Representation theory for Iwahori-Hecke algebras and Schur algebras of classical type

by

Ziqing Xiang

(Under the Direction of Daniel K. Nakano)

Abstract

We develop a theory of support varieties for Iwahori-Hecke algebras of type A, which detects natural homological properties such as the complexity of modules. Furthermore, we extend the theory to Hecke algebras of classical type with the help of Morita equivalences between Iwahori-Hecke algebras of classical type.

Schur algebras of classical type are centralizer algebras of the action of Hecke algebras of classical type on tensor spaces. We give an isomorphism theorem between Schur algebras of type B and type A, which enables us to address the questions of cellularity, quasi-hereditariness and representation type of Schur algebras of type B, and also discuss possible approach to generalize the result to type D.

Index words: Cohomology, Complexity, Iwahori-Hecke Algebra, Quantum Group, Quasi-hereditary Covers, Schur Algebra, Support Variety

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Contents

Introduction

This thesis consists of two part. The first part develops a theory of support theory for Iwahori-Hecke algebras of classical type, which is based on [NX]. The second part gives an isomorphism theorem between Schur algebras of type B and type A, and discuss its corollaries, which is based on [LNX]. Throughout the thesis, let k denote a field of characteristic p.

Part I: Support theory for Iwahori-Hecke algebras

Support varieties have been developed in a variety of contexts that involve categories which are Frobenius (i.e., where injectivity and projectivity are equivalent) and have a monoidal tensor structure. The monoidal tensor structure generally arises from a Hopf structure on an underlying algebra. Examples of such categories include modules for finite group schemes, quantum groups and Lie superalgebras (cf. [FP1, FP2] [FPe], [NPal], [BNPP]). More recently, the key properties of support varieties have be used to create axiomatic support theory and tensor triangular geometry. Very little is known about extracting geometric properties from Frobenius categories where there is no underlying coproduct.

In this thesis, we will develop a support variety theory for the Iwahori-Hecke algebra for the symmetric group (i.e., type A), and for Iwahori-Hecke algebras for other classical groups. In general, the module category for Iwahori-Hecke algebras lacks a tensor structure.

This presents major difficulties in executing important constructions. Our modified theory of support varieties differs from approaches proposed using the Hochschild cohomology (cf. [L]). In those contexts, varieties can be defined, however it is not clear how (i) these varieties can be computed and (ii) how they can be used in the general theory. It is anticipated that our methods along with several recent developments in extending the theory in type A to other Weyl groups (cf. [DPS2, DPS3]) might lead to a general finite generation results entailing the cohomology ring and the creation of a general theory of supports with realizations for arbitrary Iwahori-Hecke algebras.

The first part of the thesis is organized as follows. In Chapter 1, we introduce the Hecke algebras of classical type that will be used throughout the thesis (including Part II). The following chapter, Chapter 3, provides the definition and details of the results on transfer and its relationship to cohomology for all types. In Chapter 4, using the explicit description of the cohomology ring $R_d := \mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_d), \mathbb{C})$ due to Benson, Erdmann and Mikaelian [BEM] we show that (i) $R_\lambda := \mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_\lambda), \mathbb{C})$ is finitely-generated and (ii) $\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_\lambda)}(\mathbb{C}, M)$ is finitely generated as a R_λ-module for any composition λ. Here M is a finite-dimensional $\mathcal{H}_q(\Sigma_\lambda)$-module. The results above allow one to use the ideas involving branching to Young subgroups from Hemmer and Nakano [HN1] to construct support varieties for any $\mathcal{H}_q(\Sigma_d)$. These ideas were important for the recent proof of the Erdmann-Lim-Tan Conjecture [ELT] by Cohen, Hemmer and Nakano [CHN] that involved computing the complexity of the Lie module. Our results rely heavily on the work of Dipper and Du (cf. [DD, Du]) that provides the technical machinery to prove many of the results in this section.

Following the seminal work of Alperin, we define complexity for $\mathcal{H}_q(\Sigma_d)$-modules in Chapter 5. The main point of this section is to utilize the representation theory of the Iwahori-Hecke algebras to demonstrate that the complexity of a module is in fact equal to the dimension of its support variety (as defined in Chapter 4). As an application we prove that the complexity of any module is less than the complexity of the trivial module. Note that without a tensor structure (as in our case) this is a non-trivial fact. Subsequently, in Chapter 6, we compute the complexities and varieties for Young and permutation modules, which extends the earlier work in [HN1] for symmetric groups to Iwahori-Hecke algebras of type A.

In Chapter 7, we construct a new invariant for Specht modules called the graded dimension and relate this graded dimension to the product of cyclotomic polynomials. This definition in conjunction with results for relative cohomology allows us to show that the vertex of the Specht module satisfies certain numerical constraints. As a by-product, we are able to explicitly compute the vertex of Specht modules for a certain class of partitions. Finally, in Chapter 8, we apply our results for Iwahori-Hecke algebras of type A with various Morita equivalences to construct support varieties for Iwahori-Hecke algebras of types B/C and D, and show that the complexity of modules for these algebras is equal to the dimension of the corresponding varieties. Several open questions of further interest are posed at the end of the paper.

Part II: Coordinate constructions of q-Schur algebras

Schur-Weyl duality has played a prominent role in the representation theory of groups and algebras. The duality first appeared as method to connect the representation theory of the general linear group GL_n and the symmetric group Σ_d. This duality carries over naturally to the quantum setting by connecting the representation theory of quantum GL_n and the Iwahori-Hecke algebra $\mathcal{H}_q(\Sigma_d)$ of the symmetric group Σ_d.

Let $U_q(\mathfrak{gl}_n)$ be the Drinfeld-Jimbo quantum group. Jimbo showed in [Jim] that there is a Schur duality between $U_q(\mathfrak{gl}_n)$ and $\mathcal{H}_q(\Sigma_d)$ on the d-fold tensor space of the natural representation V of $U_q(\mathfrak{gl}_n)$. The q-Schur algebra of type A, $S_q^{\mathrm{A}}(n, d)$, is the centralizer algebra of the $\mathcal{H}_q(\Sigma_d)$-action on $V^{\otimes d}$.

It is well-known that the representation theory for $U_q(\mathfrak{gl}_n)$ is closely related to the representation theory for the quantum linear group GL_n. The polynomial representations GL_n coincide with modules of $S_q^{\mathrm{A}}(n,d)$ with $d > 0$. The relationship between objects are depicted as below:

$$\begin{array}{ccccccc} k[M_q^{\mathrm{A}}(n)]^* & \hookleftarrow & U_q(\mathfrak{gl}_n) & & & & \\ \downarrow & & \downarrow & & & & \\ k[M_q^{\mathrm{A}}(n)]_d^* & \simeq & S_q^{\mathrm{A}}(n,d) & \rightharpoonup & V^{\otimes d} & \leftharpoondown & \mathcal{H}_q(\Sigma_d) \end{array}$$

The algebra $U_q(\mathfrak{gl}_n)$ embeds in the dual of the quantum coordinate algebra $k[M_q^{\mathrm{A}}]$; while $S_q^{\mathrm{A}}(n,d)$ can be realized as its d-th degree component. The reader is referred to [PW] for a thorough treatment of the subject.

The Schur algebra $S_q^{\mathrm{A}}(n,d)$ and the Iwahori-Hecke algebra $\mathcal{H}_q(\Sigma_d)$ are structurally related when $n \geqslant d$.

- There exists an idempotent $e \in S_q^{\mathrm{A}}(n,d)$ such that $eS_q^{\mathrm{A}}(n,d)e \simeq \mathcal{H}_q(\Sigma_d)$;
- An idempotent yields the existence of Schur functor $\mathrm{Mod}(S_q^{\mathrm{A}}(n,d)) \to \mathrm{Mod}(\mathcal{H}_q(\Sigma_d))$;
- $S_q^{\mathrm{A}}(n,d)$ is a (1-faithful) quasi-hereditary cover of $\mathcal{H}_q(\Sigma_d)$[1]

The second part of the thesis aims to investigate the representation theory of the q-Schur algebras of type B that arises from the coideal subalgebras for the quantum group of type A. We construct, for type B = C, the following objects in the sense that all favorable properties mentioned in the previous section hold:

$$\begin{array}{ccccccc} k[M_{Q,q}^{\mathrm{B}}(n)]^* & \hookleftarrow & U_{Q,q}^{\mathrm{B}}(n) & & & & \\ \downarrow & & \downarrow & & & & \\ k[M_{Q,q}^{\mathrm{B}}(n)]_d^* & \simeq & S_{Q,q}^{\mathrm{B}}(n,d) & \rightharpoonup & V_{\mathrm{B}}^{\otimes d} & \leftharpoondown & \mathcal{H}_{Q,q}^{\mathrm{B}}(d) \end{array}$$

[1]The algebra $S_q^{\mathrm{A}}(n,d)$ is 1-faithful under the conditions that q is not a root of unity or if q^2 is a primitive ℓth root of unity then $\ell \geqslant 4$.

For our purposes it will be advantageous to work in more general setting with two parameters q and Q, and construct the analogs $k[M^{\mathrm{B}}_{Q,q}(n)]$ of the quantum coordinate algebras. Then we prove that the dth degree component of $k[M^{\mathrm{B}}_{Q,q}(n)]^*$ is isomorphic to the type B q-Schur algebras. The coordinate approach provide tools to study the representation theory for the algebra $k[M^{\mathrm{B}}_{Q,q}(n)]^*$ and for the q-Schur algebras simultaneously. The algebra $U^{\mathrm{B}}_{Q,q}(n)$, unlike $U_q(\mathfrak{gl}_n)$, does not have an obvious comultiplication. Therefore, its dual object, $k[M^{\mathrm{B}}_{Q,q}(n)]$, should be constructed as a coalgebra; while in the earlier situation $k[M^{\mathrm{A}}_q(n)]$ is a bialgebra.

In Chapter 9, an isomorphism theorem between the q-Schur algebras of type B and type A (under an invertibility condition) is established:

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, d-i). \tag{1}$$

One can view this as a "lifting" of the Morita equivalence (via the Schur functor) Theorem 7.1.1(a) proved by Dipper-James [DJ4].

As a corollary of our isomorphism theorem, we obtain favorable properties for our coideal Schur algebras, see Chapter 11-Chapter 12. In particular, with the Morita equivalence we are able to show that $S^{\mathrm{B}}_{Q,q}(n,d)$ is a cellular algebra and quasi-hereditary. Moreover, in § 11.3, we are able give a complete classification of the representation type of $S^{\mathrm{B}}_{Q,q}(n,d)$. In the following section (Chapter 12), we are able to demonstrate that under suitable conditions, the Schur algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is a quasi-hereditary one-cover for $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$. We also exhibit how the representation theory of $S^{\mathrm{B}}_{Q,q}(n,d)$ is related to Rouquier's Schur-type algebras that arise from the category $\mathcal{O}$ for rational Cherednik algebras.

In the one-parameter case (i.e., $q = Q$), the algebra $U^{\mathrm{B}}_q(n)$ is the coideal subalgebra $\mathbf{U}^\imath$ or $\mathbf{U}^\jmath$ of $U_q(\mathfrak{gl}_n)$ in [BW] (see also [ES]). To our knowledge, there is no general theory for finite-dimensional representations for the coideal subalgebras (cf. [Le] for establishing their Cartan subalgebras), and in some way our paper aims to establish results about "polynomial" representations for $U_q^{\mathrm{B}}(n)$. The corresponding Schur algebras therein are denoted by $\mathbf{S}^{\imath}$ or $\mathbf{S}^{\jmath}$ to emphasize the fact that they arise from certain quantum symmetric pairs of type A III/IV associated with involutions $\imath$ or $\jmath$ on a Dynkin diagram of type A_n. Namely, we have the identification below:

$$U_q^{\mathrm{B}}(n) \equiv \begin{cases} \mathbf{U}_r^{\jmath} & \text{if } n = 2r+1; \\ \mathbf{U}_r^{\imath} & \text{if } n = 2r, \end{cases} \qquad S_q^{\mathrm{B}}(n,d) \equiv \begin{cases} \mathbf{S}^{\jmath}(r,d) & \text{if } n = 2r+1; \\ \mathbf{S}^{\imath}(r,d) & \text{if } n = 2r. \end{cases}$$

There are many cases when the Morita equivalence will hold, in particular when (i) q is generic, (ii) q is an odd root of unity, or (iii) q is an (even) ℓth root of unity if $\ell > 4d$.

There are several generalizations of the q-Schur duality for type B. A comparison of the algebras regarding the aforementioned favorable properties will be given in Chapter 13. Since all these algebras are the centralizing partners of certain Iwahori-Hecke algebra actions, they are different from the ones appearing in the Schur duality (see [Hu2]) for type B/C quantum groups, and are different from the coordinate algebras studied by Doty [Do].

Part I

Support varieties of Iwahori-Hecke algebras

Chapter 1

Iwahori-Hecke algebras of classical type

1.1 Multiparameter Iwahori-Hecke algebras

Let (W, S) be a Coxeter system. For each $s \in S$, let $q_s \in k^\times$. Assume that $q_s = q_t$ if s and t are conjugate in W. For every $w \in W$, let $q_w := q_{s_1} \dots q_{s_t}$ for a reduced expression $w = s_1 \dots s_t$, which is well-defined. Let $\mathbf{q} := (q_w)_{w \in W}$.

The *Iwahori-Hecke algebra* of (W, S) with parameter $\mathbf{q}$, denoted by $\mathcal{H}_{\mathbf{q}}(W, S)$, is the free k-module with basis

$$\{T_w : \ w \in W\}$$

and with multiplication defined by

$$T_w T_s := \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ q_s T_{ws} + (q_s - 1) T_w, & \text{otherwise,} \end{cases}$$

for $w \in W$ and $s \in S$, where $\ell : \ W \to \mathbb{N}$ is the length function of (W, S).

If $q_s = q \in k^\times$ for all $s \in S$, we write $\mathcal{H}_q(W,S) := \mathcal{H}_{\mathbf{q}}(W,S)$. When S is understood from the context, we write $\mathcal{H}_{\mathbf{q}}(W) := \mathcal{H}_{\mathbf{q}}(W,S)$.

For a subset $J \subseteq S$, let (W_J, J) be the Coxeter subsystem of (W,S), and $\mathbf{q}_J := (q_w)_{w \in W_J}$. The Iwahori-Hecke algebra $\mathcal{H}_{\mathbf{q}_J}(W_J, J)$ is a subalgebra of $\mathcal{H}_{\mathbf{q}}(W,S)$, and we write $\mathcal{H}_{\mathbf{q}_J}(W_J) := \mathcal{H}_{\mathbf{q}_J}(W_J, J)$. Subalgebras arising in this way are called *parabolic subalgebras*.

There is an automorphism $^\#$ and an antiautomorphism * of $\mathcal{H}_{\mathbf{q}}(W,S)$ defined by:

$$T_w^\# := (-1)^{\ell(w)} q_w (T_{w^{-1}})^{-1}, \quad \text{and } T_w^* := T_{w^{-1}}.$$

The maps $^\#$ and * are both involutions. We will also use the dual $^\vee$ defined by:

$$T_w^\vee := q_w^{-1} T_{w^{-1}}.$$

1.2 Some modules over Iwahori-Hecke algebras

We will use right modules over Iwahori-Hecke algebras, unless stated otherwise. The algebra $\mathcal{H}_{\mathbf{q}}(W,S)$ has two distinguished one-dimensional modules:

(i) the *trivial module* k, where T_w acts as q_w, and

(ii) the *alternating module* sgn, where T_w acts as $(-1)^{\ell(w)}$.

The trivial module and alternating module coincide exactly when $q_s = -1$ for all simple reflection s. When $\mathbf{q} = (1)_{w \in W}$ these specialize to the usual trivial and alternating modules of the group algebra kW.

For any $\mathcal{H}_{\mathbf{q}}(W,S)$-module M, one can define a dual (left) module $M^* := \mathrm{Hom}_k(M,k)$, where the action of $\mathcal{H}_{\mathbf{q}}(W,S)$ is given by $h \cdot f : m \mapsto h^* m$ for $h \in \mathcal{H}_{\mathbf{q}}(W,S)$ and $f \in M^*$.

In general, the tensor product of two $\mathcal{H}_{\mathbf{q}}(W,S)$-modules is not an $\mathcal{H}_{\mathbf{q}}(W,S)$-module, since $\mathcal{H}_{\mathbf{q}}(W,S)$ is not a Hopf algebra. However, the automorphism # lets us define, for each

$\mathcal{H}_q(W,S)$-module M, a new module $M^{\#}$ with the same underlying vector space and with action given by $h \cdot m := h^{\#}m$ for $h \in \mathcal{H}_{\mathbf{q}}(W,S)$ and $f \in M^*$. This is denoted by

$$M \otimes \mathrm{sgn} := M^{\#},$$

which specializes for $\mathbf{q} = (1)_{w\in W}$ to tensoring with the alternating module.

1.3 Iwahori-Hecke algebras of type A

Let Σ_d be the *symmetric group* on d letters, and S be the set of simple transpositions in Σ_d. The pair (Σ_d, d) is a Coxeter system. The *Iwahori-Hecke algebra of type A*, denoted by $\mathcal{H}_{\mathbf{q}}^{\mathrm{A}}(d-1)$, is defined to be $\mathcal{H}_{\mathbf{q}}(\Sigma_d, S)$.

Since all simple transpositions are conjugate in Σ_d, there exists some $q \in k^\times$ such that $q_s = q$ for all $s \in S$. Therefore, we may write $\mathcal{H}_q^{\mathrm{A}}(d-1)$ instead of $\mathcal{H}_{\mathbf{q}}^{\mathrm{A}}(d-1)$.

For $1 \leqslant i \leqslant d-1$, let $s_i := (i, i+1) \in \Sigma_d$, and $T_i := T_{s_i}$. It is known that, $\mathcal{H}_q^{\mathrm{A}}(d-1)$ is generated by $T_1, \ldots, T_{d-1}$ subject to the following relations:

(i) Braid relations:

$$\begin{cases} T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & 1 \leqslant i \leqslant d-2, \\ T_iT_j = T_jT_i, & |i-j| > 1; \end{cases}$$

(ii) Quadratic relation:

$$(T_i - q)(T_i + 1) = 0, \quad 1 \leqslant i \leqslant d-1.$$

Let l be the smallest integer such that $1 + q + \cdots + q^{l-1} = 0$, and set $l := \infty$ if no such integer exists. If $q \in k^\times$ is a primitive jth root of unity, then $l = j$. Furthermore, if q is not root of unity, then $\mathcal{H}_q(\Sigma_d)$ is semisimple. Note that $\gcd(l,p) = 1$ when $l \neq \infty$.

1.3.1 Partitions and compositions

Let $\Lambda(d) := \{\lambda \vDash d\}$ be the set of all *compositions* of d, and let $\Lambda^+(d) := \{\lambda \vdash d\}$ be the set of all *partitions* of d. Given two compositions $\lambda, \mu \in \Lambda(d)$ (resp. partitions), let $\mu \vDash \lambda$ (resp. $\mu \vdash \lambda$) if μ is finer than λ. A partition/composition λ of d is called *l-parabolic* if every part of λ is either 1 or l, and it is *simple l-parabolic* provided that exactly one part of λ is l and all other parts are 1's.

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is called *l-restricted* if $\lambda_i - \lambda_{i+1} \leqslant l - 1$ for all i. The set of the l-restricted partitions of d will be denoted by $\Lambda^+_{\text{res}}(d)$. A partition λ is called *l-regular* if its *transpose* λ' is l-restricted. The set of all l-regular partitions of d is denoted by $\Lambda^+_{\text{reg}}(d)$.

1.3.2 Parabolic subalgebras

For a composition $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda(d)$, let Σ_λ be the corresponding *Young subgroup* of Σ_d, that is $\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots$, and let S_λ be the simple transpositions in Σ_λ. Associated to this Young subgroup, there is a corresponding parabolic subalgebra of $\mathcal{H}^{\mathrm{A}}_q(\Sigma_d)$:

$$\mathcal{H}_q(\Sigma_\lambda) := \mathcal{H}_q(\Sigma_\lambda, S_\lambda).$$

Clearly,

$$\mathcal{H}_q(\Sigma_\lambda) \cong \mathcal{H}^{\mathrm{A}}_q(\lambda_1 - 1) \otimes \mathcal{H}^{\mathrm{A}}_q(\lambda_2 - 1) \otimes \cdots .$$

1.3.3 Simple modules and Specht modules

In this subsection, we assume that $p = 0$. We refer the reader to [DJ1] and [Mat] for details about the representation theory of $\mathcal{H}_q(\Sigma_d)$. The major classes of representations parallel those for the modular representation theory of the symmetric group. For each $\lambda \in \Lambda^+(d)$, there is a *q-Specht module* of the Iwahori-Hecke algebra $\mathcal{H}_q(\Sigma_d)$, denoted by S^λ. If $\lambda \in \Lambda^+_{\text{reg}}(d)$, then S^λ has a unique simple quotient, denoted by D^λ. One obtains a complete collection of non-isomorphic simple modules D^λ for $\lambda \in \Lambda^+_{\text{reg}}(d)$ for $\mathcal{H}_q(\Sigma_d)$-module in this way. These simple modules are self-dual and absolutely irreducible.

For a composition $\lambda \in \Lambda(d)$, set

$$x_\lambda := \sum_{w \in \Sigma_\lambda} T_w.$$

Define the *permutation module* $M^\lambda := x_\lambda \mathcal{H}_q(\Sigma_d)$. One also has the isomorphism $M^\lambda \cong \text{ind}_{\mathcal{H}_q(\Sigma_\lambda)}^{\mathcal{H}_q(\Sigma_d)} k$. Given $\lambda \in \Lambda^+(d)$, there is a unique indecomposable direct summand of M^λ containing S^λ that is the *Young module* Y^λ. All other summands are Young modules whose partitions are strictly greater than λ in the dominance ordering. Furthermore, $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$.

The simple $\mathcal{H}_q(\Sigma_d)$-modules can also be indexed by $\Lambda^+_{\text{res}}(d)$. For $\lambda \in \Lambda^+_{\text{res}}(d)$ denote the corresponding simple module by D_λ. It is a fact that $D_\lambda = \text{soc}_{\mathcal{H}_q(\Sigma_d)}(S^\lambda)$. The relationship between these two labellings is given by:

$$D^\lambda \cong D_{\lambda'} \otimes \text{sgn} \quad \text{for any } \lambda \in \Lambda^+_{\text{reg}}(d). \tag{1.1}$$

We remark that that tensoring with the alternating module turns Specht modules into dual Specht modules and vice-versa (cf. [J1, 6.7], [Mat, Exer. 3.14]):

$$S^\lambda \otimes \text{sgn} \cong (S^{\lambda'})^* := S_{\lambda'}. \tag{1.2}$$

1.4 Iwahori-Hecke algebras of type B/C

The Coxeter group of type B/C, denoted by $W^{\text{B}}(d)$, can be realized as the *signed symmetric group* on $\{\pm 1, \ldots, \pm d\}$. Let $s_0^{\text{B}} := (1, -1)$, $s_i := (i, i+1)(-i, -i-1)$ and $S := \{s_i : 0 \leqslant$

$i < d\} \subseteq W^{\mathrm{B}}(d)$. The *Iwahori-Hecke algebra of type B/C*, denoted by $\mathcal{H}_{\mathbf{q}}^{\mathrm{B}}(d)$, is defined to be $\mathcal{H}_{\mathbf{q}}(W_d, S)$.

The generators in S are in two conjugacy classes: the conjugacy class of s_0, and the conjugacy class of $s_1, \ldots, s_{d-1}$. Let $Q := q_{s_0}$ and $q := q_{s_1}$. We write $\mathcal{H}_{Q,q}^{\mathrm{B}}(d) = \mathcal{H}_{\mathbf{q}}^{\mathrm{B}}(d)$.

Let $T_0^{\mathrm{B}} := T_{s_0}$ and $T_i := T_{s_i}$. It is known that, $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$ is generated by $T_0^{\mathrm{B}}, T_1, \ldots, T_{d-1}$ subject to the following relations:

(i) Braid relations:

$$\begin{cases} T_0^{\mathrm{B}} T_1 T_0^{\mathrm{B}} T_1 = T_1 T_0^{\mathrm{B}} T_1 T_0^{\mathrm{B}}, & \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \leqslant i \leqslant d-2, \\ T_i T_j = T_j T_i, & |i-j| > 1; \end{cases}$$

(ii) Quadratic relations:

$$\begin{cases} (T_0^{\mathrm{B}} - Q)(T_0^{\mathrm{B}} + 1) = 0, & \\ (T_i - q)(T_i + 1) = 0, & 1 \leqslant i \leqslant d-1. \end{cases}$$

We have an embedding of type A Iwahori-Hecke algebra into type B Iwahori-Hecke algebra corresponding to the embedding $W^{\mathrm{A}}(d-1) \hookrightarrow W^{\mathrm{B}}(d)$:

$$\mathcal{H}_q^{\mathrm{A}}(d-1) \hookrightarrow \mathcal{H}_{Q,q}^{\mathrm{B}}(d),$$
$$T_i \mapsto T_i, \quad 1 \leqslant i \leqslant d-1.$$

1.5 Iwahori-Hecke algebras of type D

The Coxeter group of type D, denoted by $W^{\mathrm{D}}(d)$, can be realized as the *even signed symmetric group* on $\{\pm 1, \ldots, \pm d\}$. Let $s_0^{\mathrm{D}} := (1, -1)(2, -2)$, $s_i := (i, i+1)(-i, -i-1)$ and

$S := \{s_i : 0 \leqslant i < d\} \in W^{\mathrm{B}}(d)$. Note that $s_0^{\mathrm{D}} = s_0 s_1 s_0$, therefore, $W^{\mathrm{D}}(d)$ is a subgroup of $W^{\mathrm{B}}(d)$. The *Iwahori-Hecke algebra of type B(C)*, denoted by $\mathcal{H}_{\mathbf{q}}^{\mathrm{B}}(d)$, is defined to be $\mathcal{H}_{\mathbf{q}}(W^{\mathrm{D}}(d), S)$.

The generators S are in a single conjugacy class. Let $q := q_{s_1}$. We write $\mathcal{H}_q^{\mathrm{D}}(d) := \mathcal{H}_{\mathbf{q}}^{\mathrm{D}}(d)$.

Let $T_0^{\mathrm{D}} := T_{s_0^{\mathrm{D}}}$, and $T_i := T_{s_i}$ for $1 \leqslant i \leqslant d-1$. It is known that, $\mathcal{H}_q^{\mathrm{D}}(d)$ is generated by $T_0^{\mathrm{D}}, T_1, \ldots, T_{d-1}$ subject to the following relations:

(i) Braid relation:

$$\begin{cases} T_0^{\mathrm{D}} T_i = T_i T_0^{\mathrm{D}}, & i = 1 \text{ or } 3 \leqslant i \leqslant d-1, \\ T_0^{\mathrm{D}} T_2 T_0^{\mathrm{D}} = T_2 T_0^{\mathrm{D}} T_2, & \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \leqslant i \leqslant d-2, \\ T_i T_j = T_j T_i, & |i-j| > 1; \end{cases}$$

(ii) Quadratic relation:

$$\begin{cases} (T_0^{\mathrm{D}} - q)(T_0^{\mathrm{D}} + 1) = 0, & \\ (T_i - q)(T_i + 1) = 0, & 1 \leqslant i \leqslant d-1. \end{cases}$$

We have an embedding of type A Iwahori-Hecke algebra into type D Iwahori-Hecke algebra corresponding to the embedding $W^{\mathrm{A}}(d-1) \hookrightarrow W^{\mathrm{D}}(d)$:

$$\begin{aligned} \mathcal{H}_q^{\mathrm{A}}(d-1) &\hookrightarrow \mathcal{H}_q^{\mathrm{D}}(d), \\ T_i &\mapsto T_i, \quad 1 \leqslant i \leqslant d-1, \end{aligned}$$

and we have an embedding of type D Iwahori-Hecke algebra into type B Iwahori-Hecke algebra corresponding to the embedding $W^{\mathrm{D}}(d) \hookrightarrow W^{\mathrm{B}}(d)$:

$$\begin{aligned} \mathcal{H}_q^{\mathrm{D}}(d) &\hookrightarrow \mathcal{H}_{Q=1,q}^{\mathrm{B}}(d), \\ T_0^{\mathrm{D}} &\mapsto T_0^{\mathrm{B}} T_1 T_0^{\mathrm{B}}, \\ T_i &\mapsto T_i, \quad 1 \leqslant i \leqslant d-1. \end{aligned}$$

Chapter 2

Transfer maps and cohomology

In this chapter, we fix a Coxeter system (W, S), and the sets I, J and K which are subsets of S.

2.1 Parabolic subgroups

Let I and J be subsets of K. We abuse the notation by letting W_K/W_J, $W_I\backslash W_K$ and $W_I\backslash W_K/W_J$ denote the sets of *distinguished left/right/double coset representatives*, respectively. The are the minimal length elements in the corresponding cosets. For a distinguished double coset representative $w \in W_I\backslash W_K/W_J$, the intersection $W_I^w \cap W_J$ is always a parabolic subgroup of W_K, where $W_I^w := w^{-1}W_I w$. More precisely, there exists a unique subset of K, denoted by $I^w \cap J$, such that $W_{I^w \cap J} = W_I^w \cap W_J$.

2.2 Induction, restriction and conjugation functors of modules

Let $I \subseteq K$. For a $\mathcal{H}_{\mathbf{q}}(W_I)$-module M, the *induction* of M from I to K, denoted by $\operatorname{ind}_I^K M$, is the $\mathcal{H}_{\mathbf{q}}(W_K)$-module

$$\operatorname{ind}_I^K M := M \otimes_{\mathcal{H}_{\mathbf{q}}(W_I)} \mathcal{H}_{\mathbf{q}}(W_K).$$

Let $J \subseteq K$. For a $\mathcal{H}_{\mathbf{q}}(W_K)$-module M, the *restriction* of M from K to J, denoted by $\operatorname{res}_J^K M$, is the $\mathcal{H}_{\mathbf{q}}(W_J)$-module

$$\operatorname{res}_J^K M := M \otimes_{\mathcal{H}_{\mathbf{q}}(W_J)} \mathcal{H}_{\mathbf{q}}(W_J).$$

Let $w \in W_I \backslash W_K / W_J$. For a $\mathcal{H}_{\mathbf{q}}(W_I)$-module M, the *conjugation* of M by w, denoted by $\operatorname{con}_I^w M$, is the $\mathcal{H}_{\mathbf{q}}(W_{I^w \cap J})$-module

$$\operatorname{con}_I^w M := M \otimes_{\mathcal{H}_{\mathbf{q}}(W_I)} T_w \subseteq M \otimes_{\mathcal{H}_{\mathbf{q}}(W_I)} \mathcal{H}_{\mathbf{q}}(W_K).$$

Theorem 2.2.1 (Mackey Decomposition, [Jo, Theorem 2.29])**.** *For a $\mathcal{H}_{\mathbf{q}}(W_I)$-$\mathcal{H}_{\mathbf{q}}(W_I)$-bimodule M,*

$$\operatorname{res}_J^K \operatorname{ind}_I^K M \cong \bigoplus_{w \in W_I \backslash W_K / W_J} \operatorname{ind}_{I^w \cap J}^J \operatorname{con}_I^w M.$$

2.3 Transfer, restriction and conjugation maps

For an algebra A and an A-A-bimodule M, let

$$M^A := \{m \in M : \ am = ma, \text{for all } a \in A\}.$$

Definition 2.3.1. Let M be a $\mathcal{H}_{\mathbf{q}}(W_K)$-$\mathcal{H}_{\mathbf{q}}(W_K)$-bimodule.

(i) Let $J \subseteq K$. The inclusion gives the *restriction map* $\operatorname{res}_{K,J} : \ M^{\mathcal{H}_{\mathbf{q}}(W_K)} \hookrightarrow M^{\mathcal{H}_{\mathbf{q}}(W_J)}$.

(ii) ([Jo, L]) Let $I \subseteq K$. The *transfer map* $\mathrm{tr}_{I,K} : M^{\mathcal{H}_{\mathbf{q}}(W_I)} \to M^{\mathcal{H}_{\mathbf{q}}(W_K)}$ is

$$\mathrm{tr}_{I,K}(m) := \sum_{w \in W_I \backslash W_K} T_w^{\vee} m T_w.$$

(iii) Let $w \in W_I \backslash W_K / W_J$. The *conjugation map* $\mathrm{con}_{I,w} : M^{\mathcal{H}_{\mathbf{q}}(W_I)} \to M^{\mathcal{H}_{\mathbf{q}}(W_{I^w \cap J})}$ is

$$\mathrm{con}_{I,w}(m) := T_w^{-1} m T_w.$$

Theorem 2.3.2 ([Jo, Theorem 2.30])**.** *For* $m \in M^{\mathcal{H}_{\mathbf{q}}(W_I)}$,

$$\mathrm{res}_{K,J} \mathrm{tr}_{I,K}(m) = \sum_{w \in W_I \backslash W_K / W_J} \mathrm{tr}_{I^w \cap J, J}(T_w^{\vee} m T_w).$$

2.4 Transfer, restriction and conjugation maps on $\mathrm{Ext}^{\bullet}$

For $\mathcal{H}_{\mathbf{q}}(W_K)$-modules M and N, and $h_1, h_2 \in \mathcal{H}_{\mathbf{q}}(W_K)$, let

$$(h_1 f h_2)(m) := f(m h_1) h_2.$$

This makes $\mathrm{Hom}_k(M, N)$ a $\mathcal{H}_{\mathbf{q}}(W_K)$-$\mathcal{H}_{\mathbf{q}}(W_K)$-bimodule. Moreover,

$$\mathrm{Hom}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M, N) = \mathrm{Hom}_k(M, N)^{\mathcal{H}_{\mathbf{q}}(W_I)}.$$

Therefore, the transfer map, restriction map and conjugation map in Definition 2.3.1 induces maps on extension groups:

$$\begin{aligned}
\mathrm{tr}_{I,K} &: \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M,N) \to \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_K)}(M,N),\\
\mathrm{res}_{K,J} &: \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_K)}(M,N) \to \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_J)}(M,N),\\
\mathrm{con}_{I,w} &: \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M,N) \to \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_{I^w \cap J})}(M,N).
\end{aligned}$$

Proposition 2.4.1. *Let M_1, M_2, M_3 be three $\mathcal{H}_{\mathbf{q}}(W_K)$-$\mathcal{H}_{\mathbf{q}}(W_K)$-bimodules, and $I, J \subseteq K$. The following statements hold.*

(i) *For $\alpha \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_K)}(M_1, M_2)$,*

$$\mathrm{tr}_{I,K}\,\mathrm{res}_{K,I}(\alpha) = \alpha\,\mathrm{tr}_{I,K}\,1_{\mathcal{H}_{\mathbf{q}}(W_I)},$$

where $1_{\mathcal{H}_{\mathbf{q}}(W_I)}$ is the identity element in the $\mathcal{H}_{\mathbf{q}}(W_I)$-$\mathcal{H}_{\mathbf{q}}(W_I)$ bimodule $\mathcal{H}_{\mathbf{q}}(W_I)$.

(ii) *For $\alpha \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M_1, M_2)$, $\beta \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_K)}(M_2, M_3)$,*

$$\beta \circ \mathrm{tr}_{I,K}(\alpha) = \mathrm{tr}_{J,K}(\mathrm{res}_{K,J}(\beta) \circ \alpha).$$

(iii) *For $\alpha \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_K)}(M_1, M_2)$, $\beta \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M_2, M_3)$,*

$$\mathrm{tr}_{I,K}(\beta) \circ \alpha = \mathrm{tr}_{I,K}(\beta \circ \mathrm{res}_{K,I}(\alpha)).$$

(iv) *For $I \subseteq J \subseteq K$, and $\alpha \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M_1, M_2)$,*

$$\mathrm{tr}_{J,K}\,\mathrm{tr}_{I,J}(\alpha) = \mathrm{tr}_{I,K}(\alpha).$$

(v) *For* $\alpha \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M_1, M_2)$,

$$\mathrm{res}_{K,J}\,\mathrm{tr}_{I,K}(\alpha) = \sum_{w \in W_I \backslash W_K / W_J} \mathrm{tr}_{I^w \cap J, J}(T_w^{\vee} \alpha T_w).$$

Proof. (i) Since α commutes with $\mathcal{H}_{\mathbf{q}}(W_K)$,

$$\mathrm{tr}_{I,K}\,\mathrm{res}_{K,I}(\alpha) = \sum_{w \in W_I \backslash W_K} T_w^{\vee} \alpha T_w = \alpha \sum_{w \in W_I \backslash W_K} T_w^{\vee} T_w = \alpha\,\mathrm{tr}_{I,K}\,1_{\mathcal{H}_{\mathbf{q}}(W_I)}.$$

(ii) Since β commutes with $\mathcal{H}_{\mathbf{q}}(W_K)$,

$$\beta \circ \mathrm{tr}_{I,K}(\alpha) = \beta \circ \sum_{w \in W_I \backslash W_K} T_w^{\vee} \alpha T_w = \sum_{w \in W_K / W_I} T_w^{\vee} (\mathrm{res}_{K,I}(\beta) \circ \alpha) T_w = \mathrm{tr}_{I,K}(\mathrm{res}_{K,I}(\beta) \circ \alpha).$$

(iii) The result follows from similar arguments in (ii).

(iv) For every $w_1 \in W_I \backslash W_J$ and $w_2 \in W_J \backslash W_K$, it is clear that $w_1 w_2 \in W_I \backslash W_K$. So,

$$\mathrm{tr}_{J,K}\,\mathrm{tr}_{J,I}(\alpha) = \sum_{w_2 \in W_J \backslash W_K} \sum_{w_1 \in W_I \backslash W_J} T_{w_1 w_2}^{\vee} \alpha T_{w_1 w_2} = \sum_{w \in W_I \backslash W_K} T_w^{\vee} \alpha T_w = \mathrm{tr}_{I,K}(\alpha).$$

(v) Viewing α as an element in $\mathrm{Hom}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M_1', M_2)$ for some $\mathcal{H}_{\mathbf{q}}(W_I)$-module M_1', the result follows from applying Theorem 2.3.2 to $\mathrm{Hom}_{\mathcal{H}_{\mathbf{q}}(W_I)}(M_1', M_2)$. ☐

Proposition 2.4.2. *Let* $w \in W_I \backslash W_K$ *such that* $W_I^w = W_J$. *For every* $\alpha \in \mathrm{Ext}^{\bullet}_{\mathcal{H}_{\mathbf{q}}(W_K)}(M, N)$,

$$\mathrm{con}_{I,w}\,\mathrm{res}_{K,I}(\alpha) = \mathrm{res}_{K,J}(\alpha).$$

Proof. Since $T_w \in \mathcal{H}_{\mathbf{q}}(W_K)$, T_w commutes with α, so

$$\mathrm{con}_{I,w}\,\mathrm{res}_{K,I}(\alpha) = T_w^{-1}\,\mathrm{res}_{K,I}(\alpha) T_w = T_w^{-1} T_w\,\mathrm{res}_{K,J}(\alpha) = \mathrm{res}_{K,J}(\alpha).$$ ☐

2.5 Finiteness theorem

Theorem 2.5.1. *Let $I \subseteq K$, and assume that $\operatorname{tr}_{I,K} 1_{\mathcal{H}_{\mathbf{q}}(W_I)}$ is invertible in $\mathcal{H}_{\mathbf{q}}(W_K)$. Let M be a Noetherian $\mathcal{H}_{\mathbf{q}}(W_I)$-module. If $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_I)}(k, M)$ is Noetherian over $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_I)}(k, k)$, then $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, M)$ is Noetherian over $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, k)$.*

Proof. For a $\mathcal{H}_{\mathbf{q}}(W_I)$-module N, let

$$T(N) := \ker(\operatorname{tr}_{I,K} : \operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_I)}(k, N) \to \operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, N)).$$

Since $\operatorname{tr}_{I,K} 1_{\mathcal{H}_{\mathbf{q}}(W_I)}$ is invertible, by Proposition 2.4.1(i), $\operatorname{tr}_{I,K}$ splits $\operatorname{res}_{K,I}$. Therefore,

$$\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_I)}(k, N) \cong \operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, N) \oplus T(N).$$

Let $(L_i)_{i \in \mathbb{N}}$ be an ascending chain of $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, k)$-submodules of $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, M)$. Then,

$$\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_I)}(k, k)L_i = \operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, k)L_i \oplus T(k)L_i = L_i \oplus T(k)L_i.$$

This gives an ascending chain of $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_I)}(k, k)$-submodules of $\operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, M)$, which must terminate by the Noetherian assumption. Proposition 2.4.1(iii) shows that $T(k)L_i \subseteq T(M)$. On the other hand $L_i \subseteq \operatorname{Ext}^\bullet_{\mathcal{H}_{\mathbf{q}}(W_K)}(k, M)$. Thus, $L_i \cap T(k)L_j = 0$, hence the original ascending chain $(L_i)_{i \in \mathbb{N}}$ terminates as well. □

For type A and characteristic 0, Theorem 2.5.2 below gives a criteria when $\operatorname{tr}_{I,K} 1$ is invertible.

Theorem 2.5.2. *Suppose that the characteristic of k is 0. If λ is a maximal l-parabolic subcomposition of n then $\operatorname{tr}_{\lambda,n} 1$ is invertible in $\mathcal{H}_q(\Sigma_n)$.*

Proof. See [Du, Theorem 2.7]. □

Chapter 3

Cohomology and support varieties of type A Iwahori-Hecke algebras

For the remainder of Part I, we choose $k := \mathbb{C}$.

We focus on type A in this chapter. Every Young subgroup Σ_λ is a parabolic subgroup W_I of $W := \Sigma_d$

For a composition $\lambda \vDash d$, the Young subgroup Σ_λ is a parabolic subgroup of Σ_d. This means that, for $W = \Sigma_d$ with S being the set of simple transpositions, there exists some subset $J_\lambda \subseteq S$ such that $W_{J_\lambda} = \Sigma_\lambda$. We write $\mathrm{tr}_{\lambda,\mu} := \mathrm{tr}_{J_\lambda,J_\mu}$ and $\mathrm{res}_{\lambda,\mu} := \mathrm{res}_{J_\lambda,J_\mu}$.

Let $q \in k^\times$ be a primitive lth root of unity, and suppose that $q \neq 1$. For a composition λ, let $\mathcal{H}_q(\Sigma_\lambda)$ be the corresponding Iwahori-Hecke algebra. Let $R_\lambda := \mathrm{Ext}^\bullet_{\mathcal{H}_q^{\mathrm{A}}(\Sigma_\lambda)}(k,k)$ be the cohomology ring of $\mathcal{H}_q^{\mathrm{A}}(\Sigma_\lambda)$ under the Yoneda product. For a natural number d, set $R_d := R_{(d)}$ to be the cohomology ring with respect to the trivial composition (d). Assume that all modules for $\mathcal{H}_q(\Sigma_\lambda)$ are finite-dimensional. Let $\mathrm{mod}(\mathcal{H}_q(\Sigma_\lambda))$ be the category of finite dimensional $\mathcal{H}_q(\Sigma_\lambda)$-modules. For an $M \in \mathrm{mod}(\mathcal{H}_q(\Sigma_\lambda))$, set

$$\mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_\lambda), M) := \mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_\lambda)}(k, M),$$

which is an R_λ-module.

3.1 Restriction maps on cohomology

Given a simple l-parabolic subcomposition ν of λ, $R_\nu \cong R_l$ and

$$R_\nu = \begin{cases} k[x_\nu] \otimes \Lambda[y_\nu], & l > 2, \\ k[y_\nu], & l = 2, \end{cases}$$

for some x_ν and y_ν such that $\deg x_\nu = 2l-2$ and $\deg y_\nu = 2l-3$. Set $x_\nu := y_\nu^2$ when $l = 2$. The ring R_ν has a reduced commutative subring

$$\widetilde{R}_\nu := k[x_\nu].$$

According to Proposition 2.4.2, we could choose x_ν and y_ν for all simple l-parabolic $\nu \vDash \lambda$ compatibly such that $\operatorname{con}_\nu^w x_\nu = x_{\nu^w}$ and $\operatorname{con}_\nu^w y_\nu = y_{\nu^w}$ for where w is the double coset representative in $\Sigma_\nu \backslash \Sigma_\lambda / \Sigma_\nu$ and $\nu^w \vDash \lambda$ is the unique simple l-parabolic subcomposition such that $\Sigma_\nu^w = \Sigma_{\nu^w}$.

Theorem 3.1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vDash n$ be a composition and set $\lambda/l := (\lfloor \lambda_1/l \rfloor, \ldots, \lfloor \lambda_m/l \rfloor)$. Moreover, let $\mu \vDash \lambda$ be a maximal l-parabolic subcomposition. The following statements hold.*

(i) *The restriction map $\operatorname{res}_{\lambda,\mu}$ induces an isomorphism*

$$\operatorname{res}_{\lambda,\mu} : R_\lambda \xrightarrow{\sim} \left(R_{\nu_1} \otimes \cdots \otimes R_{\nu_{|\lambda/l|}} \right)^{\Sigma_{|\lambda/l|}}$$

where $\nu_1, \ldots, \nu_{\lfloor n/l \rfloor}$ are all simple l-parabolic subcomposition of μ. Moreover, the induced isomorphism

$$\operatorname{res}_{\lambda,\mu} : R_\lambda \xrightarrow{\sim} \left(R_l^{\otimes |\lambda/l|} \right)^{\Sigma_{\lambda/l}}$$

is independent of the choice of μ.

(ii) *Under the isomorphism above,*

$$\mathrm{res}_{n,\lambda} : \left(R_l^{\otimes \lfloor n/l \rfloor}\right)^{\Sigma_{\lfloor n/l \rfloor}} \to \left(R_l^{\otimes |\lambda/l|}\right)^{\Sigma_{\lambda/l}}$$

is the restriction of the projection map $R_l^{\otimes \lfloor n/l \rfloor} \to R_l^{\otimes |\lambda/l|}$.

Proof. (i) Since $\mathcal{H}_q(\Sigma_\lambda) \cong \bigotimes_{i=1}^m \mathcal{H}_q(\Sigma_{\lambda_i})$, it follows by Künneth theorem that $R_\lambda \cong \bigotimes_{i=1}^m R_{\lambda_i}$. Hence, it is enough to prove the result of $\mathrm{res}_{\lambda,\mu}$ for the trivial partition $\lambda = (n)$. Let $\nu \vdash n$ be an l-parabolic partition conjugate to μ. The isomorphism induced by $\mathrm{res}_{\lambda,\nu}$ has been proved in [BEM]. By Proposition 2.4.2, $\mathrm{res}_{\lambda,\mu}$ and $\mathrm{res}_{\lambda,\nu}$ induce the same isomorphism.
(ii) Let $\nu \vdash n$ be an l-parabolic partition conjugate to μ. Then, $R_\lambda \cong R_\mu^{\Sigma_{|\mu/l|}} \cong R_\nu^{\Sigma_{|\nu/l|}}$. So, it suffices to prove the result for partition λ where every part is a multiple of l, and μ is a maximal l-parabolic partition. Since $\mathrm{res}_{(n),\mu} = \mathrm{res}_{\lambda,\nu} \circ \mathrm{res}_{(n),\lambda}$ and the restriction map $\mathrm{res}_{(n),\mu}$ is given by projection, the result follows. $\square$

For a composition λ, we set

$$\widetilde{R}_\lambda := \mathrm{res}_{\lambda,\mu}^{-1} \left(\widetilde{R}_{\nu_1} \otimes \dots \widetilde{R}_{\nu_{|\lambda/l|}}\right)^{\Sigma_{\lambda/l}},$$

where $\nu_1, \dots, \nu_{|\lambda/l|}$ are all simple l-parabolic subcomposition of some maximal l-parabolic subcomposition μ of λ. This definition matches the previous definition $\widetilde{R}_\nu$ before for simple l-parabolic ν. By Theorem 3.1.1, $\widetilde{R}_\lambda$ does not depend on the choice of μ, and it is a commutative reduced subring of R_λ. Moreover, R_λ is a finitely generated $\widetilde{R}_\lambda$-module.

3.2 Finite generation of cohomology

Theorem 3.2.1. *Let* λ *be a partition of* d.

(i) *R_λ is a Noetherian k-algebra.*

(ii) *If $M \in \mathrm{mod}(\mathcal{H}_q(\Sigma_\lambda))$ then $\mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_\lambda), M)$ is a Noetherian R_λ-module.*

Proof. (i) We can conclude this statement by applying the Künneth theorem and [BEM, Theorem 1.1].

(ii) First consider the case when $\lambda = (l)$. Then one can directly prove using explicit projective resolutions for $\mathcal{H}_q(\Sigma_l)$ (cf. [KN, 5.1]) that for any simple $\mathcal{H}_q(\Sigma_l)$-module S, $\mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_l), S)$ is a Noetherian R_l-module. Now using induction on the composition length and the long exact sequence in cohomology, it follows that the statement of (b) holds for a Noetherian $\mathcal{H}_q(\Sigma_l)$-module M.

Next consider the case when $\lambda = (l^a, 1^s)$. Any simple $\mathcal{H}_q(\lambda)$-module is an outer tensor product $S = S_1 \boxtimes S_2 \boxtimes \cdots \boxtimes S_a \boxtimes k^{\boxtimes s}$ By the Künneth theorem,

$$\mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_\lambda), S) \cong \mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_l), S_1) \otimes \mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_l), S_2) \otimes \cdots \otimes \mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_l), S_a).$$

which is a Noetherian R_λ-module from the preceding paragraph. By an inductive argument on the composition length, the statement holds for $R_{(l^a,1^s)}$.

Now consider the case when $\lambda = (d)$ and let $\mu = (l^a, 1^s)$ be a maximal l-parabolic partition of λ. According to Theorem 2.5.2, $\mathrm{tr}_{\mu,\lambda} 1$ is invertible in $\mathcal{H}_q(\Sigma_\lambda)$. Therefore, by Theorem 2.5.1, the statement for R_λ follows from the statement for R_μ.

Finally, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition of d. Any simple $\mathcal{H}_q(\lambda)$-module is an outer tensor product $S = S_1 \boxtimes S_2 \boxtimes \cdots \boxtimes S_t$ By the Künneth theorem and the fact that the Noetherian statement holds for $\mathcal{H}(\lambda_j)$ for $j = 1, 2, \ldots, t$, one can conclude that $\mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_\lambda), S)$ is a Noetherian R_λ-module. Again by an inductive argument one can now conclude the statement of part (ii). ☐

3.3 Support theory

Set $W_\lambda := \operatorname{MaxSpec} \widetilde{R}_\lambda$. According to Theorem 3.2.1 the set W_λ is an affine homogeneous variety. Given $M \in \operatorname{mod}(\mathcal{H}_q(\Sigma_\lambda))$, define the (relative) *support variety* $W_\lambda(M)$ as the variety of the annihilator ideal, $J_{\mathcal{H}_q(\Sigma_\lambda)}(k, M)$, in $\widetilde{R}_\lambda$ for its action on $\mathrm{H}^\bullet(\mathcal{H}_q(\Sigma_\lambda), M)$. These support varieties are closed, conical subvarieties of W_λ.

For each $\mu \vDash \lambda$, there exists a restriction map in cohomology $\operatorname{res}^*_{\lambda,\mu} : W_\mu \to W_\lambda$ which is induced by the inclusion of $\mathcal{H}_q(\Sigma_\mu) \subseteq \mathcal{H}_q(\Sigma_\lambda)$. We can now formulate a definition for the support varieties for modules in $\operatorname{mod}(\mathcal{H}_q(\Sigma_\lambda))$.

Definition 3.3.1. Let $M \in \operatorname{mod}(\mathcal{H}_q(\Sigma_\lambda))$.

(i) The *support variety* of M is defined as

$$V_\lambda(M) := \bigcup_{\mu \vDash \lambda} \operatorname{res}^*_{\lambda,\mu}(W_\mu(M)).$$

(ii) In the case when $\lambda = (d)$,

$$V_{\mathcal{H}_q(\Sigma_d)}(M) := V_{(d)}(M) = \bigcup_{\mu \vDash (d)} \operatorname{res}^*_{(d),\mu}(W_\mu(M)).$$

By using the functoriality of the restriction map and the fact that the restriction maps are finite maps, one can state the following proposition.

Proposition 3.3.2. *Let W be closed subvariety of W_ν.*

(i) $\dim W = \dim \operatorname{res}^*_{\lambda,\nu}(W)$.

(ii) $\operatorname{res}^*_{\lambda,\nu} = \operatorname{res}^*_{\lambda,\mu} \circ \operatorname{res}^*_{\mu,\nu}$.

Next we present below several elementary properties of these support varieties. The proofs from [Ben, §5.7] can be used to verify these facts.

Proposition 3.3.3. *Let* $M_j \in \operatorname{mod}(\mathcal{H}_q(\Sigma_d))$ *for* $j = 1, 2, 3$. *Then*

(i) *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence in* $\operatorname{mod}(\mathcal{H}_q(\Sigma_\lambda))$. *If* Σ_3 *is the symmetric group on three letters and* $\sigma \in \Sigma_3$ *then*

$$V_\lambda(M_{\sigma(1)}) \subseteq V_\lambda(M_{\sigma(2)}) \cup V_\lambda(M_{\sigma(3)}).$$

(ii) $V_\lambda(M_1 \oplus M_2) = V_\lambda(M_1) \cup V_\lambda(M_2)$.

The following proposition gives a simplification of the formulas given in Definition 3.3.1 via maximal l-parabolic subcompositions.

Proposition 3.3.4. *Let* $\mu \vDash \lambda$ *be a maximal* l*-parabolic subcomposition, and let* $M \in \operatorname{mod}(\mathcal{H}_q(\Sigma_\lambda))$.

(i) *For every maximal* l*-parabolic subcomposition* $\mu \vDash \lambda$, $W_\lambda(M) = \operatorname{res}^*_{\lambda,\mu}(W_\mu(M))$.

(ii) *The support variety of* M *is*

$$V_\lambda(M) = \bigcup_{\substack{l\text{-parabolic} \\ \mu \vDash \lambda}} \operatorname{res}^*_{\lambda,\mu}(W_\mu(M)).$$

(iii) *For any maximal* l*-parabolic subcomposition* $\mu \vDash \lambda$,

$$V_\lambda(M) = \operatorname{res}^*_{\lambda,\mu}(V_\mu(M)).$$

Proof. (i) Consider the transfer map

$$\operatorname{tr}_{\mu,\lambda} : J_{\mathcal{H}_q(\Sigma_\mu)}(k, M) \to J_{\mathcal{H}_q(\Sigma_\lambda)}(k, M)$$

and the restriction map

$$\mathrm{res}_{\lambda,\mu} : J_{\mathcal{H}_q(\Sigma_\lambda)}(k, M) \to J_{\mathcal{H}_q(\Sigma_\mu)}(k, M)$$

According to Proposition 2.4.1(i) and Theorem 2.5.2, $\mathrm{tr}_{\mu,\lambda} \circ \mathrm{res}_{\lambda,\mu} = a\ \mathrm{id}$ as an endomorphism of $J_{\mathcal{H}_q(\Sigma_\lambda)}(k, M)$ for some unit $a \in \mathcal{H}_q(\Sigma_\lambda)$. One has $\mathrm{res}^*_{\lambda,\mu} \circ \mathrm{tr}^*_{\mu,\lambda} = \mathrm{id}$ as an endomorphism of $W_\mu(M)$, from which the result follows.

(ii) For each $\mu \vDash \lambda$, let μ' be a maximal l-parabolic subcomposition of μ. Therefore, by (i),

$$\begin{aligned} V_\lambda(M) &= \bigcup_{\mu \vDash \lambda} \mathrm{res}^*_{\lambda,\mu}(W_\mu(M)) \\ &= \bigcup_{\mu \vDash \lambda} \mathrm{res}^*_{\lambda,\mu} \mathrm{res}^*_{\mu,\mu'}(W_{\mu'}(M)) \\ &= \bigcup_{\substack{l\text{-parabolic} \\ \mu' \vDash \lambda}} \mathrm{res}^*_{\lambda,\mu'}(W_{\mu'}(M)). \end{aligned}$$

(iii) The result follows from (ii) and the fact that every l-parabolic subcomposition of λ is contained in a given maximal l-parabolic subcomposition up to conjugacy. □

3.4 Varieties and induction

The following proposition states how relative support behave under induction.

Proposition 3.4.1. *Let ν, μ, λ be three compositions such that $\mu \vDash \lambda$ and $M \in \mathrm{mod}(\mathcal{H}_q(\Sigma_\mu))$.*

(i) $W_\lambda(\mathrm{ind}_\mu^\lambda M) = \mathrm{res}^*_{\lambda,\mu}(W_\mu(M))$.

(ii) $W_\nu(\mathrm{ind}_\mu^\lambda M) = \bigcup_{\alpha \vDash \mu} \bigcup_{w \in w_{\alpha,\mu,\nu}} T_w^\# \mathrm{res}^*_{\nu^w,\alpha}(W_\alpha(M))$, *where*

$$w_{\alpha,\mu,\nu} := \{w \in \Sigma_\mu \backslash \Sigma_\lambda / \Sigma_\nu : \alpha = \nu^w \cap \mu\}.$$

(iii) $V_\lambda(\operatorname{ind}_\mu^\lambda M) = \operatorname{res}^*_{\lambda,\mu}(V_\mu(M))$.

Proof. (i) This follows by applying Frobenius reciprocity:

$$\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_\lambda)}(k, \operatorname{ind}_\mu^\lambda M) \cong \operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_\mu)}(k, M).$$

(ii) The result follows from the following calculation.

$$\begin{aligned}
W_\nu(\operatorname{ind}_\mu^\lambda M) &= W_\nu\left(\bigoplus_{w\in\Sigma_\mu\backslash\Sigma_\lambda/\Sigma_\nu} \operatorname{ind}^\nu_{\mu^w\cap\nu}\operatorname{con}^w_\mu M\right) && \text{Theorem 2.2.1}\\
&= \bigcup_{w\in\Sigma_\mu\backslash\Sigma_\lambda/\Sigma_\nu} W_\nu(\operatorname{ind}^\nu_{\mu^w\cap\nu}\operatorname{con}^w_\mu M) && \text{Proposition 3.3.3(ii)}\\
&= \bigcup_{w\in\Sigma_\mu\backslash\Sigma_\lambda/\Sigma_\nu} \operatorname{res}^*_{\nu,\mu^w\cap\nu}(W_{\mu^w\cap\nu}(\operatorname{con}^w_\mu M)) && \text{(i)}\\
&= \bigcup_{w\in\Sigma_\mu\backslash\Sigma_\lambda/\Sigma_\nu} T_w^\# \operatorname{res}^*_{\nu^w,\nu^w\cap\mu}(W_{\nu^w\cap\mu}(M))\\
&= \bigcup_{\alpha\models\mu}\ \bigcup_{w\in w_{\alpha,\mu,\nu}} T_w^\# \operatorname{res}^*_{\nu^w,\alpha}(W_\alpha(M)).
\end{aligned}$$

(iii) We proceed with the following calculation.

$$\begin{aligned}
V_\lambda(\operatorname{ind}_\mu^\lambda M) &= \bigcup_{\nu\vDash\lambda} \operatorname{res}^*_{\lambda,\nu}(W_\nu(\operatorname{ind}_\mu^\lambda M)) && \text{Definition 3.3.1}\\
&= \bigcup_{\nu\vDash\lambda} \operatorname{res}^*_{\lambda,\nu} \bigcup_{\alpha\vDash\mu} \bigcup_{w\in w_{\alpha,\mu,\nu}} T_w^\# \operatorname{res}^*_{\nu^w,\alpha}(W_\alpha(M)) && \text{(ii)}\\
&= \bigcup_{\nu\vDash\lambda} \bigcup_{\alpha\vDash\mu} \bigcup_{w\in w_{\alpha,\mu,\nu}} \operatorname{res}^*_{\lambda,\nu} T_w^\# \operatorname{res}^*_{\nu^w,\alpha}(W_\alpha(M))\\
&= \bigcup_{\nu\vDash\lambda} \bigcup_{\alpha\vDash\mu} \bigcup_{w\in w_{\alpha,\mu,\nu}} T_w^\# \operatorname{res}^*_{\lambda^w,\nu^w} \operatorname{res}^*_{\nu^w,\alpha}(W_\alpha(M))\\
&= \bigcup_{\nu\vDash\lambda} \bigcup_{\alpha\vDash\mu} \bigcup_{w\in w_{\alpha,\mu,\nu}} T_w^\# \operatorname{res}^*_{\lambda,\alpha}(W_\alpha(M)) && w\in\Sigma_\lambda\\
&= \bigcup_{\nu\vDash\lambda} \bigcup_{\alpha\vDash\mu} \bigcup_{w\in w_{\alpha,\mu,\nu}} \operatorname{res}^*_{\lambda,\alpha}(W_\alpha(M)) && T_w^\# \in \mathcal{H}_q(\Sigma_\lambda)\\
&= \bigcup_{\alpha\vDash\mu} \bigcup_{\substack{\nu\vDash\lambda\\ w\in w_{\alpha,\mu,\nu}}} \operatorname{res}^*_{\lambda,\alpha}(W_\alpha(M))\\
&= \bigcup_{\alpha\vDash\mu} \operatorname{res}^*_{\lambda,\alpha}(W_\alpha(M)) && \text{for } \nu=\alpha,\ 1\in w_{\alpha,\mu,\nu}\\
&= \operatorname{res}^*_{\lambda,\mu}(V_\mu(M)). && \text{Definition 3.3.1}
\end{aligned}$$

We end this section with a result that will useful for computing support varieties in the case when one has some information about the vertex of a module. In particular this will be applied in case of Young vertices.

Proposition 3.4.2. *Let $\mu \vDash \lambda$. Suppose that M is an $\mathcal{H}_q(\Sigma_\lambda)$-module and N is an $\mathcal{H}_q(\Sigma_\mu)$-module such that $M | \operatorname{ind}_\mu^\lambda N$ and $N | \operatorname{res}_\mu^\lambda M$. Then $V_\lambda(M) = \operatorname{res}^*_{\lambda,\mu}(V_\mu(N))$.*

Proof. Using Proposition 3.3.3(ii) and Proposition 3.4.1(iii), we obtain

$$V_\lambda(M) \subseteq V_\lambda(\operatorname{ind}_\mu^\lambda N) = \operatorname{res}^*_{\lambda,\mu}(V_\mu(N)).$$

It follows from the Definition 3.3.1 that

$$\mathrm{res}^*_{\lambda,\mu}(V_\mu(N)) \subseteq \mathrm{res}^*_{\lambda,\mu}(V_\mu(M)) \subseteq V_\lambda(M). \qquad \square$$

Chapter 4

Rates of growth

4.1 Complexity of modules

Let $\{d_n\}_{n\geqslant 0}$ be a sequence of nonnegative integers. The *rate of growth* $r(d_\bullet)$ of this sequence is the smallest nonnegative integer c for which there exists a positive real number C such that $d_n \leqslant C \cdot n^{c-1}$ for all $n \geqslant 1$. If no such d exists, set $r(d_\bullet) := \infty$.

Alperin [A, §4] first defined the notion of complexity of modules for finite groups. We can also state this for Iwahori-Hecke algebras. Our goal will be to relate the complexity to the dimension of the support varieties defined in the previous chapter.

Definition 4.1.1. Let $M \in \operatorname{mod}(\mathcal{H}_q(\Sigma_d))$ and let

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

be the minimal projective resolution of M. The *complexity* $c_{\mathcal{H}_q(\Sigma_d)}(M)$ of M is defined as $r(\dim P_\bullet)$.

4.2 Complexity and growth rate of $\mathrm{Ext}^\bullet$

For Iwahori-Hecke algebras, the conventional proofs to relate the dimension of the support variety to (i) the rate of growth of certain extension groups and (ii) the complexity of the module do not work because of the absence of the tensor product (i.e., a comultiplication on $\mathcal{H}_q(\Sigma_d)$).

We first prove that the complexity can still be interpreted as the rate of growth of certain Ext-groups related to taking the direct sum of simple, Specht, Young and permutation modules.

Theorem 4.2.1. *Let $M \in \mathrm{mod}(\mathcal{H}_q(\Sigma_d))$. The following quantities are equal.*

(i) $c_{\mathcal{H}_q(\Sigma_d)}(M)$;

(ii) $r(\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda\in\Lambda^+_{reg}(d)} D^\lambda, M))$;

(iii) $r(\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda\vdash d} S^\lambda, M))$;

(iv) $r(\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda\vdash d} Y^\lambda, M))$;

(v) $r(\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda\vdash d} M^\lambda, M))$.

Proof. (i) = (ii). This follows by using the standard arguments (cf. [Ben, Prop. 5.3.5]).
(iii) $\leqslant$ (ii), (iv) $\leqslant$ (ii). One can apply [Ben, Prop. 5.3.5] to deduce these statements.
(ii) $\leqslant$ (iii). This will be proved by using induction on the dominance order of partitions $\trianglelefteq$. Set $s := r(\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda\vdash d} S^\lambda, M))$. If λ be maximal with respect to $\trianglelefteq$ then $S^\lambda = D^\lambda$. Consequently,

$$r(\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(D^\lambda, M)) \leqslant s.$$

By induction suppose that for every $\mu \triangleright \tau$, we know $r(\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(D^\mu, M)) \leqslant s$. We need to show that $r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(D^\tau, M)) \leqslant s$. There exists a short exact sequence of the form

$$0 \to N \to S^\tau \to D^\tau \to 0 \tag{4.1}$$

with N having composition factors of the form D^μ with $\mu \rhd \tau$. Therefore,

$$r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(D^\tau, M)) \leqslant \max\{r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(S^\tau, M)), r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(N, M))\} \leqslant s.$$

(iii) $\leqslant$ (iv). This statement will be proved in a similar fashion as above. Set

$$y := r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda \vdash d} Y^\lambda, M)).$$

Let λ be maximal with respect to $\trianglelefteq$ so $Y^\lambda = S^\lambda$ and $r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(S^\lambda, M)) \leqslant y$. Suppose that for any $\mu \rhd \tau$, $r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(S^\mu, M)) \leqslant y$. It will suffice to show that $r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(S^\tau, M)) \leqslant y$. There is a short exact sequence of the form

$$0 \to S^\tau \to Y^\tau \to Z \to 0 \tag{4.2}$$

with Z having a Specht filtration with factors of the form S^μ with $\mu \rhd \tau$. Consequently,

$$r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(S^\tau, M)) \leqslant \max\{r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(Y^\tau, M)), r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(N, M))\} \leqslant y.$$

(iv) = (v). The statement follows because every Young module appears as a summand of a permutation module, and the summands of the permutation modules are Young modules. □

4.3 Complexity and support varieties

We now can relate the complexities of modules in $\operatorname{mod}(\mathcal{H}_q(\Sigma_d)))$ to the dimension of their support varieties. Furthermore, every module in $\operatorname{mod}(\mathcal{H}_q(\Sigma_d))$ has complexity less than or equal to the complexity of the trivial module. Note that for Hopf algebras this is an easy consequence of tensoring a minimal projective resolution of the trivial module by the given module M.

Corollary 4.3.1. *Let* $M \in \operatorname{mod}(\mathcal{H}_q(\Sigma_d))$*. Then,*

(i) $c_{\mathcal{H}_q(\Sigma_d)}(M) = \dim V_{\mathcal{H}_q(\Sigma_d)}(M)$,

(ii) $c_{\mathcal{H}_q(\Sigma_d)}(M) \leqslant c_{\mathcal{H}_q(\Sigma_d)}(k)$.

Proof. (i) Since $\operatorname{res}^*_{d,\lambda}$ is a finite map, $\dim \operatorname{res}^*_{(d),\lambda} W_\lambda(M) = \dim W_\lambda(M)$. Next by using the argument given in [Ev, p. 105-106] one has

$$r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_\lambda)}(k, M)) = \dim W_\lambda(M).$$

Then,

$$\begin{aligned}
c_{\mathcal{H}_q(\Sigma_d)}(M) &= r\left(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}\left(\bigoplus_{\lambda \vdash d} M^\lambda, M\right)\right) && \text{Theorem 4.2.1}\\
&= \max_{\lambda\vdash d} r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(M^\lambda, M))\\
&= \max_{\lambda\vdash d} \{r(\operatorname{Ext}^\bullet_{\mathcal{H}_q(\Sigma_\lambda)}(k, M))\} && \text{Frobenius reciprocity}\\
&= \max_{\lambda\vdash d} \{\dim W_\lambda(M)\}\\
&= \max_{\lambda\vdash d} \{\dim \operatorname{res}^*_{(d),\lambda} W_\lambda(M)\}\\
&= \dim \bigcup_{\lambda\vdash d} \operatorname{res}^*_{(d),\lambda} W_\lambda(M) && \text{Proposition 3.3.2}\\
&= \dim V_{\mathcal{H}_q(\Sigma_d)}(M) && \text{Definition 3.3.1.}
\end{aligned}$$

(ii) From part (i),

$$c_{\mathcal{H}_q(\Sigma_d)}(M) = \dim V_{\mathcal{H}_q(\Sigma_d)}(M) \leqslant \dim V_{\mathcal{H}_q(\Sigma_d)}(k) = c_{\mathcal{H}_q(\Sigma_d)}(k). \qquad \square$$

Chapter 5

Permutation modules and Young modules

5.1 Support varieties of permutation modules

In this chapter we will use our established properties on complexity and support varieties, in addition to the theory of Young vertices, to give an formula for the complexities of the permutation modules $\{M^\lambda\}$ and the Young modules $\{Y^\lambda\}$. This is accomplished by first determining their support varieties as images of the map $\text{res}^*_{d,\lambda}$ (resp. $\text{res}^*_{d,\rho(\lambda)}$ for some partition $\rho(\lambda)$) on the support varieties of the trivial module.

Let $\lfloor\ \rfloor$ denote the floor function. Note that the Krull dimension of the cohomology ring $\text{H}^\bullet(\mathcal{H}_q(\Sigma_d)), k)$ or equivalently $\dim V_{\mathcal{H}_q(\Sigma_d)}(k)$ is $\lfloor d/l \rfloor$. We can now determine the complexity and support varieties for the permutation modules M^λ:

Proposition 5.1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_s) \models d$ and M^λ be a permutation module for $\mathcal{H}_q(\Sigma_d)$. Then:*

(i) $V_{\mathcal{H}_q(\Sigma_d)}(M^\lambda) = \text{res}^*_{(d),\lambda}(V_{\mathcal{H}_q(\Sigma_\lambda)}(k))$;

(ii) $c_{\mathcal{H}_q(\Sigma_d)}(M^\lambda) = \sum_{i=1}^{s} \lfloor \lambda_i / l \rfloor$.

Proof. The statement (i) follows immediately from Proposition 3.4.2 since $M^\lambda \cong \operatorname{ind}_\lambda^d k$ and k is a direct summand of $\operatorname{res}_\lambda^d M^\lambda$ by Theorem 2.2.1.

One can deduce (ii) follows from (i) since the map $\operatorname{res}^*_{(d),\lambda}$ preserves dimension (Proposition 3.3.2) and $\dim(V_{\mathcal{H}_q(\Sigma_\lambda)}(k))$ is determined by Theorem 3.1.1. □

5.2 Support varieties of Young modules

Dipper-Du [DD, 5.8 Theorem] determines the vertex of the Young module Y^λ for $\mathcal{H}_q(\Sigma_d)$ as $\mathcal{H}_q(\Sigma_{\rho(\lambda)})$ where $\rho(\lambda)$ is constructed as follows. Notice that any $\lambda \vdash d$ has a unique l-adic expansion of the form:

$$\lambda = \lambda_{[0]} + \lambda_{[1]} l, \tag{5.1}$$

where $\lambda_{[0]}$ is an l-restricted partition of d and $\lambda_{[1]}$ is a partition. Define the partition:

$$\rho(\lambda) := \left(l^{|\lambda_{[1]}|}, 1^{|\lambda_{[0]}|} \right). \tag{5.2}$$

The partition $\lambda_{[0]}$ can be obtained by successively striping horizontal rim l-hooks from λ, and $|\lambda_{[1]}|$ is the number of such hooks removed. The following theorem demonstrates that the complexity of the Young module Y^λ is $|\lambda_{[1]}|$.

Theorem 5.2.1. *Let $\lambda \vdash d$ with Y^λ the corresponding Young module for $\mathcal{H}_q(\Sigma_\lambda)$. Then*

(i) $V_{\mathcal{H}_q(\Sigma_d)}(Y^\lambda) = \operatorname{res}^*_{d,\rho(\lambda)}(V_{\mathcal{H}_q(\Sigma_{\rho(\lambda)})}(k))$.

(ii) $c_{\mathcal{H}_q(\Sigma_d)}(Y^\lambda) = |\lambda_{[1]}|$.

Proof. Part (i) follows from Proposition 3.4.2. In order to prove (ii) take the dimension on both sides of (i) and recall from Proposition 3.3.2 that $\operatorname{res}^*_{(d),\rho(\lambda)}$ preserves dimension, and the dimension of the support variety of the trivial module is also determined in Theorem 3.1.1.

□

As a consequence of the aforementioned theorem, we recover the well-known fact that Y^λ is projective exactly when λ is p-restricted. Furthermore, from Theorem 5.2.1(ii), one can see that for a block $\mathbb{B}$ of weight w, there are Young modules in $\mathbb{B}$ of every possible complexity $\{0, 1, \ldots, w\}$. The following result characterizes Young module of complexity one.

Corollary 5.2.2. *A nonprojective Young module Y^λ has complexity one if and only if λ is of the form $(\mu_1 + l, \mu_2, \ldots, \mu_s)$ where $(\mu_1, \mu_2, \ldots, \mu_s)$ is l-restricted.*

Proof. From Theorem 5.2.1, $\lambda_{[1]} = (1)$ precisely when the l-adic expansion of λ has the form $\lambda_{[0]} + (1)l$. □

In all known cases that the complexity has been computed for permutation, Young, Specht and simple modules, the answers for the symmetric group in characteristic $p > 0$ coincide with the answer for the Iwahori-Hecke algebra in characteristic zero at a pth root of unity.

5.3 Support varieties of blocks

In this section we will apply our prior computation for Young modules to give an explicit description for the location of the support varieties for modules in a block $\mathbb{B}$ of $\mathcal{H}_q(\Sigma_d)$. For a Specht module $\mathcal{H}_q(\Sigma_d)$-module, S_λ, let $\mathbb{B}_\lambda$ be the block of $\mathcal{H}_q(\Sigma_d)$ containing S_λ. We remark that all the composition factors of a given Specht module lie in the same block. Note that by Nakayama rule, $\mathbb{B}_\lambda = \mathbb{B}_\mu$ if and only if λ and μ have the same l-core.

Let $d = c_{[0]} + c_{[1]}l$ be the unique l-adic expansion of d, so $0 \leqslant c_{[0]} < l$, and $d = a_{[0]} + a_{[1]}l$

is another expansion, with $0 \leqslant a_{[0]}$. Then $a_{[0]} \geqslant c_{[0]}$ and $a_{[1]} \leqslant c_{[1]}$ and

$$\mathcal{H}_q(\Sigma_{(l^{a_{[1]}},1^{a_{[0]}})}) \leqslant \mathcal{H}_q(\Sigma_{(l^{c_{[1]}},1^{c_{[0]}})}).$$

Now suppose $\mathbb{B}_\mu$ is a block of $\mathcal{H}_q(\Sigma_d)$ with weight w and l-core $\tilde{\mu} \vdash d - lw$. Let $lw = c_{[1]}l$ and

$$\rho_{\max} = (1^{d-lw}, l^w) \vdash d. \tag{5.3}$$

Let $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots)$. The algebra $\mathcal{H}_q(\rho_{\max})$ is the Young vertex for Y^μ where $\mu = (\tilde{\mu}_1 + lw, \tilde{\mu}_2, \ldots)$. Furthermore, if $\mathbb{B}_\lambda = \mathbb{B}_\mu$, then $\mu \trianglerighteq \lambda$ and the Young vertex of Y^λ is of the form

$$\rho(\lambda) = (1^{a_{[0]}}, l^{a_{[1]}}),$$

where $a_{[0]} \geqslant d - lw$ and $a_{[1]} \leqslant w$. Therefore,

$$\mathcal{H}_q(\Sigma_{\rho(\lambda)}) \leqslant \mathcal{H}_q(\Sigma_{\rho_{\max}}).$$

That is, the Young vertices for the Young modules in a block are all contained in a unique vertex $\mathcal{H}_q(\rho_{\max})$, which is the vertex for the Young module $Y^{\tilde{\mu}+(lw)}$.

Define the support of the block to be $V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda) := V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu \in \mathbb{B}_\lambda} D^\mu)$ We now give a precise location for the support variety for a block of the Iwahori-Hecke algebra $\mathcal{H}_q(\Sigma_d)$.

Theorem 5.3.1. *Let $\mathbb{B}_\lambda$ be a block of $\mathcal{H}_q(\Sigma_d)$ of weight w and let M be a finite-dimensional module in $\mathbb{B}_\lambda$. Let $\rho := \rho_{\max}$ for the block $\mathbb{B}_\lambda$. Then:*

(i) $V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda) = V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu \in \mathbb{B}_\lambda} S^\mu) = V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu \in \mathbb{B}_\lambda} Y^\mu)$;

(ii) $\mathrm{res}_{(d),\rho}(V_{\mathcal{H}_q(\rho)}(k)) = V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda)$;

(iii) $V_{\mathcal{H}_q(\Sigma_d)}(M) \subseteq \mathrm{res}_{(d),\rho}(V_{\mathcal{H}_q(\rho)}(k))$;

(iv) $c_{\mathcal{H}_q(\Sigma_d)}(M) \leqslant w$.

Proof. (i) Since S^μ has a filtration with sections being irreducible modules and Y^μ has a filtration with sections being Specht modules, one has using the definition of support in § 3.3,

$$V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda) \supseteq V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu\in\mathbb{B}_\lambda} S^\mu) \supseteq V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu\in\mathbb{B}_\lambda} Y^\mu).$$

For the other inclusion, one needs to apply the ordering of factors on these filtrations. From Theorem 4.2.1, we have exact sequences of the form

$$0 \to N \to S^\tau \to D^\tau \to 0 \tag{5.4}$$

where the composition factors in N are of the form D^μ with $\mu \rhd \tau$. By induction we can assume that $V_{\mathcal{H}_q(\Sigma_d)}(N) \subseteq V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu\in\mathbb{B}_\lambda} S^\mu)$ and Proposition 3.3.3, it follows that

$$V_{\mathcal{H}_q(\Sigma_d)}(D^\tau) \subseteq V_{\mathcal{H}_q(\Sigma_d)}(S^\tau) \cup V_{\mathcal{H}_q(\Sigma_d)}(N) \subseteq V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu\in\mathbb{B}_\lambda} S^\mu).$$

Therefore, $V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu\in\mathbb{B}_\lambda} S^\mu) = V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda)$. A similar inductive argument using (Eq. (4.2)) can be used to prove that

$$V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu\in\mathbb{B}_\lambda} Y^\mu) = V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda) = V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\mu\in\mathbb{B}_\lambda} S^\mu).$$

(ii) From (i), $V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda) = V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda\in\mathbb{B}_\mu} Y^\lambda)$. Now by analysis prior to the statement of the theorem,

$$V_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda\in\mathbb{B}_\mu} Y^\lambda) = V_{\mathcal{H}_q(\Sigma_d)}(Y^\rho) = \mathrm{res}_{(d),\rho}(V_{\mathcal{H}_q(\rho)}(k)).$$

(iii) This follows because for any M in $\mathbb{B}_\lambda$, $V_{\mathcal{H}_q(\Sigma_d)}(M) \subseteq V_{\mathcal{H}_q(\Sigma_d)}(\mathbb{B}_\lambda)$ by Proposition 3.3.3.
(iv) This follows by considering dimension and applying (ii) and (iii). □

Chapter 6

Specht modules, vertices, and cohomology

In this chapter, we will consider the question of computing vertices for Specht modules. This will entail introducing a graded dimension for Specht modules, in addition to, considering the relative cohomology for Iwahori-Hecke algebras of Young subgroups.

6.1 Weights of partitions

For a partition λ and a natural number l, the *l-weight* of λ, denoted by $\mathrm{wt}_l\lambda$, is the number of l-hooks that we could consecutively remove from the partition λ to reach the *l-core* of λ, denoted by $\mathrm{core}_l\lambda$. For a natural number n, we set the l-weight of n to be the l-weight of the trivial partition (n), so $\mathrm{wt}_l n = \mathrm{wt}_l(n) = \lfloor n/l \rfloor$. For a partition λ, let $|\lambda|$ be the sum of parts in λ. It is clear that $|\lambda| = |\mathrm{core}_l\lambda| + l\mathrm{wt}_l\lambda$. We say that λ has *small l-core* if $|\mathrm{core}_l\lambda| < l$.

Lemma 6.1.1. *Let λ be a partition and l be a natural number. The number of hooks whose lengths are multiple of l is* $\mathrm{wt}_l(\lambda)$*, the l-weight of λ.*

Proof. We will prove the result with the help of l-abacus of partition λ. Suppose that

$\lambda = (\lambda_1, \ldots, \lambda_r)$, and let $b_i := \lambda_i - i + r$. The beads of the l-abacus occupies positions b_i. Hooks of length multiple of l are in bijection with the moves of a bead at b_i to an unoccupied position $b_i - lk$, which is in the same runner with b_i, for some $k \geqslant 1$. The number of such moves is exactly the l-weight of λ. □

6.2 Dimensions of Specht modules

For an integer n, let the t-integer be $[n]_t := \frac{1-t^n}{1-t}$. When $t = 1$, one applies limits to obtain $[n]_1 = n$. We will now define a graded version of the dimension for Specht modules (also referred to as the *graded dimension*) that involves the divisibility of cyclotomic polynomials. For a partition λ, let

$$\dim_t S^\lambda := \frac{\prod_{i=1}^{|\lambda|}[i]_t}{\prod_{i\in I}[h_i]_t},$$

where I is the set of all hooks of λ and h_i is the hook length of the hook i. By hook length formula, we have $\dim_1 S^\lambda = \dim S^\lambda$. The graded dimension of the partition λ is the generic degree of the partition λ up to a power of t [Car, §13.5], and the graded dimension is a polynomial with nonnegative integer coefficients [Mac, §III.6].

Theorem 6.2.1. *Let $\Phi_l(t)$ be the l-th cyclotomic polynomial in t. Then,*

$$\dim_t S^\lambda = \prod_l \Phi_l(t)^{\mathrm{wt}_l|\lambda| - \mathrm{wt}_l\lambda} = \prod_l \Phi_l(t)^{\mathrm{wt}_l|\mathrm{core}_l\lambda|},$$

where l runs over all natural number. In particular,

$$\dim S^\lambda = \prod_{p,r} p^{\mathrm{wt}_{p^r}|\lambda| - \mathrm{wt}_{p^r}\lambda} = \prod_{p,r} p^{\mathrm{wt}_{p^r}|\mathrm{core}_{p^r}\lambda|},$$

where p runs over all primes and r runs over all natural number.

Proof. Let l be an arbitrary natural number. When $l = 1$, there are no factors $\Phi_1(t)$ in

$\dim_t S^\lambda$, and $\mathrm{wt}_1|\lambda| - \mathrm{wt}_1\lambda = 0$. Now assume that $l \geqslant 2$. Applying Lemma 6.1.1 to the trivial partition $(|\lambda|)$, the number of times $\Phi_l(t)$ dividing the numerator of $\dim_t S^\lambda$ is $\mathrm{wt}_l(|\lambda|) = \mathrm{wt}_l|\lambda|$. Similarly, applying Lemma 6.1.1 to partition λ, the number of times $\Phi_l(t)$ dividing the denominator $\dim_t S^\lambda$ is $\mathrm{wt}_l\lambda$. Therefore, $\Phi_l(t)$ divides $\dim_t S^\lambda$ exactly $\mathrm{wt}_l|\lambda| - \mathrm{wt}_l\lambda$ many times.

When one specializes to $t = 1$, the result follows from the fact that $\Phi_{p^r}(1) = p$ when p is a prime, and $\Phi_n(1) = 1$ when n is not a prime power. □

6.3 Relative cohomology

In this subsection, we follow the constructions in [Ho] and provide a discussion of relative cohomology for Iwahori-Hecke algebra. Let M be a $\mathcal{H}_q(\Sigma_d)$-module, and let $\lambda \vDash d$ be a composition. A *relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective resolution of M* is a resolution of M consisting of relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective $\mathcal{H}_q(\Sigma_d)$-modules and that splits as resolution of $\mathcal{H}_q(\Sigma_\lambda)$-modules. Among all such resolutions, there exists a *minimal resolution*, that is one where there kernels contain no relatively projective summands. The growth rate of the minimal relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective resolution of M is called the complexity of M, denoted by $c_{d;\lambda}(M) := c_{(\mathcal{H}_q(\Sigma_d),\mathcal{H}_q(\Sigma_\lambda))}(M)$.

All relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective resolutions are homotopic to each other, and the relative Ext between two $\mathcal{H}_q(\Sigma_d)$-modules M and N is defined as

$$\mathrm{Ext}^\bullet_{(\mathcal{H}_q(\Sigma_d),\mathcal{H}_q(\Sigma_\lambda))}(M,N) := \mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{H}_q(\Sigma_d)}(P^\bullet_\lambda, N)),$$

where $P^\bullet_\lambda$ is any relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective resolution of M.

Using the same argument as in the proof of self-injectivity of group algebras, one can show that relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective modules are also relatively $\mathcal{H}_q(\lambda)$-injective modules.

Therefore, all relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective resolutions with finite length must have length 0. In particular, $c_{d;\lambda}(M) = 0$ if and only if M is relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective. As in the ordinary cohomology case, we showed in Theorem 4.2.1 that we can test projectivity of a module M by calculating $\mathrm{Ext}^\bullet_{\mathcal{H}_q(\Sigma_d)}(D, M)$ for all simple modules D. The same result holds for relative cohomology as well. More precisely, a $\mathcal{H}_q(\Sigma_d)$-module M is relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective if and only if $\mathrm{Ext}^n_{(\mathcal{H}_q(\Sigma_d),\mathcal{H}_q(\Sigma_\lambda))}(D, M) = 0$ for all simple $\mathcal{H}_q(\Sigma_d)$-module D and $n \geqslant 1$.

The fact above gives us the following lemma.

Lemma 6.3.1. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of $\mathcal{H}_q(\Sigma_d)$-modules. If any two of M_1, M_2 and M_3 are relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective, then so is the third.*

Proof. Let M_i and M_j be the two modules that are relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective, and let M_k be the third module. The relative complexities of M_i and M_j are zero, so for positive integer n and simple $\mathcal{H}_q(\Sigma_\lambda)$-module D, $\mathrm{Ext}^n_{(\mathcal{H}_q(\Sigma_d),\mathcal{H}_q(\Sigma_\lambda))}(D, M_i) = \mathrm{Ext}^n_{(\mathcal{H}_q(\Sigma_d),\mathcal{H}_q(\Sigma_\lambda))}(D, M_j) = 0$. Using the long exact sequence of cohomologies, we get $\mathrm{Ext}^n_{(\mathcal{H}_q(\Sigma_d),\mathcal{H}_q(\Sigma_\lambda))}(D, M_k) = 0$. Therefore, the relative complexity of M_k is zero, and M_k is relatively $\mathcal{H}_q(\Sigma_\lambda)$-projective. □

An interesting problem would be to determine whether a suitable support variety theory can be established for the relative cohomology $(\mathcal{H}_q(\Sigma_d), \mathcal{H}_q(\Sigma_\lambda))$.

6.4 Vertices for some Specht modules

We begin this section by discussing blocks and relatively projectivity.

Theorem 6.4.1. *Let $\mathbb{B}_\lambda$ be the block of $\mathcal{H}_q(\Sigma_d)$ indexed by a partition $\lambda \vdash d$. Every module M in $\mathbb{B}_\lambda$ is relatively $\mathcal{H}_q(\Sigma_\rho)$-projective for $\rho = (l^{\mathrm{wt}_l \lambda}, 1^{|\mathrm{core}_l \lambda|})$*

Proof. According to Theorem 5.2.1, every Young module in the block $\mathbb{B}^\lambda$ is relatively $\mathcal{H}_q(\Sigma_\rho)$-projective. Young modules have a Specht filtration. By an induction similar to Theorem 4.2.1

and Lemma 6.3.1, all Specht modules in $\mathbb{B}_\lambda$ are $\mathcal{H}_q(\Sigma_\rho)$-projective. Since Specht modules in $\mathbb{B}_\lambda$ admits filtrations by simple modules in $\mathbb{B}_\lambda$, by an inductive argument similar to Theorem 4.2.1 and Lemma 6.3.1, all simple modules in $\mathbb{B}_\lambda$ are relatively $\mathcal{H}_q(\Sigma_\rho)$-projective. Therefore, by Lemma 6.3.1, all modules in $\mathbb{B}_\lambda$ is relatively $\mathcal{H}_q(\Sigma_\rho)$-projective. □

By using the previous result on relative projectivity and information about the graded dimension one can obtain information about the vertex for Specht modules.

Theorem 6.4.2. *Let λ be a partition, and $\rho_a := (l^a, 1^{|\lambda|-al})$. Assume that l is prime. Then the vertex of S^λ is ρ_a for some a that satisfies*

$$\mathrm{wt}_l\lambda - \sum_{r\geqslant 2} \mathrm{wt}_{l^r}|\mathrm{core}_{l^r}\lambda| \leqslant a \leqslant \mathrm{wt}_l\lambda.$$

In particular, if λ has small l^r-core for $r \geqslant 2$ then $a = \mathrm{wt}_l\lambda$.

Proof. It is shown in [DD, Section 1.8] that the vertex of an arbitrary module, particularly S^λ, is of form ρ_a for some natural number a.

Let $\bar{\rho}_a := (1^{la}, l^{\mathrm{wt}_l|\lambda|-a}, 1^{\mathrm{core}_l|\lambda|})$. Since S^λ is $\mathcal{H}_q(\Sigma_{\rho_a})$-projective,

$$\mathrm{res}^{|\lambda|}_{\bar{\rho}_a} S^\lambda \mid \mathrm{res}^{|\lambda|}_{\bar{\rho}_a} \mathrm{ind}^{|\lambda|}_{\rho_a} S^\lambda .$$

The right hand side of the equation above is a free $\mathcal{H}_q(\Sigma_{\bar{\rho}_a})$-module, so the left hand side $\mathrm{res}^{|\lambda|}_{\bar{\rho}_a} S^\lambda$ is a projective $\mathcal{H}_q(\Sigma_{\bar{\rho}_a})$-module, and has dimension divisible by $l^{\mathrm{wt}_l|\lambda|-a}$. Note that one can verify that the projective modules in $\mathcal{H}_q(\Sigma_l)$ have dimension divisible by l by using their realization as Young modules.

So, according to Theorem 6.2.1,

$$\mathrm{wt}_l|\lambda| - a \leqslant \sum_{r\geqslant 1} \mathrm{wt}_{l^r}|\mathrm{core}_{l^r}\lambda|.$$

Therefore,

$$a \geqslant \mathrm{wt}_l|\lambda| - \sum_{r\geqslant 1} \mathrm{wt}_{l^r}|\mathrm{core}_{l^r}\lambda| = \mathrm{wt}_l\lambda - \sum_{r\geqslant 2} \mathrm{wt}_{l^r}|\mathrm{core}_{l^r}\lambda|.$$

Theorem 6.4.1 insures that the module S^λ, which is in the block $\mathbb{B}_\lambda$, is relatively $\mathcal{H}_q(\rho_{\mathrm{wt}_l\lambda})$-projective. If ρ_a is the vertex of S^λ, then $\rho_a \vDash \rho_{\mathrm{wt}_l}$, which implies that $a \leqslant \mathrm{wt}_l\lambda$. □

As a consequence of Theorem 6.4.2, one can compute the vertices for Specht modules for partitions whose some of the parts is less than l^2.

Corollary 6.4.3. *Let λ be a partition. Assume that l is prime. If $|\lambda| < l^2$, then the vertex of S^λ is $(l^{\mathrm{wt}_l\lambda}, 1^{|\mathrm{core}_l\lambda|})$.*

Proof. For every $r \geqslant 2$, $|\mathrm{core}_{l^r}\lambda| \leqslant |\lambda| < l^r$, hence λ has small l^r-core. The result follows from Theorem 6.4.2. □

Remark 6.4.4. For the group algebra of symmetric groups $k\Sigma_d$, [Lim] calculated the vertex and the support variety of S^λ for many partitions, in particular, when $|\lambda| < p^2$, where p is the characteristic of k, This can be used in conjunction with the realization of the cohomological support varieties as rank varieties to compute the complexity of Specht modules for symmetric groups. For Iwahori-Hecke algebras, the question of realization of the support varieties as rank varieties remains open, as well as the computation of support varieties for S^λ when $|\lambda| < l^2$.

Chapter 7

Cohomology and support variety of classical type

7.1 Morita equivalences

We will apply our results for Iwahori-Hecke algebra for other classical groups. Our discussion will follow the one given in [BEM, Section 6]. Recall that $\mathcal{H}_q^{\mathrm{A}}(d)$, $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$ and $\mathcal{H}_q^{\mathrm{D}}(d)$ denotes Iwahori-Hecke algebras of type A_d, B_d and D_d, respectively. Consider the following polynomials

$$f_d^{\mathrm{B}}(Q,q) := \prod_{i=1-d}^{d-1}(Q+q^i) \tag{7.1}$$

and

$$f_d^{\mathrm{D}}(q) := 2\prod_{i=1}^{d-1}(1+q^i). \tag{7.2}$$

We summarize the various Morita equivalence theorems for $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$ and $\mathcal{H}_q^{\mathrm{D}}(d)$ (cf. [DJ4, (4.17)], [P, (3.6) (3.7)]).

Theorem 7.1.1. (i) *If $f_d^{\mathrm{B}}(Q,q)$ is invertible in k, then*

$$\mathcal{H}^{\mathrm{B}}_{Q,q}(d) \sim_{\mathrm{Morita}} \bigoplus_{i=0}^{d} \mathcal{H}^{\mathrm{A}}_q(i-1) \otimes \mathcal{H}^{\mathrm{A}}_q(d-i-1).$$

(ii) *If $f_d^{\mathrm{D}}(q)$ is invertible in k and d is odd, then*

$$\mathcal{H}^{\mathrm{D}}_{q}(d) \sim_{\mathrm{Morita}} \bigoplus_{i=0}^{(d-1)/2} \mathcal{H}^{\mathrm{A}}_q(i-1) \otimes \mathcal{H}^{\mathrm{A}}_q(d-i-1).$$

(iii) *If $f_q^{\mathrm{D}}(d)$ is invertible in k and d is even, then*

$$\mathcal{H}^{\mathrm{D}}_{q}(d) \sim_{\mathrm{Morita}} \mathcal{A}(d/2) \oplus \bigoplus_{i=0}^{d/2-1} \mathcal{H}^{\mathrm{A}}_q(i-1) \otimes \mathcal{H}^{\mathrm{A}}_q(d-i-1).$$

where $\mathcal{A}(d/2)$ is specified in [Hu1, 2.2, 2.4].

7.2 Support theory for $\mathcal{A}(m)$

Let d be even and set $m = d/2$. The algebra $\mathcal{A}(m)$ as defined in [Hu1] is an example of a $\mathbb{Z}_2$-graded Clifford system (cf. [Hu1, Section 4]). Set $B = \mathcal{A}(m)$ and B_+ be the augmentation ideal of B. Furthermore, let $A = \mathcal{H}_q(\Sigma_{(m,m)})$ be the subalgebra in B corresponding to B_1 (in the Clifford system), and A_+ be its augmentation ideal. Then $B \cdot A_+ = A_+ \cdot B$. Now one can consider the quotient $\overline{B} = B//A \cong \mathbb{C}[\mathbb{Z}_2]$ (the group algebra of the cyclic group of order 2).

From [GK, 5.3 Proposition], one can apply the spectral sequence and the fact that $\overline{B}$ is a semisimple algebra to show that

$$\mathrm{H}^\bullet(B,\mathbb{C}) \cong \mathrm{H}^\bullet(A,\mathbb{C})^{\mathbb{Z}_2}. \tag{7.3}$$

In fact one can show that $\mathrm{H}^\bullet(A,\mathbb{C})$ is an integral extension of $\mathrm{H}^\bullet(B,\mathbb{C})$. If M is a finite-dimensional B-module, we will declare that $V_B(M) := V_A(M)$ which is defined in Definition 3.3.1.

Next we will compare the notion of complexity in $\mathrm{mod}(B)$ versus $\mathrm{mod}(A)$. Since B is a free A-module, any projective B-resolution restricts to a projective A-resolution, thus $c_B(M) \geqslant c_A(M)$. On the other hand, by [Hu1, 4.4 Corollary], all simple B-modules are summands of simple A-modules induced to B. By applying the characterization of complexity given in Theorem 4.2.1(i)(ii) and Frobenius reciprocity, one obtains $c_B(M) = c_A(M)$.

7.3 Support theory for type B and type D

Let $\mathcal{E}_d$ be the algebras and $f_d := f_d^{\mathrm{B}}(Q,q)$ (resp. $f_d^{\mathrm{D}}(q)$) be the polynomials as described in Theorem 7.1.1 under the Morita equivalence with $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ (resp. $\mathcal{H}^{\mathrm{D}}_{q}(d)$). For notational convenience, set

$$\mathcal{H}^{\Phi}_{\mathbf{q}}(d) := \begin{cases} \mathcal{H}^{\mathrm{B}}_{Q,q}(d), & \Phi = B, \\ \mathcal{H}^{\mathrm{D}}_{q}(d), & \Phi = D. \end{cases} \tag{7.4}$$

Let $F : \mathrm{Mod}(\mathcal{H}^{\Phi}_{\mathbf{q}}(d)) \to \mathrm{Mod}(\mathcal{E}_d)$ be functor that provides the equivalence of categories when f_n is invertible. Under the equivalence of categories, one can define support varieties for modules over $\mathcal{H}^{\Phi}_{\mathbf{q}}(d)$ as follows. Let M be a finite-dimensional module for $\mathcal{H}^{\Phi}_{\mathbf{q}}(d)$. Then

$$V_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(M) := V_{\mathcal{E}_d}(F(M)).$$

The support varieties for $\mathcal{E}_d$ can be obtained by taking the support varieties for Iwahori-Hecke algebras of type A. We have the following theorem that extends Corollary 4.3.1.

Theorem 7.3.1. *Let M be a finite-dimensional module for $\mathcal{H}^{\Phi}_{\mathbf{q}}(d)$ with f_d invertible. Then*

(i) $c_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(M) = \dim V_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(M)$.

(ii) $c_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(M) \leqslant c_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(\mathbb{C}) = \lfloor \frac{d}{l} \rfloor$.

Proof. (i) Let $S = \bigoplus_i S_i$ be the direct sum of all simple $\mathcal{H}^{\Phi}_{\mathbf{q}}(d)$-modules. Using the Morita equivalence, $F(S)$ is the direct sum of all simple $\mathcal{E}_d$-modules. Furthermore, by using our results for the Iwahori-Hecke algebra for type A,

$$\begin{aligned} c_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(M) &= r(\operatorname{Ext}^{\bullet}_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(S, M)) \\ &= r(\operatorname{Ext}^{\bullet}_{\mathcal{E}_d}(F(S), F(M))) \\ &= \dim V_{\mathcal{E}_d}(F(M)) \\ &= \dim V_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(M). \end{aligned}$$

(ii) One has that

$$c_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(M) = \dim V_{\mathcal{E}_d}(F(M)) \leqslant \left\lfloor \frac{d}{l} \right\rfloor.$$

Let L be the irreducible $\mathcal{E}_d$-module such that $F(\mathbb{C}) = T$. Under the categorical equivalence, the trivial module $\mathbb{C}$ goes to the simple $\mathcal{E}_d$-module labelled by the partition $((d), \varnothing)$. The statement now follows because

$$c_{\mathcal{H}^{\Phi}_{\mathbf{q}}(d)}(\mathbb{C}) = \dim V_{\mathcal{E}_d}(T) = \left\lfloor \frac{d}{l} \right\rfloor. \qquad \square$$

By using the Morita equivalence one can prove analogs of Theorem 5.3.1 for the blocks of $\mathcal{H}^{\Phi}_{\mathbf{q}}(d)$ and obtain the location of their support varieties for various modules. One can pose an interesting question if one can (i) extend the support variety theory for Iwahori-Hecke algebra of types B_d and D_d to even roots of unity, and (ii) if a theory of support varieties can be developed for Iwahori-Hecke algebras of other Coxeter groups.

Part II

Coordinate constructions of q-Schur algebras

Chapter 8

Quantum coordinate (co)algebras

In Part II, we use a different quadratic relation in the definition of Hecke algebras:

$$(T - q^{-1})(T + q) = 0.$$

The quadratic relations used in Part I and Part II produce isomorphic Iwahori-Hecke algebras.

In this section, we fix d, and let $\Phi \in \{\mathrm{A}_{d-1}, \mathrm{B}_d, \mathrm{D}_d\}$ be the type of algebras we consider. Let

$$W^\Phi := \begin{cases} W^{\mathrm{A}}(d-1), \Phi = \mathrm{A}_{d-1}, \\ W^{\mathrm{B}}(d), \Phi = \mathrm{B}_d, \\ W^{\mathrm{D}}(d), \Phi = \mathrm{D}_d, \end{cases} \qquad \text{and} \qquad \mathcal{H}^\Phi_{\mathbf{q}} := \begin{cases} \mathcal{H}^{\mathrm{A}}_q(d-1), \Phi = \mathrm{A}_{d-1}, \\ \mathcal{H}^{\mathrm{B}}_{Q,q}(d), \Phi = \mathrm{B}_d, \\ \mathcal{H}^{\mathrm{D}}_q(d), \Phi = \mathrm{D}_d\,. \end{cases}$$

8.1 Tensor spaces

Let

$$I(n) := \begin{cases} [-r,r] \cap \mathbb{Z}, & n = 2r+1, \\ [-r,r] \cap \mathbb{Z}\backslash\{0\}, & n = 2r. \end{cases} \tag{8.1}$$

The space of d-tuples $I(n)^d$ admits actions of W^Φ. For $\mathbf{i} = (i_0, \ldots, i_{d-1}) \in I(n)^d$ and a simple reflection $s \in W^\Phi$, let

$$\mathbf{i} \cdot s := \begin{cases} (i_0, \ldots, i_{j-2}, i_j, i_{j-1}, i_{j+1}, \ldots, i_{d-1}), & s = s_j, 1 \leqslant j \leqslant d-1, \\ (-i_0, i_1, \ldots, i_{d-1}), & s = s_0^{\mathrm{B}}, \\ (-i_0, -i_1, i_2, \ldots, i_{d-1}), & s = s_0^{\mathrm{D}}. \end{cases}$$

Let V be the free k-vector space with basis $\{v_i : \ i \in I(n)\}$. The *tensor space* $V^{\otimes d}$ admits a basis $\{v_{\mathbf{i}} : \ \mathbf{i} \in I(n)^d\}$, where $v_{\mathbf{i}} := v_{i_0} \otimes \cdots \otimes v_{i_{d-1}}$. We can extend the actions of W^Φ on $I(n)^d$ to actions of $\mathcal{H}_{\mathbf{q}}^\Phi$ on $V^{\otimes d}$. For $\mathbf{i} = (i_0, \ldots, i_{d-1}) \in I(n)^d$, the type A action by T_j for $1 \leqslant j \leqslant d-1$ is given by

$$v_{\mathbf{i}} \cdot T_j := \begin{cases} v_{\mathbf{i}\cdot s_j}, & i_{j-1} < i_j, \\ q^{-1} v_{\mathbf{i}\cdot s_j}, & i_{j-1} = i_j, \\ v_{\mathbf{i}\cdot s_j} + (q^{-1} - q) v_{\mathbf{i}}, & i_{j-1} > i_j, \end{cases}$$

while the action by T_0^{B} is

$$v_{\mathbf{i}} \cdot T_0^{\mathrm{B}} := \begin{cases} v_{\mathbf{i}\cdot s_0}, & 0 < i_0, \\ Q^{-1} v_{\mathbf{i}\cdot s_0}, & 0 = i_0, \\ v_{\mathbf{i}\cdot s_0} + (Q^{-1} - Q) v_{\mathbf{i}}, & 0 > i_0, \end{cases}$$

and the action by T_0^{D} is

$$v_{\mathbf{i}} \cdot T_0^{\mathrm{D}} := \begin{cases} v_{\mathbf{i}\cdot s_0^{\mathrm{D}}}, & -i_0 < i_1, \\ q^{-1} v_{\mathbf{i}\cdot s_0^{\mathrm{D}}}, & -i_0 = i_1, \\ v_{\mathbf{i}\cdot s_0^{\mathrm{D}}} + (q^{-1} - q) v_{\mathbf{i}}, & -i_0 > i_1. \end{cases}$$

8.2 q-Schur algebras

We generalize the construction of q-Schur algebras in [PW, §3.5]. Consider the free algebra

$$k[M(n)] := k\langle x_{i,j} : \ i, j \in I(n) \rangle.$$

We equip $k[M(n)]$ with a comultiplication

$$\begin{aligned} \Delta : \ k[M(n)] &\to k[M(n)] \otimes k[M(n)], \\ x_{i,j} &\mapsto \sum_{l \in I(n)} x_{il} \otimes x_{lj}, \end{aligned}$$

which makes $k[M(n)]$ a coalgebra.

The tensor space $V^{\otimes d}$ is a right comodule for $k[M(n)]$, and the structure map is given by

$$\begin{aligned} \tau : \ V^{\otimes d} &\to V^{\otimes d} \otimes k[M(n)], \\ v_{\mathbf{i}} &\mapsto \sum_{\mathbf{j} \in I(n)^d} v_{\mathbf{j}} \otimes x_{\mathbf{j},\mathbf{i}}, \end{aligned}$$

where

$$x_{\mathbf{i},\mathbf{j}} := x_{i_0,j_0} \dots x_{i_{d-1},j_{d-1}}.$$

Let $J_{\mathbf{q}}^{\mathrm{A}}$ be the two-sided ideal generated by elements in Table 8.1. Let $J_{\mathbf{q}}^{\mathrm{B}}$ the sum of the two-sided ideal $J_{\mathbf{q}}^{\mathrm{A}}$ and the *right* ideal generated by elements in Table 8.2, and $J_{\mathbf{q}}^{\mathrm{D}}$ the sum of the two-sided ideal $J_{\mathbf{q}}^{\mathrm{A}}$ and the *right* ideal generated by elements in Table 8.3.

	$(j \sim i, m \sim l)$
$x_{li}x_{mj} - q^{-1}x_{mj}x_{li}$	$(=,<)$ or $(<,=)$
$x_{li}x_{mj} - qx_{mj}x_{li}$	$(=,>)$ or $(>,=)$
$x_{li}x_{mj} - x_{mj}x_{li}$	$(<,>)$ or $(>,<)$ or $(=,=)$
$x_{li}x_{mj} - x_{mj}x_{li} - \widehat{q}x_{mi}x_{lj}$	$(<,<)$
$x_{li}x_{mj} - x_{mj}x_{li} + \widehat{q}x_{mi}x_{lj}$	$(>,>)$

Table 8.1: Type A relation for $i, j, l, m \in I(n)$, where $\widehat{q} := q^{-1} - q$.

	$(i \sim -i, l \sim -l) = (i \sim 0, l \sim 0)$
$x_{li} - Q^{-1}x_{-l-i}$	$(=,<)$ or $(<,=)$
$x_{li} - Qx_{-l-i}$	$(=,>)$ or $(>,=)$
$x_{li} - x_{-l-i}$	$(<,>)$ or $(>,<)$ or $(=,=)$
$x_{li} - x_{-l-i} - \widehat{Q}x_{-li}$	$(<,<)$
$x_{li} - x_{-l-i} + \widehat{Q}x_{-li}$	$(>,>)$

Table 8.2: Type B relation for $i, j, l, m \in I(n)$, where $\widehat{Q} := Q^{-1} - Q$.

	$(j \sim -i, m \sim -l)$
$x_{li}x_{mj} - q^{-1}x_{-m-j}x_{-l-i}$	$(=,<)$ or $(<,=)$
$x_{li}x_{mj} - qx_{-m-j}x_{-l-i}$	$(=,>)$ or $(>,=)$
$x_{li}x_{mj} - x_{-m-j}x_{-l-i}$	$(<,>)$ or $(>,<)$ or $(=,=)$
$x_{li}x_{mj} - x_{-m-j}x_{-l-i} - \widehat{q}x_{-mi}x_{-lj}$	$(<,<)$
$x_{li}x_{mj} - x_{-m-j}x_{-l-i} + \widehat{q}x_{-mi}x_{-lj}$	$(>,>)$

Table 8.3: Type D relation for $i, j, l, m \in I(n)$, where $\widehat{q} := q^{-1} - q$.

It is straightforward to verify that $J_{\mathbf{q}}^{\Phi}$ is a coideal of $k[M(n)]$. Let

$$k[M_{\mathbf{q}}^{\Phi}(n)] := k[M(n)]/J_{\mathbf{q}}^{\Phi}.$$

Note that this is only a coalgebra, but not an algebra. The right $k[M(n)]$-comodule $V^{\otimes d}$ is automatically a right $k[M_{\mathbf{q}}^{\Phi}(n)]$-comodule.

The *type* Φ *quantum Schur algebra* $S^{\Phi}_{\mathbf{q}}(n,d)$ is defined to be

$$S^{\Phi}_{\mathbf{q}}(n,d) := k[M^{\Phi}_{\mathbf{q}}(n)]^*_d.$$

Then, $V^{\otimes}_d$ is a left $S^{\Phi}_{\mathbf{q}}(n,d)$-module.

It is routine to check that the left action of $S^{\Phi}_{\mathbf{q}}(n,d)$ and the right action of $\mathcal{H}^{\Phi}_{\mathbf{q}}$ commute. Moreover,

$$S^{\Phi}_{\mathbf{q}}(n,d) \cong \mathrm{End}_{\mathcal{H}^{\Phi}_{\mathbf{q}}}(V^{\otimes d}). \tag{8.2}$$

When $Q = q$, it is known that $S^{\mathrm{B}}_{q,q}(n,d)$ admits a geometric realization (cf. [BKLW]) as well as a Schur duality, which is compatible with the type A duality as follows:

$$\begin{array}{cccccccc} k[M^{\mathrm{A}}_q(n)]^* & \twoheadrightarrow & k[M^{\mathrm{A}}_q(n)]^*_d & \simeq & S^{\mathrm{A}}_q(n,d) & \curvearrowright & & \curvearrowleft & \mathcal{H}^{\mathrm{A}}_q(d-1) \\ & & & & \cup & & V^{\otimes d} & & \cap \\ & & & & S^{\mathrm{B}}_{q,q}(n,d) & \curvearrowright & & \curvearrowleft & \mathcal{H}^{\mathrm{B}}_{q,q}(d) \end{array}$$

Remark 8.2.1. The category of homogeneous right $k[M^{\Phi}_{\mathbf{q}}(n)]$-comodules of degree d is equivalent to the category of left $S^{\Phi}_{\mathbf{q}}(n,d)$-modules.

8.3 A combinatorial realization of $S^{\mathrm{B}}_{Q,q}(n,d)$

It is well-known that the algebra $S^{\mathrm{B}}_{q,q}(n,d)$ admits a geometric realization via isotropic partial flags (cf. [BKLW]). This flag realization of $S^{\mathrm{B}}_{q,q}(n,d)$ admits a combinatorial/Iwahori-Hecke algebraic counterpart that generalizes to a two-parameter upgrade (cf. [LL]), i.e.,

$$S^{\mathrm{B}}_{Q,q}(n,d) = \bigoplus_{\lambda,\mu\in\Lambda^{\mathrm{B}}(n,d)} \mathrm{Hom}_{\mathcal{H}^{\mathrm{B}}_{Q,q}}(x_\mu\, \mathcal{H}^{\mathrm{B}}_{Q,q}, x_\lambda\, \mathcal{H}^{\mathrm{B}}_{Q,q}), \tag{8.3}$$

where

$$\Lambda^{\mathrm{B}}(n,d) = \begin{cases} \big\{\lambda = (\lambda_i)_{i\in I(n)} \in \mathbb{N}^n \;\big|\; \lambda_0 \in 1+2\,\mathbb{Z}, \lambda_{-i} = -\lambda_i, \sum_i \lambda_i = 2d+1\big\} & \text{if } n = 2r+1; \\ \big\{\lambda = (\lambda_i)_{i\in I(n)} \in \mathbb{N}^n \;\big|\; \lambda_{-i} = -\lambda_i, \sum_i \lambda_i = 2d\big\} & \text{if } n = 2r. \end{cases} \tag{8.4}$$

Note that in [LL], the set $\Lambda^{\mathrm{B}}(2r,d)$ is identified as a subset of $\Lambda^{\mathrm{B}}(2r+1,d)$ through the embedding

$$(\lambda_i)_{i\in I(n)} \mapsto (\lambda_{-r},\dots,\lambda_{-1},1,\lambda_1,\dots\lambda_r).$$

For any $\lambda \in \Lambda^{\mathrm{B}}(n,d)$, let W_λ be the parabolic subgroup of W^{B} generated by the set

$$\begin{cases} S - \{s_{\lambda_1}, s_{\lambda_1+\lambda_2},\dots,s_{\lambda_1+\dots+\lambda_{r-1}}\} & \text{if } n = 2r; \\ S - \{s_{\lfloor\frac{\lambda_0}{2}\rfloor}, s_{\lfloor\frac{\lambda_0}{2}\rfloor+\lambda_1},\dots,s_{\lfloor\frac{\lambda_0}{2}\rfloor+\lambda_1+\dots+\lambda_{r-1}}\} & \text{if } n = 2r+1. \end{cases} \tag{8.5}$$

For any subset $X \subset W$, $\lambda,\mu \in \Lambda^{\mathrm{B}}(n,d)$ and a Weyl group element g, set

$$T_X = \sum_{w\in X} T_w, \quad T^g_{\lambda\mu} = T_{(W_\lambda)g(W_\mu)}, \quad x_\lambda = T^1_{\lambda\lambda} = T_{W_\lambda}. \tag{8.6}$$

The right $\mathcal{H}^{\mathrm{B}}_{Q,q}$-linear map below is well-defined:

$$\phi^g_{\lambda\mu} : x_\mu\,\mathcal{H}^{\mathrm{B}}_{Q,q} \to x_\lambda\,\mathcal{H}^{\mathrm{B}}_{Q,q}, \quad x_\mu \mapsto T^g_{\lambda\mu}. \tag{8.7}$$

The maps $\phi^g_{\lambda\mu}$ with $\lambda,\mu \in \Lambda^{\mathrm{B}}(n,d)$, g a minimal length double coset representative for $W_\lambda\backslash W^{\mathrm{B}}/W_\mu$ forms a linear basis for the algebra $S^{\mathrm{B}}_{Q,q}(n,d)$. The multiplication rule for $S^{\mathrm{B}}_{Q,q}(n,d)$ is given in [LL], and it is rather involved in general. Here we only need the following facts:

Lemma 8.3.1. *Let $\lambda,\lambda',\mu,\mu' \in \Lambda^{\mathrm{B}}(n,d)$, and let g,g' be minimal length double coset repre-*

sentatives for $W_\lambda \backslash W^{\mathrm{B}}/W_\mu$*. Then*

(a) $\phi^g_{\lambda\mu}\phi^{g'}_{\lambda'\mu'} = 0$ *unless* $\mu = \lambda'$*;*

(b) $\phi^1_{\lambda\mu}\phi^g_{\mu\mu'} = \phi^g_{\lambda\mu'} = \phi^g_{\lambda\mu}\phi^1_{\mu\mu'}$.

8.4 Dimension of q-Schur algebras

It is well-known [PW] that $S^{\mathrm{A}}_q(n,d)$ have several k-bases indexed by the set

$$\left\{ (a_{ij})_{ij} \in \mathbb{N}^{I(n)^2} \,\middle|\, \sum_{(i,j)\in I(n)^2} a_{i,j} = d \right\},$$

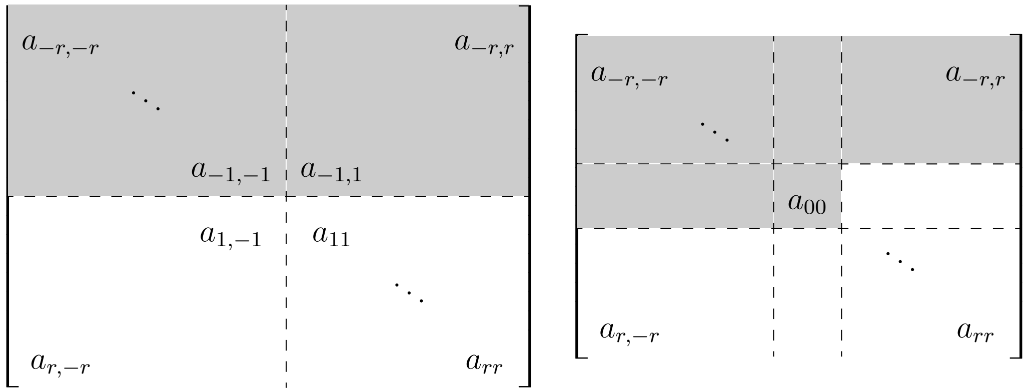
and hence the dimension is given by

$$\dim_k S^{\mathrm{A}}_q(n,d) = \binom{n^2+d-1}{d}. \tag{8.8}$$

In [LL, Lemma 2.2.1] a dimension formula is obtained via several bases of $S^{\mathrm{B}}_{Q,q}(n,d)$ with the following index set:

$$\left\{ (a_{ij})_{ij} \in \mathbb{N}^{I_-} \,\middle|\, \sum_{(i,j)\in I_-} a_{i,} = d \right\}, \quad I_- = \begin{cases} [-r,-1]\times I(n) & \text{if } n = 2r; \\ ([-r,-1]\times I(n)) \cup (\{0\}\times[-r,-1]) & \text{if } n = 2r+1. \end{cases} \tag{8.9}$$

That is, $I_- \subset I(n)^2$ correspond to the shaded region below:



if $n = 2r$ if $n = 2r + 1$

Consequently,

$$\dim_k S^{\mathrm{B}}_{Q,q}(n,d) = \binom{|I_-| + d - 1}{d} = \begin{cases} \binom{2r^2+d-1}{d} & \text{if } n = 2r; \\ \binom{2r^2+2r+d}{d} & \text{if } n = 2r+1. \end{cases} \tag{8.10}$$

In the following we provide a concrete description for the 2-dimensional algebra $S^{\mathrm{B}}_{Q,q}(2,1)$.

Proposition 8.4.1. *The algebra $S^{\mathrm{B}}_{Q,q}(2,1)$ is isomorphic to the type A Iwahori-Hecke algebra $\mathcal{H}_{Q^{-1}}(\Sigma_2)$.*

Proof. The index set here is $I(2) = \{-1, 1\}$. The coalgebra $k[M^{\mathrm{B}}_{Q,q}(2)]_1$ has a k-basis $\{a = x_{-1,-1}, b = x_{-1,1} = x_{1,-1}\}$. Note that $x_{11} = a + (Q - Q^{-1})b$. The comultiplication is given by

$$\begin{aligned} \Delta(a) &= \sum_{k=\pm 1} x_{-1,k} \otimes x_{k,-1} = a \otimes a + b \otimes b, \\ \Delta(b) &= b \otimes a + (a + (Q - Q^{-1})b) \otimes b = b \otimes a + a \otimes b + (Q - Q^{-1}) b \otimes b. \end{aligned} \tag{8.11}$$

Hence, the algebra structure of $S^{\mathrm{B}}_{Q,q}(2,1) = k[M^{\mathrm{B}}_{Q,q}(n)]^*_1$ has a basis $\{a^*, b^*\}$ such that

$$\begin{aligned}
&a^*a^*(a) = (a \otimes a)^*(\Delta(a)) = 1, \quad a^*a^*(b) = (a \otimes a)^*(\Delta(b)) = 0, \\
&a^*b^*(a) = 0 = b^*a^*(a), \quad a^*b^*(b) = 1 = b^*a^*(b), \\
&b^*b^*(a) = 1, \quad b^*b^*(b) = (Q - Q^{-1}),
\end{aligned} \tag{8.12}$$

Therefore, the multiplication structure of $S^{\mathrm{B}}_{Q,q}(2,1)$ is given by

$$a^*a^* = a^*, \quad a^*b^* = b^* = b^*a^*, \quad b^*b^* = (Q - Q^{-1})b^* + a^*. \tag{8.13}$$

□

Remark 8.4.2. We expect that the algebra $S^{\mathrm{B}}_{Q,q}(2,d)$ is isomorphic to $k[t]/\langle P_d(t)\rangle$ for some polynomial $P_d \in k[t]$, for $d \geqslant 1$.

Chapter 9

The isomorphism theorem

The entire section is dedicated to the proof of an isomorphism theorem (Theorem 9.1.1) between the Schur algebras of type B and type A that is inspired by a Morita equivalence theorem due to Dipper and James [DJ4].

9.1 The statement

Recall from We define a polynomial $f_d^{\mathrm{B}} \in k[Q, q]$ by

$$f_d^{\mathrm{B}}(Q, q) = \prod_{i=1-d}^{d-1} (Q^{-2} + q^{2i}). \tag{9.1}$$

We remark that at the specialization $Q = q$, the polynomial $f_d^{\mathrm{B}}(Q, q)$ is invertible if (i) q is generic, (ii) q^2 is an odd root of unity, or (iii) q^2 is a primitive (even) ℓth root of unity for $\ell > d$.

Theorem 9.1.1. *If $f_d^{\mathrm{B}}(Q, q)$ is invertible in the field k, then we have an isomorphism of k-algebras:*

$$\Phi : S^{\mathrm{B}}_{Q,q}(n, d) \to \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2 \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, d-i). \tag{9.2}$$

Example 9.1.2. For $n = 2, d = 1$, Theorem 9.1.1 gives the following isomorphism

$$S^{\mathrm{B}}_{Q,q}(2,1) \cong (S^{\mathrm{A}}_q(1,0) \otimes S^{\mathrm{A}}_q(1,1)) \oplus (S^{\mathrm{A}}_q(1,1) \otimes S^{\mathrm{A}}_q(1,0)) \cong k1_x \oplus k1_y,$$

where $1_x, 1_y$ are identities. We recall basis $\{a^*, b^*\}$ of $S^{\mathrm{B}}_{Q,q}(2,1)$ from Proposition 8.4.1. The following assignments yield the desired isomorphism:

$$a^* \mapsto 1_x + 1_y, \quad b^* \mapsto -Q^{-1}1_x + Q1_y. \tag{9.3}$$

We note that it remains an isomorphism if we replace $-Q^{-1}1_x + Q1_y$ in Eq. (9.3) by $Q1_x - Q^{-1}1_y$.

9.2 Morita equivalences between Iwahori-Hecke algebras

Following [DJ4], we define elements $u_i^{\pm} \in \mathcal{H}^{\mathrm{B}}_{Q,q}(d)$, for $0 \leqslant i \leqslant d$, by

$$u_i^+ = \prod_{\ell=0}^{i-1}(T_\ell \dots T_1 T_0^{\mathrm{B}} T_1 \dots T_\ell + Q), \quad u_i^- = \prod_{\ell=0}^{i-1}(T_\ell \dots T_1 T_0^{\mathrm{B}} T_1 \dots T_\ell - Q^{-1}). \tag{9.4}$$

It is understood that $u_0^+ = 1 = u_0^-$.

For $a, b \in \mathbb{N}$ such that $a + b = d$, we define an element $v_{a,b}$ by

$$v_{a,b} = u_b - T_{w_{a,b}} u_a^+ \in \mathcal{H}^{\mathrm{B}}_{Q,q}(d), \tag{9.5}$$

where $w_{a,b} \in \Sigma_{a+b}$, in two-line notation, is given by

$$w_{a,b} = \begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & a+b \\ b+1 & \cdots & b+a & 1 & \cdots & b \end{pmatrix}.$$

Finally, when $f_d^{\mathrm{B}}(Q,q)$ is invertible, Dipper and James constructed an idempotent

$$e_{a,b} = \tilde{z}_{b,a}^{-1} T_{w_{b,a}} v_{a,b}, \tag{9.6}$$

for $a+b=d$, where $\tilde{z}_{b,a}$ is some invertible element in $\mathcal{H}_q(\Sigma_a \times \Sigma_b)$(see [DJ4, Definition 3.24]). Below we recall some crucial lemmas used in [DJ4].

Lemma 9.2.1. *Let $a, b \in \mathbb{N}$ be such that $a+b=d$. Then:*

(a) *The elements $u_d^{\pm}$ lie in the center of $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$,*

(b) *For $a+b > d$, $u_b^- \mathcal{H}_{Q,q}^{\mathrm{B}}(d) u_a^+ = 0$.*

(c) *For $a+b=d$, $e_{a,b} \mathcal{H}_{Q,q}^{\mathrm{B}}(d) e_{a,b} = e_{a,b} \mathcal{H}_q(\Sigma_a \times \Sigma_b)$ and $e_{a,b}$ commutes with $\mathcal{H}_q(\Sigma_a \times \Sigma_b)$,*

(d) *For $a+d=d$, $e_{a,b} \mathcal{H}_{Q,q}^{\mathrm{B}}(d) = v_{a,b} \mathcal{H}_{Q,q}^{\mathrm{B}}(d)$,*

(e) *There is a Morita equivalence*

$$\mathcal{H}_{Q,q}^{\mathrm{B}}(d) \sim_{Morita} \bigoplus_{i=0}^{d} e_{i,d-i} \mathcal{H}_{Q,q}^{\mathrm{B}}(d) e_{i,d-i}.$$

9.2.1 The actions of u_d^+ and u_d^-

Consider the following decompositions of V into k-subspaces:

$$V = V_{\geqslant 0} \oplus V_{<0} = V_{>0} \oplus V_{\leqslant 0},$$

where

$$V_{>0} = \bigoplus_{1\leqslant i\leqslant r} kv_i, \quad V_{\geqslant 0} = \begin{cases} \bigoplus_{0\leqslant i\leqslant r} kv_i, & \text{if } n = 2r+1 \\ V_{>0} & \text{if } n = 2r, \end{cases} \tag{9.7}$$

$$V_{<0} = \bigoplus_{-r\leqslant i\leqslant -1} kv_i, \quad V_{\leqslant 0} = \begin{cases} \bigoplus_{-r\leqslant i\leqslant 0} kv_i, & \text{if } n = 2r+1 \\ V_{<0} & \text{if } n = 2r. \end{cases} \tag{9.8}$$

Hence, one has the following canonical isomorphisms:

$$S_q^{\mathrm{A}}(\lceil n/2\rceil, d) \simeq \mathrm{End}_{\mathcal{H}_q^{\mathrm{A}}(\Sigma_d)}(V_{\geqslant 0}^{\otimes d}), \quad S_q^{\mathrm{A}}(\lfloor n/2\rfloor, d) \simeq \mathrm{End}_{\mathcal{H}_q^{\mathrm{A}}(\Sigma_d)}(V_{<0}^{\otimes d}). \tag{9.9}$$

In the following, we introduce two new bases $\{w_I^+\}$ and $\{w_I^-\}$ for the tensor space to help us understand the $u_d^{\pm}$-action. First define some intermediate elements, for $0 \leqslant i \leqslant r, j \in \mathbb{N}$:

$$w_{i(j)}^+ = \begin{cases} q^{-j}v_{-i} + Qv_i, & i \neq 0, \\ (q^{-2j}Q^{-1} + Q)v_i, & i = 0, \end{cases} \quad \text{and} \quad w_{i(j)}^- = \begin{cases} q^{-j}v_{-i} - Q^{-1}v_i, & i \neq 0, \\ 0, & i = 0. \end{cases} \tag{9.10}$$

For a nondecreasing tuple $I = (i_1, \ldots, i_d) \in ([0, r] \cap \mathbb{Z})^d$, we further define elements w_I^+ and w_I^- by

$$w_{(i)}^+ = w_{i(0)}^+, \quad w_{(i)}^- = w_{i(0)}^-, \tag{9.11}$$

and then inductively (on d) as below:

$$w_I^+ = w_{(i_1,\ldots,i_{d-1})}^+ \otimes w_{i_d(j)}^+, \quad w_I^- = w_{(i_1,\ldots,i_{d-1})}^- \otimes w_{i_d(j)}^-, \quad \text{where} \quad j = \max\{k : i_{d-k} = i_d\} \tag{9.12}$$

For arbitrary $J \in ([0, r] \cap \mathbb{Z})^d$, there is a shortest element $g \in \Sigma_d$ such that $g^{-1}J$ is nondecreasing and set

$$w_J^+ = w_{g^{-1}J}^+ T_g, \quad w_J^- = w_{g^{-1}J}^- T_g \tag{9.13}$$

Lemma 9.2.2. *For* $I \in ([0, r] \cap \mathbb{Z})^d$,

$$v_I u_d^+ = w_I^+, \quad v_I u_d^- = w_I^-. \tag{9.14}$$

Proof. For non-decreasing I, the result follows from a direct computation. For general I, there exists a shortest element $g \in \Sigma_d$ such that Ig^{-1} is non-decreasing. Then, by Lemma 9.2.1(a),

$$v_I u_d^{\pm} = v_{Ig^{-1}} T_g u_d^{\pm} = v_{Ig^{-1}} u_d^{\pm} T_g = w_{Ig^{-1}}^{\pm} T_g = w_I^{\pm}.$$ □

Example 9.2.3. Let $d = 7$ and let $I = (0, 1, 1, 2, 3, 3, 3)$. We have

$$w_I^+ = w_{0(0)}^+ \otimes w_{1(0)}^+ \otimes w_{1(1)}^+ \otimes w_{2(0)}^+ \otimes w_{3(0)}^+ \otimes w_{3(1)}^+ \otimes w_{3(2)}^+.$$

For $J := (0, 2, 1, 1, 3, 3, 3) = I s_3 s_2$,

$$w_J^+ = w_I^+ T_3 T_2.$$

Example 9.2.4. In the following we verify Lemma 9.2.2 for small d's. Let $d = 2, I = (1, 1)$ and hence $w_I = w_{1(0)}^+ \otimes w_{1(1)}^+$. Since $u_2^+ = (T_1 T_0^{\mathrm{B}} T_1 + Q)(T_0^{\mathrm{B}} + Q)$, we can check that indeed

$$v_I u_2^+ = (v_1 \otimes v_1)(T_1 T_0^{\mathrm{B}} T_1 + Q)(T_0^{\mathrm{B}} + Q) = (v_1 \otimes w_{1(1)}^+)(T_0^{\mathrm{B}} + Q) = w_I^+.$$

Now we define K-vector spaces

$$W_{\geqslant 0}^d = V^{\otimes d} u_d^+, \quad W_{<0}^d = V^{\otimes d} u_d^-.$$

By Lemma 9.2.1(a), u_d^+ and u_d^- are in the center of $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$, hence $W_{\geqslant 0}^d$ and $W_{<0}^d$ are naturally $\mathcal{H}_{Q,q}^{\mathrm{B}} B(d)$-module via right multiplication. Moreover, $wT_0 = Q^{-1} w$ for all $w \in W_{\geqslant 0}^d$ and

$wT_0 = -Qw$ for all $w \in W^d_{<0}$.

Lemma 9.2.5. *We have* $W^d_{\geqslant 0} = V^{\otimes d}_{\geqslant 0} u^+_d$ *and* $W^d_{<0} = V^{\otimes d}_{>0} u^-_d$.

Proof. We only give a proof for the first claim, and a proof for the second claim can be obtained similarly. For $1 \leqslant i \leqslant d$,

$$\begin{aligned} &(V^{\otimes(i-1)}_{\geqslant 0} \otimes V_{<0} \otimes V^{\otimes(d-i)})u^+_d \\ =&(V_{>0} \otimes V^{\otimes(i-1)}_{\geqslant 0} \otimes V^{\otimes(d-i)})T_0T_1T_2 \dots T_{i-1}u^+_d \\ =&(V_{>0} \otimes V^{\otimes(i-1)}_{\geqslant 0} \otimes V^{\otimes(d-i)})Q^{-1}T_1T_2 \dots T_{i-1}u^+_d \qquad \text{Lemma 9.2.1(a) and } T_0u^+_d = Q^{-1}u^+_d \\ \subseteq& V^{\otimes i}_{\geqslant 0} \otimes V^{\otimes(d-i)}u^+_d. \qquad V^{\otimes i}_{\geqslant 0} \text{ is a } \mathcal{H}_q(\Sigma_i)\text{-module} \end{aligned}$$

Next, an induction proves that for $0 \leqslant i \leqslant d$,

$$V^{\otimes i} \otimes V^{\otimes(d-i)} = V^{\otimes i}_{\geqslant 0} \otimes V^{\otimes(d-i)},$$

from which the result follows. □

Lemma 9.2.6. *Let* $p_d : V^{\otimes d} \to V^{\otimes d}_{\leqslant 0}$ *be the projection map. For* $I \in ([0,r] \cap \mathbb{Z})^d$ *and* $J \in ([1,r] \cap \mathbb{Z})^d$, $p_d(w^+_I) = c_I v_{-I}$ *and* $p_d(w^-_J) = c_J v_{-J}$ *for some invertible elements* $c_I, c_J \in K^\times$.

Proof. When I, J are non-decreasing, and when $d = 2$, the result follows from a direct computation. For general I (or J), there exists a shortest element $g \in \Sigma_d$ such that Ig^{-1} (or Jg^{-1}) is non-decreasing. The result follows from an induction on the length of g. □

Lemma 9.2.7. (i) *The map* $v_I \mapsto w^+_I$ *gives an isomorphism of* $\mathcal{H}_q(\Sigma_d)$*-modules* $V^{\otimes d}_{\geqslant 0} \to W^d_{\geqslant 0}$.

(ii) *The map* $v_I \mapsto w^-_I$ *gives an isomorphism of* $\mathcal{H}_q(\Sigma_d)$*-modules* $V^{\otimes d}_{<0} \to W^d_{<0}$.

Proof. Since u_d^+ (resp. u_d^-) is in the center of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ by Lemma 9.2.1(a), the map $v_I \mapsto w_I^+$ (resp. $v_I \mapsto w_I^-$) is clearly $\mathcal{H}_q(\Sigma_d)$-equivariant. Surjectivity of this map follows from Lemma 9.2.5, and injectivity of this map follows from Lemma 9.2.6. □

9.2.2 The actions of $v_{a,b}$

Lemma 9.2.8. *For $a+b=d$, $V^{\otimes d}v_{a,b} = (V_{>0}^{\otimes b} \otimes V_{\geqslant 0}^{\otimes a})v_{a,b}$.*

Proof. It follows from Eq. (9.5) and Lemma 9.2.5 that

$$V^{\otimes d}v_{a,b} = (V^{\otimes b} \otimes V^{\otimes a})u_b^- T_{w_{a,b}} u_a^+ = (V_{>0}^{\otimes b} \otimes V^{\otimes a})u_b^- T_{w_{a,b}} u_a^+ = (V_{>0}^{\otimes b} \otimes V^{\otimes a})v_{a,b}.$$

For $b < i \leqslant d$,

$$
\begin{aligned}
& T_0T_1T_2\dots T_{i-1}v_{a,b} \\
=& T_1^{-1}\dots T_b^{-1}(T_b\dots T_1T_0T_1T_2\dots T_b)(T_{b+1}\dots T_{i-1})u_b^- T_{w_{a,b}}u_a^+ && \text{Eq. (9.5)} \\
=& T_1^{-1}\dots T_b^{-1}(T_b\dots T_1T_0T_1T_2\dots T_b)u_b^-(T_{b+1}\dots T_{i-1})T_{w_{a,b}}u_a^+ && u_b^- \text{ commutes with } T_{b+1},\dots \\
=& T_1^{-1}\dots T_b^{-1}(u_{b+1}^- + Q^{-1}u_b^-)(T_{b+1}\dots T_{i-1})T_{w_{a,b}}u_a^+ && \text{Eq. (9.4)} \\
=& Q^{-1}T_1^{-1}\dots T_b^{-1}u_b^-(T_{b+1}\dots T_{i-1})T_{w_{a,b}}u_a^+ && \text{Lemma 9.2.1} \\
=& Q^{-1}T_1^{-1}\dots T_b^{-1}(T_{b+1}\dots T_{i-1})u_b^- T_{w_{a,b}}u_a^+ && u_b^- \text{ commutes with } T_{b+1},\dots \\
=& Q^{-1}T_1^{-1}\dots T_b^{-1}(T_{b+1}\dots T_{i-1})v_{a,b}. && \text{Eq. (9.5)}
\end{aligned}
$$

Then, for $b < i \leqslant d$,

$$
\begin{aligned}
&(V_{>0}^{b} \otimes V_{\geqslant 0}^{\otimes(i-b-1)} \otimes V_{<0} \otimes V^{\otimes(d-i)})v_{a,b} \\
=&(V_{>0} \otimes V_{>0}^{\otimes b} \otimes V_{\geqslant 0}^{\otimes(i-b-1)} \otimes V^{\otimes(d-i)})T_0T_1T_2 \dots T_{i-1}v_{a,b} \\
=&Q^{-1}(V_{>0} \otimes V_{>0}^{\otimes b} \otimes V_{\geqslant 0}^{\otimes(i-b-1)} \otimes V^{\otimes(d-i)})T_1^{-1} \dots T_b^{-1}(T_{b+1} \dots T_{i-1})v_{a,b} \\
\subseteq&(V_{>0}^{\otimes b} \otimes V_{>0} \otimes V_{\geqslant 0}^{\otimes(i-b-1)} \otimes V^{\otimes(d-i)})(T_{b+1} \dots T_{i-1})v_{a,b} && V_{>0}^{\otimes(b+1)} \text{ is a } \mathcal{H}_q(\Sigma_{b+1})\text{-module} \\
\subseteq&(V_{>0}^{\otimes b} \otimes V_{\geqslant 0}^{\otimes(i-b)} \otimes V^{\otimes(d-i)})v_{a,b}. && V_{\geqslant 0}^{\otimes(i-b)} \text{ is a } \mathcal{H}_q(\Sigma_{i-b})\text{-module}
\end{aligned}
$$

An induction shows that for $b \leqslant i \leqslant d$,

$$V^{\otimes b} \otimes V^{\otimes(b-i)} \otimes V^{\otimes(d-i)}v_{a,b} = V_{>0}^{\otimes b} \otimes V_{\geqslant 0}^{\otimes(i-b)} \otimes V^{\otimes(d-i)}v_{a,b},$$

from which the result follows. □

For $a + b = d$, consider the projections

$$p_{a,b} : V^{\otimes d} \to V_{\leqslant 0}^{\otimes a} \otimes V_{<0}^{\otimes b},$$

Lemma 9.2.9. *For $a + b = d$, let*

$$p'_{a,b} : V^{\otimes d} \to V^{\otimes a} \otimes V_{<0}^{\otimes b}$$

be the projection map. Then, for $I \in ([0, r] \cap \mathbb{Z})^a$ and $J \in ([-r, r] \cap \mathbb{Z})^b$,

$$p'_{a,b}((v_J \otimes v_I)T_{w_{a,b}}) = c_{I,J}v_I \otimes p_b(v_J)$$

for some invertible $c_{I,J} \in K^\times$, where p_b is defined in Lemma 9.2.6. Moreover,

$$p'_{a,b}((w_J^- \otimes v_I)T_{w_{a,b}}) = c_{I,J}c_Jv_I \otimes v_{-J}$$

for some invertible $c_{I,J}, c_J \in K^\times$.

Proof. First note that $(v_J \otimes v_I)T_{w_{a,b}} = c_{I,J}(v_J \otimes v_I)w_{a,b} + \sum_{g<w_{a,b}} c_g(v_J \otimes v_I)g$ for some invertible $c_{I,J} \in K^\times$ and some $c_g \in K$, where $g < w_{a,b}$ under the Bruhat order. Hence,

$$\begin{aligned} p'_{a,b}((v_J \otimes v_I)T_{w_{a,b}}) &= p'_{a,b}(c_{I,J}(v_J \otimes v_I)w_{a,b} + \sum_{g < w_{a,b}} c_g(v_J \otimes v_I)g) \\ &= c_{I,J}p'_{a,b}(v_I \otimes v_J) + \sum_{g < w_{a,b}} c_g p'_{a,b}((v_J \otimes v_I)g) \\ &= c_{I,J}p'_{a,b}(v_I \otimes v_J) = c_{I,J}v_I \otimes p_b(v_J). \end{aligned}$$

By Lemma 9.2.6, $p_b(w_J^-) = c_J v_{-J}$ for some $c_J \in K^\times$. Therefore, $p'_{a,b}((w_J^- \otimes v_I)T_{w_{a,b}}) = c_{I,J}v_I \otimes p_b(w_J^-) = c_{I,J}c_J v_I \otimes v_{-J}$. □

Lemma 9.2.10. *For* $I \in ([0,r] \cap \mathbb{Z})^a$ *and* $J \in ([1,r] \cap \mathbb{Z})^b$, $p_{a,b}((v_J \otimes v_I)v_{a,b}) = cv_{-I} \otimes v_{-J}$ *for some* $c \in K^\times$.

Proof.

$$\begin{aligned} p_{a,b}((v_J \otimes v_I)v_{a,b}) &= p_{a,b}((v_J \otimes v_I)u_b^- T_{w_{a,b}} u_a^+) && \text{Eq. (9.5)} \\ &= p_{a,b}((w_J^- \otimes v_I)T_{w_{a,b}} u_a^+) && \text{Lemma 9.2.2} \\ &= p_{a,b}(p'_{a,b}((w_J^- \otimes v_I)T_{w_{a,b}})u_a^+) \\ &= p_{a,b}(c_{I,J}c_J(v_I \otimes v_{-J})u_a^+) && \text{Lemma 9.2.9} \\ &= p_{a,b}(c_{I,J}c_J w_I^+ \otimes v_{-J}) && \text{Lemma 9.2.2} \\ &= c_{I,J}c_J p_a(w_I^+) \otimes v_{-J} \\ &= c_{I,J}c_I c_J v_{-I} \otimes v_{-J}. && \text{Lemma 9.2.6} \end{aligned}$$

□

Lemma 9.2.11. *For $a+b=d$, the map $v_I \otimes v_J \mapsto (v_J \otimes v_I)v_{a,b}$ gives an isomorphism of $\mathcal{H}(\Sigma_a) \otimes \mathcal{H}(\Sigma_b)$-modules $V^a_{\geqslant 0} \otimes V^b_{>0} \to V^{\otimes d} v_{a,b}$.*

Proof. Since

$$T_i v_{a,b} = \begin{cases} T_{i+a}, & 1 \leqslant i \leqslant b, \\ T_{i-b}, & b+1 \leqslant i \leqslant a+b-1, \end{cases}$$

the map is $\mathcal{H}(\Sigma_a) \otimes \mathcal{H}(\Sigma_b)$-equivariant. The injectivity follows from Lemma 9.2.10, and the surjectivity follows from Lemma 9.2.8. □

9.3 The proof of the isomorphism theorem

Finally, we are in a position to prove the isomorphism theorem.

Proof of Theorem 9.1.1.

$$\begin{aligned}
&S^{\mathrm{B}}_{Q,q}(n,d) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(V^{\otimes d}) \\
&= \mathrm{End}_{\bigoplus_{0\leqslant i\leqslant d} e_{i,d-i}\mathcal{H}^{\mathrm{B}}_{Q,q}(d)e_{i,d-i}}(V^{\otimes d} e_{i,d-i}) && \text{Lemma 9.2.1(e)} \\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{e_{i,d-i}\mathcal{H}^{\mathrm{B}}_{Q,q}(d)e_{i,d-i}}(V^{\otimes d} e_{i,d-i}) \\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_i)\otimes\mathcal{H}^{\mathrm{A}}_q(\Sigma_{d-i})}(V^{\otimes d} v_{i,d-i}) && \text{Lemma 9.2.1(c)(d)} \\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_i)\otimes\mathcal{H}^{\mathrm{A}}_q(\Sigma_{d-i})}(V^{\otimes i}_{\geqslant 0} \otimes V^{\otimes(d-i)}_{>0}) && \text{Lemma 9.2.11} \\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_i)}(V^{\otimes i}_{\geqslant 0}) \otimes \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_{d-i})}(V^{\otimes d-i}_{>0}) \\
&= \bigoplus_{0\leqslant i\leqslant d} S^{\mathrm{A}}_q(\lceil n/2\rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2\rfloor, d-i). && \text{Eq. (9.9)}
\end{aligned}$$

□

9.4 Simple modules

As an immediate consequence of the Morita equivalence theorem one obtains a classification of irreducible representations for $S^{\mathrm{B}}_{Q,q}(n,d)$.

Theorem 9.4.1. *If $f^{\mathrm{B}}_d(Q,q)$ is invertible in the field k then there is a bijection*

$$\{\text{Irreducible representations of } S^{\mathrm{B}}_{Q,q}(n,d)\} \leftrightarrow \{(\lambda,\mu) \vdash (d_1,d_2) \mid d_1+d_2=d\},$$

where number of parts of λ and μ is no more than n. In particular, the standard modules over $S^{\mathrm{B}}_{Q,q}(n,d)$ are of the form $\nabla(\lambda)\boxtimes\nabla(\mu)$, where $\nabla(\lambda)$ (resp. $\nabla(\lambda)$) are standard modules over $S^{\mathrm{A}}_q(n,d_1)$ (resp. $S^{\mathrm{A}}_q(n,d_2)$).

Chapter 10

Schur functors

10.1 Schur functors

For type A it is well-known that, provided $n \geqslant d$, there is an idempotent $e^{\mathrm{A}} = e^{\mathrm{A}}(n,d) \in S_q^{\mathrm{A}}(n,d)$ such that $e^{\mathrm{A}} S_q^{\mathrm{A}}(n,d) e^{\mathrm{A}} \simeq \mathcal{H}_q(\Sigma_d)$, and a Schur functor

$$F_{n,d}^{\mathrm{A}} : \mathrm{Mod}(S_q^{\mathrm{A}}(n,d)) \to \mathrm{Mod}(\mathcal{H}_q(\Sigma_d)), \quad M \mapsto e^{\mathrm{A}} M. \tag{10.1}$$

In the following proposition we construct the Schur functor for $S_{Q,q}^{\mathrm{B}}(n,d)$ when $\lfloor n/2 \rfloor \geqslant d$.

Proposition 10.1.1. *If $\lfloor n/2 \rfloor \geqslant d$ then there is an idempotent $e^{\mathrm{B}} = e^{\mathrm{B}}(n,d) \in S_{Q,q}^{\mathrm{B}}(n,d)$ such that $e^{\mathrm{B}} S_{Q,q}^{\mathrm{B}}(n,d) e^{\mathrm{B}} \simeq \mathcal{H}_{Q,q}^{\mathrm{B}}(d)$ as k-algebras, and $e^{\mathrm{B}} S_{Q,q}^{\mathrm{B}}(n,d) \simeq V^{\otimes d}$ as $(S_{Q,q}^{\mathrm{B}}(n,d), \mathcal{H}_{Q,q}^{\mathrm{B}}(d))$-bimodules..*

Proof. Recall $\Lambda^{\mathrm{B}}(n,d)$ from Eq. (8.4) and $\phi_{\lambda\mu}^{g}$ from Eq. (8.7). Let $e^{\mathrm{B}} = \phi_{\omega\omega}^{1}$, where

$$\omega = \begin{cases} (0,\ldots,0,\underbrace{1,\ldots,1}_{2d},0\ldots,0) \in \Lambda^{\mathrm{B}}(2r,d) & \text{if } n = 2r; \\ (0,\ldots,0,\underbrace{1,\ldots,1}_{2d+1},0\ldots,0) \in \Lambda^{\mathrm{B}}(2r+1,d) & \text{if } n = 2r+1. \end{cases} \tag{10.2}$$

Note that such ω is well-defined only when $r = \lfloor n/2 \rfloor \geqslant d$. By Lemma 8.3.1, we have

$$e^{\mathrm{B}} \phi^g_{\lambda\mu} e^{\mathrm{B}} = \begin{cases} \phi^g_{\lambda\mu} & \text{if } \lambda = \omega = \mu; \\ 0 & \text{otherwise.} \end{cases} \tag{10.3}$$

Since W_ω is the trivial group, $x_\omega = 1 \in \mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ and hence $\phi^g_{\omega\omega}$ is uniquely determined by $1 \mapsto T_g$. Therefore, $e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d) e^{\mathrm{B}}$ and $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ are isomorphic as algebras.

Now from Section 8.3 we see that there is a canonical identification

$$V^{\otimes d} \simeq \bigoplus_{\mu \in \Lambda^{\mathrm{B}}(n,d)} x_\mu \, \mathcal{H}^{\mathrm{B}}_{Q,q} \simeq \bigoplus_{\mu \in \Lambda^{\mathrm{B}}(n,d)} \mathrm{Hom}_{\mathcal{H}^{\mathrm{B}}_{Q,q}}(x_\omega \, \mathcal{H}^{\mathrm{B}}_{Q,q}, x_\mu \, \mathcal{H}^{\mathrm{B}}_{Q,q}), \tag{10.4}$$

and hence the maps $\phi^g_{\omega\mu}$, with $\mu \in \Lambda^{\mathrm{B}}(n,d)$, g a minimal length coset representative for W^{B}/W_μ, forms a linear basis for $V^{\otimes d}$. Again by Lemma 8.3.1, we have

$$e^{\mathrm{B}} \phi^g_{\lambda\mu} = \begin{cases} \phi^g_{\omega\mu} & \text{if } \lambda = \omega; \\ 0 & \text{otherwise.} \end{cases} \tag{10.5}$$

Hence, $e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d)$ has a linear basis $\{\phi^g_{\omega\mu}\}$ where $\mu \in \Lambda^{\mathrm{B}}(n,d)$, g a minimal length double coset representative for $W_\omega \backslash W^{\mathrm{B}}/W_\mu$. Therefore $V^{\otimes d}$ and $e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d)$ are isomorphic as $(S^{\mathrm{B}}_{Q,q}(n,d), \mathcal{H}^{\mathrm{B}}_{Q,q}(d))$-bimodules. □

We define the Schur functor of type B by

$$F^{\mathrm{B}}_{n,d} : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)), \quad M \mapsto e^{\mathrm{B}} M. \tag{10.6}$$

Define the inverse Schur functor by

$$G^{\mathrm{B}}_d : \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)) \to \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)), \quad M \mapsto \mathrm{Hom}_{e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d) e^{\mathrm{B}}}(e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d), M). \tag{10.7}$$

In below we define a Schur-like functor $F^\flat_{n,d} : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d))$ using Theorem 9.1.1, under the same invertibility assumption: recall Φ from Eq. (9.2), let

$$\epsilon^\flat = \epsilon^\flat_{n,d} = \Phi^{-1}(\bigoplus_{i=0}^{d} e^{\mathrm{A}}(\lceil n/2\rceil, i) \otimes e^{\mathrm{A}}(\lfloor n/2\rfloor, d-i)). \tag{10.8}$$

Note that $\epsilon^\flat S^{\mathrm{B}}_{Q,q}(n,d)\epsilon^\flat \simeq \bigoplus_{i=0}^{d} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1})$, and hence left multiplication by $\epsilon^\flat$ defines a functor $\mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\bigoplus_{i=0}^{d} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1}))$. Hence, we can define

$$F^\flat_{n,d} : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)), \quad M \mapsto \mathcal{F}_H^{-1}(\epsilon^\flat M), \tag{10.9}$$

where $\mathcal{F}_H$ is the Morita equivalence for the Iwahori-Hecke algebras given by

$$\mathcal{F}_H : \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)) \to \mathrm{Mod}\Big(\bigoplus_{i} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1})\Big). \tag{10.10}$$

Under the invertibility condition, one can define an equivalence of categories induced from Φ as below:

$$\mathcal{F}_S : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}\Big(\bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2\rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2\rfloor, d-i)\Big). \tag{10.11}$$

In other words, we have the following commutativity of functors:

Proposition 10.1.2. *Assume $\lfloor n/2\rfloor \geqslant d \geqslant i \geqslant 0$ and that f^{B}_q is invertible. The diagram below commutes:*

$$\begin{array}{ccc} \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) & \xrightarrow{\mathcal{F}_s} & \mathrm{Mod}\Big(\bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil n/2\rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2\rfloor, d-i)\Big) \\ \downarrow F^\flat_{n,d} & & \downarrow \bigoplus_{i=0}^{d} F^{\mathrm{A}}_{\lceil n/2\rceil, i} \otimes F^{\mathrm{A}}_{\lfloor n/2\rfloor, d-i} \\ \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)) & \xrightarrow{\mathcal{F}_H} & \mathrm{Mod}\Big(\bigoplus_{i=0}^{d} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1})\Big) \end{array} \tag{10.12}$$

Remark 10.1.3. We expect that Proposition 10.1.2 still holds if we replace the functor $F^{\flat}_{n,d}$ therein by $F^{\mathrm{B}}_{n,d}$.

10.2 Existence of idempotents

We construct additional idempotents in Schur algebras of type B that will be used later in Section 11.3.

Proposition 10.2.1. *There exists an idempotent $e \in S^{\mathrm{B}}_{Q,q}(n',d)$ such that $eS^{\mathrm{B}}_{Q,q}(n',d)e \simeq S^{\mathrm{B}}_{Q,q}(n,d)$ if either one of the following holds:*

(a) $n' \geqslant n$ *and* $n' \equiv n \pmod 2$*;*

(b) $n' = 2r' + 1 \geqslant n = 2r$*.*

Proof. We use the combinatorial realization in Section 8.3. For (a) we set

$$e = \sum_{\gamma} \phi^{1}_{\gamma\gamma}, \tag{10.13}$$

where γ runs over the set

$$\Lambda^{\mathrm{B}}(n',d)|_n = \begin{cases} \{\gamma = (0,\dots,0,\underbrace{*,\dots,*}_{n},0\dots,0) \in \Lambda^{\mathrm{B}}(n',d)\} & \text{if } (a) \text{ holds;} \\ \{\gamma = (0,\dots,0,\underbrace{*,\dots,*}_{r},1,\underbrace{*,\dots,*}_{r},0\dots,0) \in \Lambda^{\mathrm{B}}(n',d)\} & \text{if } (b) \text{ holds.} \end{cases} \tag{10.14}$$

By Lemma 8.3.1 we have

$$e\phi^{g}_{\lambda\mu}e = \begin{cases} \phi^{g}_{\lambda\mu} & \text{if } \lambda,\mu \in \Lambda^{\mathrm{B}}(n',d)|_n; \\ 0 & \text{otherwise.} \end{cases} \tag{10.15}$$

It follows by construction that $eS^{\mathrm{B}}_{Q,q}(n',d)e$ and $S^{\mathrm{B}}_{Q,q}(n,d)$ are isomorphic as algebras. □

10.3 Existence of spectral sequences

Let A be a finite dimensional algebra over a field k and e be an idempotent in A. Doty, Erdmann and Nakano [DEN] established a relationship between the cohomology theory in $\mathrm{Mod}(A)$ versus $\mathrm{Mod}(eAe)$. More specifically, they construct a Grothendieck spectral sequence which starts from extensions of A-modules and converges to extensions of eAe-modules.

There are two important functors involved in this construction. The first functor is an exact functor from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(eAe)$ denoted by $\mathcal{F}$ (that is a special case of the classical Schur functor) defined by $\mathcal{F}(-) = e(-)$. The other functor is a left exact functor from $\mathrm{Mod}(eAe)$ to $\mathrm{Mod}(A)$, denoted $\mathcal{G}$ defined by $\mathcal{G}(-) = \mathrm{Hom}_A(Ae, -)$. This functor is right adjoint to $\mathcal{F}$.

In [DEN], the aforementioned construction was used in the quantum setting to relate the extensions for quantum GL_n to those for Iwahori-Hecke algebras. For $\lfloor n/2 \rfloor \geqslant d$ there exists an idempotent $e \in S^{\mathrm{B}}_{Q,q}(n,d)$ such that $\mathcal{H}^{\mathrm{B}}_{Q,q}(d) \cong eS^{\mathrm{B}}_{Q,q}(n,d)e$. Therefore, we obtain a relationship between cohomology of the type B Schur algebras with the Iwahori-Hecke algebra of type B.

Theorem 10.3.1. *Let $\lfloor n/2 \rfloor \geqslant d$ with $M \in \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d))$ and $N \in \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d))$. There exists a first quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{S^{\mathrm{B}}_{Q,q}(n,d)}(M, R^j\mathcal{G}(N)) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(eM, N).$$

where $R^j\mathcal{G}(-) = \mathrm{Ext}^j_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(V^{\otimes d}, -)$.

We can also compare cohomology between $S^{\mathrm{B}}_{Q,q}(n,d)$ and $S^{\mathrm{B}}_{Q,q}(n',d)$ where $n' \geqslant n$ since there exists an idempotent $e \in S^{\mathrm{B}}_{Q,q}(n',d)$ such that $S^{\mathrm{B}}_{Q,q}(n,d) \cong eS^{\mathrm{B}}_{Q,q}(n',d)e$ thanks to Proposition 10.2.1.

Theorem 10.3.2. *Let $M \in \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n',d))$ and $N \in \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d))$. Assume that either*

(a) *$n' \geqslant n$ and $n' \equiv n \pmod 2$;*

(b) *$n' = 2r' + 1 \geqslant n = 2r$.*

Then there exists a first quadrant spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{S^{\mathrm{B}}_{Q,q}(n',d)}(M, R^j\mathcal{G}(N)) \Rightarrow \mathrm{Ext}^{i+j}_{S^{\mathrm{B}}_{Q,q}(n,d)}(eM, N).$$

where $R^j\mathcal{G}(-) = \mathrm{Ext}^j_{S^{\mathrm{B}}_{Q,q}(n,d)}(eS^{\mathrm{B}}_{Q,q}(n',d), -)$.

Chapter 11

Cellularity, quasi-hereditariness and representation type of q-Schur algebras of type B

11.1 Cellularity

We start from recalling the definition of a cellular algebra following [GL]. A k-algebra A is *cellular* if it is equipped with a cell datum $(\Lambda, M, C, *)$ consisting of a poset Λ, a map M sending each $\lambda \in \Lambda$ to a finite set $M(\lambda)$, a map C sending each pair $(\mathfrak{s}, \mathfrak{t}) \in M(\lambda)^2$ to an element $C^\lambda_{\mathfrak{s},\mathfrak{t}} \in A$, and an k-linear involutory anti-automorphism $*$ satisfying the following conditions:

(C1) The map C is injective with image being an k-basis of A (called a *cellular basis*).

(C2) For any $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, $(C^\lambda_{\mathfrak{s},\mathfrak{t}})^* = C^\lambda_{\mathfrak{t},\mathfrak{s}}$.

(C3) There exists $r_a(\mathfrak{s}', \mathfrak{s}) \in k$ for $\lambda \in \Lambda, \mathfrak{s}, \mathfrak{s}' \in M(\lambda)$ such that for all $a \in A$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$,

$$aC^{\lambda}_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{s}' \in M(\lambda)} r_a(\mathfrak{s}', \mathfrak{s}) C^{\lambda}_{\mathfrak{s}',\mathfrak{t}} \pmod{A_{<\lambda}}.$$

Here $A_{<\lambda}$ is the k-submodule of A generated by the set $\{C^{\mu}_{\mathfrak{s}'',\mathfrak{t}''} : \ \mu < \lambda; \mathfrak{s}'', \mathfrak{t}'' \in M(\mu)\}$.

For a cellular algebra A, we define for each $\lambda \in \Lambda$ a *cell module* $W(\lambda)$ spanned by $C^{\lambda}_{\mathfrak{s}}$, $\mathfrak{s} \in M(\lambda)$, with multiplication given by

$$aC_{\mathfrak{s}} = \sum_{\mathfrak{s}' \in M(\lambda)} r_a(\mathfrak{s}', \mathfrak{s}) C'_{\mathfrak{s}}. \tag{11.1}$$

For each $\lambda \in \Lambda$ we let $\phi_\lambda : W(\lambda) \times W(\lambda) \to k$ be a bilinear form satisfying

$$C^{\lambda}_{\mathfrak{s},\mathfrak{s}} C^{\lambda}_{\mathfrak{t},\mathfrak{t}} \equiv \phi_\lambda(C_{\mathfrak{s}}, C_{\mathfrak{t}}) C^{\lambda}_{\mathfrak{s},\mathfrak{t}} \pmod{A_{<\lambda}}. \tag{11.2}$$

It is known that the type A q-Schur algebras are always cellular, and there could be distinct cellular structures. See [AST] for a parallel approach on the cellularity of centralizer algebras for quantum groups.

Example 11.1.1 (Mathas)**.** Let $\Lambda = \Lambda^{\mathrm{A}}(d)$ be the set of all partitions of d, and let $\Lambda' = \Lambda'(d)$ be the set of all compositions of d. For each composition $\lambda \in \Lambda'$, let Σ_λ be the corresponding Young subgroup of Σ_d. Set

$$x_\lambda = \sum_{w \in \Sigma_\lambda} T_w \in \mathcal{H}_q(\Sigma_d).$$

It is known the q-Schur algebra admits the following combinatorial realization:

$$S^{\mathrm{A}}_q(n, d) = \mathrm{End}_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda \in \Lambda'} x_\lambda \mathcal{H}_q(\Sigma_d)) = \bigoplus_{\lambda, \mu \in \Lambda'} \mathrm{Hom}_{\mathcal{H}_q(\Sigma_d)}(x_\mu \mathcal{H}_q(\Sigma_d), x_\lambda \mathcal{H}_q(\Sigma_d)).$$

The finite set $M(\lambda)$ is given by $M(\lambda) = \bigsqcup_{\mu \in \Lambda'} \mathrm{SSTD}(\lambda, \mu)$, where

$$\mathrm{SSTD}(\lambda, \mu) = \{\text{semi-standard } \lambda\text{-tableaux of shape } \mu\}. \tag{11.3}$$

For $\mu \vdash d$, denote the set of shortest right coset representatives for Σ_μ in Σ_d by

$$D_\mu = \{w \in \Sigma_d \mid \ell(gw) = \ell(w) + \ell(g) \text{ for all } g \in \Sigma_\mu\}. \tag{11.4}$$

Let $\mathfrak{t}^\lambda$ be the canonical λ-tableau of shape λ, then for all λ-tableau $\mathfrak{t}$ there is a unique element $d(\mathfrak{t}) \in D_\lambda$ such that $\mathfrak{t}d(\mathfrak{t}) = \mathfrak{t}$. The cellular basis element, for $\lambda \in \Lambda, \mathfrak{s} \in \mathrm{sstd}(\lambda, \mu), \mathfrak{t} \in \mathrm{sstd}(\lambda, \nu)$, is the given by

$$C^\lambda_{\mathfrak{s},\mathfrak{t}}(x_\alpha h) = \delta_{\alpha,\mu} \sum_{s,t} T_{d(s)^{-1}} x_\lambda T_{d(t)} h, \tag{11.5}$$

where the sum is over all pairs (s, t) such that $\mu(s) = \mathfrak{s}, \nu(t) = \mathfrak{t}$.

Example 11.1.2 (Doty-Giaquinto)**.** The poset Λ is the same as in Example 11.1.1, and we have $\Lambda = \Sigma_d \Lambda^+$. It is known that the algebra $S_q^{\mathrm{A}}(n, d)$ admits a presentation with generators $E_i, F_i (1 \leqslant i \leqslant n-1)$ and $1_\lambda (\lambda \in \Lambda)$. The map $*$ is the anti-automorphism satisfying

$$E_i^* = F_i, \quad F_i^* = E_i, \quad 1_\lambda^* = 1_\lambda.$$

For each $\lambda \in \Lambda$ we set $\Lambda^+_\lambda = \{\mu \in \Lambda^+ \mid \mu \leqslant \lambda\}$. Note that Λ^+_λ is saturated and it defines a subalgebra $S_q(\Lambda^+_\lambda)$ of $S_q^{\mathrm{A}}(n, d)$ with a basis $\{\overline{x}_\mathfrak{s} \mid 1 \leqslant \mathfrak{s} \leqslant d_\lambda\}$ for some $d_\lambda \in \mathbb{N}$. Let $x_\mathfrak{s} \in S_q^{\mathrm{A}}(n, d)$ be the preimage of $\overline{x}_\mathfrak{s}$ under the projection $S_q^{\mathrm{A}}(n, d) \to S_q(\Lambda^+_\lambda)$ that is the identity map except for that it kills all 1_μ where $\mu \nleqslant \lambda$. The finite set $M(\lambda)$ is given by

$$M(\lambda) = \{1, 2, \ldots, d_\lambda\}. \tag{11.6}$$

Finally, for $\lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in M(\lambda)$, we set

$$C^{\lambda}_{\mathfrak{s},\mathfrak{t}} = x_{\mathfrak{s}} 1_{\lambda} x^{*}_{\mathfrak{t}}. \tag{11.7}$$

We show that the isomorphism theorem produces a cellular structure for $S^{\mathrm{B}}_{Q,q}(n,d)$ using any cellular structure on the q-Schur algebras of type A. For any n, d we fix a cell datum $(\Lambda_{n,d}, M_{n,d}, C_{n,d}, *)$ for $S^{\mathrm{A}}_{q}(n,d)$. Define

$$\Lambda^{\mathrm{B}} = \Lambda^{\mathrm{B}}(n,d) = \bigsqcup_{i=0}^{d} \Lambda_{\lceil n/2 \rceil, i} \times \Lambda_{\lfloor n/2 \rfloor, d-i}, \tag{11.8}$$

as a poset with the lexicographical order. For $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda^{\mathrm{B}}$, we define M^{B} by

$$M^{\mathrm{B}}(\lambda) = \bigsqcup_{i=0}^{d} M_{\lceil n/2 \rceil, i}(\lambda^{(1)}) \times M_{\lfloor n/2 \rfloor, d-i}(\lambda^{(2)}). \tag{11.9}$$

The map C^{B} is given by, for $\mathfrak{s} = (\mathfrak{s}^{(1)}, \mathfrak{s}^{(2)}), \mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}) \in M_{\lceil n/2 \rceil, i}(\lambda^{(1)}) \times M_{\lfloor n/2 \rfloor, d-i}(\lambda^{(2)}) \subset M^{\mathrm{B}}(\lambda)$,

$$(C^{\mathrm{B}})^{\lambda}_{\mathfrak{s},\mathfrak{t}} = (C_{\lceil n/2 \rceil, i})^{\lambda^{(1)}}_{\mathfrak{s}^{(1)},\mathfrak{t}^{(1)}} \otimes (C_{\lfloor n/2 \rfloor, d-i})^{\lambda^{(2)}}_{\mathfrak{s}^{(2)},\mathfrak{t}^{(2)}}. \tag{11.10}$$

Finally, the map $*$ is given by

$$* : (C_{\lceil n/2 \rceil, i})^{\lambda^{(1)}}_{\mathfrak{s}^{(1)},\mathfrak{t}^{(1)}} \otimes (C_{\lfloor n/2 \rfloor, d-i})^{\lambda^{(2)}}_{\mathfrak{s}^{(2)},\mathfrak{t}^{(2)}} \mapsto (C_{\lceil n/2 \rceil, i})^{\lambda^{(1)}}_{\mathfrak{t}^{(1)},\mathfrak{s}^{(1)}} \otimes (C_{\lfloor n/2 \rfloor, d-i})^{\lambda^{(2)}}_{\mathfrak{t}^{(2)},\mathfrak{s}^{(2)}}. \tag{11.11}$$

Corollary 11.1.3. *If the invertibility condition in Theorem 9.1.1 holds, then $S^{\mathrm{B}}_{Q,q}(n,d)$ is a cellular algebra with cell datum $(\Lambda^{\mathrm{B}}, M^{\mathrm{B}}, C^{\mathrm{B}}, *)$.*

Proof. Condition (C1) follows from the isomorphism theorem; while Condition (C2) follows directly from Eq. (11.11). Condition (C3) follows from the type A cellular structure as follows: for $a_1 \in S_q^{\mathrm{A}}(\lceil n/2 \rceil, i)$ and $a_2 \in S_q^{\mathrm{A}}(\lfloor n/2 \rfloor, d-i)$,

$$a_1 (C_{\lceil n/2 \rceil, i})^{\lambda^{(1)}}_{\mathfrak{s}^{(1)},\mathfrak{t}^{(1)}} \equiv \sum_{\mathfrak{u}^{(1)} \in M_{\lceil n/2 \rceil, i}(\lambda^{(1)})} r^{(1)}_{a_1}(\mathfrak{u}^{(1)}, \mathfrak{s}^{(1)}) (C_{\lceil n/2 \rceil, i})^{\lambda^{(1)}}_{\mathfrak{u}^{(1)},\mathfrak{t}^{(1)}} \pmod{S_q^{\mathrm{A}}(\lceil n/2 \rceil, i)(<\lambda^{(1)})},$$

$$a_2 (C_{\lfloor n/2 \rfloor, d-i})^{\lambda^{(2)}}_{\mathfrak{s}^{(2)},\mathfrak{t}^{(2)}} \equiv \sum_{\mathfrak{u}^{(2)} \in M_{r, d-i}(\lambda^{(2)})} r^{(2)}_{a_2}(\mathfrak{u}^{(2)}, \mathfrak{s}^{(2)}) (C_{\lfloor n/2 \rfloor, d-i})^{\lambda^{(2)}}_{\mathfrak{u}^{(2)},\mathfrak{t}^{(2)}} \pmod{S_q^{\mathrm{A}}(\lfloor n/2 \rfloor, d-i)(<\lambda^{(2)})}.$$

That is, for $a = a_1 \otimes a_2 \in S_q^{\mathrm{A}}(\lceil n/2 \rceil, i) \otimes S_q^{\mathrm{A}}(\lfloor n/2 \rfloor, d-i) \subset S_q^{\mathrm{B}}(n,d)$,

$$a (C^{\mathrm{B}})^{\lambda}_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\substack{\mathfrak{u} = (\mathfrak{u}^{(1)}, \mathfrak{u}^{(2)}) \\ \in M_{\lceil n/2 \rceil, i}(\lambda^{(1)}) \times M_{\lfloor n/2 \rfloor, d-i}(\lambda^{(2)})}} r^{\mathrm{B}}_a(\mathfrak{u}, \mathfrak{s}) (C^{\mathrm{B}})^{\lambda}_{\mathfrak{u},\mathfrak{t}} \bmod S_q^{\mathrm{B}}(n,d)(<\lambda),$$

where $r^{\mathrm{B}}_a(\mathfrak{u}, \mathfrak{s}) = r^{(1)}_{a_1}(\mathfrak{u}^{(1)}, \mathfrak{s}^{(1)}) r^{(2)}_{a_2}(\mathfrak{u}^{(2)}, \mathfrak{s}^{(2)})$ is independent of $\mathfrak{t}$. □

11.2 Quasi-hereditary structure

Following [CPS1], a k-algebra A is called *quasi-hereditary* if there is a chain of two-sided ideals of A:

$$0 \subset I_1 \subset I_2 \subset \ldots \subset I_n = A$$

such that each quotient $J_j = I_j / I_{j-1}$ is a hereditary ideal of A/I_{j-1}. It is known [GL] that if A is cellular and $\phi_\lambda \neq 0$ (cf. Eq. (11.2)) for all $\lambda \in \Lambda$ then A is quasi-hereditary.

An immediate corollary of our isomorphism theorem is that $S^{\mathrm{B}}_{Q,q}(n,d)$ is quasi-hereditary under the invertibility condition. We conjecture that this is a sufficient and necessary condition and provide some evidence for small n.

Corollary 11.2.1. *If the invertibility condition in Theorem 9.1.1 holds, then $S_q^{\mathrm{B}}(n,d)$ is quasi-hereditary.*

Proof. Let ϕ^{A}_ν with $\nu \in \Lambda_{r,j}$ be such a map for $S_q^{\mathrm{A}}(r,j)$. Fix $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_{\lceil n/2 \rceil, i} \times$

$\Lambda_{\lfloor n/2\rfloor,d-i} \subset \Lambda^{\mathrm{B}}$, $\mathfrak{s} = (\mathfrak{s}^{(1)}, \mathfrak{s}^{(2)}), \mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}) \in M_{\lceil n/2\rceil,i}(\lambda^{(1)}) \times M_{\lfloor n/2\rfloor,d-i}(\lambda^{(2)}) \subset M^{\mathrm{B}}(\lambda)$, we have

$$\begin{aligned} C^{\lambda}_{\mathfrak{s},\mathfrak{s}} C^{\lambda}_{\mathfrak{t},\mathfrak{t}} &= (C_{\lceil n/2\rceil,i})^{\lambda^{(1)}}_{\mathfrak{s}^{(1)},\mathfrak{s}^{(1)}} (C_{\lceil n/2\rceil,i})^{\lambda^{(1)}}_{\mathfrak{t}^{(1)},\mathfrak{t}^{(1)}} \otimes (C_{\lfloor n/2\rfloor,d-i})^{\lambda^{(2)}}_{\mathfrak{s}^{(2)},\mathfrak{s}^{(2)}} (C_{\lfloor n/2\rfloor,d-i})^{\lambda^{(2)}}_{\mathfrak{t}^{(2)},\mathfrak{t}^{(2)}} \\ &\equiv \phi^{\mathrm{A}}_{\lambda^{(1)}}(C^{(1)}_{\mathfrak{s}}, C^{(1)}_{\mathfrak{t}}) \phi^{\mathrm{A}}_{\lambda^{(2)}}(C^{(2)}_{\mathfrak{s}}, C^{(2)}_{\mathfrak{t}}) C^{\lambda}_{\mathfrak{s},\mathfrak{t}} \bmod S^{\mathrm{B}}_{q}(n,d)(<\lambda). \end{aligned}$$

□

Recall that in Proposition 8.4.1 we see that $S^{\mathrm{B}}_{Q,q}(2,1) \simeq \mathcal{H}^{\mathrm{A}}_{Q^{-1}}(\Sigma_2)$. In the following we show that the known cellular structure (due to Geck/Dipper-James) fails when $f^{\mathrm{B}} = Q^{-2}+1$ is not invertible.

Example 11.2.2. Let $S^{\mathrm{B}}_{Q,q}(2,1) \simeq \mathcal{H}^{\mathrm{A}}_{Q^{-1}}(\Sigma_2) = k[t]/\langle t^2 - (Q^{-1} - Q)t + 1\rangle$. We have

$$\Lambda = \left\{\lambda = \square\square \rhd \mu = \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right\}, \quad M(\lambda) = \{\mathfrak{t} = \boxed{1}\boxed{2}\}, M(\mu) = \left\{\mathfrak{s} = \begin{smallmatrix}\boxed{1}\\ \boxed{2}\end{smallmatrix}\right\}.$$

The cellular basis elements are

$$C^{\lambda}_{\mathfrak{t}\mathfrak{t}} = \sum_{w\in\Sigma_2} Q^{-}\ell(w) T_w = 1 + Q^{-1}t, \quad C^{\mu}_{\mathfrak{s}\mathfrak{s}} = \sum_{w\in\Sigma_1\times\Sigma_1} Q^{-}\ell(w) T_w = 1.$$

Firstly, we have $C^{\mu}_{\mathfrak{s}\mathfrak{s}} C^{\mu}_{\mathfrak{s}\mathfrak{s}} = 1 = C^{\mu}_{\mathfrak{s}\mathfrak{s}}$ and hence ϕ_μ is determined by $\phi_\mu(C_{\mathfrak{s}}, C_{\mathfrak{s}}) = 1$, which is nonzero. For λ we have

$$C^{\lambda}_{\mathfrak{t}\mathfrak{t}} C^{\lambda}_{\mathfrak{t}\mathfrak{t}} = 1 - Q^{-2} + (Q^{-2}+1)Q^{-1}t \equiv (Q^{-2}+1) C^{\lambda}_{\mathfrak{t}\mathfrak{t}} \bmod A_{<\lambda}.$$

That is, ϕ_λ is determined by $\phi_\mu(C_{\mathfrak{t}}, C_{\mathfrak{t}}) = (Q^{-2}+1)$, which can be zero when $f^{\mathrm{B}} = Q^{-2}+1 = 0$. Therefore, $S^{\mathrm{B}}_{Q,q}(2,1)$ is not quasi-hereditary in an explicit way.

One can also see that $S^{\mathrm{B}}_{Q,q}(2,1)$ is not quasi-hereditary because if it were then it would have finite global dimension. However, $\mathcal{H}^{\mathrm{A}}_{Q^{-1}}(\Sigma_2)$ is a Frobenius algebra with infinite global dimension.

Conjecture 11.2.3. The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is quasi-hereditary if and only if $f^{\mathrm{B}}_d(Q,q)$ is invertible.

11.3 Representation Type

Let A be a finite-dimensional algebra over a field k. A fundamental question one can ask about A is to describe its representation type. The algebra A is *semisimple* if and only if every finite-dimensional module (i.e., $M \in \mathrm{mod}(A)$) is a direct sum of simple modules. This means that indecomposable modules for A are simple. If A admits finitely many finite-dimensional indecomposable modules, A is said to be of *finite representation type*. If A does not have finite representation type A is of *infinite representation type*.

A deep theorem of Drozd states that finite dimensional algebras of infinite representation type can be split into two mutually exclusive categories: tame or wild. An algebra A has *tame representation type* if for each dimension there exists finitely many one-parameter families of indecomposable objects in $\mathrm{mod}(A)$. The indecomposable modules for algebras of tame representation type are classifiable. On the other hand, the algebras of *wild representation type* are those whose representation theory is as difficult to study as the representation theory of the free associative algebra $k\langle x, y\rangle$ on two variables. Classifying the finite-dimensional $k\langle x, y\rangle$-modules is very much an open question.

11.3.1 Summary: Type A results

The following results from [EN, Theorem 1.3(A - C)] summarize the representation type for the $\bar{q}$-Schur algebra for type A over k. Assume that $\bar{q} \in k^{\times}$ has multiplicative order l and $\bar{q} \neq 1$.

Theorem 11.3.1. *The algebra $S^{\mathrm{A}}_{\bar{q}}(n,r)$ is semisimple if and only if one of the following holds:*

(i) $n = 1$;

(i) $\bar{q}$ *is not a root of unity;*

(ii) $\bar{q}$ *is a primitive* l*th root of unity and* $r < l$;

(iii) $n = 2$, $p = 0$, $l = 2$ *and* r *is odd;*

(iv) $n = 2$, $p \geqslant 3$, $l = 2$ *and* r *is odd with* $r < 2p + 1$.

Theorem 11.3.2. *The algebra $S^{\mathrm{A}}_{\bar{q}}(n,r)$ has finite representation type but is not semi-simple if and only if $\bar{q}$ is a primitive lth root of unity with $l \leqslant r$, and one of the following holds:*

(i) $n \geqslant 3$ *and* $r < 2l$;

(ii) $n = 2$, $p \neq 0$, $l \geqslant 3$ *and* $r < lp$;

(iii) $n = 2$, $p = 0$ *and either* $l \geqslant 3$, *or* $l = 2$ *and* r *is even;*

(iv) $n = 2$, $p \geqslant 3$, $l = 2$ *and* r *even with* $r < 2p$, *or* r *is odd with* $2p + 1 \leqslant r < 2p^2 + 1$.

Theorem 11.3.3. *The algebra $S^{\mathrm{A}}_{\bar{q}}(n,r)$ has tame representation type if and only if $\bar{q}$ is a primitive lth root of unity and one of the following holds:*

(i) $n = 3$, $l = 3$, $p \neq 2$ *and* $r = 7, 8$;

(ii) $n = 3$, $l = 2$ *and* $r = 4, 5$;

(iii) $n = 4$, $l = 2$ *and* $r = 5$;

(iv) $n = 2$, $l \geqslant 3$, $p = 2$ *or* $p = 3$ *and* $pl \leqslant r < (p+1)l$;

(v) $n = 2$, $l = 2$, $p = 3$ *and* $r \in \{6, 19, 21, 23\}$.

11.3.2 Representation type of algebras related to type A Schur algebras

In this section we summarize some of the fundamental results that are used to classify the representation type of Schur algebras. The first proposition can be verified by using the existence of the determinant representation for $S_{\bar{q}}^{\mathrm{A}}(n, r_1)$ (cf. [EN, Proposition 2.4B]).

Proposition 11.3.4. *If $S_{\bar{q}}^{\mathrm{A}}(n, r_1) \otimes S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ has wild representation type then $S_{\bar{q}}^{\mathrm{A}}(n, r_1 + n) \otimes S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ has wild representation type.*

Next we can present a sufficient criteria to show that the tensor product of type A Schur algebras has wild representation type.

Proposition 11.3.5. *Suppose that the Schur algebras $S_{\bar{q}}^{\mathrm{A}}(n, r_1)$ and $S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ are non-semisimple algebras. Then $S_{\bar{q}}^{\mathrm{A}}(n, r_1) \otimes S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ has wild representation type.*

Proof. First note that $S_{\bar{q}}^{\mathrm{A}}(n, r)$ is a quasi hereditary algebra and if $S_{\bar{q}}^{\mathrm{A}}(n, r)$ is not semisimple then it must have a block with at least two simple modules.

Suppose that S_1, S_2, S_3 are three simple modules in $S_{\bar{q}}^{\mathrm{A}}(n, r_1)$ with $\mathrm{Ext}^1_{S_{\bar{q}}^{\mathrm{A}}(n,r_1)}(S_1, S_2) \neq 0$ and $\mathrm{Ext}^1_{S_{\bar{q}}^{\mathrm{A}}(n,r_1)}(S_2, S_3) \neq 0$. Note that via the existence of the transposed duality,

$$\mathrm{Ext}^1_{S_{\bar{q}}^{\mathrm{A}}(n,r_1)}(S_i, S_j) \cong \mathrm{Ext}^1_{S_{\bar{q}}^{\mathrm{A}}(n,r_1)}(S_j, S_i)$$

for $i, j = 1, 2, 3$. Similarly, let T_1, T_2 be two simple modules for $S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ with $\mathrm{Ext}^1_{S_{\bar{q}}^{\mathrm{A}}(n,r_2)}(T_1, T_2) \neq 0$. Then the Ext^1-quiver for $S_{\bar{q}}^{\mathrm{A}}(n, r_1) \otimes S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ will have a subquiver of the form as in Figure 11.1 below. This quiver cannot be separated into a union of Dynkin diagrams or extended Dynkin diagrams. Consequently, $S_{\bar{q}}^{\mathrm{A}}(n, r_1) \otimes S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ must has wild representation type.

The other case to consider is when the blocks of $S_{\bar{q}}^{\mathrm{A}}(n, r_1)$ and $S_{\bar{q}}^{\mathrm{A}}(n, r_2)$ have at most two simple modules. Let $\mathcal{B}_j$ be a block of $S_{\bar{q}}^{\mathrm{A}}(n, r_j)$ for $j = 1, 2$ with two simple modules.

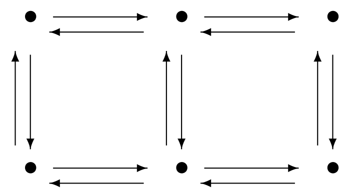


Figure 11.1:

There are four simple modules in $\mathcal{B}_1 \otimes \mathcal{B}_2$ and the structure of the projective modules are the same as regular block for category $\mathcal{O}$ for the Lie algebra of type $\mathrm{A}_1 \times \mathrm{A}_1$ (cf. [FNP, 4.2]). The argument in [FNP, Lemma 4.2] can be use to show that $\mathcal{B}_1 \otimes \mathcal{B}_2$ has wild representation type. □

The results in [EN, Theorem 1.3(A - C)] entail using a different parameter $\bar{q}$ than the parameter q in our paper. The relationship is given by $\bar{q} = q^{-2}$ or equivalently $q^2 = (\bar{q})^{-1}$ with $S_q^{\mathrm{A}}(n, d) \cong S_{\bar{q}}^{\mathrm{A}}(n, d)$. This means that

- q is generic if and only if $\bar{q}$ is generic,
- q^2 is a primitive lth root of unity if and only if $\bar{q}$ is a primitive lth root of unity;
- if q is a primitive $2s$-th root of unity if and only if $\bar{q}$ is a primitive s-th root of unity;
- if q is a primitive $(2s + 1)$-th root of unity if and only if $\bar{q}$ is a primitive $(2s + 1)$-th root of unity.

Now let $n' \geqslant n$. By Proposition 10.2.1, under suitable conditions on n' and n, there exists an idempotent $e \in S_{Q,q}^{\mathrm{B}}(n', d)$ such that $S_{Q,q}^{\mathrm{B}}(n, d) \cong eS_{Q,q}^{\mathrm{B}}(n', d)e$. By using the proof in [EN, Proposition 2.4B], one has the following result.

Proposition 11.3.6. *Let $n' \geqslant n$ with $n' \geqslant n$ and $n' \equiv n \pmod 2$.*

(a) If $S_{Q,q}^{\mathrm{B}}(n, d)$ is not semisimple then $S_{Q,q}^{\mathrm{B}}(n', d)$ is not semisimple.

(b) If $S^{\mathrm{B}}_{Q,q}(n,d)$ has wild representation type then $S^{\mathrm{B}}_{Q,q}(n',d)$ has wild representation type.

11.3.3 Type B Results

Throughout this section, let $S^{\mathrm{B}}_{Q,q}(n,d)$ be the q-Schur algebra of Type B under the condition that the polynomial $f^{\mathrm{B}}_d(Q,q) \neq 0$. Moreover, assume that $q^2 \neq 1$ (i.e., $q \neq 1$ or a primitive 2nd root of unity). One can apply the isomorphism in Theorem 9.1.1 to determine the representation type for $S^{\mathrm{B}}_{Q,q}(n,d)$ from the Type A results stated in Section 11.3.1.

Theorem 11.3.7. *The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is semisimple if and only if one of the following holds:*

(i) $n = 1$;

(ii) q is not a root of unity;

(iii) q^2 is a primitive lth root of unity and $d < l$;

(iv) $n = 2$ and d arbitrary;

Proof. The semisimplicity of (i - iii) follow by using Theorem 9.1.1 with Theorem 11.3.1. The semisimplicity of (iv) follows by Theorem 9.1.1 and the fact that $S^{\mathrm{A}}_q(1,d)$ is always semisimple.

Now assume that q^2 is a primitive lth root of unity, $d \geqslant l$, $n \geqslant 3$ and $l \geqslant 3$. Consider the case when $n = 3$. From Theorem 9.1.1,

$$S^{\mathrm{B}}_{Q,q}(3,d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(2,i) \otimes S^{\mathrm{A}}_q(1,d-i). \tag{11.12}$$

If $d \geqslant l$ then $S^{\mathrm{A}}_q(2,l)$ appears as a summand of $S^{\mathrm{B}}_{Q,q}(3,d)$ (when $i = d - l$). For $l \geqslant 3$, $S^{\mathrm{A}}_q(2,l) \simeq S^{\mathrm{A}}_{\bar{q}}(2,l)$ is not semisimple. It follows that $S^{\mathrm{B}}_{Q,q}(3,d)$ is not semisimple for $d \geqslant l$. One can repeat the same argument for $n = 4$ to show that $S^{\mathrm{B}}_{Q,q}(4,d)$ is not semisimple for

$d \geqslant l$. Now apply Proposition 11.3.6(a) to deduce that $S^{\mathrm{B}}_{Q,q}(n,d)$ is not semisimple for $n \geqslant 3$ and $d \geqslant l$. □

Theorem 11.3.8. *The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ has finite representation type but is not semisimple if and only if q^2 is a primitive lth root of unity with $l \leqslant d$, and one of the following holds:*

(i) $n \geqslant 5$, $l \leqslant d < 2l$;

(ii) $n = 3$, $p = 0$ *and* $l \leqslant d$;

(iii) $n = 3$, $p \geqslant 2$ *and* $l \leqslant d < lp$;

(iv) $n = 4$, $p = 0$, $l = 2$ *and* $d \geqslant 4$ *with* d *odd.*

(v) $n = 4$, $p \geqslant 3$, $l = 2$ *and* $4 < d \leqslant 2p - 1$ *with* d *odd.*

The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ has tame representation type if and only if

(vi) $n = 3$, $l = 2$, $p = 3$ *and* $d = 6$;

(vii) $n = 3$, $l \geqslant 3$, $p = 2$ *or* 3 *and* $lp \leqslant d < l(p+1)$;

(viii) $n = 4$, $l = 2$, $p = 3$ *and* $d = 7$.

Proof. We first reduce our analysis to the situation where $n = 3$ and 4. Assume that $n \geqslant 5$ so $\lceil n/2 \rceil \geqslant 3$ and $\lfloor n/2 \rfloor \geqslant 2$. By Theorem 11.3.1, the algebras $S^{\mathrm{A}}_q(2,l)$ and $S^{\mathrm{A}}_q(i,l+j)$ are not semisimple for $i \geqslant 3, j \geqslant 0$, and hence neither are $S^{\mathrm{A}}_q(\lceil n/2 \rceil, l+j)$ and $S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, l)$ for $n \geqslant 5, j \geqslant 0$. Therefore, $S^{\mathrm{A}}_q(\lceil n/2 \rceil, l+j) \otimes S^{\mathrm{A}}_q(\lfloor n/2 \rfloor, l)$ has wild representation type by Proposition 11.3.5. It follows that $S^{\mathrm{B}}_{Q,q}(n,d)$ has wild representation type for $d \geqslant 2l, n \geqslant 5$. When $l \leqslant d < 2l$ and $n \geqslant 5$, one can use Theorem 9.1.1 in conjunction with Theorem 11.3.2 to prove that $S^{\mathrm{B}}_{Q,q}(n,d)$ has finite representation type.

Now consider the case when $n = 3$. The isomorphism (11.12) indicates that we can reduce our analysis to considering $S^{\mathrm{A}}_q(2,r)$. From this isomorphism and Theorem 11.3.2, one can verify that (i) when $p = 0$ then $S^{\mathrm{B}}_{Q,q}(3,d)$ has finite representation type (but is not semisimple) for $l \leqslant d$, (ii) when $p > 0$ then $S^{\mathrm{B}}_{Q,q}(3,d)$ has finite representation type (but is not semisimple) for $l \leqslant d < lp$, and (iii) when $p > 0$, $S^{\mathrm{B}}_{Q,q}(3,d)$ has infinite representation type for $d \geqslant lp$.

For $n = 3$, one can also see that under conditions (vi) and (vii), $S^{\mathrm{B}}_{Q,q}(3,d)$ has tame representation type. Moreover, one can verify that $S^{\mathrm{B}}_{Q,q}(3,d)$ has wild representation type in the various complementary cases.

Finally let $n = 4$. From Proposition 11.3.5, $S^{\mathrm{A}}_{q}(2,l) \otimes S^{\mathrm{A}}_{q}(2,l)$ and $S^{\mathrm{A}}_{q}(2,l) \otimes S^{\mathrm{A}}_{q}(2,l+1)$ has wild representation type for $l \geqslant 3$. Therefore, $S^{\mathrm{B}}_{Q,q}(4,d)$ has wild representation type for $d \geqslant 2l$ and $l \geqslant 3$. For $l = 2$, the same argument can be used to show that $S^{\mathrm{B}}_{Q,q}(4,d)$ has wild representation type for d-even and $d \geqslant 4$.

This reduces us to analyzing $S^{\mathrm{B}}_{Q,q}(4,d)$ when $l = 2$ and $d \geqslant 4$ is odd. By analyzing the components of $S^{\mathrm{B}}_{Q,q}(4,d)$ via the isomorphism in Theorem 9.1.1, one can show that for d odd: (i) $S^{\mathrm{B}}_{Q,q}(4,d)$ has finite representation type (not semisimple) for $4 \leqslant d \leqslant 2p-1$ and $p \geqslant 3$, (ii) $S^{\mathrm{B}}_{Q,q}(4,d)$ has finite representation type (not semisimple) for $d \geqslant 4$ and $p = 0$, (iii) $S^{\mathrm{B}}_{Q,q}(4,d)$ has wild representation type for $d \geqslant 2p+1$ for $p \geqslant 5$, and (iv) $S^{\mathrm{B}}_{Q,q}(4,d)$ has wild representation type for $d \geqslant 2p+3$ for $p = 3$. One has then show that $S^{\mathrm{B}}_{Q,q}(4,7)$ for $p = 3$, $l = 2$ has tame representation type since the component $S^{\mathrm{A}}_{q}(2,6) \otimes S^{\mathrm{A}}_{q}(2,1)$ has tame representation type and the remaining components have finite representation type.

□

Note that for the case $\bar{q} = 1$ (i.e., $q^2 = 1$) one obtains the classical Schur algebra for type A, one can use the results in [Erd] [DN1] [DEMN] to obtain classification results in this case for $S^{\mathrm{B}}_{Q,q}(n,d)$.

Chapter 12

Quasi-hereditary covers

In this section we first recall results on 1-faithful quasi-hereditary covers due to Rouquier [Rou]. Then we demonstrate that our Schur algebra is a 1-faithful quasi-hereditary cover of the type B Iwahori-Hecke algebra via Theorem 9.1.1. Hence, it module category identifies the category $\mathcal{O}$ for the rational Cherednik algebra of type B, see Theorem 12.3.3. A comparison of our Schur algebra with Rouquier's Schur-type algebra is also provided.

12.1 1-faithful covers

Let $\mathcal{C}$ be a category equivalent to the module category of a finite dimensional projective k-algebra A, and let $\Delta = \{\Delta(\lambda)\}_{\lambda\in\Lambda}$ be a set of objects of $\mathcal{C}$ indexed by an interval-finite poset structure Λ. Following [Rou], we say that $\mathcal{C}$ (or $(\mathcal{C}, \Delta)$) is a *highest weight category* if the following conditions are satisfied:

(H1) $\mathrm{End}_{\mathcal{C}}(\Delta(\lambda)) = k$ for all $\lambda \in \Lambda$;

(H2) If $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) \neq 0$ then $\lambda \leqslant \mu$;

(H3) If $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$ then $M = 0$;

(H4) For each $\Delta(\lambda) \in \Delta$ there is a projective module $P(\lambda) \in \mathcal{C}$ such that $\ker(P(\lambda) \to \Delta(\lambda))$ has a Δ-filtration, i.e., finite filtrations whose quotients are isomorphic to objects in Δ.

Let A-mod be the category of finitely generated A-modules. The algebra A is called a *quasi-hereditary cover* of B if the conditions below hold:

(C1) A-mod admits a highest weight category structure $(A\text{-mod}, \Delta)$;

(C2) $B = \mathrm{End}_A(P)$ for some projective $P \in A$-mod;

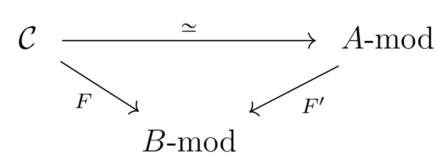
(C3) The restriction of $F = \mathrm{Hom}_A(P, -)$ to the category of finitely generated projective A-modules is fully faithful.

Quasi-hereditary covers are sometimes called highest weight covers since the notion of highest weight category corresponds to that of split quasi-hereditary algebras [Rou, Theorem 4.16]. We also say that (A, F) is a quasi-hereditary cover of B. Moreover, a category $\mathcal{C}$ (or the pair $(\mathcal{C}, F)$) is said to be a quasi-hereditary cover of B if $\mathcal{C} \simeq A$-mod for some quasi-hereditary cover (A, F) of B.

Following [Rou], a quasi-hereditary cover A of B is *i-faithful* if

$$\mathrm{Ext}^j_A(M, N) \simeq \mathrm{Ext}^j_B(FM, FN) \quad \text{for} \quad j \leqslant i, \tag{12.1}$$

and for all $M, N \in A$-mod admitting Δ-filtrations. Furthermore, a quasi-hereditary cover $(\mathcal{C}, F)$ of B is said to be i-faithful if the diagram below commutes for some quasi-hereditary cover (A, F') of B:



Rouquier proved in [Rou, Theorem 4.49] a uniqueness theorem for the 1-faithful quasi-hereditary covers which we paraphrase below:

Proposition 12.1.1. *Let B be a finite projective k-algebra that is split semisimple, and let $(\mathcal{C}_i, F_i)$ for $i = 1, 2$ be 1-faithful quasi-hereditary covers of B with respect to the partial order $\leqslant_i$ on* $\mathrm{Irr}(B)$*. If $\leqslant_1$ is a refinement of $\leqslant_2$ then there is an equivalence $\mathcal{C}_1 \simeq \mathcal{C}_2$ of quasi-hereditary covers of B inducing the bijection* $\mathrm{Irr}(\mathcal{C}_1) \simeq \mathrm{Irr}(B) \simeq \mathrm{Irr}(\mathcal{C}_2)$.

12.2 Rational Cherednik algebras

Let (W, S) be a finite Coxeter group, and let A_W be the corresponding rational Cherednik algebra over $\mathbb{C}[h_u; u \in U]$ as in [Rou], where $U = \bigsqcup_{s \in S} \{s\} \times \{1, \dots, e_s\}$ and e_s is the size of the pointwise stabilizer in W of the hyperplane corresponding to s. If $W = W^{\mathrm{B}}(d)$ and $S = \{s_0, s_1\}$ then $U = \{(s_i, j) \mid 0 \leqslant i, j \leqslant 1\}$. In this case we assume that

$$h_{(s_1,0)} = h, \quad h_{(s_1,1)} = 0, \quad h_{(s_0,i)} = h_i \quad \text{for} \quad i = 0, 1. \tag{12.2}$$

Remark 12.2.1. In [EG] the rational Cherednik algebra $\mathbf{H}_{t,c}$ is defined for a parameter $t \in \mathbb{C}$, and a W-equivariant map $c : S \to \mathbb{C}$. The two algebras, A_W and $\mathbf{H}_{t,c}$, coincide if $t = 1$, $h_{(s,0)} = 0$ and $h_{(s,1)} = c(s)$ for all $s \in S$.

Let $\mathcal{O}_W$ be the category of finitely generated A_W-modules that are locally nilpotent for $S(V)$. It is proved in [GGOR] that $(\mathcal{O}_W, \Delta_W)$ is a highest weight category of $\mathcal{H}(W)$-mod

$$\Delta_W = \{\Delta(E) := A_W \otimes_{S(V) \rtimes W} E \mid E \in \mathrm{Irr}(W)\},$$

See [Rou, 3.2.1–3] for the partial order $\leqslant$ on $\mathrm{Irr}(W)$. Let $\Lambda_2^+(d)$ be the poset of all bipartitions of d on which the dominance order $\trianglelefteq$ is given by $\lambda \trianglelefteq \mu$ if, for all $s \geqslant 0$,

$$\sum_{j=1}^{s} |\lambda_j^{(1)}| \leqslant \sum_{j=1}^{s} |\mu_j^{(1)}|, \quad |\lambda^{(1)}| + \sum_{j=1}^{s} |\lambda_j^{(r)}| \leqslant |\mu^{(1)}| + \sum_{j=1}^{s} |\mu_j^{(r)}|. \tag{12.3}$$

For $\lambda \in \Lambda_2^+(d)$, set

$$W_\lambda^{\mathrm{B}}(d) = C_2^d \rtimes (\Sigma_{\lambda^{(1)}} \times \Sigma_{\lambda^{(2)}}), \tag{12.4}$$

Set

$$I_\lambda(1) = \{1, \dots, |\lambda^{(1)}|\}, \quad I_\lambda(2) = \{|\lambda^{(1)}| + 1, \dots, d\}. \tag{12.5}$$

Following [Rou, 6.1.1], there is a bijection

$$\Lambda_2^+(d) \to \mathrm{Irr}(W^{\mathrm{B}}(d)), \quad \lambda = (\lambda^{(1)}, \lambda^{(2)}) \mapsto \chi_\lambda = \mathrm{ind}_{W_\lambda^{\mathrm{B}}(d)}^{W^{\mathrm{B}}(d)}(\chi_{\lambda^{(1)}} \otimes \phi^{(2)} \chi_{\lambda^{(2)}}), \tag{12.6}$$

where χ_λ is the irreducible character of $W^{\mathrm{B}}(d)$ corresponding to λ, and $\phi^{(2)}$ is the 1-dimensional character of $C_2^{I_\lambda(2)} \rtimes \Sigma_{I_\lambda(2)}$ whose restriction to $C_2^{I_\lambda(2)}$ is det and the restriction to $\Sigma_{I_\lambda(2)}$ is trivial.

Rouquier showed that the order $\leqslant$ is a refinement of the dominance order $\trianglelefteq$ under an assumption on the parameters h, h_i's for the rational Cherednik algebra as follows:

Lemma 12.2.2. *[Rou, Proposition 6.4] Assume that $W = W^B(d)$, $h \leqslant 0$ and $h_1 - h_0 \geqslant (1-d)h$ (see Eq. (12.2)). Let $\lambda, \mu \in \Lambda_2^+(d)$. If $\lambda \trianglelefteq \mu$, then $\chi_\lambda \leqslant \chi_\mu$ on $\mathrm{Irr}(W)$.*

Remark 12.2.3. The assumption in Lemma 12.2.2 on the parameters is equivalent to $c(s_0) = h_1 \geqslant 0$ using Etingof-Ginzburg's convention.

Let KZ_W be the KZ functor $\mathcal{O}_W \to \mathcal{H}(W)$-mod. We paraphrase [Rou, Theorem 5.3] in our setting as below:

Proposition 12.2.4. *If $W = W^{\mathrm{B}}(d)$ and $\mathcal{H}(W) = \mathcal{H}_{Q,q}^{\mathrm{B}}(d)$, then $(\mathcal{O}_W, KZ_W)$ is a quasi-hereditary cover of $\mathcal{H}(W)$-mod. Moreover, the cover is 1-faithful if $(q^2+1)(Q^2+1) \neq 0$.*

It is shown in [Rou] that under suitable assumptions, $\mathcal{O}_{W^{\mathrm{B}}(d)}$ is equivalent to the module category of a Schur-type algebra $S^{\mathrm{R}}(d)$ which does not depend on n using the uniqueness property Proposition 12.1.1. Below we give an interpretation in our setting.

Let $\Lambda_2(d)$ be the set of all bicompositions of d. In [DJM] a cyclotomic Schur algebra over $\mathbb{Q}(q, Q, Q_1, Q_2)$ for each saturated subset $\Lambda \subset \Lambda_2(d)$, which specializes to cyclotomic Schur algebras $S_Q(\Lambda)$ over k is defined (see Section 13.2). Moreover, in [Rou] an algebra $S_Q(\Lambda)$ is defined that is Morita equivalent to $S_Q(\Lambda)$ as given below:

$$S^{\mathrm{R}}(d) := \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(P_d), \quad P_d := \bigoplus_{\lambda \in \Lambda_2^+(d)} m_\lambda \, \mathcal{H}^{\mathrm{B}}_{Q,q}(d). \tag{12.7}$$

where m_λ is defined in Eq. (13.11). Note that $S^{\mathrm{R}}(d)$ does not depend on n. Set

$$F^{\mathrm{R}}_d(-) = \mathrm{Hom}_{S^{\mathrm{R}}(d)}(P_d, -) : S^{\mathrm{R}}(d)\text{-mod} \to \mathcal{H}^{\mathrm{B}}_{Q,q}(d)\text{-mod}. \tag{12.8}$$

Proposition 12.2.5. *[Rou, Theorem 6.6]*

(a) *The category* $\mathrm{Mod}(S^{\mathrm{R}}(d))$ *is a highest weight category for the dominance order;*

(b) $(S^{\mathrm{R}}(d), F^{\mathrm{R}}_d)$ *is a quasi-hereditary cover of* $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$*;*

(c) *The cover* $(S^{\mathrm{R}}(d), F^{\mathrm{R}}_d)$ *is* 1*-faithful if*

$$(q^2+1)(Q^2+1) \neq 0, \quad \text{and} \quad f^{\mathrm{B}}_{Q,q}(d) \cdot \prod_{i=1}^{d} (1 + q^2 + \cdots + q^{2(i-1)}) \neq 0. \tag{12.9}$$

The category $\mathcal{O}$ for the type B rational Cherednik algebra together with its KZ functor can then be identified by combining Propositions 12.1.1, 12.2.4 and 12.2.5. In other words, the following diagram commutes if Eq. (12.9) holds:

$$\begin{array}{ccc} \mathcal{O}_{W^{\mathrm{B}}(d)} & \xrightarrow{\simeq} & S^{\mathrm{R}}(d)\text{-mod} \\ & {\scriptstyle KZ_{W^{\mathrm{B}}(d)}} \searrow \quad \swarrow {\scriptstyle F^{\mathrm{R}}_d} & \\ & \mathcal{H}^{\mathrm{B}}_{Q,q}(d)\text{-mod} & \end{array}$$

12.3 1-faithfulness of $S^{\mathrm{B}}_{Q,q}(n,d)$-mod

Let ℓ be the multiplicative order of q^2 in $k^\times$. In this section we use the following assumptions:

$$f^{\mathrm{B}}_d(Q,q) = \prod_{i=1-d}^{d-1} (Q^{-2} + q^{2i}) \in k^\times, \quad r := \lfloor n/2 \rfloor \geqslant d, \quad \ell \geqslant 4. \tag{12.10}$$

As a consequence, there exists a type B Schur functor by Proposition 10.1.1. For type A, it is known in [HN2] that the q-Schur algebra is a 1-faithful quasi-hereditary cover of the type A Iwahori-Hecke algebra if $\ell \geqslant 4$. Moreover, Theorem 9.1.1 applies and hence we will see shortly that $S^{\mathrm{B}}_{Q,q}(n,d)$ is a 1-faithful quasi-hereditary cover of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$. Furthermore, Proposition 12.1.1 implies that we have a concrete realization for the category $\mathcal{O}$ for the type B rational Cherednik algebra together with its KZ functor using our Schur algebra.

Corollary 12.3.1. *If $f^{\mathrm{B}}_d \in k^\times$, then $S^{\mathrm{B}}_{Q,q}(n,d)$-mod is a highest weight category.*

Proof. It follows immediately from the isomorphism with the direct sum of type A q-Schur algebras that $S^{\mathrm{B}}_{Q,q}(n,d)$-mod is a highest weight category. □

In below we characterize a partial order for highest weight category $S^{\mathrm{B}}_{Q,q}(n,d)$-mod obtained via Corollary 12.3.1 and the dominance order for type A. Denote the set of all N-step partitions of D by $\Lambda^{\mathrm{A}}(N,D)$. Set

$$\Delta^{\mathrm{A}}_{N,D} = \{\Delta^{\mathrm{A}}(\lambda) \mid \lambda \in \Lambda^{\mathrm{A}}(N,D)\}. \tag{12.11}$$

Now $\Delta^{\mathrm{A}}_{N,D}$ is a poset with respect to the dominance order $\trianglelefteq$ on $\Lambda^{\mathrm{A}}(N,D)$. It is well known that for all nonnegative integers N and D, $(S^{\mathrm{A}}_q(N,D)\text{-mod}, \Delta^{\mathrm{A}}_{N,D})$ is a highest weight category.

Recall $\mathcal{F}_S$ from Eq. (10.11) and $\Lambda^{\mathrm{B}}(n,d)$ from Eq. (11.8). Set

$$\Delta^{\mathrm{B}}_{n,d} = \{\Delta^{\mathrm{B}}(\lambda) := \mathcal{F}^{-1}(\Delta^{\mathrm{A}}(\lambda^{(1)}) \otimes \Delta^{\mathrm{A}}(\lambda^{(2)})) \mid \lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda^{B}(n,d)\}. \tag{12.12}$$

Now $\Delta^{\mathrm{B}}_{n,d}$ is a poset with respect to the dominance order (also denoted by $\trianglelefteq$) on $\Lambda^{B}(n,d) \subset \Lambda^{+}_{2}(d)$. Hence, $(S^{\mathrm{B}}_{q}(n,d)\text{-mod}, \trianglelefteq)$ is a highest weight category.

Lemma 12.3.2. *Assume that $S^{\mathrm{B}}_{Q,q}(n,d)$ is a quasi-hereditary cover of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$. If Eq. (12.10) holds, then the cover is 1-faithful.*

Proof. Write $A = S^{\mathrm{B}}_{Q,q}(n,d), B = \mathcal{H}^{\mathrm{B}}_{Q,q}(d), S' = S^{\mathrm{A}}_{q}(\lceil n/2 \rceil, i), S'' = S^{\mathrm{A}}_{q}(\lfloor n/2 \rfloor, d-i)$ for short. We need to show that, for all M, N admitting Δ^{B}-filtrations,

$$\mathrm{Ext}^{i}_{A}(M,N) \simeq \mathrm{Ext}^{i}_{eAe}(F^{\mathrm{B}}_{n,d}M, F^{\mathrm{B}}_{n,d}N), \quad i \leqslant 1.$$

Recall $\mathcal{F}_S$ from Eq. (10.11). Write $\mathcal{F}_S M = \bigoplus_i M'_i \otimes M''_i$ and $\mathcal{F}_S N = \bigoplus_i N'_i \otimes N''_i$ for some $M'_i, N'_i \in \mathrm{Mod}(S')$ and $M''_i, N''_i \in \mathrm{Mod}(S'')$. From construction we see that all M'_i, M''_i, N'_i, N''_i admit Δ^{A}-filtrations since M, N have Δ^{B}-filtrations.

For $\lfloor n/2 \rfloor \geqslant d \geqslant i \geqslant 0$, we abbreviate the type A Schur functors (see Eq. (10.1)) by $F' = F^{\mathrm{A}}_{\lceil n/2 \rceil, i}, F'' = F^{\mathrm{A}}_{\lfloor n/2 \rfloor, d-i}$. Since the type A q-Schur algebras are 1-faithful provided $\ell \geqslant 4$, for $j \leqslant 1$ we have

$$\begin{aligned} \mathrm{Ext}^{j}_{S'}(M'_i, N'_i) &\simeq \mathrm{Ext}^{j}_{\mathcal{H}_q(\Sigma_{i+1})}(F'M'_i, F'N'_i), \\ \mathrm{Ext}^{j}_{S''}(M''_i, N''_i) &\simeq \mathrm{Ext}^{j}_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M''_i, F''N''_i). \end{aligned} \tag{12.13}$$

We show first it is 0-faithful. We have

$$
\begin{aligned}
\mathrm{Hom}_A(M,N) &\simeq \mathrm{Hom}_{\bigoplus_{i=0}^d S'\otimes S''}\Big(\mathcal{F}_S M, \mathcal{F}_S N\Big)\\
&\simeq \bigoplus_{i=0}^{d} \mathrm{Hom}_{S'}(M_i', N_i') \otimes \mathrm{Hom}_{S''}(M_i'', N_i'')\\
&\simeq \bigoplus_{i=0}^{d} \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})}(F'M_i', F'N_i') \otimes \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M_i'', F''N_i'')\\
&\simeq \bigoplus_{i=0}^{d} \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})\otimes\mathcal{H}_q(\Sigma_{d-i+1})}\Big(F'M_i' \otimes F'M_i'', F''N_i' \otimes F''N_i''\Big)\\
&\simeq \bigoplus_{i=0}^{d} \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})\otimes\mathcal{H}_q(\Sigma_{d-i+1})}(\mathcal{F}_H F^\flat_{n,d} M, \mathcal{F}_H F^\flat_{n,d} N)\\
&\simeq \mathrm{Hom}_B(F^\flat_{n,d} M, F^\flat_{n,d} N).
\end{aligned}
\tag{12.14}
$$

Note that the second last isomorphism follows from Proposition 10.1.2. For 1-faithfulness, we have

$$
\begin{aligned}
\mathrm{Ext}^1_A(M,N) &\simeq \bigoplus_{i=0}^{d} \big((\mathrm{Ext}^1_{S'}(M_i', N_i') \otimes \mathrm{Hom}_{S''}(M_i'', N_i''))\\
&\qquad \oplus (\mathrm{Hom}_{S'}(M_i', N_i') \otimes \mathrm{Ext}^1_{S''}(M_i'', N_i''))\big)\\
&\simeq \bigoplus_{i=0}^{d} \big((\mathrm{Ext}^1_{\mathcal{H}_q(\Sigma_{i+1})}(F'M_i', F'N_i') \otimes \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M_i'', F''N_i''))\\
&\qquad \oplus (\mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})}(F'M_i', F'N_i') \otimes \mathrm{Ext}^1_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M_i'', F''N_i''))\\
&\simeq \bigoplus_{i=0}^{d} \mathrm{Ext}^1_{\mathcal{H}_q(\Sigma_{i+1})\otimes\mathcal{H}_q(\Sigma_{d-i+1})}(\mathcal{F}_H F^\flat_{n,d} M, \mathcal{F}_H F^\flat_{n,d} N)\\
&\simeq \mathrm{Ext}^1_B(F^\flat_{n,d} M, F^\flat_{n,d} N).
\end{aligned}
\tag{12.15}
$$

□

Theorem 12.3.3. *Assume that $W = W^B(d)$, $h \leqslant 0$, $h_1 - h_0 \geqslant (1-d)h$ (see Eq. (12.2)) and $(q^2+1)(Q^2+1) \in k^\times$. If Eq. (12.10) holds, then there is an equivalence $\mathcal{O}_W \simeq S^{\mathrm{B}}_{Q,q}(n,d)$-mod* *of quasi-hereditary covers. In other words, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_W & \xrightarrow{\ \simeq\ } & S^{\mathrm{B}}_{Q,q}(n,d)\text{-mod} \\ & {\scriptstyle KZ_W}\searrow \quad \swarrow{\scriptstyle F^{\flat}_{n,d}} & \\ & \mathcal{H}^{\mathrm{B}}_{Q,q}(d)\text{-mod} & \end{array}$$

Proof. The theorem follows by combining Propositions 12.1.1, 12.2.4 and Lemmas 12.2.2, 12.3.2. □

Remark 12.3.4. The uniqueness theorem for 1-faithful quasi-hereditary covers also applies on our Schur algebras and Rouquier's Schur-type algebras. That is, the following diagram commutes provided Eq. (12.9) and Eq. (12.10) hold:

$$\begin{array}{ccc} S^{\mathrm{R}}(d)\text{-mod} & \xrightarrow{\ \simeq\ } & S^{\mathrm{B}}_{Q,q}(n,d)\text{-mod} \\ & {\scriptstyle F^{\mathrm{R}}_d}\searrow \quad \swarrow{\scriptstyle F^{\flat}_{n,d}} & \\ & \mathcal{H}^{\mathrm{B}}_{Q,q}(d)\text{-mod} & \end{array}$$

Chapter 13

Variants of q-Schur algebras of type B/C

It is interesting that the type A q-Schur algebra admits quite a few distinct generalizations in type B/C in literature. This is due to that the type A q-Schur algebra can be realized differently due to the following realizations of the tensor space $(k^n)^{\otimes d}$: (1) A combinatorial realization as a quantized permutation module (cf. [DJ2]); (2) A geometric realization as the convolution algebra on GL_n-invariant pairs consisting of a n-step partial flag and a complete flag over finite field (cf. [BLM]).

In the following sections we provide a list of q-Schur duality/algebras of type B/C in literature, paraphrased so that they are all over k, and with only one parameter q. These algebras are all of the form $\mathrm{End}_{\mathcal{H}_q^{\mathrm{B}}(d)}(V^{\otimes d})$ for some tensor space that may have a realization $V^{\otimes d} \simeq \bigoplus_{\lambda \in I} M^\lambda$ via induced modules. Considering the specialization at $q = 1$, we have

$$M^\lambda\big|_{q=1} = \mathrm{ind}_{H_\lambda}^{W^{\mathrm{B}}(d)} U, \quad H_\lambda \leqslant W^{\mathrm{B}}(d) \text{ is a subgroup}, \quad U \text{ is usually the trivial module.}$$

We summarize the properties of the q-Schur algebras in the following table:

	Coideal q-Schur Algebra $S^{\mathrm{B}}_q(n,d)$	Cyclotomic Schur Algebra $S_q(\Lambda)$	Sakamoto-Shoji Algebra $S^{\mathrm{B}}_q(a,b,d)$
Index set I	compositions $\lambda = (\lambda_i)_{i\in I(n)}$ with constraints on λ_i	bicompositions $\lambda = (\lambda^{(1)}, \lambda^{(2)})$	unclear
Subgroup H_λ	$W^{\mathrm{B}}(\lambda_0) \times \Sigma_{(\lambda_1,\dots\lambda_r)}$	$(C_2^{\lvert\lambda^{(1)}\rvert} \times C_2^{\lvert\lambda^{(2)}\rvert}) \rtimes \Sigma_\lambda$	unknown
Module U	trivial	nontrivial	
Schur duality	$(U^{\mathrm{B}}_q(n), \mathcal{H}^{\mathrm{B}}_q(d))$	unknown	$(U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b), \mathcal{H}^{\mathrm{B}}_q(d))$
Cellularity	new [LNX]	known [DJM]	unknown
Quasi-heredity	new [LNX]	known [DJM]	unknown
Schur functor	new [LNX]	unknown	unknown

For completeness a more involved q-Schur algebra (referred as the q-Schur^2 algebras) of type B is studied in [DS]. We also distinguish the coideal q-Schur algebras from the slim cyclotomic Schur algebras constructed in [DDY].

13.1 The coideal Schur algebra $S^{\mathrm{B}}_{Q,q}(n,d)$

To distinguish $S^{\mathrm{B}}_q(n,d)$ from the other variants we call them for now the coideal Schur algebras since they are homomorphic images of coideal subalgebras.

For the equal-parameter case, a geometric Schur duality is established between $\mathcal{H}^{\mathrm{B}}_q(d)$ and the coideal subalgebra $U^{\mathrm{B}}_q(n)$ as below (cf. [BKLW]):

$$\begin{array}{ccccccc} U^{\mathrm{B}}_q(n) & & & & & & \\ \downarrow & & & & & & \\ S^{\mathrm{B}}_q(n,d) & \curvearrowright & T^{\mathrm{B}}_{\mathrm{geo}}(n,d) \simeq (k^n)^{\otimes d} \simeq T^{\mathrm{B}}_{\mathrm{alg}}(n,d) & \curvearrowleft & \mathcal{H}^{\mathrm{B}}_q(d) \end{array}$$

Note that a construction using type C flags is also available, and it produces isomorphic Schur algebras and hence coideals. A combinatorial realization $T^{\mathrm{B}}_{\mathrm{alg}}(n,d)$ as a quantized permutation module is also available along the line of Dipper-James.

For the case with two parameters, the algebra $S^{\mathrm{B}}_{Q,q}(n,d)$, when n is even, was first introduced by Green and it is called the hyperoctahedral q-Schur algebra [Gr2]. A two-parameter upgrade for the picture above is partially available - a Schur duality is obtained in [BWW] between the two-parameter Iwahori-Hecke algebra $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ and the two-parameter coideal $\mathbb{U}^{\mathrm{B}}_n$ over the tensor space $\mathbb{Q}(Q,q)$; a two-parameter upgrade for $T^{\mathrm{B}}_{\mathrm{rm}}(n,d)$ is studied in [LL] - while a two-parameter upgrade for $T^{\mathrm{B}}_{\mathrm{geo}}(n,d)$ remains unknown since dimension counting over finite fields does not generalize to two parameters naively.

To our knowledge, this is the only q-Schur algebras for the Iwahori-Hecke algebras of type B that admit a coordinate algebra type construction and a notion of the Schur functors with the existence of appropriate idempotents.

13.2 Cyclotomic Schur algebras

The readers will be reminded shortly that the cyclotomic Iwahori-Hecke algebra $\mathbb{H}(r,1,d)$ of type $G(r,1,d)$ is isomorphic to $\mathcal{H}^{\mathrm{B}}_q(d)$ at certain specialization when $r=2$. For each saturated subset Λ of the set of all bicompositions, Dipper-James-Mathas (cf. [DJM]) define the cyclotomic Schur algebra $\mathbb{S}(\Lambda)$:

$$S_q(\Lambda) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_q(d)}\, T(\Lambda),$$

where $T(\Lambda)$ is a quantized permutation module that has no known identification with a tensor space. While a cellular structure (and hence a quasi-heredity) is obtained for $S_q(\Lambda)$, it is unclear if it has a double centralizer property. We also remark that there is no known identification of $T^{\mathrm{B}}_{\mathrm{alg}}(n,d)$ with a $T(\Lambda)$ for some Λ.

Let $R=\mathbb{Q}(q,Q,Q_1,Q_2)$. The cyclotomic Iwahori-Hecke algebra (or Ariki-Koike algebra) $\mathbb{H}=\mathbb{H}(2,1,d)$ is the R-algebra generated by $T^{\Delta}_0,\dots,T^{\Delta}_{d-1}$ subject to the relations below, for

$1 \leqslant i \leqslant d-1, 0 \leqslant j < k-1 \leqslant d-2$:

$$(T_0^{\Delta} - Q_1)(T_0^{\Delta} - Q_2) = 0, \quad (T_i^{\Delta} + 1)(T_0^{\Delta} - q_{\Delta}) = 0, \tag{13.1}$$

$$(T_0^{\Delta} T_1^{\Delta})^2 = (T_1^{\Delta} T_0^{\Delta})^2, \quad T_i^{\Delta} T_{i+1}^{\Delta} T_i^{\Delta} = T_{i+1}^{\Delta} T_i^{\Delta} T_{i+1}^{\Delta}, \quad T_k^{\Delta} T_j^{\Delta} = T_j^{\Delta} T_k^{\Delta}. \tag{13.2}$$

Next we rewrite the setup in . *loc. cit.* using the following identifications:

$$q_{\Delta} \leftrightarrow q^{-2}, \quad T_i^{\Delta} \leftrightarrow q^{-1} T_i. \tag{13.3}$$

Under the identification, the Jucy-Murphy elements are, for $m \geqslant 1$,

$$\begin{aligned} L_m &= (q_{\Delta})^{1-m} T_{m-1}^{\Delta} \dots T_0^{\Delta} \dots T_{m-1}^{\Delta} \\ &= (q T_{m-1}^{\Delta}) \dots (q T_0^{\Delta}) \dots (q T_{m-1}^{\Delta}) \\ &= T_{m-1} \dots T_0 \dots T_{m-1}. \end{aligned} \tag{13.4}$$

Then the cyclotomic relation is

$$(q^{-1} T_0^{\mathrm{B}} - Q_1)(q^{-1} T_0^{\mathrm{B}} - Q_2) = 0, \quad \text{or} \quad (T_0^{\mathrm{B}} - q Q_1)(T_0^{\mathrm{B}} - q Q_2) = 0. \tag{13.5}$$

This is equivalent to our Iwahori-Hecke relation at the specialization below:

$$Q_1 = -q^{-1} Q, \quad Q_2 = q^{-1} Q^{-1}. \tag{13.6}$$

In summary we have the following isomorphism of k-algebras.

Proposition 13.2.1. *The type B Iwahori-Hecke algebra* $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$ *is isomorphic to the cyclotomic Iwahori-Hecke algebra* $\mathbb{H}(2,1,d)$ *at the specialization* $Q_1 = -q^{-1} Q, Q_2 = q^{-1} Q^{-1}$.

For a composition $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{N}^\ell$ of ℓ parts write

$$|\lambda| = \lambda_1 + \ldots + \lambda_\ell, \quad \text{and} \quad \ell(\lambda) = \ell. \tag{13.7}$$

A bicomposition of d is a pair $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of compositions such that $|\lambda^{(1)}| + |\lambda^{(2)}| = d$. We denote the set of bicompositions of d by $\Lambda_2 = \Lambda_2(d)$. A bicomposition λ is a bipartition if $\lambda^{(1)}, \lambda^{(2)}$ are both partitions. The set of bipartitions of d is denoted by $\Lambda_2^+ = \Lambda_2^+(d)$.

Following [DJM], the cyclotomic Schur algebras can be defined for any saturated subset Λ of the set $\Lambda_2(d)$ of all bicompositions of d. That is, any subset Λ of Λ_2 satisfying the condition below:

$$\text{If } \mu \in \Lambda, \nu \in \Lambda_2^+(d) \text{ and } \nu \rhd \mu, \text{ then } \nu \in \Lambda. \tag{13.8}$$

For each Λ we define a cyclotomic Schur algebra $\mathbb{S}(\Lambda) = \mathrm{End}_{\mathbb{H}}\left(\bigoplus_{\lambda \in \Lambda} m_\lambda \mathbb{H}\right)$, where

$$m_\lambda = u^+_{\ell(\lambda^{(1)})} x_\lambda, \quad u^+_{\ell(\lambda^{(1)})} = \prod_{m=1}^{\ell(\lambda^{(1)})} (L_m - Q_2), \quad x_\lambda = \sum_{w \in \Sigma_\lambda} T_w, \tag{13.9}$$

and $\Sigma_\lambda = \Sigma_\lambda^{(1)} \times \Sigma_\lambda^{(2)}$ is the Young subgroup of Σ_d. The specialization $S_Q(\Lambda)$ of $\mathbb{S}(\Lambda)$ at $Q_1 = -q^{-1}Q, Q_2 = q^{-1}Q^{-1}$ is then given by

$$S_Q(\Lambda) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_{Q,q}} \left(\bigoplus_{\lambda \in \Lambda} m_\lambda \, \mathcal{H}^{\mathrm{B}}_{Q,q} \right), \tag{13.10}$$

where

$$m_\lambda = (L_1 - q^{-1}Q^{-1}) \ldots (L_{\ell(\lambda^{(1)})} - q^{-1}Q^{-1}) x_\lambda. \tag{13.11}$$

Let $\mathcal{T}_0(\lambda, \mu)$ be the set of semi-standard λ-tableaux of type μ, that is, any $T = (T^{(1)}, T^{(2)}) \in \mathcal{T}_0(\lambda, \mu)$ satisfies the conditions below:

(S0) T is a λ-tableau whose entries are ordered pairs (i, k), and the number of (i, j)'s appearing is equal to $\mu_i^{(k)}$;

(S1) entries in each row of each component $T^{(k)}$ are non-decreasing;

(S2) entries in each column of each component $T^{(k)}$ are strictly increasing;

(S3) entries in $T^{(2)}$ must be of the form $(i, 2)$.

We note that the dimension of the cyclotomic Schur algebra Λ is given by

$$\dim S_Q(\Lambda) = \sum_{\substack{\lambda \in \Lambda_2^+(d) \\ \mu,\nu \in \Lambda}} |\mathcal{T}_0(\lambda, \mu)| \cdot |\mathcal{T}_0(\lambda, \nu)|. \tag{13.12}$$

It is then define a "tensor space" $T_Q(\Lambda) = \bigoplus_{\lambda \in \Lambda} m_\lambda \mathcal{H}_{Q,q}^{\mathrm{B}}$ which has an obvious $S_q(\Lambda)$–$\mathcal{H}_q^{\mathrm{B}}(d)$-bimodule structure.

Example 13.2.2. Let

$$\Lambda_{a,b} = \Lambda_{a,b}(d) = \{\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2(d) \mid \ell(\lambda^{(1)}) \leqslant a, \ell(\lambda^{(2)}) \leqslant b\}. \tag{13.13}$$

Recall that the dominance partial order in $\Lambda_2^+(1)$ is given by $\mu_2 = (\square, \varnothing) \rhd \mu_1 = (\varnothing, \square)$, and hence $\Lambda_{0,1}(1), \Lambda_{1,1}(1)$ are saturated, while $\Lambda_{1,0}(1)$ is not. The cardinality of $|\mathcal{T}_0(\mu_\bullet, \mu_\bullet)|$ is given as below:

$$|\mathcal{T}_0(\mu_1, \mu_1)| = 1 = |\mathcal{T}_0(\mu_2, \mu_1)| = |\mathcal{T}_0(\mu_2, \mu_2)|, \quad |\mathcal{T}_0(\mu_1, \mu_2)| = 0.$$

Note that $\mathcal{T}_0(\mu_1, \mu_2)$ is empty since the only μ_2-tableau of type μ_1 is $(\varnothing, \boxed{1}\boxed{2})$, which violates (S3). Hence, the dimensions of these cyclotomic Schur algebras are

$$S_q(\Lambda_{0,1}(1)) = 1, \quad S_q(\Lambda_{0,1}(1)) = 3.$$

For $d = 2$, the dominance order in $\Lambda_2^+(2)$ is given by

$$\lambda_5 = (\square\square, \varnothing) \rhd \lambda_4 = \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing\right) \rhd \lambda_3 = (\square, \square) \rhd \lambda_2 = (\varnothing, \square\square) \rhd \lambda_1 = \left(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right).$$

The sets $\Lambda_{0,2}(2), \Lambda_{1,2}(2)$, and $\Lambda_{2,2}(2)$ are saturated. The cardinality of $|\mathcal{T}_0(\lambda_\bullet, \lambda_\bullet)|$ is given in the following table

type\shape	λ_5	λ_4	λ_3	λ_2	λ_1
λ_5	1	0	0	0	0
λ_4	1	1	0	0	0
λ_3	1	1	1	0	0
λ_2	1	0	1	1	0
λ_1	1	1	2	1	1

Hence, the dimensions are

$$\dim S_q(\Lambda_{0,2}(2)) = 3, \quad \dim S_q(\Lambda_{1,2}(2)) = 7, \quad \dim S_q(\Lambda_{2,2}(2)) = 15.$$

Recall that $\dim S_q^{\mathrm{B}}(2, d) = d + 1$ for all d, hence the algebras S_q^{B} and $S_q(\Lambda)$ small ranks do not match in an obvious way.

13.3 Sakamoto-Shoji Algebras

The cyclotomic Iwahori-Hecke algebra $\mathbb{H}(r, 1, d)$ does admit a Schur-type duality (cf. [SS]) with the algebra $U_q(\mathfrak{gl}_{n_1} \times \ldots \times \mathfrak{gl}_{n_r})$ where $n_1 + \ldots + n_r = n$. Hence, it specializes to the following double centralizer properties, for $a+b=n$:

$$\begin{array}{ccccc} U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b) & & & & \\ \downarrow & & & & \\ S_q^{\mathrm{B}}(a,b,d) & \curvearrowright & T(a,b,d) = (k^n)^{\otimes d} & \curvearrowleft & \mathcal{H}_q^{\mathrm{B}}(d) \end{array}$$

We will see in Eq. (13.17) that T_0^{B} acts as a scalar multiple on $T(a,b,d)$, which is different from our T_0^{B}-action § 8.1. Consequently, the duality is different from the geometric one. We could not locate an identification between $S_q^{\mathrm{B}}(a,b,d)$ and $S_q(\Lambda)$ for some Λ in the literature.

Now we set up the compatible version of the cyclotomic Schur duality introduced in [SS]. Let $R' = \mathbb{Q}(Q, q', u_1, u_2)$, and let $\mathbb{H}_{d,2}$ be the the R'-algebra generated by $a_1, \ldots, a_d$ subject to the relations below, for $2 \leqslant i \leqslant d, 1 \leqslant j < k-1 \leqslant d-1$:

$$(a_1 - u_1)(a_1 - u_2) = 0, \quad (a_i - q')(a_i + (q'))^{-1} = 0, \tag{13.14}$$

$$(a_1a_2)^2 = (a_2a_1)^2, \quad a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}, \quad a_ka_j = a_ja_k. \tag{13.15}$$

With the identifications below one has the following result.

$$a_i \leftrightarrow T_{i-1}, \quad q' \leftrightarrow q^{-1} \tag{13.16}$$

Proposition 13.3.1. *The type B Iwahori-Hecke algebra $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$ is isomorphic to the algebra $\mathbb{H}_{d,2}$ at the specialization $u_1 = -Q, u_2 = Q^{-1}$.*

Let $T_Q(a,b,d) = V_{a,b}^{\otimes d}$ where $V_{a,b} = k^a \oplus k^b$ is the natural representation of $U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b)$ with bases $\{v_1^{(1)}, \ldots, v_a^{(1)}\}$ of k^a and $\{v_1^{(2)}, \ldots, v_b^{(2)}\}$ of k^b. The tensor space $T_Q(a,b,d)$ admits an obvious action of the type A Iwahori-Hecke algebra generated by $T_1, \ldots, T_{d-1}$. The T_0^{B}-

action on $T(a,b,d)$ is more subtle as defined by

$$T_0^{\mathrm{B}} = T_1 \circ^{-1} \ldots \circ T_{d-1} \circ^{-1} S_{d-1} \circ \ldots \circ S_1 \circ \varpi \in \mathrm{End}(T(a,b,d)), \tag{13.17}$$

where ϖ is given by

$$\varpi(x_1 \otimes \ldots \otimes x_d) = \begin{cases} -Qx_1 \otimes \ldots \otimes x_d & \text{if } x_1 = v_i^{(1)} \text{ for some } i; \\ Q^{-1}x_1 \otimes \ldots \otimes x_d & \text{if } x_1 = v_i^{(2)} \text{ for some } i, \end{cases} \tag{13.18}$$

and that S_i is given by

$$S_i(x_1 \otimes \ldots \otimes x_d) = \begin{cases} T_i(x_1 \otimes \ldots \otimes x_d) & \text{if } x_i, x_{i+1} \text{ both lies in } k^a \text{ or } k^b; \\ \ldots x_{i-1} \otimes x_{i+1} \otimes x_i \otimes x_{i+2} \otimes \ldots & \text{otherwise.} \end{cases} \tag{13.19}$$

Define

$$S_{Q,q}^{\mathrm{B}}(a,b,d) = \mathrm{End}_{\mathcal{H}_{Q,q}^{\mathrm{B}}(d)}\left(T_Q(a,b,d)\right). \tag{13.20}$$

It is proved in [SS] that there is a Schur duality as below:

$$\begin{array}{ccccc} U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b) & & & & \\ \downarrow & & & & \\ S_q^{\mathrm{B}}(a,b,d) & \curvearrowright & T(a,b,d) & \curvearrowleft & \mathcal{H}_q^{\mathrm{B}}(d) \end{array}$$

Example 13.3.2. Let $a = b = 1, d = 2$. Then $T_Q(1,1,2)$ has a basis $\{v := v_1^{(1)}, w := v_1^{(2)}\}$.

The T_0^{B}-action is given by

$$\begin{aligned}(v\otimes v)T_0^{\mathrm{B}} &= -Qv\otimes v,\\ (v\otimes w)T_0^{\mathrm{B}} &= -Qv\otimes w,\\ (w\otimes v)T_0^{\mathrm{B}} &= Q^{-1}(w\otimes v+(q^{-1}-q)v\otimes w),\\ (w\otimes w)T_0^{\mathrm{B}} &= Q^{-1}w\otimes w.\end{aligned}$$

Note that this is essentially different from the T_0^{B}-action for the coideal Schur algebra given in § 8.1.

13.4 Slim cyclotomic Schur Algebras

The slim cyclotomic Schur algebra $S_{(u_1,\dots,u_r)}(n,d)$ introduced in [DDY] is another attempt to establish a Schur duality for the cyclotomic Iwahori-Hecke algebra $\mathbb{H}(r,1,d)$. When $r=2$, the algebra $S_{(u_1,u_2)}(n,d)$ has the same dimension as the coideal q-Schur algebra $S^{\mathrm{B}}_{Q,q}(2n,d)$; while there is no counterparts for the algebra $S^{\mathrm{B}}_{Q,q}(2n+1,d)$.

It is conjectured in [DDY] that there is a weak Schur duality between the cyclotomic Iwahori-Hecke algebras and certain Hopf subalgebras $U_q(\widehat{\mathfrak{s}l}_n)^{(t)}$ of $U_q(\widehat{\mathfrak{gl}}_n)$ for an integer t to be determined. In our setting it can be phrased as follows:

$$\begin{array}{ccccccccc} U_q(\widehat{\mathfrak{gl}}_n) & \supsetneq & U_q(\widehat{\mathfrak{s}l}_n)^{(t)} & & & & & & \\ & & \downarrow & & & & & & \\ & & S_q^{\hat{\mathrm{A}}}(n,d) & \rightarrow & S_{(q,q)}(n,d) & \curvearrowright & \Omega^{\otimes d} & \curvearrowleft & \mathcal{H}_q^{\mathrm{B}}(d)\end{array}$$

Here $S_{(q,q)}(n,d)=\mathrm{End}_{\mathcal{H}_q^{\mathrm{B}}(d)}(T_{(q,q)}(n,d))$ is the centralizer algebra of the $\mathcal{H}_q^{\mathrm{B}}(d)$-action on a finite dimensional q-permutation module $T_{(q,q)}(n,d)$, while Ω is the (infinite-dimensional) natural representation of $U_q(\widehat{\mathfrak{gl}}_n)$.

We remark that it is called a weak duality in the sense that there are epimorphisms $U_q(\widehat{\mathfrak{s}l}_n)^{(t)} \twoheadrightarrow S_{(q,q)}(n,d)$ and $\mathcal{H}_q^{\mathrm{B}}(d) \twoheadrightarrow \mathrm{End}_{S_{(q,q)}(n,d)}(\Omega^{\otimes d})$; while it is not a genuine double centralizer property.

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