BELIEFS AND KNOWLEDGE OF MIDDLE GRADES TEACHERS REGARDING

FUNCTIONS: TWO CASE STUDIES

by

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(Under the Direction of Denise S. Mewborn)

ABSTRACT

The study of teaching mathematics is a complex endeavor. In fact, attempting to

understand how teachers orchestrate mathematically sound, engaging, and meaningful lessons

often generates more questions than answers.   For many years, mathematics educators

conducted quantitative studies—intended to show a correlation between the instructional

decisions made by teachers and the number of content courses taken in college; however, no

such correlation emerged. In the 1980s, researchers began to examine more affective or

cognitive issues—such as attitudes, beliefs, or knowledge—using methods that were more

qualitative in nature. The purpose of this study was to examine the beliefs that middle grades

mathematics teachers hold about teaching functions, the knowledge that middle grades

mathematics teachers have regarding functions, and the interplay between beliefs and knowledge

during classroom instruction. This study is timely and appropriate given that there is not a large

body of literature on middle grades mathematics education.

Two middle grades mathematics teachers, Melodie and Rachel, taught Algebra I in

different (yet similar) schools within the same district. The two teachers took part in completing

an initial survey, three hour-long interviews, a card sorting activity that dealt with families of

functions, selecting a favorite definition for the term *function*, and modeling a function using a

Calculator-Based Ranger. Classroom observations were completed and artifacts were collected

while each teacher provided instruction on quadratic functions. Data analysis was on-going

throughout the study, and a theoretical framework was derived from literature pertaining to

teacher beliefs, teacher knowledge, and teacher authority.

Melodie's belief that functions are the cornerstone of high school mathematics, coupled

with her deep and flexible knowledge of mathematics, allowed her to teach procedures as well as

to treat concepts. Her students were engaged in inquiry-based activities on a regular basis, and

technology was a staple in her classroom. Rachel was a conveyor of direct instruction and

believed that good mathematics teaching consisted of step-by-step instructions for her students to

follow. She taught functions because they were part of her district's Algebra I curriculum

because her high school colleagues told her that functions were important in later mathematics

courses.

INDEX WORDS: Middle School Mathematics Teaching, Reform, Teacher Beliefs,
                  Mathematical Knowledge of Teachers, Teaching Functions

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DEDICATION

I would like to dedicate this body of work to my grandmother, Azalee Dixon Wynne. She served as my third parent and taught me that even though there is no such thing as perfection, striving to do the best job possible and maintaining a good reputation are paramount to achieving success in work and in life.

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I truly respect and admire the work and scholarship of the members of my committee. I first met Denise Spangler Mewborn in April of 1996. She served as my advisor during my master's program. Denise is a true scholar who approaches situations with common sense, intellect, and zeal. After I completed an action research study about journal writing in the mathematics classroom, Denise encouraged me to co-author an article about the experience and submit it for publication. She spent countless hours editing drafts and making suggestions for ways to improve the content. Had it not been for Denise and all of her words of encouragement, I am not sure that I would have ever attempted to have this work (nor any other) published in the *Mathematics Teacher*. I believe that the experience I gained from working with Denise and the confidence I found from having a publication gave me the confidence to pursue a Ph.D. in Mathematics Education. Of course, Denise continued her support by serving as my major professor, empowering me as a friend, and harassing me (in the most loving manner) to turn in another chapter of the dissertation during my hectic teaching schedule. Likewise, Jeremy Kilpatrick was instrumental in making me aware of mathematics curricula in other countries as well as reading research studies with a critical eye. EMAT 9630 was one of the most enlightening experiences of my doctoral program thanks to Jeremy and his probing questions. Although I never took a class taught by Nicholas Oppong, I had the great pleasure of working as his research assistant during my year of residency. Through this research opportunity, I was fortunate enough to work with and interview mathematics teachers going through the process of earning National Board Certification. I soon came to realize that this process was one of the

most meaningful forms of professional development and self-study. Nicholas was always kind

to me, and I truly enjoyed working with him. Last but not least, David Edwards was excellent in

teaching teachers to embrace the ideas of calculus and to infuse instruction with technology. I

was somewhat hesitant about using the TI-89 in my instruction, but he showed me ways of

making explorations richer—thus increasing one's conceptual understanding of the topics in

calculus.

My friends and colleagues have encouraged me and supported me throughout this

dissertation. I am so lucky to have friends in Augusta, Athens, and Atlanta who care about me as

a person and care about the work that I do. In fact, I am purposely not going to include names

because there are so many, and I do not want to leave anyone off the list. As for colleagues, I

would like to thank the Mathematics Department of Roswell High School and of Thomson High

School. My seven years at Thomson High School allowed me to "cut my teeth" as a beginning

mathematics teacher. I will always remember the teaching, the fun conversations at break and

during lunch, and the overall closeness of our department. Likewise, my colleagues at Roswell

High School are consummate professionals and incredible people. I continue to be amazed at the

level of professionalism and the level of rigor that this department holds. There is no doubt that

Roswell is my home away from home, and that these mathematics teachers are my family away

from my family.

I owe so much to my parents—way beyond the confines of this dissertation. My mother

Joan is probably one of the most decent, loving people that I know. She was (and continues to

be) a great influence on making me the person I am today. My father Billy taught me to always

take care of business and to be the best gentleman possible. There is no doubt that I am a

product of my environment.

I have mentioned friends, colleagues, and family. And then there's Maude—better known as Nancy Blue Williams. She is so "out of the box" that I cannot place her into a single category, so I have chosen to include her in every category. I first met Nancy in the early 1990s at Augusta State University. We were taking a Number Theory course from Fred Maynard, and we got into the habit of studying together—along with Bill and Wendy. After the course was over, the four of us continued to be friends, and Nancy's house was the favorite hang-out. Over time, Nancy and I began graduate school together, we taught together, we co-authored articles together, we planned and presented professional talks together—we even shared an apartment for several summers when we moved from Augusta and lived in Athens. I can recall all of the talks and laughter during dinner at the Boll Weevil, while driving back and forth to Athens after teaching all day, or while planning the next day's lesson or the next article. Nancy and her family always treated me like family, and for that I will always be grateful.

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CHAPTER 1

INTRODUCTION

Mathematics education as a research domain began as early as the 19th century when the Committee of Ten on Secondary School Studies put forth a plan to revitalize mathematics and mathematics teaching (Kilpatrick & Silver, 2000). This event served as the springboard for such efforts as Frank McMurray's address to the National Education Association in 1904 whereby he insisted that mathematics instruction be geared toward (and limited to) mathematics necessary for adults; the inception in 1908 of the International Commission on the Teaching of Mathematics, whose purpose was to carefully examine mathematics curricula and teacher preparation programs; and the formation of the National Council of Teachers of Mathematics (NCTM) in 1920. The NCTM has produced a myriad of publications related to the teaching of mathematics and has coordinated local, state, regional, and national meetings whereby mathematics educators at all levels are able to come together to share teaching ideas, to reflect on their work, and to tackle issues (both practical and theoretical) of mathematics teaching. Although various individuals and groups have offered suggestions for mathematics teaching over the years, none of their publications has combined the structure, scope, and detail of the *Professional Standards for Teaching Mathematics* (NCTM, 1991). This document "called upon teachers to create conditions that would allow learners to focus on important aspects of content and the connections between mathematics and other subject areas, and between various areas within mathematics" (Dossey & Usiskin, 2000, p. 5).

Research in mathematics teaching has flourished and has taken many directions as researchers have proposed various rationales for their individual studies, approached the studies from a myriad of perspectives, and used an assortment of methodologies for data analysis and reporting. Even so, Brophy (1986) insisted that "research on classroom teaching, including research on school mathematics instruction, is still in its infancy" (p. 328). Nearly two decades after Brophy's remark, Ball, Lubienski, and Mewborn (2001) reported that the work students and teachers do together in the classroom lies at the core of mathematics education, but their review of the literature found that research on teachers' knowledge and how it affected student learning was sparse. Koehler and Grouws (1992) made the following remark regarding the quality of instruction:

> Although there are multiple perspectives from which research on teaching can be approached and multiple interpretations of the teaching act, one underlying theme that needs to be more adequately addressed in all research on mathematics teaching, regardless of the philosophical perspective brought to the work, is the notion of quality of instruction. Although there is general agreement that quality of mathematics instruction is important, it is generally not directly addressed in most research studies. It seems to be a variable that researchers have been reluctant to tackle head-on. (p. 124)

Although this statement was made over a decade ago, it still rings true today. Amid public concern for improving the quality of mathematics education, Prichard and Bingaman (1993) defined *good* mathematics teaching as "an inexact blend of a teacher's knowledge of mathematics, pedagogy, and psychology" (p. 217).

Teacher Knowledge

The questions "Where did the subject matter go?" and "What happened to the content?" were posed by Shulman (1986a) after reviewing a body of literature about teaching in which he found little or no focus on subject matter knowledge—he later referred to this lack of emphasis on content as the "missing paradigm" problem. In decades past, researchers in mathematics

education (e.g., Begle, 1972; Eisenberg, 1977) attempted to establish a link between teachers'

content knowledge of mathematics and student achievement by calculating the correlation

between the number of mathematics courses taken by teachers and student learning. Their

analysis of the data revealed that no such correlation existed. In later years, various research

studies indicated that subject matter knowledge does, however, influence teachers' pedagogical

practices, and that increasing content knowledge is necessary for improving mathematics

teaching (Ball & Bass, 2000; Even, 1993, Fennema & Franke, 1992; Goldsmith & Shifter, 1997).

However, Kilpatrick (2003) warned that "teachers need to know mathematics in a special way so

that they can use it in teaching, just as engineers and accountants need to know mathematics in a

special way so that they can use it in their work" (p. 2). This "special way" to which Kilpatrick

was referring is pedagogical content knowledge. Shulman (1986b) provided a framework for

discussing the knowledge that teachers hold by differentiating the types of knowledge by the role

they play in instruction. In particular, he defined *pedagogical content knowledge* as follows:

> The understanding of how particular topics, principles, strategies, and the like in specific subject areas are comprehended or typically misconstrued, are learned and likely forgotten. Such knowledge includes the categories within which similar problem types or conceptions can be classified (what are the ten most frequently encountered types of algebra word problems? Least well-grasped grammatical constructions?), and the psychology of learning them. (p. 26)

The work of Carpenter, Fennema, Peterson, and Carey (1988) extended Shulman's definition to

include "knowledge of the conceptual and procedural knowledge that students bring to the

learning of a topic, the misconceptions about the topic that they may have developed, and the

stages of understanding that they are likely to pass through" (p. 386). Ball (1990) emphasized

that teachers "must understand mathematics deeply themselves…and flexibly enough so that

they can interpret and appraise students' ideas, helping them to extend and formalize intuitive

understandings and challenging incorrect notions" (p. 458). Ma (1999) equated the pedagogical content knowledge of a mathematics teacher to the knowledge of city streets, road closings, and landmarks that a taxi driver must possess, and she argued that teachers must be able to reorganize their knowledge in order to navigate the sometimes murky waters of the classroom—such as students' questions.

Even (1993) claimed that "even though it is usually assumed that teachers' subject-matter knowledge and pedagogical content knowledge are interrelated, there is little research evidence to support and illustrate the relationships" (p. 95). In an effort to better understand this disparity, Even studied 152 prospective secondary mathematics teachers' knowledge of functions—paying particular attention to how functions were defined and represented, the teachers' basic repertoire of functions that would be included in the high school curriculum, and the strength of teachers' conceptions of inverse functions and composition of functions. Even's analysis of the data indicated that these preservice teachers understood that functions could be represented by equations or formulas, believed that the graphs of the functions were to be continuous and smooth, and thought that an infinite number of functions would pass through two fixed points. She warned that a limited or underdeveloped knowledge of mathematics may "contribute to the cycle of discrepancies between concept definition and concept image of functions in students" and that "an important step in improving teaching should be better subject-matter preparation for teachers" (p. 113). Similarly, Wilson (1994) constructed a case study of one preservice secondary teacher's knowledge of functions. He found that his participant, Molly, saw little significance in holding deep and flexible knowledge of functions, and she believed that functions were to be taught as special types of relations and had no relationship to the real world.

Teacher Beliefs

Prichard and Bingaman's (1993) definition of "good mathematics teaching" contained a third component—namely psychology. Beliefs, attitudes, emotions, and other affective domains fall under this heading, and scholars have begun to undertake studies through which these domains are examined and reported. In particular, research on teachers' beliefs about mathematics, mathematics teaching, and students' learning of mathematics has flourished in the past 2 decades and has taken many directions. Thompson's (1982) investigation of three junior high school mathematics teachers' conceptions of mathematics and mathematics teaching was significant in that it was one of the first studies of teachers' beliefs completed in mathematics education. She argued that there is reason to believe that a relationship exists between one's conception of mathematics and one's teaching of mathematics, but "very little is known about the role that teachers' conceptions of the subject matter and its teaching might play in the genesis and evolution of instructional practices characteristic of their teaching" (p. 4). Thompson warned that "failure to recognize the role that the teachers' conceptions might play in shaping their behavior is likely to result in misguided efforts to improve the quality of mathematics instruction in schools" (p. 262).

Cooney, Shealy, and Arvold (1998) suggested that "teachers' beliefs about mathematics and how to teach mathematics are influenced in significant ways by their experiences with mathematics and schooling long before they enter the formal world of mathematics education" (p. 306). Although Cooney and his colleagues focused their study on preservice secondary mathematics teachers, Raymond (1997) found similar results with a beginning elementary school teacher and asserted that "although beginning elementary school teachers often enter the teaching profession with nontraditional beliefs about how they should teach, when faced with constraints

of actual classroom teaching, they tend to implement more traditional classroom practices" (p. 573). She also found inconsistencies between the teachers' professed beliefs and their classroom practices.

Rationale

The study of the professional knowledge necessary for teachers to plan and orchestrate meaningful mathematics lessons has taken many different paths. Eisenberg (1977) and Begle (1972) attempted to find a correlation between the amount of mathematics a teacher had taken and student performance, and analysis of the data indicated that there was hardly any correlation between the two factors. Leinhardt, Zaslavsky, and Stein (1990) and Wilson (1994) devoted a great deal of their work in the 1990s to examining the content knowledge of secondary teachers—mainly the teachers' knowledge about functions and how this knowledge is seen in their instruction. They concluded that teachers had a limited knowledge of functions and tended to have a single "best" representation for a function. Likewise, Thompson (1982, 1984, 1992) and Ball and Bass (2000) spent time during the last 2 decades of the 20th century examining the affective issues that had been neglected in mathematics education up to that point—namely the beliefs that teachers hold about mathematics, mathematics pedagogy, and students. In fact, the aforementioned studies (as well as similar studies in mathematics education) may be categorized as studies about "teacher knowledge" or as "beliefs" studies. Since good mathematics teaching is a blend of teacher knowledge and teacher beliefs, it is reasonable to conduct an integrated study that examines both factors. Given that studies in mathematics education are intended to contribute to and advance the current body of literature in the field, it only makes sense to include mathematics as a central component of the study. In turn, knowledge of mathematics alone does not guarantee a meaningful mathematics lesson. The beliefs that teachers hold about

mathematics and mathematics instruction influence their actions and decisions (Brown & Baird, 1993; Cooney & Shealy, 1997; Harvey, Prather, White, & Hoffmeister, 1968; Thompson, 1984).

A majority of beliefs studies have focused on elementary teachers (Collier, 1972; Raymond, 1997), and a few studies have been conducted using high school teachers (e.g., Cooney et al., 1998). Likewise, teacher knowledge studies have focused on elementary teachers (Ball, 1990; Carpenter, Fennema, Peterson, & Carey, 1988) or secondary teachers (Haimes, 1996; Livingston & Borko, 1990). In either case, middle school teachers seem to be the excluded middle. In fact, middle school teachers are often criticized because of their *perceived* lack of content knowledge. Regardless of their pedagogical knowledge, how can middle school teachers possibly orchestrate a meaningful and effective mathematics lesson if they lack the appropriate content knowledge? If strides are to be made in improving teacher quality, teacher educators must be aware of the status of teachers' knowledge and beliefs about mathematics and mathematics instruction.

The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000) were harbingers of the changing face of middle school mathematics. The nation was deemed "at risk" because school-aged children seemed under-challenged in mathematics classrooms where curricula varied widely and student expectations were minimal. The two documents set forth a cohesive, rigorous set of mathematical goals for all students at all grade levels. The NCTM (1989) described the middle school mathematics curriculum as "a bridge between the concrete elementary school curriculum and the more formal mathematics curriculum of the high school" (p. 102). The exploration of algebraic concepts in informal ways was advocated as a means for helping middle school students with their transition from the arithmetic of the elementary grades to more abstract

concepts found in the middle grades, and for building a foundation for the subsequent study of formal algebra at the secondary level. In contrast, the NCTM (2000) reported in *PSSM* that the algebra strand of the curriculum begins in the elementary grades and increases in intensity as students progress through Grades 6, 7, and 8.   The following quotation captures the NCTM's stance on middle grades algebra:

> Students in the middle grades should learn algebra both as a set of concepts and competencies tied to the representation of quantitative relationships and as a style of mathematical thinking for formalizing patterns, functions, and generalizations. In the middle grades, students should work more frequently with algebraic symbols than in the lower grades. It is essential that they become comfortable in relating symbolic expressions containing variables to verbal, tabular, and graphical representations of numerical and quantitative relationships. Students should develop an initial understanding of several different meanings and uses of variables through representing quantities in a variety of problem situations. They should connect their experiences with linear functions to their developing understandings of proportionality, and they should learn to distinguish linear relationships from nonlinear ones. In the middle grades, students should learn to recognize and generate equivalent expressions, solve linear equations, and use simple formulas. (p. 223)

These methods and ideas about algebra are intended for *all* students. The mathematical foundation from the elementary grades, if laid properly, facilitates students' transition into more formalized algebraic thinking, and it precludes algebra from becoming the notorious "gatekeeper" to more advanced studies of mathematics. Algebra should be tied to the other mathematics strands such as geometry or statistics in order to form a cohesive middle school curriculum and should be infused with hand-held and computer-based technology.

The purpose of this study was to examine the knowledge that middle school algebra teachers have regarding functions, the beliefs they hold about the teaching of functions, and how these two components influence algebra instruction. The following research questions guided the study:

1. How do middle school teachers understand and conceptualize functions?

2. What beliefs do middle school teachers hold about teaching functions?

3. How does middle school algebra teachers' knowledge of functions and their beliefs about teaching functions affect their teaching practices?

This study of middle school algebra teachers is important and timely. It is imperative that teacher educators be cognizant of the content knowledge and beliefs about teaching functions that middle school teachers hold. In particular, their knowledge of functions should be examined under the proverbial microscope because taking a functional approach to Algebra I instruction is now being advocated by certain experts in mathematics education, and this approach is now seen in a many textbooks as well. Teacher educators must also know teachers' beliefs about teaching functions because these beliefs (What is a function? Are functions useful? How can functions be represented? How does the concept of a function fit into the broad scope of mathematics? Are there times in a teacher's personal or professional lives where functions are necessary?) influence their actions and choices in the classroom.

Significance

The notion of functions is important for middle school since the study of functions is no longer restricted to the secondary curriculum. The NCTM (2000) asserted that middle grades students must be able to (1) "represent, analyze, and generalize a variety of patterns with tables, graphs, words, and when possible, symbolic rules"; (2) "relate and compare different forms of representation for a relationship"; and (3) "identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations" (p. 222). If students are to be proficient with providing multiple representations for functions, this proficiency will only occur if teachers encourage and expect such work from their students. It is important for mathematics educators to be aware of the knowledge that middle school teachers possess regarding functions. On one

hand, it is conceivable that some teachers may understand a function to be an equation or a rule such that when a value of x is chosen and directly substituted into the equation, then the output is the y-value corresponding to x. Moreover, they may view the graph of the function as a mere visual representation of the rule. On the other hand, some teachers may view functions as a tool for representing or modeling data, and the rule, the graph, and the table are simply three different representations for the data set. Teachers who make the former argument would see the function as a means for generating points or data, whereas for those making the later argument the data precedes the function.

The NCTM (2000) reported that teachers are "information providers, planners, consultants, and explorers of uncharted mathematical territory" and that they must "develop their own professional knowledge using research, the knowledge base of the profession, and their own experiences as resources" (p. 370). The path that a teacher travels from novice to expert can be meandering, rocky, and encumbered with detours, but this line of research has potential for promoting more efficient teaching strategies, better connecting of mathematics lessons to previous topics or other disciplines, making explicit the goals of the lessons, and understanding what experiences students need in order to construct meaning out of mathematics. Inservice teachers need and deserve professional development opportunities that build upon their current knowledge and belief structures as well as extends and challenges those structures. Mewborn (2003) suggested that professional development must be grounded in sound learning theories, should be organized so that teachers are expected to revisit topics found in school mathematics and gain further insights into their conceptual underpinnings and interconnections, and must provide teachers an opportunity to enhance their ability to listen to students and the ideas that the students bring to the mathematics classroom.

Regardless of the grade in which a student enrolls and earns algebra credit, it appears that mathematical modeling is a permanent fixture in college and university mathematics. Some institutions, such as the University of Georgia, have purged the traditional college algebra course from the curriculum and have replaced it with mathematical modeling. Other postsecondary schools, such as Augusta State University, have incorporated mathematical modeling into their programs and are offering a modeling course to those students who need a mathematics course beyond college algebra. Although the content of the course may vary depending upon the institution, the purpose of the course is the same: to teach students to examine and interpret real world phenomena through mathematical means. The NCTM (2000) asserted that linear functions and mathematical modeling were to be emphasized in the middle grades and should continue throughout Grades 9 – 12. As such, middle grades teachers have an obligation to engage their students in activities that are meaningful and will prepare students for secondary and postsecondary mathematics.

The NCTM's (2000) *Principles and Standards for School Mathematics* was written as a visionary document intended to describe an idealized mathematics classroom. Reports such as the report of the Video Study portion of the Third International Mathematics and Science Study (TIMSS, Stigler & Hiebert, 1999) and *A Nation at Risk* (National Commission on Excellence in Education, 1983) serve as reminders that mathematics education in the United States must undergo a significant metamorphosis if students are to enter universities, technical colleges, or the work force with the depth and breadth of mathematical knowledge and literacy required to be viable, productive citizens capable of making reasonable decisions. By painting a picture of the ideal mathematics classroom, the NCTM in *PSSM* raised the proverbial bar in mathematics education by putting forth "a comprehensive and coherent set of goals for mathematics for all

students" (p. 6) that stakeholders in mathematics education may use as a resource for developing curricula, assessments, and meaningful activities, and as a stimulus for discourse among teachers, administrators, local boards of education, legislators, and national policy makers. Realizing this vision of school mathematics is dependent upon the role that mathematics teachers play. The NCTM further asserted that "teaching mathematics well is a complex endeavor, and there are no easy recipes for helping all students learn or for helping all teachers become effective" (p. 17). Effective teachers have deep and flexible knowledge about mathematics, mathematics pedagogy, and mathematics learning. Effective teachers challenge their students while supporting their ideas, and they seek continual professional development to enhance and use their professional knowledge.

CHAPTER 2

A REVIEW OF THE LITERATURE

A vision of where mathematics education research has been is necessary to fully appreciate the current body of research in the field as well to provide direction for further research. As for instruction, there is no definitive answer for what constitutes "good mathematics teaching," but there is no doubt that the teacher plays a crucial role in facilitating the competence and confidence that students gain in mathematics classrooms. In her quest to better understand the influence of the teacher, Thompson (1992) reported that the pedagogical practices of teachers are influenced by their conceptions about mathematics. In particular, Thompson noted that the ways "teachers interpret and implement curricula is influenced significantly by their knowledge and beliefs" (p. 128). The purpose of this chapter is to review the research on mathematics teaching and to highlight some of the findings. Emphasis is placed on studies completed during the last 2 decades of the 20th century.

A Synthesis of Research on Teaching Mathematics

The practice of teaching mathematics and its far-reaching effects came to the forefront of research in mathematics education during the latter part of the 20th century. Koehler and Grouws (1992) assumed the awesome responsibility of compiling a review of research related to mathematics teaching practices. They suggested four levels of complexity in research pertaining to teachers and their instructional practices as a means of categorizing the various types of

research (for organizational purposes) and as a way to track the progress made in research on teaching during the previous decades. The levels are described in the paragraphs that follow.

Level I research is the simplest of the four types and typically focuses on teacher effectiveness. The opinions of colleagues and supervisors were the basis for identifying effective teachers, and certain characteristics of these teachers were noted and were later used as benchmarks to measure the effectiveness of other teachers. In other words, if Teacher A were labeled "effective" by her principal and the teacher possessed Characteristic A, then other teachers possessing Characteristic A would also be labeled effective. This type of research focused on the *teacher* as opposed to the act of *teaching* (Gomez, 1994; Koehler & Grouws, 1992; Mayer-Smith, Moon, & Wideen, 1994; Shaw, 1996).

Level II studies are more commonly known as process-product research. The idea that teacher behavior affects student behavior (and vice versa) served as the basic premise for this line of research. Numerous hours of classroom observations were necessary in order to collect data through extensive written detail and meticulous documentation of the teacher's instructional practices as well as the ensuing work of the students. Teacher behavior, such as posing questions, responding to student questions, presenting examples, lecturing, using manipulatives or technology, scaffolding, reviewing previous topics, incorporating cooperative learning, and developing new concepts was investigated. This level is similar to Level I in that emphasis is placed on the teacher rather than students, and when student achievement was measured, it was primarily done using standardized tests scores (Evertson, Anderson, Anderson, & Brophy, 1980; Good, Grouws, & Ebmeier, 1983).

Research at Level III is a departure from the two previous levels in that greater emphasis is placed on characteristics such as the race or gender of the students than on characteristics of

the teachers. Furthermore, student achievement is no longer a matter of examining test results. Rather, researchers began to delve into affective domains such as the attitudes, beliefs, or confidence levels of students (Quesada & Maxwell, 1994; Schoenfeld, 1989).

Koehler and Grouws (1992) defined Level IV research as "research that has a strong theoretical foundation and is based on a model that involves many factors" (p. 117). Factors that influence students' actions or behaviors include their attitudes or beliefs about mathematics as a subject, their disposition about themselves as students of mathematics and their ability to succeed in mathematics, and their beliefs about mathematics (whether or not they believe that mathematics is helpful or meaningful in everyday life). Likewise, teacher behavior is shaped by the teacher's knowledge of the topic at hand, the anticipated difficulties that lie ahead of the students when learning the content, the pedagogical ideas for teaching the lesson, and the teacher's attitudes or beliefs about mathematics and mathematics teaching. The outcomes of student learning could be situated in the student's individual actions or behaviors, which in turn are influenced (whether directly or indirectly) by the actions or behaviors of the teacher (Underhill, 1988).

The four levels presented above provide a frame for classifying research on mathematics teaching. The levels may also be thought of as a timeline for research in that Level I studies were the most primitive and dealt primarily with the characteristics of the teacher. Level II evolved from Level I and represented a shift from examining teacher characteristics to observing teacher behavior as seen in the process of teaching mathematics. One-dimensional, student-focused studies were the tradition in Level III, which then evolved into multi-dimensional studies about the beliefs and attitudes of students and teachers and their interplay—referred to as Level IV research.

Multiple Perspectives for Examining Teaching

From this point forward, I emphasize those studies that can be classified as Level IV research and whose purpose was to examine mathematics teaching practices. My goal is to examine the body of literature regarding the act of teaching and its effects. It is appropriate to categorize the present study as a Level IV study because its intent was to examine the influence of teachers' knowledge and beliefs regarding their instructional practices. Furthermore, the findings from earlier studies provided a theoretical framework for presenting and interpreting the data.

Cognitively Guided Instruction

The purpose of the Cognitively Guided Instruction (CGI) program is to provide teachers with a means for understanding how children think, incorporate these strands of thought into their classrooms, and then allow teachers ample time to reflect on their teaching as a result of using the thoughts of their students. Chambers and Hankes (1994) reported the following six characteristics of the CGI program:

1. The curriculum is problem-solving-based rather than focusing on procedures or rote algorithms.

2. The problems that are posed to students are constructed with the students and their previous experiences in mind.

3. Discourse among students is encouraged, and sharing solutions to problems is expected.

4. Students are expected to use more than one method for solving problems.

5. Mathematics is taught as an integrated whole rather than a series of compartmentalized topics. In this sense, mathematical problem solving seems natural and not separated from the real world.

6. Through frequent, yet varied assessment, teachers are better able to understand the thought processes of their students and make better instructional decisions.

The goal of the CGI program was not only to change the face of the mathematics classroom for the students, but to alter the existing knowledge and beliefs of practicing mathematics teachers.

Shulman (1986a) provided a framework for discussing the knowledge that teachers hold by differentiating the types of knowledge by the role that they play in instruction. In particular, he defined *pedagogical content knowledge* as follows:

> The understanding of how particular topics, principles, strategies, and the like in specific subject areas are comprehended or typically misconstrued, are learned and likely forgotten. Such knowledge includes the categories within which similar problem types or conceptions can be classified (what are the ten most frequently encountered types of algebra word problems? Least well-grasped grammatical constructions?), and the psychology of learning them. (p. 26)

The work of Carpenter, Fennema, Peterson, and Carey (1988) extended Shulman's definition to include "knowledge of the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions about the topic that they may have developed, and the stages of understanding that they are likely to pass through" (p. 386).

The Expert-Novice Approach

The CGI approach described above depended upon first understanding the students and the knowledge that they bring with them to the mathematics classroom, and then focus was given to the teachers and how they used the students' knowledge to orchestrate a mathematics lesson. Fennema and Franke (1992) suggested that in addition to understanding the knowledge of students, researchers must also understand the cognition of teachers as well. They further suggested that "one approach to understanding hierarchical knowledge has been to study experts and novices as they solve problems" (p. 152).

Leinhardt (1989) compared the mathematics lessons of expert teachers with those lessons of novice teachers. She defined *novice* as a student teacher enrolled in his or her last semester of coursework but currently engaged in the act of student teaching. She assumed that an *expert* was a practicing teacher whose students' growth scores were in the top 15% for at least 3 years during a 5-year period. Leinhardt interviewed, observed, and videotaped 4 expert teachers and 2 novice teachers over 3 ½ months—paying particular attention to the agendas of the teachers, the overall structure and flexibility of the mathematics lessons taught, and the types of explanations that these teachers provided when clarifying concepts or procedures for students. She found that expert teachers tended to "weave a series of lessons together to form an instructional topic in ways that consistently build upon and advanced material introduced in prior lesson" and they "construct lessons that display a highly efficient internal structure, one that is characterized by fluid movement from one type of activity to another, by minimal student confusion during instruction, and by a transparent system of goals" (p. 73). In contrast, novice teachers' lessons were more fragmented, the transition from one mathematical topic to the next was not as smooth or efficient as that of an expert teacher, and the novices were more prone to compromise their goals than the experts were.

A study similar to that of Leinhardt was conducted by Livingston and Borko (1990) in which they contrasted the review lessons of two student teachers to the review lessons of their respective mentor teachers. Livingston and Borko's interpretation of the data showed that the expert teachers were better at explaining mathematics problems and identifying the relationships across the collection of problems. The novice teachers tended to present mathematics as a set of prescribed rules or procedures, and they did not focus on the conceptual underpinnings of the subject matter at hand. Furthermore, the expert teachers "entered their lessons with flexible

working plans supported by well-developed, integrated, and easily accessible schemata," and

they "were able to improvise successfully, conducting reviews that were both responsive and

comprehensive" (p. 384).

Research studies such as the ones discussed above are intended to make teacher educators

aware of many of the effective teaching practices demonstrated by expert teachers. In turn,

teacher educators may begin to help novice teachers weave these scripts and actions into their

own tapestry of mathematics teaching.

Teacher Knowledge

Effective mathematics teaching is a "complex endeavor" in which there is no "easy

recipe" (NCTM, 2000, p. 17). Mathematics educators continue to grapple with the issue of what

knowledge an individual must possess in order to be an effective teacher of mathematics.

NCTM took the following stance in *Principles and Standards for School Mathematics*:

> Teachers need several different kinds of mathematical knowledge—knowledge about the
> whole domain; deep flexible knowledge about curriculum goals and about the important
> ideas that are central to their grade level; knowledge about the challenges students are
> likely to encounter in learning these ideas; knowledge about how the ideas can be
> represented to teach them effectively; and knowledge about how students' understanding
> can be assessed. (p. 17)

Shulman (1986a) asked the questions, "Where did the subject matter go?" and, "What happened

to the content?" after reviewing a body of literature about teaching in which he found little or no

focus on subject matter knowledge—later referring to this lack of emphasis on content as the

"missing paradigm" problem (pp. 5-6). Various research studies have suggested that subject

matter knowledge does influence teachers' pedagogical practices and that increasing content

knowledge is necessary for improving mathematics teaching (Ball & Bass, 2000; Even, 1993;

Fennema & Franke, 1992; Goldsmith & Shifter, 1997). However, Kilpatrick (2003) warned that

"teachers need to know mathematics in a special way so that they can use it in teaching, just as

engineers and accountants need to know mathematics in a special way so that they can use it in their work" (p. 2). This "special way" of knowing to which Kilpatrick was referring is known as pedagogical content knowledge. Ball and Bass (2000) defined pedagogical content knowledge as "clusters that embed knowledge of mathematics, of students, and of pedagogy" (p. 89). In this section, selected studies pertaining to content knowledge and pedagogical content knowledge of mathematics are discussed and many of the findings from the literature are shared.

Even (1993) studied 152 prospective secondary mathematics teachers' knowledge of functions. She found that these preservice teachers understood that functions could be represented by equations or formulas, believed that the graphs of the functions were to be continuous and smooth, and thought that an infinite number of functions could pass through two fixed points. Even warned that a limited or undeveloped knowledge of mathematics may "contribute to the cycle of discrepancies between concept definition and concept image of functions in students" and that "an important step in improving teaching should be better subject-matter preparation for teachers" (p. 113). Ball (1990) sought to understand the knowledge of division of fractions held by 252 preservice elementary and secondary mathematics teachers by using a single questionnaire item and one interview task. Her analysis of the data showed that few secondary teacher candidates and no elementary candidates were able to provide an appropriate representation for division of fractions (other than dividing round food such as pizza or cake).

Carpenter et al. (1988) studied 40 first-grade teachers and their pedagogical content knowledge regarding students' solutions to word problems that involved addition and subtraction. In order to measure the knowledge of these elementary teachers, Carpenter and his colleagues examined their ability to differentiate between different types of word problems,

understand the various strategies students used for solving each problem, and predict the method individual students chose when solving the problems. They reported that a majority of the teachers experienced difficulty in classifying the word problems by type. Although a few of the first-grade teachers were able to differentiate between the problems, their struggle came later when they were asked to provide a reason for how they were able to differentiate one word problem from another. Likewise, most of these teachers were able to recognize problems according to whether or not addition or subtraction was involved, but fewer teachers were able to categorize the problems according to common strategies used by first-grade students. Carpenter et al. suggested that the teachers were not successful with differentiating between types of word problems and choosing student strategies for solving word problems because of "the lack of variability on the measures of teachers' general knowledge of problems and strategies."

In order to better understand the evolution of teachers' content knowledge, Wilson (1994) studied a preservice secondary mathematics teacher Molly and her knowledge of functions. Data were collected through written assessments in which Molly was expected to interpret functional situations, seven hour-long interviews, classroom observations, and researcher-created activities with functions (choosing a favorite definition of function from a list of common definitions used throughout the last century, engaging in dialogue about teaching functions from vignettes, organizing cards containing various functions, and solving problems in context). Molly conceptualized a function as a numerical operation. Furthermore, Wilson reported that she sorted the function cards by their representation (graph, equation, etc.) rather than by family (linear, quadratic, etc.). Likewise, when solving a contextual problem, Molly relied on a numerical approach rather than examining a graph or an equation. I used Wilson's historical

definitions for functions and the function card sort activity as a means of gathering data about middle schools teachers' knowledge about functions.

<div align="center">Teacher Beliefs</div>

Thompson's (1982) work was one of the first studies of teachers' beliefs in mathematics education. She investigated three junior high school mathematics teachers (Jeanne, Kay, and Lynn) by examining their conceptions of mathematics and mathematics teaching. Her intent was to identify concepts, perspectives, and beliefs that constituted the teachers' conceptions. In particular, Thompson sought to answer the question, "How are teachers' professed beliefs, views, and preferences about mathematics and mathematics teaching reflected in their instructional practices?" (p. 4). The common focus of research studies prior to Thompson's was predominately the behavior of the teacher rather than the teachers' thoughts. She argued that there is reason to believe that a relationship exists between one's conception of mathematics and one's teaching of mathematics, but "very little is known about the role that teachers' conceptions of the subject matter and its teaching might play in the genesis and evolution of instructional practices characteristic of their teaching" (p. 4). Thompson warned that "failure to recognize the role that the teachers' conceptions might play in shaping their behavior is likely to result in misguided efforts to improve the quality of mathematics instruction in schools" (p. 262).

Thompson (1982) used the method of case studies to report on each teacher's conceptions of mathematics, conceptions of mathematics teaching, and criteria for judging effectiveness of instruction. She found that, for the most part, teachers' preferences and views of mathematics were reflected in their teaching practices. All three teachers believed that mathematics was relevant to daily life and served as an important tool for solving problems, but none of them, however, incorporated applications into their lessons. The participants cited lack of interest in

the application, lack of familiarity with the application, and deficiencies in the students'

mathematical backgrounds as reasons for not teaching applications. Also, the differing views of

mathematics, what constitutes mathematical understanding, and the purpose or benefit of lesson

planning held by the teachers related to their views about teaching. The most striking

inconsistencies that Thompson found related to teachers' beliefs about teaching were

encouraging student participation, using a wide variety of instructional approaches, and realizing

their goals in the context of mathematics education. Adherence to lesson plans, reduction of

potential discipline problems, general dissatisfaction with teaching, reliance on the textbook, lack

of familiarity with alternative explanations, and following the path of least resistance were the

reasons given for these inconsistencies. Thompson reported that "the differences among the

teachers in their views about teaching that seemed to be most directly related to differences in

their characteristic behavior lay in their views about their own role and the students' role, the

need to plan their lessons, the desirability of several specific pedagogical practices, and the

appropriate locus of control in the teaching process" (p. 267).

Cooney, Shealy, and Arvold (1998) suggested that "teachers' beliefs about mathematics

and how to teach mathematics are influenced in significant ways by their experiences with

mathematics and schooling long before they enter the formal world of mathematics education"

(p. 306). Cooney and his colleagues examined the belief structures of 4 preservice secondary

mathematics teachers (Greg, Sally, Henry, and Nancy) as they completed the last 2 years of their

teacher preparation coursework (including student teaching). The beliefs data collected through

surveys, classroom observations, written assignments, and interviews were analyzed using

Green's (1971) multidimensional perspective of the structure of beliefs. Green reported the

following:

> We may, therefore, identify three dimensions of belief systems. First, there is the quasi-logical relation between beliefs. They are primary or derivative. Secondly, there are relations between beliefs having to do with their spatial order or their psychological strength. They are central or peripheral. But there is a third dimension. Beliefs are held in clusters, as it were, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs. Each of these characteristics of belief systems has to do not with the content of our beliefs, but with the way we hold them. (pp. 47 – 48).

An analysis of the data revealed that each of the four teachers wanted affirmation for what he or she believed was the role of a good mathematics teacher. For example, Greg believed that the purpose of teaching mathematics was to prepare students to enter the world of work, and over time began to see how the use of technology (which he initially did not value) could facilitate his goals for teaching mathematics. Greg also valued the thoughts, opinions, and suggestions of his classmates—many of which he held as peripheral beliefs that he later assimilated into his repertoire of centrally held beliefs. Cooney and his colleagues also reported that "a teacher's movement from conceptualizing knowledge as something emanating from external beings toward conceptualizing knowledge as something emanating from interrelationships between self and others is an important consideration in conceptualizing teachers' professional development" (p. 329).

Beliefs about Mathematics and Mathematics Teaching

Raymond (1997) investigated the relationship between a beginning elementary teacher's professed beliefs about mathematics and its instruction, and the teacher's actual teaching practices. Data collection lasted approximately 10 months and consisted of six hour-long interviews, five classroom observations, clusters of artifacts such as lesson plans, a concept map activity, and a questionnaire about mathematics beliefs. Raymond began her data analysis by categorizing her data as beliefs (later subdivided into beliefs about the nature of mathematics, the learning of mathematics, or the teaching of mathematics), teaching practice (later subdivided into

tasks, discourse, environment, or evaluation), or influences on beliefs and practice (later

subdivided into social teaching norms, immediate classroom situation, prior school experiences,

or other). The data regarding beliefs about mathematics content, teaching, and learning were

categorized as traditional, primarily traditional, an even mix of traditional and nontraditional,

primarily nontraditional, and nontraditional (p. 556). Finally, Raymond used the work of Ernest

(1989b) as a means for analyzing and discussing teachers' beliefs about the nature of

mathematics, Kuhs and Ball (1986) for analyzing beliefs about teaching mathematics, and

Underhill (1988) for examining beliefs about learning mathematics. I used Raymond's scale of

traditional to nontraditional when discussing teachers' beliefs about mathematics as well as about

mathematics instruction.

Raymond organized and presented the case in five parts: (a) background and setting, (b)

description and categorization of Joanna's beliefs (beliefs about mathematics, learning, teaching,

and influences), (c) Joanna's teaching practices (classroom environment, classroom discourse,

mathematical tasks and evaluations, and planning), (d) inconsistencies between Joanna's

professed beliefs and classroom practice, and (e) the influence of beliefs on practice. She found

that Joanna believed that her teacher preparation program had minimal impact on her

instructional practices and moderate impact on her beliefs. Joanna's beliefs about mathematics

were traditional, whereas her beliefs about teaching and learning mathematics were rather

nontraditional, and her beliefs about teaching and learning mathematics were the most

inconsistent with her actual practice. Raymond pointed out that "although beginning elementary

school teachers often enter the teaching profession with nontraditional beliefs about how they

should teach, when faced with constraints of actual classroom teaching, they tend to implement

more traditional classroom practices" (p. 573).

Collier (1972) conducted a study intended to measure prospective elementary school teachers' beliefs about mathematics and mathematics instruction. The participants were categorized by their academic records and were then placed into one of four groups: Group I, no prior enrollment in college mathematics courses; Group II, completion of one mathematics course; Group III, completion of two mathematics courses; and Group IV, completion of two mathematics courses and a methods course. The participants responded to a list of 80 questions by rating them on a six-point scale, where 1 represented "strongly disagree" and 6 represented "strongly agree." The items themselves were designed to measure a formal-informal dimension of teachers' beliefs about mathematics and mathematics instruction. Using quantitative methods such as a two-way ANOVA and individual *t*-tests to analyze the data, Collier concluded that prospective teachers enter elementary teacher education programs with neutral beliefs about the program itself, they do not view mathematics as formal or informal, and their beliefs about mathematics instruction are neutral. After two college mathematics courses their beliefs about the program remain neutral, but their views of mathematics become somewhat informal and their views of mathematics instruction are still neutral. Upon completion of two college mathematics courses and a methods course students continued to have an informal view of mathematics and a moderately informal view of mathematics instruction.

<div align="center">Distinguishing Beliefs from Knowledge</div>

Fennema and Franke (1992) claimed that mathematics, mathematical representations, current theories of learning mathematics, and mathematics pedagogy are the components of teachers' professional knowledge. They also insisted that "teachers' beliefs . . . have a profound effect on the decisions that they make, which in turn determine to a large extent what students learn in their classrooms" (p. 156). Unfortunately, "research on affect in mathematics education

continues to reside on the periphery of the field" (McLeod, 1992, p. 575). If research on

teaching and learning is to continue to influence teachers and students, then research on beliefs

must become a more central and focused concern of the mathematics education community.

McLeod argued that affect is generally more difficult to observe, measure, and describe than

knowledge.

The absence of a theoretical framework in which to ground studies of beliefs could have

prevented researchers from advancing this line of inquiry. Answering the call to create a

theoretically-grounded model of belief systems, Nespor (1987) provided a conceptualization of

beliefs that was an extension of research in cognitive science and cognitive psychology. The

model proposed by Nespor consists of six structural features of beliefs, and these features serve

to distinguish beliefs from other forms of knowledge. The six features are existential

presumption, alternativity, affective and evaluative loading, episodic structure, non-

consensuality, and unboundedness. Nespor described her model prior to discussing her study of

the beliefs of eight seventh- and eighth-grade teachers of English, mathematics, American

history, and Texas history.

Existential Presumption

This feature of Nespor's (1987) model takes into account that the individual believer has

assumptions or beliefs about an entity either existing or not existing. Examples of such entities

include God, witchcraft, fate, the fountain of youth, love at first sight, and the like. Abelson

(1979) reported that "to insist that some entity exists implies an awareness of others who believe

it does not exist," and that "these entities are usually central organizing categories in the belief

system, and as such, they may play an unusual role which is not typically to be found in the

concepts of straight knowledge systems" (p. 357). In Nespor's view, "the reification of

transitory, ambiguous, conditional, or abstract characteristics into stable, well-defined, absolute, and concrete entities is important because entities tend to be seen as immutable—as beyond the teacher's control and influence" (p. 318).

Two of Nespor's participants were mathematics teachers who had strong beliefs about ability, maturity, and laziness on the part of the students. One teacher believed that proficiency in mathematics was possible only through drill and practice, and that a lack of proficiency was a sign of the student being too lazy to complete assignments. The other teacher believed that learning mathematics was dependent upon the maturity of the student and rejected the idea that learning could take place without the consent of the student. Nespor's analysis of the data led her to assert that "these were not simply descriptive terms, they were labels for entities thought to be embodied by the students" (p. 318).

Alternativity

The recognition of alternative views of the world as well as alternative realities is a second characteristic of an individual's system of beliefs. Nespor (1987) defined *alternativity* as "conceptualizations of ideal situations differing significantly from present realities" (p. 319). In other words, the believer has a clear vision of an ideal situation and believes that radical changes must take place to correct the deficiencies in the current situation. Abelson (1979) suggested that achieving an idealized state "is not a matter of finding the sequence of rules to apply to a starting state to reach a goal; it is a matter of rejecting the old rules and finding new ones which achieve the goal state" (pp. 357 – 358). As such, beliefs systems are a means of setting goals and defining tasks, whereas knowledge systems are realized when goals and the means for achieving the goals are well-defined. Examples include the tenets of a utopian society, Hitler's vision of

creating the perfect human with blond hair and blue eyes, or the belief that embracing all aspects of the Bible will procure one's place in heaven.

One English teacher in Nespor's (1987) study described her ideal classroom teaching situation in terms of what she wished her own school experiences had been like—fun and friendly. Despite all of her efforts, she was unable to transform this vision into a reality (at least in her opinion). In this case, the teacher's beliefs were overriding concerns in which she wanted her students to feel comfortable and relaxed—sometimes at the expense of the teacher not being able to cover all of the material she had planned to discuss since she would spend considerable time re-explaining assignments or confronting behavior management issues (according to Nespor).

Copes (1982), influenced by the work of Perry (1970), suggested that there are four positions from which a person may view the world: *absolutism*, *multiplism*, *relativism*, and *dynamism*. Copes condensed Perry's original nine positions into those four, which he judged applicable to learning and teaching mathematics. Copes noted that "most persons interviewed over time seem to move through these positions in order, although some of them backtrack" (p. 38). A teacher who views mathematics from an absolutist perspective sees it as a collection of facts and believes that every problem has a solution. Moreover, this person believes that his or her role as a teacher is to deliver the material and be the "authority" with regard to mathematics. Multiplism is characterized by a multitude of mathematical systems that may be contradictory, but equally valid. If a teacher's perspective is multiplisitic, then he or she believes that everyone has a right to his or her own set of axioms and sees mathematics as a collection of strings and symbols rather than rigid facts. A relativist acknowledges the existence of many systems of mathematics, but does not believe that all are equally valid. For example, a relativist may

believe that there is a "best" way to construct a proof—citing validity, consistency, or historic value (to name few) as his or her reason for saying this approach is the best approach. If a person holds a dynamistic view of mathematics, then he or she is committed to a particular system and tends to believe that knowledge is a personal construction that relies upon the experiences of the individual.

Affective and Evaluative Loading

Nespor (1987) reported that belief systems are often linked to affective and evaluative components—such as feelings, moods, and personal evaluations. These components are based on preferences of the individual, and they tend to function independently of other cognitive processes—unlike systems of knowledge. According to Abelson (1979), belief systems contain large categories of concepts that may be labeled as "good" or "bad " (or at least leading to a situation that is good or bad). In Abelson's view, "the concepts of 'good' and 'bad' might for all intents and purposes be treated as cold cognitive categories just like any other categories of a knowledge system" (p. 358). On the other hand, "when the good and bad entities for the system have motivational force rather than simply categorical status, unique consequences for belief systems are even more likely to emerge" (p. 358). In this context, motivational force is taken to mean an alteration of the system itself caused by the activation of some affective or evaluative component.

Nespor's (1987) analysis of the data led her to believe that "a less obvious arena in which affect is important is that of teachers' conceptions of subject matter" (p. 319). Three of the four history teachers in her study believed that teaching history effectively entailed engaging students in meaningful activities such as examining history as a cohesive body of knowledge rather than as a series of isolated events, and they believed it was important to help students develop a

plethora of practical skills—such as how to outline a chapter or how to organize a notebook. Little emphasis was given to the memorization of dates or to the recitation of passages from historical documents. In addition, these middle grades teachers did not focus a great deal of attention on material that was not a precursor to later studies (such as Texas history) or on material that would be taught a second time (American history was later taught at the high school level). These findings stand as testimony that affective and evaluative components influence how much or how little energy teachers will expend on planning, orchestrating, and reflecting on lessons and activities.

Episodic Structure

Belief systems, in Nespor's (1987) view, are primarily composed of "episodically stored" material that was deposited as a result of a personal experience, folklore, or propaganda. Similarly, Abelson (1979) stated that "beliefs often derive their subjective power, authority, and legitimacy from particular episodes or events, . . . and these critical events then continue to color or frame the comprehension of events later in time" (pp. 358 – 359). In contrast, knowledge systems do not depend on personal or cultural episodes, but rather they rely on general facts and principles.

Several of the teachers in Nespor's (1987) study cited prior professional or personal experiences as being influential in their current teaching practices. The English teacher who was briefly discussed above wanted her classroom to be "fun and friendly," and in all likelihood this ideology was based on her vivid memories of being a student. One of the mathematics teachers previously mentioned earned an undergraduate degree in agricultural education and then proceeded to teach mathematics to technical students in the Job Corps. Although he no longer teaches technical mathematics per se, this professional experience probably led him to believe

that mathematics should be taught in such a way that students see its practicality and usefulness. These cases and others like them suggest that "critical episodes are probably at the root of the fact that teachers learn a lot about teaching through their experiences" (p. 320).

Non-Consensuality

Simply put, beliefs systems are not consensual. Nespor (1987) argued that "belief systems consist of propositions, concepts, arguments, or whatever that are recognized—by those who hold them or by outsiders—as being in dispute or as in principle disputable" (p. 321). Abelson (1979) put forth that the "generation gap" exemplifies this principle whereby younger generations tend to blame older adults for being insensitive to their needs and oppressive to their activities, and in turn the older adults blame younger generations for the corruption of society.

The line of distinction between knowledge and beliefs becomes blurred within the context of consensuality. Strictly speaking, an individual cannot decide whether or not a particular belief is consensual or not if he or she is unaware that alternatives exist. So does this invisibility constitute a strand of knowledge or a belief? One possible way to answer this question is to examine the origins of each. According to Nespor (1987), "knowledge accumulates and changes according to relatively well-established canons of argument" (p. 321). In contrast, belief systems are more static and less malleable than knowledge systems. When a change does occur with a belief system, "it is more likely to be a matter of conversion or gestalt shift that the result of argumentation or a marshalling of evidence" (p. 321).

Unboundedness

Abelson (1979) described belief systems as "open" and he remarked that "it is unclear where to draw a boundary around the belief system" (p. 359). Nespor (1987) shared in Abelson's description and further defined belief systems as "loosely-bounded systems with

highly variable and uncertain linkages to events, situations, and knowledge systems" (p. 321). In other words, because beliefs are grounded in and derived from the personal experiences of the individual, it is difficult (if not impossible) to decide how relevant or applicable one's beliefs are to alleged real-world situations.

Nespor purported that beliefs have "stable core applications," and he stated that beliefs may be "extended in radical and unpredictable ways to apply to very different types of phenomena" (p. 321). In contrast, knowledge systems may be applied in a myriad of arenas and are limited in the sense that knowledge is expanded through the rigor of logical reasoning or scholarly argument. As such, unboundedness is also taken to mean that if an individual develops meaning for a situation within a particular context and this individual bases his meaning on their system of beliefs, then other individuals (with their own separate critical episodes) would not assign the same meaning or relevance to the same situation.

Unboundedness as a distinct feature of belief systems merits attention because belief systems are not totally devoid of the self-concept of the individual. In fact, the self-concept that an individual develops over time has far-reaching, ever-expanding (and sometimes transparent) boundaries. On the other hand, knowledge systems typically exclude the notion of self. In my study I differentiated belief from knowledge by deciding whether or not the idea was personal or unique to the individual teacher.

<div align="center">Characteristics of Beliefs</div>

Abelson (1979) asserted than an individual holds beliefs with varying degrees of importance—meaning one holder may be deeply committed to a certain point of view, whereas a second believer may hold the same view, but sees it as mere circumstance. Although this feature was not present in Nespor's model, Abelson argued that "the ability of belief systems to stir and

express the passions of believers is an essential feature not to be found in knowledge systems well worth our groping theoretical efforts to try to understand it" (p. 364).

McLeod (1992) suggested a framework that divides studies of affect in mathematics education into three strands—namely beliefs, attitudes, and emotions. His approach to beliefs was similar to the stance taken in earlier sections of this chapter, and much of the same literature was referenced. At this point, it is wise to address the attitudes strand in the study of affect. It is not uncommon for the lay person, as well as the scholar, to use the terms *belief* and *attitude* interchangeably. McLeod defined attitudes as "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability" (p. 581). Attitudes are formed as the "result from the automatizing of a repeated emotional reaction to mathematics" or from "the assignment of an already existing attitude to a new but related task" (p. 581). It seems reasonable to say that an individual holds certain beliefs, but the individual must have some attitude towards the belief. In Green's (1971) view, attitudes are characterized as beliefs about one's beliefs. McLeod noted that beliefs and attitudes are stable structures and both vary in their levels of intensity.

Green (1971) claimed that beliefs are not independent of one another. In other words, beliefs collectively form beliefs systems, however the beliefs themselves are never held in isolation. He also asserted that beliefs are held in clusters and that belief systems have a quasi-logical structure. In this context, *quasi-logical structure* means that the particular ordering of the beliefs within the belief system has relatively little relation to the objective logical relations between the beliefs themselves. Green also suggested that two types of beliefs exist within a quasi-logical structure: primary beliefs and derived beliefs. Simply put, primary beliefs are those most basic beliefs that are taken as given, whereas derived beliefs are those beliefs that are

derived from other beliefs. Since belief systems are quasi-logical rather than stable and rigid, Green proposed that "there is no reason to rule out, in principle, the possibility that belief systems might change in respect to the arrangement of primary and derivative beliefs" (p. 45).

Nespor (1987) reported that beliefs are formed as the result of critical episodes, but how are beliefs altered? It seems reasonable to say that if certain episodes are the catalysts for the formation of beliefs, then other, more crucial episodes will act as the agents for changing a specific belief. In fact, Green (1971) suggested that beliefs change due to circumstances. Green further suggested that modifying one's beliefs may be rather simple or quite difficult— depending on how the belief is held. In his view, beliefs are either centrally or peripherally held. Green proposed a concentric circle model in which the innermost circle represented central beliefs (those beliefs held with the largest degree of certitude and require the least amount of logical reasoning are the most difficult to alter) and the outer circles represented peripheral beliefs (those beliefs that are held with less strength and are more susceptible to change and debate—with the weakest ones on the perimeter).

<div align="center">Algebra and Functions in School Mathematics</div>

The conception of school algebra has taken many forms since its steady inclusion into the U.S. curriculum beginning in the 1700s. Kilpatrick and Izsák (2008) reported that for most of the 19th Century "school algebra remained an extension and generalization of school arithmetic built largely by induction on a base of numerical quantities and operations on them" (p. 5). Algebra textbooks from the same era reflected this notion of algebra as generalized arithmetic by making a majority of the problems operational (factoring, finding roots, expressing powers) with somewhat less emphasis on equations and formulas. The beginning of the 20th Century ushered in a global interest in the concepts and ideas of calculus. In turn, functional thinking became a

staple in secondary school mathematics since calculus was an obvious and natural extension of functions and their graphs. Felix Klein was a proponent of functions in school mathematics, and in 1904 he asserted that "the function idea graphically represented should form the central notion of mathematical teaching" (quoted by Kilpatrick & Izsák, 2008, p. 6). This endorsement soon prompted authors to retrofit their algebra textbooks with more exercises on graphing.

The tenets of the new math movement from the 1950s to the 1970s caused the focus of school algebra to shift from generalized arithmetic to algebraic structures and proofs. Students grappled with these abstractions and struggled to understand the ideas behind the theorems associated with these algebraic concepts. Likewise, teachers' attempts to convince their students of the significance and relevance of algebraic proofs were futile. Although the new math era was eclipsed by a back-to-basics approach in the 1970s, some of the early abstract ideas endured—including equations and inequalities being taught as open sentences, variables as symbols to name elements in a set, and functions as a set of ordered pairs with certain properties.

As the conception of school algebra evolves with time, what instructional issues do algebra teachers face? Usiskin (1988) argued that (1) the extent to which students should be held accountable for manipulating algebraic symbols by hand, and (2) the role of functions in the algebra curriculum are the two fundamental issues regarding algebra instruction. In addition, Usiskin (1988) provided a framework for discussing these and other issues as they related to algebra instruction. The framework consists of four conceptions:

1. *Algebra as generalized arithmetic*: In this conception, algebra is viewed as a means for mathematically describing a relationship between sets of numbers. For example, the equation $S = 5.35h + 20$ could depict a person's salary S given that they are paid \$20 for showing up to work and \$5.35 for each hour h that they work.

2. *Algebra as the study of procedures for solving certain kinds of problems*: In this conception, variables are either unknowns or constants, and certain procedures are carried out for solving a problem. For example, if students know that point O is between points P and Q on $\overline{PQ}$, $PO = x + 5$, $OQ = x - 7$, $PQ = 18$, then students may find the length of each segment by writing the equation $(x + 5) + (x - 7) = 18$, solving the equation for x, and then substituting $x = 10$ into the original expressions for each segment length.

3. *Algebra as the study of relationships among quantities*: $V = \pi r^2 h$ is a formula for calculating the volume of a cylinder, but the formula also describes the relationship among three varying quantities. This idea is different than Conception 2 because we are not solving for one of the variables. Likewise, this idea is different than Conception 1 because in this context, the variable actually varies.

4. *Algebra as the study of structures*: In this conception, algebra consists of pre-existing structures that contain certain characteristics—such as the ring of polynomials. Consider the polynomial $x^2 - 6x + 9$. It is obvious that this polynomial is not a function and cannot be graphed, it is not an equation since it is impossible to solve for x, yet this structure may be re-written as $(x - 3)^2$, and different values of x may be directly substituted into both expressions to verify their equivalence.

Given that algebra is a permanent fixture in the mathematics curriculum and that the conception of algebra continues to change even today, Usiskin (1988) remarked that "no longer is it worthwhile to categorize algebra solely as generalized arithmetic, for it is much more than that" (p. 18). In fact, Chazan and Yerushalmy (2003) insisted that teachers must help students develop

"a feeling for what methods are appropriate with what strings of symbols . . . and appreciate what those methods are meant to accomplish" (p. 123) rather than merely expecting students to solve isolated algebra problems devoid of meaning.

Teachers, administrators, parents, policy makers, and mathematics educators have expended vast amounts of time, energy, and resources discussing the teaching and learning of algebra. Debates continue, even today, concerning when algebra should be taught, which students should have access to algebra, what the algebra curriculum should look like, and how algebra lessons should be orchestrated in order to ensure a balance of students' procedural and conceptual understanding of algebra. The National Council of Teachers of Mathematics (NCTM) charged writing committees consisting of practicing teachers, mathematics educators, and mathematicians with the daunting task of creating a vision for school mathematics as well as a vision of algebra. Two documents from NCTM that merit discussion are *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000).

The 1989 Standards

The NCTM (1989) touted "students' knowledge of numbers, computation, estimation, measurement, geometry, statistics, probability, patterns and functions, and the fundamental concepts of algebra" as the ideal mathematics curriculum for Grades 5 – 8 (pp. 65 – 66). This strong and straightforward assertion was necessary since, historically, many of the aforementioned topics were relegated to the back of mathematics textbooks and were often omitted by the teachers because of either lack of time or lack of interest. The middle school curriculum set forth by NCTM was to be that of a "broad, concept driven curriculum, [and] one

that reflects the full breadth of relevant mathematics and its interrelationships with technology"

(p. 66).

Of the thirteen curriculum standards presented for Grades 5 – 8, emphasis will be given

to two—namely, patterns and functions, and algebra. In a summary of changes in content,

NCTM (1989) suggested the following:

Curriculum standard	Increased attention	Decreased attention
Functions and patterns	•Identifying and functional relationships; •Developing and using tables, graphs, and rules to describe situations; •Interpreting among different mathematical representations.	•Topics seldom in the current curriculum.
Algebra	•Developing an understanding of variables, expressions, and equations; •Using a variety of methods to solve linear equations and informally investigate inequalities and nonlinear equations.	•Manipulating symbols; •Memorizing procedures and drilling on equation solving.

In the NCTM's (1989) view, the study of patterns and functions should be integral

components of the middle school classroom—including middle school algebra. Achieving

proficiency with patterns "requires students to recognize, describe, and generalize patterns and

build mathematical models to predict the behavior of real-world phenomena that exhibit the

observed pattern" (p. 98). Investigations of patterns help students to hone their reasoning skills

and to see that mathematics is to be valued as a worthwhile human endeavor. Likewise, the

study of functions should require that students "describe and represent relationships with tables,

graphs, and rules" (p. 98). Teaching students to be adept at appreciating and presenting multiple

representations for functions ensures their flexibility in modeling data and facilitates their

conceptual understanding. Although patterns and functions are to be treated in Grades K – 4 on

the periphery, they are to become more centralized in Grades 5 – 8 as students attain a greater

level of mathematical (as well as social) maturity. Work with patterns and functions in Grades 5

– 8 is intended to be a natural extension of students' prior experiences in the lower grades and

should culminate in their ability to represent these ideas symbolically (e.g., domain and range).

The NCTM (1989) described the middle school mathematics curriculum as "a bridge

between the concrete elementary school curriculum and the more formal mathematics curriculum

of the high school" (p. 102). Exploring algebraic concepts in informal ways helps middle school

students in their transition from the arithmetic of the elementary grades and builds a foundation

for the subsequent study of formal algebra. The NCTM made the following recommendations

for the study of algebra in Grades 5 – 8:

> Students should…
> - Understand the concept of variable, expression, and equation;
> - Represent situations and number patterns with tables, graphs, verbal rules, and equations and explore the interrelationships of these representations;
> - Analyze tables and graphs to identify properties and relationships;
> - Develop confidence in solving linear equations using concrete, informal, and formal methods;
> - Investigate inequalities and nonlinear equations informally;
> - Apply algebraic methods to solve a variety of real-world and mathematical problems. (p. 102)

Engaging middle school students in problem-solving activities such as "exploring a

concrete situation to determine patterns, constructing a table of data, looking for ways to

generalize the situation described by the table, asking questions about how the variables are

related, making a graphical representation, and looking for maximum and minimum points, or

points where the graph intersects the axes" (NCTM, 1989, p. 104) facilitates a deeper and richer

understanding of algebraic concepts. The context of these problems may be real-world situations

or idealized situations as either type will serve to advance students' ability to work with

algebraic representations and to use mathematics as a tool for modeling situations. Problem

solving also provides an opportunity for students to work in groups, engage in mathematical discourse, and use technology to verify or refute conjectures. Teachers should provide their students with "opportunities to explain, conjecture, and defend one's ideas orally and in writing" (p. 78) to facilitate a deeper understanding of mathematics.

The 2000 Standards

Patterns and functions are cogs that help drive the algebra machine. Over the course of time, this study of patterns may serve as a springboard into situations (natural or artificial), where students are expected to observe constant rates of change—culminating into a meaningful and intense study of linear functions. The NCTM (2000) suggested that "students should solve problems in which they use tables, graphs, words, and symbolic expressions to represent and examine patterns of change" (p. 223) as a means of gaining proficiency with functions. Such situations could be the catalysts for discourse among students, further explorations aided by technology, taking a specific context and generalizing the mathematics to other situations, mathematical modeling, making predictions, and relating mathematics to other disciplines. The notions of slope, x- and y-intercepts, points, lines, domain, range, degree, and zeros are natural extensions from the study of linear functions, and these notions may also lead into a discussion of families of functions that may not necessarily be linear.

Facility with solving equations and variables is important to the study of algebra in the middle grades. The NCTM (2000) argued that "most students will need extensive experience in interpreting relationships among quantities in a variety of problem contexts before they can work meaningfully with variables and symbolic expressions" (p. 225). This competency is acquired gradually as students engage in problem-solving activities that require rules, tables, or graphs (or any combination of the three) to model the problem or to arrive at a conclusion. Although

unique, these three methods for displaying data provide students with an opportunity to examine the equivalence of seemingly different algebraic representations and to prompt discussion about which representation is most appropriate for the situation at hand. The attention that algebra has received in the past has been less than favorable because of its procedural nature as well as its reputation as mere symbolic manipulation. In reality, symbolic manipulation is one of the necessary evils in the study of algebra, but that is not to say that this manipulation must be devoid of meaning. NCTM insisted that "symbolic manipulation can be enhanced if it is based on extensive experience with quantities in contexts through which students develop an initial understanding of the meanings and uses of variables and an ability to associate symbolic expressions with problem contexts" (p. 227).

Analyzing situations and representing real-life data lie at the heart of mathematical modeling. The availability of graphing calculators and computer software for performing routine calculations, displaying graphs, and modifying parameters has made the modeling and interpreting of real-world phenomena less daunting. Topics such as direct variation, scatterplots, and lines of best fit are ways in which students may engage in mathematical modeling or investigating quantitative relationships. Mathematical modeling must be a shared activity between students and teachers, and as such, students must be engaged in the mathematics, and teachers must have a virtual arsenal of questions intended to guide students in a feasible direction rather than using the "teach by telling" approach.

Although the NCTM stepped up to the proverbial plate by formulating sets of standards for curriculum and evaluation in 1989, for teaching in 1991, for assessment in 1995, and re-vamping and synthesizing the aforementioned documents in 2000, there is still work to be done. In April 2006, President George W. Bush called for the creation of the National Mathematics

Advisory Panel (NMAP). After 20 months of examining research studies and deliberating issues in mathematics education, the Panel produced its final report, *Foundations for Success*. The Panel identified the following topics as paramount to the study of algebra: symbols and expressions, linear equations, quadratic equations, functions, algebra of polynomials, and combinatorics and finite probability (National Mathematics Advisory Panel, 2008). The panel also cited fluency with whole numbers, fractions, and geometry and measurement as benchmarks for the critical foundations of algebra.

<div align="center">Discussion</div>

The gap between content knowledge and pedagogical knowledge has hardly narrowed. Ball and Bass (2000) reported three issues that mathematics education researchers must address before this gap can be bridged or eliminated: (1) The mathematical knowledge that a teacher must have for teaching effectively must be identified; (2) the ways in which teachers understand and conceptualize the mathematics that they teach must be understood; and (3) how teachers use their knowledge of mathematics in their teaching practices must be highlighted. Certainly, closing the gap would help to eliminate some of the problems that have plagued teacher education programs and would have far-reaching effects in the practice of teaching. Mewborn (2003) suggested professional development opportunities as a means for improving the quality of instruction among practicing teachers. She recommended that professional development must be grounded in sound learning theories, should be organized so that teachers are expected to revisit topics found in school mathematics and gain further insights into their conceptual underpinnings and interconnections, and must provide teachers an opportunity to enhance their ability to listen to students and the ideas that the students bring to the mathematics classroom.

A second reason for avoiding the study of the quality of instruction among mathematics educators could be that the approach commonly used was insufficient for examining multiple factors. The studies that were mentioned above dealt with the issue of knowledge—either on the part of the student or on the part of the teacher (content or pedagogical). It is reasonable to say that knowledge is not the only factor that may affect instruction; there are affective issues that influence one's teaching. For example, the beliefs that a teacher holds about mathematics, how mathematics should be organized and taught, and how students learn mathematics could certain influence the teaching act. Further research is needed in order to gain a better understanding of teachers' beliefs and how these beliefs infiltrate the mathematics classroom.

How and on what mathematics teachers focus their attention determines what is taught in the classroom as well as the instructional practices used in teaching. Teachers make sense of their instructional practices through the lens of what they already know and believe. As such, the role of knowledge and beliefs in teaching is the focus of the current research study. Since knowledge and beliefs can be the focus as well as the lens for examining the act of teaching, it is necessary to study the interplay of the two as a means for better understanding why some teachers are more effective than others.

CHAPTER 3

METHODOLOGY

The field of mathematics education has grown and flourished in the last two centuries, and today researchers in the field are responsible for carrying the proverbial torch of inquiry and further advancing the field with the fervent hope of improving the quality of mathematics teaching and, in turn, the quality of student learning. Quantitative methods have proven helpful to researchers interested in testing hypotheses, precisely measuring a phenomenon, or discovering correlations between variables. The focus of the present research study was eighth-grade Algebra I teachers, their knowledge of functions, and their beliefs about teaching functions. Qualitative research is the method of choice for undertaking a topic that is more psychological than quantifiable. Of course, a research study that is qualitative in nature is only as good as the instruments that the researcher used to collect data, the methods used for the data analysis, and the manner in which the findings were reported to stakeholders. The purpose of this chapter is to discuss the selection of the research participants, data collection, data analysis, and other methodological issues.

Participants

The three participants chosen for this research study were employed by the same school system, but they taught Algebra I in different middle schools. Merriam (1998) argued that a researcher must "select a sample from which the most can be learned" (p. 61). I used purposeful selection in recruiting Melodie, Hannah, and Rachel to be participants. In Patton's (1990) view,

purposeful sampling focuses on selecting cases that are rich and will provide a plethora of information in order to illuminate the question at-hand.

In searching for research participants, I emailed the county mathematics curriculum coordinator, middle school principals, and various middle school mathematics teachers who were designated building math contacts (a math contact in a middle school functions much like a department chair in a high school) during the fall of 2004. In the email I explained the purpose of my research study, the types of data collection instruments that I would use during the study, and a tentative timeline during which data would be collected. I also stated that I was looking for teachers with at least 3 years of teaching experience who were currently teaching Algebra I to students in eighth grade. This school district employed approximately 90 middle grades mathematics teachers. Once I received replies to my initial email with the names of 5 middle school mathematics teachers who might be viable candidates, I then contacted those teachers via email and explained my research in much the same way as I did to the administrators from whom I received the teachers' names. I then compiled a list of 3 teachers who consented to take part in the research study and contacted those individuals at school via telephone. I sought teachers who had earned undergraduate degrees in middle grades education and who were articulate and adept at communicating about mathematics and mathematics teaching. After speaking with these 3 teachers over the telephone and discovering through conversation that at least two of them were from states where the requirements for teaching middle grades mathematics was different, and at least one teacher was certified to teach high school mathematics, I decided to relax the criterion about holding a middle grades certificate in order to allow for greater contrast in the study. I also expanded the criterion regarding the participant teaching Algebra I to eighth grade students to include seventh grade students as well. I made the concession because the county school system

took the stance that on-level eighth grade students must take Algebra I as their mathematics course. Subsequently, there were cohorts of seventh grade students who enrolled in Honors Algebra I and then took Honors Geometry in eighth grade.

Data Collection

The data used in this study were collected primarily through one survey, three semi-structured interviews, and approximately 15 classroom observations of each teacher. Informal conversations, teacher artifacts, and the viewing of and reflecting upon one's own teaching on videotape also contributed to the data. The paragraphs that follow provide a detailed description of each data collection episode.

Prior to conducting the first face-to-face interview, I requested that Melodie, Hannah, and Rachel complete an initial survey (Appendix A). The survey was administered in early January, and the questions on the survey were designed to elicit such cursory data as educational background, areas of teacher certification, number of years of teaching experience, and affiliation with professional organizations. I also used the survey as one means for gaining access to each participant's opinions about mathematics teaching, mathematics learning, high stakes testing, and collaboration with other mathematics teachers. The teachers were also asked to identify activities such as lectures, completion of worksheets, cooperative learning episodes, integration of hand-held or computer technology, and whole-group discussions as typical or atypical for their mathematics lessons. The data collected through this survey served as an impetus for subsequent interview questions as well as foci for classroom observations.

I interviewed each research participant three times. Given that a goal of this study was to understand how teachers understand and teach functions, a sequence of semi-structured interviews seemed an obvious choice. DeMarrais (2003) defined an interview as "a face-to-face

verbal interchange in which one person, the interviewer, attempts to elicit information or expressions of opinions or belief from another person or persons" (p. 67), and she argued that qualitative interviews are advantageous to researchers who "wish to gain in-depth knowledge from participants about particular phenomena, experiences or set of experiences" (p. 67). The first interview lasted approximately an hour and a half and was conducted in late January or early February (depending upon the participant's availability). The purpose of this interview was to build a rapport with each teacher and to gain insight into each teacher's general beliefs about mathematics and mathematics teaching. Although I asked all three participants several common questions (Appendix B), other questions were tailored to individual responses to the interview questions as well as responses to the initial survey. I also asked each teacher to define *function* prior to beginning instruction with quadratic functions. I was interested in know how each participant defined it for their class and the vocabulary their would use in their definition. During the last 30 minutes of the interview, each research participant was given a stack of 23 cards and was asked to engage in a think-aloud sort—similar to that of Wilson's (1994) study of preservice secondary teachers' understanding of functions. The cards (Appendix C) were created to address families of functions (constant, linear, quadratic, cubic) as well as various representations (equations, graphs, tables, word problems). I purposely did not give detailed instructions to the participants about how I wanted the cards sorted because I wanted each participant to make sense of the cards for herself and to organize them in a way that made sense to her. When I created the stack of cards, I also introduced several miscellaneous functions into the deck (e.g., the greatest integer function) as well as examples of relations that were not functions (e.g., an equation for a circle). After the sorting, each teacher was asked to classify her

stacks of cards in some way and to describe how she determined which card went into each stack.

Prior to the second interview, I observed each participant teach a unit on quadratic equations and functions. Since neither the on-level Algebra I nor the Honors Algebra I curriculum had a unit designated to address the notion of functions, I chose to observe the unit on quadratics given the fact there was a function component to the unit. Moreover, the quadratics unit was rich with potential connections that teachers could help students make. Each unit spanned approximately 3 weeks, and all lessons within the unit were videotaped. During the observations, I took detailed field notes that were later expanded. I used the field notes to keep a log of warm-up problems, questions posed during class by either the teacher or by a student, and theories or concepts that were discussed in class. I also recorded notes to myself that would remind me to probe a particular topic in the subsequent interviews. These classroom observations afforded me greater insight into each participant's beliefs about teaching functions, and I was able to make comparisons between many of the professed beliefs about teaching that each participant claimed in her initial interview and what actually happened in her lessons. The video recording of each lesson provided a permanent record of what transpired and facilitated the participants' reflection on her teaching in subsequent interviews.

The second interview lasted approximately one and a half hours. I devoted the first half to delving into how each participant made instructional and curricular decisions, getting her reaction to the strengths and limitations of the unit on quadratic equations and functions, and seeking clarification on responses to some of the questions from the initial interview. Appendix D contains select questions from Interview 2. As before, many of the questions were unique to each participant. Each participant and I spent the second half of the second interview jointly

viewing short segments of teaching episodes, which afforded me an opportunity to highlight certain statements or actions of the teacher and to further probe her teaching of the unit. It also gave each teacher an opportunity to reflect on her pedagogical practices and to brainstorm about the strengths and the limitations of her lessons.

The third interview lasted approximately one and a half hours and consisted of three parts: joint viewing of lesson segments followed by a debriefing, follow-up questions from the two previous interviews, and two activities that examined each teacher's conceptions of functions. Each participant was asked to watch two or three pre-selected segments of a lesson she had taught. I purposely chose segments that focused on conceptual teaching (or sometimes the lack thereof), teacher responses to student questions, or content that students find complex or nontrivial. My intent was to further probe each participant's knowledge of functions and to gain understanding as to how that knowledge was held. For example, while I observed and taped Melodie's Algebra I class, she commented to her students that she enjoyed teaching the process of completing the square because of its complex nature. During the debriefing, I asked Melodie what made completing the square a complex procedure and why she had chosen to demonstrate the procedure rather than provide a conceptual basis through an activity such as the use of Algebra Tiles™ (ETA-Cuisenaire). The middle segment of the interview consisted of follow-up questions to the two previous interviews as well as hypothetical questions. Although the follow-up questions were unique to each participant, I asked each participant the same hypothetical questions—all of which related to errors or misconceptions common to Algebra I students (Appendix E). In the last part of the interview, I gave each research participant a list of eight definitions for the term *function* as defined in textbooks (without authors) from the 20th Century (Appendix F). Each teacher was asked to select a definition that resonated with her way of

conceptualizing a function and to provide a rationale for why that definition was her top choice. Cooney and Wilson (1996) insisted that creating a research agenda on teachers' knowledge and beliefs must include "an analysis of the content domain" (p. 132) and that taking a historical perspective was reasonable "given the specificity of functions" (p. 132). At the conclusion of the interview, I asked each teacher to model eight graphs by walking (Appendix G). She constructed each graph using the TI-83 Plus graphing utility in conjunction with a Computer-Based Ranger (CBR). Each graph was to represent time versus distance in feet. The CBR was used to collect the data each participant created during the walk, and those data were subsequently displayed on the calculator's screen. Seven of the eight graphs depicted functions, and I purposely included one graph that was a vertical line.

<div align="center">Data Analysis</div>

Bogdan and Biklen (1982) defined qualitative data analysis as "working with data, organizing it, breaking it into manageable units, synthesizing it, searching for patterns, discovering what is important and what is to be learned, and deciding what you will tell others" (p. 145). The data used in this study were analyzed in several phases. The initial data analysis occurred in tandem with the data collection because each phase of data collection was dependent upon the participant's responses and my interpretations of the responses from previous data collection episodes. The subsequent interviews were informed by the major themes that emerged from previous interviews, activities, and classroom observations. Dey (1993) referred to this process of reading the data, making annotations, and finding a focus for the analysis as *inductive analysis*. Patton (1990) explained that with inductive analysis themes and categories "emerge out of the data rather than being imposed" (p. 390). Near the end of the data collection, it was necessary for Hannah to excuse herself from the research project before the final interview. She

informed me that she was leaving her current middle school and was seeking a teaching position in another school district. Prior to her announcement that she would no longer be part of this study, I realized that Hannah's case would lend very little to this report. During my classroom observations I noticed that Hannah did not seem very purposeful in the selection of her assignments nor did she seem to focus on planning her lessons. During the two interviews that I conducted with Hannah, I was suspicious of her responses to several of my questions. She did not seem sincere or strongly tied to her opinions. I became concerned that she may have been answering the interview questions to assuage me rather than discussing her actual beliefs or actual teaching practices. Because Hannah was only present for part of the study and I wanted to avoid the Hawthorne Effect by introducing the limited data that I had collected on Hannah, I decided to include only Melodie and Rachel in this report.

After collecting all of the data and completing the initial analysis, I then reread all interview transcripts, field notes, and artifacts for Melodie and Rachel. I recorded any chunks or pieces of data that seemed relevant to my research questions on index cards and kept those cards sorted into piles according to participant. I further subdivided each stack into the categories of beliefs about mathematics, beliefs about teaching functions, and knowledge of functions. At the end of this data analysis phase, I scanned the data for examples to lend support to these categories as well as examples that were counter to the themes I had identified. This phase of the data analysis was the springboard from which I was able to write case studies for each participant.

During the final phase of the data analysis, I re-examined the data and linked my findings to previous studies. Raymond's (1997) model provided a means for interpreting each participant's beliefs about mathematics and mathematics teaching. Also, the work of Shulman

(1986b) provided a means for discussing and interpreting each teacher's knowledge of mathematics as well as mathematics pedagogy. Finally, I used the work of Vinner and Dreyfus (1989) to discuss and interpret each teacher's conception of function.

<div align="center">Limitations of the Study</div>

Merriam (1998) provided the following warning: "Drawing on tacit knowledge, intuition, and personal experience, people look for patterns that explain their own experiences as well as events in the world around them" (p. 211). As a teacher of mathematics I bring many personal experiences and beliefs to the proverbial table. Although I made a concerted effort to avoid judging these participants in terms of what good mathematics teaching should resemble, I have no doubt that my personal attitudes and beliefs somehow affected how I viewed Melodie and Rachel. Moreover, I was teaching the same course from the same textbook during the time I collected data.

A second limitation is the fact that both participants taught in relatively ideal settings. The schools boasted affluent neighborhoods, high test scores, teachers' names on waiting lists to teach at either school, and involved parents. The schools could be selective in the hiring process and could attract outstanding mathematics teachers. This study did not involve typical middle school mathematics classrooms.

A third limitation of this study was the exclusion of the third research participant, Hannah. Although I believe I made a wise decision to exclude her case since she was not as purposeful in her decision making as I wanted, the two remaining cases may be interpreted as polar opposites, although that was not my intent.

CHAPTER 4

MELODIE AND RACHEL: TWO CASES

Melodie and Rachel taught mathematics in the same school district. Although they taught at different middle schools, the student demographics were similar, and these two teachers were expected to follow the same Algebra I curriculum guide and pacing chart. In this chapter, I present an overview each teacher's beliefs about teaching functions as well a survey of her content knowledge regarding functions. In the latter part of the chapter, I present an interpretation of the teachers' beliefs about functions as well as their content knowledge regarding functions, and the implications for their mathematics instruction. It might be reasonable to assume a high degree of similarity between Melodie and Rachel in their instruction given the setting and district expectations; however, their stories are unique.

Melodie

Prior to becoming a mathematics teacher, Melodie attended college at a large, state-sponsored university where she majored in mathematics and minored in education. Subsequently, she earned a master's degree in middle grades education from a regional state-sponsored university. Melodie's teaching license allowed her to teach Grades 6 – 12. During her 16-year career as an educator, she has taught mathematics to middle school and high school students. In particular, she has spent the last 10 years teaching Algebra I to students in the seventh- or eighth-grade. In 2003, she received National Board Certification in Early Adolescence—Mathematics.

Throughout her tenure as a mathematics teacher, Melodie has been a member of professional organizations such as NCTM as well as her state's chapter of the NCTM. In addition to attending local, state, regional, and national conferences, she has presented or co-presented at several of these conferences. In 1997 she (along with several of her colleagues from her school district) piloted *Math Vertical Teams* (MVT), a framework proposed by the College Board that facilitates collaboration between middle schools and high schools and encourages more students to enroll in Advanced Placement mathematics courses. According to the College Board, skills and concepts leading to Advanced Placement courses must be cultivated over many years of middle school and high school mathematics instruction. One explicit goal of a MVT is to "develop a continuum of skill building from one grade level to the next" (College Board, n.d.). Although I have received training for Math Vertical Teams, I did not attend a MVT meeting with Melodie. After taking part in this program, the College Board recruited Melodie as a consultant and trainer for other schools or districts.

At the time of this study, Melodie was assigned to teach seventh-grade Honors Algebra I, eighth-grade Honors Algebra I, and Honors Geometry at Moore Middle School. In addition to teaching three courses, she also served as the school's math contact—which was roughly equivalent to the department chair's role at a high school. Moore Middle School is located in an affluent, rapidly growing suburban area that serves approximately 1000 students in Grades 6 – 8. In 2005, 92% of sixth graders, 92% of seventh graders, and 87% of eighth graders met or exceeded the standards measured by a state-mandated test in mathematics. Likewise, 93% of the school's population met or exceeded the standards measured by the state end-of-course test (EOCT) for Algebra I.

Class Structure

     Melodie began each class period by discussing a famous mathematician. During my first observation, she discussed Pythagoras and listed several of his contributions to mathematics. She then asked the students what famous theorem they believed Pythagoras had influenced, and many students were able to remember $a^2 + b^2 = c^2$ (only the conclusion of the Pythagorean Theorem). Subsequently, Melodie drew a set of axes on the board depicting the first quadrant and then drew a line segment whose endpoints were located at the origin and the point (3, 4). It was obvious that these seventh-grade students in Honors Algebra I had used the conclusion of the Pythagorean Theorem previously given how easily they calculated the length of the hypotenuse, c. Knowing that her students had to take the square root of a number to find $c = 5$, Melodie used this calculation as an opportunity to preview cube roots, fourth roots, and so forth. During some of my later observations, Melodie discussed other mathematicians such as Gauss and the Fundamental Theorem of Arithmetic, Fermat and Fermat numbers, Archimedes, Euclid, and Hypatia. These mathematicians were mentioned in the textbook, but Melodie used history of mathematics books as well as the internet to find more details about each one.

     On a typical day, Melodie would instruct the students to take out their homework assignments after discussing the day's mathematician. Sometimes she would have already prepared an overhead transparency containing all of the answers to the homework, and at other times she would either call out the answers or have the students share their answers. When I first began observing Melodie, she was teaching a unit on quadratic equations and functions. Prior to my first observation, she had assigned the students a set of exercises in which they were asked to find the zeros of a quadratic function analytically. All of the problems were expressed in the standard form ($y = ax^2 + bx + c$), and factoring was the method of choice for finding the

corresponding zeros. As a segue into new material, Melodie wrote the equation $(x-2)^2 = 16$ and asked the class how they would approach the problem. A female student suggested that she FOIL the binomial, move all the nonzero terms to the left-side of the equation, and then use factoring to find that $x = 6$ or $x = -2$. Melodie asked the class members if they agreed with the answers, and then asked if anyone could think of an alternative approach. One of the male students suggested that she "de-square" both sides of the equation and solve by isolation. After getting the same solutions with the square root process as with factoring, the same male student commented that 16 was a perfect square, and he wanted to know how to solve the equation if the constant were not a perfect square. Melodie created the equation $(x-3)^2 = 18$ and found the solutions to be $x = 3 \pm \sqrt{18}$. It was obvious that these students had a working knowledge of radicals, but instead Melodie requested that they write the decimal approximation of the two answers rather than leaving them in exact form—mainly because the textbook answers were written as decimals.

Prior to this unit on quadratics, Melodie had already discussed linear functions as well as absolute value functions with the class. When she began discussing features of the graphs of quadratic functions such as the direction in which the parabola opens, the vertex, and the axis of symmetry, the students were able to call upon their previous knowledge of functions. Without using the formula $x = -\dfrac{b}{2a}$, Melodie asked her students to predict the vertex for the function $y = x^2 + 3$. After most of the students predicted that the vertex would be located at (0, 3), Melodie had the students verify their findings by using their graphing calculators. Prior to clearing $y_1 = x^2 + 3$ in the calculator, Melodie asked the students to enter $y_2 = (x + 3)^2$ and compare the two graphs in the same viewing rectangle. By comparing the vertices of y_1 and y_2

with the vertex of $y = x^2$, the students were able to find the respective transformations that each

function would undergo and subsequently generalize the results to other functions. Melodie then

asked the students to graph the function $y = x^2 - 9$ and to examine the *x*-intercepts using the

"trace" feature of the graphing calculator. Almost immediately, a male student noticed that the

x-coordinate of the vertex (0, -9) was located at the midpoint of the two *x*-intercepts along the *x*-

axis.

In subsequent lessons, Melodie continued to engage her students with inquiry-based

activities and to pose probing questions to help the students investigate and develop a deep,

conceptual understanding of quadratic functions and their transformations. Melodie's teaching

style, coupled with the students' facility with hand-held technology, made concepts such as

reflections of the graphs of functions across the *x*-axis, vertical stretches and compressions,

symmetry about a line, and the notion of zeros of a function accessible to all students.

Beliefs about Teaching Functions

Melodie believed that the study of mathematics is "recognizing patterns" and "taking a

physical object and discussing different parts of the object" (Interview 1, 1/25). In particular, she

labeled the study of algebra "generalized arithmetic" and pointed out that "it's more than just

manipulating symbols—I think it's really just a general means of expressing a quantity"

(Interview 1, 1/25). Although Melodie's teaching schedule consisted of all honors mathematics

courses, she believed that mathematics should be accessible to all students—regardless of a

student's mathematical background or preparation. During our initial interview, she remarked "I

was trying to convince someone just the other day that he shouldn't be held back from learning

all that someone can learn." Her comment was situated within the context of brainstorming how

students with deficiencies in arithmetic could understand (both procedurally as well as

conceptually) topics studied in algebra.

Melodie acknowledged that there is no simple recipe for planning and orchestrating

conceptually rich, rigorous mathematics lessons; however, she was able to sugest a few of the

key ingredients:

> To me good mathematics teaching is…examples that are well thought out
> ahead of time…and a variety of instructional methods. I definitely think that I get
> that variety with the use of technology. And when I say *technology*, I mean the
> table and the values in the table. Also, giving them [the students] a chance to
> work together and talk with one another—you know, let them work separately,
> but then collaborate on what they are doing. I think good mathematics teaching is
> explaining why it is what we are doing, to the point where they can explain why to
> each other and to help each other—not just to say this is the answer, but why this
> is the answer. (Interview 1, 1/25)

Melodie's desire for her students to gain both a procedural and conceptual understanding of

functions was evidenced in a multitude of ways during our work together. When I first met

Melodie, she explained that she had just finished teaching a unit involving simple percent

problems and commented that this was her least favorite unit to teach because the students were

just following steps. She perceived that her students did not "understand why they are doing

what they are doing" and "they don't really understand what a percent means" (Interview 1,

1/25). Conversely, her Honors Geometry students had just studied the golden ratio—one of her

favorite lessons because, as she said, "I like for them to know why the symbols end up the way

they do" (Interview 1, 1/25). During one of my observations, Melodie had the students in her

class use their graphing calculators to generate ten sets of three numbers in the interval [-5, 5].

She then directed the students to list ten quadratic functions using the sets of numbers generated

previously as the coefficients *a*, *b*, and *c* for $f(x) = ax^2 + bx + c$. At this point, the students

graphed all ten of their individual quadratic functions using their calculators and noted which

parabolas had *x*-intercepts and which graphs never crossed the *x*-axis. Melodie solicited

equations for quadratic functions that had *x*-intercepts as well as those that did not from her

students, and then asked if they noticed any patterns. One of the male students suggested that if

a and *c* have the same sign then the function would have no real zeros. Although Melodie's

activity did not afford her students the opportunity to truly "discover" the formula for the

discriminant, it served as a springboard for discussing the discriminant as a tool. In an interview

after the lesson, Melodie commented that her students had "gotten used to the fact that we do

experiments in class," and that they "automatically go to the 'what ifs' and say, 'Why don't we

try this?'" (Interview 1, 1/25).

Various types or families of functions were emphasized in the Algebra I curriculum in

Melodie's school district. In the Honors Algebra I class, she introduced linear functions early in

the school year and then extended that basic notion to include absolute value functions and

quadratic functions. She provided a brief overview of exponential or logarithmic functions as

the school year progressed. According to Melodie, she taught functions because "it is a major

topic throughout Algebra I" (Interview 1, 1/25) and "it pulls together the graphing and the table

of values—which is really complicated for them" (Interview 2, 2/28).

The pedagogical practice of taking a previously learned concept and generalizing that

concept to other mathematical ideas was prevalent in both my discussions with Melodie and in

my observations of her teaching. Prior to my observations of the unit on quadratic functions, she

had taught her students about absolute value functions of the form $y = a|x - h| + k$. In that unit

the students had been introduced to the idea that functions may be classified by the family to

which they belong as well as the transformations they may undergo relative to the parent

function. According to Melodie, she used "the technology first, and then we write a description

of the change" rather than giving her students a formula in terms of *h* and *k* (Interview 1, 1/25).

Many of her lessons also related back to this idea of *h* and *k*. During my third observation,

Melodie requested that her students use their graphing calculators to graph the function $y = x^2$ in

y_1, and in turn labeled this function the *parent*. Subsequently, the students examined the graphs

of $y = -x^2$, $y = 2x^2$, $y = .2x^2$, $y = (x+2)^2$, and $y = x^2 - 2$. At this point, Melodie wrote

$y = a(x-h)^2 + k$ as the generic equation for all quadratic functions. It was obvious from the

reactions of the students that they made the connection between the *h* and *k* in the new quadratic

model and the *h* and *k* learned during their previous study of absolute value functions in the first

semester.

Melodie believed that teaching functions in Algebra I is important. Although she readily

admitted that students sometimes struggle with the concept of a function as well as some of the

notation associated with functions, she was not daunted by teaching functions because "there are

many real-world examples that you can use" (Interview 2, 2/28). During the same interview, she

remarked "There's a lot of different things going on and a lot of knowledge built up at that point,

and you can really see some of what they've learned starting to be applied." That was her main

reason for gaining pleasure from teaching functions to her students. She also shared this sense of

importance and pleasure of teaching functions with her colleagues—as evidenced in the

following quotation:

> I am working with first-year teachers, and when we got to rational functions we were able to apply the *h-k* thing with the translation of the graph to even rational functions. So the first-year teachers thought that was really neat. It wasn't necessarily written in our book that way, but we kind of presented it that way to the kids and it made a lot of sense. I guess teaching functions is important because the concept applies—the overlying concept of translating the function and reflecting the function and making it wider or steeper applies across all functions. So that's the big idea. (Interview 2, 2/28)

Melodie believed that teaching functions was one of the rare opportunities in which she could really probe her student's thinking. Focusing on the procedure rather than on the concept was a habit that Melodie strived to eradicate.

Knowledge of Functions
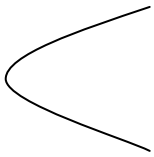
I conducted my first interview with Melodie approximately 2 days before she began teaching the unit on quadratic functions. She commented that she had planned the pacing for the unit but had not created any lesson plans (written or otherwise) for teaching quadratic functions. Although it was obvious that Melodie had not had an opportunity to review the content, I asked her to provide a definition for a function. After thinking for a moment, she replied, "A function is something that can be represented in lots of different ways…a function is about numbers—you can graph it, you can make a list of values of the function" (Interview 1, 1/25). During one of my later classroom observations, a male student in Melodie's class remembered the term *function* but was unable to recall the definition. Melodie reminded the class that for every value of x, there corresponds one and only one value y. She understood that functions may be represented in a variety of ways (equations, graphs, tables), but her formal definition of *function* was a relational one. I provided Melodie with a list of seven seemingly different definitions for the term during our final interview (Appendix F) and requested that she identify the one that most closely aligned with her definition. Initially, she selected the definition of Larson, Boswell, Kanold, and Stiff (2001) that denoted a function as a rule relating two quantities. It was not surprising that this definition resonated with Melodie because Larson and his colleagues are the authors of the Algebra I textbook used by the school district. After re-examining these definitions, she then stated that she was unable to select a single definition and claimed that the

definition given by Demana and Waits (1990) was appropriate as well. This definition (similar

to the one she provided to her male student) is more relational.
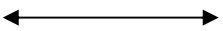
Melodie recognized that a function could be represented in multiple ways—not just as an

equation. During our second interview, she professed to value the equation of the function most

because graphing calculators can generate both a graph and a table of values from the equation.

Subsequently, I observed Melodie's class complete an inquiry-based activity in which motion

problems (such as throwing a ball in the air) were explored. I overheard Melodie explain to one

of her female students that the graph was more significant than the equation because the graph

served as a visual representation for the path of the ball and that the graph could provide data

(such as the height of the ball or the time at which the ball was at a certain position) without the

student having to do any computations. Melodie recognized that best representations depend

upon need and context.
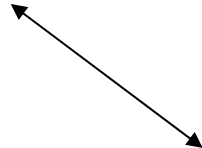
Melodie appeared to have a deep, flexible understanding of functions and their families.

At the end of our first interview, I requested that she sort the cards containing relations and

functions (Appendix C) in a way that made sense to her. In the end, she created the following

sorts:

"Not Functions"
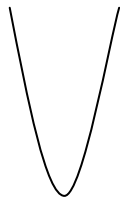
$(x-4)^2 + (y+1)^2 = 5$	x	y	
	2	6	
	-4	5	
	0	5	
	6	9	
	-4	1	
	1	3	
	5	-6	

"Linear Functions"
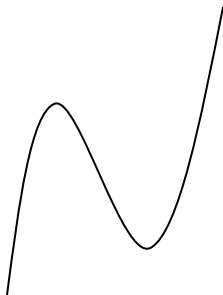
$y = -3$	x	y	The speed limit on GA 400 between mile markers 5 and 27 is 65 miles per hour. If Maude has	
	5	-1		
	-3	-1		
	1	-1		

			decided to maintain that speed limit by setting her car's cruise control, how fast is she driving as she passes a police car at stationed at mile marker 13?	
	8	-1		
	0	-1		
	-2	-1		
	-5	-1		

	x	y	A stockbroker charges \$45 to handle any transaction. In addition, he charges \$.45 per share traded. Find the cost of the broker selling 1300 shares of stock.	
$y = -3x + c$	0	5		
	2	3		
	3	2		
	4	1		
	5	0		

"Quadratic Functions"

	x	y	A parking lot is to be formed by fencing in a rectangular plot of land except for an entrance 12 meters wide on one side. Find the dimensions of the lot of greatest area if 300 meters of fencing is to be used.	
$y = \dfrac{1}{2}(x+2)^2 - 4$	-2	5		
	-1	0		
	0	-3		
	1	-4		
	2	-3		
	3	0		
	4	5		

"Cubic Functions"

	x	y	From a rectangular piece of cardboard of dimensions 8 x 15, four congruent squares are to be cut out, one each corner. The remaining cardboard is then folded into an open box. What size squares should be cut out if the volume of the resulting box is to be maximized?	
$y = \dfrac{1}{2}x^2 - \dfrac{2}{3}x^3 + 2 - x$	-3	-26		
	-2	-7		
	-1	0		
	0	1		
	1	2		
	2	9		
	3	28		

"Piece-Wise Defined Functions"

	A direct-dial long distance call costs \$2.25 for the first two minutes and \$1.03 for each addition minute or fraction thereof. Write the cost function C of a call in terms of minutes m.
$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$	

"Other"

$$y = \frac{x^2 - 4}{x + 2}$$

Melodie also demonstrated knowledge of independent versus dependent variables used in an activity in which she was given a graph and was asked to model the graph of time versus distance by walking (see Appendix G for the graphs). When discussing Walk 1 (the function is decreasing, constant, and then increasing from left to right), Melodie recognized that to create the graph she must first move closer to the CBR to decrease the distance between herself and the CBR, allow the distance to be constant for several seconds, and then walk away from the CBR to increase the distance. Similarly, she recognized that Walk 7 (the vertical line) was impossible because "you cannot be in all of those different places at one time" (Interview 3, 5/23). Melodie was able to describe each type of walk with accuracy and without hesitation.

<div align="center">Rachel</div>

Rachel graduated from a large, state-sponsored university with a degree in elementary education. She began her career teaching English, mathematics, science, and social studies to fifth-grade students. After five years of teaching fifth graders, Rachel began teaching seventh-grade—which was still housed in an elementary school at that time. During Rachel's tenure at the elementary school, her principal was asked to leave the elementary school in order to open Lucerne Middle School in the same school district. At the request of her principal, Rachel transferred to Lucerne Middle and began teaching mathematics only. She explained, "I went with her there and just taught math because that's what I thought I did best and I've been doing it ever since—seventh- or eighth-grade" (Interview I, 1/21). The state from which she earned her degree did not offer an emphasis in middle grades education; however, earning a degree in

elementary education allowed a candidate to be granted certification for Grades K – 8. Rachel's preparation for teaching mathematics consisted of two college algebra courses as well as several methods courses for teaching mathematics in elementary school. Over the last few years, she had taken part in district-wide courses such as Teaching Algebra in the Middle Grades and Teaching Geometry in the Middle Grades in order to gain greater mathematical and pedagogical knowledge. In 2002 she earned National Board Certification in Early Adolescence—Mathematics.

As part of her ongoing professional development, Rachel had attended a variety of workshops and meetings about teaching mathematics, observed other teachers (both formally and informally) in her building while they provided mathematics instruction, and created collaborative working relationships with colleagues to help with the planning and implementation of various Algebra I lessons. According to Rachel, her greatest future challenge was earning certification to teach high school mathematics. Her school district had mandated that all middle grades mathematics teachers providing instruction in Algebra I (or above) must earn a high school credential by the end of 2007 if they were to continue to teach mathematics courses in which students may earn high school credit. The notion of earning high school certification was baffling to Rachel because "according to the county, I shouldn't be teaching algebra—although I've been doing it very successfully for 15 years" (Interview I, 1/21). After reviewing Rachel's college transcripts, the state determined that she did not have a sufficient number of college mathematics credits for full certification, but she could earn certification by passing the Praxis II test. The Praxis II test measures preservice teachers' mathematical knowledge as well as general and subject-specific pedagogical knowledge. Rachel commented,

"I don't know that I'll pass the Praxis II, and if I don't it'll be a shame for the county because they'll lose a really good Algebra I teacher" (Interview I, 1/21).

At the time of this study, Rachel's teaching assignment consisted of teaching Pre-Algebra and Algebra I to students at Lucerne Middle School. This middle school is situated in a newly created township that lies in a rapidly growing suburban area. The school provides education to approximately 1100 students in Grades 6 through 8—a majority of whom came from seemingly educated, affluent households. In 2005, 89% of sixth graders, 92% of seventh graders, and 92% of eighth graders met or exceeded the standards measured by a state-mandated test in mathematics. Likewise, 100% of the students met or exceeded the state standards measured by the EOCT in Algebra I.

Class Structure

Rachel typically began her Algebra I class by having the students work problems that were written on an overhead transparency. All of the problems were routine, and more often than not they did not relate to the concepts or skills being taught in the current unit of instruction. Rachel explained that most of the warm-up problems were "usually practice for the state-produced test" (Interview 2, 5/17) and that these problems were mandated by the school's administration. As the students were working the warm-up problems, Rachel used the time (approximately 3 to 4 minutes) to take attendance and then visit each student's desk to check homework. The bottom portion of the overhead transparency was usually covered by something opaque to hide the answers to the warm-up problems, and after Rachel finished checking homework she would then reveal the answers to the problems.

The way in which Rachel provided answers to the previous night's homework assignment varied from day to day. She typically prepared an overhead transparency for the homework

assignments that required students to identify different pieces of information that would eventually lead to a final answer. For example, the students were given a teacher-created worksheet and were expected to graph several quadratic functions expressed in standard form (i.e., $y = ax^2 + bx + c$). The directions on the worksheet stated that prior to graphing the function, the student must decide whether the parabola opens up or open down, determine the location of the vertex, write an equation for the axis of symmetry, and generate a table of values. Since the answer to each problem was a culmination of several pieces of information rather than a final product, Rachel made a transparency of her answer key to the worksheet so that the students could check each component of the problem. Answers to homework assignments that were more procedural and had a final answer (or set of answers) were read aloud by individual students. Rachel used this practice of calling on individual students to share an answer to a homework problem primarily when solving quadratic equations.

After providing answers to the previous night's homework assignment and answering any questions posed by the students, Rachel typically turned off some of the lights in the classroom prior to turning on the overhead projector and beginning the next lesson. She would usually announce something like "Okay, today we are going to graph quadratic functions" to provide an overview and a focus for the day's lesson. Moreover, she gave the students a hand-out each class period that contained an outline of the notes as well as the examples that she would present. For the most part, the notes consisted of a list of steps or procedures for completing a problem. During Rachel's initial lesson on graphing quadratic functions, she wrote the following steps on the overhead projector: (1) Write it in function form ($y = $ _____), (2) Find the axis of symmetry using $x = \dfrac{-b}{2a}$, (3) Make a table with two values bigger and smaller, (4) Graph. The students seemed to struggle more with the arithmetic than the graphing. Rachel thought it

necessary to review order of operations as well as the difference between -3^2 and $(-3)^2$. In subsequent examples, she continued to remind the students of the process or would pose questions such as "The first thing we do is what?" to help the students recall the steps.

If there was any time remaining at the end of the period, Rachel would allow the students to begin working the next day's assignment. On occasion, some students would request that the teacher work a problem from the assignment that they perceived to be too difficult or too lengthy prior to the students trying the problem themselves. Rachel would also walk around the classroom and provide individual assistance to students.

Beliefs about Teaching Functions

According to Rachel, mathematics is "the study of numbers, patterns, sequences, and how they relate to each other" (Interview 1, 1/21). She believed that the study of mathematics is necessary for developing logical ways of thinking and for solving real-world problems. During the same interview, I asked Rachel to differentiate between mathematics in general and algebra—her initial response being "I guess I kind of think of it as the same thing." She continued by saying that "in algebra you might use variables to represent different situations and manipulate them to figure out problems, to solve things, to predict trends, or to solve problems." After pondering this question further, Rachel did remark that algebra is more abstract than mathematics because "you can't always draw a picture of it or you can't always relate it to apples and oranges and dividing—like a bag of candy and dividing it into four" (Interview 1, 1/21). In order to strengthen her argument, she cited simplifying square roots and factoring as two topics from the Algebra I curriculum that are difficult to visualize. These two topics were probably at the forefront of Rachel's thoughts since both were used extensively in this unit on quadratic equations and functions. After reflecting on her treatment of simplifying radicals,

Rachel admitted that "I don't teach it where I have to draw it. And I know some teachers do, but

I don't." Likewise, she realized with factoring that "you can show them [the students] by using

blocks and stuff like that—I don't do that either" (Interview 1, 1/21). I learned from the initial

participant survey that factoring was one of Rachel's favorite topics to teach in Algebra I, and as

I observed her classes I saw multiple lessons in which factoring was reviewed and discussed.

Rachel's preference was to have students complete factoring puzzles rather than using

manipulatives. She told me that throughout her years of teaching Algebra I, the puzzles made

factoring fun and the students were able to gain proficiency. As such, she did not see the use of

Algebra Tiles™ (ETA-Cuisenaire) as being essential to the students' understanding or enjoyment

of the lesson on factoring. She also mentioned that "time is always a factor in middle school"

(Interview 1, 1/21).

Although factoring was one of Rachel's favorite topics, she readily admitted that her

least favorite topic in the Algebra I curriculum was quadratic functions—mainly because the

students found such graphing difficult. When I asked her why she thought that, she remarked

that "it's just all the calculations—all of the fractions, and the kids get so frustrated" because

"one mistake, and it's kind of screwed up" (Interview 1, 1/21). In an attempt to make graphing

quadratic functions more palatable for her students last year, Rachel created a song titled *The

Twelve Days of Algebra* (a parody of *The Twelve Days of Christmas*). The song began with "On

the first day of Algebra, my teacher gave to me. . .," and continued with Rachel providing certain

numerical values—culminating in a graph that resembled a parabola. Also, she commented

twice that she enjoyed having students graph functions using the graphing calculator. During my

observations I noticed that she allowed students to borrow a graphing calculator from her

classroom set of TI-83 Plus Silver Edition calculators, but the students were mainly encouraged

to use them for computations (such as squaring a fraction) while making a table of values rather than to use the graphing feature.

In Rachel's school district, the first semester of Algebra I consisted primarily of topics such as solving linear equations and inequalities, graphing lines, writing the equation of a line, and solving systems of two equations with two unknowns. The notion of a quadratic function was taught near the middle of the second semester. In the interim, the authors of the textbook (Larson, Boswell, Kanold, & Stiff, 2001) devote a single section to the definition of a function, function notation, and means for evaluating functions at a specified value of the variable. Although the authors chose to use function notation such as $f(x)$ throughout the remainder of the textbook, Rachel continued to name the function y (or something similar). The following quotation provides greater insight into her belief about teaching function notation:

> I teach them that [$f(x)$] and put it on a test, but I tell them that it's the same thing as y equals. That's probably not the right thing to do, but it's what I do because they look at it as—they are learning all new stuff and they think $f(x)$—what the hell is this? It looks so weird. I do try to make everything as simple as possible. I don't water the material down, but I don't—I'm not one who teaches with huge words. I really teach middle grades students, and I think that's their vocabulary. It doesn't say a lot for me, but I do try to make it really easy for them. So if I'm teaching quadratic functions it [function notation] might be on there, but it's like "Okay you guys, it is just like if you have y equals, how would you do that?" You have $f(x)$, and you're finding the function of 2. When I plug 2 in for x, you are going to see what it does—where it takes us and that's the solution. . . .Kids ask right away, "Can I just make it y equals?" because they want to make it easy. And they are finally understanding how to graph $y = \frac{3}{4}x - 6$, and then to throw in function notation—that kind of messes them up a little bit. (Interview 1, 1/21)

In other words, Rachel believed that a great deal of the terminology and notation associated with the study of functions might serve as a barrier to the students' understanding (and perhaps even enjoyment) of this topic. Subsequently, Rachel volunteered to me that using mathematical terminology during instruction was a challenge for her because she never knew where such words were used in later mathematics courses.

During all of the interviews that I conducted with Rachel, she consistently commented that she did not know the role that functions played in the high school curriculum. She credited her participation in a Math Vertical Team with making her realize the significance of teaching functions in Algebra I. Overall the MVT has helped Rachel to better focus her lessons and to re-prioritize certain topics in the Algebra I curriculum. For example, she no longer spent several days having her students work with negative exponents because the high school teachers re-taught that topic in Algebra II—even though negative exponents were in the textbook and were listed as part of the county's Algebra I curriculum.

It was obvious from Rachel's classroom instruction as well as from the artifacts I collected (outlines of class notes, worksheets, and so forth) that she was giving her students the means to acquire a procedural (if not rote) understanding of functions. As we discussed some of the differences between teaching students in the on-level Algebra I class and teaching the students in an honors level class, Rachel commented, "I think I did more activities, and they were able to learn through activities, whereas the on-level students really need step-by-step instruction on everything" (Interview 2, 5/17). She believed that her students had a good time doing activities in class, but she suspected that the students were missing "the big picture" and not understanding the purpose behind the activity.

Knowledge of Functions

During our initial interview, Rachel defined a function as "a relation where each value of *x* matches with only one value for *y*" (Interview 1, 1/21) from memory prior to teaching a lesson on quadratic functions. However, Rachel soon replaced this relational definition with a more equational definition. When I asked her in a subsequent interview how she would react to a

student in her class who claimed that he or she did not understand the notion of a function, she

explained it to me as follows:

> Like, okay, I might say that a function is kind of like this formula or something, and you can put something in, and you see what is going to come out. So, you have this function—it can be anything. It can be easy—it can be $y = 3x$. So then you want to find out what the function of 5 is when you plug it in it. Well, the function of 5 is going to be 15 because $3 \bullet 5 = 15$. So, that would be the function of this number in here, and so for a quadratic you have that it's a little more complicated of a function, but the function of this number is to get this number back. (Interview 2, 5/17)
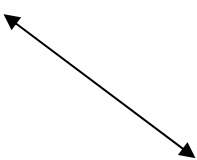
It is apparent that Rachel saw a function as merely a rule or an equation that has an input and

some type of output. Her detailed explanation also suggested that she thought of an input value

as playing some role or serving some purpose (i.e., serves some function) in obtaining the output

value. She held strong to her original definition during our final interview when she was

presented with a list of seven definitions from various textbooks (Appendix F), and she selected

the relational definition by Larson et al. (2001). During one of my classroom observations,

Rachel had completed her lesson and was walking around the classroom providing individual

assistance to her students as they began their homework assignment. I overheard her tell a

female student that "every input has to be different—every x has to be different" in response to

the girl's statement that she did not understand functions.

The Algebra I curriculum in Rachel's school district focused on linear and quadratic

functions, but left the topic of higher-order polynomial functions for Algebra II. During the card

sorting activity (Appendix C), Rachel seemed to have no difficulty identifying equations and

graphs (and for the most part word problems) that were linear or quadratic, but she seemed to

struggle with functions that did not fall into those categories. She sorted the cards into piles and
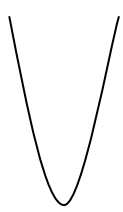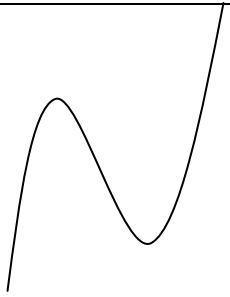
classified her piles as follows:

"Not a Function"

"Linear"

$y = -3$	The speed limit on GA 400 between mile markers 5 and 27 is 65 miles per hour. If Maude has decided to maintain that speed limit by setting her car's cruise control, how fast is she driving as she passes a police car at stationed at mile marker 13?	
$y = -3x + c$	A stockbroker charges $45 to handle any transaction. In addition, he charges $.45 per share traded. Find the cost of the broker selling 1300 shares of stock.	

"Solve by Factoring"

$y = \dfrac{1}{2}(x+2)^2 - 4$	$(x-4)^2 + (y+1)^2 = 5$	$y = \dfrac{x^2 - 4}{x + 2}$

"Max/Min Problems"

A parking lot is to be formed by fencing in a rectangular plot of land except for an entrance 12 meters wide on one side. Find the dimensions of the lot of greatest area if 300 meters of fencing is to be used.		From a rectangular piece of cardboard of dimensions 8 x 15, four congruent squares are to be cut out, one each corner. The remaining cardboard is then folded into an open box. What size squares should be cut out if the volume of the resulting box is to be maximized?	

"Don't Know"

x	y
5	-1
-3	-1
1	-1
8	-1
0	-1
-2	-1
-5	-1

x	y
0	5
2	3
3	2
4	1
5	0

x	y
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5

x	y
-3	-26
-2	-7
-1	0
0	1
1	2
2	9
3	28

x	y
2	6
-4	5
0	5
6	9
-4	1
1	3
5	-6

$$y = \frac{1}{2}x^2 - \frac{2}{3}x^3 + 2 - x$$

$$y = \frac{x^2 - 4}{x + 2}$$

A direct-dial long distance call costs $2.25 for the first two minutes and $1.03 for each addition minute or fraction thereof. Write the cost function C of a call in terms of minutes m.

When given a relation expressed as a set of ordered pairs, Rachel was able to identify the values of x as the elements of the domain and the y-values as the elements of the range. Furthermore, she understood that in any function of the form $y = f(x)$, the value of y depended on the value of x (which was independent). However, she was unable to use the concept of independent versus dependent variables within a context. In particular, she was unable to walk any of the graphs in Appendix G using the CBR. I attempted to engage Rachel in the activity by asking her whether position depended on time or whether time depended on position. After a moment, she recognized that her position during the walk was dependent on the time that had elapsed, but she was still unable to make a connection between what she knew about independent and dependent variables and walking the paths in the activity. She did recognize that Walk 7 (the vertical line) was not a function because the Vertical Line Test for functions would fail. I explained to her that in Walk 7 there was no way for an object to be multiple distances away from the CBR at one time. She seemed to grasp that concept but still did not want to reattempt

the walk activity. We ended our interview by Rachel insisting, "I didn't like this very much" (Interview 3, 5/23).

An Interpretation of the Beliefs Held by Melodie and Rachel

Raymond (1997) defined mathematics beliefs as "personal judgments about mathematics formulated from experiences in mathematics, including beliefs about the nature of mathematics, learning mathematics, and teaching mathematics" (p. 552). The experiences to which Raymond was referring could include prior experiences as a student of mathematics, interactions with or perceptions of former mathematics teachers, expectations of a teacher education program, or past episodes from teaching students mathematics. Depending on the nature of these prior experiences, one teacher could view mathematics as a fixed set of facts and procedures that is free of ambiguity, whereas another teacher might think of mathematics as a personal journey in which topics are investigated and knowledge is constructed along the way. In order to capture the range of views a teacher may have about mathematics, Raymond developed criteria for categorizing teachers' beliefs about the nature of mathematics, the learning of mathematics, and the beliefs held about teaching mathematics as traditional, primarily traditional, even mix of traditional and nontraditional, primarily nontraditional, and nontraditional as follows:

Beliefs about the nature of mathematics	Beliefs about learning Mathematics	Beliefs about teaching mathematics
Traditional • Mathematics is an unrelated set of procedures and rules. • Mathematics is absolute and applicable.	Traditional • Students are passive receivers of knowledge. • Students learn mathematics by working alone. • Students gain mastery from repeated drill. • There is a single "best" way to learn mathematics. • The ability to perform a procedure indicates mastery. • The textbook and worksheets are the sole resources for learning.	Traditional • The teacher is a lecturer and dispenses knowledge. • The teacher assigns seatwork. • The teacher encourages correct answers without explanation. • Mathematical topics are taught in isolation. • The teacher emphasizes mastery of an algorithm. • Instruction comes directly from the textbook. • Assessment is in the form

	• Learning mathematics is strictly dependent upon the teacher.	of a test or quiz.
Primarily traditional • Mathematics is primarily an unrelated set of procedures and rules. • Mathematics is primarily absolute and applicable.	Primarily traditional • Students primarily engage in repeated drill for mastery. • Performing a procedure is primary evidence of mastering a concept. • The teacher has great responsibility for ensuring learning than the student. • The textbook and worksheets are primarily the resources for learning. • Students work individually—occasionally working on homework in a group. • Students are primarily passive receivers of knowledge—raising questions from time to time.	Primarily traditional • The teacher primarily lectures. • The teacher primarily encourages correct answers with explantion. • The teacher primarily teaches from the textbook. • The teacher has opportunities for students to engage in problem solving.
Even mix • Mathematics is unchanging, but interconnected. • Mathematics is both absolute and dynamic, both applicable and aesthetically pleasing.	Even mix • Students learn through a combination of problem solving and procedure. • Students have a conceptual and procedural knowledge of content. • Students engage in an even blend of individual work and group work. • Mathematics can be learned in many ways. • Learning mathematics is a shared responsibility of the teacher and the student. • Repeated drill helps with procedures and exploration helps with understanding.	Even mix • The teacher uses multiple approaches in teaching—including performance tasks. • The teacher expects both process and product. • The teacher expects both procedures and concepts. • The teacher both lectures and facilitates activities. • The teacher supports individual work as well as group work. • The teacher uses the textbook as well as outside resources.
Primarily nontraditional • Mathematics is primarily absolute but interconnected. • Mathematics is about problem solving. • Mathematics is primarily surprising and aesthetically pleasing.	Primarily nontraditional • Mathematics is learned primarily through problem solving. • Students primarily learn by working with others. • Learning is demonstrated more by the ability to explain rather than just perform an algorithm. • Students are largely responsible for their own	Primarily nontraditional • The teacher primarily facilitates, but uses direct instruction from time to time. • The teacher expects the process somewhat more than the final answer. • The teacher encourages understanding over memorization. • The teacher incorporates

	learning.	· problem solving into most lessons. · The textbook is used on a limited basis.
Nontraditional · Mathematics is dynamic and is a growing body of knowledge. · Mathematics is surprising and aesthetically pleasing.	Nontraditional · The student is an explorer of mathematics. · Students learn mathematics through problem solving. · Students learn mathematics in the absence of textbooks and worksheets. · Students learn through cooperative activities. · Each student learns mathematics in a way unique to him to her.	Nontraditional · The teacher facilitates activities and asks probing questions. · The teacher encourages the sharing of ideas. · The teacher values the process. · The teacher is not reliant upon the textbook. · The teacher only provides problem solving. · The teacher makes group work the norm in the classroom.

Melodie explained in her initial interview that mathematics was the study of patterns. Although her belief about the nature of mathematics was quite traditional, her teaching practices can be described as an even mix of traditional and nontraditional. There were occasions on which Melodie would position herself at the whiteboard or at the overhead projector and present material in the form of a lecture, but her mathematics lessons were also infused with questions. The students' responses to these questions afforded Melodie the opportunity to understand how her students were thinking about functions. This technique of questioning also promoted the students' engagement in the mathematics. I noticed during my classroom observations that Melodie always came to class with a prepared (almost scripted) lesson plan, yet she was comfortable answering student-generated questions—even allowing these questions to guide her instruction. Melodie expected her students to be active learners, and she had fostered an atmosphere in which students were comfortable taking risks and asking what-if or why questions. Similarly, Thompson (1984) reported that her research participant Kay believed that mathematics teachers should pose stimulating questions and should be receptive to the guesses

and conjectures of students. Melodie's even mix of traditional and nontraditional teaching methods allowed her to comfortably move between conveyor of information and facilitator. As such, she was able to empower her students to have ownership in the mathematics and to create a sense of shared responsibility for understanding the mathematics between herself and the students.

Melodie also believed that good mathematics teaching was comprised of a variety of teaching methods—a belief that was also echoed by Kay from Thompson's (1984) study. Initially, Melodie's approach to teaching mathematics seemed traditional as she presented the names and mathematical contributions of famous mathematicians. However, I soon realized over the course of several observations that this approach was quite nontraditional in that these teacher-focused episodes showed the study of mathematics as an inquiry-based human endeavor rather than as a mere collection of pre-existing facts and procedures that must be memorized by students. Providing a human face to mathematics was also evidenced in Melodie's classroom activities. The students were expected to complete practice problems from the textbook or from a worksheet, but she also engaged her students in many inquiry-based activities. In a similar vein Cooney et al. (1998) highlighted Greg's belief that mathematics must involve reasoning and problem solving rather than just reliance upon memorization or algorithms. Most of the activities that I observed in Melodie's class dealt with using the graphing calculator to investigate families of functions. The students were able to construct their own knowledge of functions and to make connections to previous concepts—such as linear functions and the absolute value function. Moreover, they were able to connect solutions of a quadratic equation to the x-intercepts of the graph of the corresponding quadratic function. It was apparent that Melodie viewed the graphing calculator as a tool for exploration rather than a vehicle for

computation. She also encouraged her students to work collaboratively rather than in isolation. Melodie's even mix of traditional and nontraditional beliefs about teaching allowed her to relinquish control of the lesson and place the mathematics in the proverbial hands of the students. Although Thompson (1984) did not explicitly mention collaborative grouping in her study, she did report that Kay insisted on having her students grapple with mathematical concepts and that the teacher should assume a supporting role at times rather than always showing students how to arrive at a particular solution. As for Melodie, the collaborative, inquiry-based activities in her lessons prompted the students to engage in discourse about mathematics and to look for generalizations or patterns among topics—which relates back in to Melodie's traditional belief that mathematics deals largely with patterns.

Although the teaching of functions was mandated by her district's curriculum, Melodie also believed that functions were paramount in the high school curriculum. She readily admitted that she enjoyed teaching functions because her students were able to solve real-world problems or model real-world data. Although Lloyd and Wilson (1998) made no mention of Mr. Allen's enjoyment of the topic, they did report that Mr. Allen asserted that functions were important because of their wide and varied applications—much like Melodie. This primarily nontraditional belief that mathematics was useful and enjoyable is a departure from her original traditional view of the nature of mathematics, but it allowed Melodie to appreciate and value the idea that functions could be represented by equations, graphs, tables, or words. Melodie's students were expected to choose a best representation when solving real-world problems using functions and then provide an argument for why one representation was more appropriate than another. This expectation was a departure from Lloyd and Wilson's (1998) study, in which Mr. Allen was aware of multiple representations for functions but viewed graphs as the single best

representation. The NCTM (2000) recommended that "students need to work on problems that may take hours, days, and even weeks to solve" in order to hone their problem-solving skills, and that some of the problems "should be open-ended with no right answer" (p. 6). Melodie, aligning her views with those of the NCTM, believed that having students create conjectures, gather evidence in the form of data, and present a reasonable argument was at the heart of doing mathematics.

Rachel revealed in our first interview that mathematics primarily deals with numbers and recognition of patterns in numbers. However, she did point out that algebra was different from general arithmetic in that algebra is more abstract in nature, and involves greater manipulation of symbols or variables, yet both algebra and mathematics may be used when solving real-world problems or making predictions using real-world data. It is apparent that Rachel viewed mathematics from an arithmetic standpoint. Furthermore, she believed that algebra was an abstract entity separate from mathematics rather than a branch of mathematics or an extension of arithmetic. Wilson (1994) reported a similar differentiation between the theoretical branches of mathematics and more "down-to-earth" mathematics made by Molly, a preservice teacher. Rachel's traditional beliefs about the nature of mathematics (as well as about algebra) fit her primarily traditional beliefs about the nature of mathematics instruction. She believed that she was an excellent mathematics teacher because she devoted vast amounts of time to planning her lessons, circumvented confusion in her classroom by providing students with step-by-step guidance on how to work through a set of mathematics problems, and avoided mathematical symbols or terminology that she believed were beyond her students. Although Jeanne from Thompson's (1984) study was markedly different than Rachel in that Jeanne relied heavily on symbolism and terminology during instruction, the two teachers held congruent beliefs that

mathematics is a collection of facts and that the mathematics teacher is primarily responsible for demonstrating how to carry out mathematical processes. Similarly, Wilson (1994) reported that Molly believed it was necessary to teach procedures in an organized fashion. Rachel embraced the role of conveyor of mathematical knowledge and procedures. All of the assignments from the textbook that I noted during my observations and all of the teacher-generated worksheets that I collected as artifacts consisted of sets of routine problems that required an algorithm (or series of algorithms) to complete. Although the students were involved in mathematics by performing a procedure to solve a problem, they were never engaged in assignments that required exploration or peer collaboration. This primarily traditional mode of instruction prevented the students from taking an active role in the learning process and placed the onus for learning mathematics on the teacher. Likewise, the students were afforded no opportunity to grapple with concepts or to construct meaning for the topics being addressed in class. I noticed during my observations that Rachel's students were comfortable interacting with her because of her relaxed manner and fair treatment of students (Cooney & Shealy, 1997). In Rachel's classroom, these interactions tended to be requests to do more examples or to do a few problems from the current homework assignment because the problems appeared to be different than the examples just discussed in class.

In Rachel's view, students should enjoy learning mathematics because of activities that are fun and enjoyable. During the initial interview she commented that it was challenging to depict algebra problems or concepts pictorially. When I asked her about using manipulatives (such as Algebra Tiles) to explore the notion of completing the square, Rachel commented that she did not see great value in using these tiles and that the students gained little pleasure or enjoyment from using them in class. These results are in stark contrast to those of Raymond

(1997), who found that elementary teachers believed in hands-on activities and advocated the use of manipulatives. Moyer (2002) determined that some teachers used manipulatives as a reward for good behavior, whereas others viewed them as a means for making concepts less abstract. Instead, Rachel believed that the reinforcement of a concept through a game or through a puzzle was a better use of time. Not only did this strongly held traditional belief about teaching mathematics limit her students' access to the conceptual underpinnings of the content, but such beliefs also limited Rachel to proclaiming a best approach for teaching a topic rather than approaching the teaching of a topic from multiple perspectives. This limited scope of teaching might prove to be problematic for students requiring different and varying modalities for learning mathematics. Rachel also equated the term *fun* with mathematical topics that students found less daunting—including problems whose solutions could be easily verified or problems that did not require more than a few steps and in which there was a minimal amount of arithmetic. Although she told me during the initial interview that she made a concerted effort to not tell her students that a particular topic is difficult or that they would not enjoy it, Rachel commented to her students at the end of a lesson on solving quadratic equations by graphing, "Tomorrow we're going to learn another way to solve quadratic equations. You'll probably like it better—graphing is a big pain in the butt." By verbalizing her dislike of solving quadratic equations via graphing, Rachel inadvertently colored the notion her students held (or might hold in the future) about solving quadratic equations using a graph (Even, 1993). This imposition of beliefs from teacher to students might also preclude a future class discussion about which method for solving quadratic equations (isolation, factoring, graphically, completing the square, quadratic formula) is most appropriate based on the form in which the equation was originally expressed. This type of decision making and classroom discourse is important if students are to

learn to take risks in mathematics and for providing students an opportunity to express themselves mathematically.

Rachel believed that the notion of a function plays an important role in the high school mathematics curriculum. Prior to establishing a rapport with colleagues at the high schools that her former students attend, Rachel had introduced functions to her students only because they were part of her district's curriculum guide and because several chapters of the textbook were devoted to functions. The source of this primarily traditional, yet loosely held belief about teaching functions was either Rachel's counterparts at the high schools, the textbook, or the curriculum—all external sources of authority. Rachel had come to realize that she did not particularly enjoy teaching quadratic functions, because they were difficult for students. In particular, she strongly held her belief that students get frustrated because of the amount of arithmetic involved as well as the fact that if one point on a parabola is calculated or plotted incorrectly, then the entire graph is affected. Rachel's loosely held beliefs about teaching functions coupled with her dislike for teaching quadratic functions could negatively affect her instruction or negatively influence her students' attitudes toward learning functions. Furthermore, she might find the role of technology such as the graphing calculator tending more toward a means for making calculations easier rather than a tool for exploration and discovery. Haimes (1996) reported similar findings in his study. The teacher allowed students to use hand-held technology for computation, but little discovery took place. In Rachel's effort to make the learning of functions as easy and as enjoyable for her students as possible, she had purposely chosen to use avoid function notation such as $f(x)$ when writing the function in equation form. Although naming the function y was Rachel's way of linking the study of quadratic functions to the textbook's earlier treatment of linear equations, this instructional decision might not guide

students to achieve an understanding of the role of the dependent variable, the independent variable, and the relationship between them. Rachel believed that operations with functions took precedence over conceptually understanding the structure and notation of functions (Sfard, 1991). Furthermore, always naming the function y and using x as the independent variable limited students' notion that a function or a variable can be named using letters that are indicative of the context—such as using C if there is a need to create a cost function or t if there is a dependence on time.

<div align="center">An Interpretation of the Knowledge of Functions</div>

A teacher's ability to make sound instructional decisions regarding content, to anticipate the challenges faced by his or her students, and to interpret questions or misconceptions posed by student questions or work is dependent upon the teacher's content knowledge (Lloyd & Wilson, 1998). Shulman (1986b) referred to this type of knowledge as pedagogical content knowledge. He defined content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). In order to gain a better understanding of the content knowledge Melodie and Rachel and held regarding functions, I asked them to complete an assortment of tasks (described earlier). The work of Vinner and Dreyfus (1989) produced the following six categories for the definition of a function:

1. *Correspondence*: A function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set.

2. *Dependence relation*: A function is a dependence relation between two variables. In other words, the value of y depends upon the chosen x.

3. *Rule*: A function is a rule. There is typically no mention of neither the domain nor the codomain. The function "connects" the value of x with the value of y.

4. *Operation*: A function is an operation or a numerical manipulation.

5. *Formula*: A function may appear as a formula, an algebraic expression, or an equation.

6. *Representation*: A function is understood to be a graph or symbolic representation.

It stands to reason that the way a teacher envisions what a function is affects the way in which the topic is treated in the classroom. In fact, Lloyd and Wilson (1998) argued that this construct presents "a distinction between the formal definition an individual holds for a given concept and the way that he or she thinks about the concept" (p. 251).

Melodie demonstrated her knowledge of functions during the final interview by selecting from the list of definitions (Appendix F) that a function is "a rule that establishes a relationship between two quantities called the input and the output, and that for each input there is exactly one output." This was also the definition of *function* that she provided to her students during instruction. An analysis of Melodie's comments from the initial interview also revealed that she realized that there were times when functions might be presented as formulas as well as by other representations. Her pedagogical content knowledge was fairly consistent with her content knowledge in that she knew and then presented functions as a combination of rules and correspondences. Although Melodie did not formally define functions through formulas or other representations, she did present them as equations, graphs, and tables throughout her lessons. Furthermore, she was able to encourage her students to graph functions with their graphing calculators and to examine all three representations. Subsequently, the students typically decided which representation was the most appropriate given the context of the problem. Surprisingly, Melodie seemed to abandon her own formal definition of *function* almost immediately in favor of working with multiple representations.

Melodie possessed a deep understanding of elementary functions and their families. During the card-sorting activity, she separated the cards into stacks according to the family in which they belonged. She subsequently created "not functions," "linear functions," "quadratic functions," "cubic functions," "piece-wise defined functions," and "other" as the family labels. Although she spent more time examining the tables of values and the word problems than the equations and graphs, she was still able to determine an appropriate family for each card. Although Lloyd and Wilson (1998) created more sophisticated functions for their card sort with Mr. Allen because he was a high school mathematics teacher, his experience with the card sort mirrored that of Melodie in her relative ease in recognizing certain equations or graphs versus spending a greater amount of time with tables and word problems. After Melodie explained the families to me, I asked her why the linear stack seemed to have more cards than any of the others. She explained that even though some of the function representations were constant functions, they graphed as lines nonetheless. The only function that Melodie seemed to be unable to classify was the ratio of a quadratic function and a linear function. It is possible that she did not remember the word *rational* as a type of function, but she did realize that the function would have a discontinuity at $x = -2$. Her knowledge of function families helped Melodie to highlight salient features of functions such as the degree of the rule, the end behavior of the graph, and the consistency of the data in the table.   In turn, many of her students were able to predict graphs of functions that would undergo a series of transformations, and then use the graphing calculator to verify their predictions. Melodie's content knowledge also ensured a consistent treatment of functions during instruction as well as a proper use of terminology and notation. In essence, Melodie's content knowledge helped to inform her pedagogical content

knowledge—culminating in a conceptually rich and mathematically meaningful unit on quadratic functions.

In the walking activity (Appendix G), Melodie was able to successfully position herself an appropriate distance from the CBR in order to walk and create all of the corresponding graphs. It was apparent that she understood that her distance from the CBR was dependent upon the time—which was the independent variable. Conceptualizing functions as dependence relations is important in helping students use functions to model situations. Moreover, this knowledge enhanced Melodie's ability to ask guiding questions as her students grappled with real-life contextual situations that might be simplified by the use of a function. Although Melodie's textbook did not explicitly mention a function as a dependence relation, her content knowledge allowed her to supplement the textbook exercises with activities and novel problems that stimulated the thinking of the students and prompted peer-to-peer discourse about mathematics.

In the initial interview, Rachel stated that a function is a relation in which each value of x corresponds to a single value of y. This notion of a function as a rule was also consistent in the final interview when Rachel identified (from Appendix F) a function as "a rule that establishes a relationship between two quantities called the input and the output, and that for each input there is exactly one output." Although Rachel's understanding that a function is a rule allowed her to present functions as a rule to her students, her pedagogical content knowledge was limited in that she did not present functions as entities that might be represented using multiple representations. In other words, Rachel's students developed the notion that a graph is merely the graph of the function, or that a table of values is generated by substituting values into the function, but the function itself is the equation that gives rise to the graph and to the table of values. It is not

surprising that Rachel knew that functions are also operations. She remarked during the second interview that one may understand the function of a number by merely doing a substitution and getting the subsequent output value. The notions of a function being a rule and a function being an operation are not mutually exclusive. Similarly, Ball (1990) found that prospective elementary teachers also lacked a conceptual understanding of concepts even though they had facility with performing calculations. Neither Rachel nor her students used the graph to find the output values after selecting a value for the input. Likewise, she generated a table of values by hand through a series of operations that was dictated by the equation. Typically, the table of values served as a means for organizing (and then plotting) points to draw the graph. Rachel treated functions in a very one-dimensional way in her classroom because of her lack of deep and flexible content knowledge regarding functions. Her students came to know an equation as the function rather than as a single representation of the function. Furthermore, promoting the notion of a function as merely an operation precluded the students from understanding the dependent nature of functions. This approach conflicts with Kieran's (1992) insistence that students be able to view algebraic structures as mathematical objects rather than just prescribed processes. Rachel also tended to treat functions as pre-existing entities from which data could be derived versus presenting the students with data and then creating a rule or an equation that served as a model for the data as well as a means for making predictions.

Rachel's participation in the card-sorting activity revealed that she was able to use the Vertical Line Test to determine whether or not the graph of a relation represented a function. Although her students needed an occasional reminder about the definition of function, I never observed Rachel connect the premise of the Vertical Line Test to the formal definition of function. Rachel's students came to know the Vertical Line Test as a separate vehicle for

deciding whether the graph determined a function and not as a visual means for ensuring a graph's consistency with the definition of function given in class. In contrast, Mr. Allen from Lloyd and Wilson's (1998) study relied on the Vertical Line Test as a quick check for a graph's compliance. Rachel's content knowledge regarding families of functions was limited to constant and linear functions. Although she was teaching a unit on quadratic functions at the time of this study, she did not recognize that $y = \frac{1}{2}(x+2)^2 - 4$ was an equation (in vertex form) representing a quadratic function. In fact, she determined that it could be solved by factoring—along with the equation of a circle as well as a rational function. The quadratic functions that Rachel presented in class as well as the problems from the textbook and worksheets were all expressed in standard form rather than vertex form or intercept form. It follows that Rachel's ability to recognize quadratic functions was limited to functions expressed in standard form. Likewise, she was unable to classify any of the functions according to a table of values. Although Molly from Wilson's (1994) study had to grapple with functions represented in tabular form, Molly at least plotted the points and connected them with a smooth curve, whereas Rachel simply declared she was unable to classify the tables of values. Rachel never encouraged her students to use a graphing calculator to examine the graphs of any function nor to check the table of values. Rachel's limited knowledge of functions coupled with her limited pedagogical content knowledge for teaching functions might severely limit her students' understanding of what a function is, how functions may be represented in a variety of ways, and when technology can serve as a tool for exploration rather than as a means for performing arithmetic. Although Rachel introduced the notion of slope to her students during the unit on linear functions, she never used tables of values to connect the idea of slope as a constant rate of change versus a rate that may change, as could have been discussed during the unit on quadratic functions.

Rachel's obvious discomfort during the function walk activity suggested that her understanding and appreciation for functions was confined to the textbook rather than being situated in a context such as time versus distance. In turn, Rachel's instruction was limited to routine problems that were procedural rather than affording her students opportunities to view functions as a means for modeling or interpreting data. Carpenter and his colleagues (1988) reported similar findings among elementary teachers and their pedagogical content knowledge. Melodie's students might only understand x as the independent variable and y as the dependent variable rather than grappling with a situation and having to decide which quantity is dependent upon another.

A Comparison of Melodie and Rachel

Melodie believed that the nature of mathematics is pattern recognition and that algebra is a general means by which to express a quantity. In order to teach mathematics effectively and efficiently, she insisted that lessons had to be planned thoroughly and that the mode of instruction must be varied. One way she was able to accomplish her vision of good mathematics teaching was by beginning each class with a biographical overview of a famous mathematician. This tactic allowed her students to view mathematics as a human endeavor that has evolved with time rather than as a fixed collection of facts and procedures that were created by their teacher or someone else. Melodie varied her lessons by providing an even mix of teacher-centered instruction, opportunities for students to work collaboratively as well as individually, and activities in which students investigated mathematics as well as practiced a procedure. Like Melodie, Rachel viewed mathematics as the study of numbers and patterns, but she pointed out that algebra is a special segment of higher-level mathematics in which variables are manipulated. Rachel's typical mode of instruction was to provide students with a photocopied outline of the

day's notes, list a sequence of steps for solving problems in a set, demonstrate several problems of the same variety, and use terms that resonated with the student vernacular. This type of instruction was consistent with her belief that good mathematics teaching consisted of giving students step-by-step instructions for how to solve a problem and by avoiding mathematical terminology or symbolism that might be too overwhelming for her students.

Melodie's even mix of traditional and nontraditional teaching methods allowed her students to be active participants in the learning of mathematics. During teacher-focused instruction, she introduced new concepts and helped the students connect the current mathematical topic to concepts previously studied. At other times, the students worked collaboratively to investigate a mathematical concept and then engaged in a whole-class debriefing in which findings from the investigation were discussed and compared. Regardless of whether she was lecturing or the students were engaged in an inquiry-based activity, Melodie was a constant poser of questions aimed at assessing students' conceptual understanding of the mathematics. In turn, if Melodie was not forthcoming with the reason behind a procedure, her students would begin posing their own questions to understand the mathematics. In contrast, Rachel's traditional methods of teaching mathematics placed most of the onus for student learning on the teacher. Although her intentions were admirable, Rachel's role as conveyor of knowledge never afforded her students an opportunity to share in the responsibility for the learning of mathematics. Unlike Melodie, most of the questions Rachel posed were designed to elicit responses from students about the next step in a problem rather than probing the students' conceptual understanding of the mathematics. In turn, the students became more reliant upon Rachel for working through a seemingly difficult problem rather than relying upon themselves and making an attempt at solving a problem. According to the NCTM (2000), being a risk-taker

is critical for learning and understanding mathematics. It is this type of learning and understanding that will "enable students to solve the new kinds of problems they will inevitably face in the future" (p. 21).

Melodie saw functions as the cornerstone of high school mathematics. She enjoyed teaching quadratic functions since they are useful in solving real-world problems and may be represented in multiple ways. She was a proponent of using the graphing calculator to display the equation, the graph, and the table that corresponds to a given function. Subsequently, she would ask her students to choose a best representation of the function in a given context and justify why one representation was more appropriate than another. Melodie claimed that she also enjoyed teaching the concept of a function because the nature of the material allowed her to focus on concept and procedure rather than just procedure. In contrast, teaching quadratic functions was one of Rachel's least favorite topics in the Algebra I curriculum. She enjoyed teaching topics that the students seemed to enjoy learning and could understand relatively easily. In her view, students found graphing quadratic functions difficult because of all of the arithmetic involved in calculating the vertex as well as other points on the parabola. In order to ease the situation, Rachel allowed her students to use graphing calculators to perform much of the arithmetic. She tended to emphasize the equation as the representation for the function and then used the equation to generate a table of values following by the graph. Although Rachel allowed her students to use the graphing calculator for arithmetic, she never encouraged them to use technology as a means for checking the reasonableness of a graph or for performing an investigation—unlike Melodie, who used the graphing calculator as a tool for exploration. The issue of multiple representations was never explicitly presented or discussed in Rachel's Algebra

I class. She taught graphing quadratic functions to her students in isolation rather than exploring the use of these functions to solve real-world problems or as a means for modeling a set of data.

In terms of content knowledge, Melodie had a deep understanding of functions. She initially defined a function as a rule that related two quantities, but that definition did not preclude her from advocating multiple representations in her classroom. She demonstrated through the card-sorting activity that she had a solid knowledge of families of functions as well as multiple means for representing each family. These function families were not confined to the constant, linear, and quadratic functions she taught in Algebra I; she also had facility with cubic functions, piecewise-defined functions, and rational functions. Melodie's knowledge of functions (coupled with her belief that functions are important and that students should be actively engaged) allowed her to promote student discourse about the best model for data. This knowledge also allowed Melodie to respond to student questions thoughtfully and intelligently. It was not uncommon for Melodie to change the direction of her original lesson plan to accommodate the questions of her students, and it was her knowledge of content and pedagogy that gave her the confidence to do so. In comparison to Melodie, Rachel's content knowledge regarding functions was quite limited. During the card-sorting activity, she was able to identify functions that were constant and linear as well as graphs that were not functions, but she struggled with all of the others. In fact, Rachel did not even identify the equation $y = \frac{1}{2}(x + 2)^2 - 4$ as that of a quadratic function. Her trouble was probably due to the fact that her textbook expressed most quadratic functions in standard form ($y = ax^2 + bx + c$) rather than vertex form. Rachel's limited knowledge of functions (coupled with her belief that students must be given step-by-step instruction) made her lessons about quadratic functions routine. Likewise, her lessons were primarily devoid of any conceptual underpinnings of the material.

Melodie's understanding of independent versus dependent variables was evidenced through her rapid completion of the graph-walking activity. Melodie had not used a CBR before, but she expressed an interest in her school purchasing classroom sets. This notion of dependence is critical for understanding the nature of a function. Rachel was not able to complete the activity. Although she was able to identify time as the independent variable and position as the dependent variable (after being prompted), she was unable to interact with the CBR to create the corresponding graphs. In fact, Rachel insisted that the activity was frustrating rather than helpful.

Melodie viewed her colleagues, the textbook, and the county curriculum guide as resources for teaching, but she depended upon her own content knowledge and pedagogical knowledge to guide her instructional choices. She had the self-confidence to view herself as the authority in her classroom and used her broad view of functions to plan and orchestrate her lessons on quadratic functions. In contrast, Rachel did not understand the significance of teaching functions. She taught quadratic functions only because they were mandatory according to the county's Algebra I curriculum guide. The examples that she used during class were chosen to closely resemble the problems in the textbook. She began using the term *function* with her students rather than just *equation* when teaching quadratic functions at the request of her high school counterparts during a Math Vertical Team meeting. Rachel's authority was external when it came to making instructional decisions about quadratic functions. The NCTM (2000) argued that teachers of mathematics should use "the available textbooks, support materials, technology, and other instructional resources effectively and tailoring these resources to their particular situations so that their goals are met for mathematics instruction" (p. 374); however,

the teacher should not forego her own authority when making choices that affect students and

their learning of mathematics.

CHAPTER 5

SUMMARY AND IMPLICATIONS

Summary

The study of mathematics teaching is a complex endeavor. In fact, attempting to understand how teachers orchestrate mathematically sound, engaging, and meaningful lessons often generates more questions for future research than answers. According to the NCTM (2000), "students learn mathematics through the experiences that teachers provide" (p. 16). The classroom teacher plays an integral role in engaging students in mathematics, helping them construct meaning for and make connections between mathematical ideas, and influencing their disposition about mathematics as a discipline as well as an intellectual tool. Less apparent is the explanation behind how teachers of mathematics make instructional decisions that subsequently drive their pedagogical practices. Mathematics educators such as Begle (1972) and Eisenberg (1977) were among many educational researchers who used quantitative methods to assess the correlation between teachers' knowledge of mathematics (primarily measured by the number of college mathematics courses they had taken) and student achievement. Although an analysis of both sets of data revealed no such correlation, Shulman (1986a) insisted that researchers must continue to focus on content knowledge and how that knowledge is transformed into content for instruction.

Thompson (1982) argued that there is a relationship between one's conceptions (beliefs, views, attitudes, preferences) of mathematics and one's teaching of mathematics, but "very little

is known about the role that teachers' conceptions of the subject matter and its teaching might play in the genesis and evolution of instructional practices characteristic of their teaching" (p. 4). Thompson's dissertation and subsequent scholarly endeavors (1984, 1992) called for more research on affective issues that might influence teachers' pedagogical practices. Many studies revealed that the beliefs teachers hold about the nature and learning of mathematics significantly influence their classroom practices and instructional choices (Brown & Baird, 1993; Cooney, Shealy, & Arvold, 1998; Ernest, 1989b; Fennema & Franke, 1992; Raymond, 1997). Although there is no single definition or description of good mathematics teaching, effective mathematics teachers provide students with opportunities to experience mathematics. In turn, these experiences are influenced by the teacher's beliefs.

Historically, studies on teachers' instructional practices have focused either on the mathematical knowledge of the teacher or on the beliefs held by the teacher. However, Thompson (1992), informed by the work of Ernest (1988a), cautioned researchers that "although important, knowledge of mathematics does not account for differences in practice across mathematics teachers" (p. 131), and "teachers' approaches to mathematics teaching depend fundamentally on their systems of beliefs, in particular on their conceptions of the nature and meaning of mathematics, and on their mental models of teaching and learning mathematics" (p. 131). Rather than continuing to treat the beliefs that teachers hold and the knowledge (content and pedagogical) that teachers possess in separate research studies, it seems reasonable to examine these two factors together and their influence on classroom instruction. Therefore, the purpose of the present study was to answer three questions: (1) How do middle school teachers understand and conceptualize functions? (2) What beliefs do middle school teachers hold about teaching functions? (3) How does middle school algebra teachers' knowledge of functions and

their beliefs about teaching functions influence their teaching practices? Research studies abound that focus on elementary teachers and their beliefs about mathematics (Ball, 1990; Collier, 1972; Raymond, 1997; Thompson, 1992). Likewise, there is an abundance of literature regarding elementary teachers' knowledge (Ball, Lubienski, & Mewborn, 2001; Ball & Bass, 2000; Fennema & Franke, 1992; Ma, 1999; Mewborn, 2003) and a lesser amount of research regarding the knowledge of high school teachers (Cooney & Wilson, 1996; Eisenberg, 1977; Even, 1993; Haimes, 1996; Wilson, 1994). However, research studies intended to examine the thinking and practices of middle school mathematics teachers are sparse.

Two teachers who taught Algebra I in different (yet similar) middle schools in the same district were the participants in this study. Data about these teachers were collected through their completing an initial survey, engaging in three hour-long interviews, doing a card-sorting activity that dealt with families of functions represented in a variety of ways, selecting the most appropriate definition for the term *function* from a list of classic definitions, and doing a modeling activity in which the participant was asked to walk the shape of a pre-constructed graph using a graphing calculator and a calculator-based ranger. In the interim, each teacher was observed teaching a unit on quadratic functions for approximately 3 weeks. Classroom artifacts such as quizzes, tests, worksheets, graphic organizers, and activities were collected.

An analysis of the data was performed almost immediately after collecting it since each phase of interviews was dependent upon the previous phase. After I read all interview transcripts, field notes, and artifact comments, I coded and categorized the data. Subsequently, the data were sorted by "beliefs about mathematics," "beliefs about teaching functions," and "knowledge of functions." I used Raymond's (1997) model for characterizing the teachers' beliefs about mathematics and the teaching of mathematics. The work of Shulman (1986b)

helped me characterize each teacher's knowledge of mathematics and pedagogy. Each teacher's

concept of function was based on criteria defined by Vinner and Dreyfus (1989).

<div align="center">Conclusions</div>

Planning and orchestrating a conceptually rich and meaningful mathematics lesson can be

a daunting task for the novice mathematics teacher as well as for the veteran. There is no doubt

that external factors such as the textbook, the curriculum guide, collaboration with colleagues,

and mandates from the school or district administration influence the daily instructional practices

of mathematics teachers. Likewise, factors that are both internal and unique to each teacher also

play a major role in teachers' pedagogical practices. Consistent with the findings of Brown,

Cooney, and Jones (1990), Sfard (1991), McLeod (1992), Thompson (1992), Raymond (1997),

and Cooney et al. (1998), the findings of this study suggest that the instructional practices of

middle school mathematics teachers are influenced by the beliefs they hold about mathematics

and about its teaching. In particular, traditional beliefs about mathematics may lead to

instruction that is characterized by treating topics in isolation rather than building connections

among ideas, presenting concepts as a sequence of steps rather than examining the conceptual

components, and expecting students to complete a set of routine problems by which they mimic

the process demonstrated by the teacher rather than engaging in some type of inquiry-based

activity in which they grapple with a task and share in the responsibility of understanding.

The knowledge of content and knowledge of pedagogy that a teacher brings to the

proverbial table also affects mathematics instruction (Ball, 1990; Brown & Borko, 1992; Even,

1993; Ball & Bass, 2000; Ball et al., 2001). The results of these studies suggest that teachers'

subject matter knowledge affects the structure of lessons, the assignments and activities of the

students, and the use of the textbook as well as the course curriculum. However, Mewborn

(2003) warned that some teachers' mathematics lessons are neither rigorous nor conceptually deep since "many teachers do not possess this deep and rich knowledge of mathematics" (p. 47). One of the middle school teachers in the present study had a relatively strong mathematical background, whereas the other had knowledge that was quite limited. A knowledge of mathematics allows teachers to view mathematical ideas from multiple perspectives, create multiple representations for a mathematical entity, and argue the value of one idea or one representation over another in a particular context. Content knowledge also equips teachers with the means for facilitating classroom discourse about mathematics and affords them sufficient confidence to take risks in terms of assigning student-focused activities and allowing students to offer suggestions for solving problems. A more global understanding of mathematics may prompt teachers to help students connect various topics in mathematics and demonstrate how certain topics are applicable beyond the confines of a routine procedure.

There is no simple answer for why teachers make the instructional decisions that they make. The findings of this study support the notion that teachers' beliefs about and knowledge of mathematics work together (or sometimes in opposition to one another) during the planning and teaching stages of instruction. It is impossible to say whether beliefs play a more vital role in mathematics instruction than content knowledge, and vice versa. In fact, the argument is rather circular. If a teacher believes that a particular topic in the curriculum is important, then that topic will be probably be treated in greater detail; however, teaching a lesson involving a particular mathematical topic also relies heavily upon the teacher having an understanding and appreciation of the topic at hand. Conversely, knowledge of a topic must be constructed before a teacher can decide how much value a particular topic possesses. A study of this nature was suggested by Cooney and Wilson (1996) as follows: "Research on teachers' thinking that

neglects the importance of these contexts [beliefs and knowledge] runs the risk of studying trees but having no basic understanding of the forest in which these trees grow" (p. 155). This study adds another dimension to the growing body of research on teacher cognition.

The knowledge and beliefs that teachers hold about mathematics and mathematics instruction may also determine the role of the student. Teachers who possess deep, flexible knowledge of mathematics and believe that students should be active participants in the learning process are likely to design opportunities and experiences for students to interact with the teacher, their peers, and the mathematics. Likewise, these teachers are more apt to allow student-generated questions or comments to help determine the direction of the lesson rather than maintaining the original lesson plan that may not take into account the needs of the students. Although straying from a preplanned lesson may be a daunting, it may also become a journey that is more fruitful for the students in their quest to grapple with, understand, and appreciate mathematics.

<div align="center">Implications for Teacher Education</div>

The findings from this study have several implications for teacher education. First, teacher educators must devote a greater amount of time providing preservice teachers opportunities to explore and understand their beliefs about mathematics and mathematics teaching. Melodie was able to balance many of her lessons by having a small segment of teacher-delivered instruction, a segment of inquiry-based learning (typically using hand-held technology), and a segment that provided closure to the day's lesson. Rachel's lessons tended to be more teacher-focused, and a majority of her instruction time was devoted to providing notes and working example problems. Although this format is effective if the teacher is primarily interested in assessing a procedure, it is challenging to maintain the students' interest on a daily

basis. Furthermore, this lack of engagement in the mathematics does not necessarily facilitate enduring understanding of a concept. Philippou and Christou (2002) reported that preservice teachers' "mathematical beliefs systems were repeatedly found to make a difference in determining the level of their motivation and persistence in the face of difficulties" and that these views "influence their general pedagogical outlook, the learning climate they will contribute to, and specifically their choice of teaching strategies and learning activities" (p. 212). As a means of expressing and analyzing these beliefs, preservice teachers could write a philosophy of teaching mathematics paper rather than the typical philosophy of education paper. In this paper preservice teachers might discuss their goals for teaching mathematics, the ways in which these goals will be met and assessed, their perception of learning mathematics, and their beliefs about the roles of teacher and students. This paper could evolve and be revised as preservice teachers progress through their sequence of mathematics education courses. Not only could this paper serve preservice teachers by making their beliefs more explicit, but teacher educators could also monitor the influence of their mathematics education program and adjust practices accordingly.

Second, mathematics educators should consider making closer links between methods courses and pure mathematics courses. A teacher's knowledge of advanced mathematics is of little value if she is unable to create student-centered lessons that facilitate students' conceptual understanding. For example, a teacher's ability to solve a quadratic equation using the process of completing the square does not necessarily mean that the teacher is able to present the topic in a conceptually rich manner. The role of the teacher educator is to situate a preservice teacher's knowledge of mathematics in a pedagogical arena—perhaps showing the preservice teacher how to use Algebra Tiles™ to depict why this process consists of geometrically completing the final piece of a square. Likewise, it is difficult for a teacher to embrace various methods for teaching

mathematics if she lacks content knowledge. Goldsmith and Shifter (1997) reported that "if teachers do not have a strong enough grasp of the mathematics that they teach, they may not be able to engage their students in an exploration of mathematical ideas beyond calling attention to a variety of possible solution strategies" (p. 33). As some states continue to push topics that have been thought of as high school topics into the middle grades to grant students greater access to Advanced Placement courses, preservice middle school mathematics teachers need a wide variety of mathematics courses as well as courses in mathematics pedagogy. Inservice middle school mathematics teachers may need greater support through professional development courses as well as through school-based mathematics coaches. Mewborn (2003) cited professional development opportunities designed to enhance teachers' knowledge of mathematics as crucial since "teachers need to revisit the mathematics they are teaching to gain insight into the conceptual underpinnings of topics and the interconnections among topics" (p. 49).

The NCTM (2000) encourages middle school teachers to help students "develop facility with using patterns and functions to represent, model, and analyze a variety of phenomena" (p. 227). Likewise, the authors of *PSSM* argue that high school students should be able to "create and use tabular, symbolic, graphical, and verbal representations and to analyze and understand patterns, relations, and functions with more sophistication than in the middle grades" (p, 297). It is incumbent on middle school teachers to begin a foundation on which students' knowledge and appreciation of functions may be constructed. In turn, mathematics teacher educators are charged with the arduous task of equipping both preservice and inservice teachers with deeper and more flexible knowledge of functions as well as strategies for helping future students construct their own deep, flexible knowledge of functions. Teachers should be aware that functions may be represented in a variety of ways, that every function falls into some family, and

that it shares characteristics with its relatives. Chazan (2008) posited that "with the shift in the conceptualization of algebra, tables of values and graphs on the Cartesian plane are now a larger part of the subject (p. 27). This "shift" to which Chazan referred was algebra once being thought of as generalized arithmetic versus a modern conceptualization of algebra as the study of structures (Kilpatrick & Izsák, 2008). Rather than focusing exclusively on operations with functions and the graphing of functions, teachers must help make students aware that functions are tools by which we may model real-world data. Preservice and inservice teachers need expanded support in using graphing utilities to easily maneuver between various representations and using each representation appropriately. There should be greater discourse about the role of technology in the mathematics classroom (e.g., graphing calculators being used to perform arithmetic versus being used a tool for exploration).

Teachers at all levels need greater guidance in posing questions that assess students' understanding of the mathematics rather than procedural knowledge. Well-constructed questions can pull an otherwise disinterested student into the lesson, stimulate conversation among peers, and provide teachers with feedback on their instruction. Posing questions is not necessarily an easy task, and teachers need support in this endeavor. Sanchez (2001) found that "one way of understanding teachers' commitment to focusing on conceptual understanding is to look at the kinds of questions they feel compelled to ask their students" (p. 134). Not only was Melodie proficient at posing questions to her students verbally, but she also crafted tasks in which students were led to discover concepts in mathematics. In contrast, Rachel posed questions only about the next step in a problem. Although knowing the next step in a problem may be useful at that time, better questioning techniques could have enhanced her mathematics lessons by probing student thinking and by potentially stimulating greater interest in the subject matter.

A Call for Future Research

In this study I examined the beliefs and knowledge of two middle school mathematics teachers regarding functions. The same study might be conducted with middle school teachers regarding proofs in geometry. As Algebra I continues to move into the middle grades, so does geometry. As students are granted greater access to a formal geometry course in Grade 8, more middle school teachers are finding themselves the instructor of record for geometry. Proofs can be daunting for student and teacher alike. There is no doubt that teachers of middle school geometry will need support in providing quality instruction, but how will the support be created? Prior to mathematics educators creating professional development classes for middle school geometry teachers, it is imperative that teachers' beliefs about teaching proofs are examined and their knowledge of geometry is assessed. The findings from such a study should inform the type, duration, and scope of the professional development. As with students, it is important to understand what a teacher already knows prior to creating a plan for what is taught.

In the last section I suggested that teachers at all levels focus on asking better questions on a more frequent basis. Recall that Melodie was a perpetual question poser, and in turn her students would ask probing questions of her. This ritual of asking questions fostered an atmosphere in which students were comfortable generating their own questions (and sometimes even their own answers via a peer). A study should be conducted in which the questioning techniques of the teacher are examined to see if the nature of the questions influences the types of questions that students pose during class. The study would have to be longitudinal (probably over the course of an entire school year) and care would have to be taken to document all questions. Subsequently, students would have to be interviewed to probe why they asked the question, how they decided upon the wording, and so forth.

Math Vertical Teams is a framework presented and packaged through the College Board. In fact, school districts (and even some individual schools) across the country pay for a College Board consultant to deliver training on how to create and sustain a MVT. At various times throughout this study, Melodie and Rachel made references to their respective Math Vertical Teams—without any solicitation from me. Melodie is actually a contracted College Board consultant and has worked with teachers and school administrators on ways vertical teaming can enhance the quality of mathematics instruction and increase enrollment in Advanced Placement courses. Rachel's MVT consisted of teachers from her school, another middle school, and the high school that received students from both middle schools. It was through this involvement that Rachel gained a greater awareness that functions are important in later mathematics courses. Studies need to be conducted by which researchers investigate the influence of vertical teaming on mathematics instruction. How is vertical teaming different from cross-school collaboration? How often do these teams meet, and how meaningful are the meetings? Regardless of the question, the mathematics education community needs greater insight into Math Vertical Teams and the potential impact on instruction.

Concluding Remarks

The pedagogical decisions that mathematics teachers make on a day-by-day, hour-by-hour, or even minute-by-minute basis are influenced by many different, seemingly unrelated factors. In this study I intended to examine how the beliefs and knowledge held by two middle school Algebra I teachers worked in concert to influence their instruction of functions. This study was also intended to pave the way for future professional development of middle school mathematics teachers. Although I cannot provide a panacea for the perceived ills of mathematics education in the United States, this study has at least added to the existing body of literature

about mathematics teacher education and may prompt future research that will advance us one step closer to our overarching goal: improve mathematics instruction and facilitate student learning.

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Appendix A

INITIAL PARTICIPANT QUESTIONNNAIRE

Name	
School Name	
School Address	
Email Address	
School Phone	
Home Phone	
School Fax	
Home Fax	
Certificate Area (name & level)	

1.   What is your title? (Check all that apply)

Teacher	
Lead teacher	
Department chair	
Other (please specify):	

2.   Indicate your sex: _____ Male _____ Female

3.   Indicate your date of birth_____

4.   For how many years have you taught Algebra I to students in grade 7 or 8?_____
        Professional Development

5.   Indicate the professional organization(s) with which you are currently involved.

        _____ NCTM _____ GCTM _____ MAA

        _____ Other: (please specify)_____

6.   Indicate the role(s) you may have played in one or more of these professional
        organizations.

        _____ Attended conferences _____ Served on organizing committee

        _____ Presented at conferences _____ Elected officer

Appendix A (continued)

7.	Which of the following have occurred during your teaching career?

_____ I was/am pursuing or have received another academic degree

_____ I was/am writing or have written a teaching-related journal article

_____ I was/am involved in writing a teacher-related book or textbook

_____ I was/am hosting a radio or television program related to teaching

_____ I was/am involved in grant-writing or securing funds for education

_____ I was/am teaching undergraduate or graduate courses at a college or university

_____ I was/am on writing or development teams for QCC revisions

_____ I was/am on writing or development teams for End of Course or graduation tests

Teacher Opinion

8.	Please provide your opinion about each of the following statements.

	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
a. Students learn mathematics best in classes with students of similar abilities.	1	2	3	4	5
b. The testing program in my state/district dictates what mathematics content I teach.	1	2	3	4	5
c. I enjoy teaching mathematics.	1	2	3	4	5
d. I consider myself a "master" mathematics teacher.	1	2	3	4	5
e. I have time during the regular school week to work with my colleagues on mathematics curriculum and teaching.	1	2	3	4	5
f. My colleagues and I regularly share ideas and materials related to mathematics teaching.	1	2	3	4	5
g. Mathematics teachers in this school regularly observe each other teaching classes as part of sharing and improving instructional strategies.	1	2	3	4	5

Appendix A (continued)

9. How familiar are you with the *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics? (Check one box)

Not at all familiar	
Somewhat familiar	
Fairly familiar	
Very familiar	

10. Please indicate the extent of your agreement with the overall vision of mathematics education described in the *Principles and Standards for School Mathematics*. (Check one box)

Strongly disagree	
Disagree	
Neutral	
Agree	
Strongly Agree	

11. To what extent have you implemented recommendations from the *Principles and Standards for School Mathematics* in your mathematics teaching? (Check one box)

Not at all	
To a minimal extent	
To a moderate extent	
To a great extent	

12. Which degrees have you earned that are listed below?

Bachelors	**Yes**	**No**
Masters	**Yes**	**No**
Ed.S.	**Yes**	**No**
Doctorate	**Yes**	**No**

13. In what year did you last take a formal course for college credit in…

_____ Mathematics   _____ The Teaching of Mathematics

Appendix A (continued)

14. Which of the following activities take place during mathematics lessons that you teach? Place a check in the column under the heading "Most typical day" to indicate what happens with the greatest frequency. Mark the "Least typical day" in the same manner for the least frequent activities.

	Least Typical Day	Blend	Most Typical Day
Lecture			
Students completing textbook/worksheet problems			
Students doing hands-on/inquiry activities using technology or manipulatives			
Students reading about mathematics			
Students working in small groups			
Students using graphing calculators			
Students using computers			
Students using other technologies			
Whole-group discussions about mathematics			
Test or quiz			
None of the above			

Describe a "typical" mathematics lesson in your classroom:_____

_____

_____

_____

_____

Given the current Algebra I curriculum, my favorite topic/lesson/chapter to teach is_____

_____

and my favorite topic/lesson/chapter to teach is_____

_____

Appendix A (continued)
Teacher Background

15.   Please indicate how well prepared you believe you are to do each of the following in your mathematics instruction. (Check one box on each line)

	Not Adequately Prepared	Somewhat Prepared	Fairly Well Prepared	Very Well Prepared
Take students' prior knowledge into account when planning lessons				
Develop students' conceptual understanding of mathematics				
Provide Deeper coverage of fewer mathematics concepts				
Help students make connections between mathematics and other disciplines				
Lead a class of students using investigative strategies				
Assessment of student progress				
Manage a class of students engaged in hands-on/discovery-based work				
Have students work in cooperative learning groups				
Listen to/ask questions as students work in order to gauge their understanding				
Use the textbook as a resource rather than as the primary instructional tool				
Encourage students' interest in mathematics				
Use computers/calculators for drill and practice				
Use computers/calculators for mathematics learning games				
Use computers/calculators to collect and/or analyze data				
Use computers/calculators to demonstrate mathematical principles				

Appendix A (continued)

16. In the past 12 months, have you: (check one box on each line)

	Yes	No
Taught any in-service workshops in mathematics or mathematical teaching?		
Mentored another teacher as part of a formal arrangement that is recognized or supported by the school or district, not including supervision of student teachers?		
Received any local, state, or national grants or awards for mathematics teaching?		
Served on school or district mathematics curriculum committee?		
Served on school or district mathematics textbook selection committee?		

17. In the past 3 years, have you participated in any of the following activities related to mathematics or the teaching of mathematics? (check one box on each line)

	Yes	No
Taken a formal college/university mathematics course.		
Taken a formal college/university course in the teaching of mathematics.		
Observed other teachers teaching mathematics as part of your own professional development (formal or informal).		
Met with a local group of teachers on a regular basis to study/discuss mathematics teaching issues.		
Attended a workshop or meeting about mathematics teaching		

Appendix A (continued)

18. How would you rate your level of need for professional development in each of these area?

	None Needed	Minor Need	Moderate Need	Substantial Need
Deepen my own mathematics content knowledge.				
Understand student thinking in mathematics.				
Learning how to use technology in mathematics instruction.				
Learning how to use inquiry/investigation-oriented teaching strategies.				
Learning how to assess student learning in mathematics.				
Learning how to teach mathematics in a class that includes students with special needs.				

** This instrument was developed using items contained on the *2000 National Survey of Mathematics and Mathematics Education* and the *Mathematics Presidential Awardees* Questionnaires, created by Horizon Research. This instrument may be found at http://2000survey.horizon-research.com/instruments/teacher.php.
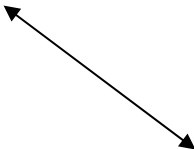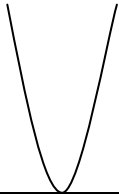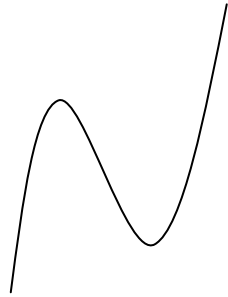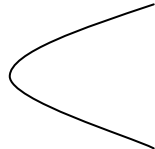
Appendix B

Interview Protocol 1

1. Tell me about your education background.

2. For how long have you been teaching Algebra I?

3. What made you decide to be a mathematics teacher?

4. If a student were to say, "What is mathematics?", how would you respond to that question?

5. What is algebra to you?

6. What is your favorite lesson to teach, and why? Your least favorite, and why?

7. What is good mathematics teaching to you?

8. Where do you see functions fitting into the Algebra I curriculum?

9. What is a function to you?

10. What is the role of technology when teaching functions?

Appendix C

Function Card Sort

$y = -3$	<table><tr><td>x</td><td>y</td></tr><tr><td>5</td><td>-1</td></tr><tr><td>-3</td><td>-1</td></tr><tr><td>1</td><td>-1</td></tr><tr><td>8</td><td>-1</td></tr><tr><td>0</td><td>-1</td></tr><tr><td>-2</td><td>-1</td></tr><tr><td>-5</td><td>-1</td></tr></table>	The speed limit on GA 400 between mile markers 5 and 27 is 65 miles per hour. If Maude has decided to maintain that speed limit by setting her car's cruise control, how fast is she driving as she passes a police car at stationed at mile marker 13?	

$y = -3x + c$	<table><tr><td>x</td><td>y</td></tr><tr><td>0</td><td>5</td></tr><tr><td>2</td><td>3</td></tr><tr><td>3</td><td>2</td></tr><tr><td>4</td><td>1</td></tr><tr><td>5</td><td>0</td></tr></table>	A stockbroker charges $45 to handle any transaction. In addition, he charges $.45 per share traded. Find the cost of the broker selling 1300 shares of stock.	

$y = \dfrac{1}{2}(x+2)^2 - 4$	<table><tr><td>x</td><td>y</td></tr><tr><td>-2</td><td>5</td></tr><tr><td>-1</td><td>0</td></tr><tr><td>0</td><td>-3</td></tr><tr><td>1</td><td>-4</td></tr><tr><td>2</td><td>-3</td></tr><tr><td>3</td><td>0</td></tr><tr><td>4</td><td>5</td></tr></table>	A parking lot is to be formed by fencing in a rectangular plot of land except for an entrance 12 meters wide on one side. Find the dimensions of the lot of greatest area if 300 meters of fencing is to be used.	

$y = \dfrac{1}{2}x^2 - \dfrac{2}{3}x^3 + 2 - x$	<table><tr><td>x</td><td>y</td></tr><tr><td>-3</td><td>-26</td></tr><tr><td>-2</td><td>-7</td></tr><tr><td>-1</td><td>0</td></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>9</td></tr><tr><td>3</td><td>28</td></tr></table>	From a rectangular piece of cardboard of dimensions 8 x 15, four congruent squares are to be cut out, one each corner. The remaining cardboard is then folded into an open box. What size squares should be cut out if the volume of the resulting box is to be maximized?	

$(x-4)^2 + (y+1)^2 = 5$	<table><tr><td>x</td><td>y</td></tr><tr><td>2</td><td>6</td></tr><tr><td>-4</td><td>5</td></tr><tr><td>0</td><td>5</td></tr><tr><td>6</td><td>9</td></tr><tr><td>-4</td><td>1</td></tr><tr><td>1</td><td>3</td></tr><tr><td>5</td><td>-6</td></tr></table>		

$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$	$y = \dfrac{x^2 - 4}{x + 2}$	A direct-dial long distance call costs $2.25 for the first two minutes and $1.03 for each addition minute or fraction thereof. Write the cost function C of a call in terms of minutes m.

Appendix D

Interview Protocol 2

1. Thinking about the major topics that you've taught this year, what are the three most important topics, and why?

2. What guides your teaching more than anything else (e.g., what guides your curriculum and instruction decisions)?

3. If you had a student that claimed, "I just don't get this whole function thing", how would you address that comment?

4. If a beginning teacher were to come and say, "I don't know where to begin planning this unit on functions", what advice would you give to this teacher? Or, could you give them a rationale for why teaching this unit is important?

5. What is the role of technology when teaching a unit on functions?

Appendix E

Interview Protocol 3

1) Tell me how your students react when you are demonstrating "why" something works or when you do a proof—a proof of the distance formula for example.

2) Why do we teach math for conceptual understanding? And, why do we care?

3) When you use certain formulas in class, let's say $A = \dfrac{1}{2}bh$, do your students ever ask where the ½ comes from, or do you ask them?

4) Suppose you had the equation $x(x + 5) = 7$ and the students gave $x = 7$ or $x = 2$ as solutions to the equation. How would you address this student?

Appendix F

Seven Definitions of Functions

If for each value of a variable x there is determined a definite value or set of values of another variable y, then y is called a *function* of x for those values of x (Townsend, 1915).

Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and variable element y of F is called is called a *functional relation* in y if, for all x in E, there exists a unique y in F which is in the given relation with x (Bourbaki, 1939).

An algebraic expression involving one or more letters is a *function* of the letter or letters involved (Hawkes, Luby, & Touton, 1909).

Thus, $2x + 3$ and $x^2 + 5x - 6$ are functions of one letter, x; $x^2 - 2xy + y^2$ and $x^3 + y^3$ are functions of two letters, x and y. The letters of a function are usually referred to as variables.

If two variables, x and y, are so related that to each value of x (the independent variable) there corresponds a definite value or set of values of y (the dependent variable), y is called a *function* of x (Betz, 1931).

A function is a set of ordered pairs in which each first component is paired with exactly one second component (Dolciani, Wooten, Beckenbach, & Sharron, 1983).

A function is a relation with the property: If (a, b) and (a, c) belong to the relation, then $b = c$. The set of all first entries of the ordered pairs is called the domain of the function, and the set of all second entries is called the range of the function (Demana & Waits, 1990).

A *function* is a rule that establishes a relationship between two quantities, called the input and the output. For each input, there is exactly one output (Larson, Boswell, Kanold, & Stiff, 2001).

Appendix G

Computer-Based Ranger Activity

Walk 1:	Walk 2:
Walk 3:	Walk 4:
Walk 5:	Walk 6:
Walk 7:	Walk 8: