CONSTRUCTION OF ALGEBRAIC REASONING AND MATHEMATICAL CARING RELATIONS

by

AMY J. HACKENBERG

(Under the Direction of Leslie P. Steffe)

ABSTRACT

The purpose of this study was to understand how sixth graders construct algebraic reasoning in interaction with a teacher-researcher who endeavors to enact mathematical caring relations (MCR) with them. Reasoning quantitatively—the purposeful functioning of schemes in quantitative contexts in order to model relationships between known and unknown quantities and determine unknowns—was taken as a foundation for students' work toward constructions and solutions of basic linear equations (of the form $ax = b$). From a teacher's perspective, MCR was formulated as an orientation to balance stimulation and depletion, or increases and decreases in levels of energy and feelings of well-being, in student-teacher interactions aimed toward mathematical learning. From a student's perspective, participating in MCR was articulated as being open to the teacher's interventions in one's mathematical activity and pursuing one's questions of interest.

As teacher-researcher I taught two pairs of sixth graders at a rural middle school in Georgia in a constructivist teaching experiment from October 2003 to May 2004. All teaching episodes were videotaped. Teaching practices included posing problems that involved multiplicative relationships between fractional quantities, adapting problem situations to harmonize with and challenge students' ways of operating, and tracking students' affective responses to and engagement with this interactive activity. In a retrospective analysis of videotapes, I constructed second-order models that accounted for changes students made in their mathematical ways of operating and in their affective responses to mathematical interaction.

MCR was beneficial in facilitating students' engagement in mathematical activity and influential on the construction of the students', as well as the teacher's, self-concepts in relation to doing and communicating about mathematics. Students' multiplicative structures were significant resources in their construction of schemes for making improper fractions and for solving problems that could be solved with basic linear equations. The students had difficulty constructing fractions as multiplicative operations on quantities. These results have implications for how teachers and researchers conceive of student-teacher mathematical interaction and of the constitution of algebra learning for middle school students.

INDEX WORDS: Affect, Algebra, Algebraic Reasoning, Caring Relations, Care Theory, Engagement, Fraction Scheme, Interaction, JavaBars, Linear Equations, Mathematics Learning, Multiplicative Reasoning, Operations, Quantity, Quantitative Reasoning, Radical Constructivism, Scheme Theory, Self-concept, Teaching Experiment, Units-Coordination

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DEDICATION

I dedicate this work the many fabulous teachers I have been fortunate to learn from throughout my life, including my very first teachers, my parents.

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Although ultimately the writing is done alone, many people other than the author contribute to the emergence of a dissertation. The reason my journey through this process has been so satisfying is undoubtedly due to all who have assisted me along the way. I begin by thanking the four students, Carlos, Michael, Bridget, and Deborah, who agreed to participate in the teaching experiment. I feel confident that I learned more from them than they did from me. I also express deep gratitude to the sixth-grade teachers, school personnel, and middle school administrators who were so generous in allowing this research to take place at their school. Inevitably, schedules were disrupted, but the teachers were uncommonly supportive of the students' participation, and the administrators unfailingly facilitated my work with the students.

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CHAPTER 1: INTRODUCTION

The Story of the Study

> How can it be described? How can any of it be described? The trip and the story of the trip are always two different things….One cannot go to a place and speak of it; one cannot both see and say, not really. One can go, and upon returning make a lot of hand motions and indications with the arms….All that unsayable life! That's where the narrator comes in… (Moore, 1998, p. 237).

On the one hand, this study is a story of four sixth-grade students' mathematical learning. This story tracks the evolution of their quantitative reasoning as a basis for their construction of algebraic reasoning. Specifically, it tracks the evolution of their multiplicative reasoning with fractional quantities as a basis for their solutions of problems that can be solved with basic linear equations (of the form $ax = b$). From this point of view, the story joins and extends previous stories of students' mathematical ways of operating in constructivist teaching experiments.

On the other hand, this study is a story of how four sixth-grade students and their teacher (the researcher) participated in mathematical interaction with certain characteristics. This story tracks my evolving understanding of the students' mathematical ways of operating and their affective states during our interactions, my efforts to pose situations that might allow them to expand their ways of operating or experience balance in their affective states, and the students' receptivity to these efforts. From this point of view, the story focuses on my learning to act in particular ways as a teacher and on the students' cognitive and affective responses to our interactive mathematical activity.

Neither view tells the entire story. Both are needed, because, above all, this study is a story of interaction between mathematical learning and mathematical caring relations, between cognition and affect in the process of learning and caring, and between teacher and student, each of whom are learning from and orchestrating learning for the other.

Problem Statement

In the United States, algebra is considered important to learn because it is viewed as a web of knowledge and skills that yield intellectual power, and as a social and political institution (Kaput, 1998) in which success is a "gateway" for full economic, political, and cultural participation in society (Moses & Cobb Jr., 2001). Both views stem from and contribute to increasingly common practices of mandating "algebra for all," moving algebra courses into middle schools, and developing algebraic reasoning across all grades (Blume & Heckman, 2000; National Council of Teachers of Mathematics [NCTM], 2000a, 2000b). However, many students continue to find algebra incomprehensible and have difficulty passing algebra courses (NCTM, 2000a; RAND Mathematics Study Panel, 2002). One reason for such failures in learning is that school algebra courses are often based on conventional algebraic conceptions, not on students' mathematical ways of operating that can be brought forth and considered as algebraic.

Participating in courses where *students'* algebraic reasoning is not nurtured may have ramifications for students beyond "simply" not learning algebra. Participating in such courses can contribute to students' conceptions of themselves as nonmathematical—i.e., to students' constructions of weak self-concepts as doers of mathematics (Larochelle, 2000; Steffe, 2000). This problem is of considerable importance in mathematics education, since

> …the rejection of one's sense of oneself as an agent of mathematical or scientific action or interaction may create a disturbance in the self as a center of subjective awareness that can handicap further learning in mathematics or science. In fact, it can lead easily to a rejection of those parts of one's experience (Steffe, 2000, p. 94).

Many researchers in mathematics education have investigated students' affective responses to mathematical activity (e.g., Hoyles, 1982; McLeod, 1992; Thompson & Thompson, 1989) and beliefs about mathematics (e.g., Kloosterman, 1996; Kouba & McDonald, 1991; Schoenfeld, 1989). But those in mathematics education who have focused on the construction of self with respect to learning and doing mathematics have often done so in relation to gender (e.g., Mendick, 2003; Turkle & Papert, 1992; Walshaw, 2001) or race (e.g., Martin, 2000; Stinson, 2004). Some researchers (e.g., Bartholomew, 2002; Boaler, 2000) have conducted studies about how students construct their identities as mathematical knowers with respect to participation in school mathematics classrooms, but few have researched the construction of self in relation to the production of mathematical knowledge as Larochelle and Desautels have done with scientific knowledge (e.g., Larochelle, 2000; Larochelle & Desautels, 1991).

Students who construct their self-concepts as nonmathematical in the process of their mathematical education may believe that their mathematics teachers do not care about them—at least in the sense that their teachers don't seem to help them connect mathematical activity to their lives. Simultaneously, teachers may view these students as not caring about mathematics. So participating in courses where students' algebraic reasoning is not nurtured is one reason that students and mathematics teachers may fail to develop caring relations (Noddings, 2002). In caring relations, a teacher (a carer) engenders experiences in which a student (a cared-for) feels satisfaction of her or his need to be cared for. That is, the teacher poses situations that harmonize with the student's ways of thinking and that challenge the student to extend those ways of thinking, and the student responds in a way that indicates reception of the teacher's care—often by continuing or renewing activity in these situations or variations of them (Noddings, 1984, 2001, 2002). The focus on *relations* means that caring is not solely a virtue of a teacher or student but a characteristic of interaction in which both teacher and student participate.

Initiating *mathematical* caring relations involves a teacher in understanding and bringing forth students' reasoning (rather than requiring that students learn only conventional mathematical conceptions), assessing how students experience provocations in learning mathematics, and formulating situations that may allow students to make changes in their ways of reasoning and affective responses to mathematical activity. From a student's perspective, participating in mathematical caring relations involves being open to the teacher's interventions in the student's mathematical activity and pursuing questions and ideas of interest. Sustained involvement in mathematical caring relations with a teacher may open possibilities for learning and may impact how students construct themselves in relationship to mathematical knowledge. That is, participating in mathematical caring relations may influence students' cognitive growth as well as their self-concepts as learners and doers of mathematics. So concerted efforts to enact mathematical caring relations with students who are learning to reason algebraically (prior to or during formal instruction in algebra) may improve the experience of students in algebra courses.

Researchers on algebra learning have worked to improve the experience of students in algebra courses by approaching algebraic activity as generalizing patterns of whole number arithmetic (e.g., Brizuela & Schliemann, 2004; Falkner, Levi, & Carpenter, 1999; Peck & Jencks, 1988; Pirie & Martin, 1997), problem solving (e.g., Bednarz & Janvier, 1996; Dettori, Garuti, & Lemut, 2001), transitioning from enacting processes to conceiving of structural relationships among mathematical objects (e.g., Kieran, 1992; Lins, 2002; Sfard & Linchevski, 1994), and coordinating representations (e.g., Kaput, 1989; Moschovich, Schoenfeld, & Arcavi, 1993). Other researchers have advocated functional approaches to algebra (e.g., Chazan, 2000;

Heid, 1996; Kieran, Boileau, & Garancon, 1996) and endeavored to understand the nature of students' difficulties in operating on unknowns (e.g., Carraher, Schliemann, & Brizuela, 2001; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994) and in formulating and using equations (e.g., Clement, 1982; MacGregor & Stacey, 1993; Swafford & Langrall, 2000).

Little research on algebra learning has been done from an ontogenetic point of view, in which researchers attempt to understand how students' mathematical ways of operating grow from modifications and reorganizations that they make in their previous ways of operating during mathematical interaction. Such an approach provides a basis for developing caring relations because of the sustained efforts to connect with "where students are" mathematically: to understand and base activity on *students'* ways of operating. However, even research that takes an ontogenetic approach (e.g., Steffe, 2003; Tzur, 1995) does not explicitly analyze mathematical caring relations to understand how they may influence mathematical learning (though not necessarily be sufficient to explain it).

An ontogenetic approach to algebra learning is based on fundamental tendencies for students to engage in counting and measuring activity (Thompson & Saldanha, 2003), which can grow into reasoning about relationships among quantities (cf. Lobato & Siebert, 2002; Thompson, 1995). In turn, quantitative reasoning frequently requires multiplicative reasoning, because relationships among quantities are often multiplicative (e.g., the height of one person is seven-eighths of the height of another person). Quantitative reasoning is a rich basis for algebra, because algebra involves reasoning about known and unknown quantities and analyzing the relationships between them (cf. Bednarz & Janvier, 1996; Chazan, 2000; Dossey, 1998). However, unlike an ontogenetic approach, previous research on algebra learning based on quantitative relationships does not take algebraic reasoning as the conceptual structures that students construct as they operate on knowns and unknowns and the relationships between them.

Research Questions

The purpose of this study was to understand how sixth graders reason quantitatively as a basis for beginning to reason algebraically in interaction with a teacher who initiates and endeavors to maintain mathematical caring relations with them. The two central research questions for the study were:

1) In what ways can a teacher and students establish mathematical caring relations during extended interaction aimed toward engendering algebraic reasoning?
2) What does it mean to establish mathematical caring relations with regard to the evolution of students' orientation to and engagement in quantitative and algebraic activity in the context of a teaching experiment?

Because mathematical caring relations require attention to mathematical learning, three more questions were asked to support the first two:

3) What is the nature of sixth graders' multiplicative structures, given that they are reasoning multiplicatively with whole numbers?
4) How can a teacher interact with these students so that they generate multiplicative reasoning with fractional quantities?
5) How can a teacher interact with these students to bring forth their construction of linear equations and their solutions?

Rationale

In the problem statement I have already identified three central reasons for the study. First, students struggle with learning algebra, so research is needed to understand what it means to learn algebra—indeed, to understand what constitutes algebraic learning for students of middle and high school ages. Second, whether or not they struggle with algebra, students often construct themselves in relation to mathematical knowledge in ways that curtail their continued mathematical learning, so research is needed to explore what kinds of mathematical experiences promote students' construction of themselves as able doers of mathematics—indeed, as makers of mathematical knowledge. Third, sometimes as a result of struggling to learn algebra (and to teach it), students and their mathematics teachers may fail to develop mathematical caring relations, which in turn can close down further learning for both teachers and students. So research is needed to understand what constitutes caring relations between mathematics teachers and their students as a way to improve the experiences of both teachers and students in interaction aimed toward learning algebra. In this section I discuss a fourth, fifth, and sixth reason for the study: to contribute to improving the teaching of algebra, to extend the research program of Steffe and colleagues, and to respond to calls for models of ethics and social interaction compatible with constructivism (von Glasersfeld, 2000).

Improving the Teaching of Algebra: Reason 4

In the last five years there have been several national calls to improve the teaching of algebra in order to address the difficulties students have with understanding algebra and passing algebra courses (e.g., National Commission on Mathematics and Science Teaching for the 21st Century, 2000; RAND Mathematics Study Panel, 2002). To improve the teaching of algebra requires a better understanding of how students learn to reason algebraically. It also requires forging connections for teachers, students, and researchers between students' ways of operating and what teachers and researchers regard as conventional algebraic conceptions. An ontogenetic analysis of students' algebraic reasoning is a powerful approach to understanding how students learn algebra and to viewing algebraic conceptions as growing out of students' mathematical activity. This research can help algebra teachers and researchers reconsider what might constitute middle school algebra courses so that students' experiences in such courses are improved.

So a fourth reason for my study is to add to the knowledge base for middle and high school teachers who are teaching algebra. My research contributes ways of thinking about students' quantitative and algebraic activity that teachers may find useful in conceptualizing their students' current ways of operating and their vision for their students' progress. My research may also contribute to how teachers think about student-teacher interaction in this process so that students' experiences of learning algebra may improve. Included in this reason is a directly personal aspect. As a former middle and high school mathematics teacher, I wanted an opportunity to construct knowledge about students' ways of operating and their learning in a far more detailed and coherent manner than I engaged in while teaching full time. Having done so helps me build a foundation for working with other mathematics teachers in the future.

Extending a Research Program: Reason 5

The reasons for this study are connected to the work of both classroom teachers and researchers. As I noted in describing "the story of the study," this research on understanding students' quantitative and algebraic reasoning in the context of caring relations is an extension and elaboration of research done to understand students' mathematical ways of operating in constructivist teaching experiments—specifically, of the research program of Steffe and colleagues. Their ontogenetic analyses of elementary school children's counting schemes and number sequences (Steffe, 1994; Steffe, von Glasersfeld, Richards, & Cobb, 1983), conceptions of multiplication and division (Steffe, 1988, 1992), and reorganization of whole number knowledge into fractional knowledge (Olive & Steffe, 2002; Steffe, 2002a, 2003) form a strong conceptual basis for my study. My study extends their research program into work with older (middle school) students in a "new" area (algebraic reasoning). My study also elaborates on their research by focusing explicitly on the social interaction between a student and a teacher-researcher in a constructivist teaching experiment as recommended by Cobb (2000), and by conjoining the study of cognitive and affective aspects of learning in these experiments as recommended by Simon (2000) and begun by Tzur (1995).

Considering how sixth-grade students' algebraic reasoning emerges from their current mathematical ways of operating requires understanding how they operate in realms Steffe and colleagues have studied—in particular how they operate multiplicatively with whole numbers and how they construct fractions. Multiplicative reasoning lays a basis for the construction of fractions; students who do not reason multiplicatively do not produce fractions in a conventional mathematical sense (Steffe, 2002a; Thompson & Saldanha, 2003). Working with fractions *as quantities* has been particularly fruitful in students' construction of fraction schemes (Steffe, 2002a, 2003; Thompson & Saldanha, 2003). In turn, these schemes can form a basis for reasoning algebraically (Steffe, 2001). Thus my approach to bringing forth students' algebraic reasoning out of their multiplicative reasoning in fractional contexts and with fractions as quantities has a well-researched foundation. Embedding this work in the context of mathematical caring relations explicates an area that is underdeveloped in the previous ontogenetic analyses of Steffe and colleagues; though mathematical caring relations are not sufficient to explain mathematical learning, they may contribute to understanding and sustaining it.

Responding to Calls for Models of Ethics and Social Interaction: Reason 6

The theoretical framework for the research of Steffe and colleagues and for my study is based significantly on von Glaserfeld's (1985; 1995) theory of knowing, radical constructivism.

This theory has been criticized for ignoring or inadequately accounting for the role of social interaction in mathematical learning (e.g., Cobb, 2000; Lerman, 1996; Phillips, 1995). Other scholars have urged stronger articulation of an ethical stance in connection to the theory (e.g., Auerswald, 1995; Lewin, 2000). For example, Lewin posits that radical constructivism already has an implicit ethics, one that values the autonomous learner and the spontaneity of learning and in fact "coincides completely with the hegemonic values of our time" (p. 48). He believes this ethics must be made explicit. In response, von Glasersfeld (2000) has called for the development of theoretical models of social interaction and ethics that are compatible with radical constructivism.

I agree with Lewin's plea for explicitness, and I believe the models von Glasersfeld calls for must not be independent: Social interaction in some sense posits an ethics, and ethical questions involve decisions about how to act in interaction with other people. While this study in no way develops such models, I believe caring relations can contribute to the development of them in both areas. The focus on caring relations grows out of an ethics of care (Gilligan, 1982; Noddings, 1984, 2002), in which moral commitments revolve around maintaining and enhancing relationships between people rather than primarily preserving individuals' rights within a community. In describing one way that caring relations are enacted in extended mathematical interaction between students and their teacher, my study provides a basis for theorizing about the connections between caring and learning, and about the ethical commitments that are intertwined with coming to know. Thus my study is my initial contribution to von Glasersfeld's call.

CHAPTER 2: A MODEL OF MATHEMATICAL LEARNING AND CARING RELATIONS[1]

> Every moment of our life is relationship….At this moment my relationship is to the rug, to the room, to my own body, to the sound of my voice. There is nothing except my being in relationship at each second (Beck, 1989, p. 77).

Basic Orientations

Interaction and Learning

A basic theoretical orientation of this study is that interaction is necessary for learning, if learning is taken broadly as the construction of ways of operating. As Steffe (1996) has said, noninteractive construction "has no meaning" (p. 85). In fact, interaction is unavoidable—by virtue of being alive, people are constantly in interaction with their environment, commonly thought of as other living creatures such as people, plants, and dogs, as well as nonliving entities such as chairs, apples, and music.[2] People's thoughts and ideas are also constantly in interaction. Together, these two types of interaction—between people and their environment, and among thoughts within a person—constitute two nonintersecting, mutually dependent realms of interaction (p. 86), both essential to learning. *Social* interaction, taken to be between people, is not sufficient for mathematics learning (cf. Confrey, 1995; Steffe, 1996), but it is crucial for it: Student-teacher and student-student interactions provoke learning for both teachers and students.

Another basic theoretical orientation of this study is that any social interaction can trigger depletion or stimulation of the people involved. I define *depletion* as a feeling of being taxed in some way, usually accompanied by a decrease in energy or a diminishment of overall well-being. I take *stimulation* to mean a feeling of being excited or awake, usually accompanied by a boost

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[1] The material of this chapter is adapted from Hackenberg (2005) with permission from the publisher.
[2] From an observer's point of view, a person's environment is everything the observer sees that is not the person.

in energy or a stronger sense of aliveness. In some social interactions, these feelings are negligible or pass unnoticed, and the prevailing feeling is one of neutrality or "evenness." In other social interactions, stimulation and depletion fluctuate more obviously, or one greatly outweighs the other. In social interaction geared toward learning, sustaining some level of depletion may be necessary for subsequent feelings of stimulation. Yet in mathematical interaction between teachers and students, depletion may dominate for a variety of reasons, such as students' attitudes toward mathematics (Hart & Allexsaht-Snider, 1996; Schoenfeld, 1989), students' considerable difficulty with the subject (RAND Mathematics Study Panel, 2002), teachers' instructional approaches (Boaler, 2000), the structure of mathematics classes (Boaler, 1998), and common beliefs about the nature of the subject (Henrion, 1997).

Caring Between Teachers and Students

Although continuous feelings of stimulation are not possible, prolonged feelings of depletion may contribute to comments from both students and teachers that the other party doesn't care. Some students will say that teachers don't care about students' lives or ways of thinking, which implies a lack of awareness about or valuation of the students. These students don't feel *cared for* by their teachers. Some teachers will say that students don't care about—that is, don't demonstrate interest in—learning, a particular subject matter, or school. These teachers don't feel cared for *as teachers* because they don't experience their students' responsiveness to and engagement with the activities they orchestrate and the questions they pose. These feelings may not correspond with the other party's intentions—teachers may care very much about their students, and students may be curious about and interested in learning, but teachers and students often fail to develop caring relations (Noddings, 1984, 2002). Such failures can interrupt or stunt learning for both students and teachers.

In Noddings' (1984, 2002) notion of caring relations, a teacher (a carer) orchestrates experiences in which a student (a cared-for) feels that her or his need for care in student-teacher interactions is satisfied. For Noddings, enacting care as a teacher means stretching students' worlds and working cooperatively with students so that they realize competence in their worlds. Receiving care as a student means responding to the teacher in a way that indicates some level of satisfaction of the student's need for care—i.e., renewed interest or activity, an increase in commitment or energy, or even a "glow of well-being" (2002, p. 28). A student's response "completes" (1984, p. 65)[3] the caring relations because they cannot transpire between teacher and student without the involvement of both parties. In addition, the student's response is how the student cares back for the teacher: Reception of the teacher's care is what the teacher "needs most to continue to care" (1984, p. 181). Thus caring is not simply a feeling, or a virtue of a teacher or student. Caring is an orientation to co-create and participate in social interactions that are responsive to the cognitive and affective states and needs of both teachers and students.

Caring in Mathematical Interaction

I conceive of enacting care in *mathematical* interaction as inseparable from engendering mathematical learning. Mathematics teachers may act as carers in general, but they start to act as *mathematical* carers when they hold their work of orchestrating mathematical learning for their students together with an orientation to balance feelings of depletion and stimulation that may accompany student-teacher mathematical interactions. From a student's perspective, participating in mathematical caring relations (MCR) involves being open to the teacher's interventions in the student's mathematical activity and pursuing questions and ideas of interest. In this chapter, I elaborate on MCR as my theoretical model for student-teacher mathematical

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[3] Noddings uses the word "completes" to indicate that *both* carer and cared-for must be involved for caring relations to occur, not to imply that in on-going interaction caring relations are ever fully finished.

interaction. Because MCR cannot be enacted without aiming for mathematical learning, I also describe my model of mathematical learning and describe how these two models are intertwined.

Mathematical Learning

Following Piaget (1970a) and von Glasersfeld (1995), as well as scholars who rely on them (e.g., Confrey, 1995; Steffe, 2002a; Thompson, 1994), I define mathematical learning as a process in which a person makes relatively permanent modifications or reorganizations in her[4] ways of operating in response to perturbations (disturbances) brought about by her current ways of operating. These modifications and reorganizations occur in the context of both types of interaction (between the person and her environment, as well as among her thoughts), and they are characterized by *logical necessity*: At some level (not necessarily within the learner's awareness), the modifications and reorganizations are necessary, reasonable, and perhaps useful to the learner as she acts within her experiential reality. The outcome of mathematical learning is a new way of operating adapted from a previous way of operating. An outcome is more or less permanent if, from the learner's point of view (again, not necessarily at the level of awareness), it continues to be useful in on-going interaction—that is, if it solves situations not previously solved and can serve in further learning.

One Step Back: The Theory of Radical Constructivism

My definition of mathematical learning is rooted in radical constructivism, a theory of rational knowing articulated by Ernst von Glasersfeld (1985, 1995) and based on the work of Piaget and other scholars. It has two central premises: People actively build up knowledge, rather than passively receive it; and cognition serves in the organization of experiential reality, not in the discovery of an ontological reality. The second premise is why von Glasersfeld chose to call

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[4] In sentences with a singular noun I alternate use of the feminine or masculine possessive pronoun instead of using the more cumbersome "her or his," or the increasingly popular plural possessive pronoun "their."

his theory "radical" (1990), because it requires giving up *that we can know* whether human knowledge comes to match some pre-existing reality (with a capital R). The difficulty with assuming that such a Reality can be known, let alone that humans could know that they know it, was recognized eons ago (von Glasersfeld, 1995): To ascertain whether human knowledge matches the way the world "really" is, a person would have to be able to compare her knowledge with a "true" representation of the world and assess the degree of match. But this comparison is precisely one people cannot make because as humans, we cannot ever step out of our own bodies and minds, out of our ways of perceiving and conceiving. So in contrast to Reality I refer to a person's experiential reality—the reality that a person knows through his perceptions and conceptions in on-going interaction.

My stronger motive in taking "one step back" to address the theory that underlies my model of mathematical learning is my desire to comment on the word "construct." The word is used to signify the *active* nature of knowing in radical constructivism, which relates to the first premise of the theory. The premise is that people are (consciously or not) active in the process of coming to know—in the process of conceiving, and even of perceiving. Using and modifying their ways of operating, people actively make regularities in the flow of their experience so as to organize their experiential realities, even when they make sense of someone else's organization. This activity is what I (and, undoubtedly, others before me) intend to communicate by the using the word "construct" in reference to how people know and learn.

Schemes and Operations

Continuing to define terms, I use the phrase "way of operating" to refer to a range of repeatable activity in which a person engages, such as a student regularly telling the teacher that his head hurts when a problem seems difficult, or a student consistently stating reciprocal multiplicative relationships between two quantities. In contrast, I use the word “operation” in a more specific, Piagetian (1970a) sense. An operation is a mental action, such as cutting an item out from its background (a unitizing operation), or repeating an item in imagination to create a plurality of items (an iterating operation). Operations are the components of conceptual schemes.

A conceptual scheme is a goal-directed way of operating consisting of a situation, an activity, and a result (Piaget, 1970a; von Glasersfeld, 1995). To initiate the activity of a scheme, a situation must be perceived or recognized by a person as similar in some way to previous situations in which the person used the scheme. This perception or recognition is the result of assimilation (Piaget, 1964; von Glasersfeld, 1995), the basis for construction—and modification—of schemes. The perceived situation then triggers the activity of the scheme, which may be mental or physical or both. The result of a scheme is an outgrowth of the activity, and the person generally anticipates that the result will be expected or satisfying in some way.

Even though schemes involve operative activity, they can have figurative or enactive aspects. For example, drawing a picture or gesturing with one’s hands may accompany the use of any scheme. However, schemes that have a significant figurative aspect require sensorimotor (often visual) material for their implementation to the extent that without the material, the person cannot engage in the activity of the scheme. Enactive schemes have a crucial experiential component—a person cannot carry out the scheme without some physical activity. A scheme that does not rely on “external” visual material or explicit physical activity still likely involves “private” imagery of some kind, which may be in or outside of awareness, and upon which the person operates in using the scheme. Private imagery is based on *re-presentation* of past experience (such as figurative or enactive activity) in visualized imagination. To re-present is to engage in mental acts that bring prior experience to consciousness without actually carrying out the acts materially (von Glasersfeld, 1995),[5] and such re-presentation effectively provides "mental material" for operating.

Acts of Learning

Assimilation. From an observer's point of view, the assimilatory act necessary to initiate the activity of a scheme may appear to involve a transfer of information from an external source to a person. But as von Glasersfeld (1995) explains, "The cognitive organism perceives (assimilates) only what it can fit into the structures it already has" (p. 63). He points out that this statement is still made from an observer's point of view and that "…when an organism assimilates, it remains unaware of, or disregards, whatever does not fit into the conceptual structure it possesses" (p. 63). Thus assimilation enables a person to initiate the activity of a scheme in a situation even if to an observer the situation seems different from previous ones in which such a scheme was activated.

Generalizing assimilations. Assimilation as primarily recognition of a particular situation or an aspect of a person's environment is not, from my point of view, an act of learning. However, I view a *generalizing* assimilation as an act of learning because of its power and prevalence. An assimilation is generalizing if the learner uses a scheme in situations that contain sensory material novel for the scheme and the scheme is modified (Steffe & Wiegel, 1994), but the modification does not result in a new scheme. This type of assimilation is a reconceptualization of the notion of transferring knowledge from one situation to another, not obviously similar situation, and it probably occurs more often than accommodations do (L. P. Steffe, personal communication, April 15, 2003). For example, a fourth-grade student who uses his counting scheme for discrete whole numbers in the context of making continuous lengths two

[5] "Re-presentation" is used instead of "representation" to indicate the active nature of mentally presenting again to oneself some prior activity or experience.

times longer, three times longer, four times longer, etc., than a unit length has modified his counting scheme for use in a continuous context but has not produced a new scheme.[6] To an observer the student appears to have "transferred" his ways of operating with discrete numbers to operating with continuous lengths through a generalizing assimilation.

Accommodations. More powerful modifications or reorganizations of schemes are accommodations, which can be categorized as primarily developmental or primarily functional.[7] Developmental accommodations involve maturation of the central nervous system, one of four factors Piaget (1964) used to explain changes in a person's conceptual structures.[8] These accommodations reconstitute a scheme on a new level and reorganize the scheme at that level (Steffe & Wiegel, 1994). Because the "jump" in level of operating may be significant, developmental accommodations might be characterized as *vertical* acts of learning. An example of a developmental accommodation is a sixth-grade student who is able to draw a whole candy bar given a drawing of three-fifths of it, when previously she was unable to initiate activity in such situations. The contexts for such sudden jumps are varied—sometimes they occur in the context of schooling and sometimes after a break, such as a vacation (see, e.g., Steffe, 2002b).

Functional accommodations of a scheme occur while using it (Steffe, 2002a) and vary in the extent to which the scheme is modified. Functional *metamorphic* accommodations are significant reorganizations of schemes that occur over time in the context of using current schemes, so they produce vertical learning. For example, consider a sixth-grade student who has, periodically over time, worked on problems like this one: Share a cake fairly among yourself and 14 friends; then share your piece fairly with a latecomer and determine what fraction of the cake

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[6] This brief example is summarized from Steffe's (2002a) analysis of the work of Jason, a fourth grader.
[7] See Steffe (1991b) for a more detailed discussion of different types of accommodations.
[8] The other three factors are experience, social transmission, and self-regulation.

the latecomer gets. Let's say his usual solution to such a problem is that the latecomer's amount is one-half of one-fifteenth. He hasn't been able to determine any other fraction name for the piece and when he has been asked how many of the latecomer's pieces he needs to make the original cake, he has responded by covering the cake with copies of the latecomer's piece, which has resulted in anywhere from 18 to 25 pieces covering the cake. If this time he comes to view the problem as a multiplicative situation (i.e., he multiplies 15 times 2 and can justify how this helps him solve the problem), he has likely made a functional metamorphic accommodation in his multiplying and fraction schemes.[9] The degree to which the accommodation is considered metamorphic is the degree to which the new way of operating contrasts with previous ways of operating to solve similar problems, as well as the relative permanence of the new way of operating in other problem situations.

Other functional accommodations (nonmetamorphic) yield more modest modifications. Often such accommodations involve the coordination of current schemes and operations. For example, say a sixth-grade student uses her multiplying and dividing schemes for whole numbers in determining the number of sixths in one-third, when previously she was unable to make this determination. If she reasons that there are six one-sixths in a unit whole, and she divides six by three to determine that two one-sixths will be one-third of the unit whole, then she has made a functional accommodation in her fraction scheme. Because of the relatively small modifications they produce, generalizing assimilations and nonmetamorphic functional accommodations might be characterized as *horizontal* acts of learning.

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[9] Functional metamorphic accommodations account for problem solving, if problems are conceived of as situations in which a person does not have immediate recourse to a solution path and may need to engage in some level of "open search."

What Drives Acts of Learning?

Perturbations. Modification or reorganization of a scheme may occur when a person's current schemes produce an unexpected result: The person does not achieve her intended goal. This "disturbed" state of affairs is one example of a perturbation and is often accompanied by a sense of disappointment or surprise. A perturbation may also arise if a person intends or wishes but is unable to initiate activity in a particular situation, a potentially frustrating state of affairs. A third example of a perturbation may occur if a person's activity in a situation—or result from the activity—seems incongruous to *another* person, who points out the incongruity. The extent to which the actor also comes to experience her own activity or result as incongruous determines the extent to which a perturbation happens.

As von Glasersfeld (1995) emphasizes, a person's "unobservable expectations" (p. 66) are instrumental in initiating a perturbation because what is crucial is the degree to which the unexpected result, inability to act, or incongruity "matter" to a person at an intentional or unintentional level. This aspect of perturbations means they are not always consciously conflictive: An unexpected result, an inability to act, or someone else's sense of incongruity may remain largely unnoticed by a person and yet have some impact on a person's subsequent activity—perhaps in a vague sense of unease or a sort of heightened interest. Thus even perturbations that are mostly outside of immediate awareness involve an affective aspect. This affective aspect is a major point of connection between mathematical learning and mathematical caring, which I will discuss toward the end of this chapter. As a person (consciously or unconsciously) eliminates perturbations, or equilibrates, the perturbation has the potential to trigger an act of learning.

Reflective abstraction. Although a perturbation opens the way for an act of learning to occur, in this view of learning the central mechanism to "go from a state of less sufficient knowledge to a state of higher knowledge" (Piaget, 1970a, pp. 12-13) is reflective abstraction. Piaget identified three types of reflective abstractions (cf. Piaget, 2001; von Glasersfeld, 1995), all of which require re-presentation of past experience. I briefly discuss two types of reflective abstraction here.

One type of reflective abstraction involves a projection (*réfléchissement*), in which mental operations or schemes developed at one level are abstracted and applied to a higher level, as well as a cognitive reorganization of these operations or schemes (*réflexion*). So this type drives accommodations (and vertical learning in particular). This type is also what drives *interiorization* of schemes, in which the results of operating can be taken as input for further operating. Interiorization entails being able to act on re-presentations of activity but it does not require enacting the entire process in visualized imagination (Leslie P. Steffe, personal communication, July 14, 2003). Whether someone who makes this type of reflective abstraction becomes aware of this activity (even of the cognitive reorganization) is uncertain.

In contrast, awareness is likely required for a second type of reflective abstraction, often referred to as *reflected* abstraction (*abstraction réflechie*), which involves thematization of the results of mental activity conducted in retrospect (von Glasersfeld, 1995). Reflected abstractions drive a person's conscious recognition of, and reflection on, her patterns of mathematical activity. *Externalization* of schemes and operations is useful in opening possibilities for reflected abstraction to occur, because externalization entails producing representations (often visual) that stand in for a person's schemes and operations. Making such representations (drawing, using notation, etc.) may allow students to reflect on their ways of operating (i.e., reflect on representations of their reflectively abstracted activity) and thematize them.

What Is Learned?

A model of mathematical learning necessarily involves the learning of students *and their teachers* (Steffe & Wiegel, 1996), because a model addresses how teachers can learn to bring forth and sustain students' learning. However, what students and teachers learn in interaction is not identical. One goal of model building is to delineate these realms of learning.

What do students learn? Students learn to order, comprehend, explain, and manage their experiential realities so as to achieve a sense of prediction and control, as well as an ability to question, evaluate, and justify their ways of operating. This kind of knowledge is first-order knowledge (Steffe & Wiegel, 1996), sometimes referred to as *students' mathematics* (Steffe & Tzur, 1994). In constructing their mathematics, students also learn to generate and operate upon standard symbolic notation that stands in for enacting their schemes and operations. For example, a student who makes eleven-sevenths of a rectangular bar by partitioning the bar into seven parts, copying one of those parts and make a new bar of eleven of those parts might represent his activity in standard fraction notation, as 11/7. Through generating and operating upon standard notation, students learn to participate and communicate in the dominant mathematics culture.[10]

What do teachers learn? Teachers also increase their first-order knowledge of mathematics through teacher-student interactions. In the midst of these interactions, a teacher acts as a first-order observer because he or she "does not intentionally analyze the mental structures of the child relative to his or her own mental structures" (Steffe & Thompson, 2000a,

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[10] I use "dominant mathematics culture" to refer to traditional ways of knowing, representing, and communicating mathematics, including both verbal communication and written notation.

p. 202). However, building first-order knowledge is insufficient to describe teachers' learning if teachers want to learn how their students think and learn. To explain their experiences with students' activity, teachers learn to make models based on their own first-order knowledge as well as their analysis of their students' first-order knowledge and learning (Steffe & Wiegel, 1996). Steffe and Tzur (1994) refer to these second-order models as the *mathematics of students*. When building the mathematics of their students, teachers act as second-order observers who "…focus specifically on explaining the child's learning relative to [their] own purposes, intentions, and contributions to mathematical interactions" (Steffe & Thompson, 2000a, p. 202).

Mathematical Caring Relations

In this view of mathematical learning, both students and teachers learn, although they do not learn the same things. In MCR, both students and teachers care and are cared-for, although not in the same ways. To elaborate on teachers' and students' participation in MCR, I first give brief background on the origins of care theory and the use of it in research.

Background on Care Theory

Caring is a word long-associated with teachers' work (Eaker-Rich & Van Galen, 1996), although care theory and an ethic of care emerged prominently in the 1980's with the research of Gilligan (1982) and philosophical writing of Noddings (1984). For these two scholars, caring was not sentimental—not merely a feeling or a notion of "being nice." They formulated caring as cognitive and emotional work centered on acting from a basis of nurturing relations between people (in contrast to acting from a basis of preserving individuals' rights, sometimes referred to as an ethic of justice). Gilligan's work formed the foundation for a body of research that used care theory to understand the moral development of girls and women (e.g., Belenky, et al., 1997; Brown & Gilligan, 1993; Lyons, 1987).

Critics of care theory have accused it of reinforcing gender roles (female as carer, male as cared-for) or essentializing women as "natural" carers (Diller, 1996). But some scholars have incorporated power and politics into care theory in order to examine gender in relation to the construction of knowledge (e.g., Rose, 1994), to analyze an elementary teacher's simultaneous exercise of power and care (Noblit, 1993), to investigate the relationships between teachers and students of different ethnicities or sexual orientations (e.g., King, 1996; Kissen, 1996; Valenzuela, 1999), and to explore the complexity of generating school-wide norms of care (e.g., Van Galen, 1996).

Care theory has also been used to address teaching and teacher education: to analyze ethical decisions in science teaching (Tippins, Tobin, & Hook, 1993), to describe collaboration between a classroom teacher and a researcher (Hunsaker & Johnson, 1992), and to study the preparation of elementary teachers (e.g., Acker, 1995; Goldstein & Lake, 2000; Rogers & Webb, 1991). Researchers in mathematics education have used care theory to discuss the inseparability of the intellectual and moral in teaching (Ball & Wilson, 1996; Vithal, 2003), the development of trust in a professional development project (Sztajn, White, Hackenberg, & Allexsaht-Snider, 2004), and the nature of in-service teacher education (Sztajn, 2004). However, the use of care theory and caring relations has not been well-developed in studies in mathematics education (Vithal, 2003), and certainly not in studies that focus on students' mathematical learning.

How Do Teachers Participate in MCR?

Mathematics teachers engage in three central activities to enact MCR with their students. First, they pose situations that harmonize with students' mathematical ways of operating and affective responses to mathematical activity. This activity relates to Noddings' (2002) discussion of teachers' "engrossment" (p. 28), a keen level of attention in getting to know students' needs, preferences, and goals. Second, mathematics teachers pose situations that challenge students—that open opportunities for them to engage in acts of learning and thereby expand their mathematical ways of operating. This activity is akin to Noddings' notion of stretching students' worlds and experiencing a "motivational shift" (p. 28) to help students realize competence in those worlds. Third, mathematics teachers track students' affective responses to mathematical activity as indications of whether they feel mathematically cared for, and then teachers make adjustments in the situations that they pose to reinitiate harmonizing with and challenging students. In Noddings' terms, teachers continually refine their engrossment with their students in order to refine their care for them.

Harmonizing with students. To harmonize with students, the teacher attempts to take on students' ways of operating—their mathematical realities—as if they were the teacher's own. Doing so means that the teacher decenters, or sets her own mathematical ways of operating temporarily to the side in order to focus on students' current schemes and operations. Decentering does not imply that the teacher loses sight of her own schemes and operations, but instead makes space to see mathematical realities that may differ significantly from her own. The teacher also decenters from her own affective responses to consider students' affective responses to mathematical activity. The teacher begins to understand students' ways of operating and responses by posing experientially real situations, maintaining a playful orientation (Steffe & Wiegel, 1994), tracking the stimulation and depletion the students seem to experience, and engaging in conversation with students about their work.

In this process, the teacher makes conjectures about a student's current schemes and the kinds of mathematical activity that fall within a student's *zone of potential construction* (ZPC) (Steffe, 1991a). The ZPC of a student is a time-sensitive concept for ways of operating that a student might construct based on the student's current schemes and operations.[11] The ZPC is time-sensitive because it constantly changes as the student changes, and because some ways of operating might be within the student's ZPC in the "short term" (say, the next month) or the "long term" (say, the next year). Posing problems that fall within a student's short-term ZPC is usually best known in retrospect—a confirming sign is that the student can use her current ways of operating to solve the problem, although doing so may still be a (sometimes significant) challenge. However, even if a student encounters problems that fall outside of her short-term ZPC, the teacher expects that the student may experience at least some stimulation because of the attention to and valuation of the her ways of operating in their interactions.

Challenging students. Aware that feelings of depletion are endemic to processes of learning and may be necessary for significant stimulation, the teacher does not avoid posing situations that challenge students. In fact, to mathematically care for students, the teacher must do so! However, the teacher also monitors and attempts to alleviate prolonged feelings of depletion (or even excessive stimulation) that may harm students' experience of mathematical care. To open new possibilities for students' mathematical thinking, the teacher considers situations that are at the "edge" of what students can currently do (at the "edges" of their short-term ZPCs), as well as situations that might allow students to learn (to make accommodations in their current schemes and operations). The goal is not necessarily to guide students to operate as the teacher does. The goal *is* to engender ways of operating that students themselves may not imagine and that the teacher believes may lead students to more sophisticated conceptions and solutions of problem situations.

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[11] The primary difference between the ZPC and Vygotsky's zone of proximal development (1962) is that the ways of operating a teacher posits in a student's ZPC are based on the mathematics of previous students. Vygotsky's concept seems to focus on the student learning the mathematics of adults (cf. Goldstein, 1999).

Bringing forth figurative and operative activity. Two general techniques a teacher can use to harmonize with and to set challenges for students are designing problem sequences that require a combination of figurative and operative activity, and helping students externalize their schemes and operations. For example, the teacher may conjecture that to solve certain problems a student is using a scheme with significant figurative or enactive aspects, so the student has not yet interiorized this scheme. That is, the student has not made a reflective abstraction in order to take the results of his figurative or enactive activity as material for further operating. If the teacher believes that such interiorization is within the student's short-term ZPC, the teacher might work toward it by first asking the student to solve problems with figurative material (harmonizing with the student's current ways of operating) and then posing problems where the figurative material is covered up or absent (challenging the student to operate in visualized imagination, on re-presentations of the figurative or enactive activity). This technique can provoke interiorization in particular, or acts of learning in general.

The interplay between figurative and operative activity can be useful "in the other direction" as well, to provoke *externalization*. For example, the teacher may conjecture that a student has interiorized a scheme or operation but is not aware of the pattern of the activity of the scheme, its usefulness in solving problems, or its connection to other ways of operating—the student has not made a reflect*ed* abstraction. The teacher might open the possibility for the student to make a reflected abstraction by asking the student to generate pictures (drawing by hand or using computer software) to show her solution and her thinking. Asking the student about the pictures can induce the student to reflect on and become more aware of her mental activity. In this way the teacher works with the student from a primarily operative realm (the student's enactment of her scheme) to a more explicitly figurative realm (the student's drawings that stand in for her reasoning).[12]

Making modifications. During this process of harmonizing with and challenging students, the teacher carefully observes students' responses and continually makes modifications in her efforts. For example, what the teacher initially conjectures will harmonize with students instead may challenge them, and what the teacher conjectures will challenge them may fall far outside of their ZPCs or, alternatively, may fail to challenge them at all. Similarly, what the teacher initially suspects will stimulate students may deplete them, and what the teacher suspects may temporarily deplete students may induce a blockage in their activity or, alternatively, pass by them unnoticed. In making modifications, the teacher adapts to her on-going construction of her students as *internalized others*. An internalized other is one person's construction of another person, which includes the other's ways of operating, personal tendencies, preferences and aversions, and affective responses. Teachers continually refine their internalized others of their students in order to maintain, enhance, or reinitiate mathematical caring relations with them.

Although mathematics teachers-as-carers are always aiming for their students to learn, mathematical caring relations can be maintained even if specific acts of learning do not occur during any given interaction. For example, a student may not make an accommodation in a scheme for some time, but through the teacher's attention to stimulation and depletion (which may be outside of the student's awareness), the student may come to sustain engagement through perturbations—even through those that are consciously conflictive. Sustaining engagement through perturbations means remaining mentally active and operative in the midst of affective

[12] Note that in my view, all operative activity involves a figurative aspect—something to operate on—which may be private imagery outside of the person's awareness. However, not all figurative activity stems from operative activity.

and cognitive disturbances, which opens possibilities for further learning. In this way mathematical caring and mathematical learning are entwined but distinct.

How Do Students Participate in MCR?

Receiving care. When a student's response to a teacher's mathematical care includes consideration of a new situation the teacher has posed, or a sense of interest or aliveness, the student participates in MCR and cares back for the teacher. This response is what the teacher receives. The reciprocity is subtle: Students are not expected to care back for teachers in the same way that teachers care for students because, for example, students are not expected to monitor and assess the depletion and stimulation that teachers experience in their mathematical activity. What is required from the student to complete the caring relations is openness to the teacher's suggestions, which provides experiences of stimulation for teachers. In this way teachers are cared for by their students and, according to Noddings (2002), receive the "natural reward of teaching" (p. 182).[13]

Students who participate in MCR (and not all do) are likely to feel overtly recognized, to have a sense of being seen by the teacher. They may feel that they are being listened to, that their ideas are valued, and, perhaps, that they are understood. As a result, the students may experience stimulation—may feel energized or stronger in some way. Students may also feel some uncertainty or confusion in response to *experiential provocations* set by the teacher. The word provocation is intended to indicate how a student may experience a teacher's attempt to set a

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[13] An expanded note on who cares is important given the controversy around the notion of caring in feminist theory. Since women have traditionally been care-givers and women teach in K-12 classrooms in far greater numbers than men, promoting teachers-as-carers may maintain stereotypes that caring is simply what women "naturally" do. Thus an emphasis on caring in teaching and learning may perpetuate exploitation of women, or at least the essentializing of them. I respond to this criticism similarly to Noddings (2001, 2002): To say that female teachers enact caring with their students more than male teachers is absurd, unknowable, and not my project. However, I don't want to deny the care-giving history of women. Most important, I don't wish to throw out notions of care simply because they have been associated with women. Instead, I'd rather explicate the work associated with caring (cf. Rose, 1994) so that rather than be regarded as solely "natural," caring might be seen as valuable learned behavior that is in the province of all people. In this sense I depart somewhat from Noddings' reliance on "natural" with regard to caring.

challenge (to provoke a perturbation). The provocations can be stimulating if the student finds that it's possible—or even satisfying—to resolve them (although some may not be resolved quickly!). Such experiences may help students sustain or increase their engagement in mathematical activity, both that engendered by the teacher and by themselves.

Constructing oneself as a social-mathematical being. In addition to affecting students' feelings, participating in MCR may influence a student's *mathematical self-concept*—his construction of himself as a doer and learner of mathematics.[14] One aspect of a mathematical self-concept is a student's construction of himself as a person who is aware of his ability to do and communicate about mathematics with his teacher or peers—that is, his construction of himself as a *social-mathematical being*.[15] Generating this awareness involves constructing others. Students do construct internalized others of their mathematics teachers, but students do not have a need to elaborate their construction in the way that teachers do. Students *are* commonly concerned with, or at least affected by, their construction of their teacher's perceptions and conceptions of them as mathematical learners and thinkers. This sense of the teacher's regard can influence a student's construction of himself as a social-mathematical being. Because in MCR a central focus of the teacher is on tailoring mathematical activity to students' ways of operating and affective responses, students may be likely to view themselves as engaging in activity that is held in high regard by the teacher. Thus students may be likely to construct themselves as social-mathematical beings, at least in relationship to the teacher.

Trusting the teacher. Participating in MCR requires a certain level of trust in the teacher. Although definitions of trust have changed over time and continue to be debated, trust is

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[14] A "positive" or "productive" mathematical self-concept can be seen as one part of the definition of "productive disposition" in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001, p. 131).

[15] Constructing both a mathematical self-concept and the self as a social-mathematical being are part of von Glasersfeld's notion of the self as a locus of subjective awareness (1995; cf. Steffe, 2000).

commonly conceived of as a willingness to be vulnerable to another person based on assessments of the person's competence, reliability, openness, or concern (Tschannen-Moran & Hoy, 1998). Students may or may not consciously assess these characteristics in their mathematics teachers. But they do have to be willing to take on some risk to engage in learning, because making changes in one's ways of operating opens the possibility of being in situations where one doesn't know what do to, with accompanying feelings of vulnerability, embarrassment, uncertainty, confusion, worry—all of which are potentially depleting.[16] To participate in MCR, students have to develop some level of trust that the teacher is posing situations that are in some way beneficial to the students, even if they don't or can't see the point of the teacher's activity in the moment. Ideally, students trust that the teacher's activity is in the interest of the students' overall growth. Of course, many students do not exhibit this level of trust with their teachers and may have difficulty participating in MCR.

What Drives MCR?

Teacher's activity. A student who does not easily enter into MCR may come to complete such relations through extended interaction with a mathematics teacher-as-carer. This comment is not meant to imply that teachers' work toward addressing their students' needs for care in mathematical interaction is simple or constant: The fluctuation of feelings of stimulation and depletion in social interaction aimed toward mathematical learning is sometimes subtle, often rapid, and not solely in the teacher's control. Furthermore, it is conceivable that one student's need for care may be satisfied in ways that are almost directly opposite to another's. For example, some students may need a great deal of significant challenges (e.g., situations that are intended to provoke vertical learning) in order to feel mathematically cared for, while others may

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[16] Since teachers enacting MCR are also learners, these teachers also must trust their students in certain ways, but the trust is different in nature from students' trust in their teacher. This issue is related to the power K-12 teachers have over their students by virtue of age and societal structure.

require such challenges much less frequently. Despite the difficulty in addressing different students' needs simultaneously, it is likely that the teacher's practice of decentering will facilitate students' overall sense of being cared for and students' development of trust in the teacher.

But what happens if students don't participate reciprocally in completing the relations? Well, they might not—caring relations can fail to be enacted due to the carer, cared-for, or other aspects of the situation. Just as teachers may teach but students may not learn, teachers may try to enact mathematical care and students may not respond in ways described here (cf. Noddings, 2001). Or at least, student responses to a teacher's care may not be obvious to the teacher, or may include overt resistance to the teacher's intentions and actions. In those cases, a mathematics teacher-as-carer recognizes that she and the students are not in harmony in some way. She tries to understand better what activities and interactions are stimulating for the students. It is important to note that such activities might be radically different from the teacher's initial ideas and might involve cessation of mathematical activity, or quite different mathematical activity, for some time. In reinitiating her efforts to harmonize and challenge students, the teacher may also attempt to search with the students for mathematical activity and interaction that interests them.

Why care mathematically? Since the establishment of MCR is not guaranteed even with the best intentions, and because enacting MCR clearly requires effort, a natural question is "why care." Noddings (2002) posits that *the need for care* is universal, even if that need is satisfied quite differently over the span of a person's life, let alone across cultures and time periods. So one response to the question "why care mathematically" is that all students need some form of mathematical care in order to develop as mathematical thinkers, let alone as people who view themselves as mathematical thinkers in relation to others—who construct themselves as social-mathematical beings. Furthermore, working to satisfy this need for mathematical care may open further opportunities for mathematical learning for both students and teachers.

Still, the decision to care is the teacher's because *caring for others* is variable, dependent on situations and conditions (Noddings, 2002). Sometimes mathematics teachers-as-carers respond because they want to—that is part of why they have become teachers: They are intrigued by the experiences of their students and how those experiences change in student-student and teacher-student interactions, so giving mathematical care is satisfying to them, in harmony with their intentions. At other times, the same teachers meet resistance to caring mathematically. For example, students may seem particularly scattered or disinterested despite the teacher's decentering efforts, teachers are consistently tired and busy, and the act of trying to harmonize with and open possibilities for their students' mathematical ways of operating is a significantly complex job. Teachers may persist as mathematical carers in these situations because of their own memories and images of being mathematically cared for and of caring mathematically; because of beliefs that through acting out of duty they will return to a more harmonious state of mathematical caring; or even because they see caring through resistance as fully in the scope of enacting mathematical care.[17]

Connections Between Mathematical Learning and MCR

The foregoing discussion indicates ways in which mathematical learning and mathematical caring relations mutually support each other. In the last section of this chapter I

[17] However, just because caring through resistance can maintain caring relations does not mean that a carer always must—or can—enact it! Noddings (2002) notes "Care theory does not attempt to develop a model of moral education that can produce people who will behave virtuously no matter how bad the world that surrounds them" (p. 9). That is, for a variety of reasons, sometimes a carer cannot—or chooses not—to enact care. Setting such boundaries may actually maintain the carer who, because of his orientation, will likely return to initiating, maintaining, and enhancing caring relations in the future. This caveat also holds for enacting mathematical care.

elaborate on the close links between acts of learning and caring at the level of schemes, as well as parallels between how teachers act as mathematical carers and as learners with their students.

At the Level of Schemes

Since perturbations—central triggers for acts of learning—involve an affective response to the unexpected, uncertain, or incongruous, experiencing a perturbation can be accompanied by feelings of both depletion and stimulation. Feeling depleted is one reaction to a perturbation, especially if a person senses that she does not know what to do to eliminate it, or that such activity will be particularly onerous. Not immediately knowing what to do in a mathematical situation is a common and necessary part of learning, but protracted lack of knowing, or burdensome activity, or inability to act in a situation, can deplete a student. Depletion may manifest as simply a sense of fatigue, or more strongly as emotional states like dread, dislike, irritation, anger, etc. If a feeling of depletion is too great or extended for too long, a student may feel overwhelmed, which may impede engagement in mathematical activity either immediately or in the future.

However, perturbations can also provide stimulation in the form of a challenge, particularly if a person senses that she can meet or satisfy that challenge in some way, or that such activity itself will be enjoyable. Meeting a challenge in a mathematical situation may mean that the student assimilates the situation and can initiate some activity, even if she may not foresee a particular result. Stimulation may manifest as continued engagement in activity, or it may include more obvious emotional experiences of satisfaction, excitement, flow (Csikszentmihalyi, 1990), eager anticipation, and even joy. If a feeling of stimulation is sufficient, the student's subsequent interest in or curiosity about acting in the situation may prolong mathematical activity and open new opportunities for learning.

If, over time, feelings of stimulation outweigh feelings of depletion, the student may feel mathematically cared for. Thus experiencing mathematical care is not tied to the elimination of perturbations or the resolution of problems, although both may be satisfying to students. In a given mathematical situation, students may participate in MCR through carrying out the activity of a scheme, producing a result that makes sense to them, or reflecting on that result alone or in discussion with others.

In the Teacher's Activity

Recall that teachers' learning in student-teacher interaction involves building second-order models of their students. To do so, teachers must cultivate awareness of how they and students interact, and of the consequences of different choices for and habitual ways of interacting (Steffe, 1996). Such consequences include increasing or decreasing students' feelings of stimulation and depletion as well as opening or foreclosing possibilities for students' acts of learning. Thus building second-order models is tied to enacting mathematical care.

Decentering from one's own ways of operating as a teacher is required both to build second-order models and to initiate MCR. In decentering, teachers practice "close listening" (Confrey, 1998a; cf. Davis, 1997) and observing of students' ways of operating. Teachers test out activities and pose problems that they conjecture students can solve with their current schemes and operations. The teacher's main goal in listening and observing is not to confirm her own mathematical thinking but to make images of and conjectures about the students' mathematics. In effect, the teacher is trying to learn mathematical ways of operating from students in order to construct the mathematics of her students. The teacher does not expect her students to think as she does, and she does expect to be surprised (in the sense of delighted, stimulated, and challenged) by how students think. In this sense she works to harmonize with students, a central aspect of enacting mathematical care.

However, to build second-order models teachers also must act. Based on their images of and conjectures about students' ways of operating, teachers pose situations to provoke vertical and horizontal acts of learning. Such situations can expand students' mathematical realities in ways that students have likely not envisioned, which can be stimulating for students and addresses the aspect of challenging them in enacting mathematical care. But because such work can also be taxing, student-teacher interactions focused solely on vertical and horizontal learning have the potential to sustain depletion. So students and teachers also engage in situations that are not intended to bring forth specific acts of learning, but that can allow students to build confidence—and pleasure!—in their ways of operating. Through acting in these situations students may feel successful, believe that their ways of operating are valued and useful, and therefore experience stimulation. As teachers continue to observe and reflect on the consequences of these interactions with students, teachers can further refine their models, a process similar to refining their enactment of mathematical care.

Thus teachers learn to build second-order models of students' mathematics through the same processes by which they enact mathematical care: decentering to harmonize with students and posing challenges to open new possibilities for them. In this process teachers make conjectures about students' current schemes and operations, just as in enacting mathematical care teachers assess students' feelings of depletion and stimulation. In turn, teachers' second-order models can help them refine their enactment of mathematical care and their attempts to provoke mathematical learning. So a second-order model can be thought of as a dynamic "mechanism'" by which mathematical caring relations and mathematical learning can be engendered.

CHAPTER 3: CONCEPTUAL FRAMEWORK FOR QUANTITATIVE AND ALGEBRAIC REASONING

> What is important [in quantitative reasoning] is relationships among quantities. In that regard, quantitative reasoning bears a strong resemblance to the kind of reasoning customarily emphasized in algebra instruction (Thompson, 1993, p. 165).

In this chapter I present my orientation to quantitative reasoning as a basis for algebraic reasoning. I focus on the construction and solution of linear equations as a central goal of learning to reason algebraically, and I use *reasoning* to refer to the purposeful functioning of a person's schemes and operations in a variety of contexts. So quantitative reasoning is the purposeful functioning of schemes and operations in the context of quantities and quantitative relationships. In conjunction, I review previous research on students' learning of algebra that both supports and contrasts with my orientation. Through this embedded literature review, I demonstrate ways in which my approach to studying how students begin to reason algebraically grows out of and differs from work done by other researchers.

Basic Conceptions of Linear Equations

I approached my study with at least three basic conceptions about linear equations: beliefs about the reasoning that underlies writing and solving a linear equation like $ax = b$, about the value of rooting such reasoning in quantitative situations, and about ways of bringing forth this reasoning. In this section I give a brief overview of these conceptions.

$ax = b$ is a Statement of Division

Linear equations are often identified and studied by their form—for example, $x + a = b$, $ax = b$, $ax + b = c$, $ax + b = cx + d$, etc. In my study I focused on equations of the form $ax = b$

because I believe they are a more complex construction than equations of the form $x + a = b$, and because all other linear equations can be built from these two forms. The equation $ax = b$ is essentially a statement of division. Considering its construction and solution requires understanding how a student produces division, which entails understanding the student's multiplying schemes and multiplicative reasoning. Furthermore, any statement of division inherently involves reasoning with fractions: While fractions may be implicit or disguised in solving an equation like $4x = 28$, they soon become explicit in solving an equation like $3x = 7$.[18] So both multiplicative and fractional reasoning are implicated in studying how students construct and solve linear equations like $ax = b$.

Rooting the Formulation of $ax = b$ in Quantitative Situations

Many educators and researchers decry teaching equation solving as manipulation of algebraic notation devoid of using such notation to represent problem situations (e.g., Chazan, 2000; Davis, 1985; Moses & Cobb, 2001). These positions reflect desires for mathematics to be meaningful to students as well as the findings that instruction in manipulation of notation is often unsuccessful (Kieran, 1989). Yet students tend to separate manipulation of notation from using that notation to model problems (Fey, 1989; Kieran, et al., 1996; Sfard, 1995). As Rosnick and Clement (1980; Clement, 1982) have demonstrated, even for college students majoring in engineering, algebraic symbols often function as labels for a group (e.g., *S* represents students, *P* represents professors) rather than as representations of a *number* of something (e.g., *S* represents *number of* students, *P* represents *number of* professors). This conception of symbols as labels is one foundation of the "reversal error," in which students write the equation $6S = P$ for the situation "there are six times as many students as professors at the university."

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[18] One could argue that fractional thinking is in fact *not* implicit in solving $4x = 28$, since taking 1/4 of 28 involves fraction multiplication. However, fractional reasoning can be "avoided" in this case by using whole number multiplication—thinking about what number times 4 is 28.

Emphasizing quantitative reasoning as a basis for algebraic reasoning means that symbols (numbers as well as letters) represent known and unknown *quantities* and that reasoning about relationships among quantities is the foundation for constructing and solving equations. This emphasis is consistent with MacGregor and Stacey's (1993) finding that students construct cognitive models for comparing quantities as a basis for formulating linear equations and with Sfard's (1995) recommendation that students should reason with underlying mathematical operations in algebraic problems rather than manipulate symbol strings devoid of "operational underpinnings" (p. 30). So following them as well as Thompson (1993), I aim to root motivation to write and solve a linear equation like $ax = b$ in a student's goal to multiplicatively relate an unknown and known quantity in order to determine the unknown quantity.

Investigating Schemes and Operations that Underlie the Formulation of $ax = b$

Because $ax = b$ is a statement of division, its construction and solution can be based in quantitative situations that require *reversible reasoning*. Consider these three examples:

> Example 3.1, Juice Problem: Twenty-eight ounces of juice is four times the amount that you drank; how much did you drink?
>
> Example 3.2, Peppermint Stick Problem: A 7-inch peppermint stick is three times longer than another stick; how long is the other stick?
>
> Example 3.3, Tree Problem: Three-fourths of a decameter is two-thirds of the height of a tree. How tall is the tree?

In the Juice Problem, most sixth-grade students would not see a need to write a linear equation to find the unknown amount of juice—they could "just" use their whole number multiplication knowledge.[19] But in the other two examples, many sixth-grade students might not find it so

[19] Given that they could solve the problem, of course! Some sixth grade students might not yet be able to do so.

obvious to determine the unknown quantities. So one impetus for students to construct and solve a linear equation like $ax = b$ is to compress their reasoning with quantities into a symbolic way of operating that they can use to solve *any* problem in which the goal is to determine an unknown in a quantitative situation involving reversible multiplicative relationships.

I want to emphasize that in my study I aimed to explore the schemes and operations that *underlie* writing an equation like $(2/3)x = 3/4$ and solving it in order solve the Tree Problem (Example 3.3). So rather than take a linear equation as a given, I was trying to understand how constructing and solving linear equations like $ax = b$ might grow out of students' ways of operating in quantitative situations. This aim is the hallmark of ontogenetic approach to studying how students construct new mathematical ways of operating based on their previous ways of operating. In an ontogenetic approach, algebra is taken to be students' ways of operating that a teacher-researcher can recognize as algebraic. My ontogenetic approach, combined with my beliefs in rooting activity in quantitative situations involving reversible multiplicative relationships, meant that understanding and engendering students' construction of reasoning with quantities was essential.

Quantitative Reasoning as a Basis for Algebraic Reasoning

To explore students' reasoning with quantities as a basis for the construction and solution of linear equations requires distinguishing between quantitative and algebra reasoning. In this section I outline such distinctions and review other researchers' approaches to studying and bringing forth students' algebraic reasoning.

Definition of Quantitative Reasoning

Thompson (1993) defines quantitative reasoning as "… the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (p. 165), where the structure or network is created by conceiving of new quantities in relationship to others (Thompson, 1993, 1994). This definition is fairly high-powered, since conceiving of a situation structurally might be considered algebraic. So I view my definition of quantitative reasoning, the purposeful functioning of schemes and operations in the context of quantities and quantitative relationships, as a little more basic than Thompson's. My definition admits that when reasoning quantitatively, people may operate on known quantities to produce an unknown quantity without necessarily being aware of what I might call a structural relationship between all quantities involved. However, my definition is certainly compatible with Thompson's definition, since the purposeful functioning of schemes and operations is a mechanism for analyzing a situation into a quantitative structure. This mechanism allows a person to elaborate relationships between unknown and known quantities with goals of determining unknown quantities as well as modeling or analyzing these relationships. [20]

Defining quantitative reasoning in these ways calls for defining *quantity*. Thompson (1994) defines a quantity as a conceptual entity that consists of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign numerical value to the dimension. Quantities can be reasoned about in the absence of knowing their actual numerical value—that is, a quality of an object becomes quantified when a person can view the quality as an entity or unit consisting of some number of equal units (cf. Steffe, 1991c).[21] For example, the "tallness" of a person becomes a quantity like height when one can imagine the person's tallness as a unit subdivided into some number of units that *could* be determined by, say, the process of counting those units. In this sense, the person's height can be reasoned about *as a quantity*

[20] This amplification of my definition is based on Dewey's (1989) definition of reasoning as being concerned with linking what is unknown and known through an elaboration of the relationships between them.

[21] Note that the notion of viewing a quality—such as a plurality—as a unit consisting of some number of units without as of yet specifying *how many* units is fundamental for the construction of number and counting schemes. In this sense, number can be regarded as a quantity (cf. Steffe, 1991c).

(rather than a quality) without actually knowing the value of the height. Furthermore, the person's height is part of a quantitative structure that consists of a quantitative relationship with a unit quantity. A quantity such as person's height might be determined by an additive or multiplicative relationship with another height, rather than by counting unit quantities. In my study, I aimed for students to explore multiplicative relationships between quantities, such as my height of five feet is seven-eighths of another person's height, and to reason with those relationships to determine unknown quantities, such as the other person's height.

Distinctions between Quantitative and Algebraic Reasoning

Drawing sharp boundaries between quantitative and algebraic reasoning is difficult—these two types of reasoning tend to overlap and intertwine, depending on one's perspective on the nature of algebra and one's conception of what a particular student is doing in reasoning through a quantitative situation. Furthermore, as Dan Chazan (2000) has noted, algebra is notoriously difficult to pin down, particularly if the desire is to move beyond primarily manipulating notation to simplify expressions and solve equations (cf. Davis, 1985). However, with respect to problems that can be solved by using an equation like $ax = b$, I posit three central distinctions between reasoning quantitatively versus algebraically to solve them.

Abstraction of schemes and operations into conceptual structures. Quantitative reasoning involves constructing schemes and operations that can be used to operate on known quantities in a situation—often to make new quantities from given quantities. For example, to solve the Peppermint Stick Problem (Example 3.2), a student needs to find a way to divide seven inches by three, which can be a significant problem! A variety of schemes might be constructed to solve such a problem. For example, a student might partition each of the seven inches into more "mini"-parts and then experiment, repeatedly selecting some number of those mini-parts and iterating that number of mini-parts three times to test whether he has made another length of seven inches. Although this student may have a constructed a scheme that he can use to divide some *other* number of inches into three equal parts, I would be unlikely to attribute to him a *conceptual structure* of dividing any number of inches by three.

Von Glasersfeld defines a concept as “a kind of place-holder or variable for some of the properties in the sensory complex we have abstracted from our experiences of particular things” (1991, p. 49), where the place-holder is usually embedded in language. By conceptual structure, I mean the abstraction of a “program of operations” from the experiences of using particular schemes that includes an awareness of how the schemes are composed (their structure) and an ability to operate with this awareness. So attributing a conceptual structure to a student means inferring the student is operating on or with the structure of his schemes. The student described above might have a *concept* of dividing by three as consisting of partitioning a length into three equal parts, any one of which can be copied and repeated two more times to make a length the same size as the original length. But, based on the experimental nature of his activity in solving the Peppermint Stick Problem (Example 3.2), I wouldn’t infer that he had an awareness of how his scheme was composed nor that he was operating on the structure of the scheme.

In contrast, another student who constructs a scheme for dividing seven inches into three equal parts might partition each of the seven inches into three equal parts and take one of those parts from each of the seven inches. If this student abstracted the structure of her scheme as dividing each unit of a length into three parts in order to divide the entire length into three equal parts, and then used that structure to divide *any* number of units into *any* number of parts, I would likely attribute a conceptual structure of “dividing a composite unit by another composite unit" to her. I would call her way of operating algebraic reasoning because of her awareness and use of the (multiplicative) structure of her scheme.[22]

Operation on unknowns as well as knowns. Operating explicitly on unknowns is often seen as one of the hallmarks of algebra (cf. Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). However, operating on unknown quantities is often left implicit in reasoning quantitatively. For example, in solving the Tree Problem (Example 3.3), a student reasoning quantitatively might conceive of the problem in a "sequential" manner: Divide the 3/4 decameter into two equal parts, each of which is one-third of the height of the tree. Then use three of those parts to make the height of the tree, thereby reversing the operations of making "two-thirds" and proceeding from known to unknown.

A student reasoning more algebraically might conceive of the problem in a more "simultaneous" manner as network of two quantities, each related multiplicatively to the other. He might envision three-fourths of a decameter as both a quantity in its own right and as a quantity that forms two-thirds of the tree's height. Thus both the 3/4 decameter and two-thirds of the unknown height could be partitioned into two parts, each part being one-half of the three-fourths of a decameter and (simultaneously) one-third of the tree's height. In this sense, the tree's height, three-thirds, will also be three one-halves of the 3/4 decameter. The simultaneous nature of this way of conceiving the problem is one basis for the construction of reciprocal relationships, in which quantitative relationships are seen as bi-directional and a student can appropriate any quantity as the basis by which another quantity is produced. So when quantities are related multiplicatively, reciprocity, another hallmark of algebraic reasoning, is linked to operating explicitly on both knowns and unknowns.

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[22] See also Steffe (2001) as well as Hackenberg and Tillema (2005) for discussions of how constructing schemes to compose fractions multiplicatively may be seen as algebraic, based on how students operate with the structure of their schemes.

The use of notation. The use of notation is often fetishized in algebraic activity in schools—Mason (1996) calls it the "rush to symbols" (p. 75). As I have outlined above, a great deal of algebraic reasoning may happen without explicit written notation. Furthermore, operating on notation that does not stand in for schemes and operations would not necessarily be considered algebraic (cf. Sfard, 1995), and operating on notation other than standard algebraic symbols can be quite algebraic. For example, the student I have just described solving the Tree Problem may draw a rectangle to represent three-fourths of a decameter, and another rectangle the same height to represent two-thirds of the tree's height. Both rectangles may represent what he is envisioning, and he may use this figurative material to generate his operations on both quantities or to demonstrate his mental operations retrospectively. His reasoning is algebraic in that the notation represents his current conceptual structures that he has used to operate on both knowns and unknowns.

However, his reasoning can also be characterized as quantitative, in that he is not yet operating on standard algebraic notation in lieu of operating on mental imagery or drawings. In particular, quantitative reasoning is often marked by notation that indicates a process that students have "finished" performing (mentally or materially). For example, a student might show her solution to the Tree Problem after she had worked through the problem by writing $3/4 \div 2 = A$, $A*3 = T$, where A represents "the answer to dividing the 3/4 decameter into two equal parts" and T represents the height of the tree. A student reasoning more algebraically might be able to compress that notation considerably, perhaps even *starting* by representing the given multiplicative relationship between the two quantities, $(2/3)*T = 3/4$. The degree to which the student could then operate on that notation would indicate the degree to which she had interiorized her notation into a symbolization system.

Other Researchers' Approaches to Algebraic Reasoning

In the previous section I have argued that algebraic reasoning can occur before unknowns are explicitly involved, when students operate on the structure of their schemes. This argument is basic to an ontogenetic approach to algebraic reasoning, in which students' algebraic reasoning grows out of their previous ways of operating with quantities (including numbers). A perspective that algebraic reasoning is, or should be, connected to students' previous mathematical experiences and activity is explicitly advocated by other researchers (e.g., Bednarz & Janvier, 1996; Carraher, et al., 2001; Chazan, 2000; Mason, 1996), even though many researchers do not emphasize *quantitative reasoning* as a basis for algebraic reasoning. However, some researchers are critical of this emphasis on connection and view algebraic reasoning as a fundamental break from students' previous mathematical activity (e.g., Dettori, et al., 2001; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Wheeler, 1996). In this section I consider other scholars and researchers' views on and approaches to algebraic reasoning in order to help define my own.

Generalizing arithmetical activity. As noted in previous reviews of research on algebra learning (Kieran, 1989, 1992), investigating algebraic reasoning as generalized arithmetic is a popular approach. Researchers who take this approach (e.g., Booth, 1984; Carraher, et al., 2001; Falkner, et al., 1999; Herscovics & Kieran, 1980; Peck & Jencks, 1988; Pirie & Martin, 1997) focus on students identifying and generalizing patterns from whole number computation; work to foster conceptions of the equal sign as a relation versus as a signal to compute; and emphasize unknowns as missing numbers in number sentences. Although these researchers do not explicitly advocate reasoning with quantities as a basis for constructing algebraic reasoning, they do view arithmetic and algebra as connected. For example, Booth posited that students' errors in algebra stemmed from poor learning of arithmetic, and Peck and Jencks stated that algebra "should arise logically and naturally as a consequence of children's decisions about how arithmetic works" (p. 85). As a group, their views seem to be consistent with those of mathematician Hyman Bass (1998), who has described algebra as being about arithmetic operations on basic number systems, including integers and rational numbers, as well as the study of algebraic equations that arise from these systems.

Research done from this approach makes important contributions to understanding students' construction of patterns and conceptions of equivalence. But some of the work reifies computational activity to determine missing numbers in number sentences without necessarily bringing forth students' conceptual structures. For example, a central aim of Carpenter and his colleagues (Carpenter & Levi, 2000; Falkner, et al., 1999) is to help kindergarten through second-grade students develop their conceptions of equality through solving problems like 13 + 5 = ___ + 17. Such problems may not be useful for young students who cannot yet operate on the structure of their counting schemes. Herscovics and Kieran (1980) and Pirie and Martin (1997) have worked similarly with middle and high school students to generate solution processes of basic and more complex linear equations in one variable involving solely whole numbers. This tack seems overly restrictive with older students since it does not address solving linear equations in "continuous" versus "discrete" contexts and therefore may not engender multiplicative ways of operating necessary for more complex equation solving.

Formulating equations and operating on unknowns. A diverse group of researchers—some who take a generalized arithmetic approach and some who do not—have sought to understand how students formulate and solve linear equations in one variable. Some of these researchers have proposed explanations for students' difficulties in operating on unknowns when formulating and solving equations (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994), while others have contested such difficulties (e.g., Carraher, et al., 2001; Swafford & Langrall, 2000) or researchers' explanations for them (e.g., MacGregor & Stacey, 1993).

Filloy and Rojano (1989) posited a significant distinction between arithmetic and algebra called a "didactic cut" that students experience in moving from solving equations like $ax + b = c$ to solving equations with unknowns "on both sides," like $ax + b = cx + d$. Herscovics and Linchevski (1994) reexamined this conclusion by investigating students' informal solution processes of first-degree equations. They reformulated the distinction between arithmetic and algebra as a "cognitive gap" that involved students' inability to spontaneously operate on unknowns. Although I don't deny that students may have difficulty operating on unknowns, the approach of viewing algebra as disjunct from arithmetic does not fit with my ontogenetic approach to algebraic reasoning, or with how I conceive of students' mathematical ways of operating in general (cf. Steffe, 2002a). In contrast, Carraher and his colleagues (Carraher, et al., 2001; Brizuela & Schliemann, 2004), who have a central aim to help students view algebra as growing out of arithmetic, have worked with nine-year-olds and ten-year olds who successfully write equations and operate on unknowns "on both sides" in additive contexts. However, like others who take a generalized arithmetic approach, their work does not adequately address how students might build multiplicative relationships between unknowns in continuous contexts.

Similar to Carraher and colleagues, Swafford and Langrall (2000) seek to connect algebraic activity to students' previous mathematical knowledge. They have demonstrated that prior to instruction, sixth-grade students can formulate equations to represent problem situations (see also Dettori, et al., 2001). However, students in their study rarely used the equations to solve the problems (see also Kieran, et al., 1996), preferring to reason through solutions independently of their formulations of equation. In other research on students' formulation of linear equations, MacGregor and Stacey (1993) found that students' errors (including the famous reversal error, cf. Clement, 1980; Rosnick & Clement, 1982) are rooted in students' cognitive models of situations rather than in difficulties in syntactic (i.e., notational) translation. Both of these studies suggest that research is needed to understand the ways in which students' mathematical ways of operating form a foundation for students to formulate equations that are useful to them.

Generalizing more generally: Structure and multiple representations of it. Another central approach to research on algebra learning focuses on algebraic reasoning as generalizing procedural activity into structural ways of thinking (cf. Mason, 1996; Sfard, 1995). Some researchers who take this approach adopt process-object perspectives (e.g., Kieran, 1992; Lins, 2001; Moschovich, et al., 1993; Sfard, 1995; Sfard & Linchevski, 1994). These researchers do not include a specific emphasis on quantities or quantitative reasoning. Instead, they focus on *reification*, a transition they posit that algebra students can (or must) make in coming to regard a mathematical process as a mathematical "object" that then can be taken as a given in further operating. For example, students who view $2x + 5$ as only a process (taking twice a number plus five) and not as an object (the result of this process) may find it problematic to use $2x + 5$ as an input for, say, taking half of it, or squaring it, or substituting it into another expression. From my perspective, reification is an important example of interiorization and merits a good deal of study in understanding how students reason algebraically. However, a focus only on reification does not seem to inform the generation of algebraic reasoning more broadly and seems akin to aiming only for vertical learning in interactions with algebra students.

Researchers who analyze students' coordination of multiple representations of an equation or function also sometimes use this process-object approach (e.g., Moschovich et al., 1993), while others focus more on marrying the semantic and syntactic aspects of algebra (e.g., Kaput, 1989). Researchers in this area who conduct micro-genetic analyses typically track students' progress in building knowledge structures through coordinating graphs, tables, and symbolic equations (e.g., Izsak, 2000; Moschovich et al., 1993; Schoenfeld, Smith, & Arcavi, 1993). Such research usually does not focus on quantitative reasoning, although it may when researchers engage students in using algebra to model the physical world (e.g., Izsak, 2000). Although these researchers are dedicated to understanding the representations that students produce, they are likely to focus on the match (or mis-match) between students' ways of operating and the knowledge of mathematically-knowledgeable adults, and they are unlikely to conceive of algebra as the ways of operating that students produce.

Analyzing contexts and problems. Many researchers who focus on algebraic activity as emerging from analyzing contexts and problems explicitly advocate reasoning with quantities. For example, in Confrey's (1998b) talk at a conference about the nature and role of algebra in the K-14 curriculum, she cited the advantages of using a contextual approach of reasoning with quantities over conceiving of algebra as generalized arithmetic because students may find such an approach more meaningful. At the same conference and for similar reasons, Dossey (1998) advocated quantitative reasoning in three of his four goals for school algebra.

Other educators and researchers who focus on contextual situations and problems lend more indirect support for quantitative reasoning as a basis for algebraic reasoning. From my point of view, Moses' (Moses & Cobb Jr., 2001; Moses, Kamii, Swap, & Howard, 1989) work with middle and high school students in the Algebra Project revolves around quantity and quantitative situations, but he does not identify them as such. Similarly, although Bednarz and Janvier (1996) do not discuss quantity, they have examined students' solutions to a wealth of problems involving relationships between known and unknown discrete quantities in order to better understand students' transitions between arithmetical and algebraic problem solving. In addition, Nathan and Koedinger (2000) have shown that while adults and textbooks support the position that algebraic word problems are hard, students have more trouble with "naked" computation than with contextualized problems. All contextualized problems do not invoke quantitative reasoning, but many do so; thus Nathan and Koedinger's research provides implicit support for using quantitative situations in learning and teaching algebra.

Finally, those who choose functional approaches to algebra often highlight quantitative reasoning as a basis for algebraic reasoning. For example, Chazan (2000) chose a functional approach to teaching algebra because it makes reasoning with quantities the central object of study. He and others who focus on co-varying quantities (e.g., Heid, 1996; Kieran, et al., 1996) have mined the advantages of engaging students in examining and analyzing "real world" problems. However, these researchers may jump too quickly to using variables and co-variation with students without sufficiently building up conceptions of unknowns and "static" multiplicative relationships. In fact, starting algebra instruction with variables and functions has not been shown to result in stronger achievement in or understanding of algebra (Stacey & MacGregor, 2001). So while these researchers advocate using quantities to connect with students' experiences, a functional approach may not sufficiently connect with students' current ways of operating or provide better ways to bring forth students' algebraic ways of operating.

General Constructive Resources for Quantitative Reasoning

In this section, I give an overview of three general *constructive resources*—types of reasoning or mathematical activity that I believe are involved in reasoning quantitatively as a basis for beginning to reason algebraically. All three resources—multiplicative reasoning, fractional reasoning, and reasoning with unknowns—are implicated in quantitative reasoning because relationships among quantities are frequently multiplicative, fractional, and unknown: For example, my height is seven-eighths of your height. This example indicates that there is considerable overlap between these three constructive resources. A central goal of my study was to understand in detail how these constructive resources are involved in solving problems that can be solved with a linear equation like $ax = b$.

Multiplicative Reasoning

Units-coordination. I agree with other researchers (e.g., Kamii & Housman, 2000; Kaput & West, 1994; Steffe, 1992; Thompson & Saldanha, 2003) that multiplication conceptualized only as repeated addition is not constitutively multiplicative and is insufficient for algebraic reasoning. With these researchers I conceive of multiplicative reasoning in the context of units-coordinations that students make (cf. Steffe, 1988, 1992), where the simplest multiplicative structure involves coordinating two levels of units. The repeated addition conception in which, for example, 5*4 is conceived of as 4 + 4 + 4 + 4 + 4, is primarily additive in structure (Kamii & Housman, 2000) and is problematic for conceiving of, say, (2/3)*4 ("4 repeated 2/3 times" seems odd, cf. Thompson & Saldanha, 2003). A multiplicative conception of 5*4 involves envisioning five units, each of which contain four units, so the person's concept for four is in some sense "injected" into each of the five units (see also Confrey & Lachance, 2000; Steffe, 1992; Wright, Martland, Stafford, & Stanger, 2002). From this point of view, multiplying schemes, by their very nature, have a distributive aspect.

In coordinating 5 and 4 in this way a student has made at least a *unit of units*, in that each of the five units is a unit consisting of four units. Students who can coordinate two levels of units prior to operating can often engage in strategic multiplicative reasoning.[23] For example, if they have four strings each containing five beads and they get three more such strings, they can reason that they have 35 beads in all because they have seven strings total, each containing five beads, and they know that counting by five seven times produces 35 beads. Coordinating three levels of units prior to operating is even more sophisticated and involves conceiving of 20, say, as a unit of five units each containing four units. Students who can take a *unit-of-units-of-units* structure as given can, in the bead example above, reason that they have 35 beads in all by combining 20 beads from the first four strings with 15 beads from the additional three strings, thereby uniting two units that consist of units of units. Coordinating three levels of units prior to operating—what I will often refer to as having constructed three levels of units—seems to be required for a good deal of fractional reasoning (cf. Steffe, 2002a, in press). For my study, my intention was to select two students who were solidly coordinating three levels of units prior to operating. During selection interviews I used fractional tasks and tasks involving the splitting operation to make initial assessments of students' multiplicative structures.

Splitting operation. Steffe's (2002a) splitting operation is a fundamental multiplicative operation. The splitting operation is used to solve problems like the following:

> Example 3.4, Splitting Problem: This rectangle is a drawing of my candy bar, which is five times longer than yours. Make your bar.

Students who can solve this problem engage in several operations nearly simultaneously—in fact, Steffe has called splitting a composition of partitioning and iterating. Students need to posit their bar, which stands in relationship to the given bar but is also separate from it: Their bar can be iterated five times to make the given bar, and at the same time, their bar is formed from

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[23] Note that these students have constructed at least an explicitly nested number sequence (Steffe, 1988) where the unit 1 is iterable. For these students, 5 consists of one iterated five times, rather than just being the result of a count of five items.

partitioning the given bar into five equal parts. So students who solve a problem like Example 3.4 are aware of at least one multiplicative relationship between their bar and the given bar, namely that their bar taken five times produces the given bar. They may not necessarily be aware that one-fifth of the given bar produces their bar, depending on the state of their fraction language and fraction schemes.

In a sense, students need to reason "in reverse" to solve the Splitting Problem. They must determine that they are given a whole and are to make a (fractional) part of it that stands in multiplicative relationship to the whole. So this problem is fundamentally different (more complex) than a problem in which they are asked to make one-fifth of a candy bar. In this case, making one-fifth is a result of a student's fraction scheme, whereas Steffe has commented that the concept of one-fifth is *input* to the splitting operation, even if the student does not name the part as "1/5" of a bar (Leslie P. Steffe, personal communication, February 11, 2005). So the splitting operation is crucial for constructing *reversible* multiplicative relationships among quantities. For example, in the Juice Problem (Example 3.1), students need to split 28 ounces into four equal parts in order find the unknown amount of juice.[24]

In addition, Steffe (2002a) has hypothesized that the splitting operation opens the possibility for constructing improper fractions, among other fraction schemes. One reason that splitting seems important for the construction of improper fractions is that splitting involves disembedding a posited part from the given whole. Similarly, making improper fractions requires transcending part-whole relationships so that, for example, nine-eighths stands independently from the whole, eight-eighths, as one-eighth disembedded from the whole and iterated nine times. At the same time, nine-eighths clearly stands in relationship to the whole since it is made

[24] Splitting is also involved in problems like Examples 3.2 and 3.3, as I will explain and explore in the case studies, Chapters 5 and 6.

from parts that can be iterated eight times to make the whole. In this way, nine-eighths is a unit of nine units any of which can be iterated eight times to make the whole, another unit embedded within the nine-eighths. Thus nine-eighths can be seen as a three-levels-of-units structure. When I started my study, I assumed that students who had constructed the splitting operation could already operate multiplicatively with three levels of units, or would soon be able to do so.

Distributive reasoning. Even when operating solely with whole number calculation, using multiplication distributively is a hallmark of reasoning algebraically because doing so involves operating on the structure of one's multiplying scheme and number sequence. That is, a student who can calculate 7*13 mentally by finding 7*10 and adding it to 7*3 is engaging in what Steffe likes to call "algebraic calculation." Distributive reasoning seems even more important in reasoning multiplicatively with fractions (as I explored in my study): Making two-sevenths of three-fifths of a foot can involve taking two-sevenths of one-fifth of a foot three times, or one-seventh of three-fifths of a foot two times. The latter option requires further distribution: Taking one-seventh of three-fifths of a foot means taking one-seventh of each of the three one-fifths. A simpler version of this reasoning is required in the Tree Problem (Example 3.3): finding one-half of 3/4 decameter entails finding one-half of 1/4 decameter three times. Distributive reasoning is also involved in reasoning multiplicatively with unknowns. For example, taking one-fourth of a collection of three candy bars of different lengths by taking one-fourth of each candy bar's length is a basis for a student knowing that $1/4(x + y + z) = (1/4)*x + (1/4)*y + (1/4)*z$.

Fractional Reasoning

Partitive fraction schemes. A partitive fraction scheme is the first scheme that Steffe (2002a) regards as fractional. A student who has constructed a partitive fraction scheme conceives of one-fifth of a candy bar as a part that, when repeated five times, will produce a bar exactly like the original. To make three-fifths of a candy bar, the student can partition the candy bar into five equal parts, take out one of those parts, and repeat it to make three parts. Thus the student has transcended part-whole conceptions of fractions in that the student can *disembed* a unit fractional part from the whole and iterate it to make another fractional part of the whole that is smaller than or equal to the whole. That is, the student is also tied to the whole in that most students with "basic" partitive fraction schemes do not find it possible to make, say, seven-fifths of a candy bar by iterating one-fifth of the bar beyond the whole. So in this sense, the fractional parts students make remain tied to the whole—it is as if the "mathematical world" of the student is the whole, and going beyond it does not seem to make sense to them.

Iterative fraction schemes. Partitive fraction schemes are crucial for constructing fractional knowledge. But they are insufficient for algebraic reasoning, as well as for sophisticated quantitative reasoning in multiplicative, fractional contexts, because they do not include the construction of improper fractions. Improper fractions are involved as quantities in solving problems that can be solved using $ax = b$. For example, in the both the Peppermint Stick Problem (Example 3.2) and the Tree Problem (Example 3.3), the solutions are improper fractional quantities. "Seven-thirds of an inch" or "nine-eighths of a decameter" will make little sense to students who have constructed only a partitive fraction scheme. Being able to generate *any* fraction by a whole number iteration of a part of it (e.g., to make 9/8 dm from 1/8 dm iterated nine times, or from 3/8 dm iterated three times), and yet maintain its relationship to the whole (e.g., simultaneously know that 9/8 dm is a whole, 8/8 dm, and 1/8 dm more) means fractions are freed from relying on the whole for meaning even though they stand in relation to the whole. Constructing such a scheme for improper fractions, an *iterative fraction scheme*, is a significant advance in students' fractional reasoning (Steffe, 2002a, in press) and is required to solve problems like the Peppermint Stick and Tree Problems quantitatively or algebraically.

Reversibility and reciprocity. In addition, implicit in the solution of the Tree Problem is an improper fraction that is used as a multiplicative operation—i.e., three-halves of the 3/4 decameter is the tree's height. Identifying this fractional multiplicative relationship between the two quantities involves conceiving of the relationships between the quantities bi-directionally, or reciprocally. That is, it entails knowing that if A is two-thirds of B, then B must be three-halves of A. As I have described earlier in this chapter, reasoning reciprocally is a characteristic of algebraic reasoning because it involves constructing a structural view between two quantities, operating on both known and unknown in a "simultaneous" manner. In problems like the Tree Problem, having constructed improper fractions as quantities seems necessary to be able to construct them as multiplicative operations on quantities, although not necessarily a guarantee of constructing reciprocity.

In both the Peppermint Stick and Juice Problems, a slightly simpler version of reciprocity is involved because the whole number multiplicative relationships mean that only unit fractions are involved as operations on quantities. These problems can be used to demonstrate how the reversible reasoning inherent in the splitting operation may be an important constructive resource for reciprocity. For example, a student engaged in the Peppermint Stick Problem likely has a goal to split the stick into three equal parts, any of which will be one-third of 7 inches. If the student simultaneously conceives that taking one-third of 7 inches three times produces the original stick, and taking one-third of the original stick produces the one-third of 7 inches, then the student has begun to construct a basic reciprocal multiplicative relationship. In other words, if A is three times B, then B is one-third of A. One reason for the relative simplicity of constructing reciprocity in this problem is that for some students, taking one-third and dividing by three may be closely associated, even if neither of those operations are identified explicitly with multiplying a quantity by one-third. But in the Tree Problem, taking half of the 3/4 decameter (which is one-third of the unknown height) three times to produce the tree's height may not so easily be identified with taking three-halves of 3/4 decameter. So even if students can reason reversibly with fractional quantities and relationships to solve the Tree Problem, they may not have constructed reciprocal relationships between the two quantities.

Reasoning with Unknowns

Positing unknown quantities. The splitting operation may also provide an important resource for explicitly positing unknown quantities that can be operated upon, a well-accepted characteristic of algebraic reasoning. Recall that the splitting operation involves reasoning reversibly from a whole to another quantity identical to a part of the whole (i.e., given the whole, the student is to determine another quantity that can be used in iteration so many times to make a quantity identical to the whole). So the splitting operation requires positing a hypothetical quantity (cf. Steffe, 2002a) that stands in relation to the whole and yet is distinct from it. The quantity can be iterated some number of times to make the whole, while the whole can be partitioned into some number of parts to make the quantity. This mental activity is quite similar to the mental activity involved in positing the unknown quantities in the Juice, Peppermint Stick, or Tree Problems. In these problems, the unknown must be a quantity in its own right (independent from the known quantity) and yet it most certainly is related to (can be made from) the known quantity by reasoning with the given quantitative relationship. So the unknown amount of juice can be taken four times to make the known amount of juice, the unknown length of the other peppermint stick can be taken three time to make the known length of the original peppermint stick, and two-thirds of the height of the tree will be the given height.

Notation and symbolization. In reasoning quantitatively through these problems, positing the unknown (let alone operating on it) may be implicit. So one goal in transitioning from quantitative to algebraic reasoning is to engender the explicit construction of unknowns and operations on them—to help students increase their awareness, and subsequent use of, these implicit aspects of their quantitative reasoning. Verbalization is a primary way that mental activity first becomes notated and symbolized. In my study, I attempted to use students' explanations of their solutions to problems as an opportunity to increase my own understanding of their ways of operating and to engender their awareness of their own mental activity. For example, after a student had solved a problem like the Juice Problem (Example 3.1), I might have asked the student what fractional part of the 28 ounces the student has to drink, or what they have to do to their part to make the original amount of juice. Verbal articulation of these relationships between quantities is an important start to notating with algebraic symbols and developing that notation into a symbol system, although it is no means sufficient to engender such notational or symbolic activity.

Another important early form of notation that helps student increase awareness of their ways of operating is making drawings. Drawings can be made retrospectively, in which they function as subsequent notation of mental activity and may induce reflection on that activity—i.e., drawings can help the student externalize their schemes and operations, opening the way for reflected abstractions. Alternatively, producing and operating on drawings in the process of solving problems may engender new schemes, perhaps initially with a dominant figurative component that may eventually become interiorized. Just as with students' verbalizations, students' drawings can be a central tool for the teacher in building second-order models of the students' current schemes and operations. Drawings can also communicate a student's reasoning to her peers, whose activity may be influenced by her ideas.

After drawing, a student might use algebraic symbols to notate and reflect the reasoning that is represented by his drawing (which is in some sense a second-order reflection, since the drawing was already a trace of the student's reasoning). However, as long as students notate subsequent to drawing or reasoning, their notations tend not to function as a symbolization system. The transition to writing and operating on an equation like $(2/3)x = 3/4$ to solve the Tree Problem is significant and beyond the scope of this study. Such a transition likely requires interiorization of notation so that algebraic symbols can be used in ways that are independent from the sequence of operations used to "come up with" the symbols. This transition to using a symbolization system in lieu of operating on drawings, or notating retrospectively, coincides with greater awareness of and flexibility with one's ways of operating.

Reversible Multiplicative Relationship Problems

In this last section, I describe problems that can be solved by writing and solving a linear equation of the form $ax = b$. These problems all involve multiplicative relationships between a known and unknown quantity, and they require a solver to reverse her multiplicative reasoning in order to solve them. I want to emphasize that the problems are labeled and categorized from my point of view, not according to how I infer students viewed them. I used these types of reversible multiplicative relationship (RMR) problems with my students throughout the study.

Tree Problem: Type 5

The Tree Problem is an example of the most complex RMR problem I was aiming for students to solve by the end of the teaching experiment. Recall the statement of the problem:

Example 3.3, Tree Problem: Three-fourths of a decameter is 2/3 of the height of a tree. How tall is the tree?

Problems like the Tree Problem are complex because they involve a known fractional quantity and a fractional relationship, where the numerator of the fractional relationship *does not divide* the numerator of the fractional quantity. In other words, the problem would be simpler if four-fifths of a decameter was two-thirds of the height of the tree, because two divides four and so one-third of the tree's height would be 2/5 decameter. Splitting is still involved in this easier variation, but splitting four-fifths of a decameter into two equal parts is relatively easier than splitting three-fourths of a decameter into two equal parts. In general sense, solving RMR problems of Type 5 involves being able to relate any two fractions multiplicatively—a complex construction that Steffe calls the Rational Numbers of Arithmetic (2002a).

Candy Bar Problem: Type 4

Problems that I consider one step back from the complexity of the Tree Problem involve a whole number instead of a fractional quantity:

Example 3.5, Candy Bar Problem: That collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?

This problem is still considerably complex in that the numerator of the fractional relationship does not divide the known quantity. That is, if the original collection of candy bars contained only six inch-long bars, the problem would be quite a bit easier—then three inch-long bars would be one-fifth of the new collection. Splitting the seven inch-long bars into three equal parts can be a critical activity for a student in solving the Candy Bar Problem.

Peppermint Stick Problem: Type 3

Problems like the Peppermint Stick Problem are one step back from the Candy Bar Problem because they involve a whole number quantity *and* a whole number relationship, where the relationship does not divide the quantity:

> Example 3.2, Peppermint Stick Problem: A 7-inch peppermint stick is three times longer than another stick; how long is the other stick?

The Peppermint Stick Problem highlights one of the critical activities in solving the Candy Bar Problem (splitting 7 inches into three equal parts) without involving the added complexity of a fractional relationship. That is, making the other stick in the does not explicitly require fraction schemes, although they are necessarily involved in determining the new stick's length. In a general sense, solving problems of Type 3 entails being able to relate any two whole numbers multiplicatively—a complex activity that necessitates the construction and use of fractional quantities and unit fractional relationships.

Money Problem: Type 2

All three preceding types of RMR problems are more complex than problems like the following, even though a whole number quantity and a fractional relationship are involved:

> Example 3.6, Money Problem: Monica has $21, which is 3/7 of Todd's money. How much money does Todd have?

The Money Problem is easier to solve than the preceding problems because the numerator of the fractional relationship divides the whole number quantity. So a student can use her reversible fraction scheme to solve the problem, given that her fraction scheme includes conceiving of three-sevenths as three times one-seventh, and given that she can operate with composite units as fractional amounts (e.g., $7 is one-seventh of Todd's money). Type 2 RMR problems can increase in complexity for students if improper fractional relationships are involved (e.g., $21 is 7/5 of Todd's money).

Juice Problem: Type 1

Perhaps the most basic RMR problems are "simple" statements of whole number division, such as in the Juice Problem:

> Example 3.1, Juice Problem: Twenty-eight ounces of juice is four times the amount that you drank; how much did you drink?

RMR problems of Type 1 involve splitting, as I have described earlier in this chapter. They do not explicitly involve fractions, although students may assimilate them to their fraction schemes by identifying 7 ounces as one-fourth of the total number of ounces. I intended to invite four students to participate in my study who could solve Type 1 RMR problems, and at least two students who also conceived of 28 as a unit of four units each containing seven units. This intention proved somewhat difficult to assess, as I will explain in the next chapter on methodology and methods.

CHAPTER 4: METHODOLOGY AND METHODS

> We argue that, in such an exploration [of the limits and subtleties of children's construction of mathematical concepts and operations], there is no substitute for experiencing the intimate interaction involved in teaching children (Cobb & Steffe, 1983, p. 83).

Two central goals of a researcher conducting a teaching experiment are to understand students' current mathematical ways of operating and to engender changes in those ways of operating *through teaching the students*. To realize these goals means that the researcher must accomplish what Clement calls a central goal of science: to "develop explanatory models that give satisfying explanations for patterns in observations" (2000, p. 551). These explanatory models—which I have referred to as second-order models of students' mathematics (cf. Chapter 2)—become further refined through the researcher's on-going aims to bring forth and sustain students' learning based on their current schemes and operations. The dynamic nature of these models, and their foundation in students' previous and current ways of operating, demonstrate why ontogenetic approaches to research on mathematical learning are frequently accompanied by teaching experiment methodology. In this chapter I describe the teaching experiment methodology I used in my study and then outline the details of my teaching experiment.

Constructivist Teaching Experiment Methodology

Basic Characteristics

In a constructivist teaching experiment, researchers use teaching as a scientific method for investigating student learning (Cobb & Steffe, 1983; Steffe & Thompson, 2000b). A constructivist teaching experiment is similar to a clinical interview study because researchers seek to understand and explain students' current schemes and operations. But a teaching experiment is also distinct from a clinical interview study because researchers aim to account for how students' schemes and operations change as the researchers engage students in mathematical situations that are designed to engender the construction of more powerful schemes and operations. Since the researchers intentionally intervene, I refer to them as teacher-researchers. Generally speaking, thorough explanations of students' current and changing mathematical activity cannot be formulated until after intensive interactions with students, usually over time periods that range from six weeks to a year or more (Cobb & Steffe, 1983; Steffe, 2003).

Learning from students and conceptual analysis. In a constructivist teaching experiment, teacher-researchers are interested in justifying the mathematical constructions that students make, not in justifying their own first-order mathematical knowledge or *a priori* notions of mathematics. Teacher-researchers "remain aware that we may not, and probably cannot, account for students' mathematics using our own mathematical concepts and operations" (Steffe & Thompson, 2000b, p. 268). Instead, each teacher-researcher needs to learn new mathematics to understand students' mathematics, attempting "to put aside his or her own concepts and operations and not insist that the students learn what he or she knows" (p. 274). To this end, a teacher-researcher acts responsively and intuitively in learning to think like her students—in merging with the students' experiences to the extent that is possible (Leslie P. Steffe, personal communication, April 25, 2002) and in giving students' mathematical ways of operating an independent "life" (Cobb & Steffe, 1983). This responsive teaching is particularly important at the start of a teaching experiment, but it is a persistent way of acting throughout the experiment.

Simultaneously, teacher-researchers use their own first-order mathematical knowledge and second-order knowledge about students to engage in conceptual analysis of how students *might* act in the context of mathematical interactions. Prior to the start of a teaching experiment, conceptual analysis is one basis for constructing initial conjectures (Confrey & Lachance, 2000) and hypothetical learning trajectories (HLTs) (Simon, 1995; Steffe, 2003). Initial conjectures and HLTs guide teacher-researchers' work with students but they are subject to significant articulation and revision based on how students act in teaching episodes. Teaching experiments involve overarching conjectures as well as numerous local conjectures that are formed in the context of teaching episodes. The latter are sometimes more "sensed" than fully articulated and become elaborated or discarded during retrospective analysis. Overall, teacher-researchers strive to set conjectures aside temporarily when working with students in order to make the most progress possible with them—that is, to see and facilitate students' ways of operating without being "blinded" by looking for evidence related to the conjectures.

Teaching practices during the experiment. Out of the interplay between learning mathematics from students and engaging in conceptual analysis, teacher-researchers design and modify problem situations, formulate and test conjectures, and, over time, revise HLTs. As the teacher-researcher's models of the students become refined, teacher-researchers increasingly tailor problem situations and HLTs to these students' ways of operating. Teaching practices include presenting students with problem situations, assessing students' responses as indications of students' current schemes and operations, and determining new problem situations that might allow students to construct potentially more powerful schemes and operations. In my use of teaching experiment methodology, I was also explicitly concerned with monitoring students' affective responses to our interactions as indications of their levels of stimulation and depletion during our mathematical interactions. So I made modifications in problem situations and HLTs based on my inferences about students' cognitive *and* affective states.

Although a teacher-researcher engages in responsive teaching throughout a teaching experiment, as the experiment progresses the teacher-researcher acts in two other critical ways: as a learning-theorist and as a reflective researcher. The teacher-researcher acts as a learning theorist by revising and testing overarching and local conjectures to understand how generative and flexible students are in their ways of operating and how students transition to new ways of operating. Later in the experiment, the teacher-researcher also acts as a reflective researcher by looking back over the path she and students have taken to consider where they might go next. This activity involves revising HLTs, again tailoring them to these particular students. Acting as a responsive teacher, learning-theorist, and reflective researcher all contribute to the construction of *working* second-order models of students' ways and means of operating.

Roles of witness-researchers. Witness-researchers help the experiment to run smoothly by assisting the teacher-researcher in videotaping each teaching episode with two cameras: one camera to capture the interaction between the teacher-researcher and the students, the other camera to record the students' computer or written work. These videotaped records are crucial for the last phase of a teaching experiment, retrospective analysis that occurs after teaching episodes have ceased.

However, witness-researchers play a more critical role in teaching experiments by providing another perspective during all three phases of the experiment: the actual teaching episodes, on-going analysis that occurs between episodes during the course of the experiment, and retrospective analysis. While observing the teaching episodes, witness-researchers may formulate questions to test a particular conjecture that the teacher-researcher, enmeshed in the interaction, cannot immediately see. The teacher-researcher always has the prerogative to ask the witness-researcher to wait, because the teacher-researcher may be following up on another line of investigation. But the witness-researcher, as an “outside” observer of the interaction, can sometimes see aspects of students’ schemes and operations—or affective states—that the teacher-researcher cannot or does not see, and thereby can help the teacher-researcher to make significant progress with the students. Witness-researchers also provide valuable triangulation of interpretations in on-going and retrospective analysis.

Analysis

On-going. On-going analysis occurs in between teaching episodes. Central activities of on-going analysis involve making local conjectures about students’ current ways of operating and designing new problems and problem sequences for the next teaching episode. If breaks of more than a few days occur between a set of teaching episodes, on-going analysis may also entail revising HLTs, which sometimes entails significant changes in direction or approach. Viewing video records of a set of teaching episodes can be useful toward this end, although teacher-researchers should not expect to engage in significant video analysis while the teaching episodes are in progress. Regardless, teacher-researchers may keep a record—a research journal—of their musings and ideas during on-going analysis in order to provide a trace of the path they and students took through teaching episodes. Witness-researchers provide critical alternative perspectives in on-going analysis: Discussions with them of students’ current schemes and operations, as well as ways of operating that may be within their ZPCs, are highly valuable for the teacher-researcher as she prepares for subsequent teaching episodes and formulates working second-order models of students.

Retrospective. The goals of a constructivist teaching experiment include building *analytic* second-order models of students’ mathematics and producing “actual” learning trajectories based on second-order knowledge (Steffe, 2003; Steffe & Thompson, 2000b). Learning trajectories are considered actual not in the sense of being what students are "really" doing but in the sense of being constructed retrospectively to explain the states and activities of students after an extended period of mathematical interaction. They are based on revised conjectures and HLTs, and most importantly on the experience of working with students over time.

Reviewing video records and engaging in discussions of retrospective conjectures about students' mathematics are critical in retrospective analysis. Many teacher-researchers create a chronology of the path that they took with each student through the experiment by carefully reviewing videos chronologically and taking notes (cf. Cobb & Whitenack, 1996). The notes focus on what students are able to do, changes in how students operate, and constraints the teacher-researcher experiences with the students. Because of my focus on mathematical caring relations, my notes also included a trace of when students seemed stimulated or interested, when students seemed to become depleted or disinterested, as well as transitions between those states. The chronology may need to be revisited and revised several times before the teacher-researcher can identify a series of critical moments through the experiment that allows the teacher-researcher to tell a "story" of each student's progress. Using theoretical constructs from previous ontogenetic research and tools such as scheme theory, the teacher-researcher formulates explanatory constructs to account for each story. So each story, often written in the form of a case study (Stake, 2000), is essentially an analytic model of each student's ways of operating.

It is important to emphasize two aspects of creating an analytic model: looking across many episodes with the same student, and looking across many students' ways of operating, either in actual interaction or from previous research. That is, models gain credence to the extent that they can account for a student's ways of operating over a significant period of time (e.g., an academic year). Reading a single protocol or viewing a single video excerpt of that student's activity may yield many interpretations, but doing those activities in the context of numerous episodes can allow the researcher to construct an interpretation that is consistent across the entire data set. An explanatory model is more powerful—or viable—than a previous model if it can account for the observations the previous model accounted for and additional observations for which previous model did not provide explanation (cf. Thompson, 2000). In this sense one can speak of superceding second-order models of students' mathematics.

Models also gain credence to the extent that they can account for a student's ways of operating in relation to the second-order models of other students—either those within the same teaching experiment or those from previous teaching experiments. For example, in my study I was particularly cognizant of Jason, a student who participated in Steffe and Olive's Project, *Children's Construction of the Rational Numbers of Arithmetic* (Steffe & Olive, 1990). Steffe's model of Jason's ways of operating (Steffe, 2002a, 2002b, 2003, in press) provided an important contrast to help me refine my analytic models of two of my students, because I could use Jason as an *epistemic subject* (cf. von Glasersfeld, 1995, p.72; see also Piaget, 1970b, p. 139). That is, I could use Steffe's abstraction of the cognitive core of Jason's ways of operating to judge how my students' ways of operating were similar to and different from Jason's. Doing so helped me to consider my students as epistemic subjects—to abstract explanations of their cognitive cores with respect to their quantitative and algebra reasoning.

The Scientific Nature of Teaching Experiments

I have just described two ways in which analysis in a teaching experiment—making analytic second-order models of students' mathematics—entails notions of credibility or "validity." That is, looking across many episodes to make a consistent and comprehensive model within the data set, and looking across students to hone a model in relation to models of other students' mathematics, are both related to traditional conceptions of reliability of observations and external validity. Given the current controversy over the nature of scientific educational research (e.g., Berliner, 2002; Erickson & Gutierrez, 2002; Feuer, Towne, & Shavelson, 2002; Pellegrino & Goldman, 2002), and given the controversy over traditional methodological concepts like validity (e.g., Lather, 1993; Richardson, 1993; Scheurich, 1996), I now explore the scientific nature of teaching experiments a bit further.

In defense of small numbers. Teaching experiments that involve a small number of students have been criticized for being unrealistic in not conforming to the constraints of schooling. In this sense, these teaching experiments have lacked "face validity" in the eyes of some. Perhaps as a result, whole class teaching experiments have also become prevalent (e.g., Cobb, Wood, & Yackel, 1990; Confrey & Lachance, 2000; Simon, 1995). However, Confrey and Lachance argue that confining research to whole-class settings can constrain knowledge-building and slow reform efforts; relaxing constraints may be instrumental in making room for more speculative research. I would argue that relaxing constraints may also facilitate the development of fresh visions for how schooling could be. In particular, I claim that to understand mathematical learning and mathematical caring relations well in the way I have described in Chapter 2, intensive work with a small number of students is crucial.

Furthermore, the supposed need for large numbers of students in order to be "doing science" is based on the notion that scientific statements can be made *only* from generalizing from a large number of observations—i.e., on inductive inference. The problem with viewing inductive inference as the only basis for science is that it leads to the learning paradox (cf. Steffe, 1991b). That is, if doing science is only inductive inference from a (large) set of observations, then how can a scientist create something new—a higher or different level of thinking about an issue or area—out of ways of thinking and observing that are tied to the ways of thinking or observing that a scientist already has (cf. Piaget, 1970a; von Foerster, 1979)?

Piaget answered this question largely through reflective abstraction as a mechanism for the coordination of sensory, motor, and operative experiences. As von Glasersfeld (1995) says,

> Coordination…is a strictly internal affair and, therefore, is always subjective to the coordinator. This applies…to every observer, be it a scientist or a simple bystander. Like all cognizing organisms, they [scientists or philosophers] draw conclusions from their own sensorimotor and conceptual experience, and any explanation of their conclusions, i.e., their "knowledge," must be in terms of *internal* events and cannot draw on elements posited elsewhere (p. 72).

In other words, conducting a large number of observations is not what makes knowledge scientific (although certainly in research based on inferential statistics, a large number of observations can be critical to using statistical models). Instead, the scientist's coordination of her experiences is what produces (or does not produce) knowledge that a community of scientists—or bystanders, or philosophers—calls scientific. If a scientist makes such coordinations in the service of creating a conjecture or explanation that, if it were true, would account for a phenomenon, the scientist is engaging in what Charles Peirce called *abduction*, which can be based on a single case (cf. Clement, 2000; von Glasersfeld, 2001). Of course, that does not mean that such creative scientific acts are not subject to a great deal of rational scrutiny (based on criteria such as simplicity and consistency) and further testing with other cases in order to refine a particular model so that it seems to account for new observations (Clement, 2000).

Reliability reconceived. Instead of asking when an observation is reliable (a question that is a bit spurious in a radical constructivist framework—reliable *to whom* must be considered since there is no absolute reliability that can be known), a teacher-researcher might ask: How can or do I experience recurrence in my observations? Since a teacher-researcher in a constructivist teaching experiment posits that each person has a reality of her own and thus cannot simply be instructed so that she operates the way someone else (such as the teacher-researcher) does, the teacher-researcher seeks to explain and account for aspects of how the person operates. In other words, in observing students' activity the teacher-researcher sees ways of operating that are *not* the teacher-researcher's—i.e., ways of operating that the students themselves contribute. If the teacher-researcher observes a student solving a problem, the teacher-researcher needs to test out the conjecture "the student can solve these kinds of problems" by trying out another (similar but not identical) situation with the student. That is, the way of operating must be repeatable for it to be characteristic of the student and to form a part of an analytic model of the student's mathematics. In this way, during the teaching experiment I corroborated my observations of my students, and in doing so, my observations might be seen as reliable.

Validity reconceived: Viability. Some researchers in the learning sciences call for reconceiving classical notions of validity in terms of viability (e.g., Clement, 2000), in which an explanatory model is judged by its plausibility, empirical support (from data), rational support (such as coherence with previous models), and "tests over time" (p. 560) with regard to data from new situations. The latter criterion might be called external viability, which is related to classical notions of external validity, or generalizability. My discussion of looking across many episodes with the same students demonstrates efforts to produce plausible, empirically-grounded models of students' mathematical ways of operating. My discussion of looking across models of students—even those outside my study—shows how rational support can be built for second-order models in a teaching experiment.

Developing rational support for a model is linked to building external viability of other models—that is, searching for rational support for a current model may impact the external viability of previous models (perhaps provoking modifications in them). Piaget's notion of the epistemic student, which I have already mentioned as the abstraction of the cognitive core of some aspect of a student's ways of operating, is useful in illustrating this linkage. For example, I might think that a student in my study was epistemically similar to Jason in Steffe and Olive's Rational Numbers of Arithmetic Project (1990; Steffe, 2002a). To investigate this idea, I would try to corroborate the conceptual boundaries of what each student can do—note that with Jason I would have to investigate via reading and perhaps consulting with the researchers on the project. I would give problems to my student similar to those Jason solved and those he did not. Then, once I knew well the "boundaries" of these two students, I would see what I might do so that my student might change her boundaries. To the extent that my student acts like Jason in interaction with me to bring forth such changes is the extent to which Steffe's model of Jason's ways of operating may be useful (or generalizable) in this new context.

Ultimately, of course, in qualitative research, notions of reliability, validity, or viability hinge on the quality of thinking and internal consistency of the teacher-researcher, since a researcher doing qualitative research is, in a significant sense, the research instrument (cf. Peshkin, 1988; Richardson, 2000; Weis & Fine, 2000). Fortunately, teacher-researchers do not have to "go it alone." Both von Glasersfeld (1995) and Noddings (2002) emphasize the role others play in developing one's own thoughts and characteristics. Von Glasersfeld speaks of intersubjective knowledge as the highest, most reliable level of experiential reality" (p. 119),[25] and Noddings believes that "how good I can be depends, insubstantial part, on how you treat me"

[25] Intersubjective knowledge is knowledge that a person finds to be viable in her own experiential reality as well as in that of another's experiential reality—in this sense, the knowledge gains a second-order viability (von Glasersfeld, 1995, p. 120).

(p. 2).[26] So the quality of thinking of others who participate in the research (such as witness-researchers) is critical, because their challenges to the teacher-researcher's interpretations can greatly enhance the viability of the teacher-researcher's model. The witness-researchers help provide rational scrutiny in the form of questioning the teacher-researcher's conjectures, and they provide a new look at the empirical support (the videotaped data) by offering their own interpretations based on their experiences and observations.

My Teaching Experiment

Basic Set-up

I taught two pairs of sixth-grade students at a rural middle school in a northern Georgia county from October 30, 2003 to May 12, 2004. The students were invited to participate after selection interviews in September and early October of 2003, which I will discuss later in this section. The students and I met twice weekly in thirty-minute sessions for two to three weeks, followed by a week off. The sessions occurred during school hours in a room on the sixth-grade hall in the middle school. Although every intention was made to minimize interruptions to academic classes by meeting during nonacademic classes, some interruptions were unavoidable. Most sessions included the use of JavaBars (Biddlecomb & Olive, 2000), a computer software program, and all sessions were videotaped with two digital cameras. Two witness-researchers were present at nearly all sessions. The two videotapes from each teaching episode were mixed electronically into a single video file where the video of the computer or written work was inset into the video of the interaction between teacher-researcher and students. These video files were used in retrospective analysis.

[26] Noddings uses "good" here largely to refer to virtues like congenial, sensitive, or polite, but stretching "good" to include astuteness or internal consistency in thought does not seem to go beyond the intentions of her work.

Room configuration. The room where we met for all but the month of May was an unused classroom at the end of the hall where sixth-grade classrooms were located. The students and teacher-researcher sat at a round table upon which were (generally) a laptop computer, a mouse and mousepad, and two microphones that were connected to a sound mixer. Cameras were located as shown in the diagram (see Figure 4.1). Witness-researchers sometimes appeared within the field of the interaction camera, although they tried to remain outside of view while still being close enough to observe the students' mathematical activity. For the month of May we relocated to a different classroom but retained the same configuration.

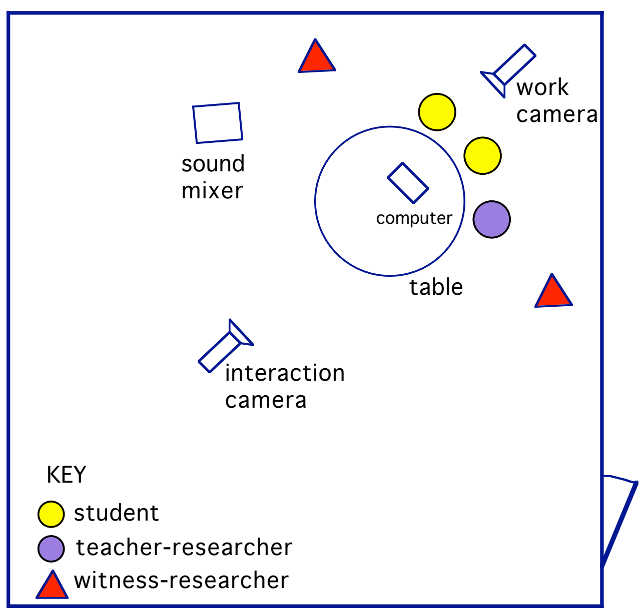

Figure 4.1, Room set-up.

JavaBars. The computer program that we used throughout the experiment, JavaBars (Biddlecomb & Olive, 2000), allows students to draw rectangles (bars) of variable dimension. Using the PIECES menu, students can partition bars "by hand" into "pieces" (e.g., they can estimate where to mark the bar to show, say, one-fourth of the bar). Or, students can use the PARTS menu to partition a bar into some number of equal parts (e.g., they can dial to 10 and mark the bar into 10 equal parts). Students can partition a partition (e.g., partition each of the 10 parts into, say, three equal parts), CLEAR a partition (erase all marks on the bar), BREAK a bar into its parts, CUT a piece or part off of a bar, or PULL OUT a piece or part from a bar. When students use the function PULL OUT, the part simultaneously remains in the bar and is disembedded from the bar (see Figure 4.2).

Figure 4.2, One-fourth of a bar pulled out from the bar.

Students can COPY bars or parts of bars, JOIN parts or bars together, or use REPEAT to iterate a part or bar some number of times. Students also can FILL a part of a bar, or a whole bar, to change it to a different color. In addition, JavaBars (Biddlecomb & Olive, 2000) has a MEASURE function: If students set a unit bar as the unit of measure, the computer will give the fractional "measure" of any bar made from that unit bar (see Figure 4.3). Finally, students can use the COVER function to draw a rectangular cover over a bar that they have made (covers are movable and removable).

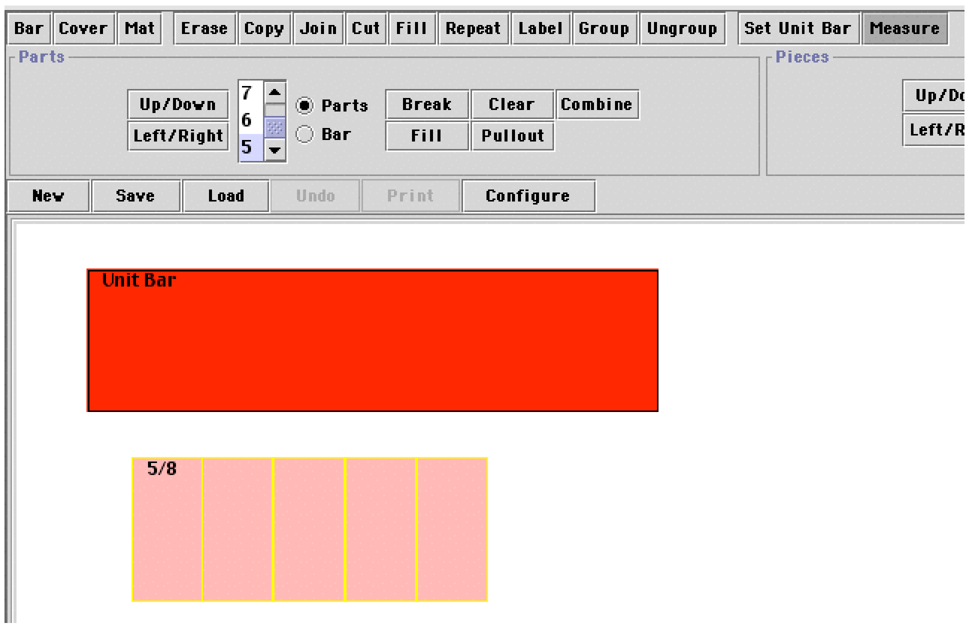

Figure 4.3, Five-eighths of the unit bar, colored (filled) a different color and measured.

Initial Overarching Conjectures

I started the experiment with two initial, overarching conjectures. First, I conjectured that emphasizing mathematical caring relations would facilitate students' engagement with mathematical activity. I expected that these students, if they felt mathematically cared for, would learn to sustain mathematical activity while in a state of perturbation—even one that was consciously conflictive—because of their level of engagement. Sustaining mathematical activity through a consciously conflictive state of perturbation may assist students in developing greater ability to eliminate perturbations by making accommodations, as opposed to eliminating perturbations by abandoning activity or engagement. I refer to the ability to eliminate perturbations by making accommodations in one's schemes and operations as the ability to reestablish *viability* (von Glasersfeld, 1987a, 1987b) of one's schemes and operations.[27] So my conjecture was that participation in MCR would strengthen students' ability to reestablish viability in their mathematical environments, and therefore would be beneficial for students' mathematical learning. In turn, I believed that students' reestablishment of viability would affect their level of engagement. So the relationship between engagement and reestablishment of viability seemed cyclical, and I thought of MCR as an entrée into this cycle.

My second conjecture was that students' multiplicative structures with whole numbers would significantly constrain and open possibilities for their construction of quantitative and algebraic reasoning. That is, I expected that students' coordination of units prior to operating, as well as in activity, would be fundamental in differentiating how students reasoned multiplicatively with fractional quantities and solved reversible multiplicative relationship (RMR) problems (problems that can be solved via basic linear equations of the form $ax = b$, as I described in Chapter 3). My intentions were to select one pair of students who coordinated three levels of units prior to operating, and another pair of students who coordinated at least two levels of units prior to operating (but may not have given solid indication of coordinating three levels of units). Steffe (2002a) has posited that the reasoning of students who coordinate only two levels

[27] Note that the use of viability here is not different from its use as a criteria for judging the quality of scientific research: Ways of operating (or models or theories) are viable "as long as our experience can be successfully fitted into them" (von Glasersfeld, 1987b, p. 139), i.e., as long as they do not "knock against" constraints in our experiential worlds and therefore need to be discarded or modified (p. 140).

of units prior to operating is constrained by the lack of a splitting operation, which was one basis for creating tasks for selection interviews.

Selection interviews. In September and October of 2003 I and the witness-researchers conducted twenty-minute selection interviews with 20 sixth-grade students out of a pool of approximately 100 students, all of whom had the same classroom mathematics teacher.[28] I observed four of this teacher's five mathematics periods, which were organized by level of achievement, and I consulted with this teacher to identify students to interview. As I have stated, for my study I sought students who were multiplicative reasoners—who were coordinating two or three levels of units prior to operating. I also wished to select students who exhibited a range of school mathematics achievement and a willingness to communicate with me and the other researchers verbally. Finally, I intended to work with a mix of girls and boys and with students from whom I sensed that I would learn a great deal (cf. Kvale, 1996; Stake, 2000).

During the one-on-one interviews, which were not tape-recorded or video-recorded, I used mostly fractional tasks (see Appendix A) to gain initial understanding of students' multiplicative structures and fractional reasoning and their general manner in interaction with me. The witness-researchers and I discussed the interviews, attempting to identify students who fit well together cognitively in order to create pairings. So from our notes about students' mathematical activity and overall manner in interaction with us, we made inferences about their multiplicative and fractional reasoning, as well as their potential to work with specific other students. Once I and the witness-researchers had identified potential pairs of students, we met with the principal and counselor to discuss their perceptions of the students' relative stability at the school, their attendance records, and the feasibility of them missing an occasional academic

[28] Another witness-researcher was also planning to be a teacher-researcher with four other students who did not reason multiplicatively or who coordinated at most two levels of units prior to operating. Hence a total of eight students were being selected.

class. After this meeting, I and a witness-researcher met with eight students (my four participants and four other students) to invite them to participate in the study.

The students. My students ended up being a pair of girls and a pair of boys, although I did not intend them to be paired by gender. (In fact my initial pairings were mixed in gender, but one student was planning to move from the district before the end of the year and another decided she did not want to participate.) The girls, both Caucasian, were in the top two mathematics periods in the sixth grade. The boys, one Caucasian and one Asian-American, were in the middle level mathematics period. In their initial interviews, all four students demonstrated that they had constructed at least a partitive fraction scheme (Steffe, 2002a). I made this inference because, given a cardboard rectangle that represented a candy bar, all four students marked an estimate for the fair share for *one* out of six people sharing the bar. They indicated that to test whether the share was indeed fair they could cut off their estimate and repeat it six times; if the share was fair, the result should be the whole candy bar. They all used fraction language, calling the fair share one-sixth of the candy bar.

All four students also demonstrated a splitting operation—that is, they could solve problems like Example 3.4: This rectangle is a drawing of my candy bar, which is five times longer than yours; make your bar. They also demonstrated some reversible reasoning because they solved a Type 1 RMR problem like Example 3.1, the Juice Problem (Twenty-eight ounces of juice is four times the amount that you drank; how much did you drink?). However, one of the boys, Carlos,[29] did not initially reverse his reasoning in that he wanted to multiply the given quantity by the given multiplicative relationship. A witness-researcher posed some other Type 1 RMR problems with the relationship "two times" and "four times," which Carlos successfully

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[29] The names of the four students are pseudonyms.

solved. The other boy, Michael, as well as the girls, Bridget and Deborah, divided right away when solving their Type 1 RMR problems, but all experienced difficulty drawing pictures to show their solutions. I interpreted the students' ability to split and solve Type 1 RMR problems as an indication that they were coordinating at least two levels of units prior to operating.

The girls seemed more advanced than the boys because during the selection interviews they solved problems that required them to reverse the operations of their fraction schemes—RMR problems of Type 2 like Example 3.6, the Money Problem (Twenty-one dollars is three-sevenths of my money; how much money do I have?). The girls also demonstrated that they had constructed recursive partitioning operations (Steffe, 2002a, 2003). That is, each girl shared a sub sandwich fairly among herself and four friends, shared her piece fairly with three latecomers, and determined the fractional part of the whole sub sandwich that the three latecomers received. I inferred that the girls were making and splitting composite units, which meant they were coordinating three levels of units prior to operating. This interpretation was to prove simplistic during the course of the experiment.

My initial impressions of the students' affective responses during the selection interviews suggest some variations in responsiveness to me and to the mathematical tasks. One of the girls, Bridget, seemed a bit reticent in working on problems, but at other times she indicated genuine interest in a problem with an expression (like "oh!"). The other girl, Deborah, was soft-spoken, worked intently, and verbalized her responses thoroughly. Michael, one of the boys, considered problems for long periods of time and seemed slightly amused by the whole experience. The other boy, Carlos, was generally friendly and seemed eager to participate. Many of these qualities persisted during the experiment, although others (like Deborah's soft-spoken manner) were not characteristic of the students once we were no longer strangers!

CHAPTER 5: MICHAEL AND CARLOS

In this case study, I first describe how Michael and Carlos solved problems involving reversible multiplicative relationships (RMR problems)[30] in February and March, including heuristic characterizations of their schemes and operations. Then I analyze key moments from November through March in the boys' construction of these schemes and operations, as well as key moments in March through May during which the boys participated in activity to extend these schemes and operations toward solving more complex RMR problems or making more sophisticated solutions of RMR problems. The key moments are organized into five constructive resources that seem pivotal in accounting for boys' ways of operating with RMR problems.

The description and analysis of the boys' schemes and operations in each constructive resource might be thought of as one of the main "results" of mathematical caring relations (MCR), where I use the word "results" in a similar way to how it's used in scheme theory: the outcome of a way of operating. But here the way of operating is the teacher's, rooted in the teacher's orientation to enact MCR. Focusing only on results leaves out a good deal of the story. So entwined with the description and analysis of the boys' schemes and operations, I describe and analyze the enactment of MCR between each boy and me, their teacher. This description and analysis might be thought of as the "situation" and "activity" of MCR. I conclude the chapter by using the five constructive resources to give an account of the boys' solutions to RMR problems in February and March.

[30] As I have noted in Chapter 3, the descriptions of RMR problems are given based on how I conceive of the problems, not how I infer that the students conceived of the problems.

Michael Solving Problems Involving Reversible Multiplicative Relationships

Making the New Collection of Candy Bars in February

Michael first solved RMR problems of Type 4 on February 18th. Recall that Type 4 RMR problems involve a fractional relationship and whole number quantity where the numerator of the relationship does not divide the quantity.[31] Carlos, Michael's partner, was absent this day. About midway through the episode I asked Michael to make seven inch-long candy bars from a unit bar (see Figure 5.1).

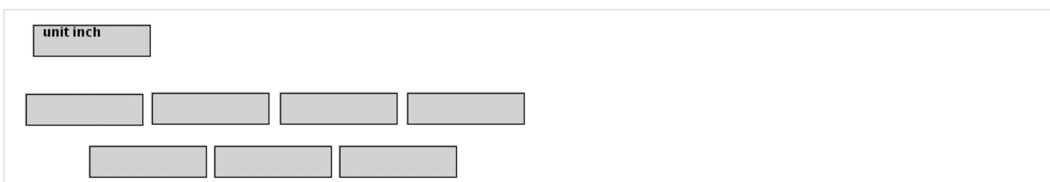

Figure 5.1, Seven inch-long candy bars, with unit inch shown.

Then I posed the following problem:

Task 5.1, the Candy Bar Problem: That collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?

Michael dragged the seven bars to align them end-to-end and told me that he was just doing so to "try something out" (see Figure 5.2).

[31] Michael had solved RMR problems of Type 2 in November. Recall that Type 2 RMR problems involve a fractional relationship and whole number quantity where the numerator of the relationship *does* divide the quantity.

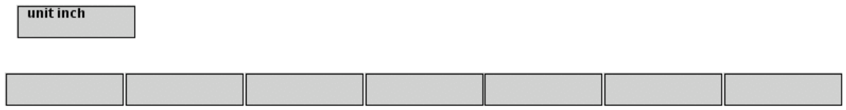

Figure 5.2, Seven inch-long candy bars aligned in a row.

From his activity of dragging the bars into a row, I infer that he conceived of the collection of bars as a single unit consisting of seven inch-long units, possibly because he was focused on the length of the collection. From his comment about "just trying something out," I infer that he did not immediately know what to do, and that he had entered a state of perturbation.

Protocol 5.1: Michael's solution of the Candy Bar Problem on 2/18/04.[32]

M: So something should be able to divide in three [8-second pause]. Two wouldn't work, 'cause it's odd [laughs a little]. Have to divide 'em into something.
T: Oh okay, all right.
M: There's seven there, so. [M dials to 3 in the PARTS menu and *partitions each inch-long bar into three equal parts*.] Three-fifths… [He adjusts each bar slightly, realigning them with the mouse.]
T: Yeah, so that's three-fifths of another collection.
W: You want him to make the other collection?
T: Mm-hmm [yes].
M: *There's twenty-one there anyway, divided by three would be seven.*
[M drags two of the 3-part bars down from the line of 3-part bars. He breaks the third 3-part bar in the line-up into its three parts, and he drags one of those parts down next to the two 3-part bars he dragged out of line, see Figure 5.3.]

[32] In the protocols, M stands for Michael, C for Carlos, T for the teacher-researcher (me), and W for a witness-researcher—recall that two witness-researchers were present during almost all teaching episodes. The protocols are numbered sequentially by chapter number (e.g., 5.1, 5.2, etc.). Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher-researcher's perspective. Ellipses (…) indicate a sentence or idea that seems to trail off. Four periods (….) denote omitted dialogue. Italics indicate verbalizations or activity which figure prominently in my analysis of the student's ways of operating or which are important in understanding my evolving model of the student. Since italics are used in this way, underlining indicates emphasis in speech.

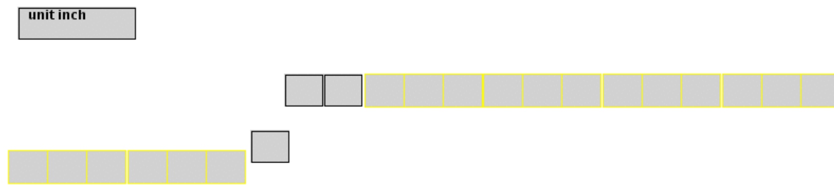

Figure 5.3, Making one 7-part bar.

[M joins the two 3-part bars and the part to make a 7-part bar. Then M drags this 7-part bar beneath the line of remaining 3-part bars and aligns the left end of 7-part bar with the left end of the remaining 3-part bars, see Figure 5.4.]

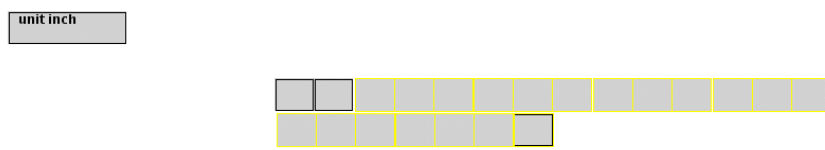

Figure 5.4, One 7-part bar aligned beneath the line of remaining 3-part bars.

[Under his breath, M starts counting at the eighth part in the line of remaining 3-part bars (top line in Figure 5.4). He counts from 1 to 7.]

M: I did it!

T: Okay, so…

M: And I know how many there are now. So, I'll just join these. [M makes two more 7-part bars out of the remaining 3-part bars and lines them up in a stack, see Figure 5.5.]

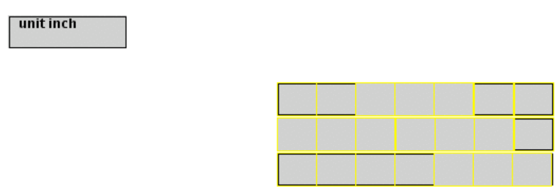

Figure 5.5, Three 7-part bars.

T: So have you made the other collection?

M: Twenty-one. [He rearranges the three 7-part bars into a line.] So that's, so each would have seven, so two times seven would be fourteen. [*He repeats one of the 7-part bars to make two more 7-part bars*.] Can't make them together though [i.e., the five 7-part bars will not fit in one line on the screen, see Figure 5.6]. So there would be thirty-five in all.

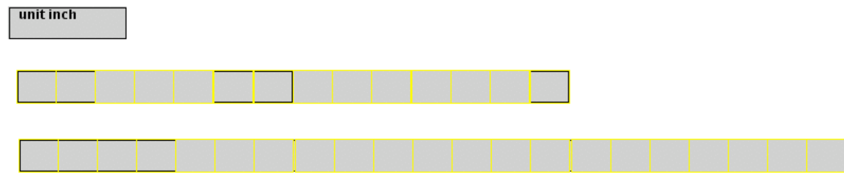

Figure 5.6, The whole new collection of candy bars.

The beginning of the protocol shows that Michael did not have a scheme established for operating in this situation and was using his current schemes to reason through it. His first comment in the protocol, that he was looking "to divide in three," indicates that he had formed a goal to divide the seven bars into three equal parts because each of these parts would be one-fifth of the new collection. This goal shows that Michael assimilated the problem using his reversible fraction scheme.[33] That is, Michael knew that if the seven bars were *three*-fifths of another collection of bars, then dividing the seven bars into three equal parts would produce three one-fifths of the new collection, any of which could be used to make the new collection. The formation of this goal indicates that for Michael, three-fifths meant one-fifth three times. It also shows that Michael could reverse his scheme for making three-fifths by dividing three-fifths by three to produce one-fifth.

However, Michael's subsequent comments indicate that he didn't know how to achieve his goal and was searching for a way. In other words, Michael had entered a state of perturbation where the perturbing element seemed to be the *seven* bars that he could not immediately divide into three equal parts. Michael's statement that "two wouldn't work" indicates his initial search for a way to accomplish his goal. Based on his activity in some previous episodes in January in

[33] I infer that Michael had a reversible *iterative* fraction scheme at this point. I will develop the rationale for this inference through this case study.

which Michael solved problems such as making a 2/2-bar into a 3/3-bar without clearing the halves,[34] I infer that he considered partitioning each of the seven bars into two equal parts. But he found that doing so wouldn't allow him to make three equal parts out of the seven bars. It's unclear from the videotape whether his reason, "because it's odd," referred to the seven original bars, or his goal of making three equal parts, or both. But it is clear that using *two* did not allow him to complete his goal.

Evoking his past experience in January of making a 2/2-bar into a 3/3-bar was an important basis for Michael's elimination of his perturbation. That is, Michael began to eliminate his perturbation by partitioning each of the seven bars into three equal parts, and then finished eliminating his perturbation when he determined the fruitfulness of this way of operating for accomplishing his goal. In this process, Michael modified his splitting operation (Steffe, 2002) (and the partitioning operation involved in splitting) to include a distributive operation. This modification was a functional accommodation in his splitting operation, and it was distributive on two levels. First, Michael distributed his operation of partitioning by three across each of the seven bars (this insertion of units into units was the activity of his multiplying scheme in a continuous context). Second, Michael partitioned *each* of the seven bars into three equal parts in order to divide the *entire* seven bars into three equal parts. The distributive aspects of his activity were novel—they are what Michael learned, and this learning made it possible for him to solve the problem. I make the claim that he learned based on the fact that this modification, "splitting distributively," was relatively permanent for him during the rest of the teaching experiment.

However, I *don't* claim that Michael was aware of the distributive pattern of his activity—that dividing each of the seven bars into three equal parts would allow him to divide

[34] To solve this problem, Michael partitioned each of the halves into six equal parts. I discuss his activity with these kinds of problems in the third constructive resource section of this chapter, "New Uses of Multiplying Schemes: Constructive Resource 3, January and March."

the entire seven bars into three equal parts. Nor did he explicitly indicate that taking one-third (one part) from each of the seven bars would yield one-third of the entire seven bars. Instead, Michael spoke in terms of whole number multiplication and division when he began to recognize the fruitfulness of his partitioning activity. That is, he noted that he had made 21 parts, and that he could divide the 21 parts by three to make three equal bars consisting of seven parts each. So in effect, Michael "converted" the composite unit of seven inches into a composite unit consisting of number of parts (21) that he *could* split into three equal parts using his splitting operation. In this sense, commutativity became evident in his activity—seven bars each split into three parts yielded 21 parts that could be rearranged into three bars each split into seven parts.

In this process, Michael used his multiplying scheme in a new way—in a continuous context and in service of his reversible fraction scheme. Thus as a result of modifying his splitting operation, Michael learned that multiplication in a continuous context (a units-coordination of seven and three in this case) was relevant for him to complete his goal to find one-fifth of the new collection of candy bars given three-fifths of it. By using his multiplying scheme in service of enacting his reversible fraction scheme, I infer that he constructed a new scheme I will call a *reversible multiplying scheme with fractions*, characterized by the embedding of a multiplying scheme within a reversible fraction scheme. His excited comment "I did it!" followed upon counting to check whether he had indeed made three equal bars from the original seven bars. The comment indicates he was rather thrilled to have accomplished his goal and corroborates that this way of operating was novel for him.

In addition, Michael's subsequent announcement, "And I know how many [parts] there are now [in all]," shows his confidence in what he had done. However, because he did not immediately make more bars, I intervened to find out whether he thought he had solved the problem. My question about whether he had made the new collection of candy bars may have oriented him to consider the larger goal of making the new collection, because he promptly made two more 7-part bars. Shortly after this protocol, in explanation of his work he said, "I took seven, seven, and seven, and that made the three [three-fifths]. So then I repeated [to make] two [more] of them." I infer that "two of them" referred to two more fifths of the new collection.

Determining the Length of the New Collection in February

In Michael's work to this point, reference to the unit bar as an inch-long quantity was not significant. To explore whether Michael could determine the *length* of the new collection, I questioned him about the 35 parts he had made.

Continuation of Protocol 5.1: Michael's solution of the Candy Bar Problem on 2/18/04.

T: Thirty-five—now thirty-five what—what are those little pieces?
M: Thirty-five…sevenths.
T: Thirty-five sevenths—is that what this little piece is of the unit bar? [T points to one of the little pieces, meaning to ask if each little piece is one-seventh of the unit bar.]
M: Wait! [Laughs a little.] That was twenty-one, so twenty-eight, and then thirty-five. But you said that [the 21 parts] would be three-fifths so it [the new collection] would be five-fifths.
T: Okay! So,
M: Thirty-five—wait—thirty-five—thirty-fifths.
[Both W and T tell M that he's doing great, and some talk ensues with W about how M knew how to do what he's done so far. Then T reorients M to think about the length of the unit bar and the length of the original collection of bars.]
T: I wonder how long the collection is now?
M: It would be…*three can't go into thirty-five equally, so it would have to be a fraction*, so…it would have to be eleven and two-thirds.
T: How'd you get that so fast?
M: 'Cause *three will go in there only eleven times because eleven times three equals thirty-three, and there'd be two left over, and since the unit bar would be divided by three, it would be two-thirds.*
T: Oh, my! Wow, M! [Both T and W are impressed—almost don't know what to say.]

Michael's comment that he had made thirty-five sevenths indicates that sevenths were relevant to him in the problem. Together with his initial aligning of the seven bars into one long bar, his comment supports the conjecture that he viewed the original collection of candy bars as one bar partitioned into sevenths. Naming the new collection thirty-five thirty-fifths is consistent with Michael's focus on the 35 parts and on having made five-fifths of the new collection from the initial three-fifths of the new collection. That is, the new collection of bars was the "whole" for Michael when he first thought about the length of this collection. However, after reviewing with me that the unit bar was an inch in length, he was reoriented to think in terms of inches—or at least in terms of unit bars. I infer that he knew that three parts made a unit bar because he said, "the unit bar would be divided by three." I also infer that he knew he could not make a whole number of unit bars out of 35 parts because he said, "three can't go into thirty-five equally." Thus his response of eleven and two-thirds likely meant that the new collection was eleven unit bars and two-thirds of a unit bar, although at this point it's not clear whether to him eleven unit bars and two-thirds of a unit bar meant the same thing as thirty-five thirds of a unit bar.

Variations and Adaptations in Michael's Ways of Operating in March and May

During the remainder of the teaching experiment, Michael used his reversible multiplying scheme with fractions to solve RMR problems of Types 3 and 4, i.e., RMR problems with whole number quantities and both fractional and whole number relationships. In some of these cases he did not partition *all* of the inches in enacting the scheme. For example, on March 10th, Michael and Carlos worked on this task:

> Task 5.2: An 8-centimeter peppermint stick is marked to show the 8 centimeters. It is 3/4 of another peppermint stick; make the other stick and tell how long it is.

Michael formed a goal to divide the 8-centimeter bar into three equal parts so "we can add on one more." But instead of partitioning each of the 8 centimeters into three equal parts, he partitioned only the two rightmost centimeters into three equal parts. Then he copied the 8-

centimeter bar with this further partitioning, broke it into all of its parts,[35] and joined two 1-centimeter bars and two 1/3-centimeter bars, indicating that continuing to rearrange and join in this way would produce three equal parts from the 8-centimeter bar (see Figure 5.7).

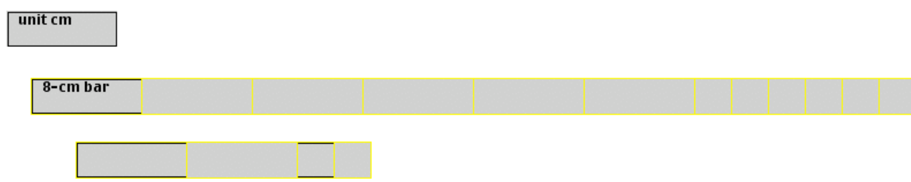

Figure 5.7, An 8-cm bar and 1/3 of it made from two 1-cm bars and two 1/3-cm bars.

By abbreviating his partitioning activity in this way, Michael further demonstrated his distributive ways of operating when solving RMR problems. To split the 8-centimeter bar into three equal parts he conceived of the bar as a 6-centimeter bar and two 1-centimeter bars. I infer that he mentally split the 6-centimeter bar into three equal parts (each consisting of a 2-centimeter bar) and also split each of the two 1-centimeter bars into three equal parts. Then one-third of the 8-centimeter bar was, as Michael stated, "two of the regular centimeters and…two thirds of one centimeter." Although I don't claim that he was aware of this distributive pattern in his activity, I do claim that his activity in solving Task 5.2 in March corroborates my assertion that he learned to split distributively in February when he solved the Candy Bar Problem.

Michael also adapted his reversible multiplying scheme with fractions to solve RMR problems of Type 5, with fractional quantities and fractional relationships. The central adaptation

[35] When BREAK is used to break a bar into all of its parts, I refer to these now-separated parts as *bars*. For example, the 8-centimeter bar consists of eight 1-centimeter parts, and when broken, becomes eight 1-centimeter bars.

was his use of recursive partitioning to determine the length of the new bar. For example, on May 12th, Michael and Carlos worked on the following task in the context of a homemade racecar contest between two teams from a science class, the Lizards and the Cobras:

> Task 5.3: The Lizards' car goes 1/2 of a meter. That's 2/3 of how far the Cobras' car went. Can you make how far the Cobras' car went and tell how far they went?

Michael immediately partitioned the 1/2-meter bar into two equal parts. By this point in the teaching experiment the boys were showing the original and new quantities separately, so to show the Cobras' distance separately from the Lizards' distance, Michael pulled out one-half of the 1/2-meter bar and repeated it to make a 3-part bar (see Figure 5.8).

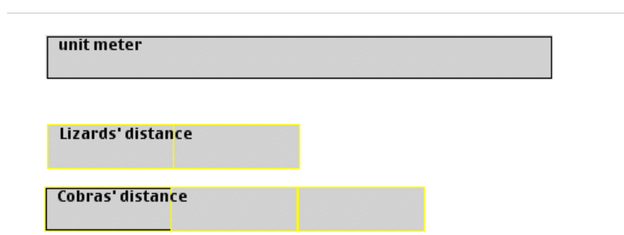

Figure 5.8, The Lizards' distance (1/2-m bar, above) and Cobras' distance (3-part bar, below).

Michael determined that the Cobras' car went three-fourth of a meter "because we divided that one [the 1/2-meter bar] into two." That is, in imagination he recursively partitioned the "missing" half meter into two equal parts, knew that four parts would make up the unit meter, and so knew that each part of the Cobras' distance would be one-fourth of a meter.

The Enactment of MCR with Michael in February

The interaction with Michael on February 18th was pivotal in enacting MCR with him because it helped me understand the power of his ways of operating and the satisfaction that he seemed to derive from his activity. Posing the Candy Bar Problem was not necessarily my intention for that episode because I was not sure that such a problem was within Carlos's short-term ZPC. So, Carlos's absence from school that day, of which I was unaware until I went to pick up the boys from their classrooms, meant I had an opportunity to adapt my plans in order to investigate my hunches[36] about the power of Michael's ways of operating.

Despite these hunches, I was not sure how Michael would handle the Candy Bar Problem. Somewhat to my surprise, the problem provided a situation that seemed to be in harmony with his ways of operating—i.e., he could use his current schemes in assimilating the problem. The problem also seemed to challenge him to modify and extend those ways of operating. The challenge and extension came from the opportunity the problem provided for Michael to modify his splitting operation, opening the possibility for him to coordinate his reversible fraction and multiplying schemes in constructing a new scheme, a reversible multiplying scheme with fractions. In short, the Candy Bar Problem seemed to fall right into Michael's ZPC. So in posing the Candy Bar Problem, I enacted mathematical care for Michael.

Evidence that he received my mathematical care came from his concentrated engagement during the episode in working through this problem and others like it. Michael tended to consider problems thoughtfully and to make observations about aspects of problems that interested him and that went beyond what I or the witness-researchers had specifically asked. At this point in

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[36] I use the word "hunch" rather than the more formal word "conjecture" to convey the experience of exploring my ideas with and about the students during the teaching experiment. The word "conjecture" implies a more conscious and intentional state than my experience of this exploration. Although I did make explicit conjectures (cf. Chapter 4), much of the time I felt like I was acting on suspicions or intuitions—hunches.

the teaching experiment, he had already expressed enthusiasm for working in JavaBars.[37] He seemed to get a good deal of pleasure from the "fitting together" both of bars in the microworld and of his ideas. His manner in this episode was a good example of the satisfaction and enjoyment that regularly accompanied his participation in our interactive mathematical activity.

However, this episode also marked his developing satisfaction and enjoyment from acting in a way that he seemed to be able to tell was satisfying (indeed, pleasing) to me and to the witness-researchers. That is, I conjecture that this episode was pivotal in Michael's construction of his teacher's (my) perception of him, which I have suggested may be influential in a student's construction of himself as a social-mathematical being (cf. Chapter 2). Based on my responses and the responses of the witness-researcher—verbal praise as well as our somewhat surprised and pleased attitudes—Michael seemed to be able to tell that what he was doing was impressive to *us* and that we held a high opinion of his work. I infer that his construction of our sense of him increased his own satisfaction, enjoyment, and engagement in his activity. In turn, through his engagement in the problems I posed, as well as his own independently-contributed observations, he cared back for me as his teacher.

Thus our interaction during this episode was an establishment of MCR that allowed both of us to continue to enact it. I learned more about how to harmonize with his ways of operating and to envision new possibilities for challenging and extending those ways of operating. In particular, I made an assessment that solving RMR problems of Type 5, with fractional quantities and fractional relationships, was within his short-term ZPC. I also conjectured that he was ready to work toward more algebraic solutions of RMR problems. So the episode was pivotal in advancing my working model of Michael's mathematics. The episode was also significant in my

[37] As noted in Chapter 4, JavaBars was developed by Biddlecomb and Olive (2000). I take the liberty of not including a citation after every use of the word in Chapters 5 and 6.

construction of Michael as an internalized other, in that I developed my understanding of his preferences and sense of himself as a social-mathematical being.

In addition, the February 18th episode appeared to mark a shift in Michael's motivation for attending our sessions. Up to this point in the teaching experiment, he seemed to like me just fine and like mathematics well enough to participate in the project. But on his initial information sheet in November, in response to the prompt "One thing that interests or excites me about this project is…" he wrote, "having a partner." He and Carlos got along well, and Michael tended to be a little shy or at least quiet with the research team. An example of Michael's preference for coming with Carlos occurred during the episode on February 2nd, the first time Carlos was absent. Michael seemed a little reluctant to come that day and somewhat uncomfortable being the only kid in the room. So up until the episode on February 18th, Michael appeared to participate largely because he liked being with Carlos.

After this point, a switch seemed to occur. Michael seemed to become more aware that we sometimes had to slow down for Carlos and that in doing so, Michael did not always have an opportunity to operate as on February 18th, in the ways that were so pleasing to him. In general Michael was highly patient with his friend and did not blurt out responses or indicate frustration—in fact, he often used the wait time to reconsider his own ideas and search for more efficient ways to approach problems, another by-product of the growing power of his schemes and his personal tendencies. But Michael's motivation to attend the episodes seemed to become rooted more deeply in his enjoyment of his mathematical activity and to depend less on the presence of Carlos.

Carlos Solving Problems Involving Reversible Multiplicative Relationships

Making the Other Cake Independently in March

At the beginning of March, just after Michael's solution of the Candy Bar Problem in mid-February, the boys worked on more RMR problems of Type 4, like the Candy Bar Problem. These problems proved quite challenging for Carlos. By using his own schemes, he could sometimes assimilate Michael's activity or comments, but doing so didn't mean that Carlos's schemes were the same as Michael's or that Carlos could act in the absence of Michael's activity. Thus, I infer that in March of 2003, solving Type 4 RMR problems was within Carlos's *long-term* ZPC. As a result, I began to pose RMR problems of Type 3, with whole number quantities and whole number relationships, where the relationship did not divide the quantity. These problems were also challenging for Carlos, but appeared to be within his short-term ZPC.

Carlos's first independent solution of a Type 3 RMR problem occurred on March 24th with the following task:

> Task 5.4, Cake Problem: Make a cake that's three of those unit bars long (each unit bar is 1 foot). That cake is twice as long as another cake. Make the other cake.

In contrast to my intuitive posing of the Candy Bar Problem with Michael, I posed Task 5.4 very deliberately to Carlos after he had demonstrated difficulty in solving other Type 3 RMR problems. Because Carlos had not coordinated his reversible fraction scheme with his multiplying scheme as Michael had, and because Carlos repeatedly demonstrated a lacuna in operating with improper fractions (to be discussed later in this chapter), he did not seem to have constructed ways of operating for solving many RMR problems. However, I knew that Carlos had constructed a splitting operation (Steffe, 2002) and so, combined with the ease most students have with the relationship "two times" (as opposed to another whole number of times), I conjectured that he would be able to solve Task 5.4.

After I posed the problem, Carlos made a 3-foot cake by repeating a copy of the unit bar to make three bars joined end-to-end. The cake was slightly longer than the screen's width.

Protocol 5.2: Carlos's solution of the Cake Problem on 3/24/04.

C: So it has to be twice as long?
T: Mm-hmm [yes]. This cake [points to the 3-foot cake] is twice as long as another cake. [M comments that he knows what to do. T acknowledges that this is hard for him—to know and not say—and assures him that he'll get a chance next.]
C: Twice as long as another cake. So then…[There is a 15-second pause during which C drags the 3-foot cake a little on the screen.] So then I have to bring it back down?
T: Mm, okay. [T is uncertain about what his comment means. C makes a copy of the 3-foot cake.] Yeah, so you want this cake [points to the copy] to be twice as long as the cake you're going to make.
C: Okay, so then…
W: There's twice as much cake as the one you've got, in there.
C: You have to have twice as much as the cake?
T [points to the copy]: This cake, that you started with, is going to have twice as much cake as the one you're going to make.
C [softly]: Okay. [During an 18-second pause, C drags the copy of the 3-foot cake back and forth beneath the first 3-foot cake. Then *he partitions the middle foot into two equal parts*, see Figure 5.9.]

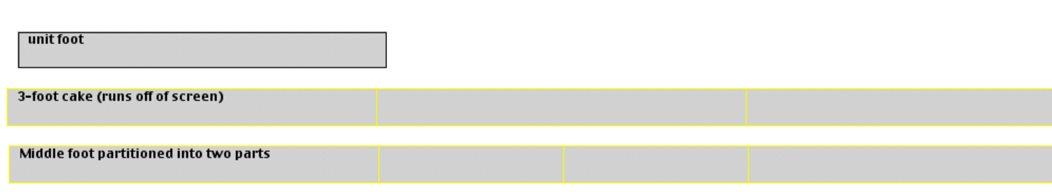

Figure 5.9, The middle foot of the 3-foot cake partitioned into two equal parts.

[He pauses and *then partitions the first foot and the third foot each into two equal parts*. He pulls out and joins the second, third, and fourth parts, see Figure 5.10.]

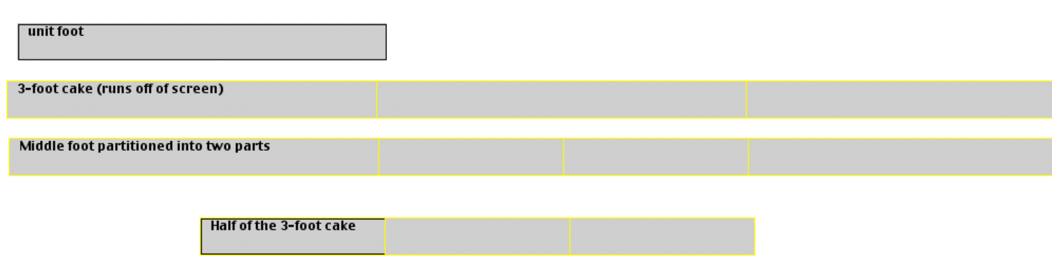

Figure 5.10, The new cake (bottom) consisting of three 1/2-foot parts.

[While he joins them, W laughs softly and T smiles. W says he can’t believe it.]
T: How’d you know to do that, C?
C: Since it’s three times as much, you have to split it in half, and then you split it over there and right here, that’d be six. And then, since, one, since this is [inaudible, may have said “divided in”] two, so you just…hold on. [He drags his new cake on top of the 3-foot cake and tries to indicate how two new cakes make up the 3-foot cake by matching up the new cake with the left end of the 3-foot cake and then dragging the new cake to match it up with the right end of the 3-foot cake. He has some trouble demonstrating because the 3-foot cake is longer than the screen’s width.]

Carlos’s two pauses during which he dragged the 3-foot cake as well as his restatements of the question indicate that he did not have a scheme established to use in this situation and had entered a state of perturbation. Despite his demonstration of a solid splitting operation during his selection interview, reasoning reversibly with composite units to solve Type 1 RMR problems had been an initial challenge for him during the interview and in early November teaching episodes. So in working on Task 5.4, his question “You have to have twice as much as the cake?” may have indicated that that he was considering whether *the cake he was to make* was twice as much as the 3-foot cake, or whether it was the other way around. Because I believed that he was attempting to clarify this issue, I restated the problem. After the 18-second pause, Carlos used his splitting operation to “split it [the 3-foot cake] in half,” splitting the middle 1-foot part in the same way he could have operated if the foot marks were not present. But then he appeared to modify his partitioning activity because he also partitioned both the first and third 1-foot parts into two equal parts.

Despite this partitioning activity, I don't claim that Carlos modified his splitting operation here as Michael had in the Candy Bar Problem in mid-February, because whether Carlos felt a logical necessity to split each of the three 1-foot parts into two equal parts is not entirely clear. In fact, his activity may have been initiated by watching what Michael did in these situations. My conjecture is that Carlos split the first and third 1-foot parts as a result of a second scheme that he was just beginning to use with RMR problems, his multiplying scheme to coordinate parts in a continuous context. To split the 3-foot cake into two equal parts, Carlos may have formed a goal to make the 3-foot cake into a number of equal parts that was divisible by two. Splitting each foot into two equal parts made six parts, which could be divided by two. Although Carlos did not explain his work this way in this problem, he did explain his work—or sometimes Michael's work—this way in episodes following this one.

I also contend that Carlos used his partitive fraction scheme[38] implicitly near the end of the protocol to show why what he had done solved the problem. That is, by dragging his new cake twice along the 3-foot cake, he showed that the 3-foot cake was twice as much as the new cake. But he also showed that the new cake was half of the 3-foot cake, because for Carlos "half" meant that repeating twice would make the whole. This demonstration is a typical instance of his use of his partitive fraction scheme to "check" or "explain" his solutions to RMR problems. Because solving this problem seemed to have been a significant challenge and achievement for Carlos, and because Michael had been waiting patiently for his turn to work on a problem, I did not pursue asking Carlos about the length of the new cake at this time.

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[38] Recall that based on their selection interviews I inferred each student had constructed at least a partitive fraction scheme (cf. Chapter 4).

Making a Two-Unit Bar into Three Equal Parts in March

By partitioning each of the three 1-foot parts into two equal parts in the Cake Problem, Carlos demonstrated a possible start to modifying his splitting operation. To test this possibility, in the next teaching episode on March 29th I posed the following problem for him:

> Task 5.5: That 2-foot candy bar is three times longer than your candy bar. Make your candy bar and tell how long it is.

Carlos began by copying the 2-foot bar and breaking it into two 1-foot bars. By pulling out a 1-foot bar, he ended up with three 1-foot bars. This event was something of an accident, but Carlos decided and stated that he actually needed three feet. Perhaps based on his work in the previous episode, he partitioned each of the three 1-foot bars into two equal parts, broke the bars into their parts (six small bars), and dragged one of these small bars directly beneath the 2-foot bar. Immediately he seemed to think something was wrong.

Protocol 5.3: Carlos making a 2-foot bar into three equal parts on 3/29/04.

C: Ah—darn it, I lost it! [He repeats one of his small bars to get four of them joined, clears this bar, and partitions it into two equal parts. This activity remakes the 2-foot bar.] Start all over.
T: Okay, that's fine. Yeah. [10-second pause] So what are you thinking about, C?
C: *I'm thinking about splitting these [gestures to the 2-foot bar], and then splitting them into little pieces so I can add them all up into three pieces.*
T: Okay. And you want this bar [points to the 2-foot bar] to be how many times longer than the bar you're making?
C: Three times. Okay, now I know what to do. [*He clears the 2-foot bar, partitions the unmarked bar into three equal parts, breaks them apart, and drags out one of the parts*. W laughs.]
T: Oh, that works doesn't it?
C [Continues to drag the one part, one-third of the 2-foot bar, beneath the original 2-foot bar and aligns their left ends]: Yep.
T: That was really good! But now I want you to try it again, and you can't clear it.
C: Aw! [We both laugh a little.]
T: But that was a really good idea. But now try it and see if you can make that bar but you can't clear.

[C makes a new copy of the 2-foot bar and "measures" his first solution, the one-third of 2-foot bar that he just made, along this new 2-foot bar. He uses CUT to cut off the appropriate amount, see Figure 5.11. T and W laugh.]

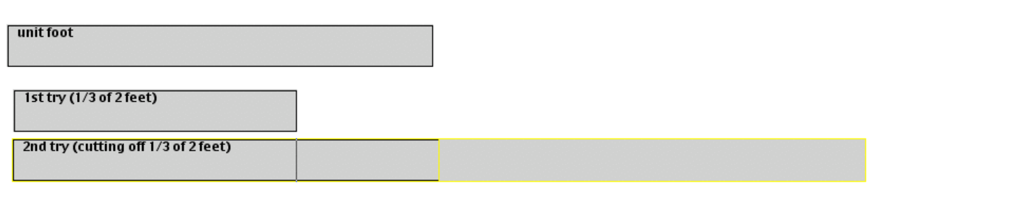

Figure 5.11, Using the first solution (1/3 of a 2-ft bar) to measure the second solution.

C: There you go!
T: That was pretty swift! [more laughter] Very creative. Can you try it one more time? And this time I want you to erase this guy [points to the one-third of 2-foot bar, C's first solution].
W: How would you do it if you couldn't use CUT?
C: Mmm, I'm not sure!
T: Let's try it again—I bet you could [do it].
[C erases everything on the screen except for the 2-foot bar and the unit bar. He breaks a copy of the 2-foot bar into two 1-foot bars and then *partitions each 1-foot bar into three equal parts. Then he breaks everything apart so he has six separate, small bars*. He drags the second through sixth small bar out of the alignment and then drags the second small bar back up, aligning it end-to-end with the first small bar, see Figure 5.12. *He joins these two small bars*.]

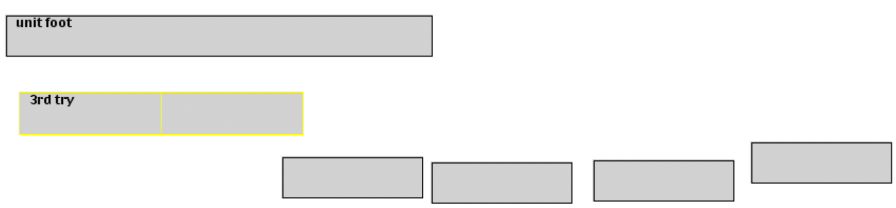

Figure 5.12, Carlos's third solution, a 2-part bar consisting of two 1/3-foot parts.

T: Oh! So let's see—what'd you do to do that?
C: Well since there were two pieces [two feet], then you said I couldn't clear it or use cut, so then I had to divide, first I had to break them, and *then I multiplied two by three and got six*, and since there were supposed to be three of the two pieces, I just did that.
T: Oh, I see, okay. So you put each one [points to the first and then the second foot in the original 2-foot bar] into three, because you were thinking two times three is six? [C

nods.] Oh! And so how long is your bar [points to C's 2-part bar]—remember we said this is our unit foot [points to the unit bar]—how long is your bar?

C: Two-thirds. Of a foot.

At the start of this protocol, Carlos's dominant scheme in solving these problems was his splitting operation, as shown by him clearing the 2-foot bar and then partitioning it into three equal parts. My first attempt to create a situation in which he would have to modify his splitting operation did not work because one-third of the 2-foot bar (Carlos's first solution) was still on the screen and he could "measure" its length along a new copy of the 2-foot bar. However, the second attempt, in which I asked him to erase his first solution (the one-third of 2-foot bar) and the witness-researcher asked him not to use CUT, provoked an initial modification of his splitting operation as I conjecture started to occur when he solved Task 5.4 on March 24th (cf. Protocol 5.2). That is, Carlos had formed a goal to split each foot up into "little pieces" in order to "add them all up into three pieces"—in order to make the 2-foot bar into three equal parts. But making specifically *three* "little pieces" in each foot did not immediately occur to him. When it did occur to him on his third solution of the problem, I infer that he made a functional accommodation in his splitting operation (and the partitioning operation involved in splitting) to include a distributive operation. That is, in order to partition the entire 2-foot bar into three equal parts, Carlos partitioned each of the two parts of the 2-foot bar into three equal parts. So at the end of March, Carlos learned to split distributively, at least in making three equal parts in a 2-part bar.

As with Michael, I don't claim that Carlos was explicitly aware of the distributive pattern of his activity, nor did Carlos explicitly take one part from each of the two parts of the 2-foot bar to make one of three equal parts of the entire 2-foot bar. But I do claim that, as with Michael, Carlos's accommodation included using his multiplying scheme in a new way. That is, multiplication became relevant for Carlos in this continuous context: Multiplying two and three allowed him to produce a total number of parts that could be divided by three. So in effect, using his multiplying scheme he created a composite unit, a bar consisting of six equal parts rather than two, upon which he could use his splitting operation. As with Michael, commutativity seems evident in Carlos's activity in that two equal parts each containing three equal parts make six equal parts that can be reorganized into three equal parts each containing two equal parts.

Because Carlos used this way of operating in other problems in subsequent episodes, I infer that this modification was a relatively permanent change. For example, in the next episode on March 31st, he made one-third of two-fifths of yard by distributing his partitioning operation across each fifth—that is, he partitioned each of the two 1/5-yard parts into thirds in order to partition the entire 2/5-yard bar into thirds. The main difference in the boys' coordinations of schemes and operations seemed to be that Carlos's accommodation of his splitting operation and new use for his multiplying scheme did not involve an explicitly fractional context. His reversible fraction scheme was not involved, so he did not use his multiplying scheme in service of using his reversible fraction scheme, as Michael had done in the Candy Bar Problem. Thus Carlos's scheme for coordinating two and three equal parts in the same bar can be called a *reversible multiplying scheme* (without fractions).

As seen in Protocol 5.3, Carlos seemed to know immediately that the bar he made was two-thirds of a foot, but how he knew is uncertain since I did not probe his response. Perhaps because Carlos worked on each foot of the 2-foot bar separately, he knew that he had made each foot into thirds and had taken two of them. In that case, he would have used his fraction scheme to know that each small bar that he had made was one-third of a foot, and joining two of them produced two-thirds of a foot. It is also possible that he knew his goal was to make thirds of the 2-foot bar and he could see that he had two parts, so he responded two-thirds. In this latter case, his response would indicate to me that he had relied on part-whole meanings of fractions to determine a length that, coincidentally, was correct.

Carlos might have been able to use his reversible multiplying scheme in solving problems like Task 5.1, the Candy Bar Problem.[39] However, we did not make a significant return to RMR problems of Type 4 during the remainder of the teaching experiment, so it is as yet unknown how general his way of operating was or could have become. In working on RMR problems of Type 5, with fractional quantities and fractional relationships, like Michael, Carlos could use recursive partitioning to determine the length of the resulting bar. Yet in almost all instances Carlos agreed with Michael's assessment of the length of the resulting bar and did not independently state the length. I infer that Carlos could make sense of what Michael said because Carlos had constructed recursive partitioning. But in these cases Carlos generally did not produce independent responses about the length of the resulting bar, in part because the "cognitive load" of working with RMR problems in fractional contexts seemed to be considerable for him.

The Enactment of MCR with Carlos in March

The episodes on March 24th and 29th were important in my enactment of MCR with Carlos because they demonstrate my redoubled efforts in March to see him as distinct from Michael and give reason to Carlos's ways of operating (cf. Duckworth, 1996). Making these efforts also meant I began to address more seriously Carlos's tendency to follow Michael's responses and activity. For example, in Protocol 5.2 on March 24th, asking Carlos to make a cake such that a 3-foot cake was twice as long (the Cake Problem, Task 5.4) represented a significant simplification of the problems I had been posing for the boys, which had been more like Task 5.2 (an 8-centimeter peppermint stick is 3/4 of another stick; make the other stick and tell how long

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[39] Doing so likely would have required further coordination with his reversible fraction scheme.

it is). Making this shift was an initial attempt to harmonize better with Carlos as distinct from Michael. As shown in Protocol 5.3 on March 29th, my interactions with Carlos during his solutions of Task 5.5 (a 2-foot bar is three times another bar; make the other bar) became more finely tuned to my conceptions of his current schemes and operations. In contrast, some of my previous (and, somewhat unfortunately, later) attempts to harmonize with and challenge him were too blunt or coarse.

For example, when he was searching for a way to solve Task 5.5, I did not immediately outlaw clearing the 2-foot bar. In my working model of Carlos, I was confident that he had constructed a splitting operation, and I wanted him to have the opportunity to use this operation so that his activity would be logically necessary for him and so that he could feel autonomous. Thus, when he cleared the 2-foot bar and partitioned it into three equal parts, I both accepted and praised what he had done. In that sense I decentered from my own ways of operating to see his ways of operating as not just valid but as significantly powerful in his mathematical reality. My intention was to communicate that I thought his ideas were valuable and worthwhile.

However, I also intended to challenge his ways of operating because I had a hunch that he could solve the problem by splitting distributively. So I banned clearing the 2-foot bar. Then, when in his second solution he again subverted confronting the problem in a way that might bring forth distributive splitting, I still praised his work as creative—it was! I and the witness-researcher then introduced further constraints to sustain his perturbation and (hopefully) provoke an accommodation in his splitting operation. So throughout the interaction I harmonized with what Carlos wanted to do, responding to his productions affirmatively and enthusiastically. But I also set challenges to open possibilities for him to expand his ways of operating, responding to his productions with new restrictions on what he wanted to do. This dual response characterizes well the enactment of MCR with Carlos—or any student.

In turn, Carlos considered the restrictions I set on Task 5.5, even when he didn't know what to do to solve the problem, thereby receiving my mathematical care for him. In part this perseverance was a personality trait: Carlos tended to be an open, good-natured, and enthusiastic participant even when he had no immediate way to operate. However, he also tended to say that he forgot or that his "head hurt" when he (from my point of view) felt pushed out of his ZPC. Because he said neither of these typical comments during his work on Task 5.5 (cf. Protocol 5.3), I infer that, although he was in a state of perturbation, it was "bearable" (Steffe & Tzur, 1994; Tzur, 1995). Furthermore, he seemed to trust (implicitly) that the restrictions I made were not capricious or arbitrary, even when he did not see the point of them (beyond the fact that they foiled his current ideas for acting in the situation). His open manner indicates that he believed his work was acceptable to me and that he sensed that I believed he could do even more with the problem. So as with Michael, Carlos's construction of his teacher's (my) perception of him seemed to play a role in our enactment of MCR. And through his willingness to work with my interventions, he cared back for me as his teacher.

Whether Carlos's construction of himself as a social-mathematical being was significantly affected by our interactions during the March 29th episode is unclear, because it is hard to assess to what extent he was *aware* of operating more powerfully. I certainly was aware of it: A central implication of MCR in the March 29th episode was that Carlos's modification of his splitting operation to include a distributive operation appeared to open new ways of operating for him in other situations, such as making one-third of two-fifths of a yard in the next episode, and even making thirds of bars with more units than two. For example, at the end of this same episode Carlos made one-third of a 7/9-yard bar distributively. He appeared to conceive of the 7/9-yard bar as a 6/9-yard bar and a 1/9-yard bar. He determined one-third of the 6/9-yard bar (two 1/9-yard bars) and made one-third of the "leftover" 1/9-yard bar by partitioning that ninth into three equal parts and taking out one part. So to make one-third of seven-ninths of a yard, Carlos made a bar consisting of two-ninths of a yard and one-third of one-ninth of a yard.

This situation marked the first time Carlos independently partitioned distributively, distributing his operation of taking one-third across whole numbers of units (in this case, ninths) and fractional parts of units (thirds of ninths), in service of a goal to make one-third all the units. His activity was strikingly similar to Michael's activity in Task 5.2 (an 8-centimeter stick is 3/4 of another stick; make the other stick and tell how long it is), although Carlos's task in this case was a fraction composition problem, not a RMR problem, and so involved only distributive partitioning, not distributive splitting. Carlos's activity corroborates my claim that he learned to split (and therefore partition) distributively in the March 29th episode, and that this learning, facilitated by our MCR, opened the way for an expansion of his mathematical reality. However, the extent to which he was aware of, or the extent to which his mathematical self-concept was affected by, this change in his ways of operating remains uncertain.

Reversible Reasoning: Constructive Resource 1, November

In the next five sections of this case study I explore key moments in the boys' constructions of their ways of operating to solve RMR problems. The key moments are organized into five constructive resources: reversible reasoning, activity with improper fractions, new uses of multiplying schemes, fraction composition activity, and fractions as operations. These resources occur approximately chronologically through the teaching experiment (see Figure 5.13). In some cases, the key moments demonstrate that Michael or Carlos constructed a scheme or operation that was useful in their solutions of RMR problems in February and March. In other cases, the key moments demonstrate that Michael or Carlos experienced a constraint in constructing a way of operating that I conjecture would have been a resource in solving RMR problems. In the case of the four and fifth constructive resources, the key moments show activity that opened possibilities for the boys to modify their schemes in order to solve more complex RMR problems (e.g., of Type 5) or to move toward more algebraic solutions of them.

Nov 03	Dec 03	Jan 04	Feb 04	Mar 04	Apr 04	May 04
C and M **reason reversibly** with whole numbers and proper fractions. M also does so with **improper fractions**.	Both boys work on **activity with improper fractions**.	M reasons with large **improper fractions** and constructs a **new use for his mult. scheme**, a co-measure-ment scheme.	M solves the Candy Bar Problem, thereby constructing distributive splitting and a reversible multiplying scheme with fractions.	C makes a 2-foot bar into 3 equal parts, thereby constructing distributive splitting and a **new use for his mult. scheme**, a reversible multiplying scheme.	Both boys work on **fraction composition activity** (begun at the end of March).	Neither boy demonstrates the construction of **fractions as operations** (also evident in March and April).

Figure 5.13, Timeline of the boys' five constructive resources.

Reversible Reasoning with Proper Fractions

The first teaching episode with both boys occurred on November 4th. Carlos had attended his first teaching episode on October 30th, with a different partner who did not ultimately participate in the teaching experiment. On October 30th, Carlos operated reversibly with his fraction scheme only when he started out with a unit fraction. For example, if two pencils was one-twelfth of all the pencils in a box, Carlos determined that 24 pencils were in the box. But if two pencils was two-fifths of all the pencils in the box, Carlos believed that 10 pencils were in the box. On November 4th, I posed the following problem, largely for Carlos:

Task 5.6: Three pencils is 3/5 of the pencils in a box. How many pencils are in the box?

As I posed the problem, I pulled out three pencils from the 13 that were on the table. Both boys stared at the group of three pencils, Michael resting his finger on his lips and Carlos occasionally looking up into space. Ten seconds after I posed the question, Michael's eyes traveled over to me and he gave a little laugh. I was soon to learn that Michael's little laugh signaled that he had figured out a problem to his satisfaction.

Carlos said that "we don't have enough pencils" to solve Task 5.6. I asked Michael whether he thought we had enough, and he nodded. But Carlos illustrated his claim by placing three more groups of three pencils beside the initial group of three pencils, leaving a lone pencil sitting in the middle of the table. He said, "Put them like this 'cause you still need one more pair." I reminded Carlos that the original group of three pencils was *three*-fifths and not one-fifth, and I asked him if that made a difference. Approximately 10 seconds later, Carlos spread out the original group of three pencils, joined two more pencils to the group of three so that all five pencils were evenly spaced, and stated that each pencil would be one-fifth (of the number of pencils in the box). Michael smiled and nodded in agreement.

Although Michael made almost no verbalization during Carlos's activity, his belief that we had enough pencils and his agreement with Carlos's resolution of Task 5.6 indicates that Michael knew if three pencils was three-fifths of the amount in the box, then one pencil was one-fifth and five pencils would be five-fifths (the total amount in the box). Following Task 5.6, Michael drew a whole candy bar given a drawing of three-fifths of it. In explaining his solution he said, "I divided that [the bar] into three and added two more thirds." So I infer that for

Michael, three-fifths meant one-fifth three times. Given three-fifths, he could divide by three to determine one-fifth (which was simultaneously one-third of the original amount of candy) and iterate one-fifth two more times to make five-fifths (which was simultaneously two more thirds of the original amount of candy). In doing so, Michael used his splitting operation on the three-fifths to produce a piece he could use to make five-fifths, and he demonstrated that unit fractions were iterable for him at least within the whole. Based on Michael's activity in both discrete and continuous contexts, I can conclude that he had entered the teaching experiment with at least a reversible partitive fraction scheme *with splitting*.

In contrast, in Task 5.6 Carlos initially thought of the three pencils as one-fifth of the number in a box and so intended to make five groups of three pencils for the total number in the box (which meant that there were not enough pencils on the table). Emphasizing that the three pencils were *three*-fifths was enough of an intervention to make him reconsider his result. Carlos's partitive fraction scheme meant that for him, three-fifths was three one-fifths. I infer he used that conception to determine that each pencil was one-fifth of the total number in the box. Then, using his reversible fraction scheme with unit fractions, he could ascertain that only two more pencils were needed to make five-fifths, the total amount in the box. This modification in his reversible partitive fraction scheme in discrete contexts was relatively permanent because he successfully and swiftly solved other problems like Task 5.6 in subsequent teaching episodes. Also, like Michael, following Task 5.6 Carlos made a whole candy bar given three-fifths of it by partitioning the given candy into three equal parts and adding on two more parts. So I can infer that in this episode he used his splitting operation with proper fractions in order to reason reversibly, and I can also attribute a reversible partitive fraction scheme with splitting to him.

Reversible Reasoning with Improper Fractions

Difficulty in splitting an improper fractional quantity. During the same teaching episode on November 4th, I posed a problem to test out the boys' reversible reasoning with improper fractions and composite units:

> Task 5.7: Twelve pencils is 4/3 of the pencils in a box. How many pencils are in the box?[40]

In posing the task I made a group of 12 pencils, which Carlos dragged directly in front of him. Then the boys sat in silent concentration for 20 seconds.

> Protocol 5.4: Difficulty in determining how many pencils in the box on 11/4/03.
>
> M [gives a little smile and glances at me]: One and one-third.
> T: So four-thirds is one and one-third—you're right. So I wonder how many pencils will be in the box if twelve pencils is four-thirds of all the pencils in the box. [Both boys are gazing at the 12 pencils. C fiddles with them a little bit.]
> M: Ah—thirty-six!
> T: You think it's thirty-six?
> M: Well no—gotta be sure. [He stares out into space in concentration. Twelve seconds pass.]
> C: I think I got it. [He puts his head down on the table.] I'm not real sure though.
> T: Okay, so, M you think it's thirty-six?
> M [shrugs]: I don't know.
> T: Not sure? Okay, what do you think C?
> C: Thirty-six.
> T: You think thirty-six also? So what did you think C, to get thirty-six?
> C: *Since it was a mixed number, that I could probably just do, multiply twelve by three.*
> T: Is that what you were thinking M—something like that?
> [M does not respond to T's question. Instead, *he separates out a group of three pencils and continues to make four groups of three*. C helps him, see Figure 5.14.]

[40] In retrospect, posing this task was a jump for both boys—it was a rather blunt move to engage them in problems that required them to *reverse* the operations they might use to make improper fractional quantities prior to *making* such improper fractional quantities. But, as this episode marked my first interaction with the boys, I was exploring.

Figure 5.14, Four groups of three pencils each.

T: Oh! So how much is this [gestures to the first group of three]—if the whole thing is four-thirds, how much is one of these?
C: Four-thirds.
T: Each one of these [each group of three pencils] would be how much of the box?
C: One-third.
T: Oh. So three pencils is one-third. So if three pencils is one-third of a box, how many will be in a box?
C: Twelve.
T [6-second pause]: But twelve is more than a box, right?
M: Thirty-nine.
T: How'd you get thirty-nine?
M: *Twelve times three equals thirty-six but then there's three more, so thirty-nine.*

In this situation, both boys seemed to consider the 12 pencils to be one-third of the pencils in the box, rather than four-thirds. It was as if "four-thirds" was a mysterious number to them, and so they dropped back to a significantly easier problem involving a unit fraction, as Carlos had done before with proper fractions. Carlos's explanation that he thought he could multiply 12 and 3 confirms his tendency to use his reversible partitive fraction scheme with *unit* fractions in composite unit situations. It is also an early example of Carlos's inclination to believe and agree with Michael's responses. Alone, Carlos may not have arrived at 36 as his response, but when Michael did, Carlos was able to impute meaning to an answer of 36 (as 12 times 3), based on his usual ways of operating in these kinds of reversible situations.

Yet neither Carlos nor Michael was certain of this response. Carlos explicitly indicated he was not sure. Furthermore, his explanation included the idea that "it was a mixed number," which shows that he knew something was different in this problem—he had not simply heard me say "one-third" instead of "four-thirds." Michael demonstrated his uncertainty by separating the 12 pencils into four groups of three. His activity indicates that using his splitting operation in this situation to make four equal parts was relevant to him—probably because for him four-thirds was three-thirds and one-third more. Carlos identified one group of three pencils as one-third of the box and so may have made sense of Michael's partitioning activity. Thus for both boys, four-thirds seemed to have some meaning as four one-thirds. However, identifying three pencils as one-third did *not* seem to help Carlos determine the total number in the box. Michael's response of 39 corroborates that he was trying to coordinate the three-thirds of a box and one-third more in thinking about the number of pencils. But it also indicates that there were some lacunas in his thinking with regard to determining four-thirds of the number of pencils versus the number of pencils in a box, and with regard to 12 pencils as one-third of the box rather than four-thirds.

Splitting in continuous and discrete fractional contexts. Although through coaching (not shown Protocol 5.4) both boys were able to say that nine pencils were in a box, Michael was not convinced, and I was not convinced that Carlos would have solved the task without coaching. So, using the following task, I backed up to test their reversible reasoning with improper fractions in a continuous context:

Task 5.8: Seven-fifths of a candy bar is drawn on the paper. Make the candy bar. Michael marked his bar immediately into seven equal parts and drew a dark mark after the first five parts to indicate where the candy bar would end (see Figure 5.15).

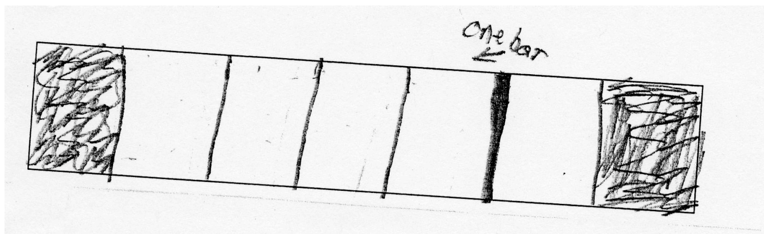

Figure 5.15, Michael's identification of 5/5 of the candy bar given 7/5 of it.

Carlos first marked his bar into five equal parts and then switched to make seven equal parts. The motivation for his switch is unclear, although Carlos made it just as Michael's work was becoming visible. Carlos indicated that the candy bar would be the first five parts. In explanation he said, "It's another mixed number…. Subtract seven from five and it would be five-fifths and that would be one whole bar." Both boys identified one part as one-fifth of the candy bar.

Because this work seemed to go relatively well, I decided to return to the pencil context with the following problem:

Task 5.9: Fourteen pencils is 7/5 of the pencils in a box. How many pencils are in a box? After a 17-second pause, Michael took two pencils away from the 14 that were on the table while Carlos slid the rest of the pencils together. As Carlos separated out two more pencils, Michael offered back the two pencils he had taken.

Protocol 5.5: Finding the number of pencils in the whole box on 11/4/03.

M [referring to C, who appeared to be pairing the pencils]: That's what I was thinking.
T [to M]: That's what you were going to do?
[C puts one more pencil with the two pencils M separated out. Then *C continues making groups of three pencils, but his fifth group has only two pencils in it*.]
T: Oh I see. So C, tell me what you were trying to do there.
C: *I was trying to split it into three equal parts, I mean five*.
T: Five equal parts, okay. Was that what you were thinking about M?
[M shakes his head no.]
T: What were you going to do?
M: What was the question [laughs]?
T: So the question was, these are fourteen pencils, and that's seven-fifths of a box of pencils. How many pencils will be in a box?
[While T repeats the question, *M rearranges the pencils into groups of two*. C helps him.]
T: Oh, so how many equal parts do you have now?
M: *Ten*. No wait—
C [softly]: Seven equal parts.
T: Seven equal parts. And M, you said ten—what were you thinking about when you said ten?
M [*gestures with his hand to the end of the first five groups of two*]: *There's the whole bar* [gives a little laugh].

During this protocol Michael seemed to reason reversibly with seven-fifths, and he seemed to connect this activity to his solution of Task 5.8 (i.e., to a continuous context). At the beginning of the protocol I infer that Michael recognized what Carlos was doing when Carlos appeared to be making groups of two pencils each. That is, Michael thought Carlos was making seven equal groups, each of which would be one-fifth of the pencils in the box. Thus I infer that for Michael, seven-fifths meant seven equal parts that were each one-fifth of the whole. Using his splitting operation, Michael intended to partition the seven-fifths, or 14 pencils, into seven equal parts to identify one-fifth, or two pencils.

Then Michael sat silently while it turned out that Carlos was attempting to divide the 14 pencils into five equal groups. Michael may have been wondering if Carlos's way would work, and doubting what he himself had thought. I infer that Michael doubted himself because when I asked about his idea, he asked me to repeat the question. Once I did so, Michael readily solved the problem using his reversible fraction scheme, which was based on using his splitting operation on the seven-fifths to make seven equal parts, each one-fifth of the number of pencils in a box. In fact, he was so focused on solving the problem that he stated the number of pencils in the box (10) when I asked a different question, about the number of equal groups he'd made. Michael seemed to see this problem as similar to Task 5.8 because he called the first five groups of two pencils "the whole bar." For him, five-fifths made "the whole bar," which in this case was the whole box of pencils. His reversible reasoning with improper fractions in a discrete context was relatively permanent because in the next episode Michael swiftly solved a similar problem when eight pencils was four-thirds of the number in a box. So, based on his activity in both continuous and discrete contexts, I can attribute an initial construction of a reversible iterative fraction scheme to him during this first teaching episode.

Carlos also may have connected Tasks 5.8 and 5.9 because he tried to divide the 14 pencils into five equal parts just as he had initially partitioned the candy bar into five equal parts. This activity indicates that for Carlos, seven-fifths involved making fifths (five equal parts) out of whatever material was available. However, *seven*-fifths did not seem to have meaning for him as one-fifth seven times. It did seem to have meaning as a "mixed number:" He could interpret seven-fifths as five-fifths and two more fifths as he did in his solution of Task 5.8. So it is likely he could make seven-fifths of a whole if he had enough material available (e.g., two candy bars, or a lot pencils with one-fifth of a box identified as two pencils). But he could not as yet take seven-fifths of a whole as a starting point for determining one-fifth of a whole, at least in a discrete context with composite units. Once Michael solved Task 5.9, Carlos identified the five groups of two pencils as five-fifths. But this identification is not at all sufficient to infer that Carlos could produce the solution to this problem or had constructed a reversible iterative fraction scheme. Although I did not know it at the time, or even for a long time afterwards, Carlos's activity with these two problems in this episode foretold a great deal about his ways of operating with fractions larger than one, which I discuss in the next section, "Activity with Improper Fractions: Constructive Resource 2, November through January."

Splitting to Solve RMR Problems with Whole Numbers

Testing Carlos's reversible reasoning with composite units. In the next episode on November 6th, I wanted to determine whether the boys could use their splitting operations to solve RMR problems of Type 1—i.e., to split composite units. I conjectured that these problems would be relatively easy for Michael based on his solution of a Type 1 RMR problem during his selection interview and his splitting of fractional quantities in the November 4th episode. Carlos's difficulty in reasoning reversibly during his selection interview combined with his difficulty splitting fractional quantities on November 4th made me uncertain about how he would operate with composite units. So on November 6th I posed this task:

> Task 5.10: You have some money and I have some money. Say you have $39. That's three times the amount of money I have. How much money do I have?

Michael promptly divided 39 by 3 while Carlos multiplied 39 by 3. When a witness-researcher and I asked the boys to resolve the issue, Michael told Carlos that the 39 dollars was three times more than Ms. H's (my) money. Carlos replied, "So you should actually times that [the 39 and the 3]" and seemed undeterred from his solution.

I asked each of them to draw a picture to help decide the matter. Michael drew 39 dollars in a long vertical stack and made cut points separating the stack into three groups of 13 dollars each. He wrote "1/3" next to each group and also wrote down "13 each" (see Figure 5.16).

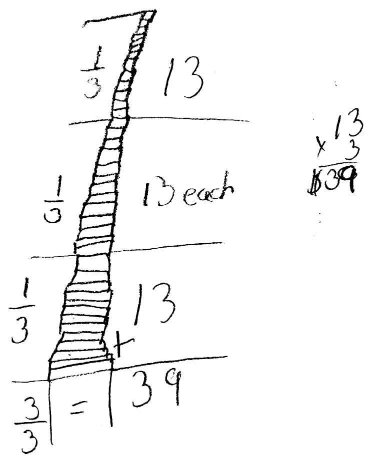

Figure 5.16, Michael's splitting of $39 into three equal groups of $13 each.

Michael's solution confirmed his use of his splitting operation to reason reversibly with composite units and is an early indication that he conceived of 39 as a unit of units of units—a unit of 3 units each of which contained 13 units. Michael also seemed to connect his splitting activity to his reversible fraction scheme (i.e., $39 was three-thirds and he was to make one-third by dividing by 3).

Carlos drew a shaded rectangle to represent 39 dollars and then drew two more of these, so he had three rectangles in a vertical stack, each representing 39 dollars (see Figure 5.17). I infer that he used his iterating operation on 39 dollars to produce the other amount of money. He did not seem to be able to split 39 into a unit of units of units, so his solution opens the possibility that conceiving of 39 as a unit of units of units was not yet available to him.

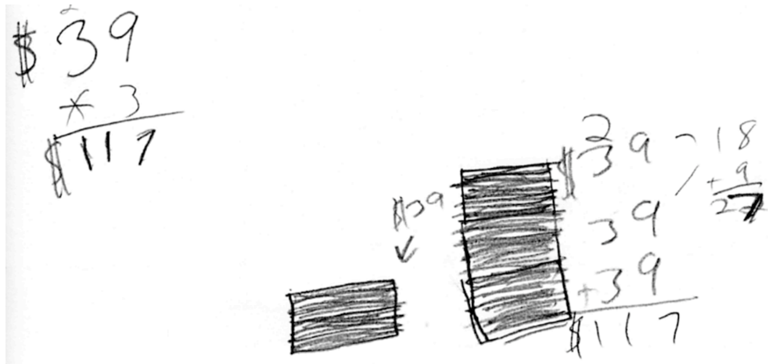

Figure 5.17, Carlos's iteration of $39 three times to produce $117.

Engendering Carlos's reversible reasoning with composite units. At this point the witness-researcher intervened with a new problem:

Task 5.11: Carlos, you have $21, and Michael has some money too. Your money is three times as much as Michael's. Draw a picture of how much Michael has.

Michael immediately and softly said "seven," and then the boys began to draw. The witness-researcher asked Carlos who had more money, and Carlos pointed to Michael. When the witness-researcher reminded Carlos that his money was three times more than Michael's, Carlos said he himself had more money—he had $63 (21 times 3). Both I and the witness-researcher reminded Carlos that he had $21, but Carlos repeatedly insisted that he had $63, saying, "Twenty-one, then you times it by three, and you get sixty-three." So despite our efforts to provoke a perturbation that Carlos might resolve by abandoning his multiplication of 21 and 3 and instead splitting the 21 dollars, he seemed convinced that iterating 21 three times solved the problem satisfactorily. Based on my belief that Carlos could split, I intervened with a new problem, Task 5.12, without coming to any further resolution of Task 5.11:

Task 5.12: The witness-researcher has $42. He has twice as much money as you have. Who has more money and how much do you have?

Protocol 5.6: Carlos's solution of a RMR problem of Type 1 on 11/6/03.

T: So let's try another one. So this time—let's see, [I point to one of the witness-researchers] he's got…forty-two dollars, okay? Whoo-hoo! He's rich. Okay, C? He has twice as much money as you have. So he has twice as much. So who has more, you or him?
C: *Him* [*gestures toward the witness-researcher*].
T: Oh. So figure out how much you have.
[During a 5-second pause C looks out, squints a little, says hmm. Then on his paper *he writes long division for 42 divided by 2*. M has written $21 next to a vertical sum of 21 and 21 equaling 42. M looks over at C's paper. C has written $21.]
T: So, what do you think C?
C: I have twenty-one dollars.
T: What do you think, M?
M: Twenty-one.
T: Twenty-one. Hmm! Okay we agree this time. Huh—interesting. So C, why do you say twenty-one?
C: 'Cause since I have two, since I have twice, *since he has twice as much, more than I do, so then I have to divide it by two to get my money.*
T: Oh, I see. Is that what you were thinking M?
[M nods yes.]

Two factors appear to have facilitated Carlos's use of his splitting operation to solve this problem. One factor was the relative ease of using "two times" as opposed to another whole number of times. Most students have built up a good deal of familiarity with and intuition about two times and halving that they haven't necessarily built up with other multiplicative relationships. A second factor was that the direction of the relationship was reversed. Carlos was now figuring out his own money as opposed to someone else's, which may have allowed him to reverse the direction of his operations. Carlos's explanatory comment, "since he has twice as much, …I have to divide it by two," indicates that he was involved in partitioning the 42 dollars into two equal parts rather than iterating it two times to solve the problem. Carlos's adaptation to use his splitting operation in solving problems like Task 5.12 was not limited to the relationship "two times," because during this episode and in the next episode on November 11th he solved other RMR problems of Type 1 with the relationship "five times."

Carlos's switch in how he operated on the known amount of money in an effort to determine the unknown amount of money is significant in terms of algebraic reasoning. His tendency was to start with the known amount and consider the unknown amount as so many times more than the known. In this sense the problems were "unbounded" for Carlos—he could take the known and iterate it any number of times (multiplying to compute the unknown).[41] Thinking about a *known* amount that was twice as much as an unknown amount caused him to reverse his usual thinking—to conceive of the *known* as so many times more than the unknown. It was as if he realized that the "answer" (the larger quantity that was produced by iteration of a smaller quantity) was what he knew. So he had to determine the part of the known that would, when iterated the requisite number of times, produce the known. The problem became bounded in the sense that Carlos had to "go inside" the known amount to determine the unknown amount within it. Carlos still operated on the known amount to produce the unknown amount. But in making this switch, I see the kernel of building an image of an unknown in relationship to a known (cf. Steffe, 2002a), an image that is not required in thinking in a "forward" direction.

Initiating MCR at the Start of the Teaching Experiment

In these early teaching episodes, to learn about the boys' mathematical ways of operating and their personalities and preferences, I engaged in two basic strategies: *coaching* them when they encountered difficulty, and *abandoning* tasks without final resolution in order to pose a different one that was (hopefully) more responsive to their ways of operating. When the boys encountered difficulty with a problem, my first recourse was to restate the problem, as shown

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[41] Note that his multiplying activity here does not necessarily indicate he had constructed a multiplicative structure that included coordinating three levels of units.

early on in Protocol 5.4 when I restated Task 5.7 (12 pencils is 4/3 the number in the box), or to emphasize an aspect of it, as shown in Protocol 5.6 when I emphasized $42 as *twice* as much as Carlos's amount of money. Then my inclination was to begin a dialogue with the boys that I characterize as coaching. Coaching sometimes took the form of leading them through parts of the problem, as I did toward the end of Protocol 5.4 when I said, "So if three pencils is one-third of a box, how many will be in a box?"

While some coaching was reasonable in exploring the boys' ways of operating and the interventions that might allow the boys to operate, leading the boys through a problem in a piecemeal fashion generally did not allow them to produce a way of operating that was logically necessary for *them*. That is, in breaking down the problem in certain ways, *I* was the one structuring the problem—they were not producing this structure. This kind of "over-coaching" occurred following Protocol 5.4 (not shown in the protocol), when I tried to lead the boys toward solving Task 5.7. Through this over-coaching, both boys were able to state that there were nine pencils in the box, but as I have stated, I believed that I rather than they had imputed structure to the problem in order to engender a solution.

As an alternative to over-coaching, abandoning tasks in order to pose another one that the student might solve more independently, originally, and in its entirety, became appealing but was not very easy to do: Throughout the teaching experiment, my first instinct as a former classroom teacher was to coach, and sometimes to over-coach. For example, in abandoning the discrete context of Task 5.7 for a continuous context of candy bars (Task 5.8, Seven-fifths of a candy bar is drawn; make the candy bar), Michael expanded his ways of operating with his reversible fraction scheme. Doing so seemed to allow him to solve a similar problem in a discrete context, Task 5.9 (Fourteen pencils is seven-fifths of the pencils in a box, cf. Protocol 5.5). Similarly, abandoning the relationship “three times” in Task 5.11 ($21 is three times as much money as his) for the relationship “two times” in Task 5.12 ($42 is two times your money), as well as switching the direction of the relationship was significant for Carlos. It seemed to help him use his splitting operation with composite units to solve a Type 1 RMR problem (cf. Protocol 5.6). So both coaching and abandoning tasks were “tools” I was using to initiate MCR with the boys.

In this process, I learned that it was relatively easy to harmonize with Michael’s ways of operating, as they often seemed closer to my own in the problems we’d attempt. In contrast, it was relatively harder to harmonize with Carlos’s ways of operating because they did not include some operations and schemes that I took for granted—so I needed to decenter more dramatically. However, it was relatively hard to challenge Michael at this point; early on he seemed to be able to operate or adapt his ways of operating quite swiftly in almost all situations. It would be deceptive to say that it was relatively easy to challenge Carlos. It *was* easy to pose problems that he found difficult, but this did not mean that it was easy to pose an “appropriate” challenge, one that was within his short-term ZPC but went *just* beyond his current schemes. At this early stage in our interactions, the boys cared back by continuing to attend our sessions and by persevering in mathematical situations that they had not before encountered.

Activity with Improper Fractions: Constructive Resource 2, November through January

Based on the boys’ reversible reasoning in early November, I have inferred that Carlos had constructed a “strong” partitive fraction scheme—strong because it included a splitting operation. I have also inferred that Michael had constructed an iterative fraction scheme, since he was able to operate reversibly with improper fractions (i.e., split improper fractions). However, I needed to confirm Michael’s construction of an iterative fraction scheme and investigate whether Carlos could construct (or had constructed) an iterative fraction scheme. So, from November through January, the boys and I worked on activity with fractions larger than one. This activity allowed me to learn that a splitting operation was not sufficient for Carlos to construct an iterative fraction scheme and that Michael had considerable power in operating with improper fractions, because nonunit fractions seemed to be available to him as iterable units.

Making and Comparing Fractions Larger than One

Activity during the teaching episode on November 18th corroborated that Michael had constructed an iterative fraction scheme, but whether Carlos had also done so remained questionable. The boys had successfully compared an 8/7-bar with a 12/11-bar by assessing that the parts in the 8/7-bar were bigger. Michael had noted that only “a teeny bit” was added to the 11/11-bar. Then I posed the following “prediction” problem:

Task 5.13: Which do you think will be bigger, 15/14 or 14/13?

Protocol 5.7: Comparing improper fractions on 11/18/03.

C [softly, after about 4 seconds]: Fifteen-fourteenths.
[M laughs a little and glances at me.]
T [softly]: Mmm. Fifteen-fourteenths or fourteen-thirteenths, what d’you think?
M [confidently]: Fifteen-fourteenths.
T: You think that will be bigger?
C: Yeah.
T: Okay, all right, let’s make them! So, who wants to make fifteen-fourteenths?
C [yawning, points to M]: He can.
M [simultaneously with C, raising his hand]: Me.
T: Okay, M, you make fifteen-fourteenths.
[M erases what’s on the screen.]
T: And then C, you’ll make—which one will you make?
C: Fourteen-thirteenths.
T: Okay. And then we’ll see.
[M draws a new unit bar and makes a copy. He partitions the copy into 14 parts, pulls out one part, and places the part adjacent to the right end of the 14/14-bar. Then M rests his chin in his hands. C confirms that he is to make fourteen-thirteenths. He copies the unit bar and partitions the copy into 14 parts.]
M: *No—no—no!* [M gestures with his hands toward the screen.] *You made fourteenths—[looks at C] yours is thirteenths* [gives a little laugh].
C [to T]: *Didn’t you say fourteen-thirteenths?*
T: Fourteen-thirteenths. So how many—

C: Oh yeah [as if he understands now].
T: Oh, you can always clear if you want to.
C: *Yeah, I keep on forgetting*. [C clears the bar and partitions it into 13 parts.]
T: Okay, what'd you have now?
C: *Now I have thirteen-thirteenths*. [*He pulls out one of the 13 parts and places it adjacent to the right end of the 13/13-bar*. M notes that the bars are almost the same length, but both boys confirm visually that C's bar is longer. When I ask why that is, C says his bar has bigger pieces, and *M notes that it's just like the problem of comparing eight-sevenths and twelve-elevenths*.]

Michael's iterative fraction scheme. This protocol shows that not only could Michael make fifteen-fourteenths without incident, he also could critique Carlos's production of fourteen-thirteenths. Thus Michael gave every indication of having constructed an iterative fraction scheme, although further support for this claim will occur later in this section. As shown in this protocol by how Michael made his fraction, for him fifteen-fourteenths certainly meant one-fourteenth more than fourteen-fourteenths. But based on his response to Carlos, I infer that for Michael fifteen-fourteenths simultaneously meant one-fourteenth iterated 15 times. I make this inference because Michael said to Carlos "yours is thirteenths." I take this comment to mean that Michael thought of fourteen-thirteenths as consisting of thirteenths, specifically as 14 one-thirteenths, or one-thirteenth iterated 14 times. Both of these conceptions of improper fractions are necessary in order to attribute an iterative fraction scheme to a student.

Carlos's lack of an iterative fraction scheme. Carlos had made improper fractions such as eight-sevenths from a 7/7-bar by pulling out one-seventh and joining it to the 7/7-bar. But to make fourteen-thirteenths Carlos partitioned the unit bar into the largest number stated in the fraction, 14 parts instead of 13 parts. Michael's intervention provoked Carlos to enter a state of perturbation, which in turn engendered his need to clarify the goal (what he was to make) with me. It is likely that Carlos resolved the perturbation by realizing (he would say "remembering") that *fourteen*-thirteenths meant to make a total of 14 parts, each of which was one-thirteenth of the unit bar. Carlos's comment that he "kept on forgetting" confirms that he tended to interpret fourteen-thirteenths as thirteen-fourteenths unless someone pointed him toward making thirteenths. His tendency to interpret fourteen-thirteenths as thirteen-fourteenths may involve a reluctance to extend beyond the unit bar, which is characteristic of students with a partitive fraction scheme (Steffe, 2002, in press). Yet when prompted, Carlos *could* extend beyond the unit bar in making bars representing improper fractions, which is not characteristic of a student with only a partitive fraction scheme.[42]

A Lacuna in Carlos's Ways of Operating with Fractions Larger than One

There seemed to be a lacuna in Carlos's ways of operating with fractions larger than one, which recurred frequently in other episodes. That is, Carlos could make a fraction like fourteen-thirteenths in activity, when he was oriented to "remember" that fourteen-thirteenths meant 14 parts, each of which were one-thirteenth of the unit bar. But he did not seem to take fourteen-thirteenths as a number "in its own right," and so he made thirteen-fourteenths (which he did take as a fraction, using his strong partitive fraction scheme).

Investigating further. In November and December I did not understand well Carlos's ways of operating with fractions larger than one, but I could see that he did not seem to have constructed an iterative fraction scheme, even though he could split. So I used Michael's absence in the teaching episode on December 4th as an opportunity to investigate further and to attempt to engender perturbations through which Carlos might construct an iterative fraction scheme.

I started the December 4th episode by posing this problem:

Task 5.14, Magic Cake Problem: This bar is a magic mint chocolate chip ice cream cake—it can only be divided into five pieces and it fills back in when you take out a

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[42] Students with only a partitive fraction scheme usually cannot make a fraction like fourteen-thirteenths, and sometimes they even object to making such a fraction (cf. Steffe, 2002).

piece. Seventeen people come to a party, and each person wants a slice. Can you show how much cake they'd get?

At first, Carlos partitioned the cake into five parts and then partitioned the first of the five parts into 17 parts. Then he said he messed up, cleared his cake, partitioned it into 17 parts, and pulled out one part as the share for one person at the party. I told him he would be right if we split the cake among 17 people, but I reminded him that the cake could only be divided into five pieces. Carlos cleared the bar and partitioned the cake back into five parts. "So since it keeps on refilling I can just take out one piece and make seventeen," he said. He did so, pulling out one of the five parts and copying it to make 12 of them. He then joined the 12 parts onto the original cake. When I asked him for a fraction name for the amount of cake he'd made, he said, without hesitation, "seventeen-fifths." I asked him how many unit cakes were in seventeen-fifths of the cake, and he said, "three…and two more," meaning three unit cakes and two-fifths more.

I infer that in solving the Magic Cake Problem, Carlos tried to figure out a way to coordinate "seventeen" and "fifths." When he rejected his first try, he then ignored the fifths and tried to make sure all 17 people got a share of the cake by partitioning it into 17 parts. His comment about the cake refilling seemed to free him from the need to stay within the amount of cake given (the unit cake) so that he could make 17 shares that were *more* cake than a single cake. Furthermore, his response that seventeen-fifths of the cake was three unit cakes "and two more" showed that when requested he could "break down" seventeen-fifths of a unit bar in terms of whole unit bars and fractional parts of unit bars.

In subsequent episodes, Carlos seemed to make fractions larger than one with much less frequent conflation. For example, during the next teaching episode December 9th, he made nine-sevenths without first partitioning the bar into nine parts. During that same episode, given seven-fifths of the unit bar he made the unit bar without first partitioning the 7/5-bar into five parts, as he had done on November 4th. So he demonstrated a reversibility with improper fractions that I had not previously seen him enact. However, I still cannot attribute an iterative fraction scheme to him at this point because he operated as he had in Protocol 5.7 repeatedly throughout the teaching experiment.

Not enough pieces. For example, one of the best views of Carlos's lacuna with improper fractions occurred in late March, in the teaching episode on March 24th. The boys were working on problems that involved a distributive operation, where they were to make different-sized candy bars and take a fractional amount of the whole collection. Carlos took one-fifth of three different bars by taking one-fifth of each of the three bars, and Michael took five-sevenths of four different bars by taking one-seventh of each of the four bars and repeating that amount five times. Both seemed convinced that taking the same fractional amount of each of the bars produced that fractional amount of all the bars. So it seemed like each boy had a distributive operation in this context. Then I posed this problem to Carlos:

> Task 5.15: Make five different-sized candy bars. Can you make 7/5 of that whole collection?

Carlos made and colored five different candy bars. Then he partitioned each bar into *seven* parts, (see Figure 5.18).

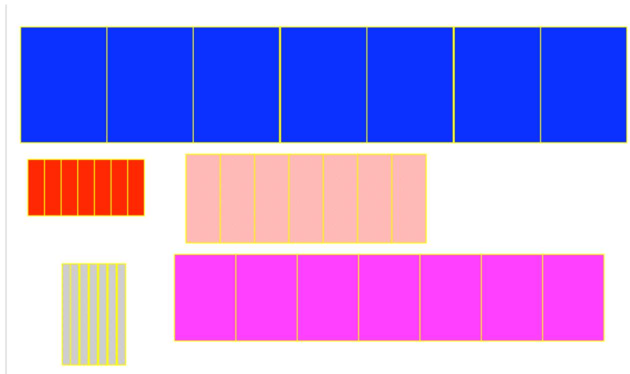

Figure 5.18, Five different bars, each partitioned into seven equal parts.

Protocol 5.8: Carlos's reluctance to extend beyond the whole on 3/24/04.

M [gives a little laugh and then pauses]: I was wondering if you could add…
T: Mmm. [To C, who is pulling out one part from his largest, black-colored bar] So let's see, how many—what did you make—what's this part [points to the first part in his largest black bar] of the whole black bar?
C: *One-seventh.*
T: Oh.
C: Oh [smiles].
T: I said to make seven-fifths, right?
C: Yeah [smiles].
T: Hmm. So if you're going to make seven-fifths, [pause] do you want one-seventh?
C: *No, I need more. But I thought that all these are sevenths, right?*
T: Well, we want to make seven-*fifths*.
C: Oh—just out of one of them?
T: Seven-fifths of the whole thing.
M: Clear them [smiles; his suggestion is to clear the bars in order to make fifths].
T: Seven-fifths of the whole thing.
C: Oh! okay! [C clears all five bars and erases the 1/7-part that he had pulled out of the largest bar. He *partitions each bar into five parts*. Whispers] All right. [C breaks all the bars into all of their parts—there are 25 parts in all on the screen. Then he starts to pull one part away from each of the five bars.]
M: This is going to be too big [the screen is not big enough to show all the parts]. [C starts to arrange four of the five parts on top of the large black part, so the black part forms a "placemat," see bottom left of Figure 5.19.]

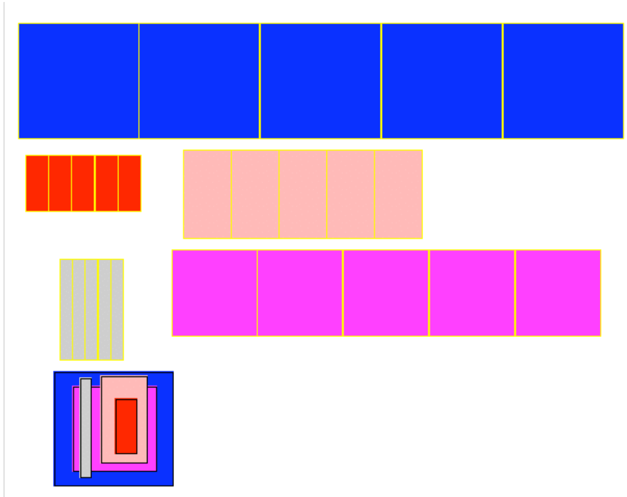

Figure 5.19, Rearranging parts to show 1/5 of the five bars (bottom left).

M: Well, not unless you put it like that.
T: So what have you made right here [points to the five parts C is arranging]?
C: *This is one-seventh.*
T: One-seventh?
C: *No wait—one-fifth.* [C continues to rearrange the parts so they fit.]
M: You need to make six more of those.
C: *But I don't have enough pieces, right?*
T: Oh—that's a good point.
M: That's why you need to—[stops, glances at T, and gives a little laugh].
C: *That's why I split them into sevenths.*
T: Oh, I see.
C: *'Cause I was just going to make them five.*
T: Oh I see, I see. But what would that make, if you had split it into sevenths and just made five?
C: *That would be…five-sevenths!*
[M notes that they made five-sevenths in the last problem. T confirms his observation.]
T [to C]: It's a good point that we don't have enough [pieces]. What are you going to do?
M [whispering]: Copy!
C: Yeah we've got to copy some of them. [C continues making his arrangements. He uses the fifth part from each of the five bars to make two more parts for each of the five bars. Then he continues to make his arrangements, making seven arrangements in all, each consisting of one part from each of the five bars.]

At the time it occurred, Carlos's activity with Task 5.15 allowed me to better understand his ways of operating with fractions larger than one. Up to this point, he had not explicitly verbalized his concern that he didn't have enough pieces. I told him he had a good point because at that moment I understood more clearly how problematic it was for him to go beyond the whole in making an improper fraction. His activity in this episode corroborates that seven-fifths was not a fractional number for him in the sense that he did not take it as a unit. To take seven-fifths as a unit means taking it as a unit of units of units, because from this point of view seven-fifths consists of seven one-fifth units, each of which retain their identity as fifths since each can be iterated five times to make the whole (the unit bar). So seven-fifths is a unit consisting of seven units, each of which is a unit in relation to another unit (the whole) from which seven-fifths is disembedded but to which seven-fifths stands, necessarily, in relation.

Because Carlos was concerned about not having enough material (candy) to make more than a whole set of bars in Task 5.15, I conjecture that he had not constructed this unit-of-units-of-units structure for fractions larger than one. That is, Carlos could make seven-fifths if he "heard" the fraction as seven one-fifths, because he could iterate one-fifth seven times and call the result seven-fifths. But, a priori, the relationship between seven-fifths, the one-fifth part, and the whole was not available to him. That is, he could explain seven-fifths as two fifths more than the whole, as he did in the November 4th teaching episode *after* he had drawn the bar (and perhaps glimpsed Michael's drawing). But prior to making seven-fifths, Carlos did not "imagine" that two-fifths more than five-fifths and one-fifth iterated seven times were the same amount. In this sense, making five-sevenths instead of seven-fifths was a reasonable response to Task 5.15 because Carlos *could* imagine a priori how five-sevenths was related to the whole: The whole was a unit of seven units and five-sevenths was one of those units iterated five times.[43]

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[43] Note that while five-sevenths can be viewed or taken as a unit of five units any of which could be iterated seven times to produce the whole, this three-levels-of-units view does not seem to be necessary to make five-sevenths, at least in a partitive sense. Those students who have not yet constructed three levels of units in fractional contexts may not be able to take five-sevenths as a unit (of units of units), even though they can make the fraction.

This conjecture would explain why he was able to solve magic cake problems (cf. Task 5.14). Since the "magic" of the cake ensured that there was plenty of cake available, he could continue to pull out parts until he had the number he needed. He could also retrospectively analyze the 17/5-bar that he'd made, determining the number of unit bars and number of fifths it contained. But doing so did not mean that Carlos had constructed these relationships *prior* to making the bar. So I claim that Carlos's scheme for making fractions larger than one was an extension of his partitive fraction scheme, but it was not an iterative fraction scheme because Carlos was not really making improper fractions—he was making long fraction bars that did not seem to have an a priori relationship with the whole. I elaborate on this claim in the context of examining the power of Michael's iterative fraction scheme.

Making and Reasoning with "Large" Improper Fractions[44]

Carlos's lack of an iterative fraction scheme is corroborated by his tendency *not* to coordinate whole numbers of units and fractional parts of units in making or reasoning with large improper fractions such as 43/6. This tendency was a significant difference between Carlos and Michael. The contrast between the boys' ways of operating with large improper fractions became apparent in early December, but a good example of it occurred on January 14th, after we had been playing the "fraction comparison game" for part of our time together over several teaching episodes. In the fraction comparison game, I gave each boy a slip of paper with a fraction written on it. Each boy took a turn making his fraction and drawing a cover over it while the other boy looked away.[45] When both had finished making their fractions, they told each other what they

[44] I refer to "large" improper fractions as fractions that are a good deal more than one, like 21/5 or 7/2 or 72/12. "Small" improper fractions are "just over" one, like 16/15 or 3/2 or 101/100.

[45] Making a cover in JavaBars entails selecting COVER and dragging a rectangle around the bars to be hidden. Covers can be moved or erased to show what's underneath.

had made and tried to determine which fraction was bigger, and why, before removing the covers to check their responses by visual comparison.

On January 14th, I gave Carlos a slip with “25/6” written on it, and Michael a slip with “21/5” written on it. Carlos partitioned a copy of the unit bar into six equal parts and pulled out all six parts. I reminded him that he could use REPEAT if it was easier—that is, I envisioned him joining up all those pieces and having a hard time, because the JOIN feature sometimes malfunctioned. Carlos erased five of the six parts he had pulled out and repeated one part to make a 19-part bar. He joined the 19 parts to the 6/6-bar and made a long cover that spanned the entire width of the screen. To make 21/5, Michael repeated a copy of the unit bar so he had four bars joined end-to-end. He partitioned each bar into five equal parts, pulled out one part, and joined it onto the long bar. Then he made a giant cover that took up the rest of the screen. Both boys looked at the screen and told each other their fractions.

Protocol 5.9: Comparing 25/6 and 21/5 on 1/14/04.

T: All right, so, what d’you think—whose is bigger?
C: Mine!
T: Yours?
M: No wait.
T: Let’s think about it.
M [pointing to C’s paper]: Twenty-six fifths, right? [M looks at C’s paper.]
T: Twenty-five sixths.
M: Okay, *then there would be four*. [*M taps along the table with his right fingers four times*, as if iterating a bar four times.] Mine would be bigger [points to himself]!
T: Oh, M thinks his is bigger. What do you think C?
M [showing C his paper]: Twenty-one fifths. ‘Cause five would go in—*no wait, they’re four, so it would have to depend on*…it’s that one [points to C’s paper].
T: You think this one’s [C’s] is bigger?
C [*taking the mouse, ready to erase the covers*]: Okay let’s check.
T: Wait-wait-wait-wait. [To C] what do you think?
C: I think it’s mine.
T: Okay now, [to C] how many unit bars are in yours?
C: Four.
T: Four. How many unit bars are in M’s?
C: Same thing.

M [simultaneously with C, holding up four fingers]: Four.
T [to C]: What's left over in yours?
C: One [glances at M's paper] and one.
T: One what?
C: One-sixth.
T: Okay. What's left over in—
M: One-fifth.
T [nods slightly]: One-fifth.
M: So, [points to C] his is bigger.
T: Okay, so yours [to C] has the one-sixth on the end there, right? After four unit bars?
C: Yeah, right.
T: And yours [to M]—
M: *Four and one-sixth, four and one-fifth*. So his is bigger.
T: Oh! So you're saying one-sixth—the one-sixth versus the one-fifth makes the difference, right?
[M nods yes.]
T: So do you [to C] agree that one-sixth is bigger than one-fifth?
C: No. [C shakes his head no emphatically, smiling.]
T: Oh you don't agree with that?
C: I think one-fifth.
T: Oh. Wait a minute, wait a minute, before we check let's make sure we agree. So—
M: *Oh! I see [smiling], yeah, mine is bigger*.
C: Yeah.
T: Oh, oh, so you think—so which one's bigger now?
C: His [points to M].
T: Twenty-one fifths?
M: I just noticed that.
[C removes the covers in order to check. They confirm that 21/5 is bigger than 25/6.]

The way that Michael made his fraction, twenty-one fifths, indicates that his scheme for making improper fractions included five-fifths as an iterable unit. That is, rather than think of twenty-one fifths solely as one-fifth repeated 21 times, Michael could conceive of it as five-fifths repeated four times and then one-fifth more. In fact, in subsequent episodes, Michael made twenty-*five* fifths by iterating four-fifths six times and adding one more fifth *and* by iterating five-fifths five times. His activity allows me to infer that nonunit fractions were iterable units for Michael. The iterability of nonunit fractions in producing improper fractions is confirmation that Michael had constructed an iterative fraction scheme—in fact, it is a characteristic of an "advanced" iterative fraction scheme since "more" than unit fractions are iterable.

Further corroboration of Michael's construction of nonunit fractions as iterable in the production of improper fractions occurred in this protocol when he compared 21/5 with 25/6. His reasoning in making this comparison involved determining the number of unit bars in each fraction, which was possible because both five-fifths and six-sixths were iterable units for him. Then, since there were an equal number of unit bars in each fraction, he compared the "leftover" fractional parts—in this case, one-fifth and one-sixth. I view his assessment that four and one-sixth was bigger than four and one-fifth as a minor conflation of sixths and fifths. This conflation does not detract from the power of his iterative fraction scheme, because once I asked about comparing one-sixth and one-fifth, he quickly assessed that twenty-one fifths would have to be bigger since he knew one-fifth was bigger than one-sixth.

The way Carlos made his fraction, twenty-five sixths, was consistent with his "extended" partitive fraction scheme: Carlos iterated a unit fraction (1/6) the number of times necessary (19) to produce twenty-five one-sixths. The way he made the fraction, the absence of a reason for his initial claim that his fraction was bigger, and his eagerness early on to check his claim via visual assessment (by removing the covers) all give no indication that for him twenty-five sixths was the same as some number of unit bars and some number of sixths. Upon questioning, just as in the Magic Cake Problem, Carlos could determine the "structure" of the improper fractions—even Michael's fraction, which Carlos had not made. But he did not initiate this activity in making a comparison. It's also unclear that my intervention (later in the protocol) of asking Carlos if he agreed that one-sixth was bigger than one-fifth was logical for him. I knew he could answer such a question—he knew that fifths were bigger than sixths. But it is unlikely he would have produced that question himself in order to compare the two fractions. Thus I still cannot conclude that he was coordinating three levels of units prior to operating, or that had constructed an iterative fraction scheme, as of January 14th.

Further evidence of the power of Michael's iterative fraction scheme occurred two episodes later, on January 28th. I posed the following challenge as the last problem in the episode:

Task 5.16: Can you use twelfths to make a bar a little bit longer than 79/11?

It soon became apparent that using the unit bar we had made, 79/11 would be longer than the width of the screen. So I asked the boys if we should change the problem so that it would fit. I said, "How about instead…" and I paused. "Forty-six elevenths," Michael suggested. What was striking about his suggestion is that it showed the "structural" way he was conceiving of these improper fractions. For him, 79/11 was seven unit bars (since eleven-elevenths was an iterable unit for him) and two more elevenths. So he suggested 46/11 because it would be four unit bars (which would fit on the screen) and two more elevenths. Thus 79/11 and 46/11 were similar to him structurally: Each was some number of unit bars and two more elevenths.

MCR During Activity with Improper Fractions

The most obvious example of MCR during this activity occurred in my attempt to adapt to and explore Carlos's ways of operating with fractions larger than one in the teaching episode on December 4th. By beginning the session with the Magic Cake Problem, Task 5.14, my intention was to test to what extent Carlos had constructed improper fractions and to provide an opportunity for him to construct an iterative fraction scheme. In addition, during this episode I was concerned with Carlos's construction of my perception of him; that is, I wanted to ensure that Carlos believed his work to be acceptable to me and that he knew I regarded him as a strong thinker. I had this goal partly *because* I believed that he had a sense that I viewed Michael as a strong thinker, and I wanted to encourage his independence from Michael. So, for example, I made a point of praising Carlos's activity in partitioning the cake into 17 equal parts (it was a good solution—all 17 people would get a fair share), but I also presented a challenge by reminding him that the cake was to be divided only into five equal parts.

During that episode Carlos responded to my mathematical care for him by working on the tasks and trying to articulate reasons for his responses. Only once, when I pressed him on the reason for a response, did he trail off and say, "my head starts to hurt now." My work with him was only partially successful in engendering either his or my learning. Carlos seemed to make bars to represent fractions like fifteen-fourteenths and seventeen-sixteenths more consistently after this episode, but I cannot conclude that he constructed an iterative fraction scheme. In addition, my working model of his mathematics did not deepen that much. Because I knew that improper fractions were still problematic for him, at times we returned to magic cake problems like Task 5.14 (e.g., on February 11th). These problems were not that fruitful in addressing Carlos's lacuna—the problems did not engender a perturbation for him because he could solve them without coordinating three levels of units prior to operating. Yet I claim that the episode was a good example of MCR because it demonstrated my attempt to harmonize with and learn more about his ways of operating, and because of his receptivity to our activity.

However, on March 24th when Carlos showed the same lacuna again in making improper fractions (cf. Task 5.15, Protocol 5.8), I admit that I felt a bit chagrinned. I paused before commenting because I was searching for an intervention that might be more meaningful to him than the previous interventions I had made throughout the year. This pause was an example of MCR in the moment of this episode, since it was part of my quest for how to better adapt to Carlos. The pause was also lucky, because through it I learned that Michael had picked up on questioning a person about his or her work so that the person might come to a resolution of some difficulty (that the questioner perceived), rather than telling the person what to do in a particular situation. I often said, “I wonder,” in posing a question to the boys. So Michael’s initial comment in the protocol, “I was wondering if you could add…” was quite similar to how I began my comments. I conjecture that Michael was thinking about the need to add on two more fifths in making seven-fifths and was trying to suggest that notion to Carlos without telling him. Thus, from my point of view, Michael was implicitly attempting to enact MCR for Carlos, an event I did not expect in the teaching experiment.

In addition, as shown in the protocol, Michael’s statement that “you need to make six more of those” helped me learn more about Carlos’s ways of operating because it triggered Carlos’s comment about not having enough pieces to make seven-fifths, and for that reason aiming to make five-sevenths. Hearing his comment moved me out of my chagrinned state and back into a fascinated one because I understood better the logical necessity of his ways of operating with fractions larger than one. So I felt more in harmony with his ways of operating than I had felt previously. I also felt renewed energy to formulate situations that might provoke a perturbation for him. At the time of the episode, I recognized that this situation did not seem to engender a change in his scheme for making fractions larger than one, but my new insight into the logical necessity of his ways of operating only augmented my fascination about what might.

This fascination with feeling in greater harmony with Carlos’s ways of operating and with searching for what might engender adaptations had deepened considerably since the teaching episode on January 14th, when the boys compared 21/5 and 25/6. During that episode I certainly felt fascinated by the boys’ activity—it was a common state for me. But I also felt at a loss for what might induce Carlos to think “structurally” about improper fractions, since he could respond to my questions about the number of unit bars in an improper fraction but did not seem to have a logical necessity to conceive of improper fractions in this way on his own. In the teaching experiment at this point, I had not yet constructed the idea that improper fractions were not fractional numbers for Carlos because he did not yet coordinate three levels of units prior to operating. Basically I could see that the boys were operating differently, and that at times Carlos did not act as powerfully as Michael. On January 14th (about one-third of the way through the teaching experiment) I didn't see the implications of these differences, but by March 24th (about two-thirds of the way through the teaching experiment), I certainly had seen more of them. Perhaps that change in perspective is what made my learning on March 24th so fascinating to me.

In addition, the episode on January 14th is a good example of the playful attitude the boys held toward each other and toward the problems, which contributed to a gentle, good-natured atmosphere in their sessions. They kidded each other in a low-key way by drawing really large covers over their improper fraction bars, trying to "fool" the other person or at least disguise the length they'd made. They seemed to enjoy the fraction comparison game, to the point that Carlos especially would enthusiastically say "yes!" every time I asked if they wanted to do one more. And they both tried to justify their choices for the larger fraction. By engaging with and communicating their enjoyment of this activity, they cared back for me.

New Uses of Multiplying Schemes: Constructive Resource 3, January and March

An improper fractional quantity can be the solution to Type 4 RMR problems like the Candy Bar Problem (Task 5.1). So engendering activity with improper fractions was important for the boys to be able to produce and interpret such a solution. However, the boys' schemes for operating with fractions larger than one were probably more significant in indicating the nature of the boys' units-coordinations in whole number and fractional contexts. That is, in November through January I learned that Michael coordinated three levels of units *prior to operating*.

Carlos coordinated two levels of units prior to operating and sometimes appeared to make three levels of units *in activity*. This difference had implications for how the boys used their multiplying schemes to solve RMR problems of Types 3 and 4, as I explore in this section.

Using Multiplying Schemes in a Continuous Context

Making a 2/2-bar into a 3/3-bar. In the episode on January 14th, I posed this problem:

Task 5.17: Make a 2/2-bar. Can you make that bar into a 3/3-bar, without clearing the halves?

I led up to this task by first posing problems like making a 4/4-bar into an 8/8-bar without clearing the fourths. After the boys had made the 8/8-bar by partitioning each fourth into two equal parts, I asked them to color the bar to show the original 4/4-bar. Michael pronounced these problems "easy," and both boys seemed to solve them swiftly using their recursive partitioning operations. However, attempting to make a 3/3-bar from a 2/2-bar evoked a different response!

Protocol 5.10: Making a 2/2-bar into a 3/3-bar on 1/14/04.

[For 5 seconds after T poses the task, both boys gaze at the screen.]
C: Okay, I'm stumped [smiles widely].
T [laughs with C]: Well, yeah, it's tricky isn't it? [M is smiling too.] Let's think about it a little bit. Think about how you could make that so you could have thirds there.
C: Well you can't clear it, can we pull out?
T: Umm. Well, you gotta make just this bar [points to the 2/2-bar]. [M has a finger on his lips, staring at the screen in deep concentration.] We don't want to make more bar [gestures beyond the 2/2-bar]. We just want to make this bar, into three-thirds.
W [after a 9-second pause during which both boys stare at the screen]: *You might have to do something different than thirds*.
T: Yeah, you might have to use something different than thirds, that's true.
M: *Wait!* [M smiles and glances at me.]
T: Mmm, M has an idea?
C [softly]: *Try six*.
M: Yeah, *six, two times three is six*.
T: Oh, you want to try six?
M [*partitioning each half into six equal parts*]: And then—
C [softly]: Sixes.
M: Oh I know!
T: So where are the thirds?

M [pointing to the bar]: *Each two—no wait, no wait. Each—*
C: Each, no wait—
M: *Four.*
C [*almost simultaneously with M*]: *Four.*
T: Can you color them? Can you color a third for me? [The boys choose a color.]
M: So there would be—one, two, three,…
C: Four of them.
M: Six of them, or no, twelve. So four. [M colors the first four parts, see Figure 5.20.]

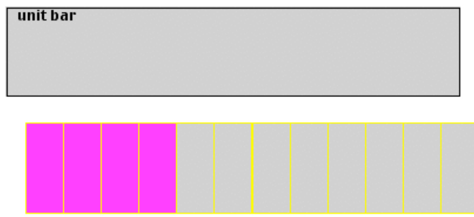

Figure 5.20, The first four parts of the bar colored to show 1/3 of the bar.

T: So is that a third?
M: Uh-huh [yes].
C: Yeah. [C wants to color the bar to show the rest of the thirds, and does so.]
T: So what's one of these little pieces?
M & C [simultaneously]: One-twelfth.
T: So how many twelfths are in a third?
M: *Four.*
C [*glances at M*]: *Four.*
T: And what about in a half—how many twelfths are in a half?
C: Six.
M: Yeah.
[They make preparations to start another problem by erasing some of the bars on the screen and making a new copy of the unit bar.]
M: *These are fun.*

With Task 5.17, both boys had entered a state of perturbation, indicated by their intent gazing at the computer screen, Carlos's admission of being stumped, and Michael's silent concentration. The witness-researcher's comment that they might have to "do something different than thirds" was intended to prompt the boys to consider making a number of parts other than three (total) that would allow them to show thirds. This comment may have had some effect on both Carlos and Michael. When Carlos murmured "six," I infer that he was thinking about a way to “combine” two and three. Carlos may or may not have intended to *insert* units within units, but six was certainly “something different than thirds” that involved two and three.

Michael’s idea about six seemed more elaborated. That is, when Michael said “wait!”, I infer that he had an idea about how to coordinate two parts and three parts that involved inserting units of three into each half of the bar. I make this claim because of Michael’s agreement with Carlos about “six” and Michael’s explanation that “two times three is six.” This idea was “something different than thirds” because it involved making six parts. It was also a novel use of the activity of his multiplying scheme (inserting units into units) in a continuous context. The novelty of this way of operating is confirmed by Michael’s excitement in exclaiming “wait!” and by his comment at the end of the protocol that “these are fun.” Possible further corroboration that he intended to make six total parts comes later in the protocol when Michael was determining the number of parts needed to show one-third. He said, “each two—no wait, no wait. Each—four.” He may have anticipated that if he made six parts total in the bar, then two parts would be one-third. Since he instead had made twelve parts total, he had to adjust this anticipated result.

In fact, because Michael said, “two times three is six,” I expected him to make a total of six parts. My explanation for him making 12 parts total has two sources. First, Carlos’s suggestion of “six” happened nearly simultaneously with Michael’s explanation about using six and may have oriented him to insert six parts into each half. Second, I asked Michael if he wanted to try six. At the time, I meant “try six parts total.” But Michael likely interpreted my question to mean that he should insert six parts in each half. In any case, doing so allowed him to solve the problem and demonstrated that he had used his multiplying scheme in service of his iterative fraction scheme. I claim that his solution of Task 5.17 was the beginning of him making a functional accommodation in his multiplying and fraction schemes: Inserting units into units was a way for him to coordinate two different fractional parts in the same bar, even when one fractional part was not a whole number multiple of the other.

Making a 5/5-bar into a 4/4-bar. During the remainder of the January 14th episode, Michael refined his new use of his multiplying scheme in service of his iterative fraction scheme to include two features: an anticipation of the smallest total number of parts needed, and an anticipation of the number of those parts in the fraction he was aiming to make. For example, later in the episode I asked the boys to make a 5/5-bar into a 4/4-bar without clearing the fifths. Carlos proposed partitioning each fifth into eight parts, a suggestion likely based on his immediately preceding success in partitioning each third of a 3/3-bar into eight equal parts in order to make a 4/4-bar. I asked the boys if they could use something smaller or easier to solve the problem. While Carlos proposed using two, Michael said, "Twenty. No, no wait. Something that would equal twenty." But Carlos insisted on his idea. "Try two's Michael!" he said. So Michael partitioned each fifth into two parts to make 10 parts total. When I asked where the fourths were, Michael pointed to the middle of the third and eighth parts,[46] and he partitioned each of the 10 parts again into two parts to make 20 parts total. "So that would be just like using four, like my way!" he exclaimed. Both boys identified that five twentieths made one-fourth, although Michael was the first to do so.

Carlos's way of operating in these situations seemed to involve partitioning the bar into more parts by partitioning each part (in this case, each fifth) into some number of parts. His suggestion of partitioning each fifth into eight equal parts would indeed have worked, but his suggestion of partitioning each fifth into two parts would not. His latter suggestion leads me to infer that Carlos was suggesting a number of parts into which to partition each fifth without

[46] Visual material on the screen may have helped him determine these locations because a same-sized bar above had been colored into fourths on a previous problem.

anticipating whether the partitioning activity would produce the desired result—so he was experimenting, without carrying out his experiment in thought prior to enacting it. Thus his way of operating did not seem to include anticipating either the total number of parts in the bar or the number of those parts in one part of the "target" fraction. I make this claim because had Carlos anticipated making 10 total parts *with a goal to make fourths* of the bar, he likely would have realized that partitioning each fifth into two equal parts would not work.

Michael's co-measurement scheme. In contrast, Michael's goal to make "something that would equal twenty" was likely based on coordinating four and five multiplicatively. He seemed to engage in a *thought experiment* (versus experimenting, as Carlos did) to test his idea before enacting it. I infer that he could anticipate that 20 parts would "work" because four parts in each fifth would make 20 parts in all, and 20 divided by 4 was 5, so five parts would be in each fourth of the bar. I conjecture that this way of reasoning was possible for him because he could take the 5/5-bar as a unit of units, and he could anticipate inserting units into each of those units toward a goal of reorganizing the bar as unit of *another* number of units of units. In other words, he could view the bar as a unit of *two different* units-of-units structures. Michael could take the whole bar as a unit of five units into each of which he could insert four units, and he could anticipate that the 20 units created could be considered as a unit of four units each containing five units.

Nevertheless, Michael was willing to try Carlos's suggestion of partitioning each fifth into *two* parts, probably for at least two reasons. First, Michael may not have been convinced that his way of making four-fourths by making 20 parts was the easiest way to do the problem. This reason reflects Michael's tendency to search for easier or more efficient ways to solve a problem as well as my request for an easier way to do the problem (easier than using eight). Second, Carlos made a direct recommendation to Michael to try using two, which was rare, and Michael was apt to listen to his friend.

Based on Michael's goal of making 20 parts, he adapted Carlos's suggestion by partitioning by two *twice*.[47] In this process, Michael refined the functional accommodation in his multiplying and fraction schemes that he had initiated in making a 2/2-bar into a 3/3-bar in Task 5.17. The refinement came in the *activity* of using his multiplying scheme—partitioning the bar into the smallest number of parts necessary (in this case the product of four and five) to show both fractions by partitioning each part into the requisite number of parts. The relative permanence of this refinement was evident in the next teaching episode on January 21st: To make a 2/2-bar into a 5/5-bar, Michael partitioned each half of the bar into five equal parts to make a total of 10 parts. This activity occurred just after Carlos had partitioned *each* half into 10 equal parts for a total of 20 parts, and so it exemplifies Michael's logical necessity in partitioning the bar into the smallest number of parts necessary to show both fractions. I call Michael's new scheme a *co-measurement scheme*, because it involves making a common "measurement," or specific size of part, in order to show two fractions in the same bar. When two fractions have denominators that are relatively prime, such as one-half and one-fifth, determining their smallest co-measurement entails determining a number of parts that can be taken as a unit of two parts each containing five parts *and* as a unit of five parts each containing two parts, So in this sense, a co-measurement scheme appears to be one kernel of commutativity.

Carlos's Enactive Partitioning of Partitions

On January 21st, after Michael partitioned each half of the 2/2-bar into five equal parts, Carlos said he had another way but "it's a little bit longer." He identified 95 as the "highest" and

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[47] Note that this adaptation corroborates that Michael anticipated dual units-of-units structures in the bar, because he was able to take the new 10/10-bar as a unit of 10 units into each of which he could insert units of two in order to make twentieths and thereby show fourths.

partitioned each half of the 2/2-bar into 95 parts. Both boys agreed that “anything divided by five” (any number of parts divisible by five) would work to make a 2/2-bar into a 5/5-bar. And since the dial in the PARTS menu went up to 99, 95 was the highest multiple of five on the dial. Carlos’s astute observation shows that he knew that the total number of parts in the bar would need to be divisible by five in order to show five-fifths, and that he “trusted” dividing each half into a number divisible by five would result in a total number of parts divisible by five. His observation also demonstrates his tendency to look for and adapt quickly to patterns he saw in his own or others’ ways of operating. However, Carlos did not calculate the 190 parts total parts that 95 parts in each half would produce, and he had difficulty determining the number of these parts that constituted one-fifth of the bar. At one point he suggested dividing the total number of parts by five, but in general he grew distracted while Michael determined how many one hundred-ninetieths were in one-fifth of the bar via some sophisticated strategic reasoning.

This activity confirms the inferences I made about Carlos’s way of operating in making a 5/5-bar into a 4/4-bar: Namely, Carlos’s partitioning activity did not seem to include recognition of the implications of using larger or smaller partitions—i.e., that partitioning into a large number of parts could work but was hard to handle, and that partitioning into just enough parts would be easier in terms of calculation. Not recognizing the implications means that Carlos did not anticipate the (minimum) total number of parts that might be desirable in order to show the other fraction, and therefore the number of parts needed in each given fractional part, so that his solution to the problem would be “contained” in his partitioning activity. Instead, he often seemed to have a sense or hunch that using a particular partition would work (such as 95 parts in each half). But determining the number of parts needed to show one part of the other fraction was a *separate* problem for him. Corroboration of these inferences occurred later in the January

21st episode. Carlos's had observed Michael's use of his co-measurement scheme throughout the episode. However, in making a 3/3-bar so that it showed eighths, Carlos wanted to partition each third into 24 parts. Michael objected and suggested partitioning each third into eight parts, advice that Carlos then followed.[48]

Thus Carlos's scheme for showing two fractions in the same bar appeared more experimental than Michael's co-measurement scheme and did not seem to have the same power or efficiency. I contend that the experimental nature of Carlos's scheme came from him not yet taking the whole bar as a unit of units into which he could anticipate inserting units to accomplish a goal of viewing the whole bar as a unit of a different number of units of units. For example, to make the 5/5-bar into the 4/4-bar, Carlos could certainly conceive of the 5/5-bar as a unit of five units. He could propose inserting units of eight into each fifth and he could use multiplication to determine that 40 parts would result. So he could make the bar into a unit of units of units in activity. But because Carlos did not yet have an *anticipatory* unit-of-units-of-units structure, I conjecture that he could not hold in mind the view of the bar as a unit of five units each containing eight units and flexibly "switch" to viewing the 40 parts in the whole bar as a unit consisting of four units, each of which contained of 10 units.

After further partitioning the fraction bar in these problems to produce a larger number of parts, Carlos seemed to use two different schemes to find the number of parts necessary to show one part of the "target" fraction. Sometimes he divided the total number of parts by the number of parts in the target fraction bar—e.g., divide 40 by 4 to get 10. At other times, he made an estimate for the number of parts in the target fraction and then colored this number in succession along the bar ("iterating" the part via color) to see if the estimate "worked." In these cases,

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48 Note that Carlos did seem to abstract a pattern here, probably based on observing Michael's activity: Carlos seemed to partition each part into the product of the two different numbers of parts he was trying to coordinate.

Carlos used his partitive fraction scheme to test out whether he had made the target fraction. If it didn't work, he would adjust his estimate and try again. Often Carlos would color the whole bar in this manner even if he had used division to determine the necessary number of parts. So Carlos seemed to make sense of the activity of Michael's scheme that involved partitioning the bars into smaller numbers of parts, and Carlos could use this partitioning activity to solve the problems. But he had to *enact* his partitioning idea in order to then, afterward, figure out how or whether it "worked." Carlos did not use his multiplying scheme in a way that included the logical necessity of making a specific total number of parts until March 29th, when he partitioned a 2-foot bar into three equal parts, as discussed in the opening section of this chapter.

Differences in the Boys' Ways of Operating with RMR Problems

Significance of a co-measurement scheme in solving RMR problems. During the teaching episodes on March 8th and 10th, we worked on RMR problems of Type 4 like the Candy Bar Problem (Task 5.1). To return to an example I have already mentioned in the first section of this chapter, on March 10th the boys worked for quite some time on Task 5.2.

> Task 5.2: An 8-centimeter peppermint stick is marked to show the 8 centimeters. It is 3/4 of another peppermint stick; make the other stick and tell how long it is.

As I have described, to solve this problem Michael made a copy of the 8-centimeter bar in which he had partitioned the two rightmost centimeters into thirds. Then he broke the bar into all of these parts and made a new bar by joining together two 1-centimeter bars and two 1/3-centimeter bars (cf. Figure 5.7). For Michael, that new bar represented one-third of the 8-centimeter peppermint stick and one-fourth of the other stick, and he could use it to solve the problem.

I conjecture that one important constructive resource in Michael's ability to solve Task 5.2 was his dual unit-of-units-of-units views of the 8-centimeter bar. That is, he could view the bar as a unit of eight units, any of which could be split further in order to view the bar as a unit of three units, each containing some equal number of units. In solving the problem he abbreviated his partitioning activity: He did not partition all 8 centimeters as he had partitioned all 7 inches in the Candy Bar Problem (Task 5.1). Instead, he appeared to determine how many centimeter units he needed to split further (two) in order to achieve the second view, where the three equal parts of the 8-centimeter bar consisted of both whole centimeters and thirds of centimeters. Although abbreviated, his partitioning activity still seemed to rely on making dual three-levels-of-units views of the bar, a chief characteristic of his co-measurement scheme. So his co-measurement scheme seemed to be a fundamental constructive resource for him in solving the Candy Bar Problem and other RMR problems of Type 4.

Whole number quantities and relationships. During Michael's work on Task 5.2—and throughout the teaching episodes on March 8th and 10th—the witness-researchers and I were not sure what sense Carlos was making of these rather complex situations. So directly after Task 5.2, in an effort to simplify the problem situations, a witness-researcher suggested switching to a RMR problem with whole number quantities and whole number relationships, where the relationship did not divide the quantity—that is, to a Type 3 RMR problem:

Task 5.18: Eleven centimeters is three times another bar. Make the other bar.

Michael immediately said he knew how to do it. Carlos wanted to clear the bar and then partition it into three equal parts, so he assimilated the problem to his splitting operation, as long as he could clear the marks on the bar. But I stipulated no clearing since I aimed to engender distributive splitting and wanted him to determine the length of the new bar out of his partitioning activity. I also acknowledged Michael's idea, but since he had done the last problem, I told the boys that it was Carlos's turn to try.

Although Carlos confirmed with me that his new bar would be three times shorter than the 11-centimeter bar, after a 30-second pause he partitioned the eleventh centimeter into *two* equal parts. He seemed stumped and noted that the problem was "hard." I asked him if he thought Michael's activity on Task 5.2 might help him. After some more discussion, Carlos started over, splitting the two rightmost centimeters each into three equal parts. He broke the bar into all of its parts and rearranged them into bars consisting of two 1-centimeter bars and a 1/3-centimeter bar. After making three bars in this manner (see Figure 5.21), he started to make fourth bar and noted "there'll be one [centimeter] leftover."

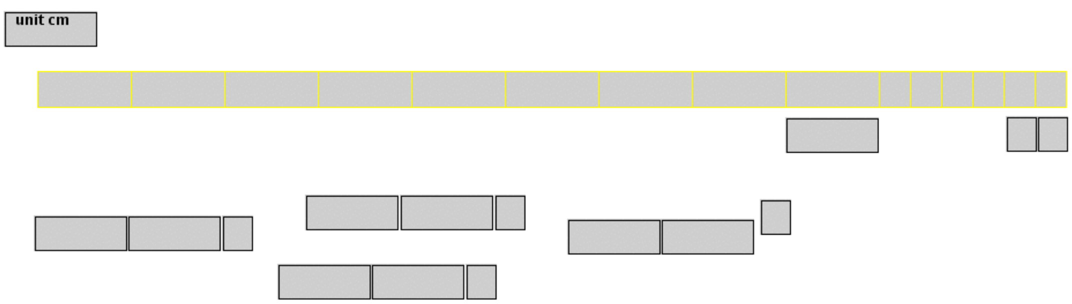

Figure 5.21, Making groups of 2 and 1/3 centimeters from an 11-centimeter bar.

Carlos's attempt to solve Task 5.18 exemplifies how he could operate similarly to Michael but without the logical necessity that characterized Michael's ways of operating. That is, given my constraint against clearing the marks on the 11-centimeter bar, Carlos seemed to experience a need to further partition each of the centimeters, as he demonstrated by partitioning the eleventh centimeter into two equal parts. However, he did not seem to have a reason to partition the centimeters in this way, let alone to partition any of the centimeters into three equal parts. He was willing to follow my suggestion of using what Michael did on Task 5.2. But in forming groups of parts that might be thirds of the 11-centimeter bar, Carlos followed Michael's activity more literally than I had expected. That is, Carlos's groups of two 1-centimeter bars and a 1/3-centimeter bar were similar (although not identical) to Michael's three equal parts of the 8-centimeter bar in Task 5.2. Given that Carlos did not partition any of the centimeters into three parts out of his own logical necessity, he very reasonably "copied" Michael because he interpreted my suggestion to include both Michael's partitioning activity and how Michael formed the equal parts for Task 5.2. Carlos did not seem to consider it problematic that he was forming *more* than three equal parts from the 11-centimeter bar by forming groups in this way.

Although with Michael's advice Carlos later rearranged the parts to form three equal groups (consisting of three 1-centimeter bars and two 1/3-centimeter bars each), the lack of logical necessity for how he operated means that Carlos did not split distributively in this episode. He did not make a modification in his splitting operation so that he could split each centimeter into three equal parts in order to split the entire 11 centimeters into three equal parts. I infer that he did not make this modification because he could not yet coordinate two different three-levels-of-units views of the 11-centimeter bar. That is, he could insert units into the 11 centimeter units, but because he didn't anticipate a view of the 11-centimeter bar as a unit of a different number of units of units, he didn't have a reason to make any particular partitioning. So Carlos tended to rely on his "nondistributive" splitting operation: Clearing the 11-centimeter bar and splitting it into three equal parts definitely solved the problem for him, but it did not allow him to determine the length of one of the parts out of his partitioning activity, let alone to coordinate two different three-levels-of-units views in the same bar.

Carlos's first attempt to split a 2-unit bar into three equal parts. Corroboration of these inferences occurred in the next teaching episode on March 24th. I posed this task for Carlos:

> Task 5.19: That 2-foot cake has a different flavor every foot. It's three times longer than another cake. Make the other cake—it has to have both flavors.[49]

Carlos copied the 2-foot cake, colored the first foot grey (for chocolate) and the second foot blue (for blueberry), and commented that we would run out of room. I took his comment to mean he was planning to iterate the cake to make a cake three times longer than the 2-foot cake, so I restated the problem. Carlos then said, "Oh, you'll have to subtract it" and mused, "it has to be three times shorter." He tried partitioning each foot into two equal parts.

> Protocol 5.11: Carlos's first attempt to split a 2-foot cake into three equal parts on 3/24/04.
>
> C: Aw! Okay now I can't think of it. Now I've got four [parts].
> T: So I have a question, can you make—let's pretend the cake is just the chocolate part [the first foot of the cake, which T encompasses with her first finger and thumb]. Could you make a cake so this chocolate part is three times longer than the chocolate part you're going to make?
> [C clears the entire 2-foot cake, partitions it back into two feet, and recolors it.]
> T: *So pretend like the blue part's not there*.
> M [to T, although he is looking at the screen, smiling]: *You're giving it away!*
> T [laughs]: Well, that's okay.
> C: *Oh, okay. So then, I can actually split this [the chocolate foot] into three pieces*.
> T: Mm, okay, why don't you try that?
> [C splits the chocolate foot into three parts and pulls out his new chocolate cake—one part—upon request.]

Once Carlos interpreted the cake he was to make as "three times shorter" than the 2-foot cake, he knew he wanted to cut the cake up into smaller parts. As he had on March 10th with Task 5.18 (11 centimeters is three times longer than another bar), to accomplish his goal he tried

[49] The reader may note that Task 5.19 is similar to Task 5.5, which Carlos solved "as an original" in the next teaching episode on March 29th, as discussed in the opening section of this chapter on Carlos (cf. Protocol 5.3). One difference between the two tasks is that in Task 5.19 coloring each foot was included to elicit distribution.

to split units of the bar (each foot) into *two* equal parts. My intervention allowed him to focus just on one (chocolate) foot, on which he could use his (nondistributive) splitting operation.

Protocol 5.11 gives further credence to my claim that Carlos had yet to take a 2-foot bar as a unit of two units, into each of which he could insert (three) units, in order to create a different unit-of-units-of-units structure (i.e., to then take the 2-foot bar as a unit of three units, each of which contained two units). In fact, when Carlos met a nearly identical problem at the start of the next teaching episode on March 29th (as already discussed, cf. Task 5.5, Protocol 5.3), it was a novel problem for him. As shown by his two initial solutions to Task 5.5 that did not involve distributive splitting, Carlos acted as if he had not worked on such a problem before, which indicates that from his point of view he had not! So in his partial solution to Task 5.19 he used his splitting operation without making any modification. But in his activity later in the March 24th episode (as already discussed, see Task 5.4 and Protocol 5.2) and in the next episode on March 29th, Carlos seemed to construct distributive splitting as well as a new use of his multiplying scheme, at least in making a 2-unit bar into three equal parts. I have called his scheme a reversible multiplying scheme rather than a co-measurement scheme because, as I discussed in the opening part of this chapter, fractions were only implicit in his ways of operating in these tasks.

MCR in the Middle of the Teaching Experiment

In Protocol 5.10 on January 14th, Michael expressed his enthusiasm for problems like Task 5.17 (make a 2/2-bar into a 3/3-bar) by stating, “These are fun.” His comment indicates that Michael found the thinking involved in these problems to be pleasing in some way. I infer that for Michael, these problems were not outside of the ways of operating that he had built up—the problems were enough in harmony with his ways of operating that he could initiate activity. Yet the problems seemed to be a challenge in that he did not immediately know how to solve them and he had to coordinate his schemes in new ways to do so. In other words, the problems were within his ZPC. For these reasons, posing such problems to Michael was a way of enacting mathematical care for him. In return, his interest and expressed enthusiasm were a way of caring back for me, of completing the caring relations between us.

Later in the January 14th episode I attempted to enact mathematical care for both Michael and Carlos by asking about partitioning the 5/5-bar into a number of parts that was smaller or easier than eight in order to make a 4/4-bar. That question was appropriate for both boys because through it I was aiming to see if they would, out of logical necessity, use their multiplying schemes to anticipate the minimum number of parts they could use. Because Michael had shown signs of operating this way in making the 2/2-bar into a 3/3-bar, my question may have helped bring forth his construction of a co-measurement scheme, refining the functional accommodation he had started to make in his multiplying and fraction schemes. So my question was a specific example of enacting mathematical care for him.

However, my question was not as fruitful for Carlos. Although he certainly could suggest a smaller number of parts (two), the question did not appear to provoke an anticipation of the minimum number of parts needed to show four-fourths in the 5/5-bar. In fact, it may have supported Carlos's notion of *experimentation* in finding a partition that would work for two reasons. First, Carlos may have interpreted that his suggestion of "two" *did* work, since Michael used two twice in succession to make the 4/4-bar. Second, I did not emphasize predicting whether partitioning each fifth into two parts would yield a bar that could be colored to show fourths, a teacher action that might have worked toward curtailing Carlos's reliance on enacting the partition to assess its success. He still might have needed to enact partitioning for some time before he could make such predictions—constructing a unit of units of units as an anticipatory structure involves vertical learning! But regular questions about predicting whether partitions would "work" might have been a useful norm to initiate in bringing forth an anticipatory use of his multiplying scheme in these situations.

In practicing mathematical care for Carlos during the teaching episode on January 21st, I made time for him to express and enact his idea about making a 2/2-bar into a 5/5-bar by partitioning each half into any number of parts divisible by five, and specifically into 95 parts. In doing so, I intended to demonstrate that his mathematical ideas were astute and worthwhile, as well as to follow through on his independent mathematical generalization. But I also missed an opportunity to challenge his mathematical reality, because I might have used Carlos's comment about the *biggest* number of parts he could use to ask about the *smallest* number of parts he could use. In retrospect, this kind of intervention would have been better mathematical care for Carlos because it might have contributed to bringing forth an anticipatory use his multiplying schemes in these situations.

In contrast with my intentions to value Carlos's ideas on January 21st, my interactions with him on March 10th with Task 5.18 (an 11-centimeter bar is three times another bar) exemplified weaker mathematical care for him. Had I better understood that he did not have an anticipatory unit-of-units-of-units structure available, I would not have disallowed "clearing" the bar at this point. Instead I would interacted with him more like I did on March 29th with Task 5.5 (a 2-foot bar is three times another bar) in Protocol 5.3, making space for him to solve the problem the way he wanted to (using his nondistributive splitting operation), and then posing further constraints. In addition, suggesting to Carlos that he follow Michael's activity in solving Task 5.2 did not help him produce a scheme of his own, based in his own logical necessity! I did not expect him to interpret my suggestion as literally as he did, or for him *not* to interpret Michael's central purpose in dividing some of the centimeters into thirds. Although this interaction with Carlos did not necessarily result in strong MCR with him in the interaction, it was quite useful in enacting MCR with him overall.

That dynamic seems to be a central feature of MCR: If a teacher intends to enact MCR in a global sense (i.e., over a long period of time) with students, the teacher may in some sense "fail" to enact it regularly with them in a local sense as the teacher comes to make a more detailed model of the students' ways of operating and the constraints the teacher experiences. My interaction with Carlos on March 10th with Task 5.18 provoked a perturbation for me, as a teacher, that led me to pose Task 5.19 (a 2-foot cake is three times another cake) on March 24th, which I envisioned as a "simpler" coordination within Carlos's short-term ZPC. When that interaction also did not seem to provoke a perturbation for him that might engender a modification in his splitting operation, it led me to "step back" further to Task 5.4, the Cake Problem (a 3-foot cake is two times another cake). As shown in Protocol 5.2, I infer that Carlos's solution to this problem marked the beginning of his modification of his splitting operation. Thus, my interactions with Carlos in the teaching episode on March 10th (and the beginning of the teaching episode on March 24th) were locally weak in MCR but were integral to enacting stronger global and local mathematical care for him in the "long run."[50]

Interestingly, Michael's developing, implicit sense of MCR is also evident on March 24th. As shown in Protocol 5.11, when I directed Carlos to focus on just the chocolate foot of cake, Michael objected by exclaiming that I was "giving it away!" I infer that he believed that my questions were supposed to enable Carlos to act in a situation when he was stumped, but that

[50] I do not want to imply that MCR are necessarily weak in an episode, or over several episodes, if particular acts of learning do not occur. I explore this notion with respect to fraction composition activity in the next section.

Carlos should have to figure out the problem—my questions should not allow him to know exactly what to do. From Michael's point of view, my intervention made the problem too easy and obvious. In some sense his objection was well-founded—Carlos was able to operate after my intervention by relying on his splitting operation rather than by modifying his splitting operation. So it is not a surprise that Carlos's activity did not seem to be meaningful to him in terms of the original problem and did not engender a new use of his multiplying scheme at that point.

Fraction Composition Activity: Constructive Resource 4, March and April

In one sense, the story of the boys could end here, with an accounting of how reversible reasoning, improper fraction activity, and new uses of multiplying schemes (Constructive Resources 1, 2, and 3) were instrumental in the boys' solutions of RMR problems of Type 4 (e.g., the Candy Bar Problem, Task 5.1, which Michael solved in February) and Type 3 (e.g., Task 5.5, make a candy bar such that a 2-foot candy bar is three times longer than it, which Carlos solved in March). However, in the remaining two months of the teaching experiment I aimed for the boys to solve RMR problems of Type 5, with *fractional* quantities and fractional relationships, such as Task 5.3 (the Lizards' car goes 1/2 of a meter, which is 2/3 of the distance the Cobras' car went; find the distance of the Cobras' car).

Working toward this aim meant engaging the boys in multiplicative activity with fractions, since composing fractions multiplicatively is involved—implicitly or explicitly—in solving problems like Task 5.3. Furthermore, multiplicative activity with fractions is entwined with conceiving of fractions as multiplicative operations on quantities. Constructing and using fractions as multiplicative operations is an essential basis for the use of reciprocal (as opposed to only reversible, cf. Chapter 3) reasoning to solve problems like Task 5.3: Finding three-halves of 1/2 meter by multiplying 3/2 and 1/2 may be considered a more algebraic solution of this problem. So the boys' activity in relation to the last two constructive resources, fraction multiplication activity and fractions as operations, continues the story of working toward solving Type 5 RMR problems, as well as toward making more algebraic solutions of any RMR problem.

The Beginning of a Fraction Composition Scheme

At the very end of March, the boys and I began to work on making compositions of fractions. On March 31st, I posed problems such as:

> Task 5.20: A man needs 2/5 of a yard of ribbon to wrap a package. I need 1/3 of how much the man has. Can you make my piece?

To solve this problem, Carlos partitioned each of his two fifths into three equal parts, pulled out one part, and repeated it to make two parts. So I infer that he used distributive partitioning, which he had constructed as part of the modification of his splitting operation in solving Task 5.5 in the previous episode on March 29th.[51] Michael agreed with Carlos's activity and resulting bar. Carlos explained that "there were two pieces and I knew that you wanted six, I mean three, so I had to times it [two] by three, and I got six, and then you divide six by three, then you get two." His explanation shows his new use of his multiplying scheme: Making six parts total allowed him to partition a 2-part bar into three equal parts. So I infer that he viewed the 2/5-bar as a unit of two units each containing three units *and* as a unit of three units each containing two units.

Then I asked Carlos what part of the yard I got. After a 10-second pause he said, "two-fifteenths." I infer that he used his recursive partitioning operation, imagining inserting units of three into each fifth, which would make fifteenths, and noting that the resulting bar was two of these fractional parts. Since in Task 5.20 I asked only about making the composition, it is not

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[51] Note that Task 5.20 is not a problem requiring reversible reasoning, so splitting—or using a reversible multiplying scheme—was not necessary.

surprising that finding the length of the result was a separate problem for Carlos. Yet his explanation for how he made the composition did not involve *fifths*. His reasoning here indicates his subsequent tendency during the experiment to proceed in two associated steps in his fraction composition activity: First make the composition by partitioning and using his multiplying scheme; second, determine its length by considering the small parts in relation to the unit whole. My conjecture is that to determine the result of the composition as an outgrowth of making it would require a more explicit distributive operation, as I will discuss later in this chapter.

Right after posing Task 5.20 I posed a similar task for Michael:

Task 5.21: A man needs 4/5 of a yard of ribbon. I need 2/3 of his piece. Can you make my piece?

Michael partitioned the first three of his four fifths into three equal parts, pulled out one part, and repeated it to make eight parts total. In explanation, he said, “because I knew that if you wanted them to be equal to something that they’d both be equal to, since there were already four, it’d be twelve, and I remembered that eight-twelfths is two-thirds.” Michael’s explanation shows his use of his co-measurement scheme in making the composition. I infer that Michael viewed the four-part bar as a unit of four units each containing three units and as a unit of three units each containing four units. He did not need to enact all of the partitioning in order to operate in the situation because he also used some knowledge of commensurate fractions (eight-twelfths was the same as two-thirds).[52] So like Carlos, in making the composition Michael also did not focus on the four parts as *fifths*, which is necessary to determine the length of the result. When I asked about the length, it took Michael about 60 seconds to arrive at an answer of eight-fifteenths. I

[52] I use the word commensurate rather than equal because commensurate implies a sense of compatibility of measurement between two fractions such as 8/12 and 2/3, similar to commensurate measures like 12 inches and 1 foot (cf. Steffe, 2003).

infer that Michael had to make a considerable readjustment in his ways of operating in order to consider the parts as fifths and use recursive partitioning to find the length.

The boys' work on these two tasks shows that although they could make compositions, they had not yet constructed a fraction composition scheme, where the result or "measure" of the composition grows out of making of the composition. In other words, they were yet to use their distributive partitioning operation in a way that would allow them to embed recursive partitioning into their activity. For example, finding two-thirds of four-fifths in this manner might involve a sub-goal to make one-third of four-fifths by taking one-third from each one-fifth, yielding four one-thirds of one-fifth, and then iterating that quantity twice to get eight one-thirds of one-fifth.[53] This reasoning does not require determining the total number of parts made during the process (circumventing the notion of making twelve parts total), and it allows recursive partitioning to be embedded in the activity (i.e., both boys knew that one-third of one-fifth is one-fifteenth and that eight one-thirds of one-fifth is eight-fifteenths). Nevertheless, the boys' solutions of Tasks 5.20 and 5.21 established the basis for moving toward the construction of a fraction composition scheme.

The Nature of the Boys' Fraction Composition Activity

Michael's distributive fraction composition activity. In the next teaching episode on April 12th, Michael made three-sevenths of three-fifths of a yard by partitioning each of the three fifths into seven parts. He pulled out one part and repeated it to make three. "There," he said. When I asked him what he had just made, he said "one-seventh." He verified his result by dragging his 3-part bar along the 3/5-yard bar seven times. I asked him if he could make three-sevenths of the 3/5-bar using what he had made. Michael said you "just do this two more times," and he repeated

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[53] Another option is to take two-thirds of one-fifth, which is two one-thirds of one-fifth, and then iterate that quantity four times.

his 3-part bar to make three 3-part bars. This activity marked an important point in the evolution of Michael's activity with fraction composition problems, because he *first* made a unit fractional amount of the fractional quantity. Then he iterated that unit fractional amount to complete the composition. In this way of operating I see the beginning of a "distributive" structure that Michael was using to make fraction compositions. Finding the length of the composition was still a separate problem for him at this point in that he did not track how many sevenths of fifths he was making as he made the composition. However, both Michael and Carlos successfully found the result to be nine thirty-fifths of a yard.

Michael did not use distribution consistently in making fraction compositions, so I cannot conclude that his distributive operation was becoming embedded in his fraction composition activity. Part of the reason he may not have progressed in this way was that by mid-April I began to ask the boys if they could state a fraction multiplication problem for the compositions they'd made, and Michael became very intent upon reconciling his knowledge of the standard computation algorithm for fraction multiplication with his productions of bars that represented a compositions of fractions. In this effort, he sometimes computed with the algorithm first, which seemed to interfere with making the bars using his distributive operation. In particular, the structural way of operating that he showed on April 12th in making three-fifths of a 3/7-yard bar seemed to get lost in his attempt to make the final result he had computed with the algorithm.

Carlos's enactive fraction composition activity. Carlos's fraction composition activity had an "experimental" character, similar to his activity in coordinating two different fractional parts in the same bar. As discussed in the previous section, Carlos's coordination of two and three equal parts in the same bar was not explicitly fractional, and whether Carlos had constructed a more general scheme for making any number of equal parts in a multiple-unit bar remained uncertain. I conjecture that his ability to view a bar as a unit of two different units-of-units structures was not yet generalized. Since the boys' evolving fraction composition schemes rested heavily on making these dual three-levels-of-units views, it is not surprising that Carlos's fraction composition activity seemed experimental.

A good example of this experimental nature occurred during the teaching episode on April 14th in making one-fourth of five-sevenths of a yard. Carlos made a 5/7-yard bar and then immediately partitioned each of the five sevenths into four equal parts to make 20 parts total. He pulled out one part from the 20-part bar and repeated it to make a 4-part bar. He iterated the 4-part bar by dragging it along the 20-part bar (the 5/7-yard bar) to assess the number of times it would fit. When he got to the fourth positioning of the 4-part bar and saw that he had not exhausted the 20-part bar, he stopped. "Okay, now I know what to do," he said. Then he tried the same technique with a 6-part bar. "Nope. Darn it! It still doesn't work," he commented, when the 6-part bar only fit three full times and a little bit more. He then tried again with a 5-part bar and was successful in fitting it four times along the 5/7-bar. When I asked him why the 5-part bar worked, he said he didn't know. Smiling, he added, "I'll let Michael say it [explain]." I also asked Carlos why he had partitioned each of the parts of the original 5/7-yard bar into four parts to start the problem. "Since there were, since you said we wanted one-fourth, [using] four [to partition] would be a good estimate," he said.

This activity shows that Carlos's way of operating to make fraction compositions consisted of "automatically" partitioning each fractional part of the given fraction into the number of parts in the fractional part that he wanted to make. His verbalized reason for doing so, that it "would be a good estimate," shows two major aspects of his way of operating. First, it shows a procedural aspect (cf. Norton, 2004) in that Carlos had seen Michael operate this way successfully and thus "believed" in this partitioning activity without necessarily having his own logical necessity to do it. Second, it shows that Carlos's general way of operating, as with his coordination of two different fractional parts in the same bar, was to partition the bar into a smaller number of parts and then use his partitive fraction scheme to check whether the partitioning "worked." In later episodes, Carlos sometimes offered a reason for the number of parts he made. The reason was based on dividing the total number of parts he'd made by the number necessary to produce a unit fraction (i.e., in the example above, dividing 20 parts by 4 to get five parts). However, even when he gave such an explanation, he tended to use his partitive fraction scheme to check, iterating what to him was an "estimate" of the correct resulting bar. It seemed as if he did not fully trust his solution until he could verify it using a scheme he *did* implicitly trust (his partitive fraction scheme). In this sense he had to *enact* his activity, rather than carry it out in thought prior to acting.

My explanation for Carlos's ways of operating with fraction composition relies on the same difficulty he had throughout the teaching experiment: He could not yet take a unit of units of units as an anticipatory structure. To make one-fourth of five-sevenths of a yard based on coordinating four and five equal parts in the same (5/7-yard) bar would require Carlos to have constructed a co-measurement scheme so that he could flexibly switch between two three-levels-of-units views of the 5/7-yard bar. To make one-fourth of five-sevenths of a yard with a general fraction composition scheme seems to require dual three-levels-of-units views in a more complex way. That is, the operations of a co-measurement scheme are involved in transforming five equal parts into four equal parts *and* the smallest parts created are units embedded within five units that stand in relation to a unit bar that is to some extent "outside" of the immediate action (i.e., the 5/7-yard bar must stand as a unit in relation to the unit bar). So I conjecture that a co-measurement scheme is constructive resource for making fraction compositions, although it's uncertain whether that scheme sufficient for enlarging the field of activity in making fraction compositions in order to coordinate fractional parts (like 5/7 yard) in relation to the unit bar. Since Carlos had not yet constructed a co-measurement scheme, it's not surprising that his fraction composition activity was enactive rather than distributive.

Efforts to Bring Forth Distribution

Because I came to understand the importance of using a distributive operation at a more explicit level in solving fraction composition problems, I attempted to engender the boys' use and awareness of distributive activity in producing compositions. Each boy demonstrated a distributive operation at the start of the teaching episode on March 24th, when they took fractional amounts of different-sized bars (see, e.g., Task 5.15 and Protocol 5.8). In other words, both Carlos and Michael operated with, and verbalized their belief in, the notion that taking one-fifth of three different-sized bars would be the same as taking one-fifth of all the bars together. But neither boy seemed to have embedded a distributive operation *that was sufficiently explicit to them* into their ways of operating with fraction composition.

My first attempt at engendering more explicit use of distribution occurred during the teaching episode on April 14th when I endeavored to ask questions about what fractional part Michael or Carlos had taken of *each* fractional part of the original quantity. For example, the episode started with Michael making one-fifth of seven-ninths of a yard of ribbon. After Michael made the composition and found the result, I asked, "How many fifths did you take of each ninth?" We then reviewed Michael's path through the problem. When I tried to repeat my question, it became: " How many fifths of a ninth did you take?" In that moment I did not fully realize my question had changed. But since Michael's response of "seven" indicated to me that my question did not elicit what I was after, I moved to a problem for Carlos. Then I became engrossed in the enactive nature of Carlos's fraction composition activity, and I did not attempt more questions like this during the episode.

I returned to bringing forth a more explicit distributive operation in the next teaching episode on April 26th, this time with more finely honed questions *and* colored candy ribbon. Carlos began the episode with the following problem:

> Task 5.22: You have 5/9 of a yard of magic candy rope, which changes flavor every ninth. Color it to show all the flavors. I want 1/4 of yours, but I want all the flavors. Make my piece and tell how much of the yard I get.

After Carlos made a 5/9-yard bar with every ninth a different color, he partitioned each ninth into four parts and pulled out one part from each of the ninths. When he tried to join the parts, the JOIN button on JavaBars kept "eating" parts. So he ended up repeating one of the parts to make a 5-part bar and then coloring it to show all the colors (see Figure 5.22).

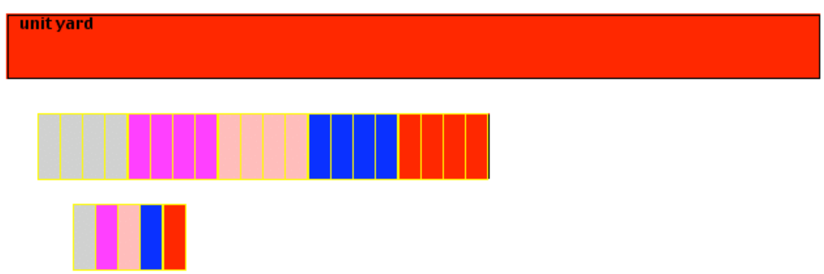

Figure 5.22, One-fourth of a 5/9-yard bar, showing all the colors.

Protocol 5.12: Carlos's response to an attempt to engender distribution on 4/26/04.

T: So, C, how'd you know to take five pieces?
C: *Since there are five of 'em…I'm still not really sure, but…*[He iterates the 5-part bar via dragging it along the 5/9-yard bar.] I'm just checking. [As he positions the 5-part bar for the fourth time, it lines up with the right end of the 5/9-yard bar.] Yeah, it's right. Okay. Now how did I know how to do that?
T [nodding yes]: Yeah.
C: *Since there were four pieces in each one and there were five pieces, I timesed—no wait. Yeah, four times five equals twenty, so I divided twenty by four, and I got five.*
T: Oh, I see. That's a really good way to do it. [pause] How many fourths did you take of each of these [points at one ninth]?
C: One-fourth.
T: One-fourth of each those, huh? *So one-fourth of each of these pieces [pointing at the five ninths in succession] gives you one-fourth of the whole thing? Is that true?*
C: *Yep.*
T: Oh, I see. [T asks C what he started with, and he says ninths.] So you took a fourth of each of those ninths?
M [softly]: Or five thirty-sixths.
T: So what is a fourth of each of those ninths?
C: A fourth of each of these ninths? Let's see.
M [very softly]: Thirty-six—ha [ends in a little laugh].
T: Mmm, M has an idea. [They all laugh.]
C [pointing at M, objecting playfully]: He keeps on saying it!
T: So what do you think C?
C: Thirty-sixths.
T: You think? How come?
C [smiling]: 'Cause M said it.
T: Oh, 'cause M said it. Let's see if we can figure out why.
C: Okay. [More laughter] 'Cause you times four times nine.
T: Oh, okay, so how long is the piece you made?
C: It would be…five thirty-sixths.

Although Carlos initially made his 5-part bar "distributively" because of the changing colors, he did not demonstrate a distributive operation in his explanation of his solution to the problem: "four times five equals twenty, so I divided twenty by four, and I got five." He also could respond appropriately to my questions about how many fourths of each ninth he took, but that did not mean that my questions and the changing colors provoked an adaptation to use an explicit distributive operation in his ways of operating with fraction composition. Michael acted similarly in solving a similar problem that I posed for him. Later in the episode, when we dropped the changing colors, both boys operated without any explicit distributive activity in making compositions. So I infer (and I inferred at the time of the experiment) that my attempted interventions did not engender changes in the boys' ways of operating with fraction composition.

MCR During Fraction Composition Activity

In one sense, during our work with fraction composition situations I enacted MCR well with boys because in early episodes (March 31st, April 12th) I focused them on making thirds of fifths or sevenths of fifths. This focus meant that the use of recursive partitioning to determine the size of the parts in terms of the unit bar was essentially the same over several problems, and they could concentrate on the number of parts to make. I also built up to more complex compositions (nonunit fractions of nonunit fractions) by first working on—and returning to—taking unit fractions of nonunit fractions (e.g., Tasks 5.20 and 5.22). In addition, I tried to engender independent mathematical activity (Steffe & Wiegel, 1994) by initiating the "fraction composition game," where each boy chose a fraction from a family (like ninths or fifths) in order to produce a fraction composition problem.

However, there were also some lacunas in how I operated as a mathematical carer, due to my lack of knowledge (not my intentions). During the teaching episode on March 31st, I focused on the boys making thirds of fifths but I jumped quite fast to having them make two-thirds of seven-fifths of a yard—a very complex problem, particularly for Carlos, because of the improper fractional quantity. Also, during the teaching episodes from March 31st to April 28th I did not pose enough problems that involved taking *nonunit* fractions of unit fractions (e.g., two-thirds of one-fifth of a yard). *Not* posing such problems was a big oversight for the boys' construction of a fraction composition scheme with an explicit distributive operation. The reason that taking nonunit fractions of unit fractions is so important, I came to understand, is that solving such a problem builds directly on a unit fractional composition scheme that in turn grows from recursive partitioning. For example, both boys could take one-fifth of a 1/7-bar and know it was a 1/35-bar, so both boys had a *unit* fraction composition scheme. Taking three-fifths of a 1/7-bar would be an expansion for them but not the same stretch that taking one-fifth of a 3/7-bar is, since the latter problem requires distributive partitioning. My conjecture is that posing problems like making three-fifths of a 1/7-bar would have been particularly useful for Carlos.

Nevertheless, I did endeavor to support and expand both boys' ways of operating in fraction composition situations. On April 14th, I supported Carlos's experimentation in making one-fourth of a 5/7-yard bar by testing out different bars. For example, I asked him if he thought his 4-part bar would work, and I agreed with him that he'd have to try something else when the 6-part bar did not work. By observing and cueing off of him, I was trying to learn about what motivated his activity and thus harmonize with his ways of operating. However, because I thought he had the potential to expand his work beyond an enactive scheme, I also searched for ways to act that might engender an adaptation. My attempts focused on asking him to explain his actions and steering him toward using multiplication to coordinate whole numbers of parts as he had done in making one-third of two-fifths of a yard (Task 5.20).[54] In turn, Carlos responded to my interest in his activity with a willingness to work on these problems.

Similarly, over several teaching episodes, I attempted to support Michael in his quest to correlate the standard computation algorithm for fraction multiplication with making compositions in JavaBars. I trusted that his resolution of this matter would involve him in expanding his ways of operating with fraction composition. Toward this end I encouraged his questioning and tried to draw out details about his logical necessity for operating. Michael's

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[54] Doing so was not necessarily helpful in engendering a more explicit distributive operation, as I have noted, but it was one way to make fraction compositions that seemed to build on Carlos's current ways of operating.

responses during this part of the teaching experiment included protesting when I announced we *wouldn't* meet the week of April 17th because I would be attending a conference, and continuing to search for easier ways to solve fraction composition problems.

One of my main attempts to expand both boys' ways of operating in fraction composition situations was to engender an explicit distributive operation within their ways of operating. These attempts were not that successful in terms of bringing forth acts of learning. But I contend that they were an example of MCR—in fact, a good example of the maintenance of MCR *without* the occurrence of a particular act of learning. My strategies to induce distributive activity included asking questions about the fractional part they took of each part of the original bar, and using color to highlight the physical action of taking a fractional part of each part of the original bar in making that fractional part of the whole bar. The boys responded to these strategies without any evident depletion or fatigue—and in fact often with a sense of stimulation or enthusiasm. In doing so they cared back for me. But neither strategy proved effective in provoking adaptations. Out of the relative "failure" of these strategies, I formulated new ideas for engendering the use and awareness of a distributive operation in these contexts. But just as important, I understood better the possibility of enacting MCR without "successfully" engendering a particular act of learning.

Fractions as Operations: Constructive Resource 5, March through May

Conceiving of a fraction as a multiplicative operation on a quantity surfaced as an explicit issue in relationship to solving RMR problems in March, when I investigated whether the boys had constructed reciprocal, multiplicative relationships between two quantities. This activity was a new facet of our work together toward solving RMR problems of Types 3, 4, and 5. For example, in solving the Candy Bar Problem (Task 5.1) in February, Michael did not seem to conceive of the new collection as five-thirds of the original collection, nor did I explicitly ask him what he needed to multiply the original collection by to produce the new one, and vice-versa. Conceiving of multiplicative relationships between quantities in RMR problems like the Candy Bar Problem is important for progressing toward more algebraic solutions of them. So in this final constructive resource section, I explore the boys' activity with fractions as multiplicative operations during the last three months of the teaching experiment.

No Fractions as Operations when Solving RMR Problems in March

During the March 10th teaching episode, recall that I posed Task 5.2 (an 8-centimeter candy bar is 3/4 of another candy bar; make the other bar and tell how long it is). Before making an "exact" solution, the boys worked on an "estimated" one. I asked Carlos to estimate how long he thought the other candy bar would be by iterating the centimeter-long unit bar. Then we cleared that estimate and determined how to make the 8-centimeter bar from it. Eventually Carlos divided the estimate into four equal parts and noted that three of those parts would (or should) make the 8-centimeter bar. The estimated bars remained on the screen while Michael made the "exact" length of the other candy bar and they both determined that the exact length was 32/3, or 10 and 2/3, centimeters.[55] Then the witness-researcher and I posed a question that I acknowledged was "big":

Task 5.23: What do you have to multiply 32/3 by to get 8?

Protocol 5.13: Michael's abstraction of whole numbers as operations on 3/10/04.

M: *Divide or?*
T: Well, you might have to do some dividing and some multiplying, but, you could think about it both ways. So, if you started with thirty-two thirds [points to the 32/3-centimeter bar], can you think of what you'd have to do to make the eight centimeters [points to the 8-centimeter bar]? We thought about this at the beginning. [T is

[55] Note that Carlos determined this length with coaching and by counting up whole centimeters and thirds of centimeters using graphics in his visual field.

referring to initial work on this problem with the "estimated" other candy bar. Both boys stare silently at the screen for twelve seconds. T suggests it might help to clear the 32/3-centimeter bar, which M does. Then T repeats the question.]

M: *Divide it.*

T: Divide it how?

M: *Oh, I know how. Thirty-two divided by eight would be four, then times two would be eight.*

T: Oh, okay. So are you going to divide this [points to the 32/3-centimeter bar] into eight pieces?

M [pause]: Yeah. [Pause] And it should be to there [gestures with his hands to the middle of the 8-centimeter bar, 4 centimeters, since he knows that 32 divided by 8 is 4]. *Well no, wait. That would be thirds [i.e., it looks like to get a piece 4 centimeters long would entail dividing the 32/3-centimeter bar into three, not eight, pieces].*

T: Hmm. So remember the eight centimeters was three-fourths of this guy [the 32/3-centimeter bar]. So I wonder what you'd have to—so you [M] said divide it by eight?

M: Mm-mm [no].

[*T asks C what he did to make the estimate and he says he doesn't remember*. They look on the screen and *M notes that C divided the estimate into four parts*.]

T: Why would you divide it into four?

M: 'Cause then there'd be eight, eight centimeters in each. Eight-<u>thirds</u> in each.

T: Oh, eight-<u>thirds</u> in each. And then what would you do with that eight-thirds?

M: Oh! Times three.

T: Oh, times three.

W: What would you take thirty-two thirds times to make eight? [3-second pause] What fraction would you multiply—

M [interrupts W's question]: *Thirty-two [thirds] divided by four, would be eight-thirds. Then we timesed it by three, would equal eight.*

W: So if you divide by four and multiply by three what fraction is that?

M: Eight, or eight over one.

T: Yeah, eight over one is the answer; you get eight-over-one centimeters [nodding], exactly. But what he's asking is, when you divide something by four and you multiply by three—and you do both those things, right?—do you know what fraction you're multiplying by?

M: Four over one.

T: *Well you're dividing by four and multiplying by three, so if you put those together, do you know what fraction you're multiplying by?*

[C says something about twelfths.]

M [frowning]: *I'm confused.*

In working on Task 5.23, initially Michael operated solely with whole numbers. When he said, "Thirty-two divided by eight would be four then times two would be eight," he was coordinating 32 and 8 rather than 32/3 and 8. This explanation was unrelated to the boys' solution of Task 5.2 and shows Michael's attempt to "transform" 32 into 8 using division and multiplication. Through coaching to reflect on what Carlos had done in estimating a solution to Task 5.2, Michael was able to articulate that he'd divide 32/3 centimeters by four and multiply the resulting 8/3 centimeters by three to make the 8-centimeter bar from the 32/3-centimeter bar.

However, the questions I and the witness-researcher asked about what fraction he would multiply 32/3 by to get 8 did not make sense to him or to Carlos. In fact, whether Carlos had abstracted the *whole number* operations he would perform on the 32/3-centimeter bar to make the 8-centimeter bar, despite his work on the estimated bar at the start of the episode, was not clear. What did seem clear was that neither boy had constructed fractions as multiplicative operations: To them, dividing by four and multiplying by three did not mean multiplying by three-fourths. Taking division and multiplication by whole numbers as multiplication by a fraction means allowing that multiplication by can produce a quantity *smaller* than the original—a potentially difficult conception given experience with whole number multiplication. At the end of the episode Michael indicated this difficulty explicitly in relationship to Task 5.18 (an 11-centimeter bar is three times another bar). I asked them what they would multiply 11 by to make the other bar, and Michael said, "But you can't get a lower number by timesing." I agreed that that was indeed a problem. "Unless you did eleven times zero," Michael added.

In Protocol 5.13, Michael appeared to take the 32/3-centimeter bar as a unit made of four units each of which contained eight 1/3-centimeter bars, and simultaneously he took the 8-centimeter bar as a unit of three units each of which contained eight 1/3-centimeter units. I infer that doing so allowed him to abstract the operations of dividing by four and multiplying by three to make the 8-centimeter bar from the 32/3-centimeter bar. Yet, since he did *not* see dividing by four and multiplying by three as multiplying by three-fourths, I conjecture that something more than an anticipatory unit-of-units-of-units structure is required for the construction of fractions as multiplicative operations.

To take three-fourths of a quantity as a multiplicative operation seems to require taking one-fourth of a quantity as a multiplicative operation, not just an operation of division. That is, constructing *unit* fractions as multiplicative operations seems like a first step in constructing any fraction as a multiplicative operation. One-fourth of a quantity—say a unit quantity—can be created by dividing the unit into four equal parts and taking one of those parts. Four times any of those parts will produce the unit. The part may be conceived of as one-fourth of the quantity, a new quantity "in its own right," which I refer to as a "result," the result of a fraction scheme. In an operational sense, the part is also "1/4 times 1," where 1 is 1 unit of the quantity. So in an operational sense, four times the part is "4 times 1/4 times 1." These operational relationships are usually left implicit when the quantity is a unit quantity (or when the quantity itself is implicit).

Michael and Carlos taught me that conceiving of one-fourth as a result of a scheme does not necessarily induce the conception of one-fourth as a multiplier on a quantity. One hypothesis is that for one-fourth to be a multiplicative operation a student has to construct an equivalence between dividing a quantity by a whole number and multiplying it by a unit fraction. I explore this hypothesis further in the next chapter with Deborah, who mid-way through the teaching experiment seemed to conceive of unit fractions as multiplicative operations.

An Attempt to Engender the Initial Construction of Fractions as Operations

Throughout the rest of the teaching episodes in March, I began to ponder situations that might provoke the boys to think of taking one-fourth of a quantity as multiplying by one-fourth, rather than only as dividing by four. So on April 14th, just after starting fraction composition problems, I began the episode by asking the boys to make one-fifth of a yard of candy rope.

Protocol 5.14: Working toward fractions as operations on 4/14/04.

T: Okay, so C can you make something seven times as long as that?
[C copies the 1/5-yard bar and repeats it to make seven 1/5-yards.]
T: So how long is that?
C: Seven-fifths.
T: Yeah, so what multiplication problem did you just do C?
C: I just did…five…one-fifth times—no wait [pause] yeah, times seven. [T asks M if he agrees, and he nods.]
T: All right, so M can you make something seven times as short as this guy [points to the 1/5-yard bar]?
[M partitions the 1/5-yard bar into seven parts and pulls out one part.]
T: How much is this [points to M's little pulled part]?
C: Of that one [points to the 1/5-yard bar] or that [points to his 7/5-yard bar]?
T: Of the unit yard.
M: One thirty-fifth.
C: *Yeah, times it by seven, seven times five*.
T [to M]: So what multiplication problem did you do?
M: Well, um, *to equal one-fifth it'd be one thirty-fifth times seven*.
T: Okay, one thirty-fifth times seven would equal one-fifth. I wonder if you could start with one-fifth, though. *One-fifth times what would give you one thirty-fifth?*
M [brings his hand up to his chin]: *You can't times something to do that [get smaller]. So you'd have to…[looks up toward the ceiling]. Not unless you could times it by negatives*.
T [softly]: Oh, by negatives.
M: But we haven't learned that!
T: *What about if you times it by a fraction—could you times it by a fraction?*
M: *Uh-uh* [*no*]. [C also says no, and then has some overlapping talk with M.]
M: *Because whenever it's timesed it'll just goes that way [gestures to the right] a little more. Can't go that way [points to the left]*.
T: Oh it'll just get bigger. Oh I see. *What happens if we did half of one-fifth—what would that be?*
C: *One-tenth*.
T: So, is that a fraction multiplication problem, half of one-fifth equals one-tenth?
M: *Oh yeah, <u>of</u> [smiles]*.
C [smiles]: Yeah, huh.
T: So what fraction multiplication problem would that be, to make one-tenth?
M: *Oh! One-seventh of one-fifth*.
T: Oh, one-seventh of one-fifth. So what is one-seventh times one-fifth?
M: *It'd be [6-second pause] one thirty-fifth! So it'd be right*.

In this protocol, at first Michael insisted that you couldn't multiply to make something get smaller "unless you could times it by negatives", and he rejected multiplying by a fraction to accomplish such a goal. Then he seemed to make a connection between taking a fraction of a fraction and fraction multiplication in considering one-half of one-fifth equals one-tenth as a fraction multiplication problem. His change of mind is signaled by him stating “Oh yeah, *of*.” Because of his emphasis on “of,” I infer that he relied largely on his classroom mathematics teacher telling him that in math, “of” means multiply.[56] So I was not convinced that Michael’s change of mind represented a construction of one-seventh as a multiplicative operation, but I was heartened that he may have started to rethink the notion that multiplication can only make bigger.

I was even less sure that the activity had had an impact on Carlos’s construction of fractions as operations, although he demonstrated some growth in other areas. He made seven-fifths by iterating one-fifth seven times (although his doing so still does not convince me that he had constructed an iterative fraction scheme at this point!). He also demonstrated his unit fraction composition scheme in his justification for Michael’s production of one thirty-fifth and in his prompt response to my question of taking half of one-fifth. However, Carlos seemed to cue off of Michael’s response regarding multiplication to make one thirty-fifth. That is, Michael’s rejection of multiplying one-fifth by a fraction to make one thirty-fifth may have influenced Carlos to make a similar rejection. And when Michael changed his idea about the matter, Carlos promptly smiled and agreed with him. Although in subsequent episodes both boys stated fraction multiplication problems for taking a fraction of a fraction, whether they had begun to construct fractions as multiplicative operations remained uncertain.

[56] All four students had the same classroom mathematics teacher (cf. Chapter 4), and at some point during the teaching experiment, all reported being told that in math, “of” means multiply.

No Fractions as Operations when Solving Fractional RMR problems in May

At the beginning of the last teaching episode on May 12th, I posed the following problem:

Task 5.24: Two groups of students, the Lizards and the Cobras, engaged in a racecar building and racing contest. The Lizards' car goes 1 meter. That's 2/3 of how far the Cobras' car went. How far did the Cobras' car go?

This task occurred just prior to Task 5.3, mentioned in the opening section of this chapter: the Lizards' car goes 1/2 of a meter, which is 2/3 of how far the Cobras' car went. Carlos solved Task 5.24 by partitioning a copy of the unit meter into two equal parts, pulling out one part, and repeating it to make three parts. Michael was the first one to say that the Cobras' car went 1 and 1/2 meters. Then I asked this question:

Task 5.25: What do you have to take times the Lizards' distance to make the Cobras' distance?

In the previous episode on Monday May 10th, the boys had worked briefly on a similar problem about the multiplicative relationship between two fractional parts of bars, but not in an explicitly quantitative context. They had run out of time because we had met for only a short while that day.[57] In particular, although Michael had commented that the problem was complicated, he had been determined to figure it out before Wednesday (May 12th). So when I posed Task 5.25, Michael smiled, indicating that he recognized the problem.

Protocol 5.15: Efforts to determine multiplicative relationships on 5/12/04.

M [after a 12-second pause]: *Do we have to times?*
T: Yeah—what could you take times this one [points to the 1-meter bar] to make this one [points to the 3/2-meter bar]?
[W asks what they did to make the other bar.]
M: Divided it in half.

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[57] In fact we weren't going to meet at all (end-of-year testing schedules had interfered that day), but Michael had insisted on meeting, even for only a short time.

T: Yeah, you divided it. How much did you take [points to half of the 1-meter bar]?
M: *Oh, divided it and then multiplied. Multiplication is related.*
T: Yeah, that's true, that's true. [C stares out a bit blankly.] How many pieces did you divide it into?
[M holds up his first two fingers.]
T: Two, and then what did you do?
M: *Added one.*
[T reminds them that C made the other distance separately and asks what he did.]
M: *Times three.*
W: If you divide by two and take that times three, what fraction is that?
M [after a 5-second pause, softly]: What?
[W repeats his question.]
M: *One-third.*
C: *One-third plus…*
[T redirects them to think about what they could multiply the meter by to get one-half of a meter.]
C [softly]: By two.
T: Would you get that [encompasses the 1/2-meter bar with thumb and forefinger]—if you multiplied that [encompasses the 1-meter bar with thumb and forefinger] by two?
C: *No, you'd have to divide.*
[T continues to ask about multiplying by a fraction.]
W: Tell me what it means to take half of something—if you're going to take half of a candy bar, what do you do?
C: Take half of it.
W: Yeah, but how do you do it?
M: *Of means times.*
[More discussion ensues, and T points out that two questions are at hand: what do you take times the 1-meter bar to make a 1/2-meter bar, and what do you take times the 1-meter bar to make the Cobras' distance.]
M: *I just lost it! I had it! I'm right on it!* [*He takes his hands up to his temples.*]
T: You are; you're right on the edge, I know it.
W: Can I ask another question?
M: *Three-halves!*

This protocol shows that the work that we did on April 14th did not provoke changes in the boys' conceptions of even unit fractions as multiplicative operations. Neither Carlos's nor Michael's ways of operating allowed them to determine what you had to take times the 1-meter bar to produce a 1/2-meter bar, let alone to answer Task 5.25. In particular, Carlos seemed disengaged during much of the episode because, I infer now, these questions were outside of his short-term ZPC. Michael's response of "three-halves" appeared promising, and I responded quite enthusiastically to it. In explanation Michael said, "'cause I was thinking of one that would equal a half, two-thirds, and I was like, six over something, like that. Three times four." I didn't understand his rationale, so I probed further. Michael noted that two-thirds times three-halves would give you "that," referring vaguely to the screen but not specifying a bar. So I interpret his response of three-halves to mean that he was thinking about what he could multiply two-thirds by to make 1 meter, not that he was articulating a multiplicative operation that would make 1 meter into three-halves of a meter.

Because Task 5.24 was a rather special case where the initial distance was 1 meter, I next posed Task 5.3, where the initial distance was 1/2 meter (and the Cobras' car went two-thirds of that distance). After the boys had solved Task 5.3, I followed up with Task 5.25 again. The boys were silent for 7 seconds. Then Michael responded "three-halves," and we all laughed. The reason for his response is uncertain: Because I had been impressed with his response at the end of Protocol 5.15, he might have been encouraged to repeat it, especially because visually the bars the boys had made to solve Task 5.3 were in an analogous arrangement to the bars they had made to solve Task 5.24, (see Figure 5.23).

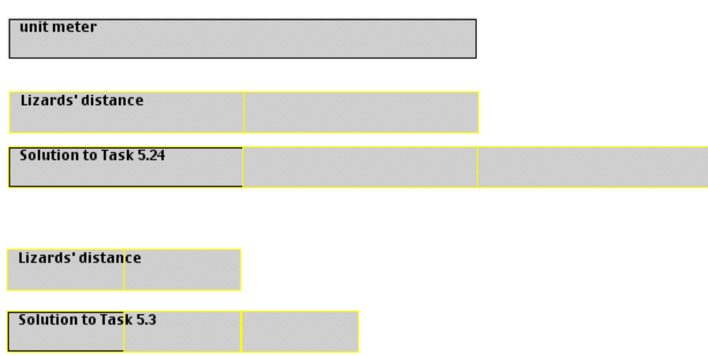

Figure 5.23, Solutions to Task 5.24 (above) and Task 5.3 (below).

Possible corroboration of the lack of *logical* necessity for saying three-halves was Michael's subsequent question about whether they had solved the problem from last time. Of course, he may not have fully remembered the problem from last time. Possible corroboration *for* the initial construction of three-halves as a multiplicative operation was the ease with which Michael confirmed that one-half times three-halves gave three-fourths of a meter. Michael's ways of operating during the remainder of this episode confirm that he had made some connections with regard to finding two fractions that multiplied to make one, but they did not allow me to conclude that he had constructed fractions as multiplicative operations on quantities. Unfortunately, this episode was the last one in the teaching experiment, so I did not have an opportunity to investigate further.

MCR in Relation to Fractions as Operations

The work the boys and I did with fractions as multiplicative operations was certainly challenging for them, and questions like Task 5.23 (what do you take times 32/3 cm to make 8 cm?) on March 10th seemed to be outside of Carlos's short-term ZPC.[58] At the time of the teaching experiment I sensed that Task 5.23 was a "big" question; now I understand that it was radically too complex then for Carlos and on the edge of what Michael could do at that point. Does this retrospective knowledge mean that the boys and I did not enact MCR well at this time? No, for the same reason that keeps surfacing: Locally MCR may be tenuous in the service of adaptations of both the teacher and students toward stronger MCR "in the long run." If I were to teach the boys again, given what I know now and with the premise that they had not made substantial changes in their ways of operating since May of 2004, posing Tasks 5.2 and 5.23 to Carlos would indeed be enacting weak MCR with him. Or, if I were to teach a student who

[58] In fact the task upon which it was based, Task 5.2, was also just beyond his short-term ZPC, given that on March 10th he had not yet modified his splitting operation to include a distributive operation.

operated much like Carlos and I did not make adaptations, our MCR would likely be weak. But at the time, posing these tasks, *and then making adaptations from them in response to the boys' activity*, was the only way I had to learn about the boys' conceptions (or lack of conceptions) of fractions as operations.

Our work on April 14th was a good example of MCR in the context of fractions as operations because it seemed to open the possibility that at least Michael might construct the notion that multiplying a quantity by a fraction could produce something *smaller* than the original quantity. It is doubtful that he made this construction, based on his difficulty in thinking about what to multiply by to take half of something on May 12th (cf. Protocol 5.15). In addition, upon retrospective analysis, my questions on April 14th may have been inadequate because they did not involve the boys in justifying why one-seventh times one-fifth yielded one thirty-fifth by, say, iterating one thirty-fifth seven times to "make back" one-fifth. Still, on April 14th my questions in Protocol 5.14 seemed to be an improvement over just trying to get the boys to verbalize what they had done to make the other quantity, as I had tried throughout March (cf. Protocol 5.13, Task 5.23) and as I tried again in May (cf. Protocol 5.15, Task 5.25).

However, unlike my attempts to engender the boys' use of their distributive operation in making fraction composition, my attempts to bring forth the construction of fractions as multiplicative operations were often rather depleting for the boys, especially for Carlos. This aspect of our work together was especially evident on May 12th, when Carlos was largely silent and seemed disengaged at times while Michael wrestled with problems about multiplicative relationships between two distances. Quite plainly, Carlos did not have a way to operate in response to these problems due to not yet having an anticipatory unit-of-units-of-units structure. That is, it is difficult to see how Carlos might have constructed, say, three-halves as an operation on a quantity when it is questionable whether he had constructed three-halves as the result of an iterative scheme. Michael engaged consistently during this episode and the previous one on May 10th, but even for him, these problems seemed to be "at the edge" of his short-term ZPC. Yet despite the fact that energy seemed low during both episodes, Michael was quite sad on May 12th that our work together was over. "I'm going to miss you guys," he said, as we finished the last teaching episode.

Accounting for the Boys' Solutions of Problems Involving RMR

In this final section, I provide an account of how the boys' construction (or lack of construction) of the five constructive resources helps to explain their solutions of the RMR problems that I described in the opening sections of the chapter: Michael's solution of the Cake Problem (Task 5.1) in February, and Carlos's solution of Task 5.5 in March, in which he made a 2-foot bar into three equal parts. A crucial difference between the boys' ways of operating in solving RMR problems was the number of levels of units they could coordinate prior to operating, which influenced their use of more than one units-coordination within the same bar or quantity. This difference is a primary explanatory tool in my evolving models of the boys' mathematics and figures prominently in my account of their solutions to Tasks 5.1 and 5.5.

Michael's Solution of the Cake Problem in February

Reversible reasoning: Constructive Resource 1. In solving the Cake Problem, Michael used reversible reasoning with fractions to form his initial goal to divide the seven inch-long bars into three equal parts. His construction of fractions as iterable units and a splitting operation was important in his formation of this goal. That is, since for him three-fifths of a quantity meant one-fifth of it three times, he aimed to partition the given quantity into three equal parts, any of which could be iterated three times to make the given quantity, or five times to make the new quantity. This way of operating is a hallmark of at least a reversible partitive fraction scheme *with splitting*, but based on his activity in November (cf. Task 5.13, Protocol 5.7), I can conclude that Michael had constructed a reversible *iterative* fraction scheme.

New uses of multiplying schemes: Constructive Resource 3.[59] To accomplish his initial goal required Michael to eliminate the perturbation that he experienced in wanting to divide *seven* bars into three equal parts. The seven bars was an element that blocked his usual way of operating. That is, if the original collection had been one bar that was three-fifths of another collection, Michael could have used his splitting operation to make three equal parts. And if the original collection of bars had been some number of bars divisible by three, he could have used his splitting operation with composite units, as he had in solving the pencil problems (cf. Task 5.9, Protocol 5.5). But seven bars was clearly problematic.

Michael eliminated his perturbation by partitioning the seven bars so that he could, in effect, use his splitting operation with composite units. That is, he partitioned each of the seven bars into three equal parts so that the collection was a unit of seven units each containing three units, or 21 units. The number of units meant he could view the original collection as a unit of three units each containing seven units. I have called this novel activity a modification of his splitting operation to include a distributive operation, since he split the entire collection of bars into three equal parts by splitting each of the bars into three equal parts. This way of operating allowed Michael to accomplish his initial goal in the problem. Distributive splitting requires coordinating three levels of units prior to operating because it rests on viewing a quantity as a unit consisting of two different units of units—keeping in mind one view while flexibly switching to the other view. Thus Michael's construction of a co-measurement scheme in

[59] For both Michael and Carlos, I discuss Constructive Resource 3 prior to Constructive Resource 2, in order to follow the trajectory of the boys' reasoning.

January, when he made a 2/2-bar into a 3/3-bar (cf. Task 5.17, Protocol 5.10) and then a 5/5-bar into a 4/4-bar, was a key constructive resource in splitting the original collection of candy bars in the Candy Bar Problem.

Both Michael's reversible iterative fraction scheme and his duel three-levels-of-units views of quantities were involved in making the new collection of candy bars. That is, for him each 7-part bar was one-fifth of the new collection, and thus he needed two more fifths (two more 7-part bars) to make five-fifths, the entire new collection of bars. Thus he appeared to conceive of the new collection as a unit of five units, each containing seven units. Because he could project this unit structure into his activity (i.e., anticipate it in thought), he could state "and I know how many there are now" *before* he had made the entire new collection in JavaBars.

Activity with improper fractions: Constructive Resource 2. Interpreting this new collection of candy bars that consisted of 35 small parts was made possible by Michael's "strong" iterative fraction scheme. For him, an improper fraction like 21/5 was not just a unit of 21 units, any of which could be iterated five times to make the whole bar, so that 21/5 stood in relation to the whole and was also greater than it by 16 one-fifths. For him, 21/5 was four units of five one-fifth units plus one more fifth—that is, in conceiving of 21/5 he formed units of five units as "wholes" and in doing so created a "structural view" of the fraction (cf. Protocol 5.9).

So in the Candy Bar Problem, once Michael thought of each of the 35 small parts as thirds of a unit bar (i.e., thirds of an inch), he formed units of three of those thirds as a whole bar (an inch) and projected these units of three into the 35 parts he'd made. For this reason he commented that "three can't go into 35 equally, so it would have to be a fraction." And, using his multiplying scheme, he concluded that 11 units of three units would be 33 units, with two units leftover to make 35. Thus he called the collection 11 and 2/3 unit bars (or inches) long. Because

Michael did not explicitly articulate the collection as 35/3 units long, I cannot claim that he saw the collection as both 35/3 as well as 11 and 2/3 at this point. But I can claim that his activity with improper fractions in November was an important constructive resource in his interpretation of the length of the new collection.

Fraction composition activity: Constructive Resource 4. Despite the fact that taking five-thirds of seven inches is one way to interpret Michael's activity in making the new collection, he certainly didn't view what he had done as a problem involved fraction multiplication. He also did not appear to view taking one-third of seven inches (his initial goal) as a fraction multiplication problem. That is, he did not explicitly take one-third of seven inches by taking one-third of an inch and iterating that amount seven times. Doing so would mean that he was aware of the distributive pattern of his activity and would *not* require that he explicitly calculate the total number of parts he'd made (21) in order to find one-third of seven. Doing so would also open the possibility that determining the length of the new collection could grow out of making it, since one-third of each inch seven times would yield seven one-third inches, or seven-thirds of an inch. Since Michael did not participate in fraction composition activity until the end of March, a full month and a half after he solved the Candy Bar Problem in mid-February, it is not a surprise that his solution did not include fraction composition. However, given that I cannot conclude that Michael constructed a general fraction composition scheme in March and April, I am fairly sure that even at the end of the teaching experiment he would not have conceived of solution to a problem like the Candy Bar Problem in terms of composing fractions multiplicatvely.

Fractions as operations: Constructive Resource 5. Michael certainly conceived of his solution to the Candy Bar Problem as making five-fifths of the new collection given three-fifths of it. But he gave no indication of bi-directional (reciprocal) multiplicative relationships between the original and new collections of candy bars: He did not seem to conceive of his solution as five-*thirds* of the original collection, seven inches, so I infer that such a relationship was implicit for him at best. In short, since Michael had not yet constructed fractions as multiplicative operations at the end of the teaching experiment in May (cf. Protocol 5.15), it is not a surprise that he did not demonstrate reciprocal multiplicative relationships between quantities in solving the Candy Bar Problem in February. The construction of such relationships can greatly compress the activity necessary to solve RMR problems and can be seen as more algebraic than ways of operating that proceed in a more sequential manner, from known to unknown (cf. Chapter 3).

Carlos Making the Two-Part Bar into Three Equal Parts (Task 5.5) in March

Reversible reasoning: Constructive Resource 1. Like Michael, in solving Task 5.5 Carlos formed a goal to divide the 2-foot bar into three equal parts. But his main "tool" to accomplish his goal was his (nondistributive) splitting operation, which was hard to use on a bar marked into two equal parts. That is, if the 1-foot mark was not shown, Task 5.5 would not have been problematic for Carlos—and indeed, his first solution to Task 5.5 involved clearing the 1-foot mark so that he could use his splitting operation. Alternatively, if the bar had been partitioned into three equal parts, or a multiple of three parts, the problem would also have been easy for Carlos, in that he could use splitting with composite units to solve the problem. For example, if the problem had been that a 12-part bar was three times longer than another bar, I claim that Carlos would have divided the 12 parts by three in order to get four parts. I make this claim based on Carlos's use of his splitting operation with composite units to reason reversibly at the start of the teaching experiment in November (cf. Tasks 5.10 and 5.11). However, having a 2-unit bar and a goal to divide it into three equal parts was a perturbation for Carlos, with *two* as the perturbing element.

New uses of multiplying schemes: Constructive Resource 3. To eliminate his perturbation, like his partner Carlos partitioned the 2-foot bar so that he could use his splitting operation with composite units. That is, he partitioned the each of the 1-foot parts into three parts so that he had six parts, which he could certainly divide into three equal parts. Thus the 2-foot bar was a unit of two units into each of which he had inserted three units, and through his partitioning activity he could view it as a unit of three units each containing two units. Thus like Michael, Carlos modified his splitting operation to include a distributive operation that relied on making three levels of units because he took the 2-foot bar as a unit that could contain two different units-of-units structures.

Unlike Michael, Carlos had not constructed a co-measurement scheme in January (cf. Task 5.17), so he did not have an explicit constructive resource to draw upon in solving Task 5.5. Perhaps for this reason, and because even at the end of the teaching experiment I cannot conclude that he had constructed an anticipatory unit-of-units-of-units structure, he did not seem to generalize his reversible multiplying scheme beyond dividing a 2-part bar, or perhaps a multiple-part bar, into three equal parts. The experimental nature of his activity in making fraction compositions is evidence for this inference. In fact, given that Carlos was not yet coordinating three levels of units prior to operating, his solution of Task 5.5 was a truly impressive achievement!

Activity with improper fractions: Constructive Resource 2. This constructive resource was not required for Carlos to solve Task 5.5, which is not surprising given that I cannot conclude that Carlos constructed improper fractions during the teaching experiment. His lack of construction of an iterative fraction scheme helps explain why he did not independently solve problems such as Task 5.18 (11 centimeters is three times another bar; make the other bar) and

Task 5.2 (an 8-inch bar is 3/4 of another bar; make the other bar), because solving such problems and constructing improper fractions both seem to require coordinating three levels of units prior to operating. Not yet being able to do so meant that Carlos could not produce duel three-levels-of-units views of the bars, such as an 11-centimeter bar as a unit of 11 units, some or all of which could be further partitioned so as to view the bar as a unit of three units each containing some equal number of units. Not coordinating three levels of units prior to operating was also the lacuna in Carlos's activity with fractions larger than one (cf. Task 5.15, Protocol 5.8). So my conjecture is that not yet having an anticipatory unit-of-units-of-units structure meant that Carlos's ability to solve RMR problems of Types 3, 4, and 5, as well as his ability to construct improper fractions, was seriously curtailed.

Furthermore, at least one other link exists between Tasks 5.18 and 5.2 and activity with improper fractions: The solutions to both problems were improper fractional quantities that Carlos may have been able to make but that may not have much significance for him. That is, Carlos could likely have made eleven-thirds by iterating one-third 11 times or by joining eight more one-thirds onto a unit whole, three-thirds, but doing so did not mean that eleven-thirds was a fractional number for him. So in this additional sense, Carlos's lacuna in his ways of operating with improper fractions may have added to the complexity of solving RMR problems whose results were improper fractional quantities.

Fraction composition activity: Constructive Resource 4. Like Michael, Carlos did not view his solution to Task 5.5 as a fraction multiplication problem—i.e., as multiplying two by one-third. And there is no reason why he should have, since we began to work on fraction composition activity just after Carlos's solution to Task 5.5. So, even though I have called Carlos's modified splitting operation a *distributive* splitting operation, that does not mean that I claim that he was aware of splitting each foot into three equal parts in order to split the entire 2-foot bar into three equal parts, nor that he was aware of taking one-third of each of 1-foot part in order to take one-third of the 2-foot bar. Had he been aware of the distributive pattern in his activity, he would not have needed to calculate the total number of parts he'd made (six) in order to find one-third of two. And, as I have discussed with Michael, being aware in this way would have allowed Carlos to determine the length of his resulting bar, two-thirds of a foot, directly from making it. As I have described, it is not entirely clear how Carlos arrived at the length of his bar (cf. Protocol 5.3). Engendering Carlos's awareness of using a distributive operation in the construction of a fraction composition scheme that he might use in solving RMR problems awaits his future constructive activity.

Fractions as operations: Constructive Resource 5. In fact, whether Carlos conceived of his solution to Task 5.5 as fractional at all (excepting the fractional length of the resulting bar) is debatable. Like Michael, he gave no explicit indication of bi-directional (reciprocal) multiplicative relationships between the 2-foot bar and the 2/3-foot bar. Based on his partitive fraction scheme with splitting, he might have been able to state that the 2/3-foot bar was one-third as long as the 2-foot bar and that three times the 2/3-foot bar would make the 2-foot bar. That is, for Carlos one-third meant a part that could be iterated three times to make the whole. Still, based on his activity in March, April, and May with questions that involved fractions as operations, I cannot conclude that Carlos had begun to construct fractions as multiplicative operations, let alone that they were a constructive resource for him in solving Task 5.5 in March. As I have noted with Michael, the construction of fractions as operations opens the way to constructing reciprocal relationships that can be a significant move toward solving RMR problems more algebraically.

CHAPTER 6: DEBORAH AND BRIDGET

As with the case study of the boys, in this case study I first describe how Deborah and Bridget solved problems involving reversible multiplicative relationship (RMR problems) in May, including characterizations of their schemes and operations.[60] Then I analyze key moments from November through May in the girls' construction of these schemes and operations. The key moments, again organized into five constructive resources, also highlight constraints I experienced with the girls in engendering solutions to RMR problems. Intertwined with this analysis, I describe the enactment of mathematical caring relations (MCR) between each girl and me, their teacher. It is important to note that Deborah, Bridget, and I followed a different trajectory through the teaching episodes than the boys and I did. In part this difference was due to my perception of cognitive differences between the pairs in their initial interviews in September and October of 2003 (cf. Chapter 4). In general, the girls' trajectory involved less explicit attention to using their multiplying schemes in continuous contexts, but more attention to solving RMR problems of Type 5, with fractional quantities and fractional relationships, and to conceptualizing, notating, and operating on unknown quantities.

Deborah Solving Problems Involving Reversible Multiplicative Relationships

Solving RMR Problems with Unit Fractional Quantities in May

Deborah solved RMR problems of Type 5, involving fractional quantities and fractional relationships, in May. During that month (i.e., for four teaching episodes), each girl worked on a separate laptop computer, and both computers were positioned side-by-side on the table. For the

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[60] As with the boys, the nature of the characterizations at the start of this chapter will be heuristic, although I take advantage of some of the concepts I built up in the boys' case study.

last two teaching episodes on May 10th and 12th, I differentiated problems that I posed to Deborah and Bridget in an attempt to adapt to my perception of significant differences in their ways of operating. On May 12th, I used the same context for all problems, a homemade racecar contest between two teams from a science class, the Lizards and the Cobras. At the start of the episode, I posed a problem to Bridget and then posed this problem to Deborah:

> Task 6.1: The Lizards' car goes 1/2 of a meter. That's 3/4 of how far the Cobras' car went. Can you make how far the Cobras' car went and tell how far it went?

Deborah immediately copied the unit meter she had made, partitioned it into two equal parts, and partitioned the first part into three equal parts. She pulled out one of those parts and repeated it to make a 4-part bar, see Figure 6.1.

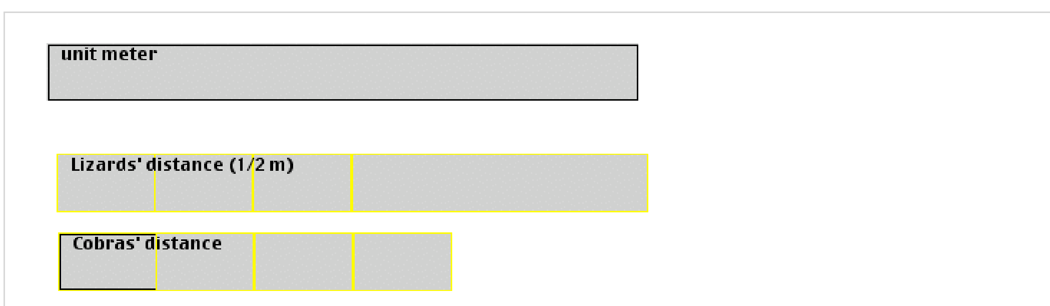

Figure 6.1, The Lizards' distance (1/2 meter, above) and Cobras' distance (4-part bar, below).

While both girls worked, I challenged them to be able to "know how much it is" *while* they made the Cobras' car's distance (as opposed to making the distance and figuring out its measure afterwards). "I already know," Deborah announced immediately. She said that the Cobras' car went four-sixths of a meter because the Lizard's car went three-sixths of a meter. I asked her what she had to take times the Lizards' distance to make the Cobras' distance. She promptly said, "three-fourths—I mean four-thirds." In explanation, she pointed to the Lizards' distance and stated that it was three-fourths of the Cobras' distance.

Solving Task 6.1 appeared to be easy for Deborah. At this point in the teaching experiment, she had constructed a reversible iterative fraction scheme similar in nature to Michael's. I infer that she used this scheme in making the new distance and used her recursive partitioning operation in tracking the size of the parts she had made in relation to the whole. That is, since the known distance was three-fourths of the unknown distance, Deborah partitioned the known distance into three equal parts, each of which were one-third of the known distance *and* one-fourth of the unknown distance. Then she iterated one of those parts four times to make the unknown distance. To determine that each of those parts were sixths of a meter, she used her recursive partitioning operation—implicitly imagining that she had partitioned both half meters into three parts for a total of six parts. Because knowing the size of the parts was so immediate for her, it seemed as if her recursive partitioning operation was embedded in her ways of operating with these RMR problems, at least when she started with a unit fractional quantity.

Another hallmark of Deborah's solution was that she had constructed what I would call reciprocal relationships between the two bars that she made. In fact, she was the only student of the four to state these relationships swiftly and to use them at least sometimes in operating. Her justification for knowing that the Cobras' distance was four-thirds of the Lizard's distance indicates that she had constructed a multiplicative relationship that did not rely on reference to material parts. In other words, she did *not* explain that the Cobras' distance was four-thirds of the Lizards' distance because the Cobras' distance was four equal parts and the Lizard's three. Instead, she relied on the given multiplicative relationship, that the Lizards' distance was three-fourths of the Cobras' distance. Her explanation leads me to infer that she knew the following: If Bar A was three-fourths of Bar B, then Bar B was four-thirds of Bar A.

Operating with NonUnit Fractional Quantities in May

In the (same) teaching episode on May 12th, my intention was to investigate whether Deborah could solve RMR problems just as fluidly when starting with *nonunit* fraction quantities. So I posed this problem for her:

> Task 6.2: The Lizards' car goes 2/3 of a meter. That's 3/4 of how far the Cobras' car went. Can you make how far the Cobras' car went and tell how far it went?

While Deborah began her activity, I spoke with Bridget about the problem on which she was working. Since we had only one video camera to capture two computer screens, I don't have a recorded visual image of what Deborah was doing until 70 seconds after I posed Task 6.2. However, I infer that during the first 35 seconds that Bridget and I spoke, Deborah made two-thirds of a unit meter. My inference is based on Deborah's interjection at this point to confirm with me that two-thirds of a meter was three-fourths of the Cobras' distance. During the next 35 seconds that Bridget and I spoke, Deborah partitioned each of the two thirds into six equal parts, which became evident when the camera captured Deborah's screen. Because in this section I am focused on how Deborah operated, I omit dialogue with Bridget during the following protocol.[61]

Protocol 6.1: Deborah making the Cobras' distance on 5/12/04.

> [While T poses a new problem for B, D *pulls out one of the parts and repeats it to make a 16-part bar*, see Figure 6.2. She rests her forehead in her left hand while she uses the mouse with her right hand, appearing to be in deep concentration.]

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[61] In the protocols, D stands for Deborah, B for Bridget, T for the teacher-researcher (me), and W for a witness-researcher. As before, the protocols are numbered sequentially by chapter number (e.g., 6.1, 6.2, etc.). Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher-researcher's perspective. Ellipses (…) indicate a sentence or idea that seems to trail off. Four periods (….) denote omitted dialogue. Italics indicate verbalizations or activity which figure prominently in my analysis of the student's ways of operating or which are important in understanding my evolving model of the student. Since italics are used in this way, underlining indicates emphasis in speech.

D: Okay [sits up].

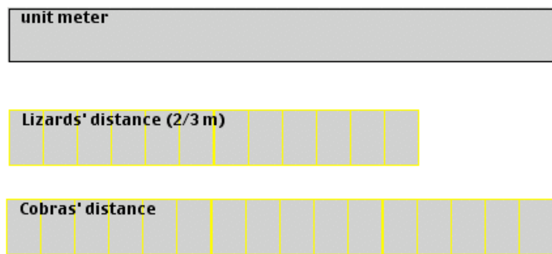

Figure 6.2, The Lizards' distance (2/3 meter) and Cobras' distance (16-part bar).

T [looking at D's screen]: Whoa—holy smokes! What is all that?

D [pointing to the partitioned 2/3-bar]: Since this is two-thirds of a meter, and *I divided it into twelve pieces because I needed three-fourths.*

T: Oh! Okay.

D: And *I put sixteen-twelfths here* [points to the 16-part bar].

T: Sixteen-twelfths.

D: Because that, that is—*'cause I knew each third is four pieces. So four times four, because you need four thirds for this one [points to the 16-part bar, the Cobras' distance].*

T: Mmm [nods].

D: Er—yeah [looks away]. Well, sort of. Yeah [smiles].

T: Mmm. So how much of a meter did the Cobras go?

D [pause]: Three-fourths. That's what you said. [D drags the 16-part bar below the unit meter and aligns the left ends of the bars.]

T: Well, no I only said—

D: You said this [dragging the 2/3-bar on top of the unit meter and aligning the left ends of the bars] is two-thirds. [T agrees.]

T & D: And that's three-fourths [T points to the 2/3-bar].

D: See, see, *and that's [points to the 16-part bar], that's four-fourths.*

T: Oh I see. But I wonder how much of a meter the Cobras went?

D: Oh. Um. [Softly] let's see…[D drags the unit meter out from underneath the 2/3-bar. For 30 seconds D participates in some surrounding talk but also stares at the screen.] *Sixteen-eighteenths of a meter.*

T: How d'you figure that out?

D: *'Cause I looked and there could be eighteen of these little pieces [in the unit bar], and there's sixteen [in the Cobras' distance].*

T: Oh. Could you figure it out without doing all that? *Could you reason through why this [points to one part] would have to be an eighteenth?*

[D stares at the screen and then eyes T.]

T: I bet you could!

[*D shakes her head no.*]

T: See if you can.
D: *I can't!*
T: Oh, I know you can.
D: *I can't!*
[T suggests that D think about how she divided up her 2/3-bar, and then T talks with B about her problem.]

This protocol exemplifies certain aspects of Deborah's ways of operating to solve these problems. Like Michael, Deborah used her reversible iterative fraction scheme to make the new distance by forming a goal to partition the known distance into three parts. That is, she knew that if the 2/3-bar was three-fourths of another bar, then she needed to divide the 2/3-bar into three equal parts, each of which would be one-fourth of the unknown distance. In order to enact this goal, she used her multiplying scheme: She aimed to partition each part of the 2-unit bar into some number of parts that would allow her to divide the total number of parts by three.

In contrast with Michael, distributive splitting was rather veiled in Deborah's activity. In order to divide the 2/3-bar into three parts, she did *not* divide each of the two one-thirds into three equal parts. Furthermore, there is no evidence to conclude that she had a goal to divide the 2/3-bar into six parts, thereby dividing each third into six parts. A possible explanation for her partitioning the 2/3-bar into precisely *twelve* parts total (instead of six or some other multiple of three) is that she was not entirely sure how many parts to aim for, but she knew she was dealing with thirds and fourths. So, using twelve parts total was a "safe bet"—twelve could be divided into three *or* four parts. Thus, rather than coordinating two and three based on a goal to make a 2-unit bar into three equal parts, she may have coordinated three and four based on a goal of making thirds and fourths in the same bar. However, in solving subsequent problems during this episode, she adapted her activity to routinely partition the first quantity into the smallest number of parts necessary to make the second quantity, which means her subsequent activity seemed to entail distributive splitting. In any case, making twelve parts total allowed her to divide by three and thus to conclude that, "each third is four pieces."

Perhaps the most revealing comment of the protocol is her explanation of why she made a 16-part bar: "so four times four, because you need four thirds for this one [the Cobras' distance]." This comment highlights that in one sense Deborah knew "more" in May than Michael did in February when he solve the Candy Bar Problem. Deborah conceived of the three equal parts of the 2/3-bar as *thirds*, and knew that she needed four of those thirds to make the unknown quantity. Yet she also called the 16-part bar "four-fourths," since each of the three equal parts of the 2/3-bar was also one-fourth of the Cobras' distance. The operational aspects of her reciprocal reasoning were not fully articulated in that both four-thirds and four-fourths seemed to refer to the end result, the 16-part bar. The degree to which she was aware of these two views as multiplicative operations—that the 16-part bar was both four-thirds *of the 2/3-bar* and four-fourths *of itself*—is uncertain. Nevertheless, viewing the unknown quantity as both four-thirds and four-fourths was a great strength in her ways of operating with these problems because of the possibilities that it opened for her.

This protocol is also an example of how finding the measure (in this case the length) of the new quantity was a separate problem from making the new quantity. That is, when solving RMR problems with *nonunit* fractional quantities, Deborah did not determine the measure of the new quantity as an outgrowth of her ways of making it. She often used recursive partitioning to do so, although in Protocol 6.1 she operated visually. I conjecture that her refusal to reason through how she knew the length of the 16-part bar was sixteen-eighteenths of a meter did not involve a lack of operations, but instead had to do with a certain fatigue and edginess that I will address when I discuss the enactment of MCR with Deborah.

Despite the strengths of Deborah's ways of operating with Type 5 RMR problems, as of the end of the teaching experiment she had yet to interiorize a scheme for solving them. During the remainder of the teaching episode on May 12th, as she solved RMR problems with nonunit fractional quantities, determining the length of the result became more closely associated with making the Cobras' distance than it was in Protocol 6.1. However, when I asked her to solve a problem like Task 6.2 by imagining it (rather than actually making the bars), she mentally calculated the *product* of the two fractions using her computational algorithm for fraction multiplication, mused for awhile, and rejected the result. At the time, this response was surprising to me. Given the strength of her ways of operating, I had assumed that she *was* interiorizing her ways of operating for solving these problems into a general scheme. Further corroboration that she had not done so occurred near the end of the episode. I had suggested that she return to making bars, but I wondered if she could solve a new problem (like Task 6.2) "without doing everything" (i.e., without making all the partitions). She worked diligently, as she usually did, but put her head down after about 4 minutes and said she couldn't do it. "This is stressful," she commented, which was somewhat of an unusual remark for her.

Encountering a Block in Solving the Box Problem in February

During the February 18th teaching episode, nearly three months earlier, was the first time Deborah encountered an RMR problem of Type 5. This episode was the last of seven (i.e., approximately a month's time, from mid-January to mid-February) during which she and Bridget had left the context of JavaBars to work on operating with unknown quantities and notating their operations using paper-and-pencil drawings and algebraic symbols. Although both girls were amenable to working outside of the computer context, Deborah did not particularly like to draw pictures (especially when she thought she already knew "the answer" to a problem). My main goal at the start of the episode was to generate a need to notate operations. Toward this end I posed problems with different measurement units, such as finding the girls' heights, and eventually any person's height, in inches, centimeters, millimeters, and meters.

At the end of the same episode (February 18th) I posed the following problem:

> Task 6.3, the Box Problem: Two groups of students, the Cobras and Lizards, have a box-stacking contest. The Cobras' tower is 3/4 of a decameter tall, and that's 2/3 of the Lizards' tower. How tall is the Lizards' tower?

Deborah, in part because of the earlier work with different measurement units, immediately wanted to convert decameters to meters (and knew that a decameter was ten times a meter). But I indicated that we could just use decameters for this problem. Deborah stated flatly that she couldn't figure it out, while Bridget drew a picture to represent the situation. Although I suggested twice to Deborah that a drawing picture might help, she persisted in trying to work computationally. Over the approximately 5 minutes we spent on the problem she moaned; said that her brain wasn't working; tapped her forehead while murmuring, "think, think, think;" and exclaimed that she didn't do anything when I tried to elicit what she had been thinking about. She appeared to be in a state of perturbation that was consciously conflictive and quite uncomfortable, and she did not find a way of operating to eliminate the perturbation.

Deborah had entered the teaching experiment with a strong multiplying scheme with whole numbers and a recursive partitioning operation, and very early in our interactions she demonstrated that she had constructed an iterative fraction scheme and a reversible iterative fraction scheme. Yet in working on the Box Problem, these schemes and operations seemed almost entirely blocked or suppressed, and she was unable to initiate activity that satisfied her, let alone that solved the problem. She seemed to have no visualized images for working on the problem and at that time was unable to generate some. Seeing Bridget achieve some success with the problem via reasoning with a drawing may have only exacerbated Deborah's relative paralysis in the situation and persistence in *not* drawing.

My basic explanation of Deborah's block is that she gained a great deal of satisfaction—indeed, even a sense of identity—from being able to calculate quickly with numbers. Her calculational facility with whole numbers was *not* algorithmic but highly sophisticated: She had embedded a distributive operation in her multiplying schemes with whole numbers so as to regularly and impressively reason strategically.[62] The sophistication of this reasoning leads me to infer that she had a rich set of images when operating with whole numbers, but her aversion to drawing indicates that these images were probably entirely implicit in her thinking. In particular, her response to the Box Problem indicates that she had not yet developed that level of sophistication—or private imagery—with her fractional calculation (and did not know the standard computational algorithm for fraction division, or if she did, she did not recognize the Box Problem as a potential situation of fraction division). Working for an extended period of time in a paper-and-pencil context, as we had done from mid-January to mid-February, combined with her aversion to drawing, seemed to only reinforce her tendency to solve problems via numerical computation. In working on the Box Problem, being unable to generate imagery in order to find a viable calculational avenue seemed to be a significant factor that prolonged her state of perturbation and blocked ways of operating she might have used to solve the problem.

Solving the Candy Bar Problem

Interestingly, in January, a month before her block in solving the Box Problem, Deborah demonstrated that she could solve RMR problems in quite sophisticated ways. I originally posed

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[62] For example, to determine 78 divided by 13, Deborah reasoned that 7 would be too big because 7 times 13 was 70 + 21. She determined that the answer was 6, because 6 times 13 was 60 + 18 = 78. Deborah's calculational facility with whole numbers was a good example of what Leslie P. Steffe calls "algebraic calculation."

the Candy Bar Problem that Michael solved on February 18th (cf. Task 5.1) to the girls during the teaching episode on January 14th:

> Task 6.4, the Candy Bar Problem: That collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?

When I posed the problem to the girls I did not have a clear understanding of the schemes and operations that might be required to solve it. So I posed it largely to test out my intuitive sense that the girls were ready to work on these kinds of RMR problems. Deborah's initial reaction was to protest, "you can't divide seven into three!" Then she brought her hands to her mouth and shrugged in embarrassment—she said that of course, you could, but you would have to "get into decimals." Her initial protest shows that, using her reversible fraction scheme she immediately formed a goal to divide the seven inch-long bars into three equal parts (since they were three-fifths of the other collection), but she lacked an immediate way of operating to achieve that goal. As with Michael, the *seven* bars seemed to be the perturbing element.

Both girls seemed stumped. Deborah mentioned "2.3333333…" which was what 7 divided by 3 meant to her, but I had asked them to *make* the new collection. About 45 seconds later, I suggested to them that each bar in this collection was three-fifths of another bar in the new collection. Soon after, Bridget voiced an idea that was not related to my suggestion, but Deborah acted on the suggestion. Using JavaBars, she partitioned one of the inch-long bars into thirds, pulled out one part, repeated it to make two parts, and joined the two parts to the original bar. She called her 5-part bar one bar of the new collection—and then my computer lost battery power. I was ready to quit for the day, but both girls seemed excited to explore their ideas. When a witness-researcher slid paper and pencils over to them, Deborah used the distributive idea I had suggested in order to produce the entire new collection (and determine its length, see Figure 6.3).

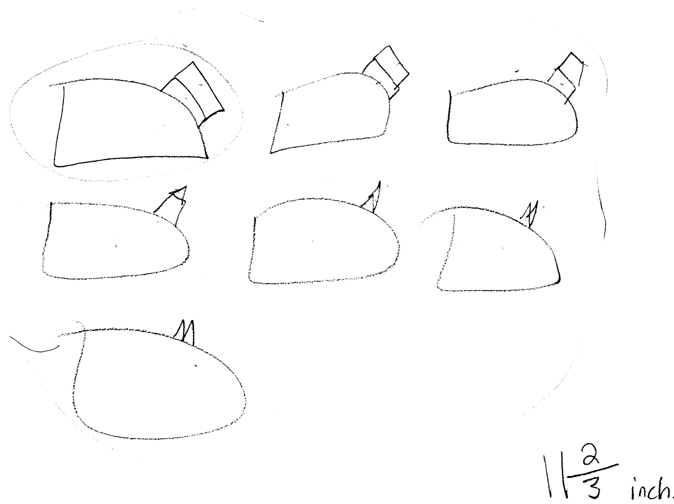

Figure 6.3, Deborah's drawing of the new collection of candy bars. The pillow-like shapes represent the original bars that the girls had made in JavaBars, and the small rectangles or triangles represent two more thirds of each bar added onto each bar.

Deborah's picture shows what she had started to do in the microworld to make the new collection, conceiving of each bar of the old collection as three-fifths of a bar in the new collection. Although she did not produce this solution path as independently as Michael had produced his, Deborah's assimilation of my suggestion, which relied on a distributive operation, shows the sophistication of her ways of operating. It also shows that she could operate on drawn imagery when starting from figurative material in the microworld.

The Enactment of MCR with Deborah in May, February, and January

I conjecture that Deborah's resistance in Protocol 6.1 to my request that she reason through why the parts she had made were eighteenths can be partially explained by two aspects of our MCR that recurred throughout the teaching experiment. The first aspect involves how I sequenced problems for her based on my working model of her ways of operating, and the second involves some of her personal tendencies in interaction with her partner and with me.

As I have stated, my goal for Deborah in the May 12th episode was to see if she would construct a general scheme for solving RMR problems with nonunit fractional quantities and relationships. My plan for accomplishing this goal was for her to solve several problems like Task 6.2 (the Lizards' car goes 2/3 of a meter, which is 3/4 of what the Cobras' car went). This plan was inadequate for enacting MCR well with Deborah because it was based on my conjectures that she was very close to constructing such a scheme and that repeated experiences in solving problems like Task 6.2 were all she needed to complete her construction. In particular, my planned sequence of problems rested on the conjecture that Deborah's distributive operation was solidly embedded into her ways of operating with fractions.

Deborah's distributive activity *was* considerably strong. As I have described, she reasoned distributively with whole number multiplication and division; she had embedded recursive partitioning into her scheme for solving RMR problems of Type 5 when starting with a unit fractional quantity (cf. Task 6.1); and she could assimilate my suggestions to operate distributively in sophisticated ways (cf. the Candy Bar Problem, Task 6.4). Furthermore, she had demonstrated distributive activity in making fraction compositions (which I discuss later in this chapter). But the sequence of tasks in the May 12th episode did not seem to open the way for her to construct a general scheme for solving any fractional RMR problem because it did not help her to build her use or awareness of distributive activity in these situations. I conjecture that one consequence of this relative lack of mathematical care was her resistance, and ultimately her fatigue and stress—her depletion—during the episode.

A second aspect of our MCR that likely contributed to her feelings of depletion involved personal tendencies. As I have mentioned, Deborah's mathematical self-concept seemed to include an identity-level satisfaction gained from her strong calculational abilities. Despite a show of confidence at times through mild boasting, she often displayed a *lack* of confidence in relation to others who she perceived might threaten her self-concept as a top mathematical knower. Bridget occasionally seemed to pose this kind of threat to Deborah, particularly when (as was the case during the May 12th episode) Bridget engaged in novel or adept activity that earned her praise. Ironically, Bridget's ways of operating often precluded her from solving problems that Deborah could solve swiftly. In fact, Bridget had been largely depleted during the previous three episodes in May because I had posed problems that were outside of her short-term ZPC. On May 12th, much to my relief, Bridget seemed to be finally working fluidly without depletion. Deborah's observations of her partner's relative "success," combined with her own work on problems that were quite challenging for her, may have left her feeling low.

At times I also may have threatened Deborah's mathematical self-concept, because throughout the teaching experiment I posed problems in which I asked for something besides a computational response (e.g., a drawing) or for which she didn't have recourse to calculation. Early on (by December), if Deborah did not immediately understand what to do to solve a problem, her response to me was "you're confusing me." This comment, which could be mildly to overtly accusatory, communicated to me that she expected to understand immediately and had little tolerance for not knowing what to do. Her expectation was undoubtedly based on a history of understanding mathematical work in school very quickly. Thus, if she didn't understand immediately, from her point of view it was probably the teacher's fault. She was the only student to respond regularly in that way to me, although Bridget adopted the response during her most depleted times over the course of the teaching experiment.

Deborah's suspicion of me as a teacher, combined with her tendency to subtly put down Bridget when, from her point of view, Bridget seemed to be posing a threat to her status as a knower, tended to dampen my response to Deborah's mathematical activity. I still expressed interest and enthusiasm to Deborah, and I still praised her work. But I could feel that my response to her was not quite as open as my response to the other students. At some level I believed that she did not need praise from me in the way she may have earned it from many of her teachers. I believed that she needed to become aware of some limitations in her own ways of operating that might cause her to reconsider what she thought she knew, or to try something new (e.g., working visually), in order to operate more powerfully. Of course, I knew that approaching such changes would be very uncomfortable for her, making her feel vulnerable by jeopardizing her status, in her eyes, as a top mathematical knower.

In addition, approaching such changes was uncomfortable for me as a teacher. Deborah's exclamations about being confused produced tension in me because, although I knew that feelings of discomfort are often essential for learning, I did not want to pose situations that were unduly stressful for her. In general, enduring a student's "necessary" depletion can be depleting for a teacher with a focus on MCR, who will question whether the level of depletion is appropriate, and if not, what might be done to alleviate it. Furthermore, I aimed to engender a feeling of trust between us, and I sensed a lack of it in her exclamations.

These kinds of dynamics between Deborah and me occurred during the February 18th teaching episode with the Box Problem (Task 6.3). In one sense I acted inadequately as a mathematical carer for Deborah, since constraining her to use decameters, given her aversion to drawing, left her with no recourse for activity with the fractional quantity and fractional relationship.[63] In another sense, as mentioned above, I believed that she needed to confront some limitations in her tendency to use a calculational approach in order to consider the value of

[63] I conjecture that the *fractional quantity* was the central perturbing element in the problem for Deborah.

making drawings, which might allow her to operate more powerfully. So from this perspective I enacted mathematical care. In a more global sense, posing the Box Problem at that juncture was probably not appropriate mathematical care for either girl because they had not constructed some of the schemes (e.g., fraction composition schemes) that they needed to solve the problem.

In contrast with the teaching episodes on May 12th and February 18th, where Deborah showed evidence of depletion outweighing stimulation, solving the Candy Bar Problem (Task 6.4) on January 14th seemed to be an experience of considerable stimulation for both Deborah and Bridget. This episode probably represents the best example of their abilities to care back for me by engaging in problems beyond my expectations. Deborah often independently demonstrated energy and excitement during teaching episodes. But in this case, her commitment and interest was particularly strong, perhaps because she built on Bridget's enthusiasm. Since the episode shows Deborah's facile assimilation of my suggestion, and therefore the sophistication of her current schemes, it also shows the relative ease of making unwarranted assumptions about her constructive activity—and as a result, the relative challenge of enacting MCR with her.

Bridget Solving Problems Involving Reversible Multiplicative Relationships

Solving RMR Problems with Unit Fractional Quantities in May

In May, Bridget also solved RMR problems of Type 5, involving fractional quantities and fractional relationships. During the May 10th episode, she solved a few problems with the relationship two-thirds between a *unit* fractional known quantity and an unknown quantity (e.g., 1/2 meter is 2/3 of another distance). She could solve such problems when she enacted her work in JavaBars, but at the end of the episode she experienced difficulty solving the following problem with the constraint that she *not* make it in JavaBars:

Task 6.5: A cubic liter is 1000 grams of water (1000 cubic centimeters). That's 2/3 of the water in another container. How much water is in the other container?

Note that Task 6.5 is a Type 2 RMR problem; Bridget had solved such problems, albeit with a smaller known quantity, in her selection interview without needing to draw pictures (cf. Chapter 4). Because at this point she could not solve Task 6.5 without making bars to represent it, I inferred that her current scheme for solving RMR problems relied on figurative material for its implementation—that is, she had not yet interiorized the operations she was enacting on the drawings in JavaBars. So, for the next teaching episode on May 12th, I set about designing a sequence of tasks that might engender this interiorization.

My basic plan was to pose several problems with unit fractional quantities and the *same* fractional relationship. I would encourage Bridget to solve the first few with JavaBars and then ask her to solve one *without* making bars. We went through three rounds of this sequence of problems, first with the relationship two-fifths between known and unknown quantity, then with the relationship three-fourths, and finally with four-fifths. In round one, with the relationship two-fifths, this problem was the third that I posed to her:

Task 6.6: The Lizards' car goes 1/7 of a meter. That's 2/5 of how far the Cobras' car went. Can you make how far the Cobras' car went and tell how far it went?

Bridget partitioned a copy of the unit meter into seven equal parts. Then she copied this 7/7-bar, partitioned each seventh into two parts and colored the first five parts (see Figure 6.4).

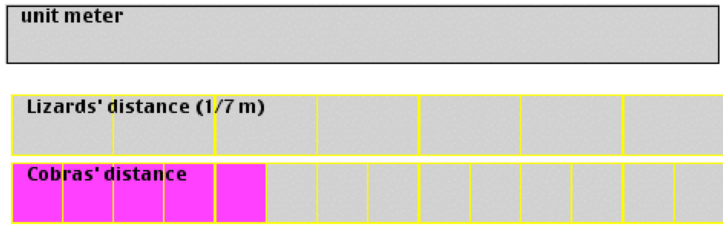

Figure 6.4, Five parts colored to show the Cobras' distance.

Protocol 6.2: Bridget's solutions of unit fractional Type 5 RMR problems on 5/12/04.

T: Ooh, what d'you do there?
B: One of them is two-fifths [points with the mouse to one of the seven parts of the unit meter]. *So um, you divide the bottom one, well, you divide it into two, each of them [points to the 7/7-bar], and you get fourteen [parts]. So I just put a bar with fourteen on it, and then five-fourteenths would be like, the way the other team…*
T: What the Cobras went?
B: Yeah.
T: Wow. So I wonder if you could do this one in your head. [*B frowns a little and raises her eyebrows, glancing uncertainly at T*.] So this time the Lizards went one-eleventh of a meter.
B [*looks worried*]: Mm-hmm!
T: And that's two-fifths of what the Cobras went.
B [*continues to look worried*]: Mm-hmm!
T: Can you tell how far the Cobras went?
B [*a little plaintively*]: *No*.
T: Oh I bet you can. See if you can imagine it.
B [rubs her eye]: So they went one-eleventh?
T: Mm-hmm [yes].
B [*after 5 seconds*]: *Huh? Five twenty-two*.
T [whispers excitedly]: How d'you get that?
B: *You can divide each eleventh into two, and you have twenty-two, and you got to have five instead of two*.
T: Awesome, awesome B!

I infer that Bridget used her reversible fraction scheme to solve Task 6.6 and its variation shown in the protocol. That is, she knew that if she had a quantity that was two-fifths of another quantity, then dividing it into two equal parts would produce fifths of the other quantity, and three more of those fifths were needed to make five-fifths, the other quantity. Furthermore, she used recursive partitioning to determine the size of the parts she had created in relation to the unit meter. Her solution of the variation on Task 6.6, starting with one-eleventh of a meter, demonstrates that she could perform her solutions in visualized imagination, and thus that she may have begun interiorizing her operations.[64] In fact, I infer that she was abstracting a pattern from her operations to solve RMR problems of Type 5 with unit fractional quantities.

In this activity, I purposefully did not ask Bridget the questions that I asked Deborah about the multiplicative relationships between the two distances. Bridget sometimes responded correctly to such questions. But based on our interaction during the teaching episodes on May 5th and 10th, I believed that the questions made little sense to her and that she was mostly following Deborah's responses. For example, in explaining her solution of "five twenty-two" Bridget said, "you got to have five [parts] instead of two [parts]," which confirms for me that her reasoning did not include reciprocal relationships. That is, I infer that she conceived of the two parts of the Lizards' distance as two-fifths of the Cobras' distance, but not explicitly as two-halves of the Lizards' distance. Thus her solution of Task 6.6 (and its variation) did not seem to include conceiving of the Cobras' distance as five-halves of the Lizards'. So, based on this inference, not asking her about multiplicative relationships between the distances was appropriate. However, at the time of the experiment I had not made such an explicit inference. I did not probe further primarily because of my perception of her depletion during the three preceding episodes, which I will address in my discussion of enacting MCR with Bridget.

Throughout the May 12th episode, Bridget operated similarly with the other rounds of problems (with the relationship three-fourths between known and unknown quantity, and then

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[64] Note that Bridget's response of "five twenty-two" instead of "five twenty-seconds" was not uncharacteristic of her; she often did not articulate her fraction language, which may reflect her coordination of whole numbers of parts (22 parts, and 5 of those parts) in the context of fractional activity, rather than fractional parts (twenty-seconds).

with the relationship four-fifths). Furthermore, she was aware, to some extent, of the nature of the problems she was solving, and seemed to feel a degree of comfort and control from being able to solve them. I make this claim because for the last problem of the episode, I suggested that the Lizards went three-tenths of a meter. Bridget immediately said, “Uh-oh, what?!” So she appeared to recognize that starting with three-tenths of a meter was quite a different problem. I stated that three-tenths of a meter was two-fifths of the other distance. She began the problem by making a 3/10-meter bar. But 90 seconds later she commented, “This is hard,” and after 60 more seconds she said, “I don’t get it.” I suggested she think about what she had done when she had started with, say, 1/3 meter. She returned to that problem and told me that she “took it [the 1/3-meter bar] out, divided it by two, and added three.” While I interacted with Deborah, Bridget cleared the 3/10-meter bar, partitioned it into two parts, pulled out one part, repeated it to make three parts, and joined the partitioned 3/10-bar and the 3-part bar (see Figure 6.5).

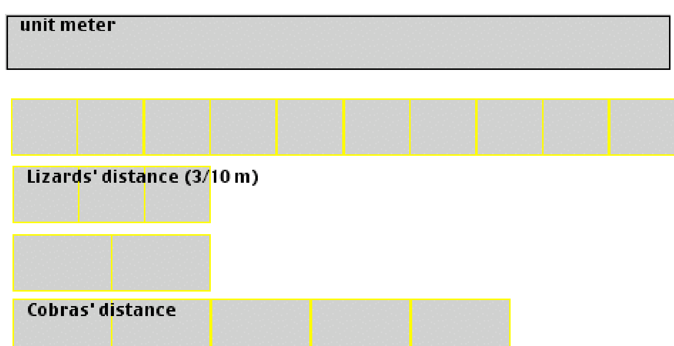

Figure 6.5, The 3/10-meter bar, partitioned into two parts, joined with three more of those parts.

When Bridget explained her work to me, she said she knew that four-fifths of the Cobras' distance was six-tenths of a meter (because two-fifths of the Cobras' distance was three-tenths of a meter), but she didn't know what five-fifths was. She seemed convinced that she had indeed made the Cobras' distance, but she didn't know its measure in meters.

I praised Bridget's work but challenged her to figure out the length. By visually inspecting the bars, she determined that she could divide each tenth of the 3/10-meter bar into two equal parts to make six skinny parts, 15 of which made the Cobras' distance. But she concluded that the Cobras' distance was fifteen-nineteenths of a meter because she mis-counted the number of skinny parts that would fit into the unit meter (rather than relying on her recursive partitioning operation to determine the size of the parts). So I infer that she was far from constructing a general scheme to solve RMR problems of Type 5. A major constraint was that she did not seem to have constructed a way to partition a 3-part bar into two equal parts without relying on visual comparison and estimation.

Solving the Box Problem in February

Unlike Deborah, Bridget seemed to find drawing pictures useful, and she independently drew a picture right after I posed the Box Problem, Task 6.3, during the February 18th teaching episode. To represent the Cobras' height,[65] three-fourths of a decameter, Bridget drew a rectangle partitioned horizontally into three equal parts. She stated that the other tower was bigger because the Cobras' height was (only) two-thirds of the Lizards' height. Then she partitioned the middle of the three equal parts of the Cobras' height into two equal parts and crossed out the lines that indicated the fourths of a decameter. To represent the Lizards' height she drew another rectangle consisting of three rectangles, each equal to half of the Cobras' height (see Figure 6.6).

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[65] I use "Cobras' height" to refer to the height of the Cobras' tower, and similarly for the Lizards.

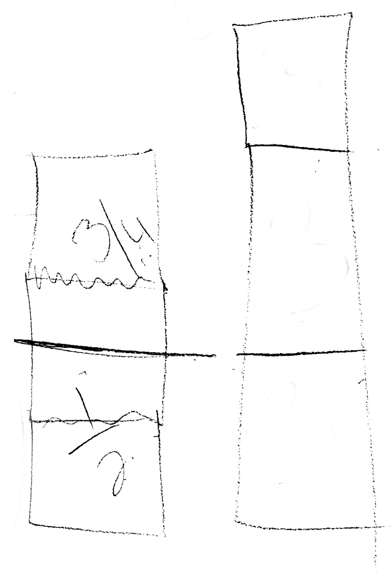

Figure 6.6, Bridget's drawing of the Cobras' height (3/4 dm, left rectangle), which was 2/3 of the Lizards' height (right rectangle).

Upon questioning, she articulated that one of the two equal parts of the Cobras' height was one-third of the Lizards' height. I infer that she used her reversible fraction scheme to draw the height of the Lizards' tower in relation to the height of the Cobras' tower.

I then challenged her to find the Lizards' height. She said that it was one decameter "because you've got one extra fourth." But immediately she thought that was wrong, and then she seemed stumped. I tried to focus her on the three-fourths of a decameter and what she had done to it in making the Lizards' height. (Meanwhile, Deborah had entered her blocked state and was moaning.) Then I intervened more strongly with Bridget, asking her what half of the Cobras' height was. Bridget responded that "you can't divide three fourths into half—well you can, but you'd get one point five over two." I probed to see if she could "figure it out as a fraction." After some time, she drew a rectangle to represent a whole decameter and partitioned it into fourths. She then partitioned each fourth into two equal parts and said "three-eighths" (see Figure 6.7).

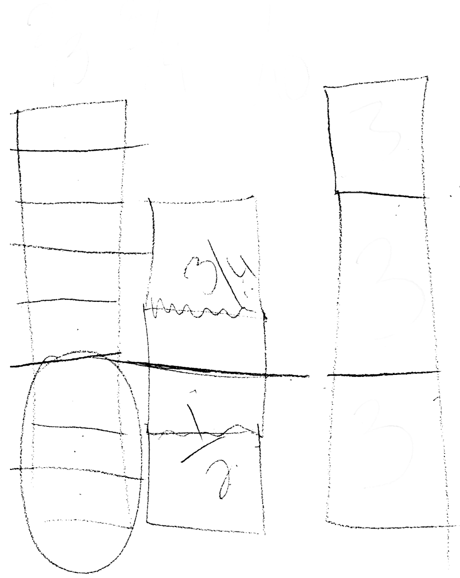

Figure 6.7: Bridget's determination of half of 3/4 of a decameter.

I was impressed and was going to continue with Bridget, but a witness-researcher prompted me to respond to Deborah's depletion. So I began to curtail the episode, asking Deborah about what she had tried and complimenting both of them on a good start to a hard problem. "Wait a minute," Bridget said excitedly, pointing to the Lizards' height, "This is—this is nine-eighths!"

Bridget's work here is significant because she was able to use recursive partitioning to determine the height of the tower, obviously with considerable coaching from me. My coaching assisted her in forming a goal to find half of 3/4 dm, which she had indicated was relevant through how she made the Lizards' height, but which she had not explicitly articulated.

However, she independently contributed the drawing of the decameter and the further partitioning of it to accomplish this goal. She also independently iterated the 3/8 dm three times to find the Lizards' height, which may indicate the use of an iterative fraction scheme. At the time of this teaching episode, I inferred that although Bridget's reversible fraction scheme was quite solid, she had not yet constructed a fraction composition scheme. Now, during retrospective analysis, I can make a more detailed inference: At this point, I infer that Bridget had yet to modify her splitting operation in a way that would allow her, without coaching, to enact her reversible fraction scheme to solve a problem like the Box Problem.

Solving the Candy Bar Problem in January

Interestingly, in January, about a month earlier, Bridget appeared to independently initiate the use of a distributive operation in order to use her reversible fraction scheme to solve the Candy Bar Problem, Task 6.4. Bridget's solution of this problem on January 14th contrasted with Deborah's solution (and Michael's) and seemed less generative algebraically. But Bridget's solution was still quite powerful because of the creative and spontaneous way she used distribution. Just as my computer was losing its battery power, Bridget communicated that she wanted to split the seventh candy bar into six equal parts and give one of these parts to each of the other six candy bars. I didn't immediately understand her plan, but she was itching to explain. So, once the witness-researcher supplied paper for her and Deborah, Bridget demonstrated her idea by drawing (see Figure 6.8).

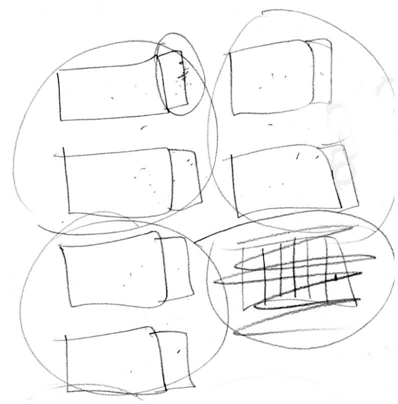

Figure 6.8, Distributing the seventh candy bar among the other six bars.

After she distributed the seventh bar among the other six bars, she noted that the six “big” bars were “three out of five.” She circled pairs of bars, and she drew two more pairs (see Figure 6.9).

Figure 6.9, Completing the drawing of the new collection of bars.

Bridget initiated partitioning the seventh bar into six equal parts in service of enacting her reversible fraction scheme: Her goal was to divide the seven inch-long bars into three equal parts, and she didn't know how to do so (again, seven was a perturbing element). But she *could* divide six bars into three equal parts. So, she effectively transformed the situation into a problem about six bars. If six bars were three-fifths of another collection, then two bars were one-fifth, and she knew she would need to add four more bars, or two-fifths more. With some coaching, Bridget also determined the length of the new collection, but the kernel of her creativity involved making the new collection. At the end, both she and Deborah seemed very excited that they had come up with the same result despite using different ways to solve the problem.

The Enactment of MCR with Bridget in May, February, and January

The final episode of the teaching experiment on May 12th represents perhaps the best enactment of MCR with Bridget after many episodes of poor enactment. We reached a nadir three teaching episodes earlier, on May 3rd, when Bridget was unable to operate in solving a variety of problems and seemed shut down emotionally. That day I had initiated a "problem-solving workshop" because of my conjecture that the girls were feeling constrained in some ways and needed to feel free to solve problems however they wanted to (e.g., for Deborah that would probably mean without drawing pictures). That episode was also the first of the four episodes in May in which the girls worked on separate computers. The problems I posed at the start of the May 3rd teaching episode involved improper fractional amounts and the coordination of two different quantities (e.g., a runner runs a mile in 6 minutes; find how long will it take him to run 13/8 of a mile). I now infer these problems were outside of Bridget's short-term ZPC. Although I understood very soon that Bridget was struggling, in the moment I did not know how to adapt the problems so that she could act more independently, as well as feel more autonomous and in control. I resorted to heavy coaching, which was laborious for both of us. Bridget's mood lifted during the episode, and I also posed problems toward the end that Bridget could solve using her current schemes. But I felt worn out afterwards.

Our interactions in this episode taught me that weakened MCR can put a heavy burden on the teacher as well as the student. In this episode with Bridget, I was empathetic, patient, and persistent. I decentered enough to understand that she was experiencing great difficulty and significant depletion, and I endeavored to find ways *within the problem situation* that might allow her to operate. But because those ways had to do with *my* perception of what breaking down the problem might involve (i.e., for someone with operations like mine), my suggestions were not very effective for her and indicated that I did not decenter enough cognitively. Thus my suggestions did not alleviate her depletion very well, and the longer she remained in a depleted state, the more depletion I felt!

One might say that in this situation I tried to use "general" caring relations (i.e., caring relations without a focus on mathematical learning) with Bridget, in terms of decentering to consider her emotional experience (of darkness and frustration), her general cognitive experience (of not being able to operate, not understanding), but *not* her specifically mathematical cognitive experience (of, for example, being bothered by finding one-eighth of a 6-part bar, let alone finding thirteen-eighths of it). In the moment of interaction with a student I do not always have available the explanatory constructs that I formulate during retrospective analysis (such as not yet coordinating two different three-levels-of-units views of a quantity). And I cannot always think of how to change a problem so that depletion might be alleviated (such as asking how long it would take the runner to run two-thirds of a mile instead of thirteen-eighths of a mile). However, relying solely on general caring relations is insufficient when trying to bring forth mathematical learning because "GCR" does not seem to help a teacher address a student's specifically mathematical cognitive experience so that her mathematical activity might continue.

My interaction with Bridget on May 3rd set off a rather fervent search for better ways to communicate with her mathematically—I was in a consciously-conflictive state of perturbation! That the teaching experiment was nearly over only augmented my fervor and discomfort: I felt abashed that I was having such difficulty harmonizing with Bridget at this late date. Out of that search came a gradual strengthening of our MCR over the final three episodes of the teaching experiment. I began to take more seriously that certain numbers bothered Bridget—particularly large improper fractions like 13/8 or 17/5. I also conjectured that, despite her ability to follow and often make sense of Deborah's activity, Bridget's activity with Type 5 RMR problems was restricted to unit fractional quantities and relied on figurative material. Hence I carefully planned a sequence of tasks for her on May 12th that involved only unit fractional quantities and opportunities for her to operate in visualized imagination at points during the episode.

These planned actions of mine could not be considered fully "successful" mathematical care for Bridget unless she experienced some alleviation of her extended period of depletion, or received them with some sense of openness. In fact, she responded quite positively to the sequence, both cognitively and affectively. I have already described the cognitive aspect: how I consider this episode to be her initial construction of a scheme for solving Type 5 RMR problems with unit fractional quantities. Affectively, she went through an interesting transformation. Her prompt activity when I posed initial problems indicates that she felt confident that she could *make* solutions in the microworld. However, her responses to my requests for her to solve such a problem *in her imagination* evolved dramatically over the episode.

At my first request, shown in Protocol 6.2, Bridget looked worried—she frowned, raised her eyebrows, and spoke "no" plaintively. However, she solved the problem in less than 15 seconds and could justify her solution. My second request came about 4.5 minutes later, after she had solved, using JavaBars, two more RMR problems with the relationship three-fourths between known and unknown quantity. I said, "I wonder if you could do this one by imagining it." "Probably not," she responded, but made no other protest and solved the problem, with justification, in approximately 15 seconds. My third and final request came about 5 minutes later, after she had used JavaBars to solve two more RMR problems with the relationship four-fifths. "See if you can do this one in your head," I said. "Okay," said Bridget. Again she had a response in approximately 15 seconds but realized it wasn't quite right and asked to do it on the computer. This change in her responses from unconfident and worried to agreeable and willing was a marked example of the reciprocity of MCR. While perhaps she did not reach a place of overt stimulation, she was open and active and did not seem tired throughout the entire episode. In this way, Bridget completed our MCR and cared back for me.

This interaction taught me a lot about Bridget's ways of operating, but it also helped me see Bridget's personal tendencies in a new light. During the teaching experiment, I persisted in the view that Bridget had schemes nearly equally as powerful as Deborah's, only she seemed to lack confidence. Unlike Deborah, Bridget did *not* seem to expect to understand everything nearly instantly—her mathematical self-concept appeared weaker than her partner's, and sometimes she seemed sure that she *wouldn't* understand. Although occasionally Bridget would adopt Deborah's phrase that I was "confusing her," in general she did not look to blame me (or anyone else, except perhaps herself) if she felt confused. I believed that her apparent lack of confidence, combined with Deborah's tendency to dominate interactions because of her personality and her strong ways of operating, meant that Bridget just wasn't getting enough of a chance to express her ideas. This view was not completely unwarranted: Deborah's swift calculational ability could curtail Bridget's activity and sometimes dampen her spirit. But my view of her also blocked me from understanding that her seeming lack of confidence and reticence were entirely entwined with less powerful ways of operating with, specifically, RMR problems.

My interactions with Bridget when she solved the Box Problem, Task 6.3, and the Candy Bar Problem, Task 6.4, are good examples of my concerted attempts to understand and draw out her activity and ideas. They support my view both then and now that she had powerful and creative ways of operating! And they also show her enthusiasm and engagement—she more than Deborah was the instigator of continuing to solve the Candy Bar Problem well beyond the time I had intended to end the episode. However, in both of these interactions with Bridget, I was operating largely from my view that she needed space and time to express her ideas more than from an understanding of the ways in which she might be constrained by her ways of operating. So, although gratifying for me (because of her creative solutions) and hopefully beneficial to her (through creating these solutions), these two interactions in January and February did not allow me to learn what I learned from her in May.

Reversible Reasoning: Constructive Resource 1, October

In the next five sections, I explore key moments in the girls' construction of their ways of operating to solve RMR problems of Type 5 in May. The key moments are organized into the same five constructive resources as in the boys' case study (reversible reasoning, activity with improper fractions, new uses of multiplying schemes, fraction composition activity, and fractions as operations), which occur approximately chronologically through the teaching experiment (see Figure 6.10). As with the boys, the key moments demonstrate that Deborah or Bridget constructed a way of operating (i.e., a resource) that was useful in her solutions of RMR problems in May, or that either girl experienced a constraint in constructing a way of operating that I conjecture would have been a useful resource in solving RMR problems in May.

Oct 03	Nov 03	Dec 03	Jan 04	Feb 04	Mar 04	Apr 04	May 04
Both girls **reason reversibly** with whole numbers and fractions but they don't make **improper fractions**.	Both girls work on **activity with improper fractions**; D reasons with large improper fractions.	D constructs a **new use for her mult. scheme**, a co-measurement scheme.	Both girls solve the Candy Bar Problem. We leave JavaBars to work on unknowns. D shows she has ***unit* fractions as operations**.	B solves the Box Problem but D does not. We return to JavaBars to work on fraction composition.	Both girls work on **fraction composition activity**. B constructs a **new use for her mult. scheme** a co-measurement scheme.	B has not constructed **fractions as operations**. D demonstrates a reversible multiplying scheme with fractions and the use of reciprocal relationships in solving RMR problems.	

Figure 6.10, Timeline of the girls' five constructive resources.

Splitting to Solve RMR problems

Splitting whole numbers. In contrast with the boys, both girls solved RMR problems of Type 1, where the whole number relationship divided the whole number quantity, without incident during the first teaching episode on October 30th. For example, on that day I posed this problem:

Task 6.7: Sara's stack of CDs is 65 cm tall. That's 5 times the height of Roberto's stack of CDs. What would you do to find out how tall Roberto's stack is?

Both girls immediately knew that they needed to divide 65 centimeters by five, so I asked them to draw a picture to show why they had to divide. Bridget drew a rectangle to represent Sara's stack, 65 centimeters tall, and partitioned it horizontally into five equal parts.

Protocol 6.3: Splitting whole numbers to solve a Type 1 RMR problem on 10/30/03.

B: So hers would be this [*points her pencil tip to the lowest of the five parts* and draws a line from it], *so you would divide sixty-five by five*. [She writes 5 and then crosses it out as she says] that's one [writes 1, see Figure 6.11]. That's how much his is.

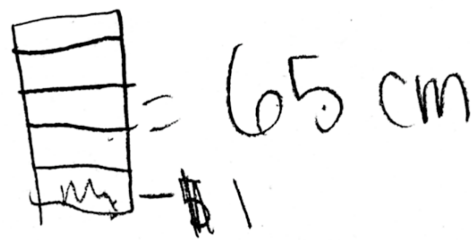

Figure 6.11, Bridget's drawing of 65 cm split into five equal parts.

T: I see.
[D has been drawing and watching B.]
D [very softly]: *I—I don't know*.
T: Okay, okay. So why don't you go ahead and continue working; you can figure out how tall his stack is.
B: I know how tall his stack is.
D [almost simultaneously with B]: I know [raises her pencil].
T: Oh, you <u>do</u> know—oh okay. [All laugh.] So let's look at D's picture too. So D how did you draw your picture?
[D *has drawn a rectangle labeled 65 and a small rectangle next to it labeled 5. The small rectangle is about 1/5 the height of the large rectangle,* see Figure 6.12.]
D [very softly]: Hers is sixty-five centimeters [points to the large rectangle with her pencil eraser] and *his [points to the small rectangle] is five times less* [shrugs a little].

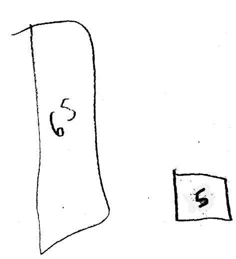

Figure 6.12, Deborah's drawing of 65 cm and a rectangle 1/5 of that height.

T: So what's that five mean there [points to the small rectangle]?

D [very softly]: Five times less—I don't know.
T: Oh it means five times less? Okay.
[D laughs a little, seems a little embarrassed.]
T: So B, you said this [points to the bottom part of B's five part rectangle] was five but you changed it to one; can you tell me about that? [*D erases the 5 in her small rectangle and changes it to 13*.]
B: *'Cause that's one-fifth out of five, so that would be like one out of five*.
T: Oh, I see. And D, you just changed yours to what?
D: *Thirteen centimeters*.
[B smiles and agrees.]

In this protocol, Bridget was more articulate than Deborah in partitioning the 65-centimeter rectangle into five equal parts and referring to one of those parts as one-fifth of the 65 centimeters. Bridget's picture supports my inference that she used her splitting operation to solve this problem. Furthermore, the ease with which she split the composite unit of 65 centimeters into five units each of which contained thirteen units (centimeters) is an indication that she may have conceived of 65 as a unit of units of units. Deborah's picture and comments are less elaborated; she was very quiet during this episode which I was to learn was not characteristic of her. (It was, after all, the very first meeting.) I take Deborah's comment "I—I don't know" to mean that she didn't know how to draw a picture to show why she knew to divide. However, Deborah's reference to her small rectangle as "five times less" indicates that she viewed the problem multiplicatively and knew that dividing by five would produce something five times smaller. Erasing the "five" and writing "thirteen" in her small rectangle shows the relationship between the heights of Sara's and Roberto's stacks and is an indication that she also may have conceived of 65 as a unit of units of units.

Splitting proper fractions. To further explore the girls' reversible reasoning, in this same episode I posed a RMR problem of Type 2 with a whole number quantity and a proper fractional relationship, where the numerator of the fraction divided the whole number quantity:

Task 6.8: Tanya has $16, which is 4/5 of what David has. How much does David have?

Both girls solved the problem within 5 seconds, and again I requested a picture.

Protocol 6.4: Splitting to solve a Type 2 RMR problem on 10/30/03.

D [softly]: Uh-oh [in response to T's request to draw a picture. Everyone laughs. Then both girls write on their paper.]

D [murmuring]: Four times five…[She finishes writing after 10 seconds and looks at B. *D has written 16 ÷ 4 = 4 in long division format and next to it, 4 x 5 = 20.*]

B [softly, while writing and drawing]: *Sixteen, and then that'd be four each.* [B finishes writing and drawing after about seven more seconds. *She has drawn a rectangle partitioned horizontally into five parts. She has written 4 on the left side of each part and 16 on the right side of the rectangle next to the top four parts,* see Figure 6.13.]

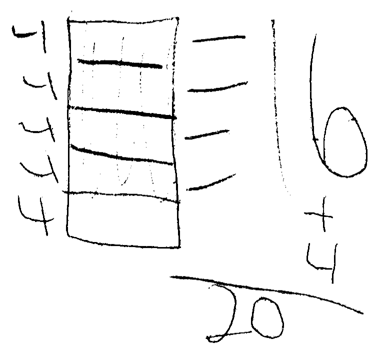

Figure 6.13, Bridget's drawing of $16 as 4/5 of another amount of money.

T: Oh. B, why don't you explain your picture?

B: Okay, well, you have *five as your denominator so four-fifths* [*shades the top four parts of her drawing*], *we already knew what four-fifths was. I divided sixteen by four, which I got four. So that's how I got the four for each one.*

T: I see. And so you ended up with twenty. What about you D—what were you thinking?

D: *I said, since it's four-fifths, sixteen divided by four equals four, and then four times five equals twenty.*

T: Okay, and that's a lot like what B did.

B: Yeah, *I didn't really have to draw my picture though. It was kind of easy.*

Although neither girl fully articulated the fractional aspects of their results (i.e., that four dollars was one-fifth of David's money and that 20 dollars was five-fifths of it), based on their activity in solving Task 6.8 and other problems in this episode I infer that they viewed the situation in these terms. For both girls, four-fifths of a quantity was one-fifth of it four times, and so they could divide four-fifths by four to get one-fifth of the quantity. So both girls could use their splitting operation to split four-fifths of a unit into four equal parts, each of which was one-fifth of the unit. This use of their splitting operation was more sophisticated than their use of it solely with whole numbers and indicates that unit fractions were iterable for them, at least within the unit. From their solutions of Task 6.8 I can infer that both girls had constructed at least a reversible partitive fraction scheme *with splitting*.

After splitting four-fifths into four equal one-fifths, they could use any of the one-fifth parts to find all five-fifths of the quantity by multiplying by five, as Deborah did. Bridget's picture shows that she added four more dollars (one-fifth) onto 16 dollars (four-fifths). But her comment "that's how I got the four for each one" indicates that she saw each fifth of David's money as four dollars, so I infer that multiplying by four dollars by five was available to her as a solution to the problem. In any case, the girls found the problem "easy." Because of the evidence of their strong reversible fraction schemes in this episode, we did not spend time on the "pencil problems" that I posed to the boys (cf. Tasks 5.6, 5.7, 5.9). Instead, in the next teaching episode on November 4th, the girls and I started exploring the JavaBars program and focused on activity involving improper fractions, including reversible reasoning with improper fractions.

Early Trends in the Establishment of MCR

A significant description of MCR can hardly be based on the first episode of the teaching experiment, when I was just getting to know the girls. However, I will point out two early trends in our interactions related to making drawings. First, the contrast between the girls' attitudes toward drawing is apparent here. Deborah did not seem to feel comfortable drawing to solve either Task 6.7 or 6.8, largely because she didn't seem to see a need for it—she could solve the problems easily without drawing. In addition, she didn't know what to do to draw a solution (it was not how she was accustomed to thinking about such problems), so as shown in Protocol 6.4, to solve Task 6.8 she didn't draw. As I have already indicated in the first section of this chapter, these themes would become magnified with her, particularly in the seven episodes from mid-January to mid-February in which we worked outside of the JavaBars microworld in a paper-and-pencil context. In contrast, Bridget felt comfortable drawing, although whether she saw it as useful here is debatable, since she said she didn't need to draw a picture for Task 6.8. In this episode, her drawings were probably more useful *for me* than for her, allowing me to understand more about her ways of operating. But, as already discussed in her solution of the Box Problem (Task 6.3) in February, she was comfortable with using drawings to help her solve a problem.

Despite the fact that their drawings were useful to me, I didn't know very well how to probe their drawings and reasoning in order to further understand them *and* to push for articulation that might be useful in externalizing the girls' schemes and operations. As I have discussed in Chapter 2, externalizing ways of operating is important in opening the possibility for retroactive thematization of ways of operating—i.e., for reflected abstractions. For example, I could recognize that Bridget's drawings and comments indicated that she viewed the solution to Task 6.7 as one-fifth of 65 centimeters, but I didn't ask good questions about it to probe her fraction scheme with composite units. Now I would ask both girls questions like: So 13 centimeters is what fractional part of the 65 centimeters? Why is it one-fifth? What would be two-fifths of the 65 centimeters? Three-fifths? How do you know that? Questions like these are even more important in Task 6.8, where the fraction language in the girls' explanations was left almost completely implicit. After asking these kinds of questions with Task 6.8, I would (now, in retrospect) re-ask Deborah to draw a picture to show the relationships in the problem.

There is nothing magic about these questions. But asking them has the potential to help the girls articulate their schemes and operations, bringing out implicit aspects of them. As I have discussed in Chapter 3, making ways of operating explicit is an important element in moving from quantitative to algebraic reasoning. In addition, assessing the girls' responses to these kinds of questions would help me to create more elaborated initial models of the girls' current schemes (in this case, of their splitting operations and their reversible fraction schemes). By creating more detailed models, I would have a better foundation from which to enact MCR—to harmonize with and open new possibilities for their ways of operating.

Activity with Improper Fractions: Constructive Resource 2, October and November

Although during the first episode on October 30th both girls demonstrated that they had constructed at least a reversible partitive fraction scheme with splitting, neither girl indicated that she had constructed improper fractions. So during the month of November, I worked with them to see if they would construct an iterative fraction scheme. In this process, I confirmed with Bridget what I learned from Carlos, namely that a splitting operation does not appear to be sufficient for the construction of an iterative fraction scheme because it does not seem to require (or induce) coordinating three levels of units. I also learned that like Michael, Deborah could operate in sophisticated, "structural" ways with large improper fractions because she had interiorized three levels of units and seemed to take nonunit fractional quantities as iterable units.

Making Meaning for Seven-fifths in October

On October 30th, both girls seemed uncertain about what to do to solve this problem:

> Task 6.9: The rectangle drawn on this piece of paper represents a candy bar. Can you make a drawing of 7/5 of that candy bar?

Bridget frowned and Deborah looked intently at the paper. Then they each gave a little laugh. “I have no clue,” said Bridget. Deborah slowly picked up a pencil and drew a rectangle beneath the candy bar that was the same length and width, but she did not close off the right end. She murmured that she needed to make it bigger. Next, without making marks, she gestured along the candy bar with her pencil and seemed to be judging how long a fifth of the candy bar was.

During this time Bridget announced that she wasn’t sure if what she was going to do was right, but she was going to do it anyway. She marked the original bar into five equal parts and then drew a longer bar underneath it. In marking the new bar into parts, she ended up with six parts total. She counted the parts and added another part onto the left end (see Figure 6.14).

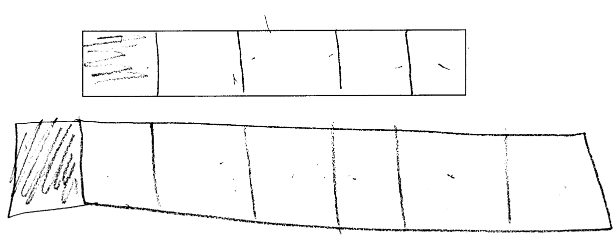

Figure 6.14, Bridget’s drawing of 7/5 of the candy bar, with the leftmost part added last.

When I asked about her drawing she explained that she divided the candy bar into five pieces “because you have fifths,” and then she “added two more pieces so it’d be seven.”

Protocol 6.5: Engendering the construction of improper fractions on 10/30/03.

T: Okay. So each one of these [points to the parts of the new bar] is a fifth?
B: *Um, no, I don’t know.*
T: Oh, okay, okay. But you had five pieces and then you said you added two more.
B: Yeah.
T: Okay. What about you D, what are you thinking?

D: Well *I said seven-fifths is one and two-fifths*, so I drew one whole candy bar and I didn't close it so I could draw more, and then I looked to see what two-fifths would look like on this [points to the original] candy bar, and I added that on.
T: Oh, okay.
….
[W intervenes to ask how D knows that she has made seven-fifths—he doesn't see the pieces. Everyone laughs. *T asks if D can show all the pieces. D picks up her pencil and pauses for about 20 seconds. Then she draws in marks showing seven equal parts in her new candy bar*. W then asks them to color in one-fifth of the candy bar. Each girl colors one part of her top 5/5-bar.]
W: What about those pieces in the bottom candy bar?
D [smiles]: *Uh-oh*. [Everyone laughs.]
W: How big are those pieces—are they fifths or sevenths or ninths or—
D: *Sevenths*.
B: *Yeah …. 'Cause they turned into seven pieces instead of five pieces*.
[T asks about one piece from each bar; the girls agree that they are supposed to be the same amount, equal. W asks them to shade in one of the seven pieces, which they do.]
W: How much is that piece of the top candy bar?
B: The top bar?
D: *Uh-oh*. [She pulls at her ear. During an 8-second pause B gives a little laugh. D addresses T.] *Are we allowed to do math on this sheet?*
T: Oh yeah, sure.
[D begins to write on her paper.]
B [After five more seconds]: I have no clue. I'm thinking you're doing something with the two fractions, but I don't know. 'Cause that's one-seventh [points to the shaded part of the new bar] and that's one-fifth [points to the shaded part of the original bar]. I don't know. [T has been making listening noises as B has been speaking.] You sound like my mom. [B and T laugh.]
[D looks up and then both girls say softly that they don't know.]

The girls' activity in this protocol indicates that seven-fifths was yet to mean one-fifth iterated seven times, one important hallmark of an iterative fraction scheme. For Deborah, seven-fifths seemed to mean one and two-fifths. However, I infer that this meaning did not simultaneously entail the meaning of seven one-fifths, because she spent 20 seconds looking at her paper and murmuring before she produced seven equal parts in her new candy bar. Furthermore, the seven equal parts in the new bar took on a status of sevenths, despite the fact that she had conceived of the new bar in terms of a whole and two more fifths.

Bridget did not explicitly use one and two-fifths in her activity, although she did speak of adding “two more pieces.” Shortly after her qualifying statement that she wasn’t sure about what she was doing, she explained that she had to make fifths. So I infer that because she was to make seven-*fifths,* she made fifths in her candy bar. She also seemed to know that seven-fifths would be more than a whole bar, probably because she knew that a whole bar was five-fifths. However, the most I can infer from her activity is that she drew seven parts in the new bar because she was to make *seven*-fifths; those parts did not retain a status of fifths of the original bar except when they were embedded in the original bar. Once she had made seven parts in her new bar, like Deborah, the parts had the status of sevenths. Therefore in this protocol, neither girl showed evidence of operating in a way that meant she had constructed one-fifth as a unit that was iterable *beyond the whole*.

As a neophyte teacher in a teaching experiment, I did not know how to address my perception of the girls’ difficulties. I did not even know *whether* to address them, since a central purpose of this first episode was to establish some benchmarks. (At this point I also did not understand many of the implications of their activity with improper fractions!) So I started to move to a new problem. The witness-researcher quickly intervened, reminding me that this was *teaching*. He asked the girls to pretend the candy bar was magic, meaning every time they took a piece out, the candy would fill back in. Then he asked them to make three-fifths of the candy bar. Each girl marked a fresh copy of the candy bar into five parts and drew a new bar below it that spanned three of those parts, marking the three equal parts in their new bars when prompted. The witness-researcher asked them to shade one of the parts and tell how much it was of the candy bar. Bridget said it was one-fifth and Deborah agreed.

Continuation of Protocol 6.5: Engendering the construction of improper fractions on 10/30/03.

W: Okay. *Now I want you to take that piece and make seven of them. Make a new candy bar and draw seven of those pieces.*
[Each girl draws a long bar beneath the original candy bar and the 3/5-bar they just drew. They each mark all seven parts in their bars.]
W: Now shade in the first piece. [Both girls do so.] How much is that piece of the top candy bar?
B [*no hesitation, but softly, with a slight frown*]: *One-fifth.*
T: Mmm. You think so?
[*D smiles broadly*.]
W: Why is that?
B: *'Cause that* [*the shaded part of the 7-part bar*] *is equal to that* [*the first part of the original candy bar*].
[W asks them to shade in the last piece of the 7-part bar. Both girls say that that piece is also one-fifth of the top bar. B provides justification by saying the last piece is also equal to the first piece of the top bar.]
W: *So how many fifths do you have in that big candy bar?*
D [pause, looks down at her paper]: *Seven.*
B: *Yeah.*
W: Seven-fifths?
D & B: Yeah.
W: Okay, case closed.

In this protocol, the witness-researcher built on his assumption that the girls conceived of three-fifths as three one-fifths to build toward conceiving of seven-fifths as seven one-fifths. This assumption was well-founded, since both girls had constructed at least a partitive fraction scheme with splitting (as discussed in the previous section). Once they had each drawn a bar that was seven of those parts long, Bridget seemed to easily identify one part as one-fifth of the top bar. Deborah did not comment, but she smiled broadly, and then said that the big candy bar consisted of seven-fifths (of the top bar). Thus the girls appeared to make meaning for seven-fifths of a candy bar as seven parts, each part being one-fifth of the candy bar. Protocol 6.5 and its continuation *potentially* marked the girls' initial construction of an iterative fraction scheme.

Iterating Beyond the Whole

To build on our activity in the October 30th episode, my goal was for the girls to make fractions larger than one by iterating beyond the whole. So during the teaching episode on

November 6th, I asked the girls to use, for example, twelfths to make a bar a little bit longer than an 11/11-bar, and then to state the length of the new bar. However, I conjectured that this kind of problem might stump them at first, so I made a sequence of problems leading up to it. First I planned to ask them to use twelfths to make a bar that was *the same length* as an 11/11-bar, and then to use twelfths to make a bar a little *shorter* than an 11/11-bar. If they had trouble with the latter question, I planned to ask them to make a bar a specific amount (such as two-twelfths) shorter than an 11/11-bar. Then I would request a specific amount (such as two-twelfths) longer. I posed such a problem to Deborah after I asked her to make a 13/13-bar:

Task 6.10: Can you make a bar 2/15 longer than that 13/13-bar?

Deborah looked at the screen for about 37 seconds. During this time a witness-researcher and I reminded Bridget to think about it too, and she said she knew how to do it. Then Deborah made a copy of the unit bar, partitioned it into 15 equal parts, and gestured with the mouse to the two rightmost parts of the 15/15-bar. Because the girls were just learning the program, I asked her what she wanted to do. She indicated that she wanted to "add more." I reminded her that she could use PULL OUT to make more parts. So she pulled out the rightmost part of the 15/15-bar, dragged it adjacent to the right end of the 15/15-bar, and initiated using REPEAT to make two 1/15-parts joined together. I then asked Bridget how long Deborah's bar was.

Protocol 6.6: Naming fractions larger than one on 11/6/03.

B [looking intently at the screen for 5 seconds]: *Seventeen-seventeenths. That one there, 'cause there's seventeen pieces.*
T: D, what do you think—how long is this bar that you just made?
D [yawns]: *It's seventeen-fifteenths.*
T: Oh. Wait a minute. We have disagreement here.
B: Well, *yeah, 'cause there's fifteen on the bottom, yeah, and sevent—yeah.*
D: *It's more than one.*
B: *Seventeen-fifteenths of the unit bar.*
T: Oh, okay, so in terms of the unit bar this is seventeen-fifteenths. Oh, okay.
W: Why is that true B?

B: *Because that bar has fifteen pieces and she added two more of that [15/15-bar] to that…* [points to the 13/13-bar. She hits her elbow on the table and grimaces. W and T ask her how much each little part of D's bar is of the unit bar, and B says one-fifteenth.]

Deborah's iterative fraction scheme. Based on her pause of 37 seconds, Deborah did not seem to immediately know what to do to solve Task 6.10. I infer that she formed a goal to add two 1/15-parts to the right end of her 15/15-bar, but she wasn't entirely certain how to get these two parts. Part of her uncertainty may have involved operating within the microworld, since she had just started using JavaBars one episode earlier (on November 4th). Once she pulled out a 1/15-part from the 15/15-bar and used it to make the 15/15-bar longer by two-fifteenths, it seemed unproblematic for her that the length of the new bar was seventeen-fifteenths. Her justification, that seventeen-fifteenths is "more than one" allows me to infer that at this point a bar two-fifteenths longer than a unit bar was *both* a whole unit bar and two-fifteenths more *and* seventeen-fifteenths of the unit bar.

Holding this "dual view" of a bar two-fifteenths longer than a unit bar was a change from Deborah's view of seven-fifths of a candy bar on October 30th (cf. Protocol 6.5 and its continuation). I account for this change by inferring that in Protocol 6.5, Deborah began to construct a unit fraction as a unit that could be disembedded from and iterated beyond the whole to make a new unit (of unit fractions) that still stood in relation to the whole, but in a way that transcended part-whole relationships. She seemed to solidify that construction in working on Task 6.10: One-fifteenth of a unit bar could be iterated 17 times to produce a bar that was a unit made of 17 units, each of which gained their identity from being able to be iterated 15 times to make another unit (the whole). So at this point Deborah operated with three levels of units: Seventeen-fifteenths was a unit of 1/15-units that contained another unit of those units, fifteen-fifteenths. Thus I can attribute an iterative fraction scheme to her.

Bridget's lack of iteration beyond the whole. In contrast, during the 5-second pause at the start of Protocol 6.6, Bridget seemed to count the number of parts in Deborah's 17-part bar, and she called the bar seventeen-seventeenths. So I infer that for Bridget, parts of a bar gained their identity from their relationship to the "current whole" *of which they had just become a part*, rather than in relation to the whole *from which they had been a part*. This meaning of fractions is basically partitive. Thus Bridget's unit fractions were not yet iterable outside of the whole, and she did not seem to coordinate three levels of units in this context. However, as soon as she heard Deborah's response, Bridget made sense of it and could elaborate that the bar was seventeen-fifteenths "'cause there's fifteen on the bottom, yeah, and sevent[een]." Her comment may mean that Bridget knew, just as when she made seven-fifths on October 30th, that there were 17 parts in the new bar and each one was the same size as each of the 15 parts in the unit bar. Thus the bar could be called seventeen-fifteenths. Bridget may also have been thinking about the written notation for seventeen-fifteenths since she stated that fifteen was on the "bottom." Finally, her comment may indicate that Bridget was also constructing an iterative fraction scheme—after all, she stated confidently that the bar was seventeen-fifteenths *of the unit bar*, and she responded appropriately to probes about the size of the parts of the bar.

But based on this protocol I can't attribute an iterative fraction scheme to Bridget for two reasons. First, she did not independently produce the name of the bar as seventeen-fifteenths. (If she went on to do so in other situations, that might confirm an initial construction of an iterative fraction scheme here.) Making sense of Deborah's response—which could imply that Bridget coordinated three levels of units retrospectively—does not imply that she could coordinate three levels of units prior to operating. Second, she did not indicate any explicit relationship between the 17/15-bar and the whole. That is, just because she could see the bar as 17 1/15-parts did not mean that this quantity was a whole unit bar and two more fifteenths, as it was for Deborah. In other words, in the protocol I don't find corroboration that Bridget "kept track of" the whole in the process of explaining the bar to be seventeen-fifteenths, even though she said "of the unit bar." So I can't claim that she was coordinating three levels of units even retrospectively, which means I can't conclude, at this point, that she had constructed an iterative fraction scheme.

A Lacuna in Bridget's Fraction Scheme

Not taking the whole as a unit. Bridget's reliance on the "current whole" to identify bars longer *and shorter* than a unit bar persisted into the next teaching episode on November 11th. At the start of this episode, I asked Deborah to make a 21/21-bar and Bridget to use fiftieths to make a bar a little bit shorter. Bridget made a 50/50-bar, broke the bar, and took away two 1/50-parts. When I asked the length of her bar, she said "forty-eight forty-eighths." Rather than address her response directly, I asked Deborah to make a bar a little smaller than the 21/21-bar using thirty-fifths. Deborah made a 35/35-bar, and I asked how many thirty-fifths she had. Both girls responded "thirty-five." I asked how long Bridget's bar was before she took two parts away. "Fifty-fiftieths," both said. Then Deborah said, "So isn't it [B's bar] forty-eight fiftieths?" Bridget again readily agreed: "Oh yeah," she said, "'Cause I just took two off of a 50/50-bar."

Despite her facile correction and explanation, Bridget's activity here indicates that she was not taking the "whole" as a unit, which corroborates my conjecture after Protocol 6.6 that she was not yet coordinating three levels of units. Based on her activity on October 30th in taking three-fifths of the candy bar as three one-fifths (cf. Protocol 6.5) and four-fifths of David's money as four one-fifths (cf. Protocol 6.4), I infer that she had constructed a partitive fraction scheme with splitting. That is, for her, unit fractions were iterable within the whole, and she operated fluidly with two levels of units. But at this point, "fixing" the whole as a unit that could stand as a reference in relation to a unit fraction iterated a number of times seemed problematic for Bridget, even when the iterated unit fraction did not exceed the whole.

Similar to Carlos, Bridget was able to complete magic cake problems (cf. Task 5.14) with ease (and she did so later in the November 11th teaching episode). However, throughout the teaching experiment she continued to demonstrate a lacuna in her ways of operating with improper fractions. This lacuna seemed similar to Carlos's lacuna of not yet coordinating three levels of units, but its appearances were somewhat subtler. In general Bridget made small improper fractions, and reasoned reversibly with them, without incident. That is, unlike Carlos, she never partitioned a bar into sevenths if she was going to make seven-fifths, nor did she partition a bar into fifths if the bar was seven-fifths of a unit bar and she was going to make the unit bar. Yet, like Carlos, Bridget had difficulty making and reasoning with large improper fractions. As I have discussed in Chapter 5 with regard to Carlos, I conjecture that this difficulty stems from not coordinating three levels of units prior to operating, even though Bridget seemed to be able to make three levels of units in activity (cf. Protocols 6.3 and 6.4), and sometimes retrospectively in making sense of Deborah's responses.

A good example of Bridget's lacuna occurred in the February 9th teaching episode, when the girls and I were visualizing a long peppermint stick that was some number of centimeters in length. We called this length *P*. Then I told them that another stick was fifteen-fourths as long as the first peppermint stick. I asked them what they needed to do to the first stick to make the second one. Deborah said something about making *P* three times. Then Bridget said, "you got to keep going, you got to go fourteen more of the original one." Deborah disagreed, demonstrating that three peppermint sticks of length *P* and three-fourths of it more would make the other stick.

Bridget's notion of needing 14 more of *P* indicates she conflated "whole" lengths of *P* and fourths of *P* in making fifteen-fourths of *P*. She seemed to conceive of fifteen-fourths as 15 equal parts that could be created by iterating one part 15 times, but she did not form units of four of those parts as equal to *P* because I infer that she had not yet interiorized three levels of units. So she iterated *P* instead of one-fourth of *P*. In this sense she operated with two levels of units: Fifteen-fourths was a unit of 15 units but the relation to the whole was absent. Upon further probing, she certainly could say that three lengths of *P* would be twelve-fourths of *P*, and she seemed to understand Deborah's idea to add three more fourths of *P* to the twelve-fourths of *P* in order to make fifteen-fourths of *P*. But this ability to make sense of Deborah's response did not mean that Bridget could yet produce these ideas independently.

Bridget's experience of improper fractions. Another brief example may illuminate Bridget's experience of improper fractions as "odd" or "bothersome" numbers, even through May. At the start of the May 3rd teaching episode I posed the following problem:

Task 6.11: An apple costs 75 cents. How much does 5/3 of an apple cost?

"Oh my gosh!" Bridget exclaimed, with some annoyance in her voice, while Deborah said the problem was easy. Bridget then said "two and one-third?" as Deborah solved the problem (nearly instantly). Bridget seemed to be trying to make sense of five-thirds, while Deborah could take five-thirds as given and operate with it to solve the problem. So five-thirds did not seem to be a fractional number for Bridget, and she endeavored to interpret it as a mixed number. But it was not automatic for her to do so—she knew that five-thirds was more than one, but she did not know how much more. In contrast, Deborah did not need to think of five-thirds as a mixed number in her operative activity—five-thirds was one-third five times, and she could use that to swiftly determine that five-thirds of an apple cost five times 25 cents, or 1 dollar and 25 cents.

Activity with Small Versus Large Improper Fractions

Bridget's ways of operating with small improper fractions. As I have stated, Bridget's lacuna with improper fractions appeared to be somewhat subtler than Carlos's lacuna. For example, early in the November 11th teaching episode, Bridget compared 70/69 and 80/79. She determined that 70/69 was longer because "sixty-ninths are bigger than seventy-ninths, and you add one to both of them." She made this statement directly following a comparison that Deborah made of 21/20 and 20/19, in which Deborah said that 20/19 was bigger because 1/19 was bigger than 1/20. Thus Bridget may have cued off of Deborah's response, even though in solving her problem Bridget did not follow Deborah's response exactly. Furthermore, Bridget seemed to have established that 70/69 was made from sixty-ninths—70 of them—and was 1/69 more than a unit bar. Similarly, 80/79 was made from seventy-ninths—80 of them—and was 1/79 more than a unit bar. So, at least with small improper fractions, Bridget appeared to operate as if she had constructed an iterative fraction scheme, which was a difference between her and Carlos.

Deborah's reasoning to compare large improper fractions. However, later in the same episode (November 11th), Bridget did not operate in this "structural" way with large improper fractions, while Deborah did. Near the end of the episode I asked Deborah to use tenths to make a bar a little longer than 34/11. From a 4/11-bar already on the screen she pulled out a 1/11-part and repeated it to make 34 parts total. Then she made a 10/10-bar and aligned its left end with the left end of the 34/11-bar. She repeated the 10/10-bar to make three of them, and accidentally repeated the bar one more time—that is, she expressed dismay, indicating that she did not intend to repeat the 10/10-bar a fourth time. She broke the bar into parts, and by visual inspection erased parts until her tenths bar was a little longer than the 34/11-bar, (see Figure 6.15).

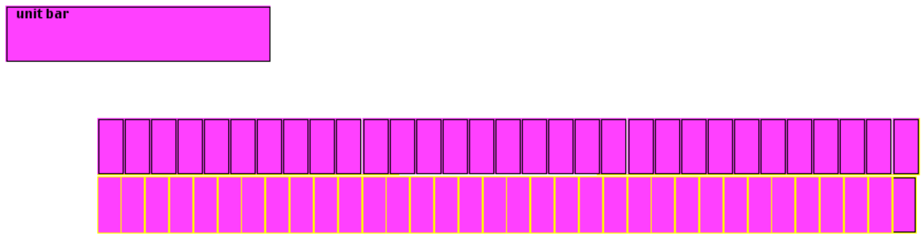

Figure 6.15, A bar made of tenths (above) that is a little longer than a 34/11-bar (below).

Determining the length of Deborah's bar in relation to the unit bar took some time. First Deborah said her bar was "three unit bars and a little smidge;" then Bridget said it was "three and one-fifth;" finally Deborah said it was "three and one-tenth and a little smidge," but she indicated that the smidge was part of the extra 1/10-part.

Protocol 6.7: Explaining why 31/10 is bigger than 34/11 on 11/11/03.

T: So why is thirty-one tenths bigger than thirty-four elevenths?
B: Because the tenths are smaller pieces and you have—
D: No the tenths are—[B looks at D]—bigger pieces!
B: *Bigger pieces, and then the other ones are smaller pieces, and then you have, however many are on the top in that one—wha—what?*
[D giggles.]
B [putting her hand to her forehead]: *I'm confused.*
D: No it's because—*it's because ten times three is thirty, and then plus another tenth, and then eleven times three is thirty-three, plus another eleventh, but tenths are bigger than elevenths, so it's a smidge more than the elevenths [pinches the air with two fingers]—that what the smidge is!* [Everyone laughs.]
[T asks B if D's explanation makes sense. Nodding, B says yes.]

Deborah's ways of *making* 34/11 and 31/10 did not indicate that she operated with large improper fractions similarly to Michael, because she iterated a single 1/11-part 34 times to make 34/11 and seemed to be working via visual inspection in making the tenths bar. (Even when she repeated the 10/10-bar to make three of them, she may have judged that she needed three bars by looking at the 34/11-bar, rather than by reasoning.) But her explanation of why 31/10 was bigger than 34/11 clearly demonstrates that she could coordinate unit bars and parts of unit bars in order to compare two large improper fractions. Her explanation opens the possibility that both ten-tenths and eleven-elevenths were iterable units for her, and that she could use these units to make (or mentally break apart) 31/10 and 34/11. So at this point I can conclude that Deborah could use her interiorized three levels of units, combined with iterable nonunit fractions, to reason through a comparison of large improper fractions. During the rest of our November episodes, Deborah only grew more efficient in reasoning with large improper fractions in this structural way.

Bridget's attempt to explain a comparison of large improper fractions. Bridget agreed with Deborah's explanation. But nothing in the interaction caused me to conjecture that she could operate in this way. So I asked her to use tenths to make a bar a little longer than 25/11. Unfortunately, this problem was poorly posed because Bridget could solve it simply by visual inspection, erasing parts from Deborah's 31/10-bar and 34/11-bar. In fact, both Deborah and I encouraged her to adapt Deborah's bars! Doing so precluded finding out to what extent Bridget might coordinate unit bars and parts of unit bars in making large improper fractions. When Bridget finished making her bar, she knew, by counting parts, that it was 23/10. I asked her to explain why 23/10 was bigger than 25/11, and Deborah seemed eager to explain. After I asked Deborah to wait, Bridget looked at the screen for about 14 seconds and then made an explanation that relied on tenths being bigger than elevenths. Covering up the twenty-third tenth in her bar with my thumb, I pointed out that *twenty-two* tenths would *not* be bigger than 25/11, and I wondered why *twenty-three* tenths was bigger.

Deborah became very excited, wanting to speak, waving her hand in the air. When I asked her to hang on, she put her hand down and fidgeted in her seat, smiling. During the next 10 seconds, Bridget did not offer any further explanation. Eventually Deborah made funny faces at the screen, as if she was going to burst. Both Bridget and I laughed, and I relented, allowing Deborah to explain. Her explanation was similar to the one she gave for 34/11 and 31/10 in Protocol 6.7. I tried to slow her down, checking with Bridget to see if Deborah's explanation made sense to her. Bridget said "yeah" repeatedly but was quiet and seemed a bit glum.

This situation confirms Deborah's solid iterative fraction scheme and Bridget's lack of one at this point, because Bridget's comparison of the two fractions relied solely on the size of the parts without a coordination of whole numbers of unit bars and parts of unit bars. (My interventions as a teacher were also inadequate to help her construct such a scheme, which I address later in this section when I discuss MCR with the girls.) Her attempted explanation here corroborates my hypothesis that she was not coordinating three levels of units prior to operating in these contexts, and that in particular, the unit she did not "include" was the whole. That is, she conceived of 25/11 as a unit of 25 units, each of which were elevenths, and similarly 23/10 was a unit of 23 units, each of which were tenths. But the significance of elevenths as one of eleven units that made up a whole (or tenths as one of ten units that made up a whole) appeared "lost."

Bridget's progress in making large improper fractions. However, Deborah's explanations may have had some effect on Bridget because at the start of the very next teaching episode on November 13th, I asked Bridget to make 47/15. She made a 15/15-bar, repeated the bar to make two bars, and noted that she had thirty fifteenths. Then she pressed REPEAT one more time to make three bars, which she said was forty-five fifteenths. Although at that point Deborah told Bridget that she only needed two (fifteenths) more, Bridget quite likely would have determined what to add on her own. This activity is significant because it indicates that she may have been beginning to conceive of fifteen-fifteenths as an iterable unit. It also highlights another difference between Carlos and Bridget, since fifteen-fifteenths was *not* an iterable unit for Carlos during the teaching experiment. Yet despite Bridget's progress in making large improper fractions in a more structural way in this episode, like Carlos she did not yet use this structure to make comparisons of the kind Deborah made in Protocol 6.7 with 34/11 and 31/10.

Therefore, in the next episode on November 18th, we worked directly on visualizing how to make large improper fractions. For example, I asked Bridget to imagine making 37/7 and tell how she'd do it. She explained how to make sevenths in a unit bar. So I asked her how many unit bars she would need, and she responded "five and two more," meaning five unit bars and two sevenths more. In retrospect, as I have discussed with Carlos, I am not convinced that such direct work contributes to the interiorization of three levels of units in the context of improper fractions. Since when requested, *both* Bridget and Carlos could break large improper fractions into a structure of unit bars and parts of unit bars, making three levels of units in activity may be sufficient to respond accurately to such questions. Doing so is not the same as initiating the use of this structure independently in problem situations, as both Deborah and Michael did, and may not contribute to inducing such activity.

MCR During Activity with Improper Fractions

The two strongest enactments of MCR during activity with improper fractions occurred in the first teaching episode on October 30th, when the witness-researcher assisted the girls in exploring seven-fifths (Task 6.9), and on November 6th, when I modulated tasks for the girls toward solving problems like using twelfths to make a bar a little bit longer than an 11/11-bar. On October 30th, the witness-researcher harmonized with an aspect of the girls' mathematical ways of operating (their partitive fraction schemes with splitting) in order to open possibilities for them to expand their ways of operating (construct seven-fifths as a number made of one-fifth seven times, cf. continuation of Protocol 6.5). His interaction with them seemed to have some impact, since both girls called the parts of their bars fifths instead of sevenths, and Deborah may have determined that both one and two-fifths *and* seven-fifths consist of seven one-fifths. In this sense the girls expanded their (mental) activity with fractions larger than one, and thus expanded an aspect of their mathematical realities. One reason the witness-researcher could successfully enact MCR with them at this point had to do with his work with other students who operated with improper fractions in ways that seemed similar to the girls' ways of operating early in the episode (cf. Protocol 6.5).

During the November 6th teaching episode, I also enacted MCR with the girls, but I did not have a breadth of experience to draw upon as the witness-researcher did. My MCR was not based on previous interactions with other students but on the few interactions I had already had with Deborah and Bridget and my own conceptual analysis of my goals for them. That is, I created a plan to work toward a task like using twelfths to make a bar a little longer than an 11/11-bar because I anticipated that such a task would be difficult (both because of its novelty and because it required iterating beyond the whole). My plan of backing up to make a bar a little bit shorter or a specific amount shorter (which built directly on their partitive fraction schemes), and then a bar a specific amount longer, seemed to allow them to approach the "target" task. Thus I was trying to anticipate their responses and work out some ways of adapting to them that would allow them to solve the problems without excessive coaching.

In both of these episodes, the girls were receptive to and operated with the problem situations. So they completed the caring relations with the witness-researcher and with me (they cared back). However, at this early stage in the teaching experiment it is likely that the novelty of the sessions, the fact that the witness-researcher and I were relative strangers to them, the chance of playing with the computer software, the opportunity to leave their classrooms for awhile, among other factors fostered their engagement as much as our modulation of questions and tasks.

Still, such modulation can't be underrated in enacting MCR. In the next episode on November 11th, I began to enact MCR *less* well with Bridget, in part because my questions and tasks were not sufficiently tuned to affective aspects of our interactions or to her ways of operating with large improper fractions. Just after Protocol 6.7, when Deborah had explained why 31/10 was bigger than 34/11, I asked Bridget if what Deborah had said made sense. This question was not appropriate here, nor over the several episodes that I continued to use it, because for several reasons Bridget would always reply that it did make sense.

First, during the November episodes Bridget and Deborah's interactions became somewhat competitive, and I infer that Bridget did not want to appear *not* to understand something Deborah had done. Second, Bridget may not have wanted to appear not understand in front of me and the witness-researchers. That is, she was likely implicitly concerned with our constructions of her (our internalized other of her), because her construction of our perceptions of her impacted her construction of herself as a social-mathematical being. At some level, she wanted our impressions of her to be strong, and it is likely that from her perspective, saying that Deborah's activity did not make sense would not contribute to that kind of an impression. Third, and perhaps most important, Bridget very often *could* make sense of what Deborah had done—she could assimilate Deborah's productions to her own ways of operating, and sometimes she could explain them (in fact, she worked hard to do so throughout the teaching experiment). But, as with Carlos's ability to make sense of Michael's mathematical activity, making sense of Deborah's activity did not mean that Bridget had herself constructed the same schemes and operations. So asking the question of "whether it made sense" curtailed my further understanding of *Bridget's* ways of operating so as to harmonize with them and to open new possibilities for her to expand those ways of operating.

In addition, after Deborah explained why 34/11 was bigger than 31/10, I was aware that Bridget needed an opportunity to work on a similar problem and produce such an explanation. As I have already stated, the problem I then posed to her was flawed in that she could operate solely with the visual material already on the screen to make both a 25/11-bar and the number of tenths that would be required to make a bar just longer than the 25/11-bar. Instead of making the bars right away, one technique I did not take advantage of enough at this point was asking the girls to predict how many unit bars they needed in order to make a particular improper fraction. Asking for such a prediction might have oriented Bridget to incorporate such "structural" thinking into her ways of operating, or at least might have helped her explain why 23/10 was bigger than 25/11 in this particular situation.

Instead, I tried to enact MCR for Bridget by keeping at bay Deborah's growing eagerness to speak in order to give Bridget time to think and respond. But I infer that the time was rather "empty" for Bridget. That is, probing Bridget's initial explanation and then making "wait time" for Bridget was not sufficient for her to elaborate her explanation, let alone to reason with unit bars and parts of unit bars. Bridget's quiet manner and glum look at the end of the episode indicate that she experienced some depletion during the interaction—that she felt low. In contrast, Deborah was clearly stimulated and excited. These problems were within her ZPC, and I seemed to be enacting MCR with her. Like Michael, Deborah found it relevant to reason with unit bars and parts of unit bars because she had interiorized three levels of units and because nonunit fractions seemed to be iterable for her beyond the whole.

At the start of the November 18th teaching episode, I enacted MCR a little better with Bridget in that I asked her to imagine how to make improper fractions using unit bars and fractions of unit bars. As I have stated, I am not sure that such "direct" work engenders the interiorization of three levels of units—it may only provide an orientation for action in the local context of that episode. Indeed, in terms of the entire experiment, our work during that episode did not seem to have much effect on Bridget's activity with improper fractions. Bridget's somewhat annoyed response in May to "five-thirds" in Task 6.11 indicates that she still experienced depletion in dealing with improper fractions. However, in November, asking her to imagine and describe how she would make large improper fractions was my adaptive response to my conception of her difficulties and, at that point, an example of enacting MCR with her.

New Uses of Multiplying Schemes: Constructive Resource 3, December and March

From the girls' activity with improper fractions in October and November, I conjecture that a central difference between the girls was their units-coordination prior to operating. Like Michael, Deborah coordinated three levels of units prior to operating, which allowed her to construct a strong iterative fraction scheme (strong because nonunit fractions seemed to be iterable for her). Somewhat like Carlos, Bridget seemed to make three levels of units *in activity* or retrospectively but coordinated only two levels of units prior to operating, and thus did not construct an iterative fraction scheme at this time. Coordinating three levels of units opened possibilities for Deborah to construct a co-measurement scheme similar to Michael's, which I have claimed to be an important resource in solving RMR problems. Not coordinating three levels of units prior to operating seemed to delay Bridget in making a similar construction.

Making a 4/4-bar into a 6/5-bar

On December 2nd, both girls used recursive partitioning to solve problems like making a 2/2-bar into a 7/6-bar without clearing the half mark. Then I posed the following problem, which met with a great deal of concern and surprise:

Task 6.12: Make a 4/4-bar into 6/5 of the unit bar, without clearing the fourths.

Protocol 6.8: Making a 4/4-bar into a 6/5-bar on 12/2/03.

D: What!? I don't *think* so!
T [to D]: You don't think so?!
B [staring intently at the 4/4-bar on the screen]: Six-fifths.
D: Can we clear it?
T: Nope! [D and T laugh.]
B: Wait a minute! *You couldn't divide them in twenty and then take 'em away, like, something, something like that*…Okay, you wanted six-fifths. *Well, you could pull out [one part] and make it a fifths bar instead of a fourths bar* [*smiling a little, as if she is kidding*. Everyone laughs.]
D: That'd still be fourths!
T: Yeah, we want it to be six-fifths of the unit bar.
….
D: *Oh! I got it—I think I got it*.
T [to D]: You think you've got an idea?
B: *I don't*.
[T tries to stall to give B more time, but *B sits back in her chair and shakes her head*.]
T: D, what's your idea?
D: *You divide it into five pieces in each fourth [points along the 4/4-bar] so that would make it twenty total*.
T: Oh wait, [to B] did you say something about twenty?
B: *Yeah I did, I said five and four is twenty*.
D: Yeah, so you split each of these pieces [the fourths] into five and then you'd take, like, *four of those twentieths would be a fifth*.
T: Oh!
B: *Exactly*. *I said take away four*.
D: So then you'd have to add, you'd have to add, wait.
B: *You'd find out how much is a fifth and then you'd make six of them*.
[D makes her idea by *partitioning each fourth into five parts*. She pulls out one part and repeats it to make four parts.]
B: *That's a fifth, right?*
T: Why is that a fifth?
B: Wait a minute.
D: Yeah, it's a fifth. [D joins the 4-part bar to the 20/20-bar.]
B: *Yeah, 'cause five and four is twenty and then you've made the whole bar*.

Deborah's initial modifications. In this situation, I claim that Deborah began to modify her partitioning operation in order to coordinate fourths and fifths in the same bar. Deborah's goal was to make the bar into fifths, and since I had banned clearing the bar, she could not "just" partition the bar into five equal parts. Her solution involved partitioning *each* of the fourths into five equal parts in order to partition the whole bar into five equal parts. So I infer that she modified her partitioning operation to include a distributive operation, although I don't claim that she was aware of the distributive pattern of her activity.

But I do claim that the modification of her partitioning operation was entwined with her construction of a new use her multiplying scheme in service of her iterative fraction scheme. That is, Deborah knew that inserting five parts into each fourth gave a total of 20 parts, which she could use to view the bar as a unit of two different units-of-units structures. Deborah's activity indicates that she viewed the bar as a unit consisting of four units each of which could be divided into five units (which certainly allowed her to show the fourths in the bar). However, she also viewed the resulting 20-part bar as a unit consisting of five units each of which would be partitioned into four units, which would allow her to identify one-fifth of the bar (four-twentieths), and thus also six-fifths (by pulling out and adding on four-twentieths). So I can also see commutativity in her use of her multiplying scheme to create these dual views of the bar: The 20-part bar was made from both four equal parts each containing five equal parts and five equal parts each containing four. Thus I claim that Deborah's solution of Task 6.12 marks her initial construction of a co-measurement scheme similar to Michael's scheme.[66]

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[66] Note that Task 6.12 was harder than the tasks of this nature that I posed to the boys (eg., Task 5.17) because the girls' task involved making improper fractions, as opposed to "just" making a 4/4-bar into a 5/5-bar. The central reason for this difference was that I posed these problems to the girls before I posed such problems to the boys, and I did not realize the complexity the improper fractions added until after I worked with the girls. With the boys, I revised the problems to omit this complexity (which was especially important for Carlos).

Bridget's idea of "twenty." Yet Bridget was the first to have an idea of "dividing in twenty," which indicates that multiplying four and five was relevant for her in the situation, and which may have helped Deborah formulate her solution. Because Bridget retreated from her idea, joking about pulling out one part from the 4/4-bar to make it "a fifths bar" (identified as fifths only because it would be five equal parts, not because it was five-fifths of the unit bar), she did not seem to know how to use "twenty" to solve the problem. Furthermore, when Deborah began to explain her solution, Bridget did not seem to know why *specifically* 20 parts would be useful: Even after Deborah said that four-twentieths would be one-fifth, Bridget said, "you'd find out how much is a fifth and then you'd make six of them." So at this point Bridget seemed uncertain of the number of twentieths that would constitute one-fifth. I infer that she had not yet constructed a way to shift flexibly between two three-levels-of-units views of the bar. This lack of construction is not a surprise given my hypothesis that Bridget had not yet interiorized three levels of units even though she could use them in activity or retrospectively.

Interestingly, when Deborah first explained her solution, Bridget agreed with it by saying, "Exactly. I said take away four." If "take away four" referred to pulling out four parts and joining them onto the bar, then her comment may indicate she knew the number of twentieths necessary to make one-fifth. But during the remainder of the protocol Bridget seemed uncertain about the matter. It's possible that she was waiting for corroboration from Deborah. So when Deborah confirmed that the 4-part bar was one-fifth, Bridget attempted an explanation: "'Cause five and four is twenty and then you've made the whole bar." I infer that she knew that five groups of four parts each would "make the whole bar," so one group of four parts would be one-fifth of the bar. This way of operating corroborates that Bridget could use her multiplying scheme to justify Deborah's solution and could coordinate three levels of units in activity. At least retrospectively, Bridget may have conceived of the bar as a unit of five units each containing four units. However, similar to her ways of operating with large improper fractions, she seemed to be at her edge in coordinating three levels of units in one "view" of the bar. Coordinating *two different* three-levels-of-units views of the bar *in advance of operating* seemed just beyond her current schemes.

Making a 3/3-bar into a 9/8-bar

Deborah's accommodation in her multiplying and fraction schemes. To test whether either girl had modified her partitioning operation in solving Task 6.12, I posed this task:

Task 6.13: Make a 3/3-bar into a 9/8-bar without clearing the thirds.

Deborah's immediate response was, "Oh man, you're trying to hurt our brains!" We all laughed, but I infer that her comment was significant: These problems were challenging for her and she had not yet constructed a scheme to solve them. Bridget held her fingers to her temple, closed her eyes, and said something about getting a spark. Five seconds later she noted, "We can do the same thing we did up there," (in solving Task 6.12). Then she said, "I know three goes into nine, and nine's like part of the problem, but that don't mean anything really." So Bridget also did not have a way of operating ready-at-hand to solve these problems.

Protocol 6.9: Making a 3/3-bar into a 9/8-bar on 12/2/03.

D: *What's sixteen divided by three?*
B: Eight.
T: I don't know; what is it?
B: Wait a minute, no it's not eight.
D: *It's not even is it. I mean, it's not a whole number.*
B: No.
[Fifteen seconds elapse. *B sits back, occasionally murmuring about the problem.*]
D: *Oh I got it, I got it.*
[W intervenes to ask B how they did the last problem. B doesn't remember, so T drags down the 6/5-bar from Task 6.12 and asks D if she remembers what she did.]
D: *Yeah, but it's not the same.*

[At T's request, D repeats what she did to solve Task 6.12. T asks B if that helps and B says no. D repeats that she's got the (new) problem, and *B shakes her head no, eyebrows raised, to indicate she doesn't have an idea*.]

B: *You should get me in the afternoon*.

[W recommends that they take a little more time to think it out. B tries again, but does not know what to do. W and T say that they can work on it next time. But T invites D to tell how she'd start the problem.]

D: Okay, *you divide each of these thirds into eight*.

B: *That's exactly what I was thinking about. I was going to say something about twenty-four*, but I got…

[D nods at B. T suggests that B make it. B partitions each third into eight parts.]

T: Now how does that help us?

B: That's twenty-fourths. And then, nine-eighths.

D: *Three of these little twenty-fourths [points to the bar] equal one-eighth*.

B: Yeah, so you would—like, *if you could clear it, you could make, um, eight more, so you'd have nine-eighths*.

D: *Actually, you just take three of those twenty-fourths*.

T: Well let's see, so where are the eighths? [Both girls stare intently at the screen.]

D: The eighths are three twenty-fourths [points to the first three parts of the 24/24-bar. The girls identify the first few eighths by pointing.]

T: *How many eighths do we have here?*

B: *Twenty-four—wait a minute, eight*.

D: Eight.

[T encourages B to finish making nine-eighths of the unit bar. B pulls out one twenty-fourth and repeats to make three of them.]

T: How do you know that [is one-eighth]?

B: *Because eight and three is twenty-four, and then three is in one bar*.

D [simultaneous with B]: They're twenty-fourths.

Deborah's question about dividing 16 by 3 was revealing. From it I infer that she visualized the bar as sixteen-sixteenths by partitioning each eighth into two equal parts. She wanted to see if sixteen-sixteenths would allow her to show the three-thirds with which we had started. But it didn't because, as she said, "it's [16 divided by 3 is] not a whole number." So she mentally partitioned the eighths into thirds to make twenty-fourths, and that *did* allow her to show the original three-thirds (i.e., 24 divided by 3 *is* a whole number). From my point of view, she used her partitioning operation distributively, inserting three units into each of the eight units of the 8/8-bar in order to get a total number of parts divisible by three, which in turn allowed her to view the bar as a unit of three units each containing eight units. So in effect she partitioned each eighth into three equal parts in order to partition the whole bar into three equal parts. In doing so, Deborah demonstrated the construction of a co-measurement scheme that I claim she had begun to make in solving Task 6.12. That is, she embedded her multiplying scheme into her iterative fraction scheme, a functional accommodation in the schemes that allowed her to show two fractions in the same bar.

It's not unusual that Deborah first thought about dividing each eighth into two equal parts in the process of making this accommodation—both Michael and Carlos also divided fractional parts into two equal parts in related contexts. But it *is* unusual that she took the target bar, nine-eighths (which she had to imagine), and tried to alter it so as to make the original bar, three-thirds. My current explanation of her ability to do so is as follows: Deborah's ways of operating in this situation relied more obviously on a scheme for making commensurate fractions than the boys' ways of operating in similar situations. Deborah could swiftly transform 9/8 into 18/16 or 27/24, at least numerically. So, since she could imagine these fractions as "the same" as nine-eighths, she could use them to work "backwards" to show the original three-thirds.

This explanation is strengthened by her comment that this problem was "not the same" as the previous problem, Task 6.12. I infer that she saw the problems as different because she solved them in different ways. In Task 6.12, she may not have needed to *search* for a total number of parts with which to relate four and five: Using 20 parts may have come almost automatically to her, or she may have cued off of Bridget's suggestion to divide into 20. Once Deborah mentally made 20 total parts in the bar, again she may have had the experience of knowing that four-twentieths was one-fifth based on her knowledge of commensurate fractions. In subsequent episodes, Deborah solved problems like Tasks 6.12 and 6.13 by searching for a total number of parts that could be partitioned to show both fractions, so her construction of a co-measurement scheme seemed relatively permanent.

Bridget's lack of a way to operate. In contrast with her partner, Bridget's way of operating did not change much from Protocol 6.8 to Protocol 6.9, and she did not seem to embed her multiplying scheme into her fraction scheme. After Deborah said she'd divide each third into eight parts, Bridget agreed or recognized that making 24 parts was relevant in the situation (which shows Bridget's use of her multiplying scheme). But unlike in Protocol 6.8, in Protocol 6.9 she did not verbalize this idea, so if she thought about it prior to Deborah's statement, she again did not seem to know how to use the idea to solve the problem. Instead, she communicated her discomfort over not having an idea when Deborah did by recommending that I work with her in the afternoon (the implication being that she was more awake and alert then).

Bridget's activity after she made the 24 parts corroborates that she did not know exactly how to use them to make nine-eighths. That is, she said, "if you could clear it, you could make, um, eight more, so you'd have nine-eighths." This comment demonstrates her lack of coordination of three levels of units prior to operating. At this moment, nine-eighths was nine equal parts, regardless of their relationship to the unit bar (i.e., in this case she communicated an intention to iterate the entire unit bar nine times). In November she had appeared to make small improper fractions like nine-eighths without conflation. But in those cases, the eighths were "singular" units, while in Task 6.13 the eighths themselves were units of units. I infer that under the cognitive load of this units-coordination, combined with some emotional stress from the interaction, she dropped back to coordinating only two levels of units.

Nevertheless, Bridget *could* make sense of Deborah's explanation that three twenty-fourths made one-eighth. But her justification for why, "Because eight and three is twenty-four, and then three is in one bar," is a bit ambiguous, in part because she dropped any fraction language that might have clarified her statement. Since she responded correctly to my question about the number of eighths in the 24/24-bar, I infer that in her explanation she meant that three *twenty-fourths* are in one *1/8*-bar, since eight times three *twenty-fourths* is a "whole" 24/24-bar. So her activity at the end of the protocol corroborates my claim that Bridget could make three levels of units in activity or in retrospect. However, I cannot conclude that she had made an accommodation similar to Deborah's, let alone constructed a co-measurement scheme.

Bridget's Continued Attempts to Operate

Partitioning into more parts. Unlike Carlos who liked to experiment by partitioning into more parts to solve problems that involved making two fractions in the same bar, Bridget tended be more reluctant to act. One reason for this difference may have been Bridget's reticence to try out in front of Deborah an idea that might not work. As discussed in the MCR part of the last section, Bridget may have disliked being seen *not* to understand when she believed Deborah always understood. Carlos seemed to believe similarly in Michael's understanding, but perhaps because they were closer friends and had developed stronger trust, Carlos seemed more willing to try out his ideas in front of Michael.

Another reason for this difference between Bridget and Carlos was that the problems I posed to Bridget involved improper fractions—which made them significantly harder for her than the problems I posed for Carlos. The improper fraction aspect of the problems may have severely dampened her activity. For example, in the next episodes on December 4th I asked Bridget to make a 4/4-bar into an 8/7-bar. Bridget said good-naturedly "Man, you always give us these problems!" Deborah, despite expressing excitedly that she knew how to do the problem, agreed with Bridget and accused me of trying to hurt their brains. I asked Deborah to wait, making time for Bridget to think. After about 30 seconds Bridget said she had "a little bit of something" and suggested making eight parts total, indicating that she could do so by partitioning each fourth into two equal parts. I reminded her that we were trying to make eight-*sevenths*. My intervention likely interfered with any focus she may have had on making sevenths out of the 4/4-bar. It also indicates a significant missed opportunity on my part to build on her idea, which I discuss in the MCR part of this section.

Explaining Deborah's activity. In the next teaching episode on December 9th I asked the girls to make an 8/8-bar into a 13/12-bar—a potentially (though not necessarily) more difficult task than previous ones since eight and twelve are not relatively prime. After about 30 seconds of silence, Deborah had an idea to solve the problem that was similar to how she solved Task 6.13 (cf. Protocol 6.9): divide each of the eighths into three parts to make 24 parts total, which would allow her to show both eighths and twelfths.

Bridget sat back with her head resting on her hand while Deborah explained her solution. I was only growing more concerned about Bridget's inability to operate in these situations, but I myself didn't have a way of operating that might have engendered her autonomous activity. I tried to involve her by asking her why Deborah's solution would work. Bridget's initial response was both vague and glum. However, when I asked Deborah why she had made 24 parts, Bridget seemed to make more sense of Deborah's activity. To Deborah I said, "So you were looking for something that…", and Bridget finished my sentence by saying "was divisible by both [8 and 12]." So she could at least impute meaning to Deborah's aim to make 24 parts, although that does not imply that she had constructed two different three-levels-of-units views of the bar (i.e., as a unit of 8 units each containing 3 units and as a unit of 12 units each containing 2 units).

Posing a problem such as making a 2/2-bar into a 3/3-bar at this point might have opened the way for Bridget to construct a co-measurement scheme—or at least to solve a problem of her own! But instead I asked the girls if there was *another way* to make an 8/8-bar into a 13/12-bar. Deborah suggested using 48 parts total, and so we worked on how many of those parts would need to be added to make thirteen-twelfths. This activity was not very stimulating for either girl, and we seemed to reach a standstill (which I describe further when I discuss MCR in this section). Bridget said she was sleepy. I was at a loss, hesitating, trying to figure out what to do to help Bridget become more operational. Finally Deborah asked me, pointedly and a little suspiciously: Do *you* know [how many forty-eighths to add on]? She stated that she knew.

Bridget's Construction of a Co-Measurement Scheme

Making halves and thirds in the same bar. Bridget did not coordinate two fractions in the same bar using her multiplying scheme until we were well into working on fraction composition problems, near the end of March. At that time, I did not realize the significance of constructing dual three-levels-of-units views of a quantity, and it was a happy accident that in an effort to work on the girls' distributive operations, I posed this problem during the March 24th episode:

> Task 6.14: I need 1/3 of a yard of ribbon from a yard of ribbon. Can you make my piece and cut it off (since I will use it up)? Now, my friend needs 1/2 of a yard of ribbon. Can you make her piece from what's left?

Deborah began to solve the problem by pulling out a 1/3-yard part from the 2/3-yard bar and repeating it to make three-thirds of a yard. I anticipated that she was going to clear the 3/3-yard bar and partition it into two equal parts, so I emphasized that she could only use what was left (the 2/3-yard bar) to make the half of a yard.

Protocol 6.10: Bridget making a 1/2-yard bar from a 2/3-yard bar on 3/24/04.

D: But I can't!
B: *I can do it! I can do it!*
[T re-states that they can only use the 2/3-yard bar to make what the friend needs.]
D: That's not fair.
B: *I can do it!*
T: You can do it? Okay, try.
B [taking the mouse from D and erasing the 3/3-yard bar]: *Okay, couldn't you, um, divide that one* [*referring to the 2/3-yard bar, possibly to the second one-third of it*] *in half*?
T: Okay.
D: No. It's divided in half right there [points to the mark that divides the 2/3-yard bar into two equal parts].
B: That's just one-third of the whole thing. [*B partitions the second one-third into two equal parts*.] *You divide that in half and then his half would be…*[*B colors the first 1/3-yard and the first half of the second 1/3-yard,* see Figure 6.16.] That's half.

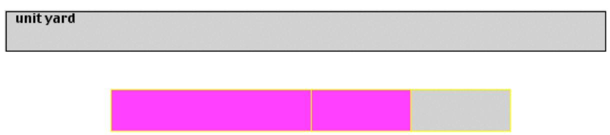

Figure 6.16, One-third and half of 1/3 of a yard identified as 1/2 of a yard.

D: Are you <u>sure</u>?
T: How'd you know that?
B: *'Cause you know if that's two pieces then—*
D: *Oh I know how she knew!*
T: Hang on, hang on, hang on. [T puts her hand on D's arm.]
B: *'Cause you have three total pieces and you divide each piece in two,* you would have four. And each piece in two would be eight. You divide eight by two and that'd be four.
[Some discussion ensues about whether there would be eight or six pieces after dividing each of three parts into two pieces. Both girls say there would be six pieces.]
B: *Then half of six is three, so you would count—there'd be two there [points to the first one-third that is colored blue]. Look [she partitions the first 1/3-yard into two parts], there'd be half there—that'd be three. And if you pulled out and repeated [one part] that would be the whole bar.*
[D announces that she knew that too. T asks B why she divided each third into two.]
B: Yeah, *'cause you wanted to make it even so you divide it in half.*

This protocol demonstrates that Bridget made halves and thirds in the same bar. Initially she may have thought only of dividing the second 1/3-yard part into two equal parts in order to make 1/2 yard from the 2/3-yard bar. But to justify why her action would allow her to identify 1/2 yard, she explained that she was really dividing each 1/3-yard part (including the imaginary one-third that had been erased) into two equal parts for a total of six parts. In this sense she independently and creatively partitioned distributively, partitioning each third into two equal parts in order to partition the whole bar into two equal parts. She had distributed parts of bars on January 14th in solving the Candy Bar Problem (Task 6.4), but up to this point she had not used distributive partitioning with a goal of showing two fractions in the same bar. So I infer that here she made an initial modification of her partitioning operation to include a distributive operation.

I also infer that distributive partitioning opened a new possibility for the use of her multiplying scheme in service of her strong partitive fraction scheme: Partitioning each of the three 1/3-yard parts into two equal parts gave her six parts, which could be divided by two to find the number of parts that would make up each 1/2-yard part.[67] Her final comment that "you wanted to make it even so you divide it in half" may corroborate that she had formed a goal to make three equal parts (the thirds) into an even number of parts, since an even number of parts could be divided by two. (It may also indicate that she was trying to make two "even" parts from the entire yard, which was why she initially divided only the second 1/3-yard part into two parts.) Calling upon her multiplying scheme *may* indicate that Bridget viewed the unit yard as a unit of three units, each of which contained two units, *and* as a unit two units, each of which contained three units. If so, her activity here would indicate that Bridget had begun to operate with three levels of units in fractional contexts—a significant advance.

Bridget's accommodation in her multiplying and fraction schemes. In the next teaching episode on March 29th, Bridget demonstrated that she was constructing a scheme more general

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[67] Bridget's discussion of four and eight parts may have been triggered by the four small parts in her visual field. I infer that her intentions were to explain the total number of parts created from partitioning each of three thirds into two equal parts, and then taking half of all of those. So I view her talk of four and eight as a minor conflation.

than coordinating halves and thirds in the same bar. In this episode I continued to pose problems like Task 6.14:

Task 6.15: There's a chocolate bar a yard long! I eat 1/4 of it; make my piece and cut it off. The witness-researcher wants 1/5 of a yard; can you make his piece from what's left?

Bridget made a 3/4-yard bar and announced that she knew how to solve the problem. Deborah said that she did as well, and I asked Bridget to go ahead. Bridget partitioned each of the three fourths into five parts. She noted, "if you had another one [fourth] you'd have twenty pieces in all." Then she said that one-fifth would be three parts. I asked her to pull out the one-fifth.

Protocol 6.11: Bridget making a 1/5-yard bar from a 3/4-yard bar on 3/29/04.

B: *Pull three [parts] out?*
T: *However many you need.*
B: *Yeah I need three, right?*
D: *Yep.*
[B pulls out one part and repeats it to make three parts.]
T: Okay, is that a fifth of the whole yard?
B: Yeah—no. *Wait a minute. Of the whole yard?*
T: The whole yard [traces finger along the unit yard]. He wants a fifth of the whole yard.
D: *You need another one.* [D reaches for the mouse and uses REPEAT to make one more part for a total of four parts.]
B: *You need another one because I forgot twenty. One-fifth of twenty.*
[T asks D how she knew what to do to solve the problem.]
D: Five times four equals twenty.
T: But how'd you know to use that—how'd you know to put five in each one?
D: Because he wants fifths. And it was divided into four.
B: *You need to get a number that's divisible by four and the five because you have four, like, pieces.*

This protocol shows Bridget using her multiplying scheme to determine the total number of parts she needed in order to identify two fractions in the same bar—as she explained, the total number of parts had to be divisible by both of the number of parts she was coordinating, in this case four and five. Making 20 parts in the bar by partitioning each fourth into five parts certainly allowed her to show fourths, so I infer that she viewed the 20-part bar as a unit of four units each of which could contain five units. In addition, she anticipated that 20 parts would allow her to show fifths, because she could conceive of 20 parts as a unit of five units each of which would have to contain four units. Thus her activity in this protocol confirms that in the previous episode (cf. Protocol 6.11) Bridget had begun to make a functional accommodation in her strong partitive fraction scheme and her multiplying scheme, using her multiplying scheme within her fraction scheme in order to produce two fractions in the same bar, at least in these contexts.

Yet Bridget initially "lost" the unit bar when finding one-fifth of the yard. Although she intended to use 20 parts, she found one-fifth of the 15 parts in her visual field. In part this conflation may have been due to her orientation after a month of activity with fraction composition problems (i.e., the girls had been working on problems like finding one-fifth of three-fourths of a yard.) But in part this conflation was characteristic of her tendency (still) to drop one level of unit in operating—in particular, to drop the "whole." It seemed as if the 3/4-yard bar temporarily became the central unit under consideration—Bridget's "entire universe" in the problem—rather than a unit within the unit bar. Since this bar consisted of 15 parts, it was easy to find one-fifth of it because five groups of three parts made 15. However, she recognized that she had "forgotten" the 20 parts and had intended to take one-fifth of 20. And since throughout the protocol, she seemed confident that using 20 parts allowed her to identify both the original fourths and the 1/5 yard, I can still conclude that Bridget was constructing a co-measurement scheme to show two fractions in the same bar by coordinating two three-levels-of-units views of the bar. The implications of this construction for Bridget's coordination of three levels of units prior to operating was as yet unknown.

MCR in Engendering New Uses of Multiplying Schemes

Enacting MCR well with the girls while we worked on problems to engender new uses of their multiplying schemes was challenging because I found it difficult to harmonize well with the girls' current schemes. One of my biggest mistakes in posing problems was to involve improper fractions. That is, I could have asked them to make a 4/4-bar into a 5/5-bar instead of a 6/5-bar (cf. Task 6.12). As I have already noted, this modification would have allowed Bridget to focus on coordinating the two fractions without the considerable "distraction" of making an improper fraction. Although Task 6.12 was within Deborah's ZPC, I believe she too could have benefited from first engaging in these modified problems, because she might have experienced a stronger conceptual build. At least I used my experiences in December with the girls to adapt the problems for the boys in January—that in itself is an example of mathematical care for the boys.

Weakened MCR with Bridget in December. My interactions with Bridget in December are a good example of insufficient decentering to "see" *her* ways of operating, which led to increasing difficulties both cognitively and affectively for her—and for me—and thus weakened our MCR. On December 2nd, when she was the first to mention "twenty" in solving Task 6.12 (cf. Protocol 6.8), I made a significant assumption. I assumed that Bridget's ways of operating in these situations were not much different than Deborah's but that Bridget didn't wasn't confident in her ideas. As a result, as shown in Protocol 6.9 when the girls were solving Task 6.13, I acted as if Bridget just needed more time to think of an idea. Furthermore, I encouraged her to enact Deborah's idea, which made sense given my assumption. But doing so did not help Bridget construct a scheme so that she could operate independently from Deborah, based on her own logical necessity. In this sense, I acted similarly with Bridget as I did with Carlos when I prompted him to enact Michael's idea in making a bar such that an 11-centimeter bar was three times longer than it (cf. Task 5.18).

At the start of the next teaching episode on December 4th, I spoke to the girls about not needing to find the answer to problems immediately, noting that it was okay to take time to think. I was trying to set a more relaxed norm because Deborah's rapid and excited manner when she solved problems seemed like a pressure on both of them to work swiftly. I was also trying to communicate that it was okay not to know what to do right away. My talk was certainly well-intentioned but largely ineffective. I was soon to understand that Deborah's need to know quickly and first was quite trenchant—a few minutes of talk from me was not going to change that. Furthermore, she *did* often know how to solve a problem quickly: It was stimulating for her, and it was important for her to express that excitement. For Bridget, my talk likely had little impact because she was still faced with the discomfort of not having a way to operate in these situations when Deborah clearly did. That is, just because I told Bridget that it was okay not to know what to do did not help her construct a way of operating!

In fact, my talk may have very well been for myself, because *I* clearly did not know what to do—what kinds of tasks to pose—that might open further possibilities for Bridget. During the teaching episodes on December 4th and 9th, I worked to give Bridget more time—to protect her from my perception of the pressure she felt from Deborah's rapidity and certainty. But I continued to pose the same kinds of questions (e.g., make a 4/4-bar into an 8/7-bar), which did not facilitate Bridget's ability to operate (let alone to make an accommodation in her multiplying and fraction schemes). I also did not build on her idea of partitioning the 4/4-bar to make an 8/8-bar. In general, I was accepting but not adaptive: I accepted (and empathized with the fact) that Bridget found these problems difficult and thus had different mathematical needs than Deborah seemed to have. But I did not (know how to) adapt our mathematical interaction to meet those mathematical needs so that Bridget might have become more operational in these situations.

Bridget's responses in December. Bridget's responses to my weak mathematical care for her over these episodes show well the effects of deteriorating MCR. At the beginning of our work on problems like Task 6.12, Bridget demonstrated that she was trying and thinking hard about the situations. For example, on December 2nd Bridget offered her idea of "twenty" even though she clearly wasn't sure about it (cf. Protocol 6.8), and she engaged in Deborah's solution by agreeing emphatically with it ("Exactly," she said, "I said take away four"). During the next (December 4th) teaching episode Bridget "complained" but in a good-natured way about making a 4/4-bar into an 8/7-bar by saying, "Man, you always give us these problems." Then she searched for a way to solve the problem and offered a "little" idea that was quite astute (partitioning each of the fourths into two parts to make eight parts total), given that she had not constructed a way of operating to solve this kind of problem.

Throughout our interaction, however, Bridget's level of depletion appeared to increase. She seemed disturbed over not solving the problems when Deborah did and began to cover up her lack of solution ideas. For example, in Protocol 6.9, when she admitted that she didn't know how to solve the problem, she mentioned that I should work with her in the afternoon (when she was more awake and alert). In the same protocol, once Deborah mentioned partitioning each third into eight parts, Bridget emphasized that that was *exactly* what she was thinking. Since she did not seem to know how 24 parts might help her to make nine-eighths, her emphasis on "exactly" seemed to be more about attempting to stand on par with Deborah than about knowing how to solve the problem. So by December 9th, Bridget's body language (sitting back, head resting on hand) is not very surprising! Her manner seemed to indicate that she was tired of being asked to do these problems that she couldn't (yet) do or understand, particularly when Deborah seemed to do and understand them so swiftly. I infer that she was experiencing depletion that outweighed stimulation.

Deborah's responses in December. During our mathematical interactions in December, Deborah seemed to experience the problems I posed as quite challenging since she repeatedly asked if I was "trying to hurt" their brains. However, when she figured out how to solve them she grew quite excited. Her excitement seemed to be part of her personality—she easily got excited about knowing or figuring out something, perhaps in part because it confirmed her self-concept as a mathematical knower. But her excitement also seemed to occur because solving these *particular* problems was enjoyable or satisfying to her. Unlike Michael, she did not state that these problems were "fun." Yet she communicated a sense of fun. So she seemed to derive stimulation from our activity, and thus I seemed to be enacting MCR with Deborah.

At the same time, my relations with Deborah were also affected by my deteriorating MCR with Bridget, which was particularly apparent during the December 9th teaching episode. As I have described, I was in a state of perturbation with Bridget. While we worked on how to make the 8/8-bar into a 13/12-bar using forty-eighths, I was at a loss, searching for a way to better interact with Bridget. I infer that Deborah began to wonder why I repeatedly asked her to explain her reasoning, and why I seemed hesitant. The energy of our interaction seemed to dissipate because of my uncertainty and Bridget's feelings of depletion. So when Deborah asked me, "Do *you* know?", I infer that she interpreted my hesitation as not knowing how to solve the problem. In that moment my construction of Deborah's perception of me altered: I believe I lost some of Deborah's respect or trust. She seemed suspicious of me, as though she thought that she was the only one of the three of us who knew what was going on. So I infer that during that episode her internalized other—her construction of me as a teacher—changed. This change led to further complications in our enactment of MCR during the rest of the teaching experiment.

MCR with Bridget in March. In fact, my perturbation with Bridget over coordinating two fractions in the same bar was resolved in late March, but not through my plans or intentions. By posing problems like Tasks 6.14 and 6.15 without the intention of (or initial recognition of!) bringing forth an accommodation in her multiplying and fraction schemes, it may seem that I enacted MCR for Bridget in an "accidental" way. Yet I contend it was more intuitive than accidental: I saw some need for these problems in the context of fraction composition activity, and I was not wrong about their potential usefulness for Bridget. Locally, as shown in both Protocols 6.10 and 6.11, I enacted MCR intentionally for Bridget in making room for her to show and articulate her ways of operating to solve the problems (which meant restraining Deborah at times, cf. Protocol 6.11). "Globally" in terms of the progress of the experiment, I was as yet unaware of enacting stronger MCR with Bridget in March in a way that was related to our weakened MCR in December.

In response, Bridget seemed quite stimulated, as shown by her confident manner in announcing that she could solve both Tasks 6.14 and 6.15. Bridget also persisted in her solution to Task 6.14 despite Deborah's skeptical reception of it during Protocol 6.10 ("Are you *sure*?" Deborah asked, in a manner more suspicious than curious.) Furthermore, on March 29th, directly following Bridget's solution of Task 6.15 in Protocol 6.11, she commented that these problems were "easy." At some level (likely outside of her awareness), it was probably a cognitive and emotional relief for her to solve problems that were fundamental to much of our mathematical activity during the teaching experiment—to potentially make some progress toward using and perhaps interiorizing three levels of units.

Fraction Composition Activity: Constructive Resource 4, March

Unlike in the boys' case study, the story of the girls' solutions of RMR problems cannot end here because the first three constructive resources are not sufficient to account for their ways of operating with RMR problems in May. Their story continues because of my over-arching goal for them to solve RMR problems of Type 5 by the end of the teaching experiment—problems like the Box Problem (Task 6.3).[68] Recall that when working on that problem during the February 18th teaching episode, Bridget had difficulty determining half of 3/4 decameter, and Deborah seemed almost entirely blocked from operating with the known fractional quantity and fractional relationship. This episode was the last of seven episodes from mid-January to mid-February during which we had worked with paper and pencil outside of the computer context, notating and operating on unknown quantities. After my work with the girls on the Box Problem, the witness-researchers and I hypothesized that some of the girls' operations with quantities had become suppressed. So, during the next teaching episode on March 1st we returned to JavaBars to work on fraction composition problems. Making fraction compositions seemed necessary for solving problems like the Box Problem and an opportunity to re-engage the girls operatively.

Operations Suppressed

At the beginning of the March 1st episode I posed this problem:

> Task 6.16: Joanne ran 5/8 of a mile. Her sister ran half of that. Make how far her sister ran and tell how far it is.

After making a 5/8-mile bar, both girls had ideas about how to make the sister's distance. Deborah wanted to partition the middle eighth into two equal parts. Bridget wanted to clear the 5/8-mile bar in order to partition it into two parts, and she did so. However, neither girl knew

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[68] I had this goal for all four students, but the girls progressed more quickly than the boys, perhaps because Carlos's lacuna with improper fractions was somewhat more trenchant than Bridget's. I also may have pushed the girls harder because of my perception of cognitive differences between the pairs in their selection interviews.

how to determine the length of the new bar. Eventually Deborah said it was "two and a half eighths," which she then converted to five-sixteenths numerically, by multiplying two and one-half as well as eight by two. Bridget said it looked like one-third; she copied the unit mile and partitioned it into three parts to compare, verifying that the sister's distance was close to one-third of a mile.

Their solutions to Task 6.16 confirm my inference that some of their operations were suppressed after the seven episodes without the computer. While Deborah's solution was not wrong, it did not incorporate operating on the quantities in the microworld. When I asked Deborah if she could show on the computer why two and one-half eighths was the same as five-sixteenths, I was quite surprised that she did *not* partition each eighth into two parts and use her recursive partitioning operation. Bridget's way of operating was less surprising: She used her partitive fraction scheme to make her solution, perhaps because she had not yet constructed distributive partitioning. (Recall that Task 6.16 took place *prior to* her work at the *end* of March on Tasks 6.14 and 6.15). However, she too had entered the teaching experiment able to make compositions of unit fractions using recursive partitioning—i.e., she could find, say, how much one-half of one-eighth of a quantity was of the whole quantity. During the March 1st episode eventually both girls reactivated recursive partitioning and made compositions of unit fractions.

The Nature of the Girls' Fraction Composition Activity

Bridget's coordination of whole numbers of parts. In the first two teaching episodes concerned with fraction composition activity (March 1st and 3rd), Bridget tended to clear the bar and partition it into the requisite number of parts (as she did in Task 6.16). Her need to do so corroborates that she had not yet modified her partitioning operation to include a distributive operation. (I contend that she did not do so until the end of March, as discussed in the previous section.) However, she began to see that clearing the bar had significant limitations in that it did not allow her to easily determine the measure of the composition. So she began to abandon clearing and partitioning in favor of coordinating whole numbers of parts of bars. An accompanying characteristic of her way of operating was a "loss" of the whole (the unit bar) as a unit of reference when determining the result of the composition.

Her activity in solving a problem near the end of the March 3rd episode exemplifies this characteristic. During the episode we had worked on problems similar to Tasks 6.16, where the composition involved taking a unit fraction of a proper fraction. Then I posed this task:

> Task 6.17: I need 3/4 of a yard of ribbon to tie up a package. My sister needs 2/5 of my piece. Can you make it and tell how much she needs?

Bridget said the problem was easy. She copied the unit yard and colored three-fourths of it blue. Then she expressed an idea about "dividing into fifteen." After some discussion with Deborah and me, she pulled out a 1/4-yard part, used it to make a 3/4-yard bar, and partitioned each fourth into five parts (see Figure 6.15).

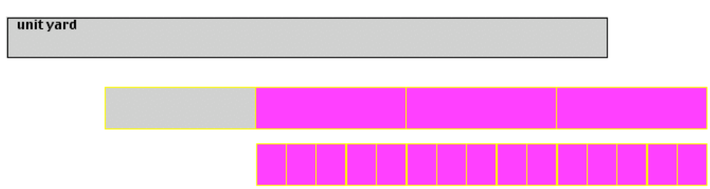

Figure 6.15, Each fourth of three-fourths of a unit yard partitioned into five equal parts.

Protocol 6.12: Bridget's coordination of whole numbers of parts on 3/3/04.

B: And then, that would be fifteen, three, she'd have six.
T: Six! Six what?
B: Six—[pauses and laughs] six-fifteenths.
D [almost simultaneously with B]: Six-twentieths.
B: Or twentieths or something.
D: Six-twentieths of the whole bar, six-fifteenths of yours.
B: Yeah. Six-fifteenths of yours, that's what I mean.
T: Oh. Six-fifteenths of mine. And why is it six-twentieths of the whole bar?
B: Because they're all divided by five.
D [simultaneously with B]: Because you have to add five more. Because you didn't have the whole yard.

As shown in Protocol 6.12, Bridget's intention to make 15 parts total allowed her to make five equal parts out of a three-part bar because 15 could be divided by both five and three. By intending to make 15 parts, I infer that she could insert five units into each of the three units of the 3/4-bar and then "regroup" so that the 15-part bar consisted of five units each containing three units. So she seemed to operate with three levels of units with respect to the 15-part bar: It was a unit of three units each containing five units, and she could reorganize it to see it as a unit of five units each containing three units.[69] However, she did *not* seem to take the 3/4-bar as a unit in relation to the unit bar because she called the 15 parts she had made fifteenths. Even though a 4/4-bar was in her visual field, it was as if her "mathematical "world" was the 3/4-bar. She used her multiplying scheme in this world to find that one-fifth was three parts out of 15, and two-fifths was six parts out of 15. But for her these parts were not integrally related to the unit bar, which indicates she did not take the unit bar as a unit to which the 3/4-bar stood in relation (as a unit). Thus she did not seem to coordinate operating with three levels of units in the 15-part bar with operating *and* operating with three levels of units in the fractional context of the problem.

[69] I infer that her activity here was an important constructive resource in the construction of a co-measurement scheme at the end of March (cf. Protocols 6.10 and 6.11). The reason I don't call her way of operating here a co-measurement scheme is that doing so would mean she intentionally coordinated thirds and fifths. I conjecture that in Task 6.17, the 3/4-bar as "thirds" was implicit at best in Bridget's activity.

However, that Bridget made sense of, and agreed with, Deborah's response of six-twentieths indicates that Bridget could subsequently make three levels of units in the fractional context of the problem, as she reviewed the bar she had made. That is, Bridget *could* take the 3/4-bar as a unit in relation to the unit bar after she made the composition, and she could use her recursive partitioning operation to relate the small parts to the unit bar. What seemed difficult for her was to hold these two coordinations—of whole numbers of units within the bar at hand and of fractional parts in the larger context of the problem—together. As a result, Bridget's ways of operating with fraction composition tended to be a "two step" process: make the composition by coordinating whole numbers of parts, and then determine the measure of it (in relation to the unit bar) as a "separate" (though associated) problem. In this sense, her recursive partitioning operation seemed associated but not embedded in her ways of making fraction compositions.

Deborah's initial construction of a fraction composition scheme. In Deborah's solutions to problems like Task 6.17, finding the measure of the bar seemed to grow more directly out of making the bar than it did for Bridget. Given Deborah's interiorization of three levels of units, it is not a surprise that she coordinated three levels of units in fractional contexts while making compositions and found measures of the results (in relation to the unit bar) routinely and accurately. However, Deborah soon determined that using the standard algorithm to multiply fractions was relevant in these situations because, as she said, "of means times." So, at least during the teaching episodes on March 1st, 3rd, and 8th, she tended to compute mentally with the algorithm and then tried to make the result. In contrast with Michael, Deborah seemed somewhat uninterested in correlating the result from the algorithm with making the bar—if they didn't match, her inclination was to dismiss the picture and believe the computation. Since she could calculate so quickly and accurately, at times making the bar seemed a bit burdensome for her.

Yet over time she engaged more with making compositions and less with computing, except as a check on what she'd made.

Like Bridget, Deborah sometimes coordinated whole numbers of parts in making fraction compositions. In addition, partly as a result of her personal tendency to compute and then make the bar, she often made compositions by iterating single parts, which could hide the structure that she imputed to the situations. For example, on March 10th, Deborah worked on this problem:

> Task 6.18: Make 11/9 of a yard of ribbon. Your friend wants 4/3 of that piece. Make it and tell how much your friend needs.

Deborah divided the first one-ninth into three equal parts.[70] Then she pulled out one small part and repeated it to make 44 parts. When I asked how much one of those parts was of the unit yard, Bridget responded "One thirty-third." Deborah insisted that one thirty-third was wrong. After five seconds, she stated that it was one twenty-seventh, so the new piece of ribbon was 44/27 of a yard. In explanation of how she had made the composition, she said, "take one out of each of those boxes [the eleven one-ninths] and that equals eleven and that would be one-third, so I multiplied that by four 'cause she needs four."

This explanation is significant for two reasons. First, Deborah indicated that she was (at least sometimes) using a distributive operation to make a unit fractional amount of the given fractional quantity. To make one-third of the entire eleven-ninths of a yard she imagined taking one-third from each of the eleven ninths. So 11 small parts was one-third of the whole 11/9-yard bar. As I have discussed in relation to the boys' fraction composition activity, this way of operating is pivotal in the construction of a general fraction composition scheme because it allows recursive partitioning to be embedded in the scheme for making the composition: That is,

[70] Partitioning only one-ninth instead of all 11 is further evidence that fractional parts were iterable units for her.

one-third of each ninth is one twenty-seventh, so together the 11 small parts is eleven twenty-sevenths. I conjecture that Deborah's use of an explicit distributive operation in making one-third of eleven-ninths of a yard is a central reason why she emphatically rejected Bridget's idea that each part was one thirty-third and concluded that each part was one twenty-seventh.

Second, Deborah used her iterative fraction scheme to make four-thirds by iterating one-third four times, but she did not use the tools of JavaBars to reflect this construction (i.e., she did not repeat one part to make 11 parts and then repeat the 11 parts 4 times.) In fact, she did not often independently use the tools of JavaBars to demonstrate "structural" ways of operating with fraction composition problems, perhaps because she found making the bars somewhat tedious—or perhaps because making the bars remained largely an illustration of the end results of her mental activity. Nevertheless, from Deborah's explanation of her solution of Task 6.18 I infer that she had made an *initial* construction of a general fraction composition scheme. Unfortunately, Deborah's clarity in making fraction compositions was not consistent across the next few episodes. I attribute this inconsistency to her personal preferences for using the standard fraction multiplication algorithm, her willingness to fall back on coordinating whole numbers of parts in concert with Bridget, and my own difficulty as a teacher-researcher in adequately organizing fraction composition problems so that Deborah might have found them less tiring and more useful. I discuss this difficulty further when I address MCR in this section.

Efforts to Bring Forth Distribution

Deborah's use of a distributive operation in solving Task 6.18 (making 4/3 of 11/9 yd) on March 10th may have been influenced by my efforts to bring forth distributive ways of operating in fraction composition situations in early March. These intentions were not always successful. As I have noted with the boys, using color (e.g., a special ribbon that changes color every ninth, cf. Task 5.22) to try to compel distribution seemed generally ineffective in bringing forth a distributive operation in a student's ways of operating if such an operation was not already present. So my use of color during these episodes may have been ineffective for Bridget, who had not demonstrated a consistent distributive operation in her mathematical activity.

However, the use of color may have helped Deborah activate and externalize her distributive operation in fraction composition situations. For example, during the March 8th episode, the girls worked on making three-halves of five-sevenths of a yard of ribbon, where the ribbon changed color every seventh of a yard. Bridget partitioned each of the five sevenths into two parts to make a 10-part bar that now changed color every two parts. Deborah knew right away that in the new ribbon, each of the five colors would be applied to *three* parts. So I infer that she made three-halves of each of the five one-sevenths in order to make three-halves of the entire 5/7-yard bar. In contrast, to make the new ribbon Bridget planned to add five more parts onto the 10-part bar, because the old ribbon had 10 parts and she had to add half of those more. To determine how to color the new ribbon, Bridget divided the total number of parts, 15, by 5. So for her, color seemed to be an "extra" aspect of the problem rather than something she might use to make the composition.

Yet, since Deborah's distributive operation did not seem consistently embedded in her fraction composition activity, and since I was uncertain whether Bridget had constructed a distributive operation at all, on March 24th I posed this problem:

Task 6.19: Make four different-sized candy bars. Can you make 5/7 of that collection?

Like the boys (cf. Task 5.15), both girls operated with—and verbalized belief in—the notion that taking one-seventh of each bar meant taking one-seventh of the entire collection of bars. Then making five-sevenths of the collection just involved copying one-seventh of it five times. Their facility in solving Task 6.19 informed me that having distributive operation in one context did not mean it was usable, let alone embedded in, other ways of operating.

This lack of connection is not surprising given the units-coordination involved in making fraction compositions that is *not* involved in taking fractional parts of separate bars. In taking five-sevenths of four bars, coordinating three levels of units may be required. Making sevenths in each bar means creating each bar as a unit of seven units. Forming a quantity of one-seventh from each bar as equal to one-seventh of all the bars, and taking that quantity five times, means that five-sevenths of the four bars is a unit of five units, each containing four units (one-seventh from each of the four bars). Completing this task does not seem to require students to “hold in mind” the size of these smallest units in relation to the *entire* four bars—taking the four bars as a unit in relation to another larger unit is unnecessary.

Making a fraction composition, such as one-seventh of four-fifths of a bar, seems to require the above coordination of three levels of units *and* the coordination of three levels of units in relation to the unit bar, five-fifths, to which four-fifths stands in relation as a unit. So making sevenths in each of the four one-fifths means creating the 4/5-bar as a unit of four units each consisting of seven small units. But maintaining four-fifths as a unit in relation to the unit bar means considering seven of these smallest units as a unit, *five* of which constitute the unit bar. In this duel coordination of three levels of units within the four-fifths and with respect to the unit bar, a student can “track” the making of thirty-fifths, not twenty-eighths.

Since I hoped to encourage the use and embedding of each girl’s distributive operation in her fraction composition activity, for the last two episodes in March (the 24^{th} and the 29^{th}) the witness-researchers helped me design other tasks that might bring out distributive ways of operating. I have already discussed these tasks (e.g., Task 6.14 and 6.15) as the site of Bridget’s modification of her partitioning operation and construction of a co-measurement scheme. Although work on these problems appeared to engender Bridget's progress in operating with three levels of units in fractional contexts, it did not appear to appreciably increase either girl's use of a distributive operation in fraction composition activity.

MCR During Fraction Composition Activity

Starting work on fraction composition problems back in the context of JavaBars was a prime example of enacting MCR with Deborah and Bridget because in doing so I responded to Bridget's specific difficulty with fraction composition in solving the Box Problem, Task 6.3, on February 18th; Deborah's feelings of depletion in drawing pictures; and my perception of the suppression of their operations. However, enacting MCR within our work on fraction composition was challenging because it was my first time working in this area with a pair of students (i.e., fraction composition activity with the girls took place in March, about one month before similar activity with the boys.) So I struggled to understand the girls' ways of operating as well as the nature of operations I was trying to bring forth in their activity. As with the boys, with the girls I did not sufficiently explore how to take nonunit, proper fractional amounts of unit fractions. But unlike my work with the boys, I had an even harder time sequencing problems well for the girls so that they might experience a conceptual or "structural" build. Many of my difficulties involved not yet understanding the nature of their distributive operations or the distributive activity that seemed necessary to construct a general fraction composition scheme.

A good example of the difficulty I had in sequencing of problems for Bridget occurred on March 1st. After she had struggled with Task 6.16 (make 1/2 of 5/8 of a mile), she might have benefited from another problem that involved finding one-half of a fractional quantity. Instead, I moved on to taking other unit fractions of proper fractional quantities (e.g., make 1/3 of 4/5 of a mile), which were a significant challenge for her given that she had not yet constructed distributive partitioning. With Deborah, the difficulty I had in sequencing tasks was somewhat less obvious because she could operate in most of the situations. But being able to operate did not mean that she was constructing more powerful schemes. Because I knew she could operate in many of the composition situations, I tended to pose complex problems for her (e.g., Task 6.18, make 4/3 of 11/9 of a yard) without sufficient build and with the intention that she would construct a consistent, general fraction composition scheme.

Perhaps as a result of problem sequencing that was not sufficiently tailored to either of their ways of operating, both girls showed evidence of depletion during our work on fraction composition. I conjecture that Bridget's depletion stemmed from trying to operate in situations without having built up adequate schemes and operations to do so. In particular, not having yet interiorized three levels of units so that she could coordinate three levels of units with whole numbers of parts as well as within the larger fractional context of problems was a consistent constraint. In both the March 1st and 3rd episodes, Bridget expressed, with some dismay, that she always "thinks these problems wrong." So she seemed to recognize that something about her thinking in these situations was not useful or "off." She rallied, trying to operate with her strong partitive fraction scheme and her multiplying scheme with whole numbers. But I conjecture that relying on these schemes alone was cognitively draining for her. Without the compression afforded by embedding her multiplying scheme into her fraction scheme, as Deborah had done in December, solving fraction composition problems was cognitively rather onerous.

Perhaps as an outgrowth of this cognitive load, Bridget's sense of control and autonomy when operating in these situations appeared weak. For example, when Deborah and Bridget came up with three different fractions for the length of a composition during the March 3rd teaching episode, I asked which length they thought was right. "Whichever you think is right," Bridget glibly responded. To some extent she was kidding, but her comment indicates that she did not feel "in charge" of determining the validity of her and Deborah's results—deciding what was right was out of her control. This comment also indicates a certain lack of interest or involvement in the situation, probably because she had a sense of not being able to operate.

Deborah also displayed depletion during some of the fraction composition activity. Waiting as Bridget struggled to make a composition that was easy (from Deborah's point of view) was depleting for Deborah, who was generally eager to work on the computer. But although she would tackle significantly more complex composition problems (e.g., that involved improper fractions, like Task 6.18), often with excitement and intensity, they were hard for her. Such problems did not necessarily help her to construct a consistent structure in her ways of operating with compositions, so they may have seemed burdensome and may have provoked Deborah's negative responses to new problems (e.g., she would often insist that she couldn't do them). In retrospect, my sense is that Deborah didn't see an overall goal in what we were doing—she could compute fraction multiplication swiftly and accurately, and so making simple compositions seemed pointless and making hard compositions was overly difficult. At the start of the March 10th teaching episode she asked when we would be switching computer programs, a sign that she was not fully interested or engaged in this work.

However, the girls also showed evidence of stimulation during fraction composition activity. Once Bridget began to coordinate whole numbers of parts during the March 3rd teaching episode, she was often eager to solve problems because she had a way to make the compositions. Similarly, Deborah sometimes got very excited and absorbed in this activity. Toward the middle of March, she may have *started* to appreciate the power of making the compositions rather than just computing them, since around the March 10th teaching episode she began to curtail her computational activity and focused her attention on making bars.

Nevertheless, both witness-researchers and I felt uneasy about the girls' fraction composition activity. Although at the time none of us articulated that the lack of an *embedded* distributive operation was problematic, we believed our uneasiness involved the lack of distributive activity. So toward the end of March we designed problems to bring forth the girls' distributive operations. Making these changes was an example of enacting MCR with the girls because the problems harmonized with the girls' ways of operating and challenged them (e.g., Tasks 6.14 and 6.15). The changes seemed beneficial for Bridget since she modified her partitioning operation and began to construct a co-measurement scheme (cf. Protocols 6.10 and 6.11), marking an expansion of her mathematical reality. It's uncertain whether this enactment of MCR had specific effects on Deborah.

Fractions as Operations: Constructive Resource 5, January, March through May

The issue of fractions as multiplicative operations on quantities arose in the girls' activity in January, in relation to operating on an unknown quantity. As the teaching experiment progressed, I began to understand the significance of conceiving of fractions as operations in the construction and solution of linear equations. For example, in the Box Problem (Task 6.3) two-thirds of an unknown length is the same amount as a known length, 3/4 decameter. While it is possible to solve this problem without conceiving explicitly of the fractional multiplicative relationships involved (as Bridget did on February 18th), my hypothesis is that moving toward more algebraic solutions—such as constructing and solving a linear equation—rests upon conceiving of fractions as multiplicative operations in at least three ways. First, explicitly operating on the unknown length means conceiving of "two-thirds of the length" as a multiplication of that length by two-thirds, and as the result of that operation—i.e., as a new length in its own right. Second, abstracting operations on the known and unknown lengths in the solution process involves "seeing" dividing by two and multiplying by three as multiplying by three-halves, which in turn is related to constructing reciprocity. Third, *notating* these operations as $(2/3)x = 3/4$ or $x = (3/4)*(3/2)$ requires fractions as operations, if the notation is to stand in for mental operations. So in this final constructive resource section, I explore the girls' activity with fractions as operations in January and during the last three months of the teaching experiment.

Operating on Unknowns in January

After the girls had worked so exuberantly on the Candy Bar Problem (Task 6.4) with pencil drawings on January 14th, I began to work with them on conceiving of, operating on, and notating unknowns without the computer program. So in the next teaching episode on January 21st, I asked the girls to think about quantities that they didn't know the exact measure of but could find out. Since we didn't know the exact measure, I asked how we could represent the quantities, and they suggested notating them with letters. Then we considered quantitative changes to these unknown quantities. For example, Bridget thought of the unknown quantity "how many hairs on your head." The girls let *H* stand for this quantity. I asked them how they would write down twice as many hairs. Both girls wrote *H**2.

Not long after, the witness-researcher intervened with this question:

Task 6.20: *L* is how long room is. How would you write 3/4 of that?[71]

Both girls said "*L* minus one-fourth," although Deborah amended her response to "*L* minus one-fourth *of the length*." The witness-researcher asked them to write that down, so Bridget wrote "*L* – 1/4," and Deborah wrote "*L* – 1/4 of." We asked the girls what the length of the room was, and

[71] The girls frequently used their initials, B and D, as the letters to represent unknowns. For readability, particularly in protocols, I have changed the letters representing unknowns to letters *other than* B and D throughout this section.

Bridget said, emphatically, "*L*!" Deborah then amended her written statement to "*L* – 1/4 of *L*." I asked Bridget what that would look like. She said, "You just have the room and you divide it into four, and you take three out." She noted that she could draw it (rather than write it with notation).

Since both girls thought of removing one-fourth to make three-fourths, I infer that neither of them conceived of "three-fourths of" as a multiplicative operation. Deborah repeatedly insisted that she was removing not just one-fourth but one-fourth *of the length of the room* in order to make three-fourths of the length of the room. Although Bridget's verbal and written notation did not explicitly reflect a similar conception (she said and wrote "*L* – 1/4"), her explanation about dividing the length of the room into four parts and taking out three indicates that she conceived of taking out three-fourths *of the length of the room*. From their activity to this point I infer that neither had constructed taking a fractional part of a quantity as a multiplicative operation—i.e., they had not built the idea that "1/4 of *L*" means to multiply *L* by one-fourth.

As the episode continued, the witness-researcher and I asked the girls about one-fourth of *L*. Both girls seemed a little exasperated at what they perhaps thought were silly questions.

Protocol 6.13: Determining one-fourth of an unknown length on 1/21/04.

T: How do you figure out one-fourth of *L*?
[D makes a little "argh" sound and rests her chin on the table.]
B: *You measure the room!*
D: You measure *L*! You measure *L*!
B: *The length of the room and you minus one-fourth of it, and then you minus two more fourths.*
D: *Actually you times it by one-fourth.*
B [*to D*]: *Really?* [Both girls look at T.]
T: Oh!
B: *I thought you take out one-fourth.*
D: *Yeah but you have to times it by one-fourth to know what it was.*
B: *Oh yeah, you have to times one-fourth by something else.*
D: Times *L*.
W: Suppose I say you can't use subtraction [to find three-fourths of *L*].
B: *Well then you divide it into four.* Then you take off—not subtract—you <u>take off</u>.
[Everyone laughs.]

W: You can't even take off. [They all discuss taking out one-fourth and leaving three-fourths or taking out three-fourths and leaving one-fourth.]
B: *See 'cause, you don't have to multiply. See if you have this room and you divide it into four, you wouldn't have to multiply one-fourth because you could just go ahead and take out three*. [T encourages B to draw her idea. *B draws a rectangle partitioned into four parts and shades three of them.* She says it's *L* minus three-fourths.] But he said we couldn't use subtraction. So it would be…
D [softly]: *L times one-fourth times two*.
B: Wait a minute, if you already took three-fourths out, you could <u>add</u> three-fourths.
D: *No you could take L times one-fourth, and then you get the answer to that, and then you say that answer plus that answer plus that answer*. [D writes down "*L* x 1/4 = *A*." Underneath she writes "*A* + *A* + *A* = 3/4 of *L*." *B* notes that you could make it shorter by writing down "*A*3*."]

How Deborah knew that she could multiply by one-fourth to find one-fourth of the length of the room is not clear, particularly when she did not seem to think that she could multiply by three-fourths to find three-fourths of the length of the room. Her classroom mathematics teacher had told her that "of" means multiply. However, my conjecture is that Deborah did not rely on this meaning of "of" here *because* she did not think to multiply the length of the room by three-fourths to find three-fourths of *L*. One explanation is that Deborah had constructed an equivalence between dividing by a whole number and multiplying by a *unit* fraction, so she knew that dividing the length by four and multiplying it by one-fourth would give the same amount. I infer that she constructed this equivalence because she knew that four times a length that had been divided by four would produce the length. Simultaneously, using her iterative fraction scheme, she knew that *one-fourth times* the length taken four times would also produce the length. This explanation is plausible, although it depends heavily on notions of equivalence rather than operations of fraction schemes.

Bridget seemed surprised by Deborah's idea. She seemed to play along with it by stating "oh yeah, you have to times one-fourth by something else," but it's not clear what she meant by that comment. Furthermore, she did not seem to see multiplication as necessary. She maintained dividing by four and taking parts out was sufficient to make one-fourth of *L* or three-fourths of *L*. So I infer that Bridget had not yet constructed unit fractions as multiplicative operations. However, she had an idea to shorten Deborah's notation of $A + A + A$ to $A*3$. Combined with her ideas about dividing by four to make fourths, I *can* infer that whole numbers were multiplicative operations for Bridget. In this protocol she also made an important contribution to our work: She wasn't sure about how to notate three-fourths of *L* with algebraic symbols, but she could draw it. In subsequent episodes I built on her idea of drawing before notating with algebraic symbols.

A Basic Conjecture Foiled in March

Although the research team and I were not fully aware of it, a basic—and rather implicit—conjecture of ours for engendering the construction of fractions as multiplicative operations was not tenable. In retrospect, our basic conjecture was that bringing forth students' verbalizations of their operational activity would provoke reflection on that activity which would, in turn, engender the interiorization of fractions as operations. So in January, when the girls demonstrated difficulty in notating "three-fourths of the length of the room," our response was to ask them to articulate how they would make that quantity. We believed that describing (verbally) how they would make it would facilitate them abstracting what "taking three-fourths of something" might mean. In fact, they did *not* promptly respond "divide the something by four and multiply that amount by three," although they arrived at this idea somewhat indirectly by the end of Protocol 6.13. And in any case, abstracting out whole number operations from making three-fourths of a quantity still does not necessarily entail constructing the notion that taking three-fourths of quantity involves *multiplying* the quantity by three-fourths.

In fact, even abstracting the whole number *multiplicative* operations involved in taking a fraction of a quantity was not obvious to the girls. Nearly two months after Protocol 6.13, in the

March 24th teaching episode, the girls worked on making and determining the length of seven-fifths of three-fourths of a yard. As they completed the problem, a witness-researcher asked them what they had to do to take seven-fifths of something. Deborah said you have to divide into five and add two-fifths. Bridget said you have to copy the whole and add two-fifths. The witness-researcher asked about multiplying by something, but that did not elicit a response. So I asked what they needed to do with one-fifth. Deborah said, "times it by two and add it to the end." Eventually I elicited that multiplying the one-fifth by seven would produce seven-fifths. The heavy coaching that this response required indicates that neither girl had abstracted "taking seven-fifths of something" as a *multiplicative* operation. At the time, I did not see the implications of this brief interaction, and I continued to act upon our basic conjecture when working on RMR problems from March 31st through May 12th.

Fractions as Operations when Solving RMR Problems in April

Deborah's construction of fractions as operations. The first evidence of Deborah's construction of nonunit fractions as operations occurred during the April 26th teaching episode. We were working in JavaBars with squares that represented quilt blocks. I posed this situation:

> Task 6.21: One-fourth of a quilt block is blue (make it separate from the block). That amount is 2/3 of the gold part in another quilt block.

My intention had been to work through the problem slowly—to ask if they could first show two-thirds of the gold part, for example. But before I even asked a question, Deborah copied the blue 1/4-bar, partitioned it into two parts, pulled out one part, and joined it on (see Figure 6.16).

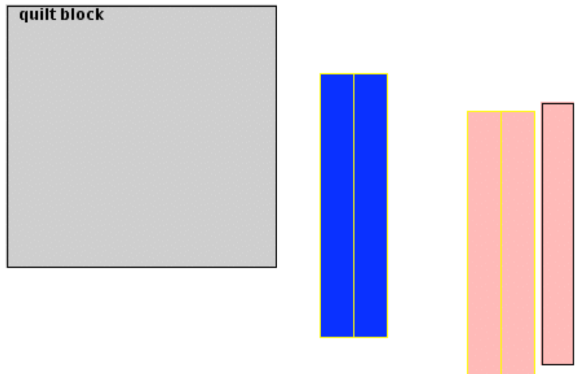

Figure 6.16, The blue part of a quilt block (1/4, left) and the gold part (almost made, right).

While Deborah worked, Bridget said, "What are you doing? Please tell me." She may have been able to make the gold part using her reversible partitive fraction scheme with splitting, but evidently she needed more time to consider the situation. When I asked Bridget whether what Deborah had done made sense to her, Bridget verbalized Deborah's process. Then I moved into questions about the multiplicative relationships between the parts.

Protocol 6.14: Evidence of Deborah's construction of fractions as operations on 4/26/04.

T: What do you have to take times the blue part to make the gold part?
D: Mmm…times it…
B: What? [T repeats the question.] You got to times it by—
D: *Blue part times one and one-half.*
T: Oh. Blue part times one and one-half. Why?
B: *Because it's one and one-half* [points to the gold part].
[*Both girls rest their heads on their arms on the table and remain that way during the rest of this protocol.*]
T: Oh—so what fraction is one and one-half if you just write it as a fraction?
B: *What!?*
D: *Three-halves.*
T: So if we take three-halves times the blue part we get the gold part, right? What times the gold part will give you the blue part?
B: *Divide—you got to divide.*
T: Could you do times? Could you times the gold part by a fraction?

D: *Times one-third—no, times two-thirds.*
B: Yeah.

This protocol shows that Deborah could state multiplicative relationships between the two fractional parts when they were in her visual field. My current conjecture is that to do so, she used her iterative fraction scheme and construction of unit fractions as operations, in coordination with her ability to take any fractional part as a unit to which another fractional unit could be compared. With regard to multiplying the blue part to make the gold part, Deborah may have abstracted from her activity: She actually made the gold part so that it was one-half of the blue part more than the blue part. I infer that she could take this activity to mean that the entire blue part and one-half of the blue part would constitute the gold part. Because Deborah had constructed unit fractions as operations, for her one-half of the blue part was one-half times the blue part. So the entire blue part was one and one-half times the gold part. By using her iterative fraction scheme, she knew one and one-half was the same number as three-halves.

With regard to multiplying the gold part to make the blue part, my conjecture is that Deborah could "switch" perspectives, taking the gold part as her primary focus, a unit made of three units. The blue part could be made from iterating one of those units twice. Since she had already constructed one-third as a multiplicative operation, I infer that she initially thought about multiplying the gold part by one-third—that would, after all, accomplish the goal of partitioning the gold part into three equal parts. But then she knew she needed two of those one-third parts to make the blue part, which led her to conclude that two-thirds was the multiplier.

"How come bars have to have so many names?" Bridget asked this question on March 31st, as she worked on RMR problems of Type 3, with whole number quantities and whole number relationships where the relationship did not divide the quantity. The question seems representative of her response to questions about multiplicative relationships between quantities.

It may reflect that Bridget found it difficult to take any bar (or fractional part of a bar) as a unit in relation to another bar (or fractional part of a bar), which is not surprising because doing so seems to require at least coordinating three levels of units prior to operating.

In Protocol 6.14, Bridget seemed to make sense of Deborah's response of "one and one-half" because she could make a visual assessment that the gold part was half as much more as the blue part. But it is not clear that she had constructed the notion that *multiplying* the blue part by one and one-half would make the gold part. In addition, not having yet constructed an iterative fraction scheme was a constraint for her in conceiving of one and one-half as identical to three-halves. To start from the gold part and make the blue part, I infer that Bridget thought of *dividing* the gold part because she knew that the blue part was smaller than the gold part. Her comment about dividing also indicates that she had likely constructed dividing as the "opposite" of multiplying. Once again, she may have used the bars in her visual field to make sense of Deborah's response—the blue bar was partitioned into two equal parts and the gold bar was three of those parts. However, once again it's unlikely that independently she would have produced *multiplying* by two-thirds as a way to make the blue part from the gold part.

As their work on Task 6.21 progressed, Bridget used recursive partitioning to find the fractional "size" of the gold part (in relation to the quilt block) before Deborah. But Bridget seemed lost when Deborah verified that the multiplicative relationships "worked." That is, when I asked what they could take times one-fourth (the size of the blue part) to make three-eighths (the size of the gold part), Deborah verified that 1/4 times 3/2 produced 3/8. Bridget agreed, but her agreement tells me little about how she was operating at that point (i.e., she may have agreed primarily or even solely for social reasons.) Going in the other direction (what do you have to take times 3/8 to make 1/4) required a little bit more coaching with Deborah, but she found that

3/8 times 2/3 gave 6/24, which she identified as the blue part based on her knowledge of commensurate fractions.

In response to this interaction, Bridget said, "I don't get it." I tried to summarize and bring out the relationships Deborah had identified, and they both started waving their hands. Bridget said it was a "reversible thing" and Deborah mentioned "integer." I offered the word "reciprocal," which seemed meaningful to both of them. Stating the word reciprocal may have allowed Bridget to regain a degree of comfort because she had heard about reciprocals in her math class (even though she seemed to have not yet constructed reciprocal relationships). I told them both that they had done algebra today, which Bridget thought was "cool."

Producing Multiplicative Relationships Between Quantities in May

Deborah stating reciprocal relationships. In Deborah's progress through the remaining five teaching episodes, identifying multiplicative relationships between parts of quilt block or bars representing quantities became almost automatic and not confined to the two-thirds and three-halves relationships. As I have described at the beginning of this chapter, on May 12th after she solved Task 6.1 (the Lizards' car goes 1/2 of a meter which is 3/4 of the Cobras' car's distance), she knew immediately that "you multiply the Lizards times four-thirds to get the Cobras because the Lizards' length is three-fourths of the Cobras' length." So I infer that she had begun to construct fractions as operations more generally as a basis for generating reciprocal relationships. In other words, for Deborah, knowing that distance *L* was three-fourths of distance *C* meant that distance *C* was a unit of four units, any of which were one-fourth of *C* and could be iterated three times to produce distance *L*. But in a seemingly simultaneous sense, distance *L* was a unit of three units, any of which were one-third of *L* and could be iterated four times to produce distance *C*. Thus distance *C* would be four-thirds of distance *L*. I infer that her interiorization of three levels of units was necessary for this construction, but as I have discussed in relation to Michael, this units-coordination may not be sufficient for it.

Yet although Deborah could identify these relationships with ease, the significance of them for her at the time was uncertain. For example, after working on several problems with the relationship three-fourths between known and unknown quantity, she never multiplied the known quantity by four-thirds to determine the unknown quantity. She may have believed that I wanted her to make the bars. However, on May 12th she had several opportunities to take shortcuts when I asked her to try to solve a problem in her head or “without making everything.” These opportunities seemed burdensome to her, as I have discussed at the start of this chapter. So although she could state reciprocal relationships after solving these problems, she did not yet use the relationships as a “tool” for compressing her ways of operating or for further operating. My current conjecture is that, as powerful a reasoner as Deborah was, she had not yet become aware of how she could use reciprocal relationships in the service of further operating.

Attenuating questions about multiplicative relationships with Bridget. Over the remaining five teaching episodes, Bridget responded to some of the multiplicative relationship questions when I started with the relationship two-thirds between known and unknown quantities, probably because she had a good memory, could cue off of Deborah, and visually could make some sense of these relationships as she had on April 26th. However, her difficulty in producing multiplicative relationships between quantities was evident on May 10th. At the beginning of that episode, Bridget found the amount of money needed for the soda machine if one-fifth of a dollar was one-third of the money needed. To solve the problem, she made fifteenths (partitioning her 1/5-dollar bar into three parts) and then made two more 3/15-dollar bars to make a 9/15-dollar bar. With coaching she could see that 9/15 was 3 times 1/5 (or 3/15), but the other direction (what could she take times 9/15 to make 3/15) did not make sense to her: "I don't get it," she said. Her use of fifteenths undoubtedly compounded her difficulty in determining multiplicative relationships between the two quantities. But it also seems clear that she had not constructed fractions as operations, and that simply asking her about them at this point would not provoke such a construction. So during the May 10th and 12th episodes, I stopped asking her questions about multiplicative relationships between quantities.

MCR in Relation to Fractions as Operations

Switching contexts in January and April. Leaving the computer to work on conceptions of and operations on unknowns in January and February was an attempt to enact MCR in trying to expand the girls' mathematical realities toward notating quantitative relationships. Furthermore, based on their excitement and engagement on January 14th when they solved the Candy Bar Problem, Task 6.4, the research team and I believed the girls' were ready this kind of move. However, during the seven episodes we spent in this activity, enacting MCR with them was very challenging. As already noted, some of their quantitative operations became suppressed, which is a telling sign of weakening MCR. The girls' responses varied quite a bit in these episodes, but they were not as interested and energetic as they were on January 14th. Deborah in particular could get depleted over the issue of making drawings.

On April 26th, we switched contexts within JavaBars to using quilt blocks as a basis for fraction composition and RMR problems. I made this switch because the girls seemed worn out over the linear ribbon and candy bar contexts. The research team and I conjectured that using a "quasi"-area context to open the possibility for them to rethink their ways of operating with fraction compositions might be a refreshing and useful change. And, to some extent, this change was successful, in that the girls used cross-partitioning to make fraction compositions.[72] However, as shown in Protocol 6.14, even with the new context the girls sometimes had their heads down for extended periods of time, which indicates they still experienced a sense of fatigue or depletion.[73]

Bridget and Deborah's depletion in May. In general, during the episodes from April 26th to May 10th, I found it very challenging to enact MCR with Bridget. I continued to pose questions about multiplicative relationships between quantities, and Bridget was largely reliant on Deborah for responses to such questions. So Bridget's depletion became particularly evident when each girl worked on her own computer for the four episodes in May, because Bridget did not always have a way to operate independently on her computer while Deborah generally did. Bridget's depletion during these episodes is quite understandable: It is very taxing to try to respond to questions for which one has not developed sufficient schemes.

As usual, the low points in MCR can open possibilities for better enactment. The witness-researchers and I decided it would be best to differentiate problems for the two girls during the last two teaching episodes (May 10th and 12th), which, combined with my search for how to interact with Bridget so as to alleviate her depletion, led to improved MCR between Bridget and me. My search, which I have described in this chapter's opening section on Bridget, included curtailing questions that involved fractions as multiplicative operations. My motivation to do so rested on my perception of the difficulties she experienced with such questions and my aim to engender more fluid operating without getting slowed or stopped by questions that stumped her. At the time I did not have an explanation for why these questions were so difficult for her.

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[72] That is, for a short time we worked on making, say, one-third of two-fifths by "fifthing" vertically and "thirding" horizontally.

[73] I don't attribute this fatigue or depletion entirely to our interactions, of course. End-of-the-year testing and a desire to be outside in the warm weather rather than inside for several more weeks of school are two other obvious factors that might have led to drooping heads in April!

Even though Deborah demonstrated some feelings of stimulation in working on problems during the episodes from April 26th to May 12th, our MCR was also not that strong, mostly because it was so accidental. Deborah had made some significant progress in constructing fractions as operations, but my interactions with her may not have had much to do with her constructions. For example, I am fairly certain that her construction of unit fractions as operations (cf. Protocol 6.13) was mostly independent of our activity together. Even more troubling, I didn't know what to do to help her build on the sophisticated ways of operating she had constructed. She seemed to find my repeated requests for her to identify multiplicative relationships in RMR problems during May to be answerable, but responding to such questions did not seem to help her to re-organize or expand her ways of operating with RMR problems.

Two final bright points. Near the end of the teaching experiment, both girls and I experienced some stimulation in our interactions. The bright point for Deborah in our enactment of MCR was that she seemed more at ease with drawing (at least on the computer) as a way to solve problems and as a way that could tell her more about her solution to a problem. In particular, during the teaching episode on May 5th, she appeared deeply engaged in making some creative pictures in JavaBars to show and extend her solution of a problem. She remarked that she understood (at least in the context of a specific problem) what I meant by explaining or showing with her picture. Two episodes later on May 12th, the bright point for Bridget in our enactment of MCR was that her depletion seemed alleviated by my adaptation of problem situations and careful sequencing of tasks. The change in her responses to requests to solve RMR problems mentally—recall she went from rather worried to more assured—attests to stronger MCR between us during that last episode of the teaching experiment.

These two bright points had two similar characteristics: a sense of greater communication between each girl and me, and a sense that each girl had expanded her mathematical world in some way. With Deborah, I had a feeling that she had had an insight into why I had been asking for pictures the whole year, and she in turn seemed to appreciate that making pictures had some power to inform her reasoning. With Bridget, I had a feeling that I was asking her questions that made sense to her and yet were still challenging, and she in turn began to construct a scheme to solve Type 5 RMR problems with unit fractional quantities.

Accounting for the Girls' Solutions of Problems Involving RMR

In this final section, I provide an account of how the girls' construction (or lack of construction) of the five constructive resources helps to explain their solutions of RMR problems in May, as described in the opening sections of the chapter. I focus on Deborah's solution of Task 6.2, in which she determined how far the Cobras' car traveled if the Lizards' car's distance of 2/3 meter was three-fourths of the Cobras' distance, and on Bridget's solution of Task 6.6, in which she determined how far the Cobras' car traveled if the Lizards' distance of 1/7 meter was two-fifths of the Cobras' distance. As with the boys, a crucial difference between the girls' ways of operating in solving RMR problems was the levels of units they coordinated prior to operating, and in particular, the use of more than one units-coordination within the same situation. This difference is a primary explanatory tool in my evolving models of the girls' mathematics and figures prominently in my account of their solutions to Tasks 6.2 and 6.6.

Deborah's Solution of Task 6.2 in May

Reversible reasoning: Constructive Resource 1. Like Michael, Deborah used reversible reasoning with fractions in dividing the known length, 2/3 meter, into three equal parts. Her construction of fractions as iterable units and a splitting operation was crucial in this way of operating. That is, for Deborah three-fourths was one-fourth three times, which meant that she could use her splitting operation on three-fourths to make one-fourth, and then iterate one-fourth four times to make the Cobras' distance. Because solving the problem does not require iterating one-fourth beyond the whole, four-fourths, only a reversible partitive fraction scheme with splitting seems to be necessary to reason reversibly in this situation. However, early in the teaching experiment Deborah demonstrated that she had a reversible *iterative* fraction scheme (cf. Protocols 6.6 and 6.7), which was significant in her reasoning in other ways.

New uses of multiplying schemes: Constructive Resource 3.[74] To implement her reversible reasoning with fractions, Deborah needed to split a 2-part bar into three equal parts. At this point in the teaching experiment, doing so was a perturbation for her, although not in the sense that she felt at a loss for what to do. That is, in constructing a co-measurement scheme, Deborah had embedded her multiplying scheme into her iterative fraction scheme (cf. Protocols 6.8 and 6.9), so she knew that she needed to partition the 2/3-meter bar into more parts in order to be able to view it as a unit of three equal parts, each containing some number of parts. However, I infer that she was not completely certain of the total of number of parts necessary, because she partitioned each of the thirds into six parts to produce 12 parts total.

There is nothing wrong with how she operated! But using 12 parts total indicates that she may have been coordinating thirds and fourths in the same bar rather than splitting the 2/3-meter bar into thirds. In fact, by partitioning into 12 parts total she could have either taken three-fourths of the resulting 12-part bar (a fraction composition problem) *or* operated on the bar as three-fourths of another bar (a reversible multiplicative reasoning problem, Task 6.2). Since later in this episode, when I requested that she try to solve an RMR problem without making the bars,

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[74] As with the boys, I discuss Constructive Resource 3 prior to Constructive Resource 2, in order to follow the trajectory of the girls' reasoning. With Deborah, I also discuss Constructive Resource 5 prior to Constructive Resource 4.

Deborah multiplied the fractions, it's quite possible that she had not completely worked out differences between composing fractions and reasoning reversibly with them in problems like Task 6.2. Nevertheless, even though making 12 parts shows a potential conflation between these two kinds of problems, Deborah clearly used the 12 parts to accomplish her goal of making thirds of the 2/3-bar in order to solve Task 6.2.

I infer that partitioning the 2/3-meter bar into 12 parts means Deborah viewed the bar as a unit of two units each containing six units and as a unit of three units each containing four units. Having interiorized three levels of units meant that she could flexibly switch between two unit-of-units-of-units views in the bar—i.e., she could hold one structure in mind while creating the other structure. Thus, like Michael, Deborah's construction of dual three-levels-of-units views of a bar in December (cf. Task 6.12 and Protocol 6.8, Task 6.13 and Protocol 6.9), which seemed to be critical for her construction of a co-measurement scheme, was an important constructive resource for her in solving Task 6.2. And like Michael, Deborah had embedded her multiplying scheme into her *reversible* fraction scheme, so I can attribute a reversible multiplying scheme with fractions to her in May.

Activity with improper fractions: Constructive Resource 2. Since the solution to Task 6.2 is a proper fraction, Deborah did not need to use her iterative fraction scheme as Michael did in Task 5.1 to determine the measure of the new quantity. But she did use it in conceiving of the multiplicative relationships between the two quantities. That is, for Deborah, one-fourth of the Cobras' distance was simultaneously *one-third* of the Lizards' distance, two-thirds of a meter. Because she knew that iterating that amount four times made the Cobras' distance, she could conceive the Cobras' distance as *four-thirds* of the Lizards' distance. I conjecture that operating reciprocally in this way involved the construction of fractions as iterable units *and* operating with three levels of units—which are also both required for the construction of an iterative fraction scheme. That is, I infer that Deborah conceived of the Lizards' distance as a unit of three units, any of which could be iterated four times to produce the Cobras' distance, and simultaneously she conceived of the Cobras' distance as a unit of four units, any of which could be iterated three times to produce the Lizards' distance. Even though Michael had a solid iterative fraction scheme, he did not readily use this kind of reciprocal reasoning, which indicates that an iterative fraction scheme may be necessary but not sufficient for constructing reciprocity.

Fractions as operations: Constructive Resource 5. Deborah's construction of bi-directional multiplicative relationships between fractional quantities seems to involve an anticipatory unit-of-units-of-units structure projected into the relationship between two bars *from the perspective of either of the bars*. From the perspective of the Cobras' distance as the "main unit," the Lizards' distance was a unit of three units contained within the Cobras' distance, in that any of the three units that made up the Lizards' distance could be iterated four times to make the Cobras' distance. From this perspective, the Lizards' distance was three-fourths of the Cobras' distance. However, from the perspective of the Lizards' distance as the "main unit," the Cobras' distance was a unit of four units related to the Lizards' distance in that any of the four units could be iterated three times to make the Lizards' distance. From *this* perspective, the Cobras' distance was four-thirds of the Lizards' distance. Perhaps not all iterative fraction schemes (even strong ones such as Michael's) include this "perspective switching" of units. Abstracting fraction as conceptual entities, not just the results of schemes, may be required to produce such perspective switching. I address this possible explanation in the next chapter.

Incidentally, Deborah's construction of reciprocal multiplicative relationships between the two distances may provide the basis for another explanation for her insertion of six (rather than three) units into each part of the 2/3-meter bar. Making 12 parts total in the 2/3-meter bar may have seemed like the most "promising" activity since that bar was three-fourths of the new bar, and the new bar was four-thirds of that bar. That is, she may have been focused on coordinating three units and four units. In this case, the notion of making three equal parts in a 2-unit bar may have been temporarily unavailable or not useful to her. This explanation would take into account the simultaneous nature of her use of bi-directional relationships between known and unknown, as opposed to the more sequential way of operating, from known to unknown, that Michael demonstrated in his solution of the Candy Bar Problem (Task 5.1).

Fraction composition activity: Constructive Resource 4. Deborah's lack of explicit distributive activity with fractions seemed to be a limitation in solving Task 6.2 because it blocked determining the length of the Cobras' distance as an outgrowth of making of it. That is, had Deborah found one-third of the 2/3-meter bar by taking one-third from each 1/3-meter part, it would have opened the possibility of embedding her recursive partitioning operation directly into her activity: One-third of 2/3 meter is 2/9 meter, so four-thirds of the Lizards' distance would be four times 2/9 meter, or 8/9 meter. In this way of operating, fraction composition might have been woven her solution—i.e., she would have completed all the operations for taking four-thirds of 2/3 meter. It is not surprising that Deborah did not use these ways of operating given that she had not constructed a general fraction composition scheme in March. But because she had made an initial construction of such a scheme, I contend that operating in these ways was within her short-term ZPC in May.

Bridget's Solution of Task 6.6 in May

Reversible reasoning: Constructive Resource 1. For Bridget, just as for the other three students, reversible reasoning with fractions was crucial in her solution of RMR problems and in particular of her solution to Task 6.6. That is, for Bridget two-fifths was one-fifth two times, so she knew that partitioning two-fifths into two equal parts produced one-fifth that could iterated five times to make five-fifths, the Cobras' distance, or iterated three times to make the amount (three-fifths) she needed to add onto the two-fifths to get five-fifths. I infer that she used her reversible partitive fraction scheme with splitting to operate in this way, and that at least within the whole, unit fractions were iterable units for her. Since she demonstrated this way of operating during the first episode of the teaching experiment (cf. Task 6.8, Protocol 6.4), I infer that she entered the teaching experiment with this scheme.

New uses of multiplying schemes: Constructive Resource 3. Like Carlos, Bridget seemed most comfortable using her basic (nondistributive) splitting operation to accomplish her goal of partitioning two-fifths of the Cobras' distance into two equal parts. That is, during the previous teaching episodes when I had posed Type 5 RMR problems involving multiple-unit quantities (e.g., a nonunit fractional quantity), she was largely unable to operate unless she could clear the markings on the bars. On May 12th I intentionally limited my problems for her to unit-fractional quantities so that she *could* operate.

I conjecture that unlike Carlos, who constructed distributive *splitting* at the end of March, Bridget constructed distributive *partitioning* (cf. Protocols 6.10 and 6.11) at that same time. That is, Carlos distributed his partitioning operation across a 2-unit quantity in order to reason *reversibly* (cf. Task 5.5, Protocol 5.3), while Bridget did not need to reason reversibly to make 1/2 yard from 2/3 yards (cf. Task 6.14) or 1/5 yard from 3/4 yards (cf. Task 6.15). Bridget's tasks were explicitly fractional and allowed me to attribute the initial construction of a co-measurement scheme to her, which I could not attribute to Carlos. But, distributive splitting may be a more powerful operation in that it requires positing a part separate from the original quantity (and yet in relation to it) that is *not* required in Bridget's distributive partitioning tasks. Once Carlos had split distributively he could partition distributively in problems that did not require splitting (e.g. in making one-third of a 2/5-yard bar, cf. Task 5.20). But it's possible that being able to *partition* distributively does not necessarily entail being able to *split* distributively.

Yet at the start of the teaching experiment Bridget's splitting operation seemed stronger than Carlos's splitting operation, so this explanation may not be sufficient—in fact, it ignores the significant complexity of the Type 5 RMR problems on which Bridget was working. Thus I conjecture that a central reason that Bridget was unable to split distributively in solving *fractional* RMR problems was that three levels of units were required in multiple ways. First they were involved in coordinating two fractional parts in the same bar—for example, making three equal parts from a 2-part bar as in Deborah's problem, Task 6.2. As I have already described, this coordination entails viewing the 2/3-bar as two different three-levels-of-units structures. Second, they were involved in determining the relationship of the smallest parts in the bar to the unit bar in order to find the measure of the new quantity. This activity entails taking the 2/3-bar as a unit of two units in relation to the unit bar, and using recursive partitioning to view the unit bar as so many of the smallest units (i.e., in Deborah's case, the unit bar was a unit of three units each containing four units, so the smallest units were twelfths).

Since throughout the teaching experiment Bridget tended to "lose" one level of unit (often the unit bar), it is not a surprise that dealing with nonunit fractional quantities in Type 5 RMR problems was too great a load for her at this point. In any case, in the context of solving Type 5 RMR problems, a distributive splitting operation did not seem available to Bridget, and the problems she could solve (like Task 6.6) did not require her to use her multiplying scheme to flexibly switch between two different three-levels-of-units views of a bar.

Activity with improper fractions: Constructive Resource 2. This constructive resource also was not required for Bridget to solve Task 6.6, which was appropriate for her then, given that I cannot conclude that she had constructed an iterative fraction scheme during the teaching experiment. In the same episode, the first problem she solved was the following: The Lizards' car travels 1/2 meter, which is 2/5 of the distance of the Cobra's car. Bridget determined that the Cobras' car went five-fourths of a meter, which she said in a questioning way. The questioning may not have referred to the improper fractional quantity, but given Bridget's tendency to question improper fractions even in May (cf. Task 6.11), it is possible that Bridget still thought "five-fourths" was a little strange—not quite legitimate, perhaps. By design, the solutions to all the rest of her problems during this episode were proper fractions, since my goal was to activate her current schemes in solving Type 5 RMR problems rather than pose problems that required schemes beyond those she had constructed. Like Carlos, I conjecture that not yet having interiorized three levels of units was a serious constraint for Bridget in solving RMR problems, and as the last episode in May demonstrates, limited the range of problems she could solve.

Fraction composition activity: Constructive Resource 4. Similar to the other students, Bridget did not seem to view her solution to Task 6.6 as a fraction multiplication problem—independently initiating the multiplication of 1/7 meter by five-halves to get 5/14 meter as a solution was well beyond her current schemes. However, she did use recursive partitioning to solve an *implicit* unit fraction composition problem: She found one-half of 1/7 meter in order to assess that her smallest parts were fourteenths of a meter. Furthermore, in solving the variation on Task 6.6, she found one-half of 1/11 meter without making bars. In this sense, Bridget appeared to embed recursive partitioning into her ways of operating with unit fractional RMR problems of Type 5. Being able to do so is one basis for my claim that she was constructing a scheme for solving Type 5 RMR problems with unit fractional quantities.

Fractions as operations: Constructive Resource 5. However, as I have described, multiplicative relationships between the two quantities in these problems appeared to be unavailable to Bridget. In Task 6.6, even though the problem stated that the Lizards' 1/7 meter was two-fifths of the Cobras' distance, it is unlikely that Bridget knew that multiplying the Cobras' distance, 5/14 meter, by two-fifths would (or should) produce 1/7 meter (let alone that five-halves times 1/7 meter was a way to arrive at 5/14 meter). Since I contend that constructing these relationships requires (at least) coordinating three levels of units prior to operating and flexibly appropriating any quantity as a unit to which another quantity can be compared, it is not surprising that fractions as operations were *not* a constructive resource for Bridget in solving Task 6.6 in May. Making this construction awaits Bridget's future mathematical activity.

CHAPTER 7: SUMMARIES, IMPLICATIONS, AND CONCLUSIONS

Through the previous two chapters, the case studies of each pair of students, I have responded to three of my five research questions. That is, I have described ways in which a teacher (myself) and students (Michael, Carlos, Deborah, and Bridget) can establish mathematical caring relations during extended interaction aimed toward engendering algebraic reasoning (question 1); I have characterized these sixth-grade students' multiplicative structures, given that they entered the teaching experiment reasoning multiplicatively with whole numbers (question 3); and I have explored how I interacted with these students in order that they might generate multiplicative reasoning with fractional quantities (question 4).

In this final chapter, I have four goals. First, I summarize my response to question 1 and consider what it means to establish mathematical caring relations with regard to the evolution of the students' orientation to and engagement in quantitative and algebraic activity in the context of the teaching experiment (question 2). Second, I draw upon my responses to questions 3 and 4 in order to consider the ways in which the students' schemes and operations were (and were not) algebraic. Doing so is my response to research question 5, how a teacher can interact with students to bring forth their construction of linear equations and their solutions. Third, I present five conclusions from my study, drawn from my responses to all research questions. Of course, conclusions are also beginnings, so, fourth, I offer suggestions for improving the study of mathematical caring relations and the study of students' algebraic reasoning, and I identify areas for further research that have been stimulated by this study.

Summaries and Implications of MCR

> "Everything we do is a structural dance in the choreography of coexistence" (Maturana & Varela, 1987, p. 248).

> "We usually think that if something is not one, it is more than one; if it is not singular, it is plural. But in actual experience, our life is not only plural, but also singular. Each one of us is both dependent and independent" (Suzuki, 1996, p. 25).

In this section, I summarize how each of the four students and I established mathematical caring relations during extended interaction aimed toward algebraic reasoning (i.e., I summarize my response to research question 1). The summaries highlight how each student and I contributed to "the choreography of our coexistence" during the teaching experiment. From the summaries I draw five themes that might be regarded as expectations for teachers who are attempting to enact MCR with their students. Then I consider the implications of MCR for students' engagement in and orientation to quantitative and algebraic activity during the teaching experiment (i.e., I respond to research question 2).

Enacting MCR

With Deborah.[75] Harmonizing with Deborah was, in some respects, rather easy because she coordinated multiple three-levels-of-units views of quantities. She could make sense of a wide range of multiplicative tasks by assimilating them to her current schemes, so she was highly adaptable. This characteristic was likely the main reason she seemed to have constructed a conception of herself as a top mathematical knower (recall that she was the only student of the four in the highest-level sixth-grade mathematics class). Both her adaptability and her mathematical self-concept meant that posing situations that challenged her to make a significant change in her ways of operating was, in some respects, rather difficult. As I have discussed, she appeared to derive considerable stimulation and self-confirmation from understanding

[75] I discuss the girls prior to the boys in order to highlight some aspects of MCR that I learned during the study.

mathematical situations and solving problems *swiftly*. In concert, she seemed to feel threatened by problems that she did not understand immediately and to have little patience with not knowing what to do or with exploring unfamiliar mathematical activity like drawing pictures (cf. the Box Problem, Task 6.3). So, despite her strengths, posing challenges for her that might engender the construction of new schemes, but were neither too easy for her nor too depleting (as in the Box Problem), was difficult.

In part, this difficulty meant that harmonizing with Deborah was not actually that easy—it was easy to *appear* to be harmonizing with her, but not easy to determine what coordinations of operations she had not yet constructed or to design problems for her that would engender such constructions. Her activity with fraction composition problems in March and with Type 5 RMR problems in May provide two good examples of this difficulty. In both cases, I believed that some of the tasks on which she and Bridget worked were quite easy for her (e.g., taking a unit fraction of a proper fractional amount, and solving Type 5 RMR problems in which the known quantity was a unit fraction, cf. Task 6.1). Solving such problems swiftly seemed, in some sense, confirming for her but not necessarily that stimulating or challenging.

Her swift solutions of such problems led me to overestimate her ability to solve any fraction composition or Type 5 RMR problem. Furthermore, based on my assessment of her affective state, I aimed to open the possibility for more significant stimulation (possibly preceded by some depletion). So, in March I tended to pose significantly harder fraction composition problems (e.g., with improper fractions, cf. Task 6.18) to her without necessarily building up to them. Similarly, in May the RMR problems that I posed presupposed that she had already embedded her distributive operation into her ways of operating in these situations. These problems did not necessarily help her to construct a robust fraction composition scheme in

March or scheme for solving any RMR problem in May, and such problems also could be depleting for her.

Thus from Deborah I learned two central aspects of enacting MCR. First, I learned that difficulty in posing challenges to a student is very likely accompanied by difficulty in harmonizing with the student. Second, I learned that a student's mathematical self-concept can impact the enactment of MCR. That is, Deborah's self-concept as someone who was smart mathematically and could do mathematics was certainly warranted, but it could get in the way of her patience and perseverance in encountering novel mathematical problems and of my assessment of her ways of operating.

With Bridget. In contrast, it seemed relatively hard to harmonize with Bridget's ways of operating. In retrospect, I can explain this difficulty based on her fraction scheme, which was not yet iterative, and her units-coordination prior to operating, which seemed to include three levels only later in the experiment (in late March). My difficulty harmonizing with Bridget stemmed from not understanding her ways of operating well at the time, even though I persistently saw that her ways of operating were different—and often seemed less powerful—than Deborah's. That is, in contrast with Deborah, Bridget was not able to assimilate as wide a range of tasks to her current schemes. But, because Deborah could be quite dominating, and because Bridget seemed reticent, I often tried to account for the differences between them by positing that Bridget did not have enough time to think or sufficient confidence in her own mathematical ideas.

By making such explanations, I did not take seriously enough the cognitive constraints that both Bridget and I experienced in her ways of operating—I had trouble decentering from my own ways of operating and my own ideas about her ways of operating. A good example of this phenomenon occurred in December, when the girls worked on problems like making a 4/4-bar into a 6/5-bar (cf. Task 6.12). Although I worked hard to provide more time for Bridget to make solutions to these types of problems, to hold Deborah's eagerness at bay, and to encourage Bridget, I have already noted that the time was rather "empty" for her. That is, giving Bridget more "think" time was *not* sufficient for her to find a way to solve such problems, let alone to coordinate dual three-levels-of-units views in the same bar (which likely would have been an act of vertical learning at that time). In response, her depletion increased considerably over the three episodes that we worked on these problems, and my depletion did as well!

My interactions with Bridget in December did not immediately trigger learning for either of us. She did not begin to coordinate two three-levels-of-units views in the same bar until the end of March (cf. Tasks 6.14, 6.15), and I did not take some of the cognitive constraints I experienced with her seriously enough until May, when again we entered a period of significant depletion in response to the tasks (Type 5 RMR problems, among other problems) that I posed to her. The learning I experienced in May was triggered by the history of our interactions over the year, combined with the view I had of Bridget when she worked alone on a computer, separate from Deborah. The significant depletion that she experienced over trying to solve problems that were outside of her short-term ZPC, and the subsequent depletion I experienced over my realization that I had not decentered sufficiently to understand some of her ways of operating throughout the teaching experiment, led me to adapt problem situations to my experience of her cognitive constraints. In turn, my adaptations appeared to engender in Bridget at least greater operativity and relief from depletion, if not an act of learning.

Thus Bridget taught me two other aspects of enacting MCR. First, I learned that attributing students' difficulties largely to affective or social issues is likely not sufficient to understand their difficulties well enough to harmonize with their current schemes, and so doing so may be depleting for both student and teacher. Second, I learned that "failing" to enact MCR over a short period of time can be critical in establishing it over a longer period of time. So in this sense, failing is also not failing.

With Michael. In comparison to my experiences with the girls, Michael and I seemed to have a relatively easier time of enacting MCR together—I seemed to pose problems that harmonized with his ways of operating and also opened new challenges for him so that he made changes in his ways of operating. Doing so seemed to be stimulating and rather satisfying for him. He rarely seemed overly depleted. Even when he waited for Carlos to solve problems that he could already solve, particularly in the spring months, he often used this time to rethink his ideas or to search for a more efficient way to solve a problem. Interestingly, in contrast with Deborah, *not* having constructed a strong mathematical self-concept seemed to work in his favor in enacting MCR with me. That is, although Michael certainly wanted to understand problems and worked hard to solve them, he seemed much more concerned with how well a solution made sense to him than with how swiftly he could arrive at such a solution. His interest in, and the satisfaction he derived from, the process of solving problems seemed to help him to receive my mathematical care and to enact mathematical care for me. Furthermore, not having a dominant image of himself as a strong doer of mathematics may have opened the way for our MCR to influence his construction of himself as a social-mathematical being.

But Michael's personal characteristics were not the only factor that facilitated our MCR. In general, throughout the teaching experiment, I tried out problems or sequences of problems with the girls before I tried them with the boys.[76] Thus I could make adaptations in problems before I used them with the boys. I could also adjust my hypothetical learning trajectory (HLT)

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[76] Recall that this lag in time (usually of about a month) occurred in part because of my perception of cognitive differences between the pair of boys and pair of girls at the start of the teaching experiment (cf. Chapter 4).

for the boys based on my experiences with the girls, even though the boys' and girls' ways of operating were not identical. These adaptations often seemed to work to the boys' advantage. For example, for the boys I eliminated the improper fraction aspect of problems that involved showing two fractional parts in the same bar, because this aspect seemed to unnecessarily complicate my main intent in posing such problems (e.g., make a 2/2-bar into a 3/3-bar, cf. Task 5.17, instead of make a 4/4-bar into a 6/5-bar, cf. Task 6.12). Michael had constructed an iterative fraction scheme and likely could have solved Task 6.12. But I believe simplifying it was beneficial for him: Abstracting a way of operating that was usable in subsequent problems like the Candy Bar Problem (Task 5.1) may have been easier from solving problems like Task 5.17 than from solving problems like Task 6.12.

So my learning with Michael confirms a theme that I identified with Deborah—namely, that a students' mathematical self-concept can influence the enactment of MCR. But my interaction with Michael also taught me that MCR may contribute to shaping a student's mathematical self-concept, and in particular a student's construction of himself as a social-mathematical being. In addition, from Michael I learned that enacting MCR as a teacher may be greatly affected by the teacher's previous interactions with other students—that posing problems with clearer intentions as a result of trying them first with the girls was a critical tool in my ability to harmonize with and open possibilities for the boys.

With Carlos. Although my interactions with Carlos did not generally carry the emotional charge that my interactions with the girls did, our enactment of MCR did not proceed as smoothly as my enactment of MCR with Michael. Somewhat like Bridget, Carlos was not always easy to harmonize with—if only because he demonstrated constraints in his ways of operating that I did not expect. I did not expect him to experience lasting difficulty with improper fractions or reversible reasoning because during his selection interviews he demonstrated that he had constructed a splitting operation (cf. Chapter 4). But his early difficulty in solving a Type 1 RMR problem ($39 is three times your money; how much do you have, cf. Task 5.10) helped me understand that his ways of operating were different from Michael's. So even though I remained quite puzzled over why Carlos persisted in making, say, thirteen-fourteenths when he was intending to make fourteen-thirteenths (cf. Task 5.13), and even though throughout the teaching experiment I continued to pose problems for him that were outside of his short-term ZPC, I tended to take the constraints I experienced with him more seriously than I did with Bridget.

Still, I had difficulty harmonizing with Carlos. Prior to the end of March, I didn't understand that had he not constructed dual three-level-of-units views of the same bar in order to show two fractions in the same bar, and that his persistent lacuna in making improper fractions was quite real to him—he was literally worried about not having enough material to make, say, seven-fifths of a collection of candy bars (cf. Task 5.15). At least up to this point I also had difficulty posing appropriate challenges for Carlos. I can now infer that posing RMR problems like the Candy Bar Problem (Task 5.1, cf. Tasks 5.2, 5.18) were overwhelming for him cognitively, even if he tended not to demonstrate undue emotional depletion.[77] In early March he could not solve these problems with his current schemes. But posing problems like the Magic Cake Problem (Task 5.14) to address his construction of improper fractions was also not appropriate for him: He could solve these problems in December and February without making any significant changes in his schemes for dealing with fractions larger than one. My sense was that such problems seemed easy to him—and therefore not that interesting or stimulating.

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[77] Just because he tended to express less did not mean that he felt less deeply. In May when we worked for a short while in an area setting with fraction composition problems, Carlos "saw" cross products very easily and solved some challenging problems before Michael did. Carlos expressed great joy at doing so, stating good-naturedly, "Finally, I beat Michael!"

In late March, when after several episodes I seemed to pose an appropriate challenge to Carlos (that 2-foot candy bar is three times another bar; make the other bar, cf. Task 5.5), I infer that I was finally harmonizing with his ways of operating. I harmonized with him during the episode on March 29th, going along with his initial desire to clear the foot mark, and then posing a new constraint (no clearing). But I also was more in harmony with his ways of operating in a larger sense because I took more seriously that he had a powerful splitting operation and partitive fraction scheme, but had not yet constructed ways of operating that Michael had constructed (e.g., a co-measurement scheme). So from Carlos I learned that posing appropriate challenges can also signal attaining greater harmony with a student's ways of operating. This theme is quite similar to what I learned from Deborah—that difficulties in posing challenges are rooted in difficulties in harmonizing with a student. Furthermore, as with Bridget, I learned from Carlos that "failing" to enact MCR with a student can be a significant trigger toward establishing stronger MCR "in the long run." My learning with Carlos was somewhat gentler than my learning with Bridget, because in contrast with Bridget, Carlos did not appear to get as depleted when he encountered problems that he could not yet assimilate to his current schemes.

Expectations for teachers. The summaries of MCR with each student indicate five themes that teachers might expect when enacting MCR with their students (see Figure 7.1). Although I have identified the main student or students from whom I learned each theme, all themes—broad findings about the enactment of MCR—emerged from interactions with all four students. Perhaps the most important finding is the extensive cognitive learning that enacting MCR requires on the part of the teacher, which takes considerable time and energy. Patience, perseverance, and discussions with others who can help the teacher examine her assumptions about students and refine her working second-order models of them are all highly valuable in order to undertake such learning.

Theme	From work with (primarily)
Difficulties in posing mathematical challenges for students are tied to difficulties in harmonizing with students' ways of operating, and posing challenges well (just on the edge of a student's short-term ZPC) often signals harmonizing well with the student's ways of operating.	Deborah, Carlos
The nature of a student's mathematical self-concept may influence MCR, and enacting MCR in turn may influence a student's mathematical self-concept, and in particular the construction of self as a social-mathematical being.	Deborah, Michael
Taking cognitive constraints experienced with students seriously means being wary of attributing difficulties in operating mathematically primarily to affective or social factors and requires significant decentering from one's own mathematical ways of operating and from one's assumptions about students' mathematical ways of operating.	Bridget
"Failing" to enact MCR in the short-term with a student may be necessary for enacting stronger MCR in the long-term with the student (and therefore failing is not necessarily failing).	Bridget, Carlos
Adapting problem situations and sequences based on previous interactions with other students can allow a teacher to improve the enactment of MCR with current students.	Michael

Figure 7.1, Themes from the summaries of enacting MCR with the four students.

Reflecting on the Enactment of MCR

Engagement. As is evident from the case studies in the previous two chapters as well as from the summaries in this chapter, participating in MCR may have influenced students' engagement in quantitative and algebraic activity during the teaching experiment. My interactions with Bridget in May and Carlos in March likely provide the most convincing evidence that MCR could facilitate students' engagement—that is, assist them in becoming mentally active and operative. My renewed attempts to harmonize with and challenge Bridget during the fourth episode in May, after her experience of considerable depletion during previous episodes in May, seemed to reestablish MCR between us and engender her renewed activity. As I have discussed in Chapter 6, she responded to my mathematical care by moving from a state of relative disengagement from operating with RMR problems of Type 5 to a state of solving a restricted set of such problems even in visualized imagination, without making bars.

Although less emotionally dramatic, Carlos made a similar change in engagement at the end of March after I adapted to his difficulty in solving Type 4 RMR problems like the Candy Bar Problem (Task 5.1, see also Task 5.2). My mathematical care for him included shifting to Type 3 RMR problems (cf. Tasks 5.18, 5.4, 5.5) and eventually accepting his initial solutions while posing challenges so that he might produce a more powerful solution (from my point of view). In turn, he responded with engagement both in the problem in which he first split distributively (Task 5.5), as well as in subsequent problems where he used this new way of operating (e.g., fraction composition problems like Task 5.20). Prior to his accommodation, he did not demonstrate the same kind of marked depletion as Bridget, so his change in engagement may appear less powerful. But his less intense emotional response in early March to being unable to solve Type 3 and 4 RMR problems should not detract from the significance of being able to sustain engagement in new problems after his solution of Task 5.5.[78]

The implications of establishing MCR with regard to engagement in mathematical activity during the teaching experiment were different for Michael and Deborah than for their partners. Throughout the experiment, Michael seemed to be the student most consistently interested in the problems and situations I posed. This interest persisted into May, when he wanted to meet even when we had little time available. However, he was remarkable not only

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[78] In fact, this example underscores that affective responses to problems outside of a student's short-term ZPC can vary considerably.

because he tended to engage, but because he seemed to remain aware of the satisfaction he felt from operating in the episodes. Even if he didn't feel such satisfaction at every moment, he seemed to trust that he could and would feel it again. This trust may have been, largely, a personal tendency, although it seemed to increase after his solution of the Candy Bar Problem on February 18th. Overall, Michael's trust was likely significant in sustaining his engagement and facilitating our MCR, and in turn our MCR seemed to open possibilities for him to engage in the way that he enjoyed on February 18th. So with Michael, MCR seemed to facilitate his persistent cultivation of the satisfaction he felt from engaging in our interactive mathematical activity.

In contrast, throughout the teaching experiment, Deborah may have been the student most suspicious of the work we were doing—and perhaps most likely to be worried over whether she would complete it satisfactorily (i.e., whether it would confirm or threaten her self-concept as a top mathematical knower). So she often demonstrated resistance to my questions or suggestions (cf. Protocol 6.1). At other times she engaged enthusiastically in mathematical activity, which may have been facilitated by MCR. For example, when I initiated "problem-solving workshop" during the first episode in May as a result of my sense that the girls (and Deborah in particular) felt constrained, she responded with concentrated engagement for several episodes. In fact, during these episodes she expressed a greater appreciation of using pictures to solve problems, or at least saw value in demonstrating her solutions with pictures, a significant change for her. Yet perhaps because of the difficulty in establishing MCR with Deborah, the influence of MCR on her engagement in mathematical activity during the teaching experiment seems uncertain.

Orientation. My discussion of the students' engagement in response to MCR necessarily overlaps with my examination of the influence of MCR on the evolution of students' orientation to quantitative and algebraic activity during the teaching experiment. In this study, I defined orientation as "general attitude toward."[79] Stronger engagement was often was accompanied by a warmer orientation to our mathematical activity, while disengagement often marked by a cooler or indifferent orientation to our activity. In general, the evidence for MCR as an influence on students' orientation to mathematical activity appears weaker than the evidence for MCR as an influence on students' engagement.

The most compelling evidence that MCR influenced students' orientation to mathematical activity in a positive way over the course of the experiment rests with Michael. As I have described, Michael's solution of the Candy Bar Problem (Task 5.1) on February 18th seemed to contribute to his construction of himself as a social-mathematical being. In my discussion in Chapter 5, I posited that his construction of *my* (and the witness-researchers') perceptions of him as demonstrating impressive mathematical activity likely contributed to his construction of himself as someone who could do and communicate about mathematics. The apparent shift in Michael's motivation for attending our sessions after this point contributes to my inference of the awareness he seemed to be gaining of himself—and his enjoyment of this awareness. In addition, although the episode in which he solved the Candy Bar Problem was pivotal, over the course of the teaching experiment Michael's awareness and enjoyment seemed to strengthen, so that he wanted to attend our sessions regardless of a lack of partner or limited time. Thus for Michael, MCR seemed to contribute to an orientation to appreciate and enjoy mathematical activity for the sake of the activity itself.

Interactions with Bridget support the possibility that a teacher's difficulty in enacting MCR with students may engender the evolution of negative orientations to mathematical activity. Her withdrawal from activity in both December and early May indicate that when MCR

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[79] Thus my definition of a positive orientation can be considered one part of the definition of "productive disposition" in *Adding It Up* (Kilpatrick, et al., 2001, p. 131).

is weak and "fails locally," some students may come to view mathematical activity negatively. Although in general Bridget said she liked math and in particular liked the math that we did (versus the math she did in her regular mathematics classroom), at her most depleted times she would complain that she hated certain problems or that she couldn't think in the right way. Fortunately, as shown in May, such influences seemed rectifiable. She did not appear to enjoy our mathematical activity the way Michael did, but when she and I enacted MCR well on May 12th, at least within that episode she transitioned to an open, relatively positive, orientation.

My interactions with Deborah and Carlos lend less conclusive support for MCR as an influence on a student's orientation to mathematical activity over the course of the teaching experiment. Like Bridget, Deborah's orientation to our activity tended to vary. She seemed to enjoy it, but unlike Michael, she did not seem as patient with the processes of "figuring out" or "making sense." Since part of enacting MCR with her meant engaging in these processes, sometimes using means other than what she was accustomed to (e.g., drawings), participating in MCR may have produced a sense of discomfort—even anxiety—in Deborah. Indeed, at times she demonstrated impatience with the problems and situations, which seemed to surface as impatience with me (e.g., when she accused me of confusing her) or temporary withdrawal from activity. Yet at other times, such as in May when she appeared to find new appreciation for making drawings, she demonstrated a remarkably positive orientation. Because of the variability in her orientation, the influence of MCR on her orientation to mathematical activity during the teaching experiment remains debatable.

If the girls' orientations to our activity seemed rather variable, Carlos's orientation seemed rather steady, regardless of whether he was able to solve the problems that I posed in a particular session. That is, with Carlos, MCR may have had less influence, perhaps because he seemed less swayed by feelings of depletion or stimulation during our interactions. At a specific time such as the end of March, when Carlos appeared to experience stimulation over solving a pivotal problem for him (Task 5.5), his orientation toward his activity may have become more positive. But in general, day after day, Carlos was a willing participant. He did sometimes say that his head hurt or that he forgot, indicating some level of discomfort. But in general he was game for experimenting even if he did not know how to solve a problem, and he demonstrated considerable enjoyment in using the microworld. Overall he did not seem to derive a sense of satisfaction similar to Michael's, perhaps because Carlos's current schemes constrained the level of organization of his activity in relation to the problems I posed.

Implications of the Students' Activity for Algebraic Reasoning

> "…by acknowledging the several different aspects of algebra and their roots in younger children's mathematical activity, a deeply reformed algebra is not only possible but very achievable within our current capacity for change" (Kaput, 1998, p. 25).

> Research question 5: How does a teacher interact with these students to bring forth their construction of linear equations and their solutions?

My four students did not construct and solve basic linear equations of the form $ax = b$. So my most blunt and direct response to my fifth research question is: We didn't get there yet. However, that response does not adequately address the intent of the question within the study, which involves how students begin to reason algebraically based their quantitative reasoning. So, a more nuanced response to my fifth research question entails considering how the students' schemes and operations were—and were not yet—algebraic. This response, although in part speculative, addresses a central goal of this project: to consider what constitutes algebraic reasoning for sixth-grade students. In this study, accomplishing this goal requires considering how sixth graders' reasoning with quantities is a kernel of their eventual algebraic reasoning.

How Are the Students' Schemes and Operations Algebraic?

Schemes and operations themselves. In discussing distinctions between quantitative and algebraic reasoning in Chapter 3, I noted that students' construction and use of schemes themselves may be considered algebraic based on the extent to which students seem to operate on or with the structure of their schemes. During the experiment, the students' ways of operating to solve key types of problems—RMR problems as well as problems comparing fractions larger than one, showing two fractional parts in the same bar, and composing fractions—varied in the level of structure I could attribute to them. In general, Michael's and Deborah's schemes appeared more structural than their partners' schemes, which is not surprising given the differences I have described in the students' coordinations of units prior to operating.

For example, Michael's activity in comparing large improper fractions was far more structural than Carlos's activity (cf. Protocol 5.9). Michael projected a structure of whole unit bars and parts of unit bars into an improper fraction like 21/5 prior to making it, and therefore could use this structure to compare fractions; Carlos generally needed to make fractions like 21/5 by iterating a single 1/5-part and only retrospectively, when asked, could he consider the number of whole unit bars and 1/5-parts that made up the fraction. Thus Michael's activity to compare improper fractions can be seen as more algebraic than Carlos's activity. A similar discussion could be made for the boys' ways of operating to show two fractional parts in the same bar and to make fraction compositions. In general, Carlos tended to assimilate problems to his strong partitive fraction scheme, and I have argued that he was constrained to doing so by his multiplicative structure. So throughout the experiment, his schemes seemed more enactive or experimental than structural, and his reasoning more quantitative than algebraic.

Analogous differences can be drawn between Deborah and Bridget. For example, in composing fractions Deborah not only coordinated dual three-levels-of-units views within a fractional part, but she coordinated these views with the fractional part's relationship to the unit whole (cf. Protocol 6.12), sometimes by using her distributive operation to make complex compositions (cf. Task 6.18). In contrast, Bridget tended to lose the relationship of the fractional part to the whole (cf. Protocol 6.12), and so her ways of operating were not as structural as her partner's. Like Carlos, Bridget was constrained by not coordinating three levels of units. Her fraction composition activity seemed more structural than Carlos's activity—I have inferred that during her fraction composition activity in March she was in the process of constructing a co-measurement scheme, rather than solely relying on her strong partitive fraction scheme. But I still can assess Deborah's ways of operating as more algebraic than Bridget's.

One final note is important, regardless of these judgments: The schemes and operations that all students used to solve RMR problems (i.e., splitting, splitting within a reversible partitive or iterative fraction scheme, distributive splitting, and a reversible multiplying scheme with or without fractions) are schemes and operations that could be used to solve basic linear equations. The students' operations and schemes allowed them to transform an initial relationship between the known and unknown quantities into a state that produced the solution (i.e., determined the unknown quantity). So the schemes and operations can be seen as way of generating equivalent states of quantitative relationships, and therefore as a kernel of generating equivalent equations that can be used to model a quantitative relationship and determine the value of an unknown.

Distributivity. In solving RMR problems of types 3, 4, and 5, using a distributive operation in some way appears to be crucial.[80] One of the most basic problems of Type 3 is Task

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[80] This statement is true as long as the known quantity remains marked into units of measure.

5.5: This 2-foot candy bar is three times longer than another bar; make the other bar and determine its length. Unless the foot mark is cleared (cf. Carlos's first solution of Task 5.5 in Protocol 5.3), students solving this problem are faced with modifying their splitting operation to include a distributive operation—that is, splitting each foot into three equal parts and taking two of these parts in order to split the entire 2 feet into three equal parts. I have called this operation distributive splitting. Although it is by no means the only way to use a distributive operation to solve an RMR problem,[81] it seems to open new possibilities for students to create a network of quantities related multiplicatively. As I have described, students may split distributively without being aware of a distributive pattern in their activity. But distributing the operation of splitting across units of a quantity in order to split the entire quantity is a creative mathematical act which I can recognize as a kernel of an algebraic distributive operation.

Abstracting the solution of Task 5.5 into a more general way of operating means that the student can split any quantity consisting of a whole number of units into any whole number of parts. So, distributive splitting may mean that the students' number world is "more continuous" than it was before, since splitting quantities into very small numbers of parts is possible. For example, a student who has a generalized distributive splitting operation may be able to solve a problem like this one: A 2-foot bar is 50 times longer than another bar; make the other bar. A distributive splitting operation also may mean that the student develops a concept of distribution as an activity with a useful end of reorganizing a quantity by operating on its parts, rather than only as a property that the student is supposed to know in order to, say, change the expression $(1/5)(a + b)$ to $(1/5)a + (1/5)b$.

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[81] Deborah's and Bridget's uses of distributivity in solving the Candy Bar Problem (Task 6.4) on January 14th, neither of which was distributive splting, demonstrate this point.

At some point during the experiment, all four students used a distributive operation to split or partition a continuous quantity, although Carlos and Bridget did not seem to construct these operations as generally as their partners did. Michael's solution of the Candy Bar Problem (Task 5.1) is one of the best examples of distributive splitting to solve an RMR problem. In order to split the 7 inches of old collection of candy bars into three equal parts, he split each inch into three equal parts. Although I cannot claim that Michael was yet aware of the pattern of his activity as distributive—he had not yet abstracted distribution as a concept—I propose that his activity was foundational for doing so.

Positing unknowns. In reasoning reversibly to solve RMR problems, all four students were involved in positing unknowns—i.e., in imagining an unknown quantity in multiplicative relationship to a known quantity. Positing an unknown does not seem required in the same way when reasoning with multiplicative relationships between quantities in a "forward" direction, perhaps because students tend to take it for granted that (whole number) multiplication makes things bigger. That is, if you have $39 and I have three times that amount, engaging in a process to find three times $39 may not require thinking much about the unknown amount of money; by virtue of what students know about whole number multiplication, the unknown amount has to be bigger than $39.[82] In contrast, as I have examined with Carlos's work in solving RMR problems of Type 1 (cf. Tasks 5.10, 5.11, 5.12), if $39 is three times my amount of money, I have to reverse my thinking. In some sense I have to consider $39 as the end result of the multiplication and determine that my money will be a smaller amount than $39—in fact, exactly three times smaller. Positing a smaller amount that can be operated upon to produce $39 seems to be a kernel of positing unknowns that can be notated and operated upon, a hallmark of algebraic reasoning.

[82] This conception may also be linked to early conceptions of multiplication as only iteration. As I have argued in Chapter 3, multiplication as solely iteration is insufficient for algebraic reasoning, and it is not even called "multiplicative" by some researchers (e.g., Kamii & Housman, 2000).

One of the best examples of positing an unknown quantity in relation to a known quantity is Bridget's work on the Box Problem (Task 6.3), because it is so explicit. Bridget's initial drawing (cf. Figure 6.6) clearly shows that she visualized the known quantity, 3/4 dm (the Cobras' height), and determined that the unknown height (the Lizards' height) would have to be taller, since the Cobras' height was two-thirds the Lizards' height. Drawing the unknown quantity in relation to the known demonstrates that she was reasoning with a quantitative relationship. But it also shows that she had begun to consider an unknown quantity in a similar way to a known quantity: Even though it was not yet known, it could be drawn, and the drawing of it could be altered in relation to the known—i.e., operated upon.

It is important to recall the nature of Bridget's work during the six sessions that preceded her drawing of the Box Problem on February 18th. From January 21st through February 11th, Deborah and Bridget had drawn two unknown quantities in relationship to each other to represent quantitative problem situations like this one: One mountain is some number of meters tall; another mountain's height is four-thirds of the first mountain's height; draw a picture to show the two heights. This work may have influenced Bridget's representation of the Box Problem because on January 14th, prior to this work, her representation of the Candy Bar Problem (Task 6.4 and 5.1) was markedly different.

Bridget solved the Candy Bar Problem creatively (cf. Figures 6.8 and 6.9), but she did not explicitly draw (or even estimate) the unknown collection of candy bars in relation to the known collection at the beginning of the problem. Indeed, the point of the problem was to figure out how to draw the new collection from the old collection. So Bridget operated on the known collection to produce the unknown. She and Deborah appeared to know that the unknown collection would have to be longer than the known, since the known collection of 7 inches was three-fifths of the unknown collection. So I infer that they were still positing an unknown mentally—they at least implicitly knew they were making a bigger quantity. But they did not pictorially represent the unknown and known in relationship to each other. This way of operating was typical of the students throughout the teaching experiment. Mentally positing an unknown seems to be critical for later representing it as Bridget did in the Box Problem.

How Are the Students' Schemes and Operations Not Yet Algebraic?

Operating on unknowns. Even though students mentally posited unknowns in reasoning reversibly, they usually did not explicitly operate on representations of unknowns. For example, like the girls, in solving the Candy Bar Problem (Task 5.1) Michael did not represent three-fifths of the unknown length. And in solving Task 6.2 (2/3 of a meter is 3/4 of the Cobras' distance), Deborah did not represent three-fourths of the unknown distance. In all cases, students had to consider the unknown and its relationship to the known in order to operate on the known to produce the unknown. For example, because Michael knew that three-fifths of the unknown length was 7 inches, he formed a goal to split 7 inches into three equal parts. By doing so, he was implicitly conceiving of three-fifths of the unknown split into three equal parts. Similarly, because Deborah knew that two-thirds of a meter was three-fourths of the unknown distance, she knew that she needed four one-thirds of the known distance to make the unknown. Implicit in her activity was finding four one-thirds of three-fourths of the unknown distance, which would produce the unknown. But since representations of these operations were not explicit, the students' solutions tended to remain more quantitative than algebraic in this respect.

Reciprocity. One reason for not operating on knowns may have been that the students, with the exception of Deborah, had yet to construct or use reciprocal relationships when solving RMR problems. For example, in solving the Candy Bar Problem (Task 5.1), Michael did not appear to conceive of the new collection as five-thirds of 7 inches, let alone use that reciprocal relationship to solve the problem. Similarly, Carlos did not overtly take one-third of 2 feet (let alone one-third times two) in solving Task 5.5, and Bridget did not take five-halves of one-seventh of a meter to make the unknown distance in solving Task 6.6. So Michael's, Carlos's, and Bridget's reasoning in solving RMR problems was primarily "one-way," from known to unknown rather than bi-directional, moving back and forth between known and unknown. This characteristic allows me to assess their reasoning as more quantitative than algebraic.

Deborah, as I stated, was an exception. She had constructed reciprocal relationships between known and unknown quantities to the extent that she could state them after solving an RMR problem. She also seemed to use reciprocal relationships to solve RMR problems (cf. Protocol 6.1), although not to the point where she treated the solution as a fraction composition problem (i.e., not to the point where she took four-thirds of two-thirds of a meter to solve Task 6.2, for example.) Because of her ability to state and use reciprocal relationships, I view her reasoning to solve RMR problems as more algebraic than the reasoning of the other students. That is, she had constructed bi-directional relationships in a way the other students had yet to do, which allowed her to operate on a known and unknown quantity in a "simultaneous" manner, flexibly viewing either quantity as the basis for producing the other quantity (cf. Chapter 6).

Operating on algebraic notation. Finally, of course, the four students did not operate on algebraic notation to solve RMR problems. We frequently verbalized relationships between known quantities, particularly in fraction comparison and fraction composition problems. But much less frequently did we verbalize unknowns or operations upon them, let alone use written notation to represent unknowns. As I have discussed in Chapter 6, the foray that the girls and I made into notating quantitative relationships and equations with written algebraic symbols was mostly unsuccessful, since their schemes and operations seemed to become suppressed in the process. The lack of success of these activities meant that I was hesitant to try something similar with the boys, and so they and I did not leave the microworld during the experiment. I now conjecture that working operatively and engaging in written notation could proceed in greater alignment, as I suggest in the final section of this chapter. In general, the issue of written notation and symbolization is a large one that I cannot claim to have explored in this study.

Conclusions

> "It should be obvious by now that if your ending fails, your whole story fails" (Kaplan, 1997, p.167).

Fortunately, Kaplan was referring to ending a piece of fiction—specifically, a literary short story. Hopefully his statement applies less strictly to writers of educational research, because it is hard to judge (for me at least) what ending to the story of this study would constitute a success or, conversely, a failure. In this section—ostensibly, my ending—I present five conclusions from my study. The first and second conclusions respond directly to the two initial, over-arching conjectures I made prior to the study (cf. Chapter 4). The third, fourth, and fifth conclusions evolved in the progress of the teaching episodes and analysis in ways that I did not anticipate before conducting the study. In describing these findings, I hope I have achieved an ending that succeeds in bringing together major themes from the case studies as highlighted in the summaries and discussions of implications in this chapter.

Conclusion 1: MCR May Facilitate the Ability to Reestablish Viability

First conjecture. At the start of the study, I conjectured that participating in mathematical caring relations would augment students' ability to reestablish the viability of their ways of operating in their mathematical environments (cf. Chapter 4). I briefly review the train of reasoning that led to this overarching conjecture. Initially I conjectured that if, over time, a student's feelings of stimulation outweighed feelings of depletion, she would feel mathematically cared for, which would contribute to her ability to sustain engagement even through perturbations that were consciously conflictive. Being able to do so could open opportunities for her to eliminate perturbations in her environment by making accommodations in her schemes. So I conjectured that a student's participation in MCR could be beneficial for her mathematical learning. In fact, I viewed the relationship between reestablishing one's viability and sustaining engagement in mathematical activity as cyclical, with MCR as an entrée into the cycle.

Discussion. Michael's activity provides support for this overarching conjecture in the sense that, as the teaching experiment progressed, he regularly sustained engagement through perturbations that were consciously conflictive and he seemed to find the process of reestablishing his viability increasingly satisfying. Toward the end of the experiment, when he was stumped by questions about fractions as operations (e.g., what to take times one distance to get another distance), he persisted in his steady engagement, stating on Monday, May 10th that he would figure out the problem before the final episode on Wednesday, May 12th. Even though he didn't do so, he continued to persevere in solving the problem during that final meeting (cf. Task 5.25, Protocol 5.15). After retrospective analysis, I now view Task 5.25 as at the edge of Michael's short-term ZPC at that time. So the progress he made toward a solution of it is impressive and demonstrates that he had developed a strong ability to work toward reestablishing viability in his mathematical environment. Michael entered the teaching experiment with solid schemes that likely contributed to his tendencies to enjoy the process of reestablishing viability and to persevere even during consciously conflictive perturbations. I cannot claim that Michael developed the ability to reestablish viability solely from engaging in MCR. But I can infer that MCR facilitated his development and use of this ability in our mathematical interaction.

Carlos's activity provides less holistic, or possibly less dramatic, support for the conjecture. As I have mentioned, Carlos often sustained engagement in our sessions even though, retrospectively, I posit that during some periods of the experiment the problems I posed were outside of his short-term ZPC. So his perseverance may have had less to do with the enactment of MCR and more to do with his personal qualities and, perhaps, his affinity for working with his partner. Still, at times MCR may have influenced his reestablishment of viability. For example, enacting MCR with Carlos in March seemed to allow him to increase his ability to eliminate perturbations by using what he learned on March 29th (from his solution to Task 5.5) in other situations (e.g., making 1/3 of 2/5 of a yard, cf. Task 5.20, on March 31st).

The support for the first overarching conjecture from the girls is less clear. For example, enacting MCR with Bridget on May 12th seemed to facilitate her ability to reestablish viability in that session: That is, she could operate in visualized imagination with RMR problems starting with a unit fractional quantity (cf. Protocol 6.2), and her depletion from earlier episodes in May seemed alleviated. However, during much of May (and at other times like December) her ability to reestablish viability seemed to *weaken* as I struggled to harmonize with her ways of operating. One could say that these observations support my first conjecture, if one grants that from May 3rd to May 10th I was not enacting MCR with Bridget. But, I don't want to grant that—as I have discussed earlier in this chapter, part of enacting MCR with a student may include not enacting it well in a particular moment in order to enact it well over time. So not enacting MCR well locally, in a given interaction, must be incorporated in the overall project of establishing MCR in the long run. This view may imply that my first conjecture must be modified—namely, that strengthening the ability to reestablish viability through participation in MCR may include times when a student's ability to do so may weaken.

Deborah's activity supports this modified conjecture. In part due to her solid schemes, Deborah's ability to reestablish her viability in her mathematical environment was quite strong at the beginning of the teaching experiment. This ability seemed threatened at times during the experiment because her schemes did not necessarily serve her well in eliminating perturbations triggered by problems that could require drawing, or for which she did not have calculational recourse, as shown in her response in February to the Box Problem (Task 6.3). Furthermore, from my point of view, being unable to reestablish her viability was itself a perturbation for her since it threatened her mathematical self-concept. So posing situations to open possibilities for Deborah to expand her ways of operating was something of a risky business and may not have strengthened her ability to reestablish her viability during any given episode. However, her appreciation in May of the usefulness of drawings to solve problems or demonstrate solutions indicates that our MCR may have contributed, in the long run, to augmenting her ability to reestablish her viability by opening new possibilities for her in solving problems.

Closing remark. With the caveat that participating in MCR is not the only influence on a student's ability to reestablish viability in extended mathematical interaction, and with the modification that this ability may weaken at times in the process of enacting MCR, I can conclude that participating in MCR over time *may* help students strengthen their abilities to reestablish viability in their mathematical environments. The evidence for this first overarching conjecture of the study is not conclusive, however, and the issue warrants further exploration.

Conclusion 2: Units-Coordination is Critical in Solving RMR Problems

Second conjecture. At the start of the study I also conjectured that students' multiplicative structures with whole numbers would significantly constrain or open possibilities for their quantitative and algebraic reasoning, and specifically for their multiplicative reasoning with fractions and their solutions of RMR problems. This conjecture was confirmed by the activity of all students in the teaching experiment. That is, a central finding of my study is that units-coordination is critical in solving RMR problems. In particular, coordinating three levels of units prior to operating plays a fundamental role in the construction of schemes and operations to solve RMR problems of Types 3, 4, and 5.

Discussion. As I have described in the previous section on the algebraic nature of students' schemes and operations in this study, distributive splitting is a critical operation in the solution of RMR problems of Types 3, 4, and 5.[83] A somewhat more general manifestation of distributive splitting involves recursively partitioning a given quantity into a composite unit that can be split into a target number of equal parts. In distributive splitting or this more general manifestation of it, a student aims to insert units into each unit of a quantity consisting of some number of equal units. Determining the number of units to insert into each unit of the quantity comes from being able to reorganize the quantity into a *different number* of units of units. Thus, to split the quantity usefully and purposefully means flexibly switching between two different views of the same quantity, which seems to require taking the quantity itself as a unit consisting of two different units-of-units structures. So distributively splitting a quantity seems to require taking the quantity as a unit of units of units prior to operating, holding one such structure in mind while switching to the other structure.

The activity of all four students supports this analysis. Early in the teaching experiment, both Michael and Deborah appeared to coordinate three levels of units prior to operating, as demonstrated by their iterative fraction schemes and their reasoning to compare large improper fractions (cf. Protocols 5.9 and 6.7). And prior to using distributive splitting to solve RMR

[83] Unless, as shown with Bridget, the known quantity in the problem is one unit, such as a unit fractional quantity.

problems, they both constructed what I have called co-measurement schemes by solving problems that involved showing two fractions in the same bar. Solving these problems required viewing a bar as unit consisting of two different units-of-units structures. For example, in January, when Michael make a 5/5-bar into a 4/4-bar, he constructed two different views of the bar as a unit of five units each containing four units, and as a unit of four units each containing five units. Similarly, in December Deborah made a 3/3-bar into a 9/8-bar by determining how to further partition eight-eighths in order to also show three-thirds (cf. Protocol 6.9).

I have discussed how their co-measurement schemes were significant constructive resources in their solutions of RMR problems like the Candy Bar Problem (Task 5.1), which Michael solved in February, and problems similar to the Box Problem (e.g., Task 6.2), which Deborah solved in May. In solving these RMR problems of Types 4 and 5, both Michael and Deborah embedded their multiplying schemes into their reversible iterative fraction schemes. I have inferred that such activity was an accommodation of both their multiplying and fraction schemes, and that the result was the construction of a new scheme that I have called a reversible multiplying scheme with fractions.

In contrast, neither Carlos or Bridget coordinated three levels of units prior to operating, as demonstrated by their lacunas in their schemes for making fractions larger than one (cf. Protocols 5.7 and 6.6). They did not solve problems that required viewing a quantity as a unit containing two different units-of-units structures until March (cf. Protocols 5.2, 5.3, 6.10, 6.11, and 6.12), and the coordinations they made in their ways of operating at that time seemed to be less general or powerful than the coordinations their partners had constructed. For example, Carlos's solution for Task 5.5 did not explicitly involve fractions, so I cannot claim that he constructed a co-measurement scheme at that time. I have called his scheme a reversible multiplying scheme, at least for conceiving of a 2-unit quantity as three times larger than another. In contrast, Bridget coordinated two fractions within the same bar and so constructed a co-measurement scheme (cf. Protocols 6.10 and 6.11). But she seemed to use this scheme more readily in making fraction compositions than in solving RMR problems, perhaps because she did not necessarily modify her *splitting* operation, only her partitioning operation.

Furthermore, even after March, neither Carlos nor Bridget appeared to regularly coordinate two different three-levels-of-units views within a fractional part of a quantity *with the relationship of that part to the unit quantity*. For example, Bridget's "mathematical universe" tended to become the fractional part with which she was currently working (e.g., three-fourths of a yard) and thus she named fractional parts of it in relation to it, rather than in relation to the unit quantity (e.g., for her two-fifths of three-fourths of a yard was initially six-fifteenths not six-twentieths, cf. Protocol 6.12). Carlos demonstrated a certain "overload" in terms of units-coordination by making estimates of fractional parts of fractional parts and then using his partitive fraction scheme to test out whether the estimates worked (cf. Protocol 5.12).

Both of these characteristics imply that Carlos and Bridget had not yet embedded their multiplying schemes into their reversible fraction schemes. That is, they did not appear to use distributive splitting or partitioning within a larger fractional context, but only within a single bar or quantity. Carlos and Bridget still had powerful schemes and operations—they were each able to solve quite a lot of problems by relying on their *associated* (but not embedded) whole number multiplying schemes and reversible partitive fraction schemes with splitting. But I infer that the lack of embedment was significant in constraining Bridget to solving Type 5 RMR problems with unit fractional quantities in May, and in constraining Carlos to experimenting with his partitive fraction scheme to work on RMR problems of Types 3 and 4.

Closing remark. Coordinating three levels of units prior to operating seems to be crucial in the construction of a quantity as a unit consisting of two different units-of-units structures. In turn, constructing two three-levels-of-units views of a quantity appears to be a critical constructive resource in the construction of distributive splitting and in embedding one's multiplying scheme into one's reversible fraction scheme. These constructions appear to open the way to solving RMR problems of Types 3, 4, and 5.

Conclusion 3: Units-Coordination is Critical in Constructing Improper Fractions

Steffe's hypothesis. Steffe (2002a) has hypothesized that when students have constructed a splitting operation,[84] the construction of improper fractions is available to them. He states, "Upon the emergence of the splitting operation, I regard the partitive fractional scheme as an *iterative fractional scheme*" (p. 299, italics in the original). He made this statement with regard to his analysis of Jason and Laura, two students who participated in the teaching experiment, *Children's Construction of the Rational Numbers of Arithmetic* (Steffe & Olive, 1990). In particular, Jason and Laura had been unable to split or to produce improper fractions during their fourth grade. However, Jason could split at the start of his fifth grade, and he was also able to produce improper fractions (Steffe, 2003, in press). So Steffe had good reason to consider splitting as instrumental in opening the way to the construction of improper fractions. Therefore, it was a major surprise that, although both Carlos and Bridget could split, neither one seemed to construct improper fractions or an iterative fraction scheme during my teaching experiment.

Discussion. Splitting still is likely instrumental in the construction of an iterative fraction scheme. But Carlos's and Bridget's activity indicates that it is not sufficient because it does not seem to require the construction of three levels of units, which seems to be necessary to construct

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[84] Recall that if a student solves the following problem, it indicates that he has constructed a splitting operation: This rectangle is a drawing of my candy bar, which is five times longer than yours. Make your bar (cf. Example 3.4).

an iterative fraction scheme. To review, during selection interviews in September, both Carlos and Bridget demonstrated that they had constructed partitive fraction schemes and splitting operations (cf. Chapter 4). Based on Steffe's model of Jason, both Carlos and Bridget should have been able to construct an iterative fraction scheme.

Yet, as described in the case studies, Carlos and Bridget each demonstrated a persistent lacuna in making, and reasoning with, fractions larger than one. My current hypothesis is that the lacuna involved not having yet constructed three levels of units. So, for example, for Bridget 17/5 was a unit of 17 units, but she did not seem to form units of five of those units as the "whole" from which 17/5 was disembedded and yet to which 17/5 stood in relation. That is, 17/5 as a unit of 17 units, any of which could be iterated five time to make five-fifths, and 17/5 as (simultaneously) some number of five-fifths units and some number of fifths, did not seem available to Bridget prior to making 17/5. In fact, when she was to make a fraction like 17/5, sometimes she iterated one-fifth 17 times, and sometimes she iterated any available unit (e.g., five-fifths) 17 times (the latter often occurred when she was asked to take 17/5 as a given in further operating). Carlos's lacuna was similar, although he had an additional characteristic of sometimes making, say, thirteen-fourteenths, if he was supposed to make fourteen-thirteenths, because of his concern over not having enough material available (cf. Protocols 5.7, 5.8).

One way to interpret their surprising ways of operating with fractions larger than one is to examine their solutions of RMR problems of types I and II, which involved them in using their splitting operations. Bridget entered the teaching experiment splitting composite units. During the first episode on October 30th (cf. Protocol 6.3), she solved Type 1 RMR problems like Task 6.7 (Sara's stack of CDs is 65 cm tall, which is 5 times the height of Roberto's stack; what would you do to find out how tall Roberto's stack is?). During that same episode, she also split fractional quantities in reasoning reversibly with her partitive fraction scheme. So she solved Type 2 RMR problems, as shown by her solution of Task 6.8 (Tanya has $16, which is 4/5 of what David has; how much does David have?). Although Carlos could split, his initial difficulty on November 6th with Type 1 RMR problems (cf. Tasks 5.10, 5.11) shows he did not enter the teaching experiment splitting composite units. But in that same episode he learned to do so (cf. Task 5.12 and Protocol 5.6), and soon he split fractional quantities in reasoning reversibly with his partitive fraction scheme to solve Type 2 RMR problems. So both students used their splitting operations to solve RMR problems of types I and II early in the teaching experiment.

Their abilities to operate in these ways were different from Jason, who could not solve RMR problems of Type 2 before he constructed his splitting operation, even though he had a partitive fraction scheme. That is, in his fourth-grade year he could not solve a problem like Task 6.8 that involved him in reversing his partitive fraction scheme. In fact, before he had constructed a splitting operation, given a rectangle that represented two-fifths of a candy bar, Jason could not make the whole candy bar. So at that point, for Jason, two-fifths did not mean one-fifth two times. He could not split two-fifths into two one-fifths, which is not a surprise given that he had not yet constructed splitting (i.e., if he was given a bar that was two times longer than another bar, he could not make the other bar). Once he could split, he could use splitting to reverse his partitive fraction scheme *and* he could do more: make improper fractions, and reverse his making of them (e.g., make the whole bar given, say, nine-fifths of it). So for Jason, the construction of splitting seemed to be accompanied by the construction of three levels of units necessary for an iterative fraction scheme. In contrast, Carlos and Bridget could use their splitting operations to reverse their partitive fraction schemes without simultaneously constructing improper fractions, since for them the construction of splitting did not seem to be accompanied by the construction of three levels of units prior to operating.

Closing remark. My study warrants a revision of Steffe's (2002a) hypothesis. The splitting operation seems crucial in the construction of a reversible fraction scheme, where unit fractions are iterable at least within the whole. However, the construction of improper fractions seems to require the construction of three levels of units prior to operating, which does not necessarily follow from the construction of a splitting operation.

Conclusion 4: Constructing Fractions as Multiplicative Operations is Challenging

All four students had considerable difficulty seeing "taking a fraction of" an unknown *or known* quantity as multiplying that quantity by the fraction. Like Conclusion 3 above, this conclusion was unanticipated. Previous research has documented students' difficulty with operating on unknowns (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Researchers have also explored the importance and complexity of the operator "personality," or subconstruct, of fractions (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997; Freudenthal, 1983), and the challenges students have in using fractions as operators (e.g., Davis, Pearn, & Hunting 1993; Kieren & Southwell, 1979). So this conclusion confirms previous research findings.

Discussion. Deborah was the only student who seemed to construct unit fractions as multiplicative operations, which she first demonstrated in January by noting she could multiply by one-fourth to find one-fourth of the length of the room (cf. Protocol 6.12). In Chapter 6, I have given an account of her construction of one-fourth (or any unit fraction) as an operation based on constructing an equivalence between dividing a quantity by four and multiplying the quantity by one-fourth. However, that account may not be sufficiently basic enough since it relies on equivalence more than on the operations of her iterative fraction scheme. To construct a unit fraction like one-fourth as an operation may involve abstracting one-fourth as a *concept* (cf. Chapter 3) rather than conceiving of it solely as the result of a scheme.

As a concept, one-fourth would contain all the interiorized schemes the student has for making it (e.g., partition a quantity into four parts and take out one part), as well as the schemes that reverse the making of it—that make back the whole quantity from one-fourth of it (iterate that part four times). Together these operations would be abstracted as a "program" constituting the student's conceptual meaning for one-fourth. When one-fourth is a concept, reciprocal reasoning with one-fourth comes right out of the concept—one-fourth "contains" the notion of being able to be iterated four times to make back a whole because it contains reversible reasoning. So it may be that Deborah had abstracted unit fractions like one-fourth as concepts, while the other students had yet to do so. In this sense, her ways of operating with unit fractions can be considered more algebraic than the other students' ways of operating with unit fractions.

Since the notion of a fraction as a concept contains reversible reasoning, it includes a sense of simultaneous operating. That is, a concept of three-fourths contains both multiplying a quantity by three-fourths and the reversible reasoning involved in "unmaking" three-fourths of the quantity. Notions of simultaneity may not be fostered well by work in JavaBars (Biddlecomb & Olive, 2000) because in JavaBars, the only way to make three-fourths of a bar is to work sequentially. For example, to make three-fourths of a bar a student can divide a bar into four parts, take out one part, and repeat it to make three parts. Similarly, the only way to make back the whole is again to work sequentially: divide three-fourths of the bar into three parts, take out one part, and repeat it to make four parts. These operations are crucial in constructing fractions as results of schemes, but they may limit abstracting fraction schemes into concepts. Working in a microworld in which making three-fourths of four-thirds of a quantity can be performed as a simultaneous operation may assist students like Michael, who have constructed iterative fraction schemes but not unit fractions as operations, abstract fraction schemes into concepts.

Closing remark. Students' considerable difficulty with constructing fractions as operations suggests that further research is needed on what is required to make such a construction, including choices of software and activities that may facilitate the abstraction of a fraction as a concept, rather than the construction of a fraction solely as the result of a scheme.

Conclusion 5: MCR May Influence the Construction of Self-concepts

In Chapter 2 I posited that MCR may influence students' construction of themselves as people who can do and communicate about mathematics with others—as social-mathematical beings, one facet of people's mathematical self-concepts. Linking what one knows and with whom one has come to know it to one's construction of oneself is not uncommon (e.g., Bartholomew, 2002; Boaler, 2000; Belenky, et al., 1997; Desautels & Larochelle, 1991; Harding, 1991; Larochelle, 2000; Linchevski & Kutscher, 1998; Maturana & Varela, 1987; Wenger, 1998; Wortham, 2004). However, because enacting MCR entails a sustained effort to balance the energetic dynamics of student-teacher mathematical interaction, it may be particularly prominent in influencing both students' constructions of their mathematical self-concepts and teachers' constructions of themselves as mathematical carers for their students.

Discussion. My interactions with Michael provide the strongest support for this conclusion, as I have discussed with regard to our "structural dance" (Maturana & Varela, 1987) on February 18th, as representative of many other interactions between us. In particular, his construction of my view of him as an impressive doer of mathematics, combined with the satisfaction he derived from mathematical activity that was challenging but that he could reason through so that it made sense *to him*, apart from my judgment, seemed to be pivotal in his construction of himself as a social-mathematical being. In turn, his willingness to trust that I was posing problems that would allow him to feel this kind of satisfaction and that (at some level) would allow him to "know more," contributed to my construction of myself as a teacher who could enact mathematical care for a student in the ways I had envisioned theoretically. I can attest that for myself, and I can infer that for Michael, such interactions were very empowering!

My interactions with the other three students may provide weaker support for linking MCR to the construction of mathematical self-concepts. As I have described, I found it harder to harmonize with Carlos's and Bridget's ways of operating, which meant that we spent a good deal of time "failing" to enact MCR locally in order to enact it in the long-term. The times when we were not in harmony could impact the students' construction of their mathematical self-concepts, and they could also influence my self-concept as a mathematical carer. For example, in December, when Bridget seemed increasingly unable to operate in solving problems like making a 3/3-bar into a 9/8-bar (Task 6.13), I infer that her mathematical self-concept was weak (particularly in comparison to Deborah's). In turn, my inability at that time to conceive of problems that would harmonize with her ways of operating weakened my construction of myself as someone who could mathematically care for others. In contrast, periods when we harmonized well, such as during the last episode on May 12th, opened the way for her to re-conceive of herself as a doer of mathematics and to reinitiate trust in my mathematical care for her. I infer that any re-conception she made was not influenced solely by my approbation of her activity but, as with Michael, by her experience at this time that her mathematical activity made sense. These times also allowed me—particularly after the fervent search as in May—to re-conceive of my ability to learn the mathematics of my students and to enact mathematical care for them.

My interactions with Carlos during periods in which we were and were not in harmony were similar, although mitigated in emotional impact because he tended to demonstrate less emotional variability in relation to them. For example, the reappearance throughout the teaching experiment of his lacuna in making improper fractions (e.g., Protocol 5.8), as well as his difficulty with Type 3 RMR problems in March (cf. Task 5.18), definitely challenged my self-concept as a someone who could learn Carlos's mathematics and communicate with him. But his generally genial manner during these periods indicated that being stumped was not so threatening for his sense of self, even though he demonstrated a need to "save face" when he didn't know by commenting that his head hurt or that he'd forgotten. In general, even critical moments of the teaching experiment for him (e.g., his solution to Task 5.5) did not seem to affect him as strongly as Michael or Bridget appeared to be affected during some of their pivotal constructions. It is possible that, despite my praise for Carlos's ways of operating (e.g., Task 5.5, cf. Protocol 5.3), his schemes at that time in relation to the problems I posed may have limited the internal satisfaction he experienced from his activity and therefore limited the impact our interactions had on his construction of his mathematical self-concept.

Overall, my interactions with Deborah were probably the most challenging in terms of constructing oneself in relation to doing and communicating about mathematics. As I have stated, my attempts to expand her mathematical world, which inevitably caused her to confront situations where she did not immediately know what to do or did not have recourse to a calculation, could jeopardize her mathematical self-concept and cause her to feel vulnerable or insecure. Her insecurity occasionally manifested in asking comparison questions: were other students who worked with the me doing the same problems that she was; was she the "dumbest" one, etc. But overall her vulnerability seemed to result in a tendency to distrust the problems I posed and my ability as a teacher. Her suspicion, which was marked by frequent statements that I was confusing her, was challenging for me because it threatened my self-concept as a mathematical carer. So in turn, I felt quite vulnerable in interacting with her. Although at times Deborah and I enjoyed a certain level of mutual harmony, throughout the teaching experiment we both seemed to challenge each other in ways that are perhaps only empowering in retrospect.

Closing remark. Participating in MCR may impact students' constructions of their mathematical self-concepts, as well as teachers' constructions of themselves as mathematical carers. This conclusion speaks to the power of a student's construction of a teacher's views of the student as a doer of mathematics, but the teacher's regard is, generally speaking, not enough. For a student engaged in MCR to construct a productive mathematical self-concept, he appears to require experiencing a level of internal sense-making that allows him to trust (implicitly) that the teacher is giving him mathematical care and to believe the teacher's view of him as a powerful doer of mathematics. As with Conclusion 1, this conclusion warrants further study.

Suggestions

> "Expansion. That is the idea the novelist must cling to. Not completion. Not rounding off but opening out" (Gardner, 1927/1955, p. 169).

As I stated regarding Kaplan's quote about endings, I'm not sure that aphorisms about fiction writing can be imported into writing up educational research. Nevertheless, Gardner's idea of expansion seems a reasonable notion for the ending of many research studies, because doing research opens up as many questions as it may "answer." In this final section, I offer some brief suggestions for improving the study of MCR and for enhancing research on the transition between quantitative and algebraic reasoning, as well as some questions for future research.

Improving the Study of MCR

My study of MCR was highly generative as opposed to convergent (Clement, 2000), in that I attempted to formulate a theoretical perspective on student-teacher mathematical interaction and to study it. I have articulated ways of conceiving of a teacher's activity with students, such as harmonizing with and opening new possibilities for students' ways of operating, and ways of conceiving of students' responses to mathematical interaction with a teacher, such as experiencing stimulation and depletion. A central question for further study is whether these conceptions are sufficient to account for the notion of mathematical care. That is, do these theoretical conceptions sufficiently describe and explain the relatedness between teacher and student engaged in mathematically caring interaction? Do they adequately account for the structural dance between teacher and student? A deeper formulation of the development of trust in relation to the history of the teacher and student's interaction may be necessary.

In addition, I generated some observation categories with respect to MCR, such as examining students' bodily, facial, and verbal expressions as indications of their feelings of stimulation and depletion in student-teacher mathematical interaction. However, these categories are general at best, and therefore, my study of MCR is rather coarse-grained. If MCR is to prove useful as a framework for student-teacher interaction, further generative studies that attempt to produce finer observation categories are important. These studies may need to take place in both similar (e.g., small-group) and different (e.g., whole-class) settings.

Improving the Study of Moving between Quantitative and Algebraic Reasoning

Developing tasks for students like Carlos and Bridget. I have four broad suggestions for improving the study of the transition between quantitative and algebraic reasoning with respect to aiming to solve RMR problems algebraically. First, developing tasks and HLTs specifically aimed toward students like Carlos and Bridget, who have not yet constructed an iterative fraction scheme and are not yet coordinating three levels of units prior to operating, is critical.

Such tasks include *sharing problems*, in which students are asked to share, say, five candy bars fairly among three people and state how much of one candy bar much each person will get. Solving problems like these may be an important resource in constructing a distributive splitting operation. Other useful tasks are Type 2 RMR problems that push the boundaries of students' multiplication facts. For example, 72 ounces of water is four-sevenths of what Mary drank; can you determine how much water Mary drank? These problems may open the way to flexibly *splitting composite units* by reasoning strategically with whole number multiplication and division (e.g., to solve the example, students have to reason through dividing 72 by 4). Such activity is important in solving RMR problems more generally and in developing algebraic calculation. Finally, working on RMR problems of Type 3 with "broken" units (similar to the sharing situations, and similar to the Candy Bar Problem) may be useful. For example, this collection of seven inch-long candy bars is three times longer than another collection of bars; how long is the other collection? Extensive work with sharing problems and splitting composite units may open the way for students to conceive of the seven inch-long candy bars as a composite unit that they can split into three equal parts.

Coordinating two three-levels-of-units views. Second I suggest expanding the number of settings in which students might use two three-levels-of-units views of quantities, even with students who seem to have constructed dual views, such as Michael and Deborah. In my study, I did not take enough advantage of, for example, fraction addition situations as possible sites for the construction of two three-levels-of-units views. Exploring, say, what fractional part of a unit bar is constituted by one-half of the bar and one-third of the bar combined involves students in determining how to partition the bar to show both one-half and one-third in the bar at the same time. Working in many contexts that involve dual three-levels-of-units views may increase the possibilities for students to construct and use three levels of units when solving RMR problems.

Constructing reciprocity. My third suggestion involves the construction of reciprocity. As I have noted in Conclusion 4, working with JavaBars (Biddlecomb & Olive, 2000) alone may constrain the abstraction of fractions as concepts that can be foundational for constructing fractions as multiplicative operations and reciprocity. So, particularly for students who have an iterative fraction scheme like Michael and Deborah, I suggest working with both JavaBars and The Geometer's Sketchpad (GSP) (Jackiw, 2001), a program in which fractional amounts of quantities (represented by segments) can be made with dilations. In dilating a quantity by fractional amount like three-fourths, the operational activity of dividing by four and iterating by three happens "simultaneously" (and applies to each unit of the quantity, cf. Behr et al., 1997; Davis et al., 1993). In addition, to further contribute to students' abstraction of fractions as concepts, working with fractional areas as cross-products of fractional lengths may be important.

Engendering notation. Finally, I would encourage entwining written notation using standard numerals and algebraic symbols, as well as students' drawings, with their operational work in microworlds. The written notational work cannot be done all the time, in every episode, since it can interfere with students becoming operational in solving problems and constructing schemes. But I think it can be integrated into the students' activity in a way I did not accomplish. Shifting between operating in microworlds, verbalizing operational activity, making drawings, and writing with standard algebraic notation seems fruitful for students' construction of notation as a record of their ways of operating. It may also open the way to engendering this notation into a symbolization system in which the notation stands in for students' operations.[85] So a central revision that I would make in the study involves students periodically recording, in writing, knowns, unknowns, and their operations on them.

Further Research

I have identified four areas for further research from my study. As I have indicated in my discussion so far, students' construction of fractions as operations and of reciprocity merits further investigation. Questions include what operations, schemes, and concepts are necessary for these constructions, as well as what these constructions make possible in terms of algebraic reasoning. For example, how do students construct fractions as concepts? How does doing so influence the construction of fractions as multiplicative operations and reciprocity? What influence does written notation have on these constructions? To what extent is the construction of dilation important in these constructions? How do these constructions open the way to writing and solving linear equations of the form $ax = b$ in order to solve RMR problems (particularly those of Types 3, 4, and 5)? How do such constructions lead into other significant areas of algebraic reasoning, like the construction of rate and linear functions?

The second and third areas involve the study of MCR. As I have stated in my suggestions for improving the study of MCR, I am curious about how to explore closer links between student-teacher interaction, students' mathematical learning, and both students' and teachers' self-concepts: What theoretical conceptions need to be generated to construct such links? What kinds of observation categories and methodological adjustments are necessary? In addition, I am interested in what is generalizable about MCR. That is, I have constructed a way of operating from working with my four students that I believe will be useful in working with other students

[85] This kind of work has proved useful in continuing work with some of these four students, who are being taught during academic year 2004-2005 by the witness-researchers.

who seem to operate mathematically in similar ways (cf. Cobb, 2000). But what does it mean to enact MCR with a larger group of students? How does a teacher work to balance stimulation and depletion in this setting? I am curious about whether it would be important for students to take on some of the caring responsibilities for fellow students, in the sense that Michael seemed to do at times for Carlos, and how doing so might influence students' mathematical learning.

Finally, I am interested in how middle school mathematics teachers might find the orientation to caring and learning and the conclusions of this study usable, and how both MCR and a quantitative approach to algebraic reasoning might be investigated in classrooms. How do classroom teachers balance stimulation and depletion with students? How do they learn to experiment with provocations in order to use them skillfully to engender perturbations? In what ways does working to develop students' quantitative reasoning influence teachers' algebraic reasoning and their goals in teaching algebra? How does enacting MCR affect teachers' communication and sense of effectiveness with their students? How is an awareness of the nature of students' units-coordinations useful in posing quantitative and algebraic tasks for students? These questions, among many others, await further study.

In identifying these questions and areas for further research, I hope I have achieved the expansion that Gardner advocates. For although I have found a great deal of "answers" through doing this research—I have learned in ways that cannot be adequately expressed in the many pages of this document—I am even more excited about the many questions my study has opened for me to pursue in the future.

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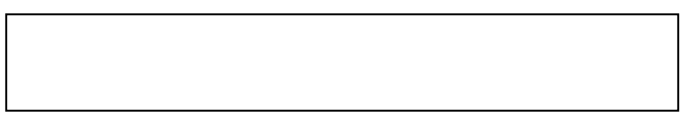
APPENDIX A: Selection Interview Guide

These questions were used with the 20 sixth-grade students during selection interviews in September and October of 2003 (cf. Chapter 4). The goals of the selection interviews were to make an initial determination of each student's fractional and multiplicative reasoning, and to assess each student's openness to the interviewer and to this mathematical activity. Questions 1 through 6a were asked of almost all students. Question 6b was asked of some students, including all four students in my study. Questions 7 and 8 were asked of only a few students, including Deborah and Bridget, but not including Michael or Carlos.

1. Here is a drawing of a candy bar.



Use a pencil to mark how you could share it fairly among five people, so that all people get an equal amount.

Are you sure that everyone would have a fair share? (If the student says yes, go to *. If not, let the student try again with another copy of the bar or make adjustments on the original bar until she or he is sure.)

*Do you think you got the pieces *exactly* right? (If the student says yes, go to **. If the student says no, let her or him adjust until she or he believes it's exactly right. If the student doesn't believe it's exactly right but it's close enough, go to **.)

Sara doesn't think you got the pieces exactly right (intention is to throw doubt on the situation). Can you think of a way to show Sara that you did? (If the student can think of a way, let her or him try it and go to *. If the student cannot think of a way, suggest as follows.) Could you use scissors to show Sara you got the pieces exactly right?

***Can you think of another way to show Sara? (Can keep asking about other ways, depending on time and intuitive impressions of the student.)

2. Here is another candy bar.

The candy bar is to be shared fairly among six people, so each person gets an equal share.) Make *one* mark to show your share.

Pretend that you can take your piece out of the bar. How could you use it to show that it's a fair share? (If the student moves finger along the bar or makes more marks, go to *.)

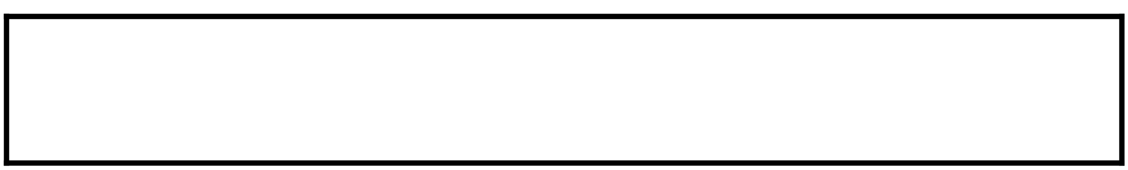
*(Give student another identical candy bar.) Use this candy bar to make your share. (If the student does not cut off her or his share, then go to this question.) What could you do to get your part separate from the candy bar? (If the student does not cut off his share with scissors, then gently suggest that.) Now how can you use that part to show it's a fair share?

3.

a. Here is a candy bar and several separate pieces that were made from identical candy bars. (Give the student a 1/3-piece, 1/4-piece, 1/6-piece, and a 1/8-piece.) The candy bar is to be shared fairly among six people, so that each person gets an equal share. Which one of these pieces would be your piece? How do you know it's going to be fair?

(Use (b) only if student does not do well on (a).)

b. Here is a candy bar and a piece that was taken from an identical candy bar. (Give the student a 1/8-piece.) The candy bar is to be shared fairly among six people so that each person gets an equal share. Is this piece a fair share? (Probe the student's justification for her or his response.)



4. The drawing below shows my piece of string. Think of a piece of string that it is *seven* times longer than mine. (Pause.) Can you draw what you're thinking of? (If the student has difficulty with "long," restate the question using "big." If the student draws one continuous line without breaks or marks, go to *.)

*How do you know that your string is seven times longer than mine? (If the student doesn't know, give student a copy of the string and go to **.)

**Could you use this to show me that your string is seven times longer than mine?

5. a. The drawing below shows my piece of string. Think of *your* piece of string so that mine is five times longer than yours. (Pause.) Can you draw what you're thinking of? (If the student iterates the string five times, go to *.)

*Is yours five times longer than mine? (The student will hopefully say yes.) Ah, but I said I wanted *mine* to be five times longer than yours. (If the student then draws a piece of string shorter than the string, go to **.)

**Can you show for sure that mine is five times longer than yours?
(If the student has a lot of trouble with (a), go to (b).)

b. Think of piece of string so that mine is *twice* as long as yours (could also try "two times longer than yours"). Can you draw what you're thinking of? Can you show for sure that mine is twice as long as (or "two times longer than") yours? (If the student can do this problem, ask the same question with three times as long or four times as long.)

6. a. The Giant Soda at the convenience store is 24 ounces. That's eight times the amount of soda that Stephanie drank. What would you do to find out how much Stephanie drank? (Ask the student to do it, if she or he suggests something that seems feasible. Probe to get at operations, behind what the student says verbally.) Can you draw a picture of what you are thinking?

b. Camika has $21. That's one-seventh as much money as Rickard has. What would you do to find out how much money Rickard has? (Ask the student to do it, if she or he suggests something that seems feasible. Probe to get at operations, behind what the student says verbally.) Can you draw a picture of what you are thinking?

7. The submarine sandwich is to be shared fairly among four people, you and three friends, so that everyone gets an equal piece.

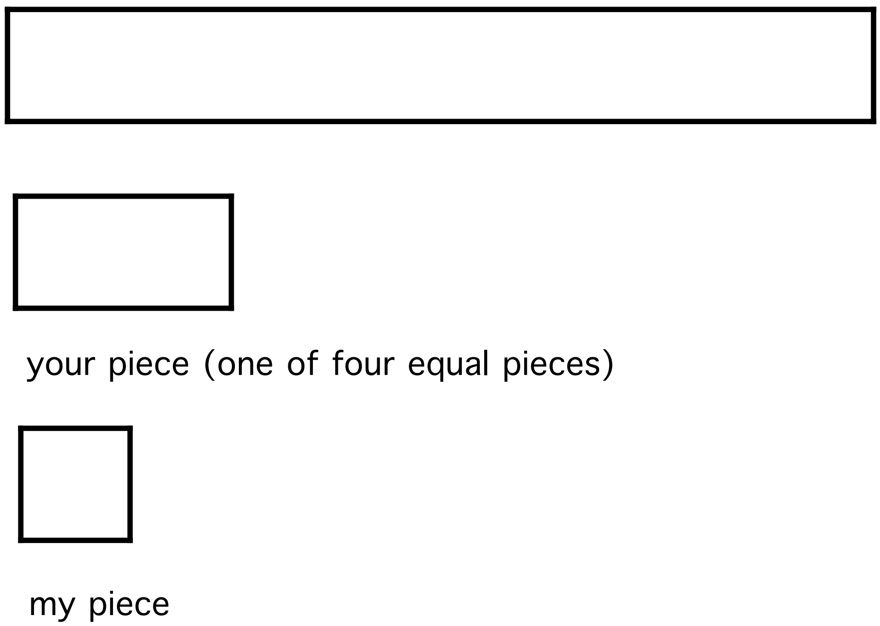
a. Make one mark to show your piece.

b. You decide to share your piece equally with me (because I'm hungry!) What part of the sandwich do I get?

c. What fraction is my piece of the whole sandwich?

(If the student has trouble with this task, try (d), which will be on a separate page for the student.)

7d. The piece below shows your piece. The next piece below shows my piece.



My piece is what part of the whole sandwich?

8. At a small party, five friends split a submarine sandwich fairly. You take one of their pieces and share it fairly among four people. How much of the whole sandwich do three of these people get?