{array}$$

Again, we sum entries to obtain the \mathbb{Z}_2 Betti numbers for $X_\Sigma(\mathbb{R})$

$$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 109 \ 4]$$

and the \mathbb{Z}_2 Betti numbers for $X_\Sigma(\mathbb{C})$.

$$[1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 8 \ 21 \ 35 \ 31 \ 16 \ 2 \ 1]$$

Example 4. *A six dimensional example.*

Let Δ be the convex hull of the twelve vertices below.

$$\{-e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, e_6, e_1 - e_6, -e_1 - e_2 - e_3 - e_4 - e_5 - e_6\}$$

The f -vector for Δ is $(12, 62, 174, 267, 207, 64)$. The polar polytope Δ^* has 64 vertices. Using `polymake` we determine that Δ^* defines a toric variety with isolated singularities. That is, $\Sigma_{\leq 5}^*$ consists of \mathbb{Z}_2 regular cones. The results of Section 4.4 and Section 4.5 give the \mathbb{Z}_2 Hodge

spaces $H_{pq}(\Sigma)$ for $p < 4$, as shown by the `torhom` computation below.

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & & 58 & 0 \\
 & & & & & 15 & 113 & 0 \\
 & & & & 1 & 38 & 96 & 0 \\
 q & & & 1 & 0 & 34 & 45 & 0 \\
 & & 1 & 0 & 0 & 10 & 10 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & p
 \end{array}$$

We do not have the theory to guarantee collapsing of the spectral sequence \overline{E}^r at \overline{E}^1 . However, using `torhom` we compute the \mathbb{Z}_2 Betti numbers for the real points $X_\Sigma(\mathbb{R})$.

$$[1 \ 1 \ 1 \ 1 \ 97 \ 322 \ 1]$$

We conclude that the spectral sequence \overline{E}^r collapses at \overline{E}^1 . Again, we use Equation (2.6) to obtain the collapsing of the spectral sequence E^r at E^2 . We can therefore compute the \mathbb{Z}_2 Betti numbers for $X_\Sigma(\mathbb{C})$ by adding along the diagonals.

$$[1 \ 0 \ 1 \ 0 \ 1 \ 10 \ 45 \ 83 \ 111 \ 113 \ 58 \ 0 \ 1]$$

CHAPTER 5

A COUNTEREXAMPLE TO THE MAXIMALITY OF TORIC VARIETIES

Theorem 2. *There exists a six dimensional projective toric variety which is not maximal.*

We define a toric variety from a matroid as in [Gel]. Matroids abstract the theory of dependence studied in linear algebra or graph theory. There are many equivalent definitions of a matroid; most are discussed in [Oxl]. A definition useful for our purposes involves a convex polytope P_M . Let $E \subset \binom{[n]}{k}$ be a collection of k element subsets of the ground set $[n] := \{1, 2, 3 \cdots, n\}$. We define a convex polytope P via

$$P := \text{conv}\{e_I = \sum_{i \in I} e_i \mid I \in E\}.$$

If each edge of P is parallel to $e_i - e_j$ for some i, j then the set E is the set of *bases* of a *matroid* M . In this case $P = P_M$ is the matroid polytope for the matroid M . Subsets of bases are *independent sets*, and the *rank* of a matroid is the cardinality of one of its bases.

Let M be the rank 3 matroid F_7 , the Fano plane. The affine dependencies of M are depicted in Figure 5.1.

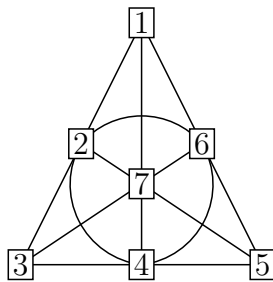


Figure 5.1: The Fano plane F_7

In this case, the set E is the set of triangles in Figure 5.1. Next, consider the projective space \mathbb{CP}^{27} whose coordinates are given by elements of E

$$\{y_{ijk} \mid ijk \in \binom{[7]}{3} \setminus \{123, 147, 156, 246, 257, 345, 367\}\}.$$

We define a linear action of the complex algebraic torus

$$T := (\mathbb{C}^*)^7 = \{(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \mid t_i \in \mathbb{C}^*\}$$

on \mathbb{CP}^{27} where the action on the coordinates is

$$(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \cdot y_{ijk} = t_i t_j t_k y_{ijk}.$$

Note that the one dimensional subtorus $\{(t, t, t, t, t, t, t) \mid t \in \mathbb{C}^*\}$ acts trivially. By restricting this action to the subtorus

$$T' := \{(t_1, t_2, t_3, t_4, t_5, t_6, 1) \mid t_i \in \mathbb{C}^*\},$$

we obtain an effective action of a six dimensional algebraic torus on \mathbb{CP}^{27} . Let $X(\mathbb{C})$ be the closure of the torus orbit

$$X(\mathbb{C}) = \overline{T' \cdot (1, 1, 1, \dots, 1)} \subset \mathbb{CP}^{27}.$$

Then, X is a six dimensional projective toric variety.

In this case, the matroid polytope $P_M = \text{conv}\{e_I \mid I \in \binom{[7]}{3} \setminus \{123, 147, 156, 246, 257, 345, 367\}\}$ is a six dimensional polytope and the moment polytope for X is the projection of P_M onto the first six coordinates. We use `torhom` [Fra1] to compute the ranks of the entries in

$E_{p,q}^2 = \overline{E}_{p,q}^1$ for X (5.1) and the \mathbb{Z}_2 Betti numbers for $X(\mathbb{R})$ (5.2).

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 15 & 0 \\
 & & & & & 22 & 0 & 0 \\
 & & & & 6 & 26 & 0 & 0 \\
 q & & & & 2 & 3 & 9 & 0 & 0 \\
 & & & & 1 & 0 & 1 & 1 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & p
 \end{array} \tag{5.1}$$

$$\left[1 \ 1 \ 1 \ 8 \ 57 \ 15 \ 1 \right] \tag{5.2}$$

By adding up the ranks of the entries in (5.1) and in (5.2), we see the spectral sequence $\overline{E}_{p,q}^r$ does not collapse at \overline{E}^1 . Moreover,

$$\sum_{i=0}^6 b_i(X(\mathbb{R})) = 84.$$

The total grading for the spectral sequence $E_{p,q}^r$ is $p + q$. Thus, the only possible nonzero higher differential is $d_{4,1}^2 : E_{4,1}^2 \longrightarrow E_{2,2}^2$. If $d_{4,1}^2 = 0$ then the spectral sequence collapses at E^2 , the \mathbb{Z}_2 Betti numbers for $X(\mathbb{C})$ are

$$[1 \ 0 \ 1 \ 0 \ 3 \ 4 \ 15 \ 26 \ 22 \ 0 \ 15 \ 0 \ 1]$$

and

$$\sum_{i=0}^{12} b_i(X(\mathbb{C})) = 88.$$

If $d_{4,1}^2 \neq 0$ then the \mathbb{Z}_2 Betti numbers for $X(\mathbb{C})$ are

$$[1 \ 0 \ 1 \ 0 \ 2 \ 3 \ 15 \ 26 \ 22 \ 0 \ 15 \ 0 \ 1]$$

and

$$\sum_{i=0}^{12} b_i(X(\mathbb{C})) = 86.$$

In either case, we obtain

$$\sum_{i=0}^6 b_i(X(\mathbb{R})) < \sum_{i=0}^{12} b_i(X(\mathbb{C})),$$

and hence X is not maximal.

Remark. *In the above example, the map $A_2^T(X) \longrightarrow H_2(X(\mathbb{R}))$ is not injective since the rank of $A_2^T(X)$ is 2 and $b_2(X(\mathbb{R})) = 1$. Hence in general, the map $f_q^{\mathbb{R}} : A_q^T(X_\Sigma) \longrightarrow H_q(X_\Sigma(\mathbb{R}))$ from (2.5) is neither injective nor surjective.*

APPENDIX A

QUESTIONS

A.1 REASONS FOR THE COUNTEREXAMPLE

In Chapter 5 we found a counterexample to the conjecture on the maximality of toric varieties. The counterexample is purely a computation, and as a result we have no insight into why the toric variety X is not maximal. We may be able to find a reason for this counterexample in terms of properties of the matroid F_7 . Figure A.1 as found in [Oxl §6.7] shows the containment of some important classes of matroids. We use the following abbreviations.

A	All matroids	R	Representable matroids
B	Binary matroids	T	Ternary matroids
G	Graphic matroids	C	Cographic matroids

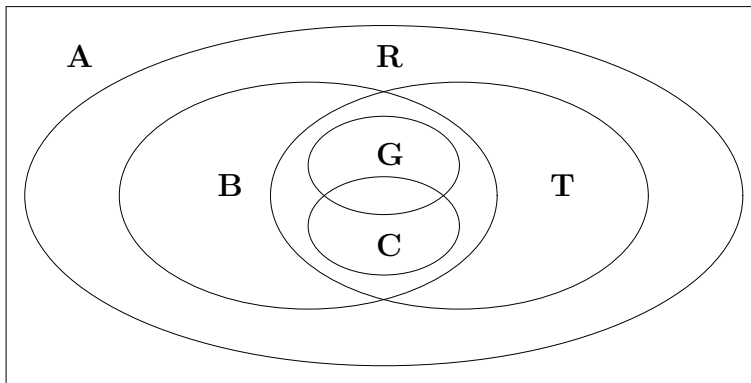


Figure A.1: Venn diagram of matroid classes

The matroid F_7 is the smallest example of a matroid which is binary (representable over \mathbb{F}_2) but not ternary (representable over \mathbb{F}_3). Possibly there are more toric varieties which are not maximal and come from matroids in the regions $(\mathbf{B} \cup \mathbf{T}) \setminus \mathbf{T}$ or $(\mathbf{B} \cup \mathbf{T}) \setminus \mathbf{B}$. Do certain

classes of matroids always give maximal toric varieties?

For instance, if a rank k matroid M on n elements is both binary and ternary, then M is orientable [Oxl §6.6 and §13.4]. In this case M determines a semi-algebraic subset of the real Grassmanian $G_{\mathbb{R}}(k, n)$ and a corresponding subset in the complex Grassmanian $G_{\mathbb{C}}(k, n)$. See [Bjö §2.4] for a discussion of the matroid stratification of the Grassmanian. The toric variety corresponding to the matroid polytope P_M is the closure of a generic torus orbit in the matroid stratum of M . It would be interesting to determine if these toric varieties are always maximal. Since the Fano plane is not realizable over any field of characteristic 0, the projective space $\mathbb{C}\mathbb{P}^{27}$ from Chapter 5 with coordinates

$$\{y_{ijk} \mid ijk \in \binom{[7]}{3} \setminus \{123, 147, 156, 246, 257, 345, 367\}\}$$

does not intersect the Grassmanian $G_{\mathbb{C}}(3, 7)$.

Due to the correspondence between combinatorial properties of a lattice polytope Δ and geometric properties of a toric variety X_{Σ} , we list many properties of the moment polytope in Chapter 5 below, as computed by `polymake` [Gaw].

Number of faces of each dimension (f-vector)	(28, 126, 245, 238, 112, 21)
Number of facets incident at each vertex	10
Number of edges incident at each vertex	9
Maximal dimension in which all faces are simplices	1
Maximal dimension in which all faces are simple polytopes	2
Maximal dimension in which all cones in Σ are simplicial	2
Volume (normalized)	232

Can we find a smaller example of a lattice polytope whose associated projective toric variety is not maximal? What combinatorial or metric properties of the polytope cause this phenomena?

A.2 THE SPECTRAL SEQUENCE (E^r, d^r)

In [Fra2] Franz conjectured that the \mathbb{Z}_2 Leray spectral sequence (E^r, d^r) for $\mu_{\mathbb{C}}$ collapses at E^2 for all projective toric varieties. Moreover, the \mathbb{Q} Leray spectral sequence for $\mu_{\mathbb{C}}$ is known to collapse in general [Tot]. In our example of Chapter 5, we do not know whether or not (E^r, d^r) collapses at E^2 . We can gain additional information about the homology of X by looking at $\widehat{E}_{p,q}^2$ as computed by `torhom` below, where $(\widehat{E}^r, \widehat{d}^r)$ is the Leray spectral sequence for $\mu_{\mathbb{C}}$ with \mathbb{Z} coefficients.

$$\begin{array}{ccccccc}
 & & & & & & \mathbb{Z} \\
 & & & & & & \mathbb{Z}^{15} & 0 \\
 & & & & & & \mathbb{Z}^{22} & 0 & 0 \\
 & & & & & & \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^3 & \mathbb{Z}^{23} & 0 & 0 \\
 q & & & & & & \mathbb{Z}^2 & (\mathbb{Z}_2)^3 & \mathbb{Z}^6 & 0 & 0 \\
 & & & & & & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & 0 & 0 \\
 & & & & & & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & p
 \end{array}$$

Note that there cannot be any higher differentials and hence $(\widehat{E}^r, \widehat{d}^r)$ collapses at \widehat{E}^2 . We have two possibilities.

$$\begin{aligned}
 H_4(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & \iff \text{the filtration is } \mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 = H_4(X, \mathbb{Z}) \\
 & \iff E^r \text{ collapses at } E^2
 \end{aligned}$$

$$\begin{aligned}
 H_4(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} & \iff \text{the filtration is } \mathbb{Z} \oplus 2\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z} = H_4(X, \mathbb{Z}) \\
 & \iff E^r \text{ does not collapse at } E^2
 \end{aligned}$$

Does (E^r, d^r) collapse at E^2 for this example? More generally, does the \mathbb{Z}_2 Leray spectral sequence for $\mu_{\mathbb{C}}$ collapse at E^2 for projective toric varieties?

A.3 THE MAPS $f_q^{\mathbb{R}}$ AND $f_q^{\mathbb{C}}$

We recall $f_q^{\mathbb{R}} : A_q^T(X_\Sigma) \longrightarrow H_q(X_\Sigma(\mathbb{R}))$ is the natural map given by the edge homomorphisms for the spectral sequence $(\tilde{E}^r, \tilde{d}^r)$. We have a similar map

$$f_q^{\mathbb{C}} : A_q^T(X_\Sigma) \longrightarrow H_{2q}(X_\Sigma(\mathbb{C}))$$

given by the edge homomorphisms for the \mathbb{Z}_2 Leray spectral sequence (E^r, d^r) . If X_Σ is a complete nonsingular toric variety then the triangle of maps

$$\begin{array}{ccc} A_q^T(X_\Sigma) & \xrightarrow{f_q^{\mathbb{C}}} & H_{2q}(X_\Sigma(\mathbb{C})) \\ & \searrow f_q^{\mathbb{R}} & \downarrow \\ & & H_q(X_\Sigma(\mathbb{R})) \end{array}$$

are all isomorphisms.

In general, neither $f_q^{\mathbb{C}}$ nor $f_q^{\mathbb{R}}$ are surjective as singular toric varieties can have non-algebraic homology. For the singular toric variety in Chapter 5, the map $f_2^{\mathbb{R}}$ is not injective and moreover the map $f_2^{\mathbb{C}}$ is injective if and only if (E^r, d^r) collapses at E^2 . I would like to understand the maps $f_q^{\mathbb{R}}$ and $f_q^{\mathbb{C}}$ in general. What conditions on Σ will guarantee that the maps $f_q^{\mathbb{R}}$ and $f_q^{\mathbb{C}}$ are injective for all q ? What conditions will guarantee $f_q^{\mathbb{R}}$ and $f_q^{\mathbb{C}}$ are surjective?

A.3.1 THE BIPYRAMID CONSTRUCTION

Example 5. *Let Δ be the 5 dimensional centrally symmetric polytope which is the convex hull of the twelve vertices*

$$\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, \pm(e_1 + e_2 + e_3 + e_4 + e_5)\}.$$

We list the ranks of the \mathbb{Z}_2 Hodge Spaces for the normal fan Σ of Δ (A.1) and for the normal fan Ξ of the bipyramid B_Δ (A.2) as computed by `torhom`.

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 19 & 4 \\
 & & & & & 25 & 11 & 6 \\
 q & & 1 & 42 & 5 & 4 & & \\
 & & 1 & 0 & 14 & 1 & 1 & \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & & & & p & &
 \end{array} \tag{A.1}$$

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 39 & 5 \\
 & & & & & 44 & 39 & 10 \\
 q & & 1 & 108 & 27 & 10 & & \\
 & & 1 & 0 & 70 & 11 & 5 & \\
 & & 1 & 0 & 0 & 14 & 2 & 1 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & p & &
 \end{array} \tag{A.2}$$

The toric variety X_{Σ^*} has isolated singularities and hence $X_{\Xi^*} = X_{\Sigma^*} \times \mathbb{P}^1$ has a one dimensional singular locus. The results of Sections 4.4 and 4.5 determine the ranks $H_{pq}(\Xi)$ for $p \leq 2$ and our `torhom` computation shows $H_{3q}(\Xi) = 0$ for $q < 3$. Using an argument as in Section 4.6, we obtain that the higher boundaries in the spectral sequence \overline{E}^r for X_Σ with target the $(2, 2)$ entry are zero. (Of course, a `torhom` computation of $b_i(X_\Sigma(\mathbb{R}))$ also gives this information for this example.)

The above example is not a rare phenomena. For each example I have computed, if Δ is a reflexive polytope with $\Sigma_{\leq k}^*$ consisting of \mathbb{Z}_2 regular cones then $H_{pq}(\Xi) = 0$ for $q < p \leq k$ and $H_{pp}(\Xi)$ has rank 1 for $p \leq k$. Our work from Sections 4.4 and 4.5 only give this result

for $p \leq k - 1$.

If $(f_0, f_1, f_2, \dots, f_{d-1})$ is the f -vector for Δ then

$$(f_0 + 2, f_1 + 2f_0, f_2 + 2f_1, \dots, f_{d-1} + 2f_{d-2}, 2f_{d-1})$$

is the f -vector for B_Δ . Moreover, the faces of B_Δ come in two types: a face of B_Δ is either a face of Δ or the cone over a face of Δ . I would like to understand the how the inclusion $\Delta \rightarrow B_\Delta$ for reflexive polytopes Δ affects the corresponding \mathbb{Z}_2 Hodge spaces. What is the relationship between the topology of X_Σ and that of X_Ξ ? Can we use the bipyramid construction to determine the collapsing of the spectral sequence \overline{E}^r for X_Σ in other cases?

A.4 THE SMITH SEQUENCE AND THE QUOTIENT SPACE $X(\mathbb{C})/c$

The Smith-Thom inequality

$$\sum_i b_i(X_\Sigma(\mathbb{R})) \leq \sum_j b_j(X_\Sigma(\mathbb{C}))$$

is a consequence of a long exact sequence on homology called the Smith sequence. We use the following notation.

$$\begin{aligned} X &= X(\mathbb{C}) && \text{a complex toric variety} \\ c : X &\longrightarrow X && \text{complex conjugation} \\ F &= X(\mathbb{R}) && \text{the fixed points of } c \\ &&& (= \text{the real points of } X) \\ \text{in} : F &\longrightarrow X && \text{the inclusion map} \\ \text{pr} : X &\longrightarrow X/c && \text{the projection map} \end{aligned}$$

Theorem 1.2.1 in [Deg] states that the following *Smith sequence* is exact

$$\dots \longrightarrow H_{p+1}(X/c, F) \xrightarrow{\Delta} H_p(X/c, F) \oplus H_p(F) \xrightarrow{\text{tr}^* + \text{in}_*} H_p(X) \xrightarrow{\text{pr}_*} H_p(X/c, F) \longrightarrow \dots$$

where tr^* and Δ are natural maps described in [Deg §1.1]. The Smith sequence gives motivation for understanding the topology of the quotient space X/c . The Smith identity follows

from the Smith sequence and states

$$\sum_i b_i(F) + 2 \sum_p \dim \operatorname{coker}(\operatorname{tr}^p + \operatorname{in}_p) = \sum_j b_j(X).$$

The counterexample in Chapter 5 shows $\operatorname{tr}^* + \operatorname{in}_*$ need not be surjective. I would like to understand the maps $\operatorname{tr}^* + \operatorname{in}_*$, Δ , and pr_* to determine homologically when the difference between $\sum_i b_i(F)$ and $\sum_j b_j(X)$ is nonzero. How large can this difference $\sum_j b_j(X_\Sigma(\mathbb{C})) - \sum_i b_i(X_\Sigma(\mathbb{R}))$ be?

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