Degenerations of Prym Varieties and Cubic Threefolds

by

Tawanda Gwena

(Under the direction of Valery Alexeev)

Abstract

We present a surprising connection between degenerations of cubic threefolds and well known regular matroids by making use of intermediate Jacobians of cubic threefolds realized as Prym Varieties. We extend previous known results of one node to 10 nodes. As a corollary we obtain a new proof of the nonrationality of a generic cubic threefold.

Index words: Prym Varieties, Cubic Threefolds, Algebraic Geometry

Degenerations of Prym Varieties and Cubic Threefolds

by

Tawanda Gwena

B.Sc., The University of Zimbabwe, Zimbabwe, 1995

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2004

Degenerations of Prym Varieties and Cubic Threefolds

by

Tawanda Gwena

Approved:

Major Professor: Valery Alexeev

Committee: Robert Varley
Elham Izadi
Roy Smith
Ed Azoff

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
August 2004

Acknowledgments

I would like to thank the algebraic geometry people at UGA especially Valery Alexeev and Robert Varley. Without them I would not have made it through. I would also like to thank the past graduate students and staff from the department.

Table of Contents

Page

Acknowledgments iv

Chapter

1 Introduction 3

2 Cubic Threefolds 8

2.1 The Moduli Space of Cubic Threefolds 8

2.2 Nodes of Cubic Threefolds 10

3 Degenerations of Prym Varieties 14

3.1 Jacobian Varieties 15

3.2 Prym Varieties 16

3.3 Degenerations of Prym varieties 19

4 Matroids 22

4.1 Introduction to Matroids 22

4.2 Nodes on Cubic Threefolds 28

5 Curves Associated to a Cubic Threefold 36

5.1 The Fano Surface and Sextic Curve 37

5.2 The Plane Quintic 41

6 Jacobians and Pryms 50

6.1 The Chow Group of a Certain Threefold 51

6.2 The Tangent Bundle 56

7 Calculations and Main Proof 61

7.1 Proof of the Main Theorem 61

7.2 Nonrationality of Cubic Threefolds 76

8 Further work 78

8.1 Matroid Related 78

8.2 Other Stable Cubic Threefolds 80

Bibliography 82

Appendix

A Computing the plane quintic curve 86

Notation

Throughout this thesis we will use the following notation.

A^i	a subgroup of CH^i, the cycles algebraically equivalent to 0
A_g	the moduli space of abelian varieties of dimension g
$CH^i(\cdot)$	the Chow group
Δ, Δ_0	the plane quintic coming from the conic bundle (depends on the choice of a line).
Δ_n	normalization of Δ
Δ', Δ'_0	the double cover of the plane quintic. Its normalizationis Δ'_n
$\mathcal{F}$	the Fano surface of a cubic threefold
$\mathcal{F}'$, $\mathcal{F}''$	subsets of $\mathcal{F}$
$J(\cdot)$	the Jacobian of a curve
L, L_0	a plane in $\mathbb{P}^4$
ℓ, ℓ_0	lines on the cubic threefold
ℓ_s	line on cubic surface corresponding to a point $s \in \mathcal{F}$
$\Gamma(\cdot)$	dual graph of a curve (a node corresponding to a curve of genus > 0 is drawn as a square)
M_g	the moduli space of curves of genus g
M_i, $M_{i,j}$	various matroids we define
P_i	nodes of the cubic threefold X
$Pr(\Delta'/\Delta)$	the (generalized) Prym variety associated to a curve Δ and its double cover Δ'
R_g	the space of unramified double covers of curves of genus g
R_{10}	a special unimodular system of vectors
$\mathcal{S}$	the segre cubic threefold
S	the sextic associated to a cubic threefold (depends on a choice of a nodes)

S_n	the normalization of S
X, X_0	a possibly nodal cubic threefold

Chapter 1

Introduction

In this dissertation we use the methods of degenerations of Prym varieties of [ABH] to establish a surprising new connection between degenerations of cubic threefolds and matroids. As in [ABH] we will work over an arbitrary algebraically closed field k of characteristic $\neq 2$.

Cubic threefolds and curves have a number of similarities. For example the Jacobian of a smooth curve over $\mathbb{C}$ is given by

$$J(C) = \frac{H^{1,0}(C)^*}{H_1(C,\mathbb{Z})}.$$

The cubic threefold analog, the intermediate Jacobian is given by the very similar

$$J(X) = \frac{(H^{2,1}(X) \oplus H^{3,0}(X))^*}{H_3(X,\mathbb{Z})}.$$

In this dissertation we show the analog of a certain result from the moduli space of curves. In the Deligne-Mumford compactification $\overline{M}_g$ the various strata are in one-to-one correspondence with dual graphs of curves ([HaM] Chapter 2). For cubic threefolds we show an analogous result where the dual graphs are replaced by matroids.

Our main object of study is cubic threefolds. These are hypersurfaces defined by one cubic equation in projective 4-space. We extend the results given by Collino and Murre in [CM] and Clemens and Griffiths in [CG] to partial compactification of moduli space of cubic threefolds where the threefolds acquire more than one ordinary double point. This also further generalizes the result in [G]. The other objects we will work with are the space of Prym varieties of dimension 5, the moduli space A_5 and its compactification, and the space of double covers R_6. In particular we prove the following

Theorem 1.1 (Main Theorem) *The unimodular system of vectors associated to a family of cubic threefolds degenerating to a cubic threefolds with nodes as constructed using [ABH] is the same as the unimodular system of vectors obtained by looking at the nodes on the limit 3-fold as vectors in the space k^5.*

This theorem has some very interesting consequences. The generalized Prym variety is an extension of an abelian variety by a torus. With this theorem we can tell the dimension of the torus and the abelian part by computing the dimension of the span of the nodes. It shows in which strata the generalized Prym varieties associated to cubic threefolds lie in the space $\overline{A}_g^{Vor}$. Another consequence is that it shows that the unimodular systems associated to degenerations of Prym varieties form a much larger class than those associated to degenerations of Jacobians of curves.

Yet another instance is the following result.

Corollary 1.2 *There is a regular map from the space of smooth cubic threefolds to A_5. This result shows that this map extends to a regular map from the space of cubic threefolds with nodes to $\overline{A}_5^{Vor}$.*

THis corollary comes from the the proof of the main theorem. Other important theorems we prove are the following

Theorem 1.3 *For a nodal cubic threefold X with associated sextic curve S and a plane quintic Δ with unramified double cover Δ' we have the following*

$$J(S_n) \simeq Pr(\Delta'_n/\Delta_n)$$

where the subscript $_n$ means normalization.

This result is a generalization of Theorem 4.22 of [CM] and of Corollary 3.26 of [CG]. Corollary 3.26 of Clemens-Griffiths is the following : *If there exists a birational map between a cubic threefold V and $\mathbb{P}^3$ then $J(V)$ (the intermediate Jacobian of V) is isomorphic to $J(C)$ for some (possibly reducible) non-singular curve C.*

This theorem in positive characteristic is Theorem 7.4. Theorem 4.22 of Collino-Murre is the special case of 1.3 where the cubic threefold has only one node. In that case S is uniquely defined and smooth so what they obtain is

$$J(S) \simeq Pr(\Delta'_n/\Delta_n).$$

As a corollary of 1.1 we give a new proof of the theorem of nonrationality of cubic threefolds which we define below.

Definition 1.4 *A variety X is rational if there exists two rational maps $\phi : X \dashrightarrow \mathbb{P}^n$ and $\gamma : \mathbb{P}^n \dashrightarrow X$ such that $\phi \circ \gamma$ and $\gamma \circ \phi$ are the identity.*

Definition 1.5 *A variety X is unirational if for some n there exists a dominant rational map*

$$\phi : \mathbb{P}^n \dashrightarrow X.$$

The theorem is:

Theorem 1.6 *Fix an algebraically closed field k of characteristic not two. Then a generic cubic threefold over k is not birational to $\mathbb{P}^3$.*

This was first proved by Clemens and Griffiths for smooth cubic threefolds over $\mathbb{C}$ and was later extended by Murre to other fields of characteristic not 2.

The importance of this theorem is that it gave one of the first examples of a unirational variety which is not rational ([CG]). This gives a counterexample to the Lüroth conjecture in dimension 3. The conjecture was known to be true in dimension 1 and 2. The classical one dimensional theorem of Lüroth is the following:

Theorem 1.7 ([Be], Theorem V.4) *Every unirational curve is rational.*

The question which arose from this was whether it was true in all dimensions. In the case of dimension 2 Castelnuovo and Enriques proved the following over an algebraically closed field of characteristic 0.

Theorem 1.8 ([Fr], Cor 10.16) *Every smooth unirational surface is rational.*

At about the same time as the Clemens-Griffiths paper two other papers with counterexamples to the Lüroth problem appeared. Iskovskikh and Manin showed that a smooth quartic 3-fold is nonrational ([IM]). Artin and Mumford constructed their counterexample. It was a unirational conic bundle X over a two dimensional base space for which $\mathbb{Z}_2 \subset H^3(X, \mathbb{Z})$. Fano had claimed he had counterexamples some 50 years before, but his proofs were not accepted as rigorous ([M] Appendix 4).

1.0.1 Outline

We will proceed as follows:

This paper works out certain matroids in two different ways.

The first way is as follows: In chapter 2 we briefly describe the moduli space of cubic threefolds and various properties of the nodes on cubic threefolds.

In chapter 4 we review matroids and give examples we will encounter later on in the study. In particular we will introduce the important matroid R_{10}. In this chapter we will also compute the matroids obtained by looking at the nodes of the cubic threefolds. This is the first way we work out matroids.

The second matroid is worked out as follows: Chapter 3 gives the background material from the paper [ABH]. This includes the various definitions and results from Degenerations of Prym varieties.

In order to use the results of chapter 3 we first need to obtain the dual graph of the double cover of a certain plane quintic curve Δ associated to each cubic threefold. Chapters 5 and 6 give some vital clues to what this double cover looks like. In chapter 5 some of the curves associated to a nodal cubic threefold are defined and studied. These include the plane quintic Δ and its double cover Δ'. It also includes the sextic curve S associated to X which gives some very important information on the Fano surfaces.

Chapter 6 gives a proof of the result $Pr(\Delta'/\Delta) = J(S_n)$. This is the important glue connecting the curves we have and will be used a lot in Chapter 7. It is very useful in eliminating a lot of cases.

In Chapter 7 we calculate the matroid using chapter 3 and prove the main theorem. In order to prove the main result we go as follows. We use the definition of the generalized Prym variety as $Pr(\Delta'/\Delta)$. First we look at what kind of plane quintic curve Δ we can obtain for each type of cubic threefold. Then we use the main result of Chapter 6 to obtain the abelian part of the generalized Prym variety $Pr(\Delta'/\Delta)$. From those two pieces of information we can describe what the double cover Δ' of Δ should be. From the definition $J(X) \simeq Pr(\Delta'/\Delta)$ we can now calculate the extension data for the generalized Prym. We compare the matroids we get from the cubic threefold in Chapter 4 and the matroid from the extension data. They are shown to be the same and thus we would have proved our theorem. In that chapter we will also prove Theorem 1.6. We will also give a brief description of previous theorems

The final chapter is on Further Questions and to what other avenues these results might take us. In particular it will deal with amalgamations of matroids and give one result known about a certain strictly semistable cubic threefold (defined in Definition 2.2).

The appendix is some *mathematica* code used to calculate plane quintics. It is not necessary in the proof of the theorem, but it does provide a place to experiment with various cubic threefolds.

Chapter 2

Cubic Threefolds

Cubic threefolds are interesting varieties for several reasons. They are the first “nontrivial simplest varieties” after surfaces. They form a good testing ground for theories in higher dimensions, such as the Lüroth problem mentioned in the previous chapter. Smooth cubic threefolds have been studied extensively (e.g [CG], [BS]). Singular cubic threefolds also have a history, but not as extensive as the smooth ones. One of the most studied singular cubic threefold is the Segre threefold (e.g [SR]). Another popular choice was the cubic threefolds with one node (e.g. [CG], [CM]). In this chapter we will review what is known about singular and smooth cubic threefolds.

2.1 The Moduli Space of Cubic Threefolds

The space of cubic threefolds is parametrized by a projective space of 34 dimensions. A cubic threefold X is semistable of there is an $SL(5)$-invariant hypersurface in $\mathbb{P}^{34}$ which does not contain X. It is stable if, in addition, it has a finite symmetry group and its orbit is closed in the space of semistable threefolds. It is strictly semistable if it is semistable but not stable.

Stable cubic threefolds are important because they are the main ingredients in the construction of the moduli space of cubic threefolds. The moduli space is a projective variety and contains the orbit space of stable threefolds as an open dense subset.

The stable and semistable threefold can have isolated singularities which we define here.

Definition 2.1 ([Al]) *A hypersurface singularity is called an A_n singularity if it is locally analytically of the form $x_1^{n+1} + x_2^2 + x_3^2 + x_4^2 = 0$.*

A D_4 singularity is locally analytically of the form $x_1^3 + x_2^3 + x_3^2 + x_4^2 = 0$.

In the study of the moduli space of cubic threefolds some interesting ones show up. A special cubic threefold which shows up in the stability theory is G given by

$$x_0x_1x_2 + x_3^3 + x_4^3 = 0.$$

An interesting family of cubic threefolds $X_{A,B}$ is given by the equation

$$Ax_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 + Bx_1x_2x_3 = 0.$$

At least one of A, B has to be nonzero. Because of this, we can rescale the constants and write X_β where $\beta = 4A/B^2$. This covers all but one of $X_{A,B}$ i.e. when $B = 0$.

Definition 2.2 *The secant cubic threefold is given by secants to the rational curve in $\mathbb{P}^4$ is given by $x_0 = s^4$, $x_1 = s^3t$, $x_2 = t^4$, $x_3 = st^3$, $x_4 = s^2t^2$. The equation of the secant cubic is ([Ha] p 103, [C1] §1)*

$$\begin{vmatrix} x_0 & x_1 & x_4 \\ x_1 & x_4 & x_3 \\ x_4 & x_3 & x_2 \end{vmatrix} = (x_2x_4 - x_3^2)x_0 - (x_4^3 - 2x_1x_3x_4 + x_1^2x_2) = 0$$

The main results of [Al] and [Yo] are the following:

Theorem 2.3 ([Al] Theorem 1.1, [Yo] Main Theorem) *A cubic threefold is stable if and only if each of its singularities are of the type A_1, A_2, A_3 or A_4*

Theorem 2.4 ([Al], Theorem 1.3) *A cubic threefold X is strictly semistable if and only if*

1. *X contains a D_4 singularity in which case X degenerates to G or*

2. *X contains an A_5 singularity in which case X degenerates to X_β $(\beta \neq 1)$ or*

3. *X contains an A_n singularity $n > 5$ but none of the planes contained its null line in which case X degenerate to a secant cubic threefold or*

4. X is a secant cubic.

The moduli space is a 10 dimensional space. The points corresponding to strictly semistable threefolds in the moduli space of cubic threefolds form a rational curve corresponding to the X_β threefolds and an isolated point corresponding to G.

We shall concern ourselves only with cubic threefolds where the nodes are of the form A_1, otherwise known as ordinary double points. Because of this condition on nodes all our cubic threefolds are irreducible. The other singularities need a much deeper analysis because the curves we obtain could be nonstable, and the Fano surface is much harder to understand. The theory as developed in [ABH] applies to curves with nodes, and so by 5.12 our threefold can only have nodes. However we will mention (in the last chapter) one interesting result concerning the secant cubic which shows what can happen at nonstable points.

2.2 Nodes of Cubic Threefolds

The nodes on a cubic threefold have some restriction on how they are configured Here we will give some basic results on the configurations.

Lemma 2.5 *A cubic threefold can have at most 10 ordinary double points.*

This is known as the Varchenko bound on the number of nodes on a cubic threefold ([Hu] Section 3.2). This bound is achieved by the special Segre Cubic threefold below.

Definition 2.6 *For any field of characteristic* $\neq 2$ *the Segre Cubic* $\mathcal{S} \subset \mathbb{P}^4$ *is given by the equation*

$$\left\{(x_0, \ldots, x_4) : \sum_{i \neq j \neq k} 2x_i x_j x_k + \sum_{i \neq j} x_i^2 x_j = 0, \quad i, j, k \in \{0, \ldots, 4\}\right\}$$

One node is given by the coordinates $(1 : 1 : 1 : -1 : -1)$ and the other nine are obtained by permuting these coordinates.

The next three lemmas describe what can and cannot happen with node of cubic threefolds.

Lemma 2.7 *No three of the double points can lie in a line without the whole line being singular.*

Proof

Once three nodes lie on a line then the whole line is on the threefold and is singular. This can be easily shown by a direct computation. Suppose without loss of generality that the nodes are the points $P_1 = (0:0:0:0:1)$, $P_2 = (1:0:0:0:1)$ and $P_3 = (-1:0:0:0:1)$ in projective 4-space with coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$. Then we write the equation of the threefold as

$$f(x_0, x_1, x_2, x_3, x_4) = \sum_{0 \leq i,j,k \leq 4} a_{ijk} x_i x_j x_k = 0.$$

We have

$$\begin{aligned} f(P_1) &= a_{444} \\ \frac{\partial f}{\partial x_0}(P_1) &= a_{044} \\ \frac{\partial f}{\partial x_1}(P_1) &= a_{144} \\ \frac{\partial f}{\partial x_2}(P_1) &= a_{244} \\ \frac{\partial f}{\partial x_3}(P_1) &= a_{344} \\ \frac{\partial f}{\partial x_4}(P_1) &= 3a_{444}. \end{aligned}$$

For all these to be zero we need $a_{i44} = 0$ for $i = 0 \ldots 4$. In a similar manner, the point P_2 and P_3 will impose the additional conditions $a_{000} = a_{001} = aa_{002} = a_{003} = a_{014} = a_{024} = a_{034} = 0$. These conditions imply that all the partial derivatives along the line $(t : 0 : 0 : 0 : 1)$ are 0 and that the line is actually in the threefold and is singular.

An alternative proof is as follows. Cut the cubic three fold with a generic 2-plane containing the three nodes. Then the intersection is a cubic curve with three collinear singularities. This means that they are not isolated, so the whole line is singular. Since the plane was general (and all such planes do cover $\mathbb{P}^4$) we get a singular line. ∎

Lemma 2.8 *If five nodes lie in a 3-subspace then four of them lie in a plane.*

Proof

This fact can be verified by direct computation similar to the one above. If five nodes do have such a configuration then the cubic threefold will no longer have isolated singularities.

An alternative proof is as follows. We cut the cubic threefold by the hyperplane in $\mathbb{P}^4$ which contains all the nodes. Then the resulting cubic surface has five nodes. By [BW] a cubic surface can have at most four isolated singularities, so there might be a small problem. So suppose the cubic surface has non-isolated singularities. In this case by [BW] (Section 2, case E,F) either the singularity locus is $\mathbb{P}^1$ which is not possible as the five nodes did not lie in a line, or the surface is a quadric and a plane. In this case four of the nodes lie in a plane along the intersection of the quadric and the plane. ∎

Lemma 2.9 *If 4 nodes lie in a plane then the whole plane is on the cubic threefold. Conversely if a cubic threefold with only ordinary double points contains a plane then the plane has exactly four nodes.*

Proof

Let our cubic threefold with four nodes in a plane be X. A general statement is that line joining two nodes on a cubic threefold is contained in the threefold because it intersects the threefold with multiplicity 4. Suppose we have four nodes in a plane(this plane is not necessarily in X). Then we have a plane containing the six distinct lines (since no three nodes are collinear) joining pairs of nodes. The intersection of a cubic threefold and a plane can have degree at most three unless the plane is contained in X. Therefore the plane is on X.

For the converse, suppose the local coordinates are $x_1, \ldots, x_4$. Then assume the plane is cut by $x_1 = x_2 = 0$. The equation of X can be written $f(x_1, x_2, x_3, x_4) = x_1 Q_1(x_1, \ldots, x_4) + x_2 Q_2(x_1, \ldots, x_4) = 0$ where Q_i are quadratic terms. The partial derivative $\frac{\partial f}{\partial x_1}$ with the conditions $x_1 = x_2 = 0$ gives a quadratic polynomial in x_3 and x_4 as follows: Set

$$\frac{\partial f}{\partial x_1} = Q_1 + x_1 \frac{\partial Q_1}{x_1} + x_2 \frac{\partial Q_2}{x_1} = 0.$$

The substitution $x_1 = x_2 = 0$ gives

$$Q_1(0,0,x_3,x_4) = 0.$$

In a similar way we get

$$Q_2(0,0,x_3,x_4) = 0$$

from $\frac{\partial f}{\partial x_2}$. So by Bezout's theorem we get four solutions to these equations. For the partial $\frac{\partial f}{\partial x_3}$ we have

$$\frac{\partial f}{\partial x_3} = x_1 \frac{\partial Q_1}{\partial x_3} + x_2 \frac{\partial Q_2}{\partial x_3}.$$

This is 0 when we impose $x_1 = x_2 = 0$. The same goes for $\frac{\partial f}{\partial x_4}$. So we have four nodes in the plane $x_1 = x_2 = 0$. There might be other nodes but they will not be on the plane. ∎

Because of these restrictions on the nodes in cubic threefold, there are other restrictions such as how many planes can be found in a cubic threefold. In [F] alternative proofs of some of the theorems above are given. In particular they give a generalization of Lemma 2.9 which applies to general hypersurfaces ([F] Proposition 1.1).

Chapter 3

Degenerations of Prym Varieties

This chapter introduces the most crucial aspects of this dissertation. In here we introduce the Jacobians of curves and Prym varieties.

We also introduce the new machinery of degenerations of Prym varieties as developed by Alexeev, Birkenhake and Lange in [ABH]. We will also quote results from several other very important papers including [A1], [A2] and [V]. The main results of this chapter are Theorem 3.2 of [ABH] and Theorem 0.1 of [V].

Degenerations of Prym varieties were developed to answer several questions of Prym varieties. One question was to answer the question of whether there is a good compactification of Prym varieties arising from nodal curves. Another question is about the map from the space R_g of unramified double covers of a curves of genus g to the space A_{g-1} of abelian varieties. Can this map be extended to a regular morphism from the compactification $\overline{R}_g$ to a suitable compactification of A_{g-1}? These questions are answered in [ABH] and we will give the results here. We will use the results in this paper.

Since we talk so much about abelian varieties we will define them here.

Definition 3.1 ([A2], 1.1) *An abelian variety over k is a connected proper group variety over k.*

A semiabelian variety G is a group variety which is an extension

$$1 \to T \to G \to A \to 0$$

of an abelian variety A by a torus T.

Over $\mathbb{C}$ an abelian variety is a quotient of $\mathbb{C}^n$ by a lattice of rank $2n$ satisfying Riemann conditons of [GH] 2.6. or [BL] chapter 4.

3.1 Jacobian Varieties

Jacobians of curves form an important class of abelian varieties. For a smooth curve C the Jacobian JC is an abelian variety which parametrizes line bundles of degree 0 on C. In the case where the field is $\mathbb{C}$ the Jacobian of a curve C can be given by ([BL] Chapter 11)

$$JC = \frac{H^0(\omega_C)^*}{H_1(C,\mathbb{Z})}$$

In this case the Abel-Jacobi map give the isomorphism $\mathrm{Pic}^0(C) \simeq JC$. The Jacobian of nodal curves have been studied at various times in the past. At least two ways have been found to generalize the notion of a Jacobian. One generalization is $\mathrm{Pic}^0 C$, the Picard variety. It is a semiabelian group variety which parametrizes invertible sheaves of degree 0. It may not be complete.

The second generalization is the *compactified Jacobian* which is a projective variety. Suppose we start with a stable curve C. Oda and Seshadri ([OS]) defines a family of such compactifications Jac_ϕ for C. Namikawa defines one variety $\overline{J}C$ (which is isomorphic to one of Jac_ϕ) and in [A1] it is shown to how to construct a theta divisor on it and obtain a pair $(\overline{J}C, \Theta)$ as a stable semiabelic pair. A stable semiabelic pair is a defined in [A2] Section 1.1.15. The Torelli map $M_g \to A_g$ from the moduli space of curves to principally polarized abelian varieties can be extended to a morphism from the Deligne-Mumford compactification $\overline{M}_g$ to $\overline{A}_g^{Vor}$, the toroidal compactificaton of A_g for the second Voronoi fan. This latter compactification appears in [A2] as the closure of A_g in the moduli of stable semiabelic pairs ([A2] section 5). The image of C in $\overline{A}_g^{Vor}$ is $(\overline{J}C, \Theta)$. This construction of $\overline{J}C$ works for degenerating families. Part of the degeneration data is listed in [ABH] as J0 and J1 and J6. It does not depend on the particular degeneration([ABH] 3.1). It is

(J0) A_0 is the abelian variety coming from the normalization of the curve C_0.

(J1) The lattice $H_1(\Gamma, Z)$ where Γ is the dual graph of C_0

(J6) A cell decomposition obtained by taking the intersection of the standard cubes $C_1(\Gamma, \mathbb{R})$ with the subspace $H_1(\Gamma, \mathbb{Z})$.

We will review the last of the degeneration data.

Definition 3.2 *The dual graph $\Gamma(C)$ of a nodal curve C is an unoriented graph where to each irreducible component a vertex is assigned. Two vertices have an edge joining them if the corresponding components intersect at a node. If a component has a node on it then the edge is a loop on the vertex corresponding to the component.*

Let $\Gamma = \Gamma(C)$ be the dual graph of C. We choose an orientation on it. The vertices $\{v_i\}_{i\in I}$ correspond to irreducible components of C and the edges $\{e_j\}_{j\in J}$ correspond to the nodes. We have a chain complex

$$0 \to C_1(\Gamma, \mathbb{Z}) \to C_0(\Gamma, \mathbb{Z}) \to 0$$

with

$$C_0(\Gamma, \mathbb{Z}) = \bigoplus_{i\in I} v_i\mathbb{Z}, \qquad C_1(\Gamma, \mathbb{Z}) = \bigoplus_{j\in J} e_j\mathbb{Z}.$$

Let $H_1(\Gamma, \mathbb{Z})$ be the first homology group. There is an embedding $H_1(\Gamma, \mathbb{Z}) \hookrightarrow C_1(\Gamma, \mathbb{Z})$. Part of the degeneration data is ([ABH] J6) the cell decomposition obtained by intersecting the standard cubes of $C_1(\Gamma, \mathbb{R})$ with the subspace $H_1(\Gamma, \mathbb{R})$. This is shown to come from a cographic unimodular system of vectors (where the relation is given in 4.1.3).

3.2 Prym Varieties

After Jacobians of curves, Prym varieties form another interesting class of abelian varieties. The set of (generalized) Pryms includes the set of (generalized) Jacobians. Up to dimension 5 every abelian variety is a Prym variety. All abelian varieties are Jacobians only up to dimension 3. Let C' be a smooth curve with involution (automorphism of order 2) ι and let $C = C'/\iota$ be the quotient curve. This induces a double cover

$$\pi : C' \to C.$$

The norm map $\mathrm{Nm} : \mathrm{Pic}C' \to \mathrm{Pic}C$ can be defined on divisors as

$$\mathrm{Nm}\left(O'_C\left(\Sigma a_i p_i\right)\right) = O_C\left(\Sigma a_i \pi\left(p_i\right)\right).$$

The kernel of the norm map has two components. The component with identity is an abelian variety, the Prym variety $P(C, \iota)$.

When the involution has 0 or 2 fixed points then the polarization is twice a principle polarization. The question is what happens when C' is a nodal curve with involution? This is discussed in [ABH] Chapter 1and we will outline it here. In this case the quotient curve C has nodes and the identity component of the kernel of the norm map is a semiabelian variety. We show how it is constructed here.

We construct the dual graph $\Gamma' = \Gamma(C')$ as above. For a curve C' define JC' to be the group of line bundles on C' whose multidegree is 0 (as in the case above). Then there is an exact sequence

$$1 \to H^1(\Gamma', k^*) \to JC' \to JN \to 0$$

where JN is the product of the Jacobians of the normalization of the components of C'.

Let $C = C'/\iota$. There is norm map $\pi_* = \mathrm{Nm} : JC' \to JC$ which is defined for nodal curves and coincides with the norm map for smooth curves above and it is surjective. We define

$$P = \ker(\mathrm{Nm} : JC' \to JC)_0.$$

It is a semiabelian variety as we now show. The norm map $\mathrm{Nm} : JC' \to JC$ induces a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_{C'} = H^1(\Gamma', k^*) & \longrightarrow & JC' & \longrightarrow & JN' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Nm}_T} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \\ 1 & \longrightarrow & T_C = H^1(\Gamma, k^*) & \longrightarrow & JC & \longrightarrow & JN & \longrightarrow & 0. \end{array}$$

Adding the kernels to these maps gives

$$\begin{array}{ccccccccc}
 & & 1 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T'_P & \longrightarrow & P' & \longrightarrow & K'_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T_{C'} & \longrightarrow & JC' & \longrightarrow & JN' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T_C & \longrightarrow & JC & \longrightarrow & JN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 0 & & 0 & &
\end{array}$$

Let the identity components of the top row be

$$P = (P')_0, \quad K_N = (K'_N)_0, \quad T_P = (T'_P)_0.$$

K_N is an abelian variety which is the Prym variety for the double cover $N' \to N$. and there is a morphism $P \to K_N$. The identity component of the kernel of this map is isomorphic to T_P and there is the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_P & \hookrightarrow & K_P & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P & = & P & & . \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & A & \to & K_N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here A is an abelian variety and $H \simeq K_P/T_P$ is a finite group. The left columns shows that P is a semiabelian variety.

Definition 3.3 *Let* $X = H_1(\Gamma', \mathbb{Z})$. *We define the map*

$$\begin{aligned} \pi^- : H_1(\Gamma', \mathbb{Z}) &\longrightarrow H_1(\Gamma', \tfrac{1}{2}\mathbb{Z}) \\ h &\longmapsto \tfrac{1}{2}(h - \iota(h)) \end{aligned}$$

and let $X^- = \pi^-(H_1(\Gamma', \mathbb{Z}))$.

The abelian part of P is A and the character group of its toric part is X^-.

3.3 Degenerations of Prym varieties

Suppose we have a 1-parameter family of Prym varieties degenerating to (C_0, ι_0) where C_0 is a stable curve with involution ι_0. In a way similar to the case of Jacobians of curves there is data obtained from the central fiber.

The question of uniqueness of the limit of the ppavs coming from degeneration of Prym varieties is answered in [ABH]. It is answered in terms of the dual graph of (C_0, ι_0).

We have $X^- \otimes \mathbb{R} \subset X \otimes \mathbb{R} \subset C_1(\Gamma', \mathbb{R})$.

Each edge e_j of Γ' defines a coordinate function z_j in $C_1(\Gamma', \mathbb{R})$. Let $m_j = 1$ if $z_j : X^- \to \mathbb{Z}$ is surjective and $m_j = 2$ if $z_j : X^- \to \frac{1}{2}\mathbb{Z}$ is surjective.

The combinatorial criterion for the uniqueness of the limit is

(*) The linear functions $m_j z_j$ define a dicing of the lattice X^-.

A lattice dicing of a real vector space is cutting up the real vector space by families of parallel hyperplanes such that the hyperplanes intersect on a lattice.

Geometrically, the condition (*) can be described as follows. Consider all translations of the hyperplanes $H_j = \{z_j = 0\}$ in $C_1(\Gamma', \mathbb{R})$ through points in X^-. Take the intersection with $X^-_{\mathbb{R}}$. This defines a cell decomposition and (*) is fulfilled if the vertices of this cell decomposition are exactly the points of the lattice X^-. (This is closely related to unimodular systems defined in section 4.1.3).

Similar to the case of degenerations of Jacobians of curves Theorem 3.2 of that paper gives equivalent conditions under which the Prym extends uniquely in the neighbourhood of the curve $[(C_0, \iota_0)]$. The theorem is

Theorem 3.4 ([ABH] Theorem 3.2) *The following are equivalent*

1. *In a neighborhood of the point* $0 = [(C_0, \iota_0)]$ *the Prym map* $\mathcal{R}_g \to A_{g-1}$ *extends to a morphism from* $\overline{\mathcal{R}}_g$ *to the second Voronoi toroidal compactification* $\overline{A}_{g-1}^{\mathrm{Vor}}$.

2. *The limit variety* $(\overline{P}_0, \Theta_0)$ *depends only on the pair* (C_0, ι_0) *and not on the choice of a one-parameter degeneration.*

3. *The cell decomposition* Δ^- *depends only on the pair* (C_0, ι_0) *and not on the choice of a one-parameter degeneration.*

4. *Condition (*) holds. In this case, the decomposition* Δ^- *coincides with the dicing.*

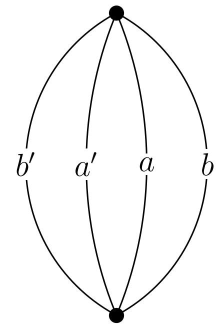
In [V] this criterion is translated to a combinatorial criterion as follows.

Theorem 3.5 ([V] Theorem 0.1) *A curve with an involution is in the indeterminacy locus of the extended Prym map if and only if the curve is a degeneration of a Friedman-Smith example with the number of edges 4 or greater.*

A Friedman-Smith example with $2n$ edges is a curve $(C, \iota) \in R_g$ such that the curve C is the union of two irreducible components, both invariant under the involution, intersecting in $2n$ points, so that the involution is base point free and interchanges these $2n$ nodes.

Example

Example of curve in locus of indeterminacy: A curve made of two components, which are invariant under involution, meeting at four points not fixed under involution is in the Friedman-Smith locus. Its dual graph is

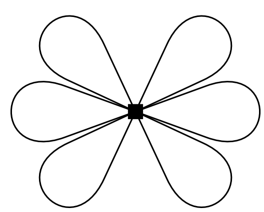


Note that in all graphs with an involution in this paper it will interchange left and right. □

So if our curve is not in the locus of indeterminacy we can obtain the dicing Δ^- using the information above. *Remark*
Additionally the strata of $\overline{A}^{\mathrm{Vor}}$ correspond to Delaunay decompositions modulo $GL(g-1,\mathbb{Z})$ by [A2] 1.2.16 and, in the principally polarized case, $\overline{A}^{\mathrm{Vor}}$ is the main component of the space $\overline{AP_g}$ constructed in [A2].

Example

Suppose we have a smooth curve with involution degenerating to an irreducible curve with 2n nodes. Let C' be the irreducible curve with $2n$ nodes which are pairwise exchanged by ι, $\iota(P_j) = P_{2a-j+1}$. The dual graph Γ has one vertex and $2a$ loops $e_1, \ldots, e_{2a}$. In the case where $n = 3$ the dual graph is



The involution acts on $H_(\Gamma', \mathbb{Z})$ by $\iota(e_j) = e_{2a-j+1}$.. In this case $X^- =< e_j - e_{2a-j+1} >$, $j = 1, \ldots n$. We compute the dicing. $X^- = \frac{1}{2} < e_1 - e_{2a}, e_2 - e_{2a-1}, \ldots >$. This gives a dicing of cubes in $\mathbb{R}^n$.

□

Chapter 4

Matroids

A very important component of this study is matroids. A lot of the information we obtain will be the matroids associated to degenerations of Prym varieties.

In this chapter we will do two things. In the first part we will introduce matroids. In the second part we will list out the matroids we will (expect to) meet.

4.1 Introduction to Matroids

Matroids can be seen as generalizations of both graph and vector space bases. The material from this section can be found in books such as [O]. Unimodular systems of vectors can be found in [DG].

Definition 4.1 *A matroid is a collection I of subsets (called independent sets) of a finite set E with the following properties*

- *The empty set is independent.*
- *Every subset of an independent set is independent.*
- *If A and B are two independent sets and $|A| > |B|$ then there exists an $a \in A$ such that $\{a\} \cup B$ is independent.*

Every maximal independent set in a matroid has the same number of elements. This number is called the *rank*. This is a direct generalization of linearly independent bases in linear algebra.

We will also show it is a generalization of graphs in the sense that every graph give rise to a matroid. However there could be several graphs which give the same matroid.

Definition 4.2 *It is generally not necessary to list all the independents sets for a matroid M. Listing just the maximal independent sets is enough as the rest are subsets of bases. These maximal independent sets are called a* basis *for M.*

We can use the idea of a basis to create the dual matroid.

Definition 4.3 *The dual matroid M^* of a matroid coming from a matroid $M = (E, I)$ is a matroid given as follows: The basis of the independent set I^* is given by $\{E - A | A \in I\}$.*

4.1.1 Constructions of Matroids

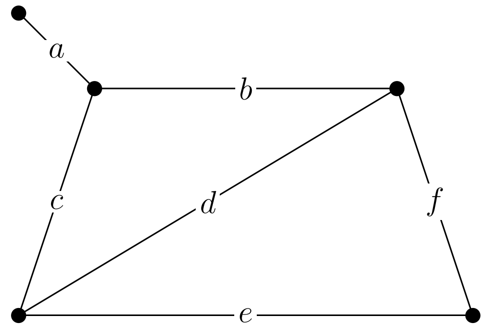
Matroids from Graphs

Given a graph G we construct a matroid as follows: Let E be the edge set of G. Let I be the collection of subsets that do not contain all the edges of a cycle of G. The maximal independent sets for a graph are called "spanning forests" (spanning trees for a connected graph). The first two condition of the matroid definition are easily met. The third is also met in the following way. Suppose we have $|A| < |B|$ with A and B in I. From the definition we have A and B are contained in trees. Since $|A| < |B|$ we have that there is at least one vertex of the graph which an edge of B touches but no edge of A does. Let b be such an edge which meets that vertex. Then $\{b\} \cup A$ is also independent.

Definition 4.4 *The matroids constructed in such a way are the graphic matroids. The graphic matroid arising from the graph G is written $M(G)$ and the dual matroid is written $M^*(G)$.*

Example

Suppose we have the graph



The set E in this case is $\{a, b, c, d, e\}$. The maximal independent sets are all subsets with three elements except for the sets $\{b, c, d\}$ and $\{d, e, f\}$. □

The dual of a graphic matroid is not necessarily graphic. The graphicness is preserved only if the graph is planar([O] Theorem 2.3.4).

Matroids from Matrices

Given a matrix A over a field k we can construct a matroid as follows: Let E be the set of column labels of A. This E is the finite set and the independent sets for the matroid are the subsets of E such that the corresponding columns are linearly independent. In this case A is said to represent the matroid constructed. The following operations preserve the matroid associated to a matrix

1. interchange two rows
2. multiply a row by a nonzero element of k
3. replace the row by the sum of that row and another
4. interchange two columns
5. multiply a column by a nonzero element of k.

If one matrix representing a matroid can be obtained from another by these row operations then the two matrices are said to be *equivalent*.

Example

Let the matrix be

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then if we label the columns as a, b and c then the matroid has the set $E = \{a, b, c\}$ and the set $I = \{\{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$. The matrix is equivalent to

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

□

4.1.2 Unimodular Systems of Vectors

Definition 4.5 *A regular matroid, or a unimodular system of vectors, is a matroid that can be represented by a matrix over* $\mathbb{R}$ *which is totally unimodular, i.e all the square submatrices have determinants in* $\{-1, 0, 1\}$. *For this class of matroids if we write the matroid as* $(I_r|A)$ *where* A *is of dimension* $r \times n$ *then the dual matroid is* $(I_{n-r}|A^T)$.

The rank of the matroid $(I_r|A)$ *is* r.

The number n is the dimension of the span of the vectors and $n+r$ is the number of vectors.

If a matroid is regular then its dual is also regular. In general a matroid is regular if it does not contain the matroids $U_{2,4}$, F_7 and $(F_7)^*$. The matroid $U_{2,4}$ is a matroid with the set E with four elements. The maximal independents sets are all subsets of 2 elements. The matroid F_7 is represented by the matrix ([O], 6.4.9)

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right)$$

where the coefficients are taken from a field of characteristic 2. In the field of characteristic 2 there may be other linear dependencies which may not occur in other fields. This matroid is called the Fano matroid.

A matroid M is uniquely representable if every pair of matrices representing M are equivalent. If we have that all the matroid we get are regular then we have the next theorem

Theorem 4.6 ([O] 10.1.4) *If a matroid is regular then it is uniquely representable over all fields.*

What this means for us is that when we obtain a matrix from a regular matroid we do not need to worry about which matrix it is as they are all equivalent.

Definition 4.7 (A cographic matroid from a graph) *The construction of a cographic matroid from a graph G is done as follows. Let e_i be the edges of the graph. Put an orientation on the edges of G and choose a spanning forest of the graph. Each of the remaining edges indexed by e_j defines a cycle α_j relative to the chosen tree. Give these cycles and orientation. For each e_i (including those from the tree) we construct the vector v_i as follows. The jth entry of v_i is defined to be 1 if edge e_i is on α_j and the directions coincide. If the directions do not coincide we put -1, otherwise we put 0. This is how the unimodular system of vectors for Jacobians of curves is constructed.*

Examples of interesting graphic regular matroids we will meet come from the graphs K_5 and $K_{3,3}$. The matroids are

$$M(K_5) = \left(\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \tag{4.1}$$

$$M(K_{3,3}) = \left(\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right). \tag{4.2}$$

One reason these two are interesting is because of the following theorem:

Theorem 4.8 ([O] 6.6.5) *A regular matroid is graphic if and only if it does not contain the submatroids* $M^*(K_{3,3})$ *or* $M^*(K_5)$*. Also a regular matroid is cographic if and only if it does not contain the submatroids* $M(K_{3,3})$ *or* $M(K_5)$*.*

This is analogous to Kuratowski's result in graph theory which states that a graph is planar if and only if it does not contain K_5 and $K_{3,3}$ as subgraphs ([O] 2.3.8).

There are three basic types of unimodular systems of vectors: the cographic, graphic, and the special one called R_{10} which is represented by the matrix

$$R_{10} = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

One property of R_{10} is that it is self dual. It also has the following very important property

Theorem 4.9 ([O] Cor 13.2.5.) R_{10} *is neither graphic nor cographic.*

Proof

R_{10} contains $M(K_{3,3})$, so it is not cographic. Its dual (which happens to be itself) contains $M(K_{3,3})$ so it does not contain $M^*(K_{3,3})$ therefore it is not graphic. ∎

A celebrated theorem of Seymour on matroid decompositions shows that all unimodular systems of vectors can be obtained from amalgams of the three types mentioned above ([DG], [O]) (Also see chapter 8).

4.1.3 Regular Matroids and Lattice Dicings

The very important connection between matroids and lattice dicings (which we make use of in Theorem 3.4) is in the following: The matrix for a unimodular system of vectors can be written as a matrix $(I_n|A)$ where the columns are the vectors. Let R be the set of column vectors. The vectors define a family $H(R)$ of of parallel hyperplanes $H(\mathbf{r}, z) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{r} = z\}, z \in \mathbb{Z}, \mathbf{r} \in R$. If $B \subset R$ is a basis for $\mathbb{R}^n$ then the intersection points of hyperplanes in $H(B)$ is a lattice L and $H(R)$ is then called a *lattice dicing*.

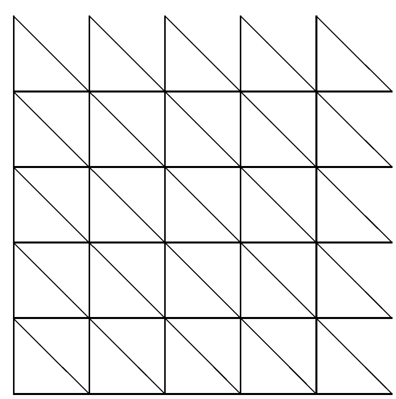
Theorem 4.10 ([ER]) *The set of intersection points of hyperplanes of $H(R)$ is L if and only if R is a unimodular system of vectors.*

Example

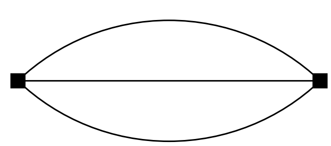
If we have the unimodular system

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

then the dicing we obtain is



The unimodular system is $M^*(G)$ for the graph G given by



It is obtained using Definition 4.7. $\square$

4.2 Nodes on Cubic Threefolds

In this section we will construct all the matroids coming from nodes of cubic threefolds in $\mathbb{P}^4$. We will use the following facts we have already seen in Section 2.2 to make this construction.

1. No three nodes lie in a line.

2. Four nodes can lie on a plane.

3. No five nodes can lie on a plane.

4. No five nodes can lie in a three space with no four in a plane.

The unimodular systems below are written in such a way that the dependence of the nodes is reflected in the dependence of the corresponding columns. We will label the matroids we obtain by M_a or $M_{a,b}$ where a is the number of nodes in X and b is a number indexing for the case there are several matroids (e.g. in the case of 4 nodes).

4.2.1 1 NODE

The unimodular system in this case is just $M_1 = [1]$

4.2.2 2 NODES

Since these are in general position the unimodular system is $M_2 = I_2$, the 2×2 identity matrix.

4.2.3 3 NODES

These are in general positon by the comments at the beginning of this section. So the unimodular system is $M_3 = I_3$.

4.2.4 4 NODES

Things get interesting from here on because there can be multiple configurations for the same number of nodes. Two things can happen to the configuration of the nodes.

1. If the four nodes do not lie in a plane then the unimodular system is $M_{4,1} = I_4$ in the same was as above.

2. If the four node do lie on a plane the unimodular system is

$$M_{4,2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

4.2.5 5 nodes

Once again only one of two things can happen.

1. If the nodes are in general position then the U-system is $M_{5,1} = I_5$.

2. If four of the nodes lie on plane then the U-system is

$$M_{5,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

   If the nodes are $P_1, \dots, P_5$ the the plane is $\{P_1, P_2, P_3, P_5\}$.

4.2.6 6 nodes

Cubic threefolds with six nodes have been studied in [CLSS]. They show that for six points in general position the linear system of threefolds singular at those 6 points is four dimensional. This is the situation where the cubic threefold has no planes. Note that *general position* means that the every subset with m points or less with $m \leq 5$ spans $\mathbb{P}^{m-1}$. The other two cases are when the nodes fail to be in general position. Suppose the nodes are $P_1, \dots, P_6$.

1. The cubic threefold has no planes. All the nodes could be such that any five span $\mathbb{P}^4$. In this case the unimodular system is

$$M_{6,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

   and this relation can be seen from the unimodular system.

2. There could be only one plane. So there are four nodes in this plane. In this case the unimodular system is

$$M_{6,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

   The plane is $\{P_1, P_2, P_3, P_6\}$.

   It is not possible to have exactly two planes. Suppose this does happen. Then without loss of generality assume the first plane has nodes $\{P_1, P_2, P_3, P_5\}$. The second plane can share only two points with the first plane. If they share P_1 and P_2 then the second plane has nodes $\{P_1, P_2, P_4, P_6\}$. If we cut the cubic threefold with the hyperplane containing the two planes then we can a cubic surface which contains two planes. So it must contain a third plane which give us the case below, or a contradiction for the two planes case.

3. There could be three planes. More explicitly the planes have nodes $\{P_1, P_2, P_3, P_5\}$, $\{P_1, P_2, P_4, P_6\}$ and $\{P_3, P_4, P_5, P_6\}$. In this case the unimodular system is

$$M_{6,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

4.2.7 7 NODES

The case of seven nodes in general position is interesting in its own right. However by [CLSS] (Cor 2.8) if a cubic threefold has 7 nodes in general position then it has nonisolated singularities. So the interesting cases for us always have at least one plane on the threefold. By Table 5.1 in Chapter 5 there are two possiblities. Suppose the nodes are $P_1, \ldots, P_7$.

1. The threefold has 2 planes. Then the two planes can meet only at a single node. If they meet at two nodes then the hypersurface spanned by the nodes cuts out a cubic surface which contains two planes. There would then be a third plane. The nodes on each of the planes are $\{P_1, P_5, P_6, P_7\}$ and $\{P_2, P_3, P_4, P_7\}$. The unimodular system is

$$M_{7,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

2. the threefold has 3 planes. They are

   $\{P_1, P_2, P_4, P_7\}$,

   $\{P_1, P_3, P_6, P_7\}$

   and

$\{P_2, P_3, P_4, P_6\}$.

$$M_{7,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4.2.8 8 NODES

Suppose the nodes are $P_1, \ldots, P_8$. Then by results of Chapter 5 Section 5.1 we see that there are exactly 5 planes. More explicitly we can show how the nodes are arranged. The space sextic curve we will use is the one which splits as $(2,1)+(1,0)+(0,1)+(0,1)$. Let the $(1,0)$ curve be a. Let the $(0,1)$ curves be b and c respectively. Assume we projected from the point P_8. The lines a and b intersect at a node. This corresponds to a node of the cubic threefold which we suppose is P_7. Similarly the intersection of a and c correspond to the node P_3. The line a correspond to a plane of X and this plane has four nodes. We already have three of them. The fourth one corresponding to the intersection of the $(2,1)$ curve with a is P_4. So the plane corresponding to a has nodes $\{P_3, P_4, P_7, P_8\}$. The plane corresponding to b has nodes $\{P_5, P_6, P_7, P_8\}$. Because it already had nodes P_7 and P_8 we included two more which correspond to the intersection of b with the $(2,1)$ curve. The plane corresponding to c has nodes $\{P_1, P_2, P_3, P_8\}$. So this forces us to see that the plane corresponding to the $(1,1)$ curve $a+b$ to have nodes $\{P_3, P_4, P_5, P_6\}$ and the last plane to have nodes $\{P_1, P_2, P_4, P_7\}$.

The planes each contain the following nodes:

$\{P_1, P_2, P_3, P_8\}$,

$\{P_1, P_2, P_4, P_7\}$,

$\{P_3, P_4, P_7, P_8\}$,

$\{P_3, P_4, P_5, P_6\}$,

$\{P_5, P_6, P_7, P_8\}$.

The unimodular system obtained is

$$M_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that there are two types of nodes. There are those which lie in two planes and those that lie in three planes.

4.2.9 9 NODES

In this case we have nine planes using the results of of Chapter 5 again. By exactly the same argument as in the 8 nodal case we obtain the planes with nodes in the following configuration.

$\{P_1, P_2, P_3, P_6\}$, $\{P_1, P_2, P_5, P_8\}$,
$\{P_1, P_3, P_4, P_9\}$, $\{P_1, P_4, P_5, P_7\}$,
$\{P_2, P_4, P_6, P_9\}$, $\{P_2, P_4, P_7, P_8\}$,
$\{P_3, P_5, P_6, P_8\}$, $\{P_3, P_5, P_7, P_9\}$,
$\{P_6, P_7, P_8, P_9\}$

In this case the unimodular system is $M_9 = M(K_{3,3})$. This and the ten nodal case show that in the world of degenerations of Prym varieties more unimodular systems than cographic ones can be obtained.

4.2.10 10 NODES

This is a very well known cubic threefold. In [FW] all the planes are listed. They are

$\{P_1, P_2, P_5, P_8\}$, $\{P_1, P_3, P_6, P_9\}$,
$\{P_1, P_4, P_7, P_{10}\}$, $\{P_1, P_2, P_3, P_4\}$.
$\{P_1, P_5, P_6, P_7\}$, $\{P_1, P_8, P_9, P_{10}\}$,

$\{P_2, P_5, P_9, P_{10}\}$, $\{P_2, P_6, P_7, P_8\}$,

$\{P_2, P_4, P_6, P_9\}$, $\{P_2, P_3, P_7, P_{10}\}$,

$\{P_3, P_4, P_5, P_8\}$, $\{P_3, P_5, P_7, P_9\}$,

$\{P_3, P_6, P_8, P_{10}\}$, $\{P_4, P_5, P_6, P_{10}\}$,

$\{P_1, P_2, P_4, P_7\}$.

The unimodular system in this case is $M_{10} = R_{10}$.

Chapter 5

Curves Associated to a Cubic Threefold

In this chapter we construct three curves associated to a cubic threefold. In the process will also describe the Fano surface. A lot of this work follows and generalizes results from Hadan ([Ha]) (for the Fano surface), Finkelnberg-Werner ([FW]) (for the sextic) and Allcock ([Al]). The three curves we construct are the plane quintic Δ, another curve Δ' which is an unramified double cover of Δ and the associated sextic curve. There is nothing canonical about any of these curves. The plane quintic curve and its double cover depend on the choice of a line (which in most places we will denote ℓ_0). The sextic curve depends on the choice of a node. There are usually several choices (unless of course the cubic threefold has only one node!). The main results are

Theorem 5.1 *1. There exists a sextic curve S (which depends on the choice of a node) with the property that the surface $\mathcal{F}$ is birational to the symmetric product of S with itself.*

2. There exists a plane quintic curve Δ which depends on the the choice of a point in $\mathcal{F}$. The point can be chosen so that Δ only has nodes coming from X.

3. There exists a curve $\Delta' \subset \mathcal{F}$ such that there is a 2-to-1 unramified map $\Delta' \to \Delta$.

Throughout this chapter we assume X is a cubic threefold with only finitely many singularites.

5.1 The Fano Surface and Sextic Curve

The family of lines on a cubic threefold are interesting objects. It was first studied by Fano and is now named after him. In this paper we will give a description of how they can be seen. A lot of the work here is from the papers [H] and [AK].

Definition 5.2 ([AK], 1.1) *Let*

$$\mathcal{F} = \{\ell{:}\ell \textit{ is a line on } X\} \subset Gr(2,5).$$

$\mathcal{F}$ is called the Fano variety of lines on X

The first thing to note is that $\mathcal{F}$ is nonempty. Cut X with a generic hyperplane. The intersection is a smooth cubic surface which has 27 lines in general.

In the case of cubic threefolds $\mathcal{F}$ is actually a surface ([AK] 1.4). Some more results are that

Lemma 5.3 *([AK] 1.4, 1.16) The Fano surface associated to a cubic surface is geometrically connected and of pure dimension 2.*

The Fano surface lives in the Grassmanian $Gr(2,5)$ of lines in $\mathbb{P}^4$ which is of dimension 6.

We now look at a special curve associated with a nodal cubic threefold X. Let X be a cubic threefold in $\mathbb{P}^4$ with at least one node P_0. Let H be a hyperplane not passing through P_0, and let Q be the intersection of H with the tangent cone to X at P_0. Let $S = Q \cap X$. This is a sextic curve lying on a smooth quadric surface.

Definition 5.4 *S is called an associated sextic of the threefold X.*

The projection $p : X \backslash \{P_0\} \to H$ maps nodes of X (not P_0) to ordinary double points of S and these are the only singularities of S. Note that since S lies on $Q \cap H$ it is of type $(3,3)$. Also, this curve is not unique as it depends on the choice of a node.

The main result linking the two concepts defined above is:

Theorem 5.5 ([H], Prop 3.10) *Let ℓ be a line in X. If $P_0 \in \ell$ then ℓ is a line $\overline{P_0 s}$ with s a point of S and conversely every $s \in S$ defines a line in X through P_0. A line ℓ not containing P_0 is mapped onto a line $\overline{\ell} = p(\ell) \in H$ not contained in Q which either connects two points of S or is a tangent at S in a smooth point of S or tangent at Q in a singular points of S. Conversely every such line $\overline{\ell}$ is the image of a line in X.*

Yet more information can be obtained from the above results. The following two results are important enough that we will mention them here.

Proposition 5.6 ([H], Prop 3.11) *The Fano surface $\mathcal{F}$ of X is birationally equivalent to the symmetric product of the sextic with itself.*

The components of the Fano surface are of two types. From the above results we can count the number of components of the Fano Surface. One type of components comes from the symmetric product of an irreducible curve with itself. The other type is from the symmetric product of two different irreducible components. The number of components of the Fano surface can now be counted easily.

Corollary 5.7 ([H], Prop 3.12) *If the sextic S of X consists of n irreducible components then the Fano surface has $\binom{n+1}{2}$ components.*

Planes in the Fano surface $\mathcal{F}$ of X are important in their own right because there is a one-to-one correspondence between planes in X and planes in $\mathcal{F}$. They can be explicitly seen in the following lemma.

Lemma 5.8 ([H] Prop 3.13,[FW] 3.1) *Any plane in X containing P_0 corresponds to an irreducible component of S which is a line (of type (1,0) or (0,1)). Any other plane in the cubic corresponds to an irreducible component of type (1,1) or a pair of crossing lines (i.e a reducible curve of type (1,1)). These correspondences are one-to-one.*

With the results above we can make a table of nodal cubic threefolds and their Fano surfaces. We first note that an (a, b) curve on a smooth quadric surface can have at most

nodes	USV	Splitting of S	g of S_n	nodes of 1 comp	Comps in F	Planes in X
1	M_1	$(3,3)$	4	0	1	0
2	M_2	$(3,3)$	3	1	1	0
3	M_3	$(3,3)$	2	2	1	0
4	$M_{4,1}$	$(3,3)$	1	3	1	0
4	$M_{4,2}$	$(3,2)+(0,1)$	2	0	3	1
5	$M_{5,1}$	$(3,3)$	0	4	1	0
5	$M_{5,2}$	$(3,2)+(0,1)$	1	1	3	1
5	$M_{5,2}$	$(2,2)+(1,1)$	1	0	3	1
6	$M_{6,1}$	$(2,1)+(1,2)$	0	0	3	0
6	$M_{6,2}$	$(3,2)+(0,1)$	0	2	3	1
6	$M_{6,2}$	$(2,2)+(1,1)$	0	1	3	1
6	$M_{6,3}$	$(2,2)+(1,0)+(0,1)$	1	0	6	3
7	$M_{7,1}$	$(3,1)+(0,1)+(0,1)$	0	0	6	2
7	$M_{7,1}$	$(2,1)+(1,1)+(0,1)$	0	0	6	2
7	$M_{7,2}$	$(2,2)+(1,0)+(0,1)$	0	1	6	3
7	$M_{7,2}$	$3\times(1,1)$	0	0	6	3
8	M_8	$(2,1)+(1,0)+2\times(0,1)$	0	0	10	5
8	M_8	$2\times(1,1)+(1,0)+(0,1)$	0	0	10	5
9	M_9	$2\times(1,0)+2\times(0,1)+(1,1)$	0	0	15	9
10	M_{10}	$3\times(1,0)+3\times(0,1)$	0	0	21	15

Table 5.1: Unimodular systems and Sextic Curves.

$(a-1)(b-1)$ nodes because it has genus $(a-1)(b-1)$. In the table below we show all the possible sextic curves we may obtain. We will also give the unimodular matroid coming from the cubic threefold with that sextic.

Because our sextic depends on the choice of a node it is possible for a cubic threefold to have more than one type sextic curve associated to it. This is also noted in [FW] Section 3. It uses the term *watchtower* for the node projected from. In particular:

1. In the case of 5 nodes there are three entries. The last two come from a cubic threefold with four nodes in a plane and one off the plane. If S is constructed by projecting from

a node on the plane we obtain the curve $(3,2)+(0,1)$ and if we project for the node off the plane we obtain the curve $(2,2)+(1,1)$.

2. A similar thing happens with the 6 node case. The second and third entry can be obtained from one threefold. In the case of the second entry the watchtower is a node on the plane. The third entry has as its watchtower a node off the plane.

3. The same thing happens in the seven node case. Entries 1 and 2 are from the same threefold. For entry 1 the watchtower is the unique node in both planes, while entry 2 is from the other types of nodes.

   Entries 3 and 4 are from the same threefold. In this case entry 3 is obtained from a node on one of the planes. Entry 4 is from a node not on any plane.

4. Same goes for eight nodes. In the first case the watchtower is a node in three planes and the second case it's a node in only two planes.

For any sextic S listed above a cubic threefold can be constructed. If we have a nodal curve of type $(3,3)$ on a smooth quadric surface Q which has at most ordinary double points we can construct a nodal cubic threefold. Start with the the following sequence.

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Q \to 0.$$

Tensor this with $\mathcal{O}_{\mathbb{P}^3}(3)$ and look at the cohomology. We obtain

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)) \to H^1(\mathcal{O}_{\mathbb{P}^3}(1)).$$

The last term is 0 by [Har] (III.5.1) and so the map $H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3))$ is surjective. Hence any $(3,3)$ curve is cut out by a cubic surface. If f is the equation of the cubic surface and q is the equation of the Q (both with homogeneous coordinates $x_0,\ldots,x_3$) then x_4q+f is the equation of a cubic threefold with a node at (0:0:0:0:1) and has S as the associated sextic.

5.2 The Plane Quintic

In the case of a smooth cubic threefold$X$ there is another curve associated to the cubic threefold. Choose a generic line ℓ in X corresponding to a point in F and let Δ'_ℓ be the subvariety of lines in X incident to ℓ. The blowup of X along ℓ has the structure of a conic bundle over $\mathbb{P}^2$ whose discriminant curve Δ_ℓ is a smooth plane quintic. This construction occurs a lot in the literature but notably in [CG] (Appendix C) and also in [Mu1] and [Mu2]. We wish to extend this concept to our nodal threefolds. However the definition of a generic line needs to be clarified and we need to deal with the fact that if X has nodes then plane quintic will also have nodes.

From now on we will drop the subscript ℓ in Δ unless it is not clear what choice of line we are using.

5.2.1 A good line

We will call a line $\ell \subset X$ a *good line* if the plane quintic Δ we obtain by projecting from this line has only nodes as singularities and the double cover Δ' is unramified. This section is dedicated to the search for good lines. It is modelled very closely on [Mu1].

The main result is the following

Theorem 5.9 *The set of good lines for a cubic threefold X form an open subset of the nonplanar components of the Fano surface.*

This theorem is a direct corollary of Lemma 5.10 and Corollary 5.11. The first proof we do needs the following definition. For a point $s \in \mathcal{F}$ we let ℓ_s be the corresponding line in the cubic threefold. Also in this section L is a plane. Let

$$\mathcal{F}' = \{s \in S : \exists L \text{ with } L \cdot X = 2\ell_s + \ell_t\}$$

$$\mathcal{F}'' = \{m \in S : \exists L \text{ with } L \cdot X = 2\ell_m + \ell_p\}$$

The following lemma is an extension of Lemma 1.4 in [Mu1]. In particular they did not (have to) deal a multicomponent Fano surface.

Lemma 5.10 *The set $\mathcal{F}'$ is a Zariski closed set of $\mathcal{F}$. Also, locally on nonplanar components of $\mathcal{F}$ it is defined by one equation, therefore it is one dimensional on such components.*

Proof

Let $\mathbb{P}^4$ be a projective space given by coordinates (x, y, z, u, v). Let X be a cubic threefold given by

$$F(x, y, z, u, v) = 0.$$

As usual let $\mathcal{F}$ be the fano surface of lines. We show in this lemma that there always exists a good line. First we look at planes through a line in $\mathbb{P}^4$. Firstly we put local coordinates on $G(2, 5)$, the Grassmanian of lines in $\mathbb{P}^4$. This has dimension $6 = 2\times(5-2)$. Suppose ℓ is a line in $\mathbb{P}^4$ not meeting the plane $(u = 0, v = 0)$. Then let $(x', y', z', 1, 0)$ be the point $\ell \cap (v = 0)$ and $(x'', y'', z'', 0, 1)$ be the point $\ell \cap (u = 0)$. From this we can use $(x', y', z', x'', y'', z'')$ as the local coordinates for the grassmanian of lines in $\mathbb{P}^4$. A point $P \in \ell$ has coordinates

$$\begin{aligned} x &= ux' + vx'' \\ y &= uy' + vy'' \\ z &= uz' + vz'' \\ u &= u \\ v &= v. \end{aligned} \tag{5.1}$$

The line ℓ lies on X if and only if $F(ux' + vx'', uy' + vy'', uz' + vz'') = 0$ for all u, v.

Now let ℓ_0 be a line on X. We look for the conditions so that we have a plane L_0 such that $X \cdot L_0 = 2\ell_0 + \ell'$ for some line ℓ'. If $T_0 = (a, b, c, 0, 0)$ is a point of $L_0 \cap (u = 0, v = 0)$ then L_0 is generated by ℓ_0 and T_0. A point $Q \in L_0$ is given by

$$Q = P + t(a, b, c, 0, 0)$$

where P is a point of ℓ_0 and u, v (in P) and t are the projective coordinates of L_0.

So

$$F(Q) = F(P) + t\left[a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z}\right] + t^2(\cdots) + t^3(\cdots) = 0. \tag{5.2}$$

As $F(P)=0$, on $X \cap L_0$ we have the line ℓ_0 given by $t=0$ and a conic given by

$$\left[a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z}\right] + t(\cdots) + t^2(\cdots) = 0.$$

This conic is degenerate (i.e $X \cdot L_0 = 2\ell_0 + \ell'$) if and only if

$$a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z} = 0. \tag{5.3}$$

This is because $t=0$ on ℓ_0 so only the first part has to be 0 as the rest of the expression is zero when t=0.

We expand and get

$$\begin{aligned}
\frac{\partial F}{\partial x}(P) &= u^2\psi_{11}(x_0', \ldots, z_0'') + uv\psi_{12}(x_0', \ldots, z_0'') + v^2\psi_{13}(x_0', \ldots, z_0'') \\
\frac{\partial F}{\partial y}(P) &= u^2\psi_{21}(x_0', \ldots, z_0'') + uv\psi_{22}(x_0', \ldots, z_0'') + v^2\psi_{23}(x_0', \ldots, z_0'') \\
\frac{\partial F}{\partial z}(P) &= u^2\psi_{31}(x_0', \ldots, z_0'') + uv\psi_{32}(x_0', \ldots, z_0'') + v^2\psi_{33}(x_0', \ldots, z_0'')
\end{aligned}$$

We get that there exists a plane L_0 such that $X \cdot L_0 = 2\ell_0 + \ell'$ if and only if

$$\det\begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{pmatrix} = \psi(x_0', \cdots, z_0'') = 0. \tag{5.4}$$

So far we have ignored the lines meeting the 2-plane $\{u=0, v=0\}$. These lines are a Zariski closed subset of $\mathcal{F}$. So at this point we have shown that $\mathcal{F}'$ is a Zariski closed subset of $\mathcal{F}$.

Now we show that this closed subset is of dimension at most one on the non-planes of the $\mathcal{F}$. Let $\ell_0 \in \mathcal{F}$ be chosen such that it does not lie in a plane component and does not contain a node of X. Assume it is given by

$$x = y = z = 0.$$

Using the notation from above we have the local coordinates of ℓ_0 are

$$x' = x'' = y' = y'' = z' = z'' = 0.$$

So we can rewrite F as

$$xf(x,y,z,u,v) + yg(x,y,z,u,v) + zh(x,y,z,u,v) = 0 \tag{5.5}$$

where

$$\begin{aligned} f &= \lambda u^2 + \lambda' uv + \lambda'' v^2 + \cdots \\ g &= \mu u^2 + \mu' uv + \mu'' v^2 + \cdots \\ h &= \nu u^2 + \nu' uv + \nu'' v^2 + \cdots . \end{aligned} \tag{5.6}$$

Let ℓ be a variable line as before. Substitute 5.1 in 5.5 and look at the coefficients of u^3, u^2v, uv^2 and v^3 and obtain

$$\begin{aligned} \phi_1(x',\cdots,z'') &= \lambda x' + \mu y' + \nu z' + \cdots = 0 \\ \phi_2(x',\cdots,z'') &= \lambda' x' + \lambda x'' + \mu y' + \mu y'' + \nu' z' + \nu z'' + \cdots = 0 \\ \phi_3(x',\cdots,z'') &= \lambda'' x' + \lambda' x'' + \mu'' y' + \mu' y'' + \nu'' z' + \nu' z'' + \cdots = 0 \\ \phi_4(x',\cdots,z'') &= \lambda'' x'' + \mu'' y'' + \nu'' z'' + \cdots = 0 \end{aligned}$$

The tangent space to F at ℓ_0 is given by the linear terms in $x', \ldots, z''$ of $\phi_1, \cdots, \phi_4$ above. We determine its dimension by looking at the rank of the matrix

$$\begin{pmatrix} \lambda & 0 & \mu & 0 & \nu & 0 \\ \lambda' & \lambda & \mu' & \mu & \nu' & \nu \\ \lambda'' & \lambda' & \mu'' & \mu' & \nu'' & \nu' \\ 0 & \lambda'' & 0 & \mu'' & 0 & \nu'' \end{pmatrix}. \tag{5.7}$$

Consider for instance the submatrix

$$\begin{pmatrix} \mu & 0 & \nu & 0 \\ \mu' & \mu & \nu' & \nu \\ \mu'' & \mu' & \nu'' & \nu' \\ 0 & \mu'' & 0 & \nu'' \end{pmatrix} \tag{5.8}$$

Its determinant is the resultant of $g(0,0,0,u,v) = 0$ and $h(0,0,0,u,v) = 0$.

If f, g and h do not have a common zero in pairs on ℓ_0 then the rank of the matrix 5.7 is 4. Otherwise we can assume $(u, v) = (1, 0)$ is a common zero for f and g and $(u, v) = (0, 1)$ is a common zero for g and h. Then $\lambda = \mu = \mu'' = \nu''$. By using the same idea as in determining the rank of 5.9 we see that $\nu \neq 0$, $\lambda'' \neq 0$ and $\nu' \neq 0$. Then the first second third and fifth column are independent and the rank is again 4. So $\mathcal{F}$ is nonsingular around ℓ_0.

Again let $T_0 = (a, b, c, 0, 0)$ and L_0 be the span of ℓ_0 and T_0. The equation 5.2 gives

$$a \cdot f(0, 0, 0u, v) + b \cdot g(0, 0, 0u, v) + c \cdot h(0, 0, 0u, v) + \text{terms in t} = 0.$$

Thefore by Equation 5.3 $\ell_0 \in \mathcal{F}'$ if and only if

$$af(0, 0, 0, u, v) + bg(0, 0, 0u, v) + ch(0, 0, 0u, v) + (\text{terms in t}) = 0.$$

By expanding $f(0, 0, 0, u, v)$, $g(0, 0, 0, u, v)$ and $h(0, 0, 0, u, v)$ and using 5.6 we see that $\ell_0 \in \mathcal{F}'$ if and only if the matrix

$$\begin{pmatrix} \lambda & \lambda' & \lambda'' \\ \mu & \mu' & \mu'' \\ \nu & \nu' & \nu'' \end{pmatrix} \tag{5.9}$$

is of rank 2. The rank of this matrix a always at least 2. If $f(0, 0, 0, u, v) = 0$, $g(0, 0, 0, u, v) = 0$ and $h(0, 0, 0, u, v) = 0$ had a common zero then at $P = (0, 0, 0, u, v)$ we would have $\frac{\partial f}{\partial x}(P) = f(0, 0, 0, u, v)$, $\frac{\partial f}{\partial y}(P) = g(0, 0, 0, u, v)$ and $\frac{\partial f}{\partial z}(P) = h(0, 0, 0, u, v)$. This contradicts the absence of a node on ℓ_0.

Let now $\ell_0 \in \mathcal{F}'$. We make a change of variable so that $f(0, 0, 0, u, v) = 0$. So now $\lambda = \lambda' = \lambda'' = 0$ and so the tangent space ot $\mathcal{F}$ at ℓ_0 is given by

$$y' = y'' = z' = z'' = 0.$$

So now the determinant of 5.8 is different from zero as $\mathcal{F}$ is smooth around ℓ_0 (because of choice and property of singularities of $\mathcal{F}$). Equation 5.5 now simplifies to

$$x^2\ell(u, v) + y \cdot g(x, y, z, u, v) + z \cdot h(x, y, z, u, v) = 0. \tag{5.10}$$

We now make explicit the equation $\psi = 0$ of 5.4 defining $\mathcal{F}'$ on the nonplanar components. Substituting the coordinates 5.1 of a point $P \in \ell$ in equation 5.10 we get $\ell \in \mathcal{F}'$ if and only if 5.3 is satisfied.

We do the substitution and compute the coefficients of u^2, uv and v^2. We are only interested in the neighbourhood of ℓ_0 and so we only look at

$$\overline{\psi} = \psi \bmod (y', y'', z', x'^2, x''^2, x'x'').$$

This gives

$$det \begin{pmatrix} 2x'\ell(1,0) & \mu & \nu \\ 2x'\ell(0,1) + 2x''\ell(1,0) & \mu' & \nu' \\ 2x''\ell(0,1) & \mu'' & \nu'' \end{pmatrix}$$

Therefore $\overline{\psi} = 2x'\{\ell(1,0)(\mu'\nu'' - \mu''\nu') - \ell(0,1)(\mu\nu'' - \mu''\nu)\} + 2x''\{\ell(0,1)(\mu\nu' - \mu'\nu) - \ell(1,0)(\mu\nu'' - \mu''\nu)\}$.

Look at the coefficents

$$\ell(1,0)(\mu'\nu'' - \mu''\nu') - \ell(0,1)(\mu\nu'' - \mu''\nu)$$

and

$$\ell(0,1)(\mu\nu' - \mu'\nu) - \ell(1,0)(\mu\nu'' - \mu''\nu).$$

From 5.8 and the fact that the determinant is not zero that not both coefficients in $\overline{\psi}$ are zero unless $\ell(u,v)$ is identically zero. This would mean that ℓ_0 is contained in a plane which is not possible by our choice.

The tanget space to $\mathcal{F}'$ is given by

$$y' = y'' = z' = z'' = \overline{\psi} = 0$$

we have the dimension of $\mathcal{F}'$ is 1. ∎

A corollary is the following.

Corollary 5.11 *The set $\mathcal{F}''$ is dimension at most one.*

This comes about because each point of $\mathcal{F}''$ is uniquely determined by a point of $\mathcal{F}'$.

A good line is a line that does not live in the subsets defined by $\mathcal{F}'$ and $\mathcal{F}''$ and does not lie in a plane of X. An easy way to determine whether a line lives in $\mathcal{F}'$ is the following. Suppose our line is given by

$$x = y = z = 0.$$

Then the equation for X can be written as

$$u^2\ell_1(x,y,z) + 2uv\ell_2(x,y,z) + v^2\ell_3(x,y,z) + 2uQ_1(x,y,z) + 2vQ_2(x,y,z) + C(x,y,z) = 0 \tag{5.11}$$

where ℓ_i are linear, Q_i are quadratic and C_i is cubic in x, y, z. The claim is that $\ell \in \mathcal{F}$ if and only if ℓ_1, ℓ_2 and ℓ_3 are linearly independent. *Proof*
Comparing equations 5.11, 5.5 and 5.6 we see that

$$\begin{aligned} \ell_1(x,y,z) &= \lambda x + \mu y + \nu z \\ \tfrac{1}{2}\ell_2(x,y,z) &= \lambda' x + \mu' y + \nu' z \\ \ell_1(x,y,z) &= \lambda'' x + \mu'' y + \nu'' z. \end{aligned}$$

So $\ell \in \mathcal{F}'$ if and only if determinant of 5.9 is zero. This follows from the comment that the rank of this matrix is two only if $\ell \in \mathcal{F}'$. ∎

So we have that a good line is a line not in $\mathcal{F}'$ or $\mathcal{F}''$ or a plane contained in X.

Now that we know how to pick a good line we construct a plane quintic.

5.2.2 Constructing the Plane Quintic

So we choose a good line ℓ_0. We can rewrite the equation of X as

$$u^2x + 2uvy + v^2z + 2uQ_1(x,y,z) + 2vQ_2(x,y,z) + C(x,y,z) = 0. \tag{5.12}$$

Let

$$N = \{L : L \text{ a plane containing } \ell_0\}.$$

This is a projective plane. We suppose ℓ_0 has equations $x = y = z = 0$. We the identify N with the plane defined by $(u = 0, v = 0)$. Let the coordinates of a point T on N be $(a, b, c, 0, 0)$. Let L_T be the span of ℓ_0 and T. The the intersection of L_T and X is a line ℓ_0 and a conic K_T. A point on L_T has projective coordinates (at, bt, ct, u, v). We can use (t, u, v) as projective coordinates on L_T. If the equation of X is given by 5.12 then the equation of K_T is given by

$$u^2a + 2uvb + v^2c + 2utQ_1(a, b, c) + 2vtQ_2(a, b, c) + t^2C_3(a, b, c) = 0.$$

The conic K_T degenerates to two lines if T is on the curve Δ in N where Δ has equation

$$\det \begin{pmatrix} a & b & Q_1(a,b,c) \\ b & c & Q_2(a,b,c) \\ Q_1(a,b,c) & Q_2(a,b,c) & C_3(a,b,c) \end{pmatrix} = 0.$$

This is clearly a quintic equation.

The next result relates the nodes of X and the nodes of C.

Lemma 5.12 *The nodes on the cubic threefold are in one-one correspondence with the nodes the plane quintic curve.*

Proof

This uses the ideas in [Sc]. Let the cubic threefold blown up along the line be $\tilde{X}$ and the plane quintic curve be $C \subset (\mathbb{P}^2, y_0)$ where y_0 is a point of Δ. There is a map $C \subset (\mathbb{P}^2, y_0) \to (\mathbb{A}^1, 0)$. Let $y_0 \in C$ and let the preimage of y_0 in $\tilde{X}$ above $(\mathbb{P}^2, y_0)$ i.e. the two lines be given locally by affine equation $xy = 0$. The map $\hat{\mathcal{O}}_{A^2,y_0} \leftarrow \hat{\mathcal{O}}_{A^1,0}$ of local C algebras takes the generator t to the the function $\delta(u, v)$ which is regular at y_0. Upstairs the versal deformation of $xy = 0$ is $xy - t = 0$. The first order deformation is $T' = C[[x, y]]/(xy, x, y)$. This is 1-dimensional with generator 1. So the versal deformation of $xy = 0$ is $xy - 1 \cdot t = 0$. Then formally the function $xy - t$ pulls back to $xy - \delta(u, v)$. The total space $(\tilde{X}, x_0)$ has local equation $xy = g(u, v)$ which can be changed to $\delta(u, v) + \tilde{x}^2 + \tilde{y}^2$ for different $\tilde{x}, \tilde{y}$. So if $\delta(u, v)$ was a singularity of type A_n then upstairs (at least formally) the singularity is of type A_n. ∎

Now that we have Δ we can give Δ'. Suppose our line ℓ_0 corresponds to the point $s_0 \in \mathcal{F}$ Define Δ' to be

$$\Delta' = \{s \in \mathcal{F} : \ell_s \cap \ell_0 \neq \emptyset \text{ and } s \neq s_0\}.$$

There is a map $q : \Delta' \to \Delta$ which takes $s \in \mathcal{F}$ to the point $T \in N$ such that L_T is spanned by ℓ_0 and ℓ_s. Because of the fact that ℓ_0 does not lie in a plane on X the argument in [BS] page 5 apples and we see that Δ' is a closed subset of $\mathcal{F}$ and $s_0 \notin \Delta'$.

The map q is two to one because each plane L_T on Δ contains two lines beside ℓ_0 and they both map to $T \in N$. By the argument of [CM] Cor 2.31 we see that Δ' is a nodal curve with exactly $2n$ singularities. Related to q is the fact that Δ' has an involution which comes from exchanging two lines which map to the same point.

Chapter 6

Jacobians and Pryms

In this chapter we prove the theorem

Theorem 6.1 *For a cubic threefold X with associated sextic curve S and a plane quintic Δ with double cover Δ' we have*

$$J(S_n) \simeq Pr(\Delta'_n/\Delta_n).$$

The objects of study in this chapter are Chow groups. These are defined as follows (following [CM]):

Definition 6.2 ([CM]) *Let X be a smooth quasiprojective variety. The Chow group is*

$$CH(X) = \oplus_i CH^i(X)$$

where $CH^i(X)$ is the group of cycle classes modulo rational equivalence of codimension i.

The subgroup A^i of $CH^i(X)$ is those classes which are algebraically equivalent to zero.

If X is a curve we define $J(X)$ to be A^0.

The main reason for studying them is because they are the direct generalization of divisors and hence it can give us important information on the variety. The main theorem we prove in this chapter is

Theorem 6.3 *Let S be an associated sextic curve as defined by Definition 5.4. Let Δ and Δ' be the plane quintic and its unramified double cover as constructed in 5.2. Let Δ_n and Δ'_n be their normalizations. Then*

$$J(S_n) \simeq P(\Delta'_n/\Delta_n). \tag{6.1}$$

This is proved in two parts. The first part gives the following general result which will be used in the second part.

Proposition 6.4 *Let V be a smooth threefold. Let $Y \subset V$ be a curve with n ordinary double points $\{y_i\}$. Let $b : V^+ \to V$ be the blowup of V along Y. Then V^+ is singular with n nodes. Let $b^+ : V' \to V^+$ be the blowup of V^+ at the nodes. Then*

$$A^2(V') = A^2(V) \oplus J(Y_n).$$

We will follow the proof as done in by Collino and Murre in [CM]. They only dealt with the case of when Y has two nodes as opposed to our more general result of arbitrarily many nodes.

6.1 The Chow Group of a Certain Threefold

Let V be a smooth threefold. Let $Y \subset V$ be a curve with n ordinary double points $\{y_i\}$. Let $b : V^+ \to V$ be the blowup of V along Y. Also let $D = b^{-1}(Y)$ be the exceptional divisor and $d_i = b^{-1}(y_i)$, $y = 1, \ldots, n$.

Proposition 6.5 *V^+ has n multiple points x_i. They are ordinary double points and $b(x_i) = y_i$. The surface D is a $\mathbb{P}^1$ fibration over Y and $\{d_i\}$ are ordinary double lines for D.*

Proof

We prove this result locally at each y_i and passing to the completion of local rings. We may then assume that V is an affine 3-space with coordinates z_1, z_2, z_3, that y_i is the origin and Y is the curve $z_1 z_2 = z_3 = 0$. Then V^+ has coordinates $(z_1, z_2, z_3; w_1, w_2)$, homogenous in w_1 and w_2 with the condition

$$w_1 : w_2 = z_1 z_2 : z_3. \tag{6.2}$$

By checking directly we see that the 6.2 defines a hypersurface in $\mathbb{A}^3 \times \mathbb{P}^1$ given by

$$w_1 z_3 = w_2 z_1 z_2$$

with exactly one double point at $(0,0,0;0,1)$. D is given in $\mathbb{A}^3 \times \mathbb{P}^1$ by the equations $z_1 z_2 = z_3 = 0$, so $b^{-1}(y_i)$ an ordinary double line on D. ∎

Let $b^+ : V' \to V^+$ be the blowup of V^+ at x_i. Let $Q_i = (b^+)^{-1}(x_i)$ and $f : V' \to V$ be the composition $b^+ b$.

Proposition 6.6 *V' is smooth and Q_i is a smooth quadric surface for each $i = 1, \ldots, n$.*

Proof

Following the same argument as previously, locally in $\mathbb{A}^4 \times \mathbb{P}^3$ V' has equations

$$\begin{aligned} z_1 z_2 &= z_3 z_4 \\ t_1 t_2 &= t_3 t_4 \end{aligned}$$

and

$$z_i t_j = z_j t_i \qquad (i \neq j, \quad i,j = 1, \ldots, 4)$$

The first one comes from the following coordinates in previous proof. We are looking locally at the node so $w_2 = 1$ and so we relabel w_1 to be z_4. The equation $z_1 z_2 / z_3 = w_1 / w_2$ gives $z_1 z_2 = z_3 z_4$.

The equation of Q_i is $z_1 = z_2 = z_3 = z_4 = 0$, $t_1 t_2 = t_3 t_4$. Let $g(t_1, t_2, t_3, t_4) = t_1 t_2 - t_3 t_4$. This gives a smooth quadric surface V' is not singular outside the nodes. For the quadric surface we have $\frac{\partial g}{\partial t_1} = t_2$, $\frac{\partial g}{\partial t_2} = t_1$, $\frac{\partial g}{\partial t_3} = -t_4$, $\frac{\partial g}{\partial t_4} = -t_3$. Since not all t_i can be zero at the same time there is no singularity on Q_i. ∎

Let E be the proper inverse image of D on V', i.e. the closure of $f^{-1}(Y) - (\cup_i Q_i)$. Let d_{i0} be the proper inverse image of d_i.

Proposition 6.7 *$E \cap Q_i = d_{i1} \cup d_{i2}$ where d_{i1} and d_{i2} are two lines on Q_i which belong to different rulings.*

Proof

With the notation from the beginning of the section we assume that D is the surface is $\mathring{A}^4$

formed by the planes $(0, z_2, 0, z_4)$ and $(z_1, 0, 0, z_4)$ so d_i is the intersection of these planes. Direct checking finishes the proof. ∎

Let $g : F \to E$ be the normalization of E. By propositions 6.7 and 6.5 E is singular only along d_{i0} which are ordinary double lines, therefore F is smooth and

$$g^{-1}(d_{i0}) = e_{i3} \cup e_{i4}.$$

Also define

$$e_{ij} = g^{-1}(d_{ij}).$$

The e_{ij} intersects one just one of the lines e_{i3} and e_{i4}, so we use the convention $e_{i1} \cap e_{i3} \neq \emptyset$ and $e_{i2} \cap e_{i4} \neq \emptyset$.

On the normalization $n : Y_n \to Y$ put $n^{-1}(y_i) = y_{i1} \cup y_{i2}$. Since D is a $\mathbb{P}^1$ fibration over Y, F is a fibration

$$pr : F \to Y_n$$

with $F_y = \mathbb{P}^1$ and $pr^{-1}(y_{ij}) = e_{ij} + e_{i,j+2}$. Let D_n be the normalization of D. Then D_n is a projective fiber bundle over Y_n and F is the blowing up of D_n along $2n$ points, the exceptional lines being e_{ij} $(j = 1, 2)$, $i = 1, \ldots, n)$. From results on the Chow ring of ruled surfaces ([Har], V.2.3) and blowing ups ([Fu] 8.3.10) we get the first line below and by [Fu] (3.3(a)) we get the second line:

$$CH^1(F) = \left(CH^1(Y_n) \oplus \mathbb{Z}z\right) \oplus \left(\oplus_i \mathbb{Z}e'_{ij}\right) \qquad (j = 1, 2) \tag{6.3}$$

$$CH^2(F) = CH^2(Y_n) \tag{6.4}$$

where z is the class of the tautological sheaf on D_n and $e'_{ij} = class(e_{ij})$.

The following results are quoted without proof because the proofs as presented in[CM] are general enough and do not require any alteration.

Lemma 6.8 ([CM] 3.6) *Let X_0, X_1, X_2 be varieties, $j_i : X_i \to \mathbb{P}^{n_i}$ closed embeddings. Assume that X_0 is smooth and give morphisms $b_i : X_i \to X_{i+1}$ $(i = 0, 1)$. Let Z_i, $(i = 1, 2, 3)$*

be effective equidimensional cycles on X_0 such that

$$\dim(b_0(Z_1)) < \dim(Z_1), \dim(b_0(Z_j)) = \dim(Z_j) \qquad (j \neq 1)$$

$$\dim(b_1 b_0(Z_2)) < \dim(Z_2),\ \dim(b_1 b_0(Z_3)) = \dim(Z_3).$$

Then the cycle $K = n_1 Z_1 + n_2 Z_2 + n_3 Z_3$ is algebraically (linearly) equivalent to zero if and only if $n_1 = n_2 = n_3 = 0$.

Lemma 6.9 ([CM] 3.7, 3.8) *Let $f : V' \to V$ be a birational morphism of smooth irreducible projective varieties of dimension d. Then*

$$CH(V') = f^* CH(V) \oplus Ker f_*$$

and f^* is injective.

Also

$$CH^d(V') = CH^d(V).$$

Using the definition of V_n to be $\{x \in V : \dim\ f^{-1}(x) \geq n\}$ we quote this lemma

Lemma 6.10 ([CM] 3.9) *For every class $z \in Ker(f_*)$ there is a representative cycle Z' such that the supp$Z' \subset f^{-1}(V_1)$.*

With the two lemmas above we can compute $CH(V')$ by computing $\mathrm{Ker}(f_*)_2$ and $\mathrm{Ker}(f_*)_1$ where

$$(f_*)_i : CH^{3-i}(V') \to CH^{3-i}(V)$$

since by lemma 6.9 $CH^3(V') = CH^3(V)$.

We first prove the two dimensional case:

Proposition 6.11

$$CH^1(V') = CH^1(V) \oplus (\oplus_{i=1}^n \mathbb{Z} q_i) \oplus \mathbb{Z} e.$$

where q_i = class Q_i and e = class (E)

First the case $i = 1$. Since the dimension drops, $f_*(E) = f_*(Q_j) = 0$ for all $j = 1 \dots n$. By the notation above, $f^{-1}(V_1) = \sum Q_i + E$. So by Lemma 6.10 if $f_*(z) = 0$ there is a representative cycle of the kind $Z' = \sum_i m_i Q_i + m_e E$. Now we show that $q_i, i = 1, \dots, n$ and e generate $\mathrm{Ker}(f_*)_2$. We use the lemma 6.8 to show linear independence of the Q_i and E and finish the proof. We apply the lemma as follows. Let $Z_1 = Q_1$, $Z_2 = m_2 Q_2 + \cdots m_n Q_n$, $Z_3 = E$. Let $X_0 = V'$, X_1 be the blowup of V^+ at $x_2, \dots x_n$ and $X_2 = V^+$. We can repeat it for each Q_i and thus we get $m_i = 0$ for $i = 1, \dots, n$ and $m_e = 0$.

Now we work on the 1-dimensional cycles and finally prove

Proposition 6.12

$$CH^2(V') = CH^2(V) \oplus CH^1(Y_n) \oplus (\oplus_i \mathbb{Z} e'_{i1}) \tag{6.5}$$

By lemma 6.10 every $z \in \mathrm{Ker}(f_*)_1$ has a representative with support in the set $E + \sum Q_i$. Every 1-cycle with support in the quadric Q_i is linearly equivalent to a combination of lines in the two rulings of the quadric. The lines d_{i1} and d_{i2} belong to different rulings in Q_i and are contained in E by 6.7. So every 1-cycle class in $\mathrm{Ker}(f_*)_1$ has a representative with support in E. From the definition of F there is a morphism $g : F \to E$ with $g(F) = E$. It follows that $g_*(CH^1(F)) \supset \mathrm{Ker}(f_*)_1$. So to compute $\mathrm{Ker}(f_*)_1$ amounts to finding $\mathrm{Ker}(g)_*$ and $\mathrm{Ker}(fg)_*$. From 6.3 the component $CH^1(Y_n)$ is generated by the fibers $F_y,\ y \in Y_n$. Therefore $(fg)_*(e'_i j) = 0$ while $(fg)_*(z) \neq 0$ because $(fg)_*(z)$ is the class of the effective cycle Y in V. So

$$\mathrm{Ker}(f_*)_1 = g_* \left(pr^* CH^1(Y_n) \oplus \mathbb{Z} z \oplus \left(\oplus_i \mathbb{Z} e'_{ij} \right) \right)$$

We now compute this group. The next two results come from [CM]

Lemma 6.13 ([CM], 3.12) *The restriction $g_* : pr^* CH^1(Y_n) \to CH^1(V')$ is injective.*

Lemma 6.14 $g_*(e'_{i2}) = g_*(e'_{i1} + pr^* \,\mathrm{class}(y_{i2} - y_{i1}))$.

Proof

We have

$$pr^*(\mathrm{class}\ y_{i1}) = e'_{i1} + \mathrm{class}\ e_{i3}$$

so

$$g_*(pr^*(\text{class } y_{i1}) - e'_{i1}) = g_*(\text{class } e_{i3}).$$

Now $g_*(e_{i3}) = d_{i0} = g_* e_{i4}$ by definition so the above equality holds. ∎

Now put $\alpha \in pr^* CH^1(Y_n)$ and let n_i $(i = 1, \dots, n)$ be integers. Then

Lemma 6.15 *If $g_*(\Sigma n_{i1} e'_{i1} + \alpha) = 0$ then $n_{i1} = 0$ for all i and $\alpha = 0$.*

Proof

First assume degree $\alpha \neq 0$. We can assume without loss of generality that it is positive and so there is some $m > 0$ such that for $m\alpha$ there is a representative cycle A which is effective and has support in $F - pr^{-1}(\Sigma y_{ij})$. ∎

From the preceding lemmas we conclude that 6.5 holds. Let Al^* be the graded group of cycles modulo algebraic equivalence. The the following exact sequence holds

$$0 \to A^*(X) \to CH^*(X) \to Al^*(X) \to 0.$$

By the same proof as in 6.12 we get the result

$$Al^2(V') = Al^2(V) \oplus Al^1(Y_n) \oplus (\oplus_i \mathbb{Z} e'_{i1}).$$

From these we obtain Proposition 6.4.

6.2 The Tangent Bundle

The idea of this section is to construct a certain threefold and use that to use the constructions from the previous section to give our main theorem. This uses the ideas from Collino-Murre ([CM]) and Murre ([Mu1])

Let ℓ_0 be a good line of the cubic threefold as given in 5. Consider the tangent bundle $T(X)$ of $X - \{x_1, \dots x_n\}$. Let $\text{Proj}(T(X))$ be the associated projective bundle. Denote by X^* its restriction to ℓ_0.

Let $j : X^* \to X$ be the structure morphism. There is a rational transformation $\psi : X^* \to X$ defined as follows. Let $x^* \in X^*$, the x^* is a line in the tangent space T_{x_*} for some $x_* = j(x^*) \in \ell_0$. The intersection $X \cdot x^* = 2x_* + x$. This x is $\psi(x^*)$. It is generically 2-to-1.

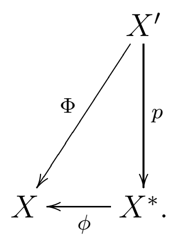
Now look at two loci in X^*.

$$\begin{aligned} Y' &= \{x^* \in X^* : x^* \text{ corresponds to } \ell_s, s \in \Delta'\} \\ Y'' &= \{x^* \in X^* : x^* \text{ corresponds to } \ell_0\}. \end{aligned}$$

Claim 6.16 ([CM], 4.5) *Y'' is a copy of $\mathbb{P}^1$ and Y' is isomorphic with Δ'. Also $Y' \cap Y'' = \emptyset$.*

The proof in [CM] goes through without any problems.

So now we identify Y' with Δ'. So the $2n$ ordinary double points of Δ' (y_{ij}, $i = 1, \ldots, n$ and $j = 1, 2$ with the involution exchanging y_{i1} and y_{i2}) correspond to the ordinary double points of Y'. Let X' be the threefold obtained by blowing up X^* along Y' and Y''. There is a diagram



Proposition 6.17 *X' is a singular threefold with exactly $2n$ singular points x_{i1}, x_{i2} ($i = 1, \ldots n$). These points are ordinary double points for X' and $p(x_{ij}) = y_{ij} \in Y' \subset X^*$.*

This follows by blowing up X^* along Y'' and applying 6.5.

Let $B = \{x \in X : x \notin \ell$ and the plane $< x, \ell_0 >$ intersects X along 3 lines meeting in one point or intersects X along ℓ_0 and K_T, where K_T is irreducible and tangent to ℓ_0 $\}$

Lemma 6.18 *Φ is a morphism, quasi-finite 2:1 out of $\Phi^{-1}(\ell_0)$. Over $X - (\ell_0 \cup B)$ the fibers of Φ consist of two distinct points.*

$\Phi(x_{ij}) = x_i$ ($j = 1, 2$). Blow up X at x_i and X' at the nodes x_{ij} to obtain V' and Z respectively.

Lemma 6.19 *There is an isomorphism*

$$Bl_{x_0}(X) \simeq Bl_S(\mathbb{P}^3)$$

This is proved by Finkelnberg ([F], Theorem 2.1) If we blow this up at the nodes we obtain a smooth threefold Z which is the blowup of X at all its nodes. Then there is a diagram

$$\begin{array}{ccc} V' & \xrightarrow{\phi} & Z \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\Phi} & X \end{array}$$

where ϕ is induced by Φ We apply the results of the previous section by taking V to be X^* blown up at Y'' and V' as given above. Then $A^2(V) = 0$ and therefore by the main result, Proposition 6.4, there is an isomorphism

$$\alpha : J(\Delta'_n) \to A^2(V').$$

This isomorphism is $\alpha = g_* pr^*$ where we use the notation of previous section.

By a similar reasoning on the right column there is an isomorphism $A^2(Z) \simeq J(S_n)$. So we have a map $\phi_* : J(\Delta'_n) \to J(S_n)$ induced by ϕ.

Also induced by ϕ is a function $\phi^* : CH^2(Z) \to CH^2(V')$ given by lifting cycles. This induces a morphism

$$\phi^* : J(S_n) \to J(\Delta'_n)$$

which is algebraic therefore is a morphism of abelian varieties. Since ϕ is generically 2 to 1 we get $\phi_*\phi^*$ is multiplication by 2.

Let q_n be the map $\Delta'_n \to \Delta_n$. There is a diagram

$$J(\Delta_n) \underset{(q_n)^*}{\overset{q_n^*}{\rightleftarrows}} J(\Delta'_n) \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} J(S_n).$$

Lemma 6.20 *The morphism* $\phi_* q_n^* = 0$.

Proof

If d_1 and d_2 are general points on Δ_n then $\phi_* q_n^*(d_1 - d_2) = \pi^*(K_{d_1} - K_{d_2})$ where K_* are the residual conics. Then this is 0 because any two points are linearly equivalent on a plane. ∎

We claim $Pr(\Delta_n'/\Delta_n) \simeq J(S_n)$ by the following steps.

Since $q_n : \Delta' \to \Delta$ is 2 to 1 we have an involution $\sigma : J(\Delta_n') \to J(\Delta_n')$ and similarly since $\phi : V' \to Z$ is 2 to 1 we have an involution $\tau : A^2(V') \to A^2(V')$. More explicitly τ is defined by

$$\tau(a) = \phi^* \phi_*(a) - a.$$

Relating these two involutions there is $\tau(a) = -\sigma(a)$ for all $a \in A^2(V') = J(\Delta_n')$ ([CM] Section 4)

Lemma 6.21 *We get* $Im(\phi^*) \subset Pr(\Delta_n'/\Delta_n)$.

Proof

Suppose $a \in Im(\phi^*)$. Then $a = \phi^*(b)$ for some b. Now using the definition of the Prym as $Ker(1+\sigma)^0$ we have $(1+\sigma)a = a - \tau(a) = a - \phi^*\phi_*(a) + a = 2\phi^*(b) - \phi^*\phi_*\phi^*(b) = 0$ ∎

Lemma 6.22 *We also get* $Im(\phi^*) \supset Pr(\Delta_n'/\Delta_n)$.

Proof

The proof in [Mu1] works.

Here is an alternative proof:. We want to compare $Im(\phi^*)$ with $Ker(1-\tau)^0$

$$Ker(1-\tau)^0 = Im(1+\tau).$$

Then we have this is equal to $Im(\phi^*\phi_*)$ because $\tau = \phi^*\phi_* - 1$. So trivially we get our result.

∎

So we have almost the result we want. The proof of Theorem 1.3 comes from applying Cor 10.10 of [Mu1].

The outline goes like this. There is a map $\phi^* : J(S_n) \to Pr(\Delta'_n/\Delta_n)$ with kernel $Ker(\phi^*)$. We need to show that

$$Ker(\phi^*) \cap Im(\phi_*) = 0.$$

This follows from the fact that $\phi^*\phi_*$ is multiplication by 2 on the Prym (which comes from that ϕ^* is onto).

This will come from the statement that the 2-torsion elements of the Prym are contained in the kernel of ϕ_*. This statement follows from the following two

1. $q_n^*(J(\Delta_n)) \subset Ker(\phi_*)$
2. $Pr(\Delta'_n/\Delta_n)_2 \subset q_n^*(J(\Delta_n))$

Chapter 7

Calculations and Main Proof

7.1 Proof of the Main Theorem

In this chapter we will make use of the results of $J(S_n) \simeq Pr(\Delta'_n/\Delta_n)$ and the descriptions of the possible plane quintics for a given number of nodes. One of the main facts we will use is the following lemma.

Lemma 7.1 *The only unramified double cover of a rational curve is the trivial double cover.*

This is proved using Hurwitz Theorem ([Har] IV.2.4).

For each nodal cubic threefold X_0 the proof steps are listed below:

1. We first give an argument on what the possible plane quintic curve Δ_0 is. There can be several, depending on the choice of the line projected from. Let Γ be the dual graph of Δ_0

2. Using the results of the previous two chapters we then give an argument for what the possible double cover Δ'_0 of the plane quintic is. Let Γ' be the dual graph of Δ'_0.

3. We then use 3.5 to show that the map is regular in that neighbourhood. This is usually obvious from the dual graph Γ'.

4. We give a degenerating family of cubic threefolds degenerating to X_0 and the associated double covers of quintic curves. Let ℓ_0 be a line in $\mathcal{S}$ (chosen as in Section 5.2.1). Choose

a smooth cubic threefold X such that it also contains ℓ_0. We have a pencil of cubic threefolds which contain ℓ_0:

$$X_{a,b} = aX + b\mathcal{S} \qquad (a:b) \in \mathbb{P}^1.$$

By restricting to some open subset S of $\mathbb{P}^1$ such that the only singular threefold is X_0 over $0 = (0:1)$ we get a family $\mathcal{X}/S$ of cubic threefolds. Let $U = S - \{(0:1)\}$. For each $s \in U$ there is a pair (Δ'_s, ι). The family of such pairs is a flat family over U. By direct calculation the limit is (Δ'_0, ι).

5. We finally compute the unimodular system and show that it is as in the list shown in Table 5.1.

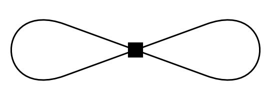
Let us first list all the quintics available to us. The column g is for the genus of the normalization.

Now we will go through the proof steps for a given number of nodes on X_0. The first few cases with one to five nodes are very similar.

7.1.1 1 node

This was done in [CM] and [CG].

The curve Δ is a plane quintic with one node. This is very clear because there is only such type of plane quintic. Also the Fano surface of the cubic threefold has only one component and the most line on a component behave the same. The doube cover Δ' is an irreducible curve of arithmetic genus 11 with 2 nodes because we know the dimension of the abelian part is 4. Its dual graph is



Clearly this generalized Prym does not lie in the locus of indeterminacy of the Prym map as it is not in the Friedman-Smith locus by 3.5.

#	degree	nodes	g
1	5	0	6
2	5	1	5
3	5	2	4
4	5	3	3
5	5	4	2
6	5	5	1
7	5	6	0
8	4+1	4	3
9	4+1	5	2
10	4+1	6	1
11	4+1	7	0
12	3+2	6	1
13	3+2	7	0
14	3+1+1	7	1
15	3+1+1	8	0
16	2+2+1	8	0
17	2+1+1+1	9	0
18	1+1+1+1+1	10	0

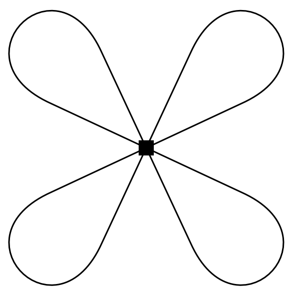
Table 7.1: A selection of plane quintic curves

The unimodular system obtained is M_1

The generalized Prym is an extension of a Jacobian $J(S)$, where S is the associated (smooth) sextic of genus 4, and a k^*.

7.1.2 2 NODES

This is very similar to the 1 nodal case above above. The curve Δ is an irreducible plane quintic with two nodes. We know the $Pr(\Delta'_n/\Delta_n)$ is 3 dimensional because S_n is a genus 3 by 5.1. So the curve $\tilde{\Gamma}$ has two nodes and the $\tilde{C}$ has 4 nodes and is irreducible. Its dual graph is

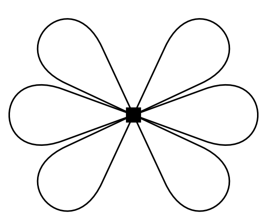


It is clearly not in the locus of indeterminacy by 3.5. We can easily calculate the unimodular system using Section 3.3. This is a special case of the example in that section with $n = 1$.

The dicing obtained comes from the unimodular system of vectors $I_2 = M_2$.

7.1.3 3 NODES

By the common argument we know the Prym of Δ'_n/Δ_n is 2 dimensional. So the curve has 3 nodes. The curve Δ'_0 has 6 nodes and is irreducible because of the dimension of the Prym variety. The dual graph is

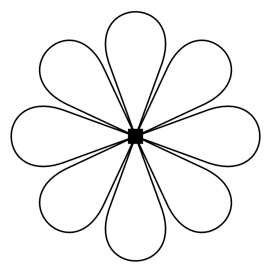


It is clearly not in the locus of indeterminacy. In this case the matroid is M_3, the 3×3 identity matrix.

7.1.4 4 NODES

There are two cases here depending on the configuration of the nodes on the cubic threefold.

1. The nodes are in general position. In this case $Pr(\Delta'_n/\Delta_n)$ is 1 dimensional. So since the plane quintic has four nodes not in a line (because the nodes in X are not in a plane) it has to be irreducible with four nodes (number 5 in the table above). The genus of the normalization is 2, so the genus of the normalization of Δ' is 3. So Δ' is irreducible with 8 nodes. Its dual graph is

   In this case the unimodular system obtained is I_4.

2. The nodes lying on a plane. We first state a result we will use extensively from now on.

   Lemma 7.2 *Suppose we have a plane in $\mathbb{P}^4$ not containing the node P (from which we will project and form S). Suppose further that this plane cuts X in a triangle. Then the image of the plane is a plane in the space $\mathbb{P}^3 \supset S$ and intersects S at 3 points in one direction and 3 in the other.*

   Proof

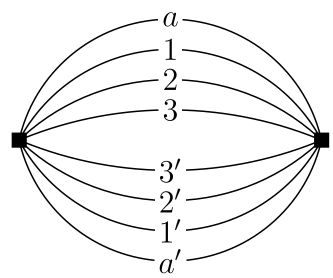
   Since the plane does not contain P it maps isomorphically onto its image. So the image is also a plane L_2. The curve S is a (3,3) curve and so $L_2 \cap S$ is 6 points with 3 from one ruling and 3 from the other. ∎

By using the above lemma we can see in which components Δ' live in.

For the case of nodes lying on a plane we have two cases depending on which component of the Fano surface we pick the good line ℓ_0. In both these case we know that $P(\Delta'_n/\Delta_n)$ has dimension 2. We also know the Fano surface is three components. The two components of interest here are the component birational to the symmetric product of the (3,2) component of the associated sextic which we will label A. The other component, B, is birational to the product of the (3,2) and (0,1) curves. These two surfaces intersect along four lines. Each such line is isomorphic to the $(3,2)$ curve. This is clear from the fact that a secant which is in both symmetric products goes through one of the nodes. This secant translates to a point in the intersection of the two corresponding Fano surfaces.

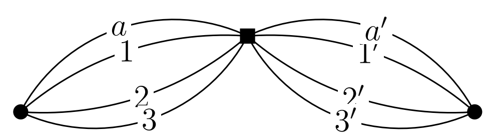
(a) The plane quintic is irreducible with four nodes (number 5 in table above). This happens when the chosen good line is skew to the plane with the four nodes. Suppose we choose a point on the plane quintic. This translates to three lines forming a triangle on X. By using the lemma above we see that the other two lines correspond to points in components A and B of the Fano surface. The nodes of Δ translate to two nodes on Δ' and these two nodes lie on one of the singular lines. So all eight nodes lie on the singular lines and join the two components of Δ'.

    So the double cover is two curves of geometric genus 2 meeting at 8 points. Its dual graph is



This is clearly not in the locus of indeterminacy by 3.5. So we can work out the unimodular system of vectors. In this case the tree we picked is indexed by letter of the alphabet. The involution interchanges edge t with t'. So $X^- =< (e_a - e_1) - (e'_a - e'_1), (e_a - e_2) - (e'_a - e'_2), (e_a - e_3) - (e'_a - e'_3) >$. This give a unimodular system of vectors $M_{4,2}$ as expected.

(b) The plane quintic is a line and a quartic (number 8 in above table). This happens when the good line hits the plane containing the four nodes at a smooth point. In this case the doube cover is curve of genus five over the quartic and two copies of $\mathbb{P}^1$ over the line. The dual graph is



The unimodular system obtained is the one expected. We compute it as follows $X^- =< (e_a - e_1) - (e_{a'} - e_{1'}), \ldots, (e_a - e_4) - (e_{a'} - e_{4'}) >$. This also give the matroid $M_{4,2}$.

In both cases the unimodular system obtained is the one in 2.

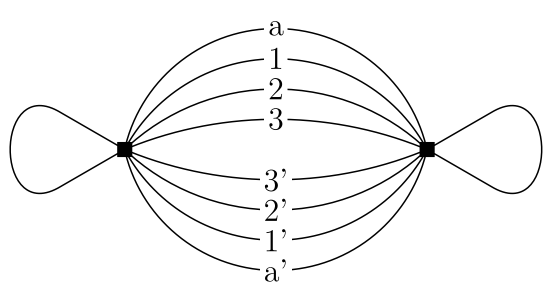
7.1.5 5 NODES

The analysis here is similar to the four node case depending on the configuration of the nodes on X. There are two cases here.

1. Supose the nodes in general position. The plane quintic curve is the curve number 6 in the above table. Then by virtue of the dimension of $J(S_n)$ the abelian part of the Prym is 0 dimensional. So the double cove is one curve with 10 nodes. The dual graph Γ_0' is (as above) like a flower with ten petals. By the same calculation as above the unimodular system obtained is I_5.

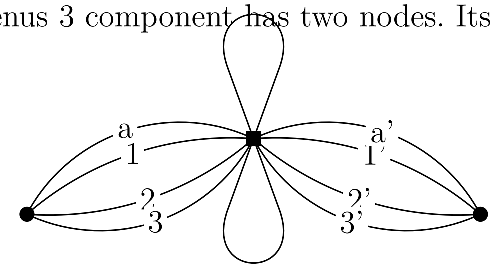
2. Now suppose four of the nodes lie on a plane. In this case JS_n is 1 dimensional by 5.1. There are then two choices depending on which component we choose the line ℓ_0. Note that the Fano surface has three components but one of them is a plane.

   (a) Suppose plane quintic is number six in the table above. This occurs when ℓ_0 is skew to the plane in X_0. So double cover is two curves meeting in 8 points and they each have a node. By an argument similar to the case of four nodes the double cover has the following dual graph.

   From this we see that it is not in the Friedman-Smith locus. The computation for the unimodular system gives the expected answer of $M_{5,2}$ in almost exactly the same way as for the 4 node case.

(b) Now suppose the plane quintic is number 9 in the table above. This occurs when ℓ_0 hits the plane with the nodes. So double cove has genus 3 curve attached to two $\mathbb{P}^1$ curves. The genus 3 component has two nodes. Its dual graph is

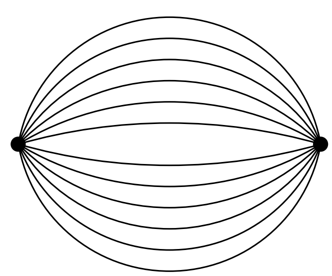


In exactly the same way as the four node case we get the unimodular system $M_{5,2}$.

7.1.6 6 NODES

There are three cases here depending on the configuration of the nodes (or equivalently, on the number of planes in X_0.)

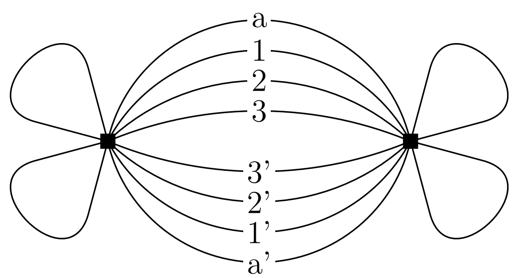
1. Suppose there are no planes in X_0. Then in this case plane quintic curve has 6 nodes (the maximum an irreducible plane quintic can have). Since $J(S_n)$ has dimension 0 this means that Δ_0' has genus 0. But the nomalization of Δ_0 is $\mathbb{P}^1$ so Δ_0' is two rational curves meeting in 12 points the dual graph is:



We make our tree was follows. Let one of the edges be e_a and its image under involution is $e_{a'}$. Then the rest of the edges are numbered $e_1, \dots e_5$ and $e_{1'}, \dots e_{5'}$. Then the unimodular system of vectors obtained is $M_{6,1}$.

2. Now suppose there is one plane. This case is very similar to what we have encountered for the case of 4 and 5 nodes. In this particular case the dimension of $J(S_n)$ is 0. There are two possibilities for the plane quintic Δ_0 depending on the choice of ℓ_0.

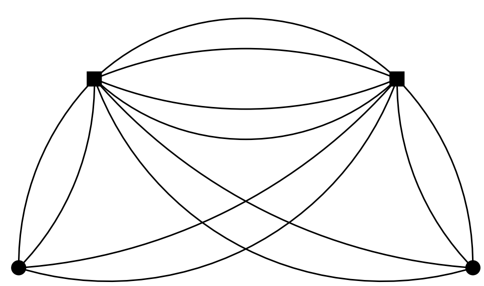
   (a) The line misses the plane. In this case Δ_0 has 6 nodes. The curve Δ_0' has two components but in this case (unlike 7.1.6) each component has two nodes. So it has the following dual graph

   In this case the unimodular system of vectors obtained using the tree above is $M_{6,2}$.

   (b) The line hits the plane. In this case Δ_0 is a quartic curve with three nodes and a line.

3. Suppose X_0 has 3 planes. The dimension of $J(S_n)$ is 1 in this case. Because of this it means the curve Δ_0 has genus at least one and has six nodes. The curve which fits the bill is number 10 in the table above. In this case the dual graph of Δ' is

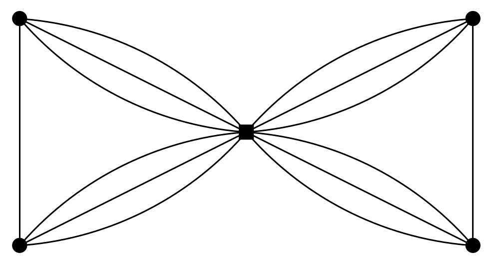


In this case the unimodular system obtained by looking at the tree is $M_{6,3}$

7.1.7 7 nodes

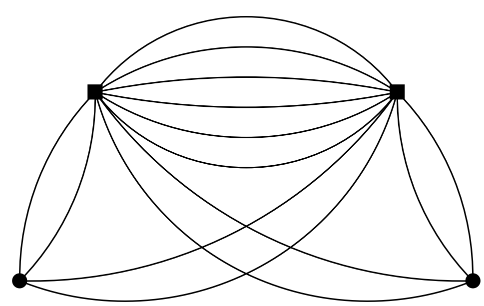
There are two cases from 4.

1. X_0 has two planes. In this case $J(S_n)$ is dimension 0. The dual graph of the double cover is



   This graph is because of the fact that the genus one curve has to have a double cover of one genus one curve otherwis the dimension of the abelian part will be too big. The unimodular system we get is $M_{7,1}$.

2. X_0 has three planes. $J(S_n)$ has dimension 0. In this case the dual graph is



The unimodular system is $M_{7,2}$.

7.1.8 8 NODES

Since the plane quintic has eight nodes we have the following possibilities.

1. The plane quintic curve is two conics plus a line.

2. The plane quintic is irreducible cubic with one node plus two lines.

In both these case we obtain the unimodular system M_8

7.1.9 9 NODES

This is very similar to the 10 node case below. Since the plane quintic has nine nodes it means that it is 3 copies of $\mathbb{P}^1$ and a conic. Since the abelian part of the Prym is 0 we get only rational components of the double cover. Over each component is two components. There is only one special component, the conic. The dual graph we obtain is the same as the dual graph for the ten nodal case with the edges a_1b_2 and a_2b_1 contracted. Everything in the ten nodal case works, including the tree. So what we obtain at the end is the matroid $M(K_{3,3})$.

The matroid we obtain is a graphic one which is not cographic. This is the first time we encounter something which does not arise from curves. The dual graph can be obtained from the 10 node one by simply contracting one pair of nodes.

7.1.10 10 NODES

This was studied in [G]. The matroid we obtain is the well known matroid R_{10}. From [Do] (5.17.4) we are given a description of the double cover of Δ_0.

A way in which we can obtain $\mathcal{S}$ is as follows.

Let $p_1, \ldots, p_5$ be points in general position in $\mathbb{P}^3$ and for each pair of points let ℓ_{ij} be the line joining p_i and p_j. Blow up $\mathbb{P}^3$ at the five points and then blow down the proper transforms of the ten lines ℓ_{ij}. The image is $\mathcal{S}$ in $\mathbb{P}^4$ ([Hu] 3.1-3.2).

There are 15 planes contained in $\mathcal{S}$ (Chapter 5). Ten of them come from proper transforms of the planes in $\mathbb{P}^3$ containing three points p_i, p_j and p_k. These are labeled Π_{ijk} $(i < j < k)$. The other five are images of the exceptional divisors coming from the blowup. These are labeled Π_i $(1 \leq i \leq 5)$. Each of the 15 planes contains 4 nodes.

$\mathcal{S}$ contains 6 two dimensional families of lines which are not planes, R_i $(1 \leq i \leq 5)$ which are proper transforms of lines through p_i and R_0 which is the family of twisted cubics through the five points. Each line in R_i goes through the five planes Π_{ijk} $(i < j < k)$. Each line in R_0 goes through Π_i ([SR] VIII 2.32).

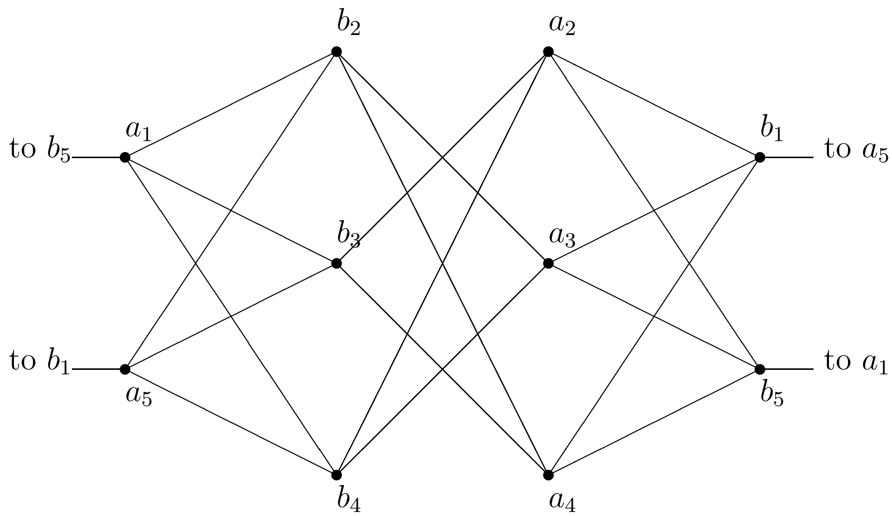
Choose a line ℓ in R_1 which does not go through a node and goes through exactly 5 planes, and project from it onto $\mathbb{P}^2$. The images of each of the five planes is a line in the degeneracy locus for $\mathcal{S}$. Each line meets each of the other four lines at a node. Each node is where the preimage is a plane containing a node on $\mathcal{S}$. Let Δ_0 be the plane quintic consisting of these 5 lines. We label each line L_i, $1 \leq i \leq 5$. The dual graph of Δ_0 is the complete graph on five vertices K_5. An explicit example of computing Δ_0 is given in the appendix.

We now describe a double cover for the curve Δ_0 above. The preimage of each point in C_0 consists of two lines, excluding ℓ, which are points in an etale double cover Δ_0' interchanged by the involution ι. The whole cover can be described as follows: Start with ten copies of $\mathbb{P}^1$, L_i^ϵ where $1 \leq i \leq 5$ and $0 \leq \epsilon \leq 1$. Each line L_i^ϵ has four points marked on it $p_{i,j}^\epsilon$ where $1 \leq j \leq 5$ and $j \neq i$. In the following the notation $(p_{i,j}^0 \sim p_{j,i}^1)$ means that the points $p_{i,j}^0$ and $p_{j,i}^1$ are identified. The double curve Δ_0' is

$$\Delta_0 = \left(\coprod_{i,\epsilon} L_i^\epsilon \right) \Big/ \left(p_{i,j}^0 \sim p_{j,i}^1\right)$$

The dual graph $\Gamma_0 = \Gamma(\Delta_0')$ is shown below. If the vertices of $\Gamma(\Delta_0)$ are labelled v_j, the the vertices a_j and b_j both map to v_j. The map of edges is given by the map of vertices.

The dual graph $\Gamma_0 = \Gamma(\Delta_0')$ is shown below. If the vertices of $\Gamma(\Delta_0)$ are labelled v_j, the the vertices a_j and b_j both map to v_j. The map of edges is given by the map of vertices.



The edges are named as follows: $e_1 = (b_3, a_2)$, $e_2 = (a_4, b_2)$, $e_3 = (a_5, b_3)$, $e_4 = (a_5, b_4)$, $e_5 = (a_5, b_1)$, $e_6 = (a_4, b_3)$, $e_7 = (b_3, a_1)$, $e_8 = (b_2, a_5)$, $e_9 = (a_1, b_4)$, $e_{10} = (b_2, a_1)$. The rest of the edges are named as follows: if edge e_i is (a_j, b_k) ((b_j, a_k) respectively) the edge e'_i is (b_j, a_k), ((a_j, b_k) respectively).

The tree used to form the basis of $H_1(\Gamma, \mathbb{Z})$ is given by the edges $e_6 = (a_4, b_3)$, $e_7 = (b_3, a_1)$, $e_8 = (b_2, a_5)$, $e_9 = (a_1, b_4)$, $e_{10} = (b_2, a_1)$ and $e'_{10} = (a_2, b_1)$, $e'_7(a_3, b_1)$, $e'_9 = (b_1, a_4)$, $e'_8 = (a_2, b_5)$.

Now we prove the lemma

Lemma 7.3 *Let Δ'_0 be as above. Then the map $\overline{R}_6 \to \overline{A_5^{vor}}$ is regular in the neighborhood of (Δ'_0, ι).*

Proof

Here we will use Lemma 3.5. Let Γ_0 be as above. Suppose Γ_1 and Γ_2 are subgraphs with $\iota(\Gamma_i) = \Gamma_i$ $(i = 1, 2)$. Then the first possible case is that Γ_1 has 2 vertices and Γ_2 has 8 vertices. If the vertices in Γ_1 correspond to the lines L_1^0 and L_1^1 (using the notation from above). The line L_1^0 and L_1^1 do not meet, so on the dual graph their corresponding vertices do not have an edge between them. Then Γ_1 is not connected so the conditions of Lemma 3.5 are satisfied. The second possible case is if Γ_1 and Γ_2 have 4 and 6 vertices respectively. Suppose without loss of generality the lines L_1^0, L_1^1, L_2^0 and L_2^1 are in Γ_1. This would imply Γ_1 is not connected because the connected subgraph with L_1^0 and L_2^1 is not connected with the subgraph with L_1^1 and L_2^0. Therefore the map is regular around the Segre threefold. ∎

So we now know we can compute the matroid just from curve Δ_0' which we now proceed to do. By the above lemma and we only need to compute $X^- \subset C_1(\Gamma, \mathbb{Z})$ as in Section 3.3 and show it is R_{10}. The following is a basis for $H_1(\Gamma, \mathbb{Z})$.

$$\begin{aligned}
h_1 &= e_6' + e_7' - e_9' + e_6 + e_7 - e_9, \\
h_2 &= e_1' + e_7' + e_9' + e_6 + e_7 + e_{10}, \\
h_3 &= e_1 - e_{10}' - e_9' + e_6, \\
h_4 &= e_2 + e_6 + e_7 + e_{10}, \\
h_5 &= e_2' - e_{10}' - e_9' + e_6 + e_7 - e_9, \\
h_6 &= e_5' - e_8' - e_{10}' - e_9' + e_6 + e_7, \\
h_7 &= e_5 - e_9' + e_6 + e_7 + e_{10} + e_8, \\
h_8 &= e_4 + e_9 + e_{10} - e_8, \\
h_9 &= e_3 + e_7 + e_{10} - e_8, \\
h_8' &= e_4' + e_9' + e_{10}' - e_8', \\
h_9' &= e_3' + e_7' + e_{10}' - e_8',
\end{aligned}$$

The basis for X^- is as follows

$$\begin{aligned}
\ell_1 &= \frac{1}{2}(h_2 - \iota(h_2)) = \frac{1}{2}(h_3 - \iota(h_3)) \\
\ell_2 &= \frac{1}{2}(h_4 - \iota(h_4)) = \frac{1}{2}(h_5 - \iota(h_5)) \\
\ell_3 &= \frac{1}{2}(h_9 - \iota(h_9)) = \frac{1}{2}(h_9' - \iota(h_9')) \\
\ell_4 &= \frac{1}{2}(h_8 - \iota(h_8)) = \frac{1}{2}(h_8' - \iota(h_8')) \\
\ell_5 &= \frac{1}{2}(h_6 - \iota(h_6)) = \frac{1}{2}(h_7 - \iota(h_7))
\end{aligned}$$

and $X^- =< \ell_1, ..., \ell_5 >$. The dicing is obtained by seeing how the edges e_j restrict to X^-. The matrix for the dicing of $X^- \otimes \mathbb{R}$ is (a_{ij}) where a_{ij} is defined to be 1 if ℓ_i contains e_j. The matrix we get is R_{10}.

7.2 Nonrationality of Cubic Threefolds

As a corollary of Theorem 1.1 we get Theorem 1.6. We use the particular case of the theorem that the unimodular system associated to a curve degenerating to the Segre Primal is R_{10}. The other theorem we will use in the proof is

Theorem 7.4 ([Mu2] Thm 3.11) *Let char$(k) \neq 2$. Let X be a nonsingular cubic threefold in $\mathbb{P}^4$, defined over k. If there exists a birational transformation between X and $\mathbb{P}^3$ then the canonically polarized Prym variety $(P(X), \Xi)$ associated with X is isomorphic, as a polarized abelian variety, to a product of canonically polarized Jacobian varieties of curves.*

The proof of Theorem 1.6 now follows easily. If the threefolds in this family were rational then by Theorem 7.4 the family of degenerating Pryms we obtain should give a cographic unimodular system. But by Theorem 1.1 (the 10 nodal case in Section 7.1.10) we get R_{10} which is not cographic. This is a contradiction.

In previous proofs of Theorem 1.6 which used degenerations, the limiting (generalized) Prym was an extension of a nontrivial Abelian variety by a torus. In [C2] the Prym is an extension by a torus k^*, and in [B] the torus is $(\mathbb{C}^*)^2$. In this proof the extension is the maximal possible as shown in the following proposition

Proposition 7.5 *The generalized Prym associated to the double cover* (Γ_0', ι) *is* $(k^*)^5$.

Proof

This follows from the fact that the generalized Prym is an extension of an abelian variety by a torus ([ABH] Section 1). The dimension of the torus is given by $h_1(\Delta_0', \mathbb{Z}) - h_1(\Delta_0, \mathbb{Z}) = 5$. The abelian part must therefore be 0. ∎

Chapter 8

Further work

Some questions come up from this work.

8.1 Matroid Related

A very basic question which arises is the following: For each unimodular system of vectors does there exist a cubic threefold with nodes at precisely the point determined by the unimodular system? The answer seem to be yes. It has been verified for up to six nodes.

8.1.1 Arithmetic of matroids

There are lots of types of matroids in matroid theory. An interesting question is what sort of matroid can arise. Clearly cographic matroids do come up. In this paper we show that R_{10} and $M(K_{3,3})$ also show up. The first simplest case is to find out if $M(K_5)$ does come up, or if not why not.

In Matroid theory there are defined what are called $n-$sums. We will briefly explain them in the case of unimodular systems of vectors. If we have one unimodular system $(I_m|A)$ and another one $(I_n|B)$. then their $0-$sum is the unimodular system given by

$$(I_m|A) \oplus_0 (I_n|B) = \left(\begin{array}{cc|cc} I_m & 0 & A & 0 \\ 0 & I_n & 0 & B \end{array} \right).$$

1-sums are defined in a similar way. The one means that we identify one dimensional subspaces in the matroid. So

$$\left(\begin{array}{cc|c} I_{m-1} & 0 & A \\ 0 & 1 & a \end{array}\right) \oplus_1 \left(\begin{array}{cc|c} 1 & 0 & b \\ 0 & I_{n-1} & B \end{array}\right) = \left(\begin{array}{ccc|cc} I_{m-1} & 0 & 0 & A & 0 \\ 0 & 1 & 0 & a & b \\ 0 & 0 & I_{n-1} & 0 & B \end{array}\right).$$

The final operation we will define is 2-sums. In this case we identify two dimensional subspaces. For unimodular systems of vectors it is

$$\begin{pmatrix} I_{m-2} & 0 & 0 & 0 & A \\ 0 & 1 & 0 & 1 & a_1 \\ 0 & 0 & 1 & 1 & a_2 \end{pmatrix} \oplus_2 \begin{pmatrix} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 1 & b_2 \\ 0 & 0 & I_{n-2} & 0 & B \end{pmatrix} = \begin{pmatrix} I_{m-2} & 0 & 0 & 0 & 0 & A & 0 \\ 0 & 1 & 0 & 0 & 1 & a_1 & b_1 \\ 0 & 0 & 1 & 0 & 1 & a_2 & b_2 \\ 0 & 0 & 0 & I_{n-2} & 0 & 0 & B \end{pmatrix}.$$

In a similar way 3-sums etc can be defined. Seymour's theorem says the following

Theorem 8.1 *Every regular matroid can be constructed by means of* $0-$ $1-$ *and* $2-$*sums starting with matroids which are either graphic, cographic or* R_{10}.

What we would like to show is that in the degenerations of Jacobians of curves these sums exist.

Theorem 8.2 *The* $0-$ $1-$ *and* $2-$*sums of unimodular systems arising from degenerations of Jacobians exist.*

Proof

For 0-sums: Suppose we have two degenerations of Jacobians of curves which give rise to two unimodular systems of vectors M_1 and M_2. Then the degeneration coming from the the degenerations of these curves identified at one point give the unimodular system $M_1 \oplus_1 M_2$. This is because the limit curve is now the two curves identified at one point. The dual graph is just the two original graphs with an edge joining them.

For 1-sums: We first have to see what 1-sums mean in terms of graphs. If we have G_1 with an edge e_1 joining vertices v_1 and v_2 and G_2 with an edge e_1' joining v_1' and v_2' we make a new graph as follows. Remove the edges e_1 and e_1' and add two new edges joining v_1 to v_1'

and v_2 to v_2'. Recall from definition 4.7 that each row of the matrix obtained is indexed by one cycle of the graph. So we choose our tree in G_1 carefully enough so that e_1 is not in the tree, and do likewise for G_2. Then the operation we have carried out above joined the two cycles into one cycle. So what we have done is the following: In the matrix for G_1 which is $\left(\begin{array}{cc|c} I_{m-1} & 0 & A \\ 0 & 1 & a \end{array}\right)$ the cycle formed by adding the edge e_1 indexed the last row. For G_2 and the edge e_2 indexed the first row. The 1-sums as defined above was exactly joining the two.

So what this means is that the 1-sum of two degenerating families of Jacobians can be made by looking for the appropriate graph.

A very similar construction is made for the 2-sum of a degenerating family of Jacobians

$\blacksquare$

In the case of degenerations of Prym varieties there is no obvious way to carry out these sums, even just on the dual graphs. So a question is: can n-sums somehow be defined as to make sense on the degenerations of Prym varieties?

The first question is which matroids do arise? In the case of Jacobians of curves the answer is clearly the cographic unimodular systems of vectors. We have shown that noncographic and nongraphic matroids do appear. The current conjecture with no basis in reality is that $M(K_5)$ cannot occur.

1-sums and 2-sums can be described quite easily for cographic matroids in terms of the graphs. Maybe this idea can be extended to the actual curves themselves.

For the unimodular systems arising from degenerations of Pryms there seems no clear way to do these sums. This would be a very interesting avenue to follow.

8.2 Other Stable Cubic Threefolds

There are a lot of questions which have come up. The first question is what can we do for the other stable threefolds?

Another question is: what can we do in the case of strictly semistable threefolds. Collino has one interesting result in that regard in the paper [C1]. He studies the secant threefold which we have met earlier. What he shows is the following.

Theorem 8.3 *Inside A_5 the locus of Jacobians of hyperelliptic curves of genus 5 is contained in the closure of the set of intermediate Jacobians of cubic threefolds.*

He shows that for a degenerating family of smooth cubic threefolds degenerating to the secant threefold the limit depends on the degeneration and the limit is the Jacobian of a hyperelliptic curve of genus 5. The hyperelliptic curve is the double cover of the secant curve ramified at 12 points.

What I would like to do is to carry out a similar procedure for other semistable threefolds.

Bibliography

[A1] V. Alexeev, *Compactified Jacobians*. Preprint. alg-geom/9608012.

[A2] V. Alexeev, Complete moduli on the presence of semiabelian group action Ann. of Math., **155** (2002) 611–708.

[ABH] V. Alexeev, Ch. Birkenhake and K. Hulek, *Degenerations of Prym Varieties*, J. Reine Angew. Math. **553**(2002), 73–116

[AN] V. Alexeev and I. Nakamura, On Mumford's construction of degenerating abelian varieties, Tohoku Math. J., **51** (1999), 399–420, alg-geom/9608014.

[Al] D. Allcock, *The Moduli Space of Cubic Threefolds*, J. Algebraic Geometry, **12**(2003), 201–223.

[AK] A. B. Altman and Steven L. Kleiman, *Foundations of the theory of Fano schemes*, Comp. Math. **34** Fasc. 1, (1977), 3–47.

[B] F. Bardelli, *Polarized Mixed Hodge Structures: on Irrationality of Threefolds via Degeneration*, Ann. Mat. Pura Appl. (4), **137** (1984), 287–369.

[Be] Arnaud Beauville *Complex algebraic surfaces 2e*, London Mathematical Society Student Texts, vol. 34, Cambridge University Press 1996.

[BL] Christina Birkenhake and Herbert Lange *Complex Abelian Varieties.* Springer, 2004.

[BS] E. Bombieri and P. Swinnerton-Dyer, *On the local Zeta function of a cubic threefold.* Ann. Scuola Norm. Sup. di Pisa **21**, 1967, 1–29.

[BW] J. W. Bruce and C. T. C. Wall, *On the classification of cubic surfaces*, J. London Math. Soc. (2),**19** (1979), 245–256.

[CG] C. Herbert Clemens and Phillip A. Griffiths, *The Intermediate Jacobian of the Cubic Threefold.* Annals of Mathematics, **95** (1972), 281–356.

[C1] A. Collino, *The fundamental group of the Fano surface I, II.* Springer Lecture Notes in Mathematics, **947** (1981), 209–220.

[C2] A. Collino, *A cheap proof of the irrationality of most cubic threefolds.* Boll. Un. Mat. Ital. (5), **16-B** (1979), 451-465.

[CM] A. Collino, J. P. Murre, *The intermediate Jacobian of a cubic threefold with one ordinary double point; an algebraic-geometric approach (I), (II).* Nederl. Akad. Wetensch. Proc. Ser. A, **(**81) no 1 (1977), 43–55, 56–71.

[CLSS] D. Coray, D. Lewis, N. Shepherd-Barron and Sir Peter Swinnerton-Dyer, *Cubic Threefolds with Six Double Points.* Number theory in progress, Vol. 1 (Zakopane-Kościelisko, 1997), 63–74, de Gruyter, Berlin, 1999.

[DG] V. Danilov and V. Grishukhin, *Maximal Unimodular Systems of Vectors.* Europ. J. Combinatorics, **20** (1999), 507–526.

[Do] R. Donagi, *The Fibers of the Prym Map.* Contemporary Mathematics, **136** (1992), American Mathematical Society, 55–126.

[ER] R. M. Erdahl and S. S. Ryshkov, *On lattice dicing.* Europ. J. Combinatorics, **15** (1994), 459–481.

[F] Hans Finkelnberg, *Small resolutions of the Segre cubic* Nederl. Akad. Wetensch. Proc. Ser. A, **(**90) no 3 (1987), 261–277.

[FW] Hans Finkelnberg and Jurgen Werner, *Small resolutions of nodal cubic threefolds* Nederl. Akad. Wetensch. Proc. Ser. A, **(**92) no 2 (1989), 185–198.

[Fr] Robert Friedman, *Algebraic Surfaces and Holomorphic Vector Bundles.* Springer, 1998.

[Fu] William Fulton, *Intersection Theory.* Ergebnisse der Math und ihrer Grenzgebiete, : 2 no 34, Springer 1998.

[G] T. Gwena, *Degenerations of Cubic Threefolds and Matroids.* To appear in Proc. AMS.

[GH] Griffiths and Harris, Principles of Algebraic Geometry. Wiley, 1978.

[H] Ingo Hadan, *Tangent Conics at Quartic Surfaces and Conics in Quartic Double Solids.* Math. Nachr **210** (2000), 127–162.

[Ha] Joe Harris, *Algebraic Geometry: A first course.* Graduate Texts in Mathematics vol. 133, Springer, 1992.

[HaM] Joe Harris and Ian Morrison, *Moduli of curves.* Graduate Texts in Mathematics vol. 187, Springer, 1998.

[Har] R. Hartshorne, *Algebraic Geometry.* Graduate Texts in Mathematics vol. 52, Springer, 1977.

[Hu] B. Hunt, *The Geometry of Some Special Arithmetic Quotients*, Lecture Notes in Mathematics, vol. 1637, Springer-Verlag 1996.

[IM] V. A. Iskovskih and Yu. I. Manin, *Three-dimensional quartics and counterexamples to the Luroth problem.* Mat. Sb. 86 (1971), 140–166.

[M] Yu. I. Manin, *Cubic forms.* North-Holland Mathematica Library, North-Holland 1986.

[Mu1] J. P. Murre, *Algebraic Equivalence Modulo Rational Equivalence on a Cubic Threefold.* Compositio Mathematica, **25** (1972), Fasc. 2, 161–206.

[Mu2] J. P. Murre, *Reduction of the Proof of the Non-Rationality of a Nonsingular Cubic Threefold to a Result of Mumford.* Compositio Mathematica, **27** (1973), Fasc. 1, 63–82.

[OS] T. Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety.* Trans. Amer. Math. Soc. **253** (1979), 1–90.

[O] J. G. Oxley, *Matroid Theory.* Oxford University Press, 1992.

[Re] *A young person's guide to canonical singularities.* Algebraic Geometry Bowdoin 1985 Vol 46 Part 1, American Mathematical Society, 345–416.

[Sc] Michael Schlessinger, *Functors of Artin Rings.* Trans. Amer. Math. Soc. **130** (1968), 208–222.

[SR] J. G. Semple and L. Roth, *Introduction to Algebraic Geometry.* Clarendon Press, Oxford, 1985.

[V] V. Vologodsky, *Locus of Indeterminacy of the Prym Map.* J. Reine Angew. Math. **553** (2002), 117–124.

[Yo] Mutsumi Yokoyama, *Stability of Cubic 3-folds.* Tokyo J. Math. **25** (2002) No.1, 85–105.

Appendix A

Computing the plane quintic curve

This code used the information from chapter 5 to construct the plane quintic curve. It helped to make and check conjectures about the curves especially for chapter 7.

Preliminaries

This first code is to do one thing. The function `myextract[a,b,n]` extracts the degree n part of the polynomial a with indeterminates the set b. For example

```
myextract[xyz+z^3+y,{y},1]
```

returns `y+xyz`.

```
multidegne[a_, b_] := Last[Sort[Apply[Plus,
  Map[Exponent[#, b] &, If[Not[SameQ[Head[a],
        Plus]], {a}, ReplacePart[a, List, 0]]], {1}]]]

multideg[a_, b_] := multidegne[Expand[a], b]

myextractne[a_, b_, n_] :=
  If[PolynomialQ[a, b],
    If[SameQ[Head[a], Plus],
      ReplacePart[

        Select[
          ReplacePart[a, List, 0], multideg[#, b] == n &
          ], Plus, 0

        ], If[multideg[a, b] == n, a, 0]
      ],
    0
    ]

myextract[a_, b_, n_] := myextractne[Expand[a], b, n]
```

The variable part

Now we define the equation of the Segre cubic threefold and the partial derivatives. This will be our running example.

For different cubic threefolds this is the one of three places we need to change. The other two places are `ch1` and `ch2`.More examples are given at the end of this chapter.

```
f[x_, y_, z_, w_] = (x^3 + y^3 + z^3 + w^3 + 1) - (x + y + z + w + 1)^3
ch1=2;
ch2=3;

fx[x_, y_, z_, w_] := D[f[x, y, z, w], x];
fy[x_, y_, z_, w_] := D[f[x, y, z, w], y];
fz[x_, y_, z_, w_] := D[f[x, y, z, w], z];
fw[x_, y_, z_, w_] := D[f[x, y, z, w], w];
```

We do a check to make sure we really have our ten nodes.

```
Solve[{f[a, b, c, e] == 0, fx[a, b, c, e] == 0, fy[a, b, c, e] == 0,
            fz[a, b, c, e] == 0,  fw[a, b, c, e] == 0}, {a, b, c, e}]
```

This returns the expected answer:

```
{{a -> -1, b -> -1, c -> -1, e -> 1}, {a -> -1, b -> -1, e -> -1, c -> 1},
{a -> -1, c -> -1, e -> -1, b -> 1}, {a -> 1, b -> 1, e -> -1, c -> -1},
{a -> 1, c -> 1, b -> -1, e -> -1}, {a -> 1, e -> 1, b -> -1, c -> -1},
{b -> -1, c -> -1, e -> -1, a -> 1}, {b -> 1, c -> 1, a -> -1, e -> -1},
{b -> 1, e -> 1, a -> -1, c -> -1}, {c -> 1, e -> 1, a -> -1, b -> -1}}
```

A point and line on the threefold

Now the hard work begins. Firstly we find a point on the threefold. The algorithm is self explanatory. The number `ch1` is used to choose one of (usually) three points.

```
ptt = {2, 1, t, t};
Clear[linec, pt]
(* find point in threefold*)
Solve[Apply[f, ptt] == 0, t]
pt = (ptt /. %)[[ch1]]
```

The object of interest here is pt. Here we get

$$pt = \left(2, 1, \frac{1}{2}\left(-7-\sqrt{13}\right), \frac{1}{2}\left(-7-\sqrt{13}\right)\right)$$

Now that we have it we find a line. This is found via Bezout's theorem. If a line hits the cubic in four points then it is on the cubic. There are two methods here. They usually both give the same answer.

```
method = 1;
(* use ch2 to choose one of six lines through point
  *)
lnet = Solve[Table[Apply[f, pt + i*{u, v, w, x}] == 0, {i, 1, 3}],
                            {u, v, w, x}];
N[lnet]
lne = lnet[[ch2]];
If[
  method == 1,
  linecc = {ucc, vcc, wcc, xcc} = (({u, v, w, x} /. lne));
  linec =
    If[Variables[u /. lne] == {},
      If[Variables[v /. lne] == {},
        If[Variables[w /. lne] == {},
          If[Variables[x /. lne] == {},
            0,
            linecc /. (Variables[x /. lne][[1]] -> 1)
            ],
          linecc /. (Variables[w /. lne][[1]] -> 1)
          ],
        linecc /. (Variables[v /. lne][[1]] -> 1)
        ],
      linecc /. (Variables[u /. lne][[1]] -> 1)
      ];
  ]
If[
  method == 2,
  (* Find an indep variable for the line *)
```

```
  uc = If[SameQ[(u /. lne),
          0], 0, Coefficient[u /. lne, Variables[u /. lne]][[1]]];
  vc = If[SameQ[(v /. lne), 0], 0, Coefficient[v /. lne,
            Variables[v /. lne]][[1]]];
  wc = If[SameQ[(w /. lne), 0], 0, Coefficient[
                  w /. lne, Variables[w /. lne]][[1]]];
  xc = If[SameQ[(x /. lne), 0], 0, Coefficient[x /. lne,
      Variables[x /. lne]][[1]]];
  linec = {uc, vc, wc, xc};
  ]

linec
```

For our example we obtain

$$linec = \left(\frac{1}{2}\left(17 - 5\sqrt{13}\right), -11 + 3\sqrt{13}, 1, -10 + 3\sqrt{13}\right).$$

So the line we have is `pt+`t`linec`.

The plane quintic

Now we go and start working out the plane quintic curve. The code below changes f to a function g in A^4 with coordinates x, y, z, w so that the line is given by $x = y = z = 0$

```
lbda = If[SameQ[uc, 1], u, lbda];
lbda = If[SameQ[vc, 1], v, lbda];
lbda = If[SameQ[wc, 1], w, lbda];
lbda = If[SameQ[xc, 1], x, lbda];

If[SameQ[lbda, u],g[x_, v_, w_, u_]=Apply[f, u*linec + pt + {0, v, w, x}]];
If[SameQ[lbda, v],g[x_, u_, w_, v_]=Apply[f, v*linec + pt + {u, 0, w, x}]];
If[SameQ[lbda, w],g[x_, v_, u_, w_]=Apply[f, w*linec + pt + {u, v, 0, x}]];
If[SameQ[lbda, x],g[u_, v_, w_, x_]=Apply[f, x*linec + pt + {u, v, w, 0}]];

gx[x_, y_, z_, w_] = D[g[x, y, z, w], x];
gy[x_, y_, z_, w_] = D[g[x, y, z, w], y];
gz[x_, y_, z_, w_] = D[g[x, y, z, w], z];
gw[x_, y_, z_, w_] = D[g[x, y, z, w], w];
```

So before we continue we need to check whether we have a good line. The code below uses [Mu1] 1.6.

```
fn[y_, z_, w_] = Expand[CoefficientList[g[x, y, z, v], x][[2]]] ;
lx  = (myextract[fn[y, z, v], {v}, 2])/v^2;
lx1 = (myextract[fn[y, z, v], {v}, 1] /. {y -> 0, z -> 0})/v;
lx2 = myextract[fn[y, z, v], {v}, 0] /. {y -> 0, z -> 0};

gn[y_, z_, w_] = Expand[CoefficientList[g[x, y, z, v], y][[2]]] ;
mx  = (myextract[gn[x, z, v], {v}, 2])/v^2;
mx1 = (myextract[gn[x, z, v], {v}, 1] /. {x -> 0, z -> 0})/v;
mx2 = myextract[gn[x, z, v], {v}, 0] /. {x -> 0, z -> 0};

hn[y_, z_, w_] = Expand[CoefficientList[g[x, y, z, v], z][[2]]] ;
nx  = (myextract[hn[x, z, v], {v}, 2])/v^2;
nx1 = (myextract[hn[x, z, v], {v}, 1] /. {x -> 0, y -> 0})/v;
nx2 = myextract[hn[x, z, v], {v}, 0] /. {x -> 0, y -> 0};

Det[{{lx, lx1, lx2}, {mx, mx1, mx2}, {nx, nx1, nx2}}]
```

If the answer is 0 then there is no point in continuing. We have to go back to the beginning and find a new line. In our example the number we get is $23328 - 6480\sqrt{13} \neq 0$. Next we compute the plane quintic curve.

```
abcdef = CoefficientList[g[x, y, z, w], w];
Aa[x_, y_, z_] = myextract[abcdef[[1]], {x, y, z}, 3];
Ba[x_, y_, z_] = myextract[abcdef[[2]], {x, y, z}, 2];
Ca[x_, y_, z_] = myextract[abcdef[[1]], {x, y, z}, 2];
Da[x_, y_, z_] = myextract[abcdef[[3]], {x, y, z}, 1];
Ea[x_, y_, z_] = myextract[abcdef[[2]], {x, y, z}, 1];
Fa[x_, y_, z_] = myextract[abcdef[[1]], {x, y, z}, 1];



Qa[x_, y_, z_] = Factor[
                 Simplify[
                 Expand[
                 Det[{{Aa[x, y, z], Ba[x, y, z]/2, Ca[x, y, z]},
                      {Ba[x, y,  z]/2, Da[ x, y, z], Ea[x, y, z]},
                      {Ca[x, y, z], Ea[x, y, z], 4Fa[x, y, z]}}
                   ] ] ] ]
```

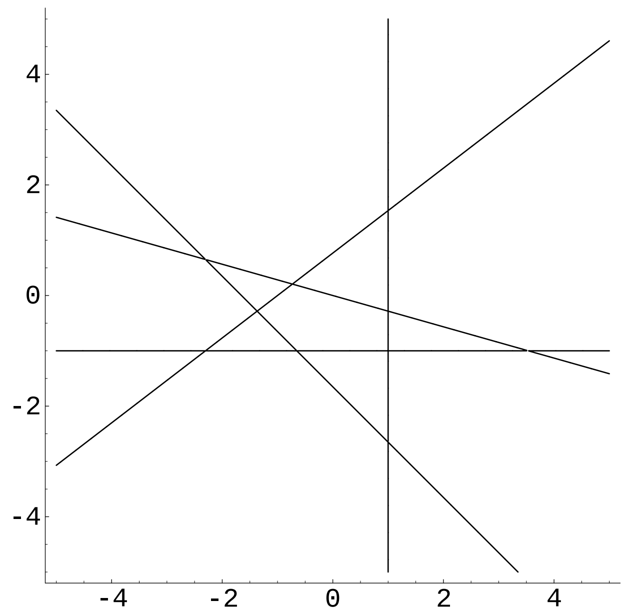
Note that the plane quintic curve is in projective 2-space. In this case it is

$$-9(-25+7\sqrt{13})(x-y)(y+z)((-3+\sqrt{13})x+y+(-3+\sqrt{13}))\times$$
$$(-2x-2y+(-1+\sqrt{13})z)(3x+(7+\sqrt{13})z)$$

Just to top things off we draw the plane quintic curve.

```
tinx = 5
scaley = 1
centx = 0
centy = 0
ImplicitPlot[Qa[x, 1, z] == 0., {x, -tinx + centx, tinx + centx}, {
  z, -tinx*scaley + centy, tinx*scaley + centy}, PlotPoints -> 2^10]
```

The graph we get is



7 NODES WITH TWO PLANES

In this case the affine equation is

$$-3wx-2wy+2xy+3wxy+wz+xz+wxz+yz-4xyz=0.$$

With the choices $ch1=2$, $ch2=3$, $ptt=(1,0,t,t)$ we obtain the plane quintic curve which is a degree 2 and a nodal degree 3 curve. With choices $ch1=2$, $ch3$ and $ptt=(2,0,t,t)$ we obtain the quintic which is a degree 4 curve with three nodes and a degree 1 curve.