Stable Pair Compactification of the Moduli Space of Two Special Families of Calabi-Yau 3-Folds and Chow Quotients of Grassmannians by Diagonal Subtori

by

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(Under the Direction of Valery Alexeev and Noah Giansiracusa)

Abstract

The moduli of two families of Calabi-Yau 3-folds are compactified via stable pairs. Some descriptions of the degenerations are given. The classical moduli of ordered points on a rational curve is generalized by using Chow quotients.

Key Words: moduli, compactification, Calabi-Yau, Chow quotients.

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B.S. Jilin University 2009

M.S. University of Idaho 2013

A Dissertation Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment
of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

AUGUST, 2019

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1. Introduction

1.1. **Background.** A moduli space is a geometric scheme/stack whose points classify isomorphic algebro-geometric objects. For many reasons, people have to deal with compactified moduli spaces, e.g. the intersection theory works much better on compact spaces.

In the study of compactifying moduli, the most classical tool is geometric invariant theory (GIT) due to D.Mumford. But in general it only gives a natural compactification with weak geometric meaning in higher dimensional cases. A nicer compactification is given by stable pairs in [KSB88] and [Ale94], which is an analogue of Deligne-Mumford's compactification $\overline{M}_g$ for moduli of genus g curves in higher dimension.

For the moduli of hyperplane arrangements, the compacitification by stable pairs has a nice description of the boundary structure. It uses a powerful tool from combinatorics: the theory of matroid polytope subdivisions of hypersimplices.

The first part of this thesis is on the compactification of a family of Calabi-Yau 3-folds. In algebraic geometry, Calabi-Yau varieties are hard to understand in general. Almost all known examples are constructed as complete intersections in some toric varieties/Grassmannians, covers or fibrations. For dimension 3, it is not known whether they have finitely many deformation types or bounded Betti numbers. Their moduli spaces are important in the study of mirror symmetry and in string theory. But the boundary structure of compactified moduli has not been studied very much. Even in dimension 2, there are not too many examples computed so far, see [MSY92], [Laz12], [AB19] and [AET19] for moduli of some K3 surfaces. And in dimension 3, a special one dimensional moduli was compactified in [HK14] by using the theory developed in [KU08]. In this thesis, we study a family of Calabi-Yau 3-folds constructed by using Pardini's abelian cover building data [Par89]. In particular, this family is a 9-dimensional complete family, i.e. smooth deformations of each variety are all abelian covers in this family. Similarly, another non-complete family of singular Calabi-Yau 3-folds can be obtained. Applying the tools of compactifying moduli of weighted hyperplane arrangements [Ale15], one can get a modular compactification of the moduli of these two families of Calabi-Yau 3-folds. We also give the description of the one parameter degenerations for generic cases.

The second part of this thesis gives a generalization of Kapranov's results to the quotients of some special subtori. This provides a compactification of the moduli of parametrized linear subspace arrangements. Chow quotients of Grassmannians by the full torus action were studied in [Kap93a]. In Kapranov's work, degenerations of cycles can be described via regular matroid polytope subdivisions of hypersimplices. Then for diagonal subtori quotients, the replacement of the matroid polytopes in discrete geometry is the discrete polymatroid polytopes, which have been studied a lot in many different backgrounds. We compute some examples to see the defining equations of the Grassmannian quotient by diagonal subtori. Also a intersection/projection map therorem is proved, which generalizes Kapranov's full torus descending map.

1.2. **Acknowledgement.** I would like to thank my advisor Valery Alexeev, who gave me the first project to work on and discussed with me patiently every week during the past several years. I learnt MMP, GIT, toric varieites and many other algebraic geometry tools from him in class and in person. I thank my advisor Noah Giansiracusa, who lead me to the road of the combination of combinatorics and algebraic geometry. Without his guidance, the second part of this thesis could not been done.

Both of them supported me on spirit when I was in a hard time at the end of my graduate student life.

I am also grateful to the discussion and help from many mathematicians, including Benjamin Bakker, Phillip Engel, Patricio Gallardo, Daniel Krashen, Colleen Robles and Robert Varley.

Part 1. **Stable Pair Compactification of the Moduli Space of Two Special Families of Calabi-Yau 3-Folds**

2. Construction of the Special Calabi-Yau 3-Folds

2.1. **Review of Abelian Covers.**

Abelian covers are generalizations of cyclic covers. The general theory of abelian covers was developed in [Par89], and for non-normal cases in [AP12].

Definition 2.1. Let G be an abelian group, and Y be a smooth variety. The *building data* for a Galois $G-$cover $\pi : X \to Y$ involves the following:
1) a line bundle L_χ for each $\chi \in G^* = \mathrm{Hom}(G, \mathbb{C}^*)$;
2) an irreducible effective divisor D_i for each cyclic subgroup $H_i \subset G$ together with ψ_i, a generator of the character group $H_i^* = \mathrm{Hom}(H_i, \mathbb{C}^*)$;
4) L_χ's and D_i's satisfying the *fundamental relation*:

(1)
$$L_\chi + L_{\chi'} \equiv L_{\chi\chi'} + \sum_i \epsilon^{H,\psi}_{\chi,\chi'} D_i,$$

where $\chi|_{H_i} = \psi_i^{a^i_\chi}$, $0 \leqslant a^i_\chi < |H_i|$, and

$$\epsilon^{H,\psi}_{\chi,\chi'} = \lfloor \frac{a^i_\chi + a^i_{\chi'}}{|H_i|} \rfloor,$$

which is either 0 or 1.

Remark 2.2. If Y is not smooth but normal, the line bundles L_χ's are replaced by rank 1 S_2 sheaves $\mathcal{F}_\chi$'s, equivalently, $\mathcal{F}^{**} = \mathcal{F}$ on a normal variety. In this case, the multiplication is

$$\mathcal{F}_\chi \times \mathcal{F}_{\chi'} \to \mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \to \mathcal{F}_\chi \otimes \mathcal{F}^{**}_{\chi'} \to \mathcal{F}_{\chi\chi'}.$$

And π can be uniquely extended from a smooth restriction $\pi_0 : X_0 \to Y_0$, see [AP12] section 1.4.

If Y is not normal but satisfies:

a. S_2 (normal=$R_1 + S_2$, i.e. regular in codim 1 and satisfying Serre's condition 2);

b. generically double crossing, i.e. outside of a closed subset of codim $\geqslant 2$, every point is either smooth or isomorphic to $xy = 0$.

In this case, the properties of the cover is given in [AP12] section 1.5. Recall that S_2 condition means that locally the depth of $R_P \geqslant \min\{2, \text{the height of } P\}$ at all prime ideals P. Equivalently, a variety is S_2 iff regular functions on it can be uniquely extended from a codimension$\geqslant 2$ subvariety. For example, an algebraic surface with a pinch point is not S_2.

In the case when $G = (\mathbb{Z}/2\mathbb{Z})^k$, the fundametal relations are reduced to the following

$$L_\chi + L_{\chi'} = L_{\chi\chi'} + \sum_{\chi(g)=\chi'(g)=-1} D_g,\ 2L_\chi = \sum_{\chi(g)=-1} D_g. \tag{2}$$

Example 2.3. 1. As the notation in definition 2.1, let $G = \mathbb{Z}/2\mathbb{Z}$, $Y = \mathbb{P}^2$, $D_1 = \sum_{i=1}^6 H_i$ where H_i's are lines in general position and $D_0 = \emptyset$. $K_X = \pi^*(K_{\mathbb{P}^2} + \frac{1}{2} \cdot 6H) = 0$ (this is $\mathbb{Q}$-equivalent). Then by (2.2), $2L_1 = D_1 = 6H$, one has $L_1 = 3H$. $H^2(\mathcal{O}_X) = H^2(\mathcal{O}_Y \oplus \mathcal{O}_Y(-3H)) = H^2(\mathcal{O}_{\mathbb{P}^2}) \oplus H^2(\mathcal{O}_{\mathbb{P}^2}(-3H)) = \mathbb{C}$. So the cover X is a K3 surface with $\binom{6}{2} = 15$ singularities. After blowing up the singularities in a chosen order, one gets a smooth K3 surface with Picard rank 16. And the dimension of the moduli of this family of K3 surfaces is 4.

2. Let $G = (\mathbb{Z}/2\mathbb{Z})^2$, $Y = \mathbb{P}^2$,

$$D_{(0,1)} = H_1 + H_2,$$

$$D_{(1,0)} = H_3 + H_4,$$

$$D_{(1,1)} = H_5 + H_6$$

are pairs of lines in general position, and $D_{(0.0)} = \emptyset$. $K_X = \pi^*(K_{\mathbb{P}^2} + \frac{1}{2} \cdot 6H) = 0$ is still trivial up to $\mathbb{Q}$-equivalence. By (2.2),

$$2L_{(0,1)} = D_{(0,1)} + D_{(1,1)} = 4H,$$

$$2L_{(1,0)} = D_{(1,0)} + D_{(1,1)} = 4H,$$

$$2L_{(1,1)} = D_{(0,1)} + D_{(1,0)} = 4H.$$

So $L_{(0,1)} = L_{(1,0)} = L_{(1,1)} = 2H$. Thus,

$$\begin{aligned} H^2(\mathcal{O}_X) &= H^2(\mathcal{O}_Y \oplus (\oplus_{j=1}^3 \mathcal{O}_Y(-2H))) \\ &= H^2(\mathcal{O}_{\mathbb{P}^2}) \oplus (\oplus_{j=1}^3 H^2(\mathcal{O}_{\mathbb{P}^2}(-2H))) \\ &= 0 \end{aligned}$$

In this case, the cover X is an Enrique surface. The moduli of this family of Enrique surfaces is complete and also dimension 4.

The building data we consider in this paper is this:

(1). $\pi : X \to Y = \mathbb{P}^3$, $G = (\mathbb{Z}/2\mathbb{Z})^4$ and $D_1, \cdots, D_8 \subset \mathbb{P}^3$ are eight hyperplanes in general position, and other D_g's are 0. Here the subindices of eight D_g's correspond to $\{(1, *, *, *)\} \subset G$.

(2). $\pi' : X' \to Y = \mathbb{P}^3$, $G' = \mathbb{Z}/2\mathbb{Z}$ and $D_1, \cdots, D_8 \subset \mathbb{P}^3$ are eight hyperplanes in general position. The sum of all the eight D_g's correspond to $1 \in (\mathbb{Z}/2\mathbb{Z})$ and other D_g's are 0.

The first case here is the construction via *almost uniform covers* by Pardini, see [AP09]. Fix $\chi^- = (1, 0, 0, 0) \in G^* = \mathrm{Hom}(G, \mathbb{C}^*) \cong (\mathbb{Z}/2\mathbb{Z})^4$, the eight nonzero D_g's are labelled by the set $\Sigma = \{g \in G, \chi^-(g) = -1\}$. One can find that the L_χ's in the building data are $L_1 = 0$, $L_2 = \cdots = L_{15} = 2H$, $L_{16} = 4H$. Indeed, when $\chi = \chi^-$, according to the fundamental relation, we have

$$2L_{\chi^-} = \sum_{g\in\Sigma, \chi^-(g)\neq 0} D_g.$$

So g runs over Σ, thus $L_{\chi^-} = 4H$. And when $\chi \neq 1, \chi^-$,

$$2L_\chi = \sum_{g\in\Sigma, \chi(g)\neq 1} D_g,$$

we get $2L_\chi = 4H$. E.g. for $\chi = (0,1,0,0)$, we have $g = (1,1,*,*)$, thus we get four nonzero D_g's.

By above we have

$$\pi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^3} \oplus_{i=1}^{14} \mathcal{O}_{\mathbb{P}^3}(-2H) \oplus \mathcal{O}_{\mathbb{P}^3}(-4H).$$

Definition 2.4. A variety X of dimension n is called *Calabi-Yau* if K_X is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \cdots, n-1$.

Since $\pi_*\Omega_X^p = \oplus_{\chi\in G^*}\Omega_Y^p(\log(D))\otimes L_\chi^{-1}$ (see [Par89] proposition 4.1 b), cohomologies can be computed by

$$H^q(X, \mathcal{O}_X) = \oplus_{\chi\in G^*} H^q(\mathbb{P}^3, L_\chi^{-1}), \tag{3}$$

and

$$H^q(X, \Omega_X^p) = \oplus_{\chi\in G^*} H^q(\mathbb{P}^3, \Omega_{\mathbb{P}^3}^p(\log(D)) \otimes L_\chi^{-1}). \tag{4}$$

Lemma 2.5. *X is a Calabi-Yau 3-fold.*

Proof. $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ because

$$h^i(\mathcal{O}_{\mathbb{P}^3}) = h^i(\mathcal{O}_{\mathbb{P}^3}(-2H)) = h^i(\mathcal{O}_{\mathbb{P}^3}(-4H)) = 0$$

for $i = 1, 2$, and

$$\begin{aligned} H^0(\mathcal{O}_X) &= H^0(\mathcal{O}_{\mathbb{P}^3}) \oplus (\oplus_{i=1}^{14} H^0(\mathcal{O}_{\mathbb{P}^3}(-2))) \oplus H^0(\mathcal{O}_{\mathbb{P}^3}(-4)) \\ &= \mathbb{C}. \end{aligned}$$

$$\begin{aligned} H^3(\mathcal{O}_X) &= H^3(\mathcal{O}_{\mathbb{P}^3}) \oplus (\oplus_{i=1}^{14} H^3(\mathcal{O}_{\mathbb{P}^3}(-2))) \oplus H^3(\mathcal{O}_{\mathbb{P}^3}(-4)) \\ &= H^0(\mathcal{O}_{\mathbb{P}^3}(-4))^* \oplus (\oplus_{i=1}^{14} H^0(\mathcal{O}_{\mathbb{P}^3}(-2))^*) \oplus H^0(\mathcal{O}_{\mathbb{P}^3})^* \\ &= \mathbb{C}. \end{aligned}$$

□

Moreover, by the set up of almost uniform covers in [AP09], one can show that they are all smooth. Here we give a proof of our special case.

Lemma 2.6. *X is smooth.*

Proof. Assume that D_1, D_2, D_3 are three hyperplanes in general position, and the corresponding characters are g_1, g_2, g_3. Let g_1, g_2, g_3 be free, i.e. there are no linear relations between g_1, g_2, g_3 other than $2g_i = 0, i = 1, 2, 3$. Locally on the affine chart $\mathbb{A}^3_{x_1,x_2,x_3}$, the G-cover breaks into four $\mathbb{Z}/2\mathbb{Z}$ covers. Let π_1, π_2, π_3 be the three ramified $\mathbb{Z}/2\mathbb{Z}$ covers and π_4 be the nonramified one. The polynomials defining the ramified cover are

$$\begin{cases} z_1^2 = x_1 \\ z_2^2 = x_2 \\ z_3^2 = x_3 \end{cases} \tag{5}$$

in $\mathbb{A}^3_{x_1,x_2,x_3} \times \mathbb{A}^3_{z_1,z_2,z_3}$. Thus X is smooth. If g_1, g_2, g_3 are not free, e.g. $g_1 \in \langle g_2, g_3 \rangle$, this situation is ruled out due to the construction of almost uniform cover. □

2.2. **Computation of Hodge Numbers.** By using Bott's formula for projective spaces, one can compute other Hodge numbers of X.

Lemma 2.7. [OSSG] *(Bott's formula for $\mathbb{P}^n$) For the complex projective space $\mathbb{P}^n$, we have:*

Case 1: If $m = 0$, then

$$h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = \begin{cases} 1 & , \ p = q, \\ 0 & , \ \text{otherwise.} \end{cases}$$

Case 2: If $m \neq 0$, then

$$h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(m)) = \begin{cases} \binom{n+m-p}{m}\binom{m-1}{p}, \ q = 0, m > p, \\ \binom{-m+p}{-m}\binom{-m-1}{n-p}, \ q = n, m < p - n, \\ 0, \ \text{otherwise.} \end{cases}$$

Lemma 2.5 was also proved in [Hua01] by using bases for cohomology modules.

Theorem 2.8. *The Hodge diamond of X is*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 9 & & 9 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} .$$

Proof. Apply the lemma to the short exact sequence ([EV92]2.3):

$$0 \to \Omega^1_{\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^3}(\log(D)) \to \mathcal{O}_D \to 0$$

for $D=\sum_{i=1}^8 D_i$, $k=1,2$ and the twists by $\mathcal{O}(-2H),\mathcal{O}(-4H)$. Then one gets the following long exact sequence:

$$0=H^2(\Omega^1_{\mathbb{P}^3}(-bH))\to H^2(\Omega^1_{\mathbb{P}^3}(\log(D)(-bH)))\ \to\oplus_{i=1}^8 H^2(\mathcal{O}_{D_i}(-bH))\to$$

$$\to H^3(\Omega^1_{\mathbb{P}^3}(-bH))\to H^3(\Omega^1_{\mathbb{P}^3}(\log(D)(-bH)))\to\oplus_{i=1}^8 H^3(\mathcal{O}_{D_i}(-bH))=0.$$

When $b=2$, one has $H^1(\mathbb{P}^3,\Omega^1_{\mathbb{P}^3}(\log(D)(-2)))=0$.

When $b=4$, $H^2(\Omega_{\mathbb{P}^3}(-4))=H^3(\Omega^1_{\mathbb{P}^3}(\log(D)(-4)))=0$, $h^2(\mathcal{O}_{D_i}(-4))=3$, $h^3(\Omega_{\mathbb{P}_3}(-4))=15$. So $h^{2,1}(X)=h^2(\Omega^1_{\mathbb{P}^3}(\log(D)(-4)))=8\times 3-15=9$. And $H^1(X,\Omega^1_X)=H^1(\mathbb{P}^3,\Omega^1_{\mathbb{P}^3})$.

So $h^{2,1}(X)=9$, and $h^{1,1}(X)=1$. □

Remark 2.9. By the Bogomolov-Tian-Todorov's unobstructedness theorem (theorem 14.10 in [GHJ12]), since $h^{2,1}(X)=9$, the deformation space of X has dimension 9. Also, the deformation space of the hyperplane arrangement $(\mathbb{P}^3,D)$ has dimension $(8-5)\times 3=9$. But the deformation space of the abelian covers and of the hyperplane arrangements are not always the same. For example, the Kummer cover of $\mathbb{P}^3$ branched along eight hyperplanes, whose abelian group is $G=(\mathbb{Z}/2\mathbb{Z})^7$, has a 121 dimensional deformation space. In general, to see whether the deformation space of D in Y and the deformation space of X have the same dimension, one can use [FP97] for general cases, or [AP09] theorem 4.3 for hyperplane arrangements.

2.3. **A Family of Singular Calabi-Yau 3-Folds.** For the double cover $\pi':X'\to\mathbb{P}^3$ branched along eight hyperplanes in a general position, X' is singular but still a Calabi-Yau 3-fold. One may consider the resolution of $f:\tilde{X}'\to X'$ which is a smooth Calabi-Yau 3-fold. Let $\tilde{\mathbb{P}}^3\to\mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ along the $\binom{8}{2}=28$ intersection lines in a chosen order, then $\tilde{X}'=X'\times_{\mathbb{P}^3}\tilde{\mathbb{P}}^3$. But in this case, the pair $(\tilde{X}',f^*\pi^*D)$ is not a stable pair any more, because $K_{\tilde{X}'}+f^*\pi^*D$ is not ample. To apply the tool of stable pair compactification, we insist on the family of the singular Calabi-Yau 3-folds.

Proposition 2.10. [GSvSZ13] *The middle Hodge numbers of X' are the same as the one of X, which are also* $(1, 9, 9, 1)$.

Remark 2.11. The period map ρ of this family of Calabi-Yau 3-folds was studied in [GSvSZ13]. They proved that ρ does not factor through any Hermitian subdomain, so there is no hope to use classical tools to construct the Baily-Borel compactification in this case. Also, all possible monodromy groups were listed in their paper. Some ongoing work of this generalized Baily-Borel compactification can be found in [GGLR17]. The algebraicity of the period map's image is also proved in [BBT18]. We do not know whether global Torelli holds for the first family. The only known global Torelli for Calabi-Yau 3-folds is for the cyclic triple cover of $\mathbb{P}^3$ branched along six hyperplanes [SX19].

3. Compactification via Stable Pairs

The modular compactification of moduli by stable pairs was constructed in [KSB88] and [Ale96]. It is also called the KSBA or MMP compactification. This compactified moduli is an analogue of Deligne-Mumford's compactification for moduli of pointed curves $\overline{M}_{g,n}$ in higher dimension cases.

We briefly review the moduli of stable varieties/pairs. Let (X, B) be a pair, i.e. X is a normal projective variety and $B = \sum b_i B_i$ is a $\mathbb{Q}$-divisor. Let $f : Y \to X$ be a *log resolution*, which means that $f_*^{-1}B \cup \{\text{exceptional locus}\}$ is a normal crossing divisor. Then we have the formula:

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + B) + \sum a_D D. \tag{6}$$

Definition 3.1. Let X be a normal variety and B be a divisor on X with $0 < b_i \leqslant 1$. The pair (X, B) is *log canonical* if it satisfies:

1. (X, B) is $\mathbb{Q}$-Cartier, i.e. $m(K_X + B)$ is Cartier for some $m \in \mathbb{Z}_+$;
2. all $a_D \geqslant -1$ in (3.1).

Example 3.2. The pair $(\mathbb{A}^2_{x,y}, \epsilon C = V(x^a - y^b))$ is log canonical iff $\mathbb{Q} \ni \epsilon \leqslant \frac{1}{a} + \frac{1}{b}$.

For non-normal case, one has to deal with the double locus. The analogue of nodal curves on singularities in higher dimensional non-normal case is the following.

Definition 3.3. A pair (X, B) is called *semi-log canonical* if

1. X is a S_2 (Serre's condition) variety;
2. $K_X + B$ is $\mathbb{Q}$-Cartier;
3. in codimension 1, X has only double normal crossings, and the double locus does not contain any irreducible components of B;

4. the pair $(\tilde{X}, \nu^{-1}(B + \text{double locus}))$ is log canonical, where $\nu : \tilde{X} \to X$ is the normalization.

Definition 3.4. A pair (X, B) is called *stable* if

1. $K_X + B$ is ample;
2. the pair is semi-log canonical.

Definition 3.5. The moduli functor M_{CY} for polarized Calabi-Yau n-folds is defined as follows: for the base S reduced and normal, $M_{CY}(S)$ is the set of flat families (X, L) with fixed Hilbert polynomial $\chi(X_s, L_s)$ for all fibers such that:

1. on every geometric fiber X_s, the relative dualizing sheaf $\omega_{X/S}$ is locally trivial over S, and $H^i(X_s, \mathcal{O}_{X_s}) = 0$, $1 \leqslant i \leqslant n-1$, where $\dim(X_s) = n$;
2. there is a relative ample line bundle L on X.

Minimal model program is involved in one parameter degenerations. Let $\pi^0 : X^0 \to S^0$ be a family over a punctured curve $S^0 = S \backslash \{0\}$. After a finite base change, the family has a unique extension at 0 to a family of stable pairs.

There are two definitions of moduli functor for stable varieties, see [Ale15] 1.4.2, 1.4.3 or [Kol17], they are equivalent in char 0 over reduced base. But the moduli functor for stable pairs is still not clear in general at this moment, see [Kol17].

Definition 3.6. [Kol17] A family of pairs of dimension n over a reduced normal scheme S is an object

$$f : (X, D) \to S$$

consisting of a morphism of schemes $f : X \to S$ and an effective Weil divisor$D = \sum d_i D_i$ satisfying:

a) $f : X \to S$ is flat, of pure relative dimension n and with geometrically reduced fibers;

b) the nonempty fibers of $\mathrm{Supp}(D) \to S$ have pure dimension $n-1$;

c) f is smooth at generic points of $X_s \cup \mathrm{Supp}(D)$ for $s \in S$.

Remark 3.7. One has to be careful when pulling back a Weil divisor, for details see 4.1 in [Kol17].

For a family of pairs $f : (X, D) \to S$, fix the relative dimension n, the common denominator m for all coefficients in D, and the volume $v = (K_s + D_s)^n$ of the fibers, then there is a subfunctor $SP^{sn}(n, m, v)$:

$$SP^{sn}(n, m, v) : \{seminormal\ S - schemes\} \to \{sets\}.$$

Recall that a subfunctor of a functor $G : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair (F, i) where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $i : F \to G$ is a natural transformation such that its components $i_M : F(M) \to G(M)$ are monic.

Theorem 3.8. *(Theorem 4.9 in [Kol17]) Let S be a reduced seminormal base scheme of char 0, and fix n, m, v. Then the coarse moduli space exists for $SP^{sn}(n, m, v)$, which is a seminormal scheme, whose irreducible components are proper over S.*

A special case of stable pairs: stable weighted hyperplane arrangement is studied in the next chapter.

Theorem 3.9. *The moduli space of the special families M_{CY}, M'_{CY} are quotients of moduli of weighted hyperplane arrangement $M_{\boldsymbol{b}}(4, 8)$ by $Stab(\chi^-) \subseteq GL(4, \mathbb{F}_2)$ and S_8, where $\boldsymbol{b} = (\frac{1}{2}, \cdots, \frac{1}{2})$.*

Proof. The automorphism of eight hyperplanes for the second family has no stablizer, so $M'_{CY} = [M_{\mathbf{b}}(4, 8) : S_8]$. For the first family, $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z})^4 = Gl(2, \mathbb{F}_2)$, but D_g are labelled by $\{g \in G = (\mathbb{Z}/2\mathbb{Z})^4, \chi^-(g) = -1\}$ so the automorphism of D_g's is $\mathrm{Stab}(\chi^-)$, $M_{CY} = [M_{\mathbf{b}}(4, 8) : \mathrm{Stab}(\chi^-)]$. □

For the abelian cover $\pi : X \to Y = \mathbb{P}^3$ (or $X' \to Y$), choose $B = \frac{1+\epsilon}{2}(D_1 + \cdots + D_8)$, where D_g's are eight hyperplanes in a general position. By $K_X + R = \pi^*(K_Y + B)$, where $R \in |L|$ for the ample line bundle $L = \pi^*(\mathcal{O}(H))$ on $\mathbb{P}^3$ is the ramified divisor, the moduli problem of polarized Calabi-Yau varieties (X, L) is the same with the moduli problem of the stable pair (Y, B). Note that here R can be chosen canonically. Since

$$\begin{aligned}
\pi_*(L) &= \mathcal{O}_{\mathbb{P}^3}(1) \otimes (\oplus_{\chi \in G^*} L_\chi^{-1}) \\
&= \mathcal{O}_{\mathbb{P}^3}(1) \otimes (\mathcal{O}_{\mathbb{P}^3} \oplus (\oplus_{i=1}^{14} \mathcal{O}_{\mathbb{P}^3}(-2H)) \oplus \mathcal{O}_{\mathbb{P}^3}(-4H)) \\
&= \mathcal{O}_{\mathbb{P}^3}(H) \oplus (\oplus_{i=1}^{14} \mathcal{O}_{\mathbb{P}^3}(-H)) \oplus \mathcal{O}_{\mathbb{P}^3}(-3H)
\end{aligned}$$

so $H^0(L) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.

So we can apply the tools from the next chapter to study M_{CY}, M'_{CY}.

4. Moduli of Weighted Hyperplane Arrangements

In the study of degenerations of hyperplane arrangements, one has to deal with matroid polytopes. A *matroid* on a finite set $[n] = \{1, \cdots, n\}$ can be defined in the following equivalent ways [Oxl06]:

Definition 4.1. A *matroid rank function* on $[n]$ is a map

$$\text{rank} : 2^{[n]} \to \mathbb{Z}_{\geqslant 0},$$

such that

1. $\text{rank}(I) \leqslant |I|$ for any subset $I \subset [n]$;
2. $\text{rank}(I) \leqslant \text{rank}(J)$ for any $I \subset J$;
3. (submodularity) $\text{rank}(I \cap J) + \text{rank}(I \cup J) \leqslant \text{rank}(I) + \text{rank}(J)$.

A matroid on $[n]$ is the set $[n]$ together with a matroid rank function.

Definition 4.2. A matroid on $[n]$ is a pair $([n], \mathcal{B})$, where $\mathcal{B} \subset 2^{[n]}$ is a set of *bases* satisfing:

1. (the base change property) for any $I, J \in \mathcal{B}$ and $i \in I \backslash J$, there exists $j \in J \backslash I$ such that $(I \backslash \{i\}) \cup \{j\} \in \mathcal{B}$;
2. no $I \subseteq J$ for different $I, J \in \mathcal{B}$.

Definition 4.3. A lattice polytope P is a *matroid (base) polytope* if it is the convex hull of e_I's, where $e_I = \sum_{i \in I} e_i$ is the sum the i-th unit vectors and I runs over the set of bases $\mathcal{B}$ for some matroid $([n], \mathcal{B})$.

A characterization of matroid polytopes was given in [GGMS87].

Proposition 4.4. *A polytope is a matroid polytope if and only if it is a lattice polytope in $\Delta(k,n)$ and its edges are parallel to $e_i - e_j$ for some i, j, where the* hypersimplex *$\Delta(k,n)$ is*

$$\Delta(k,n) = \{\boldsymbol{x} \in \mathbb{R}^n, 0 \leqslant \boldsymbol{x}_i \leqslant 1, \sum_{i=1}^{n} \boldsymbol{x}_i = k\}.$$

Another interpretation of matroid polytope is:

Proposition 4.5. [Ale15] *The matroid polytope P_V is the set of points $(x_i) \in \mathbb{R}^n$ such that the pair $(\mathbb{P}V, \sum x_i B_i)$ is log canonical and $K_{\mathbb{P}V} + \sum x_i B_i = 0$; the interior is the set of points such that $(\mathbb{P}V, \sum x_i B_i)$ is klt and $K_{\mathbb{P}V} + \sum x_i B_i = 0$.*

Definition 4.6. A *subdivision* or *tiling* of a polytope is a face matching polytope subdivision.

Definition 4.7. A pair (X, B) is called *log Calabi-Yau* if $K_X + B \equiv 0$. It is called *log general type* if $K_X + B$ is big.

Definition 4.8. [Ale15] For $\mathbf{b} = (b_1, \cdots, b_n), 0 \leqslant b_i \leqslant 1$, a $\mathbf{b}$-cut of the hypersimplex $\Delta(k,n)$ is

$$\Delta_{\mathbf{b}}(k,n) = \{\mathbf{x} \in \mathbb{R}^n, 0 \leqslant x_i \leqslant b_i, \sum_{i=1}^{n} x_i = k\}.$$

Theorem 4.9. *(Theorem 4.4.2 in [Ale15]) A log general type or log Calabi-Yau hyperplane arrangement $(\mathbb{P}^{k-1}, \sum b_i B_i)$ is log canonical if and only if $\Delta_{\boldsymbol{b}} \subset BP_M$.*

We discussed the existence of the moduli of stable pairs in the last chapter. The existence of the moduli of the stable hyperplane arrangements was also proved in [Ale15] 5.4.2:

Theorem 4.10. *For every r, n and $\mathbf{b} = (\mathbf{b_1}, \cdots, \mathbf{b_n})$, there exist a projective scheme $\overline{M}_{\mathbf{b}}(r,n)$ and a flat projective family $(\mathcal{X}, \mathcal{B}_1, \cdots, \mathcal{B}_n) \to \overline{M}_{\mathbf{b}}(r,n)$ such that every fiber is one of the stable weighted hyperplane arrangements (as stable pairs), and there are no repeating fibers.*

For the stable pair $(\mathbb{P}^{k-1}, D = \frac{1+\epsilon}{2}\sum_{i=1}^{2k} D_i)$, all matroid polytopes appear in the subdivisions of $\Delta(k, 2k)$ contain the center $(\frac{1}{2}, \cdots, \frac{1}{2})$.

Definition 4.11. A *central polytope* is a lattice subpolytope in $\Delta(k, 2k)$ (or $k\Delta(1, k)$) with $(\frac{1}{2}, \cdots, \frac{1}{2})$ (respectively $(1, \cdots, 1)$) on its boundary. A subdivision is *central* if all subpolytopes appearing in the subdivision are central.

Lemma 4.12. *For any point x of a matroid polytope P, let $S = \{I | x_I = rank(I)\}$. If $I, J \in S$, then $I \cap J, I \cup J \in S$.*

Proof. We have

$$\begin{aligned}\operatorname{rank}(I) + \operatorname{rank}(J) &= x_I + x_J \\ &= x_{I\cap J} + x_{I\cup J} \\ &\leqslant \operatorname{rank}(I \cap J) + \operatorname{rank}(I \cup J) \\ &\leqslant \operatorname{rank}(I) + \operatorname{rank}(J).\end{aligned}$$

So $x_{I\cap J} = \operatorname{rank}(I \cap J), x_{I\cup J} = \operatorname{rank}(I \cup J)$. □

Corollary 4.13. For a central subdivisions, the subindices of the hyperplanes $\{x_I \leqslant \operatorname{rank}(I), I \in \mathcal{I}\}$ cutting matroid polytopes in $\Delta(k, 2k)$ must have $|I \cap J|, |I \cup J|$ even, $\forall I, J \in \mathcal{I}$.

Proof. Otherwise, there are two hyperplanes $x_I = rank(I) = |I|/2, x_J = \operatorname{rank}(J) = |J|/2$ passing through the center $c = (\frac{1}{2}, \cdots, \frac{1}{2})$ with $|I \cap J|$ odd. By the lemma we have $c_{I\cap J} = \operatorname{rank}(I \cap J)$, but the left side is not an integer. □

Corollary 4.14. For any central matroid polytope Q in $\Delta(k, 2k)$, the possible hyperplanes cutting Q appear in pairs, i.e. can be labelled as $\{11'22'33'44' \cdots kk'\}$.

Proof. If the subindices are already in pairs, then of course the condition in corollary 4.13 is satisfied. Otherwise, there are some subindices $I \in \mathcal{I}_0$ such that $| \cap_{I\in\mathcal{I}_0} I|$ is odd. Without lost of generality, one can assume that $|\cap_{I\in\mathcal{I}_1} I|$'s are even for any $\mathcal{I}_1 \subsetneq \mathcal{I}_0$. Let $K = \cap_{I\in\mathcal{I}_0} I$, keep applying lemma 4.12, one gets c on $x_K = |K|$. But this is impossible since $|K|$ is odd. $\square$

Remark 4.15. In this case of $k = 4$, on [8] the only subsets satisfy corollary 4.13 are $I_1 = \{1234\}, I_2 = \{3456\}, I_3 = \{1357\}$. For the subpolytope defined by $\{x_{1234} \leqslant 2, x_{3456} \leqslant 2, x_{1357} \leqslant 2\} \subseteq \Delta(4,8)$, by

$$\mathrm{rank}(\mathrm{Span}(v_1, v_2, v_3, v_4)) \leqslant 2,$$

$$\mathrm{rank}(\mathrm{Span}(v_3, v_4, v_5, v_6)) \leqslant 2,$$

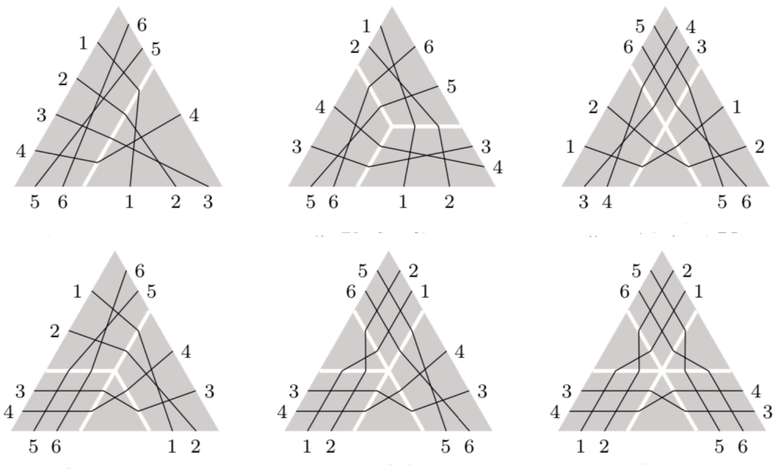
$$\mathrm{rank}(\mathrm{Span}(v_1, v_3, v_5, v_7)) \leqslant 2,$$

one gets $\mathrm{rank}(\mathrm{Span}(v_1, v_2, v_3, v_4, v_5, v_6, v_7)) \leqslant 2$ (or $\mathrm{rank}(\mathrm{Span}(v_3)$, but this inequality does not appear. So this polytope is not matroidal. (Or directly by using the proof of corollary 4.14, one gets that $c = (\frac{1}{2}, \cdots, \frac{1}{2})$ on the hyperplane $\{x_3 = 1\}$, which is impossible).

By identifying each pair in the subindices $\{x_I \leqslant \mathrm{rank}(I) = |I|/2\}$, central subdivisions of $\Delta(k, 2k)$ can be reduced to central subdivisions of a lower dimensional polytope $k\Delta(1,k)$. Then the inequalities cutting each subpolytope in $k\Delta(1,k)$ have the form $\{x_I \leqslant |I|\}$.

4.1. **An Example in Dimension 2.** ([Ale15] 6.2.1) In this subsection we consider weighted hyperplane arrangement $(\mathbb{P}^2, B = \sum_{i=1}^{6} \frac{1+\epsilon}{2} B_i)$ where each B_i is a line and they are in a general position.

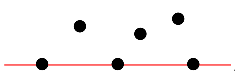
For the one parameter degenerations, up to symmetry, there are six different types.

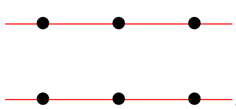
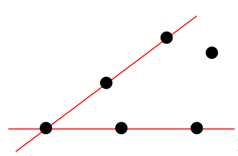


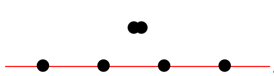
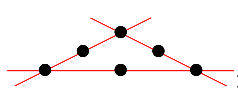
Remark 4.16. Let X_{sing} be the double cover of $\mathbb{P}^2$ branched along these six lines. And X is the blow up of X_{sing} along a chosen order of singularities. Then X is a smooth K3 surface. The minimal Baily-Borel compactification of this family of K3 surfaces was computed in [MSY92] 0.9. There are five strata in the Baily-Borel compactification:

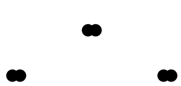
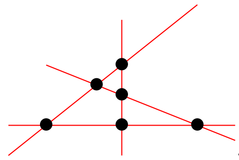
$$\overline{X}^{BB} = X_4 \cup X_3 \cup X_{2a} \cup X_{2b} \cup X_{1a} \cup X_{1b} \cup X_{0a} \cup X_{0b},$$

where X_{i*} is the i-strata of type $*$. Here are the dual graphs of degenerations of these different types.

X_3:  ;

X_{2a}:  ; X_{2b}:  ;

X_{1a}:  ; X_{1b}:  ;

X_{0a}:  ; X_{0b}:  .

4.2. **Subdivisions of $\Delta(4,8)$.**

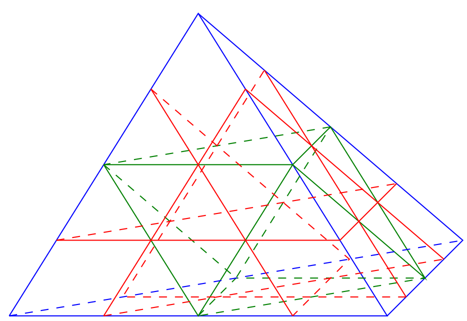
Proposition 4.17.

$$\left\{\begin{array}{c}\textit{central matroid polytope}\\ \textit{decomposition of } \Delta(k,2k)\end{array}\right\}/iso \hookrightarrow \left\{\begin{array}{c}\textit{central polytope}\\ \textit{decomposition of } k\Delta(1,k)\end{array}\right\}/iso.$$

In particular, the matroid subpolytopes of $\Delta(4,8)$ are given by some subpolytopes $4\Delta(1,4)$ via the map above.

Proof. By corollary 4.14, after pairing the subindices as $\{1,1',\cdots,k,k'\}$, one may replace the inequality $x_{i_1i_1'\cdots i_li_l'} \leqslant l$ by $x_{i_1\cdots i_l} \leqslant l$. Then use the latter inequality to cut $k\Delta(1,k)$. □

The map in the proposition above is only surjective when $k \leqslant 3$. When $k = 4$, some subpolytopes in $4\Delta(1,4)$ do not give matroid polytopes in $\Delta(4,8)$.



Proposition 4.18. *In $4\Delta(1,4)$, matroid subpolytopes of $\Delta(4,8)$ are given by:*

type I: $4\Delta(1,4)$;

type II:

a. $\{x_1 \leqslant 1\}$ *(16B, for the simplified dual graph, see the Appendix),*

b. $\{x_{12} \leqslant 2\}$ *(16C),*

c. $\{x_{123} \leqslant 3\}$ *(16A);*

type III:

a. $\{x_{12} \leqslant 2, x_{134} \leqslant 3\}$ *(5D),*

b. $\{x_1 \leqslant 1, x_{23} \leqslant 2\}$ *(5C),* *c.* $\{x_{123} \leqslant 3, x_{124} \leqslant 3\}$ *(6B),*

d. $\{x_1 \leqslant 1, x_2 \leqslant 1\}$ *(6A),*

e. $\{x_1 \leqslant 1, x_{123} \leqslant 3\}$ *(10A),*

f. $\{x_{12} \leqslant 2, x_{123} \leqslant 3\}$ *(11B),*

g. $\{x_1 \leqslant 1, x_{12} \leqslant 2\}$ *(11A);*

type IV:

a. $\{x_{123} \leqslant 3, x_{124} \leqslant 3, x_{134} \leqslant 3\}$ *(1B),*

b. $\{x_1 \leqslant 1, x_2 \leqslant 1, x_3 \leqslant 1\}$ *(1A),*

c. $\{x_1 \leqslant 1, x_{23} \leqslant 2, x_{124} \leqslant 3\}$ *(2A),*

d. $\{x_{12} \leqslant 2, x_{123} \leqslant 3, x_{134} \leqslant 3\}$ *(3B),*

e. $\{x_1 \leqslant 1, x_2 \leqslant 1, x_{13} \leqslant 2\}$ *(3A),*

f. $\{x_1 \leqslant 1, x_{123} \leqslant 3, x_{124} \leqslant 3\}$ *(5A),*

g. $\{x_1 \leqslant 1, x_2 \leqslant 1, x_{123} \leqslant 3\}$ *(5B),*

h. $\{x_1 \leqslant 1, x_{12} \leqslant 2, x_{123} \leqslant 3\}$ *(8A),*

i. $\{x_1 \leqslant 1, x_2 \leqslant 1, x_{123} \leqslant 3, x_{124} \leqslant 3\}$ *(4A),*

j. $\{x_1 \leqslant 1, x_{12} \leqslant 2, x_{13} \leqslant 2, x_{123} \leqslant 3\}$ *(6C).*

Here we list the pictures of all polytopes, where the red point is the center:

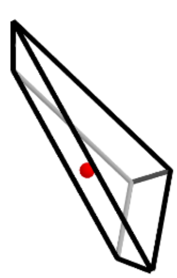


FIGURE 1. IIa

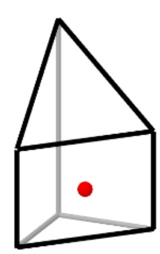


FIGURE 2. IIb

Proof. There are 33 central subpolytopes of $4\Delta(1,4)$ in total. But all other 13 central subpolytopes do not have enough defining inequalities, which are listed below. Recall that the minimal set of defining inequalities for a matroid polytope is $x_I \leqslant \text{rank}(I)$

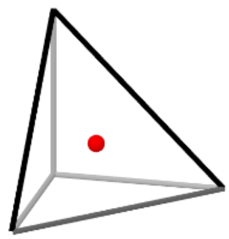


Figure 3. IIc

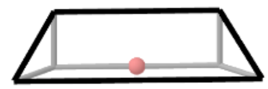


Figure 4. IIIa

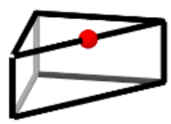


Figure 5. IIIb

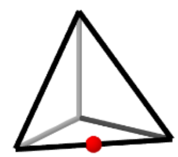


Figure 6. IIIc

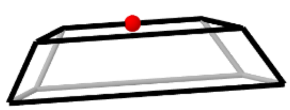


Figure 7. IIId

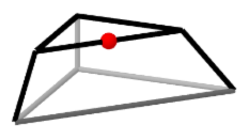


Figure 8. IIIe

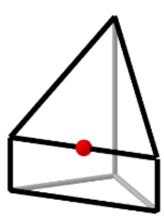


Figure 9. IIIf

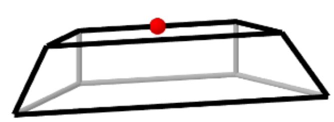


Figure 10. IIIg

for all nontrivial nondegenerate flats, see [Ale15] 4.2.2. (In general, a *flat* is a subset $I \subseteq [n]$ such that for any $j \notin I$, one has $\text{rank}(I) < \text{rank}(I \cup \{j\})$).

Figure 11. IVa

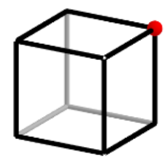


Figure 12. IVb

Figure 13. IVc

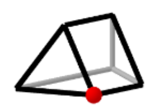


Figure 14. IVd

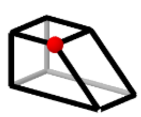


Figure 15. IVe

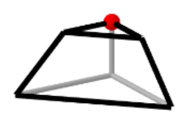


Figure 16. IVf

Figure 17. IVg

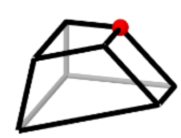


Figure 18. IVh

$\{x_{12} \leqslant 2, x_{13} \leqslant 2\}$. This means that

$$\mathrm{rank}(\mathrm{Span}\{v_1, v_1', v_2, v_2'\}) \leqslant 2$$

$$\mathrm{rank}(\mathrm{Span}\{v_1, v_1', v_3, v_3'\}) \leqslant 2$$

FIGURE 19. IVi

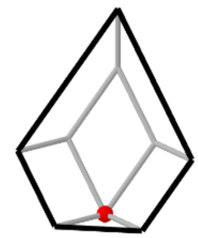


FIGURE 20. IVj

Then all these four vectors $\{v_1, v_1', v_2, v_2'\}$ span a dimension 2 space, but rank(Span$\{v_1, v_1', v_2, v_2', v_3, v_3'$
$2(x_{123} \leqslant 2)$ does not appear in the defining equations of this subpolytope. So this polytope is not matroidal.

For the other 11 central subpolytopes in $4\Delta(1,4)$, by the same reason, one has:

$\{x_1 \leqslant 1, x_{23} \leqslant 2, x_{24} \leqslant 2\} \Rightarrow \{x_{234} \leqslant 2\}$,

$\{x_{12} \leqslant 2, x_{13} \leqslant 2, x_{234} \leqslant 3\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_{12} \leqslant 2, x_{13} \leqslant 2, x_{14} \leqslant 2\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_{12} \leqslant 2, x_{13} \leqslant 2, x_{23} \leqslant 2\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_{12} \leqslant 2, x_{13} \leqslant 2, x_{124} \leqslant 3\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_1 \leqslant 1, x_{12} \leqslant 2, x_{23} \leqslant 2\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_1 \leqslant 1, x_{12} \leqslant 2, x_{13} \leqslant 2\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_{12} \leqslant 2, x_{13} \leqslant 2, x_{124} \leqslant 3, x_{134} \leqslant 3\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_1 \leqslant 1, x_2 \leqslant 1, x_{13} \leqslant 2, x_{23} \leqslant 2\} \Rightarrow \{x_{123} \leqslant 2\}$,

$\{x_{12} \leqslant 2, x_{13} \leqslant 2, x_{14} \leqslant 2, x_{123} \leqslant 3\} \Rightarrow \{x_{124} \leqslant 2\}$,

$\{x_1 \leqslant 1, x_{12} \leqslant 2, x_{13} \leqslant 2, x_{23} \leqslant 2\} \Rightarrow \{x_{123} \leqslant 2\}$.

For each central subpolytope in $4\Delta(1,4)$above, we only list one inequality which is implied by the given ones but not appear in the defining inequalities. And there is one subpolytope defined by $\{x_{12} \leqslant 2, x_{13} \leqslant 2, x_{123} \leqslant 3\}$, which means

$$\operatorname{rank}(\operatorname{Span}\{v_1, v_1', v_2, v_2'\}) \leqslant 2,$$

$$\operatorname{rank}(\operatorname{Span}\{v_1, v_1', v_3, v_3'\}) \leqslant 2,$$

$$\operatorname{rank}(\operatorname{Span}\{v_1, v_1', v_2, v_2', v_3, v_3'\}) \leqslant 3.$$

An implied fact is $\operatorname{rank}(\operatorname{Span}\{v_1, v_1'\}) \leqslant 1$, which does not appear in the defining inequalities for this polytope. So this subpolytope is also not matroidal.

□

Here type II,III polytopes have one and two interior facets. And type IV polytopes have three or four interior facets.

Lemma 4.19. *The corresponding toric varieties associated to the polytopes above are the following:*

1. IVb: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*;*

2. IIc, IIIc, IVa: $\mathbb{P}^3$*;*

3. IIIe, IVc: $\mathbb{P}^2 \times \mathbb{P}^1$*;*

4. IVe: $(\operatorname{Bl}_p \mathbb{P}^2) \times \mathbb{P}^1$*, where* $\operatorname{Bl}_p \mathbb{P}^2$ *is the blow up* $\mathbb{P}^2$ *at a point* $p \in \mathbb{P}^2$*;*

5. IIa, IIIe, IVf: $\operatorname{Bl}_p \mathbb{P}^3$*;*

6. IIb, IIIf, IIIa, IIId, IVd, IVg, IVi: $\operatorname{Bl}_l \mathbb{P}^3$*, which is the blow up of* $\mathbb{P}^3$ *along a line* l*;*

7. IIIg, IVh: $\operatorname{Bl}_l \operatorname{Bl}_p \mathbb{P}^3$*;*

8. IVj: $\operatorname{Cont}_{l_3} \operatorname{Bl}_{l_1, l_2} \operatorname{Bl}_p \mathbb{P}^3$*, first blow up* $\mathbb{P}^3$ *at a point* p*, one gets IIIe, then blow up IIIe along* l_2, l_3*, which are two red edges in the picture above proposition 4.16, then contract* l_3*, which is the only interior edge that can be contracted to the center.*

Lemma 4.20. *The doulbe locus (interior facets in $4\Delta(1,4)$) of each toric variety together with the polarization are:*

IIa,c: $(\mathbb{P}^2, \mathcal{O}(3))$*;*

IIb: $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2))$*;*

IIIa: $(Bl_1\mathbb{P}^2, \mathcal{O}(3H-E))$ *and* $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2))$*;*

IIIb: $(\mathbb{P}^2, \mathcal{O}(2))$ *and* $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2))$*,*

IIIc: two $(\mathbb{P}^2, \mathcal{O}(2))$*;*

IIId: two $(Bl_1\mathbb{P}^2, \mathcal{O}(3H-E))$*;*

IIIe: $(\mathbb{P}^2, \mathcal{O}(2))$ *and* $(Bl_1\mathbb{P}^2, \mathcal{O}(3H-E))$*;*

IIIf: $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2))$ *and* $(Bl_1\mathbb{P}^2, \mathcal{O}(3H-E))$*;*

IIIg: $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2))$ *and* $(\mathbb{P}^2, \mathcal{O}(2))$*;*

IVa: three $(\mathbb{P}^2, \mathcal{O}(1))$*;*

IVb: three $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$*;*

IVc: two $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ *and* $(\mathbb{P}^2, \mathcal{O}(1))$*;*

IVd: $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$*,* $(\mathbb{P}^2, \mathcal{O}(1)$ *and* $(Bl_1\mathbb{P}^2, \mathcal{O}(2H-E))$*;*

IVe: two $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)$ *and* $(Bl_1\mathbb{P}^2, \mathcal{O}(2H-E))$*;*

IVf: two $(Bl_1\mathbb{P}^2, \mathcal{O}(2H-E))$ *and* $(\mathbb{P}^2, \mathcal{O}(1))$*;*

IVg: $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ *and two* $(Bl_1\mathbb{P}^2, \mathcal{O}(2H-E))$*;*

IVh: two $(Bl_1\mathbb{P}^2, \mathcal{O}(2H-E))$ *and* $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$*;*

IVi: two $\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ *and two* $(\mathbb{P}^2, \mathcal{O}(1))$*;*

IVj: three $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ *and* $(\mathbb{P}^2, \mathcal{O}(1))$*.*

Remark 4.21. There are 20 nontrivial subpolytopes in $4\Delta(1,4)$ above. Midpoints of each edge of $4\Delta(1,4)$ do not appear as vertices of any of these 20 subpolytopes. Up to symmetry, central subdivisions of $4\Delta(1,4)$ without using those midpoints correspond to central subdivisions of the Lie type A_3 root polytope $Q(A_3) = \text{Conv}\{e_i - e_j, 1 \leqslant i, j \leqslant 4\}$.

Definition 4.22. A *tiling is of type* i if polytopes of type i appear.

Theorem 4.23. *There are 1176 different tilings of* $4\Delta(1,4)$*. More specifically, one has:*

type I tilings: 1 (no subdivision);

type II tilings: 2;

type III tilings: 10;

type IV tilings: 1163.

Proof. I thank Professor Alexeev who provides a program to compute this. □

For type II tilings: the two different tilings are just IIa-IIc and IIb-IIb.

The 10 type III tilings are listed:

(1) IIIa-IIIa-IIIc-IIIb-IIIb-IIId;
(2) IIIa-IIIc-IIIb-IIIe-IIId;
(3) IIIc-IIIe-IIIe-IIId;
(4) IIIa-IIIa-IIIb-IIIf-IIId;
(5) IIIa-IIIe-IIIf-IIId;
(6) IIIa-IIIc-IIIb-IIIb-IIIg;
(7) IIIc-IIIb-IIIe-IIIg;
(8) IIIa-IIIb-IIIf-IIIg;
(9) IIIe-IIIf-IIIg;
(10) IIIa-IIIb-IIIf-IIIg.

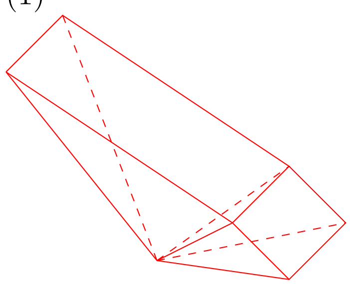
Actually, the volume of the whole $4\Delta(1,4)$ is 32 (by volume we mean the number of nondecomposable subpolytopes). Type III polytopes are of volume 5,6,10 or 11. 32 can be written as the sum of the numbers above in 7 ways. For two cases, $32 = 10 + 5 + 6 + 11$ and $32 = 5 + 5 + 5 + 6 + 11$, each has two different tilings by either use IIIf or IIIg. Moreover, there are two types of $32 = 5 + 5 + 11 + 11$, one has a smoothing IIa-IIc and the other has a smoothing IIb-IIb. So there are 10 types of III tilings.

Remark 4.24. There are seven minimal subdivisions, i.e. such that any smoothing is not a subdivision, and all these are regular (for the definition, see next section). It is easy to see that both type II tilings and III(9) are minimal. The other four are:

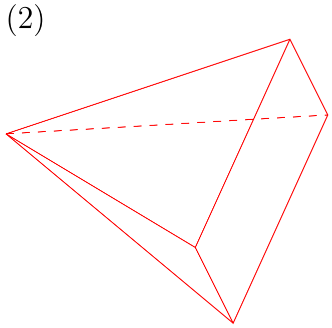
(1) IVc-IVi-IVd-IVe-IIIc-IIId-IVh;
(2) IVf-IVg-IVj-IVh-IVh;
(3) IVi-IVi-IVj-IVj-IVj-IVj;
(4) IVh-IVh-IVh-IVh.

This is also computated by Professor Alexeev's program. The following polytopes have their vertices as each subpolytope in the tiling, edges as common facets between two subpolytopes, and facets as common rays between different subpolytopes. Since all of them are convex, so all these four tilings are regular.

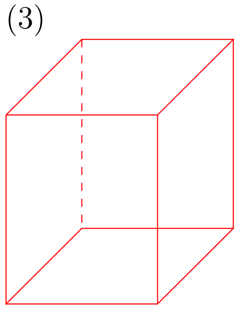
(1)



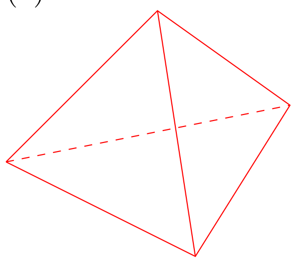
(2)



(3)



(4)



4.3. **Regular Subdivisions.**

Definition 4.25. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection removing the last coordinate. A subdivision of a polytope $P \subseteq \mathbb{R}^n$ is *regular* if it is obtained by projecting the lower faces under π of the polythedron $\tilde{P} \subseteq \mathbb{R}^{n+1}$ for some height function $h : P \to \mathbb{R}$, where h gives the lower envelope of $\tilde{P}$.

.

Regular subdivisions of hypersimplices correspond to the main component of the compactified moduli space via stable pairs.

There are some obvious non-regular subdivisions, as we show below.

There are six squares, and some of their subdivisions give non-regular subdivisions of $Q(A_3)$.

Let h be a height function, and e_{ij} be the vertex $e_i - e_j$ on $Q(A_3)$. Define $s_k, k = 1, \cdots, 6$ on each square via

$$s_1 = h(e_{12}) + h(e_{34}) - h(e_{14}) - h(e_{32})$$

$$s_2 = h(e_{14}) + h(e_{23}) - h(e_{13}) - h(e_{24})$$

$$s_3 = h(e_{32}) + h(e_{41}) - h(e_{42}) - h(e_{31})$$

$$s_4 = h(e_{43}) + h(e_{21}) - h(e_{41}) - h(e_{23})$$

$$s_5 = h(e_{13}) + h(e_{42}) - h(e_{12}) - h(e_{43})$$

$$s_6 = h(e_{24}) + h(e_{31}) - h(e_{34}) - h(e_{21})$$

Since $\sum_{k=1}^{6} s_k = 0$, when all $s_k \geqslant 0$ (or all $s_k \leqslant 0$) and at least one $s_i > 0$ (< 0 respctively), the subdivision is non-regular. So up to symmetry, there are two obvious non-regular tilings.

Regular subdivisions can be studied by tools from tropical geometry. All definitions and results in this subsection on tropical geometry are from [MS15].

Definition 4.26. A *tropical hypersurface* $\mathrm{trop}(V(f))$ is the set

$$\{w \in \mathbb{R}^n : \text{the minimum in} \, \mathrm{trop}(f)(w) \text{ is achieved at least twice}\}.$$

A *tropoical prevariety* is a finite intersection of tropical hypersurfaces.

Recall that Grassmannians can be embedded into projective spaces via Plücker maps. Explicitly, a point $L \in \mathrm{Gr}(k, n)$ can represented by a $k \times n$ matrix M_L. Then the image of L in $\mathbb{P}^{\binom{n}{k}-1}$ has coordinates $\{\det \sigma\}$ where σ runs over all $k \times k$ submatrices of M_L. The Plücker relations define $\mathrm{Gr}(k, n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ with the ideal generated by quadratics.

Definition 4.27. The *Dressian* of the matroid M is the tropical prevariety in $\mathbb{R}^{|\mathcal{B}|}/\mathbb{R}\mathbf{1}$ obtained by intersecting the tropical hypersurfaces of the Plücker quadratics.

There is a one-to-one correspondence between regular matroid polytope subdivisions and vectors in Dressians:

Lemma 4.28. *For a matroid* $([n], \mathcal{B})$*, let* $\mathbf{w} \in \mathbb{R}^{|\mathcal{B}|}/\mathbb{R}\mathbf{1}$*. Then* $\mathbf{w} \in \mathrm{Dr}(M)$ *if and only if* $\mathbf{w}$ *induces a regular matroid subdivision.*

Remark 4.29. For central subdivisions with the center not being a vertex of any subpolytopes, there is a correspondence between central regular subdivisions of Lie type A polytope $Q(A_{n-1})$ and alcoved polytopes, see [Zha16].

The computation for the entire fan of the Dressian $\mathrm{Dr}(3, 8)$ can be found in [HJS14]. The computation for $\mathrm{Dr}(4, 8)$ is still open at this moment.

5. Moduli via GIT Quotients

Let X be a projective variety and G be a reductive group acting on X. In general, the set X/G does not inherit an algebraic structure. To obtain a nice space, one has to take the GIT (geometric invariant theory) quotient developed by Mumford or the Chow/Hilbert quotient as in [Kap93a](see the second part of this thesis). The compactification via stable pairs in general is difficult to describe. But for the moduli of weighted hyperplane arrangements, one may apply the GIT theory to approach it very well. In general, GIT stability could be stronger then the stability in birational geometry.

Definition 5.1. Let V be a vector space and G be a reductive group acting on it. A point $x \in V$ is *stable* if $G \cdot x$ is closed and its stablizer is finite. A point $x \in V$ is *semistable* if $0 \notin \overline{G \cdot x}$.
For a projective variety $X \subseteq \mathbb{P}(V)$, a point $x \in X$ is *stable (respectively semistable)* if x^* is stable (respectively semistable) for any lift $x^* \in V$ of x. A point is *unstable* if it not semistable. The set of stable points, semistable points and unstable points are denoted by X^s, X^{ss} and X^{us}.

Mumford proved that $X^s \subseteq X^{ss} \subseteq X$ and each inclusion is open in [MFK94]. Points in X^s/G are G-orbits. And points in X^{ss} are G-orbit equivalent classes, where $G \cdot x_1 \sim G \cdot x_2$ iff $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$.

Note that for projective variety $X \subseteq \mathbb{P}(V)$, the GIT stability defined above is not intrinsic. It depends on the embedding. In other words, the quotient depends on how G acts on $\mathbb{P}(V)$ or the sections of ample line bundle which embed X into $\mathbb{P}(V)$.

Let $S \subset T = \mathbb{C}^n$ be a subtorus. For a polarized toric variety (X, L) of dimension n with a S-linearized ample line bundle L, there is a GIT quotient $X /\!\!/_L S$. The torus embedding induces a lattice map $M_T \to M_S$. The S-linearization is equivalent to a point $\mathbf{b} \in M_S \otimes \mathbb{R}$, in this case, the GIT quotient is just the toric variety associate to the fiber over $\mathbf{b}$. The reference for this theory can be found in [Ale15] 5.1 or [CLS11] chapter 14.

Definition 5.2. A variety X is *seminormal* if any finite bijective morphism $X' \to X$ has to be an isomorphism.

Remark 5.3. Cuspital curves or $V(x, y, x-y) \subseteq \mathbb{A}^2_{x,y}$ are not seminormal. For more discussion, see [Vit11].

Definition 5.4. ([Ale15]) A *stable toric variety* X is a seminormal union of toric varieties glued along the toric invariant boundaries.

For each matroid tiling $\Delta(k, 2k) = \cup Q_i$, there is a stable toric variety $Y = \cup Y_i$ associated to it. Let $U \subseteq \mathrm{Gr}(k, 2k) \times \mathbb{P}^{2k-1}$ be the universal cycle over $\mathrm{Gr}(k, 2k)$, i.e. over a point $[V] \in \mathrm{Gr}(k, 2k)$, the fiber is just the $(k-1)$-dimensional subspace $V = \mathbb{P}^{k-1}$ in $\mathbb{C}^{2k}$. Then there is a T-action on $Y \times_{\mathrm{Gr}(k,2k)} U$.

Definition 5.5. ([Ale15] 5.3.1) The *weighted hyperplane arrangement* $(X, B = \sum b_i B_i)$ associated to $Y = \cup Y_j$ for the weight $\mathbf{b} = (b_1, \cdots, b_{2k})$ is the GIT quotient $X = Y \times_{\mathrm{Gr}(k,2k)} U /\!\!/_{\mathbf{b}} T$, and $B_i = (Y \times_{\mathrm{Gr}(k,2k)} U) \cap H_i /\!\!/_{\mathbf{b}} T$, where $H_i = \{x_i = 0\} \subset \mathbb{P}^{2k-1}, i = 1, \cdots, 2k$.

Theorem 5.6. *([Ale15] 5.3.2) The pair (X, B) above is a stable pair.*

The universal cycle $U \subseteq \mathrm{Gr}(k, 2k) \times \mathbb{P}^{2k-1} \subseteq \mathbb{P}^{\binom{2k}{k}-1}_{p_I} \times \mathbb{P}^{2k-1}_{z_1,\cdots,z_{2k}}$ is defined by the Plücker relations together with the *U-relations*:

$$\{\sum_{j \in J} (-1)^j p_{J \setminus \{j\}} z_j = 0, \forall J \subseteq [2k], |J| = k+1\}.$$

The U-relations can be written as $M_P \cdot \vec{z} = 0$, where M_P is the $\binom{2k}{k+1}$ by $2k$ *Plücker matrix* whose entries are Plücker coordinates and $\vec{z} = (z_1, \cdots, z_{2k})$.

Theorem 5.7. *The central tilings of $k\Delta(1,k)$ from matroid subdivisions of $\Delta(k,2k)$ in proposition 4.15 give exactly the GIT quotients $Y \times U /\!/_{\mathbf{b}} T$.*

Proof. First note that $\Delta(k,2k)$ is the convex hull of the barycenters of all $(k-1)$-dimensional faces of $2k\Delta(1,2k)$. So a matroid tiling of $\Delta(k,2k)$ naturally gives a tiling of $2k\Delta(1,2k)$ by extending the hyperplanes cutting each subpolytope away from the center. Assume that the parity in corollary 4.14 is fixed, say $\{1,1',\cdots,k,k'\}$, then $\vec{z} = (z_1, z_1', \cdots, z_k, z_k')$. Also note that there is a $(k-1)$-dimensional face $F' = 2k\Delta(1,k)$ which is the convex hull of $\{2ke_{i'}, i = 1,\cdots,k\}$. Let Y be an irreducible component associated to a central matroid subpolytope $Q = \cap_{J\in\mathcal{J}}\{x_{JJ'} \leqslant |J|\}$ in the tiling of $\Delta(k,2k)$. For a linearization $\gamma = (1,\cdots,1,\epsilon,\cdots,\epsilon), 0 < \epsilon \ll 1$, we will show that $Y \times U^{ss} \subseteq \{p_{I'} \neq 0\}, I' = \{1',\cdots,k')$ the defining inequalities $x_{11'\cdots rr'} \leqslant r$ are $x_{1\cdots r} \leqslant r$ in the GIT quotient.

Step 1. The toric quotient $Y \times \mathbb{P}^{2k-1} /\!/_{\gamma} T \cong Y(Q \cap \epsilon\Delta(1,2k))$, which is the toric variety associated to the polytope $Y(Q \cap \epsilon\Delta(1,2k))$. Consider the map from the polytope to $M_T \otimes \mathbb{R}$

$$\phi : mQ \times \epsilon\Delta(1,2k) \to \mathbb{R}^{2k}/\mathbb{R}$$

$$(a,b) \mapsto a+b.$$

Let $\tilde{c} = (\frac{m}{2}\cdots,\frac{m}{2},\epsilon,\cdots,\epsilon)$ be the center of $mQ \times \epsilon\Delta(1,2k)$ and $c = \phi(\tilde{c})$. The fiber over c is $Q \cap \epsilon\Delta(1,2k)$. Actually, the fiber is the intersection of Q and the convex hull of the set

$$\{(\frac{m}{2},\cdots,\frac{m}{2},0,\cdots,0) + (-2k\epsilon e_i, 2k\epsilon e_i) \in \mathbb{R}^{2k} \times \mathbb{R}^{2k}\},$$

which is $\epsilon 2k(1,2k)$. So $Y \times \mathbb{P}^{2k-1} /\!/_{\mathbf{b}} T$ is the toric variety assciated to the central polytope $Q \cap \epsilon\Delta(1,2k)$. After rescaling, this polytope is the polytope defined by $\cap_{J\in\mathcal{J}}\{x_{JJ'} \leqslant |J|\}$ in $\Delta(1,2k)$. Moreover, a central matroid tiling of $\Delta(k,2k) = \cup_j Q_j$ induces a central tiling of $\Delta(1,2k)$, which is the rescaling of tiling $\epsilon\Delta(1,2k) = \cup_j(Q_j \cap$

$\epsilon\Delta(1,2k))$.

Step 2. $Y \times U /\!\!/_\gamma T \cong Y(Q \cap \epsilon F')$. Note that

$$\operatorname{codim}(Y \times U, Y \times \mathbb{P}^{2k-1}) = 2(2k-1) - [(2k-1) + (k-1)] = k.$$

So $\operatorname{rank}(M_P) = k$, the solution space of $M_P \cdot \vec{z} = 0$ has dimension $2k - k = k$. Thus, $\{z'_1, \cdots, z'_k\}$ generate the solution space. Thus, there is a bijection

$$\{x_{II'} = |I|\} \overset{1:1}{\longleftrightarrow} \{x_{I'} = |I|\}.$$

Then there is a central tiling of $F' = \cup_j \overline{Q}_j$. The tiling is central because here we only care those hyperplanes passing through the vertex $x_{1' \dots k'} = k$, which is the center of F'. Moreover, by the U-relation, the tiling of F' uniquely determines the tiling of $\Delta(k, 2k)$.

□

Example 5.8. Consider $Y \times_{\mathrm{Gr}(2,4)} U /\!\!/_{\mathbf{b}} T$ where Y is a generic T-orbit closure in $\mathrm{Gr}(2,4)$, whose polytope is $\Delta(2,4)$. $Y \times U /\!\!/_{\mathbf{b}} T$ lies in $Y \times \mathbb{P}^5 /\!\!/_{\mathbf{b}} T$. By theorem 5.7, $Y \times \mathbb{P}^5 /\!\!/_{\mathbf{b}} T \cong \mathbb{P}^3$, and $Y \times_{\mathrm{Gr}(2,4)} U /\!\!/_{\mathbf{b}} T \cong \mathbb{P}^1$. Actually, the Plücker matrix

$$M_P = \begin{bmatrix} p_{23} & -p_{13} & p_{12} & 0 \\ p_{24} & -p_{14} & 0 & p_{12} \\ p_{34} & 0 & -p_{14} & p_{13} \\ 0 & p_{34} & -p_{24} & p_{23} \end{bmatrix}$$

has rank 2. So the solution space V of the U-relation $M_P \vec{z} = 0$ has rank $4 - 2 = 2$, thus $\mathbb{P}(V) = \mathbb{P}^1 \subseteq \mathbb{P}^3$. In this case, there is no subdivision of $F' = 2\Delta(1,2) = \operatorname{Conv}\{(0,2,0,0), (0,0,0,2)\} \subseteq \mathbb{R}^4_{x_1, x'_1, x_2, x'_2}$, so the quotient $\mathbb{P}^1$ is polarized by $\mathcal{O}(2H)$. If $\Delta(2,4) = Q_1 \cup Q_2 = \{x_{11'} \leqslant 1\} \cup \{x_{22'} \leqslant 2\}$, then the corresponding sub-polytopes of Q_1, Q_2 in F' are the segments $\overline{Q}_1 = \operatorname{Conv}\{(0,2,0,0), (0,1,0,1)\}$ and $\overline{Q}_2 = \operatorname{Conv}\{(0,1,0,1), (0,0,0,2)\}$. The associated toric varieties are both $(\mathbb{P}^1, \mathcal{O}(H))$.

Remark 5.9. By theorem 5.7, we know that the computation we did in chapter 4 gives the variety X in the stable degenerated weighted hyperplane arrangement (X, B). For the divisor B, we have the following.

Lemma 5.10. [Ale19] *Let* $\mathbf{w} = (\mathbf{w_1}, \cdots, \mathbf{w_n}), \mathbf{0} < \mathbf{w_i} \leqslant \mathbf{1}$ *be a given weight. If* w *and* $(\epsilon, \cdots, \epsilon, 1, \cdots, 1)$ *are in the same chamber, then the quotient* $Y \times U /\!\!/_T$ *is toric.*

Corollary 5.11. The generic $B_i^j = (Y_j \times_{\mathrm{Gr}(k,2k)} U) \cap H_i /\!\!/_{\mathbf{b}} T$'s in Y_j appear in pairs (each pair of B_i's are linear equivalent). And $B^j = \frac{1}{2} \sum B_i^j$ is linear equivalent to the toric boundary of Y_i minus its double locus.

Proof. Note that $(Y_j \times \mathbb{P}^{2k-1}) \cap H_i = Y_j \times H_i$. The fiber over c is $Q \cap \epsilon P_i$, which are exterior boundaries of ϵP_i cut by $\{x_{II'} \leqslant |I|\}$. By the bijection from the U-relation, one may look at the $\{x_I \leqslant |I|\}$ on $F' \cap P_i$. Since the GIT quotients of toric boundaries are still boundaries. Note that the quotients $Y \times U /\!\!/_b T$'s are all toric. So $B_i^j = (Y_j \times_{\mathrm{Gr}(k,2k)} U) \cap H_i /\!\!/_{\mathbf{b}} T$'s form the exterior toric boundary of $Y_j \times_{\mathrm{Gr}(k,2k)} U /\!\!/_{\mathbf{b}} T$. □

6. Abelian Covers of Irreducible Components of Degenerations

6.1. Description of Abelian Covers of Each Irreducible Component for M_{CY}.

In this subsection we describe the abelian covers of each irreducible toric component.

Lemma 6.1. *(5.20 in* [KM08] *and 2.3 in* [AP12]*) Let $f : X \to Y$ be a finite dominant morphism between two equidimensional S_2 varieties. Δ_X, Δ_Y are $\mathbb{Q}$-divisors on X and Y satisfying $K_X + \Delta_X = f^*(K_Y + \Delta_Y)$. Then*
1. $K_X + \Delta_X$ is ample iff $K_Y + \Delta_Y$ is;
2. $K_X + \Delta_X$ is $\mathbb{Q}$-Cartier iff $K_Y + \Delta_Y$ is;
3. (X, Δ_X) is semi log canonial iff (Y, Δ_Y) is.

Corollary 6.2. Let X be the $(\mathbb{Z}/2\mathbb{Z})$ cover of degerations of $\mathbb{P}^3$, in this case the branched loci and some double loci overlapped, but (X, R) is a stable pair, where $R = \pi^*(B)$, B is the unions of B_i's. in corollary 5.11.

Note that if the fundamental relations in the abelian building data hold, some double loci have to appear as branch divisors.

For a fixed central lattice subpolytope Q of $4\Delta(1,4)$, let D_{int} be the divisor of double boundary loci, i.e. all the facets of Q containing the center $(1,1,1,1)$. And let $D_{int\ (*)}$ be the irreducible component of D_{int} who corresponds to the polarized toric variety $*$.

Lemma 6.3. *For each Y_i associated to the polytope in proposition 4.16, the branch divisors $D_i(')$ are listed below (the double locus is in blue):*
IIa: $D_1(') = B_1(') + D_{int}$, $D_{2,3}(') = B_{2,3}(')$;
IIb: $D_i(') = B_i(')$;
IIc: $D_{1,2,3}(') = B_{1,2,3}('), D_4(') = B_4(') + D_{int}$;

IIIa: $D_1(') = B_1(') + D_{int(\mathrm{Bl}_1\,\mathbb{P}^2)}, D_{2,3}(') = B_{2,3}(')$;

IIIb: $D_{1,2}(') = B_{1,2}(')$, $D_3(') = B_3(') + D_{int(\mathbb{P}^2)}$;

IIIc: $D_{1,2}(') = B_{1,2}(')$;

IIId: $D_i(') = B_i('), i = 1, 2, 3, 4$;

IIIe: $D_{1,2}(') = B_{1,2}(')$, $D_3(') = B_3(') + D_{int}$;

IIIf: $D_{1,2}(') = B_{1,2}(')$, *and* $D_3(') = B_3(') + D_{int\ \mathbb{P}^2}$;

IIIg: $D_{1,2}(') = B_{1,2}('), D_3(') = B_3(') + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2}$, *and* $D_4(') = B_4(') + D_{int\ \mathbb{P}^1\times\mathbb{P}^1}$;

IVa: $D_1(') = B_1(') + D_{int}$;

IVb: $D_i(') = B_i(') + D_{int(\mathbb{P}^1\times\mathbb{P}^1)} = H_i + H_i, i = 1, 2, 3,$ *where* H_i *is the* $\mathbb{P}^1 \times \mathbb{P}^1$ *fiber over the i-th* $\mathbb{P}^1$;

IVc: $D_1(') = B_1(') + D_{int(\mathbb{P}^1\times\mathbb{P}^1)}, D_2(') = B_2(') + D_{int(\mathbb{P}^2)}$;

IVd: $D_1(') = B_1(') + D_{int(\mathbb{P}^2)}, D_2(') = B_2(') + D_{int(\mathrm{Bl}_1\,\mathbb{P}^2)}$;

IVe: $D_1 = B_1 + D_{int(\mathbb{P}^1\times\mathbb{P}^1)_1}, D'_1 = B'_1 + D_{int(\mathbb{P}^1\times\mathbb{P}^1)_1}$, $D_2 = B_2 + D_{int(\mathbb{P}^1\times\mathbb{P}^1+\mathbb{P}^1\times\mathbb{P}^1)}, D'_2 = B'_2 + D_{int(\mathbb{P}^1\times\mathbb{P}^1+\mathbb{P}^1\times\mathbb{P}^1)}$, $D_3 = B_3 + D_{int(\mathrm{Bl}_p\,\mathbb{P}^2)}, D'_3 = B'_3 + D_{int(\mathrm{Bl}_p\,\mathbb{P}^2)}$, *here the subpolytope has three exterior boundary components:*

A. $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2))$,

B. $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1))$,

C. $(\mathrm{Bl}_p\,\mathbb{P}^2, \mathcal{O}(2H - E))$.

$(\mathbb{P}^1 \times \mathbb{P}^1)_1$ *is the double locus adjacent to A, and the other* $\mathbb{P}^1 \times \mathbb{P}^1$ *is the double locus parallel to A;*

IVf: $D_1(') = B_1(') + D_{int\mathbb{P}^2(e)}$, $D_2(') = B_2(')$;

IVg: $D_1(') = B_1(') + D_{int(\mathbb{P}^1\times\mathbb{P}^1)}$, *all other* $D_{2,3}(') = B_{2,3}(')$;

IVh: $D_1 = B_1 + D_{int}, D'_1 = B'_1; + D_{int}$, $D_2 = B_2 + D_{int(Bl_p\mathbb{P}^2_1+\mathbb{P}^1\times\mathbb{P}^1)}, D'_2 = B'_2; + D_{int(Bl_p\mathbb{P}^2_1+\mathbb{P}^1\times\mathbb{P}^1)}$, $D_3 = B_3 + D_{int(\mathrm{Bl}_p\,\mathbb{P}^2_1)}, D'_3 = B'_3 + D_{int(\mathrm{Bl}_p\,\mathbb{P}^2_1)}$, *here all the six* $B_i(')s$ *are* $\mathrm{Bl}_p\,\mathbb{P}^2$*'s, whose the subindices are in the decreasing order with respect to the volume. And* $\mathrm{Bl}_1\,\mathbb{P}^2_1$ *is one of the two trapezoid interior facets with smaller volume;*

IVi: $D_{1,2}(') = B_{1,2}(')$;

IVj: $D_1(') = B_1(') + D_{int}$, $D_{2,3}(') = B_{2,3}(')$.

Moreover, the covers of IIIe,IIIf,IVa,IVb,IVh,IVj are CY 3-folds, and the covers of IIb,IIc,IIIc,IIId,IVd,IVg,IVi together with the ramified loci are log CY pairs.

Proof. Fix the subindices of D_g's as below:

1=(1,0,0,0), 1'=(1,0,0,1),

2=(1,0,1,0), 2'=(1,0,1,1),

3=(1,1,0,0), 3'=(1,1,0,1),

4=(1,1,1,0), 4'=(1,1,1,1). The fundamental relation

$$2L_\chi = \sum_{\chi(g)=-1} D_g$$

must be satisfied. Now we check the fundamental relations for all subpolytopes in proposition 4.16. Without lost of generality, when the branched loci has $2k$ componets, we label them by $1, 1', \cdots, k, k'$. Note that one needs $\sum_g gD_g = 0 \in \text{Pic}(Y_i) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ for the existence of G-cover.

IIa: $B_i(') = \mathbb{P}^2 = H, i = 1,2,3$, $D_{int} = \mathbb{P}^2 = H$, $\sum gB_g = (0,0,0,1)B_1 = (0,0,0,1)H \neq 0$, so $D_i = D_i' = H + D_{int} = 2H$ (H is the hyperplane in the $\mathbb{P}^2$ corresponding to the polytope IIa. Then

$$\begin{aligned} K_X &= \pi^*(-4H + \frac{1}{2}\cdot 6 \cdot 2H) \\ &= \pi^*(2H). \\ K_X + \text{d.l.} &= \pi^*(2H + H) \\ &= \pi(3H) \end{aligned}$$

IIb: $B_{1,2}(') = \mathbb{P}^2, B_{3,4}(') = \operatorname{Bl}_1 \mathbb{P}^2$, $D_{int} = D_{int\ \mathbb{P}^1 \times \mathbb{P}^1}$. $\sum gB_g = 0$, so there is no boundary branched locus. And

$$\begin{aligned} K_X &= \pi^*(-2\operatorname{Bl}_1 \mathbb{P}^2 - 2\mathbb{P}^2 - D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + \frac{1}{2}\cdot 2(2\mathbb{P}^2 + 2\operatorname{Bl}_1 \mathbb{P}^2)) \\ &= \pi^*(-D_{int\ \mathbb{P}^1\times\mathbb{P}^1}). \\ K_X + \text{d.l.} &= \pi^*(-D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathbb{P}^1\times\mathbb{P}^1})) \\ &= 0 \end{aligned}$$

So in this case, $(X, D + \text{d.l})$ is a log CY pair.

IIc: $B_{1,2,3}(') = \operatorname{Bl}_1 \mathbb{P}^2 = s, B_4(') = \mathbb{P}^2$, $D_{int} = \mathbb{P}^2 = e$. And $B_4(') = s + D_{int}$, so

$$\begin{aligned} \sum gB_g &= (0,0,0,1)(4s+e) \\ &= (0,0,0,1)e \neq 0. \end{aligned}$$

Thus,

$$\begin{aligned} D_{1,2,3}(') &= B_{1,2,3}('), \\ D_4(') &= B_4(') + D_{int}. \\ K_X &= \pi^*(-4s - 2D_{int} + \frac{1}{2}\cdot 2(3s + D_{int} + s)) \\ &= \pi^*(-D_{int}). \\ K_X + \text{d.l.} &= 0. \end{aligned}$$

So $(X, D + \text{d.l.})$ is a log CY pair.

IIIa: $B_1(') = \operatorname{Bl}_1 \mathbb{P}^2, B_{2,3}(') = \mathbb{P}^2$. $D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \operatorname{Bl}_1 \mathbb{P}^2}$. Note that $B_1(') = D_{int\ \operatorname{Bl}_1 \mathbb{P}^2}$.

$$\begin{aligned} \sum gB_g &= (0,0,0,1)B_1 \\ &\neq 0, \end{aligned}$$

So we have

$$
\begin{aligned}
D_1(') &= B_1(') + D_{int\ \mathrm{Bl}_1\mathbb{P}^2}, \\
D_{2,3}(') &= B_{2,3}('). \\
K_X &= \pi^*(-2\mathbb{P}^2 - D_{int\ \mathbb{P}^1\times\mathbb{P}^1} - 2\,\mathrm{Bl}_1\mathbb{P}^2 + \frac{1}{2}(2(2D_{int\ \mathrm{Bl}_1\mathbb{P}^2} + 4\mathbb{P}^2)) \\
&= -D_{int\ \mathbb{P}^1\times\mathbb{P}^1}. \\
K_X + \text{d.l.} &= D_{int\ \mathbb{P}^1\times\mathbb{P}^1}.
\end{aligned}
$$

IIIb: $B_{1,2}(') = \mathbb{P}^1 \times \mathbb{P}^1$, $B_3(') = \mathbb{P}^2$, $D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathbb{P}^2}$.

$$
\begin{aligned}
\sum gB_g &= (0,0,0,1)D_{int\ \mathbb{P}^1\times\mathbb{P}^1} \\
&\neq 0.
\end{aligned}
$$

So we have

$$
\begin{aligned}
D_{1,2}(') &= B_{1,2}('), \\
D_3(') &= B_3(') + D_{int\ \mathbb{P}^2}. \\
K_X &= \pi^*(-3\mathbb{P}^1\times\mathbb{P}^1 - 2\mathbb{P}^2 + 2\mathbb{P}^1\times\mathbb{P}^1 + 2\mathbb{P}^2) \\
&= \pi^*(-D_{int\ \mathbb{P}^1\times\mathbb{P}^1}), \\
K_X + \text{d.l.} &= \pi^*(D_{int\ \mathbb{P}^2}).
\end{aligned}
$$

IIIc: $D_{int} = 2D_{int\ \mathbb{P}^2} = H$, where H is the hyperplane in the $\mathbb{P}^2$ associated to the polytope IIIc. $\sum gB_g = 0$, so $D_i(') = B_i(')$.

$$
\begin{aligned}
K_X + \text{d.l.} &= \pi^*(-4H + \frac{1}{2}\cdot 4H + 2H) \\
&= 0.
\end{aligned}
$$

So the pair (X, D) is a log CY pair.

IIId: $D_{int} = 2D_{int\,\mathrm{Bl}_1\,\mathbb{P}^2}$, $\sum gB_g = 0$, so $D_i(') = B_i(')$.

$$K_X + \text{d.l.} = \pi * (-2\mathbb{P}^1 \times \mathbb{P}^1 - 2\,\mathrm{Bl}_1\,\mathbb{P}^2_{(1)} - 2\,\mathrm{Bl}_1\,\mathbb{P}^2_{(2)} + 2\mathbb{P}^1 \times \mathbb{P}^1 + 2\,\mathrm{Bl}_1\,\mathbb{P}^2_{(1)} + 2\,\mathrm{Bl}_1\,\mathbb{P}^2_{(2)})$$

$$= 0.$$

So the pair (X, D) is log CY.

IIIe: $B_{1,2}(') = \mathrm{Bl}_1\,\mathbb{P}^2, B_3(') = \mathbb{P}^2$, $D_{int} = D_{int\ \mathbb{P}^2} + D_{int\ \ \mathrm{Bl}_1\,\mathbb{P}^2}$ and $B_3(') = D_{int}$.

$$\sum gB_g = (0,0,0,1)B_3$$
$$= (0,0,0,1)D_{int} \neq 0.$$

So we have

$$D_{1,2}(') = B_{1,2}('),$$
$$D_3(') = B_3(') + D_{int}.$$
$$K_X = \pi^*(-3\,\mathrm{Bl}_1\,\mathbb{P}^2 - \mathbb{P}^2_{out} - \mathbb{P}^2_{int} + 2\,\mathrm{Bl}_1\,\mathbb{P}^2 + \mathbb{P}^2_{out} + \mathrm{Bl}_1\,\mathbb{P}^2 + \mathbb{P}^2_{int})$$
$$= 0.$$

So X is a CY variety.

IIIf: $B_{1,2}(') = \mathrm{Bl}_1\,\mathbb{P}^2, B_3(') = \mathbb{P}^2$, $D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathbb{P}^2}$.

$$\sum gB_g = (0,0,0,1)D_{int\ \mathbb{P}^2} \neq 0.$$

So we have

$$
\begin{aligned}
D_{1,2}(') &= B_{1,2}(') \\
D_3(') &= B_3(') + D_{int\ \mathbb{P}^2} \\
K_X &= \pi^*(-2\mathbb{P}^2 - \mathbb{P}^1 \times \mathbb{P}^1 - 2\,\mathrm{Bl}_1\,\mathbb{P}^2 + 2\mathbb{P}^2 + 2\,\mathrm{Bl}_1\,\mathbb{P}^2) \\
&= \pi^*(-D_{int\ \mathbb{P}^1\times\mathbb{P}^1}), \\
K_X + \text{d.l.} &= \pi^*(D_{int\ \mathbb{P}^2}).
\end{aligned}
$$

IIIg: $B_{1,2}(') = \mathrm{Bl}_1\,\mathbb{P}^2 = s$ polarized by $\mathcal{O}(2H - E_1)$, $B_3(') = \mathrm{Bl}_1\,\mathbb{P}^2$ polarized by $\mathcal{O}(4H - E_3)$, and $B_4(') = \mathrm{Bl}_1\,\mathbb{P}^2$ polarized by $\mathcal{O}(4H - E_2)$, where E_i is the exceptional curve associated to lattice length i segment. $D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2}$. There are relations:

$$
\begin{aligned}
B_3(') &= s + D_{int\ \mathbb{P}^1\times\mathbb{P}^1}, \\
B_4(') &= s + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2}.
\end{aligned}
$$

$$
\begin{aligned}
\sum g B_g &= (0,0,0,1)(B_3 + B_4) \\
&= (0,0,0,1) D_{int} \neq 0.
\end{aligned}
$$

So we have

$$
\begin{aligned}
D_3(') &= B_3(') + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2}, \\
D_4(') &= B_4(') + D_{int\ \mathbb{P}^1\times\mathbb{P}^1}. \\
K_X &= \pi^*(-D_{int} - 2s - e_3 - e_2 + \frac{1}{2}\cdot 2(f + e_3 + e_2)) \\
&= 0,
\end{aligned}
$$

where e_2, e_3 are boundary divisor divisors $Bl_1\mathbb{P}^2$ outside with exceptional curve associated to lattice length $2, 3$ segments. So X is a CY variety.

IVa: $B_1(') = \mathbb{P}^2$, $D_{int} = 3D_{int\ \mathbb{P}^2}$. And $B_1(') = D_{int\ \mathbb{P}^2}$.

$$\sum gB_g = (0,0,0,1)B_1$$
$$\neq 0.$$

So $D_1(') = B_1(') + D_{int}$.

$$K_X = \pi^*(-4\mathbb{P}^2 + \frac{1}{2}(2 \cdot 4\mathbb{P}^2))$$
$$= 0.$$

So X is a CY vareity.

IVb: $B_{1,2,3} = H_{1,2,3}$, $\sum gB_g \neq 0$.

$$D_i(') = B_i(') + H_i, i = 1,2,3.$$
$$K_X = \pi^*(-2 \sum_{i=1,2,3} H_i + \frac{1}{2} \cdot 4 \sum_{i=1,2,3} H_i)$$
$$= 0.$$

So X is a CY variety.

IVc: $B_1(') = \mathbb{P}^1 \times \mathbb{P}^1, B_2(') = \mathbb{P}^2$, $D_{int} = 2D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathbb{P}^2}$.

$$D_1(') = B_1(') + D_{int\ \mathbb{P}^1\times\mathbb{P}^1},$$
$$D_2(') = B_2(') + D_{int\ \mathbb{P}^2}.$$
$$K_X = \pi^*(-3\mathbb{P}^1 \times \mathbb{P}^1 - 2\mathbb{P}^2 + \frac{1}{2}(4\mathbb{P}^1 \times \mathbb{P}^1 + 4\mathbb{P}^2))$$
$$= \pi^*(-D_{int\ \mathbb{P}^1\times\mathbb{P}^1}).$$
$$K_X + \text{d.l.} = \pi^*(D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathbb{P}^2}).$$

IVd: $B_1(') = \mathbb{P}^2, B_2(') = \mathrm{Bl}_1\mathbb{P}^2$, $D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathbb{P}^2} + D_{int\ \mathrm{Bl}_1\mathbb{P}^2}$.

$$D_1(') = B_1(') + D_{int\ \mathbb{P}^2},$$

$$D_2(') = B_2(') + D_{int\ \mathrm{Bl}_1\mathbb{P}^2}.$$

$$\begin{aligned} K_X &= \pi^*(-2\,\mathrm{Bl}_1\mathbb{P}^2 - 2\mathbb{P}^2 - \mathbb{P}^1\times\mathbb{P}^1 + \frac{1}{2}(2\,\mathrm{Bl}_1\mathbb{P}^2 + 2\mathbb{P}^2)) \\ &= \pi^*(-D_{int\ \mathrm{Bl}_1\mathbb{P}^2} - D_{int\ \mathbb{P}^2} - D_{int\ \mathbb{P}^1\times\mathbb{P}^1}) \\ &= \pi^*(-D_{int}). \end{aligned}$$

$$K_X + \text{d.l.} = 0.$$

So (X, D) is a log CY pair.

IVe: $B_1(') = \mathbb{P}^1\times\mathbb{P}^1$ polarized by $\mathcal{O}(1,2)$, $B_2(') = \mathbb{P}^1\times\mathbb{P}^1$ polarized by $\mathcal{O}(1,1)$, $D_3(') = \mathrm{Bl}_1\mathbb{P}^2$. $D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1(f)} + D_{int\ \mathbb{P}^1\times\mathbb{P}^1(e)} + D_{int\ \mathrm{Bl}_1\mathbb{P}^2}$, where $D_{int\ \mathbb{P}^1\times\mathbb{P}^1(f)}$ is the fiber over $\mathbb{P}^1$ and $D_{int\ \mathbb{P}^1\times\mathbb{P}^1(e)}$ is the product of the exceptional $\mathbb{P}^1$ and the base $\mathbb{P}^1$ (recall that $Y = \mathrm{Bl}_1\mathbb{P}^2\times\mathbb{P}^1 \to \mathbb{P}^1$).

$$\begin{aligned} \sum gB_g &= (0,0,0,1)(\mathbb{P}^1\times\mathbb{P}^1(f) + \mathbb{P}^1\times\mathbb{P}^1(e) + \mathrm{Bl}_1\mathbb{P}^2 \\ &\neq 0. \end{aligned}$$

Thus, one has

$$D_1(') = B_1(') + D_{int\ \mathbb{P}^1\times\mathbb{P}^1(e)},$$

$$D_2(') = B_2('),$$

$$D_3(') = B_3(') + D_{int\ \mathrm{Bl}_1\mathbb{P}^2}.$$

$$\begin{aligned} K_X &= \pi^*(-3\mathbb{P}^1\times\mathbb{P}^1(f) - 2\,\mathrm{Bl}_1\mathbb{P}^2 - 2\mathbb{P}^1\times\mathbb{P}^1(e) + \frac{1}{2}\cdot 2(2\mathbb{P}^1\times\mathbb{P}^1(f) + 2\mathbb{P}^1\times\mathbb{P}^1(e) + 2\,\mathrm{Bl}_1\mathbb{P}^2) \\ &= \pi^*(-D_{int\ \mathbb{P}^1\times\mathbb{P}^1(f)}). \end{aligned}$$

And $K_X + \text{d.l.} = \pi^*(D_{int\ \mathbb{P}^1\times\mathbb{P}^1(e)} + D_{int\ \mathrm{Bl}_1\mathbb{P}^2})$.

IVf: $B_1(') = \mathbb{P}^2, B_2(') = \mathrm{Bl}_1\,\mathbb{P}^2$ and $D_{int} = 2D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2} + D_{int\ \mathbb{P}^2(e)} = 2f + e$.

$$\begin{aligned}\sum gB_g &= (0,0,0,1)(\mathbb{P}^2 + \mathrm{Bl}_1\,\mathbb{P}^2)\\ &= (0,0,0,1)\mathbb{P}^2(e).\end{aligned}$$

Thus,

$$\begin{aligned}D_1(') &= B_1(') + D_{int\ \mathbb{P}^2(e)},\\ D_2(') &= B_2(').\\ K_X &= \pi^*(-3f - f - e - e + \frac{1}{2}(f + e + e + f))\\ &= \pi^*(-2D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2}).\\ K_X + \text{d.l.} &= \pi^*(D_{int\ \mathbb{P}^2(e)}).\end{aligned}$$

IVg: $B_1(') = \mathbb{P}^1 \times \mathbb{P}^1, B_{2,3}(') = \mathrm{Bl}_1\,\mathbb{P}^2 = b,\ D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + 2D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + a$.

$$\begin{aligned}\sum gB_g &= (0,0,0,1)B_1\\ &= (0,0,0,1)D_{int\ \mathbb{P}^1\times\mathbb{P}^1} \neq 0\end{aligned}$$

Then we have

$$\begin{aligned}D_1(') &= B_1(') + D_{int\ \mathbb{P}^1\times\mathbb{P}^1},\\ D_{2,3}(') &= B_{2,3}(').\\ K_X &= \pi^*(-2b - 2a - 2\mathbb{P}^1 \times \mathbb{P}^1 + \frac{1}{2}\cdot 2(2b + \mathbb{P}^1 \times \mathbb{P}^1))\\ &= \pi^*(-2a - \mathbb{P}^1 \times \mathbb{P}^1)\\ &= \pi^*(-D_{int}).\\ K_X + \text{d.l.} &= 0.\end{aligned}$$

So $(X, D + \text{d.l.})$ is a log CY pair.

IVh: all $B_i(')$'s $i = 1,2,3$ are $\mathrm{Bl}_1\,\mathbb{P}^2$ but the polarizations are different. Let $B_1('), B_2('), B_3(')$ be in increasing order with respect to the degree (volume of the polytope). $D_{int} = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(f)} + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2}(e) = D_{int\ \mathbb{P}^1\times\mathbb{P}^1} + f + e$, where . Moreover,

$$B_1(') = D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(f)},$$

$$B_2(') = D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(f)} + D_{int\ \mathbb{P}^1\times\mathbb{P}^1},$$

$$B_3(') = D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(f)} + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(e)}.$$

So

$$\begin{aligned}\sum gB_g &= (0,0,0,1)(3D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(f)} + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(e)} + D_{int\ PP^1\times\mathbb{P}^1})\\ &= (0,0,0,1)D_{int}.\end{aligned}$$

Then we have

$$\begin{aligned}D_1(') &= B_1(') + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(e)},\\ D_2(') &= B_2(') + D_{int\ \mathrm{Bl}_1\,\mathbb{P}^2(f)},\\ D_3(') &= B_3(') + D_{int\ \mathbb{P}^1\times\mathbb{P}^1}.\\ K_X &= \pi^*(-D_{int} - D_{out} + 3f + e + \mathbb{P}^2\times\mathbb{P}^1 + D_{int})\\ &= 0.\end{aligned}$$

So X is a CY variety.

IVi: $B_{1,2}(') = \mathrm{Bl}_1\,\mathbb{P}^2$, $D_{int} = 2D_{int\ \mathbb{P}^2} + 2D_{int\ \mathbb{P}^1\times\mathbb{P}^1}$ In this case, there is no boundary branched locus since $\sum gB_g = 0$. And $B_i(') = \frac{1}{2}D_{int}$, then

$$K_X = \pi^*(-D_{int}).$$

$$K_X + \text{d.l.} = 0.$$

So (X, D) is a log CY pair.

IVj: $B_1(') = \mathrm{Bl}_2\,\mathbb{P}^2, B_{2,3}(') = \mathrm{Bl}_1\,\mathbb{P}^2$, $D_{int} = D_{int\ \mathbb{P}^2} + D_{int\ \mathbb{P}^1\times\mathbb{P}^1(e)} + 2D_{int\ \mathbb{P}^1\times\mathbb{P}^1(f)}$, where $D_{int\ \mathbb{P}^1\times\mathbb{P}^1(e)}$ is the $\mathbb{P}^1\times\mathbb{P}^1$ parallel to $\mathrm{Bl}_2\,\mathbb{P}^2$ outside and the other two interior $\mathbb{P}^1\times\mathbb{P}^1$'s are $D_{int\ \mathbb{P}^1\times\mathbb{P}^1(f)}$.

$$\sum gB_g = (0,0,0,1)B_1.$$

Since $B_1(') = D_{int}$, so

$$\begin{aligned} D_1(') &= B_1(') + D_{int}, \\ D_{2,3}(') &= B_{2,3}('). \\ K_X &= \pi^*(-D_{int} - 2\,\mathrm{Bl}_1\,\mathbb{P}^2 - \mathrm{Bl}_2\,\mathbb{P}^2 + D_{int} + 2\,\mathrm{Bl}_1\,\mathbb{P}^2 + \mathrm{Bl}_2\,\mathbb{P}^2) \\ &= 0. \end{aligned}$$

So X is a CY variety. □

6.2. Fibration Structures of Abelian Covers of Each Irreducible Component for M'_{CY}.

Definition 6.4. [DR14] A *toric fibration* is a flat surjective morphism $f : X \to Y$ with connectied fibers, where:

1. X is a toric variety;
2. Y is a normal variety;
3. $\dim X > \dim Y$.

Lemma 6.5. *([Ewa12]) Let $N \cong \mathbb{Z}^n$, $\Sigma \subset N \otimes \mathbb{R}$ be a fan, and $X = X_\Sigma$ be the toric variety associated to it. Let $\phi : N_0 \hookrightarrow N$ be a sub-lattice. Then ϕ induces a toric fibration $f : X \to Y$ iff:*

a) $(N_0 \otimes \mathbb{R}) \cap N = N_0$;

b) for every $\sigma \in \Sigma(n)$, $\sigma = \tau + \eta$, where $\tau \in N_0$ and $N_0 \cap \eta = \{0\}$.

Remark 6.6. The dual version of the lemma 6.4 in lattice M says that a lattice polytope has parallel faces of the same dimension iff the associated polarized toric variety has a toric fibration structure.

Proposition 6.7. *For the eight different toric varieties in remark 4.16, there are toric fibration structures for each type:*

1: 3 fibrations over $\mathbb{P}^1$*;*

2: no fibration;

3: 1 fibration over $\mathbb{P}^1$ *and 1 fibration over* $\mathbb{P}^2$*;*

4: 1 fibration over $\mathbb{P}^1$ *and 1 fibration over* $\mathbb{F}_1$*;*

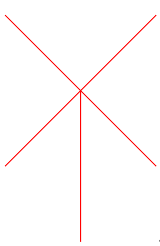
5: 1 fibration over $\mathbb{P}^2$*;*

6: 1 fibration over $\mathbb{P}^1$*;*

7: 1 fibration over $\mathbb{P}^1$ *and 1 fibration over* $\mathbb{F}_1$*;*

8: 2 fibrations over $\mathbb{P}^1$ *and fibration over* $\mathrm{Bl}_p(\mathbb{P}^1 \times \mathbb{P}^1)$.

Proof. By lemma 5.4, or the dual version, one can find parallel faces for each polytope. The parallel dimenison 1 faces give a fibration over $\mathbb{P}^1$. For parallel dimension 2 faces, the fibration is over the quotient normal fan by the dimension 1 linear subspace normal to the faces. For example, the quotient fan for 8 is


,

which is $\mathrm{Bl}_p(\mathbb{P}^1 \times \mathbb{P}^1)$. □

Lemma 6.8. *The fibers over dimension 2 bases are just* $\mathbb{P}^1$*'s. And the fibers over* $\mathbb{P}^1$*'s are:*

1: $\mathbb{P}^1 \times \mathbb{P}^1$*;*

3: $\mathbb{P}^2$*;*

4: $\mathbb{F}_1$*;*

6: $\mathbb{P}^2$*;*

7: $\mathbb{F}_1$*;*

8: general fibers $\mathbb{F}_1$ *and a special fiber* $\mathbb{P}^2 \cup (\mathbb{P}^1 \times \mathbb{P}^1)$*;*

Proof. For each polytope in proposition 5.6 which has a fibration structure, look at the parallel $\mathbb{P}^1$'s. Take a generic point on each such a $\mathbb{P}^1$ such that they form a polytope. Then each polytope gives a toric fiber as in the lemma. □

The branch loci D restricted to each fiber are listed below:

1: $2h_1 + 2h_2$;

3: $4h$;

4: $2(e + f) + 2f$;

6: $4h$;

7: $2(e + f) + 2f$;

8: general fiber $2(e + f) + 2f$, special fiber $2h_1 + 2h_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $2h$ on $\mathbb{P}^2$.

By computing the self-intersection numbers of the canonical sheaves, one has:

Proposition 6.9. *([Ale15] 6.2.1) For the family* M'_{CY}*, the double covers of each fiber branch along the restriction of* D *above are:*

1: dP surface of degree 4;

3,6: dP surface of degree 2;

4,7: a partial resolution of an A_1 *singularity on a dP surface of degree 2;*

8: general fiber: a partial resolution of an A_1 *singularity on a dP surface of degree 2; special fiber: one component dP surface of degree 4 and one component dP surface of degree 8 with one* A_1 *singularity.*

Part 2. Chow Quotients of Grassmannians by Diagonal Subtori

7. An Overview of Full Torus and Diagonal Subtorus Quotient

Some results from [Kap93a] and its generalization in higher dimension have been discussed in the previous chapters.

In particular, the compactified moduli of pointed rational curve $\overline{M}_{0,n}$ can be described by the Chow quotient, see 8.1.

Theorem 7.1. [Kap93a] *There is an isomorphism* $\overline{M}_{0,n} \cong Gr(2,n)/\!\!/_{Ch}T$*, where* $T = (\mathbb{C}^*)^n/diag(\mathbb{C}^*)$.

Instead of generalizing to hyperplane arrangements as in [HKT06] and [Ale15], one can also consider the linear maps from different spaces to a fixed projective space $\mathbb{P}^{k-1}$. Thus, the matrix which maps points to a projective space is replaced by a matrix with m different column blocks $M_{k\times r_i}$, each block giving a linear map. In this case, the full torus action for each column is replaced by a single $\mathbb{C}^*$ action on each block. So we consider the following subtorus action on the matrices/Grassmannians.

Definition 7.2. A *diagonal subtorus* $S \subset T$ has the following form:

$$S = \{(\underbrace{t_1,\dots,t_1}_{r_1}, \underbrace{t_2,\dots,t_2}_{r_2}, \dots, \underbrace{t_m,\dots,t_m}_{r_m}) \mid t_i \in \mathbb{C}^*\} \subseteq T,$$

where $\sum r_i = n$.

The the diagonal subtorus quotient $\mathrm{Gr}(k,n)/S$ is an open moduli of linear maps. To compactify this moduli, we apply Kapranov's Chow quotient. In particular, when $r_1 = \cdots = r_{m-1} = 1$, this Chow quotient gives a birational model to the space

[CGK09] and they are not isomorphic. This non-trivial assertion is proven in upcoming joint work with Noah Giansiracusa using results established in this thesis.

8. Discrete Polymatroid

Let $[n]$ be the finite set $\{1, \cdots, n\}$. We recall the three equivalent definitions of discrete polymatroid.

Definition 8.1. [HH02] A *discrete polymatroid* on the ground set $E = [n] = \{1, \cdots, n\}$ is a nonempty finite set $P \subset \mathbb{Z}_+^n$ satisfying:

D1) if $u \in P$ and $\mathbb{Z}_+^n \ni v \leqslant u$ (coordinatewisely), then $v \in P$;

D2) if $u, v \in P$ and $\sum_{i=1}^n u_i < \sum_{i=1}^n v_i$, then there exists u_i and v_i such that $u_i < v_i$ and $u_i + e_i \in P$.

Definition 8.2. [HH02] A *discrete polymatroid* on $[n]$ is a nonempty compact subset $\mathcal{P} \subset \mathbb{Z}_+^n$ of *independent vectors*, such that

D1') every integral subvector of an independent vector is independent;

D2') if $u, v \in \mathcal{P}$ with $|v| > |u|$, then there is a vector $w \in \mathcal{P}$ such that

$$u < w \leqslant u \vee v,$$

where $u \vee v = (\max\{u_1, v_1\}, \cdots, \max\{u_n, v_n\})$.

Definition 8.3. [HH02] A *discrete polymatroid* is a pair $([n], \mathrm{rank})$ such that the rank function $\mathrm{rank} : 2^{[n]} \to \mathbb{Z}_+$ satisfying

D1") (nondecreasing) $A \subseteq B \subseteq [n] \Rightarrow \mathrm{rank}(A) \leqslant \mathrm{rank}(B)$;

D2") (submodularity) $A, B \subseteq [n] \Rightarrow \mathrm{rank}(A) + \mathrm{rank}(B) \geqslant \mathrm{rank}(A \cap B) + \mathrm{rank}(A \cup B)$;

D3") $\mathrm{rank}(\emptyset) = 0$.

The discrete polymatroid is a *matroid* if it satisfies

D4") $\mathrm{rank}(A) \leqslant |A|, \forall A \subset [n]$.

Remark 8.4. In this paper, we also consider the case when the base set is a muli-set. Let A be a multi-set $[m]$ with i repeated k_i times. Then condition D4" above will be replaced by:

MD4") $\text{rank}(A) \leqslant |A|_{mult}$, where $|A|_{mult} = \sum_{i\in A} k_i$ is the cardinality of A with multiplicity. In this paper, we always assume that $k_i \leqslant k-1$ for all $i = 1, \cdots, m$.

Definition 8.5. For a polymatroid $([n], \text{rank})$, the *base polytope* associated to it is defined by

$$P_{base} = \{x \in \mathbb{R}^n : \sum_{e\in A} x_e \leqslant \text{rank}(A), \forall A \in 2^{[n]}, \sum_{i=1}^{n} x_i = \text{rank}([n])\},$$

i.e. the convex hull of vectors with maximized rank.

Proposition 8.6. *A lattice polytope P is a polymatroid base polytope if and only if the coordinates of its vertices are positive integers and the edges of P are parallel to $e_i - e_j$ for $i, j \in [n]$.*

Proof: Fink's notes.

Polymatroids can be interpreted as matroids on a multiset, we have the following lemma:

Lemma 8.7. *On a ground set $[m]$, there is a bijection between*

$$\left\{ \text{ polymatroids } \right\} \leftrightarrow \left\{ \text{ multiset matroids } \right\}.$$

Proof. let $\text{rank}_{poly}(A) = \text{rank}_{mult}(A)$, then rank_{poly} has the same submodularity with rank_{mult}. On the other side, let $k_i = \text{rank}_{poly}(i)$, by submodularity on has $\text{rank}_{mult}(A) = \text{rank}_{poly}(A) \leqslant \sum_{i\in A} k_i$, so condition MD4" is satisfied. □

Remark 8.8. For a matroid $([n], \text{rank})$, $n = \sum_{i=1}^{m} k_i$, there is a map as following:

$$\phi : [n] \to [m],$$
$$\{1, \cdots, k_1\} \mapsto 1,$$
$$\cdots$$
$$\{\sum_{i=1}^{m-1} k_i + 1, \cdots, n\} \mapsto m.$$

Then ϕ induces a polymatroid structure on $[m]$ according to the lemma above. The rank function on $[m]$ is $\text{rank}(A) = \text{rank}(\phi^{-1}(A)), A \subseteq [m]$.

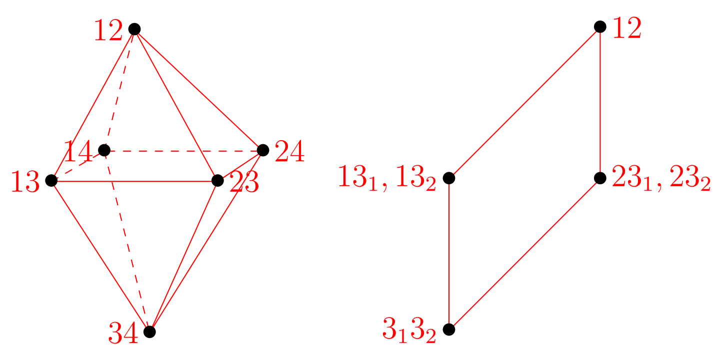
There is a moment map associated to the full torus $T = (\mathbb{C}^*)^n$:

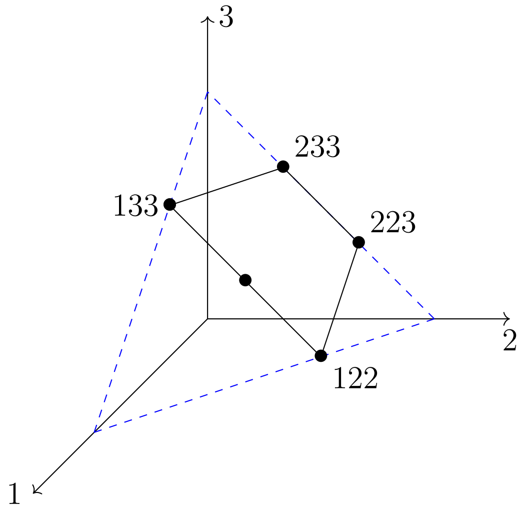
$$\mu_T : \mathbb{P}^N \to \mathbb{R}^n, N = \binom{n}{k} - 1$$
$$L \mapsto \frac{\sum_{|I|=k} p_I(L) e_I}{\sum_{|I|=k} p_I(L)}, \forall L \in \mathbb{P}^N.$$

The image of $\text{Gr}(k, n)$ under μ_T is the hypersimplex $\Delta(k, n)$

For the diagonal subtorus S, let $\iota : S \to T$ be the diagonal embedding and M_ι be the corresponding matrix. Then the moment map associated to S is $\mu_S = \iota^t \circ \mu_T := M_\iota^t \circ \mu_T$.

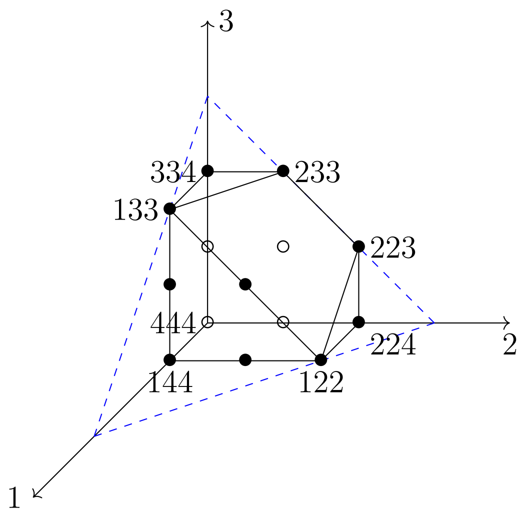
Example 8.9. Consider the set $E = \{1, 2, 3, 4\}$ and the multi-set $\tilde{E} = \{1, 2, 3_1, 3_2\}$, the matroid the map is $1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3_1, 4 \mapsto 3_2$. The matroid base polytope for (E, rank) is the hypersimplex $\Delta(2, 4)$ and the polymatroid base polytope for $(\tilde{E}, \text{rank}_m$ is the parallelogram, whose two vertices on the short diagonal with multiplicity 2.





Example 8.10.

$E = \{1, 2, 2, 3, 3\}$, $n = 5$, $m = 3$, $k = 3$.



Example 8.11.

$E = \{1, 2, 2, 3, 3, 4, 4, 4\}$, $n = 8$, $m = 4$, $k = 3$.

A *subspace arrangement* is a collection of subspaces $\mathcal{V} = \{V_1, \cdots, V_n\}$ of some vector space of finite dimension. The polymatroid associated to such an arrangement is given by $([n], \mathrm{rank}_{\mathcal{V}})$ where

$$\mathrm{rank}_{\mathcal{V}}(A) = \dim(\sum_{i \in A} V_i).$$

Definition 8.12. A polymatroid is *realizable* over a field k if it is isomorphic to $([n], \mathrm{rank}_{\mathcal{V}})$ for some k-subspace arrangement $\mathcal{V}$.

Definition 8.13. A polymatroid is *uniform* if its base polytope is the image of some hypersimplex under the map ϕ.

Proposition 8.14. *For generic* $L \in Gr(k,n)$*, one has* $\mu_S(\overline{S \cdot L}) = \iota^t \Delta(k,n)$*, which is a realizable uniform polymatroid polytope.*

Proof. This is a corollary of lemma 3.4. $\square$

9. Degenerations in the Chow Quotient by Diagonal Subtori

9.1. **Degenerations of Cycles.** Fix a homology class on a polarized projective variety $X \subseteq \mathbb{P}^M$, i.e. fix the dimension n and degree d, algebraic cycles in this class can be parametrized by a projective variety $\mathrm{Chow}(X, n, d)$ (see [GKZ08] chapter 4), which is called the *Chow variety of X of dimension n and degree d.*

Definition 9.1. [Kap93b] Let G be a group (maybe not reductive) acting on a projective X, then there is an open subset $U \subseteq X$ so that $\overline{G \cdot x}$ for $x \in U$ are in the same homology class with fixed dimension n and degree d. The *Chow quotient* $X /\!\!/_{Ch} G$ is defined as the Zariski closure of U/G in the Chow variety $\mathrm{Chow}(X, n, d)$.

Remark 9.2. a). By flatness stratification, the non-empty open subset U in the definition exists and the Chow quotient does not depend on the choice of U. In the case of a torus (or subtorus) acting on a Grassmannian, one may choose $U = \mathrm{Gr}(d, n)^\circ$, where $L \in \mathrm{Gr}(d, n)^\circ$ if the Plučker image of L has no zero coordinates.

b). In some literature people use the inverse limit of GIT quotients as the definition of Chow quotient, Kapranov's Chow quotient is isomorphic to the main component when the quotient is taken for toric varieties, extra components correpond to non-regular polytope subdivisions, see section 4 in [KSZ91]. For general non-toric cases, they are birational and homeomorphic under some restrictions, see theorem 3.8 in [Hu05].

A lattice polytope will be denoted by $\mathrm{Supp}(P)$, where P is a lattice polytope together with multiple vertices.

Using Plücker embedding $pl : X \hookrightarrow \mathbb{P}^N$, we know $X /\!\!/_{Ch} S \subseteq \mathbb{P}^N /\!\!/_{Ch} S$, the latter is the polarized toric variety corresponding to the fiber polytope $\Sigma(\Delta_N, \mathrm{Supp}(\iota^T(\Delta(d, n))))$.

As for the full torus case, cycles over the boundary of Chow quotient are still reduced for subtorus.

Lemma 9.3. *For cycles $Z = \sum c_i Z_i \in Gr(d,n)/\!/_{Ch} S$, the coefficients c_i's are either* 0 *or* 1*.*

Proof. For $Z_i = \overline{S \cdot L}$, let Γ_{Z_i} be the sublattice of $M_H = Hom(H, \mathbb{Z}) \cong \mathbb{Z}^m/diag(\mathbb{Z})$ generated by all lattice points on $\mu(Z_i)$, then $c_i = [M_S : \Gamma_{Z_i}]$. Since the projection from M_T to M_S is given by the matrix Mat_S^t, each row has only one 1 and other elements are all 0's, so the generaters go to either the image of the vertex or still generaters, i.e. $\Gamma_{Z_i} = M$, so $c_i = 1$. □

The convex theorem due to [Ati82],[GS82] gives that the image $\mu_S(\overline{S \cdot L})$ is exact the convex hull of $\iota^t(Q)$, where Q varies over all fixed components in $\overline{S \cdot L}$ of S action. Since $\overline{S \cdot L}$ is projective and toric, the fixed points of S action correspond to vertices of the polytope associated to $\overline{S \cdot L}$ with the Plücker polarization.

We say a k-subset $(i_1, \cdots, i_m), \sum i_l = k, i_l \geqslant 0$ of the multi-set $([m], k_1, \cdots, k_m)$ is *extremal* if $(i_1, \cdots, i_m)$ cannot written as the average coordinates of any two other k-subsets.

Lemma 9.4. *Let $L \in Gr(k,n)$, then the fixed points of S which lie in $\overline{S \cdot L}$ are precisely Plücker coordinates J such that J is a basis of the matroid $M(L)$ and extremal in non-zero subindices.*

Proof. Since $\overline{S \cdot L} \subseteq \overline{T \cdot L}$, by lemma 1.4 in [GGMS87], limit points in $\overline{S \cdot L}$ must be a subset of non-zero Plücker coordinates of $pl(L)$. If the subindex J of one Plücker coordinate x_J is extremal, then at most one k_i is not maximized. Without lost of generality, let $J = \{1\} \cup J_1$ and 1 is not maximized with multiplicity s_1. First, let $\lambda_j \to 0$ for $j \notin \{1\} \cup J_1$. Then the rest nonzero coordinates have only subindices in $\{1\} \cup J_1$, and 1 must be maximized in them. So one may divide by $\lambda_1^{s_1}$ then let $\lambda_1 \to 0$. Thus, the only non-zero coordinate is $J = \{1\} \cup J_1$. So extremal coordinate gives a limit point. On the other side, let x_J be a non-extremal coordinate, then $x_J \to 0$ when $\lambda_j \to 0$ for $j \in J$. $\square$

Remark 9.5. Remember that for matroid case, fixed points correspond to bases of the matroid. This fact fails in our polymatroid case. The bases of the polymatroid are all integral points on $\iota^t\Delta(k,n)$, but only extremal ones are from the fixed points.

Example 9.6. Consider $S = (\mathbb{C}^*)^2/diag$ acting on $\mathrm{Gr}(2,4)$ via

$$(\lambda_1, \lambda_2) \cdot (c_1\ c_2\ c_3\ c_4) = (\lambda_1 c_1\ \lambda_1 c_2\ \lambda_2 c_3\ \lambda_2 c_4).$$

And choose a point

$$L = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathrm{Gr}(2,4).$$

The basis of $M(L)$ are $\{12, 14, 23, 24\}$, i.e. the Plücker coordinates of L are $[1 : 0 : 1 : -1 : 0 : 1] \in \mathbb{P}^5$. S rescales L in $\mathbb{P}^5$ by $[\lambda_1^2 : 0 : \lambda_1\lambda_2 : -\lambda_1\lambda_2 : 0 : \lambda_2^2]$. But only $\{12, 34\}$ are extremal. The fixed points in $\overline{S \cdot L}$ are $[1 : 0 : 0 : 0 : 0 : 0]$ when $\lambda_1 \to 0$, and $[0 : 0 : 0 : 0 : 0 : 1]$ when $\lambda_2 \to 0$. Let the corresponding vertices in $\Delta(2,4)$ be $Q_1 = (1,1,0,0)$ and $Q_2 = (0,0,1,1)$. Thus $\iota^t(Q_1) = (2,0)$, $\iota^t(Q_2) = (0,2)$, so the orbit closure is a degree 2 rational curve in $\mathbb{P}^5$.

When

$$L' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \in \mathrm{Gr}(2,4)$$

with the same subtorus action, $(\lambda_1, \lambda_2) \cdot [1 : 1 : 0 : 0 : 0 : 0] = [\lambda_1^2 : \lambda_1\lambda_2 : 0 : 0 : 0 : 0]$. Two limit points are $[1 : 0 : 0 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0 : 0 : 0]$. The corresponding vertices are mapped to $(2,0)$ and $(1,1)$ under ι^t. So $\overline{S \cdot L'}$ in this case corresponds to a subpolytope of $\iota^t\Delta(2,4) = Conv\{(2,0),(0,2)\}$, which is a degree 1 rational curve in $\mathbb{P}^5$.

If $S = (\mathbb{C}^*)^3/diag$ and the action is

$$(\lambda_1, \lambda_2, \lambda_3) \cdot (c_1\ c_2\ c_3\ c_4) = (\lambda_1 c_1\ \lambda_2 c_2\ \lambda_3 c_3\ \lambda_3 c_4),$$

then $(\lambda_1, \lambda_2, \lambda_3) \cdot L = [\lambda_1\lambda_2 : 0 : \lambda_1\lambda_3 : -\lambda_2\lambda_3 : 0 : \lambda_3^2]$, the limit points in $\overline{S \cdot L}$ are $[0 : 0 : 0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0 : 0 : 0]$, $[0 : 0 : 0 : 0 : 0 : 1]$ and $[1 : 0 : 0 : 0 : 0 : 0]$. The corresponding vertices in $\Delta(2,4)$ are $Q_1 = (0,1,1,0)$, $Q_2 = (1,0,0,1)$, $Q_3 = (0,0,1,1)$ and $Q_4 = (1,1,0,0)$. Their images under ι^t are $(0,1,1)$, $(1,0,1)$, $(0,0,2)$ and $(1,1,0)$. So $\overline{S \cdot L}$ in this case is $\mathbb{P}^1 \times \mathbb{P}^1$ polarized by $\mathcal{O}(1,1)$.

Lemma 9.7. *The image of μ_S of each irreducible component Z_i of $Z = \sum Z_i \in Gr(d,n)/\!\!/_{Ch}S$ is a $(m-1)$-dimensional polymatroid base polytope contained in $\iota^t\Delta(d,n)$.*

Proof. By the convexity theorem, the image of Z_i under μ_S is a $(m-1)$ dimensional polytope. □

Theorem 9.8. *Let $Z = \sum Z_i \in Gr(d,n)/\!\!/_{Ch}S$, then $\mu_S(Z_i)$ form a realizable polymatroid polytope subdivision of $\iota^t\Delta(d,n)$.*

Proof. By proposition 3.6 in [KSZ91], for any cycles in the toric Chow quotient $Z = \sum Z_i = \sum \overline{S \cdot L} \in \mathbb{P}^N/\!\!/_{Ch}S$, $\bigcup \mu_S(Z_i)$ form a subdivision of $\mathrm{Supp}(\iota^T(\Delta(d,n))$. The image of the moment map μ_S is still $\mathrm{Supp}(\iota^T(\Delta(d,n))$ when μ_S is restricted to $\mathrm{Gr}(d,n)$, so for $Z = \sum Z_i \in \mathrm{Gr}(d,n)/\!\!/_{Ch}S$, $\bigcup \mu_S(Z_i$ is also a subdivision of $\mathrm{Supp}(\iota^T(\Delta(d,n))$. □

9.2. **Chow Strata.** In this subsection we give two versions of strata for S acting on $\mathrm{Gr}(k,n)$ and then show that they are exact the same stratification, which generalizes the results in [GGMS87] 2.4.

Definition 9.9. Two points $L_1, L_2 \in \mathrm{Gr}(d,n)$ are said to lie in the same *Grassmann stratum* if

$$\dim_{\mathbb{C}} \mathrm{Span}\{\pi_{L_1}(\mathrm{Span}\{e_j\}, j \in K_i)\} = \dim_{\mathbb{C}} \mathrm{Span}\{\pi_{L_2}(\mathrm{Span}\{e_j\}, j \in K_i)\}, \forall i = 1, \cdots, m.$$

Definition 9.10. Let $Z_1 = \overline{S \cdot L_1}, Z_2 = \overline{S \cdot L_2}$, L_1 and L_2 are *polymatroid equivalent* if $\mu_S(\overline{S \cdot L_1}) = \mu_S(\overline{S \cdot L_2})$.

Theorem 9.11. *Definition 3.6 and 3.7 are equivalent, the strata is called* Chow Strata *following Kapranov.*

Proof. If L_1, L_2 are in the same Grassmann stratum, then they give the same polymatroid on $[m]$. By lemma 3.4, the extremal bases give the vertices of images under μ_S, so $\mu_S(L_1) = \mu_S(L_2)$. On the other side, if $\mu_S(L_1) = \mu_S(L_2)$, then extremal mult-subsets on $[m]$ are the same. Thus, the polymatroids on $[m]$ are uniquely determined. □

The Chow strata is parametrized by the secondary polytope, which is a special case of fiber polytopes. The *fiber polytope* $\Sigma(P \xrightarrow{\pi} Q)$ associated to projection of lattice polytopes $\pi : P \to Q$ is introduced in [BS92]. We briefly review their work: Let X_P be the polarized toric variety associated to P, the action by the small torus S on X_P is encoded in the projection $\pi_S : P \to \pi_S(P) =: Q \subseteq \mathbb{R}^{\dim S}$. Then $\Sigma(P \xrightarrow{\pi} Q)$ gives a new polarized toric variety, which is exact the Chow quotient $X_P /\!/_{Ch} S$ [KSZ91]. A special case of fiber polytope is that when $P = \Delta_n$ and $Q = Conv(\mathcal{A})$ where $\mathcal{A} = \{\pi(e_i), i = 0, \cdots, n\}$, $\Sigma(P \xrightarrow{\pi} Q)$ coincides with the secondary polytope of $(Q, \mathcal{A})$ [GKZ08], which parametrizes all regular subdivisions of Q.

10. Main Examples

We compute the equations of $\mathrm{Gr}(2,4)/\!\!/_{Ch}S$ for two different diagonal subtori of T.

Example 10.1. In this example we look at the diagonal subtorus acting on $\mathrm{Gr}(2,4)$ by rescaling the last two columns with a single $\mathbb{C}^*$. The Plücker embedding maps to $\mathrm{Gr}(2,4)$ to $\mathbb{P}^5$, so there is an induced H action on the $\mathbb{P}^5$. According to KSZ, $\mathbb{P}^5/\!\!/_{Ch}H$ is the polarized toric variety corresponding to the secondary polytope $\Sigma(\Delta_5, Q)$, where Q is the projection of $\Delta(2,4)$ under the transposition of the diagonal embedding, precisely $Q = Conv\{(1,1,0),(1,0,1),(0,1,1),(0,0,2)\}$. Thus, we have a pair (Q, A), where A is the multiset of vertices of Q counting $\{(1,1,0),(0,0,2)\}$ once and $\{(1,0,1),(0,1,1)\}$ twice.

By the construction of secondary polytope, we get eight vectors in $\mathbb{R}^6$ which are vertices of $\Sigma(\Delta_5, Q)$:

$$a = (1,2,0,2,0,1),$$

$$b = (1,2,0,0,2,1),$$

$$c = (1,0,2,0,2,1),$$

$$d = (1,0,2,2,0,1),$$

$$a' = (2,1,0,1,0,2),$$

$$b' = (2,1,0,0,1,2),$$

$$c' = (2,0,1,0,1,2),$$

$$d' = (2,0,1,1,0,2).$$

The matrix $A = (a^t, b^t, c^t, d^t, a'^t, b'^t, c'^t, d'^t)$ has rank 4. So $\dim(\Sigma(\Delta_5, Q) = 4 - 1 = 3$. The relations between these vectors are

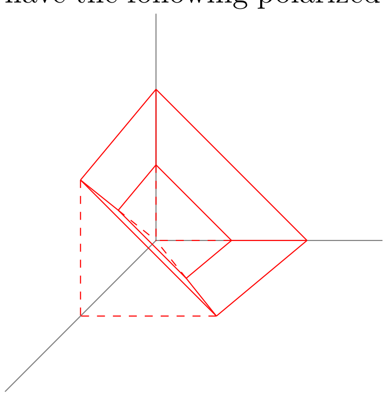
$$a + c = b + d, \tag{7}$$

$$a' + c' = b' + d', \tag{8}$$

$$2a' + b = 2b' + a, \tag{9}$$

$$2c' + d = 2d' + c. \tag{10}$$

Moreover, there are five lattice points lying on $Conv\{a, b, c, d\}$, which are mid-points of ab, bc, cd, da and ac. And there is no other lattice points on $Conv\{a', b', c', d'\}$. The third and fourth relations are saying that $aba'b'$ and $cdc'd'$ form two trapezoids. So we have the following polarized toric picture (cutting the top corner off from a pyramid):



The polarized toric variety is

$$(Bl_1 Proj(\frac{\mathbb{C}[x, y, z, w, u]}{yw - zu}), \pi^*\mathcal{O}(2) \otimes \mathcal{O}(-E)).$$

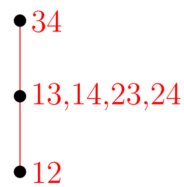
For the quotient of the Grassmannian, $\dim \mathrm{Gr}(2, 4)/\!\!/_{Ch} H = 2(4-2) - (3-1) = 2$. Let $A, B, C, D, A', B', C', D'$ be the corresponding monomials of $a, b, c, d, a', b', c'.d'$, multiply the Plücker relation $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$ by $x_{12}x_{34}x_{13}x_{24}$ and $x_{12}x_{34}x_{14}x_{23}$, we get relations

$$B' - B + \Pi = 0,$$

$$D' - \Pi + D = 0,$$

where $\Pi = x_{12}x_{34}x_{13}x_{24}x_{14}x_{23}$. So we have $B' - B + D' + D = 0$. On the polarized toric variety above, there is a non-toric linear relation between the monomials.

Example 10.2. In this example we look at $H = (\mathbb{C}^*)^2/diag(\mathbb{C}^*)$ acting on $\mathrm{Gr}(2,4)$ by recaling the first two columns with $\lambda_1 \in \mathbb{C}^*$ and the last two columns with $\lambda_2 \in \mathbb{C}^*$. Then the projected polytope Q is a segment with the midpoint repeated four times.



The vertices of $\Sigma(\Delta_5, Q)$ are:

$$a = (1, 2, 0, 0, 0, 1),$$

$$b = (1, 0, 2, 0, 0, 1),$$

$$c = (1, 0, 0, 2, 0, 1),$$

$$d = (1, 0, 0, 0, 2, 1),$$

$$v = (2, 0, 0, 0, 0, 2).$$

We can see that a, b, c, d give a polarized variety $(\mathbb{P}^3, \mathcal{O}(2))$. And taking the cone from v to a, b, c, d, each facet is a singular, e.g. on the facet v, a, b, c:

$$u_1 = a - v = (-1, 2, 0, 0, 0, -1),$$

$$u_2 = b - v = (-1, 0, 2, 0, 0, -1),$$

$$u_3 = c - v = (-1, 0, 0, 2, 0, -1).$$

u_1, u_2 cannot ($\mathbb{Z}$-)generate their mid-point. And the mid-points of each pair are the only integral points we can get.

The polarized toric variety is $(Proj(u, w^2, x^2, y^2, z^2, wx, wy, wz, xy, xz, yz), \mathcal{O}(1))$.

For the quotient of the Grassmannian, let

$$A = x_{12}x_{13}^2x_{34},$$

$$B = x_{12}x_{14}^2x_{34},$$

$$C = x_{12}x_{23}^2x_{34},$$

$$D = x_{12}x_{24}^2x_{34},$$

$$V = x_{12}^2x_{34}^2.$$

We have $x_{12}x_{34} = \sqrt{V}, x_{13}x_{24} = \sqrt{\frac{AD}{V}}, x_{14}x_{23} = \sqrt{\frac{BC}{V}}$. Plug in the Plücker relation $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$, that is

$$\sqrt{V} - \sqrt{\frac{AD}{V}} + \sqrt{\frac{BC}{V}} = 0,$$

i.e. $(V^2 - (AD + BC))^2 = 4ABCD$.

11. Intersection and Projection Maps

11.1. **Maps to Quotients of Small Grassmannians.** Recall that there is a bijection between polarized toric varieties and convex lattice polytopes in the character lattice $M_{\mathbb{R}} \cong \mathbb{R}^m$, $(X, L) \leftrightarrow P$, and the lattice points in P correspond to sections in $\Gamma(X, L)$. Moreover, we know that $\dim X = \dim P$, and $\deg(X) = \mathrm{Vol}(P)$, where Vol is the normalized volume, see [Stu96] theorem 4.16. In this section, degree means the degree under the Plücker embedding $\mathrm{Gr}(k, n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$.

Let

$$S_i = \begin{cases} S & \text{if } l_i < k_i, \\ S/\mathbb{C}_i^* & \text{if } l_i = k_i \end{cases}$$

for $i = 1, \cdots, m$. By $H/\mathbb{C}_i^*$ we mean that remove all λ_i's from the torus action.

On the generic locus of Grassmannians, there are two maps $A_i : \mathrm{Gr}^\circ(k, n) \to Gr^\circ(k - l_i, n - l_i)$ and $B_i : \mathrm{Gr}^\circ(k, n) \to Gr^\circ(k, n - l_i)$. Let $J_{\Lambda_i} : \mathbb{C}^{n-l_i} \hookrightarrow \mathbb{C}^n$ take $(x_1, \cdots, x_{n-l_i})$ to the point in $\mathbb{C}^n$ by adding 0's on the coordinates indexed by Λ_i, then for $L \in \mathrm{Gr}^\circ(k, n)$, $A_i(L) = J_{\Lambda_i}^{-1}(L)$, $B_i(L)$ is just removing the coordinates from Λ_i.

By the argument in [BMS+87], A_i, B_i descends to morphisms of the diagonal torus quotients:

$$a_i : \mathrm{Gr}^\circ(k, n)/S \to \mathrm{Gr}^\circ(k - l_i, n - l_i)/S_i$$

$$\overline{S \cdot L} \mapsto \overline{S_i \cdot A_i(L)},$$

$$b_i : \mathrm{Gr}^\circ(k, n)/H \to \mathrm{Gr}^\circ(k, n - l_i)/S_i$$

$$\overline{S \cdot L} \mapsto \overline{S_i \cdot B_i(L)}.$$

Let $\{e_i\}_{i=1}^m$ be the standard basis of $\mathbb{R}^m$, there is a hyperplane $F_{l_i} = \{p | e_i \cdot \overrightarrow{(0,p)} = l_i\}$ for each integer l_i, $0 \leqslant l_i \leqslant k_i$, which cuts $\mathbb{R}^m$ into two half spaces $F_{l_i}^+ = \{p | e_i \cdot \overrightarrow{(0,p)} \geqslant l_i\}$ and $F_{l_i}^- = \{p | e_i \cdot \overrightarrow{(0,p)} \leqslant l_i\}$.

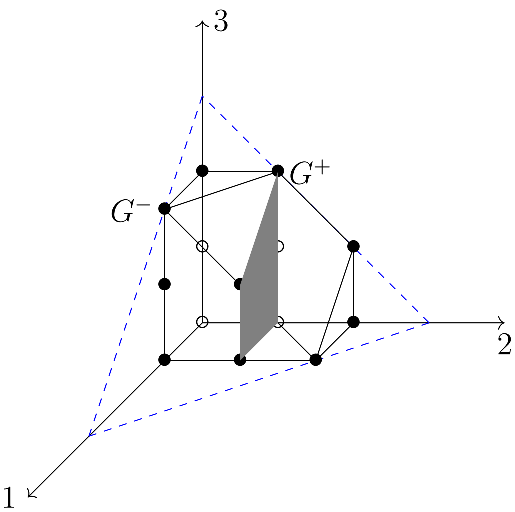
For a fixed l_i, define two subvarieties of $\mathrm{Gr}(k,n)$ as:

$$G_{l_i}^+ = \{V \in \mathrm{Gr}(k,n) | \#\{j \in \Lambda_i, V \ni e_j\} \geqslant l_i\},$$

$$G_{l_i}^- = \{V \in \mathrm{Gr}(k,n) | \#\{j \in \Lambda_i, V \ni e_j\} \leqslant l_i\}.$$

It is clear that $G_{l_i}^+ \cup G_{l_i}^- = \mathrm{Gr}(k,n)$ and $G_{l_i}^+ \cup G_{l_i}^- = F_{l_i}$, so we have:

Lemma 11.1. *The images of $G_{l_i}^+$ and $G_{l_i}^-$ under μ_S are $\delta_S(k,n) \cap F_{l_i}^+$ and $\delta_S(k,n) \cap F_{l_i}^-$.*



Example 11.2.

For the discrete polymatroid in example 6.12, and $l_2 = 1$.

From the previous section we know that for a Lie complex $Z = \overline{S \cdot L} \in \mathrm{Gr}^\circ(k,n)/S$,

Let $Z = \sum Z_i = \sum \overline{S \cdot L_i} \in \mathrm{Gr}(k,n) /\!\!/_{Ch} S$ be generalized Lie complex, then $\mu_S(Z)$ gives a regular matroid polytope decomposition of $\delta_S(k,n)$. This subdivision induces two subdivisions on $\delta_S(k,n) \cap G_{l_i}^+$ and $\delta_S(S,n) \cap G_{l_i}^-$ by restriction. For $\delta_S(k,n) \cap G_{l_i}^+$, since we only use a subset of the vertices of the $\delta_S(k,n)$, and the edges on F_{l_i} are still parallel to $e_j - e_k$ because ???, the subdivision on $\delta_S(k,n) \cap G_{l_i}^+$ must be polymatroid subdivision. Moreover, is also regular. So we have a regular polymatroid subdivision of $\delta_S(k,n) \cap G_{l_i}^+$. The same argument also holds for $\delta_S(k,n) \cap G_{l_i}$.

Theorem 11.3. *For any integer l_i, $1 \leqslant l_i \leqslant k_i$, there exist two morphisms:*
a) $\overline{a_i} : Gr(k,n)/\!\!/_{Ch} S \to Gr(k-l_i, n-l_i)/\!\!/_{Ch} S_i$;
b) $\overline{b_i} : Gr(k,n)/\!\!/_{Ch} S \to Gr(k, n-l_i)/\!\!/_{Ch} S_i$.

We use the following lemma to extend a_i, b_i to the boundary of Chow quotients. Let $(A, \mathfrak{m})$ be a discrete valuation ring, K be the fraction field of A, and X be some proper scheme. By valuation criterion for properness, any morphism $g : \mathrm{Spec}(K) \to X$ can be always extended to $\overline{g} : \mathrm{Spec}(A) \to X$. Denote $g(\mathfrak{m}) \in X$ by $\lim_{t\to\mathfrak{m}} g$.

Lemma 11.4. [GG14] *Suppose X, Y are proper schemes over a noetherian scheme S with X normal. Let $U \subseteq X$ be an open dense set and $f : U \to Y$ an S-morphism. Then f extends to an S-morphism $f : X \to Y$ if, and only if, for any DVR $(A, \mathfrak{m})$ as above and any morphism $g : \mathrm{Spec}(K) \to U$, the point $\lim_{t\to\mathfrak{m}}(fg)$ of Y is uniquely determined by the point $\lim_{t\to\mathfrak{m}} g$ of X.*

Note that for example $\mathbb{P}^2 \backslash \{0\} \hookrightarrow \mathrm{Bl}_{pt}\,\mathbb{P}^2$ does not satisfy the condition of the lemma.

Proof of theorem 2: a) According to lemma 6, we only have to show that the limit point in $\mathrm{Gr}(k-l_i, n-l_i)/\!\!/_{Ch} S_i$ does not depend on how $\mathrm{Spec}(K)$ maps into $\mathrm{Gr}(k,n)/\!\!/_{Ch} S$. Choose a non-generic point $Z = \sum \overline{S \cdot L_j} \in \mathrm{Gr}(k,n)/\!\!/_{Ch} S$, we expect that on $\mathrm{Gr}(k-l_i, n-l_i)/\!\!/_{Ch} S_i$ the special fiber over $\mathrm{Spec}(R)$ should be exact $[Z] \cap G_{l_i}^+$, which is equivalent to $\lim_{t\to\mathfrak{m}}(Z_t \cap G_{l_i}^+) = (\lim_{t\to\mathfrak{m}} Z_t) \cap G_{l_i}^+$ for $t \in \mathrm{Spec}(K)$. We prove the equality by showing that the cycles on two sides are bijective, in particular, they have the same degree under Plücker embedding.

On one hand, the limit of intersection is always a subset of intersection of limit. So we have $\lim_{t\to\mathfrak{m}}(Z_t \cap G_{l_i}^+) \subseteq (\lim_{t\to\mathfrak{m}} Z_t) \cap G_{l_i}^+$.

On the other hand, for any special fiber $Z = \sum_j \overline{S \cdot L_j}$, by intersecting with $G_{l_i}^+$. We have $Z \cap G_{l_i}^+ = \sum_j \overline{S_i \cdot L_j}$, maybe some cycles disappear, but we still have $\mu_S(Z \cap$

G_i^+) forming a polymatroid polytope decomposition of $\mu_H(G_{l_i}^+) = \delta_S(k,n) \cap F_{l_i}^+$. $\deg(Z \cap G_{l_i}^+) = \sum_j \deg(\overline{S_i \cdot L_j}) = \sum \mathrm{Vol}(\mu_S(\overline{S_i \cdot L_j})) = \mathrm{Vol}(\delta_S(k-l_i, n-l_i))$.

The proof for b) is the same but restricting to $\mu_H(G_{l_i}^-)$.

In particular, when the torus is of full dimension $S = T$, the hypersimplex $\Delta(k,n)$ has $2n$ facets: $\{x_i = 1\}_{i=1}^n$ and $\{x_i = 0\}_{i=1}^n$ [Kap93b] proposition 1.2.5. Under the moment map μ_T, they are images of the following subvarieties:

$$G_{l_i}^+ = \{V \in \mathrm{Gr}(k,n) | e_i \in V\} \cong \mathrm{Gr}(k-1, n-1);$$

$$G_{l_i}^- = \{V \in \mathrm{Gr}(k,n) | e_i \notin V\} \cong \mathrm{Gr}(k, n-1).$$

So in this case, we have:

Corollary 11.5. [Kap93b] 1.6.6 When $l_i = k_i = 1$ for all i, and $S = T$, $T_i = T/\mathbb{C}^*$ by removing the i-th $\mathbb{C}^*$, the morphisms are:
$\overline{a_i} : \mathrm{Gr}(k,n)/\!\!/_{Ch}T \to \mathrm{Gr}(k-1,n-1)/\!\!/_{Ch}T_i$;
$\overline{b_i} : \mathrm{Gr}(k,n)/\!\!/_{Ch}T \to \mathrm{Gr}(k,n-1)/\!\!/_{Ch}T_i$.

11.2. **The Doulbe Quotient Map.**

Lemma 11.6. [BHK12] *A torus T acts on a normal variety X. For a subtorus $S \subset T$, there is a canonical proper birational morphism*

$$(X/\!\!/_{Ch}T)/\!\!/_{Ch}(T/S) \to X/\!\!/_{Ch}S.$$

Lemma 11.7. [BS94] *In the lemma above, if X is a toric variety, then the morphism is surjective.*

Theorem 11.8. *The double Chow quotient first by the full torus followed by the quotient torus dominates the Chow quotient of the diagonal subtorus, i.e. $(Gr(k,n)/\!\!/_{Ch}T)/\!\!/_{Ch}(T/S) \twoheadrightarrow Gr(k,n)/\!\!/_{Ch}S$ for all diagonal subtorus S in T.*

Proof. This is immediate by theorem 2.8 in [BHK12] and [BS94]. □

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Appendices

Types a:●, b:●, c:●, d:● are nondecompsable central subpolytopes.

The inequalities in purple are not essential, i.e. they can be implied by the inequalities in black.

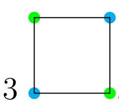
1: A: $x_1 \leqslant 1, x_2 \leqslant 1, x_3 \leqslant 1$ ●,

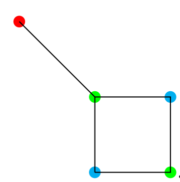
B: $x_{123} \leqslant 3, x_{124} \leqslant 3, x_{134} \leqslant 3$ ($x_{12} \leqslant 2, x_{13} \leqslant 2, x_{14} \leqslant 2, x_1 \leqslant 1$) ●.

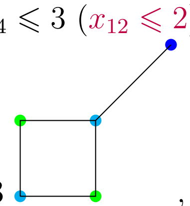
2: A: $x_1 \leqslant 1, x_{23} \leqslant 2, x_{124} \leqslant 3$ ($x_2 \leqslant 1$) .

3: A: $x_1 \leqslant 1, x_2 \leqslant 1, x_{13} \leqslant 2$,

B: $x_{12} \leqslant 2, x_{123} \leqslant 3, x_{134} \leqslant 3$ ($x_{13} \leqslant 2, x_1 \leqslant 1$) .

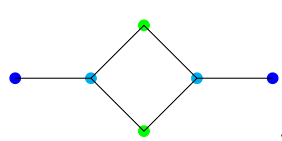
4: A: $x_1 \leqslant 1, x_2 \leqslant 1, x_{123} \leqslant 3, x_{124} \leqslant 3$ .

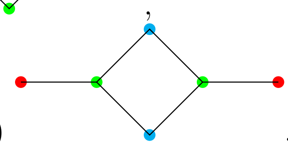
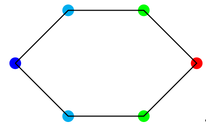
5: A: $x_1 \leqslant 1, x_{123} \leqslant 3, x_{124} \leqslant 3$ ($x_{12} \leqslant 2$) ,

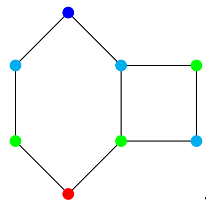
B: $x_1 \leqslant 1, x_2 \leqslant 1, x_{123} \leqslant 3$ ,

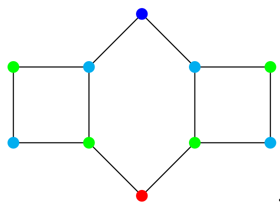
C: $x_1 \leqslant 1, x_{23} \leqslant 2$,

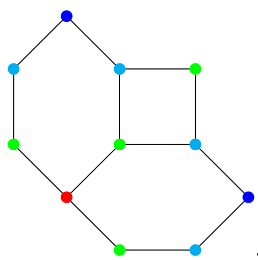
D: $x_{12} \leqslant 2, x_{134} \leqslant 3$ ($x_1 \leqslant 1$) .

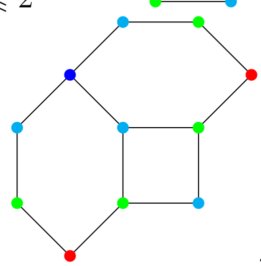
6: A: $x_1 \leqslant 1, x_2 \leqslant 1$ ,

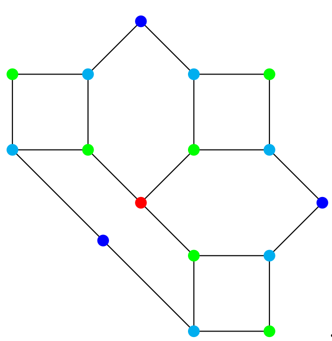
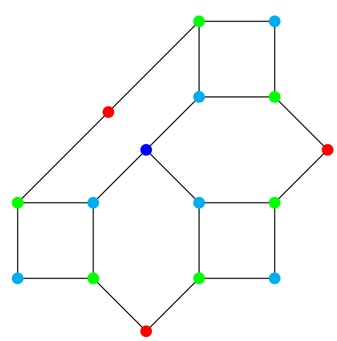
B: $x_{123} \leqslant 3, x_{124} \leqslant 3$ ($x_{12} \leqslant 2$) , C: $x_1 \leqslant 1, x_{12} \leqslant 2, x_{13} \leqslant 2, x_{123} \leqslant 3$  .

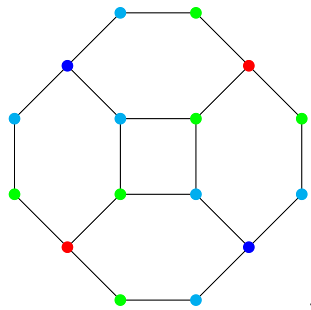
8: A: $x_1 \leqslant 1, x_{12} \leqslant 2, x_{123} \leqslant 3$  .

10: A: $x_1 \leqslant 1, x_{123} \leqslant 3$  .

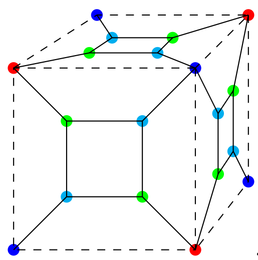
11: A: $x_1 \leqslant 1, x_{12} \leqslant 2$  ,

B: $x_{12} \leqslant 2, x_{123} \leqslant 3$  .

16: A: $x_{123} \leqslant 3$  , B: $x_1 \leqslant 1$ 



C: $x_{12} \leqslant 2$.

32: A:  .