Value at Risk for Linear and Non-Linear Derivatives

by

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(Under the direction of John T. Scruggs)

Abstract

This paper examines the question whether standard parametric Value at Risk approaches, such as the delta-normal and the delta-gamma approach and their assumptions are appropriate for derivative positions such as forward and option contracts. The delta-normal and delta-gamma approaches are both methods based on a first-order or on a second-order Taylor series expansion. We will see that the delta-normal method is reliable for linear derivatives although it can lead to significant approximation errors in the case of non-linearity. The delta-gamma method provides a better approximation because this method includes second order-effects. The main problem with delta-gamma methods is the estimation of the quantile of the profit and loss distribution. Empiric results by other authors suggest the use of a Cornish-Fisher expansion. The delta-gamma method when using a Cornish-Fisher expansion provides an approximation which is close to the results calculated by Monte Carlo Simulation methods but is computationally more efficient.

INDEX WORDS: Value at Risk, Variance-Covariance approach,
                 Delta-Normal approach, Delta-Gamma approach, Market Risk,
                 Derivatives, Taylor series expansion.

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LIST OF FIGURES

LIST OF TABLES

Introduction

Financial disasters with derivatives in the early 1990's such as Barings, Metallgesellschaft, Orange County, and Daiwa have led to considerable concern among regulators, senior management, and financial analysts about the trading activities of companies, financial institutions, and banks. The causes for those debacles, among others, range from rogue traders for Barings and Daiwa to misunderstood market risks for Metallgesellschaft and Orange County.

Jorion [12] observes increased volatility in financial markets since the early 1970's. Among many other factors the fact that more and more assets, such as bank loans, became liquid and tradable due to securitization led to more volatile markets. Therefore, controlling market risk became very important for financial and non-financial institutions, as well as regulators.

It can be argued (see [12]) that the development of Value at Risk (VaR) is a direct response to the disasters in the early 1990's. Eventually, the VaR approach became an industry-wide standard for measuring market risk. To determine how the VaR approach is different from other risk measures, Dennis Weatherstone, former chairman of J.P.Morgan, made the comment, "at close of each business day tell me what the market risks are across all businesses and locations." Saunders and Cornett [19, p.235] describe the above comment in the following manner, "what he wants is a single dollar number which tells him J.P. Morgan's market exposure the next business day especially if that day turns out to be a bad day." VaR provides a consistent

and comparable measure of risk denominated in currency units (given a confidence-level and time horizon) across all instruments, business desks, and business lines. Managers think of risk in terms of dollars of loss. That makes a VaR measure more effective, such as communicating the risk exposure to senior management.

Most VaR implementations consider linear exposure to market risks. As we will see later, linear approximations can be inaccurate for non-linear positions such as options. However, the rapid growth of derivative markets reveals that derivatives are widely used for hedging and speculation and non-linear exposures to risks are common characteristics in portfolios. Therefore, it is important to be aware of the implication of non-linear risk exposure. This paper attempts to answer the question of whether standard parametric VaR approaches, such as the delta-normal and the delta-gamma approach are appropriate for derivative positions such as forwards and options.

The outline of the paper is as follows: Chapter 2 presents the underlying risk model which illustrates the set of assumptions about the risk factors as well as derivative pricing models for forwards, futures, and European-style options. Chapter 3 provides an introduction to the VaR methodology. I will also discuss the implicit assumptions of standard VaR models.

Chapter 4 discusses the delta-normal method which is based on a first-order Taylor series expansion. I will conclude that the delta-normal method is reliable for portfolios with linear risk exposures but can lead to significant approximation errors for portfolios with non-linear risk exposures. Non-linear positions such as options will be discussed in chapter 5. The delta-gamma method provides a better approximation because it models second-order effects. Since the relationship between the value of the position and normally distributed risk factors is non-linear, the distribution of the position is non-normal. I will show that this distribution exhibits high skewness and excess kurtosis which makes a VaR calculation using the delta-normal

method inaccurate. The main difficulty with delta-gamma methods is the estimation of the quantile of the profit and loss distribution. Different methods for estimating this quantile are presented including the Cornish-Fisher expansion. In addition, a method which matches the known moments of the distribution to one of a family of distributions known as Johnson distributions and using a central χ^2-distribution to match the first three moments is presented. Empirical results by Pichler and Selitsch [17] suggest the use of a higher moment Cornish-Fisher expansion. Delta-normal approaches are appropriate for linear derivatives and for positions that reveal only weak curvature. The delta-gamma approach offers a significant improvement over the delta-normal method in the case of non-linear derivatives. In particular, the delta-gamma method, when using a Cornish-Fisher expansion, provides an approximation which is close to results calculated by Monte Carlo Simulation methods but is computationally more efficient.

CHAPTER 2

BACKGROUND

2.1 DEFINITION OF RISK

Jorion [12] defines risk as the volatility of unexpected outcomes. Of the many sources of risk a financial institution is exposed to, I will focus on financial risk and, in particular, market risk. According to J.P. Morgan [13, p.2], market risk can be defined as a "risk related to the uncertainty of a financial institution's earnings on its trading portfolio caused by changes in market conditions such as the price of an asset, interest rates, market volatility, and market liquidity." Market risk can occur in terms of interest rate, foreign exchange, and equity price risk. Those risks will be explained in more detail in the following paragraph which is based on Crouhy et al. [5, pp.178].

> *Interest rate risk* is the risk that occurs due to the fact that changes in market interest rates have effects on the value of a fixed-income security. The *equity price risk* can be further split into two sources, systematic risk and idiosyncratic risk. Systematic, or market risk, refers to an sensitivity of an instrument or portfolio value to a change in the level of broad stock market indices. Idiosyncratic, or specific risk refers to that portion of a stock's price volatility that is determined by characteristics specific to that firm. These specific risks could be its line of business, quality of management, or a breakdown in its production process. As a result of

portfolio theory, idiosyncratic risk can be diversified away, while market risk cannot. *Foreign exchange risk* is one of the major risks faced by large multinational companies. It is caused by imperfect correlations in the movement of currency prices and fluctuations in international market interest rates.

To briefly conclude, market risks arise from open, i.e. not hedged or imperfectly hedged, positions.

2.2 THE RISK MODEL

2.2.1 DISTRIBUTION OF RISK FACTORS

At first, I will define a set of assumptions for the risk factors. There will be additional assumptions when discussing the different VaR models which will be explained in chapter 3.

ASSUMPTION 1:

Continously compounded returns on risk factors or proportional changes in risk factors follow a multivariate normal distribution (which is also called a joint-normal distribution).

The following line of argument is based on Crouhy et al. [5]. Prices are assumed to be log-normally distributed, so that log-returns, R_t, during the period $(t-1, t)$ can be defined as

$$R_t = \underbrace{ln\left(\frac{S_t}{S_{t-1}}\right)}_{\text{geometric return}} = ln\left(1 + \frac{S_t - S_{t-1}}{S_{t-1}}\right) \sim \underbrace{\frac{\Delta S_t}{S_{t-1}}}_{\text{arithmetic return}},$$

where S_t (S_{t-1}) is the spot price of a arbitrary asset at time t $(t-1)$, and ΔS_t is the price change during the period, $\Delta S_t = S_t - S_{t-1}$.

This equation contains definitions for both the arithmetic return and the geometric return. According to Dowd [6, p.41], geometric returns are approximately equal to arithmetic returns when dealing with returns over short horizons, such as one day. However, the advantage of using geometric returns is twofold. First, they are more economically meaningful than arithmetic returns. If geometric returns are distributed normally (i.e. prices are log-normally distributed), then the price is non-negative. The second advantage of using geometric returns is that they easily allow extensions into multiple periods which means that returns can simply be added.

It is worth noting that assumption 1 can also be used in a slightly different way. In general, prices or levels of risk factors are assumed to be log-normally distributed. Therefore, log-returns, or in more general terms, continously compounded proportional changes are normally distributed. Bonds or interest rates are one example of how this assumption is used. It depends on whether the bond price or the underlying yield to maturity is assumed to be a risk factor. RiskMetrics suggests the use of bond prices as risk factors, whereas delta-normal or delta-gamma methods usually consider the yield to maturity as a risk factor. This reveals the problem that the price-yield relationship for a bond is convex and therefore needs to be approximated. Therefore, considering bond prices as risk factors mitigates the problem of non-linearity in the payoff function.

ASSUMPTION 2:

Returns between successive time periods are uncorrelated.

This is consistent with the efficient markets hypothesis. An efficient market is a market in which prices fully reflect available information. If so, all price changes must be due to news that, by definition, cannot be anticipated and therefore must be uncorrelated over time. According to Jorion [12], we can therefore assume that prices follow a random walk. In addition, it can reasonably assumed that returns

are identically distributed over time. This assumption is the basis for a convenient property of a VaR measure, i.e. that VaR measures can be easily translated over different time periods.

CRITICAL COMMENT ON THE ASSUMPTIONS:

There is empirical evidence (e.g. Brooks and Persand [3]) that many individual return distributions are not normal but exhibit what is called fat tails or excess kurtosis. That means that these distributions reveal a far higher incidence of large markets movements than is predicted by a normal distribution. To be more precise, fat tails should worry risk managers because they imply that extraordinary losses occur more frequently than a normal distribution would lead them believe. To conclude, a normal distribution is more likely to underestimate the risk of extreme returns, whereas primarily extreme losses are of concern to risk managers.

However, when considering a large and diversified portfolio, there is a statistical theorem that should lower concerns regarding to the assumption of risk factors being normally distributed. The central limit theorem states that the independent random variables of well-behaved distribution will possess a mean that converges, in large samples, to a normal distribution. In practice, this implies that a risk manager can assume that a portfolio has a normal return distribution, provided that the portfolio is well diversified and the risk factor returns are sufficiently independent from each other. Crouhy et al. [5, p.193] argue that there is empirical evidence doubting whether returns are sufficiently independent.

The assumption of returns on assets following a multivariate normal distribution is convenient. This refers to the fact that VaR measures can easily converted over different time horizons and confidence levels.

Nevertheless, some empirical studies suggest the class of Student-t distributions to be more adequate for modelling portfolio returns. Those distributions allow for fat

tails and are fully characterized by the mean μ, the variance σ of the portfolio return, and by a parameter called the degree of freedom (or degree of leptokurtosis), v. As v gets larger the Student-t distribution converges to the normal distribution. Jorion [12] states v to vary in the range 4 and 8. Assuming $v = 5$ and a confidence level of 99%, the VaR is 3.365 times standard deviation instead of 2.326 times standard deviation if the distribution is assumed to be normal. In this case, the effects of fat tails is clearly shown since extreme returns are more likely to happen.

2.3 Derivative Pricing Models

In the following two sections I will focus on pricing models for forward/futures contracts, and for European-style options.

2.3.1 Pricing a Forward Contract

At first, consider a forward or a futures contract on an investment asset. Forward contracts are easier to analyze than futures contracts because there is no daily settlement. However, under certain assumptions it can be argued that the forward and the futures price of an asset are very close to each other when the maturities for the contracts are the same.[1] According to Hull [10], a forward contract obligates the holder to buy (long position) or sell (short position) an asset such as a stock for a predetermined delivery price (the forward price) at a predetermined future date.

The value of a forward contract at the time it is entered is zero. Hull [10] argues that the value of a forward contract at time t, F_t, can be derived from the no-arbitrage

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[1]To be more precise, it can be shown that when the risk-free interest rate is constant and the same for all maturities, the forward price for a contract with a certain delivery date is the same as the futures price for a contract with that delivery date. This argument can be extended to cover situations where the interest rate is a known function of time.

argument as follows:[2]

$$F_t = S_t - F_T \ e^{-r\tau}, \tag{2.1}$$

where S_t is the current price of the security or commodity to be delivered (i.e. spot price), F_T is the delivery price, and r is the continously compounded risk-free interest rate, and $\tau = T - t$ as the time to maturity.

We can see that the price of a forward contract depends on the current spot price, the delivery price, the risk-free interest rate, and the time to maturity. The time to maturity and the forward price are certain which leaves the spot price of the asset and the risk-free interest rate as risk factors. From a risk management perspective, market risk managers are interested in changes of the value of the forward contract when these risk factors change. I will discuss sensitivity measures which measure the exposure of the value of derivatives to changes in their risk factors in section 2.4.

2.3.2 Pricing a European-style Option

The Black-Scholes option pricing model [1] tells us what the price of a European-style option on a non-dividend paying stock should be if it is consistent with a no-arbitrage equilibrium. This is a state that rules out profitable, riskless trades.

Hull [10] summarizes the underlying assumptions of the Black-Scholes model as follows:

1. The stock price, S_t, follows a geometric Brownian Motion.

2. The short selling of securities with full use of proceeds is permitted.

3. There are no transaction costs or taxes. All securities are perfectly divisible.

4. There are no dividends during the life of the derivative.

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[2]I will assume for simplicity that the asset underlying a forward contract provides no known dividend yield. An example is a stock that pays no dividends.

5. There are no riskless arbitrage opportunities.

6. Security trading is continous.

7. The risk-free rate of interest, r, is constant and the same for all maturities.

According to the Black-Scholes Model, the value of an European option on a non-dividend paying stock is dependent on five variables. These variables are the underlying stock price, S_t, the volatility of the underlying stock price, σ, the continously compounded interest rate, r, the time to maturity, τ, and the strike price of the option, X. Therefore, the value V_t of an option at time t is a function $V(\cdot)$ dependent on five variables.

$$V_t = V\ (S_t,\ X,\ \sigma,\ r,\ \tau).$$

The Black-Scholes valuation equation for a European call option on a non-dividend paying stock is then:

$$C_t = S_t\ N(d_1) - X\ e^{-r\ \tau}\ N(d_2), \tag{2.2}$$

where

$$d_1 = \frac{ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad \text{and}$$

$$d_2 = \frac{ln\left(\frac{S_t}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau},$$

and where $N(x)$ is the cumulative distribution function (CDF) for a variable, x, that is normally distributed with a mean of zero and a standard deviation of one. The Black-Scholes equation tells us that the price of a call option should be equal to that of a leveraged position in the underlying stock. This interpretation of the Black-Scholes option pricing equation in particular will be explained in more detail when discussing synthetic options later in this section.

2.4 The Greeks

As discussed in the previous section, both the price of a forward and the price of an option contract depend on various risk factors. As we have seen, the value of a forward contract depends on two risk factors whereas the price of an option contract depend on three risk factors.[3] Sensitivity measures of a derivative's value to the different dimensions of risk are commonly referred to as the "Greeks." Most often the Greeks are thought of only being the sensitivity measure of an option but some of them can also be calculated for other kinds of derivative contracts.

These measures result from a comparative statics analysis. Comparative statics analysis is concerned with measuring the sensitivity of a dependent variable of a function on the exogenous variables. The Greeks are first- or second-order partial derivatives of the value function with respect to each risk factor. They describe how the value of an option changes when the underlying risk factor changes by one unit leaving all else constant (ceteris paribus). For the sake of completeness, I will describe all Greeks for an option in the following paragraphs, though for our purpose I will mainly use delta and gamma. The parameters rho and vega are also important for risk management considerations, whereas theta can be ignored when modelling short time horizons.

The Greeks are "local" risk measures. The term local refers to the fact that they measure the sensitivity of a derivative's value to infinitesimal changes in market rates around current rates. Wilson [22] remarks that these representations of the payoff profile based in local measures may not be sufficient to fully characterize the payoff function for large market events. This aspect will also be discussed in greater depth in this section.

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[3]As will be discussed later, it is questionable if the passage in time must be considered as a "risk" factor. This is due to the fact that the passage in time is certain and theta-effects are small.

2.4.1   Delta

The delta of a derivative, δ, is defined as the rate of change in the value of the derivative with respect to the price of the underlying asset, S_t. In the case of a forward or an option, the delta is the slope of the curve that relates the forward or the option price to the underlying asset price.

The Delta of a Forward Contract

The delta of a forward contract is the first partial derivative of the pricing equation (equation (2.1)) with respect to the price of the underlying asset. Therefore, the delta of a forward contract is,[4]

$$\delta = \frac{\partial F_t}{\partial S_t} = 1.$$

We can see that the delta of a forward contract is always equal to one which is the reason why a forward contract is referred to as a linear derivative. I will discuss the difference between linear and non-linear payoff functions in greater depth in section 2.6.

The Delta of an Option Contract

The delta of an option, δ, is defined as the first partial derivative of the pricing equation (equation (2.2)) with respect to the underlying asset price. The delta for a long position in a call option and a long position in a put option can be expressed as follows:

$$\delta_{CALL} = \frac{\partial V}{\partial S} = N(d_1) = N\left(\frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) \text{ and} \tag{2.3}$$

$$\delta_{PUT} = \frac{\partial V}{\partial S} = N(d_1) - 1 = N\left(\frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) - 1. \tag{2.4}$$

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[4]I will assume the situation that the asset underlying a forward contract provides no known dividend yield.

The delta of a call option can have values between 0 and 1, whereas the delta of a put can have values between -1 and 0. As can be seen in equation (2.3) for a call option, delta depends on the "moneyness" of the option, represented by the fraction $\left(\frac{S_t}{X}\right)$, the time to maturity, τ, the risk-free interest rate, r, and the volatility of the underlying asset, σ. In contrast to the delta of a forward contract, the delta of an option contract changes due to changes in these factors. Theoretically, delta remains constant only for an instant.

An option's delta is the foundation for delta hedging. The idea behind delta hedging is that a position is constructed so that the deltas of the different securities offset each other. A position with a delta of zero is referred to as being delta neutral. As we have seen, delta changes due to parameter changes. Therefore, the hedge has to be adjusted regularly. This is known as rebalancing or dynamic hedging.

Figure 2.1 shows how the parameter delta of a hypothetical European-style call option changes when both the underlying asset price and time to maturity change. The data for this hypothetical option position is summarized in Table 2.1. This example illustrates multiple points including the delta of an option position is close to zero when the option is out-of-the money, whereas the delta is close to one when the option is deep in-the-money. The most important point to be considered is that the delta changes with respect to fluctuations in the variables on which it is based namely time to maturity and the underlying asset price. It can therefore be deduced, whenever delta is used for the purpose of delta hedging or the delta-normal VaR method it is crucial to keep in mind that delta must be updated frequently when parameters change. These changes in delta are larger with the passage of time and, in particular, when the option trades around its exercise price.

However, Figure 2.1 reveals another property of the parameter delta. It can be seen that large changes in delta occur when the current price in the underlying instrument is near to the exercise price. In other words, we should expect to see large

Table 2.1: Data for a Hypothetical Call Option

Parameter	Parameter Value
Strike Price (X)	$100
Volatility of the asset (σ)	40%
Risk-free interest rate (r)	2.5%
Price range of the asset (S_t)	$0 - $200
Time Horizon (τ)	0 - 1 year

changes in delta for small changes in the underlying price when the option is at-the-money. This is the definition of the Greek letter gamma, which will be explained in the next section and describes the second-order effect in the relationship between the value of the option and the asset price.

2.4.2 GAMMA

If changes in the underlying asset price are small, the option's delta provides a good approximation for the change in the option's value. However, when changes in S_t are large, and in particular when gamma is large in absolute terms, delta is highly sensitive to the price of the underlying asset. This is due to the fact that delta is a linear approximation of a non-linear function for options. Therefore, the real change in the option's value might deviate significantly from the amount predicted by a delta approximation. The gamma of a forward contract is always zero since the payoff function is linear in the price of the underlying asset, which means that $\frac{\partial^2 V}{\partial V^2} = 0$.

The gamma, Γ, of an option is the rate of change of the option's delta with respect to the price of the underlying asset. It is the second partial derivative of the
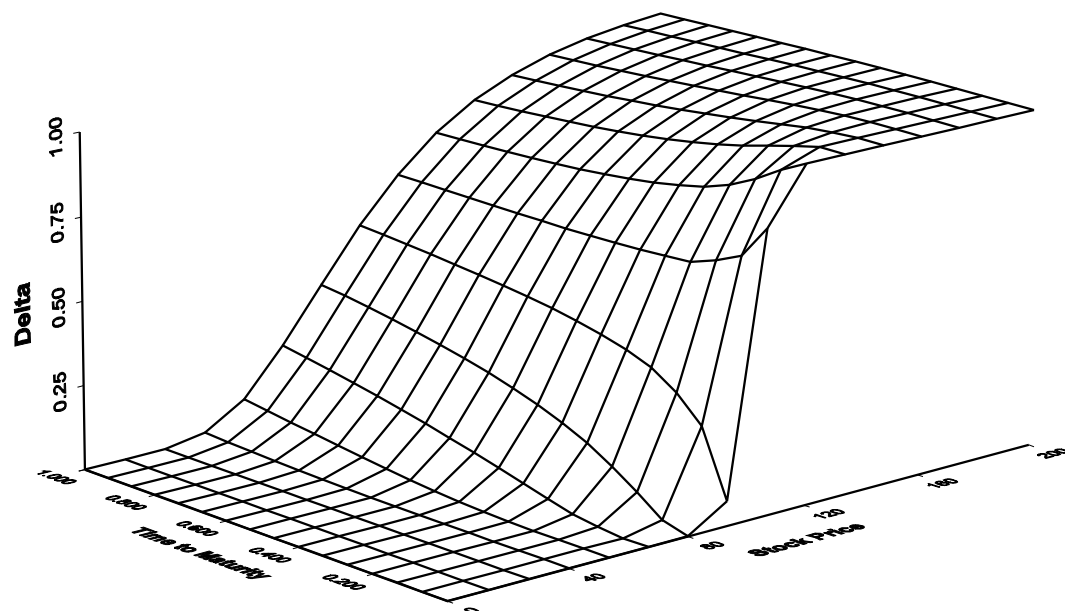
Figure 2.1: The Delta of a Call Option vs. the Asset Price and the Passage of Time

option's value with respect to the asset price. Therefore, gamma measures curvature.

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{N'(d_1)}{S_t \; \sigma \; \sqrt{\tau}}, \tag{2.5}$$

with

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} \; e^{\frac{-d_1^2}{2}},$$

where $N'(\cdot)$ is the probability density function (PDF) of a standard normal distribution. Note that gamma is the same for a call and a put option. The parameter gamma of long positions always has a positive sign whereas gamma for short positions is negative.

As can be seen in equation (2.5), gamma depends on the "moneyness" of the option, represented by the fraction $\left(\frac{S_t}{X}\right)$ in the term d_1, the time to maturity, τ, the market interest rate, r, and the volatility of the underlying asset, σ. As a result, the gamma of an option contract is sensitive to changes in these factors. To illustrate this I will use the same numerical example that was used when discussing delta. The parameters for the hypothetical European call option on a non-dividend paying stock are given in Table 2.1. Table 2.1 also presents the ranges over which the price of the underlying asset and the time to maturity will change.

As we can see in Figure 2.2, gamma reveals a bell shape with respect to the underlying asset price. This is due to the fact that gamma (see equation (2.5)) is a standard normal probability density function of the parameter d_1 scaled by the term $S_t \; \sigma \; \sqrt{\tau}$.

An option tends to have a high gamma when it is trading at-the-money. For an at-the-money option or a close to at-the-money option, gamma increases dramatically as the time to maturity decreases. The figure shows that gamma changes abruptly when the option is near to being an at-the-money option and the time to expiry is close to zero. Short-life at-the-money options have a very high gamma, meaning

that the value of the option holder's position is highly sensitive to jumps in the stock price.

I conclude now that if changes in the underlying asset price are small, the option's delta provides a good approximation for the change in the option's value. However, gamma effects must be considered when changes in S_t are potentially large, and in particular when the option trades close to its strike price or at-the-money. Otherwise, the approximation error can be significant. This is an argument against delta hedging for assets with non-linear risk exposure. Due to the fact that delta hedging focuses only on first-order effects, second-order effects are left out. Therefore, a delta-neutral position is not insensitive to large price changes in the underlying asset.

2.4.3 THETA

The theta of an option, Θ, is the rate of change in the option's value with respect to the passage of time. Theta is sometimes referred to as the time decay of the portfolio. Theta is not considered a risk factor since there is no uncertainty about the passage of time. However, theta depends on other risk factors. When the time horizon is short, such as one day, many authors (e.g. Hull [10]) argue that theta can be ignored because changes in the option's value due to a theta effect are small.

$$\Theta_{CALL} = \frac{\partial V}{\partial \tau} = -\frac{S_t \ N'(d_1) \ \sigma}{2 \ \sqrt{\tau}} - r \ X \ e^{-r\tau} \ N(d_2). \tag{2.6}$$

$$\Theta_{PUT} = \frac{\partial V}{\partial \tau} = -\frac{S_t \ N'(d_1) \ \sigma}{2 \ \sqrt{\tau}} + r \ X \ e^{-r\tau} \ N(d_2).$$

2.4.4 VEGA

The Black-Scholes option pricing model implicitly assumes that the volatility of the underlying asset is constant. In practice, volatilities change over time. That means that the value of an option is liable to change because of the movement in volatility.
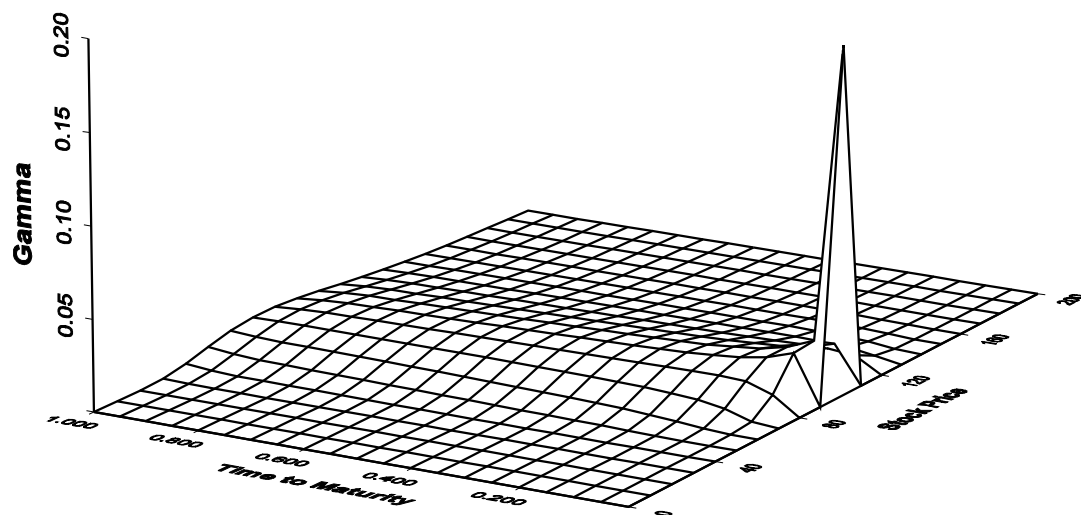
Figure 2.2: The Gamma of an Option vs. the Asset Price and the Passage of Time.

The vega of a single European-style put or call option is the rate of change if the value of the single option or portfolio changes with respect to the volatility of the underlying asset.

$$\vartheta = \frac{\partial V}{\partial \sigma} = S_t \ \sqrt{\tau} \ N'(d_1), \tag{2.7}$$

where $N'(d_1)$ is the probability density function (PDF) for a standard normal variable (see equation (2.6)).

2.4.5 Rho

The rho of an option is the rate of change in its value with respect to the risk-free interest rate, r:

$$\rho_{CALL} = \frac{\partial V}{\partial r} = X \ \tau \ e^{-r\tau} \ N(d_2). \tag{2.8}$$

$$\rho_{PUT} = \frac{\partial V}{\partial r} = -X \ \tau \ e^{-r\tau} \ N(-d_2). \tag{2.9}$$

As we have seen in equation (2.1), the price of a forward contract is also sensitive to changes in the market interest rate. The rho for a forward contract is:

$$\rho = \frac{\partial F_t}{\partial r} = -\tau \ F_T \ e^{-r\tau}.$$

Therefore, he parameter rho, ρ, describes the sensitivity of the forward price with respect to changes in the risk-free rate for this maturity.

2.5 Synthetic Options

2.5.1 Creation of Synthetic Options

Based on the Black-Scholes option pricing model, European-style options can be created synthetically. The components of such a synthetic option are called building

blocks and the whole process of replicating an option with buildings blocks, according to a certain model, is called reverse-engineering.

As mentioned, when dealing with a standard European-style call option on a non-dividend paying stock, the Black-Scholes model can be used to create an equivalent synthetic option. Recall, the Black-Scholes formula:

$$C_t = S_t \underbrace{N(d_1)}_{\delta} - \underbrace{X \; e^{-r\tau} \; N(d_2)}_{short \; position \; in \; discount-bond}$$

The synthetic option position can be constructed by buying δ shares of the underlying stock at price S_t. The stock purchase is financed by borrowing $(X \; e^{-r\tau} \; N(d_2))$ dollars. Alternatively, this could also be seen as shorting a discount-bond with a present value of $(X \; e^{-r\tau} \; N(d_2))$ and a face value $(X \; N(d_2))$. This bond matures in T and hence, has a duration of τ.

Once, this is done, one can then calculate a VaR figure for the equity and bond positions separately, and then the combination of the two gives us our mapped option position. The only novel point to watch here is that the components of the synthetic option will change with changes in the underlying price - and indeed, other factors as well - so the mapping itself needs to be regularly updated to reflect current market conditions. Mapping options is therefore a dynamic process. As we have seen when discussing the Greeks of an option, an option's delta changes with respect to changes in the price of the underlying, the passage of time, and interest rates. Changes in delta can differ severely in magnitude depending on the "moneyness" of the option, i.e. with respect to the price of the underlying asset.

Hull [10, Ch.13] remarks that the position in the stock and the option is risk-less for only a very short period of time. Theoretically, it remains only risk-less for an instant. To remain risk-less it must be adjusted or rebalanced frequently.

2.5.2 PORTFOLIO INSURANCE

According to Hull [10], portfolio insurance is defined as entering into trades to ensure that the value of a portfolio will not fall below a certain level. The value of a long portfolio of stocks can be insured against dropping below a certain level by simultaneously investing in put options. An alternative to buying an option is creating it synthetically. This strategy involves taking a time-varying position in the underlying asset such that the delta of the position is equal to the delta of the required option. Creating a synthetic put option implies that at any given time a proportion δ_{PUT} (see equation (2.4)) of the stock has been sold and the proceeds invested in risk-less assets.

As we have seen in the previous section, delta is not constant. This means that the synthetic option position has to be rebalanced frequently. Hull [10] remarks that creating a put option synthetically does not work well if the volatility of the index changes rapidly or if the index exhibits large jumps. An example of this is the stock market crash in 1987. On Monday, October 19, 1987, the Dow Jones Industrial Average Index dropped by over 500 points and portfolio managers who had chosen to insure their portfolio with synthetic put options found that they were unable to sell either stocks or index futures fast enough to protect the position. I will mention aspects of market liquidity again when discussing the implicit assumptions of standard VaR models.

As we have seen, the creation of synthetic options is based on the option's delta. Such a decomposition of an option into a risk-less position in a discount-bond and a short position in the underlying asset can also be used for modelling and calculating a VaR figure. We will see this and the limitations when discussing the delta-normal approach.

2.6 Payoff Functions

VaR is a measure for market risk. It requires an understanding of how a position's value changes when its underlying risk factors change. Whereas RiskMetrics [13, p.123] suggests a classification of positions into three different categories, namely simple linear positions, linear derivative positions, and non-linear positions, I will here categorize positions only as linear or non-linear.

Linear positions include long or short positions in underlying assets such as stocks or foreign currencies or long and short positions in linear derivative contracts such as a forward contract on a stock or on a foreign currency. Non-linear positions include an option on a stock or on a foreign currency.

Figure 2.3 illustrates these two types of positions and, in particular, how the value of a position varies with price changes of the underlying asset. This figure is based on a numerical example that contrasts the payoff function of a long call option on an asset (a non-linear exposure) with the payoff function of a long position in a forward contract (a linear exposure) on the same asset with the same time to maturity. The specific data for both positions is summarized in Table 2.2.

The straight line in Figure 2.3 represents a linear relationship (i.e. $\delta = 1$) between the position's price and underlying asset. This line represents the payoff function of the forward contract. We can see that a change in value of such a position can be expressed in terms of the delta, i.e. the slope, of the underlying security.

The other line represents the payoff function of the call option. This line clearly represents a non-linear relationship between the position's value and the underlying security. The payoff line is curved such that the position's value can change dramatically as the value of the underlying asset increases. We can see that when measuring the sensitivity of an option contract by its delta, this is only a good approximation

Table 2.2: Hypothetical Call and Forward Data.

Position	Parameter	Parameter Value
Long Call Option	Current Asset Price (S_t)	$100
	Strike Price (X)	$100
	Volatility of the Asset (σ)	40%
	Risk-free interest rate (r)	2.5%
	Time Horizon (τ)	45 days
Long Forward Contract	Current Asset Price (S_t)	$100
	Forward Price (F_T)	$100
	Risk-free interest rate (r)	2.5%
	Time Horizon (τ)	45 days

for small changes. When discussing the Greeks of options, the curvature of the line is quantified by, and referred to, as the parameter gamma.

Table 2.3 summarizes the points already made in this section and gives examples for both categories of payoff functions. The table is partly based on RiskMetrics Technical Document [13, p.124]. It is worth noting that there are two approaches to modelling interest rate securities, such as bonds or interest rate swaps. As can be seen in the table 2.3, one way to model interest rate securities is to consider the price of the bond or the swap price as the underlying risk factor. In this case, the relationship between the position's value and the underlying price is clearly linear. Another way is to consider a representative market interest rate as the underling risk factor. In this case, the relationship between a bond price and its yield to maturity is convex or clearly non-linear. For convenience, fixed-income securities are usually modelled using representative bond prices, and not yields, as risk factors to avoid a non-linear relationship which would be more difficult to model.
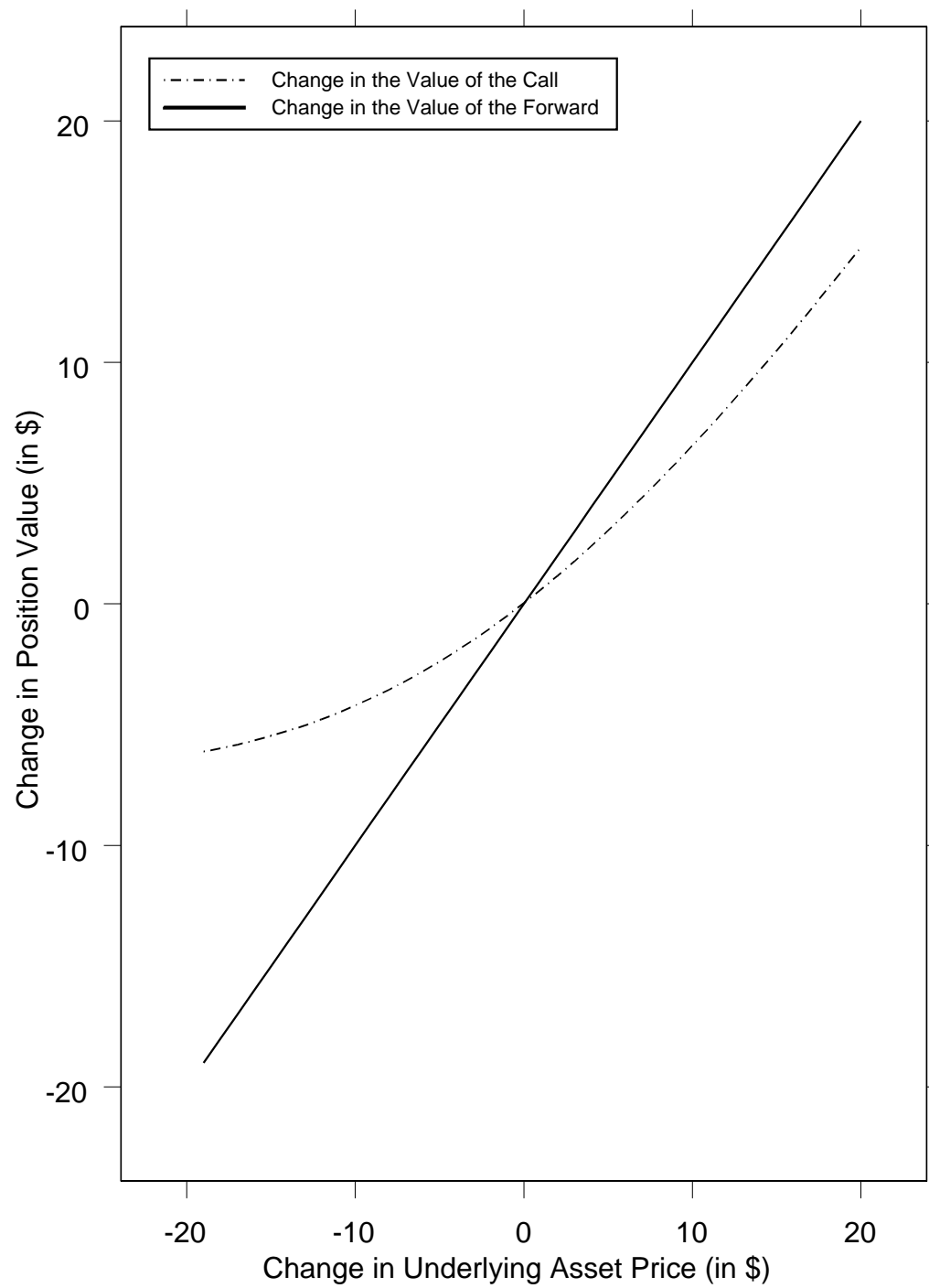
Figure 2.3: Linear and Non-Linear Payoff Functions.

Table 2.3: Relationship between Instrument and Underlying Price or Rate.

Type of Position	Security	Underlying Price/Rate
Linear	Stock	Stock Price
	Commodity	Commodity Price
	Foreign Exchange (FX)	FX Rate
	Bond	Bond Price
	Interest Rate Swap	Swap Price
	FX Forward	FX Rate
	Floating Rate Note	Money Market Price
	Forward Rate Agreement	Money Market Price
	Currency Swap	Swap Price/FX Rate
Non-Linear	Stock Option	Stock Price
	Bond Option	Bond Price
	FX Option	FX Rate

CHAPTER 3

THE VARIANCE-COVARIANCE VALUE AT RISK APPROACH

3.1 DEFINITION OF VALUE AT RISK

Jorion [12, p.22] defines VaR as "the worst expected loss over a given horizon under normal market conditions at a given probability level known as confidence level." In more formal terms, VaR describes the quantile of the projected distribution of asset or portfolio (dollar) returns over a target horizon. A target horizon could be a single day or a 10 day period for the purpose of regulatory capital reporting. If c was the selected confidence level, VaR corresponds to the $1 - c$ lower-tail level.

According to Crouhy et al. [5, p.187], VaR offers a probability statement about the potential change in the value of a single position or an entire portfolio resulting from a change in market factors over a specific period of time. It does not state, however, how much actual losses will exceed the VaR measure. Instead, it states how likely it is that the VaR measure will be exceeded.

There are two interpretations of VaR:

1. VaR = Expected profit/loss – worst case loss at a given confidence level c

2. VaR' = Worst case loss at a given confidence level c.[1]

To be more precise, the VaR over a horizon H and a given confidence level c can be expressed as:

$$\text{VaR}(H;c) = -\alpha_{1-c} \, \sigma \, V = -\alpha_{1-c} \, \sigma(\Delta V), \tag{3.1}$$

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[1]VaR' is known as the "absolute Value at Risk"

26

where the VaR measure is either calculated with σ, which is the volatility of returns, or with $\sigma(\Delta V)$, which is the volatility of dollar returns. The expression $\sigma(\Delta V) = \sigma V$ defines the relationship between both volatility definitions.

$$\text{VaR}'(H; c) = -(\alpha_{1-c}\, \sigma + \mu)\, V = -\alpha_{1-c}\, \sigma(\Delta V) + \mu\, V, \tag{3.2}$$

where the parameter α_{1-c} denotes the $1 - c$ quantile of the assumed distribution of returns, μ is the expected return, and V is the marked-to-market value of the position.

Therefore, the α_{1-c} quantile of the distribution fulfils the condition that the probability of an outcome x which is less than or equal to α_{1-c} is $1 - c$. More formally this can be expressed as:

$$\Pr(x \leq \alpha_{1-c}) = 1 - c$$

$$\Leftrightarrow N(\alpha_{1-c}) = 1 - c = 1\%,$$

where $N(\cdot)$ denotes a cumulative distribution function (CDF). Since we are interested in α_{1-c}, we need the inverse of the CDF, $N^{-1}(\cdot)$, which leads to the following expression:

$$\Leftrightarrow \alpha_{1-c} = N^{-1}(1 - c) = N^{-1}(1\%) = -2.326.$$

Therefore, the $\alpha_{1\%}$ quantile of a standard normal distribution is -2.326.

The relative VaR definition and the absolute VaR definition lead to the same result when the expected profit or loss of a position, μ, is zero. This assumption is usually made that over a short period of time, such as one day, the returns of risk factors follow a multivariate normal distribution with mean zero and a volatility of σ. As we will see later, these two VaR definitions might differ when dealing with positions that reveal a non-linear relationship to changes in their underlying risk factors. This might be the case even though returns in the underlying risk factors are normally distributed with mean zero. An example for this is an option position.

What we can see so far is that the VaR measure is equal to the standard deviation σ of a position or a portfolio, multiplied by the value of the position or portfolio V, and by the $-\alpha$ quantile of the distribution. Therefore, the term σV provides the standard deviation of the profit and loss distribution and the quantile is an alternative means of quantifying the distribution.

In these general definitions the $-\alpha$ quantile is not related to a specific distribution. This means that for computing the VaR measure of a specific position the $-\alpha$ quantile can be derived from a (standard) normal distribution or a Student-t distribution with a certain degree of freedom.

I conclude that the task of deriving a VaR figure for a single position or a complex portfolio can be scaled down to the problem of deriving a probability distribution of returns or deviations of the position or the portfolio.

3.2 Implicit Value at Risk Assumptions

In this section I will present some implicit assumptions of standard VaR models. VaR models rely on market prices. Standard VaR models implicitly assume perfect and frictionless capital markets. In particular, these models assume that markets remain liquid at all times and therefore, assets can be liquidated at prevailing market prices. This means that these models assume that market prices are achievable transaction prices. The risk that the liquidation value of a asset differs significantly from the current mark-to-market value is called liquidity risk. This risk depends on the prevailing market conditions. If assets are traded in deep markets most positions can be liquidated easily with little price impact. In the case of thin markets, such as exotic over-the-counter (OTC) derivatives markets, any transaction can quickly affect prices. In times of market distress liquidity can dry up quickly. This can lead to situations in which positions cannot be sold at all or to dramatically deflated prices.

Jorion [12, p.339] concludes that this marking-to-market approach is adequate to quantify and control risk but is questionable if VaR is supposed to represent the worst loss over a liquidation period.

VaR models are based on volatility and correlation estimates of risk factors in order to aggregate diverse risky positions. An implicit assumption of standard VaR models is that volatilities and covariances are constant throughout the forecast period. The problem is that volatility and correlation maybe highly unstable over time. However, it should be noted that when volatilities and correlations are estimated under normal market conditions, these estimates may be not reliable in times of market distress. Brooks and Persand [3] found that correlations between markets increase when volatility is high. In conclusion, VaR estimates based on normal market conditions may seriously underestimate the riskiness of a portfolio when the volatilities and correlations of risk factors change.

3.3 Value at Risk for Single and Multi Assets

In section 2.6, I separated securities into two different categories, namely linear and non-linear positions. The variance-covariance approach can be applied to linear positions, such as stocks and foreign currencies. For a more detailed list of securities which belong to this category refer to Table 2.3.

Implicitly, we have already seen the equation for calculating a VaR measure for a single position. An example would be a stock whose value depends on one risk factor, i.e. the stock price. Equation 3.1 is the formal expression for calculating a VaR measure in a single asset case.

Since a financial institution's trading portfolio usually consists of a huge number of different assets including positions in equities, bonds, and foreign currencies, it is therefore essential to know how to derive a VaR measure for an entire portfolio.

It is worth noting that the variance-covariance approach considers risk factors in terms of asset prices rather than in terms of underlying market rates, such as bond yields. This is one of the differences between the variance-covariance approach the delta-normal method which will be presented in the next section. The delta-normal method considers a portfolio of assets in terms of their underlying risk factors rather than in terms of their assets. This requires that assets which depend on more than one risk factor must be decomposed into their underlying risk factors.

However, one could model a single position as a portfolio of risk factors. An example would be an U.S. investor holding a long position in a bond issued by a foreign government and denominated in a foreign (non-dollar) currency. In this case, the value of the position is exposed to both interest rate risk and foreign exchange risk. In order to calculate the VaR of that position, correlation effects between the two risk factors would have to be considered in addition to the volatilities of the two risk factors. Here, I will consider a portfolio of risk factors in terms of its assets.

The variance-covariance approach is an extension of the portfolio model by Markowitz [15]. As long as market factors are not perfectly positively correlated assets, the risk of a portfolio of assets is not just the sum of the risks of every asset in portfolio but reveals diversification benefits (see Crouhy et al. [5, p.99]). The variance-covariance approach assumes that the distribution of changes in the portfolio value is multivariate normally distributed (MVN). The multivariate normal distribution is completely characterized by its first two moments, the mean vector and the covariance matrix. These can be estimated from historical data. Another assumption is that the value of a position has a linear relationship to the underlying risk factor (i.e. stocks, foreign currencies, futures, or forward contracts).

Dowd [6, p.63] briefly summarizes the two assumptions of the variance-covariance approach as follows:

1. Returns follow a multivariate normal distribution.[2]

2. There is a linear relationship among the value of a position or portfolio and its underlying risk factors, i.e. their prices.

To illustrate this approach, I will assume a portfolio with current marked-to-market value V. Changes in value of the portfolio, ΔV, depend on deviations in K underlying risk factors, Δf_k $(k = 1, \ldots, K)$.[3] As mentioned, one could also think of a single position which is exposed to changes in several underlying risk factors. A change in value, ΔV, can then be defined as

$$\Delta V = \sum_{k=1}^{K} \Delta f_k. \tag{3.3}$$

We can now define Δx_k as the proportional change in risk factor k[4] so that

$$\Delta x_k = \frac{\Delta f_k}{f_k}.$$

Then substitute the definition of Δx_k into equation (3.3) which leads to

$$\Delta V = \sum_{k=1}^{K} f_k \, \frac{\Delta f_k}{f_k} = \sum_{k=1}^{K} f_k \, \Delta x_k$$

I will now derive the volatility of ΔV, $\sigma(\Delta V)$, under the assumption that risk factors are log-normal distributed and therefore, log-changes follow a multivariate normal distribution as follows:

$$\sigma(\Delta V) = \sqrt{\sum_{k=1}^{K} f_k^2 \, \sigma^2(\Delta x_k) + \sum_{j=1}^{K} \sum_{h=1;j\neq h}^{K} f_j \, f_h \, cov(\Delta x_j, \Delta x_h)}, \tag{3.4}$$

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[2]To be more precise, the assumption is made that prices are log-normally distributed and therefore log-returns follow a normal distribution.

[3]Usually, K refers to the number of risk factors. Here, the risk factor is considered to be the price of the asset. Therefore, if we have N assets $K = N$.

[4]If the risk factor k is for instance a stock, Δx_k can be interpreted as the rate of return provided by the stock over a time horizon, H.

where $\sigma(\Delta x_k)$ represents the volatility of proportional changes in risk factor k and $cov(\Delta x_j, \Delta x_h)$ is the covariance[5] between changes in the risk factors j and h. Now, replacing $cov(\Delta x_j, \Delta x_h)$ by

$$cov(\Delta x_j, \Delta x_h) = \rho_{(x_j;x_h)}\ \sigma(\Delta x_j)\ \sigma(\Delta x_h),$$

where $\rho_{(x_j;x_h)}$ is the correlation coefficient[6] between the proportional changes in risk factor j and h. This leads to the following expression:

$$\sigma(\Delta V) = \sqrt{\sum_{k=1}^{K} f_k^2\ \sigma^2(\Delta x_k) + \sum_{j=1}^{K} \sum_{h=1;j\neq h}^{K} f_j\ f_h\ \rho_{(\Delta x_j;\Delta x_h)}\ \sigma(\Delta x_j)\ \sigma(\Delta x_h)}. \quad (3.5)$$

Equation 3.5 looks rather complex and can be made more concise by using matrix notation. I define $\mathbf{f}$ as a $K \times 1$ column vector containing the current value or level of each risk factor k, f_k, and $\boldsymbol{\Sigma}$ as a $K \times K$ covariance matrix for the risk factors. Therefore,

$$\sigma(\Delta V) = \sqrt{\mathbf{f'}\ \boldsymbol{\Sigma}\ \mathbf{f}}, \quad (3.6)$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2(\Delta x_1) & cov(\Delta x_1, \Delta x_2) & \dots & cov(\Delta x_1, \Delta x_K) \\ cov(\Delta x_2, \Delta x_1) & \sigma^2(\Delta x_2) & \dots & cov(\Delta x_2, \Delta x_K) \\ \vdots & \vdots & \ddots & \vdots \\ cov(\Delta x_K, \Delta x_1) & \dots & \dots & \sigma^2(\Delta x_K) \end{bmatrix} \text{ and } \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_K \end{bmatrix}.$$

Equation (3.6) is the volatility of changes in a portfolio's value. We can calculate the VaR measure using equation (3.1) which multiplies the volatility estimate just

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[5]Covariance is a measure of of the degree to which returns on two risky assets move in tandem. A positive covariance means the asset returns move together. A negative covariance means they vary inversely.

[6]The correlation coefficient is a scale-free measure of linear dependence and lies in a range between -1 to $+1$. When equal to $+1$ (-1), the two variables are said to be perfectly positively (negatively) correlated. When 0, the variables are uncorrelated.

derived with the quantile of the distribution. Alternatively, we can rearrange equation (3.5). The following transformation is based on Jorion [12, p.152].

Before substituting the formal definitions of VaR, mentioned earlier, into equation (3.5), they will be restated below and solved for the volatility $\sigma(\cdot)$:

$$\sigma(\Delta V) = \frac{\text{VaR}_p(H;c)}{-\alpha_{1-c}}$$

and

$$\sigma(\Delta x_k) = \frac{\text{VaR}_k(H;c)}{-\alpha_{1-c}}.$$

As mentioned, substituting these rearranged VaR definitions into equation (3.5) leads to

$$\text{VaR}_p = \sqrt{\sum_{k=1}^{K} \text{VaR}_k^2 + \sum_{j=1}^{K} \sum_{h=1;j\neq h}^{K} \rho_{(\Delta x_j;\Delta x_h)} \text{VaR}_j \text{VaR}_h}. \qquad (3.7)$$

It is possible to write equation (3.7) in a more convenient form using matrix notation. Assume that a portfolio consists of K different risk factors, then the VaR measure for the entire portfolio $\text{VaR}_p(H,c)$ can be calculated as follows:

$$\text{VaR}_p(H,c) = \sqrt{\text{VaR}' \, \rho \, \text{VaR}} \qquad (3.8)$$

where ρ is a symmetric $K \times K$ correlation matrix among the risk factors and VaR is a $K \times 1$ column vector consisting of individual VaR measures, VaR_k for each position in risk factor k.[7]

$$\rho = \begin{bmatrix} \rho_{(\Delta x_1;\Delta x_1)} & \rho_{(\Delta x_1;\Delta x_2)} & \cdots & \rho_{(\Delta x_1;\Delta x_K)} \\ \rho_{(\Delta x_2;\Delta x_1)} & \rho_{(\Delta x_2;\Delta x_2)} & \cdots & \rho_{(\Delta x_2;\Delta x_K)} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{(\Delta x_K;\Delta x_1)} & \cdots & \cdots & \rho_{(\Delta x_K;\Delta x_K)} \end{bmatrix} \quad \text{and} \quad \text{VaR} = \begin{bmatrix} \text{VaR}_1 \\ \text{VaR}_2 \\ \vdots \\ \text{VaR}_K \end{bmatrix}$$

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[7]All VaR measures must be calculated based on the same time horizon and the same confidence level.

It is obvious that the portfolio VaR takes diversification benefits into account due to the correlation coefficient $\rho_{(\Delta x_j; \Delta x_h)}$. As mentioned, the correlation coefficient is a scale-free measure of linear dependence which lies in a range between -1 to $+1$. When equal to $+1$ (-1), the two variables are said to be perfectly positively (negatively) correlated. When 0, the variables are uncorrelated.

The following three cases show how correlation affect the VaR measure of a portfolio. For the sake of simplicity, assume that the portfolio consists of two different assets which could be a portfolio of long positions in two stocks. For the sake of simplicity, short sales are not considered at this point.

$\rho_{(\Delta x_1; \Delta x_2)} = +1$: Perfectly positively correlated returns yield to $\text{VaR}_p = \text{VaR}_1 + \text{VaR}_2$. This shows that the portfolio VaR is equal to the sum of the individual VaR measures, meaning that there are no diversification benefits.

$\rho_{(\Delta x_1; \Delta x_2)} = 0$: Uncorrelated returns lead to $\text{VaR}_p = [\text{VaR}_1^2 + \text{VaR}_2^2]^{\frac{1}{2}}$. It is apparent that this VaR measure is less than the sum of the individual measures. This shows that diversification benefits reduce the exposure to market risks and the result reflects the fact that with assets that move independently, a portfolio is less risky than either asset.

$\rho_{(\Delta x_1; \Delta x_2)} = -1$: Perfectly negatively correlated returns lead to $\text{VaR}_p = [\text{VaR}_1 - \text{VaR}_2]$. It is obvious that this VaR measure has the smallest value among the three constellations. The two assets move inversely, and in the case that the two VaR measures have the same value, the VaR measure can be reduced to zero. This proves that, in this theoretical framework, the market risk exposure is zero.

CHAPTER 4

PARAMETRIC LINEAR APPROXIMATION VALUE AT RISK MODELS

4.1 THE DELTA-NORMAL APPROACH

4.1.1 INTRODUCTION TO DELTA-NORMAL METHODS

As we saw, the basic variance-covariance approach assumes that position's values reveal a linear relationship to their underlying risk factors. This assumptions holds for instance for stocks and positions in foreign currencies as well as for linear derivatives such as futures and forward contracts. Therefore, the variance-covariance approach is clearly appropriate for securities that exhibit such a relationship, and when changes in their underlying risk factor follow a normal distribution. The variance-covariance approach considers risk factors in terms of asset prices rather than in terms of underlying market rates, such as bond yields. The delta-normal method considers a portfolio of assets rather in terms of their risk factors than in terms of their assets. This requires that assets which depend on more than one risk factor must be split off into their underlying risk factors.[1]

I will now turn to positions belonging to the category of non-linear payoff functions. Examples for securities that reveal such a non-linear relationship are options, callable or convertible bonds, and mortgage-backed securities.

The idea and assumption behind the delta-normal approach is that any kind of relationship between changes in the position's value and the risk factors can be

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[1]Example two on page 46 illustrates the decomposition of a forward contract on a foreign currency into its risk factors.

sufficiently approximated by a first-order derivative with respect to changes in the risk factor. To be more precise, the approximation will be accomplished by using a first-order Taylor series expansion based on first-order partial derivatives of the value of the position with respect to its underlying risk factors. Such a first-order partial derivative can be interpreted as a sensitivity measure of the value of the position with respect to changes in the underlying risk factors, such as a delta of a derivative as was illustrated in chapter 2. Changes in the value of a position can then be described in terms of the underlying risk factors.

The delta-normal method can be seen as an extension of the basic variance-covariance approach. Alternatively, the variance-covariance approach can be seen as a special case of the delta-normal approach where the delta is equal to one ($\delta = 1$) and the risk factors are assets. Both approaches lead to exactly the same result when considering a linear relationship, where the first-order derivative is constant. As will be illustrated in this section, the delta-normal approximation is reliable for linear payoff functions but can lead to significant approximation errors when dealing with curved payoff functions. In the latter case, the idea is to replace the non-linear relation between asset values and underlying rates and prices, i.e. the risk factors, with a linear approximation based on the asset's delta.

The basic ideas of the delta-normal approach can be expressed mathematically as follows: If a financial instrument has an equilibrium value V which can be expressed as a function of K risk factors then the change in value, ΔV, can be approximated by a first-order Taylor series expansion of the pricing equation as follows:

$$\Delta V = \sum_{k=1}^{K} \frac{\partial V}{\partial f_k} \, \Delta f_k + \epsilon_k(1) = \sum_{k=1}^{K} \delta_k \, \Delta f_k + \epsilon_k(1). \tag{4.1}$$

The first-order partial derivative of the value function V with respect to the risk factor f_k, $\frac{\partial V}{\partial f_k}$, can be viewed as a sensitivity measure of the value of the financial instrument to changes or shocks in underlying risk factors and is usually referred to

as δ_k. Depending on the payoff function, the first-order approximation error, $\epsilon_k(1)$ is zero in the case of a linear payoff function or greater than zero in the case of a non-linear payoff function.

4.1.2 Delta-Normal Value at Risk for one Risk Factor

The least complex case is a single position that depends only on one ($K = 1$) risk factor. Hence, equation (4.1) becomes

$$\Delta V = \frac{\partial V}{\partial f_1} \, \Delta f_1 + \epsilon_1(1) = \delta_1 \, \Delta f_1 + \epsilon_1(1), \tag{4.2}$$

where, again, ΔV is a change in value of the position V, δ_1 is the rate of change of the position V with respect to the underlying risk factor f_1, and Δf_1 is the absolute change in the underlying risk factor. Further, I define Δx_1 as the proportional change in the underlying risk factor in one day so that

$$\Delta x_1 = \frac{\Delta f_1}{f_1}.$$

Substituting Δx_1 in equation (4.2) and omitting, for the sake of simplicity, the first order approximation error, $\epsilon_1(1)$, leads to

$$\Delta V \approx f_1 \, \delta_1 \, \Delta x_1.$$

The volatility of ΔV is then

$$\sigma(\Delta V) = \sigma \left(f_1 \, \delta_1 \, \Delta x_1 \right) = f_1 \, \delta_1 \, \sigma(\Delta x_1). \tag{4.3}$$

Once the standard deviation is derived, the calculation of the VaR over a holding period H and to a confidence level c is straightforward and can be obtained by substituting equation (4.3) into the formal VaR definition presented in equation (3.1) which leads to

$$\text{VaR}(H; c) = -\alpha_{1-c} \, f_1 \, \delta_1 \, \sigma(\Delta x_1).$$

Comparing this final result to the VaR expression I derived earlier for the simple variance-covariance approach, we can see that those two definitions differ in one parameter. This difference is the sensitivity measure, δ_k, which measures the contribution of a risk factor to the overall risk of a position. If it is now assumed that this approximation is equal to one which is the case when considering a single stock, a forward, or a futures contract, the result of the delta-normal approach is the same as the variance-covariance approach. It is now apparent that the variance-covariance approach is a special case of the delta-normal approach.

Example one on page 45 illustrates this method for a long position in a European-style call option on a non-dividend paying stock. In the next section, I will discuss the case when a position depends on more than one risk factor.

4.1.3   Delta-Normal Value at Risk for K Risk Factors

After discussing the case of a position which depends on one $(K = 1)$ risk factor, I will now generalize the discussion to any type of financial instrument, for which the equilibrium value, V, can be expressed as a function of K risk factors, f_k, $k = 1, \ldots, K$. Recall equation (4.1):

$$\Delta V = \sum_{k=1}^{K} \frac{\partial V}{\partial f_k} \, \Delta f_k + \epsilon_k(1) = \sum_{k=1}^{K} \delta_k \, \Delta f_k + \epsilon_k(1).$$

Defining Δx_k as the proportional change in risk factor k,

$$\Delta x_k = \frac{\Delta f_k}{f_k},$$

and substituting Δx_k in equation (4.1) leads to

$$\Delta V = \sum_{k=1}^{K} f_k \, \delta_k \, \Delta x_k \qquad (4.4)$$

As could be seen when deriving the volatility for the simple variance-covariance approach in equation (3.5), the expression becomes quite complex. The following transformations can be made more concise when using matrix notation.

I now define $\mathbf{R}$ as a $K \times 1$ column vector of proportional changes in the risk factors and δ as a $K \times 1$ column vector containing in each row the product of the current price or rate of risk factor k, f_k, and the sensitivity measure with respect to each risk factor k. More formally,

$$\mathbf{R} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_K \end{bmatrix} \quad \text{and} \quad \delta = \begin{bmatrix} f_1 \, \delta_1 \\ f_2 \, \delta_2 \\ \vdots \\ f_K \, \delta_K \end{bmatrix}.$$

Then, equation (4.7) can be expressed in matrix notation as follows

$$\Delta V = \mathbf{R}' \times \delta, \tag{4.5}$$

where ΔV is still a scalar.

Under the assumption that risk factors are multivariate log-normally distributed and hence, log-returns follow a multivariate normal distribution, the volatility of equation (4.5) can be calculated by

$$\sigma(\Delta V) = \sqrt{\delta' \, \boldsymbol{\Sigma} \, \delta}, \tag{4.6}$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2(\Delta x_1) & cov(\Delta x_1, \Delta x_2) & \ldots & cov(\Delta x_1, \Delta x_K) \\ cov(\Delta x_2, \Delta x_1) & \sigma^2(\Delta x_2) & \ldots & cov(\Delta x_2, \Delta x_K) \\ \vdots & \vdots & \ddots & \vdots \\ cov(\Delta x_K, \Delta x_1) & \ldots & \ldots & \sigma^2(\Delta x_K) \end{bmatrix}$$

The matrix $\boldsymbol{\Sigma}$ is a $K \times K$ covariance matrix that consists of all the variance terms $\sigma^2(\Delta x_k)$ of each proportional change in risk factor k on the main diagonal and the covariances among the changes of the risk factor in the off-diagonal cells of the matrix.

As was illustrated when discussing the simple variance-covariance approach, the calculation of a VaR is now straightforward. The volatility estimate, just derived, needs to be multiplied by a quantile of the probability distribution according to the chosen confidence level.

4.1.4 PORTFOLIO CONSIDERATIONS

After having discussed the two scenarios of a single position which is dependent on either one or more than one risk factor in the two previous sections, I will now illustrate the case of a portfolio of assets.

Britten-Jones and Schaefer [2] present an approach of how to approximate and aggregate changes of K risk factors over n assets. The following formula is consistent with the notation used earlier in this paper but differs from the notation in Britten-Jones and Schaefer [2]. Also, the equation itself differs slightly from the original equation presented in their paper. Here, for the sake of simplicity, the sensitivity measure theta is considered to be as one of the regular risk factor, k.

Equation 4.1 can then be adjusted for the multi-asset case as follows

$$\Delta V = \sum_{i=1}^{n} w_i \sum_{k=1}^{K} f_k \frac{\partial V_i}{\partial f_k} \Delta x_k, \tag{4.7}$$

where w_i describes the quantities of asset i held in the portfolio. As can be seen, the first-order sensitivity measures can be aggregated by using a weighted average method over the sensitivity measures with respect to the same asset i.

4.2 RISK FACTOR COVERAGE

It was illustrated in chapter 2 how derivatives such as forwards, futures, and option contracts can be priced. These pricing models reveal the risk factors on which prices of derivatives are dependent. Recall that according to the pricing models stated in

equations (4.11) and (2.2), the values of a forward contract and an option contract can be defined as follows:

$$F_t = V\ (S_t,\ r)\ \text{and}$$

$$V_t = V\ (S_t,\ \sigma,\ r,\ \tau),$$

where $V(\cdot)$ is a value function and the variables are the same as in chapter 2.

In the previous section the delta-normal approach was presented which provides a framework for approximating and relating changes in risk factors to changes in the price of the position.

Which risk factors, and therefore which Greeks should be considered when using a delta-normal approach? I will now summarize some arguments in favor or opposed to some possible candidate factors.

Obviously, delta has to be considered for both forward and option contracts. As we saw in chapter 2, delta has to be adjusted frequently for option positions and theoretically remains constant only for an instant. Furthermore, delta provides a reliable approximation only when changes in the underlying asset price are small. Wilson [22] argues that delta as a risk measure is not sufficient to manage an options book. This is due to the fact that many options books are run with an explicit strategy of being delta-neutral at all times. Paradoxically, many of the most popular VaR techniques (e.g. the delta-normal or RiskMetrics methods) recognize only directional price or delta risk. This is the reason why J.P. Morgan does not recommend that their standard RiskMetrics technique be applied to portfolios that include options.

When considering a short period of time, the parameter theta is often left out. Hull [10, p.354] argues that theta risk (which measures the expected change in value due to the passage of time) can assumed to be zero. This assumption is useful, in particular, when VaR is measured for a short period of time, such as a one-day time horizon.

The Black-Scholes model assumes that the market interest rate and the volatility of the underlying asset are constant. If this is true, no vega or rho effects have to be considered. In practice, interest rates and volatility change. To mitigate these limitations, Crouhy et al. [5, p.206] suggests modelling the change in value of an option by including sensitivity measures with respect to changes in the volatility of the underlying asset, σ, and with respect to changes in the interest rate, r, as follows:

$$\Delta V = \frac{\partial V}{\partial S}\,\Delta S + \frac{\partial V}{\partial \sigma}\,\Delta \sigma + \frac{\partial V}{\partial r}\,\Delta r$$
$$\Delta V = \delta\,\Delta S + \vartheta\,\Delta \sigma + \rho\,\Delta r.$$

This method treats implied volatilities as another risk factor, treats the vega as another delta or directional price sensitivity, and incorporates it directly into standard methods such as RiskMetrics or the delta-normal model.

Nevertheless, there is a trade-off between the accuracy of the model and the tractability of the model. If all risk factors were to be considered in a model, a large number of volatility and correlation estimates would be needed.

4.3   Advantages and Shortcomings

Linear approximation models such as the delta-normal approach have a number of attractions, such as that it keeps the linearity of the portfolio. This method provides a tractable way of handling positions with non-linearity while retaining the benefits of linear normality. The method of using a first-order approximation is plausible and rather reliable in certain scenarios. According to Jorion [12, p.68], the delta-normal approach is likely to be reliable when the portfolio is close to linear in the first place, since only then can a linear approximation be expected to produce an accurate VaR estimate.

According to Wilson [21, p.220], delta-normal methods are likely to be reliable "if the time horizon is very short, e.g. intra-day, and if the products themselves have relatively linear payoff profile [...] Thus, it may be very well suited for measuring and controlling intra-day risks of a money market or foreign exchange book with few option positions."

Jorion [12] further concludes that linear approximations can seriously underestimate the VaR for options because they ignore second-order risk factors, such as gamma risk with options and convexity with bonds. This shortcoming will be described in more detail in the next chapter.

Wilson [22] notes that the approximation error arising from a delta representation typically increases with the size of the market rate innovation (i.e. changes in risk factors). Unfortunately, large market events or movements are exactly the kind of scenario that risk managers are concerned about when calculating VaR. This can be illustrated graphically when considering a long position in a European-style call option. Consider again the numerical example from chapter 2. The option's strike price, the risk-free interest rate, and the asset's volatility are given in Table 2.1. In addition, the underlying asset trades at $100 and the call option has two days to maturity. Figure 4.1 plots the first-order Taylor approximation of the option's value compared to the "true" option value based on the Black-Scholes option pricing model for prices from $90 to $110 one day to the option's maturity. The delta approximation is based on the option's delta two days before maturity. As we can see, the approximation error increases with the size of changes in the asset price. To achieve a closer approximation, I will consider second-order effects by including the gamma of an option. This aspect will be discussed in next chapter.
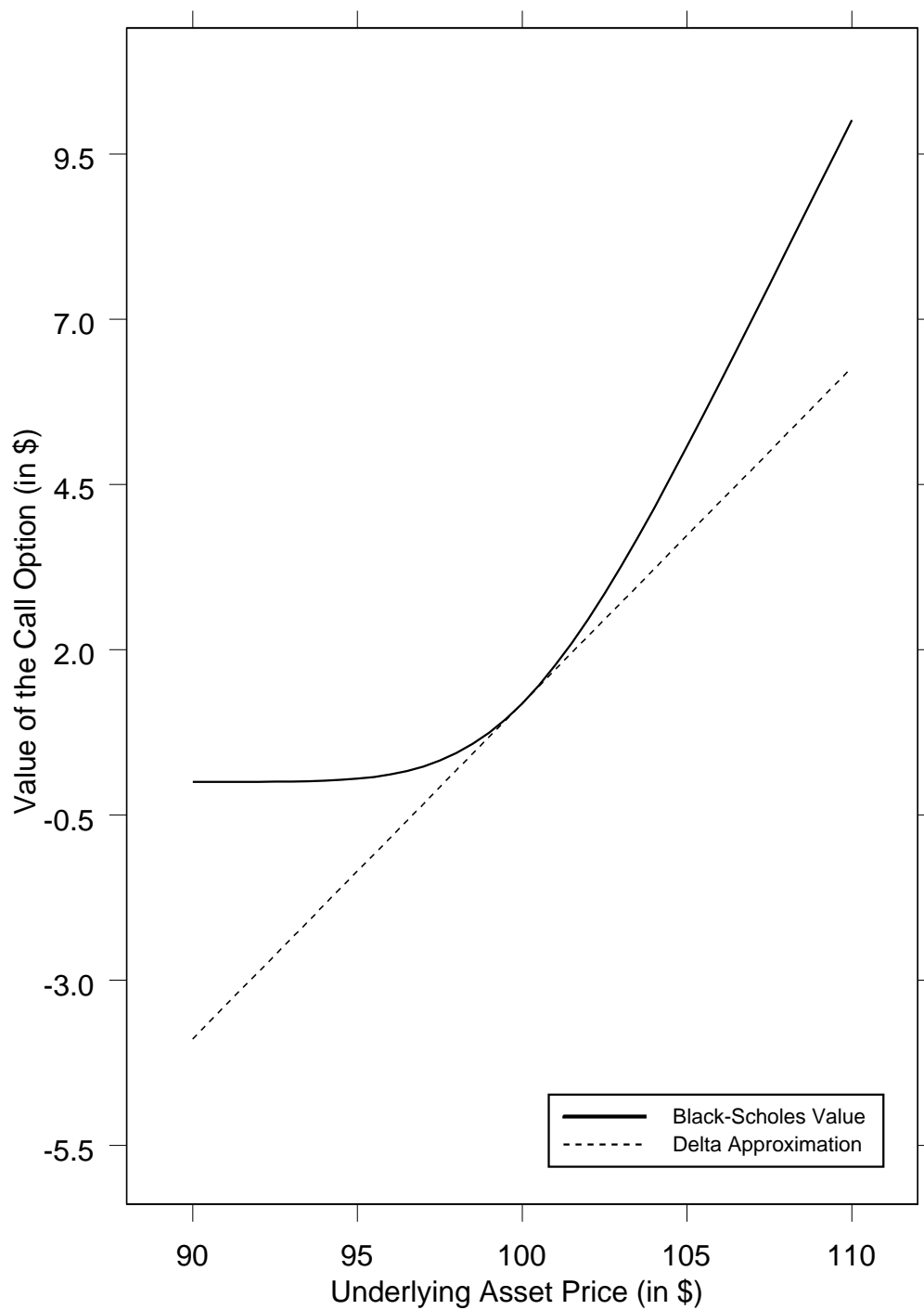
Figure 4.1: "Delta-only" Approximation vs. Black-Scholes Value of a Long Call Option Position.

4.4 Examples of the Delta-Normal Approach

4.4.1 Example 1: Single Option Position

Hull [10] gives an example for a delta-normal model. One could think of an option position consisting of one or several European-style options on the same asset, such as a stock with current price S. This position can include long and short positions on the same underlying asset.

As mentioned earlier, an option's delta describes the rate of change of the value of the option with respect to changes in the price of the underlying asset. Simultaneously, δ is defined as the rate of change of the value of the portfolio with the underlying asset S.[2] As mentioned when discussing the multi-asset case based on Britten-Jones and Schaefer [2], the delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. Let w_i denote the weight of option i.[3] The delta of the entire portfolio is given by

$$\delta = \sum_{i=1}^{n} w_i \delta_i, \tag{4.8}$$

where δ_i is the delta of the ith option.

The change in the value of the option portfolio, ΔV, can now be approximated as follows:

$$\Delta V = \delta \, \Delta S,$$

where δ is the rate of change of the value of the portfolio with S and ΔS is the change in the underlying stock price. Further, I define Δx as the proportional change in the stock price in one day so that:

$$\Delta x = \frac{\Delta S}{S}.$$

---

[2]The notation and the derivation of this example is based on Hull [10].

[3]To be more precise, the parameter w_i describes a quantity of option contracts of option i and the parameter δ_i is the delta of *the* ith option contract.

It follows that the an approximate relationship between ΔV and Δx is

$$\Delta V = S \; \delta \; \Delta x. \tag{4.9}$$

The standard deviation of ΔV is then

$$\sigma(\Delta V) = S \; \delta \; \sigma(\Delta x). \tag{4.10}$$

This leads to a VaR figure of

$$\text{VaR}(H; c) = -\alpha_{1-c} \; S \; \delta \; \sigma(\Delta x).$$

4.4.2 Example 2: Forward and Option Portfolio

The following example provides an illustration for a VaR calculation for a portfolio consisting of a long position in a forward contract on a foreign currency and a short position in a call option on the same currency. First, each single security will be modelled before turning to the problem of calculating a VaR figure for the entire portfolio of these securities. I will start with the forward contract on a foreign currency. The notation used by Hull [10] will be used for the entire example.

Forward Contracts on Foreign Currencies

Forward and futures contracts are the simplest form of derivatives. According to Table 2.3, their value is linear in the underlying spot rates and their risk can be constructed from basic building blocks.

The underlying asset in a forward on a foreign currency is a certain number of units of the foreign currency. Let S_t be the current spot price, measured in dollars, of one unit of the foreign currency and F_t be the forward price, measured in dollars, of one unit of the foreign currency. A foreign currency has the property that the holder of the currency can earn interest at the risk-free rate prevailing in the foreign country. I define r_f as the value of a foreign risk-free interest rate for a maturity

τ with continous compounding and r as the domestic risk-free interest rate for the same maturity.

According to Hull [10], the value of a forward foreign exchange contract, F_t, is given by[4]

$$F_t = S_t \, e^{-r_f \tau} \; - \; F_T \, e^{-r\tau}, \tag{4.11}$$

where F_T is the contracted forward price at maturity. We can now see that the price of a forward on a foreign currency is sensitive to changes in the spot rate, S_t and changes in the domestic and foreign currencies.

This allows us to approximate a change in value of the forward contract using a first-order Taylor series expansion, such as

$$\Delta F_t = \frac{\partial F_t}{\partial S} \, \Delta S + \frac{\partial F_t}{\partial r_f} \, \Delta r_f + \frac{\partial F_t}{\partial r} \, \Delta r. \tag{4.12}$$

Calculating the first-order partial derivatives with respect to the underlying risk factors leads to:

$$\frac{\partial F_t}{\partial S} = e^{-r\tau},$$

$$\frac{\partial F_t}{\partial r_f} = -\tau \, S_t \, e^{-r_f \tau}, \text{ and}$$

$$\frac{\partial F_t}{\partial r} = \tau \, F_T \, e^{-r\tau}.$$

Substituting these partial derivatives into equation (4.12) leads to

$$\Delta F_t = e^{-r\tau} \, \Delta S - \tau \, S_t \, e^{-r_f \tau} \, \Delta r_f + \tau \, F_T \, e^{-r\tau} \, \Delta r.$$

---

[4]The valuation equation for a forward contract on a foreign currency differs from the pricing equation initially used for pricing a forward contract. This is due to the fact that a foreign currency has the property that the holder of the currency can earn interest at the risk-free interest rate prevailing in the foreign currency.

Here, the risk factors are the spot rate as well as the domestic and foreign interest rates. Alternatively, I will replace the interest rates by the price of zero-bonds, such as U.S. T-Bills, for the same maturity. The price of a domestic zero-bond, P, is defined as $P = e^{-r\tau}$, whereas the price of a foreign zero-bond, P_f, is defined as $P_f = e^{-r_f\tau}$. I will further define price changes in these zero-bonds in terms of interest rate changes as follows:

$$\Delta P = \frac{\partial P}{\partial r} \Delta r = -\tau \ e^{-r\tau} \ \Delta r \ \text{ or } \Delta r = \frac{\Delta P}{-\tau \ e^{-r\tau}}$$

and

$$\Delta P_f = \frac{\partial P}{\partial r_f} \Delta r_f = -\tau \ e^{-r_f\tau} \ \Delta r_f \ \text{ or } \Delta r_f = \frac{\Delta P_f}{-\tau \ e^{-r_f\tau}}.$$

Now substitute these two expressions into equation (4.13) and replace ΔS with $S \frac{\Delta S}{S}$, ΔP with $e^{-r\tau} \frac{\Delta P}{P}$, and ΔP_f with $e^{-r_f\tau} \frac{\Delta P}{P}$. The resulting expression is

$$\Delta F_t = \left[S_t \ e^{-r_f\tau} \right] \frac{\Delta S}{S} + \left[S_t \ e^{-r_f\tau} \right] \frac{\Delta P_f}{P_f} - \left[F_T \ e^{-r\tau} \right] \frac{\Delta P}{P}. \tag{4.13}$$

Note that the forward position in the foreign currency can be decomposed into three cash flows which are

1. a long position in a foreign currency, $[S_t \ e^{-r_f\tau}]$,

2. a long position in a zero-bond denominated in the foreign currency, $[S_t \ e^{-r_f\tau}]$, and

3. a short position in a domestic zero-bond worth $[F_T \ e^{-r\tau}]$.

The VaR calculation for this single forward position is now straightforward and can be achieved by applying equation (3.5) to derive a volatility estimate for the forward position and then simply multiplying this volatility estimate by the quantile for the chosen confidence level.

CALL OPTIONS ON FOREIGN CURRENCIES

For the purpose of valuing a European option on a foreign currency the Black-Scholes option pricing model can be used as introduced earlier when assuming that the spot rate follows a geometric Brownian motion process similar to that for stocks. This leads to the following valuation equation[5]

$$C_t = S_t \ e^{-r_f \tau} \ N(d_1) - X \ e^{-r\tau} \ N(d_2), \tag{4.14}$$

where

$$d_1 = \frac{ln\left(\frac{S_t}{X}\right) + (r - r_f + \frac{\sigma^2}{2})\tau}{\sigma \ \sqrt{\tau}}$$

and

$$d_2 = d_1 - \sigma \ \sqrt{\tau}.$$

The parameter S_t describes the current spot exchange rate at time t, r is the continously compounded domestic interest rate for the maturity τ, r_f is the continously compounded foreign interest rate, X is still the strike price of the option, and σ is the volatility of the exchange rate.

Considering the same risk factors, chosen for modelling changes in value of a forward contract, leads to the following approximation for changes in the value of the option:

$$\Delta C = \frac{\partial C}{\partial S} \ \Delta S + \frac{\partial C}{\partial r_f} \ \Delta r_f + \frac{\partial C}{\partial r} \ \Delta r. \tag{4.15}$$

Deriving the sensitivity measures from valuation equation (4.14) with respect to the underlying spot exchange rate, S, the domestic risk-free interest rate, r, and the

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[5]This pricing equation for an option on a foreign currency differs from the option pricing equation used in chapter 2. However, a foreign currency is analogous to a stock providing a known dividend yield. The owner of a foreign currency receives a "dividend yield" equal to the risk-free interest rate, r_f, in the foreign currency.

foreign risk-free interest rate, r_f leads to:

$$\frac{\partial C}{\partial S} = \delta = e^{-r_f \tau} \ N(d_1),$$

$$\frac{\partial C}{\partial r_f} = \rho_f = -\tau \ S \ e^{-r_f \tau} \ N(d_1), \ \text{and}$$

$$\frac{\partial C}{\partial r} = \rho = -\tau \ X \ e^{-r\tau} \ N(d_2).$$

The next step is to substitute these partial derivatives into equation (4.15) and analogously to the transformations done for the forward contract, replacing the parameters ΔS, Δr_f, and Δr leads to

$$\Delta C = \left[S_t \ N(d_1) \ e^{-r_f \tau} \right] \frac{\Delta S_t}{S_t} + \left[S_t \ N(d_1) \ e^{-r_f \tau} \right] \frac{\Delta P_f}{P_f} - \left[X \ N(d_2) \ e^{-r\tau} \right] \frac{\Delta P}{P}.$$

This equation is quite similar to that for a foreign currency forward. The differences are that any building block, according to one risk factor, is multiplied either by the term $N(d_1)$ or $N(d_2)$. Both expressions are risk-neutral probabilities. For example, $N(d_2)$ represents the probability that the option will be exercised in a risk-neutral world, i.e. the option expires in-the-money. One property from the Black-Scholes option pricing model is that when the spot price becomes very large relative to the exercise price, a call option is almost certain to be exercised. Since the spot price, S_t becomes very large, both d_1 and d_2 become very large, and $N(d_1)$ and $N(d_2)$ are both close to one and the expression becomes similar to the expression for a forward contract. I will now turn to the case of a portfolio that consists of a long position in a forward contract on a foreign currency and a short position in a European option on the same foreign currency.

4.4.3   Portfolio of a Forward Contract and a Single Option

In this section I will discuss the scenario of a portfolio consisting of a long position in a forward contract on a foreign currency and a short position in a European-style

call option on the same foreign currency. I assume the two contracts are based on the same quantity of underlying currency, and have the same exercise price $(X = F_T)$. Therefore, the change in value of such a portfolio, ΔV, can be modelled as follows

$$\Delta V = \Delta F - \Delta C. \tag{4.16}$$

Substituting the expressions for ΔF and ΔC derived in this section into equation 4.16 leads to

$$\Delta V = \left[S_t\ e^{-r_f \tau} \right] \frac{\Delta S}{S} + \left[S_t\ e^{-r_f \tau} \right] \frac{\Delta P_f}{P_f} - \left[F_T\ e^{-r\tau} \right] \frac{\Delta P}{P}$$

$$- \left[S_t\ N(d_1)\ e^{-r_f \tau} \right] \frac{\Delta S_t}{S_t} - \left[S_t\ N(d_1)\ e^{-r_f \tau} \right] \frac{\Delta P_f}{P_f} + \left[X\ N(d_2)\ e^{-r\tau} \right] \frac{\Delta P}{P}.$$

After rearranging this equation, the final expression for ΔV is

$$\Delta V = \left[S_t\ e^{-r_f \tau}\ (N(d_1) - 1) \right] \frac{\Delta S}{S} + \left[S_t\ e^{-r_f \tau}\ (1 - N(d_1)) \right] \frac{\Delta P_f}{P_f}$$

$$- \left[X\ e^{-r\tau}\ (N(d_2) - 1) \right] \frac{\Delta P}{P}$$

As previously mentioned, in the case of a deep in-the-money call option the terms $N(d_1)$ and $N(d_2)$ are close to one and the entire position is nearly risk-free (i.e. $\Delta V = 0$).

This conclusion is tempting, but dangerous. We know from the Black-Scholes model that the price of a European-style call option depends on more than the risk factors used above. According to the Black-Scholes model, such an approach neglects other crucial risk factors, such as vega risk and gamma risk which in particular will be discussed in greater depth later.

CHAPTER 5

PARAMETRIC NON-LINEAR APPROXIMATION VALUE AT RISK MODELS

5.1 INTRODUCTION TO SECOND-ORDER EFFECTS

As we saw in the previous chapter, the delta-normal approach assumes that the non-linear relationship between changes in the position's value and the risk factors can be sufficiently approximated by a first-order Taylor series expansion based on first-order partial derivatives of the value of the position with respect to its underlying risk factors. This is clearly inappropriate and leads to a large approximation error when changes in the underlying risk factors are large and when the payoff function reveals strong curvature. The curvature of the payoff function of an option is measured by the parameter gamma. As we saw in chapter 2 and, in particular, in Figure 2.2, gamma tends to be high when the option trades at-the-money and the time to maturity is short. A high gamma means that the value of the option position is highly sensitive to changes in the stock price.

Standard VaR models assume that risk factors follow a multivariate normal distribution. Assuming a linear relationship between the value of the position and the risk factors implies that the profit/loss distribution of the entire position is also normally distributed. Even if changes in the underlying risk factors are normally distributed, non-linear risk exposures leads to a non-normal probability distribution for the changes in the position's value. The distribution can be skewed, meaning that the distribution is asymmetric around the mean. When gamma is positive, the probability distribution tends to be positively skewed, whereas when gamma is negative,

the probability distribution is negatively skewed. A positively skewed distribution has a thinner left tail than the normal distribution whereas a negatively skewed distribution has a thicker left tail than the normal distribution.

Since VaR is critically dependent on the left tail of a probability distribution, it is crucial to know whether the distribution is skewed or not. Compared to a normal distribution, a positively skewed distribution has a thinner left tail which pushes the VaR percentile to the right. In this case, the VaR figure will be too high. Overestimating the probability of extreme negative events is unsatisfactory, but not that alarming. However, the case of a negatively skewed distribution when normality is assumed should cause concern. Compared to a normal distribution, a negatively skewed distribution has a thicker left tail which pushes the VaR percentile to the left. Assuming normality leads to a VaR figure that is too low compared to the real probability distribution and leads to an underestimation of risk. Such underestimation of risks increases when the curvature, i.e. gamma, of an option position is high.

When using an option pricing model, such as the Black-Scholes model, gamma can be derived using equation (2.5). As already mentioned, a long position in call or put options always has a positive gamma, whereas a short position in call or put options has a negative gamma. The lack of symmetry implies that the discretion of the bias of VaR depends on whether one is net long or short in options.

Jorion [12, p. 207] mentions an economic interpretation of the skewed distribution regarding to options as follows:

> "note that the option distribution (of a long call) has a long right tail, due to the upside potential, whereas the downside is limited to the option premium."

5.2 Quadratic Models

The basic idea of the delta-gamma approach is that a second-order Taylor series expansion provides a better approximation than a first-order Taylor series expansion as was used for the delta-normal approach. A second-order Taylor series expansion is based on first-order and second-order partial derivatives of the position's value function with respect to the underlying risk factors.

5.2.1 Delta-Gamma Value at Risk for one Risk Factor

Consider a portfolio exposed to a single risk factor. A change in the value of this position, ΔV, can be approximated by using a second-order Taylor series expansion. The first partial derivative of the value function $V(\cdot)$ with respect to a change in the underlying risk factor k, $\frac{\partial V}{\partial f_k}$, is delta. Analogously, $\frac{\partial^2 V}{\partial f_k^2}$ is the second partial derivative of the value function V with respect to changes in risk factor k. When applying this approach to an option, this is an option's gamma.[1] The second-order Taylor expansion series is

$$\Delta V = \frac{\partial V}{\partial f_k} \, \Delta f_k + \frac{1}{2} \, \frac{\partial^2 V}{\partial f_k^2} \, (\Delta f_k)^2 + \epsilon_k(2), \tag{5.1}$$

where the term Δf_k ($[\Delta f_k]^2$) describes the change (squared change) in the underlying risk factor. Finally, $\epsilon_k(2)$ is defined as the "second-order" approximation error.

Equation (5.1) can be applied to a position of options. The option's value is dependent on one asset whose current price is S_t. We can make use of the Taylor series expansion and approximate the change in the value of the position ΔV by using the position's delta and gamma as the first and the second partial derivatives

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[1]The gamma of a portfolio which consists of different options on the same underlying securities can be achieved by aggregating the gammas of the single options using a weighted average. This method is similar to the approach used to aggregate deltas described in equation (4.8). When a portfolio consists of options on different underlying risk factors, correlation effects between the risk factors must be considered.

of the value function with respect to the price of the underlying asset:

$$\Delta V \approx \delta \ \Delta S + \frac{1}{2} \ \Gamma \ (\Delta S)^2. \tag{5.2}$$

Setting

$$\Delta x = \frac{\Delta S}{S}$$

reduces equation (5.2) to

$$\Delta V = S \ \delta \ \Delta x + \frac{1}{2} S^2 \ \Gamma \ (\Delta x)^2. \tag{5.3}$$

The variable Δx can be interpreted as a rate of return of the underlying asset. The variable ΔV is non-normal. According to Pichler and Selitsch [17], in the case of only one risk factor the distribution is a noncentral χ^2-distribution.

The next step is to derive the first three moments of this distribution. Assuming that Δx is normal with a mean of zero and a volatility of σ the following terms can be derived based on equation (5.3):

$$E(\Delta V) = \frac{1}{2} S^2 \Gamma \sigma^2,$$

$$E[(\Delta V)^2] = S^2 \delta^2 \sigma^2 + \frac{3}{4} S^4 \Gamma^2 \sigma^4, \text{ and}$$

$$E[(\Delta V)^3] = \frac{9}{2} S^4 \delta^2 \Gamma \sigma^4 + \frac{15}{8} S^6 \Gamma^3 \sigma^6.$$

MEAN AND VARIANCE

According to Hull[10], the next step is to define μ_V and σ_V as the mean and standard deviation of the option position so that

$$\mu_V = E\left(\Delta V\right) = \frac{1}{2} S^2 \Gamma \sigma^2 \text{ and}$$

$$\sigma_V^2 = E\left[(\Delta V)^2\right] - [E(\Delta V)]^2, \text{ with } [E(\Delta V)]^2 = \frac{1}{4} S^4 \Gamma^2 \sigma^4$$

Table 5.1: Statistical Properties of an Option vs. its Underlying Asset

Moment	Option (ΔV)	Returns	$ Returns
Return	ΔV	$\Delta x = \frac{\Delta S}{S}$	ΔS
Mean	$\frac{1}{2}S^2\Gamma\sigma^2$	0	0
Variance	$S^2\delta^2\sigma^2 + \frac{1}{2}S^4\Gamma^2\sigma^4$	$\sigma(\Delta x)^2$	$S^2\sigma(\Delta x)^2$
Skewness	$3S^4\delta^2\Gamma\sigma^4 + S^6\Gamma^3\sigma^6$	0	0

which leads to a variance term of:

$$\sigma_V^2 = S^2\delta^2\sigma^2 + \frac{3}{4}S^4\Gamma^2\sigma^4 - \frac{1}{4}S^4\Gamma^2\sigma^4 = S^2\delta^2\sigma^2 + \underbrace{\frac{1}{2}S^4\Gamma^2\sigma^4}_{\text{adjustment}} \ .$$

In contrast to the linear case presented when discussing the delta-normal approach where the gamma term is assumed to be zero, both the expected value and the variance reveal adjustment terms. Since the gamma term is non-zero, the expected value of ΔV is now clearly non-zero in contrast to the delta-normal approach where the expected value was zero. The variance is also adjusted by the term $\frac{1}{2}S^4\Gamma^2\sigma^4$.

SKEWNESS

Skewness is a measure of asymmetry in a random variable's probability distribution. The skewness coefficient of the probability distribution of ΔV, ξ_V, is defined as:

$$\xi_V = \frac{E\left[(\Delta V - \mu_V)^3\right]}{\sigma_V^3} = \frac{E\left[(\Delta V)^3\right] - 3E\left[(\Delta V)^2\right]\mu_V + 2\mu_V^3}{\sigma_V^3},$$

whereas the skewness is defined as

$$E\left[(\Delta V - \mu_V)^3\right] = E\left[(\Delta V)^3\right] - 3E\left[(\Delta V)^2\right]\mu_V + 2\mu_V^3$$
$$= 3S^4\delta^2\Gamma\sigma^4 + S^6\Gamma^3\sigma^6.$$

Table 5.1 summarizes the first three moments of the option position with the moments of the underlying asset. I will now briefly summarize the points made so far in this section:

1. Although it is assumed that the return on the underlying asset is distributed with a mean of zero, the change in the option's value is non-zero unless gamma is zero.

2. The sign of the option's mean will be determined by the relative magnitude and sign of gamma and whether one is long or short in the option.

3. The variance of the change in the option's value differs from the variance of the return on the underlying instrument by an adjustment term $\frac{1}{2}S^4\Gamma^2\sigma^4$. If gamma is zero, the position's variance is the same as in the delta-normal approach.

4. Finally, whether the change in value of an option position is positively or negatively skewed depends on whether the option position is a long or a short position.

5.2.2 Delta-Gamma Value at Risk for K Risk Factors

In this section, I will extend the univariate case presented in the previous section to the case of a portfolio consisting of K risk factors. At first, I will will present the general case when individual instruments in the portfolio may be dependent on more than one market variable. This would be the case if the portfolio contained diff swaps[2] and choosers.[3] The following expressions use the same notation as was

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[2]A diff swap is a fixed-floating or floating-floating interest rate swap. One of the floating rates is a foreign interest rate, but it is applied to a notional amount in the domestic currency. Floating-floating diff swaps are a vehicle for directly betting on spreads between different currency's interest rates.

[3]According to Hull [10], a chooser option is an option where the holder has the right to choose whether it is a call or a put at some point during its life.

used in the univariate case discussed above. I will start with deriving a second-order Taylor expansion series for the case of K risk factors as follows

$$\Delta V = \sum_{k=1}^{K} \frac{\partial V}{\partial f_k} \Delta f_k + \sum_{k=1}^{K} \sum_{j=1}^{K} \frac{1}{2} \frac{\partial^2 V}{\partial f_k \partial f_j} \Delta f_k \Delta f_j + \epsilon_k(2). \tag{5.4}$$

Analogously to the case presented in the previous section, I will define Δx_k as the proportional change in risk factor k as follows

$$\Delta x_k = \frac{\Delta f_k}{f_k}$$

and substituting Δx_k in equation (5.4) leads to

$$\Delta V = \sum_{k=1}^{K} \frac{\partial V}{\partial f_k} f_k \Delta x_k + \sum_{k=1}^{K} \sum_{j=1}^{K} \frac{1}{2} \frac{\partial^2 V}{\partial f_k \partial f_j} f_k f_j \Delta x_k \Delta x_j + \epsilon_k(2).$$

The previous equation looks rather complex and can be made more concise by using matrix notation:

$$\Delta V = \delta' \mathbf{R} + \frac{1}{2} \mathbf{R}' \mathbf{\Gamma} \mathbf{R}, \tag{5.5}$$

where the vector $\mathbf{R}$ is still a $K \times 1$ column vector consisting of the proportional changes of each risk factor k, Δx_k, and δ as a $K \times 1$ column vector containing in each row the product of the current price or rate of risk factor k, f_k, and the sensitivity measure with respect to each risk factor k. Therefore,

$$\mathbf{R} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_K \end{bmatrix} \quad \text{and} \quad \delta = \begin{bmatrix} f_1 \, \delta_1 \\ f_2 \, \delta_2 \\ \vdots \\ f_K \, \delta_K \end{bmatrix}.$$

The matrix $\mathbf{\Gamma}$ denotes the $K \times K$ matrix of gamma terms

$$\mathbf{\Gamma} = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \dots & \Gamma_{1,K} \\ \Gamma_{2,1} & \Gamma_{2,2} & \dots & \Gamma_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{K,1} & \dots & \dots & \Gamma_{K,K} \end{bmatrix} \quad \text{with} \quad \Gamma_{k,j} = \frac{\partial^2 V}{\partial f_k \partial f_j} f_k \, f_j.$$

More specifically, the gamma matrix is the second derivative, or Hessian,[4] of the portfolio's of position's value function. The gamma terms on the main diagonal $(k = j)$ are conventional gamma measures in which case gamma is defined as $\frac{\partial V^2}{\partial f_k^2}$. In this scenario, the gamma of an option position describes the change in δ_k for a change in risk factor k.

The off-diagonal terms are cross-gamma terms $(k \neq j)$. They are often ignored if the individual prices are functions of only one market price or if the cross-product effects are trivial. Wilson [22] remarks that this is a potentially dangerous assumption. For correlation-dependent products such as diff swaps or choosers, the cross-product terms can be significant and should not be ignored.[5] Jorion [12] notes that in the case of an option position when there is more than one risk factor considered, such as the asset price and the implied volatility there exists a cross-effect. This is due to the fact that the delta of an option also depends on the implied volatility.

As in the univariate case, the moments of the distribution of ΔV must be determined first. According to Pichler and Selitsch [17], we can derive the moments of the profit and loss distribution of the position as follows.

MEAN AND VARIANCE

$$\mu_V = E[\Delta V] = \frac{1}{2} \text{ tr} \left[\mathbf{\Gamma \Sigma} \right] \text{ and}$$

$$\sigma_V^2 = E \left[(\Delta V)^2 \right] - [E(\Delta V)]^2 = \delta^T \mathbf{\Sigma} \delta \; + \underbrace{\frac{1}{2} \text{ tr} \left[\mathbf{\Gamma \Sigma} \right]^2}_{\text{adjustment}} .$$

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[4]Here, the Hessian is multiplied with the current level of the risk factors k and j.
[5]See also Rouvinez [18].

The trace[6] of the matrix $[\mathbf{\Gamma\Sigma}]$ is the sum of the K eigenvalues of $[\mathbf{\Gamma\Sigma}]$ and the trace of $[\mathbf{\Gamma\Sigma}]^2$ equals the sum of the squared eigenvalues of $[\mathbf{\Gamma\Sigma}]$. As was discussed in the univariate case and in contrast to the linear case presented when discussing the delta-normal approach, the expected value is non-zero and variance also reveals an adjustment term. The adjustment term for the variance is the trace of $[\mathbf{\Gamma\Sigma}]^2$.

Skewness, Kurtosis, and higher Moments

Let X be the standardized value of ΔV

$$X = \frac{\Delta V - E[\Delta V]}{\sqrt{\sigma_V^2}}$$

which leads to higher moments of X with $r \geq 3$ as follows:

$$E\left[X^r\right] = \frac{\frac{1}{2}\, r!\, \delta^T \mathbf{\Sigma} \left[\mathbf{\Gamma\Sigma}\right]^{r-2} \delta + \frac{1}{2}\, (r-1)!\, \mathrm{tr}\left[\mathbf{\Gamma\Sigma}\right]^r}{(\sigma_V^2)^{\frac{r}{2}}}.$$

In particular, the skewness can be derived for $r = 3$ as

$$E\left[X^3\right] = \xi = \frac{3\, \delta^T \mathbf{\Sigma} \left[\mathbf{\Gamma\Sigma}\right] \delta + \mathrm{tr}\left[\mathbf{\Gamma\Sigma}\right]^3}{(\sigma_V^2)^{\frac{3}{2}}}$$

and the kurtosis can be derived for $r = 4$ as

$$E\left[X^4\right] = \eta = \frac{12\, \delta^T \mathbf{\Sigma} \left[\mathbf{\Gamma\Sigma}\right]^2 \delta + 3\, \mathrm{tr}\left[\mathbf{\Gamma\Sigma}\right]^4}{(\sigma_V^2)^2}.$$

Recall that skewness is a measure of asymmetry in a random variable's probability distribution. Kurtosis describes the degree of "peakedness" of a distribution relative to a symmetric normal distribution. If the distribution has a higher peak than the normal distribution the kurtosis is greater than three. The distribution is then said to be leptokurtic. Flat-topped distributions which have a kurtosis of less than three are referred to as platykurtic. The standard normal distribution is mesokurtic which

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[6]According to Zangari [13], the trace of a matrix is defined as the sum of the diagonal elements of the matrix.

means that the distribution is neither peaked, nor flat-topped and reveals a kurtosis of three.

Having derived the moments of the profit/loss distribution of the entire position, I will now focus in the next sections on approaches to determine the specific quantile of this distribution.

5.3 The Cornish-Fisher Expansion

The idea behind the method presented in the following paragraphs[7] is a direct approximation of the required quantile of the distribution of ΔV based on the Cornish-Fisher expansion[8] around the quantile of a normal distribution. The confidence interval parameter must be adjusted for the skewness and other distortions from normality created by the presence of the gamma factor. This adjustment can be achieved by using an approximation formula known as the Cornish-Fisher expansion which provides a relationship between the moments of a distribution and its percentiles. The Cornish-Fisher expansion is based on the statistical principle that one distribution (e.g. a chi-squared) can always be described in terms of the parameters of another (e.g. a normal). This adjustment factor needs estimates of the skewness and the kurtosis. However, suggestions regarding to the number of higher moments that need to be considered vary. Hull [10] suggests the use of the first three moments of the distribution, namely mean, variance, and skewness, whereas the Zangari [23] suggests also including kurtosis. Pichler and Selitsch [17] even suggests the use of the first six moments of the distribution. According to Zangari [23, p.9], we can simply apply the adjustment by using our new confidence parameter and then proceed as if the distribution were normal.

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[7]This approach is based on the work of Fallon [8], Pichler and Selitsch [17], and Hull [10] which recommend approximate solutions based on the Cornish-Fisher expansion.

[8]See Johnson and Kotz [11] for details.

Once the moments of the distribution are derived, the estimation of the adjusted quantile does not differ between the univariate and the multivariate case. Using the first three moments of ΔV, the Cornish-Fisher expansion estimates the q-percentile of the distribution of ΔV, w_q as

$$w_q = \alpha_q + \frac{1}{6}(\alpha_q^2 - 1)\xi_V.$$

However, using the first four moments of ΔV, the Cornish-Fisher expansion estimates the q-percentile of the distribution of ΔV, w_q as

$$w_q = \alpha_q + \frac{1}{6}(\alpha_q^2 - 1)\xi_V + \frac{1}{24}(\alpha_q^3 - 3\alpha_q)\eta_V - \frac{1}{36}(\alpha_q^3 - 5\alpha_q)\xi_V^2.$$

The parameter α_q is the q-percentile of the standard normal distribution. According to Zangari [23, p.9], we can simply apply the adjusted confidence parameter for the VaR calculations. Therefore, the VaR of this position can be calculated as follows:

$$\text{VaR}(H; c) = -w_q \, \sigma_V \text{ and} \tag{5.6}$$

$$\text{VaR}'(H; c) = -w_q \, \sigma_V + \mu_V \, V. \tag{5.7}$$

In contrast to the delta-normal approach, the definition of the absolute VaR, VaR$'$, and relative VaR, VaR, differ, since the mean of the distribution is now non-zero.

5.4   ALTERNATIVE APPROACHES

As mentioned, the profit/loss distribution of a non-linear position is non-normal. How should one estimate the quantile of this new distribution? In the academic literature, several approaches have been discussed. I will briefly mention these different approaches.

The Delta-Gamma Normal Approach

Dowd [6] discusses an approach called the delta-gamma normal approach. The essence of this method is to regard the extra risk factor $(\Delta x_k)^2$ as equivalent to another independently distributed normal variable and treat them in the same way as the first factor, Δx_k. The idea behind this approach is to preserve the normal linearity as was the case of the delta-normal approach. Hull [10] remarks that the first two moments, namely the mean and the variance, can be fitted to a normal distribution which is better than ignoring gamma altogether. The assumption that ΔV is normal is less than ideal. It is based on the assumption that the quadratic term $(\Delta x)^2$ is also normally distributed. This is wrong since a squared normally distributed variable is then itself a non-central χ^2 variable. An approach such as this relies on the assumption that ΔV is normally distributed was discussed in academic literature but is, according to Dowd [6, p.73], "clearly inadequate, being both logically incoherent and unreliable and should therefore should ruled out of court."

A moment-fitting Approach

Zangari [13] suggests an approximation using Johnson curves. Having obtained the first four moments of the portfolio's profit/loss (or return) distribution, Zangari [24] suggests finding a distribution that has the same moments but whose distribution is known exactly. He suggests:

> "that it is beneficial rather than deriving the portfolio returns exact distribution - which is intractable anyway - we find the statistical characteristics, or moments, of its distribution. [...] These moments depend only on the price of the option, the current market prices of the underlying securities, the option's Greeks, and the RiskMetrics matrix. Having obtained the first four moments of portfolios return distribution, we find

a distribution that not only has the same moments, but also has a form

that is exactly known."

Zangari [24] actually suggests matching those moments to one of a family of distributions known as Johnson distributions.[9] Matching moments means finding a distribution that has the same mean, standard deviation, skewness and kurtosis as the portfolio's return distribution. As mentioned in the quote above, this hypothetical distribution will have the same moments as the "true" profit/loss distribution, but unlike the true distribution will also have a known form. The VaR estimate can then be calculated from this known distribution.

Matching moments to a family of distributions requires that ΔV is approximated by transformed standard normal variable $Y = f^{-1}(Z)$, where Z is a standard normally distributed variable. According to Pichler and Selitsch [17], the specific choice of the transformation function f^{-1} depends on the ratio of the square root of the skewness and the kurtosis of ΔV. Furthermore, the following Johnson family distributions are sufficient to cover all possible combinations of the first four moments:

$$Z = a + b \, \log\left(\frac{Y - c}{d}\right) \quad \text{(Lognormal)},$$

$$Z = a + b \, \sinh^{-1}\left(\frac{Y - c}{d}\right) \quad \text{(Unbounded), and}$$

$$Z = a + b \, \log\left(\frac{Y - c}{c + d - R_p}\right) \quad \text{(Bounded)},$$

where a, b, c, and d are parameters with the restriction $(c < Z < c+d)$ whose values are determined by ΔV's first four moments and $f(\cdot)$ is a monotonic function.

Zangari [24] mentions that the parameter values for a, b, c, and d are chosen through a moment matching algorithm that has an analytical solution in the lognormal case. In the unbounded and bounded cases he suggests to make use of the

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[9]The name Johnson comes from the statistician Norman Johnson who described a process of matching a particular distribution to a given set of moments.

iterative algorithm by Hill et al. [9]. Pichler and Selitsch [17] conclude that the calculation of the parameter values for the α-quantile of the Johnson distribution $J(\alpha)$ can easily obtained through the following transformation

$$J(\alpha) = c + d \ f^{-1} \left(\frac{N(\alpha) - a}{b} \right),$$

where $N(\cdot)$ is a standard normal cumulative distribution function.

OTHER APPROACHES

As mentioned, in academic literature there are several other methods that suggest methods of how to calculate the quantile of the profit/loss distribution or how to determine the distribution exactly:

- Rouvinez [18] uses a trapeziodal rule to invert the characteristic function.

- Britten-Jones and Schaefer [2] suggest an approximation through a χ^2 distribution that requires to solve a nonlinear system of equations.

- Cardenas et al. [4] use the Fourier transformation.

- Wilson [21] uses a quadratic programming approach.

5.5 DELTA-NORMAL VS. DELTA-GAMMA APPROACHES

As we saw when discussing the delta-normal approach, the approximation error arising from a "delta-only" representation typically increases with the size of changes in risk factors. In contrast to the delta-normal approach, the delta-gamma approach takes the curvature of a non-linear relationship into account by relying on a second-order Taylor series expansion based on a first and second partial derivatives. The delta-gamma approach provides a better approximation.

This can be illustrated graphically by considering the same numerical example used earlier when discussing the delta-normal approach. Consider a long position in a European-style call option. The option's strike price, the risk-free interest rate, and the asset's volatility are given in Table 2.1. Assume the asset trades at $100 two days before maturity and the call option has one day to maturity. The delta-gamma approximation and the delta-normal approximation are based on the option's delta or delta and gamma two days before maturity. The delta-gamma approximation was already presented earlier in Figure 4.1. Figure 5.1 compares the delta-normal approximation with the delta-gamma approximation. The quadratic shape of the delta-gamma approximation is clearly visible. The result of a quadratic approximation is that in the case of a long option position, the delta-gamma approximation leads to values which are higher than the value given by the Black-Scholes model. In the case of a short position, the delta-gamma approximation leads to values which are lower than the values given by the Black-Scholes model. However, Figure 5.1 also shows that the second-order Taylor approximation of the option's value is closer to the "true" option value (based on the Black-Scholes option pricing model) than the delta-normal approximation.

Figure 5.2 compares the absolute approximation errors of the delta-normal approximation with the delta-gamma approximation. This graph is based on the same data as the numerical example above. The delta-gamma approximation provides an approximation that is more accurate in changes in the underlying asset price over a much larger range than the delta-normal approximation. The approximation error of a delta-gamma approximation is significantly lower than in the case of the delta-normal approach. More specifically, the approximation is reliable for smaller changes in the underlying risk factor. However, there is still an approximation error when large movements in the underlying risk factor occur.

Dowd [6] remarks that although the delta-gamma approximation is more accurate than the delta-normal method. The delta-gamma method still reveals an approximation error that might be still too large for some purposes.

However, the increased accuracy in the approximation comes at some cost such as the loss of normality which is inevitable once we move to second-order approximations. The following limitations are common to all delta-gamma approaches and do not depend on the method that is used for calculating the quantile of the distribution.

The improved accuracy by the delta-gamma method comes at the cost of at least some reduced tractability relative to the delta-normal model. As we have seen, in using any delta-gamma approach, we might lose normality in our portfolio return even if changes in the underlying risk factors are normally distributed. The moments of the profit/loss distribution in the case of the delta-normal and the delta-gamma approach are summarized in Table 5.1. Table 5.1 shows multiple points. To illustrate the following points it is necessary that gamma is non-zero, i.e. that the relationship reveals curvature.

The expected change of the value in the underlying is zero and because of the linear relationship the expected change in the value of the position in the case of the delta-normal approach is also zero. Due to the non-linear relationship, the expected value in the case of the delta-gamma approach is $\frac{1}{2}S^2\Gamma\sigma^2$. The sign of the expected value of the option position will be determined by the relative magnitude and sign of gamma and whether one is long or short in the option.[10]

The variance of a change in value in the case of the delta-gamma approach differs from the variance of the position when using the delta-normal approach by

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[10]As already mentioned, a long position in call or put options always has a positive gamma, whereas a short position in call or put options lead to a negative gamma. The lack of symmetry has the implication that the VaR now depends on whether one is long or short in the option.

Table 5.2: Statistical Properties of the Delta-Normal vs. the Delta-Gamma Approach

Moment	Delta-Normal	Delta-Gamma	Underlying Asset
Return	ΔV	ΔV	Δx
Mean	0	$\frac{1}{2}S^2\Gamma\sigma^2$	0
Variance	$S^2\delta^2\sigma^2$	$S^2\delta^2\sigma^2 + \frac{1}{2}S^4\Gamma^2\sigma^4$	$\sigma(\Delta x)$
Skewness	0	$3S^4\delta^2\Gamma\sigma^4 + S^6\Gamma^3\sigma^6$	0

an adjustment term $\frac{1}{2}S^4\Gamma^2\sigma^4$. If gamma is zero, the position's variance is the same as in the delta-normal approach.

Finally, whether the change in value of an option position in the case of using the delta-gamma approach is positively or negatively skewed depends on whether the option position is a long or a short position, whereas the distribution in the case of the delta-normal approach does not reveal asymmetry.

As we saw, non-linear risk exposure leads to a non-normal distribution in position returns. This leads to the conclusion that the incidental benefits of normality which are the ability to translate VaR figures easily from one set of VaR parameters to another and the ability to infer expected tail losses without any difficulty, are compromised or lost altogether.
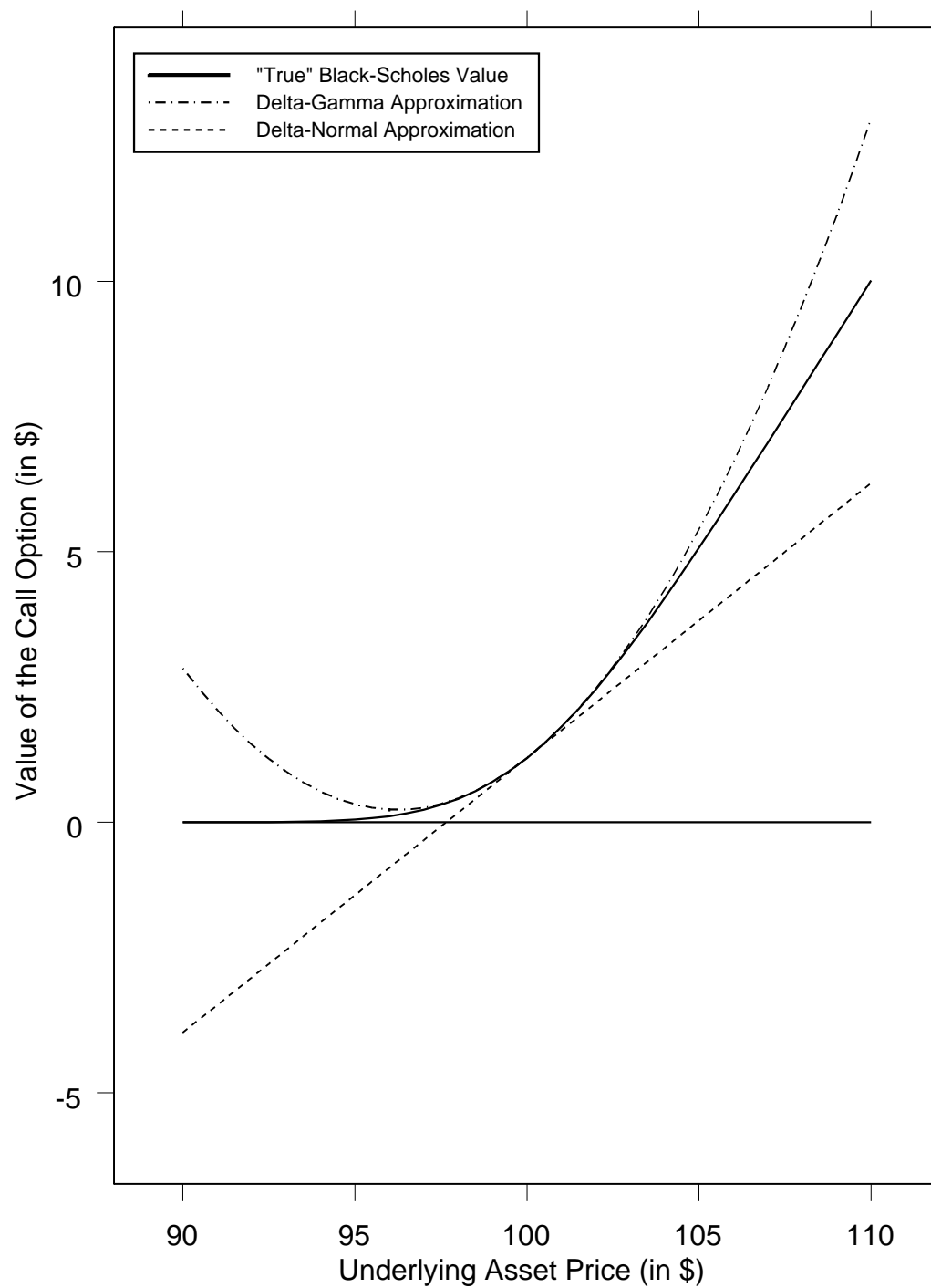
Figure 5.1: Comparison between the Delta-Normal and the Delta-Gamma Approximation with the Black-Scholes Model.
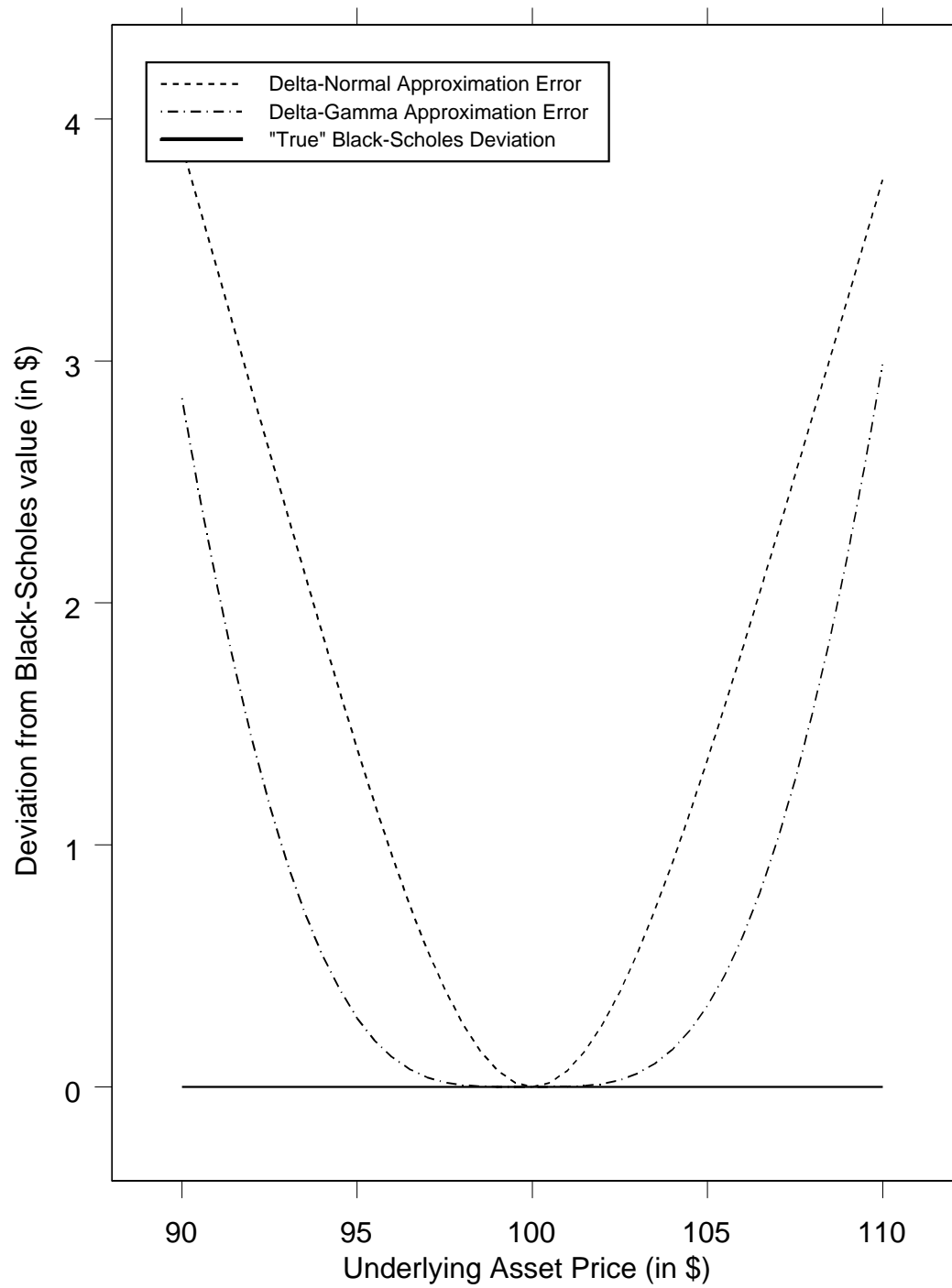
Figure 5.2: Absolute Approximation Error of the Delta-Normal vs. the Delta-Gamma Approach.

CHAPTER 6

CONCLUSION

VaR models are based on a set of assumptions regarding the distribution of the risk factors as well as on a set of implicitly made assumptions by standard VaR models. The changes in risk factors are assumed to follow a multivariate normal distribution. However, there is empirical evidence that many individual return distributions are not normal but exhibit fat tails. That means that actual distributions might reveal a far higher incidence of large market movements than is predicted by a normal distribution. A normal distribution is more likely to underestimate the risk of extreme returns.

An implicitly made assumption of standard VaR models is that VaR models rely on market prices. In particular, these models assume that markets remain liquid at all times and therefore, assets can be liquidated at prevailing market prices. In times of market distress liquidity can dry up quickly which can lead to the fact that positions cannot be sold at all or to dramatically deflated prices. This marking-to-market approach is questionable if VaR is supposed to represent the worst loss over a liquidation period.

Another implicit assumption of standard VaR models is that volatilities and covariances are constant throughout the sample period. The problem is that volatility and correlation may be highly unstable over the time. In conclusion, VaR estimates based on normal market conditions may seriously underestimate the risk of a portfolio when the correlation among risk factors changes. These simplifying assumptions

71

can lead to a inaccurate and misleading VaR estimate. Therefore, it is essential to be aware of the existence of these simplifying assumptions.

Linear approximation models have a number of attractions, such as the fact that they keep the linearity of the portfolio without adding any new risk factors. The delta-normal approach provides a tractable way of handling positions with optionality that retains the benefits of linear normality. However, the delta-normal method is only reliable for portfolios with linear risk exposure and can lead to significant approximation errors for portfolios with non-linear risk exposure. This approximation error typically increases with the size of changes in risk factors. Delta-normal approaches are appropriate for linear derivatives and for positions that reveal only weak curvature and in particular, in cases where the time horizon is very short.

In contrast to the delta-normal approach, the delta-gamma approach takes the curvature of a non-linear relationship into account as well as provides a significant improvement over the delta-normal method in the case of non-linear derivatives. In particular, the delta-gamma approximation is reliable for smaller changes in the underlying risk factor. However, there is still an approximation error when large movements in the underlying risk factor occur.

The improved accuracy by the delta-gamma method comes at the cost of at least some reduced tractability relative to the delta-normal model. Even if changes in the underlying risk factors are normally distributed, a non-linear relationship leads to a non-normal probability distribution of the changes in the position's value. Moreover, this distribution is skewed, meaning that the distribution is asymmetric and reveals excess kurtosis. This leads to the fact that the incidental benefits of normality which are the ability to translate VaR figures easily from one set of VaR parameters to another and the ability to infer expected tail losses without any difficulty, are compromised or lost.

The main difficulty with delta-gamma methods is the estimation of the quantile of the non-normal profit/loss distribution. There are many different methods discussed in academic literature for estimating this quantile. The idea discussed in this paper is a direct approximation of the required quantile based on the Cornish-Fisher expansion. The normal confidence interval parameter is adjusted for the skewness and other distortions from normality created by the presence of curvature. However, suggestions regarding to the number of higher moments that need to be considered vary. Empirical results by Pichler and Selitsch [17] suggests the use of a higher moment Cornish-Fisher expansion. In particular, the delta-gamma method, when using a Cornish-Fisher expansion, provides an approximation which is close to results calculated by Monte Carlo Simulation methods but is computationally more efficient.

c Selected confidence level.

C_t Price of a European-style Call option at time t.

d_1, d_2 Parameters of the Black-Scholes option pricing formulas.

$\mathbf{f}$ $K \times 1$ column vector containing the current value or level of each risk factor k, f_k.

f_k Value or level of risk factor k.

F_t Forward or futures price at time t.

F_T Delivery price in a forward or futures contract.

K Number of risk factors.

n Number of assets.

$N(x)$ Standard normal cumulative distribution function (CDF) for a variable x. It defines the cumulative probability that a variable with a standardized normal variable is less than x.

$N'(x)$ Probability density function (PDF) for a standard normal variable x.

P_t Price of a European-style Put option at time t.

P Price of a domestic zero-bond with time to maturity τ.

P_F Price of a foreign zero-bond with time to maturity τ.

r Continously compounded domestic risk-free interest rate.

r_f Continously compounded risk-free interest rate in a foreign country.

$\mathbf{R}$ $K \times 1$ column vector of returns in the risk factors.

S_t Price of asset underlying a derivative at a general time t.

t A future point in time.

T Time at maturity of a derivative.

V Marked-to-market value of a position.

$V(\cdot)$ Value function that relates risk factors to a value of a derivative.

w_i Quantity of asset i.

X Strike or exercise price of a European-style option.

α_{1-c} The $1 - c$ quantile of the assumed distribution.

Γ Gamma of a derivative or portfolio of derivatives.

$\mathbf{\Gamma}$ $K \times K$ Gamma matrix.

δ Delta of a derivative or portfolio of derivatives.

ΔS_t Price change during the period $t - 1$ to t. Therefore, $\Delta S_t = S_t - S_{t-1}$.

η_V Kurtosis coefficient of the position.

μ Mean return of the asset.

μ_V Mean return of the entire position.

ξ_V Skewness coefficient of the position.

ρ Depending on the context, ρ is either the rho of a derivative or describes the correlation coefficient between two variables.

ρ $K \times K$ correlation matrix.

σ Volatility (i.e. standard deviation) of an asset.

σ_V Volatility (i.e. standard deviation) of the position.

τ Time horizon $\tau = T - t$.

Θ Theta of a derivative or portfolio of derivatives.

ϑ Vega of a derivative or portfolio of derivatives.

Σ $K \times K$ covariance matrix of the risk factor returns.

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