Simplices and Sets of Positive Upper Density in $\mathbb{R}^d$

by

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(Under the Direction of Neil Lyall)

Abstract

We prove an extension of Bourgain's theorem on pinned distances in measurable subsets of $\mathbb{R}^2$ of positive upper density, namely Theorem $1'$ in [1], to pinned non-degenerate $k$-dimensional simplices in measurable subsets of $\mathbb{R}^d$ of positive upper density whenever $d \geq k+2$ and $k$ is any positive integer.

Index words: simplices, Fourier analysis, harmonic analysis, surface measure, spherical maximal function, distance sets

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# Chapter 1

# Introduction

## 1.1 Background on Main Results

Throughout this document we shall refer to a set $\Delta = \{0, v_1, \dots, v_k\}$ of points in $\mathbb{R}^k$ as a non-degenerate $k$-dimensional simplex if the vectors $v_1, \dots, v_k$ are linearly independent.

Define the *upper density* $\overline{\delta}$ of a measurable set $A \subseteq \mathbb{R}^d$ to be

$$\overline{\delta}(A) = \limsup_{N \to \infty} \frac{|A \cap B_N|}{|B_N|},$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^d$ and $B_N$ denotes the cube $[-N/2, N/2]^d$.

A result of Katznelson and Weiss [3] states that if $A$ is a measurable subset of $\mathbb{R}^2$ of positive upper density, then its distance set

$$\mathrm{dist}(A) = \{|x - y| \,:\, x, y \in A\}$$

contains all large numbers. In other words, given any sufficiently large $\lambda > 0$, there exits $x \in A$ such that on the circle centered at $x$ of radius $\lambda$ lies another point in $A$. This result was later reproven using Fourier analytic techniques by Bourgain in [1]. Bourgain also proved the following generalization, Theorem 1.1.1 below, and the following "pinned" variant, Theorem 1.1.2 below. Theorem 1.1.1 is a generalization of the result above to $k$-simplices. That is, Bourgain proved that for all sufficiently large $\lambda$, the set $A$ contains an isometric copy of $\lambda \cdot \Delta$. Theorem 1.1.2 says that for a fixed $x$ we get all large distances in some large, compact interval.

**Theorem 1.1.1** (Theorem 2 in [1]). *Let $\Delta$ be a fixed, non-degenerate, $k$-dimensional simplex. If $A$ is a measurable subset of $\mathbb{R}^d$ of positive upper density with $d \geq k+1$, then there exist $\lambda_0 = \lambda_0(A)$ such that for all $\lambda \geq \lambda_0$ one has*

$$x + \lambda \cdot U(\Delta) \subseteq A \tag{1.1}$$

*for some $x \in A$ and $U \in SO(d)$.*

**Theorem 1.1.2** (Pinned distances, Theorem 1′ in [1]). *Let $\Delta = \{0, v_1, \ldots, v_k\}$ be a fixed, non-degenerate, $k$-dimensional simplex. If $A$ is a measurable subset of $\mathbb{R}^2$ of positive upper density, then there exist $\lambda_0 = \lambda_0(A)$ such that for any given $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that*

$$A \cap (x + \lambda \cdot S^1) \neq \emptyset \tag{1.2}$$

*for all $\lambda_0 \leq \lambda \leq \lambda_1$.*

Our first new result is the following optimal strengthening of Theorem 1.1.1 above. That is, there are in fact lots of rotations $U \in SO(d)$ and lots of elements $x \in A$ such that $x + \lambda \cdot U(\Delta) \subseteq A$.

**Theorem 1.1.3** (Density of Embedded Simplices). *Let $\varepsilon > 0$ and $d \geq k+1$. If $A$ is a measurable subset of $\mathbb{R}^d$, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that*

$$\int_{SO(d)} \overline{\delta}(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k)))\, d\mu(U) > \overline{\delta}(A)^{k+1} - \varepsilon \tag{1.3}$$

*for all $\lambda \geq \lambda_0$. In particular, for each $\lambda \geq \lambda_0$ we may conclude that there exist $U \in SO(d)$ such that*

$$\overline{\delta}(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))) > \overline{\delta}(A)^{k+1} - \varepsilon \tag{1.4}$$

*and there exist $x \in A$ such that*

$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) > \overline{\delta}(A)^k - \varepsilon^{1/2}. \tag{1.5}$$

Note that if the integral in (1.3) is greater than $\overline{\delta}(A)^{k+1} - \varepsilon$, then there exists some rotation $U \in SO(d)$ such that the integrand

$$\overline{\delta}(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))) > \overline{\delta}(A)^{k+1} - \varepsilon.$$

So, (1.4) follows easily from (1.3).

Also, if (1.3) holds, then by the definition of limsup, there exists large $N$ such that

$$\int_{SO(d)} \frac{|(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))) \cap B_N|}{|B_N|} \, d\mu(U) > \overline{\delta}(A)^{k+1} - \varepsilon \tag{1.6}$$

and

$$\frac{|A \cap B_N|}{|B_N|} < \overline{\delta}(A) + \varepsilon^{1/2}. \tag{1.7}$$

We rewrite the left side of (1.6) and change the order of integration to get

$$\begin{aligned}
&\frac{1}{|B_N|} \int_{SO(d)} \int_{\mathbb{R}^d} 1_{(A \cap B_N)}(x) 1_{(A+\lambda \cdot U(v_1))}(x) \cdots 1_{(A+\lambda \cdot U(v_k))}(x) \, dx d\mu(U) \\
&= \frac{1}{|B_N|} \int_{\mathbb{R}^d} 1_{(A \cap B_N)}(x) \int_{SO(d)} 1_{(A+\lambda \cdot U(v_1))}(x) \cdots 1_{(A+\lambda \cdot U(v_k))}(x) \, dx d\mu(U) \\
&= \frac{1}{|B_N|} \int_{\mathbb{R}^d} 1_{(A \cap B_N)}(x) \cdot \mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) dx > \overline{\delta}(A)^{k+1} - \varepsilon,
\end{aligned}$$

It follows that there exists some $x \in A$ such that

$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) > \frac{\overline{\delta}(A)^{k+1} - \varepsilon}{\overline{\delta}(A) + \varepsilon^{1/2}} > \overline{\delta}(A)^k - \varepsilon^{1/2} \tag{1.8}$$

so that if (1.3) holds, then (1.5) holds.

Our main new result is the following extension of Bourgain's pinned distances theorem, Theorem 1.1.2 above, to non-degenerate $k$-dimensional simplices when $k \geq 2$.

**Theorem 1.1.4** (Density of Embedded Pinned Simplices)**.** *Let $\Delta$ be a fixed non-degenerate $k$-dimensional simplex and $\varepsilon > 0$. If $A$ is a measurable subset of $\mathbb{R}^d$ with $d \geq k+2$, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any given $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that*

$$\mu\big(\big\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\big\}\big) > \overline{\delta}(A)^k - \varepsilon \quad \textit{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1. \tag{1.9}$$

It is important to note that Theorem 1.1.4 should hold whenever $d \geq k+1$. However, extending our result to this range appears to require an essentially non-Fourier analytic approach, specifically an adaptation of the geometric arguments in Bourgain's Circular Maximal Function Theorem [2] to the configuration spaces considered in this document.

## 1.2 Outline

In Chapter 3, we give our interpretation of Bourgain's proof of Theorem 1.1.1 in the case of distances, namely the $k = 1$ case, and a proof of Theorem 1.1.2 on pinned distances in $\mathbb{R}^d$ when $d \geq 3$. These proofs are meant to serve as models for the proofs of Theorem 1.1.3 on unpinned simplices and Theorem 1.1.4 on pinned simplices given in Chapters 4 and 5. For example, in the case of distances, we must consider the average of a function on a sphere of radius $\lambda$ while in the case of simplices, we must consider appropriate multi-linear averages. Chapter 2 includes preliminary material necessary in the proofs of the Theorems 1.1.1 and 1.1.2. When we begin discussing simplices in Chapter 4, we include background material appropriate to that case.

Each of the theorems stated above follows from a dichotomy proposition wherein the set $A$ is contained in the unit cube. The dichotomy is that either our set $A$ behaves as we expect and the result holds or $A$ exhibits some unexpected behavior which is detected by the Fourier transform in that the $L^2$-norm of the Fourier transform of the characteristic function on the set $A$ is concentrated in an annulus. As such, in the proofs of some of these dichotomy propositions, we consider $L^2$-estimates on appropriate maximal averages, as outlined briefly below.

The main tool in the proof of the Theorem 1.1.1 in the case of unpinned distances is the decay of the Fourier transform of the surface measure on the sphere. Similarly, in the proof Theorem 1.1.3 on unpinned simplices, we use a decay estimate specific to the spheres we are considering along with an appropriate decomposition of the configuration space, the space of all $k$-simplices in $\mathbb{R}^d$ where $d \geq k+1$, with which we are working.

The main tool in the proof of Theorem 1.1.2 on pinned distances in $\mathbb{R}^d$, where $d \geq 3$, is the $L^2$-estimate on the spherical maximal function, discussed in Chapter 2. We also prove a similar $L^2$-estimate on appropriate "mollified" maximal averages, "mollified" meaning we are ultimately considering the average of a function $f$ on a thickened sphere. In the case of these "mollified" maximal averages, the estimate we need in order to prove Theorem 1.1.2 is stronger than what we would get from an application of the standard spherical maximal function theorem, as we shall show in Chapter 3. The is similar for the proof of Theorem 1.1.4 on pinned simplices. Here, we consider the configuration space of $k$-simplices in $\mathbb{R}^d$ for $d \geq k+2$ and corresponding multi-linear averages. Thus, the new idea that goes into the proof of Theorem 1.1.4 is the proof of a "mollified" maximal function estimate in this setting.

As mentioned in Section 1.1, Theorem 1.1.4 should hold in the case where $d = k+1$. However, this extension requires an essentially non-Fourier analytic approach. As such, in Chapter 6, we discuss a geometric approach to proving Stein's spherical maximal function theorem when $p = 2$.

# Chapter 2

# Preliminaries

## 2.1 Basic Definitions and Facts

### 2.1.1 Notation

Throughout this document, we use the following:

- $A \lesssim B$ means $A \leq C \cdot B$, for some constant $C > 0$ and quantities $A, B \in \mathbb{R}$. This is similar for $A \gtrsim B$.
- $A \ll B$ means $A \leq cB$, where $c$ is a sufficiently small constant.
- "$1_A$" denotes the characteristic function on the set $A$.
- Let $f$ be a measurable function on $\mathbb{R}^d$. Then,

$$||f||_p := \left( \int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p}.$$

- Let $f, g \in L^2(\mathbb{R}^d)$. Then $\langle f, g \rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)} dx$.

### 2.1.2 Fourier Transform

Let $d\mu$ be a complex-valued Borel measure on $\mathbb{R}^d$. We define the Fourier transform of $d\mu$ (see [7]) to be

$$\widehat{d\mu}(\xi) := \int e^{-2\pi i x\cdot\xi} d\mu(x).$$

If $f \in L^1(\mathbb{R}^d)$, then $d\mu = f(x)dx$ defines a complex-valued Borel measure on $\mathbb{R}^d$. We can thus extend the definition of the Fourier transform to all $L^1$ functions (and to Schwartz functions, in particular) on $\mathbb{R}^d$. That is,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi} dx. \tag{2.1}$$

For any function $\psi$ and a given $t > 0$, we define

$$\psi_t(x) = t^{-d}\psi(t^{-1}x). \tag{2.2}$$

We define the convolution of a function $\psi$ and a measure $\mu$ as

$$\psi * d\mu(x) = \int_{\mathbb{R}^d} \psi(x-y)d\mu(y). \tag{2.3}$$

We can extend this definition to all $L^1$ functions $f$ by identifying $f$ with the measure $d\mu = f(x)dx$. The following facts will be useful (see [7]):

(i) For any $\psi \in L^1(\mathbb{R}^d)$ and measure $d\mu$,

$$\widehat{\psi * d\mu} = \widehat{\psi}\,\widehat{d\mu}. \tag{2.4}$$

(ii) For any $t > 0$ and $\psi \in L^1(\mathbb{R}^d)$,

$$\widehat{\psi_t}(\xi) = \widehat{\psi}(t\xi).$$

(iii) (Plancherel's Theorem) For Schwartz functions $\varphi$ and $\psi$,

$$\int_{\mathbb{R}^d} \varphi(x)\overline{\psi(x)}dx = \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi. \tag{2.5}$$

### 2.1.3 The Surface Measure on the Sphere and Spherical Averages

Let $d\sigma$ denote the normalized surface measure on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ induced by Lebesgue measure. For any $t > 0$, let

$$d\sigma_t(x) := \frac{1}{t^{d-1}} d\sigma(t^{-1}x).$$

Consider

$$\widehat{d\sigma}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} d\sigma(x).$$

Then

$$\widehat{d\sigma_t}(\xi) = \widehat{d\sigma}(t\xi).$$

We can trivially bound $|\widehat{d\sigma}(\xi)|$ by 1, for all $\xi \in \mathbb{R}^d$. We will also make use of the following estimates on $|\widehat{d\sigma}(\xi)|$ and $|\nabla\widehat{d\sigma}(\xi)|$. These estimates follow from the principle of stationary phase, see for example, [5]. For all $\xi \in \mathbb{R}^d$,

$$|\widehat{d\sigma}(\xi)| \lesssim C(1 + |\xi|)^{-\frac{d-1}{2}} \tag{2.6}$$

and

$$|\nabla\widehat{d\sigma}(\xi)| \lesssim C(1 + |\xi|)^{-\frac{d-1}{2}}. \tag{2.7}$$

For a function $f$ on $\mathbb{R}^d$, let

$$\mathcal{A}_\lambda f(x) := \int_{S^{d-1}} f(x - \lambda y)d\sigma(y) \tag{2.8}$$

denote the average of $f$ over the sphere of radius $\lambda > 0$ centered at $x$. A change of variables gives

$$\mathcal{A}_\lambda f(x) = f * d\sigma_\lambda(x). \tag{2.9}$$

Let

$$\mathcal{A}_* f(x) := \sup_{\lambda>0} |\mathcal{A}_\lambda f(x)| \tag{2.10}$$

denote the spherical maximal function on $\mathbb{R}^d$. We have the following estimate on the spherical maximal function, due to Bourgain in the case where $d = 2$ (see [2]) and Stein in the case where $d \geq 3$. The standard proof in the case where $d \geq 3$ uses estimates (2.6) and (2.7), see [5].

**Theorem 2.1.1.** *If $d \geq 2$ and $p > \frac{d}{d-1}$, then*

$$||\mathcal{A}_* f||_p \lesssim ||f||_p. \tag{2.11}$$

We make use of Theorem 2.1.1 and a generalization thereof in the proof of Theorem 1.1.2 and 1.1.4.

### 2.1.4 A smooth cutoff function $\psi$ and some basic properties:

Let $\psi : \mathbb{R}^d \to (0, \infty)$ be a Schwartz function that satisfies

$$1 = \widehat{\psi}(0) \geq \widehat{\psi}(\xi) \geq 0 \qquad \text{and} \qquad \widehat{\psi}(\xi) = 0 \;\text{ for }\; |\xi| > 1.$$

Note that $\psi$ and $\widehat{\psi}$ are both nonnegative functions and $\widehat{\psi}$ has compact support.

First, we make note of the trivial observation that

$$\int \psi_t(x)\, dx = \int \psi(x)\, dx = \widehat{\psi}(0) = 1$$

as well as the fact that $\psi$ may be chosen so that

$$\left|1 - \widehat{\psi}_t(\xi)\right| = \left|1 - \widehat{\psi}(t\xi)\right| \lesssim \min\{1, t|\xi|\}. \tag{2.12}$$

Finally we record a formulation, appropriate to our needs, of the fact that for any given small parameter $\eta$, our cutoff function $\psi_t(x)$ will essentially supported where $|x| \leq \eta^{-1} t$ and is approximately constant on smaller scales. More precisely, we have the following statement.

**Lemma 2.1.1.** *Let $\eta > 0$ and $t > 0$, then*

$$\int_{|x|\geq\eta^{-1}t} \psi_t(x)\,dx \lesssim \eta. \tag{2.13}$$

*and*

$$\int\int \big|\psi_t(x-\lambda y) - \psi_t(x)\big|\,d\sigma(y)\,dx \lesssim \eta \tag{2.14}$$

*provided $t \geq \eta^{-1}\lambda$.*

*Proof.* Estimate (2.13) is easily verified using the fact that $\psi$ is a Schwartz function on $\mathbb{R}^d$ as

$$\int_{|x|\geq\eta^{-1}t} \psi_t(x)\,dx = \int_{|x|\geq\eta^{-1}} \psi(x)\,dx \lesssim \int_{|x|\geq\eta^{-1}} (1+|x|)^{-d-1}\,dx \lesssim \eta.$$

To verify estimate (2.14) we make use of the fact that both $\psi$ and its derivative are rapidly decreasing, specifically

$$\begin{aligned}\int\int \big|\psi_t(x-\lambda y) - \psi_t(x)\big|\,d\sigma(y)\,dx &\leq \int\int \big|\psi(x-\lambda y/t) - \psi(x)\big|\,d\sigma(y)\,dx \\ &\lesssim \frac{\lambda}{t}\int (1+|x|)^{-d-1}dx \lesssim \frac{\lambda}{t}.\end{aligned}$$

□

# Chapter 3

# Distance Sets

In this chapter, we prove Theorem 1.1.1 when $k = 1$. This proof is meant to serve as a model for the proof of Theorem 1.1.3 discussed in Chapter 2. We also prove Theorem 1.1.2 for $d \geq 3$. This proof is meant to serve as a model for the proof of Theorem 1.1.4 on pinned simplices.

## 3.1 Unpinned Distances

**Theorem 3.1.1.** *Let $d \geq 2$. Let $A \subset \mathbb{R}^d$ with $\overline{\delta}(A) > 0$. Then there exists $\lambda_0 = \lambda_0(A)$ such that for all $\lambda \geq \lambda_0$, there exists an $x \in A$ such that*

$$A \cap (x + \lambda S^{d-1}) \neq \emptyset. \tag{3.1}$$

Theorem 3.1.1 follows from the following dichotomy proposition for sets in the unit cube.

**Proposition 3.1.1.** *Let $\delta > 0$, $0 < \eta \ll \delta^2$, and $A \subset [0,1]^d$ such that $|A| = \delta$. Then for any $0 < \lambda \leq \eta^4$, one of the following holds:*

(i) *there exists $x \in A$ such that*

$$A \cap (x + \lambda S^{d-1}) \neq \emptyset. \tag{3.2}$$

(ii)

$$\int_{\eta^2/\lambda \leq |\xi| \leq 1/\eta^{-2}\lambda} |\widehat{1_A}(\xi)|^2 d\xi \gtrsim \delta |A|. \tag{3.3}$$

Let's first see why Theorem 3.1.1 follows from Proposition 3.1.1.

*Proof.* (Proposition 3.1.1 implies Theorem 3.1.1) Fix $\delta > 0$. Assume the theorem fails for some $A \subseteq \mathbb{R}^d$ with $\overline{\delta}(A) > \delta$. That is, for any $J$, we can choose a strictly increasing sequence $\{\lambda_j\}_{j=1}^J$ with $\lambda_{j+1} \geq \eta^{-4}\lambda_j$ such that

$$A \cap (x + \lambda_j S^{d-1}) = \emptyset,$$

for each $j$. Since $A$ has positive upper density, we can choose $N \geq \lambda_J \eta^{-4}$ such that

$$|A \cap B_N| \geq \delta |B_N|. \tag{3.4}$$

Rescale $A \cap B_N$ to a subset of $[-1/2, 1/2]^d$ and identify $[-1/2, 1/2]^d$ with $[0, 1]^d$. We abuse notation and call this rescaled set $A$. After possibly deleting some pieces, we can say that $|A| = \delta$. If $A \cap B_N$ does not contain two elements of distance $\lambda_j$ apart, then $A$ does not contain two elements of distance $\lambda_j/N$ apart. We can thus apply (ii) from the proposition for $\lambda_j/N$, for each $1 \leq j \leq J$ and we get that

$$\sum_{j=1}^{J} \int_{\eta^2 N/\lambda_j \leq |\xi| \leq N/\eta^2 \lambda_j} |\widehat{1_A}(\xi)|^2 d\xi \geq C \cdot J \cdot \delta |A| > |A|, \tag{3.5}$$

for $J > C \cdot \delta^{-1}$. On the other hand, since $\lambda_{j+1} \geq \eta^{-4}\lambda_j$, the annuli over which we are integrating above are disjoint. It follows from this disjointness property and Plancherel's identity that

$$|A| = \int_{\mathbb{R}^d} |\widehat{1_A}(\xi)|^2 d\xi \geq \sum_{j=1}^{J} \int_{\eta^2/\lambda_j \leq |\xi| \leq 1/\eta^2 \lambda_j} |\widehat{1_A}(\xi)|^2 d\xi, \tag{3.6}$$

a contradiction. □

### 3.1.1 Proof of Proposition 3.1.1

Let $A \subseteq [0, 1]^d$ and say $|A| = \delta > 0$. Let $f = 1_A$. Suppose that part (i) of the proposition is not true. Then there exists $\lambda \leq \eta^4$ such that $\mathcal{A}_\lambda f(x) = 0$, for every $x \in A$. It follows that

$$\langle f, \mathcal{A}_\lambda f \rangle = 0. \tag{3.7}$$

Define $f_1 := f * \psi_{\eta^{-1}\lambda}$. Clearly,

$$|\mathcal{A}_\lambda(f_1)(x) - f_1(x)| \leq \int |f_1(x-\lambda y) - f_1(x)| d\sigma(y).$$

It follows, then, from (2.14) that

$$|\mathcal{A}_\lambda(f_1)(x) - f_1(x)| \lesssim \eta. \tag{3.8}$$

As a result, for any $x \in A$,

$$\mathcal{A}_\lambda(f - f_1)(x) \ll \eta - f_1(x). \tag{3.9}$$

Integrating against $f$ and applying the assumption (3.7), we get that

$$\langle f, \mathcal{A}_\lambda(f - f_1)\rangle \lesssim \langle f, \eta - f_1\rangle = |A|\eta - \langle f, f_1\rangle. \tag{3.10}$$

We combine this with the following lemma.

**Lemma 3.1.1.** *Let $0 < \eta < \delta$ and $f_1 := f * \psi_{\eta^{-1}\lambda}$, then*

$$\langle f, f_1\rangle \gtrsim \delta(1 - C\eta)|A|. \tag{3.11}$$

Lemma 3.1.1 along with (3.10) gives

$$|\langle f, \mathcal{A}_\lambda(f - f_1)\rangle| \gtrsim \delta\,|A|, \tag{3.12}$$

provided $\eta \ll \delta$. The final piece in the proof of Proposition 3.1.1 is the following lemma.

**Lemma 3.1.2.** *If $f_2 := f * \psi_{\eta^2\lambda}$, then*

$$\langle f, \mathcal{A}_\lambda(f - f_2)\rangle \lesssim \eta^{2/3}\,|A|. \tag{3.13}$$

We prove Lemmas 3.1.1 and 3.1.2 below.

*Proof.* (Lemma 3.1.1) We must show that

$$\int f(x) f_1(x) dx \geq \delta^2 (1 - C\eta). \tag{3.14}$$

It follows from Parseval's identity and the fact that $0 \leq \widehat{\psi} \leq 1$ that

$$\int f(x) f_1(x) dx = \int |\hat{f}(\xi)|^2 \widehat{\psi}(\eta^{-1}\lambda\xi) d\xi \geq \int |\hat{f}(\xi)|^2 |\widehat{\psi}(\eta^{-1}\lambda\xi)|^2 d\xi = \int f_1(x)^2 dx. \tag{3.15}$$

Write $B$ for $[0,1]^d$ and let $B'$ denote $[-\eta^{-2}\lambda, 1 + \eta^{-2}\lambda]^d$. Of course,

$$\int_{\mathbb{R}^d} f_1(x)^2 dx \geq \int_B f_1(x)^2 dx.$$

Cauchy-Schwarz gives

$$\int_B f_1(x)^2 \, dx \geq \Big( \int_B f_1(x) \, dx \Big)^2. \tag{3.16}$$

So, it suffices to show that

$$\left( \int_B f_1(x) dx \right)^2 \geq \delta^2 (1 - C\eta). \tag{3.17}$$

Write

$$\int_B f_1(x) \, dx = \int_{\mathbb{R}^d} f_1(x) \, dx - \int_{\mathbb{R}^d \setminus B'} f_1(x) \, dx - \int_{B' \setminus B} f_1(x) \, dx. \tag{3.18}$$

Since $\psi$ integrates to 1, we get that

$$\int_{\mathbb{R}^d} f_1(x) \, dx = \int_{\mathbb{R}^d} f(x) dx = |A|. \tag{3.19}$$

We can apply 2.13 in Lemma (2.1.1) to get that

$$\int_{\mathbb{R}^d \setminus B'} f_1(x)^2 \, dx \leq |A| \int_{|y| \gtrsim \eta^{-2}\lambda} \psi_{\eta^{-1}\lambda}(y) \, dy \lesssim \eta |A|. \tag{3.20}$$

Finally, the fact that $\lambda \leq \eta^4$ ensures that

$$|B' \setminus B| \lesssim \eta^{-2}\lambda \lesssim \eta^2.$$

Then since $0 \leq f_1 \leq 1$, we have

$$\int_{B' \setminus B} f_1(x)\,dx \lesssim \eta^2 \leq \delta\eta. \tag{3.21}$$

Combining (3.19), (3.20), and (3.21), we get

$$\int_B f_1(x)dx \geq \delta(1 - C\eta).$$

It follows that

$$\left(\int_B f_1(x)dx\right)^2 \geq \delta^2(1 - 2C\eta),$$

as desired. $\square$

*Proof.* (Lemma 3.1.2) It follows from an application of Cauchy-Schwarz and Plancherel that

$$\left\langle f, \mathcal{A}_\lambda(f - f_2)\right\rangle^2 \leq |A| \cdot \int |\widehat{f}(\xi)|^2 |1 - \widehat{\psi}(\eta^2 \lambda\,\xi)|^2 |\widehat{d\sigma}(\lambda\,\xi)|^2\,d\xi.$$

Using the decay of $\widehat{d\sigma}(\xi)$, see (2.6), we see that $|\widehat{d\sigma}(\lambda\xi)|^2 \leq \min\{1, (\lambda|\xi|)^{-1}\}$. We combine this observation with (2.12) to get

$$|1 - \widehat{\psi}(\eta^2\lambda\xi)|^2 |\widehat{d\sigma}(\lambda\xi)|^2 \lesssim \min\{(\lambda\xi)^{-1}, \eta^4\lambda^2|\xi|^2\} \leq \eta^{4/3}. \tag{3.22}$$

It follows after an application of Plancherel that

$$\langle f, \mathcal{A}_\lambda(f - f_2)\rangle \lesssim \eta^{2/3}|A|.$$

$\square$

Since $\eta \ll \delta^2$ and

$$\left\langle f, \mathcal{A}_\lambda(f_2 - f_1)\right\rangle = \left\langle f, \mathcal{A}_\lambda(f - f_1)\right\rangle - \left\langle f, \mathcal{A}_\lambda(f - f_2)\right\rangle$$

we see that (3.12) together with Lemma 3.1.2 imply that if (3.7) holds, then

$$\left\langle f, \mathcal{A}_\lambda(f_2 - f_1)\right\rangle \gtrsim \delta|A|. \tag{3.23}$$

It then follows, via Cauchy-Schwarz and Plancherel, that

$$\int \big|\widehat{f}(\xi)\big|^2 \big|\widehat{\psi}_{\eta^2\lambda}(\xi) - \widehat{\psi}_{\eta^{-1}\lambda}(\xi)\big|^2 \, d\xi \gtrsim \delta |A|, \tag{3.24}$$

which is essentially the estimate that we are trying to prove. Now, since (2.12) implies that

$$\big|\widehat{\psi}_{\eta^2\,\lambda}(\xi) - \widehat{\psi}_{\eta^{-1}\lambda}(\xi)\big| \lesssim \eta \tag{3.25}$$

whenever $\xi$ is not contained in $\{\xi : \eta/\lambda \leq |\xi| \leq 1/\eta\lambda\}$, it indeed suffices and concludes the proof of Proposition 3.1.1.

## 3.2 Pinned Distances

In this section, we prove the following "pinned" version of Theorem 3.1.1. Just as in the case of Theorem 3.1.1, the proof of Theorem 3.2.1 is meant to serve as a model for the proof of Theorem 1.1.4 on pinned simplices.

**Theorem 3.2.1** (Pinned distances)**.** *Let $d \geq 3$. If $A$ is a measurable subset of $\mathbb{R}^d$ with positive upper density, then there exist $\lambda_0 = \lambda_0(A)$ such that for any given $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that*

$$A \cap (x + \lambda \cdot S^1) \neq \emptyset \tag{3.26}$$

*for all $\lambda_0 \leq \lambda \leq \lambda_1$.*

Similar to Theorem 3.1.1, Theorem 3.2.1 follows from the following dichotomy proposition.

**Proposition 3.2.1.** *Let $\delta > 0, 0 < \eta \ll \delta^3$, and $A \subset [0,1]^d$ such that $|A| = \delta$. Then, for any $0 < \lambda_0 < \lambda_1 \leq \eta^4$, one of the following holds:*

(i) *there exists $x \in A$ such that*

$$A \cap (x + \lambda S^{d-1}) \neq \emptyset, \tag{3.27}$$

*for all $\lambda \in [\lambda_0, \lambda_1]$.*

(ii)

$$\int_{\eta^2/\lambda_1\leq|\xi|\leq 1/\eta^2\lambda_0} |\widehat{1_A}(\xi)|^2 d\xi \gtrsim \delta^2|A|. \tag{3.28}$$

Let's see why the proposition implies the theorem.

*Proof.* (Proposition 3.2.1 implies Theorem 3.2.1) Fix $\delta > 0$. Assume the theorem fails for some $A \subseteq \mathbb{R}^d$ with $\overline{\delta}(A) > \delta$. That is, for any $J$, we can choose strictly increasing pairs $\{(\lambda_0^{(j)}, \lambda_1^{(j)})\}_{j=1}^J$ with $\lambda_0^{(j+1)} \geq \eta^{-4}\lambda_1^{(j)}$ such that

$$A \cap (x + \lambda_j S^{d-1}) = \emptyset,$$

for each $j$. Since $A$ has positive upper density, we can choose $N \geq \lambda_1^{(J)}\eta^{-4}$ such that

$$|A \cap B_N| \geq \delta|B_N|. \tag{3.29}$$

Rescale $A \cap B_N$ to a subset of $[-1/2, 1/2]^d$ and identify $[-1/2, 1/2]^d$ with $[0, 1]^d$. We abuse notation and call this rescaled set $A$. After possibly deleting some pieces, we can say that $|A| = \delta$. If for each $x \in A \cap B_N$, there exists some $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$(A \cap B_N) \cap (x + \lambda S^1) = \emptyset,$$

then, clearly, the same is the case for $A$. Thus, we can apply (ii) from the proposition for each $(\lambda_0^{(j)}/N, \lambda_1^{(j)}/N)$ and we get that

$$\sum_{j=1}^{J} \int_{\eta^2 N/\lambda_1^{(j)}\leq|\xi|\leq N/\eta^2\lambda_0^{(j)}} |\widehat{1_A}(\xi)|^2 d\xi \geq J \cdot C\delta^2|A| > |A|, \tag{3.30}$$

for $J > C\delta^{-2}$. On the other hand, since $\lambda_0^{(j+1)} \geq \eta^{-4}\lambda_1^{(j)}$, the annuli over which we are integrating above are disjoint. It follows from this disjointness property and Plancherel's identity that

$$|A| = \int_{\mathbb{R}^d} |\widehat{1_A}(\xi)|^2 d\xi \geq \sum_{j=1}^{J} \int_{\eta^2 N/\lambda_j\leq|\xi|\leq N/\eta^2\lambda_j} |\widehat{1_A}(\xi)|^2 d\xi, \tag{3.31}$$

a contradiction. $\square$

### 3.2.1 Proof of Proposition 3.2.1

Let $A \in [0,1]^d$ and say $|A| = \delta > 0$. Let $f = 1_A$. Suppose that we have a pair $(\lambda_0, \lambda_1)$ satisfying $1 \leq \lambda_0 \leq \lambda_1 \leq \eta$, but for which (i) does not hold. It follows that for all $x \in A$ there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$\mathcal{A}_\lambda(f)(x) = 0. \tag{3.32}$$

We now let $f_1 = f * \psi_{\eta^{-1}\lambda_1}$, noting the slight difference from the definition of $f_1$ given in the proof of Proposition 3.1.1. It follows from (3.32), as in the proof of Proposition 3.1.1, that for all $x \in A$ there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$\mathcal{A}_\lambda(f - f_1)(x) \lesssim \eta - f_1(x) \tag{3.33}$$

and hence that

$$\mathcal{A}_*(f - f_1)(x) \gtrsim f_1(x) - \eta, \tag{3.34}$$

for all $x \in A$, where for any Schwartz function $g$, $\mathcal{A}_*(g)$ denotes the *maximal average* defined by

$$\mathcal{A}_*(g)(x) := \sup_{\lambda_0 \leq \lambda \leq \lambda_1} |\mathcal{A}_\lambda(g)(x)|. \tag{3.35}$$

Appealing to Lemma 3.1.1, we may conclude that

$$\big\langle f, \mathcal{A}_*(f - f_1)\big\rangle \gtrsim \delta |A|. \tag{3.36}$$

Arguing as in the proof of Proposition 3.1.1 we see that everything reduces to using the $L^2$-boundedness of $\mathcal{A}_*$ together with establishing appropriate estimates for the "mollified" maximal operator

$$\mathcal{M}_\eta(f) := \mathcal{A}_*(f - f_2) \tag{3.37}$$

where $f_2 = f * \psi_{\eta^2 \lambda_0}$. Note that

$$\mathcal{M}_\eta(f) = \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \Big| \int f(x - \lambda y)\, d\mu_\eta(y) \Big| \tag{3.38}$$

where

$$d\mu_\eta = d\sigma - \psi_{\eta^2\lambda_0\lambda^{-1}} * d\sigma. \tag{3.39}$$

and hence

$$\widehat{d\mu_\eta}(\lambda\,\xi) = \widehat{d\sigma}(\lambda\,\xi)\left(1 - \widehat{\psi}(\eta^2\lambda_0\,\xi)\right). \tag{3.40}$$

Recall, Theorem 2.1.1 says that

$$\int_{\mathbb{R}^d} |\mathcal{A}_*(g)(x)|^2\,dx \lesssim \int_{\mathbb{R}^d} |g(x)|^2\,dx, \tag{3.41}$$

provided $d \geq 3$.

The other result that we need is recorded in the following proposition.

**Proposition 3.2.2** ($L^2$-decay of the "Mollified" Maximal Averages $\mathcal{M}_\eta$)**.** *Let $d \geq 3$ and $\eta > 0$. Then,*

$$\int_{\mathbb{R}^d} |\mathcal{M}_\eta(f)(x)|^2\,dx \lesssim \eta^{2/3} \int_{\mathbb{R}^d} |f(x)|^2\,dx. \tag{3.42}$$

By the sublinearity of the supremum,

$$\langle f, \mathcal{A}_*(f_2 - f_1)\rangle \geq \langle f, \mathcal{A}_*(f - f_1)\rangle - \langle f, \mathcal{A}_*(f - f_2)\rangle. \tag{3.43}$$

Since $\eta \ll \delta^3$, then by Cauchy-Schwarz and Proposition 3.2.2

$$\langle f, \mathcal{A}_*(f - f_2)\rangle \lesssim ||f||||\mathcal{A}_*(f - f_2)||_2 \lesssim \eta^{1/3}||f||_2^2 \ll \delta|A|. \tag{3.44}$$

We combine (3.36), (5.14), and (3.44) to get

$$\langle f, \mathcal{A}_*(f_2 - f_1)\rangle \gtrsim \delta||f||_2^2, \tag{3.45}$$

Now, Cauchy-Schwarz and an application of Theorem 2.1.1 give that

$$\langle f, \mathcal{A}_*(f_2 - f_1)\rangle \leq ||f||_2||\mathcal{A}_*(f_2 - f_1)||_2 \lesssim ||f||_2||f_2 - f_1||_2. \tag{3.46}$$

It follows that

$$||f_2 - f_1||_2^2 \gtrsim \delta^2 |A|,$$

which is precisely the estimate we want to prove. $\square$

### 3.2.2 Proof of Proposition 3.2.2

We will deduce the validity of Proposition 3.2.2 from the following result for the slightly more general class of operators defined for any $L > 0$ by

$$\mathcal{M}_L(f)(x) = \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \left| \int f(x - \lambda y)\, d\mu_L(y) \right| \tag{3.47}$$

where

$$\widehat{d\mu_L}(\lambda\,\xi) = m_L(\xi)\, \widehat{d\sigma}(\lambda\,\xi) \tag{3.48}$$

with the multiplier $m_L$ now any smooth function that satisfies the estimate

$$|m_L(\xi)| \lesssim \min\{1, L|\xi|\}. \tag{3.49}$$

Recall that estimate (2.12) is precisely the statement that $|1 - \widehat{\psi}(L\xi)| \lesssim \min\{1, L|\xi|\}$.

**Theorem 3.2.2.** *If $d \geq 3$,*

$$\int_{\mathbb{R}^d} |\mathcal{M}_L(f)(x)|^2\, dx \lesssim \left(\frac{L}{\lambda_0}\right)^{1/3} \int_{\mathbb{R}^d} |f(x)|^2\, dx. \tag{3.50}$$

*Proof.* It is easy to see that

$$\int_{\mathbb{R}^d} |\mathcal{M}_L(f)(x)|^2\, dx \leq \int_{\mathbb{R}^d} \sup_{\lambda_0 \leq \lambda \leq \lambda_1} |M_{L,\lambda}(f)(x)|^2\, dx. \tag{3.51}$$

where $M_{L,\lambda}$ is the Fourier multiplier operator defined by

$$\widehat{M_{L,\lambda}(f)}(\xi) = \widehat{f}(\xi)\, m_L(\xi)\, \widehat{d\sigma}(\lambda\,\xi). \tag{3.52}$$

A standard application of the Fundamental Theorem of Calculus, see for example [4], gives

$$\sup_{\lambda_0 \le \lambda \le \lambda_1} |M_{L,\lambda}(f)(x)|^2 \le 2 \int_{\lambda_0}^{\lambda_1} |M_{L,t}(f)(x)||\widetilde{M}_{L,t}(f)(x)| \, \frac{dt}{t} + |M_{L,\lambda_0}(f)(x)|^2 \tag{3.53}$$

where $\widetilde{M}_{L,t}(f) = t \dfrac{d}{dt} M_{L,t}(f)$. We further note that $\widetilde{M}_{L,t}$ is clearly also a Fourier multiplier operator, indeed

$$\widehat{\widetilde{M}_{L,t}(f)}(\xi) = \widehat{f}(\xi) \, m_L(\xi) \left( t\xi \cdot \nabla \widehat{d\sigma}(t\xi) \right). \tag{3.54}$$

We can now write

$$\begin{aligned} &\int_{\mathbb{R}^d} |\mathcal{M}_L(f)(x)|^2 \, dx \\ &\qquad \le 2 \sum_{\ell = \lfloor \log_2 \lambda_0 \rfloor}^{\infty} \int_{2^{\ell-1}}^{2^\ell} \int_{\mathbb{R}^d} |M_{L,t}(f)(x)||\widetilde{M}_{L,t}(f)(x)| \, dx \, \frac{dt}{t} \\ &\qquad\qquad + \int_{\mathbb{R}^d} |M_{L,\lambda_0}(f)(x)|^2 \, dx. \end{aligned}$$

Applying Cauchy-Schwarz to the first integral above, in the variables $x, y$, and $t$ together, followed by an application of Plancherel in two resulting integrations in $x$ as well as in the one that appears in the second integral above, we obtain the estimate

$$\int_{\mathbb{R}^d} |\mathcal{M}_L(f)(x)|^2 \, dx \le 2 \sum_{\ell = \lfloor \log_2 \lambda_0 \rfloor}^{\infty} \left( \mathcal{I}_\ell \, \widetilde{\mathcal{I}}_\ell \right)^{1/2} + \mathcal{I} \tag{3.55}$$

with

$$\mathcal{I}_\ell = \int_{2^{\ell-1}}^{2^\ell} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 |\widehat{d\sigma}(t\,\xi)|^2 \, d\xi \, \frac{dt}{t} \tag{3.56}$$

$$\widetilde{\mathcal{I}}_\ell = \int_{2^{\ell-1}}^{2^\ell} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 \left| t\,\xi \cdot \nabla \widehat{d\sigma}(t\,\xi) \right|^2 d\xi \, \frac{dt}{t} \tag{3.57}$$

and

$$\mathcal{I} = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 |\widehat{d\sigma}(\lambda_0\,\xi)|^2 \, d\xi. \tag{3.58}$$

Combining (3.49) with the decay estimate for $\widehat{d\sigma}(\xi)$ gives

$$|m_L(\xi)|^2|\widehat{d\sigma}(t\xi)|^2 \lesssim \min\{(t|\xi|)^{-1}, L^2|\xi|^2\} \le L^{2/3}t^{-2/3}. \tag{3.59}$$

which ensures, via Plancherel, that

$$\mathcal{I}_\ell \lesssim \left(\frac{L}{2^\ell}\right)^{2/3}\|f\|_2^2 \quad \text{and} \quad \mathcal{I} \lesssim \left(\frac{L}{\lambda_0}\right)^{2/3}\|f\|_2^2. \tag{3.60}$$

Using the decay of $\nabla\widehat{d\sigma}(\xi)$, statement (2.7), we have that $\widetilde{I}(\xi)$ is bounded as well. It follows immediately from this observation (and Plancherel) that

$$\widetilde{\mathcal{I}}_\ell \lesssim \|f\|_2^2. \tag{3.61}$$

Combining (3.55), (3.60), and (3.61), we get that

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{M}_L(f)(x)|^2\,dx &\lesssim \left(L^{1/3}\sum_{\ell=\lfloor\log_2\lambda_0\rfloor}^{\infty} 2^{-\ell/3} + \left(\frac{L}{\lambda_0}\right)^{2/3}\right)\int_{\mathbb{R}^d}|f(x)|^2\,dx \\ &\lesssim \left(\frac{L}{\lambda_0}\right)^{1/3}\int_{\mathbb{R}^d}|f(x)|^2\,dx\end{aligned}$$

as required. $\square$

# Chapter 4

# Unpinned Simplices

Here we prove Theorem 1.1.3, an optimal strengthening of Bourgain's theorem, Theorem 1.1.1, on unpinned simplices.

## 4.1 Density of Embedded Simplices

**Theorem 4.1.1** (Density of Embedded Simplices)**.** *Let $\Delta = \{0, v_1, \ldots, v_k\}$ be a fixed, non-degenerate k-dimensional simplex. Let $\varepsilon > 0$. If $A$ is a measurable subset of $\mathbb{R}^d$ with $d \geq k+1$, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that*

$$\int_{SO(d)} \overline{\delta}(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k)))\, d\mu(U) > \overline{\delta}(A)^{k+1} - \varepsilon \tag{4.1}$$

*for all $\lambda \geq \lambda_0$.*

Just as in the case of distances, Theorem 1.1.4 follows from the following dichotomy proposition.

**Proposition 4.1.1** (Dichotomy for Theorem 4.1.1)**.** *Let $\Delta = \{0, v_1, \ldots, v_k\}$ be a fixed, non-degenerate k-dimensional simplex such that diam$(\Delta) \leq 1$. Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^{5/2}$.*

*Let $A \subseteq [0,1]^d$ with $d \geq k+1$, then for any $\lambda$ satisfying $0 < \lambda \leq \eta^4$, one of the following must hold:*

(i)

$$\int_{SO(d)} |A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))| d\mu(U) > |A|^{k+1} - \varepsilon \tag{4.2}$$

(ii)

$$\frac{1}{|A|} \int_{\Omega_\lambda} |\widehat{1_A}(\xi)|^2 \, d\xi \gtrsim C_k \varepsilon^2 \tag{4.3}$$

*where*

$$\Omega_\lambda = \Omega_\lambda(\eta) = \{\xi \in \mathbb{R}^d \, : \, \eta^2 \lambda^{-1} \leq |\xi| \leq \eta^{-2} \lambda^{-1}\}. \tag{4.4}$$

*Proof.* (Proposition 4.1.1 implies Theorem 4.1.1) Let $\varepsilon > 0, 0 < \eta \ll \varepsilon^{5/2}$. Suppose $A \subseteq \mathbb{R}^d$ with $\overline{\delta}(A) > 0$. Suppose the conclusion of Theorem 4.1.1 fails to hold for the set $A$. That is, there exist arbitrarily large $\lambda$ such that

$$\int_{SO(d)} \overline{\delta}(A \cap (A + \lambda U(v_1)) \cap \cdots \cap (A + \lambda U(v_k)) d\mu(U) \leq \overline{\delta}(A)^{k+1} - \varepsilon. \tag{4.5}$$

That is, for any fixed $J$ we can choose an increasing sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_J$ for which the inequality above holds and $\lambda_j \leq \eta^4 \lambda_{j+1}$. By the definition of limsup, there exists an integer $N$ with $N \geq \eta^{-4} \lambda_1$ such that

$$\overline{\delta}(A)^{k+1} - \varepsilon/2 \leq \left( \frac{|A \cap B_N|}{|B_N|} \right)^{k+1} - \varepsilon/4. \tag{4.6}$$

and that

$$\int_{SO(d)} \frac{|A_N \cap (A_N + \lambda^{(j)} \cdot U(v_1)) \cap \cdots \cap (A_N + \lambda^{(j)} \cdot U(v_k))|}{N^d} d\mu(U) \leq \overline{\delta}(A)^{k+1} - \varepsilon/2 \tag{4.7}$$

holds for all $1 \leq j \leq J$, where $A_N = A \cap B_N$. To get the last inequality, we use Fatou's Lemma.

Rescale $A \cap B_N$ to a subset of $[-1/2, 1/2]^d$ and identify $[-1/2, 1/2]^d$ with $[0,1]^d$. We abuse notation and call this rescaled set $A$. If $A \cap B_N$ does not contain an isometric copy of $\lambda_j \cdot \Delta$ for any $j$, then $A$ does not contain an isometric copy of $\frac{\lambda_j}{N} \cdot \Delta$ for any $j$. Choose $J > C\varepsilon^{-2}$. Then, we can apply part (ii) of the proposition for each $1 \le j \le J$: On the one hand,

$$\sum_{j=1}^{J} \frac{1}{|A|} \int_{\Omega_{\lambda_j/N}} |\widehat{1_A}(\xi)|^2 d\xi \gtrsim J\varepsilon^2 > 1. \tag{4.8}$$

On the other hand, it follows from the choice of the increasing sequence $\{\lambda_j\}$ that the sets $\{\Omega_{\lambda_j/N}\}$ are disjoint. By the disjointness of $\{\Omega_{\lambda_j/N}\}$ and by Plancherel's theorem, we get that

$$\sum_{j=1}^{J} \frac{1}{|A|} \int_{\Omega_{\lambda_j/N}} |\widehat{1_A}(\xi)|^2 d\xi \le \frac{1}{|A|} \int_{\mathbb{R}^d} |\widehat{1_A}(\xi)|^2 d\xi = 1, \tag{4.9}$$

giving a contradiction. □

## 4.2 Background

Here, we give facts and definitions specific to the case of simplices.

### 4.2.1 The multi-linear operators $\mathcal{A}_\lambda^{(j)}$

Let $\Delta = \{0, v_1, \ldots, v_k\}$ be a fixed $k$-dimensional simplex. Without loss of generality we may assume that $|v_1| = 1$. For each $1 \le j \le k$ we introduce the multi-linear operator $\mathcal{A}_\lambda^{(j)}$, defined initially for Schwartz functions $g_1, \ldots, g_j$, by

$$\mathcal{A}_\lambda^{(j)}(g_1, \ldots, g_j)(x) = \int \cdots \int g_1(x - \lambda y_1) \cdots g_j(x - \lambda y_j)\, d\sigma^{(d-j)}_{y_1,\ldots,y_{j-1}}(y_j) \cdots d\sigma^{(d-1)}(y_1), \tag{4.10}$$

where $d\sigma^{(d-1)}$ denotes the measure on the unit sphere $S^{d-1} \subseteq \mathbb{R}^d$ induced by Lebesgue measure normalized to have total mass 1 and $d\sigma^{(d-j)}_{y_1,\ldots,y_{j-1}}$ denotes, for each $2 \le j \le k$, the normalized measure on the sphere

$$S^{d-j}_{y_1,\ldots,y_{j-1}} \subseteq y + [y_1, \ldots, y_{j-1}]^\perp \simeq \mathbb{R}^{d-j+1}$$

of radius $r_j = \mathrm{dist}(v_j, [v_1, \ldots, v_{j-1}])$ centered at $y \in [y_1, \ldots, y_{j-1}]$ with $y \cdot y_i = v_j \cdot v_i$ for all $1 \le i \le j - 1$.

The multi-linear operator $\mathcal{A}_\lambda^{(j)}$ is a natural object for us to consider in light of the observation that it could have equivalently be defined for each $1 \leq j \leq k$ using the formula

$$\mathcal{A}_\lambda^{(j)}(g_1, \dots, g_j)(x) := \int_{SO(d)} g_1(x - \lambda \cdot U(v_1)) \cdots g_j(x - \lambda \cdot U(v_j))\, d\mu(U) \tag{4.11}$$

and hence for any bounded measurable set $A \subseteq \mathbb{R}^d$, the quantity

$$\left\langle 1_A, \mathcal{A}_\lambda^{(k)}(1_A, \dots, 1_A) \right\rangle = \int_{SO(d)} |A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))|\, d\mu(U). \tag{4.12}$$

A trivial, but important, observation will be the fact that

$$\begin{aligned} &\left|\mathcal{A}_\lambda^{(j)}(g_1, \dots, g_j)(x) - g_j(x)\, \mathcal{A}_\lambda^{(j-1)}(g_1, \dots, g_{j-1})(x)\right| \\ &\leq \sup_{y_1, \dots, y_{j-1} \in \mathbb{R}^d} \int \left|g_j(x - \lambda y_j) - g_j(x)\right| d\sigma^{(d-j)}_{y_1, \dots, y_{j-1}}(y_j). \end{aligned} \tag{4.13}$$

### 4.2.2 A second averaging operator and some basic estimates

We now introduce a second averaging operator, which we also denote by $\mathcal{A}_\lambda^{(j)}$, defined initially for any Schwartz function $g$, by

$$\mathcal{A}_\lambda^{(j)}(g)(x) = \int \cdots \int \left| \int g(x - \lambda y_j)\, d\sigma^{(d-j)}_{y_1, \dots, y_{j-1}}(y_j) \right| d\sigma^{(d-j+1)}_{y_1, \dots, y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1) \tag{4.14}$$

Note that if the functions $g_1, \dots, g_{j-1}$ are all bounded in absolute value by 1, then clearly

$$\left|\mathcal{A}_\lambda^{(j)}(g_1, \dots, g_j)(x)\right| \leq \mathcal{A}_\lambda^{(j)}(g_j)(x). \tag{4.15}$$

Fix $1 \leq j \leq k$. It is easy to see, using Minkowski's inequality, that for any Schwartz function $g$ we have the crude estimate

$$\int \left|\mathcal{A}_\lambda^{(j)}(g)(x)\right|^2 dx \leq \int |g(x)|^2\, dx. \tag{4.16}$$

However, arguing more carefully one can just as easily obtain, using Plancherel's identity, the estimate

$$\int \left|\mathcal{A}_\lambda^{(j)}(g)(x)\right|^2 dx \le \int \cdots \int \Big( \int |\widehat{g}(\xi)|^2 \left|\widehat{d\sigma^{(d-j)}_{y_1,\ldots,y_{j-1}}}(\lambda\,\xi)\right|^2 d\xi \Big)\, d\sigma^{(d-j+1)}_{y_1,\ldots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1). \tag{4.17}$$

Just as in the case of unpinned distances, we made use of the decay estimate of the Fourier transform of the surface measure on the sphere, here, we will have use for the following estimate

$$\left|\widehat{d\sigma^{(d-j)}_{y_1,\ldots,y_{j-1}}}(\xi)\right| + \left|\nabla \widehat{d\sigma^{(d-j)}_{y_1,\ldots,y_{j-1}}}(\xi)\right| \le C_\Delta \big(1 + \operatorname{dist}(\xi, [y_1, \ldots, y_{j-1}])\big)^{-(d-j)/2}. \tag{4.18}$$

This is indeed a consequence of the decay of the Fourier transform of the surface measure on the sphere $S^{d-j} \subseteq \mathbb{R}^{d-j+1}$, see (2.6), (2.7) in Chapter 2.

**Lemma 4.2.1.** *Let $\eta > 0$ and $t > 0$, then*

$$\sup_{y_1,\ldots,y_{j-1}\in\mathbb{R}^d} \int \int \left|\psi_t(x - \lambda y) - \psi_t(x)\right| d\sigma^{(d-1)}_{y_1,\ldots,y_{j-1}}(y)\, dx \lesssim \eta, \tag{4.19}$$

*provided $t \ge \eta^{-1}\lambda$.*

The proof of Lemma 4.2.1 is the same as the proof of Lemma 2.1.1.

## 4.3 Proof of Dichotomy Proposition

Let $f = 1_A$ and $\delta = |A|$. Suppose that $0 < \lambda \le \eta^4$ and that (i) does not hold. Then,

$$\langle f, \mathcal{A}_\lambda^j(f, \ldots, f)\rangle \le \langle f, \delta^k - \varepsilon\rangle = (\delta^k - \varepsilon)\cdot\delta. \tag{4.20}$$

If we let $f_1 := f * \psi_{\eta^{-1}\lambda}$, then from (4.13) and (2.14), it follows for all $x \in \mathbb{R}^d$ and $1 \le j \le k$ we have

$$\left|\mathcal{A}_\lambda^{(j)}(f, \ldots, f, f_1)(x) - f_1(x)\, \mathcal{A}_\lambda^{(j-1)}(f, \ldots, f)(x)\right| \lesssim \eta. \tag{4.21}$$

We can use (4.21) repeatedly and reorganize terms to get

$$f_1(x)^k + \sum_{j=1}^{k} f_1(x)^{k-j} \mathcal{A}_\lambda^{(j)}(f, \dots, f, f - f_1)(x) \lesssim \mathcal{A}_\lambda^{(k)}(f, \dots, f)(x) + \eta. \tag{4.22}$$

Integrate the terms above against $f$ and combine the result with (4.20) to get

$$\sum_{j=1}^{k} \big\langle f f_1^{k-j}, \mathcal{A}_\lambda^{(j)}(f, \dots, f, f - f_1) \big\rangle \leq \langle f, \delta^k - f_1^k - \varepsilon/2 \rangle \tag{4.23}$$

provided $\eta \ll \varepsilon$.

We combine the above with the following lemma.

**Lemma 4.3.1.** *Let $\eta > 0$ and $f_1 := f * \psi_{\eta^{-1}\lambda}$, then*

$$\langle f, \delta^k - f_1^k \rangle \lesssim \langle f, \eta \rangle \tag{4.24}$$

Lemma 4.3.1 follows almost immediately from Lemma 3.1.1. Indeed, it suffices to establish the result when $k = 1$, namely that

$$\int f(x) f_1(x)\, dx \geq \delta^2 (1 - C\eta)\, |A| \tag{4.25}$$

since from Hölder's inequality we would then obtain

$$(\delta - C\eta)^k\, |A|^k \leq \Big( \int f(x) f_1(x)\, dx \Big)^k \leq |A|^{k-1} \int f(x) f_1(x)^k\, dx$$

from which the full result immediately follows.

Combining Lemma 4.3.1 with (4.23) we see that if $\eta \ll \varepsilon$ and (4.20) holds, then there exist $1 \leq j \leq k$ such that

$$\Big| \big\langle f f_1^{k-j}, \mathcal{A}_\lambda^{(j)}(f, \dots, f, f - f_1) \big\rangle \Big| \geq C_k\, \varepsilon |A|. \tag{4.26}$$

Hence, using (4.15) and the fact that $0 \leq f_1 \leq 1$, that

$$\big\langle f, \mathcal{A}_\lambda^{(j)}(f - f_1)\big\rangle \geq C_k\, \varepsilon |A|. \tag{4.27}$$

The final ingredient in the proof of Proposition 4.1.1 is the following

**Lemma 4.3.2** (Error term)**.** *If $f_2 := f * \psi_{\eta^2\lambda}$, then for any $1 \leq j \leq k$ we have the estimate*

$$\big\langle f, \mathcal{A}_\lambda^{(j)}(f - f_2)\big\rangle \lesssim \eta^{2/5}|A|. \tag{4.28}$$

Indeed, since

$$\big\langle f, \mathcal{A}_\lambda^{(j)}(f_2 - f_1)\big\rangle \geq \big\langle f, \mathcal{A}_\lambda^{(j)}(f - f_1)\big\rangle - \big\langle f, \mathcal{A}_\lambda^{(j)}(f - f_2)\big\rangle$$

we see that (4.27) together with Lemma 4.3.2 will imply that if $\eta \lesssim \varepsilon^{5/2}$ and (4.20) holds, then there exist $1 \leq j \leq k$ such that

$$\big\langle f, \mathcal{A}_\lambda^{(j)}(f_2 - f_1)\big\rangle \geq C_k\, \varepsilon |A|. \tag{4.29}$$

It then follows, via Cauchy-Schwarz and Plancherel, that

$$\int \big|\widehat{f}(\xi)\big|^2 \big|\widehat{\psi}_{\eta^2\lambda}(\xi) - \widehat{\psi}_{\eta^{-1}\lambda}(\xi)\big|^2\, d\xi \geq\, C_k \varepsilon^2\, |A|, \tag{4.30}$$

which is essentially the estimate that we are trying to prove and since (2.12) implies that

$$\big|\widehat{\psi}_{\eta^2\lambda}(\xi) - \widehat{\psi}_{\eta^{-1}\lambda}(\xi)\big| \ll \eta \tag{4.31}$$

whenever $\xi \notin \Omega_\lambda$, it indeed suffices and concludes the proof of Proposition 4.1.1. □

### 4.3.1 Proof of Lemma 4.3.2

It follows from an application of Cauchy-Schwarz and Plancherel that

$$\big\langle f, \mathcal{A}_\lambda^{(j)}(f-f_2)\big\rangle^2 \le |A|\cdot \int |\widehat{f}(\xi)|^2|1-\widehat{\psi}(\eta^2\lambda\,\xi)|^2 I(\lambda\,\xi)\,d\xi$$

where

$$I(\xi)=\int\cdots\int \big|\widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)\big|^2\,d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1})\cdots d\sigma^{(d-1)}(y_1). \tag{4.32}$$

While from (4.18), the trivial uniform bound $I(\xi)\ll 1$, and an appropriate "conical" decomposition, depending on $\xi$, of the configuration space over which the integral $I(\xi)$ is defined, we have

$$I(\xi)\le C_\Delta(1+|\xi|)^{-(d-j)/2}. \tag{4.33}$$

Combining this observation with (2.12) we obtain the uniform bound

$$|1-\widehat{\psi}(\eta^2\lambda\,\xi)|^2 I(\lambda\,\xi)\ll \min\{(\lambda|\xi|)^{-1/2},\eta^4\lambda^2|\xi|^2\}\le \eta^{4/5} \tag{4.34}$$

which, after an application of Plancherel, completes the proof. □

# Chapter 5

# Pinned Simplices

Here, we'll discuss the following pinned version of Theorem 1.1.3 in Chapter 4.

## 5.1 Density of Embedded Pinned Simplices

**Theorem 5.1.1** (Density of Embedded Pinned Simplices). *Let $\Delta$ be a fixed non-degenerate $k$-dimensional simplex and let $\varepsilon > 0$. If $A$ is a measurable subset of $\mathbb{R}^d$ with $d \geq k+2$, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any given $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that*

$$\mu\big(\big\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\big\}\big) > \overline{\delta}(A)^k - \varepsilon \quad \textit{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1. \tag{5.1}$$

**Proposition 5.1.1** (Dichotomy for Theorem 5.1.1). *Let $\Delta$ be a fixed, non-degenerate $k$-dimensional simplex such that $diam(\Delta) \leq 1$. Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^3$. Let $A \subseteq [0,1]^d$ with $d \geq k+2$. Then for any pair $(\lambda_0, \lambda_1)$ satisfying $0 < \lambda_0 \leq \lambda_1 \leq \eta^4$, one of the following must hold:*

(i)

$$\mu\big(\big\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\big\}\big) > |A|^k - \varepsilon \quad \textit{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1. \tag{5.2}$$

(ii)

$$\frac{1}{|A|} \int_{\Omega_{\lambda_0, \lambda_1}} |\widehat{1_A}(\xi)|^2 \, d\xi \geq C_k \varepsilon^2 \tag{5.3}$$

*where*

$$\Omega_{\lambda_0,\lambda_1} = \Omega_{\lambda_0,\lambda_1}(\eta) = \{\xi \in \mathbb{R}^d \,:\, \eta^2 \lambda_1^{-1} \leq |\xi| \leq \eta^{-2} \lambda_0^{-1}\}. \tag{5.4}$$

*Proof.* (Proposition 5.1.1 implies Theorem 5.1.1) Let $\varepsilon > 0, 0 < \eta < \varepsilon^3$. Suppose $A \subseteq \mathbb{R}^d$ with $\overline{\delta}(A) > 0$. Suppose the conclusion of Theorem 4.1.1 fails to hold for the set $A$. That is, there exist arbitrarily large pairs $(\lambda_0, \lambda_1)$ of real numbers such that for all $x \in A$ one has

$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) \leq \overline{\delta}(A)^k - \varepsilon$$

for some $\lambda_0 \leq \lambda \leq \lambda_1$.

For a fixed integer $J \gg \varepsilon^{-2}$ we choose a sequence of such pairs $\{(\lambda_0^{(j)}, \lambda_1^{(j)}\}_{j=1}^J$ with the property that $1 \leq \lambda_0^{(j)} \leq \eta^4 \lambda_1^{(j+1)}$ for $1 \leq j < J$. We now choose $N$ so that $\lambda_1^{(J)} \leq \eta^4 N$ and

$$\overline{\delta}(A)^k - \varepsilon \leq \left(\frac{|A \cap B_N|}{N^d}\right)^k - \varepsilon/2. \tag{5.5}$$

Rescale $A \cap B_N$ to a subset of $[-1/2, 1/2]^d$ and identify $[-1/2, 1/2]^d$ with $[0, 1]^d$. We abuse notation and call this rescaled set $A$. If the conclusion of the theorem does not hold for $A \cap B_N$, that is, if for all $x \in A$ one has

$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) \leq \overline{\delta}(A)^k - \varepsilon,$$

for some $\lambda_0 \leq \lambda \leq \lambda_1$, then this is similar for $A$ and $(\lambda_0^{(j)}/N, \lambda_1^{(j)}/N)$. Choose $J \geq C_k \varepsilon^{-2}$. Thus, we can apply part (ii) of the proposition for each $1 \leq j \leq J$. On the one hand,

$$\sum_{j=1}^{J} \frac{1}{|A|} \int_{\Omega_{\lambda_0^{(j)}/N, \lambda_1^{(j)}/N}} |\widehat{1_A}(\xi)|^2 d\xi \geq C_k J \varepsilon^2 > 1. \tag{5.6}$$

On the other hand, it follows from the choice of the increasing sequence $\{\lambda_0^{(j)}/N\}$ that the sets $\Omega_{\lambda_0^{(j)}/N, \lambda_1^{(j)}/N}$ are disjoint. By the disjointness of $\Omega_{\lambda_0^{(j)}, \lambda_1^{(j)}}$ and by Plancherel's theorem, we get that

$$\sum_{j=1}^{J} \frac{1}{|A|} \int_{\Omega_{\lambda_0^{(j)}/N, \lambda_1^{(j)}/N}} |\widehat{1_A}(\xi)|^2 d\xi \leq \frac{1}{|A|} \int_{\mathbb{R}^d} |\widehat{1_A}(\xi)|^2 d\xi = 1, \tag{5.7}$$

giving a contradiction. $\square$

## 5.2 Proof of Dichotomy Proposition

Suppose that we have a pair $(\lambda_0, \lambda_1)$ satisfying $1 \leq \lambda_0 \leq \lambda_1 \leq \eta^4 N$, but for which (i) does not hold. It follows that for all $x \in A$ there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$\mathcal{A}_\lambda^{(k)}(f, \ldots, f)(x) \leq \delta^k - \varepsilon. \tag{5.8}$$

We now let $f_1 = f * \psi_{\eta^{-1}\lambda_1}$, noting the slight difference from the definition of $f_1$ given in the proof of Proposition 4.1.1. It follows from (5.8), as in the proof of Proposition 4.1.1, that for all $x \in A$ there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$\sum_{j=1}^{k} f_1(x)^{k-j} \mathcal{A}_\lambda^{(j)}(f, \ldots, f, f - f_1)(x) \leq \delta^k - f_1(x)^k - \varepsilon/2 \tag{5.9}$$

provided $\eta \ll \varepsilon$, and hence that

$$\sum_{j=1}^{k} \mathcal{A}_*^{(j)}(f - f_1)(x) \geq f_1(x)^k - \delta^k + \varepsilon/2 \tag{5.10}$$

for all $x \in A$, where for any Schwartz function $g$, $\mathcal{A}_*^{(j)}(g)$ denotes the *maximal average* defined by

$$\mathcal{A}_*^{(j)}(g)(x) := \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \mathcal{A}_\lambda^{(j)}(g)(x). \tag{5.11}$$

Consequently, provided $\eta \ll \varepsilon$ and appealing to Lemma 4.3.1, we may conclude that there must exist $1 \leq j \leq k$ such that

$$\left\langle f, \mathcal{A}_*^{(j)}(f - f_1) \right\rangle \gtrsim \varepsilon |A|. \tag{5.12}$$

Let $f_2 = f * \psi_{\eta^2 \lambda_0}$. Define the following "mollified" maximal operator

$$\mathcal{M}_\eta^{(j)}(f) := \mathcal{A}_*^{(j)}(f - f_2). \tag{5.13}$$

By the sublinearity of the supremum,

$$\langle f, \mathcal{A}_*^{(j)}(f_2 - f_1) \rangle \geq \langle f, \mathcal{A}_*^{(j)}(f - f_1) \rangle - \langle f, \mathcal{A}_*^{(j)}(f - f_2) \rangle. \tag{5.14}$$

In addition to (5.12), by Cauchy-Schwarz, we get that

$$\langle f, \mathcal{A}_*^{(j)}(f_2 - f_1) \rangle \lesssim ||f||_2 ||\mathcal{A}_*^{(j)}(f_2 - f_1)||_2 \tag{5.15}$$

and

$$\langle f, \mathcal{A}_*^{(j)}(f - f_2) \rangle \lesssim ||f||_2 ||\mathcal{A}_*^{(j)}(f - f_2)||_2. \tag{5.16}$$

So, similar to the proof of Proposition 3.2.1, we see that everything reduces to establishing the $L^2$-boundedness of $\mathcal{A}_*^{(j)}$ together with appropriate estimates for the "mollified" maximal operator. The precise results that we need are recorded in the following two propositions.

**Proposition 5.2.1** ($L^2$-Boundedness of the Maximal Averages $\mathcal{A}_*^{(j)}$). *If $d \geq j + 2$, then*

$$\int_{\mathbb{R}^d} |\mathcal{A}_*^{(j)}(g)(x)|^2 \, dx \lesssim \int_{\mathbb{R}^d} |g(x)|^2 \, dx. \tag{5.17}$$

**Proposition 5.2.2** ($L^2$-decay of the "Mollified" Maximal Averages $\mathcal{M}_\eta^{(j)}$). *Let $\eta > 0$. If $d \geq j+2$, then*

$$\int_{\mathbb{R}^d} |\mathcal{M}_\eta^{(j)}(f)(x)|^2 \, dx \lesssim \eta^{2/3} \int_{\mathbb{R}^d} |f(x)|^2 \, dx. \tag{5.18}$$

Combining Proposition 5.2.1 and (5.15), we get that

$$\langle f, \mathcal{A}_*^{(j)}(f_2 - f_1) \rangle \lesssim ||f||_2 ||f_2 - f_1||_2. \tag{5.19}$$

If we combine Proposition 5.2.2 and (5.16), we get

$$\langle f, \mathcal{A}_*^{(j)}(f - f_2) \rangle \lesssim \eta^{2/3} ||f||_2^2. \tag{5.20}$$

Then, combining, (5.12), (5.19), and (5.20) we conclude that

$$||f_2 - f_1||_2^2 \gtrsim \varepsilon^2 |A|, \tag{5.21}$$

provided $\eta \ll \varepsilon^3$, which is the estimate we need to prove. $\square$

The proofs of Propositions 5.2.1 and 5.2.2 are presented in Section 5.3 below.

## 5.3 Proofs of Maximal Function Estimates

### 5.3.1 Proof of Proposition 5.2.1

We first note that Cauchy-Schwarz ensures

$$\int_{\mathbb{R}^d} |\mathcal{A}_*^{(j)}(g)(x)|^2\, dx \leq \int \cdots \int \int_{\mathbb{R}^d} \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \Big| \int g(x - \lambda y_j)\, d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j) \Big|^2 dx\, d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1).$$

Now for fixed $y_1, \dots, y_{j-1}$ we can clearly identify $[y_1, \dots, y_{j-1}]^\perp$ with $\mathbb{R}^{d-j+1}$ and $d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}$ with a constant (depending only on $d$ and $\delta$) multiple of $d\sigma^{(d-j)}$, the normalized measure on the unit sphere $S^{d-j} \subseteq \mathbb{R}^{d-j+1}$ induced by Lebesgue measure. We can write $\mathbb{R}^d = \mathbb{R}^{j-1} \times \mathbb{R}^{d-j+1}$, $g(x) = g_{x'}(x'')$, and apply *Stein's spherical maximal function theorem* for functions in $L^2(\mathbb{R}^{d-j+1})$, see Theorem 2.1.1 in Chapter 2. That is,

$$\int_{\mathbb{R}^{d-j+1}} \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \Big| \int g(x - \lambda y)\, d\sigma^{(d-j)}(y) \Big|^2 dx \lesssim \int_{\mathbb{R}^{d-j+1}} |g(x)|^2\, dx \tag{5.22}$$

whenever $d \geq j+2$. This gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \Big| \int g(x - \lambda y)\, d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y) \Big|^2 dx \\
&\qquad = C_\Delta \int_{\mathbb{R}^{j-1}} \int_{\mathbb{R}^{d-j+1}} \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \Big| \int g_{x'}(x'' - \lambda y)\, d\sigma^{(d-j)}(y) \Big|^2 dx''\, dx' \\
&\qquad \leq C \int_{\mathbb{R}^{j-1}} \int_{\mathbb{R}^{d-j+1}} |g_{x'}(x'')|^2\, dx''\, dx' = C \int_{\mathbb{R}^d} |g(x)|^2\, dx
\end{aligned}$$

with the constant $C$ independent of the initial choice of frame $y_1, \dots, y_{j-1}$. The result follows. $\square$

### 5.3.2 Proof of Propositions 5.2.2

First, note that

$$\mathcal{M}_{\eta}^{(j)}(f) = \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \int \cdots \int \left| \int f(x - \lambda y_j)\, d\mu_{\eta}^{(j)}(y_j) \right| d\sigma_{y_1,\ldots,y_{j-2}}^{(d-j+1)}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1) \tag{5.23}$$

where

$$d\mu_{\eta}^{(j)} = d\sigma_{y_1,\ldots,y_{j-1}}^{(d-j)} - \psi_{\eta^2\lambda_0\lambda^{-1}} * d\sigma_{y_1,\ldots,y_{j-1}}^{(d-j)}. \tag{5.24}$$

and hence

$$\widehat{\mu_{\eta}^{(j)}}(\lambda\,\xi) = \widehat{d\sigma_{y_1,\ldots,y_{j-1}}^{(d-j)}}(\lambda\,\xi)\,\big(1 - \widehat{\psi}(\eta^2\lambda_0\,\xi)\big). \tag{5.25}$$

We will deduce the validity of Proposition 5.2.2 from the following result for the slightly more general class of operators defined for any $L > 0$ by

$$\mathcal{M}_{L}^{(j)}(f) = \sup_{\lambda_0 \leq \lambda \leq \lambda_1} \int \cdots \int \left| \int f(x - \lambda y)\, d\mu_{L}^{(j)}(y) \right| d\sigma_{y_1,\ldots,y_{j-2}}^{(d-j+1)}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1) \tag{5.26}$$

where

$$\widehat{d\mu_{L}^{(j)}}(\lambda\,\xi) = m_L(\xi)\, \widehat{d\sigma_{y_1,\ldots,y_{j-1}}^{(d-j)}}(\lambda\,\xi) \tag{5.27}$$

with the multiplier $m_L$ now any smooth function that satisfies the estimate

$$|m_L(\xi)| \lesssim \min\{1, L|\xi|\}. \tag{5.28}$$

Recall that estimate (2.12) is precisely the statement that $|1 - \widehat{\psi}(L\xi)| \lesssim \min\{1, L|\xi|\}$.

**Theorem 5.3.1.** *If $d \geq j + 2$, then*

$$\int_{\mathbb{R}^d} |\mathcal{M}_{L}^{(j)}(f)(x)|^2\, dx \lesssim \left(\frac{L}{\lambda_0}\right)^{1/3} \int_{\mathbb{R}^d} |f(x)|^2\, dx. \tag{5.29}$$

*Proof.* An application of Cauchy-Schwarz gives

$$\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2\,dx \le \int \cdots \int \left[ \int_{\mathbb{R}^d} \sup_{\lambda_0 \le \lambda \le \lambda_1} |M_{L,\lambda}(f)(x)|^2\,dx \right] d\sigma^{(d-j+1)}_{y_1,\ldots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1). \tag{5.30}$$

where $M_{L,\lambda}$ is the Fourier multiplier operator defined by

$$\widehat{M_{L,\lambda}(f)}(\xi) = \widehat{f}(\xi)\, m_L(\xi)\, \widehat{d\sigma^{(d-j)}_{y_1,\ldots,y_{j-1}}}(\lambda\,\xi). \tag{5.31}$$

A standard application of the Fundamental Theorem of Calculus, see for example [4], gives

$$\sup_{\lambda_0 \le \lambda \le \lambda_1} |M_{L,\lambda}(f)(x)|^2 \le 2 \int_{\lambda_0}^{\lambda_1} |M_{L,t}(f)(x)||\widetilde{M}_{L,t}(f)(x)|\,\frac{dt}{t} + |M_{L,\lambda_0}(f)(x)|^2 \tag{5.32}$$

where $\widetilde{M}_{L,t}(f) = t\dfrac{d}{dt}M_{L,t}(f)$. We further note that $\widetilde{M}_{L,t}$ is clearly also a Fourier multiplier operator, indeed

$$\widehat{\widetilde{M}_{L,t}(f)}(\xi) = \widehat{f}(\xi)\, m_L(\xi)\,\big(t\xi \cdot \nabla \widehat{d\sigma^{(d-j)}_{y_1,\ldots,y_{j-1}}}(t\xi)\big). \tag{5.33}$$

We now immediately see that

$$\begin{aligned}\int_{\mathbb{R}^d} &|\mathcal{M}_L^{(j)}(f)(x)|^2\,dx \\ &\le 2 \sum_{\ell=\lfloor \log_2 \lambda_0 \rfloor}^{\infty} \int_{2^{\ell-1}}^{2^\ell} \int \cdots \int \int_{\mathbb{R}^d} |M_{L,t}(f)(x)||\widetilde{M}_{L,t}(f)(x)|\,dx\,d\sigma^{(d-j+1)}_{y_1,\ldots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1)\,\frac{dt}{t} \\ &\qquad + \int \cdots \int \int_{\mathbb{R}^d} |M_{L,\lambda_0}(f)(x)|^2\,dx\,d\sigma^{(d-j+1)}_{y_1,\ldots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1).\end{aligned}$$

Applying Cauchy-Schwarz to the first integral above, in the variables $x, y_1, \ldots, y_{j-1}$, and $t$ together, followed by an application of Plancherel in two resulting integrations in $x$ as well as in the one that appears in the second integral above, we obtain the estimate

$$\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2\,dx \le 2 \sum_{\ell=\lfloor \log_2 \lambda_0 \rfloor}^{\infty} \big(\mathcal{I}_\ell\, \widetilde{\mathcal{I}}_\ell\big)^{1/2} + \mathcal{I} \tag{5.34}$$

with

$$\mathcal{I}_\ell = \int_{2^{\ell-1}}^{2^\ell} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 I(t\,\xi)\, d\xi\, \frac{dt}{t} \tag{5.35}$$

$$\widetilde{\mathcal{I}}_\ell = \int_{2^{\ell-1}}^{2^\ell} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 \widetilde{I}(t\,\xi)\, d\xi\, \frac{dt}{t} \tag{5.36}$$

and

$$\mathcal{I} = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 I(\lambda_0\,\xi)\, d\xi \tag{5.37}$$

where, as in the proof of Lemma 4.3.2, we have defined

$$I(\xi) = \int \cdots \int \left| \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi) \right|^2 d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1) \tag{5.38}$$

and analogously now also define

$$\widetilde{I}(\xi) = \int \cdots \int \left| \xi \cdot \nabla \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi) \right|^2 d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1). \tag{5.39}$$

Combining (5.28) with (4.33), and recalling that we are assuming that $d \geq j+2$, gives

$$|m_L(\xi)|^2 I(t\,\xi) \ll \min\{(t|\xi|)^{-1}, L^2|\xi|^2\} \leq L^{2/3} t^{-2/3} \tag{5.40}$$

which ensures, via Plancherel, that

$$\mathcal{I}_\ell \lesssim \left(\frac{L}{2^\ell}\right)^{2/3} \|f\|_2^2 \quad \text{and} \quad \mathcal{I} \lesssim \left(\frac{L}{\lambda_0}\right)^{2/3} \|f\|_2^2. \tag{5.41}$$

Arguing as in the proof of estimate (4.33), we can see that estimate (4.18) for $\nabla \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)$ ensures that $\widetilde{I}(\xi)$ is bounded whenever $d \geq j+2$. It follows immediately from this observation (and Plancherel) that

$$\widetilde{\mathcal{I}}_\ell \lesssim \|f\|_2^2. \tag{5.42}$$

Combining (5.34), (5.41), and (5.42), we get that

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2 \, dx &\ll \left( L^{1/3} \sum_{\ell = \lfloor \log_2 \lambda_0 \rfloor}^{\infty} 2^{-\ell/3} + \left( \frac{L}{\lambda_0} \right)^{2/3} \right) \int_{\mathbb{R}^d} |f(x)|^2 \, dx \\ &\ll \left( \frac{L}{\lambda_0} \right)^{1/3} \int_{\mathbb{R}^d} |f(x)|^2 \, dx\end{aligned}$$

as required. □

# Chapter 6

# A Geometric Approach to Proving Stein's Spherical Maximal Function Theorem

In this chapter, we give a sketch of non-Fourier analytic proof of Stein's spherical maximal function estimate in the case where $p = 2$.

## 6.1 Set up

Let $f$ be a Schwartz function on $\mathbb{R}^d$. Let

$$\mathcal{M}f(x) := \sup_{1 \leq \lambda \leq 2} \left| \int f(x - \lambda y) d\sigma(y) \right|. \tag{6.1}$$

This is similar to the maximal function in (2.1.1), but here we are restricting our $\lambda$ to $1 \leq \lambda \leq 2$.

**Theorem 6.1.1.** *Let $f$ be a Schwartz function on $\mathbb{R}^d$. If $d \geq 3$, then*

$$||\mathcal{M}f||_2 \lesssim ||f||_2.$$

For each $x \in \mathbb{R}^d$, there exits $\lambda = \lambda(x)$ so that if

$$Tf(x) = \int_{S^{n-1}} f(x - \lambda(x)y) d\sigma(y),$$

then $\mathcal{M}f(x) \leq 2Tf(x)$, for every $x \in \mathbb{R}^d$. Otherwise, $\mathcal{M}f(x)$ is not the supremum over all such $\lambda$. Let

$$\sigma_\varepsilon(y) = \frac{1}{\varepsilon} 1_{\{y:||y|-\lambda(x)|\leq\varepsilon\}}(y)$$

and

$$T_\varepsilon f(x) = \int_{\mathbb{R}^d} f(x - \lambda(x)y) d\sigma_\varepsilon(y).$$

It follows from the way we define the surface measure d$\sigma$ that

$$||T_\varepsilon f - Tf||_2 \to 0 \text{ as } \varepsilon \to 0. \tag{6.2}$$

Thus, it suffuces to prove the following claim

**Claim 6.1.1.** *Let $f$ be a Schwartz function on $\mathbb{R}^d$. Then*

$$||T_\varepsilon f||_2 \lesssim ||f||_2,$$

*uniformly in $\varepsilon$.*

It then follows from Claim 6.1.1 and (6.2) that

$$||\mathcal{M}f||_2 \lesssim ||Tf||_2 \leq ||T_\varepsilon f||_2 + ||T_\varepsilon f - Tf||_2 \lesssim ||f||_2 + ||T_\varepsilon f - Tf||_2.$$

If we let $\varepsilon \to 0$, we get that

$$||\mathcal{M}f||_2 \lesssim ||f||_2,$$

as desired.

## 6.2 Sketch of Proof of Main Estimate

To prove Claim 6.1.1, it suffices to show

$$||T_\varepsilon T_\varepsilon^* f||_2 \lesssim ||f||_2, \tag{6.3}$$

where $T_\varepsilon^*$ is the dual operator of $T_\varepsilon$.

We have $T_\varepsilon f(x) = \int_{\mathbb{R}^d} f(x - t(x)y) d\sigma_\varepsilon(y)$. A change of variables gives

$$T_\varepsilon f(x) = \int_{\mathbb{R}^d} f(z)\sigma_\varepsilon \left( \frac{x-z}{\lambda(x)} \right) dz. \tag{6.4}$$

Let $K_\varepsilon(x,z) = \sigma_\varepsilon \left( \frac{x-z}{\lambda(x)} \right)$. Then,

$$T_\varepsilon^* f(x) = \int K_\varepsilon(z,x) f(z) dz. \tag{6.5}$$

It follows that

$$T_\varepsilon T_\varepsilon^* f(x) = \int K_\varepsilon(x,z) \int K_\varepsilon K(y,z) f(y) dy dz = \int \left( \int K_\varepsilon(x,z) K_\varepsilon(y,z) dz \right) f(y) dy. \tag{6.6}$$

Write

$$L_\varepsilon(x,y) = \int K_\varepsilon(x,z) K_\varepsilon(y,z) dz. \tag{6.7}$$

We use the following lemma of Schur (see [7]).

**Lemma 6.2.1.** *(Schur's Test) Let $\varepsilon > 0$. Let $L_\varepsilon(x,y)$ be as above so that*

$$\int_{\mathbb{R}^d} |L_\varepsilon(x,y)| dx \leq A \textit{ for each } y,$$

*and*

$$\int_{\mathbb{R}^d} |L_\varepsilon(x,y)| d(y) \leq B \textit{ for each } x.$$

*Then for $f \in L^2(\mathbb{R}^d)$ the integral defining $T_\varepsilon f$ converges for almost every $x$ and there is an estimate*

$$||T_\varepsilon f||_2 \leq \sqrt{AB}||f||_2.$$

Thus, in order to use Schur's Test, we must show

$$\int_{\mathbb{R}^d} |L_\varepsilon(x,y)|dx, \int_{\mathbb{R}^d} |L_\varepsilon(x,y)|dy$$

bounded uniformly in $y$ and $x$ respectively. We can assume that $|x-y| \leq 1$. Notice that

$$L_\varepsilon(x,y) = \frac{1}{\varepsilon^2} m(\{z : \big||z-x| - \lambda(x)\big| \leq \varepsilon, \big||z-y| - \lambda(y)\big| \leq \varepsilon\}).$$

There are two extreme cases in which these "thickened spheres" can intersect which we consider. First, these spheres can intersect transversally. In this case, it is easy to see that the measure of their intersection is bounded above by $C\varepsilon^2$.

Second, consider, for the moment, that we are in two dimensions and consider the case where the thickened circles are tangent to one another in the sense that the circle centered at $y$ sits inside the circle centered at $x$. Then, we can certainly write

$$\lambda_x - \lambda_y \gtrsim |x-y|. \tag{6.8}$$

Near the point of tangency, these circles look like parabolas $g_x(r) = \frac{r^2}{\lambda_x}$ and $g_y(r) = \frac{r^2}{\lambda_y}$, where $r \in \mathbb{R}$. Since we are thickening these circles by $\varepsilon$, then it is clear that

$$\left|\frac{r^2}{\lambda_y} - \frac{r^2}{\lambda_x}\right| \lesssim \varepsilon \tag{6.9}$$

and thus it follows from (6.8) that

$$r \lesssim \left(\frac{\varepsilon}{|x-y|}\right)^{1/2}. \tag{6.10}$$

It follows then that the measure of the intersection is bounded above by

$$C \cdot \varepsilon \cdot \left( \frac{\varepsilon}{|x-y|} \right)^{\frac{1}{2}} = \frac{\varepsilon^{\frac{3}{2}}}{|x-y|^{\frac{1}{2}}}. \tag{6.11}$$

Now, consider rotating these parabolas around the vertical $\mathbb{R}$-axis to get paraboloids in three dimensions. When we do this, we are adding another direction of length

$$\left( \frac{\varepsilon}{|x-y|} \right)^{\frac{1}{2}}. \tag{6.12}$$

Thus, to get the volume of the intersection, we multiply (6.11) and (6.12) so that

$$C \cdot \left( \frac{\varepsilon}{|x-y|} \right)^{\frac{3-1}{2}} \cdot \varepsilon = \frac{\varepsilon^{\frac{3+1}{2}}}{|x-y|^{\frac{3-1}{2}}} \tag{6.13}$$

We can continue this process so that for any $d \geq 3$, the measure of the intersection of these thickened spheres is bounded above by

$$C \cdot \left( \frac{\varepsilon}{|x-y|} \right)^{\frac{d-1}{2}} \cdot \varepsilon = \frac{\varepsilon^{\frac{d+1}{2}}}{|x-y|^{\frac{d-1}{2}}}. \tag{6.14}$$

Now, we can write

$$\int L_\varepsilon(x,y)dy \lesssim \int_{|x-y|\leq 1} \frac{1}{\varepsilon^2} \left( \varepsilon^2 + \frac{\varepsilon^{\frac{d+1}{2}}}{|x-y|^{\frac{d-1}{2}}} \right) dy \lesssim \int_{|y|\leq 1} \frac{1}{|y|^{\frac{d-1}{2}}} dy \leq \int_0^1 r^{\frac{d-1}{2}} dr \leq 1, \tag{6.15}$$

for every $x \in \mathbb{R}^d$, as desired. Similarly, $\int L_\varepsilon(x,y)dx \leq 1$, for every $y \in \mathbb{R}^d$. Note the importance that $d$ is at least 3 above. If $d = 2$, then $\frac{\varepsilon^{\frac{d+1}{2}}}{\varepsilon^2}$ is large so that the second inequality above does not hold. However, when $d \geq 3$, we can certainly say $\frac{\varepsilon^{\frac{d+1}{2}}}{\varepsilon^2} < 1$. Since (6.15) holds, it then follows from Schur's Test, Lemma 6.2.1, that $T_\varepsilon T_\varepsilon^* f(x)$ is a bounded operator from $L^2$ to $L^2$, as desired. $\square$

# Bibliography


[1] J. Bourgain, *A Szemerédi type theorem for sets of positive density in* $\mathbb{R}^k$, Israel J. Math. 54 (1986), no. 3, 307–316.

[2] J. Bourgain, *Averages in the Plane Over Convex Curves and Maximal Operators*, Journal D'Analyse Mathématique, Vol. 47 (1986).

[3] H. Furstenberg, Y. Katznelson and B. Weiss, *Ergodic theory and configurations in sets of positive density*, Mathematics of Ramsey Theory, 184-198, Algorithms Combin., 5, Springer, Berlin, 1990.

[4] L. Grafakos, *Classical Fourier Analysis*, Graduate Text in Mathematics, Volume 249, 2008.

[5] E. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ., 1993.

[6] L. Huckaba, N. Lyall, A. Magyar, *Simplices and Sets of Positive Upper Density in* $\mathbb{R}^d$, arXiv:1509.09283v2 [math.CA]

[7] T. Wolff, *Lectures on Harmonic Analysis*, University Lecture Series, Volume 29, 2003.