The compactifications of moduli spaces of burniat surfaces with $2 \leq K^2 \leq 5$

by

Xiaoyan Hu

(Under the direction of Valery Alexeev)

Abstract

A Burniat surface X is a particular surface of general type with $p_g = q = 0$, $K_X^2 = 2, 3, 4, 5$ or 6. Alexeev and Pardini constructed an explicit compactification of the moduli space of Burniat surfaces with $K_X^2 = 6$.

In this thesis, we describe compactifications of moduli spaces of Burniat surfaces with $2 \leq K_X^2 \leq 5$ obtained by adding KSBA surfaces, i.e. slc surfaces X with ample canonical class K_X. We do it in two ways: by describing all one-parameter degenerations, and by using the theory of matroid tilings by matroid polytopes.

Index words: Burniat surface, compactification, moduli space, canonical model, stable pair, MMP, matroid polytope, matroid tiling, polymake

The compactifications of moduli spaces of burniat surfaces with $2 \leq K^2 \leq 5$

by

Xiaoyan Hu

B.S., Hunan University, 2007

A Dissertation Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment
of the
Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2014

The compactifications of moduli spaces of burniat surfaces with $2 \leq K^2 \leq 5$

by

Xiaoyan Hu

Approved:

Major Professor: Valery Alexeev

Committee: Angela Gibney
Gordana Matic
Robert Varley

Electronic Version Approved:

Julie Coffield
Interim Dean of the Graduate School
The University of Georgia
August 2014

The compactifications of moduli spaces of Burniat surfaces with $2 \leq K^2 \leq 5$

Xiaoyan Hu

July 3, 2014

Acknowledgments

I would like to express my deepest gratitude to my adviser Professor Valery Alexeev. Without his excellent guidance, help and patience, I could not imagine how to get this work done. I am also grateful for the financial support he provided and opportunities he gave to me to attend conferences and workshops.

I am grateful for Professor Robert Varley for giving me valuable comments. I would also like to thank the entire University of Georgia Mathematics Department for creating an environment in which students feel comfortable to learn and grow.

Special thanks go to Weiqiang Wang, Jie Wang and Gregory Schmidt for reading and commenting on the paper related to this thesis. I want to thank Professor Edward Azoff and my undergraduate adviser Bolin Ma who often encourage, help and share stories with me.

I would like to thank my parents, my brother Jiehui and my sister Chanyan for their unconditional love and support.

Contents

Chapter 1

Introduction

Burniat surfaces are special cases of surfaces of general type with $p_g = q = 0$, $2 \leq K_X^2 \leq 6$. They were first introduced by Burniat in [Bu66]. Peters [Pet77] reinterpreted Burniat's construction using the modern language of branched abelian covers. In [LP01], Lopes and Pardini proved that a minimal surface S of general type with $p_g(S) = 0$, $K_S^2 = 6$, and bicanonical map of degree 4 is a Burniat surface. Moreover, they showed that minimal surfaces S with $p_g = 0$, $K_S^2 = 6$ and bicanonical map of degree 4 form a four-dimensional irreducible component of the moduli space of surfaces of general type.

In [KSB88], Kollár and Shepherd-Barron introduced stable surfaces and proposed a way to compactify the moduli space of surfaces of general type by adding stable surfaces (also called KSBA surfaces). They showed that the appropriate singularities to permit for the surfaces at the boundaries of moduli spaces are semi log canonical (slc) and classified all the semi log canonical surface singularities. The boundedness of slc surfaces with a fixed canonical class K^2 was settled in [Ale94]. In [Ale96a, Ale96b], Alexeev extended Kollár and Shepherd-Barron's construction to stable pairs.

In [AP09], Alexeev and Pardini constructed an explicit compactification of the moduli space of Burniat surfaces with $K^2 = 6$ by adding KSBA surfaces, i.e. slc surfaces X with ample canonical class K_X, on the boundary. They also gave a constructive algorithm for computing all stable Burniat surfaces (not necessarily from degenerations of smooth surfaces), which reduced them to computing certain tilings by matroid polytopes.

The aim of this thesis is to extend the results and methods in [AP09] from the case $K^2 = 6$ to all the remaining cases $2 \leq K^2 \leq 5$. A Burniat surface with $K^2 = d$, $2 \leq d \leq 6$, is a $\mathbb{Z}_2^2$-cover of $Y = Bl_{9-d}\mathbb{P}^2$ branched along 12 irreducible curves consisting of 9 strict preimages of lines and 3 exceptional divisors. The moduli space M^d_{Bur} of Burniat surfaces with $K^2 = d$ is a subset of dimension $d-2$ in the moduli space $\mathfrak{M}^{can}$ of canonical surfaces, where a point in M^d_{Bur} corresponds to the canonical model of a smooth Burniat surface. When $d = 6, 5$, the moduli space M^d_{Bur} is an irreducible component in $\mathfrak{M}^{can}$. Bauer and Catanese [BC10b] showed that M^4_{Bur} is a union of two irreducible subvarieties $M^4_{Bur,1}$ and $M^4_{Bur,2}$, where a general point of $M^4_{Bur,1}$ corresponds to a smooth Burniat surface, while a general surface in $M^4_{Bur,2}$ has an A_1-singularity (nodal case). Moreover, $M^4_{Bur,1}$ is an irreducible component in $\mathfrak{M}^{can}$, whereas $M^4_{Bur,2}$ is contained in an irreducible component of dimension 3 in $\mathfrak{M}^{can}$. The moduli space M^3_{Bur} is irreducible and is contained in an irreducible component of dimension 4 in $\mathfrak{M}^{can}$. M^2_{Bur} is just one point so already compact. Thus we will restrict ourselves to compactifying the moduli space $M^d_{Bur}, 3 \leq d \leq 5$.

We reduce the problem of compactifying M^d_{Bur} to the one of compactifying the moduli space of certain stable pairs (Y, D); X is an abelian cover of Y ramified in D. A point in M^d_{Bur} corresponds to the canonical model of a Burniat surface X with $K^2_X = d$. The branch data is encoded in the Hurwitz divisor D_{Hur} (see Chapter 2.2). An *abelian cover* of a variety Y with group G or a G-cover is a finite map $\pi : X \to Y$ together with a faithful action of a finite abelian group G on X such that π exhibits Y as the quotient of X by G. In the case Y is smooth and X is normal, Pardini in [Par91] described the general structure of abelian covers $\pi : X \to Y$ using the building data which we will discuss in Chapter 2.2. The work was extended to the case of non-normal abelian covers in [AP12]. In Chapters 3,4,5, we list all the interesting degenerate configurations of stable pairs (Y, D) with $K_X^2 = 3, 4, 5$, up to symmetry, and find their canonical models using the minimal model program for 3-folds. Here, interesting degenerate configurations are the ones with reducible canonical models.

The construction of the compactified coarse moduli spaces $\overline{M}^d_{Bur}$ of Burniat surfaces is an application of [Ale08], which provides a stable pair compactification $\overline{M}_\beta(r, n)$ for the moduli space of weighted hyperplane arrangements $(\mathbb{P}^{r-1}, \sum b_i B_i)$ with arbitrary weight $\beta = (b_1, ..., b_n)$, $0 \le b_i \le 1$ and $b_i \in \mathbb{Q}$. In this paper, we apply [Ale08] in the case of $\mathbb{P}^2$ and $n = 9$ with $\beta = (\frac{1}{2}, ..., \frac{1}{2})$.

Several new phenomena happen in the cases $K^2 \le 5$ as compared to the case $K^2 = 6$ in [AP09]. Most importantly, when running the minimal model program, in addition to divisorial contractions occurring in the case $K^2 = 6$, flips and flops also occur. It is also surprising that some non log canonical degenerations in the case $K^2 = 6$ correspond to log canonical degenerations in the cases $K^2 \le 5$.

We first study degenerations of stable pairs (Y, D) and apply the minimal model program to find stable limits. We summarize our main results below.

Theorem 1. *The compactified coarse moduli space $\overline{M}^d_{Bur}$ of stable Burniat surfaces, or equivalently, of stable pairs (Y, D), is of dimension $d - 2$, irreducible for $d \neq 4$, and with two components for $d = 4$. The types of degenerations, up to symmetry, are listed as below.*

(i) There are 6 types of degenerate configurations of stable pairs with reducible canonical models in the moduli space of stable pairs (Y, D) for $K_X^2 = 5$ case up to the symmetry group $\mathbb{Z}_6$ described in Chapter 3.

(ii) There are 5 types of degenerations with reducible canonical models in the moduli space of stable pairs (Y, D) for $K_X^2 = 4$ nodal case and 3 types of degenerations for $K_X^2 = 4$ non-nodal case up to the symmetry group $\mathbb{Z}_2$, described in Chapter 4.

(iii) There are only 2 types of degenerations with reducible canonical models in the moduli

space of stable pairs (Y, D) for $K^2 = 3$, described in Chapter 5.

According to the general theory of [Ale08], the unweighted stable hyperplane arrangements are described by matroid tilings of the hypersimplex $\triangle(r, n)$. For the weighted stable hyperplane arrangements, they could be described by partial tilings of the hypersimplex $\triangle(r, n)$ that cover a β-cut hypersimplex $\triangle_\beta(r, n)$.

The polytope $\triangle^d_{Bur}, d \leq 6$, is a polytope in $\mathbb{R}^{12}$ that corresponds to the stable pairs (Y, D), where $Y = Bl_{9-d}\mathbb{P}^2$. We find out all the matroid tilings of the polytope $\triangle^d_{Bur}$, $d = 3, 4, 5$, and find all possible stable pairs in the compactified moduli space of stable pairs with $K^2 = d$. All the possible stable pairs produced by this computation coincide with the degenerations listed in Chapters 3,4,5. However, for Case 8 in Chapter 4 which is a $K^2 = 4$ nonnodal case, we don't have the corresponding matroiding covering from the computation. It remains somwhat mysterious why these two methods produce different results in this special case.

Chapter 2

Preliminaries

2.1 Stable pairs

Definitions in this section come from [KM98],[Ale94, Ale96a] and [AP09, AP12].

Let X be a projective variety. Let $B = \sum b_i B_i$ be a linear combination of effective divisors, where b_i is the weight of B_i which is allowed to be an arbitrary rational number with $0 < b_i \leq 1$. The divisors B_i's are possibly reducible and possibly have irreducible components in common.

We first give the definition of singularities that are important for the minimal model program.

Definition 2. Assume that X is a normal variety. A pair (X, B) is called *log canonical* (*lc*) if

(1) $m(K_X + B)$ is a Cartier divisor for some integer $m > 0$,

(2) for every proper birational morphism $\pi : X' \to X$ with normal X',

$$K_{X'} + \pi_*^{-1} B \;\; = \;\; \pi^*(K_X + B) + \sum a_i E_i$$

one has $a_i \geq -1$. Here the E_i's are the irreducible exceptional divisors of π, and the pullback π^* is defined by extending $\mathbb{Q}$-linearly the pullback on Cartier divisors; $\pi_*^{-1}B$ is the strict preimage of B.

If char $k = 0$ then X has a resolution of singularities $\pi : X' \to X$ such that Supp $(\pi_*^{-1}B) \cup E_i$ is a normal crossing divisor; then it is sufficient to check the condition $a_i \geq -1$ for this morphism π only.

For a nonnormal variety X, we have semi log canonical as the analog of log canonical. We say a variety is *g.d.c* (has *generically double crossings*) if it is either smooth or analytically isomorphic to $xy = 0$ outside a closed subset of codimension ≥ 2.

Definition 3. A pair (X, B) is called *semi log canonical* (*slc*) if

(1) X satisfies Serre's condition S_2,

(2) X is g.d.c., and no divisor B_i contains any component of the double locus of X,

(3) $m(K_X + B)$ is a Cartier divisor for some integer $m > 0$,

(4) for the normalization $\nu : X^\nu \to X$, the pair $(X^\nu, (\text{double locus}) + \nu_*^{-1}B)$ is log canonical.

Assume that X is a curve, then (X, B) is *slc* if and only if X is at worst nodal, B_j does not contain any nodes, and for every $P \in X$ one has $\text{mult}_P B = \sum b_j \text{mult}_P B_j \leq 1$.

Definition 4. Let (X, B) be a semi log canonical pair and $f : X \to S$ a proper morphism. A pair (X^m, B^m) sitting in a diagram

$$\begin{array}{ccccc} X & & \overset{\phi}{\dashrightarrow} & & X^m \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle f^m} & \\ & & S & & \end{array}$$

is called a *minimal model* of (X, B) if

(1) f^m is proper,

(2) ϕ is a birational contraction, that is, ϕ^{-1} has no exceptional divisors,

(3) $B^m = \phi_* B$,

(4) $K_{X^m} + B^m$ is $\mathbb{Q}$-Cartier and f^m-nef, and

(5) $a(E, X, B) < a(E, X^m, B^m)$ for every ϕ-exceptional divisor $E \subset X$.

Any two minimal models of (X, B) are isomorphic in codimension one.

Definition 5. Let (X, B) be a semi log canonical pair and $f : X \to S$ a proper morphism. A pair (X^c, B^c) sitting in a diagram

$$\begin{array}{ccccc} X & & \overset{\phi}{\dashrightarrow} & & X^c \\ & {\scriptstyle f} \searrow & & \swarrow {\scriptstyle f^c} & \\ & & S & & \end{array}$$

is called a *canonical model* of (X, B) if

(1) f^c is proper,

(2) ϕ is a birational contraction, that is, ϕ^{-1} has no exceptional divisors,

(3) $B^c = \phi_* B$,

(4) $K_{X^c} + B^c$ is $\mathbb{Q}$-Cartier and f^c-ample, and

(5) $a(E, X, B) \leq a(E, X^c, B^c)$ for every ϕ-exceptional divisor $E \subset X$.

The minimal model (X^m, B^m) is usually not unique, but the canonical model $(X^c, \triangle^c)$ is unique provided $K_X + B$ is big and

$$X^c \quad = \quad \mathrm{Proj}_S \oplus_{m \geq 0} f_* \mathcal{O}_X \left(mK_X + \lfloor mB \rfloor \right).$$

The canonical model (X^c, B^c) could be obtained from (X^m, B^m) by a linear system $|d \left(K_{X^m} + B^m \right)|$ for $d \gg 0$.

The *minimal model program (MMP)* is a prodedure for finding a model for each birational equivalence class. MMP machine for pairs is as follows (We borrow the language from [Ale13]).

Input:

1. A pair (X, B) of a smooth projective variety X and a $\mathbb{Q}$- or $\mathbb{R}$-divisor $B = \sum b_i B_i$ such that $\cup B_i$ is a normal crossing divisor.

2. Or, more generally, a log canonical pair (X, B).

Outputs:

1. Either a minimal model (X^m, B^m) with nef divisor $K_{X^m} + B^m$ and dlt or log canonical singularities, or a variety X' birational to X and a Mori-Fano fibration $X' \to X$ with relatively ample $-(K_{X'} + B')$ and $\dim X < \dim X' = \dim X$.

2. If $K_X + B$ is big then also the canonical model (X^c, B^c) with ample $(K_{X^c} + B^c)$ and log canonical singularities.

In our work, we use MMP to find canonical models.

Definition 6. A (X, B) is called *stable* if it satisfies the following conditions

(1) (Singularities) the pair (X, B) is semi log canonical, and

(2) (Numerical) the divisor $K_X + B$ is ample.

Let $\beta = (b_1, ..., b_n)$, $0 < b_i \leq 1$, $b_i \in \mathbb{Q}$, be a weight. A hyperplane arrangement is a pair $(\mathbb{P}^{r-1}, \sum b_i B_i)$ with weight β, where $B_1, ..., B_n$ are hyperplanes in $\mathbb{P}^{r-1}$. The pair $(\mathbb{P}^{r-1}, \sum b_i B_i)$ is *lc* if for each intersection $\cap_{i \in I} B_i$ of codimension k, one has $\sum_{i \in I} b_i \leq k$, where $I \subset \{1, ..., n\}$. The pair $(\mathbb{P}^{r-1}, \sum b_i B_i)$ is stable if and only if it is *lc* and $|\beta| = \sum_{i=1}^{n} b_i > r$.

2.2 Abelian covers

We will recall some definitions and theorems from [Par91, AP12].

Definition 7. Let G be a finite abelian group. An *abelian cover* with Galois group G, or a G-cover, is a finite morphism $\pi : X \to Y$ of varieties which is the quotient map for a generically faithful action of a finite abelian group G.

An *isomorphism of* G*-covers* $\pi_1 : X_1 \to Y$, $\pi_2 : X_2 \to Y$ is an isomorphism $\phi : X_1 \to X_2$ such that $\pi_1 = \pi_1 \circ \phi$.

Let Y be a smooth variety and X be a normal variety. Let G be a finite abelian group and $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ is the group of characters of G. The G-action on X with $X/G = Y$ is equivalent to the decomposition:

$$\pi_* \mathcal{O}_X \quad = \quad \underset{\chi \in G^*}{\oplus} L_\chi^{-1}, \quad L_1 = \mathcal{O}_Y$$

where the L_χ's are line bundles on Y and G acts on L_χ^{-1} via the character χ.

Pardini in [Par91] showed that an abelian cover $\pi : X \to Y$, with group G, with Y smooth and complete and X normal, is determined up to isomorphism of G-cover by the building satisfying some relations.

In this thesis we will only discuss the case when $G = \mathbb{Z}_2^r, r \in \mathbb{N}$. A set of *building data* (L_χ, D_g) for the case $G = \mathbb{Z}_2^r$ described in [Par91] can be simplified as

- effective Cartier divisors D_g, $g \in G \setminus \{0\}$ (possibly not distinct),
- line bundles L_χ, $\chi \in G^*$.

Moreover the building data for the case $G = \mathbb{Z}_2^r$ need only satisfy the *fundamental relations*:

$$L_\chi + L_{\chi'} \quad \equiv \quad L_{\chi\chi'} + \sum_{g \in G} \epsilon_g^{\chi,\chi'} D_g$$

where $\epsilon_g^{\chi,\chi'} = 1$ if both $\chi(g) = \chi'(g) = -1$ and $\epsilon_g^{\chi,\chi'} = 0$ otherwise.

In particular, let $G = \mathbb{Z}_2^2 = \{e, a, b, c\}$ and $G^* = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ be the character group with $\chi_0 \equiv 1$, $\chi_1(b) = \chi_1(c) = -1$, $\chi_2(a) = \chi_2(c) = -1$, $\chi_3(a) = \chi_3(b) = -1$, and assume that Pic Y has no 2-torsion. Then the building data only need to satisfy

$$\begin{aligned} 2L_{\chi_1} &= D_b + D_c \\ 2L_{\chi_2} &= D_a + D_c \\ 2L_{\chi_3} &= D_a + D_b \end{aligned}$$

A smooth Burniat surface X with $K^2 = d$, $2 \leq d \leq 6$, which is a $\mathbb{Z}_2^2$-cover of $Bl_{9-d}\mathbb{P}^2$ is determined by building data. The general theory of abelian covers was extended to the case of non-normal X in [AP12]; it is used in [AP09]. For details of the abelian covers for the case of non-normal X we will refer to [AP12]. Now we will recall a theorem in [AP12] which is needed for our work.

For every set of building data (L_χ, D_g), [Par91, Def.2.2] defined a *standard abelian cover* explicitly by equations.

Definition 8. For a standard G-cover $\pi : X \to Y$, the *Hurwitz divisor* of π is the $\mathbb{Q}$-divisor $D_{Hur} := \sum_i \frac{m_i - 1}{m_i} D_i$, where m_i is the ramification index of D_i.

The Hurwitz formula

$$K_X \sim_{\mathbb{Q}} \pi^*(K_Y + D_{Hur})$$

shows that X is of general type if and only if $K_Y + D_{Hur}$ is big.

In this thesis, we will use $m_i = 2$ and $D_{Hur} = \sum \frac{1}{2} D_i$.

Theorem. *[AP12, Proposition 2.5]. Let $\pi : X \to Y$ be a G-cover and let D be the Hurwitz divisor of π. Then*

(i) The divisor K_X is $\mathbb{Q}$-Cartier if and only if $K_Y + D$ is $\mathbb{Q}$-Cartier.

(ii) $K_X = \pi^*(K_Y + D)$.

(iii) The variety X is slc if and only if so is the pair (Y, D).

We consider X as the pair $(X, 0)$.

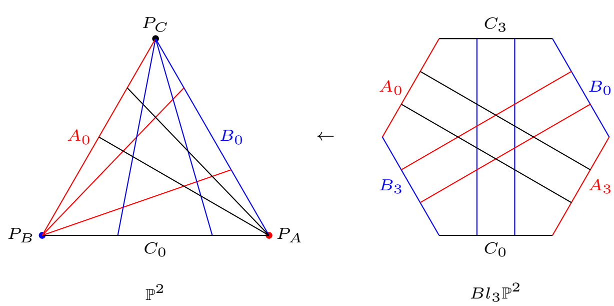
Corollary 9. *[AP12] For a G-cover $\pi : X \to Y$ with Hurwitz divisor D, X is stable if and only if the pair (Y, D) is stable.*

This corollary reduce the problem of compactifying M^d_{Bur} to the one of compactifying the moduli space of certain paris (Y, D).

2.3 Smooth Burniat surfaces

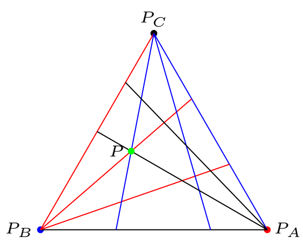
Burniat surfaces were first introduced by Burniat in 1966. A *surface of general type* is an algebraic surface with Kodaira dimension 2. A Burniat surface is a surface of general type with $p_g = q = 0, K^2 = 2, 3, 4, 5$ or 6, where $p_g(X) = h^2(\mathcal{O}_X)$ and $q(X) = h^1(\mathcal{O}_X)$. We will use the construction of Burniat surfaces in [Pet77, BHPV].

To construct a Burniat surface X with $K^2_X = 6$, we start with an arrangement of 9 distinct lines A_0, A_1, A_2, B_0, B_1, B_2, C_0, C_1, C_2 in $\mathbb{P}^2$. The lines A_0, B_0, C_0 form a non-degenerate triangle with the vertices P_A, P_B, P_C. Lines A_1, A_2 pass through P_B, B_1, B_2 pass through P_C, and C_1, C_2 pass through P_A. The other lines are in general position otherwise. Blow up $\mathbb{P}^2$ at $P_A, P_B,$ and P_C. We denote the exceptional divisors on $Bl_3\mathbb{P}^2$ by A_3, B_3, C_3 and by $A_i, B_i, C_i,\ i = 0, 1, 2$ the strict preimages of A_i, B_i, C_i on $\mathbb{P}^2$. The blowup morphism is as follows

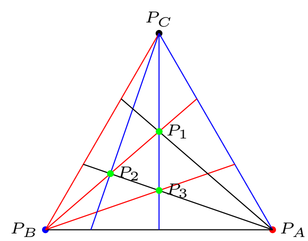


Definition 10. A Burniat surface X with $K_X^2 = 6$ is a $\mathbb{Z}_2^2$-cover of $\Sigma = \mathrm{Bl}_3\mathbb{P}^2$ for the building data $D_a = \sum_{i=0}^3 A_i$, $D_b = \sum_{i=0}^3 B_i$, $D_c = \sum_{i=0}^3 C_i$, where a, b, c are the 3 nonzero elements of $\mathbb{Z}_2^2$.

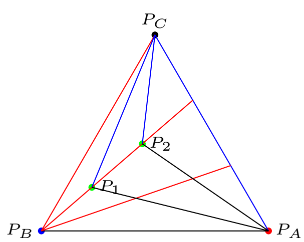
In general, a Burniat surface with $K^2 = d$, $2 \le d \le 6$, is a $\mathbb{Z}_2^2$-cover of $\Sigma = Bl_{9-d}\mathbb{P}^2$ with the building data D_a, D_b, D_c, where Σ is the blowup of $\mathbb{P}^2$ at P_A, P_B, P_C and P_i with $i = 1, 2$ or 3. The following figures from [Pet77, BC09b] are arrangements of 9 lines in $\mathbb{P}^2$ for the constuction of Burniat surfaces with $K^2 = 3, 4, 5$.



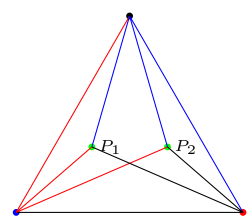
$K_X^2 = 5$



$K_X^2 = 3$



$K_X^2 = 4$ nodal



$K_X^2 = 4$ non-nodal

In particular, for $K^2 = 5$, the Hurwitz divisor is $D = \dfrac{1}{2}(D_a + D_b + D_c)$ for a $\mathbb{Z}_2^2$-cover. Let $\pi : \Sigma \to \mathbb{P}^2$ be the blowup map and A_3, B_3, C_3 and E be exceptional divisors. Then we have

$$\begin{aligned}
-3K_\Sigma &= -3\left(\pi^*K_{\mathbb{P}^2} + A_3 + B_3 + C_3 + E\right) \\
&= -3\left((-A_0 - B_0 - C_0 - 2A_3 - 2B_3 - 2C_3) + A_3 + B_3 + C_3 + E\right) \\
&= 3\left(A_0 + B_0 + C_0 + A_3 + B_3 + C_3 - E\right) \\
&= \sum_{i=0,3}(A_i + B_i + C_i) + (2A_0 + 2C_3 - E) + (2B_0 + 2A_3 - E) + (2C_0 + 2A_3 - E) \\
&= \sum_{i=0}^{3}(A_i + B_i + C_i) + (A_1 + A_2) + (B_1 + B_2) + (C_1 + C_2) \\
&= \sum_{i=0}^{3}(A_i + B_i + C_i) \\
&= 2D
\end{aligned}$$

Using the Riemann-Hurwitz formula $K_X = \pi^*(K_\Sigma + D)$, we have

$$K_X^2 = (\pi^*(K_\Sigma + D))^2 = 4 \cdot (K_\Sigma + D)^2 = 4\left(-\frac{1}{2}K_\Sigma\right)^2 = 5.$$

By the theorem in Section 2.2, we can reduce the problem of compactifying the moduli space of stable Burniat surfaces with $K^2 = d$ to compactifying the moduli space of stable pairs (Σ, D) described above.

2.4 The compactified moduli space $\overline{M}_{Bur}^6$

Let X be a variety, a compactification $\overline{X}$ is a compact variety such that $X \subset \overline{X}$ is a dense subset. More generally, we may also allow $X \subset \overline{X}$ to be non-dense. There are many compactifications one could choose, but the one we prefer is a 'modular compactification', i.e. the points in X also correspond to some geometric objects.

The moduli space of Burniat surfaces M_{Bur}^6, is a geometric space whose points represent

Burniat surfaces with $K^2 = 6$ up to isomorphism. The compactified moduli space $\overline{M}^6_{Bur}$ is constructed in [AP09, Section 5.3] as an adaption of the construction of the moduli space $\overline{M}_{\mathbf{b}}(3,9)$ of weighted hyperplane arrangements of 9 lines in $\mathbb{P}^2$ with weight $\mathbf{b} = \left(\frac{1}{2}, ..., \frac{1}{2}\right)$. We refer to [AP09] for details.

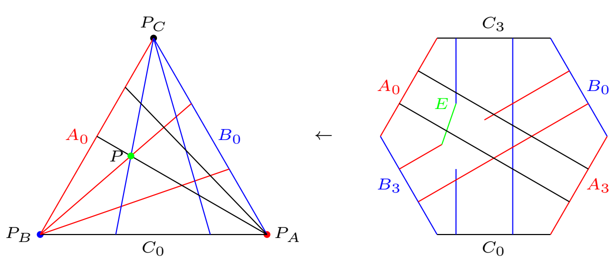
Alexeev and Pardini in [AP09] defined the polytope $\triangle^6_{Bur}$ and the moduli space M^6_{Bur}. Fix weight $\mathbf{b} = \left(\frac{1}{2}, ..., \frac{1}{2}\right)$ and a polytope $\triangle^6_{Bur}$ (see Chapter 6). We define $\overline{M}^6_{Bur}$ to be the moduli space of stable toric varieties over $\mathrm{G}^6_{Bur,\mathbf{b}}$ of topological type $\triangle^6_{Bur}$, where $\mathrm{G}^6_{Bur,\mathbf{b}}$ is the $\mathbf{b}$-cut of a certain subvariety $\mathrm{G}^6_{Bur} \subset \mathrm{G}(3,9)$ (see [AP09, Section 5.3]). Thus $\overline{M}^6_{Bur}$ parametrizes stable toric varieties $Z \to \mathrm{G}^6_{Bur,\mathbf{b}}$, and the moment polytopes of the irreducible components of $Z = \cup Z_s$ give a tiling of $\triangle^6_{Bur}$. For a stable toric variety $Z \to \mathrm{G}^6_{Bur,\mathbf{b}}$, one recovers the stable pair (Y, D) as a GIT quotient $P^6_{Bur,Z}//_{\mathbf{b}}T$, where $P^6_{Bur,Z} = P \times_{\mathrm{G}^6_{Bur,\mathbf{b}}} Z$ is the pullback of the universal family P.

Chapter 3

Burniat surfaces with $K^2 = 5$

3.1 Degenerations of Burniat surfaces with $K^2 = 5$.

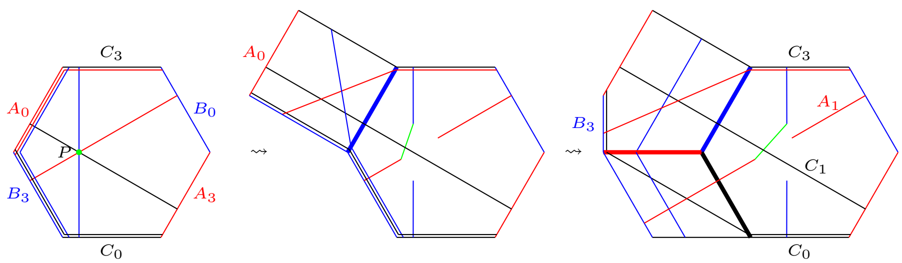
We consider degenerations of Burniat arrangements of curves on $\Sigma = \mathrm{Bl}_4\mathbb{P}^2$. When the arrangement on Σ is not log canonical, choose a generic one-parameter family of arrangements on Σ degenerating to it. Then the limit stable surface splits into several irreducible components. Below, we consider such generic degenerations. Let $\mathcal{Y}$ be the total space of the one parameter family of surfaces isomorphic to Σ with the central fiber being the degenerating arrangement. Let $\mathcal{D}$ be the divisor that is the union of D in each fiber. Write Σ_0 for the central fiber of $\mathcal{Y}$. The following figure is Σ_0.



Σ_0

Case 1. The curve A_2 degenerates to $A_0 + C_3$, B_2 degenerates to $A_0 + B_3$, and C_2 degenerates to $B_3 + C_0$ (the first figure below). Let L_P be the curve in $\mathcal{Y}$ consisting of the points

P in each fiber, which is the intersection of the curves A_1, B_1, C_1. We first blow up the total space $\mathcal{Y}$ along L_P, then blow up the resulting total space along A_0 in the central fiber. The central fiber Σ_0 becomes $Bl_4\mathbb{P}^2 \cup \mathbb{F}_1$ (the second figure below), where A_0 is the (-1)-curve in $\mathbb{F}_1$. Finally we blow up the total space along the proper transform of B_3 in the component $Bl_4\mathbb{P}^2$ of Σ_0. The resulting central fiber is a union of three components $Bl_4\mathbb{P}^2 \cup Bl_1\mathbb{F}_1 \cup \mathbb{F}_0$.



We can use the triple point formula to compute the intersection number $(K_{\Sigma_0} + D)\,|_{Y_i}.C$, where Y_i is a component of Σ_0 and C is a curve in the component Y_i.

Let us recall the triple point formula: let $\Sigma_0 = \cup Y_i$ be the central fiber in a smooth one-parameter family, and assume that Σ_0 is reduced and has simple normal crossing. Let C be the intersection $Y_i \cap Y_j$ and assume that it is a smooth curve. Denote by p_3 the number of the triple points of Σ_0 contained in C, then

$$(C|_{Y_i})^2 + (C|_{Y_j})^2 + p_3 \quad = \quad 0.$$

By the adjunction formula, we have $K_{\Sigma_0} = (K_{\mathcal{Y}} + \Sigma_0)|_{\Sigma_0}$ and $K_{Y_i} = (K_{\mathcal{Y}} + Y_i)|_{Y_i}$. Hence

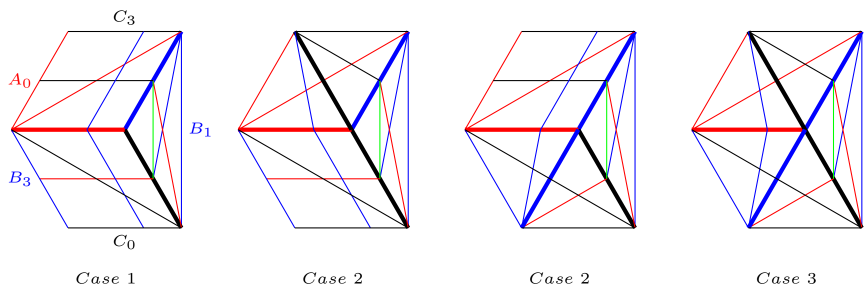
$$\begin{aligned} K_{\Sigma_0}|_{Y_i} &= ((K_{\mathcal{Y}} + \Sigma_0)|_{\Sigma_0})|_{Y_i} \\ &= K_{\mathcal{Y}}|_{Y_i} + \Sigma_0|_{Y_i} \\ &= K_{\mathcal{Y}}|_{Y_i} + 0 \\ &= (K_{Y_i} - Y_i)|_{Y_i} \\ &= K_{Y_i} + (\Sigma_0 - Y_i).Y_i - \Sigma_0.Y_i \\ &= K_{Y_i} + (\text{the double locus}) \end{aligned}$$

Therefore we have the equation

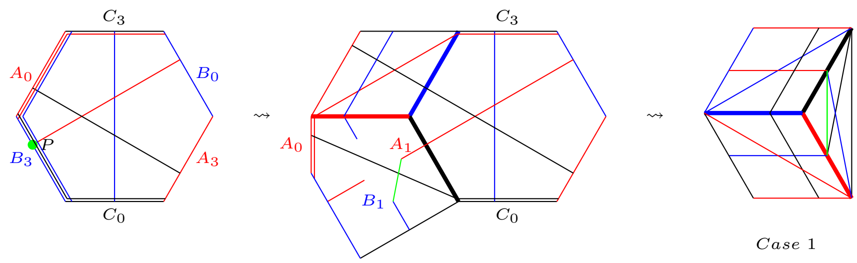
$$(K_{\Sigma_0} + D)|_{Y_i} = K_{Y_i} + D|_{Y_i} + (\text{the double locus}).$$

The intersection number $(K_{\Sigma_0} + D)|_{Bl_4\mathbb{P}^2}.C$ is 0 when the curve C is A_1, C_0, C_1 or C_3, and positive for the other curves in $Bl_4\mathbb{P}^2$. In the component $Bl_1\mathbb{F}_1$, $(K_{\Sigma_0} + D)|_{Bl_1\mathbb{F}_1}.B_3 = 0$ and $(K_{\Sigma_0} + D)|_{Bl_1\mathbb{F}_1}.C > 0$ for other curves C. We also have $(K_{\Sigma_0} + D)|_{\mathbb{F}_0}.C > 0$ for all the curves in the component $\mathbb{F}_0$. Thus $K_{\Sigma_0} + D$ is big, nef and vanishes on B_3, C_1 and C_3. The 3-fold is the minimal model of the degenerate family. Using the inversion of adjunction in [Ka07], we see that the pair $(\mathcal{Y}, \mathcal{D})$ is log terminal and $\mathcal{D}$ is an effective divisor on $\mathfrak{X}$ such that $K_{\mathcal{Y}} + \mathcal{D}$ is nef and big. By Base Point Free theorem, the linear system $|n(K_{\mathcal{Y}} + \mathcal{D})|$ is base point free for all sufficiently large $n \in \mathbb{N}$. Then we can define a birational morphism by the linear system $|n(K_{\mathcal{Y}} + \mathcal{D})|$, which contracts A_1, B_3, C_0, C_1, C_3 labeled in the third figure above. The image of the birational morphism is the canonical model of the degenerate family.

We run MMP for our resulting pair $(\mathcal{Y}, \mathcal{D})$. The surface $Bl_4\mathbb{P}^2$ becomes $\mathbb{P}^2$ after contracting A_1, C_0, C_1, C_3. The component $Bl_1\mathbb{F}_1$ becomes $\mathbb{F}_0$ after contracting B_3. The central fiber of the resulting canonical model is $\mathbb{F}_0 \cup \mathbb{F}_0 \cup \mathbb{P}^2$, which is the first figure below. For the component $\mathbb{P}^1 \times \mathbb{P}^1$, there is a further degeneration that splits to $\mathbb{P}^2 \cup \mathbb{P}^2$. We list the three possible further degenerations below which are the rest three figures. The second and third figures differ only by a permutation of colors and we classify them as the same type. Thus there are only two different degenerations, we call them Case 2 and Case 3.

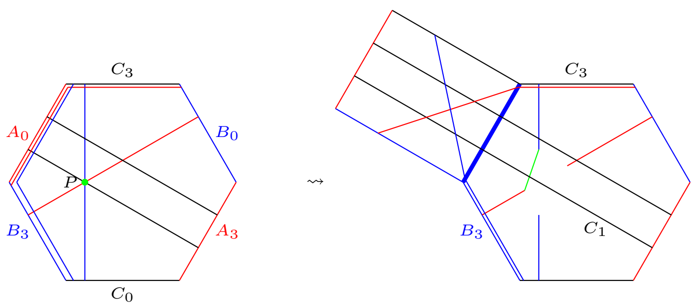


Case 1 could be obtained from another degeneration when B_1 goes to $A_0 + B_3$ and C_1 degenerates to $B_3 + C_0$ (the first figure below). We first blow up the total space $\mathcal{Y}$ along the line B_3 and then blow up along the strict image of A_0 in the component $Bl_3\mathbb{P}^2$ of the central fiber. Finally we blow up the resulting total space along the proper transform $\tilde{L}_P$ of the line L_P. The central fiber becomes $Bl_3\mathbb{P}^2 \cup \mathbb{F}_0 \cup Bl_2\mathbb{F}_0$ (the second figure blow). Running the minimal model program, we obtain the canonical model of the 3-fold with the central fiber $\mathbb{P}^2 \cup \mathbb{F}_0 \cup \mathbb{F}_0$ (the third figure below), which is the same as Case 1 above by changing the color of the building data due to the symmetry. Both degenerations could come from Case 2 in [AP09] for $K^2 = 6$, with A_1, B_1 and C_1 meeting at a point P. Case 2 in this paper could be obtained from the degeneration of Case 7 in [AP09] for $K^2 = 6$ with the point P on the boundary of the hexagon.

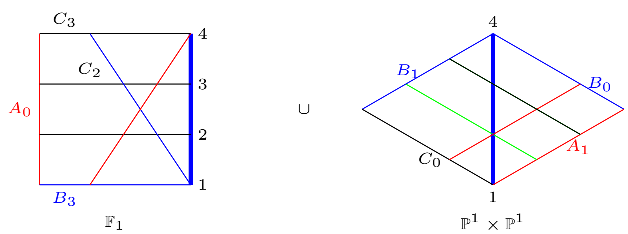


For the first figure above, if moreover the curve C_2 degenerates to $C_3 + B_0$, the canonical model is the same as Case 3 with $K^2 = 5$.

Case 4. When the curve A_2 degenerates to $A_0 + C_3$ and B_2 degenerates to $B_3 + A_0$. We first blow up the total space $\mathcal{Y}$ along the line A_0 in the central fiber, then blow up along the curve $\tilde{L}_P$, which is the proper transform of L_P. The central fiber Σ_0 becomes $Bl_4\mathbb{P}^2 \cup \mathbb{F}_1$ and the 3-fold is the minimal model of the degenerate family. This case could be obtained from Case 6 in [AP09] for $K^2 = 6$ with A_1, B_1, C_1 meeting at a point P.

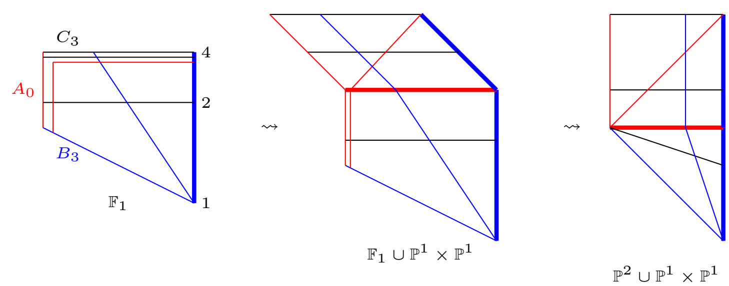


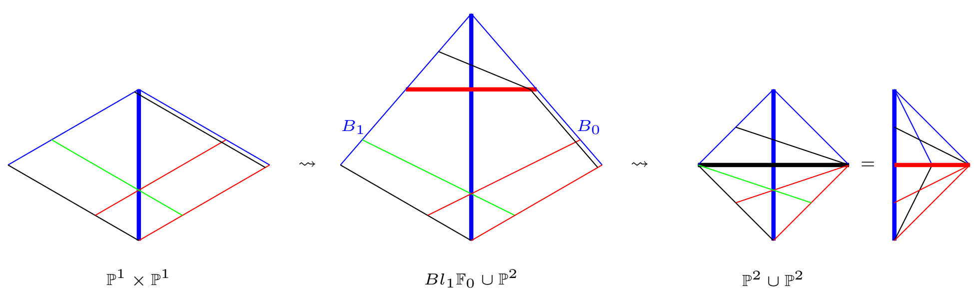
By contracting C_3 and B_3 in the component $Bl_4\mathbb{P}^2$, we get the canonical model of the 3-fold with centeral fiber $\mathbb{F}_1 \cup \mathbb{P}^1 \times \mathbb{P}^1$ and call it Case 4. In the component $\mathbb{F}_1$, curve A_0 is the (-1)-curve s, curves B_3, C_1, C_2, C_3 are fibers f, and curves A_2, B_2 are sections of the numerical type $s + f$. In the component $\mathbb{F}_0$, the double locus is the diagonal $s + f$ and all of the other curves are fibers.



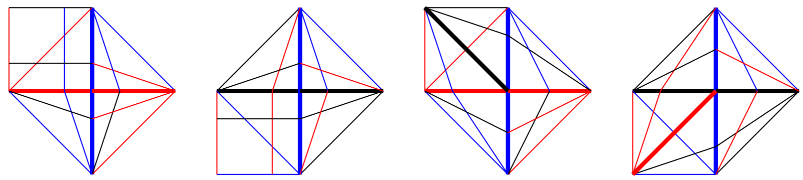
Case 4

There exists some further degenerations of Case 4. Take a one-parameter family with general fibers $\mathbb{F}_1 \cup \mathbb{F}_0$ we obtained above. In the central fiber, curve C_2 coincides with C_3, curve A_2 degenerates to $A_0 + C_3$ in the component $\mathbb{F}_1$ and the curve A_3 coincides with B_1 in the component $\mathbb{P}^1 \times \mathbb{P}^1$. The total space of the one-parameter family is a union of two nonsingular three dimensional spaces $\mathbb{A}^1 \times \mathbb{F}_1$ and $\mathbb{A}^1 \times \mathbb{F}_0$. Blowing up the total space along a line in the central fiber is the same as blowing up the line in each three dimensional space first and then gluing the two resulting surfaces together in the central fiber. Running the minimal model program, the surface $\mathbb{F}_1$ in the central fiber splits into $\mathbb{P}^2 \cup \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ in the central fiber becomes $\mathbb{P}^2 \cup \mathbb{P}^2$, which are as follows.



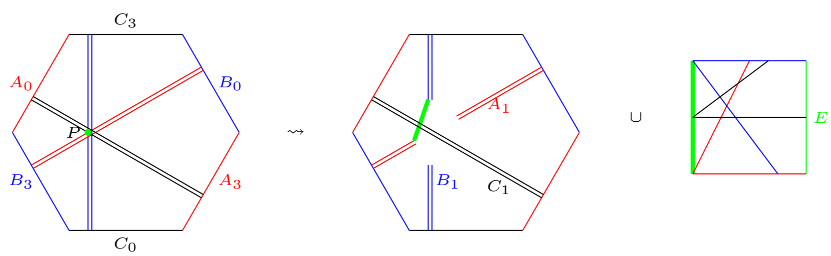


Gluing these two resulting surfaces together, we obtain a further degeneration as in the first figure below and we denote the canonical model as Case 5. Another possible degeneration is that C_2 coincides to B_3, and B_2 degenerates to $A_0 + B_3$. The central fiber of the canonical model is the second figure below, which is prevous Case 2. Since the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ could be degenerated to the section $s + f$, we have further degenerations (figures 3 and 4 below), both of which are Case 3 showed up before.

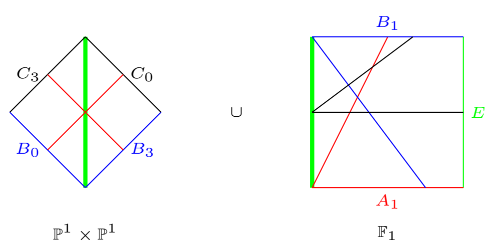


Case 5 *Case* 2 *Case* 3 *Case* 3

Case 6. When the six lines $A_1, A_2, B_1, B_2, C_1, C_2$ meet at the point P. We first blow up the total space along the curve L_P, then blow up the intersection point P of A_1, B_1 and C_1 in the component $\mathbb{P}^2$ of the central fiber, which is the intersection of A_1, B_1, C_1 in the exceptional divisor $\mathbb{P}^2$ of the blowup. The resulting central fiber contains two components $Bl_4\mathbb{P}^2 \cup \mathbb{F}_1$, which is the central fiber of the minimal model.



Running the minimal model program, we contract A_1, B_1, C_1 and get the log canonical model with the central fiber $\mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{F}_1$, where E is the (-1)-curve in $\mathbb{F}_1$. There is no further degeneration for this case.



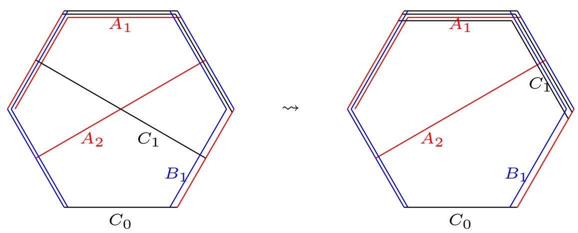
Case 6

Case 6 can be obtained from Case 9 [AP09] for $K^2 = 6$, by making A_1, B_1, C_1 intersect at one point P. For the above surface, if B_2, C_2 in the second component $\mathbb{F}_1$ degenerate to $B_1 + E$, $C_1 + E$, then it is the central fiber of canonical model of the degeneration which comes from Case 10 in [AP09] for $K^2 = 6$.

3.2 Log canonical degenerations.

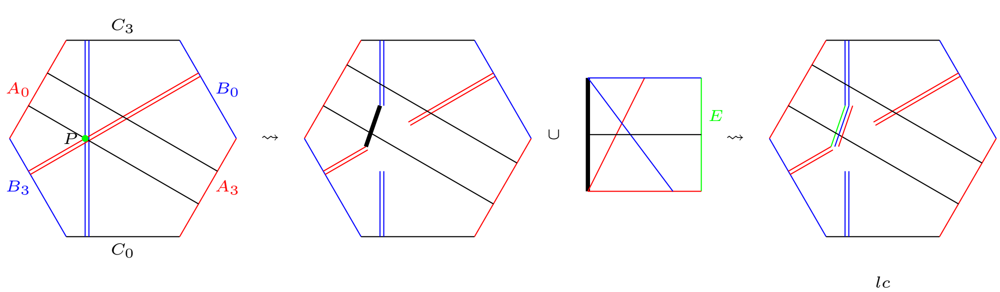
Cases 1,8 and 5 in [AP09] with $K^2 = 6$ are special. Case 5 in [AP09] with $K^2 = 6$ does not produce any degenerations with $K^2 = 5$.

For Case 5 in [AP09] with $K^2 = 6$ (the left figure below), there is no corresponding degeneration with $K^2 = 5$. Since A_1, B_1, C_1 must intersect, the resulting degeneration has an infinite automorphism group, and therefore it does not correspond to an irreducible component of a stable pair.

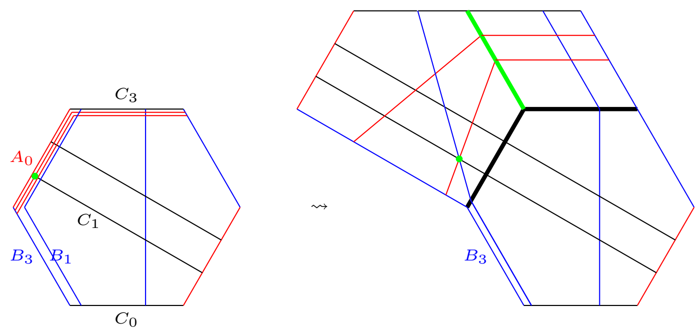


Cases 1 and 8 in [AP09] with $K^2 = 6$ produce degenerations with $K^2 = 5$. But it is surprising that the canonical models of the degenerations are irreducible and are the same as some lc degenerations. We elaborate on the special cases 1 and 8 as following.

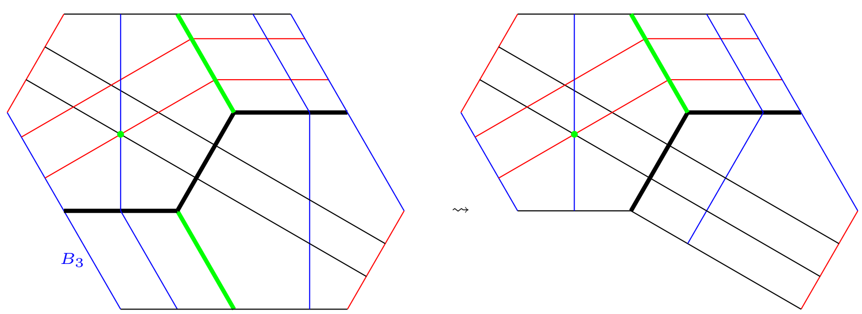
We first look at Case 8 in [AP09] with $K^2 = 6$ which is also a degeneration with $K^2 = 5$. When all of the five lines A_1, A_2, B_1, B_2, C_1 meet at a point P, we blow up the total space along the curve L_P, and the resulting central fiber is $Bl_4\mathbb{P}^2 \cup \mathbb{F}_1$. Running the minimal model program, the whole component $\mathbb{F}_1$ is contracted and the central fiber of the canonical model contains only one irrecucible component which is $Bl_4\mathbb{P}^2$.



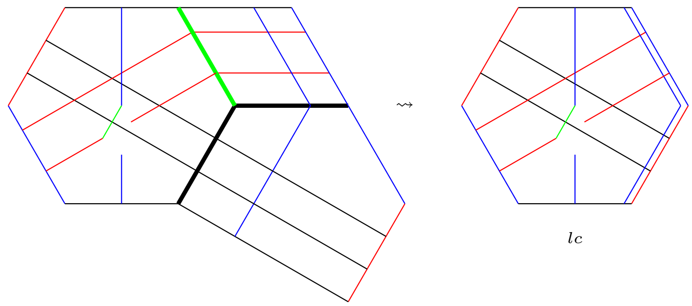
For Case 1 in [AP09] with $K^2 = 6$, we can degenerate B_1 to $A_0 + B_3$ to produce the degeneration with $K^2 = 5$, which is the first figure below. We first blow up the total space along the curve A_0 in the central fiber, then blow up the total space along the strict preimage of C_3 in the central fiber. The resulting central fiber is $Bl_3\mathbb{P}^2 \cup Bl_1\mathbb{F}_1 \cup \mathbb{F}_0$, which is the second figure below.



Consider the curve B_3 in the component $Bl_3\mathbb{P}^2$ of the central fiber, we have $\left(K_{\mathcal{Y}} + \mathcal{D}\right).B_3 = -\dfrac{1}{2} < 0$ and $K_{\Sigma_0}|_{Bl_3\mathbb{P}^2}.B_3 = 0$. When we run the minimal model program, there will be a flip for $(\mathcal{Y}, \mathcal{D})$. The normal bundle of B_3 in the total space is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The flip for $(\mathcal{Y}, \mathcal{D})$ is the Atiyah flop for $\mathcal{Y}$. The process is as follows.



After applying the flip to $(\mathcal{Y}, \mathcal{D})$, the central fiber of the resulting 3-fold space is $Bl_4\mathbb{P}^2 \cup \mathbb{F}_0 \cup Bl_2\mathbb{P}^2$. Finally we blow up the total space along the strict preimage of L_P. All of general fibers and the central fiber are all blown up at one point. Now we have $K_{\Sigma_0}.C \geq 0$ for all curves C in Σ_0. Running the minimal model program, both components $\mathbb{F}_0$ and $Bl_2\mathbb{P}^2$ in the central fiber are contracted. The central fiber becomes $Bl_4\mathbb{P}^2$, which is a lc degeneration of the general fibers $Bl_4\mathbb{P}^2$.



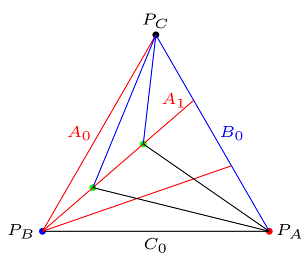
From the cases discussed above, we conclude that there are 6 types of degenerate configurations with reducible log canonical models for the moduli space of Burniat surfaces with $K^2 = 5$, up to the symmetry group $\mathbb{Z}_6$. All of the 6 cases could be obtained from the degenerating cases for $K^2 = 6$ listed in [AP09], with the additional condition that A_1, B_1, C_1 meet at a point P. All the canonical models of the degenerate configurations come from Case 1 and 8 in [AP09] with $K^2 = 6$ are irreducible. Case 5 in [AP09] with $K^2 = 6$ does not produce any degenerations for $K^2 = 5$. We give a table with the relations between cases with $K^2 = 5$ and cases in [AP09] with $K^2 = 6$. This table describes how to get cases with $K^2 = 5$ possibly from cases in [AP09] with $K^2 = 6$ with A_1, B_1, C_1 meeting at one point P. The column with $K^2 = 6$ is the list of cases in [AP09]. For each case in [AP09], we make A_1, B_1 and C_1 intersect at a point to produce the case with $K^2 = 5$ and list it next to the case with $K^2 = 6$ in the first column.

$K^2 = 6$	$K^2 = 5$	further degenerations with $K^2 = 5$
Case 1	lc	
Case 2	Case 1	Case 2,3
Case 3	Case 2	Case 3
Case 4	Case 3	
Case 6	Case 4	Case 5,3
Case 5	none	
Case 7	Case 5	Case 3
Case 8	lc	
Case 9,10	Case 6	

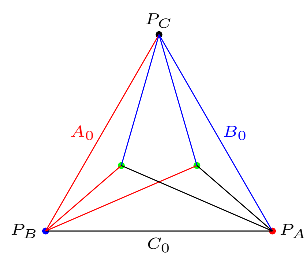
Chapter 4

Burniat surfaces with $K^2 = 4$

We consider $\mathbb{P}^2$ with 9 lines. There are two cases with two distinct points P_1, P_2 which are intersections of three lines inside the triangle. We denote [BC09b] these two cases as a "nodal case" and a "non-nodal case".



$K_X^2 = 4$ nodal



$K_X^2 = 4$ non-nodal

Let P_1 be the intersection of A_1, B_1 and C_1; P_2 be the intersection of A_1, B_2, C_2 for nodal case and A_2, B_2, C_2 for nonnodal case. Let $\Sigma = Bl_5\mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at 5 points P_A, P_B, P_C and P_1, P_2.

For $D = \frac{1}{2}(D_a + D_b + D_c)$, we have

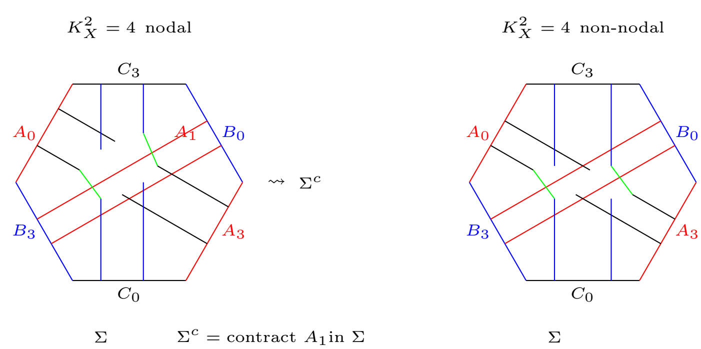
$$K_X^2 = (\pi^*(K_\Sigma + D))^2 = \left(\pi^*(-\frac{1}{2}K_\Sigma)\right)^2 = 4\left(\frac{1}{4}K_\Sigma^2\right) = 4.$$

For the nodal case, the curve A_1 in Σ is a (-2)-curve and $K_\Sigma.A_1 = 0$. The anti-canonical divisor $-K_\Sigma$ is nef but not ample, so $K_\Sigma + D = -\frac{1}{2}K_\Sigma$ is not ample which implies X is not ample. Stable Burniat surfaces X with $K_X^2 = 4$ are $\mathbb{Z}_2^2$-covers of the canonical models Σ^c of Σ with the building data $\frac{1}{2}D$.

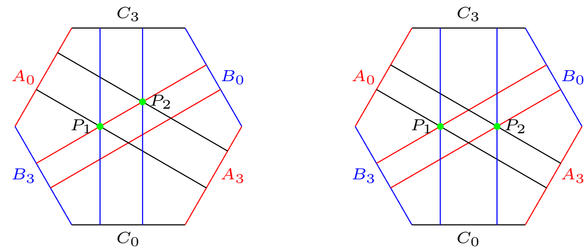
For the non-nodal case, we have that X is stable as that $-K_\Sigma$ is ample, and stable Burniat surfaces X with $K_X^2 = 4$ are $\mathbb{Z}_2^2$-covers of Σ with the building data $\frac{1}{2}D$.

Definition 11. A Burniat surface X in M^3_{Bur} is the canonical model Σ^c of a $\mathbb{Z}_2^2$-cover of $\Sigma = \mathrm{Bl}_5\mathbb{P}^2$ for the building data $D_a = \sum_{i=0}^3 A_i$, $D_b = \sum_{i=0}^3 B_i$, $D_c = \sum_{i=0}^3 C_i$, where a, b, c are the 3 nonzero elements of $\mathbb{Z}_2^2$.

To compactify the moduli space of stable pairs (Y, D), we will study the total space of one-parameter families of the surface isomorphic to Σ^c. For the nodal case, the general fiber Σ^c is the blown down of the (-2)-curve A_1 of $\Sigma = Bl_5\mathbb{P}^2$; for the non-nodal case, the general fiber Σ is $Bl_5\mathbb{P}^2$.



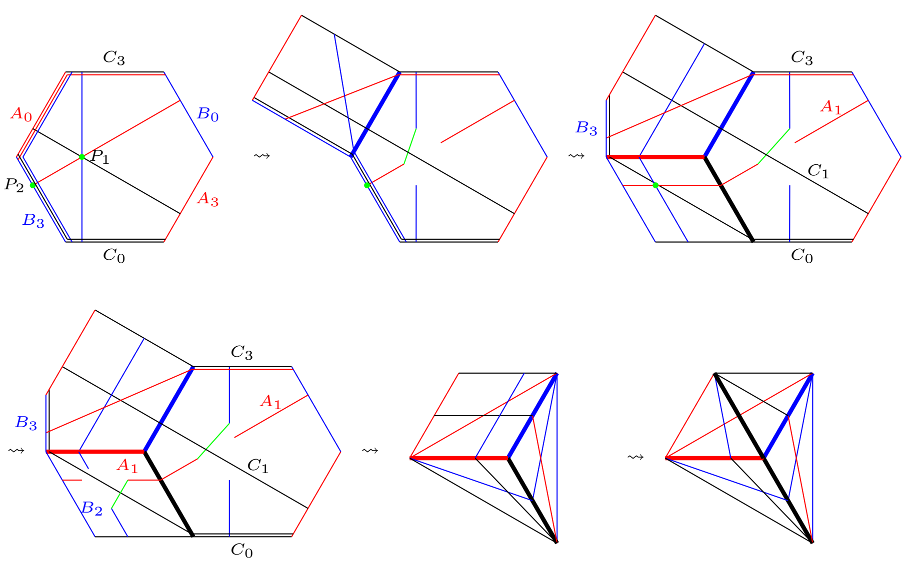
For the nodal case, the general fiber Σ^c is a singular surface with an A_1-singularity, which is obtained from $Bl_5\mathbb{P}^2$ by contracting the (-2)-curve. To see the degenerating arrangements with $K^2 = 4$, we will start with surfaces $Bl_3\mathbb{P}^2$ which are shown in the following figures.



$K_X^2 = 4$ nodal $\quad$ $K_X^2 = 4$ non-nodal

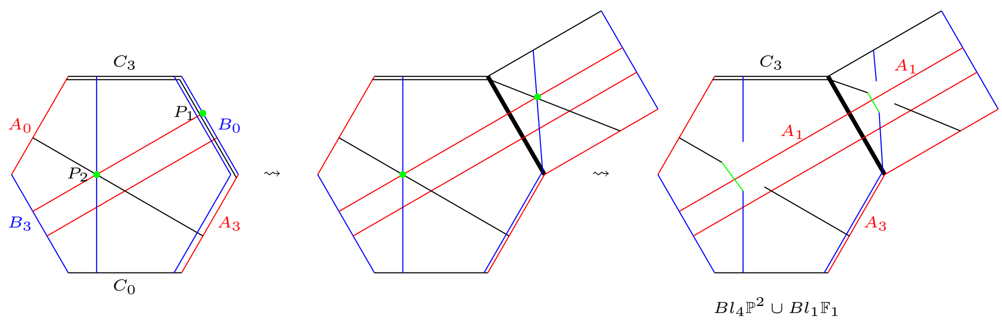
We first consider the nodal case with $K^2 = 4$.

Case 1. The curve A_2 degenerates to $A_0 + C_3$, B_2 degenerates to $A_0 + B_3$, and C_2 degenerates to $B_3 + C_0$. Blowing up the total space $\mathcal{Y}$ along the curve L_{P_1} and the curve A_0 in Σ_0, we see the general fibers are $Bl_4\mathbb{P}^2$ and the central fiber is $\Sigma_0 = Bl_4\mathbb{P}^2 \cup \mathbb{F}_1$. Next we blow up the total space along the strict preimage of B_3 in the component $Bl_4\mathbb{P}^2$ of Σ_0 and along the strict transform $\tilde{L}_{P_2}$, which results in the central fiber becoming a union of three components $Bl_4\mathbb{P}^2 \cup Bl_1\mathbb{F}_1 \cup \mathbb{F}_0$ (see the first figure of the second row below). Running the minimal model program, we get the canonical model with central fiber $\mathbb{F}_0 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ and we call it Case 1. The further degeneration is 4 copies of $\mathbb{P}^2$ and we call it Case 2.

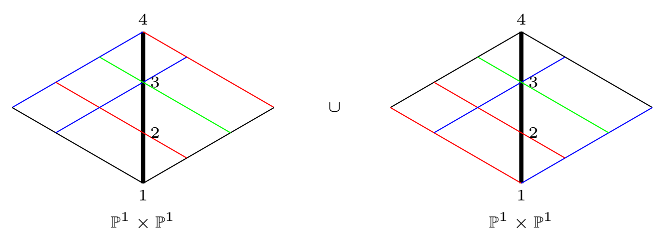


Case 1 $\quad$ *Case 2*

Case 3. Either P_1 or P_2 is on B_0 or B_3. All degenerating arrangements are the same up to rotation. WLOG, we can assume that P_2 is on B_0. To get the minimal model, we first blow up the total space $\mathcal{Y}$ along the curve B_0 in the central fiber. Let curves $\tilde{L}_{P_1}$ and $\tilde{L}_{P_2}$ be the proper transform of L_{P_1} and L_{P_2}. Then blow up the total space along $\tilde{L}_{P_1}$ and $\tilde{L}_{P_2}$. The central fiber Σ_0 becomes $Bl_4\mathbb{P}^2 \cup Bl_1\mathbb{F}_1$.



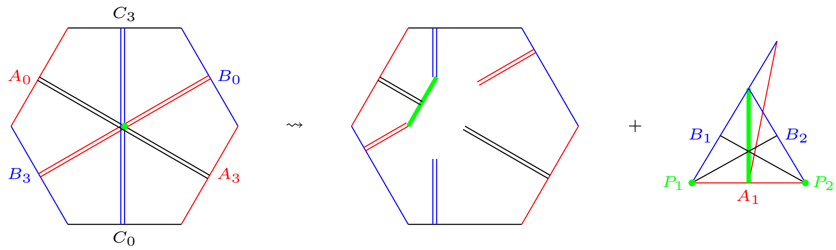
Running the minimal model program, the first figure below is obtained from the component $Bl_4\mathbb{P}^2$ by contracting 3 curves A_1, A_3, C_3. The second figure below is obtained from the component $Bl_1\mathbb{F}_1$ by contracting the curve A_1. We obtain the canonical model with the central fiber $\Sigma_0^c = \mathbb{F}_0 \cup \mathbb{F}_0$ and we denote it as Case 3. This case could be obtained from Case 4 for $K^2 = 5$.



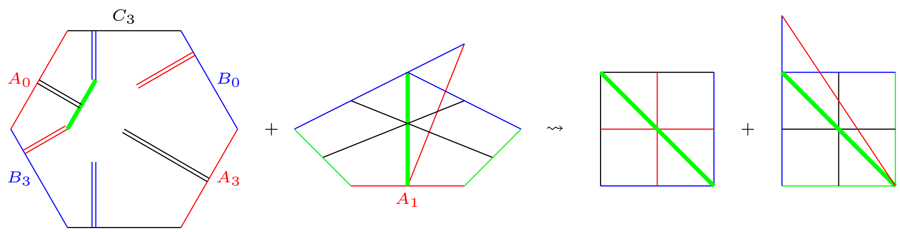
Case 3

There are further degenerations; however, the further degenerations do not produce any new cases. For example, when the point 3 in the above figures on the double locus goes to the point 4, the canonical model of the further degeneration is the same as Case 2.

Case 4. The point P_1 and P_2 coincide. We blow up the total space $\mathcal{Y}$ at the point P in the central fiber. The central fiber becomes $Bl_{4pts}\mathbb{P}^2 \cup \mathbb{P}^2$ which is as follows.

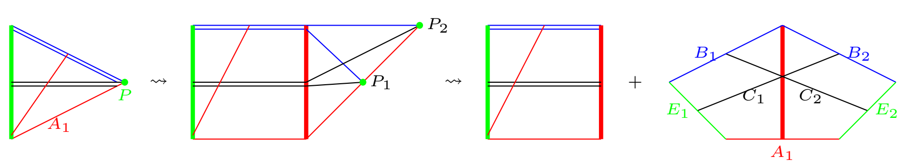


Then we blow up the total space along the proper transform $\tilde{L}_{P_1}$ and $\tilde{L}_{P_2}$. The central fiber of the minimal model is $Bl_4\mathbb{P}^2 \cup Bl_2\mathbb{P}^2$. Running the minimal model program, we contract $A_1, A_2, B_1, B_2, C_1, C_2$ in the component $Bl_4\mathbb{P}^2$ and A_1 in the component $Bl_2\mathbb{P}^2$ in the central fiber. The central fiber of the canonical model is $\mathbb{F}_0 \cup \mathbb{F}_0$ which is as follows. We denote it as Case 4.



Case 4

Case 5. The further degeneration of Case 4 above. After blowing up the total space at P, the points P_1, P_2 still could coincide in the exceptional divisor $\mathbb{P}^2$ of the blowup. We need to blow up the total space at the point P first, then P_1 and P_2 will be distinct. Now we can blow up the total space along the lines $\tilde{L}_{P_1}$ and $\tilde{L}_{P_2}$. The following figures are only the second component of the canonical model, with the first component $Bl_4\mathbb{P}^2$, which is the same as the left figure of Case 4 above.



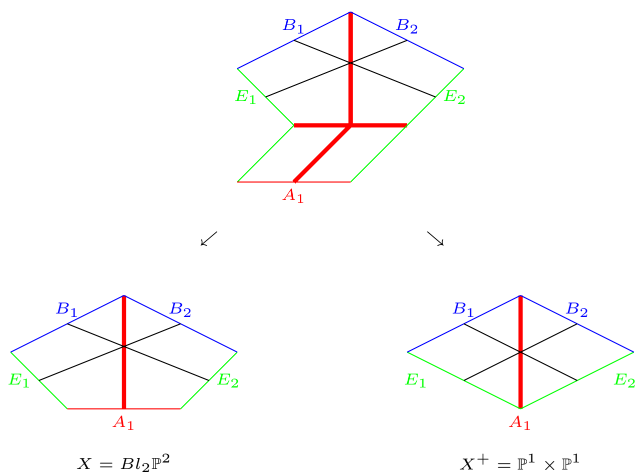
Consider the line A_1 in the component $Bl_2\mathbb{P}^2$ of the central fiber, we have

$$(K_{\mathcal{Y}} + \mathcal{D}).A_1 \;=\; -\frac{1}{2} < 0$$

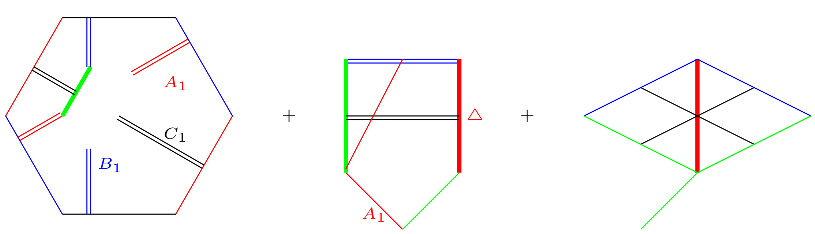
and

$$K_{\Sigma_0}|_{Bl_2\mathbb{P}^2}.A_1 \quad = \quad (K_{Bl_2\mathbb{P}^2} + \triangle)\,.A_1 = 0$$

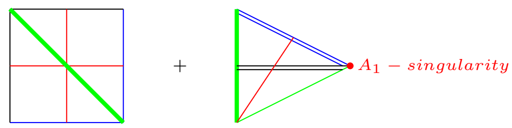
According to the minimal model program, there will be a flip for $(\mathcal{Y}, \mathcal{D})$. The normal bundle of A_1 in the total space $\mathcal{Y}_2$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The flip for $(\mathcal{Y}, \mathcal{D})$ is the Atiyah flop for $\mathcal{Y}$, with morphisms on the central fiber below.



When we apply the flip, the other component $\mathbb{F}_1$ is blown up at one point on the double locus. The resulting central fiber is a union of three components $Bl_4\mathbb{P}^2 \cup Bl_1\mathbb{F}_1 \cup \mathbb{F}_0$.



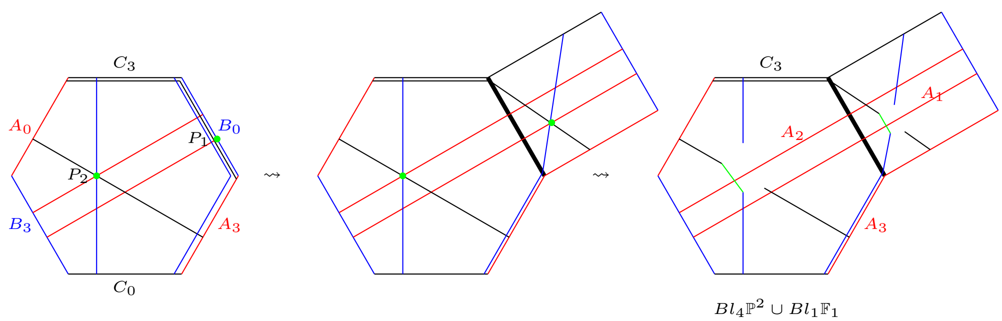
We have $(K_{\mathcal{Y}} + \mathcal{D})\,.C \geq 0$ for all the curves C in Σ_0, and in particular $(K_{\mathcal{Y}} + \mathcal{D})\,.C = 0$ for all the curves C in $\mathbb{P}^1 \times \mathbb{P}^1$. Running the minimal model program, we obtain the canonical model by contracting A_1, B_1, C_1 in $Bl_4\mathbb{P}^2$, $A_1, \triangle$ in $Bl_1\mathbb{F}_1$ and the whole component $\mathbb{P}^1 \times \mathbb{P}^1$, where $\triangle$ is the double locus. The central fiber of the resulting canonical model is $\mathbb{P}^1 \times \mathbb{P}^1 \cup S$, where S is obtained from $\mathbb{F}_2$ by contracting the (-2)-curve. So S is a surface with an A_1-singularity. We call the canonical model as Case 5.



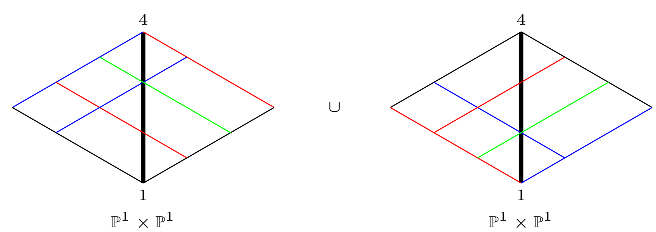
Case 5

The following cases are non-nodal cases with $K^2 = 4$.

Case 6. Similar to case 3, but the point P_1 is on B_0 instead.

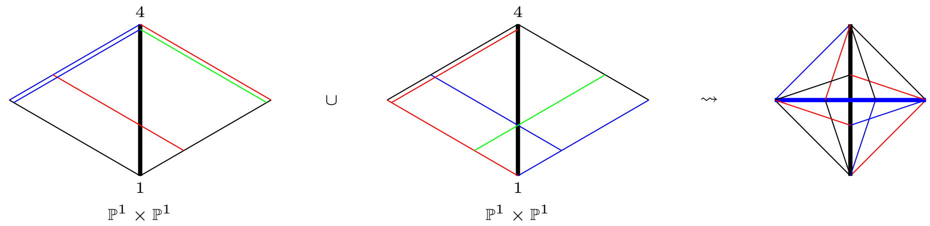


The central fiber of the resulting canonical model is $\mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \mathbb{P}^1$ and we denote is as Case 6. Case 6 is not isomorphic to Case 3, in which the central fiber is also $\mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \mathbb{P}^1$, as the line arrangements are not isomorphic.



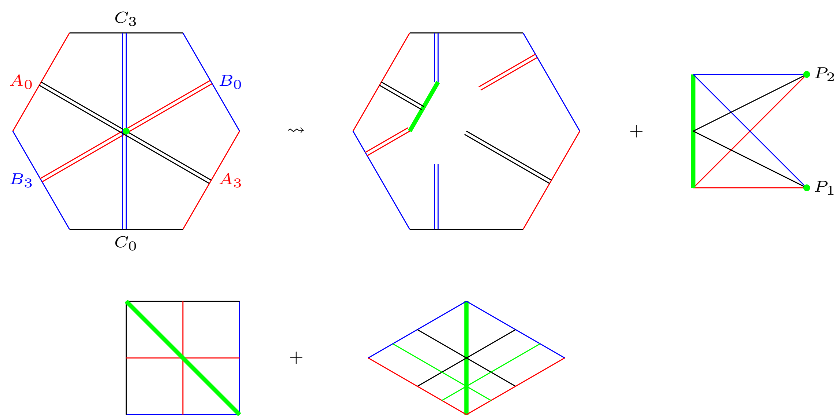
Case 6

There is a further degeneration as follows, in which the central fiber of the resulting canonical model is a union of four copies of $\mathbb{P}^2$. We denote is as Case 7.



Case 7

Case 8. For non-nodal case, lines A_1, A_2, B_1, B_2, C_1 and C_2 intersect at one point. The central fiber of the canonical model is $\mathbb{P}^1 \times \mathbb{P}^1 \cup Bl_2\mathbb{P}^2$ and we denote is as Case 8.



Case 8

In total, there are 5 types of degenerations with reducible canonical models for $K^2 = 4$ nodal case and 3 types of degenerations for $K^2 = 4$ non-nodal case up to the symmetry group $\mathbb{Z}_2$. All of them could be obtained from the cases with $K^2 = 5$.

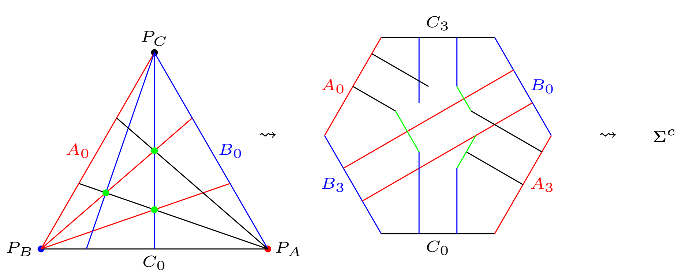
$K^2 = 5$	$K^2 = 4$	further degenerations with $K^2 = 4$
Case 1	Case 1	Case 2
Case 2	Case 2	
Case 3	Case 6	
Case 4	Cases 3,6	Cases 2,7
Case 5	Case 2,7	
Case 6	Case 4,8	Case 5

Chapter 5

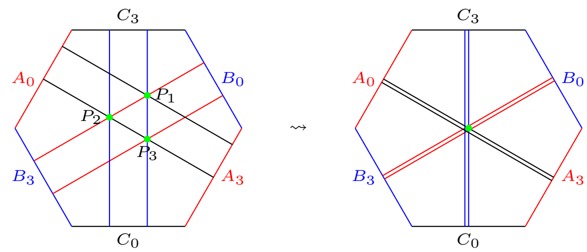
Burniat surfaces with $K^2 = 3$

Consider the surface $\mathbb{P}^2$ with 9 lines and 3 points P_1, P_2 and P_3. Here P_1 be the intersection point of $A_1 B_1, C_1$; P_2 be the intersection point of A_1, B_2, C_1; P_3 be the intersection of A_2, B_1, C_1. Let Σ is the blow-up of $\mathbb{P}^2$ at the six points P_A, P_B, P_C and P_1, P_2, P_3. A Burniat surface X in M^3_{Bur} is the canonical model of a $\mathbb{Z}_2^2$-cover of $\Sigma = Bl_6\mathbb{P}^2$ for the building data D_a, D_b and D_c.

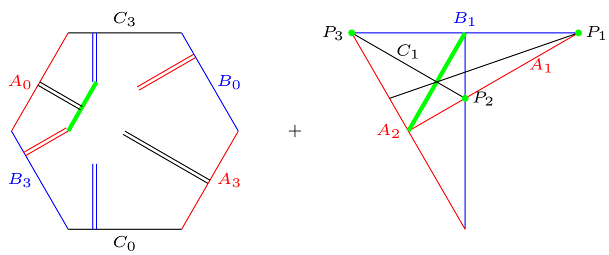
There are three (-2)-curves A_1, B_1, C_1 in Σ and $K_\Sigma + D = -\dfrac{1}{2}K_\Sigma$ is nef but not ample. The canonical model Σ^c of Σ is obtained from Σ by contracting the three (-2)-curves. Stable Burniat surfaces X with $K_X^2 = 3$ are $\mathbb{Z}_2^2$-covers of the canonical models Σ^c of Σ with the building data D. The general fiber of a one-parameter family is Σ^c and it contains three A_1-singularities. We denote the singularities obtained from contracting A_1, B_1 and C_1 by Q_1, Q_2, Q_3.



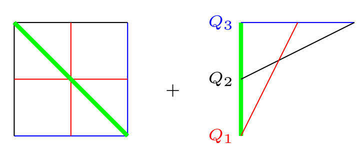
Case 1. The three points P_1, P_2 and P_3 coincide. Take a one-parameter family of Σ with the general fiber $Bl_5\mathbb{P}^2$ and the central fiber the degenerating arrangement Σ_0.



We first blow up the total space $\mathcal{Y}$ at the point P on the central fiber. The resulting central fiber is $Bl_4\mathbb{P}^2 \cup \mathbb{P}^2$.

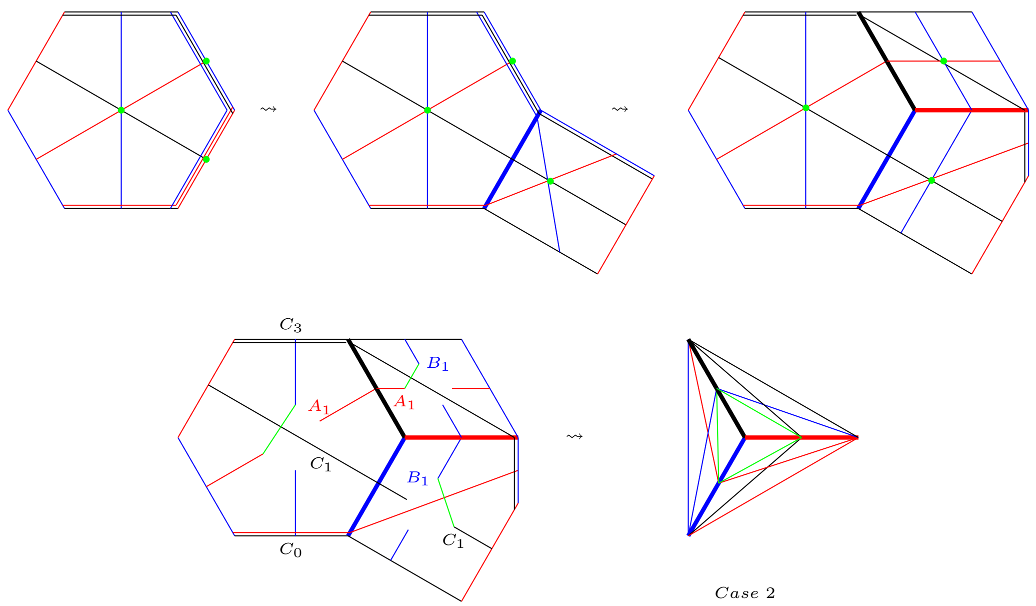


Now we blow up the total space along the curves $\tilde{L}_{P_1}$, $\tilde{L}_{P_2}$ and $\tilde{L}_{P_3}$, which are the proper transformation of L_{P_1}, L_{P_2} and L_{P_3}. The central fiber turns to be $Bl_4\mathbb{P}^2 \cup Bl_3\mathbb{P}^2$. The component $Bl_3\mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at P_1, P_2 and P_3. When we run the minimal model program, the curves A_1, B_1, C_2 in the component $Bl_3\mathbb{P}^2$ of the central fiber are contracted. In the general fiber, the curves A_1, B_1, C_1 are contracted as well. Clearly we also have that $Bl_3\mathbb{P}^2$ goes back to $\mathbb{P}^2$ in the central fiber. The general fiber of the canonical model is Σ^c, which we described at the beginning of this section, and the central fiber is $\mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{P}^2$. We denote this case as Case 1.



Case 1

Case 2. The point P_1 is on B_0 and P_2 is on A_3. We first blow up the total space along the curve A_3, then blow up along the strict preimage of B_0 in $Bl_3\mathbb{P}^2$. Finally we blow up the total space along the three curves $\tilde{L}_{P_1}$, $\tilde{L}_{P_2}$ and $\tilde{L}_{P_3}$. The central fiber of the minimal model is $Bl_4\mathbb{P}^2 \cup Bl_2\mathbb{F}_1 \cup Bl_1\mathbb{F}_0$. We obtain the central fiber $\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ of the canonical model by contracting the lines labeled in the fourth figure below. We denote this case as Case 2.



Case 2

There are only 2 types of degenerations with reducible canonical models for $K^2 = 3$. Both of them could be obtained from cases with $K^2 = 4$.

We summarize the above computations in the following statement:

Theorem 1. *The compactified coarse moduli space $\overline{M}^d_{Bur}$ of stable Burniat surfaces, or equivalently, of stable pairs (Y, D), is of dimension $d-2$, irreducible for $d \neq 4$, and with two components for $d = 4$. The types of degenerations, up to symmetry, are listed as below.*

(i) There are 6 types of degenerate configurations of stable pairs with reducible canonical models in the moduli space of stable pairs (Y, D) for $K^2 = 5$ case up to the symmetry group $\mathbb{Z}_6$ described in Chapter 3.

(ii) There are 5 types of degenerations with reducible canonical models in the moduli space of stable pairs (Y, D) *for* $K^2 = 4$ *nodal case and 3 types of degeneration for* $K^2 = 4$ *non-nodal case up to the symmetry group* $\mathbb{Z}_2$ *described in Chapter 4.*

(iii) There are only 2 types of degenerations with reducible canonical models in the moduli space of stable pairs (Y, D) *for* $K^2 = 3$ *described in Chapter 5.*

There is only one Burniat surface with $K^2 = 2$, thus the moduli space of Burniat surfaces with $K^2 = 2$ is just a single point.

Chapter 6

Matroid tilings

6.1 Vector Matroid and hyperplane arrangements

Definitions in this section are from [Ale13] and [Ja92]. Fix a field k. Consider n vectors $f_1, ..., f_n$ spanning a k-vector space W of dimension r. Call a subset $I \subset \overline{n} = \{1, ..., n\}$ a base if the vectors $\{f_i, i \in I\}$ form a basis of W.

Definition 12. A vector matroid represented by the vectors $f_1, ...f_n \in W$ is the pair $(\overline{n}, \mathcal{B})$, where $\mathcal{B}$ is the set of all bases.

Let $V = W^*$ be the dual space, and think of the vectors $f_i \in W = V^*$ as nonzero linear functions on V. Each of them defines a hyperplane $B_i \in \mathbb{P}V \simeq \mathbb{P}^{r-1}$.

Note:

1. The condition $f_i \neq 0$ assures that B_i is actually a divisor.

2. The condition that $\{f_i\}$ generates V is equivalent to $B_1 \cap ... \cap B_n = \emptyset$.

We denote $B(I)$ as the projective linear subspace of $\mathbb{P}V$: $B(I) := \cap_{i \in I} B_i$.

Lemma 13. *(i) I is independent* $\Longleftrightarrow$ *codim*$B(I) \simeq |I|$.

(ii) I is a base $\Longleftrightarrow B(I) = \emptyset$ *and* $|I| = r = \dim \mathbb{P}V + 1$.

By convention, we set codim$\emptyset = r$ and $\mathbb{P}^{-1} = \emptyset$.

A *hypersimplex* $\triangle(r, n)$ is defined to be a convex hull

$$\begin{aligned} \triangle(r,n) &= \text{Conv}(e_I | I \in \overline{n}, |I| = r) \\ &= \{(x_1, ..., x_n) \in \mathbb{R}^n | 0 \leq x_i \leq 1, \Sigma x_i = r\} \end{aligned}$$

A *tiling* is a collection of polytopes Q_j in $\mathbb{R}^n$ which is face-fitting: intersection of any two $Q_{j_1} \cap Q_{j_2}$ is either empty or is a face of both.

A *partial matroid tiling* is a tiling consisting of base polytopes in the hypersimplex $\triangle(r,n) \setminus \cup_{i=1}^{n} \{x_i = 0\}$. It does not have to cover $\triangle(r,n)$ completely.

6.2 Matroid polytopes

According to the general theory of [Ale08], the unweighted stable hyperplane arrangements are described by matroid tilings of the hypersimplex $\triangle(r, n)$. Their weighted counterparts are described by partial tilings of the hypersimplex $\triangle(r, n)$ in $\mathbb{R}^n$. In this section, we will discuss the matroid tiling of the certain polytopes $\triangle^d_{Bur}, d \leq 5$ corresponding to Burniat surfaces with $K^2 = d$.

In [AP09], Alexeev-Pardini defined the polytope $\triangle^6_{Bur}$ corresponding to Burniat surfaces with $K^2 = 6$, which is a subpolytope of a hypersimplex $\triangle(3, 9)$. They computed all stable Burniat surfaces with $K^2 = 6$ by computing matroid tilings of a certain polytope $\triangle^6_{Bur}$. We define the corresponding polytopes $\triangle^d_{Bur}, d \leq 5$ similarly to $\triangle^6_{Bur}$. We use the same method find all matroid tilings of $\triangle^d_{Bur}$, $d = 3, 4, 5$, and then find all possible stable surfaces in the main component of the compactified moduli space of Burniat surfaces with $K^2 = 5$.

Let's recall some definitions and results in [AP09, Ale08].

A *matroid polytope* $BP_V \subset \triangle(r,n)$ is the polytope corresponding to the toric variety $\overline{T.V}$ for some geometric point $[V \subset \mathbb{A}^n] \in G(r,n)(k)$. One can also describe the matroid polytopes in terms of hyperplane arrangements. Let $\mathbb{P}V \simeq \mathbb{P}^r$ and assume that it is not contained in the n coordinate hyperplane H_i (i.e. all $z_i \neq 0$ on $\mathbb{P}V$); let $B_1, ..., B_n \subset \mathbb{P}V$ be $H_i \cap \mathbb{P}V$. Then for the hyperplane arrangement $(\mathbb{P}V, \sum B_i)$, the matroid polytope BP_V is the convex hull of the points $v_I \in \mathbb{Z}^n$ for all $I \subset \overline{n}$ such that $\cap_{i \in I} B_i = \varnothing$, or in terms of inequalities as

$$BP_V = \left\{ (x_1, ..., x_n) \in \triangle(r,n) | \sum_{i \in I} x_i \leq \operatorname{codim} \cap_{i \in I} B_i,\ \forall I \subset \overline{n} \right\}.$$

For a hyperplane arrangement in general position, one has $BP_V = \triangle(r,n)$.

Let $b = (b_1, ..., b_n)$ be a weight, a *b-cut hypersimplex* is

$$\begin{aligned}
\triangle_b(r,n) &= \{(x_1, ..., x_n) | 0 \leq x_i \leq b_i, \Sigma x_i = r\} \\
&= \{\alpha \in \triangle(r,n) | \alpha \leq b\}
\end{aligned}$$

We have the theorem in [Ale08]

Theorem. *[Ale08, 2.12] The matroid polytope BP_V is the set of points $(x_i) \in \mathbb{R}^n$ such that the pair $(\mathbb{P}V, \sum x_i B_i)$ is lc and $K_{\mathbb{P}V} + \sum x_i B_i \sim_R 0$; the interior $IntBP_V$ is the set of points such that $(\mathbb{P}V, \sum x_i B_i)$ is klt and $K_{\mathbb{P}V} + \sum x_i B_i \sim_R 0$.*

A *tiling* of the b-cut hypersimplex $\triangle_b$ is a partial matroid tiling of $\triangle(r,n)$ such that $\cup BP_{M_j} \supset \triangle_b$ and such that all base polytopes BP_{M_j} intersect the interior of $\triangle_b$.

Let $(\mathbb{P}V, \sum b_i B_i)$ be a hyperplane arrangement. For a point $p \in \mathbb{P}V$, we denote by $I(p)$ the set of $i \in \overline{n}$ such that $p \in B_i$. We define $\triangle_b^p$ to be the face (possibly empty) of $\triangle_b$, where $x_i = b_i$ for all $i \in I(p)$.

Theorem. *[Ale08, 6.6] Let $(\mathbb{P}V, \sum b_i B_i)$ be a hyperplane arrangement of general type. Suppose $BP_V \cap \triangle_b \neq \emptyset$. Then $(\mathbb{P}V, \Sigma b_i B_i)$ is lc at p if and only if $BP_V \cap \triangle_b^p \neq \emptyset$.*

Now let us look at Burniat surfaces with $K^2 = 5$.

A Burniat surface with $K^2 = 5$ is a $\mathbb{Z}_2^2$-cover of $Bl_4\mathbb{P}^2$ for the data

$$D = \sum (a_i A_i + b_i B_i + c_i C_i + eE)$$

where D_a, D_b, D_c are branched divisors of the Galois cover and E is not.

Denote by $\triangle_{Bur}^d$ the polytope corresponding to Burniat surfaces with $K^2 = d$. This is a subpolytope of the hypersimplex $\triangle(3,9)$ with weight $b = (\frac{1}{2}, ..., \frac{1}{2})$. In [AP09], the polytope $\triangle_{Bur}^6$ is defined to be

$$\begin{aligned} \triangle(3,9) \supset \triangle_{Bur}^6 &= \{(a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2) \in \mathbb{R}^9 \text{ satisfying} \\ &\quad 0 \leq a_i, b_i, c_i \leq \frac{1}{2}, i = 0, 1, 2, 3; \\ &\quad \sum_{i=0}^{2} (a_i + b_i + c_i) = 3; \\ &\quad 0 \leq a_3 = c_0 + c_1 + c_2 + b_0 - 1 \leq 1/2; \\ &\quad 0 \leq b_3 = a_0 + a_1 + a_2 + c_0 - 1 \leq 1/2; \\ &\quad 0 \leq c_3 = b_0 + b_1 + b_2 + a_0 - 1 \leq 1/2; \} \end{aligned}$$

For the case $K^2 = 5$, the divisor D on Σ satisfies $K_\Sigma + D = 0$ and we got an extra equation $e = a_1 + b_1 + c_1 - 1$ comparing to $\triangle_{Bur}^6$. Since the cover $\pi : X \to Y$ is unramified over E, the coefficient $e \leq \frac{r-1}{r} = 0$, where $r = 1$ is the ramification index. Then we define $\triangle_{Bur}^5$ as follows

$$\triangle(3,9) \supset \triangle_{Bur}^5 = \triangle_{Bur}^6 \cap \{e = a_1 + b_1 + c_1 - 1 \leq 0\}$$

6.3 Matroid tilings for $K^2 = 3$ case

When lines A_0, B_0, C_0 are distinct, there is at most 4 lines could coincide in a degeneration. WLOG, let A_0 be the line coincide with three of the other lines which are among A_1, A_2, B_1, B_2. If $a_0 + a_1 + a_2 + b_1 \leq 1$, then $b_0 + b_2 + c_0 + c_1 + c_2 \geq 2$ which is the same as $b_2 + a_0 \geq 1$. the corresponding polytope is $BP_M \subset \{b_1 + a_0 \geq 1\} \cap \triangle^6_{Bur}$. Since $\triangle^d_{Bur} \subset \left\{b_2 \leq \frac{1}{2}, a_0 \leq \frac{1}{2}\right\} \subset \{b_2 + a_0 \leq 1\}$, we know $BP_M \cap \operatorname{int}\left(\triangle^d_{Bur}\right) = \emptyset$. Hence the assumed degeneration does not appear and there is no degeneration with 4 lines coincide.

Since the moduli space for $K^2 = 2$ is just a point, there is no need to look at tilings for $K^2 = 2$. We start with the tilings for $K^2 = 3$.

Let P_1 be the intersection of A_1, B_1, C_2; P_2 be the intersection of A_1, B_2, C_1; P_3 be the intersection of A_2, B_1, C_1. We will use $a_i b_j c_k$ as the abbreviation of $a_i + b_j + c_k$. The hypersimplex $\triangle^3_{Bur}$ is defined to be

$$\triangle^3_{Bur} \quad = \quad \triangle^6_{Bur} \cap \{a_1 b_1 c_2 \leq 1, a_1 b_2 c_1 \leq 1, a_2 b_1 c_1 \leq 1\}.$$

We have the following table.

$K^2 = 3$				
Type	BP_M	Case	Y_j	$4(K_Y\vert_{Y_i} + D_i)^2$
0	$\triangle^3_{Bur}$		Σ	3
1	$a_1 a_2 b_1 b_2 c_1 c_2 \leq 2$	1	$\mathbb{P}^1 \times \mathbb{P}^1$	2
2	$a_0 a_1 b_2 \leq 1$ $a_1 c_2 c_3 \leq 1$	2	$\mathbb{P}^2$	1
3	$a_0 b_0 c_0 \leq 1$	1	$\mathbb{P}^2$	1

Type 1 produces a covering corresponding to Case 1. We let BP_1 be a type 1 polytope and find all possible coverings containing BP_1 .

$$\begin{aligned} BP_1 &= \{a_1a_2b_1b_2c_1c_2 \leq 2\} \\ BP_2 &= \{a_0b_0c_0 \leq 1\} \end{aligned}$$

Calculation: Since $\sum_{i=0}^{2} a_i + b_i + c_i = 3$, $a_1a_2b_1b_2c_1c_2 \leq 2$ is equivalent to $a_0b_0c_0 \geq 1$. Let $H = \{a_1a_2b_1b_2c_1c_2 = 2\}$. Since $F = H \cap BP_1$ is a face of BP_1 and BP_1 in on one side $\{a_1a_2b_1b_2c_1c_2 \leq 2\}$ of the hyperplane H, the base polytope that can fit BP_1 along F should be on the other side $\{a_0b_0c_0 \leq 1\}$ of H and have the face in H. The only possible base polytope is BP_2. As $BP_1 \cup BP_2 = \triangle^3_{Bur}$, it is the only covering of $\triangle^3_{Bur}$ containing BP_1.

Type 2 produces a covering corresponding to Case 2:

$$\begin{aligned} BP_1 &= \{a_0a_1b_2 \leq 1, a_1c_2c_3 \leq 1\} \\ BP_2 &= \{a_2a_3b_1 \leq 1, b_0b_1c_2 \leq 1\} \\ BP_3 &= \{a_2c_0c_1 \leq 1, b_2b_3c_1 \leq 1\} \end{aligned}$$

Calculation: The base polytope BP_1 has faces F_1 on $H_1 = \{a_0a_1b_2 = 1\}$ and F_2 on $H_2 = \{a_1c_2c_3 = 1\}$. It is easy to see that $a_0a_1b_2 \leq 1$ is equivalent to $a_2a_3b_1 \geq 1$, and $a_1c_2c_3 \leq 1$ is equivalent to $a_2c_0c_1 \geq 1$. The only base polytope that can fit BP_1 along the face F_1 (respectably F_2) is BP_2 (respectably BP_3). Those three polytopes consist a covering of $\triangle^3_{Bur}$.

Type 3: there is no type 3 base polytopes that can fit $BP = \{a_0b_0c_0 \leq 1\}$ along the face $\{a_0b_0c_0 = 1\}$.

Let X be the Burniat surface with $K_X^2 = 5$ and $\pi : X \to \Sigma$, then $4K_\Sigma^2 = 3$. We can see that for each covering $\sum_i 4\left(K_Y|_{Y_i} + D_i\right) = 3$.

We give the table of coverings precisely correspond to the degenerations listed in Chapter 5.

Case	Tilings for $K^2 = 3$	From $K^2 = 4$
1	$a_1a_2b_1b_2c_1c_2 \leq 2$; $a_0b_0c_0 \leq 1$	Case 4
2	$b_0b_2c_2 \leq 1, a_2a_3b_2 \leq 1$; $a_1c_1c_2 \leq 1, b_1b_3c_1 \leq 1$; $a_0a_1b_1 \leq 1, a_1c_1c_3 \leq 1$	Case 1,3

6.4 Matroid tilings for $K^2 = 4$ case

For the nodal case, we let P_1 be the intersection of A_1, B_1, C_1 and let P_2 be the intersection of A_1, B_2, C_2. The hypersimplex $\triangle^4_{Bur}\left(nodal\right)$ is defined to be

$$\triangle^4_{Bur}\left(nodal\right) = \triangle^6_{Bur} \cap \left\{a_1b_1c_1 \leq 1, a_1b_2c_2 \leq 1\right\}.$$

We have the following table.

$K^2 = 4$ Nodal Case				
Type	BP_M	Case	Y_j	$4(K_Y\|_{Y_i} + D_i)^2$
0	$\triangle^4_{Bur}(nodal)$		Σ	4
1	$b_0b_1c_1 \leq 1$	3	$\mathbb{P}^1 \times \mathbb{P}^1$	2
2	$a_0a_2b_2 \leq 1, b_2b_3c_2 \leq 1$	1,2	$\mathbb{P}^2$	1
3	$a_1a_3b_1 \leq 1, a_1c_0c_2 \leq 1$	1	$\mathbb{P}^1 \times \mathbb{P}^1$	2
4	$b_0b_1c_2 \leq 1, a_1a_3b_1 \leq 1$ $a_1c_0c_1 \leq 1$	1,2	$\mathbb{P}^2$	1
5	$a_1a_2b_1b_2c_1c_2 \leq 2$	4, 5	$\mathbb{P}^1 \times \mathbb{P}^1$	2
6	$a_0b_0c_0 \leq 1$	4,5	$\mathbb{P}^1 \times \mathbb{P}^1$	2

Type 1 produces a covering corresponding to Case 3. Start with type 1 polytope BP_1.

$$\begin{aligned} BP_1 &= \{b_0b_1c_1 \leq 1\} \cap \triangle^4_{Bur}(nodal) \\ BP_2 &= \{b_2b_3c_2 \leq 1\} \cap \triangle^4_{Bur}(nodal) \end{aligned}$$

Calculation: base polytope BP_1 has a face F on $H = \{b_0b_1c_1 = 1\}$. Since $H = \{b_0b_1c_1 = 1\}$ is the same as $\{b_2b_3c_2 = 1\}$. There are many base polytopes with a face on H. But there is only one polytope BP_2 that fit BP_1 well. The covering $\{BP_1, BP_2\}$ corresponds to the degeneration Case 3.

Type 2 produces coverings corresponding to Case 1 and Case 2. Start with type 2 polytope BP_1.

$$\begin{aligned} BP_1 &= \{a_0a_2b_2 \leq 1, b_2b_3c_2 \leq 1\} \\ BP_2 &= \{b_0b_1c_1 \leq 1, a_2c_1c_3 \leq 1\} \\ BP_3 &= \{a_1a_3b_1 \leq 1, a_1c_0c_2 \leq 1\} \\ BP_4 &= \{a_1a_3b_1 \leq 1, a_1c_0c_2 \leq 1, b_0b_1c_1 \leq 1\} \\ BP_5 &= \{a_1a_3b_1 \leq 1, a_1c_0c_2 \leq 1, b_2b_3c_2 \leq 1\} \end{aligned}$$

Calculation: the base polytope BP_1 has faces on $H_1 = \{a_0a_2b_2 = 1\}$ and $H_2 = \{b_2b_3c_2 = 1\}$. The equation $b_2b_3c_2 \leq 1$ is equivalent to $b_2c_0a_0a_1a_2c_2 \leq 2$ which is $b_0b_1c_1 \geq 1$. The only base polytope that fit BP_1 along H_2 is BP_2. H_1 is the same as $\{a_1a_3b_1 = 1\}$, there are two cases that can cover $\triangle^4_{Bur}$ and fit BP_1 along H_1. It is easy to check that $BP_3 = BP_4 \cup BP_5$ and there are two coverings $\{BP_1, BP_2, BP_3\}$ and $\{BP_1, BP_2, BP_{3,} BP_4\}$ containing BP_1. Those two base polytopes correspond to the degeneration Case1 and Case 2.

Type 3, type 4 does not produce new coverings.

Type 5 and 6 produces coverings corresponding to Case 4,5. In Case 5, the equation $a_1b_1b_2c_1c_2 \leq 2$ could be implies by $a_1b_1c_1 \leq 1$ and $b_2, c_2 \leq \frac{1}{2}$. So the $BP_1 = \{a_0b_0c_0 \leq 1\}$ is the same as $BP_2 = \{a_0b_0c_0 \leq 1, a_1b_1b_2c_1c_2 \leq 2\}$.

For $K^2 = 4$ nonnodal case, let P_1 be the intersection of A_1, B_1, C_1 and P_2 be the intersection of A_2, B_2, C_2. The hypersimplex $\triangle^4_{Bur}\,(nonnodal)$ is defined to be

$$\triangle^4_{Bur}\,(nonnodal) \quad = \quad \triangle^6_{Bur} \cap \{a_1b_1c_1 \leq 1, a_2b_2c_2 \leq 1\}\,.$$

We have the following table.

$K^2 = 4$ Nonodal case				
Type	BP_M	Case	Y_j	$4(K_Y\|_{Y_i} + D_i)^2$
0	$\triangle^4_{Bur}\,(nonnodal)$		Σ	4
1	$b_0b_1c_1 \leq 1$	6	$\mathbb{P}^1 \times \mathbb{P}^1$	2
2	$b_0b_1c_1 \leq 1, a_1c_1c_3 \leq 1$	7	$\mathbb{P}^2$	1
3	$a_1a_2b_1b_2c_1c_2 \leq 2$	8	$\mathbb{P}^1 \times \mathbb{P}^1$	2

Type 1 produces a covering corresponding to Case 6.

$$\begin{aligned} BP_1 &= \{b_0b_1c_1 \leq 1\} \\ BP_2 &= \{b_2b_3c_2 \leq 1\}\,. \end{aligned}$$

Type 2 produces a covering corresponding to Case 7.

$$\begin{aligned}
BP_1 &= \{b_1b_3c_1 \leq 1, a_0a_1b_1 \leq 1\} \\
BP_2 &= \{b_0b_2c_2 \leq 1, a_2c_2c_3 \leq 1\} \\
BP_3 &= \{a_2a_3b_2 \leq 1, b_1b_3c_1 \leq 1, a_2c_0c_1 \leq 1\} \\
BP_4 &= \{a_2a_3b_2 \leq 1, b_2b_3c_2 \leq 1\}
\end{aligned}$$

Type 3 produces a covering corresponding to Case 8.

We give the table of coverings that precisely correspond to the degenerations listed in Chapter 5.

#	Tilings for $K^2 = 4$	From $K^2 = 5$
1	$a_1c_0c_2 \leq 1, a_1a_3b_1 \leq 1$; $a_1a_2b_2 \leq 1, b_2b_3c_2 \leq 1$; $b_0b_1c_1 \leq 1, a_2c_1c_3 \leq 1$	Case 1
2	$b_0b_1c_1 \leq 1, a_1a_3b_1 \leq 1, a_1c_0c_2 \leq 1$; $a_1a_3b_1 \leq 1, a_1c_0c_2 \leq 1, b_2b_3c_2 \leq 1$;	Case 2
3	$b_0b_1c_1 \leq 1$; $b_2b_3c_2 \leq 1$	Case 4
4	$a_1a_2b_1b_2c_1c_2 \leq 2$; $a_0b_0c_0 \leq 1$	Case 6
5	$a_1a_2b_1b_2c_1c_2 \leq 2$ $a_0b_0c_0 \leq 1, a_1b_1b_2c_1c_2 \leq 2$	Case 6
6	$b_0b_1c_1 \leq 1$; $b_2b_3c_2 \leq 1$	Case 3,4
7	$b_0b_1c_1 \leq 1, a_1a_3b_1 \leq 1$; $a_0a_2b_2 \leq 1, b_2b_3c_2 \leq 1$; $a_0a_2b_2 \leq 1, a_2c_1c_3 \leq 1, b_0b_1c_1 \leq 1$; $a_1a_3b_1 \leq 1, a_1c_0c_2 \leq 1, b_2b_3c_2 \leq 1$;	Case 5
8	$a_1a_2b_1b_2c_1c_2 \leq 2$; $a_0b_0c_0 \leq 1$;	Case 6

Case 8 is a special case we need to discuss. The base polytope $BP = \{a_1a_2b_1b_2c_1c_2\} \cap$

$\triangle^4_{Bur}$ $(nonnodal)$ contains the whole hypersimplex $\triangle^4_{Bur}$ $(nonnodal)$, while the degeneration contains two components. This stays mysterious why those two methods give different results.

6.5 Matroid tilings for $K^2 = 5$ case.

let P_1 be the intersection of A_1, B_1, C_1. The hypersimplex $\triangle^5_{Bur}$ is

$$\triangle^5_{Bur} = \triangle^6_{Bur} \cap \{a_1 b_1 c_1 \leq 1\}.$$

We have the following table.

$K^2 = 5$				
Type	BP_M	Case	Y_j	$4(K_Y\|_{Y_i} + D_i)^2$
0	$\triangle^5_{Bur}$		Σ	5
1	$a_0 a_2 b_2 \leq 1$	4	$\mathbb{P}^1 \times \mathbb{P}^1$	2
2	$a_2 a_3 b_2 \leq 1, a_2 c_0 c_2 \leq 1$	1,2,3,5	$\mathbb{P}^2$	1
3	$a_1 a_3 b_1 \leq 1, b_0 b_1 c_1 \leq 1$	5	$\mathbb{P}^2$	
4	$a_1 a_2 b_1 b_2 c_1 \leq 2$	6	$\mathbb{P}^1 \times \mathbb{P}^1$	2
5	$a_1 a_2 b_1 b_2 c_1 c_2 \leq 1$	6	$\mathbb{P}^1 \times \mathbb{P}^1$	2

Type 1 produces a covering corresponding to Case 4.

$$BP_1 = \{a_0 a_2 b_2 \leq 1\}$$
$$BP_2 = \{a_1 a_3 b_1 \leq 1\}$$

There are several polytopes with the face $a_0 a_2 b_2 = 1$ corresponding to degenerations. However, only BP_2 fit BP_1 well.

Type 2 produces coverings corresponding to Cases 1, 2, 3 and 5.

$$\begin{aligned}
BP_1 &= \{a_2c_0c_2 \leq 1, a_2a_3b_2 \leq 1\} \\
BP_2 &= \{a_1c_1c_3 \leq 1, b_0b_1c_1 \leq 1\} \\
BP_3 &= \{a_0a_1b_1 \leq 1, b_1b_3c_2 \leq 1\} \\
BP_4 &= \{a_1c_1c_3 \leq 1, b_0b_1c_1 \leq 1, a_0a_1b_1 \leq 1\} \\
BP_5 &= \{a_1c_1c_3 \leq 1, b_0b_1c_1 \leq 1, a_2a_3b_2 \leq 1\} \\
BP_6 &= \{a_0a_1b_1 \leq 1, b_1b_3c_2 \leq 1, a_1c_1c_3 \leq 1\} \\
BP_7 &= \{a_0a_1b_1 \leq 1, b_1b_3c_2 \leq 1, a_2c_0c_2 \leq 1\}
\end{aligned}$$

We start with BP_1. The complement of $a_2c_0c_2 \leq 1$ is $a_1c_1c_3 \leq 1$ and the complement of $a_2a_3b_2 \leq 1$ is $a_0a_1b_1 \leq 1$. Base polytopes BP_2 and BP_5 fit BP_1 along $a_2c_0c_2 = 1$, while BP_3 and BP_7 fit BP_1 along $a_2a_3b_2 = 1$. We also know BP_2 is $BP_4 \cup BP_5$ and BP_3 is $BP_6 \cup BP_7$. The covering $\{BP_1, BP_2, BP_3\}$, $\{BP_1, BP_4, BP_5, BP_3\}$, $\{BP_1, BP_2, BP_6, BP_7\}$, $\{BP_1, BP_4, BP_5, BP_6, BP_7\}$ correspond to degenerations Case 1, Case 2, Case 5 and Case 3.

Type 3 does not produce any new coverings.

Both Type 4 and Type 5 produce the same covering corresponding to Case 6.

We give the table of coverings that precisely correspond to the degenerations listed in Chapter 3.

#	Tilings for $K^2 = 5$	From $K^2 = 6$
1	$a_0a_2b_2 \le 1, b_2b_3c_2 \le 1;\ a_1c_0c_2 \le 1, a_1a_3b_1 \le 1;$ $a_2c_1c_3 \le 1, b_0b_3c_1 \le 1$	Case 2
2	$a_0a_2b_2 \le 1, b_2b_3c_2 \le 1; a_2c_1c_3 \le 1, b_0b_3c_1 \le 1;$ $a_1a_3b_1 \le 1, a_1c_0c_2 \le 1, b_2b_3c_2 \le 1;$ $a_0a_1b_1 \le 1, a_1c_1c_3 \le 1, b_1b_3c_2 \le 1;$	Case 3
3	$a_0a_2b_2 \le 1, b_2b_3c_2 \le 1;$ $a_1a_3b_1 \le 1, a_1c_0c_2 \le 1, b_2b_3c_2 \le 1;$ $a_0a_1b_1 \le 1, a_1c_1c_3 \le 1, b_1b_3c_2 \le 1;$ $a_1c_1c_3 \le 1, b_0b_1c_1 \le 1, a_1a_3b_1 \le 1;$ $a_0a_2b_2 \le 1, a_2c_1c_3 \le 1, b_0b_1c_1 \le 1;$	Case 4
4	$a_0a_2b_0 \le 1;\ a_1a_3b_1 \le 1;$	Case 6
5	$a_1a_3b_1 \le 1, a_1c_0c_1 \le 1;\ a_0a_2b_2 \le 1, a_2c_2c_3 \le 1;$ $a_1c_2c_3 \le 1, b_0b_1c_2 \le 1, a_1a_3b_1 \le 1;$ $a_1a_2b_2 \le 1, b_2b_3c_1 \le 1, a_0c_0c_1 \le 1;$	Case 7
6	$a_1a_2b_1b_2c_1c_2 \le 2;\ a_0b_0c_0 \le 1;$	Cases 9,10

Section 5.3 shows that degenerations listed in Section 2,3,4 are all the degenerations in our compactified moduli spaces of stable Burniat surfaces with $3 \le K^2 \le 5$.

6.6 Programming by Polymake

This section is based on the introduction of polymake on the official website www.polymake.org. Polymake is a tool one could use to study convex polytopes and polyhedra. In this paper, we use the program polymake to check the matroid tilings of the hypersimplex $\triangle^d_{Bur}$, $d = 3, 4, 5$. In this section,we will give the details of how to check that $\{BP_1, BP_2\}$ is a tiling of $\triangle^3_{Bur}$

in $\mathbb{R}^9$, where

$$\begin{aligned}
\triangle^3_{Bur} &= \{(a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2) \in \mathbb{R}^9 \text{ satisfying} \\
&\quad 0 \leq a_i, b_i, c_i \leq \frac{1}{2}, i = 0, 1, 2, 3; \\
&\quad \sum_{i=0}^{2} (a_i + b_i + c_i) = 3; \\
&\quad 0 \leq a_3 = c_0 + c_1 + c_2 + b_0 - 1 \leq 1/2; \\
&\quad 0 \leq b_3 = a_0 + a_1 + a_2 + c_0 - 1 \leq 1/2; \\
&\quad 0 \leq c_3 = b_0 + b_1 + b_2 + a_0 - 1 \leq 1/2; \\
&\quad a_1 + b_1 + c_2 \leq 1, a_1 + b_2 + c_1 \leq 1, a_2 + b_1 + c_1 \leq 1\}.
\end{aligned}$$

and

$$\begin{aligned}
BP_1 &= \{a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \leq 2\} \cap \triangle^3_{Bur} \\
BP_2 &= \{a_0 + b_0 + c_0 \leq 1\} \cap \triangle^3_{Bur}
\end{aligned}$$

An inequaliy $k_0 + k_1 x_1 + ... + k_d x_d \geq 0$ is encoded as $[k_0, ..., k_d]$ in polymake. We consider $a_0, a_1, a_2, b_0, ..., c_2$ as variables $x_1, ..., x_9$ and then all inequalities for $\triangle^3_{Bur}$ could be encoded in polymake. For the equality $\sum_{i=0}^{2}(a_i + b_i + c_i) = 3$, we use two inequalities $\sum_{i=0}^{2}(a_i + b_i + c_i) \leq 3$ and $\sum_{i=0}^{2}(a_i + b_i + c_i) \geq 3$ to describe it. We define the polytope $\triangle^3_{Bur}$ and name it as Delta_B3 in polymake as follows.

```
"polytope Delta_B3";
$inequalities=new Matrix<Rational>([
[0, 1, 0, 0, 0, 0, 0, 0, 0, 0],
[0, 0, 1, 0, 0, 0, 0, 0, 0, 0],
```

[0, 0, 0, 1, 0, 0, 0, 0, 0, 0],

[0, 0, 0, 0, 1, 0, 0, 0, 0, 0],

[0, 0, 0, 0, 0, 1, 0, 0, 0, 0],

[0, 0, 0, 0, 0, 0, 1, 0, 0, 0],

[0, 0, 0, 0, 0, 0, 0, 1, 0, 0],

[0, 0, 0, 0, 0, 0, 0, 0, 1, 0],

[0, 0, 0, 0, 0, 0, 0, 0, 0, 1],

[1/2, -1, 0, 0, 0, 0, 0, 0, 0, 0],

[1/2, 0, -1, 0, 0, 0, 0, 0, 0, 0],

[1/2, 0, 0, -1, 0, 0, 0, 0, 0, 0],

[1/2, 0, 0, 0, -1, 0, 0, 0, 0, 0],

[1/2, 0, 0, 0, 0, -1, 0, 0, 0, 0],

[1/2, 0, 0, 0, 0, 0, -1, 0, 0, 0],

[1/2, 0, 0, 0, 0, 0, 0, -1, 0, 0],

[1/2, 0, 0, 0, 0, 0, 0, 0, -1, 0],

[1/2, 0, 0, 0, 0, 0, 0, 0, 0, -1],

[3, -1, -1, -1, -1, -1, -1, -1, -1, -1],

[-3, 1, 1, 1, 1, 1, 1, 1, 1, 1],

[-1, 0, 0, 0, 1, 0, 0, 1, 1, 1],

[3/2, 0, 0, 0, -1, 0, 0, -1, -1, -1],

[-1, 1, 1, 1, 0, 0, 0, 1, 0, 0],

[3/2, -1, -1, -1, 0, 0, 0, -1, 0, 0],

[-1, 1, 0, 0, 1, 1, 1, 0, 0, 0],

[3/2, -1, 0, 0, -1, -1, -1, 0, 0, 0],

[1, 0, -1, 0, 0, -1, 0, 0, 0, -1],

[1, 0, -1, 0, 0, 0, -1, 0, -1, 0],

```
[1, 0, 0, -1, 0, -1, 0, 0, -1, 0]]);
$Delta_B3 = new Polytope <Rational> (INEQUALITIES => $inequalities);
print_constraints($inequalities);
```

Comparing to $\triangle^3_{Bur}$, we only need to define one more equation for BP_1 and BP_2 in polymake. We add [2,0,-1,-1,0,-1,-1,0,-1,-1] for BP_1 and [1,-1,0,0,-1,0,0,-1,0,0] for BP_2, which correspond to equations $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \leq 2$ and $a_0 + b_0 + c_0 \leq 1$. To check that polytopes BP_1, BP_2 cover $\triangle^3_{Bur}$, it is enough to look at the intersection of $\triangle^3_{Bur}$ and $(\mathrm{int}\,(BP_1 \cup BP_2))^c$, which can be described as some inequalities as well. It is obvious that $\triangle^3_{Bur} = BP_1 \cup BP_2$. Now we want to check BP_1 and BP_2 fit well. Since BP_1 intersect BP_2 along $H = \{a_0 + b_0 + c_0 = 1\}$, polytopes BP_1, BP_2 fit well if the faces $BP_1 \cap H$ and $BP_2 \cap H$ are exactly the same. We change our problem to checking whether $BP_1 \cap H$ and $BP_2 \cap H$ have the same vertices. The face $BP_1 \cap H$ is $\{a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 2\} \cap \triangle^3_{Bur}$ and $BP_2 \cap H$ is $\{a_0 + b_0 + c_0 \leq 1\} \cap \triangle^3_{Bur}$. The following is the program for checking the vertices of $BP_1 \cap H$ and $BP_2 \cap H$ which are completely the same.

```
$inequalities=new Matrix<Rational>([
[2, 0, -1, -1, 0, -1, -1, 0, -1, -1],
[-2, 0, 1, 1, 0, 1, 1, 0, 1, 1],
[0, 1, 0, 0, 0, 0, 0, 0, 0, 0],
[0, 0, 1, 0, 0, 0, 0, 0, 0, 0],
[0, 0, 0, 1, 0, 0, 0, 0, 0, 0],
[0, 0, 0, 0, 1, 0, 0, 0, 0, 0],
[0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
[0, 0, 0, 0, 0, 0, 1, 0, 0, 0],
[0, 0, 0, 0, 0, 0, 0, 1, 0, 0],
```

```
[0, 0, 0, 0, 0, 0, 0, 0, 1, 0],
[0, 0, 0, 0, 0, 0, 0, 0, 0, 1],
[1/2, -1, 0, 0, 0, 0, 0, 0, 0, 0],
[1/2, 0, -1, 0, 0, 0, 0, 0, 0, 0],
[1/2, 0, 0, -1, 0, 0, 0, 0, 0, 0],
[1/2, 0, 0, 0, -1, 0, 0, 0, 0, 0],
[1/2, 0, 0, 0, 0, -1, 0, 0, 0, 0],
[1/2, 0, 0, 0, 0, 0, -1, 0, 0, 0],
[1/2, 0, 0, 0, 0, 0, 0, -1, 0, 0],
[1/2, 0, 0, 0, 0, 0, 0, 0, -1, 0],
[1/2, 0, 0, 0, 0, 0, 0, 0, 0, -1],
[3, -1, -1, -1, -1, -1, -1, -1, -1, -1],
[-3, 1, 1, 1, 1, 1, 1, 1, 1, 1],
[-1, 0, 0, 0, 1, 0, 0, 1, 1, 1],
[3/2, 0, 0, 0, -1, 0, 0, -1, -1, -1],
[-1, 1, 1, 1, 0, 0, 0, 1, 0, 0],
[3/2, -1, -1, -1, 0, 0, 0, -1, 0, 0],
[-1, 1, 0, 0, 1, 1, 1, 0, 0, 0],
[3/2, -1, 0, 0, -1, -1, -1, 0, 0, 0],
[1, 0, -1, 0, 0, -1, 0, 0, 0, 1],
[1, 0, -1, 0, 0, 0, -1, 0, -1, 0],
[1, 0, 0, -1, 0, -1, 0, 0, -1, 0]]);
$BP_1H=new Polytope<Rational>(INEQUALITIES=>$inequalities);
print_constraints($inequalities);
$inequalities=new Matrix<Rational>([
[1, -1, 0, 0, -1, 0, 0, -1, 0, 0],
```

[-1, 1, 0, 0, 1, 0, 0, 1, 0, 0],

[0, 1, 0, 0, 0, 0, 0, 0, 0, 0],

[0, 0, 1, 0, 0, 0, 0, 0, 0, 0],

[0, 0, 0, 1, 0, 0, 0, 0, 0, 0],

[0, 0, 0, 0, 1, 0, 0, 0, 0, 0],

[0, 0, 0, 0, 0, 1, 0, 0, 0, 0],

[0, 0, 0, 0, 0, 0, 1, 0, 0, 0],

[0, 0, 0, 0, 0, 0, 0, 1, 0, 0],

[0, 0, 0, 0, 0, 0, 0, 0, 1, 0],

[0, 0, 0, 0, 0, 0, 0, 0, 0, 1],

[1/2, -1, 0, 0, 0, 0, 0, 0, 0, 0],

[1/2, 0, -1, 0, 0, 0, 0, 0, 0, 0],

[1/2, 0, 0, -1, 0, 0, 0, 0, 0, 0],

[1/2, 0, 0, 0, -1, 0, 0, 0, 0, 0],

[1/2, 0, 0, 0, 0, -1, 0, 0, 0, 0],

[1/2, 0, 0, 0, 0, 0, -1, 0, 0, 0],

[1/2, 0, 0, 0, 0, 0, 0, -1, 0, 0],

[1/2, 0, 0, 0, 0, 0, 0, 0, -1, 0],

[1/2, 0, 0, 0, 0, 0, 0, 0, 0, -1],

[3, -1, -1, -1, -1, -1, -1, -1, -1, -1],

[-3, 1, 1, 1, 1, 1, 1, 1, 1, 1],

[-1, 0, 0, 0, 1, 0, 0, 1, 1, 1],

[3/2, 0, 0, 0, -1, 0, 0, -1, -1, -1],

[-1, 1, 1, 1, 0, 0, 0, 1, 0, 0],

[3/2, -1, -1, -1, 0, 0, 0, -1, 0, 0],

[-1, 1, 0, 0, 1, 1, 1, 0, 0, 0],

```
[3/2, -1, 0, 0, -1, -1, -1, 0, 0, 0],
[1, 0, -1, 0, 0, -1, 0, 0, 0, 1],
[1, 0, -1, 0, 0, 0, -1, 0, -1, 0],
[1, 0, 0, -1, 0, -1, 0, 0, -1, 0]]);
$BP_2H=new Polytope<Rational>(INEQUALITIES=>$inequalities);
print_constraints($inequalities);
print $BP_1H->VERTICES;
print $BP_2H->VERTICES;
```

Output:

```
polytope > print $BP_1H->VERTICES;
1 0 1/2 1/2 1/2 1/2 1/2 1/2 0 0
1 1/2 1/2 1/2 1/2 1/2 0 0 0 1/2
1 1/2 0 1/2 0 1/2 1/2 1/2 0 1/2
1 1/2 1/2 1/2 1/2 1/4 1/4 0 1/4 1/4
1 1/2 1/4 1/4 0 1/2 1/2 1/2 1/4 1/4
1 1/2 1/2 1/2 1/2 0 1/2 0 0 1/2
1 1/2 1/2 0 0 1/2 1/2 1/2 0 1/2
1 1/2 1/2 1/4 1/4 1/2 1/4 1/4 1/4 1/4
1 1/2 1/4 1/2 1/4 1/4 1/2 1/4 1/4 1/4
1 0 1/4 1/2 1/2 1/4 1/2 1/2 1/4 1/4
1 0 1/2 1/4 1/2 1/2 1/4 1/2 1/4 1/4
1 1/2 1/2 1/2 1/2 0 0 0 1/2 1/2
1 1/2 1/2 0 0 1/2 0 1/2 1/2 1/2
1 1/2 0 1/2 0 0 1/2 1/2 1/2 1/2
1 0 1/2 1/2 1/2 1/2 0 1/2 0 1/2
```

1 0 1/2 1/2 1/2 1/4 1/4 1/2 1/4 1/4

1 0 1/2 1/2 1/2 0 1/2 1/2 0 1/2

1 1/2 1/2 0 1/2 1/2 0 0 1/2 1/2

1 1/2 0 1/2 1/2 0 1/2 0 1/2 1/2

1 1/2 0 0 0 1/2 1/2 1/2 1/2 1/2

1 0 0 1/2 1/2 1/2 1/2 1/2 0 1/2

1 0 1/4 1/4 1/2 1/2 1/2 1/2 1/4 1/4

1 0 1/2 0 1/2 1/2 1/2 1/2 0 1/2

polytope > print $BP_2H->VERTICES;

1 0 1/2 1/2 1/2 1/2 1/2 1/2 0 0

1 1/2 1/2 1/2 1/2 1/2 0 0 0 1/2

1 1/2 0 1/2 0 1/2 1/2 1/2 0 1/2

1 1/2 1/2 1/2 1/2 1/4 1/4 0 1/4 1/4

1 1/2 1/4 1/4 0 1/2 1/2 1/2 1/4 1/4

1 1/2 1/2 1/2 1/2 0 1/2 0 0 1/2

1 1/2 1/2 0 0 1/2 1/2 1/2 0 1/2

1 1/2 1/2 1/4 1/4 1/2 1/4 1/4 1/4 1/4

1 1/2 1/4 1/2 1/4 1/4 1/2 1/4 1/4 1/4

1 0 1/4 1/2 1/2 1/4 1/2 1/2 1/4 1/4

1 0 1/2 1/4 1/2 1/2 1/4 1/2 1/4 1/4

1 1/2 1/2 1/2 1/2 0 0 0 1/2 1/2

1 1/2 1/2 0 0 1/2 0 1/2 1/2 1/2

1 1/2 0 1/2 0 0 1/2 1/2 1/2 1/2

1 0 1/2 1/2 1/2 1/2 0 1/2 0 1/2

1 0 1/2 1/2 1/2 1/4 1/4 1/2 1/4 1/4

1 0 1/2 1/2 1/2 0 1/2 1/2 0 1/2

1 1/2 1/2 0 1/2 1/2 0 0 1/2 1/2

1 1/2 0 1/2 1/2 0 1/2 0 1/2 1/2

1 1/2 0 0 0 1/2 1/2 1/2 1/2 1/2

1 0 0 1/2 1/2 1/2 1/2 1/2 0 1/2

1 0 1/4 1/4 1/2 1/2 1/2 1/2 1/4 1/4

1 0 1/2 0 1/2 1/2 1/2 1/2 0 1/2

Bibliography

[Ale94] V. Alexeev, *Boundedness and K^2 for log surfaces,* Internat. J. Math. **5** (1994), no. 6., 91-116.

[Ale96a] V. Alexeev, *Log canonical singularities and complete moduli of stable pairs,* arXiv:alg-geom/9608013 (1996).

[Ale96b] V. Alexeev, *Moduli spaces $M_{g,n}(W)$ for surfaces*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1-22.

[Ale08] V. Alexeev, *Weighted grassmannians and stable hyperplane arrangements,* arXiv:0806.0881.

[AP09] V. Alexeev and R. Pardini, *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces*, arXiv:0901.4431 [math.AG] (2009).

[AP12] V. Alexeev and R. Pardini, *Non-normal abelian covers,* Compositio Mathematica **148** (2012), no. 4, 1051-1084.

[Ale13] V. Alexeev, Moduli of weighted hyperplane arrangements, with applications, Notes for the May 2013 Barcelona Advanced course "Compactifying Moduli spaces".

[BC09b] I. Bauer, F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces,* European Mathematical Society (EMS). EMS Series of Congress Reports, (2011), 49-76.

[BC10a] I. Bauer, F. Catanese, *Burniat surfaces II:secondary Burniat surfaces form three connected components of the moduli space*, Invent. Math. **180** (2010), 559-588.

[BC10b] I. Bauer, F. Catanese, *Burniat surfaces III: deformations of automorphisms and extended Burniat surfaces,* arXiv:1012.3770.

[BHPV] Barth, Hulek, Peters, Van De Ven, Compact Complex Surfaces, 2ed.

[Bu66] P. Burniat, *Sur les surfaces de genre* $P_{12} > 1$, Ann. Math. Pura Appl. **71** (1966), 1-24.

[Ja92] James Oxley, *Matroid Theory,* Oxford: Oxford University Press (1992).

[Ka07] M.Kawakita, *Inversion of adjunction on log canonicity,* Invent. Math. **167** (2007), 129-133.

[KSB88] J.Kollár and N.I. Shepherd-Barron, *Threefolds and deformations of surface singularities,* Invent. Math **91** (1998), no. 2, 299-338.

[KM98] János Kollár and Shigefumi Mori, Birational Geometry of Algebraic Varieties, Cambridge University Press(1998), ISBN 0-521-63277-3.

[LP01] M.M. Lopes, R. Pardini, *A connected component of the moduli space of surfaces with* $p_g = 0$, Topology **40** (2001), no. 5, 977-991.

[Par91] Rita Pardini, *Abelian covers of algebraic varieties,* J. Reine Angew. Math. **417** (1991), 191-213.

[Pet77] Peters, C.A.M. *On certain examples of surfaces with $p_g = 0$ due to Burniat*, Nagoya Math. J. **66** (1977), 109-119.