

THE PERIODIC SOLUTION AND THE GLOBAL ASYMPTOTIC STABILITY
FOR NORTHEASTERN PUERTO RICO ECOSYSTEM

by

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ABSTRACT

The paper deals with the following Northeastern Puerto Rico Ecosystem

$$\begin{cases} \frac{dN}{dt} = (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|}, \\ \frac{d[Chl]}{dt} = [Chl] \left[\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) - D \right], \\ \frac{dN_B}{dt} = \frac{\alpha_N N}{K_N + |N|} - \mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B. \end{cases}$$

Our aim is to approach the existence of periodic solution under nonautonomous assumption by using the method of Leray-Schauder degree, and the global asymptotically stable under autonomous assumption. Our global asymptotically stable results partly generalized [*Patrick De Leenheer, Simon A. Levin, Eduardo D. Sontag, Christopher A. Klausmeier, Global stability in a chemostat with multiple nutrients, J. Math. Biol. 52 (2006), 419–438*] from $v = D$ to $v \geq D$.

INDEX WORDS: Ecosystem, Periodic solution, Global asymptotically stable

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CHAPTER 1

INTRODUCTION

Ecosystem is an interesting topic. Recently, it attracted more and more interesting along with the global climate change and enhance of circumstance protection. In this paper, we discuss the following Northeastern Puerto Rico Ecosystem,

$$(S) \begin{cases} \frac{dN}{dt} = (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|}, \\ \frac{d[Chl]}{dt} = [Chl] \left[\mu \left(\frac{PAR^\gamma}{PAR_{min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{Bmin}^\alpha + |N_B|^\alpha} - \xi \right) - D \right], \\ \frac{dN_B}{dt} = \frac{\alpha_N N}{K_N + |N|} - \mu \left(\frac{PAR^\gamma}{PAR_{min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{Bmin}^\alpha + |N_B|^\alpha} - \xi \right) N_B, \end{cases}$$

where N is the inorganic Nitrogen in water, $[Chl]$ is Chlorophyll in water plants, N_B is the Nitrogen concentration in water plant, PAR is the sunlight function; $(I, v, \xi, \alpha, \alpha_N, K_N, \mu, PAR_{min}, N_{Bmin}, d, \gamma, D)$ are parameters, where v is the velocity of water, I is the Nitrogen concentration in upstream water, d is the depth of water, constants α_N and K_N are the units of periphyton maximum uptake rate and half saturation constant, D is the mortality and scouring rate, for example, passed away or consumption by aquatic insects and their subsequent consumption by shrimp and fish. This paper investigates the existence of periodic solution under nonautonomous assumption by Leray-Schauder degree, and the global asymptotically stable under autonomous assumption. Moreover, our global asymptotically stable results partly generalized the main results of [3] from $v = D$ to $v \geq D$.

The rest of the thesis is organized as follows. Chapter 2 introduce some preliminary. Chapter 3 gives the existence of periodic solutions for non-autonomous system. Chapter 4 mainly discuss global asymptotic stability of the Nontrivial equilibrium point. Chapter 5 introduce the future work: the application of the Ecosystem to the Northeastern Puerto Rico data.

CHAPTER 2

PRELIMINARY

At first, we discuss the sign of solutions, because the natural phenomenon should be a positive solution. So in this chapter we assume that the initial value is positive.

If (S) has a solution $(N, [Chl], N_B)$, then $[Chl]$ must be a solution of the following ordinary differential equation:

$$\frac{d[Chl]}{dt} = [Chl] \left[\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) - D \right].$$

Solving it, we get

$$[Chl](t) = [Chl](0) \cdot \exp \int_0^t \left[\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) - D \right] ds,$$

which implies that $[Chl](t)$ must be the production of initial value $[Chl](0)$ and an exponential function. Thus $[Chl](t)$ will always keep the same sign or zero with its initial value. Based on the biology interesting, we may assume that $[Chl] > 0$.

From the first equation of (S) and the positivity of $[Chl]$, we have

$$\frac{dN}{dt} = (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|} \geq Iv > 0 \text{ when } N \leq 0,$$

so it is not hard to check that $N(t)$ is an ultimate positive function, i.e. $N(t) > 0$ for large enough t . Therefore, if $N(0) > 0$, then $N(t) > 0$ for any $t > 0$.

From the third equation, we have

$$\frac{dN_B}{dt} = \frac{\alpha_N N}{K_N + N} - \mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) N_B \geq \frac{\alpha_N N}{K_N + N} > 0$$

when $N_B = 0$. Therefore, if $N_B(0) > 0$, then $N_B(t) > 0$ for any $t > 0$.

Therefore, in the following, we only need to discuss the positive solutions of (S) unless otherwise specified.

2.1 DEFINITION OF STABLE

Definition 2.1 (Limit Cycle): Limit cycle is a closed trajectory in phase space.

Definition 2.2 (Equilibrium): Equilibrium is the condition of a system in which all competing influences are balanced.

In this chapter, we mainly focus on the nonautonomous system.

Definition 2.3 (Stable Property): An equilibrium is considered **stable** (for simplicity we will consider asymptotic stability only) if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is **unstable**.

On the existence and stable property of Equilibrium Point and Limit Cycle, we prefer the following two Theorems (see [2]).

Proposition 2.4 (see [2, Theorem 1]) (First order averaging method) Consider the differential system

$$\frac{d}{dt}x(t) = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon), \quad (2.1.1)$$

where $f : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n, g : \mathbb{R} \times \Omega \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n$ are continuous functions, T -periodic in the first variable, and Ω is an open subset of \mathbb{R}^n . Define $f^0 : \Omega \rightarrow \mathbb{R}^n$ by

$$f^0(z) = \frac{1}{T} \int_0^T f(s, z) ds,$$

and assume that

- (i) f and g are locally Lipschitz with respect to x ;

(ii) there exists $b \in \Omega$ such that $f^0(b) = 0$ and a neighborhood $V \subset \Omega$ of b in which $f^0(z) \neq 0$ for all $z \in \bar{V} \setminus \{b\}$ and $d_B(f^0, V, b) \neq 0$ (where $d_B(f^0, V, b)$ denotes the Brouwer degree of f^0 in the neighborhood V of b).

Then, for $|\varepsilon| > 0$ sufficiently small, there exists an isolated T -periodic solution $\phi(\cdot, \varepsilon)$ of system (2.1.1) such that $\phi(b, 0) = b$.

Proposition 2.5 (see [2, Theorem 2]) Assume that the functions f and g of (2.1.1) are C^1 and Lipschitz respectively in a neighborhood of the limit cycle $\phi(\cdot, \varepsilon)$ given in Theorem 1 by the simple zero b of f^0 , then for $|\varepsilon| > 0$ sufficiently small if the eigenvalues of the Jacobian matrix of f^0 at b all have negative (respectively positive) real part, then the limit cycle $\phi(\cdot, \varepsilon)$ is asymptotically stable (respectively unstable).

Sometimes, we consider stable of the following n dimensional autonomous problem,

$$\frac{dx}{dt} = f(x) \in \mathbb{R}^n, \quad (2.1.2)$$

and its Equilibrium Point $x = p \in \mathbb{R}^n$.

In our system, $(N, [Chl], N_B) = (a_1, a_2, a_3)$ is an Equilibrium Point if and only if

$$\begin{cases} (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|} = 0, \\ [Chl] \left[\mu \left(\frac{1}{T} \int_0^T \frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} dt \frac{|N_B|^{\alpha-1} N_B}{N_B^\alpha + |N_B|^\alpha} - \xi \right) - D \right] = 0, \\ \frac{\alpha_N N}{K_N + N} - \mu \left(\frac{1}{T} \int_0^T \frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} dt \frac{|N_B|^{\alpha-1} N_B}{N_B^\alpha + |N_B|^\alpha} - \xi \right) N_B = 0. \end{cases}$$

Consider the system (2.1.2), and we denote the solution as $x = \phi(t, q)$.

Definition 2.6 (Lyapunov Stability): The equilibrium point p is said to be stable if given $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(t, q) - p\| \leq \epsilon$ for all $t > 0$ and for all q such that $\|p - q\| < \delta$. If δ can be chosen not only so that the solution p is stable but also so that $\phi(t, q) \rightarrow p$ as $t \rightarrow \infty$, then p is said to be asymptotically stable. If p is not stable it is said to be unstable.

Proposition 2.7 (see [1, Theorem 8.4.3]) Suppose the C^1 system (2.1.2) possesses the equilibrium point p and put $G = f_x(p)$. The origin is asymptotically stable if the real part of every eigenvalue of G is negative. It is unstable if any eigenvalue of G has a positive real part.

Note: If a linear system has a zero real part, then it is stable, but it is not asymptotically stable.

2.2 NECESSARY CONDITIONS FOR THE STABLE PROPERTY OF SYSTEM (S)

In this section, we consider the stable property of system (S).

We discuss the stable property of (S) in the following two cases:

Case (i) We assume the lightness $PAR \in [PAR_S, PAR_L]$ is a constant, where

$$PAR_S = \min_{t \in [0, T]} PAR(t) \text{ and } PAR_L = \max_{t \in [0, T]} PAR(t).$$

Case (ii) We assume the lightness PAR is a function of normal sunlight.

Notice that in the Case (ii), we consider of applying $f^0(z) = \frac{1}{T} \int_0^T f(s, z) ds$ to replace $f(s, z)$, so we only need to consider PAR (sunlight) is a constant value function, and take value in the interval $[PAR_S, PAR_L]$.

Let us consider the equilibrium point of (S) at first,

$$(I) \begin{cases} (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|} = 0, \\ [Chl] \left[\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_B^\alpha \min + |N_B|^\alpha} - \xi \right) - D \right] = 0, \\ \frac{\alpha_N N}{K_N + |N|} - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_B^\alpha \min + |N_B|^\alpha} - \xi \right) N_B = 0, \end{cases}$$

where $C_* \in [PAR_S, PAR_L]$ is a constant.

In this section, we assume $C_* = \frac{1}{T} \int_0^T \frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} dt$.

Notice that we consider positive solutions here. We make the following assumptions

$$C_* > \xi + D/\mu, \tag{2.2.1}$$

$$\alpha_N > D \left(\frac{(\xi + D/\mu) N_{B \min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}, \tag{2.2.2}$$

$$I > \frac{K_N}{\alpha_N/D \left(\frac{(\xi+D/\mu)N_{B\min}^\alpha}{C_*-\xi-D/\mu} \right)^{1/\alpha} - 1}. \quad (2.2.3)$$

Note that $[Chl] \neq 0$. Solving (I), we obtain the equilibrium point as follows

$$\begin{aligned} N_B^\alpha &= \frac{N_{B\min}^\alpha}{\frac{C_*}{\xi+D/\mu} - 1} = \frac{(\xi+D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu}, \\ N_B &= \left(\frac{(\xi+D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha} \quad (\text{by (2.2.1)}), \\ N &= \frac{K_N D N_B}{\alpha_N - D N_B} = \frac{K_N D \left(\frac{(\xi+D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}}{\alpha_N - D \left(\frac{(\xi+D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}} \quad (\text{by (2.2.2)}), \\ [Chl] &= \frac{dv(I - N)}{D N_B} = \frac{dv\left(I - \frac{K_N D N_B}{\alpha_N - D N_B}\right)}{D N_B} \quad (\text{by (2.2.3)}) \\ &= dv \left(\frac{I}{D N_B} - \frac{K_N}{\alpha_N - D N_B} \right) \\ &= dv \left(\frac{I}{D \left(\frac{(\xi+D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}} - \frac{K_N}{\alpha_N - D \left(\frac{(\xi+D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}} \right). \end{aligned}$$

Denote $\Gamma = \mu \left(C_* \frac{N_B^\alpha}{N_{B\min}^\alpha + N_B^\alpha} - \xi \right)$. Use the notation similar to (2.1.2), the Jacobian matrix is as follows

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial [Chl]} & \frac{\partial f_1}{\partial N_B} \\ \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial [Chl]} & \frac{\partial f_2}{\partial N_B} \\ \frac{\partial f_3}{\partial N} & \frac{\partial f_3}{\partial [Chl]} & \frac{\partial f_3}{\partial N_B} \end{pmatrix} \\ &= \begin{pmatrix} -v - \frac{\alpha_N}{d} \frac{[Chl]K_N}{(K_N+N)^2} & -\frac{\alpha_N}{d} \frac{N}{K_N+N} & 0 \\ 0 & \Gamma - D & [Chl]\mu C_* \frac{\alpha N_B^{\alpha-1} N_{B\min}^\alpha}{(N_{B\min}^\alpha + N_B^\alpha)^2} \\ \frac{\alpha_N K_N}{(K_N+N)^2} & 0 & -\Gamma - \frac{\mu C_* \alpha N_{B\min}^\alpha N_B^\alpha}{(N_{B\min}^\alpha + N_B^\alpha)^2} \end{pmatrix}. \end{aligned}$$

We consider the following eigenvalue problem

$$|J - \lambda I| = 0.$$

Note that $\mu \left(C_* \frac{N_B^\alpha}{N_{B \min}^\alpha + N_B^\alpha} - \xi \right) = D$ at the equilibrium point, we have

$$\begin{vmatrix} -v - \frac{\alpha_N}{d} \frac{[Chl]K_N}{(K_N+N)^2} - \lambda & -\frac{\alpha_N}{d} \frac{N}{K_N+N} & 0 \\ 0 & -\lambda & [Chl]\mu C_* \frac{\alpha N_B^{\alpha-1} N_{B \min}^\alpha}{(N_{B \min}^\alpha + N_B^\alpha)^2} \\ \frac{\alpha_N K_N}{(K_N+N)^2} & 0 & -D - \frac{\mu C_* \alpha N_{B \min}^\alpha N_B^\alpha}{(N_{B \min}^\alpha + N_B^\alpha)^2} - \lambda \end{vmatrix} = 0,$$

i.e.

$$\begin{aligned} 0 &= \left(-v - \frac{\alpha_N}{d} \frac{[Chl]K_N}{(K_N+N)^2} - \lambda \right) (-\lambda) \left(-D - \frac{\mu C_* \alpha N_{B \min}^\alpha N_B^\alpha}{(N_{B \min}^\alpha + N_B^\alpha)^2} - \lambda \right) \\ &\quad + \frac{\alpha_N K_N}{(K_N+N)^2} \left(-\frac{\alpha_N}{d} \frac{N}{K_N+N} \right) [Chl]\mu C_* \frac{\alpha N_B^{\alpha-1} N_{B \min}^\alpha}{(N_{B \min}^\alpha + N_B^\alpha)^2}. \end{aligned}$$

Denote

$$\varepsilon = \frac{\alpha_N K_N}{(K_N+N)^2} \left(\frac{\alpha_N}{d} \frac{N}{K_N+N} \right) [Chl]\mu C_* \frac{\alpha N_B^{\alpha-1} N_{B \min}^\alpha}{(N_{B \min}^\alpha + N_B^\alpha)^2}. \quad (2.2.4)$$

We have

$$\lambda \left(\lambda + v + \frac{\alpha_N}{d} \frac{[Chl]K_N}{(K_N+N)^2} \right) \left(\lambda + D + \frac{\mu C_* \alpha N_{B \min}^\alpha N_B^\alpha}{(N_{B \min}^\alpha + N_B^\alpha)^2} \right) + \varepsilon = 0.$$

Denote

$$\begin{aligned} x_1 &= v + \frac{\alpha_N}{d} \frac{[Chl]K_N}{(K_N+N)^2}, \\ x_2 &= D + \frac{\mu C_* \alpha N_{B \min}^\alpha N_B^\alpha}{(N_{B \min}^\alpha + N_B^\alpha)^2}, \end{aligned}$$

we have

$$\lambda(\lambda + x_1)(\lambda + x_2) + \varepsilon = 0. \quad (2.2.5)$$

If $\varepsilon = 0$, we can see that (2.2.5) has three nonpositive real solutions, 0 , $-x_1$ and $-x_2$.

Without loss of generality, we may assume that $x_2 > x_1 (> 0)$.

If $\varepsilon > 0$ is very small, (2.2.5) has three nonpositive real solutions around 0 , $-x_1$ and $-x_2$, but the largest one is a little smaller than 0 , the smallest one is a little smaller than $-x_2$.

Take the first derivative with respect to λ on the left hand side of (2.2.5) and let it be 0 ,

we have

$$3\lambda^2 + 2(x_1 + x_2)\lambda + x_1 x_2 = 0,$$

and we obtain the local minimum and local maximum point of $\lambda(\lambda + x_1)(\lambda + x_2)$ as following

$$\lambda = \frac{-(x_1 + x_2) \pm \sqrt{x_1^2 + x_2^2 - x_1x_2}}{3}.$$

Let $\lambda_1 = \frac{-(x_1+x_2)+\sqrt{x_1^2+x_2^2-x_1x_2}}{3} (< 0)$ and

$$\varepsilon_1 = \lambda_1(\lambda_1 + x_1)(\lambda_1 + x_2), \quad (2.2.6)$$

it is easy to see that (2.2.5) becomes

$$(\lambda - \lambda_2)(\lambda - \lambda_1)^2 + \varepsilon - \varepsilon_1 = 0, \quad (2.2.7)$$

where $\lambda_2 (< \lambda_1 < 0)$ is another solution of $\lambda(\lambda + x_1)(\lambda + x_2) + \varepsilon_1 = 0$.

Further analysis will show that, if $\varepsilon - \varepsilon_1 > 0$ is very small, (2.2.7) will have two conjugate complex solutions both have negative real part.

In fact, (2.2.7) will have two conjugate complex solutions $a \pm ib$ and a negative solution λ_3 , if $\varepsilon - \varepsilon_1 > 0$. So we have

$$\begin{aligned} 0 &= (\lambda - \lambda_3)(\lambda - (a + ib))(\lambda - (a - ib)) \\ &= (\lambda - \lambda_3)((\lambda - a) - ib)((\lambda - a) + ib) \\ &= (\lambda - \lambda_3)((\lambda - a)^2 + b^2) \\ &= \lambda^3 - (2a + \lambda_3)\lambda^2 + (a^2 + b^2 + 2a\lambda_3)\lambda - \lambda_3(a^2 + b^2). \end{aligned}$$

Compare the coefficients of the above express and (2.2.7), we have

$$\begin{aligned} 2a + \lambda_3 &= \lambda_2 + 2\lambda_1, \\ a^2 + b^2 + 2a\lambda_3 &= \lambda_1^2 + 2\lambda_1\lambda_2, \\ \lambda_3(a^2 + b^2) &= \lambda_1^2\lambda_2 - (\varepsilon - \varepsilon_1). \end{aligned}$$

When $\varepsilon - \varepsilon_1 = 0$, it is easy to see that $b = 0$ and $a = \lambda_1 (< 0)$, $\lambda_3 = \lambda_2 (< 0)$. If $\varepsilon - \varepsilon_1 > 0$ is very small, then $\lambda_3 < \lambda_2$, but $\lambda_3 > \lambda_2 + 2\lambda_1$ since $a < 0$. It is easy to see that $d\lambda_3/d\varepsilon < 0$ and $\lambda_3 \rightarrow -\infty$ as $\varepsilon \rightarrow +\infty$. So there exists a $\varepsilon_2 > \varepsilon_1 > 0$ such that, if $\varepsilon > \varepsilon_2$, then a becomes

positive. Thus (2.2.7) has two conjugate complex solutions both have positive real part. And if $\varepsilon = \varepsilon_2$ ($< \varepsilon_2$), then we will have $a = 0$ (< 0). That means

$$\begin{aligned}\lambda_3 &= \lambda_2 + 2\lambda_1, \\ b^2 &= \lambda_1^2 + 2\lambda_1\lambda_2, \\ \lambda_3 b^2 &= \lambda_1^2\lambda_2 - (\varepsilon_2 - \varepsilon_1).\end{aligned}$$

So we have

$$\varepsilon_2 = \varepsilon_1 + \lambda_1^2\lambda_2 - (\lambda_2 + 2\lambda_1)(\lambda_1^2 + 2\lambda_1\lambda_2) = \varepsilon_1 - 2\lambda_1(\lambda_2 + \lambda_1)^2. \quad (2.2.8)$$

Thus, we have the following lemma.

Lemma 2.8. Assume (2.2.1)-(2.2.3) satisfied. There exist two critical values, ε_1 and ε_2 which satisfy

(i) If $0 < \varepsilon \leq \varepsilon_1$, there are three negative eigenvalues, then system (S) is asymptotically stable near the equilibrium point provided (S) is autonomous.

(ii) If $\varepsilon_1 < \varepsilon < \varepsilon_2$, there are one negative eigenvalue, and two conjugate complex eigenvalues $a \pm ib$ both have negative real part a , then system (S) is asymptotically stable near the equilibrium point provided (S) is autonomous.

(iii) If $\varepsilon > \varepsilon_2$, there are one negative eigenvalue, and two conjugate complex eigenvalues $a \pm ib$ both have positive real part, then system (S) is unstable near the equilibrium point provided (S) is autonomous.

CHAPTER 3

THE EXISTENCE OF PERIODIC SOLUTIONS FOR NON-AUTONOMOUS SYSTEM

In this chapter, we will discuss the existence of periodic solution of (S) by Leray-Schauder degree method.

3.1 PRELIMINARY

In this section, we do some preparative for the existence of periodic solution.

At first, let us consider the following simple system periodic solution

$$(S_1) \begin{cases} \frac{dN}{dt} = h_1, \\ \frac{d[ChI]}{dt} = h_2, \\ \frac{dN_E}{dt} = h_3, \end{cases}$$

where $h = (h_1, h_2, h_3) \in (C[0, T])^3$. Denote

$$F(f)(t) = \int_0^t f(s)ds, \forall t \in [0, T].$$

The periodic condition implies that

$$\frac{1}{T}F(h_i)(T) = 0, i = 1, 2, 3.$$

Let us denote $C = (C[0, T])^3$, and define

$$P : C \rightarrow C, u \mapsto u(0); Q : C \rightarrow C, h \mapsto \frac{1}{T} \int_0^T h(s)ds, I_d : C \rightarrow C, h \mapsto h,$$

and we denote

$$K(h)(t) = F((I_d - Q)h)(t), \forall t \in [0, T].$$

Also, denote $C_T = \{u \in C \mid u(0) = u(T)\}$.

Lemma 3.1 The operator K is continuous and send bounded sets in C into relatively compact sets in C .

Proof. It is easy to check that $K(h) \in C_T$. Since

$$K(h)'(t) = (I_d - Q)h, \forall t \in [0, T],$$

K is a continuous operator from C to C_T .

Let U be an bounded set in C , then there exists $\rho \in C$, such that, for any $u \in C$

$$|u(t)| \leq \rho(t) \text{ on } [0, T].$$

We want to show that $\overline{K(U)} \subset C_T$ is a compact set.

Let $\{u_n\}$ is a sequence in $K(U)$, then there exists a sequence $\{h_n\} \in U$ such that $u_n = K(h_n)$. For $t_1, t_2 \in [0, T]$, we have that

$$|F((I_d - Q)h_n)(t_1) - F((I_d - Q)h_n)(t_2)| \leq \left| \int_{t_1}^{t_2} \rho(t) dt \right| + \frac{|t_1 - t_2|}{T} \left| \int_0^T \rho(t) dt \right|.$$

Hence the sequence $\{F((I_d - Q)h_n)\}$ is uniformly bounded and equi-continuous. By Ascoli-Arzela theorem, there exists a subsequence of $\{F((I_d - Q)h_n)\}$ which we rename the same, which is convergent in C . This completes the proof. \square

Denote $u = (N, [Chl], N_B)$. We denote $N_f(u) : C \times [0, T] \rightarrow C$ the Nemytskii operator associated to f defined by

$$N_f(u)(t) = f(t, u(t)), \text{ a.e. on } [0, T], \quad (3.1.1)$$

where $f(t, u(t)) = (f_1(t, u(t)), f_2(t, u(t)), f_3(t, u(t)))$ and

$$\begin{aligned} f_1(t, u(t)) &= (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|}, \\ f_2(t, u(t)) &= [Chl] \left[\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) - D \right], \end{aligned}$$

$$f_3(t, u(t)) = \frac{\alpha_N N}{K_N + |N|} - \mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_B|^{\alpha-1} N_B}{N_{B_{\min}}^\alpha + |N_B|^\alpha} - \xi \right) N_B.$$

Lemma 3.2 u is a solution of (S) if and only if u is a solution of the following abstract equation

$$u = P(u) + QN_f(u) + K(N_f(u)). \quad (3.1.2)$$

Proof. If u is a solution of (S), it is clear that u is a solution of (3.1.2).

Conversely, if u is a solution of (3.1.2), by the condition of the mapping P , we have

$$u(0) = P(u) + QN_f(u) + K(N_f(u))(0) = u(0) + QN_f(u),$$

then $QN_f(u) = 0$, and by the definition of K , we have

$$u(T) = P(u) + QN_f(u) + K(N_f(u))(T) = u(0) + QN_f(u) + 0 = u(0),$$

and

$$\frac{du}{dt} = (I_d - Q)N_f(u) = N_f(u).$$

Therefore u is a periodic solution of (S). \square

3.2 EXISTENCE OF PERIODIC SOLUTION

In this section, we give the existence of periodic solution.

Lemma 3.3 (see [4, Theorem 3.1]) Assume that Ω is an open bounded set in C such that the following conditions hold.

(1⁰) For each $\lambda \in (0, 1]$ the problem (S) has no solution on $\partial\Omega$.

(2⁰) The equation

$$\omega(a) := \frac{1}{T} \int_0^T f(t, a) dt = 0, \quad (3.2.1)$$

has no solution on $\partial\Omega \cap \mathbb{R}^3$.

(3⁰) The Brouwer degree $d_B[\omega, \Omega \cap \mathbb{R}^3, 0] \neq 0$.

Then (S) has a solution in Ω .

Proof. Let us consider

$$\frac{du}{dt} = \lambda N_f(u) + (1 - \lambda)QN_f(u). \quad (3.2.2)$$

For any $\lambda \in (0, 1]$, observe that, if u is a periodic solution to (S), we have necessarily

$$QN_f(u) = 0.$$

It means that (3.2.2) and $\frac{du}{dt} = \lambda N_f(u)$ has the same solutions for $\lambda \in [0, 1]$.

We denote $N(\cdot, \cdot) : C \times [0, 1] \rightarrow C$ defined by

$$N(u, \lambda) = \lambda N_f(u) + (1 - \lambda)QN_f(u),$$

where $N_f(u)$ is defined in (3.1.1). Let

$$\begin{aligned} \Phi_f(u, \lambda) &= P(u) + QN(u, \lambda) + (K \circ N(u, \lambda)) \\ &= P(u) + QN_f(u) + (K \circ [\lambda(I_d - Q)N_f + QN_f])(u), \end{aligned}$$

the fixed point of $\Phi_f(u, 1)$ is a solution for (S). Also, the problem (3.2.2) can be written in the equivalent form

$$u = \Phi_f(u, \lambda). \quad (3.2.3)$$

Since f is continuous, it is easy to see that $N(\cdot, \cdot)$ is continuous and send bounded sets into relative compact sets. According to Lemma 3.1 we can conclude that Φ_f is continuous and compact for any $\lambda \in [0, 1]$. Now from hypothesis (1⁰) it follows that (3.2.3) has no solutions for $(u, \lambda) \in \partial\Omega \times (0, 1]$. For $\lambda = 0$, (3.2.2) is equivalent to the problem

$$\frac{du}{dt} = QN_f(u),$$

and if u is a solution to this problem, we must have

$$\int_0^T f(t, u(t))dt = 0. \quad (3.2.4)$$

Hence

$$u \equiv l,$$

where $l \in \mathbb{R}^3$ is a constant. Thus, by (3.2.4)

$$\int_0^T f(t, l) dt = 0,$$

which together with hypothesis (2⁰), implies that $u = l \notin \partial\Omega$. Thus we have proved that (3.2.3) has no solution (u, λ) on $\partial\Omega \times [0, 1]$, then we get that for each $\lambda \in [0, 1]$, the Leray-Schauder degree $d_{LS}[I_d - \Phi_f(\cdot, \lambda), \Omega, 0]$ is well defined and, from the properties of that degree, we have

$$d_{LS}[I_d - \Phi_f(\cdot, 1), \Omega, 0] = d_{LS}[I_d - \Phi_f(\cdot, 0), \Omega, 0]. \quad (3.2.5)$$

Now it is clear that the following problem

$$u = \Phi_f(u, 1) \quad (3.2.6)$$

is equivalent to system (S), and (3.2.5) tell us that problem (3.2.6) will have a solution if we can show that

$$d_{LS}[I_d - \Phi_f(\cdot, 0), \Omega, 0] \neq 0.$$

Since

$$\Phi_f(u, 0) = P(u) + QN_f(u) + K(0) = P(u) + QN_f(u),$$

and then

$$u - \Phi_f(u, 0) = u - P(u) - QN_f(u) = QN_f(u).$$

By the properties of the Leray-Schauder degree, we have

$$d_{LS}[I_d - \Phi_f(\cdot, 0), \Omega, 0] = (-1)^3 d_B[\omega, \Omega \cap \mathbb{R}^3, 0],$$

where the function ω is defined in (3.2.1) and d_B denotes the Brouwer degree. Since by hypothesis (3⁰), this last degree is different from zero. This completes the proof. \square

Lemma 3.4 All the periodic solutions of (3.2.2) are uniformly bounded.

Proof. We suppose, on the contrary, i.e., we have an unbounded solution sequence $\{u_n, \lambda_n\}$.

We prove our result in two steps.

Step 1. We assume $\{[Chl]_n\}$ is unbounded in $C[0, T]$.

By the second equation, we have

$$[Chl]_n(t) = [Chl]_n(0) \cdot \exp \left\{ \lambda_n \int_0^t \left[\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) - D \right] ds \right\}.$$

Obviously, $\exp \left\{ \lambda_n \int_0^t \left[\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) - D \right] ds \right\}$ is bounded and strictly positive on $[0, T]$. So we must have a subsequence (which we still denote it as $[Chl]_n$) satisfies $[Chl]_n(0) \rightarrow +\infty$.

From the first equation, we can see that

$$(I - N_n)v - \frac{\alpha_N}{d} \frac{[Chl]_n N_n}{K_N + |N_n|} = 0 \text{ at some point } t_n \in (0, T).$$

Rewriting it, we have

$$N_n^2 + (K_N - I + \frac{\alpha_N}{dv} [Chl]_n) N_n - IK_N = 0 \text{ at some point } t_n \in (0, T),$$

so we get

$$N_n = \left[- \left(K_N - I + \frac{\alpha_N}{dv} [Chl]_n \right) \pm \sqrt{(K_N - I + \frac{\alpha_N}{dv} [Chl]_n)^2 + 4IK_N} \right] / 2,$$

then the positive solution is

$$\begin{aligned} N_n &= \left[4IK_N / \left((K_N - I + \frac{\alpha_N}{dv} [Chl]_n) + \sqrt{(K_N - I + \frac{\alpha_N}{dv} [Chl]_n)^2 + 4IK_N} \right) \right] / 2 \\ &\approx IK_N / \left(\frac{\alpha_N}{dv} [Chl]_n \right) \text{ at some point } t_n \in (0, T). \end{aligned}$$

When $Iv - \frac{\alpha_N}{d} \frac{[Chl]_n N_n}{K_N + |N_n|} \leq 0$, we have

$$(I - N_n)v - \frac{\alpha_N}{d} \frac{[Chl]_n N_n}{K_N + |N_n|} \leq -N_n v < 0,$$

so, N_n must decrease, we get

$$\frac{\alpha_N}{d} \frac{[Chl]_n N_n}{K_N + |N_n|} \leq Iv$$

which implies

$$N_n \leq \frac{IvdK_N}{\alpha_N [Chl]_n - Ivd} \leq \frac{2IvdK_N}{\alpha_N [Chl]_n} \text{ when } n \text{ is large enough.}$$

Based on the above inequality, it is easy to check that

$$N_n v \leq \frac{\alpha_N}{d} \frac{[Chl]_n N_n}{K_N + |N_n|} \text{ when } n \text{ is large enough.}$$

When n is large enough and $\frac{1}{2}Iv - \frac{\alpha_N}{d} \frac{[Chl]_n N_n}{K_N + |N_n|} \geq 0$, it is easy to check

$$(I - N_n)v - \frac{\alpha_N}{d} \frac{[Chl]N_n}{K_N + |N_n|} = \left(\frac{1}{2}Iv - \frac{\alpha_N}{d} \frac{[Chl]N_n}{K_N + |N_n|} \right) + \left(\frac{1}{2}Iv - N_n v \right) > 0,$$

then N_n increase. So we conclude that

$$N_n \geq \frac{IvdK_N}{2\alpha_N[Chl]_n + Ivd}.$$

Therefore, we have

$$\frac{IvdK_N}{3\alpha_N[Chl]_n} \leq N_n \leq \frac{2IvdK_N}{\alpha_N[Chl]_n},$$

which implies

$$\frac{C_1}{[Chl]_n(0)} \leq N_n \leq \frac{C_2}{[Chl]_n(0)}.$$

Now, we analysis the third equation. If

$$\frac{\alpha_N C_2}{K_N [Chl]_n(0)} \leq \mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) N_{Bn},$$

then we have

$$\frac{\alpha_N N_n}{K_N + |N_n|} - \mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) N_{Bn} \leq 0,$$

and N_{Bn} should decrease, then we get

$$\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) N_{Bn} \leq \frac{2\alpha_N C_2}{K_N [Chl]_n(0)}. \quad (3.2.7)$$

We claim that $\limsup_{n \rightarrow +\infty} \mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) \leq 0$ on $[0, T]$.

We suppose the contrary. Without loss of generality, we may assume that there exists $\{t_n\} \subset [0, T]$ and a positive constant ε_0 such that

$$\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha}(t_n) - \xi \right) \geq \varepsilon_0,$$

then it is easy to check that

$$N_{Bn}(t_n) \geq \left(\frac{PAR_{\min}^\gamma + PAR^\gamma}{PAR^\gamma} (t_n) N_{B\min}^\alpha \varepsilon_0 / \mu \right)^{1/\alpha} \geq (N_{B\min}^\alpha \varepsilon_0 / \mu)^{1/\alpha}.$$

Thus

$$\mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) N_{Bn}(t_n) \geq \varepsilon_0 (N_{B\min}^\alpha \varepsilon_0 / \mu)^{1/\alpha}.$$

This is a contradiction with (3.2.7).

$$\text{Therefore } \limsup_{n \rightarrow +\infty} \mu \left(\frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} \frac{|N_{Bn}|^\alpha}{N_{B\min}^\alpha + |N_{Bn}|^\alpha} - \xi \right) \leq 0 \text{ on } [0, T].$$

This is impossible for a periodic solution, because in the second equation, $[Chl]$ will keep decrease when n is large enough.

Step 2. We assume $\{[Chl]_n\}$ is bounded in $C[0, T]$.

It is easy to see that $N_n \leq I$, then $\{N_n\}$ is bounded in $C[0, T]$.

Since $\{u_n\}$ is unbounded, $\{N_{Bn}\}$ must be unbounded. We assume $N_{Bn}(t_n^*) = \min_{t \in [0, T]} N_{Bn}(t)$, $N_{Bn}(t_n) = \max_{t \in [0, T]} N_{Bn}(t) \rightarrow +\infty$ as $t_n \rightarrow +\infty$. Without loss of generality, we may assume $t_n^* \leq t_n$.

By the third equation, we have

$$\frac{dN_{Bn}}{dt} \leq \frac{\alpha_N N}{K_N + |N|} + \mu \xi N_{Bn} \leq \frac{\alpha_N I}{K_N + I} + \mu \xi N_{Bn} := a + bN_{Bn}.$$

Notice that $N_{Bn} > 0$. Thus

$$\frac{dN_{Bn}}{dt} \frac{b}{a + bN_{Bn}} \leq b.$$

Integrate from t_n^* to t_n , we have

$$\ln(a + bN_{Bn})|_{t_n^*}^{t_n} \leq bT.$$

Thus

$$\frac{a + bN_{Bn}(t_n)}{a + bN_{Bn}(t_n^*)} \leq e^{bT},$$

namely

$$a + bN_{Bn}(t_n) \leq e^{bT} [a + bN_{Bn}(t_n^*)].$$

Since $N_{Bn}(t_n) \rightarrow +\infty$ as $t_n \rightarrow +\infty$, we have

$$N_{Bn}(t_n^*) \rightarrow +\infty \text{ as } t_n \rightarrow +\infty. \quad (3.2.8)$$

By the third equation it is easy to check that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) N_B \leq \frac{2\alpha_N I}{K_N + I}. \quad (3.2.9)$$

Note that $\xi < C_*$ by (2.2.1). We get a contradiction from (3.2.8) and (3.2.9).

Therefore, all the solutions of (3.2.2) are uniformly bounded. \square

Remark 1. In the proof of Lemma 3.4, if we always have $t_n^* > t_n$, we can integrate from t_n^* to $t_n + T$. For the periodic solution, we will get the same result.

Theorem 3.5 (S) has a periodic solution if $PAR(0) = PAR(T)$ and (2.2.1)-(2.2.3) are satisfied for $C_* = \frac{1}{T} \int_0^T \frac{PAR^\gamma}{PAR_{\min}^\gamma + PAR^\gamma} dt$.

Proof. Denote $\Omega = (\mathbb{R}^+)^3 \cap B(0, R)$ where R is a large enough positive constant. By Lemma 3.4, we know that (3.2.2) has no solution on $\partial\Omega$ for any $\lambda \in (0, 1]$. For $\lambda = 0$, we know that $\omega(a) := \frac{1}{T} \int_0^T f(t, a) dt = 0$ has only one solution. Therefore $\omega(a) = 0$ has no solution on $\partial\Omega$, and $d_B[\omega, \Omega, 0] \neq 0$. According to Lemma 3.3, (S) has a solution u on $[0, T]$. Since the system is nonautonomous, the equilibrium is not a solution, the solution is a nonconstant solution which satisfies $u(0) = u(T)$. Since $PAR(0) = PAR(T)$ and the other coefficient in this system are constant (periodic- T), so we have $u'(0) = u'(T)$, then u is a periodic solution of (S). Obviously, the periodic solution is a limit cycle. \square

CHAPTER 4

THE GLOBAL ASYMPTOTIC STABILITY FOR AUTONOMOUS SYSTEM

In this chapter, we discuss the stability for autonomous system, namely, we assume PAR is a constant-valued function, and discuss the stability for solutions of (S). We show that system (S) has at most 2 equilibria, one corresponding to extinction of the organism which always exists, and possibly a second one corresponding to the presence of the organism.

Denote

$$\begin{aligned}\mathcal{P} &= \{(N, [Chl], N_B) \in \mathbb{R}^3 \mid N, [Chl], N_B \geq 0\}, \\ \mathcal{P}^+ &= \{(N, [Chl], N_B) \in \mathbb{R}^3 \mid N, [Chl], N_B > 0\}, \\ \mathcal{P}^+(M > Z) &= \{(M, Z, [Chl]) \in \mathbb{R}^3 \mid M > Z > 0, [Chl] > 0\}, \\ \mathcal{P}_1 &= \{(N, [Chl], N_B) \in \mathbb{R}^3 \mid N, N_B > 0, [Chl] = 0\}.\end{aligned}$$

4.1 UNIQUE EXTINCTION EQUILIBRIUM POINT

In this section we consider (S) on \mathcal{P}_1 .

Lemma 4.1. System (S) has a unique extinction equilibrium point on \mathcal{P}_1 .

Proof. We assume $[Chl] = 0$, then (S) becomes

$$(S_2) \begin{cases} \frac{dN}{dt} = (I - N)v, \\ \frac{dN_B}{dt} = \frac{\alpha_N N}{K_N + |N|} - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_B^{\alpha} \min + |N_B|^{\alpha}} - \xi \right) N_B. \end{cases}$$

Solving it, we get the unique equilibrium point

$$N = I, N_B = N_B^*,$$

where N_B^* is the unique nonnegative solution of

$$\frac{\alpha_N I}{K_N + I} - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B = 0.$$

Theorem 4.2. The unique extinction equilibrium point of system (S) on \mathcal{P}_1 is locally asymptotically stable.

Proof. By computation, we get the Jacobian at the equilibrium point as following

$$J_0 = \begin{pmatrix} -v & 0 \\ \frac{\alpha_N K_N}{(K_N + I)^2} & -\mu C_* \frac{(\alpha+1)N_{B \min}^\alpha |N_B^*|^\alpha + |N_B^*|^{2\alpha}}{(N_{B \min}^\alpha + |N_B^*|^\alpha)^2} + \mu \xi \end{pmatrix}.$$

We consider the following eigenvalue problem

$$|J_0 - \lambda I| = 0,$$

i.e.

$$\begin{vmatrix} -v - \lambda & 0 \\ \frac{\alpha_N K_N}{(K_N + I)^2} & -\mu C_* \frac{(\alpha+1)N_{B \min}^\alpha |N_B^*|^\alpha + |N_B^*|^{2\alpha}}{(N_{B \min}^\alpha + |N_B^*|^\alpha)^2} + \mu \xi - \lambda \end{vmatrix} = 0,$$

i.e.

$$(-v - \lambda) \left(-\mu C_* \frac{(\alpha+1)N_{B \min}^\alpha |N_B^*|^\alpha + |N_B^*|^{2\alpha}}{(N_{B \min}^\alpha + |N_B^*|^\alpha)^2} + \mu \xi - \lambda \right) = 0.$$

Thus we get two eigenvalues

$$\lambda_1 = -v, \lambda_2 = -\mu C_* \frac{(\alpha+1)N_{B \min}^\alpha |N_B^*|^\alpha + |N_B^*|^{2\alpha}}{(N_{B \min}^\alpha + |N_B^*|^\alpha)^2} + \mu \xi \leq -\mu \left(C_* \frac{|N_B^*|^\alpha}{N_{B \min}^\alpha + |N_B^*|^\alpha} - \xi \right),$$

both are negative. By Lemma 2.8, the unique extinction equilibrium point (I, N_B^*) on \mathcal{P}_1 is locally asymptotically stable on \mathcal{P}_1 . \square

4.2 NONTRIVIAL EQUILIBRIUM POINT

In this section, we consider nontrivial equilibria of system (S), i.e. equilibria with $[Chl] \neq 0$.

Lemma 4.3. System (S) has at most one nontrivial equilibrium point in \mathcal{P}^+ .

Proof. Similar to the analysis of *section 2.2*, we can see that (S) has at most one nontrivial equilibrium point in \mathcal{P}^+ . \square

Lemma 4.4. The nontrivial equilibrium point of system (S) in \mathcal{P}^+ is locally asymptotically stable if $\varepsilon < \varepsilon_2$, and it is unstable if $\varepsilon > \varepsilon_2$, where ε and ε_2 are defined in (2.2.4) and (2.2.8).

Proof. According to Lemma 2.7 and Lemma 2.8, all of the three eigenvalues at nontrivial equilibrium point of system (S) in \mathcal{P}^+ have negative real part, so it is locally asymptotically stable provided $\varepsilon < \varepsilon_2$; and two eigenvalues have positive real part provided $\varepsilon > \varepsilon_2$, then (S) is unstable. \square

Lemma 4.5. If D is large enough, then every solution of (S) tends to \mathcal{P}_1 . Moreover, (S) has no periodic solution.

Proof. By the first equation of (S) , we have $N < I$. By the third equation, we have

$$\begin{aligned} \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) &\leq \begin{cases} 2\alpha_N, & \text{if } N_B \geq 1 \\ \mu C_* \frac{1}{N_{B\min}^\alpha + 1}, & \text{if } N_B < 1 \end{cases} \\ &\leq 2\alpha_N + \mu C_* \frac{1}{N_{B\min}^\alpha + 1}. \end{aligned}$$

If $D > 2\alpha_N + \mu C_* \frac{1}{N_{B\min}^\alpha + 1}$, we have

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) - D < 0.$$

By the second equation of (S) , we have

$$\frac{d[Chl]}{dt} = [Chl] \left[\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) - D \right] < 0,$$

so it is a contradiction with the property of periodic solution for any initial value in \mathcal{P}^+ .

Therefore, there is no periodic solution. \square

Lemma 4.6. For any positive solution of (S) , we have the following estimations:

- (i) There exists a $t_0 \geq 0$ such that $N(t) \leq 2I$ for any $t \geq t_0$;
- (ii) If t is large enough, then $C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi > 0$ provided (2.2.1);
- (iii) There exists a point $t_1 \geq 0$ such that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t_1) \leq 2\alpha_N.$$

Proof. (i) We claim that there exists a $t_0 \geq 0$ such that $N(t_0) \leq 2I$.

We suppose the contrary. Assume $N(t) > 2I$ for $t \geq 0$. By the first equation of (S) , we can see

$$\frac{dN}{dt} \leq (I - N)v = -(N - I)v, \forall t \geq 0.$$

Dividing by $(N - I)$ and integrate from 0 to t , we have

$$(\ln(N - I))|_0^t \leq -vt, \forall t \geq 0.$$

Rewriting it, we have

$$N(t) \leq I + e^{-vt}(N(0) - I) \text{ for any } t \geq 0.$$

When t is large enough, it is easy to see that $N(t) \leq 2I$. It is a contrary. Therefore, there exists a $t_0 \geq 0$ such that $N(t_0) \leq 2I$.

By the first equation of (S) , it is easy to see that

$$\frac{dN}{dt} = (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|} < 0 \text{ if } N \geq I.$$

Thus, $N(t) \leq 2I$ for $t \geq t_0$.

(ii) Note that we only consider positive solution. If $C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi = 0$ at some point t_0 , then

$$\frac{dN_B}{dt} = \frac{\alpha_N N}{K_N + |N|} - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B > 0 \text{ at } t = t_0,$$

and it is easy to see that $C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi > 0$ for $t > t_0$. So it only remains to prove that there exists a point t_1 such that $C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha}(t_1) - \xi \geq 0$.

We suppose the contrary. We assume

$$C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi < 0 \text{ for } t \geq 0. \quad (4.2.1)$$

Thus

$$\frac{dN_B}{dt} > 0 \text{ for } t \geq 0,$$

and

$$\frac{d[Chl]}{dt} < 0 \text{ for } t \geq 0.$$

Thus it is easy to see that N has positive lower bound N_{\min} . Therefore

$$\frac{dN_B}{dt} \geq \frac{\alpha_N N_{\min}}{K_N + N_{\min}}.$$

Therefore $N_B \rightarrow +\infty$ as $t \rightarrow +\infty$. This combine (2.2.1) get a contradiction with (4.2.1).

Therefore there exists a point t_1 such that $C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \geq 0$, then $C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi > 0$ for large enough t .

(iii) By the third equation of (S), we have

$$\frac{dN_B}{dt} \leq \alpha_N - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B, \forall t \geq 0.$$

We claim that there exists a point $t_1 \geq 0$ such that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t_1) \leq 2\alpha_N.$$

We assume the contrary, i.e.

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B > 2\alpha_N, \forall t \geq 0. \quad (4.2.2)$$

By the third equation of (S), we have

$$\begin{aligned} \frac{dN_B}{dt} &\leq \alpha_N - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B \\ &\leq \alpha_N - 2\alpha_N = -\alpha_N, \forall t \geq 0. \end{aligned}$$

Integrating from 0 to t , we have

$$N_B(t) \leq N_B(0) - \alpha_N t, \forall t \geq 0.$$

It is a contradiction with (4.2.2).

Thus, there exists a point $t_1 \geq 0$ such that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t_1) \leq 2\alpha_N,$$

and it is easy to see that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t) \leq 2\alpha_N, \forall t \geq t_1.$$

So, there exists a positive constant C_1 such that

$$N_B(t) \leq C_1, \forall t \geq t_1. \quad (4.2.3)$$

Lemma 4.7. The ω -limit set of solution $[Chl]$ is bounded provided (2.2.1)-(2.2.3).

Proof. We prove this result in two steps.

Step A. We prove the existence of finite upper bound.

Step B. We prove the existence of positive lower bound.

Step A. We prove the existence of finite upper bound.

We assume the contrary, i.e. $\overline{\lim}_{t \rightarrow +\infty} [Chl](t) = +\infty$, then there exists a sequence $\{t_n\}$ satisfies: (1⁰) $t_n \geq n^3$ and increase, (2⁰) $[Chl](t_n) = \max_{t \in [0, t_n]} [Chl](t)$, (3⁰) $[Chl](t_n) \geq ne^{\mu C_* n^3}$.

We get a contradiction in three steps.

Step A-(i) $[Chl](t) \geq n, \forall t \in (t_n - n^3, t_n)$.

Step A-(ii) $N(t) \leq \frac{3lvdK_N}{n\alpha_N}, \forall t \in (t_n - n^3 + \frac{2}{v}, t_n)$, when n is large enough.

Step A-(iii) $\lim_{n \rightarrow \infty} \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) = 0$ for $t \in (t_n - n^3 + \frac{2}{v} + n^2, t_n)$ uniformly.

Step A-(i). We prove $[Chl](t) \geq n, \forall t \in (t_n - n^3, t_n)$.

From the second equation of (S), we have

$$\frac{d[Chl]}{dt} \leq [Chl] \mu C_*, \forall t \geq 0.$$

Dividing by $[Chl]$ and integrating from t to t_n , we have

$$(\ln[Chl]) \Big|_t^{t_n} \leq \mu C_*(t_n - t), \forall t \leq t_n.$$

Rewriting it, we have

$$[Chl](t) \geq [Chl](t_n)e^{-\mu C_*(t_n-t)} \geq ne^{\mu C_* n^3} e^{-\mu C_* n^3} = n, \forall t \in (t_n - n^3, t_n). \quad (4.2.4)$$

Step A-(ii). We prove $N(t) \leq \frac{3IvdK_N}{n\alpha_N}, \forall t \in (t_n - n^3 + \frac{2}{v}, t_n)$, when n is large enough.

When n is large enough, for each interval $(t_n - n^3, t_n)$, if there exists $t_n^\# \in (t_n - n^3, t_n)$ such that

$$N(t_n^\#) = \frac{3IvdK_N}{n\alpha_N},$$

then (4.2.4) and the first equation of (S) implies

$$\begin{aligned} N'(t_n^\#) &= \left(I - \frac{3IvdK_N}{n\alpha_N}\right)v - \frac{\alpha_N}{d} \frac{[Chl]^{\frac{3IvdK_N}{n\alpha_N}}}{K_N + \frac{3IvdK_N}{n\alpha_N}} \\ &\leq \left(I - \frac{3IvdK_N}{n\alpha_N}\right)v - \frac{\alpha_N}{d} \frac{n^{\frac{3IvdK_N}{n\alpha_N}}}{K_N + \frac{3IvdK_N}{n\alpha_N}} \\ &= \left(I - \frac{3IvdK_N}{n\alpha_N}\right)v - \frac{3Iv}{1 + \frac{3Ivd}{n\alpha_N}} \leq -Iv \text{ when } n \text{ is large enough.} \end{aligned}$$

Thus

$$N(t) < \frac{3IvdK_N}{n\alpha_N}, \forall t \in (t_n^\#, t_n).$$

Similarly, if there exists $t_n^\# \in (t_n - n^2, t_n)$ such that

$$N(t_n^\#) \leq \frac{3IvdK_N}{n\alpha_N},$$

then

$$N(t) \leq \frac{3IvdK_N}{n\alpha_N}, \forall t \in (t_n^\#, t_n).$$

We claim that there exists $t_n^\# \in (t_n - n^3, t_n - n^3 + \frac{2}{v})$ such that

$$N(t_n^\#) \leq \frac{3IvdK_N}{n\alpha_N}, \text{ when } n \text{ is large enough.}$$

We assume the contrary, i.e.

$$N(t) > \frac{3IvdK_N}{n\alpha_N}, \forall t \in (t_n - n^3, t_n - n^3 + \frac{2}{v}), \text{ when } n \text{ is large enough.}$$

By the first equation of (S), we have

$$\begin{aligned}
\frac{dN}{dt} &\leq \left(I - \frac{3IvdK_N}{n\alpha_N} \right) v - \frac{\alpha_N}{d} \frac{[Chl] \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}} \\
&\leq \left(I - \frac{3IvdK_N}{n\alpha_N} \right) v - \frac{\alpha_N}{d} \frac{n \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}} \quad (\text{by (4.2.4)}) \\
&= \left(I - \frac{3IvdK_N}{n\alpha_N} \right) v - \frac{3Iv}{1 + \frac{3Ivd}{n\alpha_N}} \leq -Iv \quad \text{when } n \text{ is large enough.}
\end{aligned}$$

Integrating from $t_n - n^3$ to t , we have

$$N(t) \leq N(t_n - n^3) - Iv [t - (t_n - n^3)], \forall t \in (t_n - n^3, t_n - n^3 + \frac{2}{v}).$$

Notice that $N(t) \leq 2I$ when n is large enough. The above inequality means

$$N(t_n - n^3 + \frac{2}{v}) \leq 2I - Iv \frac{2}{v} \leq 0.$$

It is a contradiction.

Thus, there exists $t_n^\# \in (t_n - n^3, t_n - n^3 + \frac{2}{v})$ such that

$$N(t_n^\#) \leq \frac{3IvdK_N}{n\alpha_N}, \quad \text{when } n \text{ is large enough.}$$

Therefore

$$N(t) \leq \frac{3IvdK_N}{n\alpha_N}, \forall t \in (t_n - n^3 + \frac{2}{v}, t_n), \quad \text{when } n \text{ is large enough.} \quad (4.2.5)$$

Step A-(iii). We prove $\lim_{n \rightarrow \infty} \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) = 0$ for $t \in (t_n - n^3 + \frac{2}{v} + n^2, t_n)$ uniformly.

When n is large enough, we claim that there exists $t_n^\# \in (t_n - n^3 + \frac{2}{v}, t_n - n^3 + \frac{2}{v} + n^2)$ such that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t_n^\#) \leq \frac{2\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}}, \quad \text{when } n \text{ is large enough.}$$

We assume the contrary, i.e. when n is large enough

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t) > \frac{2\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}}, \forall t \in (t_n - n^3 + \frac{2}{v}, t_n - n^3 + \frac{2}{v} + n^2).$$

By (4.2.5), when n is large enough, the third equation of (S) implies

$$\begin{aligned} \frac{dN_B}{dt} &= \frac{\alpha_N N}{K_N + |N|} - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B \\ &\leq \frac{\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}} - \frac{2\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}} \\ &= -\frac{\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}}, \forall t \in (t_n - n^3 + \frac{2}{v}, t_n - n^3 + \frac{2}{v} + n^2). \end{aligned}$$

Integrating from $t_n - n^3 + \frac{2}{v}$ to $t_n - n^3 + \frac{2}{v} + n^2$, we have

$$\begin{aligned} &N_B(t_n - n^3 + \frac{2}{v} + n^2) \\ &\leq N_B(t_n - n^3 + \frac{2}{v}) - \frac{\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}} n^2 \\ &\leq C_1 - \frac{\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}} n^2 \leq 0 \text{ when } n \text{ is large enough (by Lemma 4.6-(iii)).} \end{aligned}$$

It is a contradiction. Thus, there exists $t_n^\# \in (t_n - n^3 + \frac{2}{v}, t_n - n^3 + \frac{2}{v} + n^2)$ such that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t_n^\#) \leq \frac{2\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}},$$

when n is large enough. Combining this and (4.2.5) we have

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t) \leq \frac{2\alpha_N \frac{3IvdK_N}{n\alpha_N}}{K_N + \frac{3IvdK_N}{n\alpha_N}},$$

for any $t \in (t_n - n^3 + \frac{2}{v} + n^2, t_n)$. This combines Lemma 4.6-(ii) implies

$$\lim_{n \rightarrow \infty} \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) = 0$$

for $t \in (t_n - n^3 + \frac{2}{v} + n^2, t_n)$ uniformly. Substituting this into the second equation of (S), we

have

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) - D < 0$$

for any $t \in (t_n - n^3 + \frac{2}{v} + n^2, t_n)$, when n is large enough. This is a contradiction with

$[Chl](t_n) = \max_{t \in [0, t_n]} [Chl](t)$. Therefore, $\overline{\lim}_{t \rightarrow +\infty} [Chl](t) < +\infty$.

Step B. We prove the existence of positive lower bound.

We assume the contrary, i.e.

$$\lim_{t \rightarrow +\infty} [Chl](t) = 0.$$

Since this is a positive solution and $\lim_{t \rightarrow +\infty} [Chl](t) = 0$, there exists a sequence $\{t_n\}$ satisfies: (1⁰) $t_n \geq n$ and increase, (2⁰) $[Chl](t_n) = \min_{t \in [0, t_n]} [Chl](t)$, (3⁰) $[Chl](t_n) \leq \frac{1}{n} e^{-(D+\mu\xi)n}$.

We get a contradiction in three steps.

$$\text{Step B-(i)} \quad [Chl](t) \leq \frac{1}{n}, \forall t \in (t_n - n, t_n).$$

$$\text{Step B-(ii)} \quad N(t) \geq I_n := \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I, \text{ for any } t \in (t_n - n + \frac{1}{v} \ln n, t_n).$$

$$\text{Step B-(iii)} \quad \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_B^\alpha \min + |N_B|^\alpha} - \xi \right) > D, \text{ for any } t \in (t_n - \frac{n}{2}, t_n).$$

$$\text{Step B-(i)}. \text{ We prove } [Chl](t) \leq \frac{1}{n}, \forall t \in (t_n - n, t_n).$$

By the second equation of (S), we have

$$\frac{d[Chl]}{dt} \geq -[Chl] (D + \mu\xi), \forall t \geq 0.$$

Dividing both side by $[Chl]$ and integrating from t to t_n ,

$$(\ln[Chl])|_t^{t_n} \geq -(D + \mu\xi)(t_n - t), \forall t \leq t_n.$$

Therefore

$$\frac{[Chl](t_n)}{[Chl](t)} \geq e^{-(D+\mu\xi)(t_n-t)}, \forall t \leq t_n,$$

then

$$[Chl](t) \leq [Chl](t_n) e^{(D+\mu\xi)(t_n-t)} \leq \frac{1}{n} e^{-(D+\mu\xi)n} e^{(D+\mu\xi)n} = \frac{1}{n}, \forall t \in (t_n - n, t_n). \quad (4.2.6)$$

$$\text{Step B-(ii)}. \text{ We prove } N(t) \geq I_n := \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I, \text{ for any } t \in (t_n - n + \frac{1}{v} \ln n, t_n).$$

By (4.2.6), the first equation of (S) implies

$$\begin{aligned} \frac{dN}{dt} &= (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|} \\ &\geq \left(I - N \left(1 + \frac{\alpha_N}{ndvK_N} \right) \right) v := \left(I - N \left(1 + \frac{a}{n} \right) \right) v, \forall t \in (t_n - n, t_n). \end{aligned} \quad (4.2.7)$$

We claim that there exists a point $t_n^* \in [t_n - n, t_n - n + \frac{1}{v} \ln n]$ such that $N(t_n^*) \geq \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I$.

We assume the contrary,

$$N(t) < \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I, \forall t \in [t_n - n, t_n - n + \frac{1}{v} \ln n].$$

Rewriting (4.2.7), we have

$$\frac{d \left[\left(1 + \frac{a}{n}\right) N - I \right] / dt}{I - \left(1 + \frac{a}{n}\right) N} \geq \left(1 + \frac{a}{n}\right) v, \forall t \in (t_n - n, t_n - n + \frac{1}{v} \ln n).$$

Integrating from $t_n - n$ to t , we have

$$-\ln \left(I - \left(1 + \frac{a}{n}\right) N \right) \Big|_{t_n - n}^t \geq \left(1 + \frac{a}{n}\right) v [t - (t_n - n)], \forall t \in (t_n - n, t_n - n + \frac{1}{v} \ln n).$$

Rewriting it we have

$$\begin{aligned} I - \left(1 + \frac{a}{n}\right) N(t) &\leq e^{-(1 + \frac{a}{n})v[t - (t_n - n)]} \left[I - \left(1 + \frac{a}{n}\right) N(t_n - n) \right] \\ &\leq e^{-v[t - (t_n - n)]} I, \forall t \in (t_n - n, t_n - n + \frac{1}{v} \ln n). \end{aligned}$$

Therefore

$$N(s_n + \tau) \geq \frac{(1 - e^{-v[t - (t_n - n)]})}{(1 + \frac{a}{n})} I, \forall t \in (t_n - n, t_n - n + \frac{1}{v} \ln n).$$

It is a contradiction. Thus, there exists a point $t_n^* \in [t_n - n, t_n - n + \frac{1}{v} \ln n]$ such that $N(t_n^*) \geq \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I$.

We claim that $N(t) \geq \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I$ for any $t \in [t_n - n + \frac{1}{v} \ln n, t_n]$.

Assume $N(t_n^*) = \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I$. It is easy to see that

$$\frac{dN}{dt} \Big|_{t_n^*} \geq \left(I - \left(1 + \frac{a}{n}\right) N(t_n^*) \right) v = \frac{Iv}{n} > 0.$$

Assume $N(t_n^*) > \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I$. If there exists $t_{0n} \in (t_n^*, t_n)$ such that

$$N(t_{0n}) = \frac{1 - \frac{1}{n}}{1 + \frac{a}{n}} I = \min_{t \in [t_n^*, t_{0n}]} N(t).$$

Then $N'(t_{0n}) \leq 0$. It is easy to see that

$$\frac{dN}{dt} \Big|_{t_{0n}} \geq \left(I - \left(1 + \frac{a}{n}\right) N(t_n^*) \right) v = \frac{Iv}{n} > 0.$$

It is a contradiction. Therefore, $N(t) \geq \frac{1-\frac{1}{v}}{1+\frac{a}{n}}I$ for any $t \in [t_n - n + \frac{1}{v} \ln n, t_n]$.

Denote

$$s_n^* = t_n - n + \frac{1}{v} \ln n. \quad (4.2.8)$$

We have $t_n - s_n^* = n - \frac{1}{v} \ln n \rightarrow +\infty$. Thus

$$N(t) \geq I_n := \frac{1-\frac{1}{v}}{1+\frac{a}{n}}I, \forall t \in (s_n^*, t_n). \quad (4.2.9)$$

Step B-(iii). We prove $\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) > D$, for any $t \in (t_n - \frac{n}{2}, t_n)$, when n is large enough.

Now, let us consider the third equation of (S).

By (4.2.9) and the third equation of (S), we have

$$\frac{dN_B}{dt} \geq \frac{\alpha_N I_n}{K_N + I_n} - \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B, \text{ for any } t \in (s_n^*, t_n).$$

Denote by C_n the positive solution of $\mu \left(C_* \frac{x^\alpha}{N_{B \min}^\alpha + x^\alpha} - \xi \right) x = (1 - \frac{1}{\sqrt{n}}) \frac{\alpha_N I_n}{K_N + I_n}$. Obviously, $\{C_n\}$ is bounded. We claim that there exists $t_n^* \in (s_n^*, s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n})$ such that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t_n^*) \geq (1 - \frac{1}{\sqrt{n}}) \frac{\alpha_N I_n}{K_N + I_n}.$$

We assume the contrary.

If

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B \min}^\alpha + |N_B|^\alpha} - \xi \right) N_B < (1 - \frac{1}{\sqrt{n}}) \frac{\alpha_N I_n}{K_N + I_n}, \forall t \in (s_n^*, s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}), \quad (4.2.10)$$

then the third equation of (S) implies

$$\frac{dN_B}{dt} \geq \frac{1}{2\sqrt{n}} \frac{\alpha_N I_n}{K_N + I_n}, \forall t \in (s_n^*, s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}),$$

then

$$\begin{aligned} N_B(t) &\geq N_B(t) - N_B(s_n^*) \\ &> \frac{1}{2\sqrt{n}} \frac{\alpha_N I_n}{K_N + I_n} (t - s_n^*), \forall t \in (s_n^*, s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}). \end{aligned}$$

This means

$$N_B(s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}) > C_n.$$

By the definition of C_n , it is easy to see that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}) > (1 - \frac{1}{\sqrt{n}}) \frac{\alpha_N I_n}{K_N + I_n}.$$

This contradict with (4.2.10).

Thus, there exists $t_n^* \in (s_n^*, s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n})$ such that

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) N_B(t_n^*) \geq (1 - \frac{1}{\sqrt{n}}) \frac{\alpha_N I_n}{K_N + I_n},$$

and then

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) N_B \geq (1 - \frac{1}{\sqrt{n}}) \frac{\alpha_N I_n}{K_N + I_n}, \forall t \in (t_n^*, t_n).$$

Therefore, when n is large enough, we have

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) N_B \approx \frac{\alpha_N I}{K_N + I}, \forall t \in (s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}, t_n).$$

By (2.2.3), we have

$$\frac{\alpha_N I}{K_N + I} > D \left(\frac{(\xi + D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}.$$

Combining above results, when n is large enough, for any $t \in (s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}, t_n)$, we have

$$\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) N_B \approx \frac{\alpha_N I}{K_N + I} > D \left(\frac{(\xi + D/\mu)N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}. \quad (4.2.11)$$

Denote

$$x = \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right),$$

then

$$N_B = \left(\frac{(\xi + x/\mu)N_{B\min}^\alpha}{C_* - \xi - x/\mu} \right)^{1/\alpha}.$$

Thus we can rewrite (4.2.11) as

$$x \left(\frac{(\xi + x/\mu) N_{B\min}^\alpha}{C_* - \xi - x/\mu} \right)^{1/\alpha} > D \left(\frac{(\xi + D/\mu) N_{B\min}^\alpha}{C_* - \xi - D/\mu} \right)^{1/\alpha}, \forall t \in (s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}, t_n).$$

Obviously, $x > D, \forall t \in (s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}, t_n)$ when n is large enough. Therefore

$$x = \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_{B\min}^\alpha + |N_B|^\alpha} - \xi \right) > D, \forall t \in (s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}, t_n),$$

when n is large enough.

Thus $[Chl]$ should strictly increase in the interval $(s_n^* + \frac{2\sqrt{n}(K_N + I_n)C_n}{\alpha_N I_n}, t_n)$, where s_n^* is defined in (4.2.8). This is a contradiction with (2⁰): $[Chl](t_n) = \min_{t \in [0, t_n]} [Chl](t)$.

In conclusion, $\lim_{t \rightarrow +\infty} [Chl](t) > 0$. \square

Lemma 4.8. The ω -limit set of N and N_B are bounded provided (2.2.1)-(2.2.3).

Proof. According to Lemma 4.6 and the *Step A* of proof of Lemma 4.7, N and N_B have finite upper bound. According to the analysis of Lemma 4.7, $[Chl]$ has an upper bound $M > 1$.

We claim that there exists $t_2 > 0$ such that

$$\frac{Iv}{2} \leq \left(Nv + \frac{\alpha_N}{d} \frac{MN}{K_N + |N|} \right) (t_2).$$

We assume the contrary, i.e.

$$\frac{Iv}{2} \geq Nv + \frac{\alpha_N}{d} \frac{MN}{K_N + |N|}, \forall t > 0.$$

Then we have

$$\frac{dN}{dt} = (I - N)v - \frac{\alpha_N}{d} \frac{[Chl]N}{K_N + |N|} \geq Iv - \frac{Iv}{2} = \frac{Iv}{2}.$$

Thus

$$N(t) \geq N(0) + \frac{Iv}{2}t.$$

Therefore

$$\left(Nv + \frac{\alpha_N}{d} \frac{MN}{K_N + |N|} \right) (t) \geq vN(t) \geq \frac{Iv^2}{2}t \geq Iv, \forall t > \frac{2}{v}.$$

It is a contradiction. Therefore, there exists $t_2 > 0$ such that

$$\frac{Iv}{2} \leq \left(Nv + \frac{\alpha_N}{d} \frac{MN}{K_N + |N|} \right) (t_2).$$

It is easy to check that

$$\frac{Iv}{2} \leq \left(Nv + \frac{\alpha_N}{d} \frac{MN}{K_N + |N|} \right) (t) \leq \left(Nv + \frac{\alpha_N}{d} \frac{MN}{K_N} \right) (t), \forall t \geq t_2.$$

Thus

$$N(t) \geq \frac{Iv}{2(v + \frac{\alpha_N}{d} \frac{M}{K_N})}, \forall t \geq t_2. \quad (4.2.12)$$

By the third equation of (S) and (4.2.12), when t is large enough, we have

$$\begin{aligned} \mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_B^\alpha + |N_B|^\alpha} - \xi \right) N_B &\geq \frac{1}{2} \frac{\alpha_N N}{K_N + |N|} \\ &\geq \frac{1}{2} \frac{\alpha_N \frac{Iv}{2(v + \frac{\alpha_N}{d} \frac{M}{K_N})}}{K_N + \frac{Iv}{2(v + \frac{\alpha_N}{d} \frac{M}{K_N})}}. \end{aligned}$$

Similar to the proof of Lemma 4.6, we can get a positive lower bound of N_B such that $\mu \left(C_* \frac{|N_B|^{\alpha-1} N_B}{N_B^\alpha + |N_B|^\alpha} - \xi \right)$ also has positive lower bound when t is large enough. \square

Theorem 4.9. For any initial value in \mathcal{P}^+ , every orbit has compact closure in \mathcal{P}^+ provided (2.2.1)-(2.2.3).

Proof. Combining Lemma 4.7 and Lemma 4.8 get this result. \square

Proposition 4.10 (see [5, Theorem B]). Suppose that the following conditions hold:

- (a) $X = \mathbb{R}^3$, or \mathbb{R}_+^3 , or $[[p, q]]$,
- (b) f is monotone, i.e. $\frac{\partial f_i}{\partial x_j} \geq 0$ for $i \neq j$,
- (c) every forward semi-orbit has compact closure in X ; and
- (d) there is a unique equilibrium p of (2.1.2), and it is asymptotically stable.

Then the equilibrium point p is globally asymptotically stable.

Theorem 4.11. If $v \geq D$, and the unique equilibrium point of (S) is asymptotically stable, then it is global asymptotically stable provided (2.2.1)-(2.2.3).

Proof. We make the transformation,

$$\begin{aligned} M &= N + \frac{N_B \cdot [Chl]}{d}, \\ Z &= N_B \cdot [Chl]. \end{aligned}$$

So, we have

$$(S_3) \begin{cases} \frac{dM}{dt} = Iv - vM + \frac{v-D}{d} Z, \\ \frac{dZ}{dt} = -DZ + [Chl] \alpha_N \frac{M-Z/d}{K_N + M-Z/d}, \\ \frac{d[Chl]}{dt} = [Chl] \left[\mu \left(C_* \frac{Z^\alpha}{N_B^\alpha \min\{[Chl]^\alpha + Z^\alpha\}} - \xi \right) - D \right]. \end{cases}$$

Let us check the conditions of Proposition 4.10 one by one.

It is easy to check that the map $\mathcal{P}^+ \rightarrow \mathcal{P}^+(M > Z) : (N, N_B, [Chl]) \mapsto (M, Z, [Chl])$ is one to one and inversable, so (a) is satisfied.

Notice that $v \geq D$. It is easy to see that (S_3) is monotone, so (b) is satisfied.

According to the Theorem 4.9, (c) is satisfied.

According to the Lemma 4.3, (d) is satisfied.

In conclusion, $(M, Z, [Chl])$ is global asymptotically stable in $\mathcal{P}^+(M > Z)$. Therefore, $(N, N_B, [Chl])$ is global asymptotically stable in \mathcal{P}^+ . \square

Remark 2. (i) If $D = 0 = \xi$, it is easy to see that $\frac{d[Chl]}{dt}$ doesn't change sign, so the system will blowup as $t \rightarrow +\infty$, i.e. $[Chl](t) \rightarrow +\infty$ as $t \rightarrow +\infty$ for any positive initial value of $[Chl]$.

(ii) According to the continuity of the solutions dependent on the parameters, it's hard to get a periodic solution for (S) under autonomous assumption.

4.3 BOUNDEDNESS OF SOLUTIONS

In this section we assume $\xi = 0$. Similar to the proof of section 3, we prove that all the solution of (3.2.2) are uniformly bounded.

Proposition 4.12. All the solution of (3.2.2) are uniformly bounded.

Proof. We suppose, on the contrary. If we have an unbounded solution sequence $\{(u_n, \lambda_n)\}$.

We prove our result in two steps.

Step 1. We assume $\{[Chl]_n\}$ is unbounded in $C[0, T]$.

By the second equation, we have

$$[Chl]_n(t) = [Chl]_n(0) \cdot \exp \left\{ \lambda_n \int_0^t \left[\mu \left(C_* \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} \right) - D \right] ds \right\}.$$

Obviously, $\exp \left\{ \lambda_n \int_0^t \left[\mu \left(C_* \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} \right) - D \right] ds \right\}$ is bounded and strictly positive on $[0, T]$. So we must have a subsequence (which we still denote it as $[Chl]_n$) satisfies $[Chl]_n(0) \rightarrow \infty$. If $[Chl]_n(0) \rightarrow +\infty$, based on the proof of Lemma 3.4, it is impossible. Without loss of generality, we may assume that $[Chl]_n(0) \rightarrow -\infty$.

From the first equation, we can see that

$$(I - N_n)v - \frac{\alpha_N}{d} \frac{[Chl]_n N_n}{K_N + |N_n|} = 0 \text{ at some point } t_n \in (0, T).$$

Notice that there is no sign change periodic solution of N . If we get a negative solution of N_n . Substituting it into the third equation, there is no solution N_B .

Now, we consider positive solution. Rewriting it, we have

$$N_n^2 + (K_N - I + \frac{\alpha_N}{dv} [Chl]_n) N_n - I K_N = 0 \text{ at some point } t_n \in (0, T),$$

we get

$$N_n = \left[- \left(K_N - I + \frac{\alpha_N}{dv} [Chl]_n \right) \pm \sqrt{(K_N - I + \frac{\alpha_N}{dv} [Chl]_n)^2 + 4 I K_N} \right] / 2$$

then the positive solution is

$$N_n \approx \frac{\alpha_N}{dv} |[Chl]_n| \text{ at some point } t_n \in (0, T).$$

It is easy to check that: If N_n is a little larger, then N_n decrease, and if N_n is a little smaller, then N_n increase. Thus

$$N_n \approx \frac{\alpha_N}{dv} |[Chl]_n| \text{ for all } t \in (0, T). \quad (4.3.1)$$

Consider the third equation, we have

$$\frac{\alpha_N N_n}{K_N + |N_n|} - \mu \left(C_* \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} \right) N_{Bn} = 0 \text{ for some } t_0 \in (0, T).$$

Solving the following equation,

$$\frac{\alpha_N N_n}{K_N + |N_n|} - \mu \left(C_* \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} \right) N_{Bn} = 0 \quad (4.3.2)$$

we get the positive solution $N_{Bn} = F(N_n)$ (No sign change solution for N_B). We have: If N_{Bn} is a little larger, then N_{Bn} decrease, and if N_{Bn} is a little smaller, then N_{Bn} increase. So, $N_{Bn} \approx F(N_n)$ when n is large enough.

Combining (4.3.1) and (4.3.2), we have

$$\alpha_N - \mu \left(C_* \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} \right) N_{Bn} \approx 0 \text{ when } n \text{ is large enough.}$$

Denote

$$x = \mu \left(C_* \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} \right),$$

we have

$$N_{Bn} = \left(\frac{(\xi + x/\mu) N_{B \min}^\alpha}{C_* - x/\mu} \right)^{1/\alpha},$$

then

$$\alpha_N \approx x \left(\frac{(\xi + x/\mu) N_{B \min}^\alpha}{C_* - x/\mu} \right)^{1/\alpha} > D \left(\frac{(\xi + D/\mu) N_{B \min}^\alpha}{C_* - D/\mu} \right)^{1/\alpha} \quad (\text{by (2.2.2)}).$$

Thus $\mu \left(C_* \frac{|N_{Bn}|^{\alpha-1} N_{Bn}}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} \right) = x > D$ keep positive. This is impossible.

If we get the negative solution N_B , then substitute it into the second equation, it is easy to see that it is not a periodic solution.

Step 2. We assume $\{[Chl]_n\}$ is bounded in $C[0, T]$, and $|[Chl]_n| \leq B$.

So it is easy to see that $\{N_n\}$ is bounded in $C[0, T]$. Especially, we have $0 < N_n < I + \frac{\alpha_N}{dv} B$.

Let's consider the third equation. Notice that $\xi = 0$. It is not hard to check that

$$\mu C_* \frac{|N_{Bn}|^\alpha}{N_{B \min}^\alpha + |N_{Bn}|^\alpha} |N_{Bn}| \leq \frac{1.5\alpha_N N_n}{K_N + |N_n|} < \frac{1.5\alpha_N (I + \frac{\alpha_N}{dv} B)}{K_N + I + \frac{\alpha_N}{dv} B} \leq 1.5\alpha_N. \quad (4.3.3)$$

It is easy to see that $\{|N_{Bn}|\}$ is bounded in $C[0, T]$.

In conclusion, all the solution of (3.2.2) are bounded in C . \square

Remark 3. In the proof of Proposition 4.12, the initial value is belong to \mathbb{R}^3 , not restricted in \mathcal{P}^+ .

CHAPTER 5

FUTURE WORK

We plan to apply the Ecosystem to the Northeastern Puerto Rico data. The data are currently being collected and organized. We can apply the Markov Chain Monte Carlo (MCMC) methods to obtain the solutions in the system.

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