On Computing the Thom-Boardman Symbols for Polynomial Multiplication Maps

by

Janice Wethington

(Under the direction of Robert Varley)

Abstract

A conjecture by R. Varley states that the Thom-Boardman invariant for polynomial multiplication maps can be computed by using the Euclidean algorithm on the degrees of the polynomials. This thesis provides some history of the problem, its connection with secant maps and Gauss maps, proofs of classes of cases, and it develops a theory which gives an upper bound on the invariant that agrees with the conjectured invariant. We construct monomial ideals which have the conjectured invariant and discuss the generalization of this construction to polynomial ideals. There is also a discussion on the symmetric product of a smooth curve, along with some basic deformation theory, that is used in the proof of a proposition concerning versality of families of hyperplane sections and transversality to a stratification of the symmetric product.

Index words: Singularities, Thom-Boardman Invariants, Symmetric Product, Versal Deformations, Transversality, Secant Maps

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INTRODUCTION

The following work has its origins in the study of complex analytic maps from one complex manifold to another. In 1956, R. Thom developed a method to classify singularities of differentiable maps according to the rank of the first differential of the map and the ranks of its restrictions to submanifolds of singularities. His theory depended upon the manifold structure of the singular locus of each restriction of the map. J.M. Boardman published a paper in 1967 generalizing Thom's work to include maps whose singular loci may fail to be manifolds, or whose successive restrictions may fail to be manifolds. In effect, Boardman expanded Thom's work to almost all differentiable maps on manifolds. The Thom-Boardman classification is realized by an infinite, non-increasing sequence of nonnegative integers referred to as the Thom-Boardman invariant. When the number of nonzero terms is finite, the symbol for the invariant is usually truncated after the last nonzero entry.

Joint work at the University of Georgia by Professors M. Adams, C. McCrory, T. Shifrin, and R. Varley concerned invariants of Gauss maps of theta divisors. Their investigation revealed an fundamental connection between these Gauss maps and secant maps. Continued work by R. Varley indicated a connection between secant maps and maps defined by the multiplication of two single-variable polynomials. The multiplication maps take the coefficients of two polynomials to expressions in those coefficients that describe the coefficients of the product of the two polynomials. The classification by singularities of these polynomial multiplication maps would result in the classification of the secant maps. However, the Thom-Boardman symbol is usually difficult to compute. Even in the case of the polynomial multiplication maps the computations become extremely difficult in all but a small number of cases. Then a conversation with V. Goryunov led to Varley's conjecture that computing the Thom-Boardman symbol for polynomial multiplication reduces to computing the successive quotients and remainders for the Euclidean algorithm applied to the degrees of the two polynomials.

The first chapter of this thesis contains the definitions and constructions of the Thom-Boardman invariant and the polynomial multiplication maps. The Jacobian matrix of an ideal generated by a set of expressions $g_1, \ldots, g_n$ is the matrix formed by taking partial derivatives of those equations with respect to the variables used. The first chapter includes a discussion of the structure inherent in the first Jacobian matrices of the multiplication maps and a weighting system is introduced to help keep track of the structure of ideals that occur after sucessive steps of the Thom-Boardman computations.

The second chapter introduces some background material which, with chapter 3, motivates interest in the research. It begins with a discussion of the geometry of a particular stratification of the dth symmetric product of a nondegenerate compact complex curve C of degree d in $\mathbb{P}^n$. This chapter concludes with a proof of a proposition from [17]. Define the following:

$$I(C) = \{(p, H) \in C \times (\mathbb{P}^n)^* \text{ such that } p \in H\}$$

and

$$\mathcal{D} = \{(p, D) \in C \times C^{(d)} \text{ such that } p \in D\}.$$

Consider the following commutative diagram

$$\begin{array}{ccc} I(C) & \longrightarrow & \mathcal{D} \\ \downarrow f & & \downarrow \\ (\mathbb{P}^n)^* & \overset{i}{\hookrightarrow} & C^{(d)} \end{array}$$

The proposition states that the family of hyperplane sections of C, $I(C) \xrightarrow{f} (\mathbb{P}^n)^*$, is versal if and only if

$$(\mathbb{P}^n)^* \overset{i}{\hookrightarrow} C^{(d)}$$

is transverse to the given stratification of $C^{(d)}$.

The third chapter concerns the connection between secant maps and the polynomial multiplication maps. Conjecturally, the normal form for the secant map is complex analytically isomorphic, at the germ level, to products of polynomial multiplication maps. Then we state Varley's conjecture that the Thom-Boardman invariant for these maps can be computed using the Euclidean algorithm. It is known that each term of the Thom-Boardman symbol of a map represents the dimension of the Zariski tangent space of the fiber of the map at each stage of the restrictions. My work has indicated that not only can we compute this dimension, we can actually specify the linear equations of the tangent space.

The computation of the invariant for classes of cases is the subject of the fourth chapter. This chapter provides a proof of the stronger result for several classes of cases depending on the relationship of the degrees of the two polynomials.

Monomial ideals M with a particular choice for the Thom-Boardman invariant are constructed in chapter 5 along with a discussion about generalizing the construction to polynomial ideals P without changing the invariant. An example of such a generalization illustrates the need for imposing some nondegeneracy conditions on the coefficients of the variables. We discuss the possibilities for those conditions and pose an interesting question about them.

The sixth chapter contains a discussion of special determinants of the successive Jacobians of the ideals defined by the multiplication maps. The special symbol is realized by taking only these determinants and gives us inequalities for the full invariant for the ideal. This final chapter concludes with a brief discussion of the direction for further research on this problem.

CHAPTER 1

PRELIMINARIES

Here are some basic definitions and constructions for the Thom-Boardman symbol and polynomial multiplication maps, as well as some tools for dealing with the computations with which we are concerned. This chapter concludes with a few examples which provide insight in later discussions.

1.1 THE THOM-BOARDMAN INVARIANT TB

We want to define the symbol Σ^I, where $I = (i_1, \ldots, i_k)$ is the Thom-Boardman invariant associated to a C^∞ map of manifolds. We will follow the construction used in [3]. The construction below, originally by Thom, is given for maps between manifolds of arbitrary dimensions. The algorithm depends upon taking successive restrictions of the map to the singular locus of the previous restriction. At some stage in the algorithm, it is possible that the singular locus may fail to be a manifold. However, Boardman's work generalized Thom's work using jet spaces, which are manifolds and which allow us to perform the calculations even when the singular locus of a particular map is not a manifold, or if any of the restricted loci fails to be manifold.

We say that a point is of class Σ^i for a function f if the dimension of the kernel of the derivative of f at x is i. By the derivative of f at x we simply mean the map taking the tangent space at x to the source space to the tangent at $f(x)$ to the target. Then the subset of M of points of class Σ^i is denoted $\Sigma^i(f)$.

For example, if we think of the projection of a sphere to a plane, the map is two to one everywhere except the equator. At the equator the map is one to one. The planes tangents to the sphere are projected to planes in the target except at the equator. Planes tangent to the sphere at the equator project to lines in the target. The equator is the singular locus of this map in that regard, while the other points are considered regular points of the map. The points away from the equator are of class Σ^0, while the points on the equator are of class Σ^1.

Let J be an ideal in the algebra $\mathcal{A}$ of germs at a given point of C^∞ maps of manifolds $f : M \longrightarrow N, f = (f_1, f_2, \ldots, f_n)$, where M and N have dimensions m and n respectively. Take $x_1, \ldots, x_m$ to be local coordinates on M. The Jacobian extension, $\triangle_k J$, is the ideal spanned by J and all the minors of order k of the Jacobian matrix $(\partial f_i/\partial x_j)$, denoted δJ, formed from partial derivatives of functions f in J. Since the determinant of this matrix is multilinear and since $\partial f/\partial x' = \partial f/\partial x \cdot \partial x/\partial x'$, the Jacobian extension is independent of the coordinate system chosen, hence is an invariant of the ideal. We say that $\triangle_i J$ is critical if $\triangle_i J \neq \mathcal{A}$ but $\triangle_{i-1} J = \mathcal{A}$. That is, the critical extension of J is J adjoined with the least order minors of the Jacobian matrix of J for which the extension does not coincide with the whole algebra. If every size minor of δJ is a unit in $\mathcal{A}$, then the map was of full rank at the given point already and the critical extension is the ideal J itself. Note that $J \subseteq \triangle_i J$.

Now we shift the lower indices to upper indices of the critical extensions by the rule $\triangle^i J = \triangle_{m-i+1} J$. We repeat the process described above with the resulting ideals until we have a sequence of critical extensions of J,

$$J \subseteq \triangle^{i_1} J \subseteq \triangle^{i_2} \triangle^{i_1} J \subseteq \cdots \subseteq \triangle^{i_k} \triangle^{i_{k-1}} \ldots \triangle^{i_1} J = \mathbf{m}$$

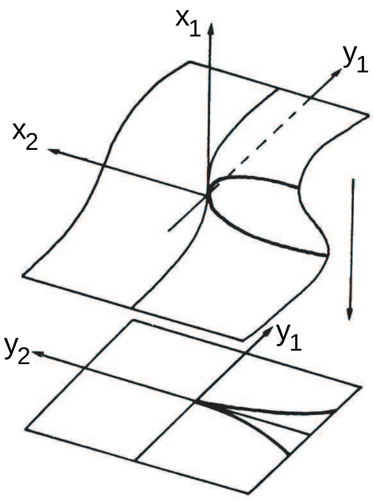
where $\mathbf{m}$ is the maximal ideal of $\mathcal{A}$. Then the Thom-Boardman symbol, $\mathrm{TB}(J)$, is given by $(i_1, i_2, \ldots, i_k)$. The purpose of switching the indices is that doing so allows us to express $\mathrm{TB}(J)$ as follows:

$$i_1 = \mathrm{corank}(J),\ \ i_2 = \mathrm{corank}(\triangle^{i_1} J),\ \ \ldots\ ,\ \ i_k = \mathrm{corank}(\triangle^{i_{k-1}} \ldots \triangle^{i_1}\ J)$$

where the rank of an ideal is defined to be the maximal number of independent coordinates from the ideal and the corank is the number of variables minus the rank.

Geometrically speaking, i_1 is the dimension of the Zariski tangent space $\mathcal{T}_0$ to the fiber in M of the map germ f at zero of this fiber as a subscheme of M defined by the ideal J, and i_j is the dimension of $\mathcal{T}_0$ to the fiber of the map germ on the subscheme of M defined by $\triangle^{i_{j-1}} \ldots \triangle^{i_1}\ J$. It follows that $\mathrm{TB}(J)$ is a non-increasing sequence of non-negative integers. Σ^I is actually the class of points having the Thom-Boardman symbol I, and $\Sigma^I(f)$ is the subset of points of the source space of f of class Σ^I. Consider the following example.

Example 1.1.1 *The Whitney Cusp*



The Whitney cusp map is realized by the map

$$(x_1, x_2) \longmapsto (x_1^3 + x_1x_2, x_2)$$

giving the ideal $J_{wc} = (x_1^3 + x_1x_2, x_2)$. Then,

$$\delta J_{wc} = \begin{pmatrix} 3x_1^2 + x_2 & x_1 \\ 0 & 1 \end{pmatrix}.$$

Since we are considering the map-germ at zero, $\operatorname{rank} \delta J_{wc}|_0 = 1$. So we take the 2x2 minors of δJ_{wc} which is just $\det(\delta J_{wc}) = 3x_1^2 + x_2$. Notice the connection between the algebra and the geometry of the situation. The zero set of this expression, $3x_1^2 + x_2$, is precisely the singular locus in the source space. This singular locus is the set of points where the tangent lines to the source map to points in the target under the derivative of the map. Then $i_1 = 1$ and

$$\triangle^1 J_{wc} = (x_1^2, x_2)$$

The fact that this is not the ideal (x_1, x_2) is reflected by the fact that the tangent space to the restriction of the cusp map has a singularity at the origin. The tangent at the origin maps to the cusp point of the target while the other tangent lines map to tangent lines of the cuspidal curve away from the cusp point.

$$\delta \triangle^1 J_{wc} = \begin{pmatrix} 2x_1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\operatorname{corank} \delta \triangle^1 J_{wc}|_0 = 1.$$

Then $i_2 = 1$ and

$$\triangle^1 \triangle^1 J_{wc} = (x_1, x_2).$$

$\mathrm{TB}(J_{wc}) = (1, 1)$. This example is related to the $\mu_{2,1}$ case given in the last section of this chapter.

1.2 Polynomial Maps

Although the following is valid over any algebraically closed field of characteristic zero, for the sake of this discussion we will always work over $\mathbb{C}$. Without any loss of generality, we may assume our polynomials are monic.

Let M_n be the set of monic complex polynomials in one variable of degree n. $M_n \cong \mathbb{C}^n$ by the map sending $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ to the n-tuple $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$.

If we take $f(x)$ of degree n as above and $g(x) = x^r + b_{r-1}x^{r-1} + \cdots + b_0$ of degree r, then the product $h(x) = fg(x)$ is a monic polynomial of the form $h(x) = x^{n+r} + c_{n+r-1}x^{n+r-1} + \cdots + c_0$ where the c_j's are polynomials in the coefficients of f and g. We can also assume that $n \geq r$. The c_j's are as shown below:

$$\begin{cases} c_{n+r-1} = a_{n-1} + b_{r-1} \\ c_{n+r-2} = a_{n-2} + b_{r-2} + a_{n-1}b_{r-1} \\ \text{and} \\ c_{n+r-j} = a_{n-j} + b_{r-j} + \sum\limits_{i+k=n+r-j} a_i b_k, \ \text{ for } \ j \leq r \\ c_{n+r-j} = a_{n-j} + \sum\limits_{i+k=n+r-j} a_i b_k, \ \text{ for } \ r < j \leq n \\ c_{n+r-j} = \sum\limits_{i+k=n+r-j} a_i b_k, \ \text{ for } \ j > n. \end{cases}$$

This gives us maps

$$\mu_{n,r} : \mathbb{C}^n \times \mathbb{C}^r \longrightarrow \mathbb{C}^{n+r}$$

defined by

$$(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{r-1}) \longmapsto (c_{n+r-1}, \ldots, c_0).$$

We are interested in the map germ $\mu_{n,r}$ at zero.

1.3 The First Jacobian of $\mu_{n,r}$

Let $A_{n,r}$ be the ideal generated by $c_{n+r-1}, c_{n+r-2}, \ldots, c_0$ defined by the multiplication map $\mu_{n,r}$. There is an interesting fact that becomes obvious when taking the Jacobian $\delta A_{n,r}$. Taking the c_j's in descending order from $n+r-1$ to 0, and the a_i's and b_i's in ascending order from 0 to $n-1$ and 0 to $r-1$ respectively, we get the following:

$$\delta A_{n,r} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots \\ b_{r-1} & 1 & 0 & \cdots & a_{n-1} & 1 & 0 & \cdots \\ b_{r-2} & b_{r-1} & 1 & \cdots & a_{n-2} & a_{n-1} & 1 & \cdots \\ \vdots & & & \ddots & & & & \ddots \end{pmatrix}$$

This is the Sylvester matrix for f and g. The rank of the Sylvester matrix for two polynomials when evaluated at the origin (in our case, $f(x) = x^n, g(x) = x^r$) is the larger of the two degrees and thus the corank is the smaller. This gives the first entry of TB($A_{n,r}$) for any $n \geq r$; $i_1 = \mathrm{corank}(\delta A_{n,r}) = r$. However, the determinants of this matrix are large and complicated and the structure of $\delta \triangle^r A_{n,r}$ is not as helpful. The following section gives us an important tool in discovering the successive entries of the Thom-Boardman symbol.

1.4 ELIMINATION

Since the Thom-Boardman symbol is independent of the choice of variables, by using the expressions with linear terms in the b's, $c_n, \ldots, c_{n+r-1}$, we may eliminate the b's and rewrite the remaining c's as functions of the a's. The new expressions are of the form:

$$\tilde{c}_n = 0, \tilde{c}_{n+1} = 0, \ldots, \tilde{c}_{n+r} = 0$$

$$\tilde{c}_0, \ldots, \tilde{c}_{n-1} \quad \text{are expressions in the} \quad a_i\text{'}s$$

$$\tilde{c}_r, \tilde{c}_{r+1}, \ldots, \tilde{c}_{n-1} \text{ have linear terms } a_0, \ \ldots \ a_{n-r-1} \text{ respectively.}$$

This gives us a map $\tilde{\mu}_{n,r} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ defined by $(a_0, \ldots, a_{n-1}) \longmapsto (\tilde{c}_{n-1}, \ldots, \tilde{c}_0)$. Let $B_{n,r}$ denote the ideal in $\mathbb{C}[a_0, \ldots, a_{n-1}]$ generated by the $\tilde{c}_j$'s obtained from the map $\tilde{\mu}_{n,r}$. The notation $B_{j,k}$ is shortened to B when there is no ambiguity. Also, since the context is clear from here on, it is preferable to commit a slight abuse of notation by dropping the use of the tilde. The first Jacobian of the ideal $B = B_{n,r}$ is the matrix

$$\delta B = \begin{pmatrix} 0 & & \cdots & & 0 & 1 & \frac{\partial c_{n-1}}{\partial a_{n-r}} & \frac{\partial c_{n-1}}{\partial a_{n-r+1}} & \cdots & \frac{\partial c_{n-1}}{\partial a_{n-1}} \\ 0 & & \cdots & 0 & 1 & \frac{\partial c_{n-2}}{\partial a_{n-r-1}} & \frac{\partial c_{n-2}}{\partial a_{n-r}} & \frac{\partial c_{n-2}}{\partial a_{n-r+1}} & \cdots & \frac{\partial c_{n-2}}{\partial a_{n-1}} \\ \vdots & & & & & & \vdots & & & \vdots \\ 1 & \frac{\partial c_r}{\partial a_1} & \frac{\partial c_r}{\partial a_2} & \cdots & & & \frac{\partial c_r}{\partial a_{n-r}} & & \cdots & \\ \frac{\partial c_{r-1}}{\partial a_0} & \frac{\partial c_{r-1}}{\partial a_1} & \frac{\partial c_{r-1}}{\partial a_2} & \cdots & & & \frac{\partial c_{r-1}}{\partial a_{n-r}} & & \cdots & \\ \vdots & & & & & & \vdots & & & \vdots \\ \frac{\partial c_0}{\partial a_0} & \frac{\partial c_0}{\partial a_1} & \frac{\partial c_0}{\partial a_2} & \cdots & & & \frac{\partial c_0}{\partial a_{n-r}} & & \cdots & \frac{\partial c_0}{\partial a_{n-1}} \end{pmatrix}$$

Recall that each c_j has at most one linear term with coefficient 1 and the rest of the terms are at least quadratic. The (k)-antidiagonal of a matrix (a_{ij}) is the set of entries on a diagonal line drawn from the entry a_{1k} to the entry a_{k1}. In other words, the sum of the indices of the relevant entries is $k + 1$.

The $(n - r)$-antidiagonal of δB contains all ones from the linear terms and no other non-zero entry of δB has a constant term. If $j > r, c_n - j$ has no terms with factors of index less than j. This shows that all entries above the $(n - r)$-antidiagonal are zero. At the origin, this matrix has corank r consistent with the Sylvester matrix discussion above.

The column that corresponds to the partial $\partial/\partial a_r$ is the first column from the left containing no ones and is important in the computations of determinants with linear terms. Indeed, it contains expressions with linear terms in the variables in order from the last entry of that column up to the first. For that reason this column is called the *critical column*. An example of this nice structure of δB is below.

Example 1.4.1 $\delta B_{5,2}$

Note that the critical column is the fourth column.

$$\begin{pmatrix} 0 & 0 & 1 & -2\,a_4 & -2\,a_3 + 3\,{a_4}^2 \\ 0 & 1 & -a_4 & -2\,a_3 + {a_4}^2 & -a_2 + 2\,a_3\,a_4 \\ 1 & -a_4 & -a_3 + {a_4}^2 & -a_2 & -a_1 + 2\,a_2\,a_4 \\ -a_4 & -a_3 + {a_4}^2 & 0 & -a_1 & 2\,a_1\,a_4 - a_0 \\ -a_3 + {a_4}^2 & 0 & 0 & -a_0 & 2\,a_0\,a_4 \end{pmatrix}$$

Notice that the triangle of zeros in the top left corner is reflected by a triangle of zeros along the bottom of the matrix just previous to the critical column. This will always happen and can be confirmed by computing the appropriate partial derivatives of the c's. We see this by noting that the c_j's giving the zeros are the ones of index lower than $n - r$. These expressions have the variables $a_0, \ldots, a_j$ in at least quadratic terms as seen in the original multiplication. They also contain terms with factors in the variables $a_r, \ldots, a_{n-1}$ from the substitution performed in the elimination process. However, they do not contain any terms with factors with index between j and r.

1.5 The Weighting Structure

We impose a weighting system on the a_i's as follows:

$$\begin{aligned} \omega(0) &= -\infty \\ \omega(k) &= 0 \quad \text{for} \quad k \in \mathbb{C}^* \\ \omega(a_i) &= n - i \quad \text{for} \quad i \in \{0, 1, \ldots, n-1\} \\ \omega(a_i a_j) &= \omega(a_i) + \omega(a_j) \end{aligned}$$

In general, the weight of a product of variables is the sum of the weights of the individuals. The weight of a sum or sum of products is defined only if all of the terms of the sum have equal weight, i.e. the sum is weighted homogeneous. Then the weight of the sum is the weight of any of the terms.

It is worth noting that the c_j's have homogeneous weight and each c_j has weight $n + r - j$. In particular, $\omega(c_{n-1}) = r + 1$ and $\omega(c_0) = r + n$. Also $\partial c_j / \partial a_i$ has weight $\omega(c_j) - \omega(a_i) = r - j + i$. Since $\omega(c_j) = \omega(c_{j-1}) - 1$, $\partial c_j / \partial a_i$ has weight $\omega(\partial c_{j-1}/\partial a_i) - 1$. The weight of the entries of matrix $\delta B_{n,r}$ are equal on each of the anti-diagonals. This tells us where linear terms are possible in the matrix. Also, there is additional structure, to be discussed in a later section, inherent in the c's that tells us which monomials can and cannot appear in each c, and thus in each partial.

1.6 EXAMPLES

This is a good place to consider some examples of the computation of the Thom-Boardman invariant, TB(J), for an ideal J given by a map $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. The example given earlier, the Whitney cusp map, provided us the opportunity to explore the process visually as well as computationally. The Whitney cusp map has the same TB-invariant as the polynomial multiplication map, $\mu_{2,1}$, shown below. There are two other examples shown here, $\mu_{4,3}$ and $\mu_{4,2}$, which provide insight into the computations.

Example 1.6.1 $\mu_{2,1}$

Consider $\mu_{2,1}$ with $f(x) = x^2 + a_1 x + a_0$, $g(x) = x + b$. Then,

$$(a_0, a_1, b) \longmapsto (a_1 + b, a_0 + a_1 b, a_0 b).$$

After eliminating the variable b,

$$B_{2,1} = (a_0 - a_1^2, -a_0 a_1).$$

$$\delta B_{2,1} = \begin{pmatrix} 1 & -2a_1 \\ -a_1 & -a_0 \end{pmatrix}.$$

The corank of $\delta B_{2,1}$ is one, so $\mathrm{TB}(B_{2,1}) = (1, \ldots)$. Also $\mathrm{rank}(\delta B_{2,1}) = 1$ so we need the 2×2 minors of $\delta B_{2,1}$ which is the determinant of the whole matrix. We adjoin $-a_0 - 2a_1^2$

to the ideal $B_{2,1}$ to get

$$\triangle^1 B = (a_0, a_1^2)$$

and

$$\delta \triangle^1 B = \begin{pmatrix} 1 & 0 \\ 0 & 2a_1 \end{pmatrix}.$$

Again, the rank and corank of the Jacobian matrix are both one. $\mathrm{TB}(B_{2,1}) = (1, 1, \ldots)$, $\det(\delta \triangle^1 B) = 2a_1$, and we get

$$\triangle^1 \triangle^1 B_{2,1} = (a_0, a_1).$$

The next corank is obviously 0 and we may stop, therefore

$$\mathrm{TB}(B_{2,1}) = (1, 1) = \mathrm{TB}(J_{wc}).$$

Example 1.6.2 $\mu_{4,3}$

$$f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

$$g(x) = x^3 + b_2x^2 + b_1x + b_0$$

$$\mu_{4,3} : (a_0, a_1, a_2, a_3, b_0, b_1, b_2) \longmapsto (c_6, \ldots, c_0)$$

After elimination of the coefficients of the polynomial g as described above,

$$\begin{cases} c_6 = c_5 = c_4 = 0 \\ c_3 = -2a_1a_3 + a_0 + 3a_2a_3^2 - a_2^2 - a_3^4 \\ c_2 = -a_0a_3 - 2a_2a_1 + 2a_2^2a_3 - a_2a_3^3 + a_1a_3^2 \\ c_1 = a_0a_3^2 - a_0a_2 - a_1^2 + 2a_1a_2a_3 - a_1a_3^3 \\ c_0 = -a_0a_1 + 2a_0a_2a_3 - a_0a_3^3. \end{cases}$$

Let $B := (c_3, \ldots, c_0)$. Then

$$\delta B_{4,3} = \begin{pmatrix} 1 & -2a_3 & 3a_3^2 - 2a_2 & -2a_1 + 6a_2a_3 - 4a_3^3 \\ -a_3 & -2a_2 + a_3^2 & -2a_1 + 4a_2a_3 - a_3^3 & -a_0 + 2a_2^2 - 3a_2a_3^2 + 2a_1a_3 \\ a_3^2 - a_2 & -2a_1 + 2a_2a_3 - a_3^3 & -a_0 + 2a_1a_3 & 2a_0a_3 + 2a_2a_1 - 3a_1a_3^2 \\ -a_1 + 2a_2a_3 - a_3^3 & -a_0 & 2a_0a_3 & 2a_0a_2 - 3a_0a_3^2 \end{pmatrix}.$$

The corank of $\delta B_{4,3}$ is 3, which gives $\mathrm{TB}(B_{4,3}) = (3, \ldots)$. And since the rank of $\delta B_{4,3}$ is 1, we adjoin the 2×2 minors of $\delta B_{4,3}$ to $B_{4,3}$ to get $\triangle^3 B_{4,3}$. In practice, we then find a generating set for $\triangle^3 B_{4,3}$ and repeat the process using this Jacobian extension. A generating set is:

$$\triangle^3 B_{4,3} = (a_3^4, 2a_2 + a_3^2, 2a_1 - a_3^3, a_0)$$

Repeating the process:

$$\delta \triangle^3 B_{4,3} = \begin{pmatrix} 0 & 0 & 0 & 4{a_3}^3 \\ 0 & 0 & 2 & 2a_3 \\ 0 & 2 & 0 & -3{a_3}^2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{corank}(\delta \triangle^3 B_{4,3}) = 1$$

$$\triangle^1 \triangle^3 B_{4,3} = (a_3^3, 2a_2 + a_3^2, a_1, a_0)$$

Again:

$$\delta \triangle^1 \triangle^3 B_{4,3} = \begin{pmatrix} 0 & 0 & 0 & 3a_3^2 \\ 0 & 0 & 2 & 2a_3 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{corank}(\delta \triangle^1 \triangle^3 B_{4,3}) = 1$$

$$\triangle^1 \triangle^1 \triangle^3 B_{4,3} = (a_3^2, a_2, a_1, a_0)$$

We continue until we get the maximal ideal $(a_0, \ldots, a_{n-1})$ of $\mathbb{C}[a_0, \ldots, a_{n-1}]$.

$$\delta \triangle^1 \triangle^1 \triangle^3 B_{4,3} = \begin{pmatrix} 0 & 0 & 0 & 2a_3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{corank}(\delta \bigtriangleup^1 \bigtriangleup^1 \bigtriangleup^3 B_{4,3}) = 1$$

$$\bigtriangleup^1 \bigtriangleup^1 \bigtriangleup^1 \bigtriangleup^3 B_{4,3} = (a_0, a_1, a_2, a_3)$$

which is the maximal ideal and hence $\mathrm{TB}(B_{4,3}) = (3,1,1,1)$. The following is an example of the class of cases $\mu_{kr,r}$ where k is a positive integer.

Example 1.6.3 $\mu_{4,2}$

$\mu_{4,2} : (a_0, a_1, a_2, a_3) \longmapsto (c_3, c_2, c_1, c_0)$

$$\begin{cases} c_3 = a_1 - 2a_2a_3 + a_3^3 \\ c_2 = a_0 - a_2^2 + a_2a_3^2 + a_1a_3 \\ c_1 = -a_0a_3 - a_1a_2 + a_1a_3^2 \\ c_0 = -a_0a_2 + a_0a_3^2 \end{cases}$$

$$\delta B_{4,2} = \begin{pmatrix} 0 & 1 & -2\,a_3 & -2\,a_2 + 3\,{a_3}^2 \\ 1 & -a_3 & -2\,a_2 + {a_3}^2 & 2\,a_2\,a_3 - a_1 \\ -a_3 & -a_2 + {a_3}^2 & -a_1 & -a_0 + 2\,a_1\,a_3 \\ -a_2 + {a_3}^2 & 0 & -a_0 & 2\,a_0\,a_3 \end{pmatrix}$$

$$\mathrm{corank}(\delta B_{4,2}) = 2$$

$$\bigtriangleup^2 B_{4,2} = ({a_3}^5,\, 3\,a_2\,a_3 - {a_3}^3,\, 3\,{a_2}^2 - 2\,{a_3}^4,\, 3\,a_1 + {a_3}^3,\, a_0)$$

$$\delta \bigtriangleup^2 B_{4,2} = \begin{pmatrix} 0 & 0 & 0 & 5\,{a_3}^4 \\ 0 & 0 & 3\,a_3 & 3\,a_2 - 3\,{a_3}^2 \\ 0 & 0 & 6\,a_2 & -8\,{a_3}^3 \\ 0 & 3 & 0 & 3\,{a_3}^2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{corank}(\delta \bigtriangleup^2 B_{4,2}) = 2$$

$$\bigtriangleup^2 \bigtriangleup^2 B_{4,2} = (a_3, a_2, a_1, a_0)$$

We have shown $\mathrm{TB}(B_{4,2}) = (2,2)$.

CHAPTER 2

THE dth SYMMETRIC PRODUCT OF A CURVE OF DEGREE d.

Let C be a smooth, nondegenerate connected compact complex curve of degree d in $\mathbb{P}^n$. We want to define the d^{th} symmetric product of C, stratify that space, and consider the inherent geometry of the stratification. The point is to understand and prove a proposition, to be given later, stated but not proved in an exposition of joint work by Malcolm Adams, Clint McCrory, Ted Shifrin and Robert Varley [17].

2.1 SOME DEFINITIONS

Definition 2.1.1 *For a nonsingular complex curve C, the d^{th}* symmetric product*, denoted $C^{(d)}$, is the d-fold product of C with itself modulo the action of S_d, the symmetric group on d elements. Hence, $C^{(d)}$ is the set of unordered d-tuples of points of C.*

Definition 2.1.2 *A* divisor *D of C is a finite formal sum: $D = m_1p_1 + \cdots + m_kp_k$, where $m_i \in \mathbb{Z}$ and $p_i \in C$ for all $i = 1, \ldots, k$. The* degree *of D is the sum of the m_i, $\deg D = \sum_{i=1}^{k} m_i$. A divisor of C is called* effective *if $m_i \geq 0$ for all $i = 1, \ldots, k$.*

We say that two linear subspaces of a finite dimensional linear space are transversal if their sum is the whole space. This leads to the following definition for maps found in [3]:

Definition 2.1.3 *Let $f : A \to B$ be a smooth map (at least $\mathcal{C}^1$) of a manifold A to a manifold B containing a submanifold B'. Then f is said to be* transversal to B' at a point *$a \in A$ if either* *i)* $f(a) \notin B'$ *or*

ii) the image of the tangent space to A *at a under the derivative* f_{*a} *is transversal to the tangent space to* B' *at* a *(i.e.* $f_{*a}T_aA + T_{f(a)}B' = T_{f(a)}B$*).*

2.2 TWO IMPORTANT PROPOSITIONS

Now we consider the following two propositions.

Proposition 2.2.1 *If* C *is a nonsingular complex curve, for any* $d \geq 1$*, the symmetric product* $C^{(d)} \cong \{$*effective divisors of degree* d *on* $C\}$ *is a complex manifold (or a nonsingular complex variety) of dimension* d*, [21].*

Proof: $C^{(d)} = C^d/S_d$ where $\sigma \in S_d$ acts on $x = (x_1, \ldots, x_d) \in C^d$ by $\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(d)})$, i.e. the map:

$$\{1, \ldots, d\} \xrightarrow{x} C^d \xrightarrow{\sigma} C^d$$

defined by:

$$i \longmapsto x_i \longmapsto \sigma(x)(i)$$

is realized by $\sigma(x)(i) = x_{\sigma(i)} = x(\sigma(i))$. If C is affine, then so is C^d/S_d [12],pg126.

Let's first consider the case when $C = \mathbb{A}^1$, affine 1-space. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{A}^1$ be the roots of the degree d monic polynomial

$$(x - \alpha_1) \cdots (x - \alpha_d) = x^d - s_1(\alpha_1, \ldots, \alpha_d)x^{d-1} + \cdots \pm s_d(\alpha_1, \ldots, \alpha_d)$$

where the s_i are the elementary symmetric functions of $\alpha_1, \ldots, \alpha_n$. We have the following commutative diagram:

$$\begin{array}{ccccc} (\alpha_1 \ldots, \alpha_d) & & (\mathbb{A}^1)^d & \longrightarrow & (\mathbb{A}^1)^d/S_d = (\mathbb{A}^1)^{(d)} \\ \downarrow & & \downarrow & \swarrow & \\ (s_1, \ldots, s_d) & & \mathbb{A}^d & & \end{array}$$

But the quotient map is invariant under the action of S_d and the map $(\mathbb{A}^1)^d/S_d \longrightarrow \mathbb{A}^d$ is, in fact, an isomorphism. Thus $(\mathbb{A}^1)^{(d)} \cong \mathbb{A}^d$.

Now we consider the case of C, a smooth curve. We have the map:

$$C^d \longrightarrow C^{(d)}$$

(ordered d-tuple) $\longmapsto$ (corresponding unordered d-tuple).

Suppose we have $x_0 = (x_1, \ldots, x_d) \in C^d$, where the x_i's are not necessarily distinct. Rename x_i by x_1 if $x_i = x_1$. Then by reordering the entries by the new indices, we can consider x_0 of the form:

$$x_0 = (x_1, \ldots, x_1, x_2, \ldots x_2, \ldots, x_k, \cdots, x_k)$$

where there are m_i entries for each distinct x_i, for $i = 1, \ldots, k$, where $m_1 \geq \cdots \geq m_k$, $\sum_{i=1}^k m_i = d$.

Consider

$$H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r} \subset S_d.$$

H is the stablizer $\mathrm{Stab}(x_0) \subset S_d$. This is the only relevant subgroup: for if we have an element $\sigma \in S_d,\ \sigma \notin H$, then there is a neighborhood U of x in $C^{(d)}$ such that $\sigma(U) \cap U$ is empty. Such a σ is not relevant to our quotient map around x_0. Take disjoint neighborhoods N_i of $x_i \in C$, for each $i = 1, \ldots, k$ of the k distinct points. We can identify each neighborhood N_i with an open unit disc $\triangle_i$ in $\mathbb{C}$ centered at a point $z_i \in \mathbb{C}$ such that $|z_i - z_j| \geq 2$ for all $i \neq j$.

Consider the product neighborhood

$$U = N_1^{m_1} \times \cdots \times N_k^{m_k}$$

with the action of H, now indentified with

$$\triangle_1^{m_1} \times \cdots \times \triangle_k^{m_k}$$

with the action of

$$S_{m_1} \times \cdots \times S_{m_k}.$$

Now

$$\triangle_1^{m_1} \times \cdots \times \triangle_k^{m_k}$$

embeds into $(\mathbb{A}^1)^d$ where the action is induced by the action of $S_{m_1} \times \cdots \times S_{m_k} \subset S_d$. So we have the following

$$\triangle_1^{m_1} \times \cdots \times \triangle_k^{m_k} \hookrightarrow (\mathbb{A}^1)^d \longrightarrow (\mathbb{A}^1)^{m_1}/S_{m_1} \times \cdots \times (\mathbb{A}^1)^{m_k}/S_{m_k}$$

But earlier we saw that $(\mathbb{A}^i)^{m_i}/S_{m_i} \cong \mathbb{A}^{m_i}$. The map becomes

$$(\triangle_1^{m_1} \times \cdots \times \triangle_k^{m_k})/(S_{m_1} \times \cdots \times S_{m_k}) \hookrightarrow \mathbb{A}^{m_1} \times \cdots \times \mathbb{A}^{m_k},$$

the right hand side of which is a complex manifold of dimension d. This implies $C^{(d)}$is a complex manifold of dimension d. ∎

Proposition 2.2.2 *Let C be a nonsingular compact complex curve. For each $k \geq 1$ and $m_1, \ldots, m_k \geq 1$ such that $m_1 + \cdots + m_k = d$, let*

$$\mathcal{S}(m_1, \ldots, m_k) = \{D \in C^{(d)} | D = m_1p_1 + \cdots + m_kp_k \textit{for distinct points} p_1, \ldots, p_k \in C\}.$$

Then

$$\mathcal{S}(\underline{m}) = \{\mathcal{S}(m_1, \ldots, m_k)\} \subseteq C^{(d)}$$

is a locally closed submanifold of dimension k. And

$$\mathcal{S} = \{\mathcal{S}(\underline{m}) \textit{with} \underline{m} \textit{running over} k \geq 1 \textit{and} m_1 \geq \cdots \geq m_k \geq 1, m_1 + \cdots + m_k = d\}$$

is a complex analytically locally trivial stratification of $C^{(d)}$, [21].

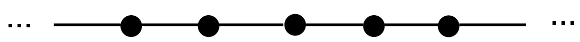
First we need to develop a language in which we may discuss the above statement. At that point, most of the proposition is easily proved. The proof of the complex analytic local triviality property for $\mathcal{S}$, however, depends on introducing the deformation theory developed by V.I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko in [3]. Once this perspective is put into place, the proof is readily seen. We start with a few definitions and examples.

Definition 2.2.3 *A subspace Z of a topological space X is* locally closed *if it is the intersection of an open and a closed subset of X.*

A general definition of a stratification of a topological space M is a decomposition of M into disjoint subsets $\{M_\alpha\}$ such that $M = \coprod M_\alpha$ and each M_α is a topological manifold which is locally closed in M.

A *stratification of a manifold* is a finite collection of sub-manifolds (the strata) following the condition that the closure of each stratum consists of itself together with the finite union of strata of smaller dimension.

Example 2.2.4 $\mathbb{R} = \mathbb{Z} \cup (\mathbb{R} - \mathbb{Z})$

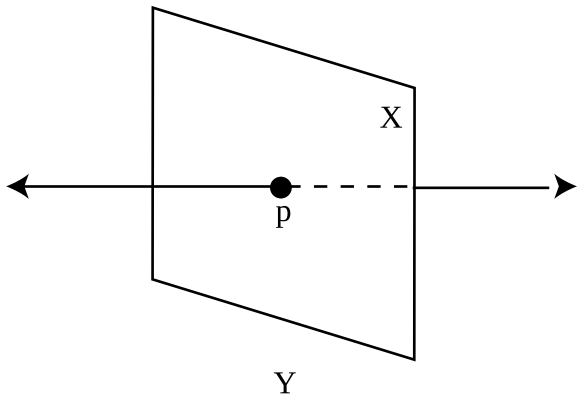


$\dim(\mathbb{Z}) = 0 \quad \dim(\mathbb{R} - \mathbb{Z}) = 1$

$\overline{\mathbb{Z}} = \overline{\mathbb{Z}} \quad \overline{\mathbb{R} - \mathbb{Z}} = \mathbb{R}.$

Example 2.2.5 $Y = (X - p) \cup (\ell - p) \cup \{p\}$

X is the plane, ℓ is the line, and p is the point of intersection of the line and the plane in the figure below. The strata are the plane minus the point (dimension 2), the line minus the point (dimension 1) and the point (dimension 0).

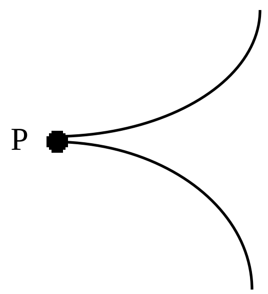


$\dim(X - p) = 2 \quad \dim(\ell - p) = 1 \quad \dim\{p\} = 0$

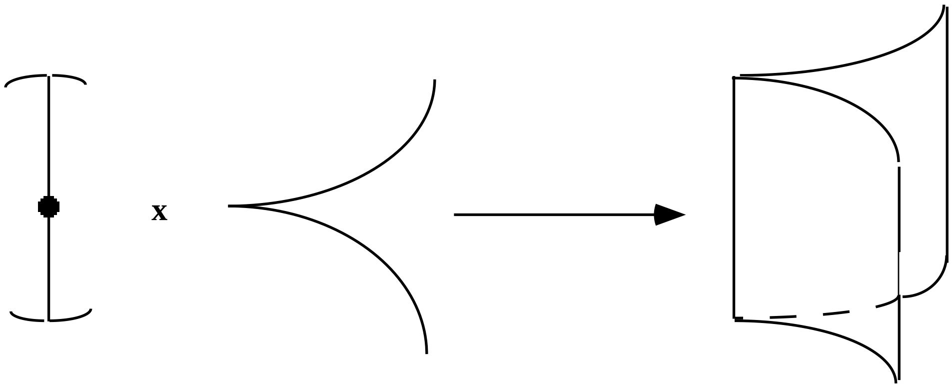
$\overline{X - p} = X \quad \overline{\ell - p} = \ell$

There is another definition of a stratification: Let X be a topological space and $X = \coprod_\alpha X_\alpha$ where X_α are locally closed and are C^0 manifolds. Then this stratification of X is said to be *locally finite* if for every $x \in X$ and there is some open neighborhood U of x so that there are only finitely many X_α such that $U \cap \overline{X}_\alpha$ is nonempty. The disjoint union is not required to be finite but must be locally finite. Our example 2.2.2 can be reconsidered in this light as the disjoint union of singleton points and open intervals.

Now consider the following topological space formed by two “curved surfaces”. A transversal slice at a point p looks like:



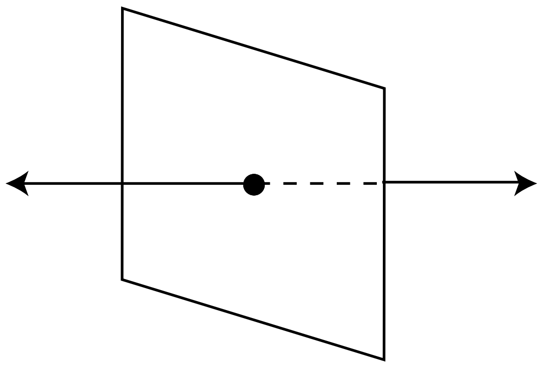
Consider an open neighborhood U of p. U is homeomorphic to a neighborhood of the stratum containing the point crossed with a slice,



This is a local triviality condition.

Definition 2.2.6 *A stratified topological space* $X = \coprod_\alpha X_\alpha$ *is* locally topologically trivial *if for every* $x \in X$ *there exists an open neighborhood* U *of* x*, an open neighborhood* τ *of* x *in* X_α *(where* $x \in X_\alpha$*), and a topological space* $\mathcal{N}$ *such that* U *is homeomorphic to* $\tau \times \mathcal{N}$.

Let's look at an example of a space which is not locally topologically trivial. Consider from example 2.2.3, a different decomposition of that space into the punctured plane and the line, $Y = (X - p) \cup l$. If we look at the point of intersection, the neighborhood looks like:



which is not a product space. But if we take the decomposition to be as in the example, (i.e. the point, the line minus the point, and the plane minus the point), then at p the tangential part τ is just the point p and $\mathcal{N}$ is the whole neighborhood. So now $Y =$

$\coprod Y_\alpha$ is locally topologically trivial. Complex analytically locally trivial is much stronger. Complex analytically locally trivial requires that U be complex analytically isomorphic to $\tau \times \mathcal{N}$.

Another useful concept is that of partitions of integers [2]:

Definition 2.2.7 *A* partition λ *of a positive integer* n *is a finite, nonincreasing sequence of positive integers* $\lambda_1, \ldots, \lambda_r$ *(the "parts") such that* $\sum_{i=1}^{r} \lambda_i = n$*. We write* $(\lambda_1, \ldots, \lambda_r)$ *for the partition denoted by* λ*, and write* $\lambda \vdash n$ *for "*λ *is a partition of* n*". Let* $\mathcal{T}(n)$ *denote the set of all partitions of* n*.*

Note we have $\underline{m} \vdash d$, since the integers m_i are positive for $i = 1, \ldots, k$, $m_1 \geq m_2 \geq \cdots \geq m_k$ and $\sum_{i=1}^{k} m_i = d$.

Now consider an ordering "$\leq$" on the partitions of an integer n given by $\lambda' \leq \lambda$ if λ' can be obtained from λ, possibly after reordering, by partitioning one or more elements of λ.

It's easy to see that "$\leq$" defines a partial ordering on $\mathcal{T}(n)$. Since n is a trivial partitioning of itself, $n \leq n$. Transitivity is an obvious consequence of the way in which one partition is obtained from another. Antisymmetry is also obvious: for if λ' is obtained from partitioning parts of λ and λ is obtained from partitioning parts of λ', then the partitions of parts must have been the trivial partitionings. This ordering is only a partial ordering on $\mathcal{T}(n)$. To see this, let $n = 15,\ \ \lambda = (7, 3, 2, 2, 1),\ \ \lambda' = (5, 3, 2, 2, 2, 1),\ \ \lambda' \leq \lambda$ and consider $\lambda'' \vdash 15$ with $\lambda'' = (5, 4, 3, 2, 1)$. Then $\lambda' < \lambda''$ by $4 = 2 + 2$. But λ'' cannot be obtained from λ, nor can λ be obtained from λ'' by any partitionings.

Consider $\mathcal{S}(\underline{m}) \subset C^{(d)}$, $\underline{m} = (m_1, \ldots, m_k)$. We defined $\mathcal{S}(\underline{m})$ to be the set $\{D = m_1 p_1 + \cdots + m_k p_k$ where the p_i are distinct points of $C\}$. There are k independent parameters, the p_i, $i = 1, \ldots, k$, needed to describe $\mathcal{S}(\underline{m})$ and this gives $\dim \mathcal{S}(\underline{m}) = k$. Since the p_i are distinct, $\mathcal{S}(\underline{m})$ can be written locally as

$$(\mathcal{S}(\underline{m}), D_0) \cong (\mathcal{S}(m_1), m_1 p_1) \times \cdots \times (\mathcal{S}(m_k), m_k p_k)$$

where varying m_ip_i in $\mathcal{S}(\underline{m})$ amounts to moving p_i along the curve C. This gives

$$\begin{aligned}
(\mathcal{S}(\underline{m}), D_0) &\cong (\mathcal{S}(m_1), m_1p_1) \times \cdots \times (\mathcal{S}(m_k), m_kp_k) \\
&\cong (C, p_1) \times \cdots \times (C, p_k) \\
&= \begin{cases} \text{ordered (or unordered) } k- \text{ tuples } (\tilde{p}_1, \ldots, \tilde{p}_k) \text{ such that} \\ \tilde{p}_i \text{ are distinct points of } (C, p_i) \end{cases} \\
&\cong (\mathbb{A}^1, 0) \times (\mathbb{A}^1, 0) \times \cdots \times (\mathbb{A}^1, 0) \\
&\cong (\mathbb{A}^k, 0).
\end{aligned}$$

(i.e. $\mathcal{S}(\underline{m})$ is a submanifold of $C^{(d)}$).

Since $C^{(d)} = C^d/S_d$, we may order the m_i so that $m_1 \geq m_2 \geq \cdots \geq m_k$ and $\underline{m} \vdash d$. We may also assume, without loss of generality, that re-ordering occurs naturally whenever necessary for the following discussion.

Consider $\underline{m} = (1, 1, \ldots, 1)$ with d ones,

$$\mathcal{S}(\underline{m}) = \{D \in C^{(d)} \text{such that} D = p_1 + \cdots + p_d \text{where the} p_i \in C \text{are distinct}\}.$$

For example, if $d = 3$ and C is a plane cubic, we get $D = p_1 + p_2 + p_3$ as the formal sum of distinct points on a line (i.e. a hyperplane of the plane) intersecting C in the most general way, nowhere tangent to C. This line, l may become tangent to C, as it sweeps through the points of C, at some point p', i.e., two points “come together” at p'. Then we have $D' = 2p' + q$. We take note that the general point of $C^{(3)}$ does not occur as $l \cdot C$.

For general d and $D' = 2p' + q_1 + \cdots + q_{d-2}$,such that $q_i \neq p'$, $\forall i = 1, \ldots, d-2$, and $q_i \neq q_j$ if $i \neq j$, $D' \in \mathcal{S}(2, 1, \ldots, 1)$ is in the closure of $\mathcal{S}(1, \ldots, 1)$. In general, the elements in the closure of $\mathcal{S}(\underline{m})$ are exactly the divisors in $\mathcal{S}(\underline{m})$ and the divisors obtained when two or more points come together. This corresponds to adding two or more of the m_i. In other words $\mathcal{S}(\underline{m}') \subseteq \overline{\mathcal{S}(\underline{m})}$ if and only if $\underline{m} \leq \underline{m}'$ as partitions of d. Notice that if $\mathcal{S}(\underline{m}') \subseteq \overline{\mathcal{S}(\underline{m})}$ with $\underline{m}' = (m_1', \ldots, m_r')$ and $\underline{m} = (m_1, \ldots, m_k)$ then $r \leq k$ so

$\dim \mathcal{S}(\underline{m}') \leq \dim \mathcal{S}(\underline{m})$. In fact, from the discussion on partitions, it can be seen that if $\underline{m} < \underline{m}'$ as partitions of d, then $r < k$.

Lemma 2.2.8 *$\mathcal{S}(\underline{m})$ is a locally closed subset of $C^{(d)}$.*

Proof: Let $U_{\underline{m}} = C^{(d)} - (\overline{\mathcal{S}(\underline{m})} - \mathcal{S}(\underline{m}))$. For each $\underline{m}$, $U_{\underline{m}}$ is open in $C^{(d)}$. $\mathcal{S}(\underline{m}) = U_{\underline{m}} \cap \overline{\mathcal{S}(\underline{m})}$ so $\mathcal{S}(\underline{m})$ is locally closed. ∎

Therefore we have that the $\mathcal{S}(\underline{m})$ are locally closed, disjoint submanifolds of $C^{(d)}$ where the closure of each $S(\underline{m})$ consists of itself and a finite union of submanifolds, $\mathcal{S}(\underline{m}')$, of smaller dimension. We have just proved that $\mathcal{S}$ is a stratification of $C^{(d)}$. Now, we must establish the complex analytical local triviality property for $\mathcal{S}$. We establish this in the next section by using a little deformation theory.

2.3 Deformations

The primary reference for this version of deformation theory is from [3].

Definition 2.3.1 *Let M, N be manifolds. A* map-germ *$M \to N$ at a point x of M is an equivalence class of maps $\varphi : U \to N$, each of which is defined on some neighborhood U of x in M, where two maps are* equivalent *if they coincide on some neighborhood V of x in M.*

Definition 2.3.2 *Let x be a point of a manifold M. A* deformation of x *is a smooth map-germ F from a manifold Λ, to M at a point 0 in Λ for which $F(0) = x$. Λ is called the* base *of the deformation.*

Let G be a Lie-group acting on M and let F and F' be two deformations of x with the same base Λ. We say F and F' are $equivalent$ if one can be carried to the other by the action of an element g_λ of G smoothly depending on λ (i.e. if $F(\lambda) = g_\lambda F'(\lambda)$ where $\lambda \longmapsto g_\lambda$ is a smooth map-germ $(\Lambda, 0) \longrightarrow (G, e)$) . Let

$$\varphi : (\Lambda', 0) \to (\Lambda, 0)$$

be a smooth map. The deformation induced from F by φ is the deformation $\varphi^* F$ of x with the base Λ' given by

$$(\varphi^* F)(\lambda') = F(\varphi(\lambda'))$$

for all λ' in Λ'.

Definition 2.3.3 *A deformation F of x is* versal *if every deformation of x is equivalent to one induced from F. A versal deformation is* miniversal *if the dimension of the base has the least possible value.*

Arnold shows that in the finite dimensional case, a deformation which is versal at first order (i.e. with respect to tangent spaces, infinitesimally versal) is versal [3],pg 151.

Example 2.3.4

Take M_m to be the set of monic polynomials $h(z)$ in $\mathbb{C}[z]$ of degree $m \geq 1$. We take G_m to be the group of translations of the form $z \mapsto z + b$ for $b \in \mathbb{C}$ acting on M_m. Consider $x_0 \in M_m$ with $x_0 = z^m$. Then a deformation of z^m with base Λ is given by a family of monic polynomials $h(z)$ of the form

$$h(z) = z^m + \lambda_{m-1} z^{m-1} + \cdots + \lambda_0$$

where $\lambda_0, \ldots, \lambda_{m-1}$ are functions on Λ vanishing at 0. Such a deformation of z^m is versal if the corresponding map $F : \Lambda \to M_m$ is transversal at z^m to the orbit $G_m z^m$. If we take a minimal transversal (one of minimum dimension to still be transversal to $G_m z^m$) we get the miniversal deformation. In our case, this amounts to taking the Tschirnhausen transformation which eliminates the z^{m-1} term of the equation. The miniversal deformation of z^m in M_m is

$$(V_m, z^m) \xrightarrow{F} (M_m, z^m)$$

where

$$V_m = \{z^m + \lambda_{m-2} z^{m-2} + \cdots + \lambda_0 \text{such that} \lambda_i \in \mathbb{C}\},$$

F is versal, and z^m represents the base point in (V_m, z^m).

Remark 2.3.5 *Thinking of miniversal deformations in this way by looking at transversality to the orbits of G, makes it clear that if we have an inducing map φ from a deformation F to a miniversal deformation F', then F is versal if and only if φ is submersive. So we actually have uniqueness at first order of inducing maps to miniversal deformations.*

It is enough to consider the deformation of mp (m-times a point, $m \geq 2$) when looking at the deformation of a divisor $D_0 = m_1 p_1 + \cdots + m_k p_k$ for two reasons. First, the deformation of a smooth point (1 times a point) is trivial. The only place anything can happen is at points where the multiplicity is greater than one so that points can "split", i.e. when m has a non-trivial partition. Also, since C is a Hausdorff space and the p_i's were taken to be distinct then on a germ level we have

$$(C^{(d)}, D_0) \cong (C^{(m_1)}, m_1 p_1) \times \cdots \times (C^{(m_k)}, m_k p_k).$$

The deformation of D_0 is the product of the deformations of $m_i p_i$ for $i = 1, \ldots, k$. Recall that

$$\begin{aligned} (C^{(m)}, mp) &\cong (\mathbb{A}^{(m)}, m \cdot [0]) \\ &\cong (\mathbb{A}^m, (0, 0, \ldots, 0)) \\ &\cong \text{monic polynomials of degree} m\}. \end{aligned}$$

New we let M_i be the set of monic polynomials in $\mathbb{C}[z]$ of degree m_i. We say that $h = (h_1, \ldots, h_k)$ is of type $\underline{m}$, denoted $h(\underline{m})$, if $h_i \in M_i$ for every $i = 1, \ldots, k$. Now define

$$M_T = \prod_{i=1}^{k} M_i = \{h(\underline{m}), k-\text{tuples (of monic polynomials) of type } \underline{m}\}.$$

Then take $G_T = \prod_{i=1}^{k} G_i$ where G_i is the group of translations $\{z \mapsto z+b_i, b_i \in C\}$ acting on M_i. Denote $(z^{m_1}, \ldots, z^{m_k})$ by $z^{\underline{m}}$. Let V_i be a miniversal deformation of $z^{m_i} \in M_i$ and take $V_T = \prod_{i=1}^{k} V_i$. Since

$$(C^{(d)}, D_0) \cong (\mathbb{A}^{m_1}, 0) \times \cdots \times (\mathbb{A}^{m_k}, 0) \cong \prod_{i=1}^{k} M_i,$$

given D_0 of type $\underline{m}$, we have the following maps of germs:

$$\begin{array}{ccccc} (C^{(d)}, D_0) & \xrightarrow{\varphi} & (V_T, z^{\underline{m}}) & \xrightarrow{F} & (M_T, z^{\underline{m}}) \\ & \searrow \sim & \uparrow g & \nearrow \sim & \\ & & \prod_{i=1}^{k}(\mathbb{A}^{m_i}, m_i[0]) & & \end{array}$$

given by $D \in (C^{(d)}, D_0)$,

$$D \overset{\varphi}{\mapsto} g(h(\underline{m})) \overset{F}{\mapsto} h(\underline{m})$$

where g is realized by applying the appropriate Tschirnhausen transformation to each h_i, and F is an inclusion. Note that the right hand side triangle formed by the maps F, g and the isomophism

$$\prod_{i=1}^{k}(\mathbb{A}^{m_i}, m_i[0]) \longrightarrow (M_T, z^{\underline{m}})$$

is not commutative. We may think of the germ $(V_T, z^{\underline{m}})$ as the germ $(\Lambda, 0)$ where

$$\Lambda = \prod_{i=1}^{k} \{\lambda^{(i)} = (\lambda^{(i)}_{m_{i-2}}, \ldots, \lambda^{(i)}_0)\}$$

and $g_i(h_i(z)) = z^{m_i} + \lambda_{m_i-2} z^{m_i-2} + \cdots + \lambda_0$, (i.e., $z^{\underline{m}}$ is the point 0 in $(V_T, z^{\underline{m}})$ and $(M_T, z^{\underline{m}})$). Then

$$(V_T, z^{\underline{m}}) \xrightarrow{F} (M_T, z^{\underline{m}})$$

is the miniversal deformation of $z^{\underline{m}} \in M_T$. Since F is transversal to the orbits of G_T,

$$(V_T, z^{\underline{m}}) \cong (M_T/G_T, z^{\underline{m}}).$$

Recall that we have a nondegenerate embedding of C into $\mathbb{P}^n$ with degree d. The incidence correspondence, $I(C)$, is the set

$$I(C) = \{(p, H) \in C \times (\mathbb{P}^n)^* \text{ such that } p \in H\}$$

while

$$\mathcal{D} = \{(p, D) \in C \times C^{(d)} \text{ such that } p \in D\}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc}
(p, H) & \longmapsto & (p, D = H \cap C) \\
\big\downarrow & \begin{array}{ccc} I(C) & \hookrightarrow & \mathcal{D} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \psi} \\ (\mathbb{P}^n)^* & \overset{i}{\hookrightarrow} & (C^{(d)}, D_0) \end{array} & \big\downarrow \\
H & \longmapsto & D = H \cap C
\end{array}$$

Putting this diagram together with the previous one gives the following diagram:

$$\begin{array}{ccccccc}
I(C) & \hookrightarrow & \mathcal{D} & & & & \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \psi} & & & & \\
(\mathbb{P}^n)^* & \overset{i}{\hookrightarrow} & (C^{(d)}, D_0) & \xrightarrow{\varphi} & V_T & \xrightarrow{F} & M_T
\end{array}$$

We want to know that φ is a submersion. But $(C^{(d)}, D_0)$ is isomorphic to

$$\prod_{i=1}^{k} (\mathbb{A}^{(m_i)}, m_i \cdot [0]) \cong \prod_{i=1}^{k} (\mathbb{A}^{m_i})$$

and each $(\mathbb{A}^{m_i}, m_i \cdot [0])$ is isomorphic to the set of monic polynomials of degree m_i in one variable. We already have the diagram:

$$\begin{array}{ccc} (C^{(d)}, D_0) & \xrightarrow{\ \varphi\ } & (V_T, z^{\underline{m}}) \\ & \searrow^{\sim} & \uparrow g \\ & & \prod_{i=1}^{k} (\mathbb{A}^{m_i}, m_i[0]) \end{array}$$

where g is the Tschirnhausen transformation which is submersive. So φ is actually a versal deformation of $z^{\underline{m}}$ with base $(C^{(d)}, D_0)$ and $\varphi : (C^{(d)}, D_0) \longrightarrow (V_T, 0)$ is a submersion.

Corollary 2.3.6 *The stratum $\mathcal{S}(m_i, \ldots, m_k)$ through D_0 in $(C^{(d)}, D_0)$ is precisely the pre-image $\varphi^{-1}(0) = \{m_1 q_1 + \cdots + m_k q_k\}$ where $q_i \in (C, p_i)$.*

Proof: Take a point of $(V_T, 0)$ that is not the origin. That point represents a divisor near D_0 in the base of a deformation. But we took $(V_i, 0)$ to be a miniversal deformation at each point and took V_T to be the product space. So the pre-image of that point is not D_0, does not have the same multiplicities, so is not in the same stratum. But φ is a submersion, so $\varphi^{-1}(0)$ is smooth, hence is the statum $\mathcal{S}(\underline{m})$ containing D_0. ∎

This gives the complex analytic local triviality property of the stratification $\mathcal{S}$, since the stratification by type of $(V_T, 0)$ is complex analytically locally trivial for the point stratum 0, and we just pull back the structure along $\mathcal{S}(\underline{m})$ at D_0.

2.4 THE PROPOSITION

Now we are ready for the statement of the promised proposition.

Proposition 2.4.1 *The family of hyperplane sections of C, $I(C) \xrightarrow{f} (\mathbb{P}^n)^*$, is versal if and only if $(\mathbb{P}^n)^* \overset{i}{\hookrightarrow} C^{(d)}$ is transverse to the stratification of $C^{(d)}$ by type.*

There is still work to do before the proof of this proposition. The proposition is stated from the algebro-geometric point of view of deformation theory which differs from the version given earlier.

An *algebro-geometric deformation* of a complex space X is a flat family $\mathcal{X} \xrightarrow{\pi} S$ together with an isomorphism $X \xrightarrow{\sim} \pi^{-1}(0)$, where $\mathcal{X}$ and S are complex spaces, π is a morphism, and 0 is a base point in the base space S.

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \Big| & & \Big\downarrow \pi \\ 0 & \in & S \end{array}$$

$\mathcal{X} \xrightarrow{\pi} S$ is said to be *versal* if given any other deformation $W \to T$ there exists a morphism $T \to S$ such that $W \cong \mathcal{X} \times_s T$.

When the possibility of misunderstanding exists, the above deformation will be called "algebro-geometric deformations", and the previous deformations are "deformations in the sense of Arnold".

Consider the diagram:

$$\begin{array}{ccccc} I(C) & \longrightarrow & \mathcal{D} & \longrightarrow & (\mathcal{X}, 0) \\ \Big\downarrow f & & \Big\downarrow & & \Big\downarrow \pi \\ (\mathbb{P}^n)^* & \stackrel{i}{\hookrightarrow} & (C^{(d)}, D_0) & \xrightarrow{\varphi} & (V, 0) \end{array}$$

where $(V, 0)$ is the germ $(V_T, 0)$ constructed earlier. We are interested in an algebro-geometric miniversal deformation, $(\mathcal{X}, 0) \xrightarrow{\pi} (V, 0)$, of the fiber over $z^{\underline{m}} = 0 \in (V, 0)$. Recall $V_T = \prod_{i=1}^k V_i$. Let $X_i \longrightarrow V_i$ be the miniversal family over the ith factor. Then

$$(\mathcal{X}, 0) \cong \prod_{i=1}^{k} (X_i \times (V_1 \times \cdots \times \hat{V}_i \times V_k), 0)$$

and $\mathcal{X} \xrightarrow{\pi} V_T$ is a miniversal deformation.

Consider the composition $\varphi \circ i : (\mathbb{P}^n)^* \longrightarrow (V, 0)$. If this composition is a submersion the differential map $T_{H_0}((\mathbb{P}^n)^*) \longrightarrow T_0(V)$ is onto, and since $(\mathcal{X}, 0) \xrightarrow{\pi} (V, 0)$ is miniversal, the deformation of D_0 over the smooth germ $((\mathbb{P}^n)^*, H_0)$ is infinitesimally versal and therefore versal. Now we are ready to prove the proposition.

Proof: Suppose i is transverse to $\mathcal{S}$ at D_0: This says that

$$im(i_*|_{H_0}) + T_{D_0}(\mathcal{S}(\underline{m})) = T_{D_0}(C^{(d)}).$$

Since φ is submersive, $\varphi_*|_{D_0}(im(i_*|_{H_0}) + T_{D_0}(\mathcal{S}(\underline{m}))) = T_0(V)$. But we know that φ maps the stratum $\mathcal{S}(\underline{m})$ to 0 in V and the tangent space of the stratum at D_0 is mapped by $\varphi_*|_{D_0}$ to 0 in $T_0(V)$.

So $im(\varphi \circ i)_*|_{H_0} = \varphi_*|_{D_0}(im(i_*|_{H_0})) = T_0(V)$ which says $(\varphi \circ i)$ is a submersion, and hence the family $I(C) \xrightarrow{f} (\mathbb{P}^n)^*$ is versal.

Now assume the family $I(C) \xrightarrow{f} (\mathbb{P}^n)^*$ is a versal deformation of the fiber $D_0 = f^{-1}(H_0)$. We have maps

$$((\mathbb{P}^n)^*, H_0) \xhookrightarrow{i} (C^{(d)}, D_0) \xrightarrow{\varphi} (V, 0)$$

There might not be a unique miniversal deformation $(C^{(d)}, D_0) \to (V, 0)$, so we introduce a map

$$\psi : ((\mathbb{P}^n)^*, H_0) \longrightarrow (V, 0)$$

which can be any map inducing the germ of f at H_0. Then by the uniqueness at first order of inducing maps to the miniversal deformation we have

$$\psi_*|_{H_0} = (\varphi \circ i)_*|_{H_0}.$$

Since ψ is versal at first order (i.e. versal), then ψ_* is onto. So

$$im(\psi_*|_{H_0}) = T_0(V) = im(\varphi \circ i)_*|_{H_0} = \varphi_*|_{D_0}(im(i_*|_{H_0}))$$

But since $\varphi_*|_{D_0}(T_D(\mathcal{S}(\underline{m}))) = 0$ we get

$$\varphi_*|_{D_0}(im(i_*|_{H_0}) + T_{D_0}(\mathcal{S}(m))) = T_0(V)$$

Since we know $\varphi_*|_{D_0} : T_{D_0}(C^{(d)}) \longrightarrow T_0(V)$ is surjective with $ker = T_{D_0}(\mathcal{S}(m))$, and since the subspaces of $T_{D_0}(C^{(d)})$ which contain $T_{D_0}(\mathcal{S}(\underline{m}))$ correspond one-to-one to the subspaces of $T_0(V)$, it follows that

$$im(i_*|_{H_0}) + T_{D_0}(\mathcal{S}(\underline{m})) = T_{D_0}(C^{(d)})$$

This is the transversality condition. ∎

CHAPTER 3

SECANT MAPS AND VARLEY'S CONJECTURE

After the statement of the proposition in the previous chapter, C. McCrory discusses what he calls a "closely related map, whose local geometry reflects the global geometry of [the incidence correspondence]." [17]. The map he describes is the secant map. He mentions the reason for studying this map is that, in the case of a nonhyperelliptic smooth curve C, the Gauss map of the theta divisor corresponds in a particular way to the secant map of the canonical embedding of C. The main focus of his work with Adams, Shifrin, and Varley was on invariants of Gauss maps of theta divisors, which leads to a question about invariants of related secant maps [1].

3.1 THE NORMAL FORM FOR SECANT MAPS

The following discussion is lifted almost entirely from the unpublished notes of R. Varley [21], some of which is of a conjectural nature.

Suppose C be a smooth, nondegenerate compact complex connected curve of degree d in $\mathbb{P}^n$. Let S be a divisor in $C^{(n)}$ and define $\bar{S}$ to be the span of S on $C \subset \mathbb{P}^n$, in other words $\bar{S} = \cap\{H \in (\mathbb{P}^n)^*$ such that $S \leq H \cdot C\}$. Let $\mathcal{U} \subset C^{(n)}$ be the open set $\{S \,\varepsilon\, C^{(n)}$ such that $\bar{S}$ is a hyperplane of $\mathbb{P}^n\}$. The secant map can be defined in terms of the embedding:

Definition 3.1.1 *The* secant map s *is the morphism,*

$$\begin{aligned} s : \mathcal{U} &\longrightarrow (\mathbb{P}^n)^* \\ S &\longmapsto \bar{S}. \end{aligned}$$

A related universal construction is the space

$$\Lambda(n,d) = \{(S,D) \in C^{(n)} \times C^{(d)} \text{ such that } S \leq D\}$$

along with the morphism

$$\begin{aligned} \pi_{n,d} : \Lambda(n,d) &\longrightarrow C^{(d)} \\ (S,D) &\longmapsto D. \end{aligned}$$

Let $\underline{\mathrm{r}} = (r_1, \ldots, r_s)$, $\underline{\mathrm{m}} = (m_1, \ldots, m_k)$, and assume $m_i > 0$, and $r_i \leq m_i$ for $i = 1, \ldots, s$, and $s \leq k$, allowing some of the r_i to be zero. A point $(S_0, D_0) \in \Lambda(n,d)$ is said to be of type $(\underline{\mathrm{r}}\ \underline{\mathrm{m}})$ if $S_0 = r_1p_1 + \cdots + r_sp_s \leq D_0 = m_1p_1 + \cdots + m_kp_k$. The germ of $\pi_{n,d}$ at a point (S_0, D_0) of type $(\underline{\mathrm{r}}\ \underline{\mathrm{m}})$ is isomorphic to a product of multiplication maps

$$\mu_{m_1-r_1,r_1} \times \mu_{m_2-r_2,r_2} \times \cdots \times \mu_{m_s-r_s,r_s} \times \mu_{m_{s+1},0} \times \cdots \times \mu_{m_k,0}.$$

The relationship of $\pi_{n,d}$ to s is that s is induced from $\pi_{n,d}$ by the inclusion

$$\begin{aligned} i : (\mathbb{P}^n)^* &\longrightarrow C^{(d)} \\ H &\longmapsto H \cdot C. \end{aligned}$$

More precisely, let $\tau : \mathcal{U} \longrightarrow \Lambda(n,d)$ denote the map $S \longmapsto (S,D)$, where $D = \bar{S} \cdot C$. Fix $S_0 \in \mathcal{U}$, let $H_0 \in (\mathbb{P}^n)^*$ be the hyperplane $\bar{S}_0$, and let $D_0 \in C^{(d)}$ be the divisor $H_0 \cdot C$. Then the pair of maps (τ, i) induces a complex analytic embedding of the secant map germ $s : (\mathcal{U}, S_0) \longrightarrow ((\mathbb{P}^n)^*, H_0)$ in the map germ

$$\pi_{n,d} : (\Lambda(n,d), (S_0, D_0)) \longrightarrow (C^{(d)}, D_0).$$

Claim 3.1.2 *The map $\pi_{n,d}$ is complex analytically locally trivial along strata.*

In other words, there is a “reduced” map germ $R_{n,d}$ such that $\pi_{n,d}$ and $R_{n,d} \times id_k$ are R-L equivalent as complex analytic map germs around corresponding base points. Here

id_k denotes the identity map on k-space, and R-L equivalence refers to the right-left equivalence found in [3].

Assuming the claim is true leads to an interesting consequence for the structure of the secant map germ when i is transverse to the stratum $\mathcal{S}(\underline{\mathrm{m}})$ at D_0. If the germ (i, H_0) is transverse to the stratification of $C^{(d)}$, then $s \sim R_{n,d} \times id_{n-d+k}$ as complex analytic map germs around corresponding base points. Therefore we get the following result:

Conjecture 3.1.3 *If $i : (\mathbb{P}^n)^* \to C^{(d)}$ is transverse at H_0 to the stratification $\mathcal{S}$, (or equivalently, if the family of hyperplane sections of C in $(\mathbb{P}^n)^*$ induces a versal deformation of the fiber $D_0 = H_0 \cdot C$ over H_0) then, as complex analytic map germs at corresponding base points,*

$$\begin{aligned} s \times id_{d-n} &\sim \pi_{n,d} \\ &\sim \mu_{m_1-r_1,r_1} \times \mu_{m_2-r_2,r_2} \times \cdots \times \mu_{m_s-r_s,r_s} \times \mu_{m_{s+1},0} \times \cdots \times \mu_{m_k,0}. \end{aligned}$$

In other words, under the right conditions, studying invariants for secant maps comes down to studying invariants for polynomial multiplication maps. In the case of the Thom-Boardman invariant, the symbol for a product of polynomial multiplication maps is the component-wise sum of the symbols for each of the factors; for some nice examples see [1]. The symbol for any identity map is a tuple of zeros, and $\mu_{k,0}$ is the identity map id_k for any $k \geq 0$. Then it follows that $\mathrm{TB}(s)$ is that component-wise sum. Here is an example of the secant map in light of the above discussion.

Example 3.1.4 $\deg(C \subset \mathbb{P}^3) = 6$

Let $S_0 = p_1 + p_2 + p_3 \ \leq \ D_0 = 3p_1 + 2p_2 + p_3$. The point in $(S_0, D_0) \in \Lambda(3,6)$ is of type

$$\begin{pmatrix} 1 & & 1 & & 1 & \\ & 3 & & 2 & & 1 \end{pmatrix}.$$

Assuming the above conjecture,

$$s(S_0) \times id_3 \sim \mu_{2,1} \times \mu_{1,1} \times \mu_{1,0}.$$

In order to get the TB-invariant for the right hand side, we add the symbols:

$$\mathrm{TB}(s(S_0)) = \mathrm{TB}(B_{2,1}) + \mathrm{TB}(B_{1,1}) + \mathrm{TB}(B_{1,0}) = (1,1) + (1,0) + (0,0) = (2,1)$$

If the computation of the TB-symbols for $\mu_{n,r}$ were easily computed for any size n and r, then the theory could be completed by a proof of Claim 3.1.2. However, the computation involves the computation of all of the minors of a certain size of a possibly very large matrix. The following conjecture proposes a simple computation of the invariant for exactly these maps.

3.2 Varley's Conjecture

Consider the polynomial multiplication map $\mu_{n,r}$ with fixed $n \geq r$. Consider the Euclidean algorithm applied to n and r:

$$\begin{aligned} n &= q_1 r + r_1 \quad 0 < r_1 < r \\ r &= q_2 r_1 + r_2 \quad 0 < r_2 < r_1 \\ &\vdots \\ r_{k-1} &= q_{k+1} r_k \quad 0 < r_k < r_{k-1} \end{aligned}$$

Let $I(n,r)$ be the tuple given by the Euclidean algorithm on n and r:

$$I(n,r) = (r, \ldots, r, r_1, \ldots, r_1, \ldots, r_k, \ldots, r_k)$$

where r is repeated q_1 times, and r_i is repeated q_{i+1} times.

Conjecture 3.2.1 *Varley's Conjecture:* $\mathrm{TB}(B_{n,r}) = I(n,r)$.

Computer calculations confirm this conjecture for all cases for which $n + r \leq 10$. The memory demands for these calculations grows exponentially with the size of n and $n - r$.

3.3 Determinants of $\delta B_{n,r}$

Let ω represent the weighting structure on $\mathbb{C}[a_0, \ldots, a_{n-1}]$ given by

$$\omega(0) = -\infty \tag{3.3.1}$$

$$\omega(k) = 0 \quad \text{for} \quad k \in \mathbb{C}^* \tag{3.3.2}$$

$$\omega(a_i) = n - i \quad \text{for} \quad i \in \{0, 1, \ldots, n-1\} \tag{3.3.3}$$

$$\omega(a_i a_j) = \omega(a_i) + \omega(a_j) \tag{3.3.4}$$

Suppose $B_{n,r} \subseteq \mathbb{C}[a_0, \ldots, a_{n-1}]$ be the ideal generated by the $c_{n-1}, \ldots, c_0$ from the $\mu_{n,r}$-map. Recall the following hold:

$$c_i = c_i(a_0, \ldots, a_{n-1}) \tag{3.3.5}$$

$$c_i(0, \ldots, 0) = 0 \tag{3.3.6}$$

$$\omega(c_{n-i}) = r + i. \tag{3.3.7}$$

Proposition 3.3.1 *The minors of any order of δJ are weighted homogeneous.*

Proof: Suppose $A = (\alpha_{ij})$ is an $n \times n$ matrix. Then $\det(A)$ is the sum of all possible combinations of products $\alpha_{1\ \sigma(1)}\alpha_{2\ \sigma(2)} \ldots \alpha_{n\ \sigma(n)}$, where $\sigma(i)$ is a permutation on n elements, in particular $\sigma(i) \neq \sigma(j)$ for $i \neq j$. Consider $\delta B_{n,r} = (\beta_{kl})$ with the rows and columns labeled $1, \ldots, n$ as usual. Then $\beta_{kl} = \partial c_j / \partial a_i$ and $k = n - j,\ l = i + 1$.

$$\begin{aligned} \omega(\beta_{kl}) &= \omega(c_{n-k}) - \omega(a_{l-1}) \\ &= r + k - (n - (l - 1)) \\ &= k + l - (n - r + 1) \\ &\quad \text{when } k + l \geq n - r + 1. \end{aligned}$$

When $k + l = n - r + 1$ then β_{kl} is a constant. This corresponds to the $(n-r)$-antidiagonal mentioned in the first chapter. When $k + l \leq n - r + 1$ the entry β_{kl} is above that antidiagonal and is therefore zero.

Let A be an $s \times s$ submatrix of $\delta B_{n,r}$ with nonzero determinant. Take the s rows of $\delta B_{n,r}$ that occur in A and name them $k_1, k_2, \ldots, k_s$ and name the columns which occur $l_1, \ldots, l_s$. Thus $\omega(\beta_{k_i l_i}) = k_i + l_i - (n - r + 1)$ for $k_i + l_i \geq n - r + 1$. Since $\omega(\alpha\beta) = \omega(\alpha) + \omega(\beta)$ and since $\{\sigma(k_i)\} = \{l_i\}$,

$$\begin{aligned} \omega(\beta_{k_1\sigma(k_1)} \cdots \beta_{k_s\sigma(k_s)}) &= s(r - n - 1) + \sum_{i=1}^{n} (k_i + \sigma(k_i)) \\ &= s(r - n - 1) + \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} l_i \end{aligned}$$

Which means that the weights of the terms of the determinant of A are dependent in exactly the same way on the sums of the indices of the entries and hence are equal. ∎

3.4 The Critical Column

Recall that the critical column corresponds to the partial $\partial/\partial a_{n-r}$ and is the first column from the left containing no ones. In fact, it contains every variable as a linear term in order of ascending weight as the entries run down the column. This follows from the fact that, given any $n, r, n \geq r$, for $0 \leq j \leq n-1$

$$\partial c_j / \partial a_{n-r} = -a_j + \text{terms of equal weight}$$

Since no two variables have equal weight, a_j is the only variable appearing in a linear term of the partial, although it may appear with a coefficient other than -1. It also can not appear in any nonlinear term of the partial, since that term would have to be of higher weight and the partials are weighted homogeneous. For the sake of illustration, recall $\delta B_{5,2}$:

$$\begin{pmatrix} 0 & 0 & 1 & -2\,a_4 & -2\,a_3 + 3\,a_4^2 \\ 0 & 1 & -a_4 & -2\,a_3 + a_4^2 & -a_2 + 2\,a_3\,a_4 \\ 1 & -a_4 & -a_3 + a_4^2 & -a_2 & -a_1 + 2\,a_2\,a_4 \\ -a_4 & -a_3 + a_4^2 & 0 & -a_1 & 2\,a_1\,a_4 - a_0 \\ -a_3 + a_4^2 & 0 & 0 & -a_0 & 2\,a_0\,a_4 \end{pmatrix}$$

The critical column is:

$$\begin{pmatrix} -2\,a_4 \\ -2\,a_3 + a_4^2 \\ -\,a_2 \\ -\,a_1 \\ -\,a_0 \end{pmatrix}$$

For general $n \geq r$, the $n-r$-antidiagonal contains all ones and thus the minors needed for the first Jacobian extension of $B_{n,r}$ are the $n-r+1$ minors. Since the jth entry of the TB-invariant is the dimension of the Zariski tangent space to the fiber of the map germ $\mu_{n,r}$ on the subscheme of $\mathbb{C}[a_0,\ldots,a_{n-1}]$ defined by $\triangle^{i_{j-1}}\ldots\triangle^{i_1}B$, the determinants with linear terms are the ones of *primary* interest at each stage. However, by definition, we must consider all of the determinants, to compute the invariant.

The way to get the determinants with linear terms is to utilize the submatrix whose anitdiagonal is the $n-r$-antidiagonal of ones and adjoin one row and one column. This means that the only linear terms that can appear in any of the determinants of the first Jacobian matrix are linear terms in the entries of the lower right hand $r\times r$ submatrix of $\delta B_{n,r}$. In later Jacobian matrices, they are the linear terms which appear outside of the rows and columns used to make the determinant 1 submatrix.

3.5 Strengthening Varley's Conjecture

Denote $\partial c_j/\partial a_{n-r}$ by $\tilde{a}_j$ since it contains a_j as its only linear term. Then the determinants of primary interest are the ones of the form:

$$d_j = \tilde{a}_j + \text{terms of degree 2 or more}$$

It is important to notice that the linear terms that appear in $\triangle^r B_{n,r}$, given the structure of the critical column, are the ones in $\tilde{a}_j$ for j going from 0 to $r-1$ and the ones already linear in $B_{n,r}$. These are the linear terms $a_0,\ldots,a_{\max\{r,n-r\}}$.

This observation gives insight into an extended version of Varley's conjecture: not only can one give the dimension of the Zariski tangent space corresponding to the Jacobian extension of B computed at each stage of the algorithm, one can also predict the linear equations which define that tangent space. The first $n-r$ variables appear as linear terms in the generators for $B_{n,r}$. In the extended conjecture, the variables appear as linear terms in each of the Jacobian extensions as follows.

Conjecture 3.5.1 *Let* $\mathrm{TB}(B_{n,r}) = (i_1, i_2, \ldots, i_k)$. *Then the a generating set of the Jacobian extension* $\triangle^{i_j} \triangle^{i_{j-1}} \cdots \triangle^{i_1} B_{n,r},\ j \le k$, *contains the variables* $a_0, \cdots, a_{n-1-i_j}$ *as linear terms.*

Remark 3.5.2 *A second observation here is that the second entry of the TB-symbol is now apparent from the above discussion. Since there are exactly* $\max\{r, n-r\}$ *linear terms in* $\triangle^{i_1}B,\ \delta \triangle^{i_1} B$ *has exactly* $\max\{r, n-r\}$ *ones along an antidiagonal and no more. Therefore* $\mathrm{rank}(\delta \triangle^{i_1} B) = \max\{r, n-r\}$, *which implies that the second entry is* $i_2 = \mathrm{corank}(\delta \triangle^{i_1} B) = \min\{r, n-r\}$. *Also, it follows that* $i_2(B_{n,r}) = i_1(B_{n-r,r})$.

CHAPTER 4

CLASSES OF CASES

The extended version of Varley's Conjecture can be proven when certain relationships hold between the degrees n and r of the polynomials. This chapter is devoted to those proofs. Notice that the goal is to prove Varley's original conjecture and that the proof of the extended version comes essentially for free. Often we are required to find a generating set of expressions for an ideal. When this occurs, one may assume that the generators can be obtained in practice by using the Groebner package in Maple.

4.1 $\mu_{n,1}$

This first case is similar to the Example 1.5.1, the Whitney cusp map. In fact, it is equivalent to the generalized Whitney cusp. For more details on this, refer to [6],[3]. Although this case is already known, it is worthwhile proving it here. The Thom-Boardman symbol $\mathrm{TB}(\mu_{n,1})$ is a tuple of n ones. This case is more easily seen using the ideal A, the ideal generated by the coefficients before eliminating variables.

Proposition 4.1.1 $\mathrm{TB}(\mu_{n,1}) = (\underbrace{1, 1, \ldots, 1}_{n\ ones})$

Proof:

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \text{ and } g(x) = x - b$$

$$\mu_{n,1} : (a_0, \ldots, a_{n-1}, b) \longmapsto (a_{n-1} + b,\ a_{n-2} + a_{n-1}b,\ a_{n-3} + a_{n-2}b, \ldots,\ a_0b)$$

$$A_{n,1} \ = \ (a_{n-1} + b,\ a_{n-2} + a_{n-1}b,\ a_{n-3} + a_{n-2}b, \ldots, a_0b)$$

as an ideal in $\mathbb{C}[a_0, \dots, a_{n-1}b]$.

The expressions give us the resultant matrix expected when taking the partial derivatives with respect to b as the first column and the derivatives with respect to the a_i's in descending order in the i's for the following columns.

$$\delta A_{n,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & & & 0 \\ -a_{n-1} & -b & 1 & 0 & \dots & & & \\ -a_{n-2} & 0 & -b & 1 & \dots & & & \\ -a_{n-3} & 0 & 0 & -b & \dots & & & \\ \vdots & & \ddots & & \vdots & & & \\ -a_1 & 0 & & \dots & & 0 & -b & 1 \\ -a_0 & 0 & & \dots & & & 0 & -b \end{pmatrix}$$

Since $\operatorname{corank}(\delta A_{n,1}) = 1$, $\mathrm{TB}(A_{n,1}) = (1, \dots)$ and we take the full determinant of the matrix for our only "minor". Notice

$$\delta A_{n,1} - (-b)I_{n\times n} = \begin{pmatrix} b-1 & 1 & 0 & 0 & \dots & & & 0 \\ -a_{n-1} & 0 & 1 & 0 & \dots & & & \\ -a_{n-2} & 0 & 0 & 1 & \dots & & & \\ -a_{n-3} & 0 & 0 & 0 & \dots & & & \\ \vdots & & \ddots & & \vdots & & & \\ -a_1 & 0 & & \dots & & 0 & 0 & 1 \\ -a_0 & 0 & & \dots & & & 0 & 0 \end{pmatrix}$$

is the companion matrix to the polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = f(x)$. Then the determinant of $\delta A_{n,1}$ is the polynomial $\pm(b^n + a_{n-1}b^{n-1} + \cdots + a_0) = \pm f(b)$ where we have "+" if n is odd and "−" if n is even. We may ignore the sign of $\det \delta A_{n,1}$ and adjoin $f(b)$ to $A_{n,1}$. Then the critical Jacobian extension of $A_{n,1}$ is

$$\triangle^1 A = (f(b),\ a_{n-1} - b,\ a_{n-2} - ba_{n-1},\ \dots, a_0 b).$$

We find a generating set for this ideal and get the equivalent ideal

$$(b^n,\ a_{n-1}-b,\ a_{n-2}-b^2,\ \dots\ ,a_1-b^{n-1},\ a_0).$$

This is not the maximal ideal $(b, a_0,\ a_1,\ \dots,\ a_{n-1})$ in the polynomial ring over $\mathbb{C}$ in those variables. We look for the critical extension of $\triangle^1 A$. The Jacobian matrix is

$$\delta \triangle^1 A_{n,1} = \begin{pmatrix} nb^{n-1} & 0 & 0 & 0 & \dots & & & 0 \\ -1 & 1 & & 0 & \dots & & & \\ -2b & 0 & 1 & & \dots & & & \\ -3b^2 & 0 & 0 & 1 & \dots & & & \\ \vdots & & \ddots & & \vdots & & & \\ -(n-1)b^{n-2} & 0 & & \dots & & 0 & 1 & 0 \\ 0 & 0 & & \dots & & & 0 & 1 \end{pmatrix}$$

This time $\operatorname{corank}(\delta \triangle^1 A_{n,1}) = 1,$ and $\operatorname{TB}(A_{n,1}) = (1, 1, \dots)$ and we must adjoin the entire determinant, $\det(\delta \triangle^1 A_{n,1}) = nb^{n-1}$, to get the critical extension. This lowers the power of b that appears in the first entry and changes $a_1 - b^{n-1}$ to simply a_1. Then our critical extension is

$$\triangle^1 \triangle^1 A_{n,1} = (b^{n-1},\ a_{n-1}-b,\ a_{n-2}-b^2,\ \dots\ ,\ a_1,\ a_0).$$

Obviously, repeating this process will continue to lower the power of the first expression listed and give us the next a_i. Each repetition gives us another 1 in the Thom-Boardman invariant. The method stops when we reach

$$\triangle^1 \triangle^1 \cdots \triangle^1 A_{n,1} = (b, a_0, \dots, a_{n-1}).$$

We had b^n as a generator after the first process, i.e. in $\triangle^1 A_{n,1},$ b^{n-1} in $\triangle^1 \triangle^1 A_{n,1},$ and b^{n-2} in $\triangle^1 \triangle^1 \triangle^1 A_{n,1}.$ Similarly, we get b^{n-j} in

$$\underbrace{\triangle^1 \cdots \triangle^1}_{j+1} A_{n,1}.$$

Therefore, we get $b^{1=n-(n-1)}$ in

$$\underbrace{\triangle^1 \cdots \triangle^1}_{n-1+1} A_{n,1},$$

and the Thom-Boardman invariant for $\mu_{n,1}$ is $(\underbrace{1,1,\ldots,1}_{n\ ones})$ for any choice of n. ∎

Notice that the linear terms appeared in order: $a_0,\ a_1,\ \ldots,\ a_{n-1}$ were present as linear terms in $A_{n,1}$ and b appeared as a linear term on the last phase of the algorithm. This proves the extended version of Varley's conjecture for $\mu_{n,1}$.

4.2 $\mu_{n,n}$

Proposition 4.2.1 $\mathrm{TB}(\mu_{n,n}) = (n)$

Proof:

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \text{ and } g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

The coefficients of the product are:

$$\begin{cases} c_{2n-1} = a_{n-1} + b_{n-1} \\ c_{2n-2} = a_{n-2} + b_{n-2} + a_{n-1}b_{n-1} \\ c_{2n-3} = a_{n-3} + b_{n-3} + a_{n-2}b_{n-1} + a_{n-1}bn - 2 \\ \vdots \\ c_j = a_{j-n} + b_{j-n} + \sum\limits_{i+k=j} a_i b_k, \text{ for } j \geq n \\ \vdots \\ c_{n-1} = a_0 b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-1}b_0 \\ c_{n-2} = a_0 b_{n-2} + a_1 b_{n-3} + \cdots + a_{n-2}b_0 \\ \vdots \\ c_0 = a_0 b_0 \end{cases}$$

The equations for elimination are:

$$\begin{cases} b_{n-1} = -a_{n-1} \\ b_{n-2} = -a_{n-2} + a_{n-1}^2 \\ b_{n-3} = -a_{n-3} + 2a_{n-2}a_{n-1} - a_{n-1}^3 \\ \vdots \\ b_{n-j} = -a_{n-j} + \text{terms of higher order} \pm a_{n-1}^j \\ \vdots \\ b_0 = -a_0 + \text{terms of higher order} \pm a_{n-1}^n \end{cases}$$

Notice that each b_i has $-a_i$ as a linear term and all the other terms involve only the a's with higher index.

The remaining coefficients are:

$$c_{n-j} = a_0(-a_{n-j}) + \cdots + a_{n-j}(-a_0 + \text{higher order terms}).$$

The only terms in c_{n-j} involving a_0 are the first term, $-a_0a_{n-j}$ and the one from the substitution of b_0, $a_{n-j}(-a_0 + \cdots)$. There are no linear terms in the remaining coefficients so there are no unit entries in the matrix. Then the first column of $\delta B_{n,n}$, the one whose entries are the partials $\partial c_{n-j}/\partial a_0$, is

$$\begin{pmatrix} -2a_{n-1} \\ -2a_{n-2} \\ -2a_{n-3} \\ \vdots \\ -2a_1 \\ -a_0 \end{pmatrix}$$

Since the rank of this matrix at the origin is zero, the size of the critical minor is 1×1. In other words, the critical minors are the entries themselves. This is enough for the proof of the conjecture. Regardless of what the other columns hold, the entries of the first column give us every variable a_i as a linear term. Adjoining those to $B_{n,n}$ gives $\triangle^n B_{n,n} = (a_0, a_1, \ldots, a_{n-1})$. ∎

4.3 $\mu_{kr,r}$, $k \geq 2$

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals in $\mathbb{C}[x_1, \ldots, x_s]$. such that $\mathrm{TB}(\mathcal{I}) = (i_1, i_2, \ldots, i_k)$ and $\mathrm{TB}(\mathcal{J}) = (j_1, j_2, \ldots, j_m)$. Define an ordering, "$\succeq$" , on the symbols as follows. Suppose $i_n = j_n$ for n from 1 to some N, and differs in the $N+1$ entry. If $i_{N+1} > j_{N+1}$, then $\mathrm{TB}(\mathcal{I}) \succ \mathrm{TB}(\mathcal{J})$. For example, $(1,1) \succ (1,0) = (1)$.

For this case, the equality $\mathrm{TB}(\mu_{kr,r}) = I(kr,r)$ is proved by proving both

$$\mathrm{TB}(\mu_{kr,r}) \succeq I(kr,r) \text{and} \, \mathrm{TB}(\mu_{kr,r}) \preceq I(kr,r).$$

The first of these two proofs is contained here, while the second proof can be found in chapter 6.

Note that we already know $i_1 = r$ from the Sylvester matrix argument and $k \geq 2$, $i_2 = \min\{kr - r, r\} = r$ (since $k \geq 2$) by the remark at the end of chapter 3, but we will not need to use these facts directly here.

Claim 4.3.1 $\mathrm{TB}(\mu_{kr,r}) \succeq I(kr,r)$.

Proof:

$$f(x) = x^{kr} + a_{kr-1}x^{kr-1} + \cdots + a_0 \text{ and } g(x) = x^r + b_{r-1}x^{r-1} + \cdots + b_0$$

After the elimination process described in the first chapter, $B_{kr,r}$ is generated by the expressions of the form:

$$\begin{cases} c_{kr-j} = a_{(k-1)r-j} + \text{higher order terms}, & j \leq (k-1)r \\ c_{kr-j} = \text{higher order terms}, & j > (k-1)r \end{cases}$$

The $a_0, \ldots a_{(k-1)r-1}$ appear as linear terms in the coefficients (and therefore in the defining equations for the Zariski tangent space). So they can be eliminated the same way the coefficients of the polynomial of lesser degree were eliminated. This is called the *reduced case*, and the reduced ideal obtained from an ideal B is denoted $\tilde{B}$. Recall that the TB-invariant is independent of the coordinates taken, so $\mathrm{TB}(B) = \mathrm{TB}(\tilde{B})$.

After this reduction, there are no linear terms in the generators of $\tilde{B}_{kr,r}$. In fact, the weights of the remaining reduced coefficients are all greater than $kr,$ the highest weight for any variable, not just the ones which were eliminated. The highest weight of the remaining variables is r. Denote the non-zero reduced expressions by

$$g_{kr+1}, \quad g_{kr+2}, \quad \ldots, \quad g_{(k+1)r}$$

where the index indicates the weights. The only variables not eliminated are

$$a_{(k-1)r}, a_{(k-1)r+1}, \ldots, a_{kr-1}.$$

Since there are no linear equations in the g_i's, the Jacobian matrix has dimension $r \times r$ and rank zero. To get $\triangle^r \tilde{B}_{kr,r}$, we use the 1×1 minors, the partials with respect to the remaining variables. But,

$$\omega(\frac{\partial g_i}{\partial a_j}) = \omega(g_i) - \omega(a_j).$$

This means that the weights of the partials cannot be lower than $kr - r + 1.$ For $k \geq 2$, no linear terms can occur in $\triangle^r \tilde{B}_{kr,r}$, thus

$$i_2 = \mathrm{corank}(\delta \triangle^r \tilde{B}_{kr,r}) = r$$

as expected.

If $k = 2$, the weights of the partial derivatives of the $\partial g_i/\partial a_j$ have weights ranging from one to $r - 1$ and it is possible that every variable appears in a linear term in those minors of the matrix associated to this step of the process. If that happens, then $\mathrm{TB}(\mu_{2r,r}) = (r, r, 0)$. If some of the variables do occur in linear terms, but not all r of them, then $i_3 > 0$. Either way, $\mathrm{TB}(\mu_{2r,r}) \succeq (r, r, 0) = I(2r, r)$.

Now, if $k > 2$ then the argument repeats. In other words, the partials of the $\partial g_i/\partial a_j$ have weights ranging from $kr - r$ to $kr - 1$ and cannot possibly produce linear terms. We can continue taking partials until the weights of the adjoined minors are less than r, that is, when $kr + 1 - tr \leq r - 1$ where t is the number of times partials have been taken at that phase of the computation. Thus $t = k$ and $\mathrm{TB}(\mu_{kr,r}) \succeq I(kr, r)$. ∎

Why can we ignore the generators of higher weights? Because the entries of the TB-symbol only change when the dimension of the Zariski tangent space falls, (i.e. when we get new variables as linear terms in the expressions for our minors). The expressions of higher weight do not affect the outcome. Notice that in the case of $\mu_{kr,r}$ we may even ignore the original g's after the first Jacobian extension.

4.4 $\mu_{2k+1,2}$

The above class of cases covers the case $\mu_{n,2}$ when n is even. The techniques used to prove it, however, do not extend to the more general case $\mu_{kr+1,r}$ except in the case where $r = 2$. This is because, for any choice of n and r with $n \geq r$, there is one lowest weight expression in B, and its partial with respect to the highest weight variable gives a unique lowest weight partial. It is easy to see that there are at least two partials of every other weight, except for the highest weight partial which has weight equal to that of one of the original expressions. It follows that only the lowest weight expression in each critical Jacobian extension is unique by weight. In the case of $\mu_{2k+1,2}$, there are only two expressions in the reduced case, $\tilde{B}$, with which to contend. Only the two lowest weights of any of the successive partials ever enter the argument. One is unique and the other provides the inequality. The case is shown here. Again, we have an inequality that we can prove using the above techniques, with a proof of equality later.

The proof of $\mathrm{TB}(\mu_{2k+1,2}) \succeq I(2k + 1, 2)$ begins the same as the above argument. We eliminate the variables of the lowest degree polynomial. In this case, the generators of $B_{2k+1,2}$ have weight 3 and up to $2k + 3$. There are $2k - 1$ distinct variables appearing in linear terms in these generators. In fact, they are, as expected, the lower indexed coefficients for the polynomial of highest degree. Denote these as $a_0, \ldots, a_{2k-2}$, (as usual). After passing to the reduced case, $\tilde{B} = (g_{2k+2}, g_{2k+3})$ where the indices indicate the weight of the reduced generators, and the remaining two variables have weights 1 and 2. Just as in the above case, the two variables do not appear in linear terms in the critical Jacobian extensions until the kth extension. The generators of $\underbrace{\triangle^2 \cdots \triangle^2}_{k} B$ have weight 2 and above, thus $\underbrace{\triangle^2 \cdots \triangle^2}_{k} B$ has possibly one linear term of weight 2 but no more. The rank of $\delta \triangle^2 \cdots \triangle^2 B$ is at most one. This proves

$$\mathrm{TB}(\mu_{2k+1,2}) \succeq (\underbrace{2, 2, \ldots, 2}_{k}, 1, 1).$$

CHAPTER 5

MONOMIAL AND POLYNOMIAL MODELS

5.1 MONOMIAL IDEALS $M(n,r)$

In this section we construct monomial ideals M with a desired Thom-Boardman symbol $I = (i_1, \dots, i_k)$ as described below.

Consider the Euclidean algorithm applied to n and r:

$$\begin{aligned} n &= q_1 r + r_1 \quad 0 < r_1 < r \\ r &= q_2 r_1 + r_2 \quad 0 < r_2 < r_1 \\ &\vdots \\ r_{k-1} &= q_{k+1} r_k \quad 0 < r_k < r_{k-1} \end{aligned}$$

Let $I(n,r)$ be the Thom-Boardman symbol given by the Euclidean algorithm on n and r:

$$I(n,r) = (r, \dots, r, r_1, \dots, r_1, \dots, r_k, \dots, r_k)$$

where r is repeated q_1 times, and r_i is repeated q_{i+1} times.

Now we construct a monomial ideal $M(n,r) \subset \mathbb{C}[x_1, x_2, \dots, x_n]$ with weighting structure consistent with the weighting structure on $B_{n,r}$, which has $\mathrm{TB}(M(n,r)) = I(n,r)$. In order to do this, we start at the last entry of $\mathrm{TB}(M(n,r))$, and consider the maximal ideal

$$\mathbf{m}_n = (x_1, x_2, \dots, x_n) \subset \mathbb{C}[x_1, \dots, x_n].$$

We define the weighting structure for $M(n,r)$ and $\mathbf{m_n}$ by taking $\omega(x_i) = i$. Then we "integrate" the first r_k variables in $\mathbf{m_n}$ with respect to x_{r_k}, i.e. we multiply each of them by x_{r_k}. (The term “integrate” is used loosely here but with good cause. This process is the opposite of taking $\triangle$ of an ideal where we take the partial derivatives.)

We repeat this process, working our way backwards through $I(n,r)$. The result is an ideal generated by a set of monomials, therefore the weighted homogeneous property of the generators of the constructed ideals is trivially guaranteed. We take g_i to be the resulting monomial with weight i. Notice that $x_{r+1},\dots,x_n$ were not affected by any of the integrations. Hence, $g_{r+i}=x_{r+i}$ for $i=1,\dots,n-r$.

By construction,

$$M(n,r)=(x_{r+1},\dots,x_n,g_{n+1},\dots,g_{n+r})$$

and

$$\mathbb{C}[x_1,\dots,x_n]\supset(x_{r+1},\dots,x_n,M(n-r,r))=(x_{r+1},\dots,x_n,\frac{\partial g_{n+1}}{\partial x_r},\dots,\frac{\partial g_{n+r}}{\partial x_r}),$$

where $M(n-r,r)$ is the monomial ideal constructed in the same way from $\mathbf{m}_{\max(n-r,r)}$ in $\mathbb{C}[x_1,\dots,x_{\max(r,n-r)}]$, but considered as an ideal in $\mathbb{C}[x_1,\dots,x_n]$.

For $n\geq i\geq j$ denote the ideal $(M(i,j),x_j,x_{j+1},\dots,x_n)\subset\mathbb{C}[x_1,\dots,x_n]$ by $M^{(n)}(i,j)$.

Example 5.1.1 $\mathrm{TB}(M(4,3))=(3,1,1,1)$

The example was chosen because we already know both symbols $\mathrm{TB}(B_{4,3})$ and $I(4,3)$ are $(3,1,1,1)$. Using the algorithm above, start with

$$\mathbf{m}_4=(x_1,x_2,x_3,x_4).$$

The last entry i_4 in $I(4,3)$ is 1 and integrating the first variable with respect to x_1 gives the ideal

$$M^{(4)}(1,1)=(M(1,1),x_2,x_3,x_4)=(x_1^2,x_2,x_3,x_4).$$

Taking the first Jacobian:

$$\delta M^{(4)}(1,1) = \begin{pmatrix} 2x_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The corank of this matrix is 1, $\det(\delta M^{(4)}(1,1)) = 2x_1$, and $\triangle^1 M^{(4)}(1,1) = \mathbf{m}_4$ hence $\mathrm{TB}(M^{(4)}(1,1)) = 1$. The next step in the construction:

$$\begin{aligned} i_3 &= 1 \\ M^{(4)}(2,1) &= (x_1^3, x_2, x_3, x_4) \\ \delta M^{(4)}(2,1) &= \begin{pmatrix} 3x_1^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

The corank of this matrix is 1, $\det(\delta M^{(4)}(1,1)) = x_1^2$, and $\triangle^1 M^{(4)}(2,1) = M^{(4)}(1,1)$ and it is easy to see that $\mathrm{TB}(M^{(4)}(2,1)) = (1,1)$.

Repeating the process with $i_2 = 1$ gives the ideal

$$M^{(4)}(3,1) = (x_1^4, x_2, x_3, x_4) \text{ with } \mathrm{TB}(M(3,1)) = (1,1,1).$$

Now for the last step in building the desired $M(4,3)$, since $i_1 = 3$ we integrate the first three expressions with respect to x_3 to get

$$M(4,3) = (x_1^4 x_3, x_2 x_3, x_3^2, x_4).$$

Then,

$$\begin{aligned} \delta M(4,3) &= \begin{pmatrix} 4x_1^3x_3 & 0 & x_1^4 & 0 \\ 0 & x_3 & x_2 & 0 \\ 0 & 0 & 2x_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \operatorname{corank}(\delta M(4,3)) &= 3 \end{aligned}$$

Adding the 2×2 minors to $M(4,3)$ gives

$$\triangle^3 M(4,3) = (x_1^4, x_2, x_3, x_4).$$

So we have constructed $M(4,3)$ such that $\mathrm{TB}(M(4,3)) = (3,1,1,1)$.

The above example illustrates an important property of the constructed monomial ideals. Notice that in the example for $j \leq i$,

$$\triangle^j\, M(i,j) = M^{(i)}(i-j,j),$$

and

$$\mathrm{TB}(M(i,j)) = (j, \mathrm{TB}(M^{(i)}(i-j,j))).$$

This turns out to be the case for arbitrary n and r.

Lemma 5.1.2 $M^{(n)}(n-r,r) = \triangle^r M(n,r)$.

Proof: Let $\alpha_j = q_{j+1}$ and let G_i be the monomials in $M(n,r)$ with the convention that $G_i = x_i x_{r_k}^{\alpha_k} \cdots x_{r_0}^{\alpha_0}$, where $r_0 = r$ and $\alpha_j = 0$ if $i > r_j$. Also note that the last $n-r$ of the G_i's are linear, corresponding to the $g_{r+1}, \ldots, g_n$ above, and the others correspond to the $g_{n+1}, \ldots, g_{n+r}$.

Then $\frac{\partial G_i}{\partial x_j}$ has one of the following forms:

$$\frac{\partial G_i}{\partial x_i} = x_{r_k}^{\alpha_k} \cdots x_{r_0}^{\alpha_0} \text{ if } i \notin \{r_0, \ldots, r_k\} \tag{5.1.1}$$

$$\frac{\partial G_i}{\partial x_{r_j}} = \alpha_j x_i x_{r_k}^{\alpha_k} \cdots x_{r_j}^{\alpha_j - 1} \cdots x_{r_0}^{\alpha_0} \tag{5.1.2}$$

$$\frac{\partial G_i}{\partial x_j} = 0 \text{ if } i \neq j, \text{ and } j \notin \{r_0, \ldots, r_k\} \tag{5.1.3}$$

$$\frac{\partial G_i}{\partial x_i} = 1 \text{ if } i \in \{r+1, \ldots, n\} \tag{5.1.4}$$

The first Jacobian looks like:

$$\delta M = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \cdots & \frac{\partial G_1}{\partial x_n} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} & \cdots & \frac{\partial G_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_n}{\partial x_1} & \frac{\partial G_n}{\partial x_2} & \cdots & \frac{\partial G_n}{\partial x_n} \end{pmatrix}$$

which has the block form:

$$\delta M = \begin{pmatrix} M' & 0 \\ 0 & I \end{pmatrix}$$

where I is the $(n-r) \times (n-r)$ identity matrix. We want to take the $n-r+1$ minors of δM and show they are in the ideal generated by the entries in the r^{th} column since the ideal so generated is the ideal $M^{(n)}(n-r, r)$. The non-zero entries of M' not in the r^{th} column are all multiples of $x_{r_0}^{\alpha_0}$ (by formula 5.2), and therefore already in the ideal with generator $\frac{\partial G_r}{\partial x_r}$. The generators of $M(n, r)$ are multiples of the elements in the r^{th} column of M'. So

$$\triangle^r M(n,r) = (\{\frac{\partial G_i}{\partial x_r}\}_{i=1}^r, x_{r+1}, \ldots, x_{r+n})$$

which is $M^{(n)}(n-r, r)$ by construction. ∎

Corollary 5.1.3 $\mathrm{TB}(M(n,r)) = I(n,r)$.

Proof:

$$\mathrm{TB}(M(n,r)) = (r, \mathrm{TB}(M^{(n)}(n-r, r))) \text{ by construction}$$

$$\begin{aligned} &= (r, \mathrm{TB}(M(n-r,r))) \text{ by definition} \\ &= (r, I(n-r,r)) \text{ by induction} \\ &= I(n,r) \text{ by definition.} \quad \blacksquare \end{aligned}$$

5.2 Generalizing to Polynomial Ideals

A natural question to ask at this point is how to perturb the monomial generators of $M(n,r)$ to polynomial generators to get a new ideal without changing the invariant $I(n,r)$. The example given above, $M(4,3)$, is helpful in this regard.

Let $M^+(4,3)$ be the ideal generated by the monomials in $M(4,3)$ with one of the expressions changed to a binomial. In particular, let $\varepsilon \neq 0$ and take

$$M^+(4,3) = (x_1^4 x_3, x_2 x_3 + \varepsilon x_1^3 x_2, x_3^2, x_4).$$

Then

$$\delta M^+(4,3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \varepsilon 3 x_1^2 x_2 & x_3 + \varepsilon x_1^3 & x_2 & 0 \\ 0 & 0 & 2x_3 & 0 \\ 4x_1^3 x_3 & 0 & x_1^4 & 0 \end{pmatrix}$$

The 2×2-minors of this matrix adjoined to $\delta M^+(4,3)$ give the ideal

$$\triangle^3 M^+(4,3) = (x_1^3, x_2, x_3, x_4).$$

The Thom-Boardman symbol for $M^+(4,3)$ is $(3,1,1)$ instead of the desired $(3,1,1,1)$. In fact, let $G(4,3)$ be as general as possible, i.e. the weighted homogeneous ideal with generators

$g_4 = x_4 + \alpha_1 x_1 x_3 + \alpha_2 {x_1}^2 x_2 + \alpha_3 {x_2}^2 + \alpha_4 {x_1}^4$

$g_5 = \beta_1 x_1 x_4 + \beta_2 {x_1}^2 x_3 + \beta_3 x_2 x_3 + \beta_4 {x_1}^3 x_2 + \beta_5 x_1 {x_2}^2 + \beta_6 {x_1}^5$

$g_6 = \gamma_1 {x_1}^6 + \gamma_2 {x_1}^4 x_2 + \gamma_3 {x_1}^3 x_3 + \gamma_4 {x_1}^2 x_4 + \gamma_5 {x_1}^2 {x_2}^2 + \gamma_6 x_1 x_2 x_3 + \gamma_7 {x_3}^2$

$$+\gamma_8\, x_2{}^3 + \gamma_9\, x_2\, x_4$$

$$g_7 = \delta_1\, x_1{}^7 + \delta_2\, x_1{}^5\, x_2 + \delta_3\, x_1{}^3\, x_2{}^2 + \delta_4\, x_1\, x_2{}^3 + \delta_5\, x_1{}^4\, x_3 + \delta_6\, x_1{}^2\, x_2\, x_3 + \delta_7\, x_3\, x_2{}^2$$
$$+\delta_8\, x_1\, x_3{}^2 + \delta_9\, x_1{}^3\, x_4 + \delta_{10}\, x_1\, x_2\, x_4 + \delta_{11}\, x_3\, x_4$$

Computations of $\mathrm{TB}(G(4,3))$ showed the invariant to be the two-tuple $(3,1)$. However, the ideal $B_{4,3}$ has TB-invariant $(3,1,1,1) = I(4,3)$. This implies that there must be some conditions on the coefficients of each of the terms that will guarantee the desired equality.

5.3 Imposing the Right Nondegeneracy Conditions

What conditions can we put on a polynomial ideal $P(n,r)$, graded by weight, to guarantee that $\mathrm{TB}(P(n,r)) = I(n,r)$? Inherent in the algorithm for finding the Thom-Boardman invariant is the fact that for any ideal J, $\mathrm{TB}(J) = (i_1, \mathrm{TB}(\triangle^{i_1} J))$. It follows that any condition placed on the generators of the ideal $P(n,r)$ must also hold for the generators for the successive Jacobian extensions of $P(n,r)$.

Also, in the proofs of the inequalities in chapter 4, sections 3 and 4, the symbol for the invariant was the same as $I(n,r)$ in the i_j position only when the variables appeared in linear terms as soon as possible by weight, provided we are working in the reduced case at each stage of the argument. Although it is clear from the discussion in $\S 4.4$ that the arguments used only work for certain choices of n and r, there is some merit in requiring that the variables appear in linear terms exactly when necessary to obtain the TB-invariant desired. Obviously a condition of this type cannot be defined only by weight, since the weights of the critical minors are not all distinct.

There is also some concern about pure powers of low weight variables appearing in low weight generators, thereby appearing as linear terms too soon in the algorithm. Experiments show that Varley's conjecture implies subtle properties for $B_{n,r}$. For example, in $B_{4,3}$ even slightly changing the coefficient of the $-2a_3a_2$ term in c_3 by any $\varepsilon \neq 0$ causes

a_3^3 to appear one extension too soon, giving a TB-symbol of $(3,1,1)$ instead of the correct symbol $(3,1,1,1)$ for $B_{4,3}$. This indicates that, for general n and r, slight changes in the ideal $B_{n,r}$ can change the TB-symbol significantly. Therefore, not only do the weights and pure powers of the variables influence the TB-invariant, but also the coefficients of the nonlinear terms.

In order to make wise predictions about the non-degeneracy conditions on a polynomial ideal required to obtain the TB-invariant $I(n,r)$, we return to the ideals $B_{n,r}$ to look for clues.

Chapter 6

Using the Critical Columns

Although the first chapter provides a definition of *the* critical column, there are often several columns which turn out to be quite important in the computation of the Thom-Boardman invariant. We need a more relaxed definition for a column to be *critical* that includes the previous one.

6.1 The Critical Columns

The *first* critical column is the one previously mentioned, i.e. the one that corresponds to taking the partials $\partial/\partial a_{n-r}$ and is the first column from the left containing no ones and containing every variable as a linear term in order of ascending weight as the entries run down the column. For each stage of the computation of the Thom-Boardman invariant, there is a critical column, which may or may not be the same as the first one. Suppose the i_j entry of the symbol has been computed. Then the *critical column for that matrix* (with corank i_j) is the $(n - i_j + 1)^{\text{th}}$-column, which corresponds to taking the partials with respect to a_{n-i_j}. Notice that according to the extended version of Varley's conjecture, a_{n-i_j} is the lowest indexed (highest weight) variable that does not appear in a linear term in the extension for which this matrix is the Jacobian matrix.

In general terms, for an ideal of weighted homogeneous polynomials, the critical column of the Jacobian matrix of that ideal is the column which corresponds to taking the partials with respect to the highest weight variable not appearing in the ideal as a linear monomial in one of the polynomial generators. An alternative definition exists in taking the column associated to the the partials with respect to the next weight variable after the lowest weight variable that did appear as a linear monomial. In the polynomial case, these two are the same. However, in a more general case, these two definitions may disagree in the critical columns – and thus in $\triangle_{spec}$ (defined next) – in a significant way.

In the case of $B_{n,r}$, the first Jacobian matrix is of a particularly nice form:

$$\begin{pmatrix} 0 & & \cdots & & 0 & 1 & \frac{\partial c_{n-1}}{\partial a_{n-r}} & \frac{\partial c_{n-1}}{\partial a_{n-r+1}} & \cdots & \frac{\partial c_{n-1}}{\partial a_{n-1}} \\ 0 & & \cdots & 0 & 1 & \frac{\partial c_{n-2}}{\partial a_{n-r-1}} & \frac{\partial c_{n-2}}{\partial a_{n-r}} & \frac{\partial c_{n-2}}{\partial a_{n-r}} & \cdots & \frac{\partial c_{n-2}}{\partial a_{n-1}} \\ \vdots & & & & & & \vdots & & & \vdots \\ 1 & \frac{\partial c_r}{\partial a_1} & \frac{\partial c_r}{\partial a_2} & & \cdots & & \frac{\partial c_r}{\partial a_{n-r}} & & \cdots & \\ \frac{\partial c_{r-1}}{\partial a_0} & \frac{\partial c_{r-1}}{\partial a_1} & \frac{\partial c_{r-1}}{\partial a_2} & & \cdots & & \frac{\partial c_{r-1}}{\partial a_{n-r}} & & \cdots & \\ \vdots & & & & & & \vdots & & & \vdots \\ \frac{\partial c_1}{\partial a_0} & \frac{\partial c_1}{\partial a_1} & 0 & & \cdots & 0 & \frac{\partial c_1}{\partial a_{n-r}} & & \cdots & \frac{\partial c_1}{\partial a_{n-1}} \\ \frac{\partial c_0}{\partial a_0} & 0 & 0 & & \cdots & 0 & \frac{\partial c_0}{\partial a_{n-r}} & & \cdots & \frac{\partial c_0}{\partial a_{n-1}} \end{pmatrix}$$

Recall that the nonzero entries of any anti-diagonal of this matrix all have the same weight, the triangle of zeros whose base runs along the bottom of the matrix has the same number of zeros as the triangle at the top left-hand corner, and there are $n-r$ ones in the antidiagonal of ones.

In fact, the structure of this matrix is even nicer than mentioned above. To see this additional structure, consider the original coefficients in terms of the a_i's and b_i's:

$$\begin{cases} c_{n+r-1} = a_{n-1} + b_{r-1} \\ c_{n+r-2} = a_{n-2} + b_{r-2} + a_{n-1}b_{r-1} \\ \text{and} \\ c_{n+r-j} = a_{n-j} + b_{r-j} + \sum_{i+k=n+r-j} a_i b_k, \text{ for } j \leq r \\ c_{n+r-j} = a_{n-j} + \sum_{i+k=n+r-j} a_i b_k, \text{ for } r < j \leq n \\ c_{n+r-j} = \sum_{i+k=n+r-j} a_i b_k, \text{ for } j > n. \end{cases}$$

The equations for elimination are:

$$\begin{cases} b_{r-1} = -a_{n-1} \\ b_{r-2} = -a_{n-2} + a_{n-1}^2 \\ b_{r-3} = -a_{n-3} + 2a_{n-2}a_{n-1} - a_{n-1}^3 \\ \vdots \\ b_{r-j} = -a_{n-j} + \text{terms of higher order} \pm a_{n-1}^j \\ \vdots \\ b_0 = -a_{n-r} + \text{terms of higher order} \pm a_{n-1}^r \end{cases}$$

Using this substitution, the expressions for the generators become:

$$\begin{cases} c_{n-1} = a_{n-r-1} + a_{n-r}(-a_{n-1}) + a_{n-r+1}(-a_{n-2} + a_{n-1}^2) + \cdots + a_{n-1}(-a_r + \cdots) \\ c_{n-2} = a_{n-r-2} + a_{n-r-1}(-a_{n-1}) + a_{n-r}(-a_{n-2} + a_{n-1}^2) + \cdots + a_{n-2}(-a_r + \cdots) \\ \vdots \\ c_{n-j} = a_{n-r-j} + a_{n-r-j+1}(-a_{n-1}) + \cdots + a_{n-j}(-a_r + \cdots) \\ \vdots \\ c_{r-1} = a_0(-a_{n-1}) + \cdots + a_{n-r-1}(-a_r + \cdots) \\ c_{r-2} = a_0(-a_{n-2} + a_{n-1}^2) + \cdots + a_{n-r-2}(-a_r + \cdots) \\ \vdots \\ c_{r-i} = a_0(-a_{n-i} + \cdots) + \cdots + a_{n-r-i}(-a_r + \cdots) \\ \vdots \end{cases}$$

We can see that the quadratic terms, the ones which give us linear terms in the derivatives, are apparent. For example, notice the anti-diagonals of the matrix for $\mu_{5,2}$ in §3.4. Not only are the weights of the anti-diagonal entries equal, the same linear term appears, up to possibly different coefficients. This is a result of the shifting of the indices by one in the quadratic terms in the successive c's. For instance, $c_{n-1} = -a_{n-r+1}a_{n-2} + \cdots$, while $c_{n-2} = -a_{n-r}a_{n-2} + \cdots$, and $c_{n-3} = -a_{n-r-1}a_{n-2} + \cdots$. Because of the weighted homogeneity of the expressions, if there are two quadratic terms involving a particular

a_i, those terms must have the same factors. Along with squares this accounts for the possible difference in the coefficients of linear terms of the derivatives; there may be two copies of $a_i a_j$ in c_k depending on r and $n - r$. It also gives us a nice structure for the Jacobian matrix that can be exploited.

6.2 $\triangle_{spec}$

Let $-\tilde{a}_i$ denote weight $n - i$ expressions which contain a_i in a linear term, and consider $\delta B_{n,r}$ as it appears in the following block form:

$$\left(\begin{array}{ccccc|cccc}
0 & 0 & \cdots & 0 & 1 & -\tilde{a}_{n-1} & -\tilde{a}_{n-2} & \cdots & -\tilde{a}_{r-1} \\
0 & \cdots & 0 & 1 & -\tilde{a}_{n-1} & -\tilde{a}_{n-2} & -\tilde{a}_{n-3} & \cdots & -\tilde{a}_{r-2} \\
 & & \ddots & & & \ddots & & & \\
1 & -\tilde{a}_{n-1} & \cdots & & -\tilde{a}_{r+1} & -\tilde{a}_{r} & -\tilde{a}_{r-1} & \cdots & -\tilde{a}_{1} \\
\hline
-\tilde{a}_{n-1} & -\tilde{a}_{n-2} & \cdots & -\tilde{a}_{r+1} & -\tilde{a}_{r} & -\tilde{a}_{r-1} & -\tilde{a}_{r-2} & \cdots & -\tilde{a}_{0} \\
\vdots & & & & & & & & \\
\vdots & & & & & & & & \\
-\tilde{a}_{n-r} & 0 & \cdots & & 0 & -\tilde{a}_{0} & & &
\end{array}\right)$$

Notice the antidiagonal containing a_0 as its only linear terms. Any entries to the right and below that antidiagonal have no linear terms. This is also inherent in the structure of the c_i's. The higher weight c_i's and the higher weight derivatives cannot include the smaller weight variables in a quadratic term. For instance, c_0 has weight $r + n$. Any term of c_0 with as a factor a_{n-1} must be at least a cubic since the rest of the term must have weight $r + n - 1$ and there are no variables of that weight.

Recall that the first critical column, which appears first on the right of the vertical bar, has all of the variables appearing in linear terms. In order to get a determinant with linear terms, the ones in the $(n - r)$ antidiagonal must be utilized along with the entries above and including the "a_0" antidiagonal. As one can see from the structure of $\delta B_{n,r}$, the only variables that can be obtained in linear terms in these determinants are $a_0, \ldots, a_{r-1}$. In particular, these linear terms can be gotten from the r determinants using only the critical column and the determinant one $(n-r) \times (n-r)$ submatrix. Denote this submatrix by U_1 for ease of notation. Denote by d_i the determinant obtained in this way such that $\partial d_i / \partial a_i =$ a constant.

The ideal obtained by adjoining only $d_0, \ldots, d_{r-1}$ to $B_{n,r}$ gives a special extension, denoted $\triangle^r_{spec} B_{n,r}$, such that $B_{n,r} \subset \triangle^r_{spec} B_{n,r} \subset \triangle^r B_{n,r}$. In general, the symbol $\triangle_{spec}$ indicates that only the determinants with linear terms coming from the critical column at that stage of the process have been adjoined to the ideal in question to get that extension.

We would like to know that we can repeat this process without cancellation of linear terms occurring in the minors or in minors corresponding to successive iterations of the algorithm.

Consider any matrix constructed by bordering U_1 by one row and one column as follows:

$$\left(\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & 1 & -\tilde{a}_{n-1} & \beta_{n-2} \\ & & \ddots & & & \vdots \\ 1 & -\tilde{a}_{n-1} & \cdots & & -\tilde{a}_{r+1} & \beta_r \\ \hline \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_{r+1} & \alpha_r & \gamma \end{array}\right)$$

The determinant of this matrix is

$$\pm(\gamma - \alpha_{n-1}\beta_r - \alpha_{n-2}\beta_{r+1} - \cdots - \alpha_r\beta_{n-1} + \text{higher order terms}).$$

In the polynomial multiplication case, γ has a linear term and each α_i and β_i has a linear term. Therefore, the only linear term which can appear in the determinant is the one from γ. We are interested in the quadratic terms as well. The only possible sources for quadratic terms are those already in γ, and those from the multiplication of linear terms from the α's and β's. The quadratic term from γ is not of interest in this argument, since the factors are of index less than $r.$ Since all of the linear terms from the first Jacobian matrix are negative, the linear term from γ is negative and the quadratic terms from the $\alpha\beta$ products are positive. In other words, the linear term and the relevant quadratic terms in the determinant have the same sign. This is all we need. In the matrix δB, the ones that appear in the antidiagonal are all positive while the linear terms in the matrix all appear with a negative coefficient. Then there is no cancellation of the important linear terms in the minors of the second Jacobian matrix. In fact, addition of the linear terms results in increasing coefficients at each step. Thus the we need not check the later phases of the computation.

6.3 TB_{spec}

One of the properities of $\triangle^r_{spec} B_{n,r}$ is that corank of its Jacobian matrix is $\min(n - r, r)$, which we know is i_2 for $\mathrm{TB}(B_{n,r})$. To see this, note that the rank of $\delta B_{n,r}$ is already known to be $n-r$, and the linear parts of the generators are $a_0, \ldots,\ a_{n-r-1}$. If $n-r \geq r$, then $\{a_0, \ldots,\ a_{r-1}\} \subseteq \{a_0, \ldots,\ a_{n-r-1}\}$, and adjoining only $d_0, \ldots, d_{r-1}$ means that $\triangle_{spec} B_{n,r}$ has no new linear terms. Hence $\mathrm{corank}(\delta \triangle_{spec} B_{n,r}) = r = \min(n-r, r)$. If $n-r \leq r$, adjoining $d_0, \ldots,\ d_{r-1}$ increases the number of linear terms to r, increasing the rank of the Jacobian matrix to r, making $\mathrm{corank}(\delta \triangle_{spec} B_{n,r}) = n-r = \min(n-r, r)$.

Define $\mathrm{TB}_{spec}(B_{n,r})$ to be the symbol computed by taking $\triangle_{spec}$ as our extension at each stage of the computation. It is possible that the coranks of the Jacobian matrices of $B_{n,r}$ with all the determinants adjoined are smaller than the coranks of the Jacobian matrices when only the special determinants are adjoined. Since the special determinants are contained in the set of all determinants, the coranks cannot be larger. Thus

$$\mathrm{TB}(B_{n,r}) \preceq \mathrm{TB}_{spec}(B_{n,r}).$$

If this were an equality, the proof of Varley's Conjecture (and the extended version) would be close at hand. All we would need to prove is that $\mathrm{TB}_{spec}(B_{n,r}) = I(n,r)$. Let's go back to an example for now:

Example 6.3.1 $\mu_{5,2}$

Using the Euclidean algorithm on 5 and 2:

$$\begin{aligned} \underline{5} &= \underline{2} \times 2 + 1 \\ \underline{2} &= \underline{1} \times 2 + 0 \end{aligned}$$

$I(5,2) = (2,2,1,1)$.

The generators for $B_{5,2}$ are:

$$\begin{aligned} c_4 &= -2a_3a_4 + a_4{}^3 + a_2 \\ c_3 &= -a_2a_4 - a_3{}^2 + a_3a_4{}^2 + a_1 \\ c_2 &= a_0 - a_1a_4 - a_2a_3 + a_2a_4{}^2 \\ c_0 &= -a_0a_3 + a_0a_4{}^2 \end{aligned}$$

The first Jacobian matrix:

$$\delta B_{5,2} = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & -2a_4 & -2a_3 + 3a_4{}^2 \\ 0 & 1 & -a_4 & -2a_3 + a_4{}^2 & -a_2 + 2a_3a_4 \\ 1 & -a_4 & -a_3 + a_4{}^2 & -a_2 & -a_1 + 2a_2a_4 \\ \hline -a_4 & -a_3 + a_4{}^2 & 0 & -a_1 & 2a_1a_4 - a_0 \\ -a_3 + a_4{}^2 & 0 & 0 & -a_0 & 2a_0a_4 \end{array}\right)$$

$$\mathrm{TB}(B_{5,2}) = (2, \ldots).$$

Using only the first critical column,

$$\begin{aligned} d_0 &= a_0 + a_2a_3 + 4a_3{}^2a_4 - 5a_3a_4{}^3 - a_2a_4{}^2 + a_4{}^5, \\ d_1 &= a_1 + 2a_3{}^2 + 3a_3a_4{}^2 - 2a_4{}^4 + a_2a_4. \end{aligned}$$

$\delta \triangle_{special} B_{5,2} =$

$$\left(\begin{array}{ccc|cc}
0 & 1 & a_4 & 4a_3 + 3{a_4}^2 & 6a_3 a_4 - 8{a_4}^3 + a_2 \\
1 & 0 & a_3 - {a_4}^2 & a_2 + 8a_3 a_4 - 5{a_4}^3 & 4{a_3}^2 - 15a_3 {a_4}^2 - 2a_2 a_4 + 5{a_4}^4 \\
0 & 0 & 1 & -2a_4 & -2a_3 + 3{a_4}^2 \\
\hline
0 & 1 & -a_4 & -2a_3 + {a_4}^2 & -a_2 + 2a_3 a_4 \\
1 & -a_4 & -a_3 + {a_4}^2 & -a_2 & -a_1 + 2a_2 a_4 \\
-a_4 & -a_3 + {a_4}^2 & 0 & -a_1 & 2a_1 a_4 - a_0 \\
-a_3 + {a_4}^2 & 0 & 0 & -a_0 & 2a_0 a_4
\end{array}\right)$$

The rank of this matrix is the same as the rank of $\delta B_{5,2}$. $\mathrm{TB}_{spec}(B_{5,2}) = (2, 2, \ldots)$. The critical column, therefore, stays the same. However, it is possible to get determinants of $\delta \triangle^2 B_{5,2}$ with linear terms $-2a_3$ and $-a_2$ by using the original antidiagonal of ones and either the first or second row of this new matrix and the critical column. In particular, one can pick up any linear term that appears in the lower left block. In fact, one may pick up any linear term that appears on the right hand side of the vertical line except the ones in row three, but they do not help since they are the same as the ones below the horizontal line. The special determinants are:

$$\begin{aligned}
d_3 &= 6a_3 + 6{a_4}^2 \\
d_2 &= 2a_2 + 8a_3 a_4 - 14{a_4}^3
\end{aligned}$$

Adjoining those to $\triangle^2_{spec} B_{5,2}$ gives $\triangle^2_{spec} \triangle^2_{spec} B_{5,2}$ with Jacobian matrix:

$$\left(\begin{array}{cccc|c}
0 & 0 & 0 & 6 & 12a_4 \\
0 & 0 & 2 & 8a_4 & 8a_3 - 42{a_4}^2 \\
0 & 1 & a_4 & 4a_3 + 3{a_4}^2 & 6a_3 a_4 - 8{a_4}^3 + a_2 \\
1 & 0 & a_3 - {a_4}^2 & a_2 + 8a_3 a_4 - 5{a_4}^3 & 4{a_3}^2 - 15a_3 {a_4}^2 - 2a_2 a_4 + 5{a_4}^4 \\
\hline
0 & 0 & 1 & -2a_4 & -2a_3 + 3{a_4}^2 \\
0 & 1 & -a_4 & -2a_3 + {a_4}^2 & -a_2 + 2a_3 a_4 \\
1 & -a_4 & -a_3 + {a_4}^2 & -a_2 & -a_1 + 2a_2 a_4 \\
-a_4 & -a_3 + {a_4}^2 & 0 & -a_1 & 2a_1 a_4 - a_0 \\
-a_3 + {a_4}^2 & 0 & 0 & -a_0 & 2a_0 a_4
\end{array}\right)$$

Notice that $\mathrm{rank}(\delta \triangle^2_{spec} \triangle^2_{spec} B_{5,2}) = 4$. $\mathrm{TB}_{spec}(B_{5,2}) = (2, 2, 1, \ldots)$. The new critical column is the last one. The variable a_4 does not appear in a linear term in

$$\triangle^2_{spec}\,\triangle^2_{spec}\,B_{5,2}.$$

And it does not appear as a linear term in the critical column of this latest matrix except in a row we cannot access. The algorithm requires that we take all of the 5×5 minors of this matrix. Since the matrix is 5×5, this coincides with taking $\triangle_{spec}$. We can take the determinant of the top 5×5 submatrix and get

$$\det_3 = -72a_3 + 432a_4^2$$

Adding at least this one gives us all we need. Since none of the others contain any new variables in linear terms, adding them in does not change the rank of the matrix, nor can it help us get any more linear linear terms. There is only one variable that has not appeared so far, a_4. As soon as we find one minor that will help us, we need not search any further. We get the following matrix for $\delta\,\triangle^1_{spec}\,\triangle^2_{spec}\,\triangle^2_{spec}\,B_{5,2}$:

$$\begin{pmatrix}
0 & 0 & 0 & -72 & 864\,a_4 \\
0 & 0 & 0 & 6 & 12\,a_4 \\
0 & 0 & 2 & 8\,a_4 & 8\,a_3 - 42\,a_4{}^2 \\
0 & 1 & a_4 & 4\,a_3 + 3\,a_4{}^2 & 6\,a_3\,a_4 - 8\,a_4{}^3 + a_2 \\
1 & 0 & a_3 - a_4{}^2 & a_2 + 8\,a_3\,a_4 - 5\,a_4{}^3 & 4\,a_3{}^2 - 15\,a_3\,a_4{}^2 - 2\,a_2\,a_4 + 5\,a_4{}^4 \\
0 & 0 & 1 & -2\,a_4 & -2\,a_3 + 3\,a_4{}^2 \\
0 & 1 & -a_4 & -2\,a_3 + a_4{}^2 & -a_2 + 2\,a_3\,a_4 \\
1 & -a_4 & -a_3 + a_4{}^2 & -a_2 & -a_1 + 2\,a_2\,a_4 \\
-a_4 & -a_3 + a_4{}^2 & 0 & -a_1 & 2\,a_1\,a_4 - a_0 \\
-a_3 + a_4{}^2 & 0 & 0 & -a_0 & 2\,a_0\,a_4
\end{pmatrix}$$

$\mathrm{TB}_{spec}(B_{5,2}) = (2,2,1,1)$ since the top 5×5 submatrix of this matrix has determinant

$$d_4 = \alpha a_4 + \text{higher order terms}$$

where α is some coefficient in $\mathbb{C}$. In other words, all of the variables appear in linear terms in the next extension. Notice that $\mathrm{TB}_{spec}(B_{5,2}) = I(5,2)$, and that this is a special case of $\mu_{2k+1,2}$. In fact, the proof of equality for Varley's Conjecture in the case of $\mu_{2k+1,2}$ is based on this same idea. Instead of using the reduced case, we simply use TB_{spec}.

Recall that $\mathrm{TB}(B_{2k+1,2}) \succeq I(2k+1,2)$, and that $\mathrm{TB}_{spec}(B_{n,r}) \succeq \mathrm{TB}(B_{n,r})$ for any choice of n, r with $n \geq r$. So

$$\mathrm{TB}_{spec}(B_{2k+1,2}) \succeq \mathrm{TB}(B_{2k+1,2}) \succeq I(2k+1,2)$$

Claim 6.3.2 $\mathrm{TB}_{spec}(B_{2k+1,2}) = I(2k+1,2)$

Obviously, proving this claim proves Varley's Conjecture for this class of cases, and the extended version is a natural product of the proof. Similarly, proving the following claim will complete another class of cases already discussed. We start with proving the following claim:

Claim 6.3.3 $\mathrm{TB}_{spec}(B_{kr,r}) = I(kr,r)$

Proof: The proof of this lies in the fact that, at every stage of the procedure, the critical column is the first critical column. The first $kr-r$ variables $a_0, \ldots, a_{kr-r-1}$ appear in linear terms in the generators for $B_{kr,r}$. Consider the matrix $\delta B_{kr,r}$ associated to the eliminated but not reduced case. There are $kr-r$ ones on the antidiagonal, and the critical column is the $kr-r+1$ column.

$$\delta B_{kr,r} = \left(\begin{array}{ccccc|cccc}
0 & 0 & \cdots & 0 & 1 & -\tilde{a}_{kr-1} & -\tilde{a}_{kr-2} & \cdots & -\tilde{a}_{r-1} \\
0 & \cdots & 0 & 1 & -\tilde{a}_{kr-1} & -\tilde{a}_{kr-2} & -\tilde{a}_{kr-3} & \cdots & -\tilde{a}_{r-2} \\
 & & \ddots & & & \ddots & & & \\
1 & -\tilde{a}_{kr-1} & \cdots & & -\tilde{a}_{r+1} & -\tilde{a}_{r} & -\tilde{a}_{r-1} & \cdots & -\tilde{a}_{1} \\
\hline
-\tilde{a}_{kr-1} & -\tilde{a}_{kr-2} & \cdots & -\tilde{a}_{r+1} & -\tilde{a}_{r} & -\tilde{a}_{r-1} & -\tilde{a}_{r-2} & \cdots & -\tilde{a}_{0} \\
\vdots & & & & & & & & \\
\vdots & & & & & & & & \\
-\tilde{a}_{kr-r} & 0 & \cdots & & 0 & -\tilde{a}_{0} & & &
\end{array}\right)$$

which has corank r. The special determinants of this matrix with linear terms are $d_0, \ldots, d_{r-1}$ as one can see from above. Then $\triangle_{spec}B_{kr,r}$ has no linear terms other than the ones already in $B_{kr,r}$. These determinants do help, however, in that they allow us to access terms in rows that have been previously inaccessible since those rows were used for a one in the antidiagonal of ones. In other words, the next Jacobian matrix is as follows:

$$\left(\begin{array}{ccccc|cccc}
0 & \cdots & 1 & \ldots & * & * & \ldots & & * \\
 & & \ddots & * & * & \ddots & & & \\
1 & * & \cdots & * & * & & & \cdots & \\
0 & 0 & \cdots & 0 & 1 & -\tilde{a}_{kr-1} & -\tilde{a}_{kr-2} & \cdots & -\tilde{a}_{r-1} \\
0 & \cdots & 0 & 1 & -\tilde{a}_{kr-1} & -\tilde{a}_{kr-2} & -\tilde{a}_{kr-3} & \cdots & -\tilde{a}_{r-2} \\
 & & \ddots & & & \ddots & & & \\
1 & -\tilde{a}_{kr-1} & \cdots & & -\tilde{a}_{r+1} & -\tilde{a}_{r} & -\tilde{a}_{r-1} & \cdots & -\tilde{a}_{1} \\
\hline
-\tilde{a}_{kr-1} & -\tilde{a}_{kr-2} & \cdots & -\tilde{a}_{r+1} & -\tilde{a}_{r} & -\tilde{a}_{r-1} & -\tilde{a}_{r-2} & \cdots & -\tilde{a}_{0} \\
\vdots & & & & & & & & \\
\vdots & & & & & & & & \\
-\tilde{a}_{kr-r} & 0 & \cdots & & 0 & -\tilde{a}_{0} & & &
\end{array}\right)$$

Where the "$*$" indicates a possibly non-zero entry which plays no role, and the top 1 appears in the r^{th} column corresponding to taking the derivative with respect to a_{r-1}. The corank of this matrix is r, so $\mathrm{TB}_{spec}(B_{kr,r}) = (r, r, \ldots)$. The determinants of this matrix include $d_r, \ldots, d_{2r-1}$, but not d_i for $i \geq 2r$. Each stage of the process simply moves us up the critical column r rows at a time, giving us the next set of r special determinants in order by index, also those variables in linear terms, and another r in $\mathrm{TB}_{spec}(B_{kr,r})$. Finally, at the last stage, the k^{th} Jacobian matrix allows us to access the $a_{kr-r}, \ldots, a_{kr-1}$

terms that were not included in $B_{kr,r}$. The next Jacobian matrix will have full rank, hence

$$\mathrm{TB}_{spec}(B_{kr,r}) = (\underbrace{r, r, \ldots, r}_{r}) = I(n, r). \quad \blacksquare$$

Now for the proof of Claim 6.3.2.:

Proof: This proof is almost complete by applying the previous methods. The first Jacobian matrix is of size $(2k+1) \times (2k+1)$ with corank 2. By working up the critical column by blocks of 2 for $k-1$ stages, we get $2k-2$ of the $2k-1$ linear terms already in $B_{2k+1,2}$ before there is any change in the rank/corank of the matrix. So, by this point, $\mathrm{TB}_{spec}(B_{2k+1,2}) = (\underbrace{2, 2, \ldots, 2}_{k}, \ldots)$ since the corank of

$$\delta \underbrace{\triangle^2_{spec} \cdots \triangle^2_{spec}}_{k-1} B_{2k+1,2}$$

is 2. However, the variables a_{2k-1}, and a_{2k-2} both appear in linear terms in determinants of this matrix. $B_{2k+1,2}$ already had a_{2k-2} as a linear term in one of its generators. But a_{2k-1} is new in that respect. The next Jacobian has corank 1, which means the critical column is now the last column. What is important to see at this point is the top of this matrix. The top row will have zero entries until the last two columns where it will have a constant and something of weight one, in that order. The weight 1 object is either a_{2k} with some coefficient, or it could be a zero entry. Either way, we cannot access that object with any antidiagonal of $2k$ ones and that column, since the top row provides the last of the ones in that antidiagonal, and the only one in that column. However, recall that the $(1, 2k+1)$ entry of the matrix is $-a_{2k-1} + a_{2k}^2$. And that entry is part of the determinant using the original $(2k+1) \times (2k+1)$ matrix along with the top row of the matrix at this stage. The next Jacobian has corank 1, but also has a nonzero determinant of weight 1, namely the one which starts

$$d = \partial / \partial a_{2k}(-a_{2k-1} + a_{2k}^2) + \cdots = 2a_{2k} + \ldots.$$

We have shown

$$\mathrm{TB}_{spec}(B_{2k+1,2}) = (\underbrace{2,\ldots,2}_{k},1,1). \quad \blacksquare$$

Notice that all the classes of cases from chapter 3 are now proved to have the invariant $I(n,r)$ as claimed. Also notice that the results were derived in part from proving $\mathrm{TB}_{spec}(B_{n,r}) = I(n,r)$ in those particular cases. It would be interesting to know how often this identity is true for other cases. Another question that arises is about the nature of $\triangle_{spec}B_{n,r}$ and how that ideal compares to $\triangle B_{n,r}$, or even to $B_{n-r,r}$. Let's see what we can say about TB_{spec}.

6.4 "Special" Results

Consider the Euclidean algorithm applied to n and r. The first computation in the algorithm is

$$n = q_1 r + r_1, \;\; 0 \leq r_1 < r.$$

If $n-r<r$, then $q_1=1$ and $\min(n-r,r) = n-r = r_1$. If $n-r \geq r$, then $q_1>1$ and $\min(n-r,r)=r$. Why mention something so obvious? It gives us a set up for a discussion on the symbol $I(n,r)$.

Take $n \geq r$ and set $j_1 = r$, $J_1 = n$. Let $j_{k+1} = \min(j_k, J_k - j_k)$ and $J_{k+1} = \max(j_k, J_k - j_k)$. Then by the aforementioned obvious fact and its application throughout the Euclidean algorithm, $I(n,r) = (j_1,\ldots,j_s)$ where $j_s = r_f$, the last nonzero remainder, and $r_f = gcd(n,r)$ which is repeated q_{f+1} times in the tuple.

Proposition 6.4.1 *Fix n, r, with $n \geq r$. Then* $\mathrm{TB}_{spec}(B_{n,r}) = I(n,r)$.

Proof: We have been thinking each of the Jacobian matrices at each phase as being the previous Jacobian with the rows corresponding to the new special determinants stacked on top. Each "stack" is also weighted homogeneous by antidiagonals since they are derivatives of weighted homogeneous expressions.

Let $I(n,r) = (j_1,\ldots,j_s)$, $\mathrm{TB}_{spec}(B_{n,r}) = (i_1,\ldots,i_{s'})$. We already know $i_1 = j_1,\ i_2 = j_2$ from earlier discussions. In fact, we know that as long as the critical column does not change from the first one, $i_m = j_m$ for $m = 1,\ldots,k$ for some $k \leq s$. So, without loss of generality, we can assume $n-r < r$. Then $B_{n,r}$ has the $n-r$ linear terms with $a_0,\ldots,a_{n-r-1}$. $\triangle^r_{spec}B_{n,r}$ is obtained by adjoining determinants $d_0,\ldots,d_{r-1}$, thus has the r linear terms with $a_0,\ldots,a_{r-1}$. The corank of $\delta \triangle^r_{spec} B_{n,r}$ is, as we know, $\min(n-r,r) = n-r$.

Now the critical column has been shifted. It is no longer the $n-r+1$ column, but rather the $r+1$ column, since there are ones sitting in each column before that one. We need to know what sits in that column. The column has shifted by $r-(n-r) = J_2 - j_2$, and all of the linear terms appearing in this column in the first Jacobian appear in determinants of $\delta \triangle^r_{spec} B_{n,r}$. Because of the anitdiagonal weighting structure, and the fact that linear terms appear when they can by weight in $\delta B_{n,r}$, the linear terms in this column run through $a_0,\ldots,a_k$ where $k = 2n-2r-1$. We already have $a_0,\ldots,a_{r-1}$ as linear terms.

If $k \leq r-1$ there are no new linear terms and we stay in that critical column and $i_3 = i_2 = j_2 = j_3$. Otherwise, there are new linear terms up to index k, and the corank of the next Jacobian is $n-k-1$. But $k \leq r-1$ means that $n-r \leq r-(n-r)$, and $j_2 \leq J_2 - j_2$, so $i_3 = j_2 = j_3$.

If $k > r-1$ then $n-r > r-(n-r)$, which means that $j_2 > J_2 - j_2$, $j_3 = J_2 - j_2 = r-(n-r)$. But the corank of the next Jacobian, as stated above, is $n-k-1 = n-(2n-2r-1)-1 = 2r-n = r-(n-r) = j_3$. Either way, $i_3 = j_3$.

We repeat this argument until we get to the last critical column. At each stage the corank of the next Jacobian must stay the same or go down by the difference between the number of distinct linear terms contained in the two extensions at that stage. When you reach the last critical column the last argument is the same as the previous ones:

you work your way up the last critical column by increments of $d = n - \operatorname{rank} S$ where S has corank $gcd(n,r)$ until you get to the top entry, which is a_{n-1} by weight. Hence $\mathrm{TB}_{spec}(B_{n,r}) = I(n,r)$.

If $gcd(n,r) = 1$, how do we know that a_{n-1} sits there instead of 0? The lowest weight determinants at each stage are obtained by using U_1, any ones from previous stages, and the lowest weight entry in the critical column of the previous matrix, which obviously comes from the previous lowest weight determinant. The first lowest weight determinant, d_{r-1} has factors of a_{n-1} in every term except the linear term and it has every variable of weight at least r in at least one expression. Since we are never considering the derivative with respect to a_{n-1} until the last column, the factor stays in the successive derivatives, and in the right entries of the critical columns until the end. Also, d_{r-1} contains the term a_{n-1}^{n-r+1}. ∎

A side note here is that Example 6.3.1 gives a nice example of the last matrix, the stacking, and the presence of a_{n-1} in the last column. Since $\mathrm{TB}_{spec}(B_{n,r}) \succeq \mathrm{TB}(B_{n,r})$, we have also just proved:

Theorem 6.4.2 $\mathrm{TB}(B_{n,r}) \preceq I(n,r)$

This is a special result (pardon the pun) for a very special ideal. As illustrated previouly, small changes in an ideal can result in changes in the Thom-Boardman invariant. In fact, they can make a difference in TB_{spec} . Recall the example in §5.2:

Let $M^+(4,3)$ be the following ideal generated by the monomials in $M(4,3)$ with one of the expressions changed to a binomial:

$$M^+(4,3) = (\ x_1^4 x_3,\ x_2 x_3 + x_1^3 x_2,\ x_3^2,\ x_4\ ).$$

Then

$$\delta M^+(4,3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 3x_1^2x_2 & x_3 + x_1^3 & x_2 & 0 \\ 0 & 0 & 2x_3 & 0 \\ 4x_1^3x_3 & 0 & x_1^4 & 0 \end{pmatrix}$$

The 2×2-minors of this matrix adjoined to $\delta M^+(4,3)$ give the ideal $\triangle^3 M^+(4,3) = (x_1^3, x_2, x_3, x_4)$ which has the Thom-Boardman symbol $(3,1,1)$ instead of the desired $(3,1,1,1)$. The highest weight variable not included in $M^+(4,3)$ as a linear term is x_3. So the critical column is the third one. And since the minors are two by two, $\triangle_{spec} M^+(4,3) = (x_1^4, x_2, x_3, x_4)$. Then $\mathrm{TB}_{spec}(M^+(4,3)) = (3,1,1,1) = I(n,r)$. We can also cook up another example where $\mathrm{TB}(J) = I(n,r)$ but $\mathrm{TB}_{spec}(J) \succ I(n,r)$.

Example 6.4.3 $\mathrm{TB}_{spec}(G(3,2)) \succ I(3,2)$

Let $G(3,2)$ be the following weighted homogeneous ideal:

$$(\frac{1}{2}x_2^2 - x_1^2x_2,\ x_1^3x_2 - \frac{1}{2}x_1x_2^2,\ x_3)$$

The first Jacobian matrix:

$$\delta G(3,2) = \begin{pmatrix} -2x_1x_2 & x_2 - x_1^2 & 0 \\ 3x_1^2x_2 - \frac{1}{2}x_2^2 & x_1^3 - x_1x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\triangle^2 G(3,2) = (x_1^3,\ x_1x_2,\ x_2 - x_1^2,\ x_2^2, x_3)$$

$$\delta \triangle^2 G(3,2) = \begin{pmatrix} 3x_1^2 & 0 & 0 \\ x_2 & x_1 & 0 \\ -2x_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\triangle^1 \triangle^2 G(3,2) = (x_1^2,\ x_2, x_3)$$

$$\triangle^1 \triangle^1 \triangle^2 G(3,2) = (x_1,\ x_2, x_3)$$

So, $\mathrm{TB}(G(3,2)) = (2,1,1) = I(3,2)$. However, the first critical column of $\delta G(3,2)$ is the second one. Then

$$\triangle^2_{spec} G(3,2) = ({x_1}^4,\ x_2 - {x_1}^2,\ x_3)$$

$$\delta \triangle^2_{spec} G(3,2) = \begin{pmatrix} 4\,{x_1}^3 & 0 & 0 \\ -2\,x_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\triangle^1_{spec} \triangle^2_{spec} G(3,2) = (x_1^3,\ x_2 - x_1^2,\ x_3)$$

$$\delta \triangle^1_{spec} \triangle^2_{spec} G(3,2) = \begin{pmatrix} 3\,{x_1}^2 & 0 & 0 \\ -2\,x_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\triangle^1_{spec} \triangle^1_{spec} \triangle^2_{spec} G(3,2) = ({x_1}^2,\ x_2,\ x_3)$$

So, $\mathrm{TB}_{spec}(G(3,2)) = (2,1,1,1) \succ I(3,2)$. This takes us right back to the question about the exact conditions to impose on the generators of a polynomial ideal to guarantee that the ideal has Thom-Boardman symbol $I(n,r)$. This is an open question that is a possible area for later investigation. However, it is beyond the scope of this paper.

6.5 Present State of Affairs

Present work includes proving certain identities that seem to hold. For instance, in the cases that we're able to confirm, it seems that $\triangle_{spec} B$ is the same as $\triangle B$. This supports the evidence above for even more cases than have been checked. If this equivalence holds for all choices of n and $r,\ n \geq r,$ then so must Varley's conjecture and its extension.

Even more compelling is that there seem to be identities in the first Jacobian which indicate that all of the determinants of that matrix are dependent on the special determinants in a consistent way. In particular, the identity

$$\frac{\partial c_{n-1-j}}{\partial a_i} = \frac{\partial c_{n-2-j}}{\partial a_{i-1}} - a_{n-1-j} \frac{\partial c_{n-1}}{\partial a_{i-1}}$$

seems to hold for all i, j in every n, r, with $n \geq r$. This relationship would indicate a simple dependency of each column entry of δB on the entries in the critical column.

There is also evidence for a close relationship between $\triangle_{spec} B_{n,r}$ and $B_{n-r,r}$, especially when working in the reduced case. Certainly they share the first and second entries in their Thom-Boardman symbols (and conjecturally all the entries). In the non-reduced case, they seem to share a lot of structure while specific relationships are unknown as of now.

BIBLIOGRAPHY

[1] M. Adams, C. McCrory, T. Shifrin, R. Varley, Invariants of Gauss Maps of Theta Divisors, *Proc. Sympos. Pure Math.*, vol 54, pp 1-8, Amer. Math. Soc., Providence, RI, 1993.

[2] G. Andrews, *Theory of Partitions*, Addison-Wesley, Reading, Mass, 1976.

[3] V. I. Arnol′d, A. N. Varchenko, S. M. Guseĭn-Zade, *Singularities of Differentiable Maps, I.*, Birkhäuser, Boston, 1985.

[4] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, *Geometry of Algebraic Curves*, vol. I, Grundlehren der Math. Wissenschaft, Springer-Verlag, New York, 1985.

[5] E. Ballico, On tangent hyperplanes to complex projective curves, *Complex Analysis and Geometry, Lecture Notes in Pure and Applied Math.*, vol 173, Marcel Dekker, 1996.

[6] T. Banchoff, T. Gaffney, C. McCrory, Cusps of Gauss Mappings, *Research Notes in Math.*, vol 55, Pitman Advanced Pub. Program, Boston, 1982.

[7] J. M. Boardman, Singularities of Differentiable Maps, *Institut des Hautes Études Scientifiques*, Publications Mathematiques, vol 33, pp 21-57, 1967.

[8] C.G. Gibson, K. Wirthmüller, A.A. du Plessis, E.J.N. Looijenga, *Topological Stability of Smooth Mappings*, Lecture Notes in Mathematics 552, Springer-Verlag, Berlin, 1976.

[9] M. Golubitsky, V. Guillemin, *Stable Mappings and Their Singularities*, Springer-Verlag, New York, 1980.

[10] P.A. Griffiths, *Introduction to Algebraic Curves*, Translations of Mathematical Monographs 76, American Mathematical Society, Providence, 1989.

[11] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley Interscience, New York, 1978.

[12] J. Harris, *Algebraic Geometry*, Graduate Texts in Mathematics 133, Springer-Verlag, Berlin Heidelberg New York, 1992.

[13] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, Heidelberg, 1977.

[14] A. Kas, M. Schlessinger, On the versal deformation of a complex space with an isolated singularity, *Math. Ann. 196*, pp 23-29, 1972.

[15] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Springer-Verlag, N.Y., 1986.

[16] H. Matsumura, *Commutative Algebra, 2. ed.*, Mathematics Lecture Notes 56, Benjamin/Cummings, Reading, Mass. 1980.

[17] Clint McCrory, *Secant Planes of Space Curves*, MSRI Preprint, no. 015-93, November 1993.

[18] C. McCrory, R. Varley, *Seminar on Singularity Theory and Deformations*, Unpublished Seminar Notes, University of Georgia, 1987.

[19] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics 1358, Springer-Verlag, 1988.

[20] I.R. Shafarevich, *Basic Algebraic Geometry*, Grundlehren der Math. Wissenschaft, Springer, Heidelberg, 1974.

[21] R. Varley, *A Normal Form for the Secant Map Germ*, Unpublished Notes, University of Georgia, 2001.