METHODOLOGICAL ISSUES IN PRODUCTION EFFICIENCY MEASUREMENTS

by

SYED WIQAR HUSSAIN

(Under the Direction of Professor C.A. Knox Lovell)

ABSTRACT

Färe and Lovell (Journal of Economic Theory; 19(1978), 150-162) have introduced a criterion which an index should satisfy in order to serve as an efficiency index. The radial measure of efficiency proposed by Farrell (*Journal of the Royal Statistical* Society 120(3) (1957), 253-290) and which is commonly used to gauge and compare the technical efficiency across production units fails to satisfy most of the conditions outlined in this criterion. As an alternate non-radial efficiency measures have been suggested. Studies have shown that these non-radial measures of technical efficiency have weaknesses of their own.

This study proposes a new non-radial measure of technical efficiency and shows that this new measure may satisfy the required criteria to a greater extent as compared to the existing technical efficiency measures. The new measure which we call the Non-radial Farrell measure of technical efficiency is constructed as a ratio of the Euclidean norm of the efficient data point to the Euclidean norm of the observed data point in a transformed data space for both input and output orientations. We provide a comparison of this newly proposed measure of technical efficiency with some of the existing technical efficiency measures and are able to show that this new measure is a generalization of the existing measures. The discussion is extended to the concepts of *cost efficiency, Revenue Efficiency* and the decomposition of cost and revenue efficiencies into technical efficiency and *allocative efficiency*. In order to support our inferences we provide numerical examples based on hypothetical data sets.

INDEX WORDS: DEA, Radial measure of efficiency, technical efficiency, non radial measure of efficiency, Euclidean norm, cost efficiency, revenue efficiency, allocative efficiency.

METHODOLOGICAL ISSUES IN PRODUCTION EFFICIENCY MEASUREMENTS

by

SYED WIQAR HUSSAIN

BSc., University of Karachi, Pakistan, 1985

MSc., University of Karachi, Pakistan, 1987

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2004

METHODOLOGICAL ISSUES IN PRODUCTION EFFICIENCY MEASUREMENTS

by

SYED WIQAR HUSSAIN

Major Professor: C. A. Knox Lovell

Committee: Christopher Cornwell
William D. Lastrapes

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2004

DEDICATION

I dedicate this work to Wali-al-Asr (A.S) for his continuous support, to my parents for their continuous prayers and to all of my teachers whose efforts have given me the knowledge to reach this end.

TABLE OF CONTENTS

Page

CHAPTER 1

INTRODUCTION AND SETTING

1. 1 Introduction

By assuming optimizing behavior microeconomics while addressing the theory of the firm, rules out the presence of any inefficiency. The real world however is not as theoretically smooth and a host of factors may leave different firms performing with varying degrees of efficiency. Differences in managerial decision making abilities, differences in organizational structures, differences in operating environment, spatial and/or regulatory, and like, define factors that affect firm's performance. Some of these factors are well under firm's control while others are not. An efficiency comparison of different performing units, made after taking into account such factors, can be used to gauge the degree of deviation of observed performance from a reference potential performance. That there is associated an opportunity cost, both social and private, with inefficient performance is a strong enough reason in itself that may motivate such comparisons. The sources of inefficiencies can be uncovered and the ways of removing these inefficiencies can be planned.

In general a meaningful comparison of any two states of affairs requires a suitable criterion on the basis of which such comparison can be made. Same is true for any productive efficiency analysis; we need to know the meaning of efficiency and we need to know how to measure it. In defining efficiency of a production unit, one can think in terms of maximum feasible output that can be produced with a given input mix. Conversely efficiency can be defined as producing a given output vector with minimum input usage. In either of these senses efficiency is referred to as *technical efficiency*, Koopmans (1951). In comparison to technical efficiency is the concept of *allocative/price efficiency* which requires the input and output mixes to be cost minimizing and/or revenue/profit maximizing respectively, given the market opportunity set. The technical and allocative efficiency together define *cost efficiency/revenue efficiency*, Debreu(1951), Farrell(1957), Fried, Lovell and Schmidt(1993).

Debreu(1951) and Farrell(1957) provided a computationally possible measure of efficiency. They defined inefficiency as the maximum possible equiproportionate reduction in all inputs while maintaining the given output vector. The Debreu-Farrell *input oriented* efficiency score of a production unit is then one minus the Debreu-Farrell inefficiency. The *output oriented* efficiency score is similarly defined in terms of the maximum possible equiproportionate expansion in all outputs for the given input vector.

In practice one can employ econometric approach to obtain the Debreu-Farrell efficiency scores. Alternatively, mathematical programming techniques can be used for the purpose. The econometric technique requires *estimation* of an efficient frontier from the given data set and computes the efficiency scores on the basis of deviations from the estimated frontier. This frontier may be *deterministic*, Afriat(1972), Richmond(1974) and Greene(1980), or it may be *stochastic*, Aigner, Lovell & Schmidt(1977) and Lovell & Schmidt(1979) and Lovell & Schmidt (1980). The mathematical programming approach, Farrell(1957) and Charnes, Cooper & Rhodes(1978), works through a series of linear programs to construct an efficient frontier and then measures the efficiency scores in terms of radial deviations from the constructed frontier. The technique has become popular with the name of Data Envelopment Analysis, Charnes, Cooper & Rhodes(1981).

Both econometric estimation and mathematical programming techniques, and their extensions and variants, have been used in a number of ways and in a number of applications. Both of them have some advantages and some drawbacks over each other. The econometric approach has the great advantage of isolating efficiency from statistical noise, an attribute lack of which marks a major drawback of mathematical programming models. On the other hand the, mathematical programming methods have the advantage of requiring lesser number of prior assumptions; most importantly it does not make any parametric assumption about the underlying technology of the production process. Which approach to employ, depends much upon the objectives of the study and sometimes upon the conditions under which the study is being conducted.

For the computation of efficiency scores the mathematical programming approach adopts the method of radial adjustment of inputs or/and outputs vectors. The scores thus obtained lack some important properties, Färe and Lovell(1978), which leave the conclusions of the analysis a little dubious. Most importantly, by including into the set of efficient data points some of the inefficient elements, the mathematical programming approach fails to neatly partition the data set into the subsets of efficient and inefficient data points. A natural extension of the research in efficiency analysis has thus been the development of an alternate, radial/non radial measure that is free from such weaknesses. Several such measures have been proposed Färe and Lovell(1978), Zieschang(1984) and Lovell, Pastor and Turner (1995), for example. All of them have weaknesses of their own.

This study carries the major objective of proposing a new non radial measure of technical efficiency which overcomes some of the weaknesses of the existing measures. We call this measure the Non-radial Farrell measure of technical efficiency due to its close resemblance to the Farrell radial technical efficiency measure. Below in this chapter we first define and discuss the technological environment in which a firm is expected to operate. Then we outline the criteria which any index is expected to fulfill to serve as an appropriate efficiency index.

In chapter 2 we provide an account of the input oriented measures of efficiency. A number of input oriented technical efficiency measures have been proposed. We use the radial input oriented measure of technical efficiency proposed by Farrell(1957), Russell non-radial measure of technical efficiency proposed by Färe and Lovell(1978) and Russell extended Farrell measure of efficiency proposed by Zieschang(1984) along with the newly proposed Non-radial Farrell measure for the purpose of comparison of desirable properties across measures. Chapter 3 extends the input oriented discussion of chapter 2 to the output oriented efficiency measures. Finally chapter 4 provides a brief summary and some possible avenues for future research in the area.

1.2 The Technological Environment

Technological environment can be defined in a number of ways. Detail discussions on the representation of technology are available in Färe and Grosskopf and Lovell(1993), Färe and Grosskopf(1994) and Färe and Primont(1995). Here we give only a brief outline necessary for our discussion.

Consider a firm that uses inputs $\boldsymbol{x} \equiv (x_1, x_2, \ldots x_N) \in \Re^N_+$ to produce outputs $\boldsymbol{y} \equiv (y_1, y_2, \ldots y_M) \in \Re^M_+$. A production technology can be characterized by an input correspondence $L: \Re^M_+ \rightarrow L(\boldsymbol{y}) \subseteq \Re^N_+$ which maps output vectors $\boldsymbol{y} \in \Re^M_+$ into subsets of input vectors, $\boldsymbol{x} \in \Re^N_+$, with $L(\boldsymbol{y})$ as the set of all input vectors that can produce $\boldsymbol{y}$. It is assumed that the technology satisfies the following[1].

L1. $L(\mathbf{0}) = \Re^N_+$, $\mathbf{0} \notin L(\boldsymbol{y})$ for $\boldsymbol{y} \geq \mathbf{0}$ where $\mathbf{0}$ is M×1 zero vector.

L2. $\boldsymbol{x} \in L(\boldsymbol{y}) \Rightarrow \nu\boldsymbol{x} \in L(\boldsymbol{y})\ \forall\ \nu \in (1, +\infty)$

---

[1] $\boldsymbol{x} \geqq \boldsymbol{x}^*$ if $x_i \geq x^*_i\ \forall$ i, $\boldsymbol{x} \geq \boldsymbol{x}^*$ if $x_i \geq x^*_i\ \forall$ i & $\boldsymbol{x} \neq \boldsymbol{x}^*$; $\boldsymbol{x} > \boldsymbol{x}^*$ if $x_i > x^*_i\ \forall$ i

L3. $L(\hat{\boldsymbol{y}}) \subseteq L(\boldsymbol{y})\ \forall\ \hat{\boldsymbol{y}} \geqq \boldsymbol{y}$

L4. L is a closed correspondence.

L5. $L(\boldsymbol{y})$ is a convex correspondence for all $\boldsymbol{y} \in \Re^{M}_{+}$

The first assumption states that any nonnegative input vector can be used to produce at least zero output and that positive output can not be produced with a zero input vector. Assumption 2 specifies weak disposability of inputs the stronger version of which is

L2s. $\boldsymbol{x} \in L(\boldsymbol{y}) \Rightarrow$ if $\boldsymbol{x}^* \geq \boldsymbol{x}$ then $\boldsymbol{x}^* \in L(\boldsymbol{y})$

The strong disposability of inputs allow for the expansion of inputs without reducing the output. In contrast, with weak disposability, output may decrease when an input is applied beyond certain limits. It thus models the case in which the marginal product of an input may become negative and is appropriate when the input use is subject to congestion. Condition 3 is the statement of the fact that if an input vector can produce $\hat{\boldsymbol{y}}$ then it can also produce $\boldsymbol{y}$ for $\hat{\boldsymbol{y}} \geq \boldsymbol{y}$. It thus reflects strong disposability in output. A weaker version of this condition is given as

L3w. $L(\nu\boldsymbol{y}) \subseteq L(\boldsymbol{y})\ \forall\ \nu \in [1, +\infty)$

L4 states that if $\boldsymbol{y}^{q} \rightarrow \boldsymbol{y}^{\circ}, \boldsymbol{x}^{q} \rightarrow \boldsymbol{x}^{\circ}$ and $\boldsymbol{x}^{q} \in L(\boldsymbol{y}^{q})\ \forall$ q, then $\boldsymbol{x}^{\circ} \in L(\boldsymbol{y}^{\circ})$. This assumption thus is needed to ensure the existence of a frontier of the technology. Assumption L1 to L4 allow us to define an isoquant of $\boldsymbol{y}$ as

$IsoqL(\boldsymbol{y}) = \{\boldsymbol{x} | \boldsymbol{x} \in L(\boldsymbol{y}) \text{ and } \lambda\boldsymbol{x} \notin L(\boldsymbol{y}), \lambda \in [0,1)\}$

The efficient subset of isoquant is described as

$EffL(\boldsymbol{y}) = \{\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y}) \text{ and } \boldsymbol{x}^* \leq \boldsymbol{x} \Rightarrow \boldsymbol{x}^* \notin L(\boldsymbol{y})\}$

In terms of figure (1.1) the curve represents the *IsoqL*(***y***).The curve and the entire area above the curve together define *L*(***y***). Finally, the segment of the curve bounded by points ***x*** and ***x**** forms the efficient subset *EffL*(***y***) of the isoquant.

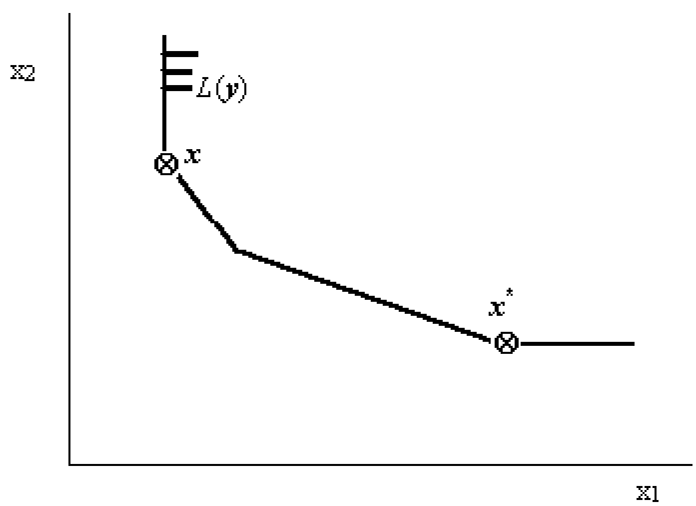

Figure 1.1. The input set

L5 states that for $\boldsymbol{x} \in L(\boldsymbol{y})$ and $\boldsymbol{x}^* \in L(\boldsymbol{y}) \Rightarrow [\alpha\boldsymbol{x} + (1 - \alpha)\boldsymbol{x}^*] \in L(\boldsymbol{y})$. This assumption is needed to define the piecewise linear technology for the purpose of computation of efficiency scores. With respect to *EffL*(***y***) we impose the following restriction on the technology;

L6. *EffL*(***y***) is bounded

This restriction is imposed to ensure that the unbounded input use for any output rate is regarded as inefficient. The above technology also has an output correspondence representation given as

P: $\Re^N_+ \rightarrow P(\boldsymbol{x}) \subseteq \Re^M_+$. The correspondence maps input vectors into the subsets of output vectors and is assumed to satisfy the following;

P1. $\boldsymbol{y} \notin P(\boldsymbol{0})$ for $\boldsymbol{y} \in \Re^M_+$, $\boldsymbol{0} \in P(\boldsymbol{x})$ for $\boldsymbol{x} \in \Re^N_+$.

P2. $\boldsymbol{y} \in P(\boldsymbol{x}) \Rightarrow \theta\boldsymbol{y} \in P(\boldsymbol{x})$ for $\theta \in [0, 1]$

P3. $P(\boldsymbol{x}) \subseteq P(\boldsymbol{x}^*)\ \forall\ \boldsymbol{x} \leqq \boldsymbol{x}^*$

P4. $P(\boldsymbol{x})$ is a closed correspondence.

P5. $P(\boldsymbol{x})$ is a convex correspondence for all $\boldsymbol{x} \in \Re^N_+$

P1 states that the zero input vector can not produce any output and that the possibility of inaction exists. P2 describes weak disposability in outputs. The stronger version of P2 is given as:

P2s. $\boldsymbol{y} \in P(\boldsymbol{x}) \Rightarrow \boldsymbol{y}^* \in P(\boldsymbol{x})\ \forall\ \boldsymbol{y}^* \leq \boldsymbol{y}$

The weak disposability of output thus allows for the output vector $\boldsymbol{y}$ to be proportionately contracted and still remain feasible. In contrast, the strong disposability of output allows for output to be freely disposed and still be feasible. P3 is the strong input disposability defined in terms of output correspondence. The weaker version of P3 is given as

P3w. $P(\boldsymbol{x}) \subseteq P(\delta\boldsymbol{x})\ \forall\ \delta \in [1, +\infty)$ and $\boldsymbol{x} \in \Re^N_+$

P4 states that if $\boldsymbol{x}^q \rightarrow \boldsymbol{x}^\circ$, $\boldsymbol{y}^q \rightarrow \boldsymbol{y}^\circ$ and $\boldsymbol{y}^q \in P(\boldsymbol{x}^q)\ \forall$ q, then $\boldsymbol{y}^\circ \in P(\boldsymbol{x}^\circ)$. This assumption is needed to ensure the existence of a frontier of the technology. Thus we can write

$P(\boldsymbol{x}) = \{\ \boldsymbol{y} | \boldsymbol{x}$ can produce $\boldsymbol{y}\}$

$IsoqP(\boldsymbol{x}) = \{\boldsymbol{y} | \boldsymbol{y} \in P(\boldsymbol{x})$ and $\delta\boldsymbol{y} \notin P(\boldsymbol{x}), \delta \in [1,+\infty)\ \}$

$EffP(\boldsymbol{x}) = \{\boldsymbol{y} | \boldsymbol{y} \in P(\boldsymbol{x})\ , \hat{\boldsymbol{y}} \geq \boldsymbol{y} \Rightarrow \hat{\boldsymbol{y}} \notin P(\boldsymbol{x})\}$

where $IsoqP(\boldsymbol{x})$ is the production isoquant and $EffP(\boldsymbol{x})$ is its efficient subset.

P5 states that for $\boldsymbol{y} \in P(\boldsymbol{x})$ and $\boldsymbol{y}^* \in P(\boldsymbol{x})$ and for $\theta \in [0,1]$, $[\theta \boldsymbol{y} - (1-\theta)\boldsymbol{y}^*] \in P(\boldsymbol{x})$.Thus for any pair of the output vectors that belong to $P(\boldsymbol{x})$, their convex combination also belongs to $P(\boldsymbol{x})$. Like in the case of the input correspondence this assumption is needed to accommodate the linear piecewise technology.

Finally since only finite rates of outputs can be produced with finite inputs, we have;

P6. $P(\boldsymbol{x})$ is bonded for $\boldsymbol{x} \in \Re^N_+$.

In terms of figure (1.2) the curve represents the $IsoqP(\boldsymbol{x})$.The curve and the entire area under the curve together define $P(\boldsymbol{x})$. Finally, the segment of the curve bounded by the points $\boldsymbol{y}$ and $\boldsymbol{y}^*$ forms the efficient subset $EffP(\boldsymbol{x})$ of the isoquant.

It is useful in many ways to characterize the above technology in terms of distance function, Shephard(1970). The input distance function is defined as follows;

Definition: The function $\psi_I : \Re^M_+ \times \Re^N_+ \rightarrow \Re_+ \cup \{+\infty\}$ defined by

$\psi_I(\boldsymbol{y}, \boldsymbol{x}) = \text{SUP}_\lambda \{\lambda \mid \boldsymbol{x}/\lambda \in L(\boldsymbol{y}), \lambda \in (1,+\infty)\}$ is called the Shephard input distance function.

In words the Shephard input distance function accounts for maximum proportional contraction to any input vector $\boldsymbol{x} \in L(\boldsymbol{y})$ without any loss of output. For $\psi_I(\boldsymbol{y}, \boldsymbol{x}) = 1$, $\boldsymbol{x} \in IsoqL(\boldsymbol{y})$ and for $\psi_I(\boldsymbol{y}, \boldsymbol{x}) > 1$, $\boldsymbol{x} \in L(\boldsymbol{y}) \setminus IsoqL(\boldsymbol{y})$. In terms of distance function we can write

$L(\boldsymbol{y}) = \{\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y}), \psi_I(\boldsymbol{y}, \boldsymbol{x}) \geq 1\}$

$IsoqL(\boldsymbol{y}) = \{\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y}) \text{ and } \psi_I(\boldsymbol{y}, \boldsymbol{x}) = 1\}$

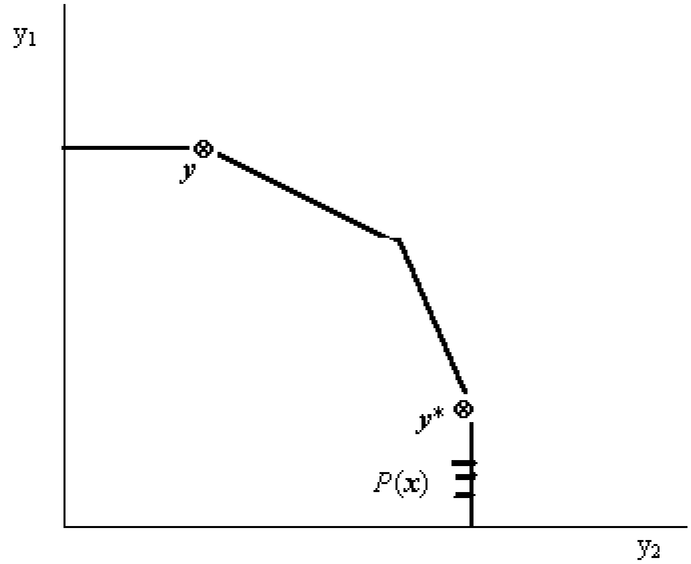

Figure 1.2. The output set

The output distance function is defined as

Definition: The function ψ_O: $\Re^N_+ \times \Re^M_+ \rightarrow \Re_+ \cup \{+\infty\}$ defined by $\psi_O(\boldsymbol{x},\boldsymbol{y})=\text{INF}_\delta\{\delta|\boldsymbol{y}/\delta \in P(\boldsymbol{x}), \delta \in (0,1]\}$ is called the Shephard output distance function.

In words the Shephard output distance function accounts for maximum proportional expansion of any output vector $\boldsymbol{y} \in P(\boldsymbol{x})$ possible with the given input vector $\boldsymbol{x} \in \Re^N_+$. For $\psi_O(\boldsymbol{x}, \boldsymbol{y}) = 1$, $\boldsymbol{y} \in P(\boldsymbol{x})$ and for $\psi_O(\boldsymbol{x}, \boldsymbol{y}) < 1$, $\boldsymbol{y} \in P(\boldsymbol{x}) \backslash IsoqP(\boldsymbol{x})$. We can thus write

$P(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x}) \text{ and } \psi_O(\boldsymbol{x}, \boldsymbol{y}) \leq 1\}$

$IsoqP(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x}) \text{ and } \psi_O(\boldsymbol{x}, \boldsymbol{y}) = 1\}$.

Besides the Shephard distance function, the directional distance function, Chambers, Chung and Färe(1998), is also sometimes used to define the functional representation of the technology.

Consider for example the case where the production process involves production of undesirable output along with the desirable output such that the desirable outputs are nulljoint with the undesirable outputs. If $\boldsymbol{y}^{U}$ represents the sub-vector for the undesirable output and $\boldsymbol{y}^{D}$ represents the sub-vector for the desirable output then

Definition: The function $\psi^{*} : \Re^{N}_{+} \times \Re^{M}_{+} \rightarrow \Re_{+} \cup \{+\infty\}$ defined by

$\psi^{*}(\boldsymbol{x}, \boldsymbol{y}^{D}, \boldsymbol{y}^{U}; \boldsymbol{g}) = SUP_{\alpha} \{\alpha \mid (\boldsymbol{y}^{D}, \boldsymbol{y}^{U}) + \alpha\boldsymbol{g} \in P(\boldsymbol{x}) \}$, where $\boldsymbol{g}$ is a pre-assigned direction vector, is called the directional output distance function.

In words the directional output distance function allows for the simultaneous variations in $\boldsymbol{y}^{U}$ and $\boldsymbol{y}^{D}$ i.e. increase in $\boldsymbol{y}^{D}$ accompanied by a simultaneous decrease in $\boldsymbol{y}^{U}$ or visa versa. Similarly we can define the directional distance function for the graph of the technology where it is needed to model the simultaneous variations in the outputs and the inputs.

Since both these topics, production with undesirable output sub-vector and graph efficiency measures, are out of the scope of this study we make use of the Shephard distance function only in the discussion that follows.

1.3 Desirable Properties of Efficiency Measures

Like any other ranking criterion, the measures of productive efficiency have to satisfy certain properties for consistent empirics. Below we give a formal definition of an efficiency index and list the desirable properties, as outlined by Färe and Lovell(1978), Russell(1988) and Färe, Grosskopf, and Lovell(1993), that such an index should possess.

Definition: A mapping $E_{I}(\boldsymbol{y},\boldsymbol{x})$: $\mathbb{R}^{M}_{+}\times \mathbb{R}^{N}_{+}\rightarrow\Re_{+}\cup\{+\infty\}$, where $\mathbb{R}^{q}_{+}=\Re^{q}_{+}/\{0\}$, with $E(\boldsymbol{y},\boldsymbol{x})\in(0,1]$, *iff* $\boldsymbol{x} \in L(\boldsymbol{y})$ and $E_{I}(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x}\notin L(\boldsymbol{y})$ is an input oriented efficiency index if it satisfies properties E1a to E6a as listed below.

E1a. $E_I(\boldsymbol{y}, \boldsymbol{x}) = 1 \Leftrightarrow \boldsymbol{x} \in EffL(\boldsymbol{y})\ \forall\ \boldsymbol{x} \in L(\boldsymbol{y})$ i.e. identifying a data point as efficient *iff* it is efficient in Koopman's sense.

E2a. If $\boldsymbol{x} \in L(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$, then $\boldsymbol{x}$ should be compared with $\boldsymbol{x}^* \in EffL(\boldsymbol{y})$.

E3a. $E_I(\boldsymbol{y}, \beta\boldsymbol{x}) = \beta^{-1} E_I(\boldsymbol{y}, \boldsymbol{x})$ for $\boldsymbol{x} \in L(\boldsymbol{y})$, $\beta > 0$ i.e. the measure should be homogenous of degree minus one in inputs.

E4a. $E_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu E_I(\boldsymbol{y}, \boldsymbol{x})$ for $\boldsymbol{x} \in L(\boldsymbol{y})$, $\nu > 0$ i.e. the measure should be homogenous of degree plus one in outputs.

E5a. $E_I(\boldsymbol{y}, \boldsymbol{x}) > E_I(\boldsymbol{y}, \boldsymbol{x}^*)$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$ and for $\boldsymbol{x} \in L(\boldsymbol{y})$ i.e. for a given output vector, the efficiency score should decrease monotonically with the increase in input use.

E6a. If $\boldsymbol{y}^* = \Omega\boldsymbol{y}$ and $\boldsymbol{x}^* = \Lambda\boldsymbol{x}$ where Ω and Λ are respectively M×M and N×N positive diagonal matrices and $L^*(\boldsymbol{y}^*) = \{\boldsymbol{x}^* \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$, then. $E_I(\boldsymbol{y},\boldsymbol{x}) = E_I(\boldsymbol{y}^*, \boldsymbol{x}^*)$ i.e. the efficiency measure is independent of units in which inputs and outputs are measured, a property defined as "commensurability" by Russell(1988).

Property E1a requires that only those data points that belong to the efficient subset of an isoquant be assigned an efficiency score equal to one. Thus for any $\boldsymbol{x} \in IsoqL(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$ the input oriented efficiency score should be less than one. Property E2a requires that $EffL(\boldsymbol{y})$ should serve as the set of reference for gauging the inefficiency of off the frontier data points. E3a, the homogeneity of degree minus one has a weaker version, the sub-homogeneity, given as

E.3aw. $E_I(\boldsymbol{y}, \beta\boldsymbol{x}) \gtreqless \beta^{-1} E_I(\boldsymbol{y}, \boldsymbol{x})$ for $\boldsymbol{x} \in L(\boldsymbol{y})$, $\beta \lesseqgtr 1$

Property E4a the homogeneity of plus one in output is likely to hold only for constant returns to scale. For non constant returns to scale, we show in chapter 2 that it is subject to violation by any measure of technical efficiency, radial or non-radial. The form of this property that is in line with the returns to scale may be written as

E4aw. $E_I(\nu \boldsymbol{y}, \boldsymbol{x}) \gtreqless \nu E_I(\boldsymbol{y}, \boldsymbol{x})$ for $\boldsymbol{x} \in L(\boldsymbol{y})$, $\nu > 0$ and for $0 < \gamma \lesseqgtr 1$

where γ is the returns to scale parameter.

Similarly a weaker version of monotonicity is given as

E.5aw. $E_I(\boldsymbol{y}, \boldsymbol{x}) \geq E_I(\boldsymbol{y}, \boldsymbol{x}^*)$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$ and for $\boldsymbol{x} \in L(\boldsymbol{y})$

The output oriented measure of efficiency is defined as under:

Definition: A mapping $E_O(\boldsymbol{x}, \boldsymbol{y})$: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, with

$E_O(\boldsymbol{x}, \boldsymbol{y}) \in (0,1]$, *iff* $\boldsymbol{y} \in P(\boldsymbol{x})$ and $E_O(\boldsymbol{x}, \boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P(\boldsymbol{x})$ is an output oriented efficiency index if it satisfies following properties;

E.1b. $E_O(\boldsymbol{x}, \boldsymbol{y}) = 1 \Leftrightarrow \boldsymbol{y} \in EffP(\boldsymbol{x})\ \forall\ \boldsymbol{y} \in P(\boldsymbol{x})$ i.e. identifying a data point as efficient *iff* it is efficient in Koopman's sense.

E.2b.If $\boldsymbol{y} \in P(\boldsymbol{x}) \backslash EffP(\boldsymbol{x})$, then $\boldsymbol{y}$ should be compared with $\boldsymbol{y}^* \in EffP(\boldsymbol{x})$**.**

E.3b. $E_O(\beta \boldsymbol{x}, \boldsymbol{y}) = \beta E_O(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{y} \in P(\boldsymbol{x})$, $\beta > 0$ i.e. the measure should be homogenous of degree plus one in inputs.

E.4b. $E_O(\boldsymbol{x}, \nu \boldsymbol{y}) = \nu^{-1} E_O(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{y} \in P(\boldsymbol{x})$ i.e. the measure should be homogenous of degree minus one in outputs.

E.5b. $E_O(\boldsymbol{x}, \boldsymbol{y}) > E_O(\boldsymbol{x}, \boldsymbol{y}^*)$ for $\boldsymbol{y}^* \leq \boldsymbol{y}$ and for $\boldsymbol{y} \in P(\boldsymbol{x})$ i.e. efficiency should increase monotonically with the increase in output for a given input vector.

E.6b. If $\boldsymbol{y}^{*} = \boldsymbol{\Omega y}$ and $\boldsymbol{x}^{*} = \boldsymbol{\Lambda x}$ where Ω and Λ are respectively M×M and N×N positive diagonal matrices and $P^{\star}(\boldsymbol{x}^{*}) = \{\boldsymbol{y}^{*} \mid \boldsymbol{y} \in P(\boldsymbol{x})\}$, then $E_{O}(\boldsymbol{x}, \boldsymbol{y}) = E_{O}(\boldsymbol{x}^{*}, \boldsymbol{y}^{*})$ i.e. the output oriented technical efficiency scores should be invariant with respect to the changes in the scale of measurement of inputs and outputs.

Like in the case of input oriented measure of technical efficiency the weaker versions of conditions E3b to E5b are given as under ;

E3bw. $E_{O}(\lambda \boldsymbol{x}, \boldsymbol{y}) \gtreqless \nu E_{O}(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{y} \in P(\boldsymbol{x})$, $\lambda > 0$ and for $\gamma \gtreqless 1$

E.4bw. $E_{I}(\boldsymbol{x}, \beta \boldsymbol{y}) \gtreqless \beta^{-1} E_{I}(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{y} \in P(\boldsymbol{x})$, $\beta \lesseqgtr 1$

E.5bw. $E_{O}(\boldsymbol{x}, \boldsymbol{y}) \geq E_{O}(\boldsymbol{x}, \boldsymbol{y}^{*})$ for $\boldsymbol{y}^{*} \leq \boldsymbol{y}$ and for $\boldsymbol{y} \in P(\boldsymbol{x})$

Färe and Lovell(1978) criteria originally included E1a, to E3a and E5a. The weaker versions of these properties and the commensurability property, E6a, originated from later works by Färe, Lovell and Zieschang(1983), Zieschang (1984), Russell(1988) and Färe, Grosskopf and Lovell(1993). Russell(1985) and Russell(1988) investigated these properties with respect to the input oriented Farrell(1957) measure and the non-radial measures proposed by Färe and Lovell(1978) and Zieschang(1984). The study concluded that none of the three measures satisfied E1a to E3a , E5a and E6a entirely and that for any of these measures one or the other condition had to be weakened. Bol(1986) remarked that no efficiency measure, radial or non-radial, could be defined that would satisfy the Färe and Lovell criteria completely.

In this study we introduce a new measure of technical efficiency and compare it with the above mentioned three technical efficiency measures with respect to the criteria E1a to E6a. The new measure is computed after transforming the data set into a format in which non-radial efficiency scores could be computed without violating the commensurability property.

CHAPTER 2

THE INPUT ORIENTED EFFICIENCY MEASURES

The objective of this chapter is to discuss efficiency measures that are constructed on the basis of deviations of an input vector from a set of reference input vectors for a given output vector. Ideally, the set of reference input vectors is the one that qualifies for the Koopmans efficiency conditions .i.e. for any vector in the set it is not possible to reduce the employment of any input and still be able to produce the same level of output without increasing the employment of at least one other input. Any input vector that satisfies this criterion is technically efficient in Koopmans sense. The input vectors that fall short of this criterion are regarded as inefficient. The construction of such input oriented technical efficiency measures require information on input and output quantities.

In comparison to these input oriented technical efficiency measures we have cost efficiency measures that are computed on the basis of behavioral assumption of cost minimization and require information on input prices in addition to the information on input and output quantities. These cost efficiency measures have the great advantage of allowing for the ranking of the trade offs available within the set of technically efficient reference vectors. The cost efficiency measures which are also called the input based overall efficiency measures can be decomposed into input oriented technical and allocative efficiency components and thus provide insight into the sources of inefficiency.

We start in section 2.1 with the Farrell input oriented measure of technical efficiency. We see that the Farrell's input oriented radial measure of technical efficiency fails to satisfy most of the properties mentioned in section 1.3 of the previous chapter. In particular, the fact that the

Farrell's technical efficiency measure uses isoquant, $IsoqL(\boldsymbol{y})$, as the benchmark criterion for identification of technically efficient input vectors and that the $EffL(\boldsymbol{y}) \subseteq IsoqL(\boldsymbol{y})$, reflects the inability of Farrell's input oriented measure of technical efficiency to satisfy Koopmans conditions. In contrast, the existing non-radial technical efficiency measures, for example the Russell's input oriented measure and the Zieschang measure are able to overcome this problem, though they share other weaknesses as we discuss below. Given these weaknesses of various measures it is of interest to broaden our search for an efficiency measure that satisfies the required criteria. We propose a new non-radial measure of technical efficiency and evaluate it on the basis of criteria outlined in section 1.3.

In sections 2.1 and 2.2 we look into the quantity based input oriented radial and non radial technical efficiency measures while maintaining the assumptions that L1, L2s, L3 and L4 hold, along with the constant returns to scale. Section 2.3 discusses the computational aspects for which purpose we make use of the convexity assumption to define the piecewise linear technology. Section 2.4 extends the discussion to the cost efficiency measure and its decomposition into the technical and allocative efficiency components. For this section we also require E6, the boundedness of $EffL(\boldsymbol{y})$. Section 2.5 relaxes the constant returns to scale and strong disposability assumptions. Section 2.6 provides a numerical example to confirm inferences drawn in earlier sections and finally the chapter ends with a summary in section 2.7.

2.1. Radial Efficiency Measures:

The input oriented radial measure of technical efficiency proposed by Farrell (1957) is defined as follows;

Definition 2.1: A function $FE_I(\boldsymbol{y}, \boldsymbol{x})$: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined by

$FE_I(\boldsymbol{y}, \boldsymbol{x}) = \min\ \{\theta_{DF} |$ for $\boldsymbol{x} \in L(\boldsymbol{y}),\ \theta_{DF}\boldsymbol{x} \in L(\boldsymbol{y}), \theta_{DF} \in (0,1]\ \}$ and $FE_I(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L(\boldsymbol{y})$,

is the Farrell's input measure of technical efficiency.

The input oriented technical efficiency score is thus obtained by computing the ratio

$FE_I(\boldsymbol{y}, \boldsymbol{x}) = \| \theta^*_{DF}\, \boldsymbol{x} \| / \| \boldsymbol{x} \|$, $\theta^*_{DF} = \min\{\theta_{DF} |$ for $\boldsymbol{x} \in L(\boldsymbol{y}), \theta_{DF}\boldsymbol{x} \in L(\boldsymbol{y}), \theta_{DF} \in (0,1], \boldsymbol{x} \in \Re^N_+\}$. In words, $FE_I(\boldsymbol{y}, \boldsymbol{x})$ measures the maximum equiproportionate reduction in all inputs that is possible without any reduction in output. $FE_I(\boldsymbol{y}, \boldsymbol{x}) = 1$ implies that the input vector $\boldsymbol{x}$ is technically efficient. A less than one value of $FE_I(\boldsymbol{y}, \boldsymbol{x})$ reflects technical inefficiency with the degree of inefficiency increasing for lower values of $FE_I(\boldsymbol{y}, \boldsymbol{x})$. Note that the Farrell's input measure of technical efficiency is equal to the inverse of the input distance function and thus fully characterizes the underlying technology because of the dual relationship between $L(\boldsymbol{y})$ and $\psi_I(\boldsymbol{y}, \boldsymbol{x})$. For $\psi_I(\boldsymbol{x}, \boldsymbol{y}) \geq 1$ we have $FE_O(\boldsymbol{x}, \boldsymbol{y}) \leq 1$.

In terms of figure (2.1), the input vector $\boldsymbol{x} \in L(\boldsymbol{y}) \backslash IsoqL(\boldsymbol{y})$. It is possible to contract this input vector radially without loosing any output. The input vector $FE_I(\boldsymbol{y}, \boldsymbol{x}) \times \boldsymbol{x} \in L(\boldsymbol{y}) \cap IsoqL(\boldsymbol{y})$, for $FE_I\ (\boldsymbol{y}, \boldsymbol{x}) \in (0, 1]$, defines the feasibility limit to this radial contraction with respect to the data point $(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 2.1: The Farrell input oriented technical efficiency measure satisfies E3a, E4a and E6a but fails to satisfy E1a, E2a and E.5a.

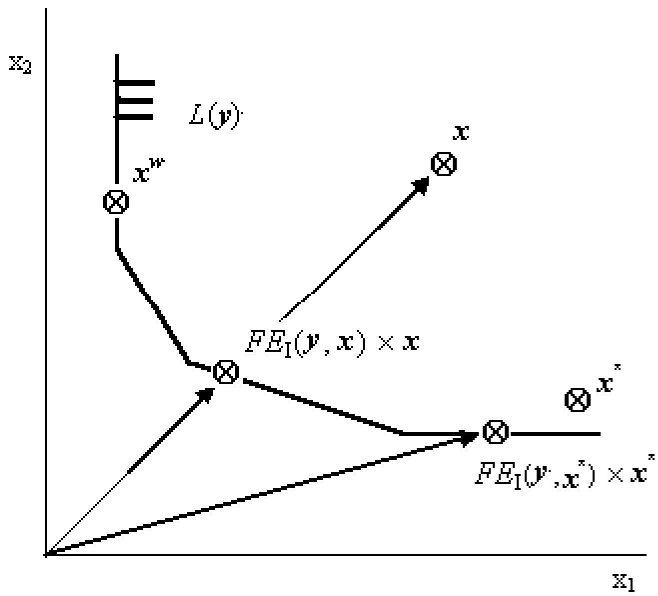

Figure 2.1. The Farrell efficiency measure

Proofs:

$FE_I(\boldsymbol{y}, \boldsymbol{x})$ does not satisfy E1a, the Identification Property.

The input oriented Farrell measure of technical efficiency assigns a value of one to any data point that belongs to $IsoqL(\boldsymbol{y})$. Since $EffL(\boldsymbol{y}) \subseteq IsoqL(\boldsymbol{y})$ and $IsoqL(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$ is not necessarily empty, $FE_I(\boldsymbol{x},\boldsymbol{y})=1$ is not a sufficient condition for Koopmans efficiency. Hence the measure does not satisfy E1a. This is shown in figure (2.1) in which the input vector $\boldsymbol{x}^w \in IsoqL(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$. By definition then we have for this data point

$FE_I(\boldsymbol{y}, \boldsymbol{x}^w) = 1$ thus violating E1a.

$FE_I(\boldsymbol{y}, \boldsymbol{x})$ does not satisfy E.2a, the "Compare to" Property

The off frontier input vector $\boldsymbol{x}^*$ in figure (2.1) which is compared with $\boldsymbol{x}^* \times FE_I(\boldsymbol{y}, \boldsymbol{x}^*)$ for efficiency score computation represents this possibility. The facts that Farrell's input measure of technical efficiency uses $IsoqL(\boldsymbol{y})$ as the frame of reference to assign an efficiency score to any data point $\boldsymbol{x} \in L(\boldsymbol{y}) \backslash IsoqL(\boldsymbol{y})$ and that the set $IsoqL(\boldsymbol{y}) \backslash EffL(\boldsymbol{y})$ is not necessarily empty leads to this situation.

<u>$FE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E.3a, the homogeneity of minus one in inputs</u>.

$FE_I(\boldsymbol{y}, \beta\boldsymbol{x}) = \min \{\theta_{DF} \mid \theta_{DF}(\beta\boldsymbol{x}) \in L(\boldsymbol{y}), \theta_{DF} \in (0, 1], \boldsymbol{x} \in \Re^N_+, \beta > 0\}$

$FE_I(\boldsymbol{y}, \beta\boldsymbol{x}) = \min \{\beta \times \beta^{-1} \theta_{DF} \mid \theta_{DF}(\beta\boldsymbol{x}) \in L(\boldsymbol{y}), \theta_{DF} \in (0, 1], \boldsymbol{x} \in \Re^N_+, \beta > 0\}$

$\Rightarrow FE(\boldsymbol{y}, \beta\boldsymbol{x}) = \beta^{-1} FE_I(\boldsymbol{y}, \boldsymbol{x})$

<u>$FE_I(\boldsymbol{x}, \boldsymbol{y})$ satisfies E.4a, the homogeneity of plus one in outputs</u>.

This follows because we have for constant returns to scale $L(\nu\boldsymbol{y}) = \nu L(\boldsymbol{y})$. Thus

$\Rightarrow FE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \min \{\theta_{DF} \mid \theta_{DF}\boldsymbol{x} \in L(\nu\boldsymbol{y}), \theta_{DF} \in (0, 1], \nu > 0, \boldsymbol{x} \in \Re^N_+ \}$

$\Rightarrow FE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \min \{(\theta_{DF} \mid \theta_{DF}\boldsymbol{x} \in \nu L(\boldsymbol{y}), \theta_{DF} \in (0, 1], \nu > 0, \boldsymbol{x} \in \Re^N_+ \}$

$\Rightarrow FE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu FE_I(\boldsymbol{y}, \boldsymbol{x})$.

<u>$FE_I(\boldsymbol{y}, \boldsymbol{x})$ does not satisfy E5a, the Monotonicity</u>

This can easily shown to be true for any technology for which $IsoqL(\boldsymbol{y}) \backslash EffL(\boldsymbol{y})$ is not empty. Figure (2.2) gives an example of piecewise linear technology where $\boldsymbol{x}/\psi(\boldsymbol{y}, \boldsymbol{x}) \in IsoqL(\boldsymbol{y}) \backslash EffL(\boldsymbol{y})$ and $\boldsymbol{x}^*/\psi_I(\boldsymbol{y}, \boldsymbol{x}^*) \in IsoqL(\boldsymbol{y}) \backslash EffL(\boldsymbol{y})$ with $\boldsymbol{x}/\psi_I(\boldsymbol{y}, \boldsymbol{x}) \geq \boldsymbol{x}^*/\psi_I(\boldsymbol{y}, \boldsymbol{x}^*)$. Violation of monotonicity follows because $FE_I(\boldsymbol{y}, \boldsymbol{x}/\psi_I(\boldsymbol{y}, \boldsymbol{x})) = FE_I(\boldsymbol{y}, \boldsymbol{x}^*/\psi_I(\boldsymbol{y}, \boldsymbol{x}^*)) = 1$. Alternatively, for inefficient input vectors $\boldsymbol{x} \in L(\boldsymbol{y})$ and $\boldsymbol{x}^* \in L(\boldsymbol{y})$ with $\boldsymbol{x} \geq \boldsymbol{x}^*$, we end up with the following

$\|\boldsymbol{x}/\psi_I(\boldsymbol{y}, \boldsymbol{x})\| / \|\boldsymbol{x}\| = \|\boldsymbol{x}^*/\psi(\boldsymbol{y}, \boldsymbol{x}^*)\| / \|\boldsymbol{x}^*\| \Rightarrow FE_I(\boldsymbol{y}, \boldsymbol{x}) = FE_I(\boldsymbol{y}, \boldsymbol{x}^*)$[2].

---

[2] Using the property of similar triangles

$FE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E.6a the commensurability:

This follows directly from the definition of Farrell measure. Alternatively, let Λ be diagonal matrices of dimensions N×N such that $\boldsymbol{x}^* = \Lambda\boldsymbol{x}$

$$FE_I(\boldsymbol{y}, \boldsymbol{x}^*) = \|\theta_{DF}\boldsymbol{x}^*\| / \|\boldsymbol{x}^*\|$$

$$FE_I(\boldsymbol{y}, \boldsymbol{x}^*) = \|\theta_{DF}\Lambda\boldsymbol{x}\| / \|\Lambda\boldsymbol{x}\|$$

$$FE_I(\boldsymbol{y}, \boldsymbol{x}^*) = |\theta_{DF}|\|\Lambda\boldsymbol{x}\| / \|\Lambda\boldsymbol{x}\| = FE_I(\boldsymbol{y}, \boldsymbol{x})$$

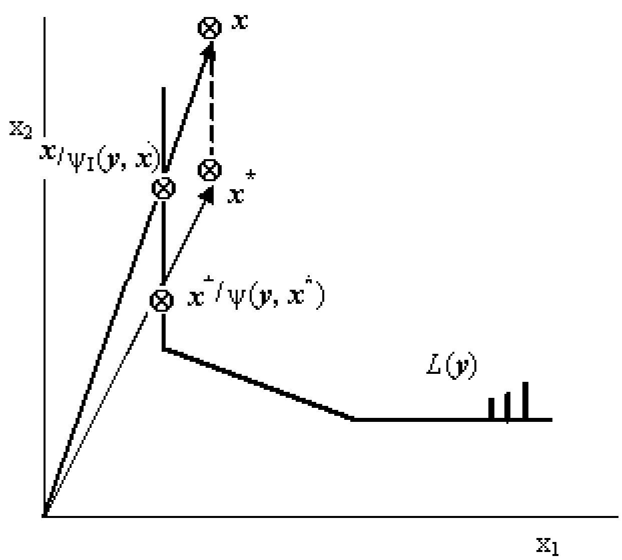

Figure 2.2. $FE_I(\boldsymbol{y}, \boldsymbol{x})$ violates monotonicity

The above discussion suggests that the homogeneity and the commensurability are the only properties that the Farrell's input oriented measure of technical efficiency fulfils for all the technologies. This gives rise to the need for searching for possible alternates which are provided by non-radial measures and which we discuss next.

2.2 Non Radial Technical Efficiency Measures

We begin this section with the Russell measure, Färe and Lovell (1978), of technical efficiency. We show that with the maintained assumptions, L1 to L4 and constant returns to scales, Russell measure satisfies the identification and the 'compare to' property but fails to satisfy strong homogeneity of minus one and strong monotonicity in inputs. Another alternative to the Farrell measure is provided by Zieschang (1984). This measure is an amalgam of Farrell and Russell measures. The measure doesn't satisfy strong monotonicity.

2.2.1 Russell Input oriented Technical Efficiency Measure:

In order to define the Russell input oriented measure of technical efficiency let Q be the set of diagonal matrices ${}^{i}\theta_{FL}$, each of dimension N with ${}^{i}\theta_{nn} \in (0,1]$ for n=1,2,…N, then

Definition: The function RE_I: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as $RE_I(\boldsymbol{y}, \boldsymbol{x}) = \min\ \{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \mid \text{for } \boldsymbol{x} \in L(\boldsymbol{y}),\ \theta_{FL}\boldsymbol{x} \in L(\boldsymbol{y}),\ \theta_{FL} \in Q\ \}$ and $RE_I(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L(\boldsymbol{y})$,

where $\mathbf{e}$ is an N×1 vector of ones, is called the Russell input oriented measure of technical efficiency.

This non-radial measure of technical efficiency minimizes the arithmetic mean of the proportionate contraction in each of the inputs in the coordinate directions. It assigns a value equal to one to a data point that belongs to $EffL(\boldsymbol{y})$. For any data point $\boldsymbol{x} \in L(\boldsymbol{y}) \backslash EffL(\boldsymbol{y})$,

$RE_I(\boldsymbol{y},\boldsymbol{x}) < 1$, with inefficiency increasing for lower values of $RE_I(\boldsymbol{y}, \boldsymbol{x})$.

Compare to the Farrell's equiproportionate contraction, the Russell measure of technical efficiency allows for the contraction of different components of input vector in different proportions. Further, due to the fact that the Russell measure minimizes the arithmetic mean of input contraction for the given output vector, the efficient subset $EffL(\boldsymbol{y})$ rather than the whole isoquant $IsoqL(\boldsymbol{y})$ is used as the set of reference for the purpose of gauging efficiency. In figure(2.3) $\boldsymbol{x} \in L(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$ is an inefficient data point. Farrell's technical efficiency measure uses $FE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})\times\boldsymbol{x} \in IsoqL(\boldsymbol{y})$ as the reference point to compute input oriented technical efficiency score. However for $FE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})\times\boldsymbol{x}$**,** contraction of input is still possible with respect to at least one input, x_1 in this case, for the given output vector. The Russell's measure, to gauge the efficiency of $\boldsymbol{x} \in L(\boldsymbol{y})$, uses the input vector as reference that belongs to the set bounded by $\theta^1_{\mathrm{FL}}\times\boldsymbol{x}$ and $\theta^2_{\mathrm{FL}}\times\boldsymbol{x}$ and minimizes $RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$. For the comparable data point for which all the nonzero elements of the Russell's diagonal matrix, θ_{FL}, have identical values, we have $RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x}) = FE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$. In contrast the non identical values in θ_{FL} suggests that some of the components of the input vector need to be contracted more than the others which makes $RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x}) < FE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$. Thus the Farrell's input oriented measure of technical efficiency is a special case of the more general Russell's input measure of technical efficiency.

Proposition 2.2: Russell's input oriented measure of technical efficiency satisfies E1a, E2a, E4a and E6a but fails to satisfy E3a and E5a for all the technologies.

<u>Proofs</u>:

<u>$RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$ satisfies E1a, the Identification Property:</u>

Suppose not. Then for some $\boldsymbol{x} \in L(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$ we have $RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x}) = 1$.

But $\boldsymbol{x} \in L(\boldsymbol{y})\backslash EffL(\boldsymbol{y}) \Rightarrow$ there exists a θ_{FL} with at least one of the diagonal elements less than one, such that $\theta_{\mathrm{FL}}\boldsymbol{x} \in EffL(\boldsymbol{y})$ and $\theta_{\mathrm{FL}}\boldsymbol{x} \leq \boldsymbol{x} \Rightarrow RE_{\mathrm{I}}(\boldsymbol{y}, \theta_{\mathrm{FL}}\boldsymbol{x}) = 1 \Rightarrow RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x}) < 1$ hence a contradiction.

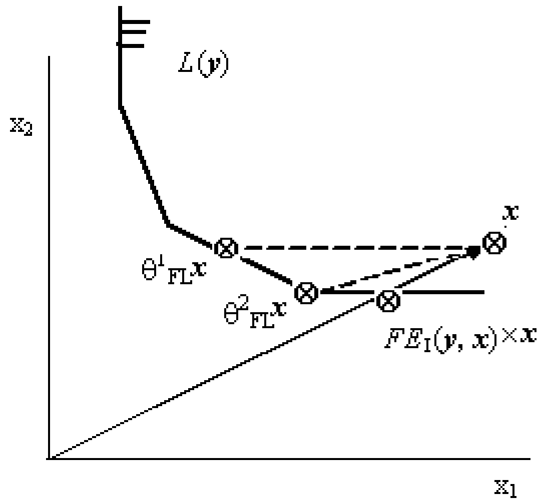

Figure 2.3. The Russell measure

Next, suppose that $\boldsymbol{x} \in EffL(\boldsymbol{y})$ and $RE_I(\boldsymbol{y}, \boldsymbol{x}) < 1$.

$RE_I(\boldsymbol{y}, \boldsymbol{x}) < 1 \Rightarrow \exists\ \theta_{FL}$ with at least one diagonal element less than one such that $\theta_{FL}\boldsymbol{x} \leq \boldsymbol{x}$ and

$\theta_{FL}\boldsymbol{x} \in L(\boldsymbol{y}) \Rightarrow \boldsymbol{x} \notin EffL(\boldsymbol{y})$, hence a contradiction.

<u>$RE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E2a, the "Compare to" Property:</u>

Suppose not. Then the Russell measure compares $\boldsymbol{x} \in L(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$ to some

$\theta_{FL}\boldsymbol{x} \in IsoqL(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$ such that $\theta_{FL}\boldsymbol{x} \leq \boldsymbol{x}$ and $\theta_{FL}\boldsymbol{x}$ assumed Russell efficient $\Rightarrow RE_I(\boldsymbol{y}, \theta_{FL}\boldsymbol{x})=1$.

But $\theta_{FL}\boldsymbol{x} \notin EffL(\boldsymbol{y}) \Rightarrow \exists\ \boldsymbol{x}^* \leq \theta_{FL}\boldsymbol{x}$ such that $\boldsymbol{x}^* \in EffL(\boldsymbol{y})$. Then we can define another N×N

diagonal matrix θ^*_{FL} with at least one diagonal element less than one such that $\theta^*_{FL}(\theta_{FL}\boldsymbol{x}) = \boldsymbol{x}^*$.

But this implies that $RE_I(\boldsymbol{y}, \theta^*_{FL}(\theta_{FL}\boldsymbol{x})) = 1 \Rightarrow RE_I(\boldsymbol{y}, \theta_{FL}\boldsymbol{x}) < 1$, hence a contradiction.

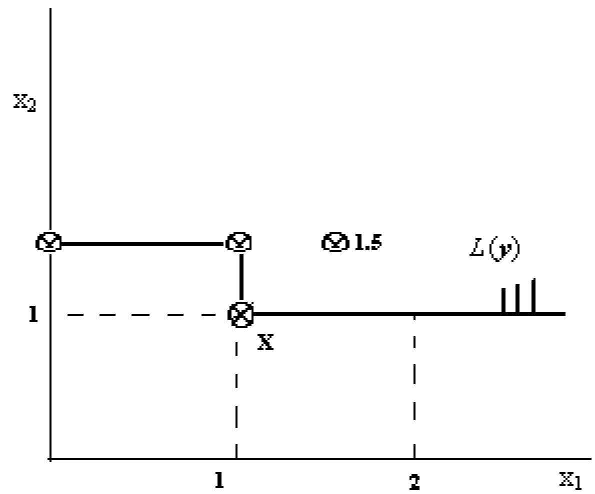

Figure 2.4. $RE_I(\boldsymbol{y}, \boldsymbol{x})$ violates monotonicity

<u>$RE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E4a, the homogeneity of plus one in outputs:</u>

This follows because for constant returns to scale we have $L(\nu \boldsymbol{y}) = \nu L(\boldsymbol{y})$. Thus

$$RE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \min \{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \mid \text{for } \boldsymbol{x} \in L(\nu\boldsymbol{y}), \theta_{FL}\boldsymbol{x} \in L(\nu\boldsymbol{y}), \theta_{nn} \in (0, 1]\}$$

$$RE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \min \{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \mid \text{for } \boldsymbol{x} \in \nu L(\boldsymbol{y}), \theta_{FL}\boldsymbol{x} \in \nu L(\boldsymbol{y}), \theta_{nn} \in (0, 1]\}$$

$$\Rightarrow RE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu RE_I(\boldsymbol{y}, \boldsymbol{x})$$

<u>$RE_I(\boldsymbol{y}, \boldsymbol{x})$ does not satisfy E5a, the Monotonicity Property</u>:

This is also shown in figure (2.4) where we have

$RE_I(\boldsymbol{y}, x_1 = 3/2, x_2 = 3/2) = RE_I(\boldsymbol{y}, x_1 = 1, x_2 = 3/2) = 1/2$.

<u>$RE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E5a, the Commensurability Property:</u>

Let $\boldsymbol{x} \in L(\boldsymbol{y})$ and let $\boldsymbol{\Lambda}$ be an N×N diagonal matrix such that $\boldsymbol{x}^* = \boldsymbol{\Lambda x}$ and where $\lambda_{nn} \in (0, +\infty)$, the nth diagonal element of $\boldsymbol{\Lambda}$ is the rescaling factor for the nth input. Further, let $\boldsymbol{\Omega}$ be an M×M diagonal matrix such that $\boldsymbol{y}^* = \boldsymbol{\Omega y}$ and where $\omega_{mm} \in (0, +\infty)$, the mth diagonal element of $\boldsymbol{\Omega}$ is the rescaling factor for the mth output. Then $\boldsymbol{x} \in L(\boldsymbol{y}) \Leftrightarrow \boldsymbol{x}^* \in L(\boldsymbol{y}^*) \Leftrightarrow \boldsymbol{\Lambda x} \in L(\boldsymbol{\Omega y})$.

We can write the Russell measure for $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ as

$RE_I(\boldsymbol{y}^*, \boldsymbol{x}^*)= \min \{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \mid \text{for } \boldsymbol{x}^* \in L(\boldsymbol{y}^*), \theta_{FL}\boldsymbol{x}^* \in L(\boldsymbol{y}^*), \theta_{nn} \in (0, 1]\}$

$\Rightarrow RE_I(\boldsymbol{\Omega y}, \boldsymbol{\Lambda x}) = \min \{1/N((\theta_1 \lambda_{11}x_1 /\lambda_{11}x_1) + (\theta_2 \lambda_{22}x_2 /\lambda_{22}x_2) + \ldots\ldots + (\theta_N \lambda_{NN}x_N/\lambda_{NN}x_N)) \mid$

$(\theta_1\lambda_1x_1, \lambda_2 \lambda_2x_2, \ldots\ldots\theta_N \lambda_Nx_N) \in L(\boldsymbol{\Omega y}), \theta_n \in (0, 1], \boldsymbol{\Omega y} \in P(\boldsymbol{\Lambda x})\}$

$\Rightarrow RE_I(\boldsymbol{\Omega y}, \boldsymbol{\Lambda x}) = \min \{1/N((\theta_1 x_1/x_1) + (\theta_2 x_2 /x_2) + \ldots + (\theta_N x_N/x_N)) \mid (\theta_1 x_1, \theta_2 x_2, \ldots\ldots\theta_N x_N) \in L(\boldsymbol{y}),$

$\theta_n \in (0, 1], \boldsymbol{x} \in L(\boldsymbol{y})\}$

$= \min \{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \mid \text{for } \boldsymbol{x} \in L(\boldsymbol{y}), \theta_{FL}\boldsymbol{x} \in L(\boldsymbol{y}), \theta_{nn} \in (0, 1]\} = RE_I(\boldsymbol{y}, \boldsymbol{x})$

2.2.2 Zieschang Input oriented Technical Efficiency Measure:

The input oriented Zieschang measure of technical efficiency is a synthesis of Farrell's and Russell's input technical efficiency measures. It is defined as follows;

Definition: The function ZE_I: $\mathbb{R}^M \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined as $ZE_I(\boldsymbol{y}, \boldsymbol{x})=$ $\{[RE_I(\boldsymbol{y}, \boldsymbol{x}/ \psi^+_I(\boldsymbol{y}, \boldsymbol{x})]/ \psi^+_I(\boldsymbol{y}, \boldsymbol{x}) \mid \text{for } \boldsymbol{x} \in L^+(\boldsymbol{y}), \boldsymbol{x}//\psi^+_I(\boldsymbol{y}, \boldsymbol{x}) \in L^+(\boldsymbol{y})\}$ and $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L^+(\boldsymbol{y})$ is called the Zieschang measure of input oriented technical efficiency.

Thus an inefficient data point is first contracted radially and if the resulting projection falls on $IsoqL^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$, a non-radial adjustment is made using maximum possible contraction of the input vector in the co-ordinate directions. In this way the Zieschang input measure of technical efficiency combines the features of Farrell and Russell measures of technical efficiency. Note that the Zieschang measure is computed for the pseudo correspondence L^+: $\Re^M_+ \rightarrow L^+(\boldsymbol{y})$ such that $L^+(\boldsymbol{y}) = L(\boldsymbol{y}) + \Re^N_+$ i. e. $L^+(\boldsymbol{y})$ contains its free disposal hull[3]. Thus in the above definition we have $\psi^+_I(\boldsymbol{y}, \boldsymbol{x}) = \text{INF}_\lambda \{\lambda \mid \lambda\boldsymbol{x} \in L^+(\boldsymbol{y}), \lambda \geq 0\}$.

Zieschang's input oriented technical efficiency measure assigns a value equal to one to any data point that belongs to $EffL^+(\boldsymbol{y})$. For any data point that belongs to $L^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$, $ZE_I(\boldsymbol{y}, \boldsymbol{x})$ is less than one with inefficiency increasing for the lower values of $ZE_I(\boldsymbol{y}, \boldsymbol{x})$.

Three possibilities exist. First, a data point belongs to $L^+(\boldsymbol{y})\backslash IsoqL^+(\boldsymbol{y})$ and its radial projection belongs to $EffL^+(\boldsymbol{y})$. Then Zieschang and Farrell measures coincide and result in efficiency score of identical magnitudes. The input vector $\boldsymbol{x}$ in figure (2.5) represents this case. In addition, for such an input vector if Farrell and Russell scores coincide, we get $ZE_I(\boldsymbol{y}, \boldsymbol{x})=FE_I(\boldsymbol{y}, \boldsymbol{x})= RE_I(\boldsymbol{y},\boldsymbol{x})$. A second possibility is that the radial projection of a data point belongs to $IsoqL^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$. Then the Zieschang measure computes the efficiency score as the product of Farrell and Russell measure. In terms of the figure, with respect to the data point $\boldsymbol{x}^*$, first $FE_I(\boldsymbol{y}, \boldsymbol{x}^*) = 1/\psi_I(\boldsymbol{y}, \boldsymbol{x}^*)$ is obtained and since $FE_I(\boldsymbol{y}, \boldsymbol{x}^*) \times \boldsymbol{x}^* \in IsoqL^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$, in the second step Russell component is computed with respect to the data point $FE_I(\boldsymbol{y}, \boldsymbol{x}^*)\times\boldsymbol{x}^*$. In our example this Russell component is

$RE_I(\boldsymbol{y}, \boldsymbol{x}^*\times FE_I(\boldsymbol{y}, \boldsymbol{x}^*)) = \min \{\sum^2_{n=1}\theta_n/2 \mid (\theta_1\times(x^*_1\times FE_I(\boldsymbol{y}, \boldsymbol{x}^*)), \theta_2\times(x^*_2\times FE_I(\boldsymbol{y}, \boldsymbol{x}^*)))\in L^+(\boldsymbol{y}),$

$\theta_m \in (0, 1], \boldsymbol{x}^* \in L^+(\boldsymbol{y}), \boldsymbol{x}^*\times FE_I(\boldsymbol{y}, \boldsymbol{x}^*)\in L^+(\boldsymbol{y}) \}$.

Then the Zieschang efficiency score for $\boldsymbol{x}^*$ is the product

$ZE_I(\boldsymbol{y}, \boldsymbol{x}^*) = FE_I(\boldsymbol{y}, \boldsymbol{x}^*)\times RE_I(\boldsymbol{y} , \boldsymbol{x}^*\times FE_I(\boldsymbol{y}, \boldsymbol{x}^*))$.

Since $FE_I(\boldsymbol{y}, \boldsymbol{x}^*) < 1$ and $RE_I(\boldsymbol{y} , \boldsymbol{x}^*\times FE_I(\boldsymbol{y}, \boldsymbol{x}^*)) < 1$, implies $ZE_I(\boldsymbol{y}, \boldsymbol{x}^*) < FE_I(\boldsymbol{y}, \boldsymbol{x}^*)$. The relationship between $ZE_I(\boldsymbol{y}, \boldsymbol{x}^*)$ and $RE_I(\boldsymbol{y}, \boldsymbol{x}^*)$ however is ambiguous. The reason is that though we have $RE_I(\boldsymbol{y} , \boldsymbol{x}^*\times FE_I(\boldsymbol{y}, \boldsymbol{x}^*)) > RE_I(\boldsymbol{y}, \boldsymbol{x}^*)$, the product $FE_I(\boldsymbol{y}, \boldsymbol{x}^*)\times RE_I(\boldsymbol{y} , \boldsymbol{x}^*\times FE_I(\boldsymbol{y},\boldsymbol{x}^*))$

[3] Note that free disposal hull and strong disposability are equivalent concepts. Thus with our maintained assumptions $L^+(\boldsymbol{y}) = L(\boldsymbol{y})$. Symbolic differences have been used to maintain uniformity with the existing literature.

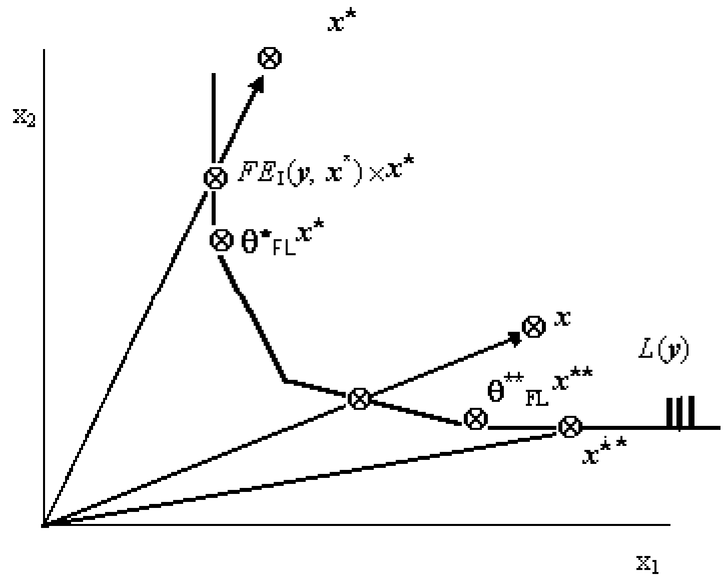

Figure 2.5. Comparing $ZE_I(\boldsymbol{y}, \boldsymbol{x})$, $FE_I(\boldsymbol{y}, \boldsymbol{x})$ and $RE_I(\boldsymbol{y}, \boldsymbol{x})$

doesn't have to have a definite relationship with $RE_I(\boldsymbol{y}, \boldsymbol{x}^*)$. Finally for the third possibility where an inefficient input vector such as $\boldsymbol{x}^{**} \in IsoqL^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$, we have $ZE_I(\boldsymbol{y}, \boldsymbol{x}^{**})$ equal to $RE_I(\boldsymbol{y}, \boldsymbol{x}^{**})$. In conclusion $ZE_I(\boldsymbol{y}, \boldsymbol{x}) \leqq FE_I(\boldsymbol{y}, \boldsymbol{x})$ and $ZE_I(\boldsymbol{y}, \boldsymbol{x}) \lesseqgtr RE_I(\boldsymbol{y}, \boldsymbol{x})$. Note however that the comparison for $ZE_I(\boldsymbol{y}, \boldsymbol{x}) < FE_I(\boldsymbol{y}, \boldsymbol{x})$ case has only been mentioned to state the computational difference between $ZE_I(\boldsymbol{y}, \boldsymbol{x})$ and $FE_I(\boldsymbol{y}, \boldsymbol{x})$ scores. Conceptually the two measures are not comparable in such situation as they use different sets of references for computing efficiency scores.

Proposition 2.3: Zieschang input oriented technical efficiency measure satisfies E.1a to E4a and E6a but it does not satisfy E5a the monotonicity.

<u>Proofs:</u>

<u>$ZE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E1a, the Identification Property:</u>

Suppose not. Then there exists an $\boldsymbol{x}$ such that $\boldsymbol{x} \in L^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$ and $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = 1$.

$\Rightarrow RE_I(\boldsymbol{y}, \boldsymbol{x} \times FE_I(\boldsymbol{y}, \boldsymbol{x})) \times FE_I(\boldsymbol{y}, \boldsymbol{x})=1$.

But for $\boldsymbol{x} \in L^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$ we have two possibilities

i) $\boldsymbol{x} \in IsoqL^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$

ii) $\boldsymbol{x} \in L^+(\boldsymbol{y})\backslash IsoqL^+(\boldsymbol{y})$

i) $\Rightarrow$ from proposition 2.1 $FE_I(\boldsymbol{y}, \boldsymbol{x}) = 1$. Then for $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = 1$ we must have

$RE_I(\boldsymbol{y}, \boldsymbol{x} \times FE_I(\boldsymbol{y}, \boldsymbol{x})) = 1$.

But from proposition 2.2 we have $RE_I(\boldsymbol{y}, \boldsymbol{x} \times FE_I(\boldsymbol{y}, \boldsymbol{x})) <1$ hence a contradiction.

ii) $\Rightarrow$ from proposition 2.1 that $FE_I(\boldsymbol{y}, \boldsymbol{x}) < 1$. Then for $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = 1$ we must have

$RE_I(\boldsymbol{y}, \boldsymbol{x} \times FE_I(\boldsymbol{y}, \boldsymbol{x})) > 1$.But this contradicts the definition of $RE_I(\boldsymbol{y}, \boldsymbol{x})$. So holds E1a for

$ZE_I(\boldsymbol{y}, \boldsymbol{x})$.

<u>$ZE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E2a, the "compare to" Property:</u>

Suppose not and that the $ZE_I(\boldsymbol{y}, \boldsymbol{x})$ compares $\boldsymbol{x} \in L^+(\boldsymbol{y})\backslash IsoqL^+(\boldsymbol{y})$ to some $\boldsymbol{x}^* \in IsoqL^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$

$\Rightarrow$ for any data point $\boldsymbol{x} \in L^+(\boldsymbol{y})$ for which $FE_I(\boldsymbol{y}, \boldsymbol{x}) \times \boldsymbol{x} \in IsoqL^+(\boldsymbol{y})\backslash EffL^+(\boldsymbol{y})$ we have

$ZE_I(\boldsymbol{y}, \boldsymbol{x}) =1$. But from proposition 2.2 we have $RE_I(\boldsymbol{y}, \boldsymbol{x} \times FE_I(\boldsymbol{y}, \boldsymbol{x})) < 1$

$\Rightarrow RE_I(\boldsymbol{y}, \boldsymbol{x} \times FE_I(\boldsymbol{y}, \boldsymbol{x})) \times FE_I(\boldsymbol{y},\boldsymbol{x})) < 1 \Rightarrow ZE_I(\boldsymbol{y}, \boldsymbol{x}) < 1$, hence a contradiction.

<u>$ZE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E3a the homogeneity of minus one in inputs:</u>

Two possibilities exist. First, the Zieschang projection of a data point $\boldsymbol{x} \in L^+(\boldsymbol{y})$ falls on $EffL^+(\boldsymbol{y})$.

Then we have $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = FE_I(\boldsymbol{y}, \boldsymbol{x})$. Then proposition 2.1 $\Rightarrow$ E3a. Second possibility is that the projection of $\boldsymbol{x} \in IsoqL^{+}(\boldsymbol{y}) \setminus EffL^{+}(\boldsymbol{y})$. Then for any $\beta > 0$ such that $\beta\boldsymbol{x} \in L^{+}(\boldsymbol{y})$ we have

$ZE_{I}(\boldsymbol{y}, \beta\boldsymbol{x}) = RE_{I}(\boldsymbol{y}, \beta\boldsymbol{x} \times FE_{I}(\boldsymbol{y}, \beta\boldsymbol{x}))\times FE_{I}(\boldsymbol{y}, \beta\boldsymbol{x})$. But from proposition 2.1 we have

$RE_{I}(\boldsymbol{y}, \beta\boldsymbol{x} \times FE_{I}(\boldsymbol{y}, \beta\boldsymbol{x})) = RE_{I}(\boldsymbol{y}, \boldsymbol{x} \times FE_{I}(\boldsymbol{y}, \boldsymbol{x}))$.

$\Rightarrow ZE_{I}(\boldsymbol{y}, \beta\boldsymbol{x}) = \beta^{-1}ZE_{I}(\boldsymbol{y}, \boldsymbol{x})$.

<u>$ZE_{I}(\boldsymbol{y}, \boldsymbol{x})$ satisfies E4a, the homogeneity of plus one in outputs:</u>

$ZE_{I}(\nu\boldsymbol{y}, \boldsymbol{x}) = RE_{I}(\nu\boldsymbol{y}, \boldsymbol{x} \times FE_{I}(\nu\boldsymbol{y}, \boldsymbol{x})) \times FE_{I}(\nu\boldsymbol{y}, \boldsymbol{x})$.

Then we have, from proposition 2.1 and 2.2

$RE_{I}(\boldsymbol{y}, \boldsymbol{x} \times FE_{I}(\boldsymbol{y}, \boldsymbol{x})) = RE_{I}(\nu\boldsymbol{y}, \boldsymbol{x} \times FE_{I}(\nu\boldsymbol{y}, \boldsymbol{x}))$ and so follows the result.

<u>$ZE_{I}(\boldsymbol{y}, \boldsymbol{x})$ fails to satisfy E5a the Monotonicity:</u>

This can be shown using figure 2.4 where

$ZE_{I}(\boldsymbol{y}, x_1 = 1, x_2 = 3/2) = ZE_{I}(\boldsymbol{y}, x_1 = 1/2, x_2 = 3/2) = 1/2$, thus violating monotonicity.

<u>$ZE_{I}(\boldsymbol{x}, \boldsymbol{y})$ satisfies E6a, the Commensurability property:</u>

This follows because $ZE_{I}(\boldsymbol{y}, \boldsymbol{x})$ is the product of Farrell and Russell measures and the two satisfy commensurability.

2.2.3 The Non-radial Farrell Input oriented Technical Efficiency Measure:

In this section we outline a two step method for obtaining an alternate non-radial input oriented technical efficiency measure for the technology described in chapter1. The first step requires transforming the data set into a form that ensures the fulfillment of the commensurability property, E6a, while computing the efficiency scores. The second step computes a technical efficiency score as the ratio of two (Euclidian) norms, with the numerator consisting of the norm of the technically efficient reference vector and the denominator consisting of the norm of the actual/observed vector. The two step procedure is as follows;

STEP 1. In this step we use the concept of a numeraire firm to transform a given data set on the production units into a format in which the output and input quantities are free of units of measurement. The transformation is such that it preserves the underlying technology and also does not change the relative position of the data points with respect to each other when one switches between units in which inputs and outputs are measured. This transformed data set then can be used to compute the non radial efficiency scores that satisfy the properties required of any efficiency index.

To get such transformation of a data set consisting of J firms, using inputs, $\boldsymbol{x} \in \Re^{N}_{+}$ to produce output $\boldsymbol{y} \in \Re^{M}_{+}$, we define a numeraire firm. Any firm can serve the purpose. Here we prefer to construct a numeraire firm by summing each of the N inputs over the entire sample and each of the M outputs over the entire sample. Thus the numeraire firm uses $\sum_j x_{nj}$ units of the nth input to produce $\sum_j y_{mj}$ units of the mth output. Using this numeraire firm, the transformation for the kth data point is as follows;

${}_s x_{nk} = x_{nk} / \sum^{J}_{j} x_{nj}$ for n=1,2…….N.

${}_s y_{mk} = y_{mk} / \sum^{J}_{j} y_{mj}$ for m=1,2…….M.

i.e. the kth firm uses ${}_s x_{nk}$% of the sample's nth input to produce ${}_s y_{mk}$% of the sample's mth output. Thus the input and the output vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are transformed into the input and the output share vectors as ${}_s\boldsymbol{x} \in \Re^{N}_{+}$ and ${}_s\boldsymbol{y} \in \Re^{M}_{+}$ respectively. This transformed data set is in the non negative n-orthant of a unit ball such that the numeraire firm belongs to its boundary and the actual data points are contained in its interior. Note that the values of ${}_s x_{nk}$ and ${}_s y_{mk}$ do not depend upon the choice of units of measurement of inputs and outputs. Thus for example measuring capital in terms of millions of dollars in place of billions of dollars has no bearing on the kth firm's share of employed capital relative to the numeraire firm's capital employment and also relative to any other firm's capital in the sample.

The transformation allows us to define the correspondence L^S: $\Re^M_+ \rightarrow L^S({}_s\boldsymbol{y}) \subseteq \Re^N_+$ which maps output share vectors ${}_s\boldsymbol{y}$ into subsets $L^S({}_s\boldsymbol{y})$ of input share vectors ${}_s\boldsymbol{x}$. Since $\boldsymbol{x} \in L(\boldsymbol{y})$ $\Rightarrow {}_s\boldsymbol{x} \in L({}_s\boldsymbol{y})$, the correspondence L^S **,** has the following implications;

1. $L^S(\mathbf{0}) = \Re^N_+$, $\mathbf{0} \notin L^S({}_s\boldsymbol{y})$ for ${}_s\boldsymbol{y} \geq \mathbf{0}$ where $\mathbf{0}$ is M×1 zero vector.

2. ${}_s\boldsymbol{x} \in L^S(\boldsymbol{y}) \Rightarrow$ if ${}_s\boldsymbol{x}^* \geq {}_s\boldsymbol{x}$ then ${}_s\boldsymbol{x}^* \in L^S({}_s\boldsymbol{y})$

3. $L^S({}_s\hat{\boldsymbol{y}}) \subseteq L^S({}_s\boldsymbol{y})\ \forall\ {}_s\hat{\boldsymbol{y}} \geqq {}_s\boldsymbol{y}$

4. L^S is a closed correspondence.

For this transformed input correspondence we can define

$IsoqL^S({}_s\boldsymbol{y}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{x} \in L^S({}_s\boldsymbol{y}) \text{ and } \lambda{}_s\boldsymbol{x} \notin L^S({}_s\boldsymbol{y}) \text{ for } \lambda \in [0, 1)\}$

$EffL^S({}_s\boldsymbol{y}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{x} \in L^S({}_s\boldsymbol{y}) \text{ and } {}_s\boldsymbol{x}^* \leq {}_s\boldsymbol{x} \Rightarrow {}_s\boldsymbol{x}^* \notin L^S({}_s\boldsymbol{y})\}$

In terms of figure(2.6) , the curve represents $IsoqL^S({}_s\boldsymbol{y})$. The area above the curve and the curve together define $L^S({}_s\boldsymbol{y})$. The segment of the curve bounded by the input share vectors ${}_s\boldsymbol{x}^*$ and ${}_s\boldsymbol{x}^{**}$ represents the $EffL^S({}_s\boldsymbol{y})$. Note in the going that the concept of data transformation that we have introduced here is different from the concept of the *data translation* that was introduced by Ali and Sieford(1990) and that involves translating the values of inputs and/or outputs of a DEA model so that a new problem is created. If this new problem has the same optimal solution as the old DEA problem then the DEA model is said to be *Translation invariant.* The concept is useful in the applications where it may be necessary or convenient to be able to handle negative data in some of the inputs or outputs. In comparison, in our case of data transformation we transform the

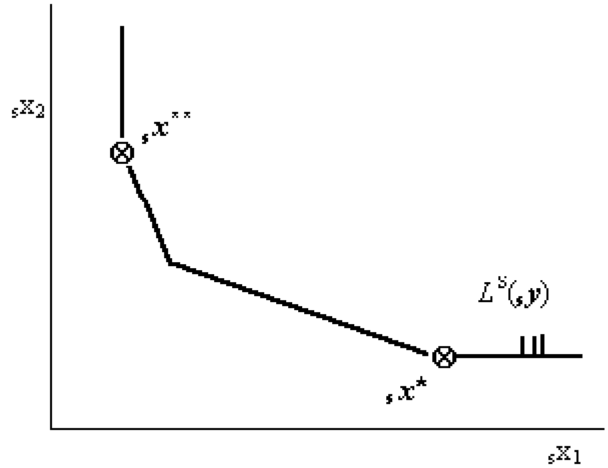

Figure 2.6. Transformed input level curve

entire data set with respect to some numeraire firm such that input and output values of each observation in the data set are expressed as proportion of the input and output values of the numeraire firm.

Step2. . We can now compute the Non-radial Farrell measure of technical efficiency using the transformed input and output vectors obtained in step one. In order to define the Non-radial Farrell measure let Q be the set of diagonal matrices $^i\theta_{NE}$, each of dimension N with $^i\theta_{nn} \in (0,1]$ for n=1,2,…N, and a vector $\boldsymbol{\varepsilon} \in \Re^N_+$, then,

Definition: The function NE_I: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$ for q = M, N, defined by $NE_I(_s\boldsymbol{y}, {_s\boldsymbol{x}}) = \min\{\psi_I(_s\boldsymbol{y}, \theta_{NE}\, {_s\boldsymbol{x}})/\psi_I(_s\boldsymbol{y}, {_s\boldsymbol{x}})|$ for $_s\boldsymbol{x} \in L^S(_s\boldsymbol{y})$, $\theta_{NE}\, {_s\boldsymbol{x}} \in L^S(_s\boldsymbol{y})$, $\theta_{NE} \in Q$, $(\theta_{NE}\, {_s\boldsymbol{x}} - \boldsymbol{\varepsilon}) \notin L^S(_s\boldsymbol{y})\}$ and $NE_I(_s\boldsymbol{y}, {_s\boldsymbol{x}}) = +\infty$ for $_s\boldsymbol{x} \notin L^S(_s\boldsymbol{y})$, is called the Non-radial Farrell measure of technical efficiency.

Thus from a set of reference vectors, such that the ith element of that set, $\theta^{i}_{NE}\ {}_{s}\boldsymbol{x} \leq {}_{s}\boldsymbol{x}$, we pick the input vector $\theta_{NE}\ {}_{s}\boldsymbol{x} \in Eff\ L^{S}({}_{s}\boldsymbol{y})$ that minimizes the ratio of the Euclidian norms, $\|\theta_{NE}\ {}_{s}\boldsymbol{x}\| / \|{}_{s}\boldsymbol{x}\|$. In this computation θ_{nn} represents the proportion by which the nth component of the input share vector ${}_{s}\boldsymbol{x}$ is to be contracted in order to obtain the technically efficient share vector $\theta_{NE}\ {}_{s}\boldsymbol{x} \in Eff\ L^{S}({}_{s}\boldsymbol{y})$. For any data point that belongs to the efficient frontier $\theta_{nn} = 1\ \forall$ n=1…N. Thus $\psi_{I}({}_{s}\boldsymbol{y}, \theta_{NE}\ {}_{s}\boldsymbol{x}) = \psi_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$ which results in $NE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) = 1$ and indicates full technical efficiency with respect to the efficiency frontier. For off the frontier data points θ_{NE} has at least one diagonal element less than one. Thus $\psi_{I}({}_{s}\boldsymbol{y}, \theta_{NE}{}_{s}\boldsymbol{x}) < \psi_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$ implying $NE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) < 1$, indicating technical inefficiency with inefficiency increasing for the lower values of $NE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$.

If θ_{NE} is equal to $\theta\mathbf{I}$, where θ is some constant and $\mathbf{I}$ is an N×N identity matrix, then

$\psi_{I}({}_{s}\boldsymbol{y}, \theta_{NE}\ {}_{s}\boldsymbol{x})/\psi_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) = \psi_{I}({}_{s}\boldsymbol{y}, \theta\mathbf{I}\ {}_{s}\boldsymbol{x})/\psi_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) = \psi_{I}({}_{s}\boldsymbol{y}, \theta_{DF}\ {}_{s}\boldsymbol{x})/\psi_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) = FE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$

Thus the Farrell radial input oriented technical efficiency measure is a special case of input oriented Non-radial Farrell measure of technical efficiency when the two are computed on the transformed data set. Further, since $NE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$ involves picking up the θ_{NE} that minimizes the value of $NE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$, we have $NE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) \leq FE_{I}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$ when the two measures are computed in the transformed data space. Note however that the two measures are comparable only for the data points that have their projections on efficient subset of the isoquant i.e. only for $\theta_{DF}{}_{s}\boldsymbol{x} \in EffL^{S}({}_{s}\boldsymbol{y})$ and $\theta_{NE}\ {}_{s}\boldsymbol{x} \in EffL^{S}({}_{s}\boldsymbol{y})$. If for any ${}_{s}\boldsymbol{x}$ we have $\theta_{DF}{}_{s}\boldsymbol{x} \in IsoqL^{S}({}_{s}\boldsymbol{y}) \setminus EffL^{S}({}_{s}\boldsymbol{y})$ and $\theta_{NE}\ {}_{s}\boldsymbol{x} \in EffL^{S}({}_{s}\boldsymbol{y})$, two measures are not comparable as they use different sets of reference.

Comparing the Non-radial Farrell measure with the Russell measure, the two measures pick up the reference vector from the same subset of $EffL(\boldsymbol{y})$. However due to the computational differences between the Russell and the Non-radial Farrell technical efficiency measure it is not unlikely for the two measures to have identical projection vectors but different technical efficiency scores. Similarly it is also not unlikely for the two measures to have identical scores with respect to a data point with different projection vectors. Since $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \leq FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$, this implies, whenever $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ we get $NE_I({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \leq RE_I({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. For $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \neq FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$, the relationship between $NE_I({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ and $RE_I({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ may go in any direction. Similar relationship holds between Non-radial Farrell and Zieschang measures of technical efficiency. Whenever $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \Rightarrow FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ and we get $NE_I({}_s\boldsymbol{y}\ {}_s\boldsymbol{x}) = ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$. For $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \neq ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$, we have $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \lesseqgtr ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$. Also for $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ and $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \neq NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$, we have $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) < ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$.

Proposition 2.4: $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies E1a to E6a but E3a.

Proofs:

$NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies E1a the Identification Property:

First suppose that ${}_s\boldsymbol{x}^* \in L^S({}_s\boldsymbol{y}) \backslash EffL^S({}_s\boldsymbol{y})$ and that $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}^*)=1$.Then since ${}_s\boldsymbol{x}^* \in L^S({}_s\boldsymbol{y}) \backslash EffL^S({}_s\boldsymbol{y})$, we can define for a radius r>0 a closed ball centered at ${}_s\boldsymbol{x}^*$ such that

$B_r({}_s\boldsymbol{x}^*) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{x} \in L^S({}_s\boldsymbol{y}),\ \|{}_s\boldsymbol{x} - {}_s\boldsymbol{x}^*\| \leqq r\}$

But this implies that $\exists\ {}_s\boldsymbol{x} \leq {}_s\boldsymbol{x}^*$ such that ${}_s\boldsymbol{x} \in B_r({}_s\boldsymbol{x}^*) \cap L^S({}_s\boldsymbol{y}) \neq \varnothing$ and a diagonal matrix θ_{NE} of order N×N with at least one diagonal element less than one can be defined such that ${}_s\boldsymbol{x} = \theta_{NE}\,{}_s\boldsymbol{x}^* \Rightarrow NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}^*) < 1$ hence a contradiction.

Next suppose that $_s\boldsymbol{x}^* \in EffL^S(_s\boldsymbol{y})$ and that $NE_I(_s\boldsymbol{y}, _s\boldsymbol{x}^*) < 1$.

Since $NE_I(_s\boldsymbol{y}, _s\boldsymbol{x}^*) < 1 \Rightarrow$ we can define a diagonal matrix θ_{NE} of order N×N, $\theta_{nn} \in (0, 1]$, with at least one diagonal element strictly less than one such that $\theta_{NE\,s}\boldsymbol{x}^* \leq {}_s\boldsymbol{x}^*$ and $\theta_{NE\,s}\boldsymbol{x}^* \in EffL^S(_s\boldsymbol{y})$. But with $\theta_{NE\,s}\boldsymbol{x}^* \in EffL^S(_s\boldsymbol{y})$, the condition $_s\boldsymbol{x}^* \in EffL^S(_s\boldsymbol{y})$ can not simultaneously occur, hence a contradiction.

Finally since for $_s\boldsymbol{x}^* \notin L^S(_s\boldsymbol{y}) \Rightarrow$ for a closed ball with $r > 0$,

$B_r(_s\boldsymbol{x}^*) = \{_s\boldsymbol{x} \mid {}_s\boldsymbol{x} \in L^S(_s\boldsymbol{y}),\ \|_s\boldsymbol{x} - {}_s\boldsymbol{x}^*\| \leqq r\}$ containing $_s\boldsymbol{x} \leq {}_s\boldsymbol{x}^*$ with $_s\boldsymbol{x} \in B_r(_s\boldsymbol{x}^*) \cap L^S(_s\boldsymbol{y}) = \varnothing$

$\Rightarrow NE_I(_s\boldsymbol{x}, {}_s\boldsymbol{y}) = +\infty$

<u>$NE_I(_s\boldsymbol{y}, _s\boldsymbol{x})$ satisfies E2a the "Compare to" Property:</u>

For any $_s\boldsymbol{x} \in L^S(_s\boldsymbol{y}) \backslash EffL^S(_s\boldsymbol{y})$ there exists a θ_{NE}, with at least one diagonal element less than one such that $NE_I(_s\boldsymbol{y}, \theta_{NE\,s}\boldsymbol{x}) = 1$. But from E1a this implies that $\theta_{NE\,s}\boldsymbol{x} \in EffL^S(_s\boldsymbol{y})$ and so holds E2a.

<u>$NE_I(_s\boldsymbol{y}, _s\boldsymbol{x})$ fails to satisfy E3a the homogeneity of minus one in inputs:</u>

This can easily be shown by the help of a simple example given in figure (2.7). In the figure $NE_I(_s\boldsymbol{y},\ _sx_1 = 1,\ _sx_2 = 5,) = 1$ and $NE_I(_s\boldsymbol{y},\ _sx_1 = 2,\ _sx_2 = 10) = 0.35 < 0.5$ thus violating strict homogeneity but still satisfying sub-homogeneity.

<u>$NE_I(_s\boldsymbol{y}, _s\boldsymbol{x})$ satisfies E4a the homogeneity of plus one in outputs:</u>

This follows because we have assumed constant returns to scale.

$NE_I(\beta_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \min\{\psi(\beta_s\boldsymbol{y}, \theta_{NE\,s}\boldsymbol{x})/\psi(\beta_s\boldsymbol{y}, {}_s\boldsymbol{x}) \mid \theta_{NE\,s}\boldsymbol{x} \in L^S(\beta_s\boldsymbol{y}), (\theta_{NE\,s}\boldsymbol{x} - \boldsymbol{\varepsilon}) \in L^S(_s\boldsymbol{y}), \beta > 0\}$

$NE_I(\beta_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \min\{[\psi(\beta_s\boldsymbol{y}, \theta_{NE\,s}\boldsymbol{x})/\psi(\beta_s\boldsymbol{y}, {}_s\boldsymbol{x})] \mid \theta_{NE\,s}\boldsymbol{x} \in \beta L^S(_s\boldsymbol{y}), (\theta_{NE\,s}\boldsymbol{x} - \boldsymbol{\varepsilon}) \notin L^S(_s\boldsymbol{x}), \beta > 0\}$

$= \beta\, NE_I(_s\boldsymbol{y}, {}_s\boldsymbol{x})$

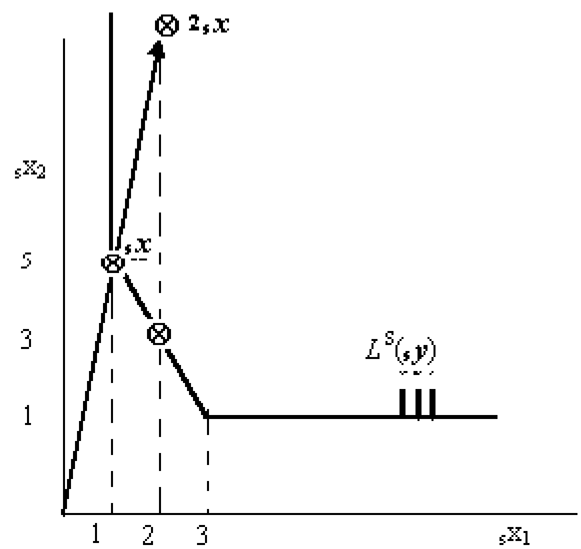

Figure 2.7. $NE_I(_s\boldsymbol{x},\ _s\boldsymbol{y})$ violates homogeneity

$NE_I(_s\boldsymbol{y},\ _s\boldsymbol{x})$ satisfies E5a the Monotonicity Property:

Let $\zeta \geq \mathbf{0}$ an N×1 vector and define

$NE_I(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x} - \zeta) = \min\{\ \psi(_s\boldsymbol{y}\ ,\ \theta_{NE}(_s\boldsymbol{x} - \zeta))/\psi(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x} - \zeta) \mid {_s\boldsymbol{x}} \in L^S(_s\boldsymbol{y}),\ \theta_{NE}(_s\boldsymbol{x} - \zeta) \in L^S(_s\boldsymbol{y}),$

$((\theta_{NE\,s}\boldsymbol{x} - \zeta) - \boldsymbol{\varepsilon}) \notin L^S(_s\boldsymbol{y})$ for $\boldsymbol{\varepsilon} \geq \mathbf{0}\}$

We need to show that $NE_I(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x} - \zeta) > NE_I(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x}\ )$

$\Rightarrow \min\{\ \psi(_s\boldsymbol{y}\ ,\ \theta_{NE}(_s\boldsymbol{x} - \zeta))/\psi(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x} - \zeta)\} > \min\{\ \psi(_s\boldsymbol{y}\ ,\ \theta^*_{NE\,s}\boldsymbol{x}\ )/\psi(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x}\ )\}$

Since ${_s\boldsymbol{x}} - \zeta \leq {_s\boldsymbol{x}} \quad \Rightarrow \theta_{NE} \geq \theta^*_{NE}$

Then we have following possibilities

a) $\theta^*_{NE}\ {_s\boldsymbol{x}}$ and $\theta_{NE}\ (_s\boldsymbol{x} - \zeta)$ coincide

$\Rightarrow \psi_I(_s\boldsymbol{y}\ ,\ \theta_{NE}(_s\boldsymbol{x} - \zeta)) = \psi_I(_s\boldsymbol{y}\ ,\ \theta^*_{NE\,s}\boldsymbol{x})$

$\Rightarrow \min\{\ \psi_I(_s\boldsymbol{y}\ ,\ \theta_{NE}(_s\boldsymbol{x} - \zeta))/\psi_I(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x} - \zeta)\} > \min\{\ \psi_I(_s\boldsymbol{y}\ ,\ \theta^*_{NE\,s}\boldsymbol{x}\ )/\psi_I(_s\boldsymbol{y}\ ,\ _s\boldsymbol{x}\ )\} \Rightarrow$ monotonicity.

b) $\theta^*_{NE}\ {}_s\boldsymbol{x}$ and $\theta_{NE}\ ({}_s\boldsymbol{x} - \zeta)$ do not coincide then

i) $\psi_I({}_s\boldsymbol{y}\ , \theta_{NE}({}_s\boldsymbol{x} - \zeta)) > \psi_I({}_s\boldsymbol{y}\ , \theta^*_{NE}\,{}_s\boldsymbol{x})$

$\Rightarrow \min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta_{NE}({}_s\boldsymbol{x} - \zeta))/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x} - \zeta)\} > \min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta^*_{NE}\,{}_s\boldsymbol{x}\ )/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x})\} \Rightarrow$ monotonicity.

ii) $\psi_I({}_s\boldsymbol{y}\ , \theta_{NE}({}_s\boldsymbol{x} - \zeta)) < \psi_I({}_s\boldsymbol{y}\ , \theta^*_{NE}\,{}_s\boldsymbol{x})$

$\Rightarrow \min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta_{NE}({}_s\boldsymbol{x} - \zeta))/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x} - \zeta)\} \gtreqless \min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta^*_{NE}\,{}_s\boldsymbol{x}\ )/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x}\ )$

ii.a) $\min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta_{NE}({}_s\boldsymbol{x} - \zeta))/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x} - \zeta)\} > \min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta^*_{NE}\,{}_s\boldsymbol{x}\ )/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x}\ ) \Rightarrow$ monotonicity.

ii.b) $\min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta_{NE}({}_s\boldsymbol{x} - \zeta))/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x} - \zeta)\} = \min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta^*_{NE}\,{}_s\boldsymbol{x}\ )/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x}\ )$.

$\Rightarrow (\theta^*_{NE}\ {}_s\boldsymbol{x} - \zeta) \in EffL(\boldsymbol{y}) \Rightarrow$ contradiction hence not possible.

ii.c) $\min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta_{NE}({}_s\boldsymbol{x} - \zeta))/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x} - \zeta)\} < \min\{\ \psi_I({}_s\boldsymbol{y}\ , \theta^*_{NE}\,{}_s\boldsymbol{x}\ )/\psi_I({}_s\boldsymbol{y}\ , {}_s\boldsymbol{x}\ )$

$\Rightarrow \exists\ {}_s\boldsymbol{x}^{**} < {}_s\boldsymbol{x}^* \in EffL(\boldsymbol{y})$ such that ${}_s\boldsymbol{x}^{**} \in EffL(\boldsymbol{y})$ which is not possible. So for (ii) only iia is possible and thus follows monotonicity.

$NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies E6a the Commensurability Property:

This follows due to step one of the procedure.

2.3 Computation

In this section we provide an account on the Data Envelopment Analysis (DEA) which is a linear programming based nonparametric technique that was first introduced by Charnes *et.al* (CCR;1978). This technique can be employed to compute technical efficiency. The elementary CCR model was based on the assumption of constant returns to scale that was later extended to variable returns to scale Banker *et. aL*(BCC, 1984). The technique basically extends the ideas proposed by Farrell(1957) to construct a sample based nonparametric, nonstochastic transformation frontier, comprising of the best practicing firms, and measures the relative performance of off the frontier firms in the sample with reference to the extent of radial deviations from this frontier. Below we outline the basic DEA model using some simplifying assumptions.

We assume strong disposability in both inputs and outputs and absence of any nondiscretionary input or output so that each firm can freely decide about the scale of input usage or output production. In line with the basic CCR model we assume constant returns to scale. Also we assume that the technology results in level sets that satisfy convexity. This assumption allows for the formulation of convex combinations of existing technologies which is needed to define the frontier. The technical efficiency of each of the producer is evaluated relative to its peer, where peer is the convex combination of the subset of the sample comprising of the best practicing firms.

Consider a sample consisting of J firms each using an input vector $\boldsymbol{x}^j \in \Re^N_+$ to produce an output vector $\boldsymbol{y}^j \in \Re^M$ where j ranges from 1,….J**.** The ratio of virtual output to virtual input can be defined as $\boldsymbol{u}^{\mathbf{T}}\boldsymbol{y}^j/\boldsymbol{v}^{\mathbf{T}}\boldsymbol{x}^j$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are the vectors of nonnegative output and input weights respectively. The objective of DEA is to find the set of optimal weights ($\boldsymbol{u}$, $\boldsymbol{v}$) that maximize the ratio of virtual output to virtual input. This is done by solving the following mathematical programming model for the kth firm in the data set.

$$\text{Max}_{v,u} \quad \boldsymbol{u}^{\mathbf{T}}\boldsymbol{y}^k/\boldsymbol{v}^{\mathbf{T}}\boldsymbol{x}^k$$

$$\text{s.t} \quad \boldsymbol{u}^{\mathbf{T}}\boldsymbol{y}^j/\boldsymbol{v}^{\mathbf{T}}\boldsymbol{x}^j \leqslant 1$$

$$\boldsymbol{v} \geqq \mathbf{0} \text{ and } \boldsymbol{u} \geqq \mathbf{0}$$

$$j = 1,2, \ldots.J$$

In words this program solves for v's and u's such that the efficiency score of the kth firm is maximized subject to the constraint that no efficiency score exceeds one, when the weights of the firm being evaluated are applied to all the firms in the sample. This program is solved repeatedly, once for each firm. If for the firm being evaluated 'max' turns out to be one then the firm is said to be efficient otherwise inefficient, with the degree of inefficiency greater for lower scores. This nonlinear mathematical program can be converted into a linear programming model as follows

$$\text{Max}_{\Lambda,\Pi} \quad \Pi^{T}\boldsymbol{y}^{k}$$

$$\text{s.t} \quad \Lambda^{T}\boldsymbol{x}^{k}=1$$

$$\Pi^{T}\boldsymbol{y}^{j} \leqq \Lambda^{T}\boldsymbol{x}^{j}$$

$$\Lambda \geqq \mathbf{0} \text{ and } \Pi \geqq \mathbf{0}$$

$$j = 1,2,\ldots..J$$

This is the basic CCR model mentioned earlier. Written in this format this model is known as the multiplier form of the DEA. $\Pi = (\mathbf{u}^{T}\boldsymbol{y}^{k})^{-1}\times\mathbf{u}$ and $\boldsymbol{\Lambda} = (\mathbf{v}^{T}\boldsymbol{x}^{k})^{-1} \times \mathbf{v}$ are known as the vectors of multipliers with vector $\boldsymbol{\Lambda}$ having dimension N×1 and the dimension of Π is M×1.This program has an input oriented envelopment counterpart given as

$$\text{Min}_{\theta_{DF},\boldsymbol{\Phi}} \quad \theta_{DF}$$

$$\text{s.t} \quad -\boldsymbol{y}^{k} + \boldsymbol{Y}\boldsymbol{\Phi} \geq \mathbf{0}$$

$$\theta_{DF}\boldsymbol{x}^{k} - \boldsymbol{X}\,\boldsymbol{\Phi} \leq \mathbf{0}$$

$$\boldsymbol{\Phi} \in \Re^{N}_{+}$$

where θ_{DF} is a scalar, $\boldsymbol{Y}$ is the M×J output matrix with columns $\boldsymbol{y}^j$, $\boldsymbol{X}$ is an N×J matrix of inputs with columns $\boldsymbol{x}^j$ and $\boldsymbol{\Phi}$ is a J × 1 vector of constants. The advantage of solving the dual over the primal is that it reduces the number of constraints from M+N to J+1. For the firm being evaluated θ_{DF} attains a value of equal to or less than one, defining the firm as efficient if θ_{DF} equals 1 and inefficient if less than one with inefficiency increasing for lower values of θ_{DF}. Again, the program is to be solved once for each firm in the sample so that we get a value of θ_{DF} for each firm in the sample.

In words this problem looks for maximum feasible radial contraction in the input vector of the firm being evaluated, for a given level of output and for a technology generated by the sample. The resulting statistic is the Farrell's input oriented technical efficiency measure.

The DEA problem for the Russell measure is given as

$$\begin{aligned} \mathrm{Min}_{\theta FL,\boldsymbol{\Phi}} \quad & 1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \\ \text{s.t.} \quad & -\boldsymbol{y}^k + \boldsymbol{Y\Phi} \geq \mathbf{0} \\ & \theta_{FL}\boldsymbol{x}^k - \boldsymbol{X\Phi} \geq \mathbf{0} \\ & \boldsymbol{\Phi} \in \Re^N{}_+ \end{aligned}$$

where θ_{FL} is an N×N diagonal matrix with the diagonal elements being the contraction constants and $\mathbf{e}$ is an N×1 vector of ones with $\mathbf{e}^T$ its transpose. Note that for the case for which all the diagonal elements of θ_{FL} have same value, the above problem becomes identical to the previous problem and thus has a Farrell outcome.

Zieschang measure is solved in two steps. In the first step we apply the linear program that solves for the Farrell Efficiency parameter. In the second step the computed Farrell efficiency statistic is used in another linear programming problem that solves for the Russell measure for the projections which have nonzero slacks associated with them.

The Non-radial Farrell measure of efficiency can be computed by solving the following linear program after transforming the data set in the share format as discussed in section 2.5 above;

$$\mathrm{Min}_{\theta_{NE},\boldsymbol{\Phi}} \quad \|\theta_{NE}\,{}_{\mathbf{s}}\boldsymbol{x}^k\| / \|{}_{\mathbf{s}}\boldsymbol{x}^k\|$$

$$\text{s.t} \quad -{}_{\mathbf{s}}\boldsymbol{y}^k + {}_{\mathbf{s}}\boldsymbol{Y\Phi} \geq \mathbf{0}$$

$$\theta_{NE}\,{}_{\mathbf{s}}\boldsymbol{x}^k - {}_{\mathbf{s}}\boldsymbol{X\Phi} \geq \mathbf{0}$$

$$\boldsymbol{\Phi} \in \Re^N_+$$

where θ_{NE} is the N×N diagonal matrix of constants with $\theta_{nn} \in (0.1]$.

2.4 Cost Efficiency and Allocative Efficiency:

Up till now we have been discussing the input oriented efficiency measures that require information on input and output quantities only. If information on input prices is also available, the analysis can be extended to quantify the broader concept of input oriented *overall efficiency* or the *Cost Efficiency*. The input oriented *overall efficiency* measure so obtained then can be decomposed into the *Allocative Efficiency* and the *Technical Efficiency* components. The purpose of this section is to explore these extensions in the input oriented efficiency concepts. We assume, as per requirement of the analysis, the availability of the information on input prices along with the quantity data sets. In what follows we continue with our basic assumptions to define and compare extended concepts of radial and non-radial efficiency measures along with an additional behavioral assumption of cost minimization. The efficiency of any sample point then is to be gauged through the attainment of its objective of cost minimization, given the input vector. If $\boldsymbol{w} \in \Re^N_+$ represents the vector of input prices then for the correspondence

L: $\Re^M_+ \rightarrow \Re^N_+$ we can write the cost function as $C(\boldsymbol{y}, \boldsymbol{w}) = \min \{\boldsymbol{w}^T\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$. The minimum exists because $L(\boldsymbol{y})$ has been assumed nonempty, bounded and closed, thereby implying continuity of $\boldsymbol{w}^T\boldsymbol{x}$. The resulting cost function has following properties[4]

C1. $C(\mathbf{0}, \boldsymbol{w}) = 0 \; \forall \; \boldsymbol{w} \in \Re^N_{++}$. $C(\boldsymbol{y}, \boldsymbol{w}) = 0 \; \forall \; \boldsymbol{y} \in \Re^M_+$ and $\boldsymbol{w} = \mathbf{0}$.

C2. $C(\boldsymbol{y}, \boldsymbol{w^*}) \geqq C(\boldsymbol{y}, \boldsymbol{w})$ for $\forall \; \boldsymbol{y} \in \Re^M_+$ and $\boldsymbol{w^*} \geq \boldsymbol{w}$.

C3a. $C(\nu\boldsymbol{y}, \boldsymbol{w}) \leqq C(\boldsymbol{y}, \boldsymbol{w})$, $\nu \in [0, 1]$. $\forall \; \boldsymbol{y} \in \Re^M_+$, $\boldsymbol{w} \in \Re^N_+$.

C3b. $C(\boldsymbol{y}, \boldsymbol{w}) \leqq C(\hat{\boldsymbol{y}}, \boldsymbol{w})$, $\hat{\boldsymbol{y}} \geq \boldsymbol{y} \; \forall \; \boldsymbol{y} \in \Re^M_+$, $\hat{\boldsymbol{y}} \in \Re^M_+$, $\boldsymbol{w} \in \Re^N_+$.

C4. . $C(\boldsymbol{y}, \theta\boldsymbol{w}) = \theta C(\boldsymbol{y}, \boldsymbol{w}) \; \forall \; \boldsymbol{y} \in \Re^M_+$, $\boldsymbol{w} \in \Re^N_+ \; \theta \geq 0$.

C5. $C(\boldsymbol{y}, \boldsymbol{w}) > 0 \Rightarrow C(\lambda\boldsymbol{y}, \boldsymbol{w}) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$

C6. $C(\boldsymbol{y}, \boldsymbol{w})$ is concave in $\boldsymbol{w} \in \Re^N_+ \; \forall \; \boldsymbol{y} \in \Re^M_+$ and continuous in prices.

C7. $C(\boldsymbol{y}, \boldsymbol{w})$ is upper semi-continuous in $\boldsymbol{y} \in \Re^M_+$ for $\boldsymbol{w} \in \Re^M_{++}$.

For the given input price vector $\boldsymbol{w}$, define the following upper contour set

$C(\boldsymbol{w}) = \{\boldsymbol{x} \mid \boldsymbol{w}^T\boldsymbol{x} \geq C(\boldsymbol{y}, \boldsymbol{w}), \boldsymbol{w}^T\boldsymbol{x^*} < C(\boldsymbol{y}, \boldsymbol{w}) \Rightarrow \boldsymbol{x^*} \notin \mathrm{C}(\boldsymbol{w})\}$

In words $C(\boldsymbol{w})$ is the set of all those input vectors that cost equal to or greater than $C(\boldsymbol{y}, \boldsymbol{w})$ and no element of the set costs less than $C(\boldsymbol{y}, \boldsymbol{w})$. From this it follows that $L(\boldsymbol{y}) \subseteq C(\boldsymbol{w})$.

The following subset of $\mathrm{C}(\boldsymbol{w})$ represents an iso-cost level

$IsoC(\boldsymbol{w}) = \{\boldsymbol{x} \mid \boldsymbol{w}^T\boldsymbol{x} = C(\boldsymbol{y}, \boldsymbol{w})\}$

[4] For proofs see Shephard (1970).

From this set we can define a subset comprising of the input vectors $\boldsymbol{x} \in L(\boldsymbol{y})$ such that the cost of producing $\boldsymbol{y} \in \Re^{M}_{+}$ is equal to $C(\boldsymbol{y}, \boldsymbol{w})$ i.e.

$CM(\boldsymbol{y}, \boldsymbol{w}) = \{\boldsymbol{x} \mid \boldsymbol{w}^{T}\boldsymbol{x} = C(\boldsymbol{y}, \boldsymbol{w}), \boldsymbol{x} \in L(\boldsymbol{y})\}$

Proposition 2.5: $\cup_{\boldsymbol{w} > \boldsymbol{0}} CM(\boldsymbol{y}, \boldsymbol{w}) \subseteq EffL(\boldsymbol{y})$

<u>Proof</u>:

Let $\boldsymbol{x} \in L(\boldsymbol{y}), \boldsymbol{y} \geq \boldsymbol{0}$. Also let $\boldsymbol{x} \in CM(\boldsymbol{y}, \boldsymbol{w})$ for $\boldsymbol{w} > \boldsymbol{0}$. If $\boldsymbol{x} \notin EffL(\boldsymbol{y}) \Rightarrow \exists\, \boldsymbol{x}^{*} \in EffL(\boldsymbol{y})$ such that $\boldsymbol{x}^{*} \leq \boldsymbol{x} \Rightarrow \boldsymbol{w}^{T}\boldsymbol{x}^{*} < \boldsymbol{w}^{T}\boldsymbol{x}$**.** But this contradicts with $\boldsymbol{x}^{*} \in CM(\boldsymbol{y}, \boldsymbol{w})$. The reverse however does not hold as $\boldsymbol{x} \in L(\boldsymbol{y})$ is necessary but not a sufficient condition for $\boldsymbol{x} \in CM(\boldsymbol{y}, \boldsymbol{w})$.

Definition: The function $OE_{I}(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = \{ C(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^{T}\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$ is called the input oriented overall efficiency measure.

An input vector $\boldsymbol{x} \in CM(\boldsymbol{y}, \boldsymbol{w}) \Rightarrow \boldsymbol{w}^{T}\boldsymbol{x} = C(\boldsymbol{y}, \boldsymbol{w}) \Rightarrow$ from proposition 2.5 $OE_{I}(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1 \Rightarrow$ for the input vector $\boldsymbol{x} \in CM(\boldsymbol{y}, \boldsymbol{w})$, $\nexists\, \boldsymbol{x}^{*} \in L(\boldsymbol{y})$ such that $\boldsymbol{w}^{T}\boldsymbol{x}^{*} < \boldsymbol{w}^{T}\boldsymbol{x}$. For any input vector $\boldsymbol{x} \in L(\boldsymbol{y}) \setminus CM(\boldsymbol{y}, \boldsymbol{w})$ we have $OE_{I}(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) < 1$ thus implying input oriented overall inefficiency or cost inefficiency. This cost inefficiency may be technical in nature or allocative or a combination of both. Finally, inefficiency is higher for lower values of $OE_{I}(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$.

Proposition 2.6: Input oriented Overall Efficiency satisfies the following, Färe, Grosskopf and Lovell(1993), for $\boldsymbol{x} \in L(\boldsymbol{y})$ and $\boldsymbol{x}^{*} \in L(\boldsymbol{y})$

$OC1.$ $OE_{I}(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \in (0, 1]$

$OC2.$ $OE_{I}(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) = \lambda^{-1} OE_{I}(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}), \lambda > 0$

$OC3$. $OE_I(\nu \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq \nu\, OE_O(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$, $\nu > 0$.

$OC4$. $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x^*}) > OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ for $\boldsymbol{x^*} \leq \boldsymbol{x}$

$OC5$. $OE_I(\boldsymbol{y}, \theta \boldsymbol{w}, \boldsymbol{x}) = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ $\theta > 0$

$OC6$. $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1$ for $C(\boldsymbol{y}, \boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{x}$.

$OC7$. $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ satisfies commensurability.

Proofs:

$OC1$. For any input vector $\boldsymbol{x} \in L(\boldsymbol{y})$ and $\boldsymbol{x}^* \in L(\boldsymbol{y})$ we have

$C(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{x}^*(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T\boldsymbol{x} \Rightarrow \boldsymbol{w}^T\boldsymbol{x}^*(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T\boldsymbol{x} \leq 1$, where $\boldsymbol{x}^*(\boldsymbol{y}, \boldsymbol{w})$ is the conditional factor demand function.. Also, from C1, $C(\boldsymbol{0}, \boldsymbol{w}) / \boldsymbol{w}^T\boldsymbol{x} = 0$.

$OC2$. $OE_I(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) = \{C(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T \lambda\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$

$\Rightarrow OE_I(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) = \{\lambda^{-1}(C(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T \boldsymbol{x}) \mid \boldsymbol{x} \in L(\boldsymbol{y})\} = \lambda^{-1}\, OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

$OC3$.Even with constant returns to scale equality may not hold unless homothetic production function is assumed. Thus $C(\nu\boldsymbol{y}, \boldsymbol{w}) \leq \nu C(\boldsymbol{y}, \boldsymbol{w}) \Rightarrow OE_I(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq \nu OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

$OC4$. This follows because for $\boldsymbol{x^*} \leq \boldsymbol{x}$ we have $\boldsymbol{w}^T\boldsymbol{x^*} < \boldsymbol{w}^T\boldsymbol{x}$. Thus $C(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T\boldsymbol{x^*} > C(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T\boldsymbol{x}$.

$OC5$. This follows from C4. Thus we have

$OE_I(\boldsymbol{y}, \theta\boldsymbol{w}, \boldsymbol{x}) = \{ C(\boldsymbol{y}, \theta\boldsymbol{w}) / (\theta\boldsymbol{w}^T)\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$

$OE_I(\boldsymbol{y}, \theta\boldsymbol{w}, \boldsymbol{x}) = \{ \theta C(\boldsymbol{y}, \boldsymbol{w}) / (\theta\boldsymbol{w}^T)\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y})\} = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

$OC6$. Suppose not $\Rightarrow OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1$ and $C(\boldsymbol{y}, \boldsymbol{w}) < \boldsymbol{w}^T\boldsymbol{x} \Rightarrow$ from proposition 2.5 that

$\boldsymbol{x} \in L(\boldsymbol{y}) \backslash CM(\boldsymbol{y}, \boldsymbol{w}) \Rightarrow OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) < 1 \Rightarrow$ a contradiction.

*OC*7. Let $\boldsymbol{x}^* \in CM(\boldsymbol{y}, \boldsymbol{w})$ so that $C(\boldsymbol{y}, \boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{x}^*$. Then

$OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = C(\boldsymbol{y}, \boldsymbol{w})/\boldsymbol{w}^T\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{x}^*(\boldsymbol{y}, \boldsymbol{w})/\boldsymbol{w}^T\boldsymbol{x} = \alpha(\boldsymbol{w}^T\boldsymbol{x})/\boldsymbol{w}^T\boldsymbol{x}, \alpha \in (0, 1]$. For an N×N diagonal matrix Λ with the diagonal elements greater than or equal to zero we can write

$\alpha(\boldsymbol{w}^T\boldsymbol{x})/\boldsymbol{w}^T\boldsymbol{x} = \alpha(\boldsymbol{w}^T(\Lambda\boldsymbol{x}))/\boldsymbol{w}^T(\Lambda\boldsymbol{x}) = \alpha((\boldsymbol{w}^T\Lambda)\boldsymbol{x})/(\boldsymbol{w}^T\Lambda)\boldsymbol{x}$. Thus $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ is not sensitive to the changes in the units in which input and costs are measured.

Definition: The function FE_A: $\mathbb{R}^M_+ \times \Re^N_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$ for q = M, N, defined by $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = \{ OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ FE_I(\boldsymbol{y}, \boldsymbol{x}) \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$ is called the Farrell input oriented allocative efficiency measure.

A value of $FE_A(\boldsymbol{y}\ \boldsymbol{w}, \boldsymbol{x}) = 1$ indicates that the data point is allocatively efficient. $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) < 1$ implies allocative inefficiency with inefficiency increasing for higher values of $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$. We have already shown in proposition 2.5 that the cost efficiency implies technical efficiency. Thus a value of $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1 \Rightarrow FE_I(\boldsymbol{y}, \boldsymbol{x}) = 1$ and we get $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1$. Since the reverse implication between $FE_I(\boldsymbol{y}, \boldsymbol{x})$ and $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ does not hold thus there is a possibility that a sample point may be technically efficient without being cost efficient, thereby indicating $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) <1 \Rightarrow FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) < 1$ and $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$. Figure (2.8) shows various possibilities for a given output vector and the input price vector. Data point $\boldsymbol{x}^* \in L(\boldsymbol{y})$ satisfies both technical and allocative efficiency conditions and thus $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1$. Data point $\boldsymbol{x} \in L(\boldsymbol{y})$ is both technically and allocatively inefficient. The component of technical efficiency is $FE_I(\boldsymbol{y}, \boldsymbol{x}) = 1/\psi_I(\boldsymbol{y}, \boldsymbol{x})$. The Farrell allocative efficiency component $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ is obtained after adjusting for Farrell technical efficiency i.e.

$FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = C(\boldsymbol{y}, \boldsymbol{w})/(\boldsymbol{w}^T \boldsymbol{x}/\psi_I(\boldsymbol{y}, \boldsymbol{x})) = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ FE_I(\boldsymbol{y}, \boldsymbol{x})$

$\Rightarrow OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \times FE_I(\boldsymbol{y}, \boldsymbol{x})$

which describes the decomposition of overall efficiency into allocative and technical components using Farrell definition of technical efficiency.

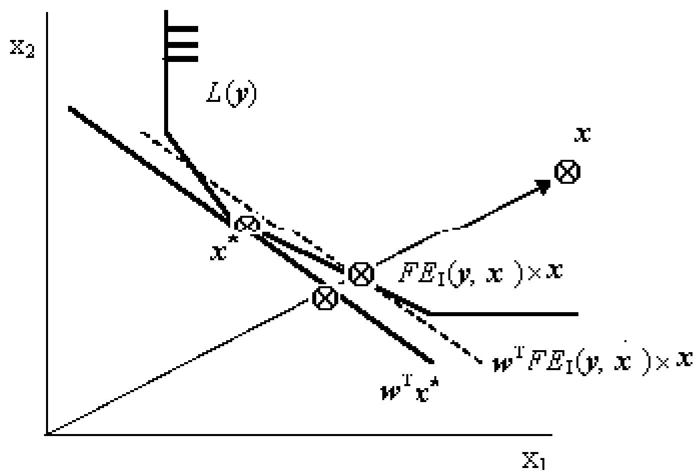

Figure 2.8. Farrell's decomposition of cost efficiency

Proposition 2.7: $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ satisfies the following

*AE*1. $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \in (0,\ 1]$

*AE*2. $FE_A(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) = FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}), \lambda > 0$

*AE*3. $FE_A(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}), \nu > 0$

*AE*4. $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \lesseqgtr FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}^*)$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$

$AE5$. $FE_A(\boldsymbol{y}, \theta\boldsymbol{w}, \boldsymbol{x}) = FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

$AE6$. $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ satisfies commensurability

Proofs:

$AE1$. Follows because $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \in (0, 1]$ and $FE_I(\boldsymbol{y}, \boldsymbol{x}) \in (0, 1]$ and $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq FE_I(\boldsymbol{y}, \boldsymbol{x})$.

$AE2$. Follows because $OE_I(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) = \lambda^{-1}OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ and $FE_I(\boldsymbol{y}, \lambda\boldsymbol{x}) = \lambda^{-1}FE_I(\boldsymbol{y}, \boldsymbol{x})$.

$AE3$. Equality holds if homothetic production function and constant returns to scale assumed. Otherwise, since $OE_I(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq \nu OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ and $FE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu FE_I(\boldsymbol{y}, \boldsymbol{x}) \Rightarrow OE_I(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ FE_I(\nu\boldsymbol{y}, \boldsymbol{x}) \leq OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ FE_I(\boldsymbol{y}, \boldsymbol{x})$ and so holds the result.

$AE4$. Follows because $OC4$ holds and because $FE_I(\boldsymbol{y}, \boldsymbol{x})$ is weakly monotonic in inputs.

$AE5$. Follows because $OE_I(\boldsymbol{y}, \theta\boldsymbol{w}, \boldsymbol{x}) = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$.

$AE6$. Follows because $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ and $FE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfy commensurability.

The non-radial Russell measure for allocative efficiency is defined as follows;

Definition: The function $RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = \{OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ RE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) | \boldsymbol{x} \in L(\boldsymbol{y})\}$, where $RE^{\mathbf{c}}_I(\boldsymbol{y},\boldsymbol{x})=\boldsymbol{w}^T\theta_{FL}\boldsymbol{x} / \boldsymbol{w}^T\boldsymbol{x}$, θ_{FL} is as defined earlier, is called the Russell input oriented allocative efficiency measure.

A value of $RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1$ implies that the sample point in question satisfies Russell allocative efficiency conditions. A value of $RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) < 1$ indicates allocative inefficiency which is higher for lower values of $RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$.

It is worth mentioning at this point one important difference between radial and non radial allocative efficiency measures which is due to the dual character of the radial technical efficiency score. In case of the Farrell measure, $FE_I(\boldsymbol{y}, \boldsymbol{x})$ represents not only the technical efficiency score but it also represents the magnitude of the (equiproportionate) adjustment in the data point in question needed to obtain the efficient vector so that the reference input vector in this case is $FE_I(\boldsymbol{y}, \boldsymbol{x}) \times \boldsymbol{x} = \theta_{DF}\boldsymbol{x}$. Same is not the case however for the non-radial measures. In the case of Russell measure, for example, the technical efficiency score, $RE_I(\boldsymbol{y}, \boldsymbol{x})$, serves as a ranking instrument only while the diagonal matrix θ_{FL} defines the adjustments in sample point in question needed to obtain the technically efficient vector. Thus where we can define the projection vector in the radial case as $FE_I(\boldsymbol{y}, \boldsymbol{x})\times\boldsymbol{x}$, the Russell projection vector is defined by $\theta_{FL}\boldsymbol{x} \neq RE_I(\boldsymbol{y}, \boldsymbol{x})\times\boldsymbol{x}$. The outcome is that the cost value $\boldsymbol{w}^T\times RE_I(\boldsymbol{y}, \boldsymbol{x})\times\boldsymbol{x}$ is different from the cost value $\boldsymbol{w}^T\times \theta_{FL}\boldsymbol{x}$. This means that for any $\boldsymbol{x} \in L(\boldsymbol{y})$ in the case of Farrell measure we have $\|\theta\boldsymbol{x}\|/\|\boldsymbol{x}\| = \boldsymbol{w}^T\theta\boldsymbol{x} / \boldsymbol{w}^T\boldsymbol{x} = FE_I(\boldsymbol{y}, \boldsymbol{x})$ which implies that the Farrell quantity based technical efficiency measure coincides with the Farrell cost based technical efficiency score. The same does not hold for the Russell measure as it is not unlikely that $\boldsymbol{w}^T\theta_{FL}\boldsymbol{x} / \boldsymbol{w}^T\boldsymbol{x}. \neq RE_I(\boldsymbol{y}, \boldsymbol{x})$. This means that to define the Russell allocative efficiency we need to use $RE^c_I(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{w}^T\theta_{FL}\boldsymbol{x} /\boldsymbol{w}^T\boldsymbol{x}$, to allow for symmetric decompositions of overall efficiency into its components. This reasoning justifies the use of cost based technical efficiency measure $RE^c_I(\boldsymbol{x}, \boldsymbol{y})$ while computing Russell input oriented allocative efficiency component and the same holds true for all the other non-radial measures as well. Figure (2.9) shows various possibilities and reflects the decomposition of overall efficiency into Russell allocative and technical efficiency components. Data point $\boldsymbol{x}^* \in L(\boldsymbol{y})$ satisfies both technical and allocative efficiency conditions and thus $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}^*) = 1$. The data point $\boldsymbol{x} \in L(\boldsymbol{y})$ is both technically and allocatively inefficient. In order to obtain the Russell allocative component of overall efficiency for this data point we first identify the Russell technically efficient input vector $\theta_{FL}\boldsymbol{x} \in L(\boldsymbol{y})$ using the quantity based technical efficiency measure The Russell allocative efficiency component $RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ is then given as $RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = (C(\boldsymbol{y}, \boldsymbol{w})/ \boldsymbol{w}^T\boldsymbol{x})/(\boldsymbol{w}^T\theta_{FL}\boldsymbol{x} / \boldsymbol{w}^T\boldsymbol{x}) = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ RE^c_I(\boldsymbol{y}, \boldsymbol{x})$ which implies following decomposition of overall efficiency;

$OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \times RE^c_I(\boldsymbol{y}, \boldsymbol{x})$

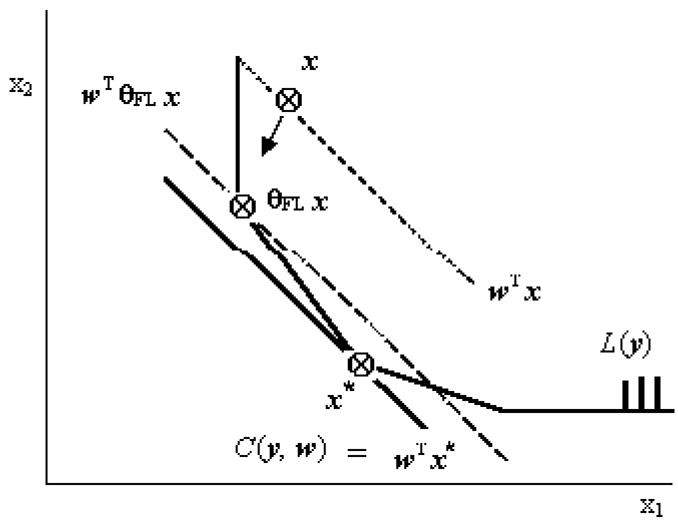

Figure 2.9. Russell decomposition of cost efficiency

Comparing the Russell allocative efficiency with that of Farrell allocative efficiency component, whenever we have $\theta_{FL} = \theta\mathbf{I}$ where θ is some constant and $\mathbf{I}$ is an N×N identity matrix, the Russell and Farrell allocative efficiency components coincide and so does the decomposition of overall efficiency across two measures. However when this is not the case the relationship between Farrell and Russell allocative efficiency components is ambiguous. This point is illustrated with the help of figure (2.10) for the data point $\boldsymbol{x} \in L(\boldsymbol{y})$. In the figure three possibilities have been shown with respect to the Russell projection vector. If the Russell projection vector coincides with the Farrell projection vector, the two are identical in terms of the allocative efficiency. If the Russell projection vector is $\theta_{FL}\boldsymbol{x}$, the allocative efficiency component associated with the Russell measure is less than Farrell allocative efficiency. On the contrary the Russell allocative efficiency component is greater than that of the Farrell allocative efficiency if the Russell projection vector is identified by $\theta^*_{FL}\boldsymbol{x}$. The source of this ambiguity is the ambiguous relationship between the cost based Russell input oriented technical efficiency and the cost based Farrell input oriented technical efficiency measures.

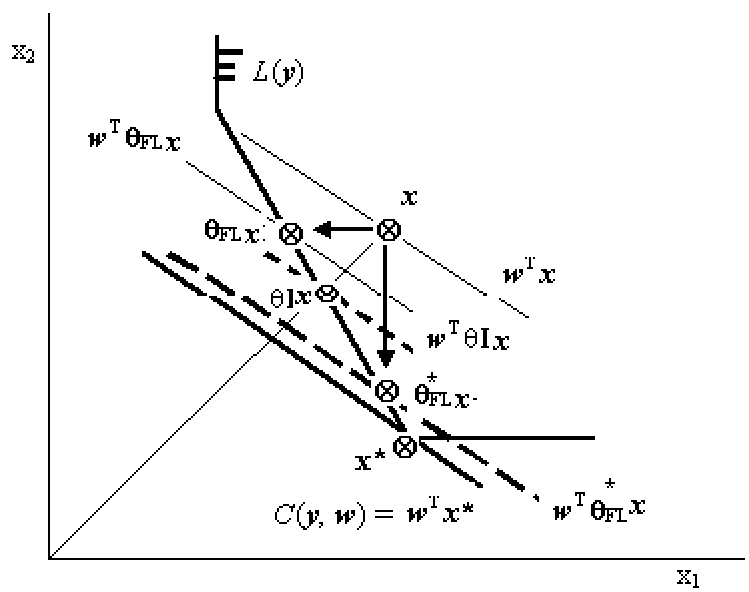

Figure 2.10. Comparing Russell and Farrell decomposition of $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

.

Proposition 2.8: The non radial price based technical efficiency measure satisfies the following;

TE^C1. $TE^{\mathbf{c}}(\boldsymbol{y}, \boldsymbol{x}) \in (0, 1]$

TE^C2. $TE^{\mathbf{c}}(\boldsymbol{y}, \lambda\boldsymbol{x}) = \lambda^{-1}\, TE^{\mathbf{c}}{}_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})\ \lambda > 0$

TE^C3. $TE^{\mathbf{c}}(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu TE^{\mathbf{c}}{}_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$

TE^C4. $TE^{\mathbf{c}}(\boldsymbol{y}, \boldsymbol{x}^*) < TE^{\mathbf{c}}{}_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$

TE^C5. $TE^{\mathbf{c}}(\boldsymbol{y}, \boldsymbol{x})$ satisfies commensurability

The weaker versions of TE^C2 and TE^C4 are given as under

TE^C2w. $TE^{\mathbf{c}}(\boldsymbol{y}, \lambda\boldsymbol{x}) \gtreqless \lambda^{-1}\, TE^{\mathbf{c}}{}_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})\ \lambda > 0$

TE^C4w. $TE^{\mathbf{c}}(\boldsymbol{y}, \boldsymbol{x}^*) \gtreqless TE^{\mathbf{c}}{}_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$

Proposition 2.8a: The Russell non radial price based technical efficiency measure satisfies TE^C1, TE^C3, TE^C5, TE^C2w and TE^C4w.

Proofs:

TE^C1 Follows because $RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x}) \in (0.\ 1]$.

TE^C2w. Let $\boldsymbol{x} \in L(\boldsymbol{y})$ such that $RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x}) = 1$. For $\lambda > 1 \Rightarrow RE_{\mathrm{I}}(\boldsymbol{y}, \lambda\boldsymbol{x}) < 1 \Rightarrow \exists\ \theta_{\mathrm{FL}}$ such that

$RE_{\mathrm{I}}(\boldsymbol{y}, \theta_{FL}(\lambda\boldsymbol{x})) = 1$ and $\boldsymbol{w}^{\mathrm{T}}\theta_{\mathrm{FL}}(\lambda\boldsymbol{x}) \gtreqless \boldsymbol{w}^{\mathrm{T}}\theta\mathbf{I}(\lambda\boldsymbol{x})$, where θ is some constant and $\mathbf{I}$ is an N×N identity matrix. RE^C2 holds because $\theta\mathbf{I}(\lambda\boldsymbol{x}) = \boldsymbol{x}$.

TE^C3. This follows because for constant returns to scale we have $RE_{\mathrm{I}}(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu\, RE_{\mathrm{I}}(\boldsymbol{y}, \boldsymbol{x})$.

TE^C4w. Suppose that $\boldsymbol{x} \in L(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$ and that $RE_{\mathrm{I}}(\boldsymbol{y}, \theta_{\mathrm{FL}}\boldsymbol{x}) = 1$. Let $\boldsymbol{x}^* \geq \boldsymbol{x}$ such that $RE_{\mathrm{I}}(\boldsymbol{y}, \theta^*_{\mathrm{FL}}\boldsymbol{x}^*) = 1$. Two possibilities exists;

i) $\theta^*_{FL}\boldsymbol{x}^*$ coincides with $\theta_{FL}\boldsymbol{x}$.

ii) $\theta^*_{FL}\boldsymbol{x}^*$ does not coincide with $\theta_{FL}\boldsymbol{x}$.

i) $\Rightarrow \boldsymbol{w}^T\theta^*_{FL}\boldsymbol{x}^* / \boldsymbol{w}^T\boldsymbol{x}^* < \boldsymbol{w}^T\theta_{FL}\boldsymbol{x}/\boldsymbol{w}^T\boldsymbol{x} \Rightarrow RE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}^*) < RE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) \Rightarrow$ monotonicity.

ii) $\Rightarrow \boldsymbol{w}^T\theta^*_{FL}\boldsymbol{x}^* / \boldsymbol{w}^T\boldsymbol{x}^* \lesseqgtr \boldsymbol{w}^T\theta_{FL}\boldsymbol{x}/\boldsymbol{w}^T\boldsymbol{x} \Rightarrow RE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}^*) \lesseqgtr RE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) \Rightarrow TE^C$4w.

TE^C5. Since $RE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies commensurability so does $RE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x})$

Proposition 2.8b: The Russell Allocative Efficiency measure satisfies the following

*RE*1. $RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \in (0,\ 1]$

*RE*2.$RE_A(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) \gtreqless RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$, $\lambda>0$

*RE*3. $RE_A(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

*RE*4. $RE_A(\boldsymbol{y}, \mathbf{r}, \boldsymbol{x}^*) \gtreqless RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$

*RE*5. $RE_A(\boldsymbol{y}, \theta\boldsymbol{w}, \boldsymbol{x}) = RE_A(\boldsymbol{y}, \mathbf{w}, \boldsymbol{x})$

*RE*6. $RE_A(\boldsymbol{y}, \mathbf{w}, \boldsymbol{x})$ satisfies commensurability

<u>Proofs</u>:

*RE*2 follows from TE^C2w while *RE*4 follows from TE^C4w. Explanations of the rest of the properties are similar to proposition 2.7.

Definition: The function $ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = \{ OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$ where $ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{w}^T\theta_{ZE}\boldsymbol{x} / \boldsymbol{w}^T\boldsymbol{x}$ and θ_{ZE} is an N×N diagonal matrix with the nth diagonal element equal to or less than one, is called the Zieschang input oriented allocative efficiency measure.

A value of $ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = 1$ implies that the sample point in question satisfies Zieschang allocative efficiency conditions. A value of $ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) < 1$ indicates allocative inefficiency which is higher for lower values of $ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$. With the allocative efficiency measure so defined, the Zieschang decomposition of the overall efficiency is given as follows;

$ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = C(\boldsymbol{y}, \boldsymbol{w})/(\boldsymbol{w}^T \boldsymbol{x} \times ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x})) = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})/ ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x})$

$\Rightarrow OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \times ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x})$

Comparing Zieschang allocative efficiency with Farrell, all the diagonal elements of θ_{ZE} are identical when Zieschang and Farrell measure of technical efficiency coincide and thus we get identical decompositions of overall efficiency for the two measures. This is the case when the Farrell projection of an input vector $\boldsymbol{x} \in L(\boldsymbol{y})$ falls on $EffL(\boldsymbol{y})$. On the other hand if the projection falls on $IsoqL(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$, some of the diagonal elements of θ_{ZE} also absorb Russell type adjustment in addition to the Farrell adjustment. Thus all the elements of θ_{ZE} are not identical in which case the Zieschang allocative efficiency component is greater than that of Farrell allocative efficiency component. Comparing the Zieschang allocative efficiency with the Russell allocative efficiency, whenever $ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) = FE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x})$, the relationship between the Russell and the Zieschang allocative efficiency components is not neatly defined due to the same reasons for which it is not neatly defined for Russell and Farrell allocative efficiency. For the data points for which $ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) < FE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x})$, we have $RE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) \leq ZE^{\mathbf{c}}_I(\boldsymbol{y}, \boldsymbol{x}) \Rightarrow RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \geq ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$.

Proposition 2.9a: The Zieschang price based technical efficiency measure satisfies properties TE^C1 to TE^C5.

Proofs:

For TE^C2 proof follows from the strict homogeneity of the Zieschang input based technical efficiency measure. For rest of the properties proofs are similar to those given in proposition 2.8a.

Proposition 2.9b: The Zieschang Allocative Efficiency measure satisfies

$ZE1$. $ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \in (0, 1]$

$ZE2$. $ZE_A(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) = ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

$ZE3$. $ZE_A(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

$ZE4$. $ZE_A(\boldsymbol{y}, \mathbf{r}, \boldsymbol{x}^*) \lesseqgtr ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$

$ZE5$. $ZE_A(\boldsymbol{y}, \theta\boldsymbol{w}, \boldsymbol{x}) = ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

$ZE6$. $ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ satisfies commensurability

Proofs:

Similar to proposition 2.7.

The Non-radial Farrell measure of allocative efficiency is defined in the same manner. However like the quantity based technical efficiency, the Non-radial Farrell measure of cost efficiency and input oriented allocative efficiency are also computed in the transformed data space. Below we first obtain the necessary transformations of the data set and then define the Non-radial Farrell overall efficiency measure. Once the overall efficiency has been defined for the transformed data set it can be used along with the technical efficiency component to obtain the allocative efficiency measure. The transformed data set with respect to any data point is $({}_s\boldsymbol{x}, {}_s\boldsymbol{y}, {}_s\boldsymbol{w})$ where ${}_s\boldsymbol{x}$ and ${}_s\boldsymbol{y}$ are as defined earlier and ${}_s\boldsymbol{w} \in \Re^N_+$ is an N×1 vector the nth element of which is the cost of the employment of the nth input incurred by entire sample i.e. ${}_sw_n = w_n\sum^J_{j=1} x_{nj}$, n= 1,2,..N. With this definition ${}_sw_n \times {}_sx_{nk}$, where ${}_sx_{nk} = x_{nk}/\sum^J_{j=1} x_{nj}$, gives the cost on the nth input incurred by the kth firm. Now we can define the following upper contour set for the transformed data

$$C^S({}_s\boldsymbol{w}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} \geq C({}_s\boldsymbol{y}, {}_s\boldsymbol{w}), {}_s\boldsymbol{w}^T{}_s\boldsymbol{x}^* > C({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) \Rightarrow {}_s\boldsymbol{x}^* \notin C({}_s\boldsymbol{w})\}$$

In words $C({}_s\boldsymbol{w})$ is the set of all those input share vectors that cost equal to or greater than $C({}_s\boldsymbol{y}, {}_s\boldsymbol{w})$ and no element of the set costs less than $C({}_s\boldsymbol{y}, {}_s\boldsymbol{w})$. From this it follows that $L^S({}_s\boldsymbol{y}) \subseteq C({}_s\boldsymbol{w})$. The following subset of $C({}_s\boldsymbol{w})$ represents an isocost level

$IsoC({}_s\boldsymbol{w}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} = C({}_s\boldsymbol{y}, {}_s\boldsymbol{w})\}$

From this set we can define a subset comprising of the input share vectors that can produce an output share vector ${}_s\boldsymbol{y}$ and cost equal to $C({}_s\boldsymbol{y}, {}_s\boldsymbol{w})$ i.e.

$CM({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} = C({}_s\boldsymbol{y}, {}_s\boldsymbol{w}), {}_s\boldsymbol{x} \in L^S({}_s\boldsymbol{y})\}$

Also from the definition of $C({}_s\boldsymbol{y}, {}_s\boldsymbol{w})$ we have

$L^S({}_s\boldsymbol{y}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{x} \in \Re^N_+, {}_s\boldsymbol{y} \in \Re^M_+, {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} \geq C({}_s\boldsymbol{y}, {}_s\boldsymbol{w})\}$

$\Rightarrow CM({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) = L^S({}_s\boldsymbol{y}) \cap IsoC({}_s\boldsymbol{w})$

An input share vector ${}_s\boldsymbol{x} \in CM({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) \Rightarrow {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} = C({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) \Rightarrow C({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) / {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} = 1 \Rightarrow$ for the input vector ${}_s\boldsymbol{x} \in CM({}_s\boldsymbol{y}, {}_s\boldsymbol{w})$, $\nexists\ {}_s\boldsymbol{x}^* \in L^S({}_s\boldsymbol{y})$ such that ${}_s\boldsymbol{w}^T{}_s\boldsymbol{x}^* < {}_s\boldsymbol{w}^T{}_s\boldsymbol{x}$.

Definition: The function $OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = \{ C({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) / {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} \mid {}_s\boldsymbol{x} \in L^S({}_s\boldsymbol{y})\}$ is called the Non-radial Farrell input oriented overall efficiency measure.

But $C({}_s\boldsymbol{y}, {}_s\boldsymbol{w}) / {}_s\boldsymbol{w}^T{}_s\boldsymbol{x} = C(\boldsymbol{y}, \boldsymbol{w}) / \boldsymbol{w}^T\boldsymbol{x} \Rightarrow OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$. Thus the overall efficiency measure associated with the Non-radial Farrell is identical to the other three measures and the transformation of the data set has no bearing on the overall efficiency scores.

Proposition 2.10: $OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})$ satisfies *OE*1 to *OE*6.

Proofs:

Since $OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ and $OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ satisfies *OE*1 to *OE*6 so does $OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})$.

The Non-radial Farrell allocative efficiency is defined for the transformed data set as follows

Definition: The function $NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = \{ OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) / NE^{\mathbf{c}}_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \mid {}_s\boldsymbol{x} \in L^S({}_s\boldsymbol{y})\}$ where $NE^{\mathbf{c}}_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = {}_s\boldsymbol{w}^T\theta_{NE}\, {}_s\boldsymbol{x} \,/\, {}_s\boldsymbol{w}^T{}_s\boldsymbol{x}$ and θ_{NE} is an N×N diagonal matrix with the nth diagonal element equal to or less than one is called the Non-radial Farrell input oriented allocative efficiency measure.

$NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = 1$ implies that the data point in question is allocatively efficient. Inefficiency associated with any data point is reflected by a value of $NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})$ less than one.

From the above definition of Non-radial Farrell allocative efficiency measure we can arrive at the following decomposition of the overall efficiency

$NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})/({}_s\boldsymbol{w}^T\theta_{NE}\, {}_s\boldsymbol{x} \,/\, {}_s\boldsymbol{w}^T{}_s\boldsymbol{x})$

$\Rightarrow NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) / NE^{\mathbf{c}}_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$

$\Rightarrow OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) \times NE^{\mathbf{c}}_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$

The above decomposition shows that whenever we can write $\theta_{NE} = \theta\mathbf{I}$,

$NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})/({}_s\boldsymbol{w}^T\theta\mathbf{I}\, {}_s\boldsymbol{x} \,/\, {}_s\boldsymbol{w}^T{}_s\boldsymbol{x})$

$= OE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})/(\theta\, {}_s\boldsymbol{w}^T\, {}_s\boldsymbol{x} \,/\, {}_s\boldsymbol{w}^T{}_s\boldsymbol{x})$

$\Rightarrow NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x}) = FE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})$.

However when this is not the case the relationship between $NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})$ and $FE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})$ is not uniquely defined because of the same reasons as discussed earlier with reference to the comparison between Russell and Farrell allocative efficiency measures. Similar relationships of $NE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{w}, {}_s\boldsymbol{x})$ hold with $RE_A({}_s\boldsymbol{y}, {}_s\boldsymbol{x}, {}_s\boldsymbol{x})$ and $ZE_A({}_s\boldsymbol{y}, {}_s, {}_s\boldsymbol{x})$. Thus for example if $\theta_{NE}\boldsymbol{x}$, the projection vector for the non-radial Farrell measure, is identical to $\theta_{FL}\boldsymbol{x}$, the projection vector for the Russell measure, then we get identical price based technical efficiency which leads to identical allocative efficiency components for the two measures. If this is not the case the relationship becomes ambiguous. Note that this is in contrast with the quantity based technical efficiency measures where we can not rule out the possibility of different quantity based technical efficiency scores associated with the Russell and the Non-radial Farrell measures despite the identical projection vectors. Finally, when $FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) = ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$,

$\Rightarrow ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \gtreqless NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$. When this is not the case we have $NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \geq ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$.

Proposition 2.11a: The Non-radial Farrell Technical Efficiency measure satisfies proposition 2.8a.

<u>Proofs</u>:

Similar to propositions 2.8a .

Proposition 2.11b: The Non-radial Farrell Allocative Efficiency measure satisfies the following;

*NE*1. $NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \in (0,\ 1]$

*NE*2. $NE_A(\boldsymbol{y}, \boldsymbol{w}, \lambda\boldsymbol{x}) \gtreqless NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$, $\lambda>0$

*NE*3. $NE_A(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \leq NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

*NE*4. $NE_A(\boldsymbol{y}, \mathbf{r}, \boldsymbol{x}^*) \gtreqless NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ for $\boldsymbol{x}^* \geq \boldsymbol{x}$

*NE*5. $NE_A(\boldsymbol{y}, \theta\boldsymbol{w}, \boldsymbol{x}) = NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$

*NE*6. $NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$ satisfies commensurability

<u>Proofs</u>:

Similar to propositions 2.8b

Computation:

Linear programming techniques can be employed to solve for the cost minimization problem $C(\boldsymbol{y}, \boldsymbol{w}) = \min \{\boldsymbol{w}^T\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y})\}$ as follows

$$\begin{aligned} \text{Min}_{\boldsymbol{x},\boldsymbol{\Phi}} \quad & \boldsymbol{w}^T\boldsymbol{x} \\ \text{s.t} \quad & \boldsymbol{x}^j - \boldsymbol{X\Phi} \geq \mathbf{0} \\ & -\boldsymbol{y} + \boldsymbol{Y\Phi} \geq \mathbf{0} \\ & \boldsymbol{\Phi} \geq \mathbf{0}, \; j = 1,2\ldots J \end{aligned}$$

This program is solved once for each firm. The solution thus obtained is used with the actual cost of the respective firm, computed on the basis of the observed input vector, to obtain the overall efficiency score. As a second step the cost based technical efficiency component is obtained by using the relevant projection vector. Thus for example $\theta\boldsymbol{x}$ is used to compute $\boldsymbol{w}^T\theta\boldsymbol{x}/\boldsymbol{w}^T\boldsymbol{x}$ in case one is interested in Farrell decomposition of overall efficiency while $\theta_{FL}\boldsymbol{x}$ is used to compute $\boldsymbol{w}^T\theta_{FL}\boldsymbol{x}/\boldsymbol{w}^T\boldsymbol{x}$ if one is interested in the Russell decomposition and so on. Finally the ratio of the overall efficiency to the relevant cost based technical efficiency gives the corresponding allocative efficiency component.

2.5. Relaxing Assumptions

While discussing the input oriented measures of efficiency in the preceding sections we continued maintaining the assumption of constant returns to scale and strong disposability. This section discusses the implication of relaxing these assumptions with respect to the properties E1a to E6a and with respect to the computation procedure.

2.5.1 Replacing Constant Returns to Scale by Non-Increasing Returns and Variable Returns to Scale

Relaxing constant returns to scale assumption can have implications in two areas of any efficiency measures; it may have implications with respect to the properties, E1a to E6a and it may have implications with respect to the computational aspects. Below we first look into these implications with respect to properties E1a to E6a and then we discuss the computational aspects of relaxing this assumption.

Non-increasing returns to scale technology implies that an equiproportionate increase in all inputs increases the outputs in the same proportion or in a lesser proportion. The implied input correspondence is thus given as

$L_{\check{N}}(\boldsymbol{y}) = \{\boldsymbol{x} \mid \boldsymbol{y} \leq \boldsymbol{Y\Phi}, \boldsymbol{x} \geq \boldsymbol{X\Phi}, \boldsymbol{\Phi} \leq \boldsymbol{0}, \mathbf{e}^T\boldsymbol{\Phi} \leq 1\}$

where $\mathbf{e}$ is a J×1 vector of ones and $\boldsymbol{\Phi}$ is the J×1, as defined earlier, intensity vector. The technology restricts the sum of the intensity vector to not to exceed one. For the technology so defined we have the following definition for Farrell input oriented technical efficiency measure.

Definition: A function $FE_{I\check{N}}(\boldsymbol{y},\boldsymbol{x})$: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined by

$FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x})= \min \{\theta_{DF} \mid \text{for } \boldsymbol{x} \in L_{\check{N}}(\boldsymbol{y}), \theta_{DF}\boldsymbol{x} \in L_{\check{N}}(\boldsymbol{y}), \theta_{DF} \in (0, 1]\}$ and $FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L_{\check{N}}(\boldsymbol{y})$, is the Farrell's input measure of technical efficiency for non-increasing returns to scale technology.

A value of $FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) = 1$ indicates that $\boldsymbol{x} \in L_{\check{N}}(\boldsymbol{y})$ is technically efficient. Technical inefficiency is indicated by value of $FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) < 1$ and lower the value of $FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x})$, greater is the technical inefficiency.

Proposition 2.12 For non-increasing returns to scale technology the Farrell input oriented measure of technical efficiency satisfies E3a, E4aw and E6a. It does not satisfy E1a, E2a, E4a and E5a.

Proofs:

Here we prove the violation of E4a. Proofs of the remaining properties are same as given in proposition 2.1

$FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x})$ fails to satisfy E.4a, the homogeneity of plus one in outputs:

Figure (2.11) shows an increase in output from $\boldsymbol{y}$ to $\lambda\boldsymbol{y}$, $\lambda = 2$, by an outward shift of the isoquant $IsoqL(\boldsymbol{y})$ to $IsoqL(\lambda\boldsymbol{y})$. If we define $f(\nu)$ as the density function of the isoquant where ν is the scale parameter and $d f(\nu)/d\nu > 0$, then $\mu\lambda = (1/f(\nu)) \times \lambda$ defines the extent of shift of an isoquant when output changes by factor λ. Thus higher the value of scale parameter, ν, higher is the isoquant density in an isoquant map and to a lesser extent shifts the isoquant in an input space for a given change in output. We can write $IsoqL(\lambda\boldsymbol{y}) = \mu\lambda IsoqL(\boldsymbol{y})$. Then for $\boldsymbol{x} \in L(\lambda\boldsymbol{y})$ we have $\theta_{DF}\boldsymbol{x} \in IsoqL(\boldsymbol{y})$ and $\mu\lambda\theta_{DF}\boldsymbol{x} \in IsoqL(\lambda\boldsymbol{y})$ with $\theta_{DF}\boldsymbol{x}$, $\mu\lambda\theta_{DF}\boldsymbol{x}$ and $\boldsymbol{x}$ lying on the same ray radiating from the origin. The efficiency score of $\boldsymbol{x}$ with respect to $IsoqL(\boldsymbol{y})$ is $FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) = \|\theta_{DF}\boldsymbol{x}\| / \|\boldsymbol{x}\|$ and the efficiency score of $\boldsymbol{x}$ with respect to $IsoqL(\lambda\boldsymbol{y})$ is $\|\mu\lambda\theta_{DF}\boldsymbol{x}\|/\|\boldsymbol{x}\| = |\mu\lambda|\|\theta_{DF}\boldsymbol{x}\|/\|\boldsymbol{x}\| = \mu\lambda\, FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x})$. For non-increasing returns to scale we have $\nu \leq 1 \Rightarrow \mu \geq 1 \Rightarrow \mu\lambda\, FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) \geq \lambda\, FE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x})$ thus violating the homogeneity of plus one in output.

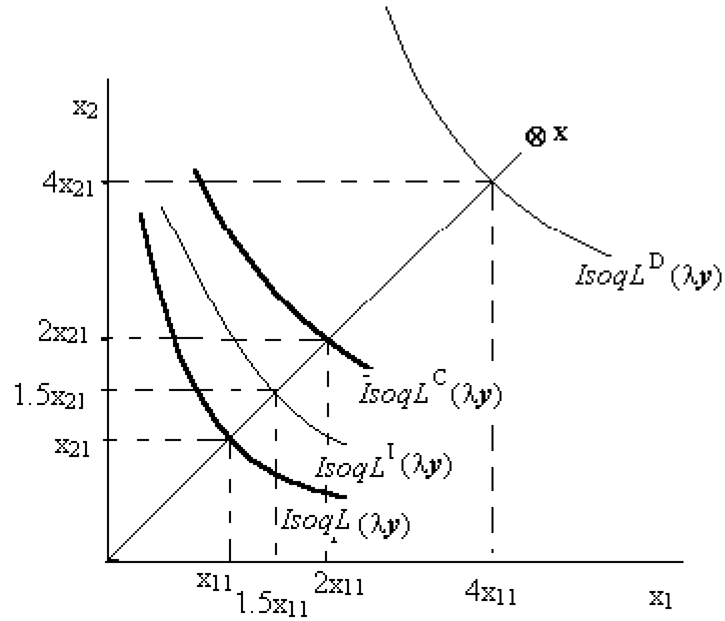

Figure 2.11 Returns to scale and output homogeneity

This is shown by the outward shift of the $IsoqL(\boldsymbol{y})$ to $IsoqL^{C}(\lambda\boldsymbol{y})$, for constant returns to scale, or beyond, for example to $IsoqL^{D}(\lambda\boldsymbol{y})$, for decreasing returns to scale. The variable returns to scale technology adds more strict restrictions on the intensity vector by forcing its sum to equal to one. Thus we get the following input correspondence

$L_V(\boldsymbol{y}) = \{\boldsymbol{x} \mid \boldsymbol{y} \leq \boldsymbol{Y\Phi}, \boldsymbol{x} \geq \boldsymbol{X\Phi}, \boldsymbol{\Phi} \leq \boldsymbol{0}, \mathbf{e}^{T}\boldsymbol{\Phi} = 1\}$

The variable returns to scale version of the Farrell measure is given as follows

Definition: A function $FE_{IV}(\boldsymbol{y}, \boldsymbol{x})$: $\mathbb{R}^{M}{}_{+} \times \mathbb{R}^{N}{}_{+} \rightarrow \Re_{+} \cup \{+\infty\}$, where $\mathbb{R}^{q}{}_{+} = \Re^{q}{}_{+}/\{0\}$, defined by

$FE_{IV}(\boldsymbol{y},\boldsymbol{x}) = \min \{\theta_{DF} | \text{ for } \boldsymbol{x} \in L_V(\boldsymbol{y}), \ \theta_{DF}\boldsymbol{x} \in L_V(\boldsymbol{y}), \theta_{DF} \in (0, 1]\}$ and $FE_{IV}(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L_V(\boldsymbol{y})$, is the Farrell's input measure of technical efficiency for variable returns to scale technology.

$FE_{IV}(\boldsymbol{y}, \boldsymbol{x}) = 1$ implies that the sample point in question is technically efficient in the Farrell sense and the technical inefficiency is reflected by values of $FE_{IV}(\boldsymbol{y}, \boldsymbol{x}) < 1$, with greater inefficiency for successively lower values of $FE_{IV}(\boldsymbol{y}, \boldsymbol{x})$.

Proposition 2.13 $FE_{IV}(\boldsymbol{y}, \boldsymbol{x})$ satisfies E3a, E4aw and E6a. It does not satisfy E1a, E2a, E4a and E5a.

Proofs

Here we prove only the violation of E4a. The proofs of the remaining properties are same as given in proposition 2.1

$FE_{IV}(\boldsymbol{y}, \boldsymbol{x})$ fails to satisfy E.4a, the homogeneity of plus one in outputs:

Extending the argument given in proposition 2.12 for variable returns to scale we have

$\mu \lesseqgtr 1 \Rightarrow \mu\lambda\, FE_{IV}(\boldsymbol{y}, \boldsymbol{x}) \gtreqless \lambda\, FE_{IV}(\boldsymbol{y}, \boldsymbol{x})$. Referring to figure 2.10, for doubling of the output, $IsoqL(\boldsymbol{y})$ may shift to $IsoqL^{I}(\lambda\boldsymbol{y})$, to $IsoqL^{C}(\lambda\boldsymbol{y})$ or to $IsoqL^{D}(\lambda\boldsymbol{y})$, depending upon the value of the scale parameter.

To define the Russell measure of technical efficiency for non-increasing returns to scale let Q be the set of diagonal matrices ${}^{i}\theta_{FL}$, each of dimension N with ${}^{i}\theta_{nn} \in (0,1]$, then

Definition: The function $RE_{I\check{N}}$: $\mathbb{R}^{M}_{+}\times \mathbb{R}^{N}_{+}\rightarrow\Re_{+}\cup\{+\infty\}$, where $\mathbb{R}^{q}_{+}=\Re^{q}_{+}/\{0\}$, defined as

$RE_{I\check{N}}(\boldsymbol{y},\boldsymbol{x})= \min\ \{1/N(\mathbf{e}^{T}\theta_{FL}\mathbf{e})|$ for $\boldsymbol{x} \in L_{\check{N}}(\boldsymbol{y}), \theta_{FL}\boldsymbol{x} \in L_{\check{N}}(\boldsymbol{y})\}$ and $RE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L_{\check{N}}(\boldsymbol{y})$

is called the Russell input oriented measure of technical efficiency for non-increasing returns to scale technology.

$RE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) = 1$ implies that the sample point in question is technically efficient in the Russell sense and the technical inefficiency is reflected by values of $RE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) < 1$, with greater inefficiency for successively lower values of $RE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x})$.

Proposition 2.14 For non-increasing returns to scale technologies the Russell input oriented measure of technical efficiency satisfies E1a, E2a E3aw, E4aw, E5a and E6a. It does not satisfy E3a and E4a.

Proofs:

For E4a proof is similar as in proposition 2.12. For the rest of the properties proofs are identical to those of proposition 2.2

Definition: The function RE_{IV}: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined as

$RE_{IV}(\boldsymbol{y},\boldsymbol{x})$= min $\{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e})|$ for $\boldsymbol{x} \in L_V(\boldsymbol{y})$, $\theta_{FL}\boldsymbol{x} \in L_V(\boldsymbol{y})$, $\theta_{FL} \in Q\}$ and $RE_{IV}(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L_V(\boldsymbol{y})$ is called the Russell input oriented measure of technical efficiency for variable returns to scale technology.

$RE_{IV}(\boldsymbol{y}, \boldsymbol{x}) = 1$ implies that the sample point in question is technically efficient in the Russell sense and the technical inefficiency is reflected by values of $RE_{IV}(\boldsymbol{y}, \boldsymbol{x}) < 1$, with greater inefficiency for successively lower values of $RE_{IV}(\boldsymbol{y}, \boldsymbol{x})$.

Proposition 2.15 For variable returns to scale technologies the Russell input oriented measure of technical efficiency satisfies E1a, E2a, E3aw, E4aw, E5aw and E6a. It does not satisfy E3a, E4a and it does not satisfy E5a.

Proofs:

Similar to propositions 2.13 and 2.2

Definition: The function $ZE_{I\check{N}}$: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined as

$ZE_{I\check{N}}(\boldsymbol{y},\boldsymbol{x})$= $\{[RE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}/\psi^+_I(\boldsymbol{y}, \boldsymbol{x})]/\psi_I^+(\boldsymbol{y}, \boldsymbol{x})|$ for $\boldsymbol{x} \in L^+_{\check{N}}(\boldsymbol{y})$, $\boldsymbol{x}/\psi^+_I(\boldsymbol{y}, \boldsymbol{x}) \in L^+_{\check{N}}(\boldsymbol{y})\}$ and

$ZE_{I\check{N}}(\boldsymbol{y},\boldsymbol{x}) = = +\infty$ for $\boldsymbol{x} \notin L^{+}_{\check{N}}(\boldsymbol{y})$ is called the Zieschang measure of input oriented technical efficiency for non-increasing returns to scale technology.

$ZE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) = 1$ implies that the sample point in question is technically efficient in the Zieschang sense and the technical inefficiency is reflected by values of $ZE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x}) < 1$, with greater inefficiency reflected by successively lower values of $ZE_{I\check{N}}(\boldsymbol{y}, \boldsymbol{x})$.

Proposition 2.16 For non-increasing returns to scale technologies the Zieschang input oriented measure of technical efficiency satisfies E1a, E2a, E3a, E4aw, E5aw and E6a. It does not satisfy E4a and E5a.

Proofs:

Similar as proposition 2.3 for E1a, E2a and E3a to E6a. The proof of E4aw is similar as given in proposition 2.12.

Definition: The function ZE_{IV}: $\mathbb{R}^{M}_{+}\times \mathbb{R}^{N}_{+}\rightarrow\Re_{+}\cup\{+\infty\}$, where $\mathbb{R}^{q}_{+}=\Re^{q}_{+}/\{0\}$, defined as

$ZE_{IV}(\boldsymbol{y},\boldsymbol{x}) = \{[RE_{IV}(\boldsymbol{y}, \boldsymbol{x}/\psi^{+}_{I}(\boldsymbol{y}, \boldsymbol{x})]/\psi^{+}_{I}(\boldsymbol{y}, \boldsymbol{x})|$ for $\boldsymbol{x} \in L^{+}_{V}(\boldsymbol{y}), \boldsymbol{x}//\psi^{+}_{I}(\boldsymbol{y}, \boldsymbol{x}) \in L^{+}_{V}(\boldsymbol{y})\}$ and $ZE_{IV}(\boldsymbol{y}, \boldsymbol{x}) =\infty$ for $\boldsymbol{x} \notin L^{+}_{V}(\boldsymbol{y})$ is called the Zieschang measure of input oriented technical efficiency for variable returns to scale technology.

$ZE_{IV}(\boldsymbol{x}, \boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Zieschang sense and the technical inefficiency is reflected by values of $ZE_{IV}(\boldsymbol{y}, \boldsymbol{x}) < 1$, with greater inefficiency for successively lower values of $ZE_{IV}(\boldsymbol{y}, \boldsymbol{x})$.

Proposition 2.17 For variable returns to scale technologies the Zieschang input oriented measure of technical efficiency satisfies E1a, E2a, E3a, E5aw and E6a. It does not satisfy E4a or its variant and E5a.

Proofs:

Similar as proposition 2.16.

To define the Non-radial Farrell measures of input oriented technical efficiency for the non-increasing returns to scale and the variable returns to scale technologies, we first need to express these technologies in the transformed data space. Thus for non-increasing returns to scale we have

$L^S_N({}_s\boldsymbol{y}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{y} \leq {}_S\boldsymbol{Y\Phi},\ {}_s\boldsymbol{x} \geq {}_S\boldsymbol{X\Phi},\ \boldsymbol{\Phi \leq 0},\ \mathbf{e}^T\boldsymbol{\Phi} \leq 1\}$

Similarly for variable returns to scale we have

$L^S_{V}({}_s\boldsymbol{y}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{y} \leq {}_S\boldsymbol{Y\Phi},\ {}_s\boldsymbol{x} \geq {}_S\boldsymbol{X\Phi},\ \boldsymbol{\Phi \leq 0},\ \mathbf{e}^T\boldsymbol{\Phi} = 1\}$

Definition: The function $NE_{I\check{N}}$: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$ for q = M, N,

defined by $NE_{I\check{N}}({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \min\{\psi_I({}_s\boldsymbol{y}, \theta_{NE\,s}\boldsymbol{x})/\psi_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \mid \theta_{NE} \in Q,\ {}_s\boldsymbol{x} \in L^S_{\check{N}}({}_s\boldsymbol{y}),\ \theta_{NE}\,{}_s\boldsymbol{x} \in L^S_{\check{N}}({}_s\boldsymbol{y}),$

$(\theta_{NE\,s}\boldsymbol{x} - \boldsymbol{\varepsilon}) \notin L^S_{\check{N}}({}_s\boldsymbol{y}),\ \forall\, \boldsymbol{\varepsilon} \in \Re^N_+\}$ and $NE_{I\check{N}}({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = +\infty$ for ${}_s\boldsymbol{x} \notin L^S_{\check{N}}({}_s\boldsymbol{y})$, is called the input oriented Non-radial Farrell measure of technical efficiency for non-increasing returns to scale.

$NE_{I\check{N}}({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = 1$ implies that the sample point in question is technically efficient and the technical inefficiency is reflected by values of $NE_{I\check{N}}({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) < 1$, with greater inefficiency for successively lower values of $NE_{I\check{N}}({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$.

Proposition 2.18 For non-increasing returns to scale technologies the Non-radial Farrell input oriented measure of technical efficiency satisfies E1a, E2a, E3a, E4aw, E5a and E6a.

Proofs:

For E1a, E2a E3a, E5a and E6a proofs are similar to proposition 2.4. E4aw follows the reasoning given in proposition 2.12.

Definition: The function NE_{IV}: $\mathbb{R}^{M}_{+} \times \mathbb{R}^{N}_{+} \to \Re_{+} \cup \{+\infty\}$, where $\mathbb{R}^{q}_{+} = \Re^{q}_{+}/\{0\}$ for q = M, N, defined by $NE_{IV}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) = \min\{\psi_I({}_{s}\boldsymbol{y}, \theta_{NE\,s}\boldsymbol{x})/\psi_I({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) \mid, \theta_{NE} \in Q, {}_{s}\boldsymbol{y} \in L^{S}{}_{V}({}_{s}\boldsymbol{y}), \theta_{NE\,s}\boldsymbol{x} \in L^{S}{}_{V}({}_{s}\boldsymbol{y}), (\theta_{NE\,s}\boldsymbol{x} - \boldsymbol{\varepsilon}) \notin L^{S}{}_{V}({}_{s}\boldsymbol{y}), \forall\, \boldsymbol{\varepsilon} \in \Re^{N}_{+}\}$ and $NE_{IV}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) = +\infty$ for ${}_{s}\boldsymbol{x} \notin L^{S}{}_{V}({}_{s}\boldsymbol{y})$, is called the input oriented Non-radial Farrell measure of technical efficiency for variable returns to scale. Q is as defined in section 2.2.3.

$NE_{IV}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient and the technical inefficiency is reflected by values of $NE_{IV}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x}) < 1$, with greater inefficiency for successively lower values of $NE_{IV}({}_{s}\boldsymbol{y}, {}_{s}\boldsymbol{x})$.

Proposition 2.19 For variable returns to scale technologies the Non-radial Farrell input oriented measure of technical efficiency satisfies E1a, E2a, E3a, E4aw, E5a and E6a.

Proofs:

For E1a, E2a, E3a, E5a and E6a proofs are similar to proposition 2.4. For E4a proof is similar to proposition 2.13.

The above discussion shows that the relaxation of returns to scale assumption has identical effects on the input oriented measures of technical efficiency. In each of these measures only property E4a is affected while the character of other properties remain unaffected.

Proposition 2.20 For non-constant returns to scale the cost efficiency satisfies *OC*1, *OC*2 and *OC*4 to *OC*7. *OC*3 is modified as follows;

*OC*3w. $OE_I(\nu\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x}) \lesseqgtr \nu\, OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$, $\nu > 0$, $0 < \gamma \gtreqless 1$, where γ is the scale parameter.

Proofs:

*OC*3w has similar explanation as given in propositions 2.12 and 2.13. For rest of the properties proofs are similar to proposition 2.6.

Proposition 2.21 For price based technical efficiency $TE^{C}3$ is modified for non-constant returns to scale as follows;

$TE^{C}3$w. $TE^{\mathbf{c}}_{I}(\nu \boldsymbol{y}, \boldsymbol{x}) \gtreqless \nu TE^{\mathbf{c}}_{I}(\boldsymbol{y}, \boldsymbol{x})$ $\nu > 0$, $0 < \gamma \lesseqgtr 1$

Other properties remain unchanged.

Proofs:

For $TE^{C}3$w proof follows the reasoning from proposition 2.12 and 2.13. For other properties proofs are same as in proposition 2.8a

Proposition 2.22 For non-constant returns to scale Farrell and Zieschang allocative efficiency measures satisfy proposition 2.7 and Russell and non-radial Farrell measures satisfy 2.8b.

Proofs:

Similar to propositions 2.7 and 2.8b respectively.

Computation

Relaxation of scale assumption from constant to non-increasing or to variable returns to scale gives rise to the need of introducing the corresponding scale parameter adjustments in the linear programming models discussed in section 2.3. These adjustments are as follows;

Farrell input oriented technical efficiency measure linear programming model for non-increasing returns to scale is written for the kth firm as

$$\begin{aligned} \mathrm{Min}_{\theta DF, \boldsymbol{\Phi}} \quad & \theta_{DF} \\ \text{s.t} \quad & \theta_{FL}\boldsymbol{x}^{k} - \boldsymbol{X\Phi} \geq \mathbf{0} \\ & -\boldsymbol{y}^{k} + \boldsymbol{Y\Phi} \geq \mathbf{0} \\ & \mathbf{e}^{T}\boldsymbol{\Phi} \leqq 1 \\ & \boldsymbol{\Phi} \in \Re^{N}_{+} \end{aligned}$$

where $\mathbf{e}$ is a J×1 vector of ones and other variables are as defined earlier. The restriction requires the sum of the intensity vector be less than or equal to one. For the variable returns to scale the weak inequality of this new constraint is replaced by the strict equality to represent the correspondence $L_V(\boldsymbol{y})$, as mentioned earlier. Thus we have

$$\begin{aligned}
\text{Min}_{\theta DF,\boldsymbol{\Phi}} \quad & \theta_{DF} \\
\text{s.t} \quad & \theta_{DF}\boldsymbol{x}^{k} - \boldsymbol{X\Phi} \geq \mathbf{0} \\
& -\boldsymbol{y}^{k} + \boldsymbol{Y\Phi} \geq \mathbf{0} \\
& \mathbf{e}^{T}\boldsymbol{\Phi} = 1 \\
& \boldsymbol{\Phi} \in \Re^{N}_{+}
\end{aligned}$$

Thus the variable returns to scale Farrell input oriented efficiency model requires the sum of the intensity vector be exactly equal to one. Identical additional constraints, $\mathbf{e}^{T}\boldsymbol{\Phi} \leqq 1$ and $\mathbf{e}^{T}\boldsymbol{\Phi} = 1$, are introduced in the three non-redial models discussed above to accommodate non-increasing and variable returns to scale assumptions respectively. No other changes in the respective models are required.

2.5.2 The Disposability Assumption

Throughout the preceding discussion we continued maintaining the strong disposability assumption besides the constant returns to scale assumption. Now we replace this assumption by the weak disposability. For our purpose let us assume that the input vector $\boldsymbol{x}$ of any sample point comprises of two sub vectors, $\boldsymbol{x}^{sd}$ and $\boldsymbol{x}^{wd}$ where $\boldsymbol{x}^{sd}$ is made up of the components that satisfy the strong disposability and $\boldsymbol{x}^{wd}$ has the components that satisfy the weak disposability. The resulting technology is now written as under;

$L_W(\boldsymbol{y}) = \{\boldsymbol{x} \mid \boldsymbol{y} \leq \boldsymbol{Y\Phi}, \boldsymbol{x}^{sd} \geq \boldsymbol{X}^{sd}\ \boldsymbol{\Phi}, \rho \boldsymbol{x}^{wd} = \boldsymbol{X}^{wd}\ \boldsymbol{\Phi}, \boldsymbol{\Phi} \geq \boldsymbol{0}, \rho \in (0, 1]\}$

This is the representation of constant returns to scale technology with the input vectors comprising both weakly disposable and strongly disposable components. The Farrell measure of input oriented technical efficiency for $\boldsymbol{x} \in L_W(\boldsymbol{y})$ can now be given as follows.

Definition: A function $FE_{IW}(\boldsymbol{y}, \boldsymbol{x})$: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined by

$FE_{IW}(\boldsymbol{y}, \boldsymbol{x}) = \min\ \{\theta_{DF} \mid \text{for } \boldsymbol{x} \in L_W(\boldsymbol{y}), \theta_{DF}\boldsymbol{x} \in L_W(\boldsymbol{y}), \theta_{DF} \in (0, 1], \ \boldsymbol{y} \in \Re^M_+ \}$ and $FE_{IW}(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L_W(\boldsymbol{y})$, is the Farrell's input measure of technical efficiency for the technology defined by $L_W(\boldsymbol{y})$.

$FE_{IW}(\boldsymbol{y}, \boldsymbol{x}) = 1$ implies that the sample point in question is technically efficient in the Farrell sense and the technical inefficiency is reflected by values of $FE_{IW}(\boldsymbol{y}, \boldsymbol{x}) < 1$, with greater inefficiency for successively lower values of $FE_{IW}(\boldsymbol{y}, \boldsymbol{x})$.

Proposition 2.23: $FE_{IW}(\boldsymbol{y}, \boldsymbol{x})$ satisfies E3a, E4a and E6a. It does not satisfy E1a, E2a and E5a.

Proofs:

Similar to proposition 2.1

To define Russell non radial measure in the above setting let Q be the set of diagonal matrices ${}^i\theta_{NE}$**,** each of dimension N with ${}^i\theta_{nn} \in (0,1]$, then

Definition: The function RE_{IW}: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as

$RE_{IW}(\boldsymbol{y}, \boldsymbol{x}) = \min\{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \mid \text{for } \boldsymbol{x} \in L_W(\boldsymbol{y}), \theta_{FL}\boldsymbol{x} \in L_W(\boldsymbol{y}), \theta_{FL} \in Q\}$ and $RE_{IW}(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L_W(\boldsymbol{y})$ is called the Russell input measure of technical efficiency for the technology defined by L_W.

Input oriented technical efficiency is associated with an input vector for which $RE_{IW}(\boldsymbol{y}, \boldsymbol{x}) = 1$. A value of $RE_{IW}(\boldsymbol{y}, \boldsymbol{x}) < 1$ implies technical inefficiency in the Russell sense which is higher for lower values of $RE_{IW}(\boldsymbol{y}, \boldsymbol{x})$.

Proposition 2.24: $RE_{IW}(\boldsymbol{y}, \boldsymbol{x})$ satisfies E1a, E2a, E4a and E6a but fails to satisfy E3a and E5a for the technology described by $L_W(\boldsymbol{y})$.

Proofs:

Similar to Proposition 2.2.

The Zieschang measure of technical efficiency for constant returns to sale technology and with weak disposable sub vector is defined as follows

Definition: The function ZE_{IW}: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as

$ZE_{IW}(\boldsymbol{y},\boldsymbol{x}) = \{[RE_{IW}(\boldsymbol{y}, \boldsymbol{x}/\psi^+_I(\boldsymbol{y}, \boldsymbol{x})] / \psi^+_I(\boldsymbol{y}, \boldsymbol{x}) |$ for $\boldsymbol{x} \in L^+_W(\boldsymbol{y}), \boldsymbol{x}/\psi^+_I(\boldsymbol{y}, \boldsymbol{x}) \in L^+_W(\boldsymbol{y}) \}$ and $ZE_W(\boldsymbol{y}, \boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin L^+_W(\boldsymbol{y})$ is called the Russell Extended Farrell measure of input oriented technical efficiency for the technology defined by $L^+_W(\boldsymbol{y})$.

Proposition 2.25: Zieschang input oriented technical efficiency measure satisfies E1a to E4a and E.6a but it does not satisfy E5a, the monotonicity property.

Proofs:

Same as proposition 2.3

In order to define the Non-radial Farrell measure of input oriented technical efficiency for the case when the input share vector ${}_s\boldsymbol{x}$ has both weakly disposable component ${}_s\boldsymbol{x}^{wd}$ and the strongly disposable component ${}_s\boldsymbol{x}^{sd}$, the transformed correspondence, $L^S_W({}_s\boldsymbol{y})$, is given as follows

$L^S_W(\boldsymbol{y}) = \{{}_s\boldsymbol{x} \mid {}_s\boldsymbol{y} \leq {}_s\boldsymbol{Y}\,\boldsymbol{\Phi}, \theta_{NEs}\,{}_s\boldsymbol{x}^{sd} \geq {}_s\boldsymbol{X}^{sd}\,\boldsymbol{\Phi}, \rho_s\,{}_s\boldsymbol{x}^{wd} = {}_s\boldsymbol{X}^{wd}\,\boldsymbol{\Phi}, \boldsymbol{\Phi} \geq \boldsymbol{0}, \rho \in (0, 1]\}$

The Non-radial Farrell measure of input oriented technical efficiency for this technology is then defined as

Definition: The function NE_{IW}: $\mathbb{R}^{M}_{+}\times \mathbb{R}^{N}_{+}\rightarrow\Re_{+}\cup\{+\infty\}$, where $\mathbb{R}^{q}_{+}=\Re^{q}_{+}/\{0\}$ for q = M, N,

defined by $NE_{IW}(_s\boldsymbol{y}, {}_s\boldsymbol{x})= \min\{\psi_I(_s\boldsymbol{y}, \theta_{NE\,s}\boldsymbol{x})/\psi_I(_s\boldsymbol{y}, {}_s\boldsymbol{x})|, \theta_{NE} \in Q, {}_s\boldsymbol{x}\in L^{S}_{W}(_s\boldsymbol{y}), \theta_{NE\,s}\boldsymbol{x}\in L^{S}_{W}(_s\boldsymbol{y})$,

$(\theta_{NE\,s}\boldsymbol{x} - \boldsymbol{\varepsilon}) \notin L^{S}_{W}(_s\boldsymbol{y}), \forall\, \boldsymbol{\varepsilon} \in \Re^{N}_{+}\}$ and $NE_{IW}(_s\boldsymbol{y}, {}_s\boldsymbol{x}) = +\infty$ for ${}_s\boldsymbol{x} \notin L^{S}_{IW}(_s\boldsymbol{y})$, is called the input oriented Non-radial Farrell measure of technical efficiency for the technology defined by $L^{S}_{W}(\boldsymbol{y})$. $NE_{IW}(_s\boldsymbol{y}, {}_s\boldsymbol{x})$ attains values equal to or less than one with $NE_{IW}(_s\boldsymbol{y}, {}_s\boldsymbol{x}) = 1$ indicating technical efficiency and technical inefficiency if $NE_{IW}(_s\boldsymbol{y}, {}_s\boldsymbol{x}) <1$. Lower vales of $NE_{IW}(_s\boldsymbol{y}, {}_s\boldsymbol{x})$ reflect higher inefficiency.

Proposition 2.26 The Non-radial Farrell measure satisfies E1a to E6a for the technology given by $L^{S}_{W}(\boldsymbol{y})$.

Proofs:

Same as in proposition 2.4.

The above discussion suggests that the disposability assumptions mainly affect the computation of the scores by adding new constraint. The properties of various measures are preserved. The discussion can easily be extended to non-constant returns to scales and weak disposability. Similar conclusions follow from these discussions as well.

Computation:

Though $FE_{IW}(\boldsymbol{y}, \boldsymbol{x})$ has same characteristics with respect to E1a to E6a as does $FE_{I}(\boldsymbol{y}, \boldsymbol{x})$, the computation however differs between the two. The fact that the input vector $\boldsymbol{x} \in L_W(\boldsymbol{y})$ has a weakly disposable sub-vector implies that $L_W(\boldsymbol{y}) \subseteq L(\boldsymbol{y})$ As a result we end up with $FE_{IW}(\boldsymbol{y}, \boldsymbol{x}) \geq FE_{I}(\boldsymbol{y}, \boldsymbol{x})$.The computation code for $FE_{IW}(\boldsymbol{y}, \boldsymbol{x})$ is written as under;

$$\text{Min}_{\theta_{DF},\boldsymbol{\Phi}} \quad \theta_{DF}$$

$$\text{s.t} \quad \theta_{DF}\boldsymbol{x}^{sdk} - \boldsymbol{X}^{sd}\boldsymbol{\Phi} \geq \mathbf{0}$$

$$\theta_{DF}\rho\boldsymbol{x}^{wdk} - \boldsymbol{X}^{wd}\boldsymbol{\Phi} = \mathbf{0}$$

$$-\boldsymbol{y}^{k} + \boldsymbol{Y\Phi} \geq \mathbf{0}$$

$$0 \leq \rho \leq 1$$

$$\boldsymbol{\Phi} \geq \mathbf{0}$$

This nonlinear program can be converted into a linear program by imposing the restriction that ρ=1. This restriction does not affect the solution values, θ_{DF} and $\boldsymbol{\Phi}$**.**

The computation codes of the non-radial measures follow changes identical to those of radial Farrell measure, mentioned above, to accommodate weak disposability.

2.6 Numerical E*x*ample

The purpose of this section is to confirm various relationships between efficiency measures discussed in the previous sections, using a numerical example based on a hypothetical data set. For simplicity we restrict this example to the case of two inputs and single output. Further, we assume constant returns to scale technology and strong disposability of outputs and inputs. The data set is given in table 2.5.1. For each data point $\boldsymbol{y}$ represents the given output vector and x_1 and x_2 are the two inputs. The transformed inputs and output variables are reported in the last three columns of the table. For the purpose of computation and comparisons of various measures we make use of these transformed values which is in line with our earlier discussion. Table 2.5.2 and figure 2.12 presents a comparison of the four quantity based input oriented technical efficiency measures. The Figure and the table confirm the inferences drawn in sections 2.2 and 2.3. Russell measure is equal to or less than the Farrell input based technical efficiency measure and the Zieschang measure completely coincides with the Farrell measure with the exception of data point 15 for which the Farrell projection falls on $IsoqL(\boldsymbol{y})\backslash EffL(\boldsymbol{y})$. Zieschang computation applies a Russell type adjustment to the Farrell projection vector in this case, thus resulting in a $ZE_I(\boldsymbol{y}, \boldsymbol{x}) < FE_I(\boldsymbol{y}, \boldsymbol{x})$.

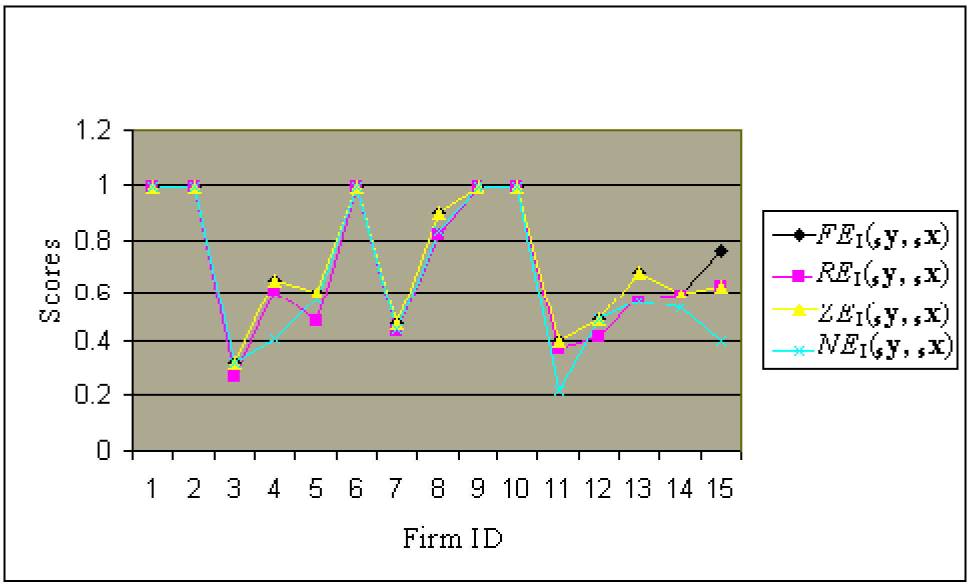

Figure 2.12, Quantity based technical efficiency

The Non-radial Farrell scores are less than or equal to the Farrell and the Zieschang scores. Results also confirm that the relationship between Russell and Non-radial Farrell measures can go in either direction. Thus for example for data point 3 the Russell score is less than that of the Non-radial Farrell score while for data point 4 the relationship is reversed.

Table 2.5.3 gives information on input oriented overall efficiency and the price based technical efficiency scores. Figure 2.13 gives comparisons of price based input oriented technical efficiency. Comparison of tables 2.5.2 and 2.5.3 confirms that only for the radial Farrell measure the quantity based technical efficiency scores coincide with the price based technical efficiency scores. Thus while the Farrell allocative efficiency scores, defined as the ratio of overall efficiency to the technical efficiency, are independent of which of the two technical efficiency concepts we use to compute the allocative efficiency, the same is not the case with the non-radial allocative efficiency scores for reasons discussed earlier.

Unlike the relationship between the Russell and the radial Farrell quantity based technical efficiency scores, the relationship between Russell and the radial Farrell price based technical efficiency is ambiguous. Also for the data points for which the Non-radial Farrell projection vector is identical to the Russell projection vector, the two measures have identical price based technical efficiency scores despite the fact that their quantity based technical efficiency scores differ, for example the data points 8 and 13. Table 2.5.4 and figure 2.14 gives comparisons on allocative efficiency scores across various measures. The figure confirms that the relationship between the radial Farrell and the Russell allocative efficiency measures can go in any direction. However, the Non-radial Farrell allocative efficiency scores are equal to or greater than the radial Farrell and the Zieschang allocative efficiency scores. For the data points for which the Russell and the Non-radial Farrell measures have identical values for the price based technical efficiency, the allocative efficiency components of the two measures are identical and so is the decomposition of the overall efficiency into technical efficiency and the allocative efficiency.

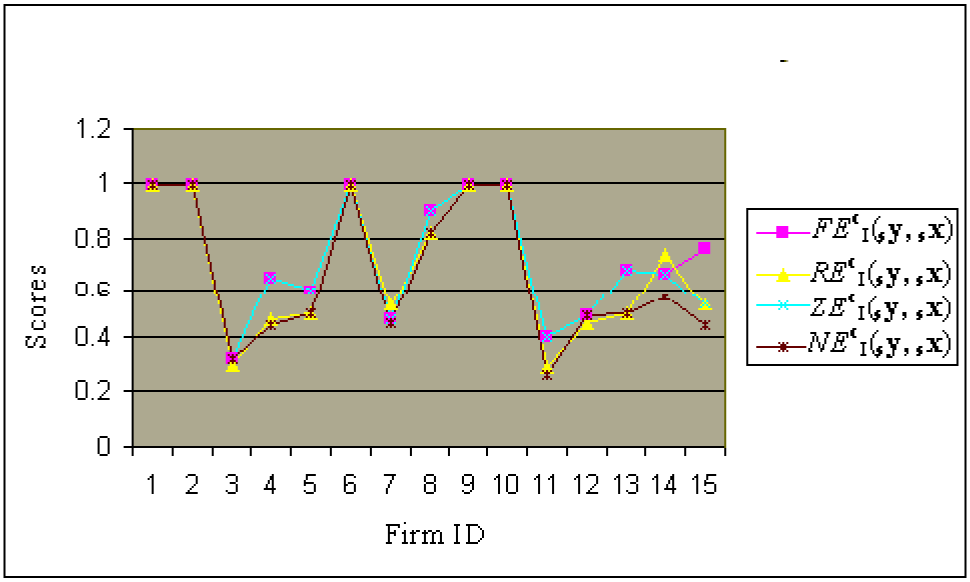

Figure 2.13, Price based technical efficiency

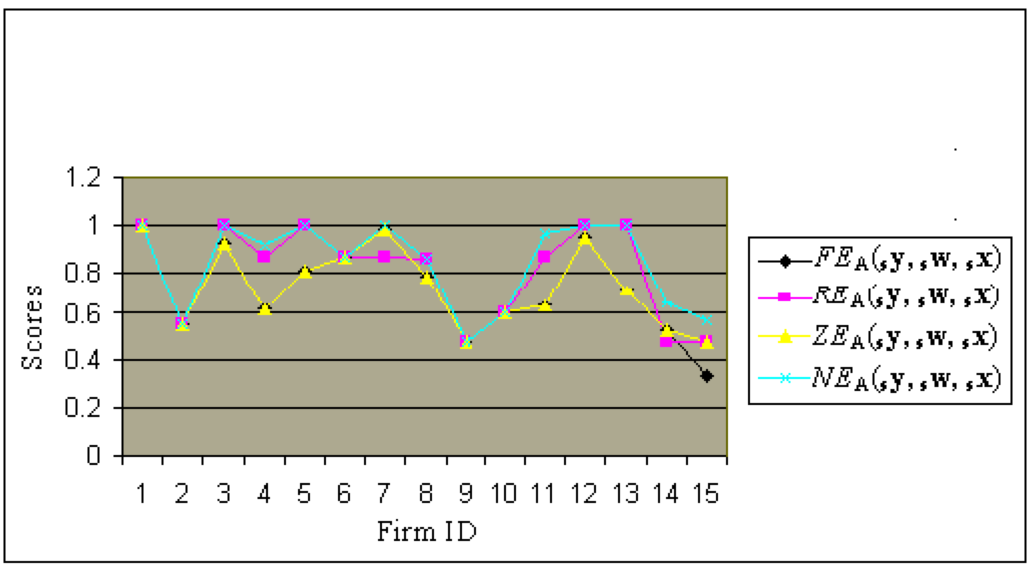

Figure 2.14, Input oriented allocative efficiency scores

Table 2.5.1 Inputs, Output and Sample Shares of Inputs and Output[5]

Firm ID	x_1	x_2	Y	$_sx_1 \times 1000$	$_sx_2 \times 1000$	$_sy \times 1000$
1	5128.205	195	100	12.46077	3.351667	20
2	11111.11	90	100	26.99834	1.546923	20
3	15000	1000	100	36.44776	17.18804	20
4	15000	150	100	36.44776	2.578206	20
5	7000	895	100	17.00895	15.38329	20
6	20000	350	300	48.59701	6.015813	60
7	34615.38	1200	300	84.1102	20.62564	60
8	13500	2000	300	32.80298	34.37607	60
9	5400	5000	300	13.12119	85.94019	60
10	7714.286	3500	300	18.74456	60.15813	60
11	144000	1500	600	349.8985	25.78206	120
12	61714.29	3500	600	149.9565	60.15813	120
13	31764.71	6800	600	77.18348	116.8787	120
14	25401.59	12000	600	61.72207	206.2564	120
15	14198.41	20000	600	34.5	343.7607	120

Table 2.5.2 Quantity Based Input Oriented Technical Efficiency

Firm ID	$FE(\boldsymbol{y}, \boldsymbol{x})$	$RE(\boldsymbol{y}, \boldsymbol{x})$	$ZE(\boldsymbol{y}, \boldsymbol{x})$	$NE(\boldsymbol{y}, \boldsymbol{x})$
1	1	1	1	1
2	1	1	1	1
3	0.31996	0.26844	0.31996	0.31996
4	0.652778	0.611111	0.652778	0.40598
5	0.603024	0.475239	0.603024	0.562653
6	1	1	1	1
7	0.461885	0.434722	0.461885	0.446998
8	0.902001	0.825305	0.902001	0.835554
9	1	1	1	1
10	1	1	1	1
11	0.397631	0.372222	0.397631	0.219736
12	0.477242	0.416431	0.477242	0.477242
13	0.681623	0.57036	0.681623	0.552762
14	0.597246	0.59536	0.597246	0.521984
15	0.760649	0.630324	0.630324	0.400565

[5] For simplification of presentation all the share vectors have been raised by a factor of 1000. Note however that such a uniform inflation has no bearing on the scores and conclusions.

Table 2.5.3 Overall Efficiency Scores and Price Based Technical Efficiency Scores

Firm ID	$OE_I(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$	$FE^C(\boldsymbol{y}, \boldsymbol{x})$	$RE^C(\boldsymbol{y}, \boldsymbol{x})$	$ZE^C(\boldsymbol{y}, \boldsymbol{x})$	$NE^C(\boldsymbol{y}, \boldsymbol{x})$
1	1	1	1	1	1
2	0.553943	1	1	1	1
3	0.294811	0.31996	0.294811	0.31996	0.31996
4	0.405177	0.652778	0.466466	0.652778	0.440386
5	0.488155	0.603024	0.488155	0.603024	0.488155
6	0.868611	1	1	1	1
7	0.452923	0.461885	0.521434	0.461885	0.452923
8	0.706133	0.902001	0.821215	0.902001	0.821215
9	0.478868	1	1	1	1
10	0.601213	1	1	1	1
11	0.252541	0.397631	0.290741	0.397631	0.262023
12	0.451538	0.477242	0.451538	0.477242	0.477242
13	0.488888	0.681623	0.488888	0.681623	0.488888
14	0.354038	0.670102	0.739323	0.670102	0.54951
15	0.250818	0.760649	0.523773	0.523773	0.445515

Table 2.5.4 Allocative Efficiency Comparisons

Firm ID	$FE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$	$RE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$	$ZE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$	$NE_A(\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{x})$
1	1	1	1	1
2	0.553943	0.553943	0.553943	0.553943
3	0.921401	1	0.921401	0.921401
4	0.620697	0.868611	0.620697	0.920051
5	0.809512	1	0.809512	1
6	0.868611	0.868611	0.868611	0.868611
7	0.980598	0.868611	0.980598	1
8	0.782851	0.859864	0.782851	0.859864
9	0.478868	0.478868	0.478868	0.478868
10	0.601213	0.601213	0.601213	0.601213
11	0.635113	0.868611	0.635113	0.963813
12	0.946139	1	0.946139	0.946139
13	0.717241	1	0.717241	1
14	0.528335	0.478868	0.528335	0.64428
15	0.329742	0.478868	0.478868	0.562985

2.6 Summary

The objective of this chapter was to propose a new input oriented technical efficiency measure and to compare it with the existing radial and non-radial measures. We started our discussion with the Farrell input oriented measure of technical efficiency and showed that it only satisfied the homogeneity property out of the required set of properties outlined in chapter one. The non radial Russell measure was shown to violate the strong homogeneity and strong monotonicity property but was shown to satisfy the sub-homogeneity and weak monotonicity. In comparison, the non-radial Zieschang measure was shown to violate the monotonicity property. Then we introduced the Non-radial Farrell input oriented technical efficiency measure and showed that it satisfied sub-homogeneity and strong monotonicity. We also introduced the input oriented overall efficiency measure and its decomposition into allocative and technical efficiency components. A comparison of decomposition of overall efficiency into its component was provided across radial and non-radial measures and various relationships between different efficiency measures were inferred. With the help of a hypothetical data set we were able to confirm the inferences that we established during our discussion. We also provided an account on effects of relaxing our basic assumptions of constant returns to scales and strong disposability. It was shown that relaxing constant returns to scale affects homogeneity of degree plus one in outputs across all measures of input oriented technical efficiency. The disposability assumption was shown to have mainly its impact on the programming code where additional constraints needed to be introduced to accommodate the sub-vector weak disposability.

CHAPTER 3

THE OUTPUT ORIENTED EFFICIENCY MEASURES

This chapter discusses the output oriented measures of technical efficiency. In contrast to the input based technical efficiency measure where the objective is to identify a technically efficient input vector for a given output vector, in the output oriented measure we seek the technically efficient output vector for a given input vector. Thus the data requirements for the output oriented measures of technical efficiency are same as that of the input based measures. However, the roles that different data components play now differ. In place of input data set, the output data set is now used to construct the efficiency frontier for the given input vector. Then in Koopmans sense the technically efficient output vector is the one for which a possibility of any feasible expansion in any of its component(s), for the given input vector, does not exist. The construction of such a measure of technical efficiency requires firstly a criterion for identification of a benchmark output vector and secondly a criterion for computation of efficiency scores of the data points in question with respect to this benchmark. The variations in these criteria give different output oriented measures of technical efficiency. The benchmarking for the purpose of comparing the inefficient data points with the efficient data points may be done on the basis of the entire isoquant such as in the case of Farrell output oriented measure of technical efficiency. Alternatively it may be based on the efficient subset of the isoquant for example the Russell efficiency measure. Similarly the computation of efficiency score of an inefficient data point may be based on the radial deviation of that data point from the relevant benchmark, like in the case of Farrell measure or it may be based on the non-radial deviation, Russell measure, Zieschang measure and the Non-radial Farrell Measure for example.

We start in section 3.1 with the Farrell output measure of technical efficiency. We see that the Farrell's output oriented radial measure of technical efficiency suffers from the same weaknesses that are present in the Farrell input based measure of technical efficiency. In particular, the fact that the Farrell's technical efficiency measure uses isoquant, $IsoqP(\boldsymbol{x})$, as the benchmark criterion for identification of technically efficient output vectors and that the $EffP(\boldsymbol{x}) \subseteq IsoqP(\boldsymbol{x})$, reflects the inability of Farrell's output oriented measure of technical efficiency to satisfy Koopmans conditions. In contrast, the existing non-radial technical efficiency measures, for example the Russell's output oriented measure and the Zieschang measure are able to overcome this problem, though they share other weaknesses as we discuss below. We also show that in comparison to the existing radial and non-radial technical efficiency measures, the Non-radial Farrell measure is likely to fulfill the criteria required of output oriented technical efficiency measure, outlined in chapter 1, to a greater extent..

In sections 3.1 and 3.2 we look into the quantity based output oriented radial and non radial technical efficiency measures while maintaining the constant returns to scale and strong disposability assumptions. Section 3.3 discusses computation. Section 3.4 extends the discussion to the output oriented overall efficiency measure and its decomposition into the technical and allocative efficiency components. Section 3.5 relaxes the constant returns to scale and strong disposability assumptions. Section 3.6 provides a numerical example which is followed by summary and conclusions in section 3.7.

3.1. RADIAL EFFICIENCY MEASURES:

The output oriented radial measure of technical efficiency is defined as follows;

Definition: A function $FE_O(\boldsymbol{x},\boldsymbol{y})$: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined by

$FE_O(\boldsymbol{x},\boldsymbol{y})= \max \{\delta_{DF} | \boldsymbol{y} \in P(\boldsymbol{x}), \delta_{DF}\boldsymbol{y} \in P(\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+ \}$ and $FE_O(\boldsymbol{x},\boldsymbol{y}) = -\infty$ for $\boldsymbol{y}$ $\notin P(\boldsymbol{x})$, is the Farrell's output measure of technical efficiency.

The output oriented technical efficiency score is thus obtained by computing the ratio $FE_O(\boldsymbol{x}, \boldsymbol{y})$ $= \| \delta^*_{DF} \boldsymbol{y} \| / \| \boldsymbol{y} \|$, where δ^*_{DF} is $\max\{\delta_{DF} | \boldsymbol{y} \in P(\boldsymbol{x}), \delta_{DF}\boldsymbol{y} \in P(\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+ \}$. In words, $FE_O(\boldsymbol{x}, \boldsymbol{y})$ measures the maximum equiproportionate increase in all outputs that is possible with the given input vector. A value of one implies that the existing output level is technically efficient. A greater than one value of $FE_O(\boldsymbol{x}, \boldsymbol{y})$ reflects technical inefficiency with the degree of inefficiency increasing for higher values of $FE_O(\boldsymbol{x}, \boldsymbol{y})$. Note that the Farrell's output measure of technical efficiency is equal to the inverse of the output distance function and thus for $\psi_O(\boldsymbol{x}, \boldsymbol{y}) \leq 1$, $FE_O(\boldsymbol{x}, \boldsymbol{y}) \geq 1$.

In terms of figure(3.1), the output vector $\hat{\boldsymbol{y}} \in P(\boldsymbol{x}) \backslash IsoqP(\boldsymbol{x})$. It is possible to expand this output vector radially without any additional input requirements. The output vector $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) \times \hat{\boldsymbol{y}} \in P(\boldsymbol{x}) \cap IsoqP(\boldsymbol{x})$, for $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) \in [1,+\infty)$, defines the feasibility limit to this radial expansion with respect to $\hat{\boldsymbol{y}} \in \Re^M_+$.

Proposition3.1: The Farrell output oriented efficiency measure satisfies E3b, E4b and E6b but fails to satisfy E1b, E2b and E.5b.

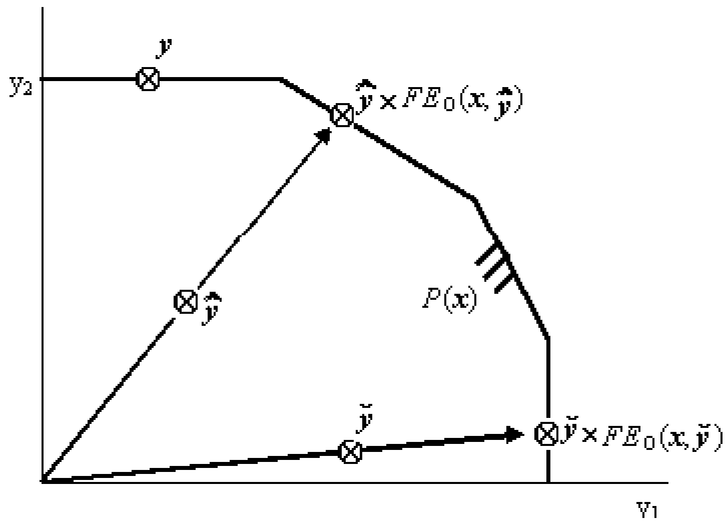

Figure 3.1 The Farrell efficiency measure

Proofs:

$FE_O(\boldsymbol{x}, \boldsymbol{y})$ does not satisfy E1b, the Identification Property:

The output oriented Farrell measure of technical efficiency assigns a value of one to any data point that belongs to $IsoqP(\boldsymbol{x})$. Since $EffP(\boldsymbol{x}) \subseteq IsoqP(\boldsymbol{x})$ and $IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ is not necessarily empty, $FE_O(\boldsymbol{x},\boldsymbol{y})=1$ is not a sufficient condition for Koopmans efficiency. Hence the measure does not satisfy E1b. This is shown in figure(3.1) in which an output vector $\boldsymbol{y} \in IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$. By definition then we have for this data point $FE_O(\boldsymbol{x}, \boldsymbol{y})=1$ thus violating E1b.

$FE_O(\boldsymbol{x},\boldsymbol{y})$ does not satisfy E.2b, the 'Compare to' Property:

The off frontier output vector $\breve{\boldsymbol{y}}$ in figure 3.1 which is compared with $\breve{\boldsymbol{y}} \times FE(\boldsymbol{x}, \breve{\boldsymbol{y}})$ for efficiency score computation represents this possibility. The facts that Farrell's output measure of technical efficiency uses $IsoqP(\boldsymbol{x})$ as the frame of reference to assign an efficiency score to any inefficient data point $\boldsymbol{y} \in P(\boldsymbol{x})\backslash IsoqP(\boldsymbol{x})$ and that the set $IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ is not necessarily empty leads to this situation.

<u>$FE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E.3b, the homogeneity of plus one in inputs</u>:

$FE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \max\{\delta_{DF} | \delta_{DF}\boldsymbol{y} \in P(\beta\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+, \beta > 0\}$

$\Rightarrow FE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \max\{\delta_{DF} | \delta_{DF}\boldsymbol{y} \in \beta P(\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+, \beta > 0\}$

$\Rightarrow FE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \beta FE_O(\boldsymbol{x}, \boldsymbol{y})$

<u>$FE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E.4b, the homogeneity of minus one in output:</u>

To show that $FE_O(\boldsymbol{x}, \boldsymbol{y})$ is homogenous of degree minus one in output let us consider an equiproportionate expansion of output vector $\boldsymbol{y} \in P(\boldsymbol{x})$, by a scale ν such that $\nu \in [1,+\infty)$ and $\nu\boldsymbol{y} \in P(\boldsymbol{x})$, then

$\Rightarrow FE_O(\boldsymbol{x}, \nu\boldsymbol{y}) = \max\{\delta_{DF} | \delta_{DF}(\nu\boldsymbol{y}) \in P(\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+\}$

$\Rightarrow FE_O(\boldsymbol{x}, \nu\boldsymbol{y}) = \max\{(\nu \times \nu^{-1})\delta_{DF} | \delta_{DF}(\nu\boldsymbol{y}) \in P(\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+\}$

$\Rightarrow FE_O(\boldsymbol{x}, \nu\boldsymbol{y}) = \nu^{-1} FE_O(\boldsymbol{x}, \boldsymbol{y})$

<u>$FE_O(\boldsymbol{x}, \boldsymbol{y})$ does not satisfy E5b, the Monotonicity Property:</u>

This can easily shown to be true for any technology for which $IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ is not empty. Figure(3.2) gives an example of piecewise linear technology where $\boldsymbol{y}/\psi(\boldsymbol{x}, \boldsymbol{y}) \in IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ and $\check{\boldsymbol{y}}/\psi_O(\boldsymbol{x}, \check{\boldsymbol{y}}) \in IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ with $\check{\boldsymbol{y}}/\psi_O(\boldsymbol{x}, \check{\boldsymbol{y}}) \geq \boldsymbol{y}/\psi_O(\boldsymbol{x}, \boldsymbol{y})$. For inefficient output vectors $\boldsymbol{y} \in P(\boldsymbol{x})$ and $\check{\boldsymbol{y}} \in P(\boldsymbol{x})$ with $\check{\boldsymbol{y}} \geq \boldsymbol{y}$, we end up with $FE_O(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{y} / \psi(\boldsymbol{x}, \boldsymbol{y})\| / \|\boldsymbol{y}\|$ and $FE_O(\boldsymbol{x}, \check{\boldsymbol{y}}) = \|\check{\mathbf{y}}/\psi(\boldsymbol{x}, \check{\boldsymbol{y}})\| / \|\check{\boldsymbol{y}}\|$ such that $FE_O(\boldsymbol{x}, \boldsymbol{y}) = FE_O(\boldsymbol{x}, \check{\boldsymbol{y}})$.

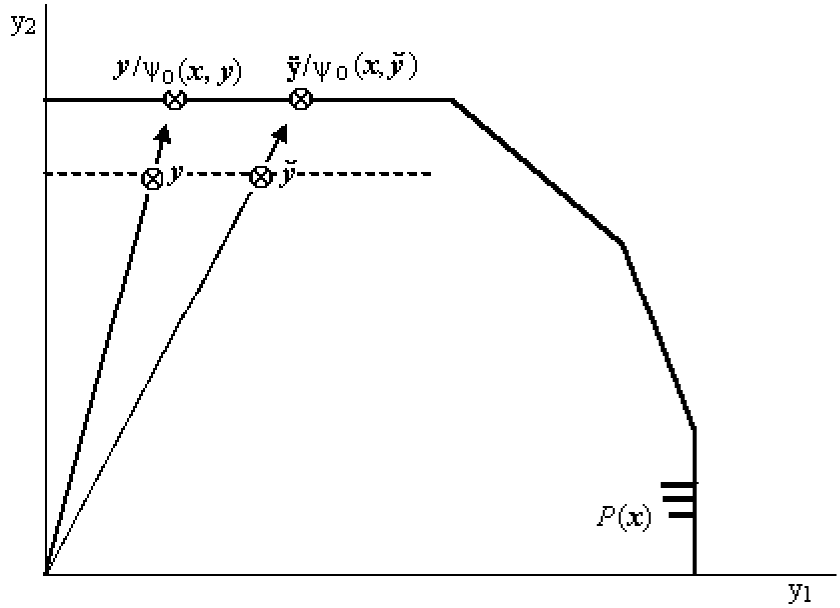

Figure 3.2, $FE_O(\boldsymbol{x}, \boldsymbol{y})$ violates monotonicity

$FE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E.6b the commensurability:

This follows directly from the definition of Farrell measure. Alternatively, let $\boldsymbol{\Omega}$ be diagonal matrices of dimensions M×M such that $\hat{\boldsymbol{y}} = \boldsymbol{\Omega y}$

$FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) = ||\delta_{DF}\hat{\boldsymbol{y}}||/||\hat{\boldsymbol{y}}||$

$FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) = |\delta_{DF}|\ ||\hat{\boldsymbol{y}}||/||\hat{\boldsymbol{y}}||$

$FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) = |\delta_{DF}|\ ||\boldsymbol{\Omega y}||/||\boldsymbol{\Omega y}||$

$\Rightarrow FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) = FE_O(\boldsymbol{x}, \boldsymbol{y})$

The above discussion suggests that the homogeneity of degree minus one and the unit invariance are the only properties that the Farrell's output oriented measure of technical efficiency fulfils for all the technologies. The failure of Farrell measure to fulfill the criteria to be satisfied by any output oriented technical efficiency measure provides room for new search. The output counterparts of the input based non-radial measures of technical efficiency discussed in chapter 2 provide natural extensions in this direction. We discuss them below maintaining the constant returns to scale and the strong disposability assumptions.

3.2 Non Radial Technical Efficiency Measures

We begin this section with the Russell measure of technical efficiency. We show that with the maintained assumptions the Russell measure satisfies all of the desirable properties mentioned in section 1.3 except strong homogeneity. Zieschang's output version of technical efficiency measure is discussed next. This measure violates strong monotonicity, though satisfying its weaker version. In comparison to Russell and Zieschang measures, the output version of Non-radial Farrell measure is shown to satisfy both weak homogeneity and monotonicity for all the technologies.

3.2.1 Russell Output oriented Technical Efficiency Measure

In order to define the output oriented Russell measure of technical efficiency let D be the set of diagonal matrices ${}^{i}\boldsymbol{\delta}_{FL}$, with the diagonal elements $\delta_{mm} \in [1, \infty)$, then

Definition: The function RE_O: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as

$RE_O(\boldsymbol{x},\boldsymbol{y})$= max $\{1/M(\mathbf{e}^T \delta_{FL} \mathbf{e}) \mid \boldsymbol{y} \in P(\boldsymbol{x}), \boldsymbol{\delta}_{FL}\boldsymbol{y} \in P(\boldsymbol{x}), \boldsymbol{\delta}_{FL} \in \text{D}\}$ and $RE_O(\boldsymbol{x}, \boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P(\boldsymbol{x})$,

where **e** is an N × 1 vector of ones, is called the Russell output oriented measure of technical efficiency.

This non-radial measure of technical efficiency maximizes the arithmetic mean of the proportionate expansion in each of the outputs in the coordinate directions. It assigns a value equal to one to a data point that belongs to $EffP(\boldsymbol{x})$. For any data point $\boldsymbol{y} \in P(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$, $RE_O(\boldsymbol{x},\boldsymbol{y})>1$, with inefficiency increasing for higher values of $RE_O(\boldsymbol{x}, \boldsymbol{y})$.

Compare to the Farrell's equiproportionate expansion, the Russell measure of technical efficiency allows for the expansion of different components of output vector in different proportions. Further, due to the fact that the Russell measure maximizes the arithmetic mean of output expansions for the given input vector, the efficient subset $EffP(\boldsymbol{x})$ rather than the whole isoquant $IsoqP(\boldsymbol{x})$ is used as the set of reference for the purpose of gauging efficiency. In figure(3.3), $\boldsymbol{y} \in P(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ is an inefficient data point. Farrell's technical efficiency measure uses $FE_O(\boldsymbol{x}, \boldsymbol{y})\times\boldsymbol{y} \in IsoqP(\boldsymbol{x})$ as the reference point to compute efficiency score. However for $FE_O(\boldsymbol{x}, \boldsymbol{y})\times\boldsymbol{y}$, expansion of output is still possible in at least one co-ordinate direction, in the direction of y_1 in this case, for the given input vector. In contrast, the Russell's measure uses the output vector as reference that belongs to the set bounded by $\delta^1_{FL} \times\boldsymbol{y}$ and $\delta^2_{FL} \times \boldsymbol{y}$ and that maximizes $RE_O(\boldsymbol{x}, \boldsymbol{y})$, where δ^i_{FL} is an M×M diagonal matrix with the mth diagonal term, $\delta_{mm}\in[1,+\infty)$, consisting of the Russell expansionary weight for the mth output. For the data point for which all the nonzero elements of the Russell's diagonal matrix, $\boldsymbol{\delta}_{FL}$, have identical values, we have $RE_O(\boldsymbol{x}, \boldsymbol{y}) = FE_O(\boldsymbol{x}, \boldsymbol{y})$. If this is not the case, we have $RE_O(\boldsymbol{x}, \boldsymbol{y}) > FE_O(\boldsymbol{x}, \boldsymbol{y})$. Thus like the input based measure, the Farrell's output oriented measure of technical efficiency is also a special case of the more general Russell's output measure of technical efficiency.

Proposition 3.2: Russell's output oriented measure of technical efficiency satisfies E1b to E3b, E5b and E6b but fails to satisfy E4b for all the technologies

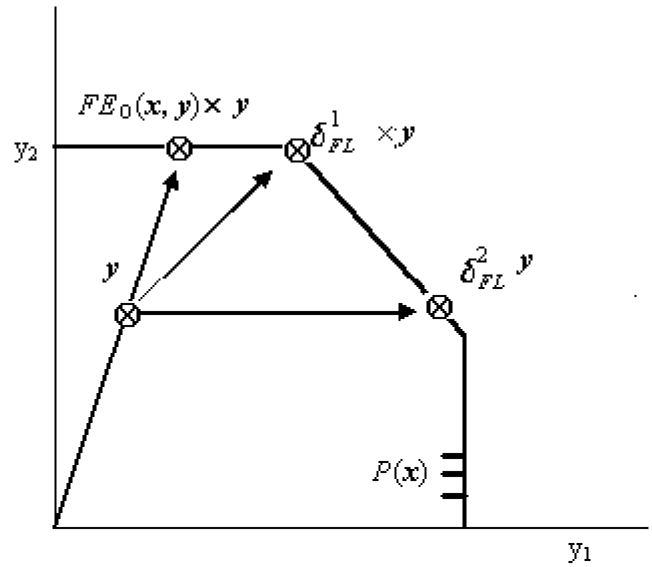

Figure 3.3, The Russell measure

.

Proofs:

$RE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E1b, the Identification Property:

Suppose not. Then for some $\boldsymbol{y} \in P(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ we have $RE_O(\boldsymbol{x}, \boldsymbol{y}) = 1$.But $\boldsymbol{y} \in P(\boldsymbol{x})\backslash EffP(\boldsymbol{x}) \Rightarrow$ there exists a δ_{FL} with at least one of the diagonal elements greater than one, such that $\delta_{FL}\boldsymbol{y} \in EffP(\boldsymbol{x})$ and $\delta_{FL}\boldsymbol{y} \geq \boldsymbol{y} \Rightarrow RE_O(\boldsymbol{x}, \boldsymbol{y}) > 1$ hence a contradiction.

Next suppose that $\boldsymbol{y} \in EffP(\boldsymbol{x})$ and $RE_O(\boldsymbol{x}, \boldsymbol{y}) > 1$.

$RE_O(\boldsymbol{x}, \boldsymbol{y}) >1 \Rightarrow \exists\ \delta_{FL}$ with at least one diagonal element greater than one such that $\delta_{FL}\boldsymbol{y} \geq \boldsymbol{y}$ and $\delta_{FL}\boldsymbol{y} \in EffP(\boldsymbol{x}) \Rightarrow \boldsymbol{y} \notin EffP(\boldsymbol{x})$, hence a contradiction.

$RE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E2b, the "Compare to" Property:

Suppose not. Then the Russell measure compares $\boldsymbol{y} \in P(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ to some $\delta_{FL}\boldsymbol{y} \notin EffP(\boldsymbol{x})$ such that $\delta_{FL}\boldsymbol{y} \geq \boldsymbol{y}$ and $\delta_{FL}\boldsymbol{y}$ being considered Russell efficient $\Rightarrow RE_O(\boldsymbol{x}, \delta_{FL}\boldsymbol{y}) = 1$. But $\delta_{FL}\boldsymbol{y} \notin EffP(\boldsymbol{x})$ $\Rightarrow \exists\, \hat{\boldsymbol{y}} \geq \delta_{FL}\boldsymbol{y}$ such that $\hat{\boldsymbol{y}} \in EffP(\boldsymbol{x})$. Then we can define another M×M diagonal matrix δ^*_{FL} with at least one diagonal element greater than one such that $\delta^*_{FL}(\delta_{FL}\boldsymbol{y}) = \hat{\boldsymbol{y}}$. But this implies that $RE_O(\boldsymbol{x}, \delta^*_{FL}(\delta_{FL}\boldsymbol{y})) = 1$.

$\Rightarrow RE_O(\boldsymbol{x}, \delta_{FL}\boldsymbol{y}) > 1$, hence a contradiction.

$RE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E3b, the homogeneity of plus one in Inputs:

This follows because for constant returns to scale we have $P(\lambda\boldsymbol{x}) = \lambda P(\boldsymbol{x})$. Thus

$$RE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \max\ \{1/M(\mathbf{e}^T\boldsymbol{\delta}_{FL}\mathbf{e}) \mid (\boldsymbol{y} \in P(\beta\boldsymbol{x}),\ \boldsymbol{\delta}_{FL}\boldsymbol{y} \in P(\beta\boldsymbol{x}),\ \boldsymbol{\delta}_{FL} \in D\}$$

$$RE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \max\ \{1/M((\mathbf{e}^T\boldsymbol{\delta}_{FL}\mathbf{e}) \mid \boldsymbol{y} \in \beta P(\boldsymbol{x}),\ \boldsymbol{\delta}_{FL}\boldsymbol{y} \in \beta P(\boldsymbol{x})\ \boldsymbol{\delta}_{NE} \in D\}$$

$$RE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \beta\max\ \{1/M(\mathbf{e}^T\boldsymbol{\delta}_{FL}\mathbf{e}) \mid (\boldsymbol{y} \in P(\boldsymbol{x}),\ \boldsymbol{\delta}_{FL}\boldsymbol{y} \in P(\boldsymbol{x}),\ \boldsymbol{\delta}_{FL} \in D\} = RE_O(\boldsymbol{x}, \boldsymbol{y})$$

Russell measure does not satisfy E.4b, the strong homogeneity, but satisfies the weak homogeneity in outputs:

This has been shown in the example of figure 3.4. Here $RE_O(\boldsymbol{x}, 2\check{y}_1, 2\check{y}_2)=1$ but $RE_O(\boldsymbol{x}, \check{y}_1, \check{y}_2) = 2.5 > 2$ thus violating strong homogeneity while still satisfying weak homogeneity.

$RE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E5b, the Monotonicity Property:

This follows because $\boldsymbol{y}^* \geq \boldsymbol{y}$ implies that for the nonzero elements of the respective expansionary diagonal elements, δ^*_{FL} and δ_{FL}, $\delta^*_m \leq \delta_m$, with strict inequality holding for at least one element. Thus we have $(1/M)\sum_{m=1}^{M}\delta_m^* < (1/M)\sum_{m=1}^{M}\delta_m$.

<u>$RE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E6b, the Commensurability Property</u>:

Let $\boldsymbol{y} \in P(\boldsymbol{x})$ and let $\boldsymbol{\Omega}$ be an M×M diagonal matrix such that $\boldsymbol{y}^* = \boldsymbol{\Omega y}$ and where $\omega_{mm} \in (0, +\infty)$, the mth diagonal element of $\boldsymbol{\Omega}$ is the rescaling factor for the mth output. Further, let $\boldsymbol{\Lambda}$ be an N×N diagonal matrix such that $\boldsymbol{x}^* = \boldsymbol{\Lambda x}$ and where $\lambda_{nn} \in (0, +\infty)$, the nth diagonal element of $\boldsymbol{\Lambda}$ is the rescaling factor for the nth input. Then $\boldsymbol{y} \in P(\boldsymbol{x}) \Leftrightarrow \boldsymbol{y}^* \in P(\boldsymbol{x}^*) \Leftrightarrow \boldsymbol{\Omega y} \in P(\boldsymbol{\Lambda x})$ We can write the Russell measure for $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ as

$RE_O(\boldsymbol{x}^*, \boldsymbol{y}^*) = \max \{\sum^M_{m=1}\delta_m/M \mid (\delta_1 y^*_1, \delta_2 y^*_2 \ldots \delta_M y^*_M) \in P(\boldsymbol{\Lambda x}), \delta_m \in [1, +\infty), \boldsymbol{y}^* \in P(\boldsymbol{x}^*)\}$

$\Rightarrow RE_O(\boldsymbol{\Lambda x}, \boldsymbol{\Omega y}) = \max \{1/M((\delta_1 \omega_{11}y_1/\omega_{11}y_1) + (\delta_2 \omega_{22}y_2/\omega_{22}y_2) + \ldots\ldots + (\delta_M \omega_{MM}y_M/\omega_{MM}y_M)) \mid (\delta_1\omega_1 y_1, \delta_2 \omega_2 y_2, \ldots\delta_M \omega_M y_M) \in P(\boldsymbol{\Lambda x}), \delta_m \in [1, +\infty), \boldsymbol{\Omega y} \in P(\boldsymbol{\Lambda x})\}$

$\Rightarrow RE_O(\boldsymbol{\Lambda x}, \boldsymbol{\Omega y}) = \max \{1/M((\delta_1 y_1/y_1) + (\delta_2 y_2/y_2) + \ldots\ldots + (\delta_M y_M/y_M)) \mid (\delta_1 y_1, \delta_2 y_2, \ldots\delta_M y_M) \in P(\boldsymbol{x}), \delta_m \in [1, +\infty), \boldsymbol{y} \in P(\boldsymbol{x})\} = RE_O(\boldsymbol{x}, \boldsymbol{y})$.

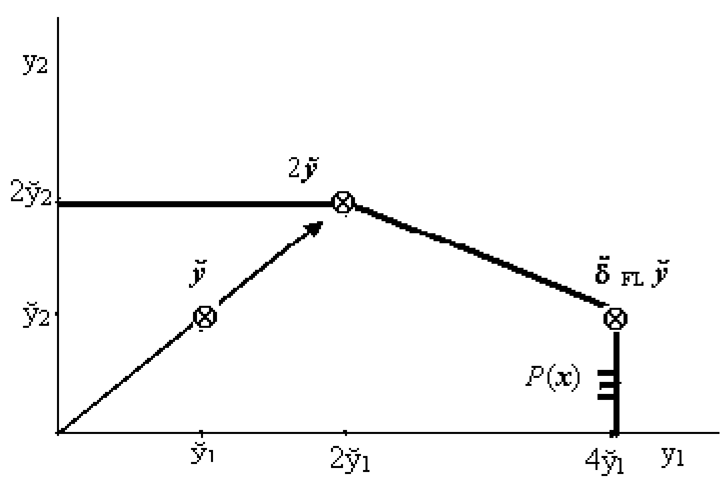

Figure 3.4, $RE_O(\boldsymbol{y}, \boldsymbol{x})$ violates monotonicity

3.2.2 Zieschang Output oriented Technical Efficiency Measure:

The output oriented Zieschang measure of technical efficiency is a synthesis of Farrell's and Russell's output technical efficiency measures. It is defined as follows;

Definition: The function ZE_O: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as $ZE_O(\boldsymbol{x}, \boldsymbol{y}) = \{[RE_O(\boldsymbol{x}, \boldsymbol{y} / \psi_O(\boldsymbol{x}, \boldsymbol{y})] / \psi_O(\boldsymbol{x}, \boldsymbol{y})|$ for $\boldsymbol{y} \in P(\boldsymbol{x})$, $\boldsymbol{y} / / \psi_O(\boldsymbol{x}, \boldsymbol{y}) \in P(\boldsymbol{x})\}$ and $ZE_O(\boldsymbol{x}, \boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P(\boldsymbol{x})$ is called the Russell Extended Farrell measure of output oriented technical efficiency for constant returns to scale technology.

Thus an inefficient data point is first expanded radially and if the resulting projection falls on $IsoqP(\boldsymbol{x}) \backslash EffP(\boldsymbol{x})$, a non-radial adjustment is made using maximum possible expansion of the output vector in the co-ordinate directions. In this way the Zieschang output measure of technical efficiency combines the features of Farrell and Russell measures of technical efficiency. Zieschang's output oriented measure assigns a value equal to one to any data point that belongs to $EffP(\boldsymbol{x})$. For any data point that belongs to $P(\boldsymbol{x}) \backslash EffP(\boldsymbol{x})$, $ZE_O(\boldsymbol{x}, \boldsymbol{y})$ is greater than one with inefficiency increasing for the increasing values of $ZE_O(\boldsymbol{x}, \boldsymbol{y})$.

Three possibilities exist. First, a data point belongs to $P(\boldsymbol{x}) \backslash IsoqP(\boldsymbol{x})$ and its radial projection belongs to $EffP(\boldsymbol{x})$.Then Zieschang and Farrell measures coincide and result in efficiency score of identical magnitudes. The output vector $\boldsymbol{y}$ in figure (3.5) represents this case. In addition, if Farrell and Russell scores coincide, we get $ZE_O(\boldsymbol{x}, \boldsymbol{y}) = FE_O(\boldsymbol{x}, \boldsymbol{y}) = RE_O(\boldsymbol{x}, \boldsymbol{y})$.A second possibility is that the radial projection of a data point belongs to $IsoqP(\boldsymbol{x}) \backslash EffP(\boldsymbol{x})$. Then the Zieschang measure computes the efficiency score as the product of Farrell and Russell measure. In terms of the figure, with respect to the data point $\hat{\boldsymbol{y}}$, first $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) = 1/\psi_O(\boldsymbol{x}, \hat{\boldsymbol{y}})$ is obtained and since $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) \times \hat{\boldsymbol{y}} \in IsoqP(\boldsymbol{x}) \backslash EffP(\boldsymbol{x})$, in the second step Russell component is computed with respect to the data point $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})\times\hat{\boldsymbol{y}}$. In terms of the figure this Russell component is $RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})) = \max \{\sum^2_{m=1}\delta_m/2 \mid (\delta_1\times(\hat{y}_1\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})), \delta_2\times(\hat{y}_2\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}))) \in P(\boldsymbol{x}), \delta_m \in [1, +\infty), \hat{\boldsymbol{y}} \in P(\boldsymbol{x}), \hat{\boldsymbol{y}}\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) \in P(\boldsymbol{x}) \}$. Then the Zieschang efficiency score for $\hat{\boldsymbol{y}}$ is the product $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})\times RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}))$.

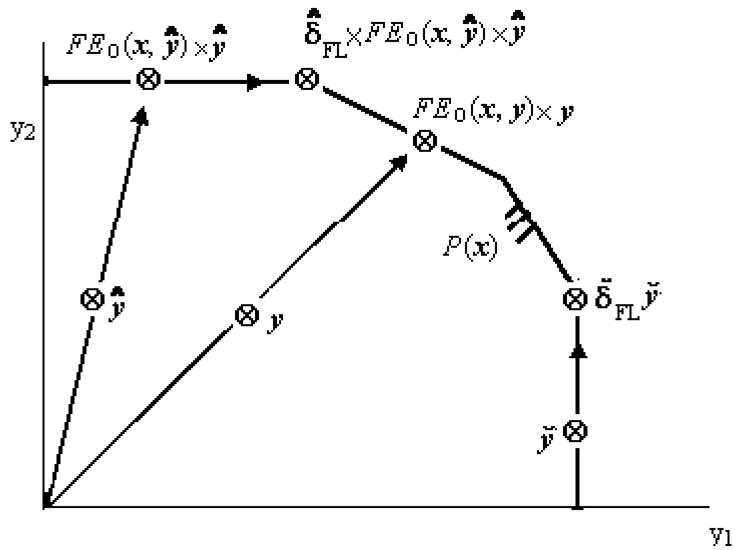

Figure 3.5. Comparing $ZE_I(\boldsymbol{y}, \boldsymbol{x})$, $FE_I(\boldsymbol{y}, \boldsymbol{x})$ and $RE_I(\boldsymbol{y}, \boldsymbol{x})$

Since $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) > 1$ and $RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})) > 1$, implies $ZE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}) > FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})$. The relationship between $ZE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})$ and $RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})$ however is ambiguous. The reason is that though we have $RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})) < RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})$, the product $FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})\times RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}\times FE_O(\boldsymbol{x}, \hat{\boldsymbol{y}}))$ doesn't have to have a definite relationship with $RE_O(\boldsymbol{x}, \hat{\boldsymbol{y}})$. Finally for the third possibility where an inefficient output vector such as $\check{\mathbf{y}} \in IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$, we have $ZE_O(\boldsymbol{x}, \check{\mathbf{y}})$ equal to $RE_O(\boldsymbol{x}, \check{\mathbf{y}})$.

In conclusion $ZE_O(\boldsymbol{x}, \boldsymbol{y}) \geqq FE_O(\boldsymbol{x}, \boldsymbol{y})$ and $ZE_O(\boldsymbol{x}, \boldsymbol{y}) \lesseqgtr RE_O(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 3.3: Zieschang output oriented technical efficiency measure satisfies E.1b to E4b and E6b but it does not satisfy E5b the monotonicity.

Proofs:

$ZE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E1b, the Identification Property:

This is ensured by the Russell adjustments for any data points for which the Radial projection does not belong to the efficient subset of the $IsoqP(\boldsymbol{x})$.

$ZE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E2b, the "compare to" Property:

This has the same explanation as for E1b above.

$ZE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E3b the homogeneity of plus one in inputs:

For the data points for which the Farrell projection belongs to the $EffP(\boldsymbol{x})$, we have $ZE_O(\boldsymbol{x}, \boldsymbol{y}) = FE_O(\boldsymbol{x}, \boldsymbol{y})$. Since we have already proved that $FE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \beta\, FE_O(\boldsymbol{x}, \boldsymbol{y})$ so the property holds for $ZE_O(\boldsymbol{x}, \boldsymbol{y})$ too. For the data points for which the Farrell projection belongs to $IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$, the Russell adjustment is given as $RE_O(\beta\boldsymbol{x}, \boldsymbol{y}\times FE_O(\beta\boldsymbol{x}, \boldsymbol{y}))$. Using propositions 3.1 and 3.2 we can write $RE_O(\beta\boldsymbol{x}, \boldsymbol{y}\times FE_O(\beta\boldsymbol{x}, \boldsymbol{y})) = RE_O(\boldsymbol{x}, \boldsymbol{y}\times FE_O(\boldsymbol{x}, \boldsymbol{y}))$

$\Rightarrow ZE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \beta ZE_O(\boldsymbol{x}, \boldsymbol{y})$

Similarly for $\boldsymbol{x} \in IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$ we have $FE_O(\boldsymbol{x}, \boldsymbol{y}) = 1$ and

$ZE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = RE_O(\beta\boldsymbol{x}, \boldsymbol{y}) = \beta RE_O(\boldsymbol{x}, \boldsymbol{y}) = \beta ZE_O(\boldsymbol{x}, \boldsymbol{y})$

$ZE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies E4b, the homogeneity of minus one in outputs:

This follows from proposition 3.1 and 3.2

$ZE_O(\boldsymbol{x}, \nu\boldsymbol{y}) = [RE_O(\boldsymbol{x}, \nu\boldsymbol{y} \times FE_O(\boldsymbol{x}, \nu\boldsymbol{y})\ ] \times FE_O(\boldsymbol{x}, \nu\boldsymbol{y}) = \nu^{-1} ZE_O(\boldsymbol{x}, \boldsymbol{y})$

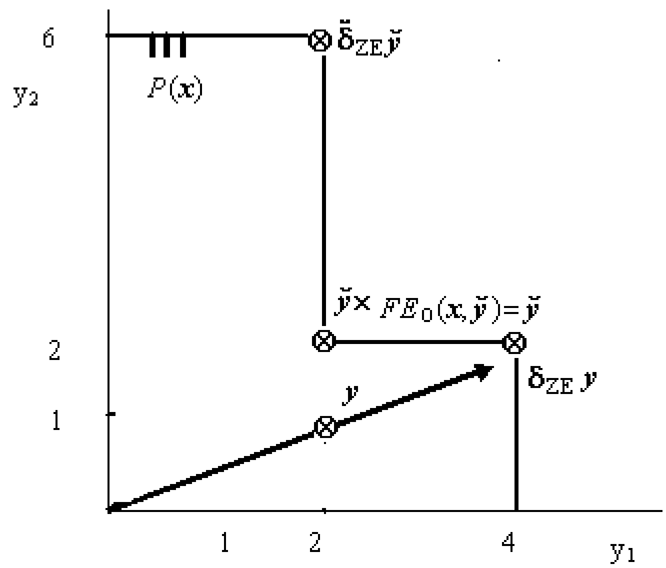

Figure 3.6, $ZE_O(\boldsymbol{x}, \boldsymbol{y})$ does not satisfy Monotonicity

$ZE_O(\boldsymbol{x}, \boldsymbol{y})$ fails to satisfy Monotonicity:

Consider the non-convex production isoquant $IsoqP(\boldsymbol{x})$ shown in figure 3.6 for two output case. For the data point $(\boldsymbol{x}, \boldsymbol{y})$ we have $ZE_O(\boldsymbol{x}, \boldsymbol{y}) = FE_O(\boldsymbol{x}, \boldsymbol{y}) = 2$. For the data point $(\boldsymbol{x}, \breve{\mathbf{y}})$ we have $FE_O(\boldsymbol{x}, \breve{\mathbf{y}})=1$ and $ZE_O(\boldsymbol{x}, \breve{\mathbf{y}})=2$. Thus $ZE_O(\boldsymbol{x}, y_1=2, y_2=1) = ZE_O(\boldsymbol{x}, y_1=2, y_2=2)$, violating strong monotonicity however still satisfying its weaker version.

3.2.3 The Non-radial Farrell Output oriented Technical Efficiency Measure

We now compute the output oriented Non-radial Farrell measure of technical efficiency using the transformed input and output share vectors, ${}_s\boldsymbol{x}$ and ${}_s\boldsymbol{y}$, obtained in step one as discussed in previous chapter. The transformation allows us to define the correspondence $P^S: \Re^N_+ \rightarrow P^S({}_s\boldsymbol{x}) \subseteq \Re^M_+$ which maps input share vectors ${}_s\boldsymbol{x}$ into subsets $P^S({}_s\boldsymbol{x})$ of output share vectors ${}_s\boldsymbol{y}$. For the technologies discussed in chapter one, the correspondence, P^S, has the following implications;

1. ${}_s\boldsymbol{y} \notin P^S(\mathbf{0})$ for ${}_s\boldsymbol{y} \in \Re^M_+$. Also, $\forall\ {}_s\boldsymbol{x} \in \Re^N_+$, $\mathbf{0} \in P^S({}_s\boldsymbol{x})$.

2. ${}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}) \Rightarrow \forall\ {}_s\hat{\boldsymbol{y}} \leq {}_s\boldsymbol{y},\ {}_s\hat{\boldsymbol{y}} \in P^S({}_s\boldsymbol{x}),\ \forall\ {}_s\boldsymbol{x} \in \Re^N_+$.

3. ${}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}) \Rightarrow {}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}^*),\ \forall\ {}_s\boldsymbol{x}^* \geq {}_s\boldsymbol{x}$

4. P^S is a closed correspondence.

For this transformed output correspondence we can define

$P^S({}_s\boldsymbol{x}) = \{{}_s\boldsymbol{y} \mid {}_s\boldsymbol{y} \text{ is feasible}\}$

$IsoqP^S({}_s\boldsymbol{x}) = \{{}_s\boldsymbol{y} \mid {}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}) \text{ and } \delta{}_s\boldsymbol{y} \notin P^S({}_s\boldsymbol{x}) \text{ for } \delta \in [1,+\infty)\}$

$EffP^S({}_s\boldsymbol{x}) = \{{}_s\boldsymbol{y} \mid {}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}) \text{ and } {}_s\hat{\boldsymbol{y}} \geq {}_s\boldsymbol{y} \Rightarrow {}_s\hat{\boldsymbol{y}} \notin P^S({}_s\boldsymbol{x})\}$

In terms of figure(3.7) , the curve represents $IsoqP^S({}_s\boldsymbol{x})$. The area under the curve and the curve together define the feasibility region. The segment of the curve bounded by the output vectors ${}_s\boldsymbol{y}^*$ and ${}_s\boldsymbol{y}^{**}$ represents the $EffP^S({}_s\boldsymbol{x})$.

To define the output oriented Non-radial Farrell technical efficiency measure let D be the set of diagonal matrices, ${}^i\boldsymbol{\delta}_{NE}$, each of dimension M with ${}^i\delta_{mm} \in [1,+\infty)$ for m=1,2,…M, and let there be a vector $\boldsymbol{\xi} \in \Re^M_+$, then,

Definition: The function NE_O: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$ for q = M, N, defined by $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = \max\{\psi_O({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}{}_s\boldsymbol{y})/\psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})|, \boldsymbol{\delta}_{NE} \in D, {}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}),\ \boldsymbol{\delta}_{NE}\,{}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}), (\boldsymbol{\delta}_{NE}\,{}_s\boldsymbol{y} + \boldsymbol{\xi}) \notin P^S({}_s\boldsymbol{x}),\ \forall\ \boldsymbol{\xi} \in \Re^M_+\}$ and $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = -\infty$ for ${}_s\boldsymbol{y} \notin P^S({}_s\boldsymbol{x})$, is called the output oriented Non-radial Farrell measure of technical efficiency.

Thus we first obtain the reference output vector $\boldsymbol{\delta}_{NE}\,{}_s\boldsymbol{y} \in EffP^S({}_s\boldsymbol{x})$.This reference output vector by definition is the one that maximizes $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ when the efficiency score is computed as the ratio of the Euclidian norms, $\| \boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y} \| / \| {}_s\boldsymbol{y} \|$. In this computation δ_{mm} represents the proportion by which the mth component of the output share vector ${}_s\boldsymbol{y}$ is to be expanded in order to obtain the technically efficient share vector $\boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y} \in Eff\, P^S({}_s\boldsymbol{x})$. For any data point that belongs to the efficient frontier $\delta_{mm} = 1\ \forall\ m=1\ldots M$.

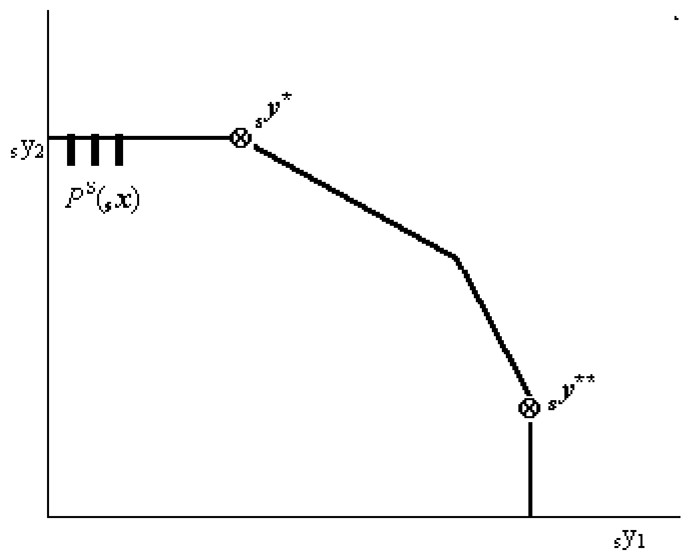

Figure 3.7. Transformed input level curve

Thus $\psi_O({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y}) = \psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ which results in $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = 1$ and indicates full technical efficiency with respect to the efficiency frontier. For off the frontier data points $\boldsymbol{\delta}_{NE}$ has at least one diagonal element greater than one. Thus $\psi_O({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y}) > \psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ implying $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) > 1$, indicating technical inefficiency with inefficiency increasing for the higher values of $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. If $\boldsymbol{\delta}_{NE}$ is equal to $\delta\mathbf{I}$, where δ is some constant and $\mathbf{I}$ is an M×M identity matrix, then

$\psi_O({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y})/\psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = \delta = FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. Also note that the Farrell measure of output oriented technical efficiency can be written as the ratio of two Euclidean norms i.e. $\psi_O({}_s\boldsymbol{x}, \delta_{DF}\, {}_s\boldsymbol{y})/\psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. Thus the Farrell output oriented technical efficiency measure is a special case of output oriented Non-radial Farrell measure of technical efficiency when the two are computed on the transformed data set. Since $\psi_O({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y})/\psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \geq \psi_O({}_s\boldsymbol{x}, \delta_{DF}\, {}_s\boldsymbol{y})/\psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \Rightarrow$ $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \geq FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. Note however that the two measures are comparable only for data points that have their projections on efficient subset i.e. only for ${}_s\boldsymbol{y} \times FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \in EffP^S({}_s\boldsymbol{x})$ and $\boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y} \in EffP^S({}_s\boldsymbol{x})$. If for any ${}_s\boldsymbol{y}$ we have $\delta_{DF}\, {}_s\boldsymbol{y} \in IsoqP^S({}_s\boldsymbol{x}) \setminus EffP^S({}_s\boldsymbol{x})$ and $\boldsymbol{\delta}_{NE}\, {}_s\boldsymbol{y} \in EffP^S({}_s\boldsymbol{x})$, two measures are not comparable as they use different sets of reference.

Comparing the Non-radial Farrell measure with the Russell measure, whenever we have $RE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \Rightarrow NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \geq RE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. For $RE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) > FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ the relationship between $RE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ and $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ may go in any direction. Further, due to the computational differences between the Russell and the Non-radial Farrell technical efficiency measure it is not unlikely that the two may have identical projection vectors but different efficiency scores as shown in figure 3.8 where the data point $({}_s\boldsymbol{x}, {}_sy_1=10\ {}_sy_2=10)$ has $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = \sqrt{5}$ and $RE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = 2$ and the identical projection vector is $({}_s\boldsymbol{x}, {}_sy_1=30, {}_sy_2=10)$. Similarly it is also possible for the two measures to have identical scores with different projection vectors.

Similar relationship holds between the Non-radial Farrell and the Zieschang measure. Whenever $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \Rightarrow FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = ZE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ and we get $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = ZE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. Also for $FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = ZE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ and $FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \neq NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$, we have $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) > ZE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$. For $FE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \neq ZE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$, we have $ZE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \leq NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$.

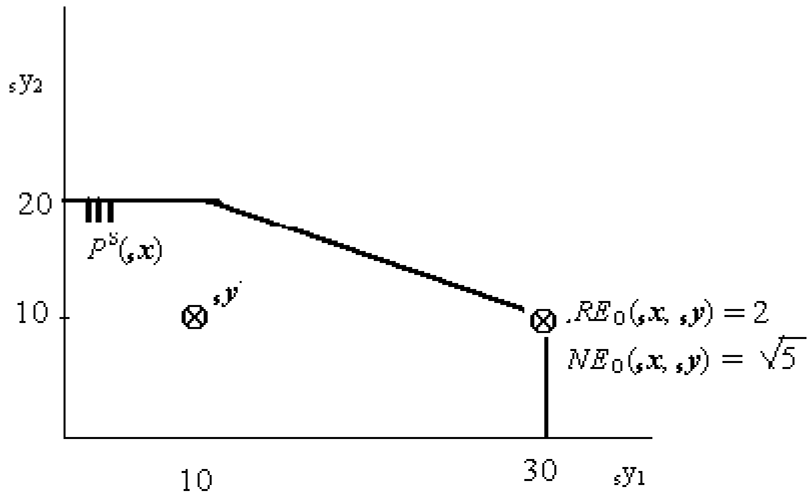

Figure 3.8. Comparing $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ with $RE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$

Proposition 3.4 $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ satisfies E1b to E6b.

Proofs:

$NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ satisfies E1b the Identification Property:

First suppose that ${}_s\boldsymbol{y}^* \in P^S({}_s\boldsymbol{x})\backslash EffP^S({}_s\boldsymbol{x})$ and that $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}^*)=1$.Then since ${}_s\boldsymbol{y}^* \in P^S({}_s\boldsymbol{x})\backslash EffP^S({}_s\boldsymbol{x})$, we can define for a radius r>0 a closed ball centered at ${}_s\boldsymbol{y}^*$ such that

$B_r({}_s\boldsymbol{y}^*)= \{{}_s\boldsymbol{y} \mid {}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}) , \|{}_s\boldsymbol{y} - {}_s\boldsymbol{y}^*\| \leqq r\}$

But this implies that $\exists\ {}_s\boldsymbol{y} \geq {}_s\boldsymbol{y}^*$ such that ${}_s\boldsymbol{y} \in B_r({}_s\boldsymbol{y}^*) \cap P^S({}_s\boldsymbol{x})$ and a diagonal matrix $\boldsymbol{\delta}_{NE}$ of order M×M, with $\delta_{mm} \in [1,+\infty)$ can be defined such that ${}_s\boldsymbol{y} = \boldsymbol{\delta}_{NE}\,{}_s\boldsymbol{y}^* \Rightarrow NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}^*) > 1$ hence a contradiction.

Next suppose that ${}_s\boldsymbol{y}^* \in EffP^S({}_s\boldsymbol{y})$ and that $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}^*) > 1$. Since $NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}^*) > 1 \Rightarrow$ we can define a diagonal matrix $\boldsymbol{\delta}_{NE}$ of order M×M, $\delta_{mm} \in [1,+\infty)$, with at least one diagonal element strictly greater than one such that $\boldsymbol{\delta}_{NE} {}_s\boldsymbol{y}^* \geq {}_s\boldsymbol{y}^*$ and $\boldsymbol{\delta}_{NE} {}_s\boldsymbol{y}^* \in EffP^S({}_s\boldsymbol{x})$. But with $\boldsymbol{\delta}_{NE} {}_s\boldsymbol{y}^* \in EffP^S({}_s\boldsymbol{x})$, the condition ${}_s\boldsymbol{y}^* \in EffP^S({}_s\boldsymbol{x})$ can not simultaneously occur, hence a contradiction.

Finally since for ${}_s\boldsymbol{y}^* \notin P^S({}_s\boldsymbol{x}) \Rightarrow$ for a closed ball with r>0

$B_r({}_s\boldsymbol{y}^*) = \{{}_s\boldsymbol{y} \mid {}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}), \|{}_s\boldsymbol{y} - {}_s\boldsymbol{y}^*\| \leqq r\}$

containing ${}_s\boldsymbol{y} \geq {}_s\boldsymbol{y}^*$ with ${}_s\boldsymbol{y} \in B_r({}_s\boldsymbol{y}^*) \cap P^S({}_s\boldsymbol{x}) = \emptyset \Rightarrow NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = -\infty$

<u>$NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ satisfies E2b the "Compare to" Property:</u>

For any ${}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}) \backslash EffP^S({}_s\boldsymbol{x})$ there exists a $\boldsymbol{\delta}_{NE}$, with at least one diagonal element greater than one such that $NE_O({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE} {}_s\boldsymbol{y}) = 1$. But from E1b this implies that $\boldsymbol{\delta}_{NE} {}_s\boldsymbol{y} \in EffP^S({}_s\boldsymbol{x})$ and so holds E.2b.

<u>$NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ satisfies E3b the homogeneity of plus one in inputs:</u>

$NE_O(\beta {}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = \max\{ \psi(\beta {}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE} {}_s\boldsymbol{y})/\psi(\beta {}_s\boldsymbol{x}, {}_s\boldsymbol{y}) \mid \boldsymbol{\delta}_{NE}({}_s\boldsymbol{y}) \in P^S(\beta {}_s\boldsymbol{x}),$

$(\boldsymbol{\delta}_{NE} {}_s\boldsymbol{y} + \xi) \notin P^S(\beta {}_s\boldsymbol{x}), {}_s\boldsymbol{y} \in P^S(\beta {}_s\boldsymbol{x}), \beta > 0\}$

$NE_I(\beta {}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = \beta \max\{\beta^{-1}[\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE} {}_s\boldsymbol{y})/\psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y})] \mid \boldsymbol{\delta}_{NE} {}_s\boldsymbol{y}/\beta \in P^S({}_s\boldsymbol{x}),$

$(\boldsymbol{\delta}_{NE} \beta {}_s\boldsymbol{y} + \xi) \notin P^S({}_s\boldsymbol{x}), {}_s\boldsymbol{y} \in P^S({}_s\boldsymbol{x}), \beta > 0\}$

$= \beta\, NE_I({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$

<u>$NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ fails to satisfy E4b the homogeneity of minus one in outputs:</u>

In terms of figure 3.9 $NE_O({}_s\boldsymbol{x}, {}_sy_1 = 20, {}_sy_2 = 20) = 1$ and $NE_O({}_s\boldsymbol{x}, {}_sy_1 = 10, {}_sy_2 = 10) = \sqrt{5} > 2$ which contradicts E4b.

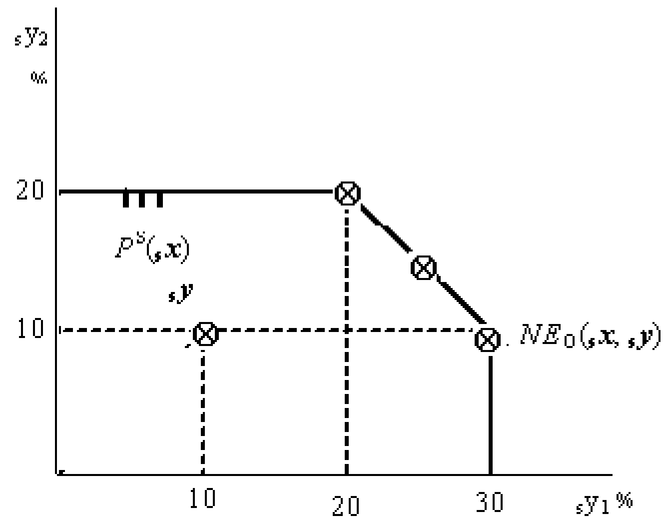

Figure 3.9. Violation of E4b by $NE_I(_s\mathbf{x}, _s\mathbf{y})$

<u>$NE_O(_s\mathbf{x}, _s\mathbf{y})$ satisfies E5b the Monotonicity Property:</u>

Let $\boldsymbol{\varepsilon} \geq \mathbf{0}$ an M×1 vector and define

$NE_O(_s\mathbf{x}, {_s\mathbf{y}} + \boldsymbol{\varepsilon}) = \max\{ \psi(_s\mathbf{x}, \boldsymbol{\delta}_{NE}(_s\mathbf{y} + \boldsymbol{\varepsilon}))/\psi(_s\mathbf{x}, {_s\mathbf{y}} + \boldsymbol{\varepsilon}) | {_s\mathbf{y}} \in P^S(_s\mathbf{x}),\ \boldsymbol{\delta}_{NE}\, {_s\mathbf{y}} \in P^S(_s\mathbf{x}), (\boldsymbol{\delta}_{NE\,s}\mathbf{y} + \boldsymbol{\varepsilon}) \in P^S(_s\mathbf{x}), ((\boldsymbol{\delta}_{NE\,s}\mathbf{y} + \boldsymbol{\varepsilon}) + \xi) \notin P^S(_s\mathbf{x})$ for $\boldsymbol{\varepsilon} \geq \mathbf{0}\}$

We need to show that $NE_O(_s\mathbf{x}, {_s\mathbf{y}} + \boldsymbol{\varepsilon}) < NE_O(_s\mathbf{x}, {_s\mathbf{y}})$

$\Rightarrow \max\{ \psi(_s\mathbf{x}, \boldsymbol{\delta}_{NE}(_s\mathbf{y} + \boldsymbol{\varepsilon}))/\psi(_s\mathbf{x}, {_s\mathbf{y}} + \xi)\} < \max\{ \psi(_s\mathbf{x}, \boldsymbol{\delta}^*_{NE\,s}\mathbf{y})/\psi(_s\mathbf{x}, {_s\mathbf{y}})\}$

We have $\boldsymbol{\delta}_{NE} \leq \boldsymbol{\delta}^*_{NE}$ and ${_s\mathbf{y}} + \boldsymbol{\varepsilon} \geq {_s\mathbf{y}}$

$\Rightarrow \{ \psi(_s\mathbf{x}, \boldsymbol{\delta}_{NE}(_s\mathbf{y} + \boldsymbol{\varepsilon})) \lesseqqgtr \psi(_s\mathbf{x}, \boldsymbol{\delta}^*_{NE\,s}\mathbf{y})$ and $\psi(_s\mathbf{x}, (_s\mathbf{y} + \boldsymbol{\varepsilon})) > \psi(_s\mathbf{x}, {_s\mathbf{y}})$

for $\psi(_s\mathbf{x}, \boldsymbol{\delta}_{NE}(_s\mathbf{y} + \boldsymbol{\varepsilon})) \leqq \psi(_s\mathbf{x}, \boldsymbol{\delta}^*_{NE\,s}\mathbf{y})$, along with $\psi(_s\mathbf{x}, (_s\mathbf{y} + \boldsymbol{\varepsilon})) > \psi(_s\mathbf{x}, {_s\mathbf{y}})$ we have the required result i.e. $\max\{ \psi(_s\mathbf{x}, \boldsymbol{\delta}_{NE}(_s\mathbf{y} + \boldsymbol{\varepsilon}))/\psi(_s\mathbf{x}, {_s\mathbf{y}} + \boldsymbol{\varepsilon})\} < \max\{ \psi(_s\mathbf{x}, \boldsymbol{\delta}^*_{NE\,s}\mathbf{y})/\psi(_s\mathbf{x}, {_s\mathbf{y}})\}$

The third possibility i.e. $\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}({}_s\boldsymbol{y} + \boldsymbol{\varepsilon})) > \psi({}_s\boldsymbol{x}, \boldsymbol{\delta}^*_{NE}\,{}_s\boldsymbol{y})$ with $\psi({}_s\boldsymbol{x}, ({}_s\boldsymbol{y} + \xi)) > \psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$

$\Rightarrow \max\{\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}({}_s\boldsymbol{y} + \xi))/\psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y} + \xi)\} \gtreqless \max\{\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}^*_{NE}\,{}_s\boldsymbol{y})/\psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y})\}$

The possibilities

$\max\{\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}({}_s\boldsymbol{y} + \boldsymbol{\varepsilon}))/\psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y} + \boldsymbol{\varepsilon})\} \geq \max\{\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}^*_{NE}\,{}_s\boldsymbol{y})/\psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y})\}$

are ruled out for negatively sloped efficiency frontier. Thus we are left with

$\max\{\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}({}_s\boldsymbol{y} + \boldsymbol{\varepsilon}))/\psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y} + \boldsymbol{\varepsilon})\} < \max\{\psi({}_s\boldsymbol{x}, \boldsymbol{\delta}^*_{NE}\,{}_s\boldsymbol{y})/\psi({}_s\boldsymbol{x}, {}_s\boldsymbol{y})\}$ implying monotonicity.

$NE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$ satisfies E6b the Commensurability Property:

This follows due to step one of the procedure.

3.3 Computation

Linear programming based, output oriented DEA models can be employed to obtain efficiency scores for the output vectors in any given data set. Like in the case of the input oriented models, the output oriented DEA models also involve constructing an efficiency frontier, reflecting the best practices in the data set. The efficiency scores are then computed on the basis of deviations from this frontier. In this section we introduce the use of the output version of the elementary CCR models to compute Farrell, Russell, Zieschang and Non-radial Farrell efficiency scores. We continue maintaining constant returns to scales and strong disposability assumptions. We also assume absence of any non-discretionary inputs or outputs so that all inputs and outputs can be varied freely as per requirement. With these assumptions made consider a sample of J firms each using an input vector $\boldsymbol{x}^j \in \Re^N_+$ to produce an output vector $\boldsymbol{y}^j \in \Re^M$ where j ranges from 1,….J.

Then for the kth sample point we can write the output oriented DEA model as

$$\begin{aligned} &\text{Max}_{\delta DF,\eta} && \delta_{DF} \\ &\text{s.t}\ \boldsymbol{x}^{k} && -\boldsymbol{X}\eta \geq \mathbf{0} \\ & && -\ \delta_{FL}\boldsymbol{y}^{k} +\ \boldsymbol{Y}\eta \geq \mathbf{0} \\ & && \eta \geq \mathbf{0} \end{aligned}$$

with $\boldsymbol{X}$ and $\boldsymbol{Y}$ as defined earlier and η is a J×1 intensity vector. The model aims to solve for δ_{DF} and elements of η.For the sample point being evaluated δ_{DF} attains a value of equal to or greater than one, with δ_{DF}=1 reflecting full efficiency and δ_{DF} >1 reflecting inefficiency with inefficiency increasing for higher values of δ_{DF}. Again, the program is to be solved once for each firm in the sample so that we get a value of δ_{DF} for each firm in the sample.

In words this problem looks for maximum feasible radial expansion in the output vector of the data point being evaluated, for a given level of input and for a technology generated by the sample. The resulting statistic is the Farrell's output oriented technical efficiency measure.

The DEA problem for the Russell measure is given as

$$\begin{aligned} &\text{Max}_{\delta FL,\ \eta} && 1/M(\mathbf{e}^{T}\delta_{FL}\mathbf{e}) \\ &\text{s.t} && -\ \delta_{FL}\boldsymbol{y}^{k} + \boldsymbol{Y}\eta \geq \mathbf{0} \\ & && \boldsymbol{x}^{k} - \boldsymbol{X}\eta \geq \mathbf{0} \\ & && \eta \in \Re^{M}{}_{+} \end{aligned}$$

where δ_{FL} is a M×M diagonal matrix with the diagonal elements comprising of Russell expansionary constants, $\mathbf{e}$ is M×1 vector of ones with $\mathbf{e}^{T}$ its transpose and η is the J×1 intensity vector.

Zieschang measure is solved in two steps. In the first step we apply the linear program that solves for the Farrell Efficiency parameter. In the second step the Russell adjustment is applied on the Farrell projection vector $\delta_{DF}\boldsymbol{y}$. If the Farrell projection vector belongs to the efficient frontier then the Russell score $RE_O(\boldsymbol{x}, \delta_{DF}\boldsymbol{y}) = 1$ and we have Zieschang score $\delta_{ZE} = \delta_{DF}$. If however $RE_O(\boldsymbol{x}, \delta_{DF}\boldsymbol{y}) > 1$ then we have $\delta_{ZE} = \delta_{DF} \times RE_O(\boldsymbol{x}, \delta_{DF}\boldsymbol{y})$.

The Non-radial Farrell scores are computed in the transformed data space and the computational ode is written as;

$$\text{Max}_{\delta_{NE},\, \eta} \quad \|\delta_{NE}\ {}_s\boldsymbol{y}^k\| / \|{}_s\boldsymbol{y}^k\|$$

$$\text{s.t} \quad -\delta_{NE}\,{}_s\boldsymbol{y}^k + {}_s\boldsymbol{Y}\eta \geq \mathbf{0}$$

$$ {}_s\boldsymbol{x}^k - {}_s\boldsymbol{X}\eta \geq \mathbf{0}$$

$$\eta \in \Re^M_+$$

where $\delta_{NE} \in [1, +\infty)$ is the M×M diagonal matrix of constant.

3.4 Revenue and Allocative Efficiencies

Up till now we have been discussing the output oriented efficiency measures that require information on input and output quantities only. If information on output prices is also available, the analysis, like in the case of input based efficiency measures, can be extended to quantify the broader concept of output oriented *overall efficiency*. The output oriented *overall efficiency* measure so obtained then can be decomposed into the *Allocative Efficiency* and the *Technical Efficiency* components. In this section we look into these efficiency concepts. We assume that the information on output prices along with the input and output quantity data sets is available. We continue with our basic assumptions to define and compare extended concepts of radial and non-radial efficiency measures along with an additional behavioral assumption of revenue maximization. The efficiency of any sample point then is to be gauged through the attainment of its objective of revenue maximization, given the input vector. If $\boldsymbol{r} \in \Re^M_+$ represents the vector of output prices then for the correspondence P: $\Re^N_+ \rightarrow \Re^M_+$, we can write the revenue function as $V(\boldsymbol{x}, \boldsymbol{r}) = \max \{ \boldsymbol{r}^T\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x})\}$. The maximum exists because $P(\boldsymbol{x})$ has been assumed, as in chapter 1, nonempty, bounded and closed, thereby implying continuity of $\boldsymbol{r}^T\boldsymbol{y}$. The resulting revenue function has following properties[6]

V1. $V(\mathbf{0}, \boldsymbol{r}) = 0 \ \forall \ \boldsymbol{r} \in \Re^M_+$. $V(\boldsymbol{x}, \boldsymbol{r}) = 0 \ \forall \ \boldsymbol{x} \in \Re^N_+$ and $\boldsymbol{r} = \mathbf{0}$.

V2. $V(\boldsymbol{x}, \boldsymbol{r}) > 0 \ \forall \ \boldsymbol{x} \geq \mathbf{0}$ and $\boldsymbol{x} \in \{\boldsymbol{x} \mid \boldsymbol{x} \in \Re^N_+, P(\boldsymbol{x}) > \mathbf{0}\}$.

$V(\boldsymbol{x}, \boldsymbol{r}) \geqq 0 \ \forall \ \boldsymbol{x} \in \Re^N_+,\ \mathbf{r} \in \Re^M_+$.

V3. $V(\lambda\boldsymbol{x}, \boldsymbol{r}) \geqq V(\boldsymbol{x}, \boldsymbol{r}),\ \lambda \geqq 1.\ \forall \ \boldsymbol{x} \in \Re^N_+, \mathbf{r} \in \Re^M_+$.

V4. . $V(\boldsymbol{x}, \theta\boldsymbol{r}) = \theta V(\boldsymbol{x}, \boldsymbol{r}) \ \forall \ \boldsymbol{x} \in \Re^N_+, \boldsymbol{r} \in \Re^M_+ \ \theta \geq 0$.

V5. $V(\boldsymbol{x}, \boldsymbol{r}) > 0 \Rightarrow V(\lambda\boldsymbol{x}, \boldsymbol{r}) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$

V6. $V(\boldsymbol{x}, \boldsymbol{r})$ is convex and continuous in prices.

V7. $V(\boldsymbol{x}, \mathbf{r})$ is upper semi-continuous in $\boldsymbol{x} \in \Re^N_+$.

For the given output price vector **r**, define the following lower contour set

$V(\mathbf{r}) = \{\boldsymbol{y} \mid \ \mathbf{r}^T\boldsymbol{y} \leq V(\boldsymbol{x}, \boldsymbol{r}), \mathbf{r}^T\hat{\boldsymbol{y}} > V(\boldsymbol{x}, \boldsymbol{r}) \Rightarrow \hat{\boldsymbol{y}} \notin V(\boldsymbol{r})\}$

In words $V(\mathbf{r})$ is the set of all those output vectors that generate revenue equal to or less than $V(\boldsymbol{x},\boldsymbol{r})$ and no element of the set generates revenue greater than $V(\boldsymbol{x}, \boldsymbol{r})$. From this it follows that $P(\boldsymbol{x}) \subseteq V(\mathbf{r})$.

The following subset of $V(\mathbf{r})$ represents an iso-revenue level

Iso$V(\mathbf{r}) = \{\boldsymbol{y} \mid \ \mathbf{r}^T\boldsymbol{y} = V(\boldsymbol{x}, \boldsymbol{r})\}$

[6] Proofs and explanations in Shephard(1970).

From this set we can define a subset comprising of the feasible output vectors that generate revenue equal to $V(\boldsymbol{x}, \boldsymbol{r})$ i.e.

$VM(\boldsymbol{x}, \boldsymbol{r}) = \{\boldsymbol{y} \mid \mathbf{r}^T\boldsymbol{y} = V(\boldsymbol{x}, \boldsymbol{r}), \boldsymbol{y} \in P(\boldsymbol{x})\}$

Also from the definition of $V(\boldsymbol{x}, \boldsymbol{r})$ we have

$P(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y} \in \Re^M{}_+, \boldsymbol{x} \in \Re^N{}_+, \mathbf{r}^T\boldsymbol{y} \leq V(\boldsymbol{x}, \boldsymbol{r})\}$

$\Rightarrow VM(\boldsymbol{x}, \boldsymbol{r}) = P(\boldsymbol{x}) \cap \mathrm{Iso}V(\mathbf{r})$

Proposition 3.5: $\cup_{\mathbf{r}>\mathbf{0}} VM(\boldsymbol{x}, \boldsymbol{r}) \subseteq EffP(\boldsymbol{x})$

Proofs:

Let $\boldsymbol{y} \in P(\boldsymbol{x}), \boldsymbol{x} \geq \mathbf{0}$. Also let $\boldsymbol{y} \in VM(\boldsymbol{x}, \boldsymbol{r})$ for $\mathbf{r} > \mathbf{0}$.

If $\boldsymbol{y} \notin EffP(\boldsymbol{x}) \Rightarrow \exists\, \hat{\boldsymbol{y}} \in EffP(\boldsymbol{x})$ such that $\hat{\boldsymbol{y}} \geq \boldsymbol{y} \Rightarrow \mathbf{r}^T\hat{\boldsymbol{y}} > \mathbf{r}^T\boldsymbol{y}$. But this contradicts with $\boldsymbol{y} \in VM(\boldsymbol{x},\boldsymbol{r})$. The reverse however does not hold as $\boldsymbol{y} \in P(\boldsymbol{x})$ is necessary but not a sufficient condition for $\boldsymbol{y} \in VM(\boldsymbol{x}, \boldsymbol{r})$. This proposition suggests that the revenue efficiency implies technical efficiency and that the reverse implication does not follow.

Definition: The function $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = \{ V(\boldsymbol{x}, \boldsymbol{r}) / \boldsymbol{r}^T \boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x})\}$ is called the output oriented overall efficiency measure.

An output vector $\boldsymbol{y} \in VM(\boldsymbol{x}, \boldsymbol{r}) \Rightarrow \mathbf{r}^T\boldsymbol{y} = V(\boldsymbol{x}, \boldsymbol{r})$ and also implies from proposition 3.5 the technical efficiency $\Rightarrow OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1 \Rightarrow$ for the output vector $\boldsymbol{y} \in VM(\boldsymbol{x}, \boldsymbol{r})$, $\nexists\, \hat{\boldsymbol{y}} \in P(\boldsymbol{x})$ such that $\boldsymbol{r}^T\hat{\boldsymbol{y}} > \boldsymbol{r}^T\boldsymbol{y}$. For any output vector $\boldsymbol{y} \in P(\boldsymbol{x}) \backslash VM(\boldsymbol{x}, \boldsymbol{r})$ we have $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) > 1$ thus implying overall inefficiency which may be technical in nature or allocative or a combination of both. Inefficiency is higher for higher values of $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$.

Proposition 3.6: Overall Efficiency satisfies the following for $\boldsymbol{y} \in P(\boldsymbol{x})$ and $\hat{\boldsymbol{y}} \in P(\boldsymbol{x})$

*OE*1 $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \in [1, +\infty)$

*OE*2. $OE_O(\lambda\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \geq \lambda OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$, $\lambda > 1$

*OE*3. $OE_O(\boldsymbol{x}, \boldsymbol{r}, \nu\boldsymbol{y}) = \nu^{-1} OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ $\nu > 0$

*OE*4. $OE_O(\boldsymbol{x}, \boldsymbol{r}, \hat{\boldsymbol{y}}) < OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ for $\hat{\boldsymbol{y}} \geq \boldsymbol{y}$

*OE*5. $OE_O(\boldsymbol{x}, \theta\, \boldsymbol{r}, \boldsymbol{y}) = OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

*OE*6. $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1$ for $V(\boldsymbol{x}, \boldsymbol{r}) = \boldsymbol{r}^{\mathrm{T}}\boldsymbol{y}$.

*OE*7. $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies commensurability.

<u>Proofs</u>:

*OE*1. For any output vector $\boldsymbol{y} \in P(\boldsymbol{x})$ we have $\boldsymbol{r}^T\boldsymbol{y}^*(\boldsymbol{x}, \boldsymbol{r}) \geq \boldsymbol{r}^T\boldsymbol{y}$

$\Rightarrow V(\boldsymbol{x}, \boldsymbol{r})/\, \boldsymbol{r}^T\boldsymbol{y} \geq 1$, where $\boldsymbol{y}^*(\boldsymbol{x}, \boldsymbol{r})$ is the revenue maximizing vector

*OE*2. This follows from V3 and the fact that for constant returns to scale we have $P(\lambda\boldsymbol{x}) = \lambda P(\boldsymbol{x})$.

*OE*3. $OE_O(\boldsymbol{x}, \boldsymbol{r}, \nu\boldsymbol{y}) = V(\boldsymbol{x}, \boldsymbol{r})/\, \mathbf{r}^{\mathrm{T}}\nu\boldsymbol{y} = \nu^{-1}V(\boldsymbol{x}, \boldsymbol{r})/\, \boldsymbol{r}^T\boldsymbol{y} = \nu^{-1}OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

*OE*4. This follows because for $\hat{\boldsymbol{y}} \geq \boldsymbol{y}$ we have $\boldsymbol{r}^T\hat{\boldsymbol{y}} > \boldsymbol{r}^T\boldsymbol{y}$.

Thus $V(\boldsymbol{x}, \boldsymbol{r})/\, \boldsymbol{r}^T\hat{\boldsymbol{y}} < V(\boldsymbol{x}, \boldsymbol{r})/\, \boldsymbol{r}^T\boldsymbol{y}$

*OE*5. This follows from V4. Thus

$OE_O(\boldsymbol{x}, \theta\, \boldsymbol{r}, \boldsymbol{y}) = \{ V(\boldsymbol{x}, \theta\, \boldsymbol{r})/\, (\theta\boldsymbol{r}^T)\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x})\}$

$OE_O(\boldsymbol{x}, \theta\, \boldsymbol{r}, \boldsymbol{y}) = \{ \theta V(\boldsymbol{x}, \boldsymbol{r})/\, (\theta\boldsymbol{r}^T)\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x})\} = OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

*OE*6. Suppose not. $\Rightarrow$ $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1$ such that $V(\boldsymbol{x}, \boldsymbol{r})/\boldsymbol{r}^T\boldsymbol{y} > 1 \Rightarrow$

$\boldsymbol{y} \in P(\boldsymbol{x})\backslash VM(\boldsymbol{x}, \boldsymbol{r})$. But this violates the revenue maximizing assumption or otherwise

$OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) > 1$

*OE*7. Let $\boldsymbol{y}^* \in P(\boldsymbol{x})$ be the revenue maximizing output vector so that

$V(\boldsymbol{x}, \boldsymbol{r}) = \boldsymbol{r}^T\boldsymbol{y}^*$. Then $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = V(\boldsymbol{x}, \boldsymbol{r})/\boldsymbol{r}^T\boldsymbol{y} = \boldsymbol{r}^T\boldsymbol{y}^*/\ \boldsymbol{r}^T\boldsymbol{y} = \mu(\ \boldsymbol{r}^T\boldsymbol{y})/\ \boldsymbol{r}^T\boldsymbol{y}$

$\Rightarrow \mu(\ \boldsymbol{r}^T\boldsymbol{y})/\ \boldsymbol{r}^T\boldsymbol{y} = \mu(\ \boldsymbol{r}^T(\boldsymbol{\Omega}\boldsymbol{y}))/\ \boldsymbol{r}^T(\boldsymbol{\Omega}\boldsymbol{y}) = \mu((\ \boldsymbol{r}^T\boldsymbol{\Omega})\ \boldsymbol{y})/\ (\boldsymbol{r}^T\boldsymbol{\Omega})\ \boldsymbol{y}$. Thus $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ is not sensitive to the changes in the units in which output and revenue are measured.

From the above it follows that the revenue or overall efficiency, defined in terms of the revenue short fall, associated with any sample point $(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ is independent of how the technical efficiency is computed. Thus all the measures of output oriented technical efficiency discussed in the previous section result in identical scores for overall efficiency with respect to any data point in question. The differences in the definition of technical efficiency across different measures however cause the decomposition of overall efficiency into its components to differ. This follows because the allocative efficiency is defined in terms of the ratio of the overall efficiency to the technical efficiency. Thus as the computation of technical efficiency varies across measures so does the ratio of overall to technical efficiency. Below we discuss the decomposition of overall efficiency into its allocative and technical components across various measures starting with the definition of Farrell output oriented allocative efficiency.

Definition: The function FE_A: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$ for

q = M, N, defined by $FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = \{\ OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})/\ FE_O(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{y} \in P(\boldsymbol{x})\}$ is called the Farrell

output oriented allocative efficiency measure.

A value of $FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1$ indicates that the data point is allocatively efficient.

$FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) > 1$ implies allocative inefficiency with inefficiency increasing for higher values of $FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$. We have already shown above that the revenue efficiency implies technical efficiency. Thus a value of $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1 \Rightarrow FE_O(\boldsymbol{x}, \boldsymbol{y}) = 1 \Rightarrow FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1$.Since the reverse implication between $FE_O(\boldsymbol{x}, \boldsymbol{y})$ and $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ does not hold thus there is a possibility that a sample point may be technically efficient without being revenue efficient, thereby indicating $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) > 1 \Rightarrow FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) > 1$ and $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$. Figure(3.10) shows various possibilities for a given input vector and the output price vector. Data point $\boldsymbol{y}^* \in P(\boldsymbol{x})$ satisfies both technical and allocative efficiency conditions and thus $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1$. Data point $\boldsymbol{y} \in P(\boldsymbol{x})$ is both technically and allocatively inefficient. The component of technical efficiency is $FE_O(\boldsymbol{x}, \boldsymbol{y}) = 1/\psi(\boldsymbol{x}, \boldsymbol{y})$. The Farrell allocative efficiency component $FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ is obtained after adjusting for Farrell technical efficiency i.e.

$FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = V(\boldsymbol{x}, \boldsymbol{r})/(\boldsymbol{r}^T \boldsymbol{y}/\psi(\boldsymbol{x}, \boldsymbol{y})) = OE(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})/ FE_O(\boldsymbol{x}, \boldsymbol{y})$

$\Rightarrow OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \times FE_O(\boldsymbol{x}, \boldsymbol{y})$

which is the decomposition of overall efficiency using Farrell definition of technical efficiency.

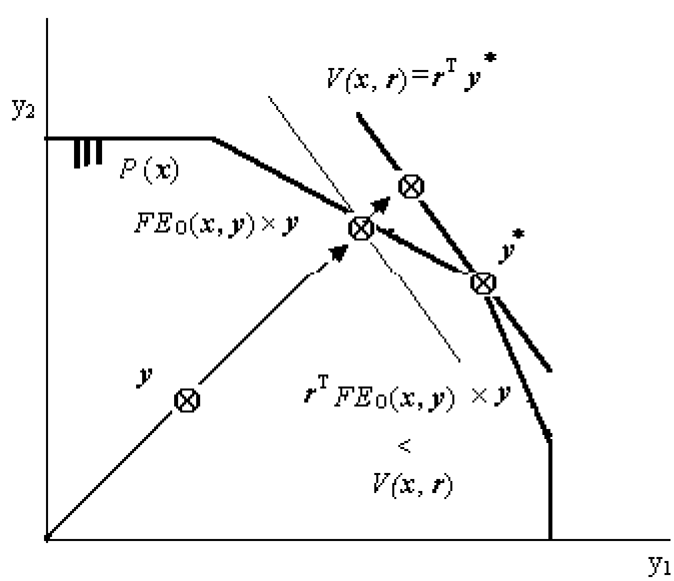

Figure 3.10. Farrell decomposition of revenue efficiency

Proposition 3.7: $FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies the following

*AE*1. $FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \in [1, +\infty)$

*AE*2. $FE_A(\lambda\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \geq FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

*AE*3. $FE_A(\boldsymbol{x}, \boldsymbol{r}, \nu\boldsymbol{y}) = FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

*AE*4. $FE_A(\boldsymbol{x}, \boldsymbol{r}, \hat{\boldsymbol{y}}) \gtreqless FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ for $\hat{\boldsymbol{y}} \geq \boldsymbol{y}$

*AE*5. $FE_A(\boldsymbol{x}, \theta\boldsymbol{r}, \boldsymbol{y}) = FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

*AE*6. $FE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies commensurability

Proofs:

*AE*1. Follows because $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \in [1,+\infty)$ and $FE_O(\boldsymbol{x}, \boldsymbol{y}) \in [1,+\infty)$ and $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \geq FE_O(\boldsymbol{x}, \boldsymbol{y})$.

*AE*2. For constant returns to scale $FE_O(\lambda\boldsymbol{x}, \boldsymbol{y}) = \lambda FE_O(\boldsymbol{x}, \boldsymbol{y})$. Then equality holds if $IsoqP(\lambda\boldsymbol{x})$ is a radial projection of $IsoqP(\boldsymbol{x})$. Otherwise the inequality follows.

*AE*3. Follows because $OE_O(\boldsymbol{x}, \boldsymbol{r}, \nu\boldsymbol{y}) = \nu^{-1} OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ and $FE_O(\boldsymbol{x}, \nu\boldsymbol{y}) = \nu^{-1} FE_O(\boldsymbol{x}, \boldsymbol{y})$.

*AE*4. Follows because *OE*4 holds and $FE_O(\boldsymbol{x}, \boldsymbol{y})$ is weakly monotonic.

*AE*5. Follows because $OE_O(\boldsymbol{x}, \theta\boldsymbol{r}, \boldsymbol{y}) = OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$.

*AE*6. Follows because $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ and $FE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfy commensurability.

The non-radial Russell measure of allocative efficiency is defined as follows;

Definition: The function $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = \{OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) / RE^{\boldsymbol{r}}_O(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{y} \in P(\boldsymbol{x})\}$, where $RE^{\mathbf{r}}_O(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{r}^T \delta_{FL} \boldsymbol{y} / \boldsymbol{r}^T \boldsymbol{y}$,is called the Russell output oriented allocative efficiency measure.

A value of $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1$ implies that the sample point in question satisfies Russell allocative efficiency conditions. A value of $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) > 1$ indicates allocative inefficiency which is higher for higher values $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$. Note that for the reasons already discussed in chapter 2, we use the price based technical efficiency measure to compute the output oriented allocative efficiency scores. Figure (3.11) shows various possibilities for a given input vector and the output price vector and reflects the decomposition of overall efficiency into Russell allocative and technical components. Data point $\boldsymbol{y}^* \in P(\boldsymbol{x})$ satisfies both technical and allocative efficiency conditions and thus $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}^*) = 1$. The data point $\boldsymbol{y} \in P(\boldsymbol{x})$ is both technically and allocatively inefficient. In order to obtain the Russell allocative component of overall efficiency for this data point we first identify the Russell technically efficient output vector $\delta_{FL}\boldsymbol{y} \in P(\boldsymbol{x})$ using the physical quantity based technical efficiency measure The Russell allocative efficiency component $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ is then given as

$RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = (V(\boldsymbol{x}, \boldsymbol{r})/ \boldsymbol{r}^T\boldsymbol{y})/(\boldsymbol{r}^T\delta_{FL}\boldsymbol{y} / \boldsymbol{r}^T\boldsymbol{y})$

$= OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})/ RE^{\mathbf{r}}{}_O(\boldsymbol{x}, \boldsymbol{y})$

which implies following decomposition of overall efficiency

$OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \times RE^{\mathbf{r}}{}_O(\boldsymbol{x}, \boldsymbol{y})$

Comparing the Russell allocative efficiency with that of Farrell allocative efficiency component, whenever we have $\delta_{FL} = \delta\mathbf{I}$ where δ is some constant and $\mathbf{I}$ is an M×M identity matrix, the Russell and Farrell allocative efficiency components coincides and so does the decomposition of overall efficiency across two measures. However when this is not the case the relationship between Farrell and Russell allocative efficiency components is ambiguous. This point is illustrated with the help of figure (3.12) for the data point $\boldsymbol{y} \in P(\boldsymbol{x})$. In the figure three possibilities have been shown with respect to the Russell projection vector . If the Russell projection vector coincides with the Farrell projection vector, the two are identical in terms of the allocative efficiency. If the Russell projection vector is $\delta_{FL}\boldsymbol{y}$, the allocative efficiency component

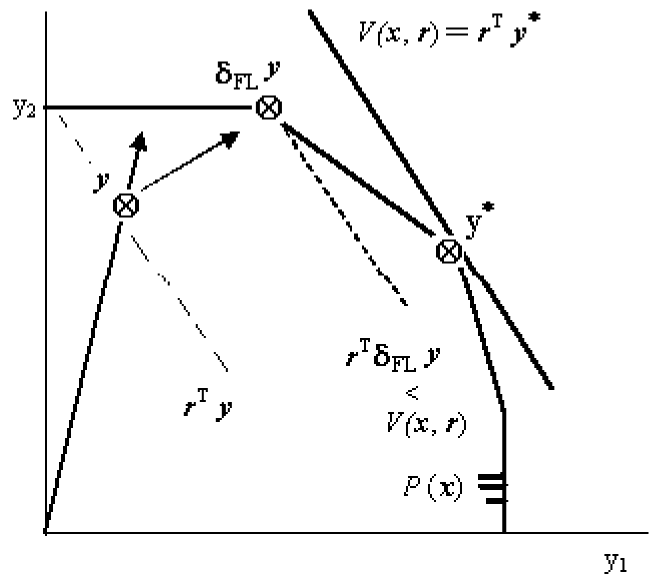

Figure 3.11. Russell decomposition of revenue efficiency

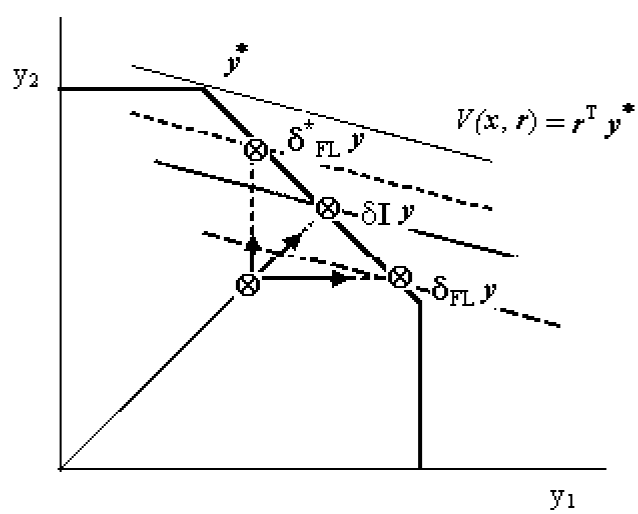

Figure 3.12. Comparing Farrell and Russell decomposition of $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

associated with the Russell measure is greater than that associated with the Farrell measure. On the contrary the Russell allocative efficiency component is lesser than that of the Farrell allocative efficiency if the Russell projection vector is identified by $\delta^{*}_{FL}\boldsymbol{y}$.

Proposition 3.8: The non radial price based technical efficiency measure satisfies the following;

$TE^{V}1$. $TE^{v}(\boldsymbol{x}, \boldsymbol{y}) \in (1, +\infty)$

$TE^{V}2$. $TE^{v}(\lambda\boldsymbol{x}, \boldsymbol{y}) = \lambda\, TE^{v}(\boldsymbol{x}, \boldsymbol{y})$

$TE^{V}3$. $TE^{v}(\boldsymbol{x}, \nu\boldsymbol{y}) = \nu^{-1} TE^{v}(\boldsymbol{x}, \boldsymbol{y}),\ \nu \in (0,1]$

$TE^{V}4$. $TE^{v}(\boldsymbol{x}, \boldsymbol{y}^{*}) < TE^{c}_{I}(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{y}^{*} \geq \boldsymbol{y}$

$TE^{V}5$. $TE^{v}(\boldsymbol{y}, \boldsymbol{x})$ satisfies commensurability

$TE^{V}3$w. $TE^{v}(\boldsymbol{x}, \nu\boldsymbol{y}) \lesseqgtr \nu^{-1} TE^{v}(\boldsymbol{x}, \boldsymbol{y}),\ \nu \in (0,1]$

$TE^{V}4$w. $TE^{v}(\boldsymbol{x}, \boldsymbol{y}^{*}) \lesseqgtr TE^{c}_{I}(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{y}^{*} \geq \boldsymbol{y}$

Proposition 3.8a: The Russell non radial price based technical efficiency measure satisfies $TE^{V}1$, $TE^{V}2$, $TE^{V}5$, $TE^{V}3$w and $TE^{V}4$w.

<u>Proofs:</u>

$TE^{V}1$ Follows because $RE_{O}(\boldsymbol{x}, \boldsymbol{y}) \in [1. +\infty)$.

$TE^{V}2$ This follows because for constant returns to scale we have

$RE_{O}(\lambda\boldsymbol{x}, \boldsymbol{y}) = \lambda\, RE_{O}(\boldsymbol{x}, \boldsymbol{y})$.

$TE^{V}3w$. . Let $\boldsymbol{y} \in P(\boldsymbol{x})$ such that $RE_{O}(\boldsymbol{x}, \boldsymbol{y}) = 1$. For $\nu < 1 \Rightarrow RE_{O}(\boldsymbol{x}, \nu\boldsymbol{y}) > 1 \Rightarrow \exists\ \delta_{FL}$ such that

$RE_{O}(\boldsymbol{x}, \delta_{FL}(\nu\boldsymbol{y})) = 1$ and $\boldsymbol{r}^{T}\delta_{FL}(\nu\boldsymbol{y}) \gtreqless \boldsymbol{r}^{T}\theta\mathbf{I}(\nu\boldsymbol{y})$, where δ is some constant and $\mathbf{I}$ is an M×M identity matrix. $RE^{V}3$ holds because $\delta\mathbf{I}(\nu\boldsymbol{y}) = \boldsymbol{y}$.

$TE^V 4w$. Suppose that $\boldsymbol{y} \in P(\boldsymbol{x}) \backslash EffP(\boldsymbol{x})$ and that $RE_O(\boldsymbol{x}, \delta_{FL}\boldsymbol{y}) = 1$. Let $\boldsymbol{y}^* \geq \boldsymbol{y}$ such that

$RE_O(\boldsymbol{y}, \delta^*_{FL}\boldsymbol{x}^*) = 1$. Two possibilities exists.

i) $\delta^*_{FL}\boldsymbol{y}^*$ coincides with $\delta_{FL}\boldsymbol{y}$.

ii) $\delta^*_{FL}\boldsymbol{y}^*$ does not coincide with $\delta_{FL}\boldsymbol{y}$.

i) $\Rightarrow \boldsymbol{r}^T\delta^*_{FL}\boldsymbol{y}^* / \boldsymbol{r}^T\boldsymbol{y}^* < \boldsymbol{r}^T\delta_{FL}\boldsymbol{y}/\boldsymbol{r}^T\boldsymbol{y} \Rightarrow RE^{\mathbf{c}}_I(\boldsymbol{x}, \boldsymbol{y}^*) < RE^{\mathbf{c}}_I(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow$monotonicity.

ii) $\Rightarrow \boldsymbol{r}^T\delta^*_{FL}\boldsymbol{y}^* / \boldsymbol{r}^T\boldsymbol{y}^* \lesseqgtr \boldsymbol{r}^T\delta_{FL}\boldsymbol{y}/\boldsymbol{r}^T\boldsymbol{y} \Rightarrow RE^{\mathbf{c}}_I(\boldsymbol{x}, \boldsymbol{y}^*) \lesseqgtr RE^{\mathbf{c}}_I(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow TE^C 4$w.

$TE^V 5$. Since $RE_O(\boldsymbol{x}, \boldsymbol{y})$ satisfies commensurability so does $RE^{\mathbf{c}}_O(\boldsymbol{x}, \boldsymbol{y})$

Proposition 3.8b: $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies the following

$AE1$. $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \in [1, +\infty)$

$AE2$. $RE_A(\lambda\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \geq RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

$AE3$. $RE_A(\boldsymbol{x}, \boldsymbol{r}, \nu\boldsymbol{y}) \lesseqgtr RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ $\nu > 0$

$AE4$. $RE_A(\boldsymbol{x}, \boldsymbol{r}, \hat{\boldsymbol{y}}) \gtreqless RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ for $\hat{\boldsymbol{y}} \geq \boldsymbol{y}$

$AE5$. $RE_A(\boldsymbol{x}, \theta\boldsymbol{r}, \boldsymbol{y}) = RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

$AE6$. $RE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies commensurability

<u>Proofs</u>:

$AE3$ and $AE4$ follow from $TE^V 3$w and $TE^V 4$w respectively. For rest of the properties proofs are similar to proposition 3.7.

Definition: The function $ZE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = \{ OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) / RE^{\mathbf{r}}_O(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{y} \in P(\boldsymbol{x})\}$ where $ZE^{\mathbf{r}}_O(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{r}^T\delta_{ZE}\boldsymbol{y} / \boldsymbol{r}^T\boldsymbol{y}$ and δ_{ZE} is an M×M diagonal matrix with the mth diagonal element equal to or greater than one, is called the Zieschang output oriented allocative efficiency measure.

A value of $ZE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = 1$ implies that the sample point in question satisfies Russell allocative efficiency conditions. A value of $ZE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) > 1$ indicates allocative inefficiency which is higher for higher values of $ZE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$.

With the allocative efficiency measure so defined, the Zieschang decomposition of the overall efficiency is given as follows;

$ZE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = V(\boldsymbol{x}, \boldsymbol{r})/(\boldsymbol{r}^T \boldsymbol{y} \times ZE^{\mathbf{r}}{}_O(\boldsymbol{x}, \boldsymbol{y})) = OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})/ ZE^{\mathbf{r}}{}_O(\boldsymbol{x}, \boldsymbol{y})$

$\Rightarrow OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) = ZE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \times ZE^{\mathbf{r}}{}_O(\boldsymbol{x}, \boldsymbol{y})$

Comparing Zieschang allocative efficiency with Farrell, all the diagonal elements of δ_{ZE} are identical when Zieschang and Farrell measure of technical efficiency coincide and thus we get identical decompositions of overall efficiency for the two measures. This is the case when the Farrell projection of an output vector $\boldsymbol{y} \in P(\boldsymbol{x})$ falls on $EffP(\boldsymbol{x})$. On the other hand if the projection falls on $IsoqP(\boldsymbol{x})\backslash EffP(\boldsymbol{x})$, some of the diagonal elements of δ_{ZE} also absorb Russell type adjustment in addition to the Farrell adjustment. Thus all the elements of δ_{ZE} are not identical in which case the Zieschang allocative efficiency component is less than that of Farrell allocative efficiency component.

Proposition 3.9a: The Zieschang revenue based technical efficiency measure satisfies properties TE^V1 to TE^V5.

Proofs:

TE^V3 follows from strict homogeneity of the Zieschang output based technical efficiency measure. For rest of the properties proofs are similar to proposition 3.8a

Proposition 3.9b: $ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies the following

$ZE1$. $ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \in [1, +\infty)$

$ZE2$. $ZE_{\text{A}}(\lambda\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \geq ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

$ZE3$. $ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \nu\boldsymbol{y}) = ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

$ZE4$. $ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \hat{\boldsymbol{y}}) \gtreqless ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ for $\hat{\boldsymbol{y}} \geq \boldsymbol{y}$

$ZE5$. $ZE_{\text{A}}(\boldsymbol{x}, \theta\boldsymbol{r}, \boldsymbol{y}) = ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

$ZE6$. $ZE_{\text{A}}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies commensurability

Proofs:

Similar to proposition 3.7.

The Non-radial Farrell measure of allocative efficiency is defined in the same manner. However like before the Non-radial Farrell measure of allocative efficiency is also computed in the transformed data space. Below we first obtain the necessary transformations of the data set and then define the Non-radial Farrell overall efficiency measure. Once the overall efficiency has been defined for the transformed data set it can be used along with the technical efficiency component to obtain the allocative efficiency measure. The transformed data set with respect to any data point is $({}_s\boldsymbol{x}, {}_s\boldsymbol{y}, {}_s\boldsymbol{r})$ where ${}_s\boldsymbol{x}$ and ${}_s\boldsymbol{y}$ are as defined earlier and ${}_s\boldsymbol{r} \in \Re^{M}_{+}$ is an M×1 vector the mth element of which is the revenue from the mth output earned by the entire sample i.e. ${}_s r_m = r_m \sum^{J}_{j=1} y_{mj}$, m= 1,2,..M. With this definition ${}_s r_m \times {}_s y_{mk}$ gives the revenue earned from the mth output by the kth firm. Now we can define the following lower contour set for the transformed data

$V({}_s\boldsymbol{r}) = \{{}_s\boldsymbol{y} \mid {}_s\boldsymbol{r}^{\text{T}}{}_s\boldsymbol{y} \leq V({}_s\boldsymbol{x}, {}_s\boldsymbol{r}), {}_s\boldsymbol{r}^{\text{T}}{}_s\hat{\boldsymbol{y}} > V({}_s\boldsymbol{x}, {}_s\boldsymbol{r}) \Rightarrow {}_s\hat{\boldsymbol{y}} \notin V({}_s\boldsymbol{r})\}$

In words $V(_s\boldsymbol{r})$ is the set of all those output share vectors that generate revenue equal to or less than $V(_s\boldsymbol{x}, {_s\boldsymbol{r}})$ and no element of the set generates revenue greater than $V(_s\boldsymbol{x}, {_s\boldsymbol{r}})$. From this it follows that $P^S(_s\boldsymbol{x}) \subseteq V(_s\boldsymbol{r})$. The following subset of $V(_s\boldsymbol{r})$ represents an isorevenue level

$\text{Iso}V(_s\boldsymbol{r}) = \{_s\boldsymbol{y} \mid {_s\boldsymbol{r}}^T {_s\boldsymbol{y}} = V(_s\boldsymbol{x}, {_s\boldsymbol{r}})\}$

From this set we can define a subset comprising of the feasible output share vectors that generate revenue equal to $V(_s\boldsymbol{x}, {_s\boldsymbol{r}})$ i.e.

$VM(_s\boldsymbol{x}, {_s\boldsymbol{r}}) = \{_s\boldsymbol{y} \mid {_s\boldsymbol{r}}^T {_s\boldsymbol{y}} = V(_s\boldsymbol{x}, {_s\boldsymbol{r}}), {_s\boldsymbol{y}} \in P^S(_s\boldsymbol{x})\}$

Also from the definition of $V(_s\boldsymbol{x}, {_s\boldsymbol{r}})$ we have

$P^S(_s\boldsymbol{x}) = \{_s\boldsymbol{y} \mid {_s\boldsymbol{y}} \in \Re^M_+, {_s\boldsymbol{x}} \in \Re^N_+, {_s\boldsymbol{r}}^T {_s\boldsymbol{y}} \leq V(_s\boldsymbol{x}, {_s\boldsymbol{r}})\}$

$\Rightarrow VM(_s\boldsymbol{x}, {_s\boldsymbol{r}}) = P^S(_s\boldsymbol{x}) \cap \text{Iso}V(_s\boldsymbol{r})$

An output share vector ${_s\boldsymbol{y}} \in VM(_s\boldsymbol{x}, {_s\boldsymbol{r}}) \Rightarrow {_s\boldsymbol{r}}^T {_s\boldsymbol{y}} = V(_s\boldsymbol{x}, {_s\boldsymbol{r}}) \Rightarrow V(_s\boldsymbol{x}, {_s\boldsymbol{r}}) / {_s\boldsymbol{r}}^T {_s\boldsymbol{y}} = 1 \Rightarrow$. for the output vector ${_s\boldsymbol{y}} \in VM(_s\boldsymbol{x}, {_s\boldsymbol{r}})$, $\nexists\ {_s\hat{\boldsymbol{y}}} \in P^S(_s\boldsymbol{x})$ such that ${_s\boldsymbol{r}}^T {_s\hat{\boldsymbol{y}}} > {_s\boldsymbol{r}}^T {_s\boldsymbol{y}}$.

Definition: The function $OE_O(_s\boldsymbol{x}, {_s\boldsymbol{r}}, {_s\boldsymbol{y}}) = \{ V(_s\boldsymbol{x}, {_s\boldsymbol{r}}) / {_s\boldsymbol{r}}^T {_s\boldsymbol{y}} \mid {_s\boldsymbol{y}} \in P^S(_s\boldsymbol{x})\}$ is called the Non-radial Farrell output oriented overall efficiency measure.

But since $V(_s\boldsymbol{x}, {_s\boldsymbol{r}}) / {_s\boldsymbol{r}}^T {_s\boldsymbol{y}} = V(\boldsymbol{x}, \boldsymbol{r}) / \boldsymbol{r}^T \boldsymbol{y} \Rightarrow OE_O(_s\boldsymbol{x}, {_s\boldsymbol{r}}, {_s\boldsymbol{y}}) = OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$.

Proposition 3.10: $OE_O(_s\boldsymbol{x}, {_s\boldsymbol{r}}, {_s\boldsymbol{y}})$ satisfies $OE1$ to $OE6$.

Proofs:

Since $OE_O(_s\boldsymbol{x}, {_s\boldsymbol{r}}, {_s\boldsymbol{y}}) = OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ and $OE_O(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies $OE1$ to $OE6$ so does $OE_O(_s\boldsymbol{x}, {_s\boldsymbol{r}}, {_s\boldsymbol{y}})$.

The Non-radial Farrell allocative efficiency is defined for the transformed data set as follows

Definition: The function $NE_A(_s\boldsymbol{x}, {_s\boldsymbol{r}}, {_s\boldsymbol{y}}) = \{ OE_O(_s\boldsymbol{x}, {_s\boldsymbol{r}}, {_s\boldsymbol{y}}) / NE^{\mathbf{r}}_O(_s\boldsymbol{x}, {_s\boldsymbol{y}}) \mid {_s\boldsymbol{y}} \in P^S(_s\boldsymbol{x})\}$ where

$NE^{\mathbf{r}}_O(_s\boldsymbol{x}, {_s\boldsymbol{y}}) = {_s\boldsymbol{r}}^T \delta_{NE}\, {_s\boldsymbol{y}} / {_s\boldsymbol{r}}^T {_s\boldsymbol{y}}$ and δ_{NE} is an M×M diagonal matrix with the mth diagonal element equal to or greater than one is called the Non-radial Farrell output oriented allocative efficiency measure.

Note that for the computation of $NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$ the first step is to identify the technically efficient output vector. By definition this is the output vector that maximizes $\|\delta_{NE\,s}\boldsymbol{y}\|/\|{}_s\boldsymbol{y}\|$, given ${}_s\boldsymbol{x}$ and ${}_s\boldsymbol{r}$. Once this vector has been obtained we compute the revenue value, ${}_s\boldsymbol{r}^T\delta_{NE\,s}\boldsymbol{y}$, of this vector. Then the ratio of the revenue value of this vector to the revenue value of the actual vector, i.e. ${}_s\boldsymbol{r}^T\delta_{NE\,s}\boldsymbol{y}/\,{}_s\boldsymbol{r}^T{}_s\boldsymbol{y}$, defines the revenue measure of output oriented Non-radial Farrell technical efficiency. This value is then used to compute $NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$. $NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y}) = 1$ implies that the data point in question is allocatively efficient. Inefficiency associated with any data point is reflected by a value of $NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$ greater than one.

From the above definition of Non-radial Farrell allocative efficiency measure we can arrive at the following decomposition of the overall efficiency

$NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y}) = OE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})/({}_s\boldsymbol{r}^T\delta_{NE}\,{}_s\boldsymbol{y}\,/\,{}_s\boldsymbol{r}^T{}_s\boldsymbol{y})$

$\Rightarrow NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y}) = OE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})/\,NE^{\mathbf{r}}_O({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$

$\Rightarrow OE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y}) = NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y}) \times NE^{\mathbf{r}}_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$

The above decomposition shows that whenever we can write $\delta_{NE} = \delta\mathbf{I}$,

$NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y}) = OE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})/{}_s\boldsymbol{r}^T\delta\mathbf{I}\,{}_s\boldsymbol{y}\,/\,{}_s\boldsymbol{r}^T{}_s\boldsymbol{y} = OE_O({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})/\delta{}_s\boldsymbol{r}^T{}_s\boldsymbol{y}\,/\,{}_s\boldsymbol{r}^T{}_s\boldsymbol{y}$

$\Rightarrow NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y}) = FE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$.

However when this is not the case the relationship between $NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$ and $FE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$ is not uniquely defined because of the same reasons as discussed earlier with reference to the comparison between Russell and Farrell allocative efficiency measures. Similar relationships of $NE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$ hold with $RE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$ and $ZE_A({}_s\boldsymbol{x}, {}_s\boldsymbol{r}, {}_s\boldsymbol{y})$. Thus for example if δ_{NE} is identical to δ_{FL} then we get identical allocative efficiency components if not the relationship is ambiguous.

Proposition 3.11a: The Non-radial Farrell Allocative Efficiency measure satisfies proposition 3.8a.

Proofs:

Similar to propositions 3.8a .

Proposition 3.11b: The Non-radial Farrell Allocative Efficiency measure satisfies the following;

$NE1$. $NE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \in [1, +\infty)$

$NE2$. $NE_A(\lambda\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y}) \geq NE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

$NE3$. $NE_A(\boldsymbol{x}, \boldsymbol{r}, \nu\boldsymbol{y}) \lesseqgtr NE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ $\nu > 0$

$NE4$. $NE_A(\boldsymbol{x}, \boldsymbol{r}, \hat{\boldsymbol{y}}) \gtreqless NE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ for $\hat{\boldsymbol{y}} \geq \boldsymbol{y}$

$NE5$. $NE_A(\boldsymbol{x}, \theta\boldsymbol{r}, \boldsymbol{y}) = NE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$

$NE6$. $NE_A(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{y})$ satisfies commensurability

Proofs:

Similar to propositions 3.8b

Computation:

Linear programming techniques can be employed to solve for the revenue maximization problem $V(\boldsymbol{x}, \boldsymbol{r}) = \max\ \{\boldsymbol{r}^T\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x})\}$ as follows

$$\begin{aligned} \text{Max}_{\eta} \quad & \boldsymbol{r}^T\boldsymbol{y} \\ \text{s.t} \quad & \boldsymbol{x}^k - \boldsymbol{X}\eta \geq \boldsymbol{0} \\ & -\boldsymbol{y}^k + \boldsymbol{Y}\eta \geq \boldsymbol{0} \\ & \eta \geq \boldsymbol{0} \end{aligned}$$

This program is solved once for each firm. The solution thus obtained is used with the actual revenue of the respective firm, computed on the basis of the observed output vector, to obtain the overall efficiency score. As a second step the revenue based technical efficiency component is obtained by using the relevant projection vector. Thus for example $\delta \boldsymbol{y}$ is used to compute $\boldsymbol{r}^T\delta\boldsymbol{y}/\boldsymbol{r}^T\boldsymbol{y}$ in case one is interested in Farrell decomposition of overall efficiency while $\delta_{FL}\boldsymbol{y}$ is used to compute $\boldsymbol{r}^T\delta_{FL}\boldsymbol{y}/\boldsymbol{r}^T\boldsymbol{y}$ if one is interested in the Russell decomposition and so on. Finally the ratio of the overall efficiency to the relevant revenue based technical efficiency gives the corresponding allocative efficiency component.

3.5. Rela*x*ing Assumptions

While discussing the output measures of technical efficiency in the preceding sections we continued maintaining the assumption of constant returns to scale and strong disposability. This section discusses the implication of relaxing these assumptions with respect to the properties E1b to E6b and with respect to the computation procedure.

3.5.1 Replacing Constant Returns to Scale b*y* Non-Increasing Returns and Variable Returns to Scale:

Non-increasing returns to scale technology implies that an equiproportionate increase in all inputs increases the output vector in the same proportion or in a lesser proportion. The implied production correspondence is thus given as

$$P_{\check{N}}(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y} \leq \boldsymbol{Y}\eta, \boldsymbol{x} \geq \boldsymbol{X}\eta, \eta \geq \boldsymbol{0}, \mathbf{e}^T\eta \leq 1\}$$

where $\mathbf{e}$ is an M×1 vector of ones and η is the as defined earlier intensity vector. The technology thus restricts the sum of the intensity vector to not to exceed one. For the technology so defined we have the following definition for Farrell output oriented technical efficiency measure.

Definition: A function $FE_{O\check{N}}$ $(\boldsymbol{x},\boldsymbol{y})$: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \to \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined by

$FE_{O\check{N}}$ $(\boldsymbol{x}, \boldsymbol{y})$= max $\{\delta_{DF} | \boldsymbol{y} \in P_{\check{N}}(\boldsymbol{x}), \delta_{DF}\boldsymbol{y} \in P_{\check{N}}(\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+\}$ and $FE_{O\check{N}}$ $(\boldsymbol{x}, \boldsymbol{y}) = -\infty$

for $\boldsymbol{y} \notin P_{\check{N}}(\boldsymbol{x})$, is the Farrell's output measure of technical efficiency for non-increasing returns to scale technology.

A value of $FE_{O\check{N}}$ $(\boldsymbol{x}, \boldsymbol{y}) = 1$ indicates that $\boldsymbol{y} \in P_{\check{N}}(\boldsymbol{x})$ is technically efficient. Technical inefficiency is indicated by value of $FE_{O\check{N}}$ $(\boldsymbol{x}, \boldsymbol{y}) > 1$ and higher the value of $FE_{O\check{N}}$ $(\boldsymbol{x}, \boldsymbol{y})$, more is the technical inefficiency.

Proposition 3.12 For non-increasing returns to scale technology the Farrell output oriented measure of technical efficiency satisfies E.3bw, E4b and E6b.It does not satisfy E1b, E2b, E3b and E5b.

Proofs:

Here we provide proof for E3b. For the rest of the properties proofs are similar to proposition 3.1.

$FE_{O\check{N}}(\boldsymbol{x}, \boldsymbol{y})$ fails to satisfy E.3b, the homogeneity of plus one in inputs:

Figure (3.13) shows an increase in input from $\boldsymbol{x}$ to $\lambda\boldsymbol{x}$, $\lambda = 2$, by an outward shift of the isoquant *IsoqP*($\boldsymbol{x}$) to *IsoqP*($\lambda\boldsymbol{x}$). Define the isoquant density in an output space as $g(\alpha)$ where α is the scale parameter and $dg(\alpha)/d\alpha < 0$. Then if we represent $\beta = 1/ g(\alpha)$ as the shift factor associated with an isoquant *IsoqP*($\boldsymbol{x}$) when $\boldsymbol{x}$ changes by λ, we can write $IsoqP(\lambda\boldsymbol{x}) = \beta\lambda IsoqP(\boldsymbol{x})$. Thus

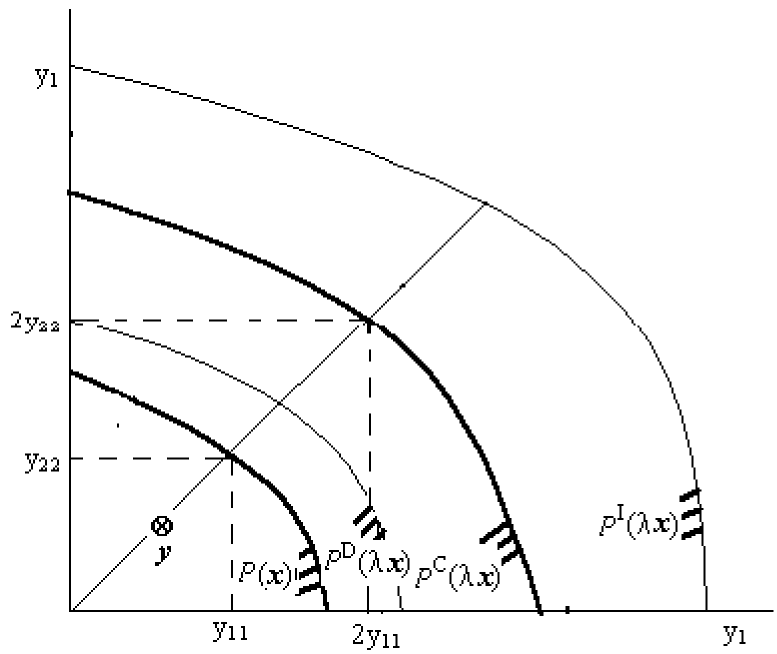

Figure 3.13. Returns to scale and input homogeneity

higher the value of α, lesser is the isoquant density in the output space and greater is the shift in an isoquant $IsoqP(\boldsymbol{x})$. Then for $\boldsymbol{y} \in P(\lambda\boldsymbol{x})$ we have $\delta_{DF}\boldsymbol{y} \in IsoqP(\boldsymbol{x})$ and $\beta\lambda\delta_{DF}\boldsymbol{y} \in IsoqP(\lambda\boldsymbol{x})$ with $\delta_{DF}\boldsymbol{y}$, $\beta\lambda\delta_{DF}\boldsymbol{y}$ and $\boldsymbol{y}$ lying on the same ray radiating from the origin. The efficiency score of $\boldsymbol{y}$ with respect to $IsoqP(\boldsymbol{x})$ is $FE_{O\check{N}}\,(\boldsymbol{y}, \boldsymbol{x}) = \|\delta_{DF}\boldsymbol{y}\| / \|\boldsymbol{y}\|$ and the efficiency score of $\boldsymbol{y}$ with respect to $IsoqP(\lambda\boldsymbol{x})$ is $\|\beta\lambda\delta_{DF}\boldsymbol{y}\|/\|\boldsymbol{y}\| = |\beta\lambda|\|\delta_{DF}\boldsymbol{y}\|/\|\boldsymbol{y}\| = \beta\lambda\; FE_{O\check{N}}\,(\boldsymbol{x}, \boldsymbol{y})$. For non-increasing returns to scale we have $\alpha \leq 1 \;\Rightarrow \beta\lambda\; FE_{O\check{N}}\,(\boldsymbol{x}, \boldsymbol{y}) \leq \lambda\; FE_{O\check{N}}\,(\boldsymbol{x}, \boldsymbol{y})$ thus violating the homogeneity of plus one in inputs.

The variable returns to scale technology adds more strict restrictions on the intensity vector by forcing its sum to equal to one. Thus we get the following production correspondence

$P_V(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y} \leq \boldsymbol{Y}\eta, \boldsymbol{x} \geq \boldsymbol{X}\eta, \eta \geq \boldsymbol{0}, \mathbf{e}^T\eta = 1\}$.

The variable return to scale version of the Farrell measure is given as follows

Definition: A function $FE_{OV}(\boldsymbol{x},\boldsymbol{y})$: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined by

$FE_{OV}(\boldsymbol{x}, \boldsymbol{y}) = \max \{\delta_{DF} \mid \boldsymbol{y} \in P_V(\boldsymbol{x}), \delta_{DF}\boldsymbol{y} \in P_V(\boldsymbol{x}), \delta_{DF} \in [1,+\infty), \boldsymbol{y} \in \Re^M_+\}$ and

$FE_{OV}(\boldsymbol{x}, \boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P_V(\boldsymbol{x})$, is the Farrell's output measure of technical efficiency for variable returns to scale

$FE_{OV}(\boldsymbol{x},\boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Farrell sense and the technical inefficiency is reflected by values of $FE_{OV}(\boldsymbol{x},\boldsymbol{y}) > 1$, with greater inefficiency for successively higher values of $FE_{OV}(\boldsymbol{x},\boldsymbol{y})$.

Proposition 3.13 $FE_{OV}(\boldsymbol{x}, \boldsymbol{y})$ satisfies E3bw, E4b and E6b. It does not satisfy E1b to E3b and it does not satisfy E5b.

Proofs:

Similar to proposition 3.12.

Definition: The function $RE_{O\check{N}}$: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined as

$RE_{O\check{N}}(\boldsymbol{x},\boldsymbol{y}) = \max \{1/M(\mathbf{e}^T\boldsymbol{\delta}_{FL}\mathbf{e} \mid (\boldsymbol{y} \in P_{\check{N}}(\boldsymbol{x}), \boldsymbol{\delta}_{FL}\boldsymbol{y} \in P(\boldsymbol{x}), \boldsymbol{\delta}_{FL} \in D\}$ and $RE_{O\check{N}}(\boldsymbol{x}, \boldsymbol{y}) = -\infty$ for

$\boldsymbol{y} \notin P_N(\boldsymbol{x})$, where D is as defined in section 3.2 and $\mathbf{e}$ is an M×1 vector of ones, is called the Russell output oriented measure of technical efficiency for non-increasing returns to scale technology.

$RE_{O\check{N}}$ $(\boldsymbol{x}, \boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Russell sense and the technical inefficiency is reflected by values of $RE_{O\check{N}}(\boldsymbol{x},\boldsymbol{y})>1$, with greater inefficiency for successively higher values of $RE_{O\check{N}}(\boldsymbol{x},\boldsymbol{y})$.

Proposition 3.14 For non-increasing returns to scale technologies the Russell output oriented measure of technical efficiency satisfies E1b, E2b, E3bw, E4bw, E5b and E6b. It does not satisfy E3b, E4b.

Proofs:

For E3bw proof is similar as in proposition 3.12. For the rest of the properties proofs are identical to those of proposition 3.2

Definition: The function RE_{OV}: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined as

$RE_{OV}(\boldsymbol{x},\boldsymbol{y})$= max $\{1/M(\mathbf{e}^T \boldsymbol{\delta}_{FL} \mathbf{e} \mid (\boldsymbol{y} \in P_V(\boldsymbol{x}), \boldsymbol{\delta}_{FL}\boldsymbol{y} \in P_V(\boldsymbol{x}), \boldsymbol{\delta}_{FL} \in D\}$ and $RE_{OV}(\boldsymbol{x},\boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P_V(\boldsymbol{x})$ is called the Russell output oriented measure of technical efficiency for variable returns to scale technology..

$RE_{OV}(\boldsymbol{x},\boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Russell sense and the technical inefficiency is reflected by values of $RE_{OV}(\boldsymbol{x}, \boldsymbol{y})>1$, with greater inefficiency for successively higher values of $RE_{OV}(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 3.15 For variable returns to scale technologies the Russell output oriented measure of technical efficiency satisfies E1b, E2b, E3bw, E4bw, E5b and E6b. It does not satisfy E4b.

Proofs:

Similar to propositions 3.13 and 3.2

Definition: The function $ZE_{O\check{N}}$: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as

$ZE_{O\check{N}}(\boldsymbol{x},\boldsymbol{y}) = \{[RE_{O\check{N}}(\boldsymbol{x}, \boldsymbol{y}/\ \psi_O(\boldsymbol{x}, \boldsymbol{y})]/\ \psi_O^{+}(\boldsymbol{x}, \boldsymbol{y})|$ for $\boldsymbol{y} \in P_{\check{N}}(\boldsymbol{x})$, $\boldsymbol{y}//\psi_O(\boldsymbol{x}, \boldsymbol{y}) \in P_{\check{N}}(\boldsymbol{x})\}$ and

$ZE_{O\check{N}}(\boldsymbol{x},\boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P_{\check{N}}(\boldsymbol{x})$ is called the Russell Extended Farrell measure of output technical efficiency for non-increasing returns to scale technology.

$ZE_{O\check{N}}(\boldsymbol{x}, \boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Russell sense and the technical inefficiency is reflected by values of $ZE_{O\check{N}}(\boldsymbol{x}, \boldsymbol{y}) > 1$, with greater inefficiency reflected by successively higher values of $ZE_{O\check{N}}(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 3.16 For non-increasing returns to scale technologies the Zieschang output oriented measure of technical efficiency satisfies E1b, E2b, E3bw, E4b, E5bw and E6b. It does not satisfy E5b.

Proofs:

Similar to proposition 3.3 for E1b, E2b and E4b to E6b. The proof for E3bw is similar to proposition 3.12.

Definition: The function ZE_{OV}: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as

$ZE_{OV}(\boldsymbol{x},\boldsymbol{y}) = \{[RE_{OV}(\boldsymbol{x}, \boldsymbol{y}/\ \psi_O(\boldsymbol{x}, \boldsymbol{y})]/\ \psi_O(\boldsymbol{x}, \boldsymbol{y})|$ for $\boldsymbol{y} \in P_V(\boldsymbol{x})$, $\boldsymbol{y}//\psi_O(\boldsymbol{x}, \boldsymbol{y}) \in P_V(\boldsymbol{x})\}$ and

$ZE_{OV}(\boldsymbol{x}, \boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P_V(\boldsymbol{x})$ is called the Russell Extended Farrell measure of output technical efficiency for variable returns to scale technology.

$ZE_{OV}(\boldsymbol{x}, \boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Russell sense and the technical inefficiency is reflected by values of $ZE_{OV}(\boldsymbol{x}, \boldsymbol{y}) > 1$, with greater inefficiency for successively higher values of $ZE_{OV}(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 3.17 For variable returns to scale technologies the Zieschang output oriented measure of technical efficiency satisfies E1b, E2b, E4b, E5bw and E6b. It does not satisfy E5b.

Proofs:

Similar as proposition 3.16.

Definition: The function $NE_{O\check{N}}$: $\mathbb{R}^{N}_{+} \times \mathbb{R}^{M}_{+} \rightarrow \Re_{+} \cup \{+\infty\}$, where $\mathbb{R}^{q}_{+} = \Re^{q}_{+}/\{0\}$ for q = M, N,

defined by $NE_{\check{N}}\ ({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y}) = \max\{\psi_{O}({}_{s}\boldsymbol{x}, \delta_{NE\,s}\boldsymbol{y})/\psi_{O}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y})|, \delta_{NE} \in D, {}_{s}\boldsymbol{y} \in P^{S}_{N}({}_{s}\boldsymbol{x}),\ \delta_{NE\,s}\boldsymbol{y} \in P^{S}_{\check{N}}\ ({}_{s}\boldsymbol{x}),$

$(\boldsymbol{\delta}_{NE\,s}\boldsymbol{y} + \xi) \notin P^{S}_{\check{N}}\ ({}_{s}\boldsymbol{x}),\ \ \forall\ \xi \in \Re^{M}_{+}\}$ and $NE_{O\check{N}}\ ({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y}) = -\infty$ for ${}_{s}\boldsymbol{y} \notin P^{S}_{\check{N}}\ ({}_{s}\boldsymbol{x})$, is called the output oriented Non-radial Farrell measure of technical efficiency for non-increasing returns to scale.

$NE_{O\check{N}}\ ({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Farrell sense and the technical inefficiency is reflected by values of $NE_{O\check{N}}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y})>1$, with greater inefficiency for successively higher values of $NE_{O\check{N}}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y})$.

Proposition 3.18 For non-increasing returns to scale technologies the Non-radial Farrell output oriented measure of technical efficiency satisfies E1b, E2b, E3bn, E4bw, E5b and E6.

Proofs:

For E1b, E2b E4bw,E5b and E6b proofs are similar to proposition 3.4. E3bn follows the reasoning given in proposition 3.12.

Definition: The function NE_{OV}: $\mathbb{R}^{N}_{+} \times \mathbb{R}^{M}_{+} \rightarrow \Re_{+} \cup \{+\infty\}$, where $\mathbb{R}^{q}_{+} = \Re^{q}_{+}/\{0\}$ for q = M, N,

defined by $NE_{OV}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y}) = \min\{\psi_{O}({}_{s}\boldsymbol{x}, \delta_{NE\,s}\boldsymbol{y})/\psi_{O}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y})|, \delta_{NE} \in D,\ {}_{s}\boldsymbol{y} \in P^{S}_{V}({}_{s}\boldsymbol{x}),\ \delta_{NE\,s}\boldsymbol{y} \in P^{S}_{V}({}_{s}\boldsymbol{x}),$

$(\delta_{NE\,s}\boldsymbol{y} + \xi) \notin P^{S}_{V}({}_{s}\boldsymbol{x}),\ \ \forall\ \xi \in \Re^{M}_{+}\}$ and

$NE_{OV}({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = -\infty$ for ${}_s\boldsymbol{y} \notin P^S{}_V({}_s\boldsymbol{x})$, is called the output oriented Non-radial Farrell measure of technical efficiency for non-increasing returns to scale.

$NE_{OV}({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Farrell sense and the technical inefficiency is reflected by values of $NE_{OV}({}_s\boldsymbol{x}, {}_s\boldsymbol{y})>1$, with greater inefficiency for successively higher values of $NE_{OV}({}_s\boldsymbol{x}, {}_s\boldsymbol{y})$.

Proposition 3.19 For variable returns to scale technologies the Non-radial Farrell output oriented measure of technical efficiency satisfies E1b, E2b, E3bw, E4bw, E5b and E6b. It does not satisfy E3b.

Proofs:

For E1b, E2b, E4b, E5b and E6b proofs are similar to proposition 3.4. E3bw does not hold for the reason discussed in proposition 3.12.

The above discussion shows that the relaxation of returns to scale assumption has identical effects on the output oriented measures of technical efficiency. In each of these measures only property E3b is affected while the character of other assumptions remain unaffected.

Proposition 3.20 For non-constant returns to scale the revenue efficiency satisfies *OE*1, *OE*3 and *OE*4 to *OE*7. *OE*2 is modified as follows;

*OE*3w. $OE_I(\lambda\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{y}) \lesseqgtr \lambda\, OE_O(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{y})$, $\lambda > 0$, $0 < \gamma \lesseqgtr 1$

where γ is the scale parameter.

Proofs:

*OE*3w has similar explanation as given in propositions 3.12 and 3.13. For rest of the properties proofs are similar to proposition 3.6

Proposition 3.21 For price based technical efficiency TE^V2 is modified for non-constant returns to scale as follows;

TE^{V}2w. $TE^{V}{}_{I}(\lambda \boldsymbol{x}, \boldsymbol{y}) \gtreqless \lambda TE^{V}{}_{I}(\boldsymbol{x}, \boldsymbol{y})$ $\lambda > 0$, $0 < \gamma \lesseqgtr 1$

Other properties remain unchanged.

Proofs:

For TE^{V}2w proof follows the reasoning from proposition 3.12 . For other properties proofs are same as in proposition 3.8a

Proposition 3.22

For non-constant returns to scale Farrell and Zieschang allocative efficiency measures satisfy proposition 3.7 and Russell and non-radial Farrell measures satisfy 3.8b.

Proofs:

Similar to propositions 3.7 and 3.8b respectively.

Computation

Relaxation of scale assumption from constant to non-increasing or to variable returns to scale gives rise to the need of introducing the corresponding scale parameter adjustments in the linear programming models discussed in section 3.3. These adjustments are as follows;

Farrell output oriented technical efficiency measure linear programming model for non-increasing returns to scale is written as

$$\begin{aligned} &\text{Max}_{\delta DF,\eta} && \delta_{DF} \\ &\text{s.t} && \boldsymbol{x}^{k} - \boldsymbol{X}\eta \geq \mathbf{0} \\ & && -\delta_{DF}\boldsymbol{y}^{k} + \boldsymbol{Y}\eta \geq \mathbf{0} \\ & && \mathbf{e}^{T}\eta \leqq 1 \\ & && \eta \geq \mathbf{0} \end{aligned}$$

where $\mathbf{e}$ is an M×1 vector of ones and other variables are as defined earlier.. The restriction requires the sum of the intensity vector be less than or equal to one. For the variable returns to scale the weak inequality of this new constraint is replaced by the strict equality to represent the correspondence $P_{V}(\boldsymbol{x})$, as mentioned earlier. Thus we have

$$
\begin{aligned}
\text{Max}_{\nu DF,\eta} \quad & \delta_{DF} \\
\text{s.t} \quad & \boldsymbol{x}^k - \boldsymbol{X}\eta \geq \mathbf{0} \\
& -\delta_{DF}\boldsymbol{y}^k + \boldsymbol{Y}\eta \geq \mathbf{0} \\
& \mathbf{e}^T\eta = 1 \\
& \eta \geq \mathbf{0}
\end{aligned}
$$

Thus the variable returns to scale Farrell output oriented efficiency model requires the sum of the intensity vector be exactly equal to one. Identical additional constraints, $\mathbf{e}^T\eta \leqq 1$ and $\mathbf{e}^T\eta = 1$, are introduced in the three non-redial models discussed above to accommodate non-increasing and variable returns to scale assumptions respectively. No other changes in the respective models are required.

3.5.2 The Disposability Assumption

Throughout the preceding discussion we continued maintaining the Strong disposability assumption besides the constant returns to scale assumption. Now we replace this assumption by the weak disposability. For our purpose let us assume that the output vector $\boldsymbol{y}$ of any sample point comprises of two sub vectors, $\hat{\boldsymbol{y}}$ and $\tilde{\mathbf{y}}$ where $\hat{\boldsymbol{y}}$ is made up of the components that satisfy the strong disposability and $\tilde{\mathbf{y}}$ has the components that satisfy the weak disposability. The resulting technology is now written as

$P_W(\boldsymbol{x}) = \{\boldsymbol{y} \mid \hat{\boldsymbol{y}} \leq \hat{\boldsymbol{Y}}\eta,\ \tilde{\mathbf{y}} = \rho\tilde{\mathbf{Y}}\eta,\ \boldsymbol{x} \geq \boldsymbol{X}\eta,\ \eta \geq \mathbf{0}\}$

This is the representation of constant returns to scale technology with the output vectors comprising both weakly disposable and strongly disposable output components.

The Farrell measure of output oriented technical efficiency for the above technology can now be given as follows.

Definition: A function $FE_{OW}(\boldsymbol{x}, \boldsymbol{y})$: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined by

$FE_{OW}(\boldsymbol{x}, \boldsymbol{y})=\max\ \{\delta_{DF} | \ \boldsymbol{y} \in P_W(\boldsymbol{x}), \delta_{DF}\boldsymbol{y} \in P_W(\boldsymbol{x}), \delta_{DF} \in [1,+\infty),\ \boldsymbol{y} \in \Re^M_+\}$ and $FE_{OW}(\boldsymbol{x}, \boldsymbol{y}) = -\infty$

for $\boldsymbol{y} \notin P_W(\boldsymbol{x})$, is the Farrell's output measure of technical efficiency for the technology defined by $P_W(\boldsymbol{x})$.

$FE_{OW}(\boldsymbol{x}, \boldsymbol{y}) = 1$ implies that the sample point in question is technically efficient in the Farrell sense and the technical inefficiency is reflected by values of $FE_{OW}(\boldsymbol{x}, \boldsymbol{y})>1$, with greater inefficiency for successively higher values of $FE_{OW}(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 3.23: $FE_{OW}(\boldsymbol{x}, \boldsymbol{y})$ satisfies E3b, E4b and E6b. It does not satisfy E1b, E2b and E5b.

Proofs:

Similar to proposition 3.1

Definition: The function RE_{OW}: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+/\{0\}$, defined as

$RE_{OW}(\boldsymbol{x},\boldsymbol{y})=\max\{(1/M(\mathbf{e}^T\boldsymbol{\delta}_{FL}\mathbf{e}\ \ |\ (\boldsymbol{y} \in P_W(\boldsymbol{x}), \boldsymbol{\delta}_{FL}\boldsymbol{y} \in P_W(\boldsymbol{x}), \boldsymbol{\delta}_{FL} \in D\}$ and $RE_{OSW}(\boldsymbol{x},\boldsymbol{y}) = -\infty$ for

$\boldsymbol{y} \notin P_W(\boldsymbol{x})$, where D and $\mathbf{e}$ are as defined earlier, is called the Russell output measure of technical efficiency for the technology defined by $P_{W.}$

Full output oriented technical efficiency is associated with a vector for which $RE_{OW}(\boldsymbol{x}, \boldsymbol{y}) = 1$. A value of $RE_{OW}(\boldsymbol{x}, \boldsymbol{y}) > 1$ implies technical inefficiency in the Russell sense which is higher for higher values of $RE_{OW}(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 3.24: $RE_{OW}(\boldsymbol{x}, \boldsymbol{y})$ efficiency satisfies E1b, E2b, E.3b, E5b and E6b. It fails to satisfy E4b for all the technologies.

Proofs:

Similar to Proposition 3.2.

Definition: The function ZE_{OW}: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$, defined as

$ZE_{OW}(\boldsymbol{x},\boldsymbol{y}) = \{[RE_{OW}(\boldsymbol{x}, \boldsymbol{y}/ \psi_O(\boldsymbol{x}, \boldsymbol{y})]/ \psi_O(\boldsymbol{x}, \boldsymbol{y})|$ for $\boldsymbol{y} \in P_W(\boldsymbol{x})$, $\boldsymbol{y}//\psi_O(\boldsymbol{x}, \boldsymbol{y}) \in P_W(\boldsymbol{x})$ } and

$ZE_{OW}(\boldsymbol{x},\boldsymbol{y}) = -\infty$ for $\boldsymbol{y} \notin P_W(\boldsymbol{x})$ is called the Russell Extended Farrell measure of output technical efficiency for the technology defined by $P_W(\boldsymbol{x})$.

Proposition 3.25: Zieschang output oriented technical efficiency measure satisfies E1b to E4b and E.6 but it does not satisfy E5b, the monotonicity property for the technology defined by $P_W(\boldsymbol{x})$.

Proofs:

Same as proposition 3.3

In order to define the Non-radial Farrell measure of output oriented technical efficiency for the case when the output vector $\boldsymbol{y}$ has both weakly disposable component $\tilde{\mathbf{y}}$ and the strongly disposable component $\hat{\boldsymbol{y}}$, we first transform the data set into the share format as discussed in section 3.2.3 to obtain $P^S_W(\boldsymbol{x})$ as follows

$P^S_W({}_s\boldsymbol{x}) = \{{}_s\boldsymbol{y} \mid {}_s\hat{\boldsymbol{y}} \leq {}_S\hat{\boldsymbol{Y}}\eta,\ \rho_s\tilde{\mathbf{y}} = {}_S\tilde{\mathbf{Y}}\eta,\ {}_s\boldsymbol{x} \geq {}_S\boldsymbol{X}\eta,\ \eta \geq \mathbf{0}\}$

The Non-radial Farrell measure of output oriented technical efficiency for this technology is then defined as follows;

Definition: The function NE_{OW}: $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$ for q = M, N,

defined by $NE_{OW}({}_s\boldsymbol{x},{}_s\boldsymbol{y}) = \max\{\psi_O({}_s\boldsymbol{x}, \boldsymbol{\delta}_{NE}\,{}_s\boldsymbol{y})/\psi_O({}_s\boldsymbol{x}, {}_s\boldsymbol{y})|, \boldsymbol{\delta}_{NE} \in D, {}_s\boldsymbol{y} \in P^S_W({}_s\boldsymbol{x}), \boldsymbol{\delta}_{NE}\,{}_s\boldsymbol{y} \in P^S_W({}_s\boldsymbol{x}),$

$(\boldsymbol{\delta}_{NE}\,{}_s\boldsymbol{y} + \xi) \notin P^S{}_W({}_s\boldsymbol{x})$, $\forall\ \xi \in \Re^M{}_+\}$ and $NE_{OW}({}_s\boldsymbol{x}, {}_s\boldsymbol{y}) = -\infty$ for ${}_s\boldsymbol{y} \notin P^S{}_W({}_s\boldsymbol{x})$, is called the output oriented Non-radial Farrell measure of technical efficiency for the technology defined by $P^S{}_W(\boldsymbol{x})$. $NE_{OW}({}_s\boldsymbol{x},{}_s\boldsymbol{y})$ attains values equal to or greater than one with $NE_{OW}({}_s\boldsymbol{x},{}_s\boldsymbol{y}) = 1$ indicating technical efficiency and technical inefficiency if $NE_{OW}({}_s\boldsymbol{x},{}_s\boldsymbol{y}) > 1$. Greater inefficiency is reflected by subsequently higher vales of $NE_{OW}({}_s\boldsymbol{x},{}_s\boldsymbol{y})$.

Proposition 3.26 For the technology defined by $P^S{}_W(\boldsymbol{x})$, the Non-radial Farrell measure satisfies E1b to E3b, E4bw, E5b and E6b. It does not satisfy E4b.

Proofs:

Same as in proposition 3.4.

Computation:

Though $FE_{OW}(\boldsymbol{x}, \boldsymbol{y})$ has same characteristics with respect to E1b to E6b as does $FE_O(\boldsymbol{x}, \boldsymbol{y})$, the computation however differs between the two. In the case of $FE_{OW}(\boldsymbol{x}, \boldsymbol{y})$ an additional constraint is required to accommodate the weak disposability condition of the sub vector of output. The computation code for $FE_{OW}(\boldsymbol{x}, \boldsymbol{y})$ is written as follows;

$$\begin{aligned} \text{Max}_{\delta DF,\eta} \quad & \delta_{DF} \\ \text{s.t} \quad & \boldsymbol{x}^k - \boldsymbol{X}\eta \geq \boldsymbol{0} \\ & -\delta_{FL}\hat{\boldsymbol{y}}^k + \hat{\boldsymbol{Y}}\eta \geq \boldsymbol{0} \\ & -\delta_{FL}\check{\mathbf{y}}^k + \rho\tilde{\mathbf{Y}}\eta = \mathbf{0} \\ & 0 \leq \rho \leq 1 \\ & \eta \geq \mathbf{0} \end{aligned}$$

This nonlinear program can be converted into a linear program by imposing the restriction that $\rho = 1$. This restriction does not affect the solution values, δ_{DF} and η.

The computation of Russell measure for the technology defined by P_W follows same changes in the constraints as does the computation of Farrell measure when the technology is P_W. Thus the linear programming code for Russell output oriented measure for technical efficiency under these conditions is written as

$$\text{Max}_{\delta FL,\, \eta} \quad 1/M(\mathbf{e}^T\delta_{FL}\mathbf{e})$$

$$\text{s.t} \quad \boldsymbol{x}^k - \boldsymbol{X}\eta \geq \mathbf{0}$$

$$-\delta_{FL}\hat{\boldsymbol{y}}^k + \hat{\boldsymbol{Y}}\eta \geq \mathbf{0}$$

$$-\delta_{FL}\tilde{\mathbf{y}}^k + \tilde{\mathbf{Y}}\eta = \mathbf{0}$$

$$\eta \geq \mathbf{0}$$

The Zieschang measure of output oriented technical efficiency is computed in two steps. The first step comprises of computing the Farrell estimates of δ_{DF} and η and in the second step Russell adjustment is made in those components of the output vector that exhibit strong disposability and have slacks associated with them. The linear programming codes for the two steps are those discussed above for the technology represented by $P_W(\boldsymbol{x})$.

Finally the linear program for Non-radial Farrell measure of technical efficiency for the technology defined by $P_W(\boldsymbol{x})$ is given as follows

$$\text{Max}_{\delta NE,\, \eta} \quad ||\delta_{NE}\ {}_s\boldsymbol{y}^k|| \,/\, ||{}_s\boldsymbol{y}^k||$$

$$\text{s.t} \quad \boldsymbol{x}^k - \boldsymbol{X}\eta \geq \mathbf{0}$$

$$-\delta_{FL}\hat{\boldsymbol{y}}^k + \hat{\boldsymbol{Y}}\eta \geq \mathbf{0}$$

$$-\delta_{FL}\tilde{\mathbf{y}}^k + \tilde{\mathbf{Y}}\eta = \mathbf{0}$$

$$\eta \geq \mathbf{0}$$

The above discussion suggests that the disposability assumptions mainly affect the computation of the scores by adding new constraint. The properties of various measures are preserved.

3.5 Numerical Example

The purpose of this section is to confirm various relationships between efficiency measures, as discussed in the previous sections, using a numerical example based on a hypothetical data set. For simplicity we restrict this example to the case of one input and two outputs. Further, we assume constant returns to scale technology and strong disposability of outputs and input. The data set is given in table 3.5.1. For each data point $\boldsymbol{x}$ represents the input and y_1 and y_2 are the two outputs. The transformed input and outputs are reported in the last three columns of the table. For the purpose of computation and comparisons of various measures of efficiency we make use of these transformed values.

Table 3.5.2 and figure 3.15 presents a comparison of the four quantity based output oriented technical efficiency measures. The first important observation is that the four measures follow almost similar patterns. Thus moving from one data point to another if increase in score is reported by one measure, the other three measures also reflect the same trend, though they differ in scores. The example also confirms that for the comparable points the Non-radial Farrell efficiency scores are greater than or equal to the Farrell efficiency scores and the Farrell scores are less than or equal to the Russell and the Zieschang scores. The example also confirms the ambiguous relationship of Non-radial Farrell efficiency measure with the Russell measures of output oriented technical efficiency.

In table 3.5.3 we have given the price based output oriented technical efficiency scores for the four measures. For each of the four measures this score is defined in terms of the ratio of the

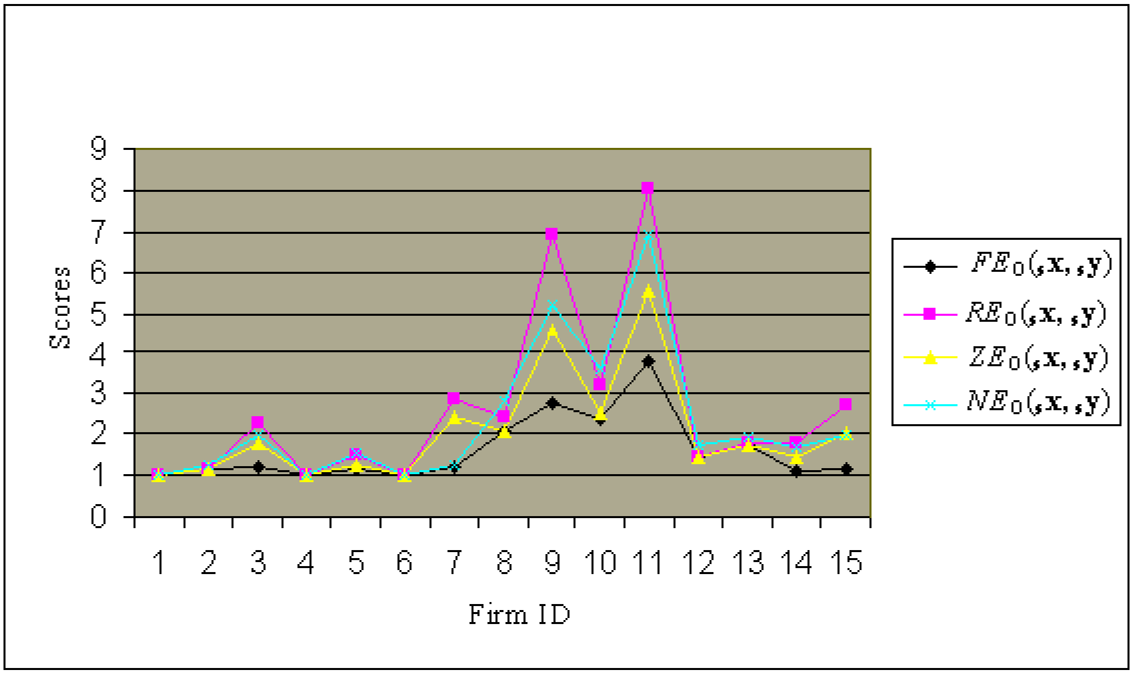

Figure 3.14. Quantity based technical efficiency scores

revenue value of the projected output vector, that is used for the computation of quantity based technical efficiency score, to the revenue value of the observed vector. Figure (3.15) shows that the price based output oriented technical efficiency measures reflect similar trends between themselves as does the quantity based technical efficiency measures.

One point is worth mentioning. There exists a possibility that while the quantity based technical efficiency scores with respect to two different efficiency measures may be different, the price based technical efficiency scores may have identical values for the two measures with respect to a data point. This happens when the two quantity based technical efficiency measures have identical projection vectors but different quantity based technical efficiency scores due to computational differences. The evidence of such a happening is provided by the data point 5 in this numerical example. For this data point the quantity based technical efficiency scores are

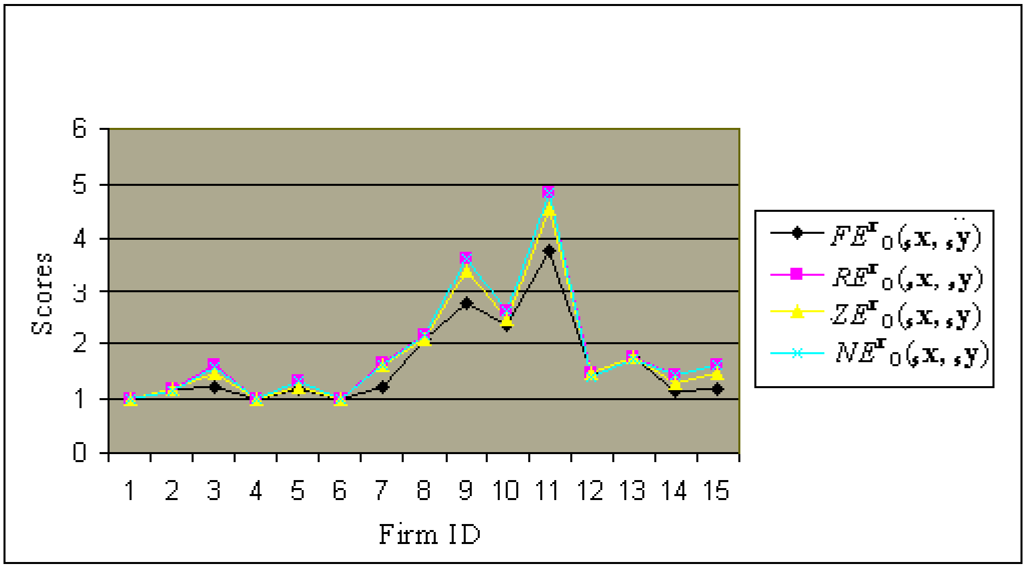

Figure 3.15. Price based technical efficiency

1.48678 and 1.543058 for Russell and Non-radial Farrell measures respectively. But due to the two of them having identical projection vectors, the price based technical efficiency is 1.314659 for each of them. The result is that the two measures have identical allocative efficiency scores despite of different quantity based technical efficiency scores. Data point 7 provides similar example with respect to the Zieschang and the Non-radial Farrell measures.

The price based output technical efficiency measures have been used to compute the allocative efficiency components, in table 3.5.4, as the ratio of the overall efficiency to the respective price based output technical efficiency. With ${}_sr_1/{}_sr_2 = 2$ the overall efficiency associated with each data point is computed as the ratio of the maximum attainable revenue to the actual revenue and is given in the first column of the table. The remaining four columns give allocative efficiency scores associated with the four efficiency measures. The four allocative efficiency scores have been plotted in figure 3.16. The figure shows that the Farrell measure overestimate allocative inefficiency as compared to the non radial measures. The figure confirms that the relationship between the three non-radial efficiency measures can go in any direction.

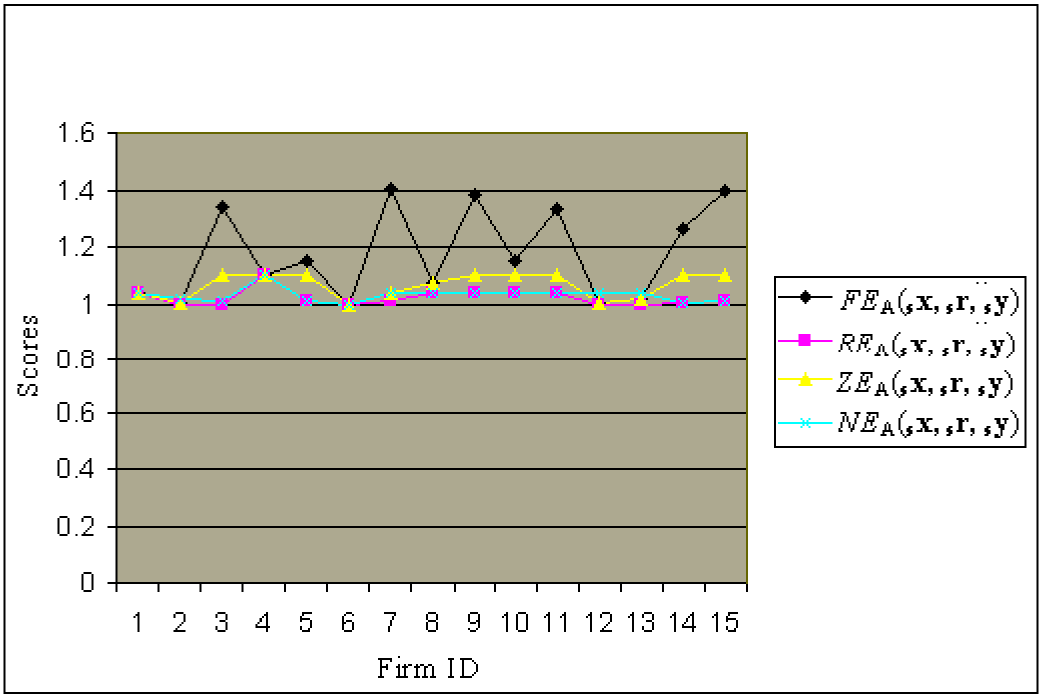

Figure 3.16. Output oriented allocative efficiency scores

.Table 3.5.1 Inputs, Outputs, Input Shares and Output Shares

Firm ID	X	y_1	y_2	${}_s x \times 1000$	${}_s y_1 \times 000$	${}_s y_2 \times 1000$
1	7	2.291503	5.485281	6.013746	6.417674	20.27708
2	12	4.928203	6.211103	10.30928	13.80212	22.96018
3	10	4.324555	1.745967	8.591065	12.11152	6.454201
4	15	7.745967	6.324555	12.8866	21.69367	23.37957
5	20	8.944272	6.472136	17.18213	25.04969	23.92512
6	38	18.32883	22.65476	32.64605	51.33245	83.74636
7	19	1.717798	12.59126	16.32302	4.810934	46.54529
8	67	16.37071	15.23155	57.56014	45.84846	56.30546
9	112	21.16601	7.211103	96.21993	59.27838	26.65681
10	82	18.11077	13.11488	70.44674	50.72175	48.48091
11	213	29.18904	12.16553	182.9897	81.748	44.9715
12	167	55.8457	68.22087	143.4708	156.4037	252.1876
13	69	16.61325	26.90725	59.27835	46.52773	99.46627
14	190	87.5681	45.49193	163.2302	245.2467	168.1671
15	143	63.91652	20.68816	122.8522	179.0072	76.47657

Table 3.5.2 Quantity based output oriented Technical Efficiency

Firm ID	$FE({}_s\mathbf{x}, {}_s\mathbf{y})$	$RE({}_s\mathbf{x}, {}_s\mathbf{y})$	$ZE({}_s\mathbf{x}, {}_s\mathbf{y})$	$NE({}_s\mathbf{x}, {}_s\mathbf{y})$
1	1	1	1	1
2	1.162919	1.163152	1.162919	1.242536
3	1.194106	2.264973	1.804513	1.976328
4	1	1	1	1
5	1.154701	1.486779	1.228815	1.543058
6	1	1	1	1
7	1.182457	2.863308	2.401626	1.233695
8	2.073686	2.393346	2.073686	2.803553
9	2.73252	6.95148	4.64061	5.235611
10	2.33809	3.19083	2.487176	3.55085
11	3.768289	8.054335	5.575259	6.936311
12	1.445652	1.450888	1.445652	1.70987
13	1.731657	1.766058	1.731657	1.90916
14	1.120449	1.799169	1.440722	1.685046
15	1.155333	2.708275	2.034879	1.963769

Table 3.5.3 Price based output oriented Technical Efficiency Measures

Firm ID	$FE^{r}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y})$	$RE^{r}\ ({}_{s}\boldsymbol{x},{}_{s}\boldsymbol{y})$	$ZE^{r}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y})$	$NE^{r}{}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{y})$
1	1	1	1	1
2	1.162919	1.164192	1.162919	1.144967
3	1.194106	1.599087	1.450954	1.580704
4	1	1	1	1
5	1.154701	1.314659	1.202609	1.314659
6	1	1	1	1
7	1.182457	1.638398	1.600163	1.600163
8	2.073686	2.141407	2.073686	2.141407
9	2.73252	3.648412	3.433055	3.648412
10	2.33809	2.587227	2.434509	2.587227
11	3.768289	4.833194	4.547903	4.833194
12	1.445652	1.449975	1.445652	1.398186
13	1.731657	1.758158	1.731657	1.695362
14	1.120449	1.408082	1.283991	1.408082
15	1.155333	1.601361	1.464958	1.601361

Table 3.5.4 Output Oriented Overall and Allocative Efficiency Measures.

Firm Id	$OE({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{r}, {}_{s}\boldsymbol{y})$	$FE_{A}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{r}, {}_{s}\boldsymbol{y})$	$RE_{A}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{r}, {}_{s}\boldsymbol{y})$	$ZE_{A}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{r}, {}_{s}\boldsymbol{y})$	$NE_{A}({}_{s}\boldsymbol{x}, {}_{s}\boldsymbol{r}, {}_{s}\boldsymbol{y})$
1	1.03704	1.03704	1.03704	1.03704	1.03704
2	1.164192	1.001095	1	1.001095	1.016791
3	1.599087	1.33915	1	1.102094	1.01163
4	1.102094	1.102094	1.102094	1.102094	1.102094
5	1.325388	1.14782	1.008161	1.102094	1.008161
6	1	1	1	1	1
7	1.659433	1.403377	1.012839	1.03704	1.03704
8	2.220725	1.070907	1.03704	1.070907	1.03704
9	3.783549	1.384637	1.03704	1.102094	1.03704
10	2.683058	1.147542	1.03704	1.102094	1.03704
11	5.012217	1.330104	1.03704	1.102094	1.03704
12	1.449975	1.00299	1	1.00299	1.03704
13	1.758158	1.015304	1	1.015304	1.03704
14	1.415078	1.262957	1.004969	1.102094	1.004969
15	1.614522	1.397451	1.008218	1.102094	1.008218

3.7 Summary

The objective of this chapter was to extend the discussion on input oriented measures of efficiency to output oriented measures. We started with the Farrell output oriented measure of technical efficiency and showed that it only satisfies the homogeneity property out of the required set of properties outlined in chapter one. The non radial Russell measure was shown to violate the homogeneity property but was shown to satisfy the sub-homogeneity. In comparison, the non-radial Zieschang measure was shown to violate the monotonicity property. Then we introduced the Non-radial Farrell output oriented technical efficiency measure and showed that it violated the homogeneity property but satisfied the sub homogeneity. We also introduced the output oriented overall efficiency measure and its decomposition into allocative and technical efficiency components. A comparison of decomposition of overall efficiency into its component was provided across radial and non-radial measures and various relationships were discussed. With the help of a hypothetical data set we were able to confirm different relationships across measures that we established during our discussion.

We also provided an account on effects of relaxing our basic assumptions of constant returns to scales and strong disposability. It was shown that relaxing constant returns to scale affects homogeneity of degree plus one in inputs across all measures. The disposability assumption was shown to have mainly its impact on the programming code where additional constraints are to be introduced to take into account the sub-vector weak disposability.

CHAPTER 4

SUMMARY AND CONCLUSIONS

Comparing economic units, using some appropriate criteria, for the purpose of ranking them more or less efficient in a relative sense, emerged as a popular area of research following the seminal paper by Farrell (1957) on the measurement of productive efficiency. Farrell's work was later developed into a linear programming based technique, Charnes, Cooper and Rhodes (1978), that came to be known as the Data Envelopment Analysis (DEA). Farrell's input oriented measure of technical efficiency is the inverse of the input distance function and it measures the maximum amount by which an input vector $\boldsymbol{x}$ can be feasibly shrunk along a ray for a given output level, $\boldsymbol{y}$. Later, Färe and Lovell(1978) suggested a set of properties that an input index was supposed to satisfy. For an input oriented measure of technical efficiency, $E_I(\boldsymbol{y}, \boldsymbol{x})$, these properties included

i) $E_I(\boldsymbol{y}, \boldsymbol{x}) = 1$ for any data point only *iff* it is efficient in Koopmans sense

ii) The feasible input vectors are compared only with the Koopmans efficient input vectors for the computation of efficiency scores

iii) $E_I(\boldsymbol{y}, \boldsymbol{x})$ is homogenous of degree minus one in inputs

iv) $E_I(\boldsymbol{y}, \boldsymbol{x})$ is strictly monotonic in input

Färe and Lovell showed that the Farrell measure of technical efficiency satisfied homogeneity condition only and as an alternate they proposed a new measure, the Russell measure. In contrast to the radial shrinking used by the Farrell measure of technical efficiency the newly proposed Russell measure would allow for the shrinking of the input vector in the coordinate directions. However in later studies this measure was shown not to satisfy the homogeneity and the monotonicity properties. Another alternate was provided by Zieschang(1984) who proposed a measure that was an amalgam of Farrell measure and the measure proposed by Färe and Lovell. It was shown by Russell(1985) and Russell(1988) that none of these measures satisfied all the conditions outlined by Färe and Lovell. Bo*L*(1986) using two examples remarked that no efficiency measure could fulfill Färe and Lovell conditions.

Given this background, the primary objective of this study was to extend the search for a technical efficiency measure in a new direction. We introduced a two step procedure the first step of which involved transforming the data set, using the concept of the numeraire firm, into a format so that the efficiency scores did not depend upon the units in which inputs and outputs were measured. In the second step a non-radial technical efficiency measure was computed as a ratio of two Euclidean norms which we named as non-radial Farrell measure of technical efficiency. In chapter 2 and chapter 3 we provided a comparison of this newly proposed measure with three of the existing measures, the Farrell measure, the Russell measure and the Zieschang measure, with respect to the Färe and Lovell criterion. Chapter 1 described the technological environment in which a firm was assumed to operate and provided a discussion on the Färe and Lovell conditions to be fulfilled by any efficiency index.

The discussion in chapter 2 was input oriented. We defined the input oriented non-radial Farrell measure of technical efficiency under assumptions of constant returns to scale and strong disposability and compared it with the other input oriented measures with respect to the Färe and Lovell criterion. The summary of these comparisons has been provided in table 4.1. The discussion was then extended in the direction of relaxing the assumptions of returns to scale and the strong disposability. Relaxing the returns to scale assumption was found to mainly affect the homogeneity property. The weak disposability assumption seems to have no bearing on the properties of the technical efficiency measures. However with a weak disposable sub vector in the data set an additional constraint is to be introduced into the computational code as shown in section 2.5.2.

In the chapter we also discussed the concepts of the cost efficiency, and allocative efficiency. It was shown that the overall efficiency score did not depend upon how the technical efficiency was defined. However the definition of the technical efficiency was shown to play an important role in the decomposition of the overall efficiency into the technical and the allocative components. The input oriented discussion on efficiency of chapter 2 was extended to the discussion on output orientated measures of efficiency in chapter 3. The summary results of this chapter for constant returns to scale are given in table 4.2. The main conclusions do not change across the two orientations.

It looks that the non-radial Farrell measure of technical efficiency proposed in this study satisfies the identification, the compare to and the monotonicity property along with sub homogeneity. Comparison of the radial and the non-radial Farrell measures of technical efficiency allows us to uncover the source of the violation of the strong homogeneity property in the non-radial measure. This investigation is possible because both of these measures can be expressed as the ratio of the Euclidean norm of the reference vector to the Euclidean norm of the actual vector. When the actual vector undergoes a proportionate change, in the case of radial Farrell measure the reference vector does not change and so remains unchanged its Euclidean norm. The Euclidean norm of the actual vector however changes in the proportion by which the actual vector has changed. The result is that the Farrell radial measure changes inversely, to the change in the actual vector, in the same proportion. In contrast to the radial measure, in the case of non-radial Farrell measure the reference vector is not necessarily static. The change in the reference vector follows because the subset of the $EffL(\boldsymbol{y})$ from which the reference vector is picked up changes. This point is illustrated in figure (4.1) where the subset of $EffL(\boldsymbol{y})$ from which the reference vector is picked up broadens from $\text{Sub}EffL(\boldsymbol{y}) = \{\boldsymbol{x}^* | \boldsymbol{x}^* \leq \boldsymbol{x}\}$ to $\text{Sub}EffL(\boldsymbol{y})_\lambda=\{\lambda\boldsymbol{x}^*|\lambda\boldsymbol{x}^*\leq\lambda\boldsymbol{x}, \lambda \geq 1\}$ as the input vector $\boldsymbol{x} \in L(\boldsymbol{y})$ increases to $\lambda\boldsymbol{x}$. Since $\text{Sub}EffL(\boldsymbol{y})$ is contained in $\text{Sub}EffL(\boldsymbol{y})_\lambda$, the old reference vector also belongs to the new reference set, and a possibility exists that it may still minimize the ratio of the two Euclidean norms in which case the strict homogeneity follows. If this is not the case then some new reference vector

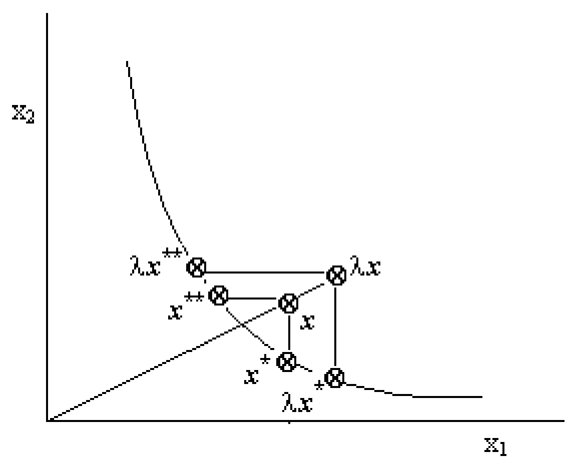

Figure 4.1. Homogeneity in non-radial measures

$\boldsymbol{x}_N \in \text{Sub}EffL(\boldsymbol{y})_\lambda \backslash \text{Sub}EffL(\boldsymbol{y})$ minimizes the ratio of the two Euclidean norms. But this possibility implies that the sub-homogeneity, and not the strong homogeneity, is satisfied. The same argument also holds for the Russell measure of technical efficiency and can be extended for the output orientation as well. Thus we see that the radial Farrell measure and the Zieschang measure of technical efficiency that preserve the original reference vector when the actual input vector undergoes a proportionate change, satisfy strict homogeneity and the Russell and the non-radial Farrell measures where the reference vector may change satisfy sub-homogeneity.

Noting that the radial Farrell measure satisfies strict homogeneity and the non-radial Farrell measure satisfies the strict monotonicity, it is our suggestion that the two measures may be combined in the same way as the radial Farrell and the Russell measures were combined by Zieschang. Construction of such a measure and investigation into its properties provide one possible direction of another research in this area. Our conjecture is that the resulting measure will satisfy all the four properties outlined by Färe and Lovell. The basis for this conjecture is that the amalgam of the radial Farrell and the Russell measure failed to satisfy the monotonicity property because it was not satisfied by the Russell measure nor it was satisfied by the radial Farrell measure. Since the non radial Farrell measure satisfies the strict monotonicity property along with the sub homogeneity, and the radial Farrell measure satisfies the strict homogeneity along with weak monotonicity, the possibility exists that combining the two together may produce the desired result.

During this research we also invested some time investigating the "most favorable" technical efficiency score from the firm's point of view[7]. If in the definition of non-radial Farrell input oriented measure of technical efficiency we replace the argument "minimum" by "maximum" we end up with the following;

Definition: The function NE_I: $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \rightarrow \Re_+ \cup \{+\infty\}$, where $\mathbb{R}^q_+ = \Re^q_+ / \{0\}$ for q = M, N,

defined by $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \max\{\psi_I({}_s\boldsymbol{y}, \theta_{NE\,s}\boldsymbol{x})/\psi_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})|$ for ${}_s\boldsymbol{x} \in L({}_s\boldsymbol{y})$, $\theta_{NE\,s}\boldsymbol{x} \in L({}_s\boldsymbol{y})$, $\theta_{NE} \in Q$, $(\theta_{NE}$

[7] This idea has not been mentioned in the main body of the research because it needs further investigation.

${}_s\boldsymbol{x}$ **- ε**) ∉ $L({}_s\boldsymbol{y})$} and $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = +\infty$ for ${}_s\boldsymbol{x} \notin L({}_s\boldsymbol{y})$, is called the most favorable Non-radial Farrell measure of technical efficiency. $NE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$.

In this definition we again end up with an efficiency score that is bounded by zero and one and that reflects properties similar to the non-radial Farrell measure discussed in chapter 2. We call this score "most favorable" from firm's point of view because the firm will have an interest in maximizing it. In contrast the actual definition as given in section 2.2 computes the least favorable score from firm's point of view. These two scores can be seen as upper and lower limits to efficiency with respect to a firm with the radial Farrell measure representing the intermediate situation. Similar upper and lower limits can be defined using the Russell measure as well after slight modifications in its definition. This idea of defining technical efficiency in terms of upper and lower limits using non-radial projections provides another potential future research and such scores may be desirable under certain circumstances when pricing information is not available and where it looks more appropriate to define an efficiency band rather than to define an efficiency score. Note that the radial Farrell and the Zieschang measures do not allow for the computation of such efficiency bands.

In the present study we discussed only input and output orientations. The extension of this discussion in the direction of graph orientation is straightforward. The transformation of input and output data set remains the first step in graph orientation as well when it comes to the question of computing the non-radial Farrell measure.

Accounting for productive efficiency in the presence of undesirable outputs, specially, when the desirable outputs are nulljoint with the undesirable output provides another potential for further research. In comparison to input and output orientated technical efficiency computations that assume weak or strong disposability and make use of Shephard input and output distance function, the graph oriented technical efficiency and, the output orientated technical efficiency with undesirable outputs, assume g-disposability and make use of the directional distance function for computation of efficiency scores.. In the present study we saw that the Shephard distance function produced identical results for Farrell radial measure of technical efficiency computed using original and transformed data set. Does the same hold for the directional distance function, and how the properties of various measures vary across these two distance concepts when applied to transformed data space is another area that may be worth investigating. In conclusion this study is only a small step in a new direction i.e. computation of efficiency scores using transformed data set and the research can be extended in many dimensions.

Table 4.1 Input oriented technical efficiency comparisons with constant returns to scale and strong disposability assumptions[8].

Properties	$FE_I(\boldsymbol{y}, \boldsymbol{x})$	$RE_I(\boldsymbol{y}, \boldsymbol{x})$	$ZE_I(\boldsymbol{y}, \boldsymbol{x})$	$NE_I(\boldsymbol{y}, \boldsymbol{x})$
(1) $E_I(\boldsymbol{y}, \boldsymbol{x}) = 1$ *iff* $\boldsymbol{x} \in EffL(\boldsymbol{y})$	No	Yes	Yes	Yes
(2) $\forall\ \boldsymbol{x} \in L(\boldsymbol{y})$, $\boldsymbol{x}$ is compared $\boldsymbol{x}^* \in EffL(\boldsymbol{y})$	No	Yes	Yes	Yes
(3) $E_I(\boldsymbol{y}, \lambda\boldsymbol{x}) = \lambda^{-1}E_I(\boldsymbol{y}, \boldsymbol{x})$, $\lambda \in (0, \infty)$	Yes	No	Yes	No
(4) $E_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu E_I(\boldsymbol{y}, \boldsymbol{x})$, $\lambda \in (0, +\infty)$	Yes	Yes	Yes	Yes
(5) $E_I(\boldsymbol{y}, \boldsymbol{x}) > E_I(\boldsymbol{y}, \boldsymbol{x}^*)\ \forall\ \boldsymbol{x} \geq \boldsymbol{x}^*$	No	No	No	Yes
(6) Commensurability	Yes	Yes	Yes	Yes
(3w) $(E_I(\boldsymbol{y}, \lambda\boldsymbol{x}) \gtreqless \lambda^{-1}E_I(\boldsymbol{y}, \boldsymbol{x})$, $\lambda \lesseqgtr 1$	Yes	Yes	Yes	Yes
. $E_I(\nu\boldsymbol{y}, \boldsymbol{x}) \gtreqless \nu E_I(\boldsymbol{y}, \boldsymbol{x})$,$\nu > 0$ and for $\gamma \gtreqless 1$. γ is returns to scale parameter	Yes	Yes	Yes	Yes
(5w) $E_I(\boldsymbol{y}, \boldsymbol{x}) \geq E_I(\boldsymbol{y}, \boldsymbol{x}^*)\ \forall\ \boldsymbol{x} \geq \boldsymbol{x}^*$	Yes	Yes	Yes	Yes

[8] "Yes" when a property is satisfied otherwise no.

Table 4.2 Output oriented technical efficiency comparisons with constant returns to scale and strong disposability assumptions.

Properties	$FE_O(\boldsymbol{x}, \boldsymbol{y})$	$RE_O(\boldsymbol{x}, \boldsymbol{y})$	$ZE_O(\boldsymbol{x}, \boldsymbol{y})$	$NE_O(\boldsymbol{x}, \boldsymbol{y})$
(1) $E_I(\boldsymbol{y}, \boldsymbol{x}) = 1$ *iff* $\boldsymbol{x} \in EffL(\boldsymbol{y})$	No	Yes	Yes	Yes
(2)$\forall\ \boldsymbol{x} \in L(\boldsymbol{y})$, $\boldsymbol{x}$ is compared $\boldsymbol{x}^* \in EffL(\boldsymbol{y})$	No	Yes	Yes	Yes
(3) $E_I(\boldsymbol{y}, \lambda\boldsymbol{x}) = \lambda^{-1}E_I(\boldsymbol{y}, \boldsymbol{x})$, $\lambda \in (0, \infty)$	Yes	Yes	Yes	Yes
(4) $E_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu E_I(\boldsymbol{y}, \boldsymbol{x})$, $\lambda \in (0, +\infty)$	Yes	No	Yes	No
(5) $E_I(\boldsymbol{y}, \boldsymbol{x}) > E_I(\boldsymbol{y}, \boldsymbol{x}^*)\ \forall\ \boldsymbol{x} \geq \boldsymbol{x}^*$	No	Yes	No	Yes
(6) Commensurability	Yes	Yes	Yes	Yes
(3w) $(E_I(\boldsymbol{y}, \lambda\boldsymbol{x}) \gtreqless \lambda^{-1}E_I(\boldsymbol{y}, \boldsymbol{x})$, $\lambda \lesseqgtr 1$	Yes	Yes	Yes	Yes
. $E_I(\nu\boldsymbol{y}, \boldsymbol{x}) \gtreqless \nu E_I(\boldsymbol{y}, \boldsymbol{x})$,$\nu > 0$ and for $\gamma \gtreqless 1$. γ is returns to scale parameter	Yes	Yes	Yes	Yes
(5w) $E_I(\boldsymbol{y}, \boldsymbol{x}) \geq E_I(\boldsymbol{y}, \boldsymbol{x}^*)\ \forall\ \boldsymbol{x} \geq \boldsymbol{x}^*$	Yes	Yes	Yes	Yes

REFERENCES:

Afriat, S. (1972), "Efficiency Estimation of Production Functions", *International Economic Review 13, 568-598.*

Aigner, D.J, and **S.F. Chu(1968),** "On Estimating the Industrial Production function", *American Economic Review, 58, 826-839.*

Aigner, D.J, C.A.K Lovell and **P**. **Schmidt(1977),** "Formulation and Estimation of Frontier Production Models", *Journal of Econometrics, 6,61-37.*

Banker, R.D., A. Charnes and ***W. W*. Cooper (1984), "**Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis", *Management Science* 30, 1078 – 1092.

Banker, R.D and **A Maindiratta (1986**), "Nonparametric Analysis of Technical and Allocative Efficiencies in Production" *Econometrica 56/5, 1315-1332.*

Bol Georg(1986), "On Technocal Efficiency Measures: A Remark**",** *Journal of Economic Theory* 38, 380-385.

Boyd, G and **R. Färe (1984**), "Measuring the Efficiency of Decision Making Units: a comment" *European Journal of Operational Research, 15, 331-332.*

Burley, H,T. (1980), "Production Efficiency in US manufacturing: a Linear Programming Approach", *Review of Economics and Statistics 62/4, 619-622.*

Caves, D, *W*, L, R. Christensen and ***W*.E. Diewert (1982**), "The Economic Theory of Index Numbers of the Measurement of Input, Output and productivity" *Econometrica 50/6, 1393-1414.*

Chambers, R. G., *Y*. Chung and R. Färe (1996), "Benefit and Distance Functions", *Journal of Economic Theory* 70, 407-418.

Chambers, R. G., *Y*. Chung and R. Färe (1998), "Profit, Directional Distance Functions, and Nerlovian Efficiency", *Journal of Optimization Theory and Applications* 98, 351-364.

Chang, K, P and ***Y*, *Y* Guh (1991), "**Linear Production Function and the Data Envelopment Analysis", *European Journal of Operational Research,52/2, 215-223.*

Charnes, A. *W.W*. Cooper and E. Rhodes(1978), "Measuring the Efficiency of Decision Making Units", *European Journal of Operational Research, 2,429-444***.**

Charnes, A, *W.W*. Cooper and E. Rhodes(1978), "An Application of Data Envelopment Analysis to Program Follow through", *Management Sciences, 27,668-697***.**

Charnes, A., *W.W.* Cooper, B. Golan*y*, L. Seiford and J. Stutz (1985), "Foundations of Data Envelopment Analysis for Pareto-Koopmans Efficient Empirical Production Functions***", Journal of Econometrics 30, 91 – 107.***

Charnes, A., *W.W.* Cooper, and R.M. Thrall, 1986, " Classifying and Characterizing Efficiencies and Inefficiencies in Data Envelopment Analysi*s", Operations Research Letter*s, 5, 105-110.
Cherch*y*e, L., T. Kuosmanen and G.T. Post (2000), "What is the Economic Meaning of FDH? A Reply to Thrall", Journal of Productivity Analysis 13, 259-263.

Charnes, A., Cooper, *W.*, Le*w*in, A. *Y.* and Seiford, L. M. (1994), "Data Envelopment Analysis Theory, Methodology and Applications", Kluwer Academic Publishers, United States.

Chung *Y.* and **R. Färe, (1995),** "Productivity and Undesirable Outputs: A Directional Distance Function Approach" Discussion Paper,*Department of Economics, Southern Illinois University, Series 95-24.*

Cooper, *W.W.*, S. Li, L.M. Seiford, K. Tone, R.M. Thrall and J. Zhu (2001),"Sensitivity and Stability Analysis in DEA/ Some Recent Developments"***,*** *Journal of Productivity Analysis 15, 217-246.*

Debreu, G (1951), "The Coefficient of Resource Utilization", *Econometrica, 19/3, 273-292*
Die*w*ert, *W*.E. (1981), "Capital and the Theory of Productivity Measurement", *American Economic Review,70/2, 260-267.*

Eichhorn, *W*., ed. (1988), "Measurement in economics: Theory and applications of Economic Indices", *Heidelberg, Physica-Verlag.*

Färe, R., S. Grosskopf and C.A.K. Lovell (1983): The Structure of Technical Efficiency, *Scandinavian Journal of Economics* 72, 41-62.

Färe, R., S. Grosskopf and C.A.K. Lovell (1985), "The Measurement of Efficiency of Production", *Kluwer Academic Publishers, Boston*.

Färe, R., S. Grosskopf and C.A.K. Lovell (1993), "Production Frontiers" Cambridge University Press.

Färe, R. and C.A.K. Lovell (1978), "Measuring the Technical Efficiency of Production", *Journal of Economic Theory 19, 150-62*.

Färe. R and **S. Grosskopf (1983),** "Measuring output Efficiency", *European Journal of Operational Research,, 13, 173-179.*

Färe. R and **S. Grosskopf (1994),** "Cost and Revenue Constrained Production" Springer-Verlag NewYork

Färe, R., L. Svensson (**1980**)**,** "Congestion of Production Factors", *Econometrica* 48,1745-1753.

Färe. R and **S. Grosskopf** and **D. Njinkeu** (**1988**), "On Piecewise Reference Technologies", *Management Sciences, 34/12, 1507-1511*

Färe. R and ***W*. Hunsaker** (**1996**)**,** "Notions of efficiency and Their Reference Sets", Management Science 32/2 273-243.

Farrell, M.J. (**1957**)**,** "The Measurement of Productive Efficiency", *Journal of the Royal Statistical Society* Series *A* 120, 253-281.

Førsund, F.R. (**1996**), "On the calculation of the scale elasticity in DEA models", ***Journal of Productivity Analysis*** **7, 283-302**.

Førsund, F.R. and **L, hgalmarason** (**1979**)**,**"Generalized Farrell Measures of Efficiency: An Application to Milk Processing in Swedish Dairy Plants" *economic Journal, 89/354, 294-315*

Frei. F. *X*. and **Patrick T. Harker** (**1999**)**,** " Projections onto Efficient Frontiers: Technical and Computational Extensions to DEA", *Journal of Productivity Analysis, 11, 275-300.*

Fried, H, O., C. A. K. Lovell and **S. S. Schmidt, eds.** (**1993**), "The Measurement of Productive efficiency: Techniques and Applications. *New York Oxford University Press.*

Fuss M. and **D. McFadden** (**1978**)**,** "Production Economics: A Dual Approach to Theory and Applications", *Volume I and Volume II , Amsterdam, North-Holland*

Gijbels, I., E. Mammen, B.U. Park and L. Simar (**1999**)**' "**On estimation of monotone and concave frontier functions", *Journal of the American Statistical Association* 94, 220-228.

Golan*y*, B. and *Y*. Roll, 1993, "Some Extensions of Techniques to Handle Non-discretionary Factors in Data Envelopment Analysis", *The Journal of Productivity Analysi*s, 4, 419-432.

Grosskopf, S. (**1986**)**, "**The Role of the Reference Technology in Measuring Productive Efficiency", *Economic Journal* 96, 499-513.

Grosskopf, S. (**1996**)**, "**Statistical Inference and Non-Parametric Efficiency: A Selective Survey", *Journal of Productivity Analysis* 7, 161-176**.**

Hall, M and **C. *W*insten** (**1959**)**,** " The Ambiguous Notion of Efficiency" *Economic Journal, 69/1, 71-86.*

Hanoch, G. and M. Rothschild (**1972**)**, "**Testing Assumptions of Production Theory: A Nonparametric Approach", *Journal of Political Economy* 80, 256-275.

Kopp J. Ra*y*mond(**1981**) "Measuring the Technical Efficiency of Production : A Comment",*Journal of Economic Theory* 25, 450-452.

Kumbhakar, S.C. and Lovell, C.A.K. (**2000**), "*Stochastic Frontier Analysi*s", Cambridge University Press, Cambridge.Land,

Lovell, C.A.K, (**1993**), 'Linear programming approaches to the measurement and analysis of productive efficiency'*, Working Paper 94–393E, College of Business Administration, University of Georgia, Athens, Georgia.*

Lovell C. A K, J.T. Pastor, J.A. Turner (**1995**)"Measuring Macroeconomic Performance in OECD: A Comparison of European and Non-European Countries" *European Journal of Operational Research,87, 507-518.*

Mcfadden, D. (**1978**)**,** "Cost revenue and Profit Functions" in *M. Fuss and D. McFadden, eds*(*1978).*

Petersen, N.C. (**1990**), "Data Envelopment Analysis on a Relaxed Set of Assumptions",*Management Science* 36, 305-314.

Russell R. Robert(1985),"Measures of Technical Efficiency**",** ***Journal of Economic Theory*** **35, 109-126.**

Russell R. Robert (**1988**)**, "**On the Axiomatic approach to the Measurement of Technical Efficiency**" in *W.* Eichhorn, ed.*Measurement in Economics: Theory and Applications of Economic Indisec.*Heidelberg, Physica-Verlag**

Russell R. Robert (**1990**)**,** "Continuity of Measures of Technical Efficiency" *Journal of economic Theory, 51:2, 255-267*

Seiford, L.M. and R.M. Thrall (**1990**)**, "**Recent Developments in DEA: The Mathematical Programming Approach to Frontier Analysis", ***Journal of Econometrics*** **46, 7-38.**

Shephard, R, *W.* (**1970**)**,** "Theory of cost and Production", *Princeton, Princeton University Press.*

Seiford, L. M. (**1996**)**,** "Data Envelopment Analysis: The Evolution of the State of the Art (1978-1995)", ***Journal of Productivity Analysis*** **7** (**2/3**)**, 99-137**.

Ruggiero, J. (**1998**)**, "**Non-Discretionary Inputs in Data Envelopment Analysis",***European Journal of Operational Research*** **111, 461-469.**

Thrall, R.M. (**1999**)**, "**What is the Economic Meaning of FDH?", ***Journal of Productivity Analysis*** **11, 243-250.**
Tulkens, H. (**1993**)**, "**On FDH Analysis: Some Methodological Issues and Applications to Retail Banking, Courts and Urban Transit", ***Journal of Productivity Analysis*** **4, 183-210.**

Valdmanis, V. (**1992**)**,** "Sensitivity analysis for DEA models", ***Journal of Public Economic*s, vol. 48,185–205**.

Varian, H.R. (1984), "The Non-Parametric Approach to Production Analysis", ***Econometrica* 52, 279-297.**

Varian, H.R. (1985), "Non-Parametric Tests of Optimizing Behavior with Measurement Error", ***Journal of Econometrics 30, 445-458.***

***W*ilson, P.*W*. (1995), "**Detecting Influential Observations in Data Envelopment Analysis", ***Journal of Productivity Analysis* 6, 27-46.**

Zieschang D.Kimberl*y*(1984), "An Extended Farrell Technical Efficiency Measure**", *Journal of Economic Theory* 33, 387-396.**

Zhu, J. (1996), "Robustness of Efficient DMUs in Data Envelopment", ***Analysis. European Journal ofOperational Research 90 451-460.***

APPENDIX

In section 2.2.3 of chapter 2 we introduced the concept of Non-radial Farrell measure of technical efficiency which is obtained after transforming the data set using the concept of the numeraire firm. Proposition 2.3 discussed the properties of the newly proposed measure in light of section 1.3 of chapter 1. One question that may be of interest is that how the radial Farrell measure, the Russell measure and the Zieschang measure of technical efficiency respond to the criteria discussed in section 1.3, when these measures are computed in the transformed data space. Below in this appendix we show that the aforementioned transformation of the data set has no bearing on the computational values and on the properties of these measures.

Let $\boldsymbol{x} \in L(\boldsymbol{y}) \backslash IsoqL(\boldsymbol{y})$ so that the technical efficiency score for $\boldsymbol{x} < 1$ with respect to any efficiency measure.

Proposition A.1. For any given data set we have

a. $FE_I(\boldsymbol{y}, \boldsymbol{x}) = FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$

b. $RE_I(\boldsymbol{y}, \boldsymbol{x}) = RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$

c. $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$

<u>Proof</u>

It has been sown in proposition 2.1, 2.2 and proposition 2.3 that the radial Farrell measure, the Russell measure and the Zieschang measure satisfy the commensurability property. Thus we can write for the Farrell measure, using property E6a $FE_I(\boldsymbol{y}, \boldsymbol{x}) = FE_I(\boldsymbol{\Omega y}, \boldsymbol{\Lambda x})$, where $\boldsymbol{\Omega}$ is a diagonal matrix of M×M the mth diagonal element, ω_{mm} of which is the inverse of the sum of the mth output over the entire sample i.e $\omega_{mm} = 1/\sum^J_j y_{mj}$ for m=1,2…….M and $\boldsymbol{\Lambda}$ is an N×N diagonal matrix with the nth diagonal element $\lambda_{nn} = 1/\sum^{J}_{j}x_{nj}$ for n=1,2……N. But by definition from section 2.2.3 we have $\boldsymbol{\Omega y} = {}_s\boldsymbol{y}$ and $\boldsymbol{\Lambda x} = {}_s\boldsymbol{x}.\Rightarrow FE_I(\boldsymbol{y}, \boldsymbol{x}) = FE_I(\boldsymbol{\Omega y}, \boldsymbol{\Lambda x}) = FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$. Same holds true for the Russell measure and the Zieschang measure. Thus we have $RE_I(\boldsymbol{y},\boldsymbol{x}) = RE_I(\boldsymbol{\Omega y}, \boldsymbol{\Lambda x}) = RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ and $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = ZE_I(\boldsymbol{\Omega y}, \boldsymbol{\Lambda x}) = ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$.

Proposition A.2

a. $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies proposition 2.1

b. $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies proposition 2.2

c. $ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies proposition 2.3

Proofs:

We use an example of the technology from Russell (1988) to prove this proposition. The technology is as described in section 2.2.1 and figure 2.4. Figure A.1 represents the same technology after transformation of the data set into share format as discussed in section 2.2.3.

a. $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies proposition 2.1

Consider data point $({}_s\mathbf{y}, {}_sx_{1k}, {}_sx_{2k}) \in EffL^S({}_s\boldsymbol{y},)$ which implies $FE_I({}_s\mathbf{y}, {}_sx_{1k}, {}_sx_{2k}) = 1$. On the other hand the data point $({}_s\mathbf{y}, {}_sx_{1k}, 1.5{}_sx_{2k}) \in IsoqL^S({}_s\boldsymbol{y},)\backslash EffL^S({}_s\boldsymbol{y},) \Rightarrow FE_I({}_s\mathbf{y}, {}_sx_{1k}, 1.5{}_sx_{2k}) = 1$ thus violating E1a. Also any data point such that $({}_s\mathbf{y}, {}_sx_{1h} > {}_sx_{1k}, {}_sx_{2h} = 1.5{}_sx_{2k})$ is compared to $({}_s\mathbf{y}, {}_sx_{1h} > {}_sx_{1k}, {}_sx_{2h} = {}_sx_{2k}) \in IsoqL^S({}_s\boldsymbol{y},)\backslash EffL^S({}_s\boldsymbol{y},) \Rightarrow$ E2a is violated. Considering data point $({}_s\mathbf{y}, 3/2{}_sx_{1k}, 3/2{}_sx_{2k}) \in L^S({}_s\boldsymbol{y},)\backslash IsoqL^S({}_s\boldsymbol{y},)$, $FE_I({}_s\mathbf{y}, 3/2{}_sx_{1k}, 3/2{}_sx_{2k}) = 2/3FE_I({}_s\mathbf{y}, {}_sx_{1k}, {}_sx_{2k}) = 2/3$ thus satisfying homogeneity of minus one in inputs. Since $FE_I(\boldsymbol{y}, \boldsymbol{x}) = FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ and $FE_I(\nu\boldsymbol{y},\boldsymbol{x}) = \nu FE_I(\boldsymbol{y}, \boldsymbol{x}) \Rightarrow FE_I(\nu{}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \nu RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ implying satisfaction of E4a. $FE_I({}_s\mathbf{y}, {}_sx_{1k}, 1.5{}_sx_{2k}) = FE_I({}_s\mathbf{y}, {}_sx_{1k}, {}_sx_{2k}) = 1 \Rightarrow$ weak monotonicity thus violating E5a. Finally commensurability is ensured because of the transformation and also by definition of $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$.

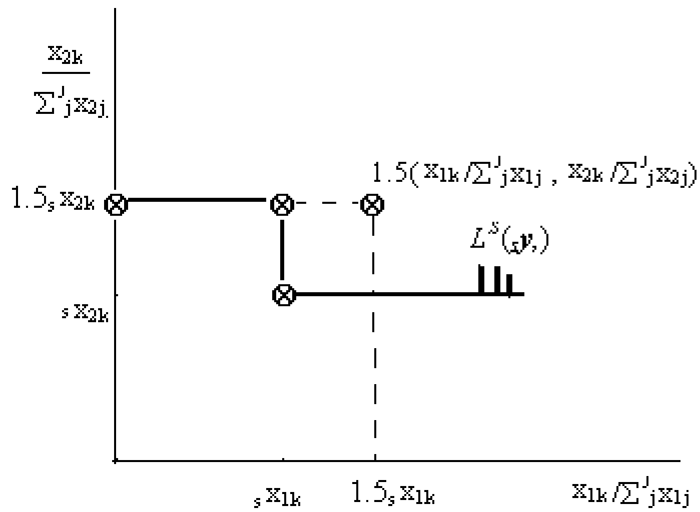

Figure A.1. Comparing efficiency measures in transformed data space.

b. $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies proposition 2.2

For the transformed data space we have

$RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \min \{1/N(\mathbf{e}^T\theta_{FL}\mathbf{e}) \mid \text{for } {}_s\boldsymbol{x} \in L^S({}_s\boldsymbol{y}), \theta_{FL\,s}\boldsymbol{x} \in L^S({}_s\boldsymbol{y}), \theta_{FL} \in Q\}$ where $\mathbf{e}$, θ_{FL} and Q are as defined in section 2.2.1. Then from proposition A1 since we have $RE_I(\boldsymbol{y}, \boldsymbol{x}) = RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ and $RE_I(\boldsymbol{y}, \boldsymbol{x})$ satisfies E1a and E2a so does $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$. $RE_I({}_s\mathbf{y}, {}_sx_{1k}, {}_sx_{2k}) = 1$ and $RE_I({}_s\mathbf{y}, 3/2{}_sx_{1k}, 3/2{}_sx_{2k}) = 1/2 < 2/3 \Rightarrow$ violation of homogeneity of minus one but satisfying sub-homogeneity. Also since we have $RE_I(\boldsymbol{y}, \boldsymbol{x}) = RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ and for constant returns to scale $RE_I(\nu\boldsymbol{y}, \boldsymbol{x}) = \nu RE_I(\boldsymbol{y}, \boldsymbol{x}) \Rightarrow RE_I(\nu{}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \nu RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ thus satisfying homogeneity of plus one in outputs. From fig A1 we have $RE_I({}_s\mathbf{y}, 3/2{}_sx_{1k}, 3/2{}_sx_{2k}) = RE_I({}_s\mathbf{y}, 0, 3/2{}_sx_{2k}) = ½ \Rightarrow$ violation of monotonicity while still satisfying weak monotonicity. For commensurability proof is similar to that given in proposition 2.2

c. $ZE_I({}_s\boldsymbol{y},\ {}_s\boldsymbol{x})$ satisfies proposition 2.3

For any data point ${}_s\mathbf{x} \in L^S({}_s\boldsymbol{y},)$ we $ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \times RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x} FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}))$ Since $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = 1$ when ${}_s\mathbf{x} \in IsoqL^S({}_s\boldsymbol{y},)$ and $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x} \times FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})) =1$ when ${}_s\mathbf{x} \in EffL^S({}_s\boldsymbol{y},) \Rightarrow ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) =1$ iff ${}_s\mathbf{x} \in EffL^S({}_s\boldsymbol{y},)$ thus satisfying E1a. $ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies E2a because $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ satisfies it and $ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ is a product of $FE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ and $RE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$. Thus any data point which is compared to the $IsoqL^S({}_s\boldsymbol{y},)\backslash EffL^S({}_s\boldsymbol{y},)$ by the Farrell measure, in Zieschang computation undergoes a Russell adjustment to ensure that E2a holds. Since $ZE_I(\boldsymbol{y}, \lambda\boldsymbol{x}) = \lambda ZE_I(\boldsymbol{y}, \boldsymbol{x})$ for constant returns to scale and $ZE_I(\boldsymbol{y}, \boldsymbol{x}) = ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x}) \Rightarrow ZE_I({}_s\boldsymbol{y}, \lambda{}_s\boldsymbol{x}) = \lambda\, ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ for constant returns to scale thus satisfying homogeneity of minus one in inputs. Same reasoning implies that $ZE_I(\nu\, {}_s\boldsymbol{y}, {}_s\boldsymbol{x}) = \nu ZE_I({}_s\boldsymbol{y}, {}_s\boldsymbol{x})$ when constant returns to scale are assumed thus satisfying E4a, the homogeneity of plus one in outputs. Violation of E5a, the monotonicity property is evident from figure A.1 where we have for any data points $({}_s\mathbf{y}, {}_s x_{1h} \leq {}_s x_{1k}, {}_s x_{2h} = {}_s x_{2k})$, $ZE_I({}_s\mathbf{y}, {}_s x_{1h} \leq {}_s x_{1k}, {}_s x_{2h} = {}_s x_{2k}) =1/2$. Finally commensurability is established from proposition A1. The above discussion shows that the transformation of the data set as proposed in section 2.2.3 does not affect the magnitudes of the scores across the Farrell, the Russell and the Zieschang measure neither it has any effects on the properties of these measures. It is also clear from section 2.6 that the computational aspects of these three measures, when computed in the transformed data space, are similar to the computational aspects when these three measures are computed in the non-transformed data space. The only difference is that in the transformed data space we are using the transformed variables in place of the original variables in the computational codes. In terms of the computed values of scores, the results are identical by virtue of proposition A.1 above.