Second Cohomology Groups for Frobenius Kernels

by

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(Under the Direction of Daniel K. Nakano)

Abstract

Let G be a simple simply connected algebraic group scheme defined over an algebraically closed field of characteristic $p > 0$. Let T be a maximal split torus in G, $B \supset T$ be a Borel subgroup of G and U its unipotent radical. Let $F : G \to G$ be the Frobenius morphism. For $r \geq 1$ define the Frobenius kernel, G_r, to be the kernel of F iterated with itself r times. Define U_r (respectively B_r) to be the kernel of the Frobenius map restricted to U (respectively B). Let $X(T)$ be the integral weight lattice and $X(T)_+$ be the dominant integral weights.

It is well known that the representations for G_1 are equivalent to the restricted representations for $\mathfrak{g} = \text{Lie}(G)$. Over the past 25 years, there has been a lot of progress in the computations of cohomology groups for Frobenius kernels, $\text{H}^\bullet(G_r, H^0(\lambda))$. Historically, the computations were done for fields of characteristic $p > h$, where h is the Coxeter number.

The computations of particular importance are $\text{H}^2(U_1, k)$, $\text{H}^2(B_r, \lambda)$ for $\lambda \in X(T)$, $\text{H}^2(G_r, H^0(\lambda))$ for $\lambda \in X(T)_+$, and $\text{H}^2(B, \lambda)$ for $\lambda \in X(T)$. The above cohomology groups for the case when the field has characteristic 2 one computed in this thesis. These computations complete the picture started by Bendel, Nakano, and Pillen for $p \geq 3$ [BNP2].

Index words: Frobenius kernels, Lie algebra cohomology, algebraic groups

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Dedication

I dedicate my thesis to my family who provided me with much love and support over the years; They always believed in me. In particular, I dedicate this to my grandparents: Edward and Jean Marquez. Their love they gave me throughout their life is always with me, which I remember through the tough times I encountered as a graduate student.

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Chapter 1

Introduction

1.1 Preliminaries

Throughout the thesis, we will follow the basic conventions provided in [Jan1]. Let G be a connected semisimple algebraic group over an algebraically closed field, k, of prime characteristic, $p > 0$; assume that G is simply connected. Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of G. For $r \geq 1$, let G_r be the rth Frobenius kernel of G. Let T be a maximal split torus in G and Φ be the root system associated to (G, T). The positive (respectively negative) roots are Φ^+ (respectively Φ^-), and Δ is the set of simple roots. Let $B \supset T$ be the Borel subgroup of G corresponding to the negative roots and let U be the unipotent radical of B. For a given root system of rank n denote the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, adhering to the ordering used in [Jan2] (following Bourbaki). In particular, for type B_n, α_n denotes the unique short simple root; for type C_n, α_n denotes the unique long simple root; for type F_4 α_1 and α_2 are the short simple roots; for type G_2, α_1 is the unique short simple root. If $\alpha \in \Phi$, and $\alpha = \sum_{i=1}^{n} m_i\alpha_i$ then the height of α is defined by $\text{ht}(\alpha)\colon = \sum_{i=1}^{n} m_i$.

Let $\mathbb{E}$ be the Euclidean space associated with Φ, and the inner product on $\mathbb{E}$ will be denoted by $\langle , \rangle$. For any root α denote the dual root by $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the fundamental weights and $X(T)$ be the integral weight lattice spanned by these fundamental weights. The set of dominant integral weights is denoted by $X(T)_+$ and the set of p^r-restricted weights is $X_r(T)$. The simple modules for G are indexed by the set $X(T)_+$ and denoted by $L(\lambda)$, $\lambda \in X(T)_+$ with $L(\lambda) = \text{soc}_G H^0(\lambda)$, where $\text{soc}_G H^0(\lambda)$ is the socle of the G-module $H^0(\lambda)$. where $H^0(\lambda) = \text{ind}_B^G \lambda$. Here λ denotes the one-dimensional B-module obtained by extending the character $\lambda \in X(T)_+$ to U trivially.

Given a G-module, M, then composing a representation of M with F results in a new representation where G_r acts trivially, where $M^{(r)}$ denotes the new module. For any λ in $X(T)$, the λ weight space of M is the $p^r\lambda$ weight space of $M^{(r)}$. One the other hand if V is a G-module on which G_r acts trivially, then there is a unique G-module M, with $V = M^{(r)}$. We denote $M = V^{(-r)}$.

1.2 Example

Let $G = GL_n(F)$, the group of $n \times n$ invertible matrices. The Lie algebra, $\mathfrak{g} = \text{Lie}(G) = sl_n$, the $n \times n$ matrices with determinant 1, corresponding to Type A_{n-1}. The maximal split torus in G is $T = \{\text{diagonal matrices}\}$. The Borel subgroup of G is $B = \{\text{lower triangular matrices}\}$. The unipotent radical of B is $U = \{\text{lower triangular matrices, with 1 on the diagonal}\}$.

Let $V = k^n$ and $e_1, \ldots, e_n$ denote the canonical basis of k^n. Note that e_i is an eigenvector for T. Let ϵ_i be the corresponding weight associated to e_i. Then $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ form a basis for $X(T)$.

W, the Weyl group, permutes $\{\epsilon_1, \ldots, \epsilon_n\}$; thus, $W \cong S_n$, the symmetric group on n letters. $\Phi = \{\epsilon_i - \epsilon_j | 1 \leq i, j \leq n, \ i \neq j\}$ and $\Phi^+ = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\}$. $\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} | 1 \leq i \leq n\}$.

1.3 History

A central question in representation theory of algebraic groups is to understand the structure and the vanishing of the line bundle cohomology, $H^n(\lambda) = \mathcal{H}^n(G/B, \mathcal{L}(\lambda))$ for $\lambda \in X(T)$. The computation of the rational cohomology groups is fundamental in the understanding of the line bundle cohomology, in particular the calculation of $\mathrm{H}^\bullet(B, \lambda)$. It is well known that the representations for G_1 are equivalent to the restricted representations for $\mathfrak{g} = \text{Lie } G$. Over the past 25 years, there has been a lot of progress in the computations of cohomology groups of Frobenius kernels. The calculations that are of significant importance are:

(1) $\mathrm{H}^n(\mathfrak{u}, k)$

(2) $\mathrm{H}^n(U_1, k)$

(3) $\mathrm{H}^n(B_1, \lambda)$, for $\lambda \in X(T)$

(4) $\mathrm{H}^n(B_r, \lambda)$, for $\lambda \in X(T)$

(5) $\mathrm{H}^n(B, \lambda)$, for $\lambda \in X(T)$

(6) $\mathrm{H}^n(G_1, H^0(\lambda))$, for $\lambda \in X(T)_+$

(7) $\mathrm{H}^n(G_r, H^0(\lambda))$, for $\lambda \in X(T)_+$

Historically, these computations were realized for large primes. In 1983, Friedlander and Parshall [FP1] calculated various cohomology groups of algebraic groups. They started with the special case when G is the general linear group with coefficients in the adjoint representation. They extended the idea further to general algebraic groups with coefficients in $V^{(r)}$, where V is a G-module. Also in this paper, they calculated G_r-cohomology groups of degrees 1 and 2 with coefficients over the trivial module and $p \neq 2, 3$.

These results were later extended by Andersen and Jantzen [AJ], determining (6) for $p \geq h$, where h is the Coxeter number (i.e. $h = \langle \rho, \alpha^\vee \rangle + 1$, more precisely, h =the height of the highest root + 1). In this paper, Andersen and Jantzen also determined (3) for $\lambda = w \cdot 0 + p\nu$ for $p > h$, where $w \in W$ and $\nu \in X(T)$. Their results originally had restrictions on the type of root system involved, which were removed by Kumar, Lauritzen, and Thomsen [KLT].

A fundamental computation related to understanding line bundle cohomology is the calculation of $\mathrm{H}^\bullet(B, \lambda)$. In [BNP2], the authors calculated $\mathrm{H}^2(B, \lambda)$ using the H^2-calculations for Frobenius kernels and Lie algebras. In 1984, Andersen [And] began to study the B-cohomology. In particular he calculated $\mathrm{H}^\bullet(B, w \cdot 0)$, where $w \in W$. More recently, Andersen and Rian [AR] proved some general results on the behavior of $\mathrm{H}^\bullet(B, \lambda)$ and developed some new techniques to enable the calculation of all B-cohomology in degree at most 3 when $p > h$. They calculated $\mathrm{H}^2(B, \lambda)$ and $\mathrm{H}^3(B, \lambda)$ explicitly for $\lambda \in X(T)$ and $p > h$. For higher cohomology groups, they proved the following theorem [AR, 3.1,6.1]:

Theorem 1.3.1. *Suppose $p > h$. Let $w \in W, \nu \in X(T)$. Then we have for all i*

(a) $\mathrm{H}^i(B, w \cdot 0 + p\nu) \cong \mathrm{H}^{i-l(w)}(B, \nu)$

(b) $\mathrm{H}^i(B, p\lambda) = 0$ *for* $i > -2 \cdot ht(\lambda)$

In the past 15 years, the computations of the cohomology listed in (1)–(7) has focused on small primes. In 1991, Jantzen [Jan2] calculated (1)–(3), (6) for $n = 1$ and all primes. Jantzen used basic facts about the structure of the root systems and isomorphisms relating the different cohomology groups. Bendel, Nakano, and Pillen used Jantzen's results to get (4),(7) for $n = 1$ and all p in [BNP1]. In 2004, Bendel, Nakano, and Pillen [BNP2] worked out (1)–(7) for $n = 2$ and $p \geq 3$.

Knowledge about the second cohomology groups is important because of the information it gives us about central extensions of the underlying algebraic structures.

1.4 Outline of Computations

In recent work, Bendel, Nakano, and Pillen [BNP2] calculated $\mathrm{H}^2(G_r, H^0(\lambda))$ for $p \geq 3$ by reducing the calculations down to $\mathrm{H}^2(\mathfrak{u}, k)$. We will use similar strategies as [BNP2] to calculate $\mathrm{H}^2(G_r, H^0(\lambda))$ when $p = 2$. For this calculation, we will first obtain some other calculations. The first step uses the following isomorphism to reduce the calculation to the B_r-cohomology.

$$\mathrm{H}^2(G_r, H^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^2(B_r, \lambda)^{(-r)}). \tag{1.4.1}$$

The use of the Lyndon-Hochschild-Serre spectral sequence reduces the problem to the B_1-cohomology. The problem is further reduced to the computation of $\mathrm{H}^2(U_1, k)$ via the isomorphism

$$\mathrm{H}^2(B_1, \lambda) \cong (\mathrm{H}^2(U_1, k) \otimes \lambda)^{T_1}. \tag{1.4.2}$$

This isomorphism tells us that the B_1-cohomology can easily be determined by looking at particular weight spaces of $\mathrm{H}^2(U_1, k)$. That is $\mathrm{H}^2(B_1, \lambda) \cong \mathrm{H}^2(U_1, k)_{-\lambda}$.

The B-cohomology completes the calculations for the second cohomology groups for $p = 2$. In [BNP1] the authors found that $\mathrm{H}^1(B, \lambda)$ is at most one-dimensional as is the case for $\mathrm{H}^2(B, \lambda)$. The second B-cohomology group was first determined in [BNP2] for $p \geq 3$ and for $p > h$ by [And] using a different method.

Theorem 1.4.1. *Let $p \geq 3$ and $\lambda \in X(T)$.*

(a) Suppose $p > 3$ or Φ is not of type G_2. Then

$$\mathrm{H}^2(B, \lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \text{ for } w \in W \text{ and } l(w) = 2, \\ k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^k \beta - p^l \alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) Suppose $p = 3$ and Φ is of type G_2. Then

$$\mathrm{H}^2(B, \lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, l(w) = 2, \\ k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^k \beta - p^l \alpha, \text{ with } 0 \leq l < k \text{ and } \alpha, \beta \in \Delta, \\ & \quad \text{where } k \neq l + 1 \text{ if } \beta = \alpha_1 \text{ and } \alpha = \alpha_2 \\ 0 & \text{else.} \end{cases}$$

In [BNP2] the authors explicitly determined $\mathrm{H}^2(\mathfrak{u}, k), \mathrm{H}^2(U_1, k), \mathrm{H}^2(B_r, \lambda), \mathrm{H}^2(G_r, \lambda)$, and $\mathrm{H}^2(B, \lambda)$ for $p \geq 3$. For $p = 2$, a similar strategy will be used by using the above isomorphisms.

The thesis will first start by reminding the reader of some results of Lie algebra cohomology. In [BNP2] the authors determined possible weights that occur involving sums of positive roots that arise in the calculations, which also hold for $p = 2$. This calculation then allows us to calculate the B_1-cohomology as both a T-module and a B-module as described in Section 5. These results can be used to give a complete answer for $\mathrm{H}^2(B_r, \lambda)$ for $p = 2$. The results for the B_r-cohomology are used to calculate both the B-cohomology and the G_r-cohomology. First, by applying the inverse limit to the B_r-cohomology results, we can calculate $\mathrm{H}^2(B, \lambda)$ in Section 6. In Section 7, by applying the induction functor to the B_r-cohomology results, we can find $\mathrm{H}^2(G_r, H^0(\lambda))$. Finishing off the $p = 2$ calculations completes the entire picture for the second cohomology groups of Frobenius kernels for all primes.

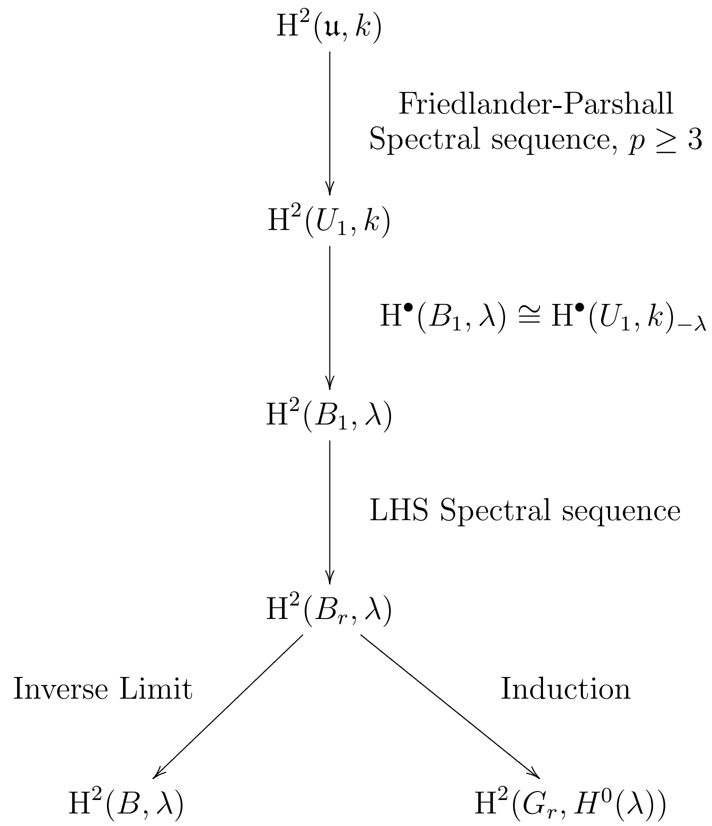

Figure 1.1: Summary of the process for the calculations

Chapter 2

Restricted Lie algebra cohomology

2.1 Observations on U_1-cohomology

Recall that $\mathrm{H}^{\cdot}(\mathfrak{u}, k)$ is computed by using the exterior algebra and the following complex:

$$k \xrightarrow{d_0} \mathfrak{u}^* \xrightarrow{d_1} \Lambda^2(\mathfrak{u})^* \xrightarrow{d_2} \Lambda^3(\mathfrak{u})^* \to \dots .$$

However, this only works when $p \geq 3$. When $p \geq 3$, then $\mathrm{H}^{\cdot}(U_1, k)$ can be computed from $\mathrm{H}^{\cdot}(\mathfrak{u}, k)$ by using the Friedlander-Parshall spectral. However, this spectral sequence only holds for $p \geq 3$. To calculate $\mathrm{H}^{\bullet}(U_1, k)$, for $p = 2$, we must take a different approach. For $p = 2$, the restricted Lie algebra cohomology is computed by the following complex [Jan1, 9.15].

$$k \xrightarrow{d_0} \mathfrak{u}^* \xrightarrow{d_1} S^2(\mathfrak{u})^* \xrightarrow{d_2} S^3(\mathfrak{u})^* \to \dots .$$

The differential d_1 is a derivation on $S^{\bullet}(\mathfrak{u}^*)$ and is thus determined by its restriction to $\mathfrak{u}^*$. We will look at the following composition of maps

$$\mathfrak{u}^* \overset{d_1}{\hookrightarrow} S^2(\mathfrak{u})^* \xrightarrow{\pi} \Lambda^2(\mathfrak{u})^*$$

where π is a surjection with kernel $\{f^2 : f \in \mathfrak{u}^*\}$ and $\partial = \pi \circ f$ being the coboundary operator for the ordinary Lie algebra cohomology, i.e. the dual of $\Lambda^2(\mathfrak{u}) \to \mathfrak{u}$ with $a \wedge b \mapsto [a, b]$. The differentials are given as follows: $d_0 = 0$ and $d_1 : \mathfrak{u}^* \to S^2(\mathfrak{u})^*$ with

$$(d_1\phi)(x_1 \otimes x_2) = -\phi([x_1, x_2])$$

where $\phi \in \mathfrak{u}^*$ and $x_1, x_2 \in \mathfrak{u}$. For the higher differentials, we identify $S^n(\mathfrak{u})^* \cong S^n(\mathfrak{u}^*)$. Then the differentials are determined by the following product rule:

$$d_{i+j}(\phi \otimes \psi) = d_i(\phi) \otimes \psi + (-1)^i \phi \otimes d_j(\psi).$$

2.2 Basic Results

In [BNP2], the authors calculated the ordinary Lie algebra cohomology, then used the first quadrant spectral sequence

$$E_2^{i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{u}, k) \Rightarrow \mathrm{H}^{2i+j}(U_1, k)$$

to calculate the restricted Lie algebra cohomology. However, this spectral sequence is only valid for $p \geq 3$. Our approach will start with the calculations of the restricted Lie algebra cohomology.

The following results will help identify some limitations on which tensor products $\phi_\alpha \otimes \phi_\beta$ or linear combinations can represent cohomology classes when char $k = 2$. Using the additive property of differentials and the fact that differentials preserve the T action, then we are interested in tensor products that have the same weight. Recall the following theorem from Jantzen, [Jan2]

Theorem 2.2.1. $\mathrm{H}^1(U_1, k) \cong \mathrm{H}^1(\mathfrak{u}, k)$

Recall the following definition from [BNP2].

Definition 2.2.2. An expression $\sum c_{\alpha,\beta}\phi_\alpha \otimes \phi_\beta \in S^2(\mathfrak{u}^*)$ is in *reduced form if* $c_{\alpha,\beta} \neq 0$ *and for each pair* (α, β) $c_{\alpha,\beta}$ *appears at most once.*

Proposition 2.2.3. *Let* $x = \sum c_{\alpha,\beta}\ \phi_\alpha \otimes \phi_\beta$ *be an element in* $S^2(\mathfrak{u}^*)$ *in reduced form of weight* γ *for some* $\gamma \in X(T)$ *and* $\gamma \notin 2X(T)$*. If* $d_2(x) = 0$*, then* $d_1(\phi_\alpha) = 0$ *for at least one* α *appearing in the sum.*

Proof. Observe for any $\alpha \in \Phi^+$, if $d_1(\phi_\alpha) = \sum c_{\delta,\gamma}\phi_\delta \otimes \phi_\gamma$, then $\mathrm{ht}(\delta) < \mathrm{ht}(\alpha)$ and $\mathrm{ht}(\gamma) <$ ht (α) for all δ, γ. For all α and β appearing in the sum for x, choose a root σ with $\mathrm{ht}(\sigma)$ being maximal. Without loss of generality, we may assume ϕ_σ appears in the second factor of the tensor product. Consider the corresponding term $c_{\alpha,\sigma}\phi_\alpha \otimes \phi_\sigma$. Computing $d_2(x)$, one of the components will be $c_{\alpha,\sigma}d_1(\phi_\alpha) \otimes \phi_\sigma$. By height considerations, ϕ_σ appears in no other terms. So it is not a linear combination of the other terms and we must have $d_1(\phi_\alpha) = 0$. $\square$

Corollary 2.2.4. *(a) Let $x \in \mathrm{H}^2(U_1, k)$ be a representative cohomology class in reduced form having weight γ for some $\gamma \in X(T)$, $\gamma \notin 2X(T)$. Then one of the components of x is of the form $\phi_\alpha \otimes \phi_\beta$ for some simple root $\alpha \in \Delta$ and positive root $\beta \in \Phi^+$ (with $\alpha + \beta = \gamma$).*

(b) Suppose $\phi_\alpha \otimes \phi_\beta$ represents a cohomology class in $\mathrm{H}^2(U_1, k)$. Then one of three things must happen either

(i) $\alpha, \beta \in \Delta$,

(ii) $\alpha \in \Delta$, then $d_1(\phi_\beta) = \sum_{\sigma_1+\sigma_2=\beta} c_{\sigma_1,\sigma_2}\ \phi\sigma_1 \otimes \sigma_2$, then $c_{\sigma_1,\sigma_2} = \pm 2$ for all decompositions of β (that is the structure constant is even), or

(iii) $\alpha = \beta$ and $\alpha \in \Phi^+$.

Proof. Part (a) follows immediately from the previous proposition and Jantzen's theorem since the first cohomology is generated by the simple roots. For part (b) let's first assume that $\alpha = \beta$, then

$$d_2(\phi_\alpha \otimes \phi_\alpha) = d_1(\phi_\alpha) \otimes \phi_\alpha + \phi_\alpha \otimes d_1(\phi_\alpha) = 2d_1(\phi_\alpha) \otimes \phi_\alpha = 0.$$

Now, assume that $\alpha \neq \beta$, then α is simple. By assumption $d_2(\phi_a \otimes \phi_\beta) = 0$. We have

$$d_2(\phi_\alpha \otimes \phi_\beta) = d_1(\phi_\alpha) \otimes \phi_\beta + \phi_\alpha \otimes d_1(\phi_\beta) = \phi_\alpha \otimes d_1(\phi_\beta).$$

Hence, $d_1(\phi_\beta) = 0$, and so either $\beta \in \Delta$ or if $\beta = \sigma_1 + \sigma_2$ then $c_{\sigma_1,\sigma_2} = \pm 2$, for all decompositions of β. □

2.3 Root Sums

As mentioned in the introduction, the computation of $\mathrm{H}^2(U_1, k)$ involves information about B_1- and B-cohomology. In this process, certain sums involving positive roots arise. Suppose $x \in \mathrm{H}^2(U_1, k)$ has weight $\gamma \in X(T)$. Then by Corollary 2.2.4, $\gamma = \alpha + \beta$ for $\alpha \in \Delta$ and $\beta \in \Phi^+$ and $\alpha \neq \beta$. Given such roots α and β, we want to know whether there exists a weight $\sigma \in X(T)$, $\beta_1, \beta_2 \in \Delta$, and integers $0 \leq i \leq p-1$ and $m \geq 0$ such that any of the following hold:

$$\alpha + \beta = 2\sigma \tag{2.3.1}$$

$$\alpha + \beta = \beta_1 + 2\sigma \tag{2.3.2}$$

$$\alpha + \beta = i\beta_1 + 2^m\beta_2 + 2\sigma. \tag{2.3.3}$$

Given γ a weight of $\mathrm{H}^2(U_1, k)$, then there is a weight $\nu \in X(T)$ such that $\mathrm{H}^2(B, -\gamma + p\nu) \neq 0$, as seen stated in the introduction from [And]. Using information on B-cohomology from Andersen [And], then γ must satisfy equation (2.3.3). Note that (2.3.2) are special cases of (2.3.3) (i.e. when $i = 0$ and $m = 0$. For more details on how these equations arise see [BNP2]. Equation (2.3.1) arises from the reduction $\mathrm{H}^2(B_1, k) = \mathrm{H}^2(U_1, k)^{T_1}$.

Remark 2.3.1. These sums noted above are only valid when 2 does not divide the index of connection.

2.4 U_1-COHOMOLOGY

We state the theorem which describes $\mathrm{H}^2(U_1, k)$ when $p = 2$. In the next chapter, we explain the proof for each type. In the following theorem, the right hand side is a list of T weights, except for $\mathfrak{u}^*$, that occur in our T-module, $\mathrm{H}^2(U_1, k)$.

Theorem 2.4.1. *As a T-module,*

(a) If $\Phi = A_n$, then

$$\mathrm{H}^2(U_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta \in \Delta \\ \alpha+\beta \notin \Phi^+}} -(s_\alpha s_\beta) \cdot 0 \oplus \bigoplus_{\substack{\alpha,\beta,\gamma \in \Delta \\ \alpha+\beta \notin \Phi^+ \\ \alpha+\beta+\gamma \in \Phi^+}} -(s_\alpha s_\beta) \cdot 0 + 2\gamma$$

(b) If $\Phi = B_n$*, then*

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta\in\Delta \\ \alpha+\beta\notin\Phi^+}} -(s_\alpha s_\beta)\cdot 0 \oplus \bigoplus_{\substack{\alpha,\beta,\gamma\in\Delta \\ \alpha+\beta\notin\Phi^+ \\ \alpha+\beta+\gamma\in\Phi^+}} -(s_\alpha s_\beta)\cdot 0 + 2\gamma$$
$$\oplus \bigoplus_{1\le i\le n-3} -(s_{\alpha_i} s_{\alpha_{n-1}})\cdot 0 + 2\alpha_n \oplus \bigoplus_{1\le i\le n-1} 2(\alpha_i + \alpha_{i+1} + \ldots + \alpha_n)$$

(c) If $\Phi = C_n$*, then*

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta\in\Delta \\ \alpha+\beta\notin\Phi^+}} -(s_\alpha s_\beta)\cdot 0 \oplus \bigoplus_{\substack{\alpha,\beta,\gamma\in\Delta \\ \alpha+\beta\notin\Phi^+ \\ \alpha+\beta+\gamma\in\Phi^+}} -(s_\alpha s_\beta)\cdot 0 + 2\gamma$$
$$\oplus \bigoplus_{1\le i\le n-3} -(s_{\alpha_i} s_{\alpha_n})\cdot 0 + 2\alpha_{n-1} \oplus \bigoplus_{1\le i\le n-1} 2(\alpha_i + \alpha_{i+1} + \ldots + \alpha_n)$$
$$\oplus -(s_{\alpha_n-1} s_{\alpha_n})\cdot 0$$

(d) If $\Phi = D_n$*,* $n \geq 4$*, then*

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta\in\Delta \\ \alpha+\beta\notin\Phi^+}} -(s_\alpha s_\beta)\cdot 0 \oplus \bigoplus_{\substack{\alpha,\beta,\gamma\in\Delta \\ \alpha+\beta\notin\Phi^+ \\ \alpha+\beta+\gamma\in\Phi^+}} -(s_\alpha s_\beta)\cdot 0 + 2\gamma$$
$$\oplus -(s_{\alpha_{n-3}} s_{\alpha_{n-1}})\cdot 0 + 2(\alpha_{n-2} + \alpha_n) \oplus -(s_{\alpha_{n-3}} s_{\alpha_n})\cdot 0 + 2(\alpha_{n-2} + \alpha_n)$$
$$\oplus \bigoplus_{1\le i\le n-3} -(s_{\alpha_{n-1}} s_{\alpha_n})\cdot 0 + 2(\alpha_i + \ldots + \alpha_{n-2})$$

(e) If $\Phi = E_6$*, then*

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta\in\Delta \\ \alpha+\beta\notin\Phi^+}} -(s_\alpha s_\beta)\cdot 0 \oplus \bigoplus_{\substack{\alpha,\beta,\gamma\in\Delta \\ \alpha+\beta\notin\Phi^+ \\ \alpha+\beta+\gamma\in\Phi^+}} -(s_\alpha s_\beta)\cdot 0 + 2\gamma$$
$$\oplus -(s_{\alpha_2} s_{\alpha_3})\cdot 0 + 2(\alpha_4 + \alpha_5) \oplus -(s_{\alpha_2} s_{\alpha_3})\cdot 0 + 2(\alpha_4 + \alpha_5 + \alpha_6)$$
$$\oplus -(s_{\alpha_2} s_{\alpha_5})\cdot 0 + 2(\alpha_3 + \alpha_4) \oplus -(s_{\alpha_3} s_{\alpha_5})\cdot 0 + 2(\alpha_2 + \alpha_4)$$
$$\oplus -(s_{\alpha_2} s_{\alpha_6})\cdot 0 + 2(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)$$

(f) If $\Phi = E_7$, then

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta\in\Delta \\ \alpha+\beta\notin\Phi^+}} -(s_\alpha s_\beta)\cdot 0 \oplus \bigoplus_{\substack{\alpha,\beta,\gamma\in\Delta \\ \alpha+\beta\notin\Phi^+ \\ \alpha+\beta+\gamma\in\Phi^+}} -(s_\alpha s_\beta)\cdot 0 + 2\gamma$$
$$\oplus -(s_{\alpha_2}s_{\alpha_5})\cdot 0 + 2(\alpha_1+\alpha_3+\alpha_4) \oplus -(s_{\alpha_3}s_{\alpha_5})\cdot 0 + 2(\alpha_2+\alpha_4)$$
$$\oplus -(s_{\alpha_2}s_{\alpha_3})\cdot 0 + 2(\alpha_2+\alpha_3) \oplus -(s_{\alpha_2}s_{\alpha_3})\cdot 0 + 2(\alpha_4+\alpha_5+\alpha_6)$$
$$\oplus -(s_{\alpha_2}s_{\alpha_3})\cdot 0 + 2(\alpha_4+\alpha_5+\alpha_6+\alpha_7) \oplus -(s_{\alpha_2}s_{\alpha_5})\cdot 0 + 2(\alpha_3+\alpha_4)$$

(g) If $\Phi = E_8$, then

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta\in\Delta \\ \alpha+\beta\notin\Phi^+}} -(s_\alpha s_\beta)\cdot 0 \oplus \bigoplus_{\substack{\alpha,\beta,\gamma\in\Delta \\ \alpha+\beta\notin\Phi^+ \\ \alpha+\beta+\gamma\in\Phi^+}} -(s_\alpha s_\beta)\cdot 0 + 2\gamma$$
$$\oplus -(s_{\alpha_2}s_{\alpha_5})\cdot 0 + 2(\alpha_1+\alpha_3+\alpha_4) \oplus -(s_{\alpha_2}s_{\alpha_3})\cdot 0 + 2(\alpha_4+\alpha_5+\alpha_6)$$
$$\oplus -(s_{\alpha_2}s_{\alpha_3})\cdot 0 + 2(\alpha_4+\alpha_5+\alpha_6+\alpha_7) \oplus -(s_{\alpha_3}s_{\alpha_5})\cdot 0 + 2(\alpha_2+\alpha_4)$$
$$\oplus -(s_{\alpha_2}s_{\alpha_3})\cdot 0 + 2(\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8)$$
$$\oplus -(s_{\alpha_2}s_{\alpha_3})\cdot 0 + 2(\alpha_4+\alpha_5)$$

(h) If $\Phi = F_4$, then

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{\substack{\alpha,\beta\in\Delta \\ \alpha+\beta\notin\Phi^+}} -(s_\alpha s_\beta)\cdot 0 \oplus -(s_{\alpha_1}s_{\alpha_3})\cdot 0 + 2\alpha_2$$
$$\oplus -(s_{\alpha_1}s_{\alpha_3})\cdot 0 + 2(\alpha_2+\alpha_3) \oplus 2(\alpha_2+\alpha_3) \oplus 2(\alpha_1+\alpha_2+\alpha_3)$$
$$\oplus 2(\alpha_2+\alpha_3+\alpha_4) \oplus 2(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \oplus 2(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$$
$$\oplus 2(\alpha_2+\alpha_3+\alpha_4)$$

(i) If $\Phi = G_2$, then

$$\mathrm{H}^2(U_1,k) \cong (\mathfrak{u}^*)^{(1)} \oplus 2(\alpha_1+\alpha_2).$$

Chapter 3

Proof of Theorem 2.4.1

3.1 Type A_n, $p \nmid n+1$

Since A_n is simply laced and $p = 2$, so all the structure constants are ± 1. Suppose $x \in \mathrm{H}^2(U_1, k)$ has weight $\gamma \in X(T)$. From the above corollary, we know that $\gamma = \alpha + \beta$ for some roots $\alpha \in \Delta$ and $\beta \in \Phi^+$, with $\alpha \neq \beta$. Since $p \nmid n+1$, then we can use equations, (2.3.1)-(2.3.3).

Proposition 3.1.1. *Let $p = 2$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. Then there is no weight $\sigma \in X(T)$ such that $\alpha + \beta = 2\sigma$.*

Proof. Consider $\alpha + \beta = 2\sum m_i\alpha_i$. Since all coefficients of the positive roots are 0 or 1 in type A_n, then $\alpha = \beta$, which contradicts the hypothesis. □

Proposition 3.1.2. *Let $p = 2$, $\alpha \in \Delta$, $\beta \in \Phi^+$, and $\alpha \neq \beta$. Then, there is no simple root $\beta_1 \in \Delta$ and $\sigma \in X(T)$ such that $\alpha + \beta = \beta_1 + 2\sigma$.*

Proof. Consider $\alpha + \beta = \beta_1 + 2\sum m_i\alpha_i$. Since $\alpha = \alpha_i$ for some i, then $\sigma \in \Delta$. Thus the only possibility for β is $\alpha_i + \alpha_{i-1}$ or $\alpha_i + \alpha_{i+1}$. Then $x \in \mathrm{H}^2(U_1, k)$ has only one component and by Corollary 2.2.4(a) $\beta \in \Delta$. So, there does not exist $\beta_1 \in \Delta$ and $\sigma \in X(T)$ such that $\alpha + \beta = \beta_1 + 2\sigma$. □

Proposition 3.1.3. *Let $p = 2$, $\alpha \in \Delta, \beta \in \Phi^+$, and $\alpha \neq \beta$. If $\alpha + \beta$ is a weight of $\mathrm{H}^2(U_1, k)$ and there exists $\beta_1, \beta_2 \in \Delta$, $\sigma \in X(T), 0 < i < p$, and $m \geq 0$ such that*

$$\alpha + \beta = i\beta_1 + 2^m\beta_2 + 2\sigma,$$

then one of the following holds

(a) $\alpha + \beta$ *is a solution to equation (2.3.1) or (2.3.2)*

(b) If $n \geq 3$*, then* $\alpha + \beta = \alpha_{i-1} + \alpha_{i+1} + 2\alpha_i$ *or* $\alpha + \beta = \alpha_{i-1} + \alpha_{i-2} + 2\alpha_i$ *or* $\alpha + \beta = \alpha_{i+1} + \alpha_{i+2} + 2\alpha_i$ *for* $i \leq n-2$*.*

Proof. First note that we only have to consider the cases $i = 0, 1$ since $p = 2$, and the σ will absorb part of β. Furthermore, since $p = 2$ if $m \geq 2$, then by choosing a different $\sigma \in X(T)$, these equation reduce down to $m \leq 1$. Also, if $i = 0 = m$, then the equation is the same as (2.3.2), which is done. If $i = 0, m = 1$, then we have that $\alpha + \beta = 2(\beta_2 + \sigma)$ and so $\sigma = 0$, but then we are back into equation (2.3.1). If $i = 1, m = 1$, then $\sigma = 0$, which is a specific case of equation (2.3.2). So, the only thing we have to check is the case when $i = 1, m = 0$. Since $\alpha = \alpha_i$, then $\sigma \in \Delta$ and $\beta = \alpha_{i-2} + \alpha_{i-1} + \alpha_i$, $\beta = \alpha_i + \alpha_{i+1} + \alpha_{i+2}$, or $\beta = \alpha_{i-1} + \alpha_i + \alpha_{i+1}$, which are the cases above. So β_1 and β_2 are either the 2 simple roots on either side of σ or to the right (left) of α. □

3.2 $p|n+1$

Note that $X(T)/\mathbb{Z}\Phi \cong \mathbb{Z}_{n+1}$ and n is odd. Moreover, $X(T)/\mathbb{Z}\Phi = \{t\omega_1 + \mathbb{Z}\Phi : t = 0, 1, \ldots, n\}$. Now

$$t\omega_1 = \frac{t}{n+1}(n\alpha_1 + (n-1)\alpha_2 + \ldots + \alpha_n).$$

By revising (3.1)-(3.3), we are now looking for $\alpha \in \Delta, \beta \in \Phi^+$ satisfying

$$\alpha + \beta = 2t\omega_1 + 2\sigma \tag{3.2.1}$$

$$\alpha + \beta = \beta_1 + 2t\omega_1 + 2\sigma \tag{3.2.2}$$

$$\alpha + \beta = i\beta_1 + 2^m\beta_2 + 2t\omega_1 + 2\sigma, \tag{3.2.3}$$

where $\sigma \in \mathbb{Z}\Phi$. Since $2t\omega_1$ must lie in $\mathbb{Z}\Phi$, $\frac{2t}{n+1} \in \mathbb{Z}$ and $2|n+1$, it follows that $\frac{t}{s} \in \mathbb{Z}$, where $s := \frac{n+1}{2}$. If $2|\frac{t}{s}$ then we are done because (3.2.1)-(3.2.3) would reduce to the original (2.3.1)-(2.3.3) with σ lying in the root lattice, and the above arguments apply. So, we can assume that $\frac{t}{s} \not\equiv 0 \pmod 2$. Consider $\alpha + \beta = \sum_{i=1}^{n} m_i\alpha_i$, then $m_i \in \{0, 1, 2\}$ for

$i = 1, 2, \ldots, n$ and m_i can be 2 for at most one i. To examine the possibilities, reduce $\frac{t}{n+1}(n\alpha_1 + (n-1)\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n)$ mod 2, so that we are looking at sequences of 0's and 1's, looking like $(1, 0, 1, 0, \ldots, 0, 1)$. One of the zeroes is a 2, and at most two other zeroes can be made into a one by using the above equations. So, we have the following:

$$\sum m_i\alpha_i = i\beta_1 + 2^m\beta_2 + (1, 0, 1, 0, \ldots, 0, 1) \pmod 2.$$

Since the roots of A_n are those when the 1's are consecutive, then $n \geq 9$ has a trivial solution.

Looking at A_3, A_5, and A_7 separately it is easy to check that no additional cohomology classes occur, and the only classes that occur are weights of the form $\alpha + \beta = s_\alpha s_\beta \cdot 0$ and $\alpha + 2\gamma + \beta = s_\alpha s_\beta + 2\gamma$, where $\alpha + \beta$ is not a root and $\alpha + \gamma + \beta$ is a root.

3.3 Type B_n

Consider the case when $\Phi = B_n$. The Lie algebra, $\mathfrak{u}$, associated with the root system B_n has some structure constants that are 2 so we have to consider these when looking at the proof. After examining the structure of the Lie algebra with the structure constants, we noticed that the structure constant is 2 when the resulting root has a coefficient of 2 in the the α_n spot and α_n is broken up between the two other roots, i.e. $[\alpha_n, \alpha_i + \alpha_{i+1} + \ldots + \alpha_n] = 2(\alpha_i + \alpha_{i+1} + \ldots + 2\alpha_n)$. For B_n, $X(T)/\mathbb{Z}\Phi \cong \mathbb{Z}_2$ where $\omega_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n)$ is a generator, which forces us to revise (3.2.1)-(3.2.3) in the following way. We are looking for $\alpha, \beta \in \Phi^+$ satisfying

$$\alpha + \beta = 2t\omega_n + 2\sigma \tag{3.3.1}$$

$$\alpha + \beta = \beta_1 + 2t\omega_n + 2\sigma \tag{3.3.2}$$

$$\alpha + \beta = i\beta_1 + 2^m\beta_2 + 2t\omega_n + 2\sigma, \tag{3.3.3}$$

where $\sigma \in \mathbb{Z}\Phi$ and $t = 0, 1$. In the resulting weight, (which is written with respect to the simple root basis), we must have at least 2 odd numbers. For simplicity, the weight will be written in the following form: $(i_1, i_2, \ldots, i_n)$, where i_j is the coefficients of the α_j term. If i_j

is even for all j, then our weight is a multiple of 2 times a weight, which it is easy to check that is always in the kernel, with extra classes. If there is only one odd number, then clearly there must be some 2's in the weight. If all of the two's occur after the odd number, then this is a root and thus it is in the previous image. On the other hand if all of the 2's occur before the odd number, then because of the pattern of the roots we can only have one 2 before it. In which case, there is only one term in the class, $\phi_{\alpha_i} \otimes \phi_{\alpha_i+\alpha_{i+1}}$. Also, there can't be more than one 2 between the 2 odd numbers in the weight. This comes from the fact that there is no way to break up this weight into a sum of a simple root and a positive root.

For the other possibilities, consider the following weight, $(0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 2)$ with the first 1 in the ith spot, which is given by $\phi_{\alpha_i} \otimes \phi_{\alpha_{n-1}+2\alpha_n}$. Using the above observation about the structure constants, this is in the kernel because the structure constant is 2 when we take the differential. Now, besides this weight, then the 1's, 2's, and 3's that show up in the weight must all be connected. In particular, the 2's and 3's must all be connected, which follows from Proposition 2.2.3 and the observation about the structure constants. In the weight, we can only have one 3, which follows from Corollary 2.2.4 and the structure of the roots. The way the roots are structured in B_n, if a 3 occurs, then if it's not in the α_n spot, then 2's must follow it and a 1 must occur before the 3 with 2 between the 1 and the 3. So that leaves us with a weight looking like $(0, \ldots, 0, 1, 2, 3, 2, \ldots, 2)$. say the 3 is in the ith spot. If there is more than one 2 that appears after the 3, then after taking the differential of $\phi_{\alpha_i} \otimes \phi_{x_\alpha}$, you see that the term $p_{\alpha_i} \otimes \phi_{x_a} \otimes \phi_{x_b}$ for some x_a and x_b can't cancel out and thus not in the kernel. So, we have that the only possible class is $(0, \ldots, 0, 1, 2, 3, 2)$ and this is in the kernel.

If $t = 0$, then the arguments in Section 3.1 apply, the only difference we have now is that in the roots, 2's exist and so 3's can appear in the weights. We also know that $(0, \ldots, 0, 1, 2, 3)$ can also happen and it's easy to check that it is in the kernel.

Including the weights that occur when Φ is of type A_n, B_n also includes the weights of the form $\alpha_i + \alpha_{i+1} + \ldots + \alpha_n$ for $1 \leq i \leq n-1$.

3.4 Type C_n

Again, $X(T)/\mathbb{Z}\Phi \cong \mathbb{Z}_2$. Using $\omega_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} + \frac{1}{2}\alpha_n$ as a generator, which forces us to revise (3.2.1)-(3.2.3) in the following way. We are looking for $\alpha, \beta \in \Phi^+$ satisfying

$$\alpha + \beta = 2t\omega_1 + 2\sigma \tag{3.4.1}$$

$$\alpha + \beta = \beta_1 + 2t\omega_1 + 2\sigma \tag{3.4.2}$$

$$\alpha + \beta = i\beta_1 + 2^m\beta_2 + 2t\omega_1 + 2\sigma, \tag{3.4.3}$$

where $\sigma \in \mathbb{Z}\Phi$ and $t = 0, 1$. Like we did with type B_n we have to check when $t = 0$ and $t = 1$. First, besides the weight $(0, \ldots, 0, 1, 0, \ldots, 2, 1)$, (with cohomology class $\phi_{\alpha_i} \otimes \phi_{2\alpha_{n-1}+\alpha_n}$ is in the kernel because $[\alpha_{n-1}, \alpha_{n-1}+\alpha_n] \equiv 0 \pmod 2$), then all the 1's, 2's, and 3's in the weight must be connected. First, if it satisfies (3.4.1), then the weight is twice a root. And it is easy to check that there are extra terms in the class besides $\phi_\alpha \otimes \phi_\alpha$ and is in the kernel. If there is one odd number in the weight, then it must be a 1 and 2's must appear. If the 2's only appear before or after the 1, then there can only be one 2 because of the structure of the roots, and there is only one term in the class with β not simple and $[\alpha, \beta] \not\equiv 0 \pmod 2$. If the 2's occur before and after then it must look like $(0, \ldots, 0, 2, 1, 2, 0, \ldots, 0)$, which you can't write as a sum of a simple root and a positive root. Unless we had $(0, \ldots, 0, 2, 1, 2)$, where there is only one way to write this, which doesn't satisfy Corollary 2.2.4. So, there must be at least two odd numbers appearing in the weight. Note that $(0, \ldots, 0, 1, 2, \ldots, 2, 1, 0, \ldots, 0)$ can't occur and if the last 1 were in the α_n spot, then it would be a root, and thus in the previous image. If there are no 3's in the weight and there is more than one 2 in the weight, then because of the structure of the roots, the last 2 must be in the α_{n-1} spot and a 1 must be in the α_n spot, which is a root. If 3's do appear in the weight, then there can only be one 3 and it must be in the α_{n-1} spot with a 1 in the α_n spot. If it's not in the α_{n-1} spot, then by calculating differentials it's easy to see that a 1 can't occur before the 3 and it were just 2's then you have a $\alpha_i \otimes \alpha_j$ in the term that can't be cancelled (with a 3 in the α_i spot and the first 2 in the α_j spot). So we have that a 3 must appear in the α_{n-1} spot. Now putting all of this together than the only possibilities for roots are (i) $(0,\ldots,0,3,1)$, which is in the cohomology since the structure constant is 2; (ii) $(0,\ldots,0,1,3,1)$ which isn't in the kernel because there is only one way to write it and doesn't satisfy corollary 3.1; (iii) $(0,\ldots,1,2,3,1)$ which isn't in the kernel after taking the differential; or (iv) $(0,\ldots 1,2,\ldots,2,3,1)$ but there are only two ways to write this and after taking the differential only one way to get $\phi_{\alpha_{i-1}}\otimes\phi_{\alpha_{n-1}}\otimes\phi_\beta$ with the structure constant not 2. So the only classes that appear in this cohomology group are those stated in Theorem 2.4.1.

Including the weights that occur when Φ is of type A_n, type C_n also includes the weights of the form $\alpha_i+\alpha_{i+1}+\ldots+\alpha_n$ for $1\le i\le n-1$ and $s_{\alpha_{n-1}}s_{\alpha_n}\cdot 0$.

3.5 Type D_n

When $\Phi = D_n$, $|X(T)/\mathbb{Z}\Phi| = 4$. If n is odd then $X(T)/\mathbb{Z}\Phi\cong\mathbb{Z}_4$ and if n is even then $X(T)/\mathbb{Z}\Phi\cong\mathbb{Z}_2\times\mathbb{Z}_2$, again forcing us to revise equations (2.3.1)-(2.3.3).

First let's look at the case when n is odd.

$$\omega_n=\frac{1}{2}(\alpha_1+2\alpha_2+\ldots+(n-2)\alpha_2+\frac{n-2}{2}\alpha_{n-1}+\frac{n}{2}\alpha_n)$$

is a generator and we are looking for $\alpha,\beta\in\Phi^+$ satisfying

$$\alpha+\beta=2t\omega_n+2\sigma \tag{3.5.1}$$

$$\alpha+\beta=\beta_1+2t\omega_n+2\sigma \tag{3.5.2}$$

$$\alpha+\beta=i\beta_1+2^m\beta_2+2t\omega_n+2\sigma, \tag{3.5.3}$$

where $\sigma\in\mathbb{Z}\Phi$ and $t=0,1$. Now, since we need $\frac{t(n-2)}{2}\in\mathbb{Z}$, then $t\equiv 0 \pmod 2$, and then need to look at the particular cases when $t=0,2$. Let's first look at the case when $t=0$. Note that the weight can have a maximum of one 3 in it because the largest coefficient is a 2, and so by using the above equations and looking at the roots there is only one 3; also, the last 2 in the weight must be in the α_n or α_{n-1} spot, but can't both be 2's. We claim that if there are no 3's involved in the weight, then there can be at most two 2's involved. First we can see that there can't be any 2's prior to the first 1 involved in the weight because there would be no way to write the weight as a sum of a positive root and a simple root. So, there can be only 2's after the first 1 in the weight. Now, again by the root structure we have that there must be another 1 and must occur in either the α_n or the α_{n-1} spot. So, there are two possibilities for weights (i) $(0,\ldots,0,1,2,\ldots,2,1)$ with $\alpha = \alpha_{n-1}$ and (ii) $(0,\ldots,0,1,2,\ldots,2,1,2)$ with $\alpha = \alpha_n$. Now when we split up all the ways of writing these weights as a sum of two positive roots, then after taking the differential, we find that in (i) $\phi_{\alpha_{n-1}} \otimes \phi_{\alpha_i} \otimes \phi_\beta$ but this can't show up again because $\alpha_{n-1} + \alpha_i$ isn't a root except when $i = n-2$. Similarly for (ii) $\phi_{\alpha_n} \otimes \phi_{\alpha_i} \otimes \phi_\beta$, where the first 2 shows up in the α_i spot, but this can't show up again because $\alpha_n + \alpha_i$ isn't a root except when $i = n-2$. Thus the only extra classes that show up are $(0,\ldots,0,1,2,2,1)$ and $(0,\ldots,0,1,2,1,2)$.

Now, let's consider the case when $t = 2$, then $\alpha + \beta \equiv (0,\ldots,0,1,1) \mod 2$. Again the root structure tells us that there can only be at most one 3 in the weight, and there must be a 1 preceding the 3, which follows by the fact that we have to write the weight as a sum of a simple root and a positive root. So, the only possibilities are (1) $(0,\ldots,0,1,3,2,\ldots,2,1,1)$, (2) $(0,\ldots,0,1,2,\ldots,2,3,2,\ldots,2,1)$ or (3) $(0,\ldots,0,1,2,\ldots,2,3,1,1)$, where the 3 is in the ith position. In each of these cases, after taking the differential, we have the following factor $\phi_{\alpha_i} \otimes \phi_{\alpha_{i-1}+\alpha_i+\alpha_{i+1}} \otimes \phi_\beta$, but this only shows up once and so can't cancel out with anything and thus not in the kernel. So, no weight in the cohomology involves a 3. So, now we are left with one of 3 possibilities left if the weight involves any 2's: (i)$(0,\ldots,0,2,\ldots,2,1,1)$, (ii)$(0,\ldots,0,1,2,\ldots,2,1,1)$, or (iii)$(0,\ldots,0,1,1,2,\ldots,2,1,1)$. The last two cases the weight is in the previous image. And is easy to check that the first case is in the kernel and thus gives us an extra cohomology class.

When n is even, then things become a little more difficult. Now we have 2 generators, $\omega_1 = \alpha_1 + \ldots + \alpha_{n-2} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n$ and $\omega_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-2}) + \frac{n-2}{4}\alpha_{n-1} + \frac{n}{4}\alpha_n$. Then we must revise our equations (3.5.1)-(3.5.3) in the following way.

$$\alpha + \beta = 2t\omega_1 + 2s\omega_n + 2\sigma \tag{3.5.4}$$

$$\alpha + \beta = \beta_1 + 2t\omega_1 + 2s\omega_n + 2\sigma \tag{3.5.5}$$

$$\alpha + \beta = i\beta_1 + 2^m\beta_2 + 2t\omega_1 + 2s\omega_n + 2\sigma, \tag{3.5.6}$$

where $i = 0, 1, m \geq 0, s, t = 0, 1, \sigma \in \mathbb{Z}\Phi$, and $\alpha, \beta \in \Phi^+$.

When $s = t = 0$, then this brings us up to the same case when $t = 0$ when n is odd. Let's first look at the case when $s = 1, t = 0$, which reduces down to the case when n is odd and $t = 2$. If $t = 1, s = 0$, then $n \leq 12$ because otherwise there are too many odd numbers in the weight. However the root structure tells us that if there are any 3's in the weight, then there must be 1's in the α_n and the α_{n-1} spots, then we have that $n \leq 6$, but since n is even, then $n = 4, 6$. When $n = 4$, then this is reduced down to the case when $s = t = 0$. When $n = 6$, then we are reduced down to the case when $s = 1, t = 0$. Now if $s = t = 1$, then notice that the only thing that changes are the α_n and the α_{n-1} spots in the weight and we have already considered these cases previously.

Including the weights that occur when Φ is of type A_n, D_n also includes the weights of the form $s_{\alpha_{\eta-3}}s_{\alpha_{n-1}}+2(\alpha_{n-2}+\alpha_n)$, $s_{\alpha_{\eta-3}}s_{\alpha_n}+2(\alpha_{n-2}+\alpha_n)$, and $s_{\alpha_{\eta-3}}s_{\alpha_{n-1}}+2(\alpha_i+\alpha_{i+1}+\ldots+\alpha_{n-2})$ for $1 \leq i \leq n-3$.

3.6 The Exceptional cases

If Φ is one of the exceptional root systems (i.e., E_6, E_7, E_8, F_4, or G_2), then determining the U_1 cohomology reduces to looking at finitely many cases. To do this, a program in GAP was written to calculate all different weights that satisfy equations (2.3.1)-(2.3.3). Then calculating all the differentials of the cohomology classes that satisfy the possible weights to see if they were in the kernel. Note that if $\Phi = E_7$, this is the only case where the index of connection is even. So, instead of running the program once (when $t = 0$), then we had to run it twice, the other time letting $t = 1$. Note, that the long simple roots in type F_4 are α_3, α_4 and the long simple root in type G_2 is α_2. A complete list of the possible weights and classes for the exceptional cases is seen in Appendix 2.

Chapter 4

B_1-cohomology

4.1 T-module structure

In this section, we compute $\mathrm{H}^2(B_1, \lambda)$ for all $\lambda \in X(T)$ as both a T-module and as a B-module. The B_1 cohomology is related to the U_1 cohomology by the Lyndon-Hochschild-Serre spectral sequence for $U_1 \trianglelefteq B_1$. Using the spectral sequence and the fact that $T_1 \cong B_1/U_1$, the following characterization follows

$$\mathrm{H}^2(B_1, \lambda) \cong \mathrm{H}^2(U_1, \lambda)^{T_1} \cong (\mathrm{H}^2(U_1, k) \otimes \lambda)^{T_1}.$$

So, it suffices to determine the $-\lambda$ weight space of $\mathrm{H}^2(U_1, k)$ relative to T_1. So, we have that $\mathrm{H}^2(B_1, \lambda) \cong \mathrm{H}^2(U_1, k)_{-\overline{\lambda}}$, where $\overline{\lambda} = \{\lambda + p\nu \in \mathrm{H}^2(U_1, k) \mid \nu \in X(T)\}$.
Furthermore it is enough to compute $\mathrm{H}^2(B_1, \lambda)$ for $\lambda \in X_1(T)$, which follows from the fact that if $\lambda = \lambda_0 + p\lambda_1$, then

$$\mathrm{H}^2(B_1, \lambda) = \mathrm{H}^2(B_1, \lambda_0 + p\lambda_1) \cong \mathrm{H}^2(B_1, \lambda_0) \otimes p\lambda_1,$$

for unique weights $\lambda_0 \in X_1(T)$ and $\lambda_1 \in X(T)$.

4.2 Case i: $\lambda = 0$

We will first calculate $\mathrm{H}^2(B_1, \lambda)$ as a T-module, when $\lambda = 0$.

Theorem 4.2.1. *Let $p = 2$. Then as a T-module, $\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)}$, except in the following cases:*

(a) If $\Phi = A_3$, then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus (\omega_1 - \omega_2 + \omega_3)^{(1)} \oplus \omega_2^{(1)}.$$

(b) If $\Phi = B_3$, then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus \omega_1^{(1)} \oplus (-\omega_1 + \omega_2)^{(1)}.$$

(c) If $\Phi = B_4$, then

$$\begin{aligned}\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} &\oplus (\omega_1 - \omega_2 + \omega_3 - \omega_4)^{(1)} \oplus (\omega_2 - \omega_4)^{(1)} \oplus (-\omega_2 + \omega_3)^{(1)} \\ &\oplus (-\omega_1 + \omega_2)^{(1)} \oplus \omega_1^{(1)}.\end{aligned}$$

(d) If $\Phi = B_n$ for $n \neq 3, 4$, then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{2 \leq i \leq n-1} (-\omega_{i-1} + \omega_i)^{(1)} \oplus \omega_1^{(1)}.$$

(e) If $\Phi = C_n$, then

$$\begin{aligned}\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} &\oplus \bigoplus_{2 \leq i \leq n-2} (-\omega_{i-1} + \omega_i - \omega_{n-1} + \omega_n)^{(1)} \\ &\oplus (-\omega_{n-2} + \omega_n)^{(1)} \oplus (\omega_1 - \omega_{n-1} + \omega_n)^{(1)}.\end{aligned}$$

(f) If $\Phi = D_4$, then

$$\begin{aligned}\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} &\oplus (\omega_1 - \omega_2 + \omega_3)^{(1)} \oplus (\omega_2 - \omega_4)^{(1)} \oplus (\omega_2 - \omega_3)^{(1)} \\ &\oplus (\omega_1 - \omega_2 + \omega_4)^{(1)} \oplus (-\omega_2 + \omega_3 + \omega_4)^{(1)} \oplus (-\omega_1 + \omega_2)^{(1)}\end{aligned}$$

(g) If $\Phi = D_n$, for $n \geq 5$, then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus \bigoplus_{i=1}^{n-3} (-\omega_i + \omega_{i+1})^{(1)} \oplus (\omega_1)^{(1)} \oplus (-\omega_{n-2} + \omega_{n-1} + \omega_n)^{(1)}$$

(h) If $\Phi = F_4$, then

$$\begin{aligned}\mathrm{H}^2(B_1,k) \cong (\mathfrak{u}^*)^{(1)} &\oplus (-\omega_1+\omega_2-\omega_4)^{(1)} \oplus (-\omega_1+\omega_2-\omega_3+\omega_4)^{(1)} \\ &\oplus (\omega_1-\omega_3+\omega_4)^{(1)} \oplus (\omega_1-\omega_2+\omega_3)^{(1)} \oplus (\omega_2-\omega_3)^{(1)} \oplus (\omega_1-\omega_4)^{(1)}.\end{aligned}$$

(i) If $\Phi = G_2$, then

$$\mathrm{H}^2(B_1,k) \cong (\mathfrak{u}^*) \oplus (-\omega_1+\omega_2)^{(1)}$$

Proof. Once the weights from Theorem 2.4.1 are expressed in terms of the fundamental weights, then check which ones are T_1 invariant (i.e. multiples of 2). For example, in type A_3, the weight $\alpha_1+\alpha_3 = 2\omega_1 - 2\omega_2 + 2\omega_3$ $\alpha_1+2\alpha_2+\alpha_3 = 2\omega_2$. Therefore, the weights $\alpha_1+\alpha_3 = s_{\alpha_1}s_{\alpha_3}\cdot 0$ and $\alpha_1+2\alpha_2+\alpha_3 = s_{\alpha_1}s_{\alpha_3}+2\alpha_2$ are in the $\mathrm{H}^2(B_1,k)$ cohomology. □

4.3 Case ii: λ arbitrary

Recall the fact that if $\lambda \neq 0$ and $\mathrm{H}^2(B_1,\lambda)\neq 0$, then λ is of the form $\lambda = w\cdot 0 + 2\nu$ for some $w\in W$ and $\nu \in X(T)$. (In our case, $l(w)=2$.) The following lemma gives the unique weight ν such that $\lambda = w\cdot 0+2\nu \in X_1(T)$.

Lemma 4.3.1. *Let $p=2$. For $w = s_{\alpha_i}s_{\alpha_j}\in W$ with $i<j$ we define*

$$\nu_w = \begin{cases} \omega_i - \omega_k + \omega_j, & \text{if } \alpha_i,\alpha_j \text{ separated by a single vertex, } \alpha_k \\ \omega_i+\omega_j, & \text{otherwise}\end{cases}$$

except in the following cases:

$$\nu_w = \begin{cases} \omega_{n-3}-\omega_{n-2}+\omega_{n-1}-\omega_n, & \text{if } \Phi = B_n, w = s_{\alpha_{n-3}}s_{\alpha_{n-1}} \\ \omega_i+\omega_{n-1}-\omega_n, & \text{if } \Phi = B_n, w = s_{\alpha_i}s_{\alpha_{n-1}},\ i\neq n-3 \\ \omega_i-\omega_{n-1}+\omega_n, & \text{if } \Phi = C_n, w = s_{\alpha_i}s_{\alpha_n},\ i\neq n-2 \\ -\omega_{n-2}+2\omega_{n-1}, & \text{if } \Phi = C_n, w = s_{\alpha_{n-1}}s_{\alpha_n}\end{cases}$$

Then $s_{\alpha_i}s_{\alpha_j}\cdot 0 + 2\nu_w \in X_1(T)$.

Proof. We are only interested in weights λ in $\mathrm{H}^2(U_1, k)$ that could add to the B_1-cohomology. For λ a weight in $\mathrm{H}^2(U_1, k)$, then write $\lambda = s_{\alpha_i} s_{\alpha_j} \cdot 0 + 2\nu$. Note that a_i and α_j are not connected except when Φ is of type C_n and $w = s_{\alpha_{n-1}} s_{\alpha_n}$. Without loss of generality we can assume $i < j$. The ν_w is determined after writing $w \cdot 0$ in the fundamental weight basis. □

Now that the unique ν_w is identified, $\mathrm{H}^2(B_1, \lambda)$ can easily be calculated for arbitrary λ.

Theorem 4.3.2. *If $p = 2$, then as a T-module,*

$$\mathrm{H}^2(B_1, \lambda) = 0$$

except in the following cases.

(a) If $\lambda = s_{\alpha_i} s_{\alpha_j} \cdot 0 + 2\nu_w \in X_1(T)$ for α_i, α_j not separated by a single vertex, then

$$\mathrm{H}^2(B_1, \lambda) = \nu_w^{(1)}.$$

(b) $\lambda = s_{\alpha_i} s_{\alpha_j} \cdot 0 + 2\nu_w \in X_1(T)$ for α_i, α_j separated by a single vertex, (i.e. j=i+2) then

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus \omega_{i+1}^{(1)}.$$

except in the following cases

(i) If $\Phi = B_n$ and $j = n$, then

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_{n-1} - \omega_n)^{(1)}$$

(ii) If $\Phi = B_n$ and $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$, then

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_{n-2} - \omega_n)^{(1)}$$

(iii) If $\Phi = B_n$ and $w = s_{\alpha_i} s_{\alpha_{n-1}}$ for $i \neq n-3$, then

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_i + \omega_n)^{(1)}.$$

(iv) If $\Phi = C_n$ *and* $w = s_{\alpha_i} s_{\alpha_n}$ *for* $i \neq n-1$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_i - \omega_{n-2} + \omega_{n-1})^{(1)}.$$

(v) If $\Phi = C_n$ *and* $w = s_{\alpha_{n-1}} s_{\alpha_n}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)}.$$

(vi) If $\Phi = D_n$ *and* $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_{n-2} - \omega_n)^{(1)} \oplus (\omega_n)^{(1)}$$

(vii) If $\Phi = D_n$ *and* $w = s_{\alpha_{n-3}} s_{\alpha_n}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_{n-2} - \omega_{n-1})^{(1)} \oplus (\omega_{n-1})^{(1)}$$

(viii) If $\Phi = E_6$ *and* $w = s_{\alpha_2} s_{\alpha_3}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_4 - \omega_5)^{(1)} \oplus (\omega_5 - \omega_6)^{(1)} \oplus \omega_6^{(1)}$$

(ix) If $\Phi = E_6, E_7, E_8$ *and* $w = s_{\alpha_2} s_{\alpha_5}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_4 - \omega_3)^{(1)} \oplus (\omega_3 - \omega_1)^{(1)} \oplus \omega_1^{(1)}$$

(x) If $\Phi = E_6, E_7, E_8$ *and* $w = s_{\alpha_3} s_{\alpha_5}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_4 - \omega_2)^{(1)} \oplus \omega_2^{(1)}$$

(xi) If $\Phi = E_7$ *and* $w = s_{\alpha_2} s_{\alpha_3}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_4 - \omega_5)^{(1)} \oplus (\omega_5 - \omega_6)^{(1)} \oplus (\omega_6 - \omega_7)^{(1)} \oplus \omega_7^{(1)}$$

(xii) If $\Phi = E_8$ *and* $w = s_{\alpha_2} s_{\alpha_3}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong \nu_w^{(1)} \oplus (\omega_4 - \omega_5)^{(1)} \oplus (\omega_5 - \omega_6)^{(1)} \oplus (\omega_6 - \omega_7)^{(1)} \oplus (\omega_7 - \omega_8)^{(1)} \oplus \omega_8^{(1)}$$

(xiii) If $\Phi = F_4$ *and* $w = s_{\alpha_1}s_{\alpha_3}$*, then*

$$\mathrm{H}^2(B_1,\lambda) \cong \nu_w^{(1)} \oplus (\omega_2-\omega_3)^{(1)} \oplus (\omega_3-\omega_4)^{(1)}$$

Proof. The relationship between the U_1 and the B_1 cohomology (as a T-module), as given in equation 1.4.2, gives us,

$$\mathrm{H}^2(B_1,\lambda) \cong \mathrm{H}^2(U_1,k)_{-\bar{\lambda}} \cong \bigoplus_{\nu} \mathrm{H}^2(U_1,k)_{-\lambda+p\nu},$$

for $\lambda \in X_1(T)$. From Lemma 4.3.1, the unique weight ν_w such that $\lambda = s_{\alpha_i}s_{\alpha_j} + p\nu_w \in X_1(T)$. So we now have that

$$\mathrm{H}^2(B_1,\lambda) \cong \bigoplus_{\sigma} \mathrm{H}^2(U_1,k)_{\lambda+2\sigma}.$$

We want to find σ such that $\lambda = -(s_{\alpha_i}s_{\alpha_j}) + 2\sigma$ is a weight of $\mathrm{H}^2(U_1,k)$.

For example, if Φ is of type B_4 and $w = S_{\alpha_2}s_{\alpha_4}$, then $\lambda = s_{\alpha_2}s_{\alpha_4}\cdot 0 + 2\nu_w$. Writing the weights $\alpha_2+\alpha_4$ and $\alpha_2+2\alpha_3+\alpha_4$ in terms of λ, we have $s_{\alpha_2}s_{\alpha_4}+2\nu_w$ and $s_{\alpha_2}s_{\alpha_4}+2\nu_w+2\alpha_3$. So, $\sigma = \nu_w$ for the first term and $\sigma = \nu_w + 2\alpha_3 = \omega_3 - \omega_4$ for the second term.

□

4.4 B-module structure

Now the T structure of $\mathrm{H}^2(B_1,\lambda)$ can be used to determine the B-module structure. Recall that $\mathrm{H}^2(B_1,\lambda)$ is a subquotient of $S^2(\mathfrak{u}^*)_{-\lambda}$. Since B acts on $\mathfrak{u}$ by the adjoint action, then for $\alpha \in \Phi$, $\mathrm{Dist}(B) = \left\langle \binom{H_i}{m}, \frac{X_\alpha^n}{n!} \right\rangle$ acts on $\mathfrak{u}^*$. Where $H_i = (d\phi_i)(1)$ where ϕ_i are a basis for $\mathrm{Hom}(G_m,T) \cong \mathbb{Z}^r$ [Jan1, II.1.11]. In particular since $\mathfrak{u}$ corresponds to the negative roots, then it's only necessary to look at $\frac{X_\alpha^n}{n!}$ for $\alpha \in \Phi^-$. The action from [Hum, 26.3] is defined by

$$\frac{X_\alpha^n}{n!}(u \otimes v) = \sum_k \left(\frac{X_\alpha^k}{k!}u \otimes \frac{X_\alpha^{n-k}}{(n-k)!}v\right). \tag{4.4.1}$$

Using the results in the previous section. If $\mathrm{H}^2(B_1,\lambda)$ has an answer consisting of only one factor, then this forms a 1-dimensional module. However, it remains to be determined if the cohomology m-dimensional submodule, and whether it is an indecomposable module. As before, we will first look at this with the trivial module, then move on to arbitrary weights.

4.5 CASE I: $\lambda = 0$

Theorem 4.5.1. *Let $p = 2$ then as a B-module, $\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)}$. Except in the following cases:*

(a) $\Phi = A_3$, then $\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M$, where M is a 2-dimensional indecomposable B-module with factors $(\omega_1 - \omega_2 + \omega_3)^{(1)}$ and $\omega_2^{(1)}$

(b) If $\Phi = B_3$, then $\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M$, where M is a 2-dimensional indecomposable B-module with factors $(\omega_1)^{(1)}$ and $(-\omega_1 + \omega_2)^{(1)}$

(c) If $\Phi = B_4$, then $\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M_1 \oplus M_2$, where M_1 is a 2-dimensional indecomposable B-module with factors $(\omega_2 - \omega_4)^{(1)}$, and $(\omega_1 - \omega_2 + \omega_3 - \omega_4)^{(1)}$, and M_2 is a 3-dimensional indecomposable B-module with factors $\omega_1^{(1)}, (-\omega_1 + \omega_2)^{(1)}, (-\omega_2 + \omega_3)^{(1)}$

(d) If $\Phi = B_n$, then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M$$

where M is an $(n-1)$-dimensional indecomposable B-module with factors, $\omega_1^{(1)}, (-\omega_1 + \omega_2)^{(1)}, (-\omega_2 + \omega_3)^{(1)}, \ldots, (-\omega_{n-2} + \omega_{n-1})^{(1)}$

(e) If $\Phi = C_n$, then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M$$

where M is an $(n-1)$-dimensional indecomposable B-module with factors $(\omega_1 - \omega_{n-1} + \omega_n)^{(1)}, (-\omega_1 + \omega_2 - \omega_{n-1} + \omega_n)^{(1)}, (-\omega_2 + \omega_3 - \omega_{n-1} + \omega_n)^{(1)}, \ldots,$ $(-\omega_{n-3} + \omega_{n-2} - \omega_{n-1} + \omega_n)^{(1)}, (-\omega_{n-2} + \omega_n)^{(1)}$.

(f) If $\Phi = D_4$, then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M_1 \oplus M_2 \oplus M_3,$$

where M_1 is a 2-dimensional indecomposable B-module with factors $(\omega_2 - \omega_4)^{(1)}$ and $(\omega_1 - \omega_2 + \omega_3)^{(1)}$; M_2 is an a 2-dimensional indecomposable B-module with factors

$(\omega_2 - \omega_3)^{(1)}$ and $(\omega_1 - \omega_2 + \omega_4)^{(1)}$; M_3 is a 3-dimensional indecomposable B-module with factors $\omega_1^{(1)}, (-\omega_1 + \omega_2)^{(1)}, (-\omega_2 + \omega_3 + \omega_4)^{(1)}$.

(g) If $\Phi = D_n$, then $\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M$, where M is a $(n-1)$-dimensional indecomposable B-module with factors
$\omega_1^{(1)}, (-\omega_1 + \omega_2)^{(1)}, (-\omega_2 + \omega_3)^{(1)}, \ldots, (-\omega_{n-3} + \omega_{n-2})^{(1)}, (-\omega_{n-2} + \omega_{n-1} + \omega_n)^{(1)}$.

(h) If $\Phi = F_4$ then

$$\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus M$$

where M is a 6-dimensional indecomposable module with factors
$(\omega_2 - \omega_3)^{(1)}(\omega_1 - \omega_2 + \omega_3)^{(1)}, (\omega_1 - \omega_3 + \omega_4)^{(1)}, (-\omega_1 + \omega_2 - \omega_3 + \omega_4)^{(1)}$,
$(\omega_1 - \omega_4)^{(1)}, (-\omega_1 + \omega_2 - \omega_4)^{(1)}$.

(i) If $\Phi = G_2$, then $\mathrm{H}^2(B_1, k) \cong (\mathfrak{u}^*)^{(1)} \oplus (-\omega_1 + \omega_2)^{(1)}$.

Proof. Using 4.4.1 and Theorem 4.2.1, then we can determine which weights combine as the factors of an indecomposable module. For example, consider the case when Φ is of type A_4 and $w = s_{\alpha_1} s_{\alpha_3}$. The cohomology classes for this w are

$$\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_2+\alpha_3} \text{ and}$$
$$\phi_{\alpha_1} \otimes \phi_{\alpha_3}$$

$$X_{-\alpha_2}^2(\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_2+\alpha_3}) = \phi_{\alpha_1} \otimes \phi_{\alpha_3}$$

and

$$X_{-\alpha_2}^2(\phi_{\alpha_1} \otimes \phi_{\alpha_3}) = 0$$

Since I can get from one cohomology class to the other class, then these factors form an indecomposable module. □

4.6 CASE II: λ ARBITRARY

Now, let's consider the cohomology for arbitrary λ.

Theorem 4.6.1. *Let $p = 2$ and $\lambda = w \cdot 0 + p\nu_w$ where $w = s_{\alpha_i} s_{\alpha_{i+2}}$, then*

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M is a 2-dimensional indecomposable B-module with $hd_B\, M \cong \omega_{i+1}^{(1)}$ and $soc_B\, M = \nu_w^{(1)}$. Except in the following cases:

(a) If $\Phi = B_n$ with $w = s_{\alpha_{n-2}} s_{\alpha_n}$, then

$$\mathrm{H}^2(B_1, \lambda) \cong M$$

where M is a 2-dimensional indecomposable B-module with $hd_B\, M = (\omega_{n-1} - \omega_n)^{(1)}$ and $soc_B\, M = \nu_w^{(1)}$.

(b) If $\Phi = B_n$ and $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$, then

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M is a 2-dimensional indecomposable B-module with $hd_B\, M = (\omega_{n-2} - \omega_n)^{(1)}$ and $soc_B\, M = \nu_w^{(1)}$.

(c) If $\Phi = B_n$ and $w = s_{\alpha_i} s_{\alpha_{n-1}}$ with $i \neq n-3$, then

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M is a 2-dimensional indecomposable B-module with $hd_B\, M = (\omega_i + \omega_n)^{(1)}$ and $soc_B\, M = \nu_w^{(1)}$.

(d) If $\Phi = C_n$ and $w = s_{\alpha_i} s_{\alpha_n}$ with $i \neq n-2$, then

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M is a 2-dimensional indecomposable B-module with $hd_B\, M = (\omega_i - \omega_{n-2} + \omega_{n-1})^{(1)}$ and $soc_B\, M = \nu_w^{(1)}$.

(e) If $\Phi = D_n$ *and* $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$ *then*

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M *is a 3-dimensional indecomposable* B*-module with factors* $(\omega_n)^{(1)}, (\omega_{n-2} - \omega_n)^{(1)}, (\omega_{n-3} - \omega_{n-2} + \omega_{n-1})^{(1)}$.

(f) If $\Phi = D_n$ *and* $w = s_{\alpha_{n-3}} s_{\alpha_n}$ *then*

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M *is a 3-dimensional indecomposable* B*-module with factors* $\omega_{n-1}^{(1)}, (\omega_{n-2} - \omega_{n-1})^{(1)}, (\omega_{n-3} - \omega_{n-2} + \omega_n)^{(1)}$.

(g) If $\Phi = E_6$ *and* $w = s_{\alpha_2} s_{\alpha_3}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M *is a 4-dimensional indecomposable* B*-module with factors* $\omega_6^{(1)}, (\omega_5 - \omega_6)^{(1)}, (\omega_4 - \omega_5)^{(1)}, \nu_w^{(1)}$.

(h) If $\Phi = E_6, E_7, E_8$ *and* $w = s_{\alpha_2} s_{\alpha_5}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M *is a 4-dimensional indecomposable* B*-module with factors* $\omega_1^{(1)}, (\omega_3 - \omega_1)^{(1)}, (\omega_4 - \omega_3)^{(1)}, \nu_w^{(1)}$.

(i) If $\Phi = E_6, E_7, E_8$ *and* $w = s_{\alpha_3} s_{\alpha_5}$*, then*

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M *is a 3-dimensional indecomposable* B*-module with factors* $\omega_2^{(1)}, (\omega_4 - \omega_2)^{(1)}, \nu_w^{(1)}$.

(j) If $\Phi = E_7$ and $w = s_{\alpha_2}s_{\alpha_3}$, then

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M is a 5-dimensional indecomposable B-module with factors $\omega_7^{(1)}, (\omega_6 - \omega_7)^{(1)}, (\omega_5 - \omega_6)^{(1)}, (\omega_4 - \omega_5)^{(1)}, \nu_w^{(1)}$.

(k) If $\Phi = E_8$ and $w = s_{\alpha_2}s_{\alpha_3}$, then

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M is a 6-dimensional indecomposable B-module with factors $\omega_8^{(1)}, (\omega_7 - \omega_8)^{(1)}, (\omega_6 - \omega_7)^{(1)}, (\omega_5 - \omega_6)^{(1)}, (\omega_4 - \omega_5)^{(1)}, \nu_w^{(1)}$.

(l) If $\Phi = F_4$ and $w = s_{\alpha_1}s_{\alpha_3}$, then

$$\mathrm{H}^2(B_1, \lambda) \cong M,$$

where M is a 3-dimensional indecomposable B-module with factors $(\omega_3 - \omega_4)^{(1)}, (\omega_2 - \omega_3)^{(1)}, (\omega_1 - \omega_2 + \omega_3)^{(1)}$.

Proof. For (i), this comes from the previous theorem for $\mathrm{H}^2(B_1, k)$ as a B-module. Following the setup found in the beginning of the section. For example, consider the case when $\Phi = A_4$ and $w = s_{\alpha_1}s_{\alpha_3}$. Then, the cohomology classes for this w are

$$\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_2+\alpha_3} \text{ and}$$
$$\phi_{\alpha_1} \otimes \phi_{\alpha_3}.$$

Consider

$$\frac{X_{-a_2}^2}{2!}(\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_2+\alpha_3}) = \phi_{\alpha_1} \otimes \phi_{\alpha_3}$$

and

$$\frac{X_{-a_2}^2}{2!}(\phi_{\alpha_1} \otimes \phi_{\alpha_3}) = 0$$

Since I can get from one cohomology class to another, then this forms an indecomposable module. A similar calculation is done with the other cases. All of the cohomology classes are found in Appendix A. The factors in the modules are those in Theorem 4.3.2. □

Remark 4.6.2. For the indecomposable B-modules in the preceding theorems, the factors are listed in order with the first factor listed is the head and the last factor listed is the socle.

Corollary 4.6.3. *Let* $p = 2$ *and* $\lambda, \gamma \in X(T)$.

(a) If $\lambda \notin pX(T)$ *and* $\lambda \neq w \cdot 0 + p\sigma$ *for some* $w \in W$ *with* $l(w) = 2$ *and* $\sigma \in X(T)$*, then* $\mathrm{H}^2(B_1, \lambda) = 0$.

(b) If $\alpha \in \Delta$*, then* $\mathrm{H}^2(B_1, p\gamma - \alpha) = 0$.

Chapter 5

B_r-Cohomology

To calculate $\mathrm{H}^2(B_r, \lambda)$ for $r > 1$, we will first state a few lemmas for specific weights before stating the final theorem.

5.1 case i: $\lambda = 0$

To calculate the B_r-cohomology, we will first look at the case when $\lambda = 0$.

Lemma 5.1.1. *Let $p = 2$ and $\lambda \in X(T)$. Then*

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1, k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

except in the following cases:

(a) If $\Phi = A_3$, then

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1, k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \omega_2,\ \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(b) If $\Phi = B_3$, then

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1, k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (\omega_2 - \omega_1),\ \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(c) If $\Phi = B_4$*, then*

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (-\omega_2 + \omega_3),\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (\omega_1 - \omega_2 + \omega_3 - \omega_4), \\ & \quad \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(d) If $\Phi = B_n$*, then*

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (-\omega_{n-2} + \omega_{n-1}), \\ & \quad \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(e) If $\Phi = C_n$*, then*

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (-\omega_{n-2} + \omega_n)\ \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(f) If $\Phi = D_4$*, then*

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (\omega_1 - \omega_2 + \omega_i),\ i \in \{3,4\}, \\ & \quad \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (-\omega_2 + \omega_3 + \omega_4), \\ & \quad \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(g) If $\Phi = D_n$, then

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (-\omega_{n-2} + \omega_{n-1} + \omega_n), \\ & \quad \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(h) If $\Phi = F_4$, then

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (-\omega_1 + \omega_2 - \omega_4), \\ & \quad \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

(i) If $\Phi = G_2$, then

$$\mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) \cong \begin{cases} p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - \alpha,\ \alpha \in \Delta,\ \gamma \in X(T), \\ p^r\gamma, & \text{if } \lambda = p^{r-l}\gamma - (-\omega_1 + \omega_2),\ \gamma \in X(T), \\ 0 & \text{otherwise} \end{cases}$$

Proof. We have

$$\begin{aligned} \mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes p^l\lambda) &\cong \mathrm{Hom}_{B_{r-l}}(k, \mathrm{H}^2(B_1,k)^{(-1)} \otimes \lambda)^{(l)} \\ &\cong \mathrm{Hom}_{B_{r-l}}(-\lambda, \mathrm{H}^2(B_1,k)^{(-1)})^{(l)}. \end{aligned}$$

with $\mathrm{H}^2(B_1,k)$ stated in Theorem 5.3, it's necessary to consider the B-socle of $\mathrm{H}^2(B_1,k)$. In general, this is the B-socle of $\mathfrak{u}^*$, which is $\sum_{\beta\in\Delta} k_\beta$ by [Jan2]. However, when the cohomology is not $\mathfrak{u}^*$, then this accounts for the additional weights. (These extra cases come from the special cases in Theorem 4.5.1.) □

With the use of this lemma gives us an easy formula to compute $\mathrm{H}^2(B_r,k)$ from $\mathrm{H}^2(B_1,k)$. However, to define this, the use the Lyndon-Hochschild-Serre (LHS) spectral sequence is necessary, which is defined for $B_1 \trianglelefteq B_r$ as follows:

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(B_r/B_1, \mathrm{H}^j(B_1,\lambda)) \Longrightarrow \mathrm{H}^{i+j}(B_r,\lambda).$$

Proposition 5.1.2. *Let* $p = 2$. *Then* $\mathrm{H}^2(B_r, k) \cong \mathrm{H}^2(B_1, k)^{(r-1)}$.

Proof. To prove this, we will use induction on r. Assume that $r > 1$. Now, let's look at $\mathrm{E}_2^{i,1} = \mathrm{H}^i(B_r/B_1, \mathrm{H}^1(B_1, k))$. By [Jan2] we know that $\mathrm{H}^1(B_1, k) = 0$ and thus $\mathrm{E}_2^{i,1} = 0$. Now applying the previous lemma with $l = 1$ and $\lambda = 0$, we have $\mathrm{E}_2^{0,2} = \mathrm{Hom}_{B_r/B_1}(k, \mathrm{H}^2(B_r/B_1, k)) = 0$. Since all differentials going into and out of $E_2^{2,0}$ are zero, and by the induction hypothesis we have that

$$\mathrm{H}^2(B_r, k) \cong \mathrm{E}_2^{2,0} = \mathrm{H}^2(B_{r-1}, k)^{(1)} \cong \mathrm{H}^2(B_1, k)^{(r-1)}.$$

□

5.2 CASE II: $\lambda = p^l\sigma$

Lemma 5.2.1. *Let* $0 \leq l < r$, *and* $\alpha \in \Delta$.

(a) Then

$$\mathrm{H}^2(B_r, -p^l\alpha) \cong \begin{cases} k & \text{if } l > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Suppose Φ *is of type* A_3. *Then*

$$\mathrm{H}^2(B_r, -p^l\omega_2) \cong \begin{cases} k & \text{if } l > 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) Suppose Φ *is of type* B_3. *Then*

$$\mathrm{H}^2(B_r, -p^l(\omega_2 - \omega_1)) \cong \begin{cases} k & \text{if } l > 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) Suppose Φ *is of type* B_4 *and* $\lambda = -p^l(-\omega_2 + \omega_3)$ *or* $\lambda = -p^l(\omega_1 - \omega_2 + \omega_3 - \omega_4)$. *Then*

$$\mathrm{H}^2(B_r, \lambda) \cong \begin{cases} k & \text{if } l > 0 \\ 0 & \text{otherwise} \end{cases}$$

(e) Suppose Φ is of type B_n. Then

$$\mathrm{H}^2(B_r, -p^l(-\omega_{n-2}+\omega_{n-1})) \cong \begin{cases} k & \textit{if } l > 0 \\ 0 & \textit{otherwise} \end{cases}$$

(f) Suppose Φ is of type C_n. Then

$$\mathrm{H}^2(B_r, -p^l(-\omega_{n-2}+\omega_{n})) \cong \begin{cases} k & \textit{if } l > 0 \\ 0 & \textit{otherwise} \end{cases}$$

(g) Suppose Φ is of type D_4 and $\lambda = -p^l(\omega_1 - \omega_2 + \omega_i)$ for $i \in \{3,4\}$ or $\lambda = -p^l(-\omega_2 + \omega_3 + \omega_4)$. Then

$$\mathrm{H}^2(B_r, \lambda) \cong \begin{cases} k & \textit{if } l > 0 \\ 0 & \textit{otherwise} \end{cases}$$

(h) Suppose Φ is of type D_n. Then

$$\mathrm{H}^2(B_r, -p^l(-\omega_{n-2}+\omega_{n-1}+\omega_n)) \cong \begin{cases} k & \textit{if } l > 0 \\ 0 & \textit{otherwise} \end{cases}$$

(i) Suppose Φ is of type F_4. Then

$$\mathrm{H}^2(B_r, -p^l(-\omega_1+\omega_2-\omega_4)) \cong \begin{cases} k & \textit{if } l > 0 \\ 0 & \textit{otherwise} \end{cases}$$

(j) Suppose Φ is of type G_2. Then

$$\mathrm{H}^2(B_r, -p^l(\omega_2-\omega_1)) \cong \begin{cases} k & \textit{if } l > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. (a) When $r = 1$, $\mathrm{H}^2(B_1, -\alpha) = 0$ by Corollary 4.6.3. Now assume $r > 1$ and consider the case that $l = 0$. We will use the LHS spectral sequence

$$E_2^{i,j} = \mathrm{H}^2(B_r/B_1, \mathrm{H}^j(B_1, -\alpha)) \Rightarrow \mathrm{H}^{i+j}(B_r, -\alpha).$$

$E_2^{i,0} = \mathrm{H}^i(B_r/B_1, \mathrm{Hom}_B(k, -\alpha)) = 0$ and $E_2^{i,2} = \mathrm{H}^i(B_r/B_1, \mathrm{H}^2(B_1, -\alpha)) = 0$, from Theorem 4.6.1. Consider $E_2^{1,1} = \mathrm{H}^1(B_r/B_1, \mathrm{H}^1(B_1, -\alpha))$. Let's first look at $\mathrm{H}^1(B_1, -\alpha)$, which is given to us in [Jan2, 3.5]. In general $\mathrm{H}^1(B_1, -\alpha) = k$ except when Φ is of type A_3, D_4, D_n in the following cases.

(i) If $\Phi = A_3$, then

$$\mathrm{H}^1(B_1, -\alpha_1) = k \oplus (-\omega_1 + \omega_3)^{(1)}$$

and

$$\mathrm{H}^1(B_1, -\alpha_3) = k \oplus (\omega_1 - \omega_3)^{(1)}$$

(ii) If $\Phi = D_4$, then

$$\mathrm{H}^1(B_1, -\alpha_1) = k \oplus (-\omega_1 + \omega_3)^{(1)} \oplus (-\omega_1 + \omega_4)^{(1)},$$

$$\mathrm{H}^1(B_1, -\alpha_3) = k \oplus (\omega_1 - \omega_3)^{(1)} \oplus (-\omega_3 + \omega_4)^{(1)},$$

and

$$\mathrm{H}^1(B_1, -\alpha_4) = k \oplus (\omega_1 - \omega_4)^{(1)} \oplus (\omega_3 - \omega_4)^{(1)}.$$

(iii) If $\Phi = D_n$, then

$$\mathrm{H}^1(B_1, -\alpha_{n-1}) = k \oplus (-\omega_{n-1} + \omega_n)^{(1)}$$

and

$$\mathrm{H}^1(B_1, -\alpha_n) = k \oplus (\omega_{n-1} - \omega_n)^{(1)}.$$

So, now we can apply [BNP1, Thm 2.8(C)] and we have that $E_2^{1,1}$ vanishes. Hence $E_2^{2,0} = E_2^{1,1} = E_2^{0,2} = 0$ in all cases and so $\mathrm{H}^2(B_r, -\alpha) = 0$.

Now assume $l > 0$. We use the LHS spectral sequence again.

$$E_2^{i,j} = \mathrm{H}^i(B_r/B_l, \mathrm{H}^j(B_l, -p^l\alpha)) \Rightarrow \mathrm{H}^{i+j}(B_r, -p^l\alpha).$$

First consider $E_2^{i,1} = \mathrm{H}^i(B_r/B_l, \mathrm{H}^1(B_l,k) \otimes -p^l\alpha)$. By [BNP1, Thm 2.8(C)] we have that $E_2^{i,1} = 0$. From the case $l = 0$, we have

$$E_2^{2,0} = \mathrm{H}^2(B_r/B_l, -p^l\alpha) \cong \mathrm{H}^2(B_{r-l}, -\alpha)^{(l)} = 0.$$

So, $\mathrm{H}^2(B_r, -p^l\alpha) = E_2^{0,2}$. From Lemma 5.1.1, one can conclude

$$E_2^{0,2} = \mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_l,k) \otimes p^l\alpha) \cong \mathrm{Hom}_{B_r/B_l}(k, \mathrm{H}^2(B_1,k)^{(l-1)} \otimes -p^l\alpha) \cong k.$$

Hence, the result follows.

For (b)-(j) the argument is analogous. As before, the case $r = 1$ follows from Corollary 4.6.3. For $r > 1$ and $l = 0$, we use the spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(B_r/B_1, \mathrm{H}^j(B_1, -\lambda)) \Rightarrow \mathrm{H}^{i+j}(B_r, -\lambda)$$

for λ defined as in the statement of the theorem for these cases. Note that in all of these cases, $\mathrm{H}^1(B_1, -\omega_i) = 0$ and so one immediately gets $E_2^{i,1} = 0$ and as before $E_2^{i,0} = 0 = E_2^{i,2}$. For the case $l > 0$, the same argument is used as the $-p^l\alpha$ case. □

5.3 CASE III: λ ARBITRARY

To compute $\mathrm{H}^2(B_r, \lambda)$ for $\lambda \notin p^r X(T)$, let's first start with some special computations when Φ is not simply laced. Define the following indecomposable B-modules, where all of the factors are listed from top to bottom:

- N_{B_n} is the two-dimensional indecomposable B-module with factors α_3 and k. Furthermore, $N_{B_n} \cong \mathrm{H}^1(B_1, -\alpha_{n-1})^{(-1)}$ [Jan2, 3.6].
- N_{C_n} is the n-dimensional indecomposable B-module with factors $\omega_1, \omega_2 - \omega_1, \omega_3 - \omega_2, \ldots, \omega_n - \omega_{n-1}$. Furthermore, $N_{C_n} \cong \mathrm{H}^1(B_1, k)^{(-1)}$. [Jan2, 3.6]
- N_{F_4} is the three-dimensional indecomposable B-module with factors $\alpha_3 + \alpha_4$, α_3, k. Furthermore $N_{F_4} \cong \mathrm{H}^1(B_1, -\alpha_2)^{(-1)}$ [Jan2, 3.6].

- N_{G_2} is the two-dimensional indecomposable B-module with factors α_2 and k.

  Furthermore $N_{G_2} \cong \mathrm{H}^1(B_1, -\alpha_2)^{(-1)}$ [Jan2, 3.7].

Notice that if N is one of the above modules, then $N \otimes \lambda$ remains indecomposable for any weight λ.

Lemma 5.3.1. *(a) If Φ is of type B_n, F_4, G_2, then $\mathrm{H}^1(U_1, N) \cong \mathrm{H}^1(\mathfrak{u}, N)$, where N is one of the modules defined above.*

(b) If Φ is of type B_n, F_4, G_2, then $\mathrm{H}^1(U_1, N)$ is defined as follows:

(i) $\mathrm{H}^1(U_1, N_{B_n})$ *has a T-basis* $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1}+\alpha_n, 2\alpha_n, \alpha_{n-1}+2\alpha_n\}$.

(ii) $\mathrm{H}^1(U_1, N_{C_n})$ *has a T-basis*

$\{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \alpha_n, \alpha_{n-1}+\alpha_n, 2\alpha_{n-1}, 2\alpha_{n-1}+\alpha_n, \alpha_{n-2}+2\alpha_{n-1}\}$

(iii) $\mathrm{H}^1(U_1, N_{F_4})$ *has a T-basis* $\{\alpha_1, \alpha_2, \alpha_4, \alpha_2+\alpha_3, 2\alpha_3, \alpha_2+2\alpha_3, 2\alpha_3+\alpha_4\}$.

(iv) $\mathrm{H}^1(U_1, N_{G_2})$ *has a T-basis* $\{\alpha_1, \alpha_1+\alpha_2, 2\alpha_2, 2\alpha_1+\alpha_2\}$.

Proof. (a) Consider the following exact sequence:

$$0 \to \mathrm{H}^1(U_1, N) \hookrightarrow \mathrm{H}^1(\mathfrak{u}, N) \to \mathrm{Hom}^s(\mathfrak{u}, N^{\mathfrak{u}})$$

where $\mathrm{Hom}^s(\mathfrak{u}, N^{\mathfrak{u}})$ is the set of all maps that are additive and satisfy the property: $\phi(ax) = a^p\phi(x)$. In our case we have that $\mathrm{Hom}^s(\mathfrak{u}, N^{\mathfrak{u}}) \cong (\mathfrak{u}^*)^{(1)}$ So, we now have the following exact sequence: $0 \to \mathrm{H}^1(U_1, N) \hookrightarrow \Lambda^1(\mathfrak{u}^*) \otimes N \to (\mathfrak{u}^*)^{(1)}$ Since $\mathfrak{u}^*$ is spanned by the negative roots, then this last map is the 0-map. Therefore, $\mathrm{H}^1(U_1, N) \cong \mathrm{H}^1(\mathfrak{u}, N)$.
(b) To calculate $\mathrm{H}^1(\mathfrak{u}, N)$, we use [Jan1, I.9.15]: $k \otimes N \to \mathfrak{u}^* \otimes N \to \Lambda^2(\mathfrak{u}^*) \otimes N$.
So, it is first necessary to calculate $\ker(\mathfrak{u}^* \otimes N \to \Lambda^2(\mathfrak{u}^*) \otimes N)$ by [Jan2, I.9.17]

$$d_i(m \otimes \psi) = \sum_j m_j \otimes (\phi_j \wedge \psi) + m \otimes d_i^1(\psi),$$

where d_i' is the differential defined in Section 2.1 and $\psi \in \Phi$. For example, when Φ is of type B_3, then N is an indecomposable B-module with factors $\alpha_3 = m_1$ and $k = m_0$. Then $x_\gamma \cdot m_0 = 0$ and

$$x_\gamma \cdot m_1 = \begin{cases} m_0 & \gamma = -\alpha_3 \\ 0 & \text{else} \end{cases}$$

Therefore, $d_0(m_0) = 0$ and $d_0(m_1) = x^*_{-\alpha_3} \otimes m_0$. Furthermore, we get that $d_1(m_0 \otimes \psi) = m_0 \otimes d_1'(\psi)$ and $d_1(m_1 \otimes \psi) = m_0 \otimes (x_{-\alpha_3} \wedge \psi) + m_1 \otimes d_1'(\psi)$. It is necessary to determine when any linear combination of these maps (both with the same map) returns 0. After checking all possible weights, the possible weights are $\{\alpha_1, \alpha_2, \alpha_3, \alpha_2+\alpha_3, \alpha_2+2\alpha_3, 2\alpha_3\}$. Now it is necessary to check which weights are in the previous image $k \otimes N \to \mathfrak{u}^* \otimes N$. The weight α_3 is in the previous image because α_3 is one of the factors of our module and $1 \otimes \alpha_3 = \alpha_3 \otimes 1$. The other calculations are similar. □

Theorem 5.3.2. *(a) If Φ is of type B_n, then*

$$\mathrm{H}^2(B_r, N_{B_n} \otimes \lambda)^{(-r)} \cong \begin{cases} \nu & \text{if } \lambda = p^r\nu - p^l\alpha, \alpha \in \Delta, 0 \leq l < r \text{ and } \alpha \in \Delta \\ & \quad \text{for } l \neq r-1 \text{ if } \alpha = \alpha_{n-1}, \text{ and } l \neq 0 \text{ if } \alpha = \alpha_n \\ \nu & \text{if } \lambda = p^\gamma - (\alpha_{n-1} + \alpha_n) \\ M \otimes \nu & \text{if } \lambda = p^r\nu - p^{r-1}\alpha_{n-1} \\ M \otimes \nu & \text{if } \lambda = p^r\nu \\ 0 & \text{otherwise} \end{cases}$$

where M is an indecomposable module with factors α_n and k.

(b) If Φ is of type C_n

$$\mathrm{H}^2(B_r, N_{C_n} \otimes \lambda)^{(-r)} \cong \begin{cases} \nu & \text{if } \lambda = p^r\nu - p^l\alpha, \text{ with } \alpha \in \Delta,\ 0 \leq l < r \text{ and} \\ & \quad \text{where } l \neq r-1 \text{ if } \alpha = \alpha_n, \alpha_{n-2} \text{ and } l \neq 0 \text{ if} \\ & \quad \alpha = \alpha_{n-1} \\ \nu & \text{if } \lambda = p^r\nu - (\alpha_{n-1} + \alpha_n) \\ M \otimes \nu & \text{if } \lambda = p^r\nu - p^{r-1}\alpha \text{ where } \alpha \in \{\alpha_{n-2}, \alpha_n\} \\ M \otimes \nu & \text{if } \lambda = p^r\nu \\ 0 & \text{else} \end{cases}$$

where M is an indecomposable module with factors α_{n-1} and k.

(c) If Φ is of type F_4

$$\mathrm{H}^2(B_r, N_{F_4} \otimes \lambda)^{(-r)} \cong \begin{cases} \nu & \text{if } \lambda = p^r\nu - p^l\alpha, \text{ with } \leq l < r \text{ and } \alpha \in \Delta \\ & \quad l \neq r-1 \text{ when } \alpha \in \{\alpha_2, \alpha_4\} \\ \nu & \text{if } \lambda = p^\gamma - (\alpha_2 + \alpha_3) \\ \nu & \text{if } \lambda = p^\gamma - (\alpha_3 + \alpha_4) \\ M \otimes \nu & \text{if } \lambda = p^r\nu - p^{r-1}\alpha \text{ for } \alpha \in \{a_2, \alpha_4\} \\ \nu & \text{if } \lambda = p^r\nu \\ 0 & \text{otherwise} \end{cases}$$

where M is the two-dimensional indecomposable module with factors α_3 and k.

(d) If Φ is of type G_2

$$\mathrm{H}^2(B_r, N_{G_2} \otimes \lambda)^{(-r)} \cong \begin{cases} \nu & \text{if } \lambda = p^r\nu - p^l\alpha, \alpha \in \Delta 0 \leq l < r-1, \\ & \quad l \neq 0 \text{ when } \alpha = \alpha_2 \\ \nu & \text{if } \lambda = p^\gamma - (\alpha_1 + \alpha_2) \\ M \otimes \nu & \text{if } \lambda = p^r\nu - p^{r-1}\alpha_2 \\ \nu & \text{if } \lambda = p^r\nu \\ 0 & \text{otherwise} \end{cases}$$

where M_1 is the two-dimensional indecomposable module with factors α_1 and k.

5.4 Statement of Theorem

With the use of the previous calculations, we can compute $\mathrm{H}^2(B_r, \lambda)$ for any r and $\lambda \in X_r(T)$.

Theorem 5.4.1. *Let $p = 2$ and $\lambda \in X_r(T)$. Then*

(a) If Φ is not of type A_3, B_n, C_n, D_n, F_4, or G_2, then

$$\mathrm{H}^2(B_r,\lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + p\nu) \text{ with } l(w) = 2 \text{ or } 0, \\ \nu^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0, \text{ with } l(w) = 2, \\ & \quad 0 \le l < r-1, \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha, \text{ with } 0 < l < r, \alpha \in \Delta \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha, \text{ with } 0 \le l < t < r, \\ & \quad \alpha, \beta \in \Delta \\ 0 & \text{else} \end{cases}$$

(b) If Φ is of type A_3, then

$$\mathrm{H}^2(B_r,\lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \text{if } \lambda = p^r\nu \\ \nu^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0, \text{ with } l(w) = 2, \\ & \quad 0 \le l < r-1, \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha, \text{ with } 0 < l < r, \alpha \in \Delta \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha, \text{ with} \\ & \quad 0 \le l < t < r, \alpha, \beta \in \Delta \\ \nu^{(r)} & \text{if } \lambda = p^r\nu + p^{r-1}\alpha_2 - p^l\alpha, \text{ with} \\ & \quad 0 \le l < r-1, \alpha \in \Delta \\ (\nu+\omega_1)^{(r)} \oplus (\gamma+\omega_3)^{(r)} & \text{if } \lambda = p^r\nu + p^{r-1}\omega_2 - p^l\alpha, \text{ with} \\ & \quad 0 \le l < r-1, \alpha \in \Delta \\ 0 & \text{else} \end{cases}$$

(c) If Φ is of type B_3. Then,

$$\mathrm{H}^2(B_r,\lambda)\cong\begin{cases}\mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r-1)} & \text{if } \lambda=p^{r-1}(w\cdot 0+p\nu) \text{ with } l(w)=0,2\\ \nu^{(r)} & \text{if } \lambda=p^r\nu+p^l w\cdot 0, \text{ with } l(w)=2,\\ & \quad 0\le l<r-1\\ \nu^{(r)} & \text{if } \lambda=p^r\nu-p^t\beta-p^l\alpha, \text{ with}\\ & \quad 0\le l<t<r, \alpha,\beta\in\Delta\\ \nu^{(r)} & \lambda=p^r\nu-p^l\alpha \text{ with } 0\le l<r,\ \alpha\in\Delta\\ (\nu+\omega_1)^{(r)}\oplus(\nu+\omega_3)^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_2-p^l\alpha, \text{ with}\\ & \quad 0\le l<r-1, \alpha\in\Delta\\ \nu^{(r)} & \lambda=p^r\nu-p^{l+1}(\alpha_2+\alpha_3)-p^l\alpha_2, \text{ with}\\ & \quad 0\le l<r-1\\ M_{B_3}^{(r)}\otimes\nu^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_2-p^l\alpha, \text{ with}\\ & \quad 0\le l<r-1, \alpha\in\Delta\\ M_{B_3}^{(r)}\otimes\nu^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_3-p^l\alpha_2 \text{ with}\\ & \quad 0\le l<r-1,\\ 0 & \text{else}\end{cases}$$

(d) If Φ is of type B_4. Then,

$$\mathrm{H}^2(B_r,\lambda)\cong\begin{cases}\mathrm{H}^2(B_1,w\cdot 0+p\nu)^{(r-1)} & \text{if } \lambda=p^{r-1}(w\cdot 0+p\nu) \text{ with } l(w)=0,2\\ \nu^{(r)} & \text{if } \lambda=p^r\nu+p^l w\cdot 0,\ \text{with } l(w)=2,\\ & \quad 0\le l<r-1\\ \nu^{(r)} & \text{if } \lambda=p^r\nu-p^t\beta-p^l\alpha,\ \text{with } 0\le l<t<r,\\ & \quad \alpha,\beta\in\Delta\\ \nu^{(r)} & \lambda=p^r\nu-p^l\alpha \text{ with } 0\le l<r,\ \alpha\in\Delta\\ \nu^{(r)} & \lambda=p^r\nu-p^{l+1}(\alpha_3+\alpha_4)-p^l\alpha_2,\ \text{with}\\ & \quad 0\le l<r-1\\ (\omega_1\oplus M_{B_4})^{(r)}\otimes\nu^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_i-p^l\alpha,\ \text{with } i\in\{1,3\},\\ & \quad 0\le l<r-1,\alpha\in\Delta\\ M_{B_4}^{(r)}\otimes\nu^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_4-p^l\alpha_3 \text{ with}\\ & \quad 0\le l<r-1,\\ 0 & \text{else}\end{cases}$$

(e) If Φ is of type B_n. Then,

$$\mathrm{H}^2(B_r,\lambda)\cong\begin{cases}\mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r-1)} & \text{if } \lambda=p^{r-1}(w\cdot 0+p\nu) \text{ with } l(w)=0,2\\ \nu^{(r)} & \text{if } \lambda=p^r\nu+p^l w\cdot 0, \text{ with } l(w)=2,\\ & \quad 0\le l<r-1\\ \nu^{(r)} & \text{if } \lambda=p^r\nu-p^t\beta-p^l\alpha, \text{ with } 0\le l<t<r,\\ & \quad \alpha,\beta\in\Delta\\ \nu^{(r)} & \lambda=p^r\nu-p^l\alpha \text{ with } 0\le l<r,\ \alpha\in\Delta,\\ & \quad \text{and } l\neq r-1 \text{ if } \alpha=\alpha_{n-1}\\ \nu^{(r)} & \lambda=p^r\nu-p^{l+1}(\alpha_{n-1}+\alpha_n)-p^l\alpha_{n-1} \text{ with}\\ & \quad 0\le l<r-1\\ M_{B_n}^{(r)}\otimes\nu^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_{n-1}-p^l\alpha, \text{ with}\\ & \quad 0\le l<r-1,\\ M_{B_n}^{(r)}\otimes\nu^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_n-p^l\alpha_{n-1}, \text{ with}\\ & \quad 0\le l<r-1,\alpha\in\Delta\\ 0 & \text{else}\end{cases}$$

(f) If Φ is of type C_n. Then,

$$\mathrm{H}^2(B_r,\lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + p\nu) \text{ with } l(w) = 0, 2 \\ \nu^{(r)} & \text{if } \lambda = p^r\nu + p^l(w\cdot 0) \text{ with } 0 \leq l < r-1, \\ & \quad l(w) = 2 \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha \text{ with } 0 \leq l < r, \alpha \in \Delta \\ & \quad \text{and } l \neq r-1 \text{ if } \alpha = \alpha_n \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha \text{ with} \\ & \quad 0 \leq l < t < r, \alpha, \beta \in \Delta \\ \nu^{(r)} & \lambda = p^r\nu - p^l(\alpha_{n-1} + \alpha_n) \text{ with} \\ & 0 \leq l < r-1, \\ M^{(r)} \otimes \nu^{(r)} & \text{if } \lambda = p^r\nu - p^{r-1}\alpha_n - p^l\alpha \text{ with} \\ & \quad 0 \leq l < r-1, \alpha \in \Delta \\ M^{(r)} \otimes \nu^{(r)} & \text{if } \lambda = p^r\nu - p^{r-1}\alpha \text{ where } \alpha \in \{\alpha_{n-1}, \alpha_n\} \\ \mathrm{H}^1(B_{r-1}, M^{(-1)} \otimes \lambda_1) \oplus & \lambda = p\lambda_1 \\ \quad \mathrm{H}^2(B_{r-1}, \lambda_1) & \\ 0 & \text{else} \end{cases}$$

with M defined as in Theorem 5.3.2.

(g) If Φ is of type D_4, then

$$
\mathrm{H}^2(B_r,\lambda)\cong\begin{cases}
\mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r)} & \textit{if } \lambda=p^{r-1}(w\cdot 0+p\nu) \\
 & \quad \textit{with } l(w)=2,0 \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu+p^l w\cdot 0,\ \textit{with } l(w)=2, \\
 & \quad 0\le l<r-1, \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu-p^l\alpha,\ \textit{with } 0<l<r, \\
 & \quad \alpha\in\Delta \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu-p^t\beta-p^l\alpha, \\
 & \quad \textit{with } 0\le l<t<r,\alpha,\beta\in\Delta \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu+p^{r-1}\alpha_2-p^l\alpha, \\
 & \quad \textit{with } 0\le l<r-1,\alpha\in\Delta \\
(\nu+\omega_1)^{(r)}\oplus & \textit{if } \lambda=p^r\nu+p^{r-1}\omega_2-p^l\alpha, \\
\quad(\nu+\omega_3)^{(r)}\oplus(\nu+\omega_4)^{(r)} & \quad \textit{with } 0\le l<r-1,\alpha\in\Delta \\
0 & \textit{else}
\end{cases}
$$

(h) If Φ is of type D_n, $n\ge 5$, then

$$
\mathrm{H}^2(B_r,\lambda)\cong\begin{cases}
\mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r)} & \textit{if } \lambda=p^{r-1}(w\cdot 0+p\nu)\ \textit{with } l(w)=2,0 \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu+p^l w\cdot 0,\ \textit{with } l(w)=2, \\
 & \quad 0\le l<r-1, \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu-p^l\alpha,\ \textit{with } 0<l<r,\ \alpha\in\Delta \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu-p^t\beta-p^l\alpha,\ \textit{with } 0\le l<t<r, \\
 & \quad \alpha,\beta\in\Delta \\
\nu^{(r)} & \textit{if } \lambda=p^r\nu+p^{r-1}\alpha_i-p^l\alpha,\ \textit{with} \\
 & \quad 0\le l<r-1, \\
 & \quad \alpha\in\Delta, i\ne n-1,n \\
(\nu+\omega_{n-1})^{(r)}\oplus(\nu+\omega_n)^{(r)} & \textit{if } \lambda=p^r\nu+p^{r-1}\omega_{n-2}-p^l\alpha, \\
 & \quad \textit{with } 0\le l<r-1\alpha\in\Delta \\
0 & \textit{else}
\end{cases}
$$

(i) If Φ is of type F_4. Then

$$\mathrm{H}^2(B_r,\lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + pv) \text{ with } l(w) = 0,2 \\ \nu^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0 \text{ with } l(w) = 2, \\ & \quad 0 \le l < r-1 \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha \text{ with } 0 \le l \le r-1,\ \alpha\in\Delta \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha \text{ with } 0 \le l < t < r \\ & \quad \alpha,\beta\in\Delta \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^{l+1}(\alpha_3+\beta) - p^l\alpha_2 \text{ with} \\ & \quad 0 \le l < r-1, \beta\in\{\alpha_2,\alpha_4\} \\ M_{F_4}^{(r)}\otimes\nu^{(r)} & \text{if } \lambda = p^r\nu - p^{r-1}\alpha_2 - p^l\alpha \text{ with} \\ & \quad 0 \le l < r-1, \alpha\in\Delta \\ M^{(r)}\otimes\nu^{(r)} & \text{if } \lambda = pr^r\nu - p^{r-1}\alpha_4 - p^l\alpha_2 \text{ with} \\ & \quad 0 \le l < r-1 \\ 0 & \text{else} \end{cases}$$

with M defined as in Lemma 5.3.1(b)

(j) If Φ is of type G_2. Then,

$$\mathrm{H}^2(B_r,\lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + p\nu) \text{ with } l(w) = 0,2 \\ \nu^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0 \text{ with } l(w) = 2, \\ & \quad 0 \le l < r-1 \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha \text{ with } 0 \le l < r,\ \alpha\in\Delta \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha \text{ with } 0 \le l < t < r \\ & \quad \alpha,\beta\in\Delta \\ \nu^{(r)} & \text{if } \lambda = p^r\nu - p^{l+1}(\alpha_1+\alpha_2) - p^l\alpha_2 \text{ with} \\ & \quad 0 \le l < r-1 \\ M_{G_2}^{(r)}\otimes\nu^{(r)} & \text{if } \lambda = p^r\nu - p^{r-1}\alpha_2 - p^l\alpha \text{ with} \\ & \quad 0 \le l < r-1, \alpha\in\Delta \\ 0 & \text{else} \end{cases}$$

Proof. We will use induction on r. For $r = 1$, the claim reduces to Theorem 4.6.1. Suppose $r > 1$. Set $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0 \in X_1(T)$ and $\lambda_1 \in X(T)$. From the LHS spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(B_r/B_1, \mathrm{H}^j(B_1, \lambda_0) \otimes p\lambda_1) \Rightarrow \mathrm{H}^{i+j}(B_r, \lambda).$$

Case 1: $\lambda_0 \not\equiv 0$ and $\lambda_0 \not\equiv -\alpha \mod pX(T)$, with $\alpha \in \Delta$.

In this case we have $E_2^{i,0} = 0$ and $E_2^{i,1} = 0$ [Jan2, 3.2]. Then, we have

$$\mathrm{H}^2(B_r, \lambda) = E^2 \cong E_2^{0,2} = \mathrm{Hom}_{B_r/B_1}(k, \mathrm{H}^2(B_1, \lambda_0) \otimes p\lambda_1.$$

by Lemma 5.1.1 this expression is zero unless $\lambda_0 = w \cdot 0 + p\nu_w$ for some $w \in W$ with $l(w) = 2$ and ν_w as given in Lemma 4.3.1. Assume λ_0 is of this form. Then by Theorem 4.6.1, the B-module $\mathrm{H}^2(B_1, \lambda_0)$ has simple socle of weight $p\nu_w$ and $E_2^{0,2}$ vanishes unless $p(\nu_w + \lambda_1) \in p^r X(T)$. This implies $\lambda_0 = w \cdot 0 + p^r\nu$ with $l(w) = 2$ and $\nu \in X(T)$. Moreover, $\mathrm{H}^2(B_r, \lambda) \cong \nu^{(r)}$ for such weights. To summarize: if $\lambda_0 \not\equiv 0$ and $\lambda_0 \not\equiv -\alpha \mod pX(T)$, with $\alpha \in \Delta$, and $r > 1$ then

$$\mathrm{H}^2(B_r, \lambda) \cong \begin{cases} \nu^{(r)} & \text{if } \lambda = p^r\nu + w \cdot 0, \text{ with } l(w) = 2, \\ 0 & \text{else} \end{cases}$$

Case 2: Suppose $\lambda_0 \equiv -\alpha \mod pX(T)$, with $\alpha \in \Delta$.

Then $\lambda_0 = p\omega_\alpha - \alpha$ by [Jan2, 3.3], except in the following cases:

- Φ is of type B_n with $\alpha = \alpha_{n-1}$, then $\lambda_0 = 2(\omega_{n-1} - \omega_n) - \alpha_{n-1}$.
- Φ is of type C_n with $\alpha = \alpha_n$, then $\lambda_0 = 2(-2\omega_{n-1} + \omega_n) - \alpha_n$.
- Φ is of type F_4 with $\alpha = \alpha_2$, then $\lambda_0 = 2(\omega_2 - \omega_3) - \alpha_2$.
- Φ is of type G_2 with $\alpha = \alpha_2$, then $\lambda_0 = 2(-\omega_1 + \omega_2) - \alpha_2$.

If $\Phi \neq C_n$ with $\lambda \equiv -\alpha_n \mod pX(T)$, then $E_2^{i,0} = 0$. By Corollary 4.6.3, it follows that $E_2^{i,2} = 0$. Therefore

$$\mathrm{H}^2(B_r, \lambda) \cong E_2^{1,1} = \mathrm{H}^1(B_r/B_1, \mathrm{H}^1(B_1, \lambda_0) \otimes p\lambda_1).$$

Now assume that Φ is of type C_n. Then by [Jan2, 3.5] we have

$$E_2^{1,1} \cong \mathrm{H}^1(B_r/B_1, p(\omega_\alpha + \lambda_1)) \cong \mathrm{H}^1(B_{r-1}, \omega_\alpha + \lambda_1)^{(1)}.$$

Now, [BNP2, 2.8] implies that $E_2^{1,1} = 0$ unless $\omega_\alpha + \lambda_1 = p^{r-1}\nu - p^{k-1}\beta$ for some $\beta \in \Delta$ and some $0 < k < r$. Moreover, in this case $\mathrm{H}^2(B_r, \lambda) \cong \nu^{(r)}$, except in the following cases:

- if $\Phi = B_n, n \neq 4, \beta = \alpha_{n-1}$ and $k = r-1$; in which case $\mathrm{H}^2(B_r, \lambda) \cong M_{B_n}^{(r)} \otimes \nu^{(r)}$
- if $\Phi = B_4, \beta = \alpha_i, i \in 1, 3$ and $k = r-1$; in which case $\mathrm{H}^2(B_r, \lambda) \cong M_{B_4}^{(r)} \otimes \nu^{(r)}$
- if $\Phi = C_n, \beta = \alpha_n$ and $k = r-1$; in which case $\mathrm{H}^2(B_r, \lambda) \cong M_{C_n} \otimes \nu^{(r)}$
- if $\Phi = F_4, \beta = \alpha_2$ and $k = r-1$; in which case $\mathrm{H}^2(B_r, \lambda) \cong M_{F_4} \otimes \nu^{(r)}$
- if $\Phi = G_2, \beta = \alpha_2$ and $k = r-1$; in which case $\mathrm{H}^2(B_r, \lambda) \cong M_{G_2} \otimes \nu^{(r)}$

If Φ is of type A_3. Here $p\omega_i - \alpha_i = \omega_2$ for $i \in \{1, 3\}$. Now applying [Jan2, 3.5(b)] yields $E_2^{1,1} \cong \mathrm{H}^1(B_r/B_1, p(\omega_1 \oplus \omega_3) \otimes p\lambda_1) \cong \mathrm{H}^1(B_{r-1}, \omega_1 + \lambda_1)^{(1)} \oplus \mathrm{H}^1(B_{r-1}, \omega_3 + \lambda_1)^{(1)}$. As before, by [BNP2, 2.8] the cohomology vanishes unless $\lambda = p^r\nu - p^k\alpha_i - \alpha$ where $i \in \{1, 3\}$. Moreover, $\mathrm{H}^2(B_r, \lambda) \cong \gamma^{(r)}$, unless $k = r-1$, in which case

$$\mathrm{H}^2(B_r, \lambda)^{(-r)} \cong \gamma \oplus (\gamma + (-1)^j(\omega_1 - \omega_3)).$$

Adding $p^r\gamma_w$ as defined in Lemma 4.3.1 to λ results in the more symmetric statement $\mathrm{H}^2(B_r, p^r\gamma + p^{r-1}\omega_2 - \alpha) \cong (\gamma + \omega_1)^{(r)} \oplus (\gamma + \omega_2)^{(r)}$.

If Φ is of type D_n, then by similar computations as to type A_3, it follows that if $\Phi = D_4$ then $\mathrm{H}^2(B_r, p^r\gamma + p^{r-1}\omega_2 - \alpha) \cong (\gamma + \omega_1)^{(r)} \oplus (\gamma + \omega_3)^{(r)} \oplus (\gamma + \omega_4)^{(r)}$ and if $\Phi = D_n$ for $n \geq 5$ we have $\mathrm{H}^2(B_r, p^r\gamma + p^{r-1}\omega_{n-2} - \alpha) \cong (\gamma + \omega_{n-1})^{(r)} \oplus (\gamma + \omega_n)^{(r)}$.

If Φ is of type B_n with $\lambda_0 \equiv \alpha_{n-1} \pmod{pX(T)}$, or Φ is of type F_4 with $\lambda_0 \equiv \alpha_2 \pmod{pX(T)}$, or Φ is of type G_2 with $\lambda_0 \equiv \alpha_2 \pmod{pX(T)}$. Then, define $\nu \in X(T)$ via $\lambda = p\nu - \alpha_{n-1}$. Then from [Jan2, 3.6] $\mathrm{H}^1(B_1, \lambda) \cong M^{(1)} \otimes \nu^{(1)}$ and so $\mathrm{H}^2(B_r, \lambda) \cong \mathrm{H}^1(B_{r-1}, M \otimes \nu)^{(1)}$ and apply Theorem 5.3.2.

Furthermore, if Φ is of type B_3, then by similar computations from above, we have that $\mathrm{H}^2(B_r, \lambda) \cong (\gamma + \omega_1)^{(r)} \oplus (\gamma + \omega_3)^{(r)}$ when $\lambda = p^r\omega_1 - p^{r-1}\alpha_1 - \alpha = p^r\omega_3 - p^{r-1}\alpha_3 - \alpha$. If Φ is of type B_4, then by the same computations as in case A_3 and the results for B_n above show that $\mathrm{H}^2(B_r, \lambda) \cong (\omega_1 + \gamma)^{(r)} \oplus (M_{B_4} \otimes \gamma)^{(r)}$ when $\lambda = p^r\omega_1 - p^{r-1}\alpha_1 - \alpha = p^r(\omega_3 - \omega_4) - p^{r-1}\alpha_3 - \alpha$

If $\Phi = C_n$ and $\alpha = \alpha_n$, then $\lambda_0 \equiv 0 \pmod{pX(T)}$, which is excluded.

Case 3: Now assume $\lambda_0 = 0$. First assume that Φ is not of type C_n. Then $E_2^{i,1} = 0$ for all i by [Jan2, 3.3]. From Lemma 5.1.1 one obtains that $E_2^{0,2} = 0$ unless $\lambda = p^r\gamma - p\alpha$, with $\alpha \in \Delta$ or if λ is one of those listed in (i)-(iv) or (vi)-(viii), then by Lemma 5.1.1(B), then $\mathrm{H}^2(B_r, \lambda) \cong \nu^{(r)}$, as claimed. Now if $E_2^{0,2} = 0$. This implies that

$$E^2 = E_2^{2,0} \cong \mathrm{H}^2(B_r/B_1, p\lambda_1) \cong \mathrm{H}^2(B_{r-1}, \lambda_1)^{(1)}.$$

If Φ is of type C_n, then by Lemma 5.1.1

$$E_2^{0,2} = \mathrm{Hom}_{B_r/B_1}(k, \mathrm{H}^2(B_1, k)^{(1)} \otimes p\lambda_1) \cong \mathrm{Hom}_{B_{r-1}}(k, \mathrm{H}^2(B_1, k) \otimes \lambda_1)^{(1)} = 0.$$

Now, consider, $E_2^{2,0} = \mathrm{H}^2(B_r/B_1, \mathrm{Hom}_{B_1}(k, p\lambda_1)) \cong \mathrm{H}^2(B_{r-1}, \lambda_1)^{(1)}$ and $E_2^{0,1} = \mathrm{Hom}_{B_r/B_1}(k, \mathrm{H}^1(B_1, k) \otimes p\lambda_1) \cong \mathrm{Hom}_{B_{r-1}}(k, M \otimes \lambda_1)^{(1)}$ and the map $d_2 : E_2^{0,1} \to E_2^{2,0}$. We want to show that d_2 is the 0-map. But, $E_2^{0,1} \neq 0$ if and only if $\lambda_1 = p^{r-1}\nu - (\omega_n - \omega_{n-1})$, and $E_2^{2,0} \cong \mathrm{H}^2(B_{r-1}, p^{r-1}\nu - (\omega_n - \omega_{n-1})) = 0$, by our previous calculations with case (i).

Now, consider $E_2^{1,1} = \mathrm{H}^1(B_r/B_1, \mathrm{H}^1(B_1, k) \otimes p\lambda_1) \cong \mathrm{H}^1(B_{r-1}, M \otimes \lambda_1)$, where M is defined as in Section 5.3 (as in [Jan2]). Since $E_2^{1,1} \neq 0$, then we must consider $E_2^{3,0}$. We want to show $d_2 : E_2^{1,1} \to E_2^{3,0}$ is the zero-map. Note that the factors of M are not in the root lattice, and so for $E_2^{1,1} \neq 0$, then $\lambda_1 \notin \mathbb{Z}\Phi$. For $\lambda_1 \notin \mathbb{Z}\Phi$, then $E_2^{3,0} = \mathrm{H}^3(B_{r-1}, \lambda_1) = 0$ by [Jan1, II.4.10]. Therefore, d_2 is the zero-map. □

Chapter 6

B–Cohomology

6.1

Using the B_r–cohomology results, the B–cohomology and the G_r–cohomology can both be computed. Here, we will calculate $\mathrm{H}^2(B,\lambda)$ for all $\lambda \in X(T)$. Cline, Parshall, and Scott [CPS] gives a relationship between the B_r-cohomology and the B-cohomology:

$$\mathrm{H}^2(B,\lambda) \cong \varprojlim \mathrm{H}^2(B_r,\lambda).$$

Assume that $\lambda \in X(T)$ with $\mathrm{H}^2(B,\lambda) \neq 0$, then $\lambda \neq 0$. Choose $s > 0$ such that

(i) the natural map $\mathrm{H}^n(B,\lambda) \to \mathrm{H}^n(B_r,\lambda)$ is nonzero for all $r \geq s$.

By choosing a possibly larger s, we can further assume that

(ii) $|\langle \lambda, \alpha^\vee \rangle| < p^{s-1}$ for all $\alpha \in \Delta$.

From Theorem 5.4.1 and condition (ii), then $\mathrm{H}^2(B_r,\lambda)$ is one-dimensional for all $r \geq s$. Furthermore, since $\mathrm{H}^2(B,\lambda)$ has trivial B-action, condition (i) implies that $\mathrm{H}^2(B_r,\lambda) \cong k$ for all $r \geq s$.

On the other hand, if there exists an integer s such that $\mathrm{H}^2(B_r,\lambda) \cong k$ for all $r \geq s$, then $\mathrm{H}^2(B,\lambda) \cong \varprojlim \mathrm{H}^2(B_r,\lambda) \cong k$. Therefore, Theorem 5.4.1 yields:

Theorem 6.1.1. *Let $p = 2$ and $\lambda \in X(T)$.*

(a) If Φ is simply laced, then

$$\mathrm{H}^2(B,\lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \le l,\ l(w) = 2, \\ k & \text{if } \lambda = -p^l\alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^t\beta - p^l\alpha, \text{ with } 0 \le l < t \text{ and } \alpha,\beta \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) If Φ is of type B_n, then

$$\mathrm{H}^2(B,\lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \le l,\ l(w) = 2, \\ k & \text{if } \lambda = -p^l\alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^t\beta - p^l\alpha, \text{ with } 0 \le l < t \text{ and } \alpha,\beta \in \Delta, \\ k & \text{if } \lambda = -p^{l+1}(\alpha_{n-1} + \alpha_n) - p^l\alpha_{n-1} \text{ with } 0 \le l, \\ 0 & \text{else.} \end{cases}$$

(c) If Φ is of type C_n, then

$$\mathrm{H}^2(B,\lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \le l,\ l(w) = 2, \\ k & \text{if } \lambda = -p^l\alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^t\beta - p^l\alpha, \text{ with } 0 \le l < t \text{ and } \alpha,\beta \in \Delta, \\ k & \text{if } \lambda = -p^{l+1}(\alpha_{n-1} + \alpha_n) \text{ with } 0 \le l, \\ 0 & \text{else.} \end{cases}$$

(d) If Φ is of type F_4, then

$$\mathrm{H}^2(B,\lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \le l,\ l(w) = 2, \\ k & \text{if } \lambda = -p^l\alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^t\beta - p^l\alpha, \text{ with } 0 \le l < t \text{ and } \alpha,\beta \in \Delta, \\ k & \text{if } \lambda = -p^{l+1}(\alpha_3 + \beta) - p^l\alpha_2 \text{ with } 0 \le l \text{ and } \beta \in \{\alpha_2,\alpha_4\}, \\ 0 & \text{else.} \end{cases}$$

(e) If Φ is of type G_2, then

$$\mathrm{H}^2(B,\lambda) \cong \begin{cases} k & \text{if } \lambda = p^l w \cdot 0, \text{ with } 0 \leq l, \ l(w) = 2, \\ k & \text{if } \lambda = -p^l \alpha, \text{ with } 0 < l \text{ and } \alpha \in \Delta, \\ k & \text{if } \lambda = -p^t \beta - p^l \alpha, \text{ with } 0 \leq l < t \text{ and } \alpha, \beta \in \Delta, \\ k & \text{if } \lambda = -p^{l+1}(\alpha_1 + \alpha_2) - p^l \alpha_2 \text{ with } 0 \leq l, \\ 0 & \text{else.} \end{cases}$$

Proof. We want condition (ii) above to be satisfied, that is:

$$| \langle \lambda, \alpha^\vee \rangle | < p^{s-1} \text{ for all } \alpha \in \Delta$$

For example, consider the case when $\Phi = A_3$ and $\lambda = p^r \nu + p^{r-1}\alpha_2 - p^l \beta$. Fix s large enough such that $s \leq r$. We want the following inequality to be satisfied.

$$\begin{aligned} & | \langle p^r \nu + p^{r-1}\alpha_2 - p^l \beta, \alpha^\vee \rangle | \\ = & |p^r \langle \nu, \alpha^\vee \rangle + p^{r-1} \langle \alpha_2, \alpha^\vee \rangle - p^l \langle \beta, \alpha^\vee \rangle | \quad < \quad p^{s-1} \end{aligned}$$

Therefore,

$$|p^{r-s+1} \langle \nu, \alpha^\vee \rangle + p^{r-s} \langle \alpha_2, \alpha^\vee \rangle - p^{l-s+1} \langle \beta, \alpha^\vee \rangle | < 1$$

Note that $\langle \nu, \alpha^\vee \rangle \in \mathbb{Z}$, $\langle \alpha_2, \alpha^\vee \rangle = -1, 2$, and $\langle \beta, \alpha^\vee \rangle = 0, -1, 2$. Since the left hand side must be < 1, then we have that $\nu = 0$. And, so we are left with

$$|p^{r-s} \langle \alpha_2, \alpha^\vee \rangle - p^{l-s+1} \langle \beta, \alpha^\vee \rangle | < 1.$$

But, since $r \geq s$ then if $\langle \alpha_2, \alpha^\vee \rangle = -1$ and I am subtracting something to result in the left hand side being greater than 1, then the inequality is not satisfied for any value of $< \langle \alpha_2, \alpha^v ee \rangle$.

Now consider, $\langle \alpha_2, \alpha^\vee \rangle = 2$ then the second term, $\langle \alpha_2, \vee \rangle$, is never greater than 1 and thus $|p^{r-s} \langle \alpha_2, \alpha^\vee \rangle - p^{l-s+1} \langle \beta, \alpha^\vee \rangle | \nless 1$. Therefore, λ doesn't satisfy condition (ii) above, and so for this particular λ, $\mathrm{H}^2(B, \lambda) = 0$. A similar computation is used for the other cases. □

Chapter 7

G_r-Cohomology

Now that the B_r-cohomology has been calculated, the G_r-cohomology of induced modules ($H^0(\lambda)$, where $\lambda \in X(T)_+$) can be determined. From [Jan1, II.12.2], one has the following isomorphism, which holds independently of the prime:

$$\mathrm{H}^1(G_r, H^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G \mathrm{H}^1(B_r, \lambda)^{(-r)}$$

for any $\lambda \in X(T)_+$. Recall the following theorem from [BNP2, 6.1], which generalizes the above isomorphism.

Theorem 7.0.2. *Let $\lambda \in X(T)_+$ and p be an arbitrary prime. Then*

$$\mathrm{H}^2(G_r, H^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G \mathrm{H}^2(B_r, \lambda)^{(-r)}.$$

7.1 $r = 1$ case

In particular, one has the following isomorphism $\mathrm{H}^2(G_1, H^0(\lambda))^{(-1)} \cong \mathrm{ind}_B^G(\mathrm{H}^2(B_1, \lambda)^{(-1)})$. Using the results from the B_1-cohomology, the G_1-cohomology can easily be determined.

Theorem 7.1.1. *Let $p = 2$*

(a) Let $\lambda = p\nu$, then

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong \mathrm{ind}_B^G(\mathfrak{u}^* \otimes \nu)^{(1)}$$

except when Φ is of type $A_3, B_n, C_n, D_n, F_4,$ or G_2. Then

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong \mathrm{ind}_B^G(\mathrm{H}^2(B_1, k) \otimes \nu)^{(1)}.$$

(b) If $\lambda \notin pX(T)$ *and* $\mathrm{H}^2(G_1, H^0(\lambda)) \neq 0$ *then* $\lambda = w \cdot 0 + p\nu$ *with* $l(w) = 2$ *and* $\nu \in X(T)$*.*

(c) Let $\lambda = s_\alpha s_\beta \cdot 0 + p\nu$ *where* $\alpha, \beta \in \Delta$ *are not connected and are not separated by a single vertex. Furthermore suppose* Φ *is not of type* B_n *with* $w = s_{\alpha_i} s_{\alpha_{n-1}}$ *and* Φ *is not of type* C_n *with* $w = s_{\alpha_i} s_{\alpha_n}$*, then*

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong H^0(\nu)^{(1)}.$$

If the B_r-cohomology involves an indecomposable B-module, then it is necessary to determine the G_1-cohomology structure separately. First consider the following modules, with the factors listed from top to bottom.

- M has factors α_i and k corresponding to $w = s_\alpha s_\beta$ where α, β are separated by a single vertex, unless otherwise noted in one of the following modules.
- If $\Phi = B_n$ and $w = s_{\alpha_{n-2}} s_{\alpha_n}$, then M has factors α_{n-1} and k
- If $\Phi = B_n$ and $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$ then M has factors α_{n-2} and k
- If $\Phi = B_n$ and $w = s_{\alpha_i} s_{\alpha_{n-1}}$ with $i \neq n-3$ then M has factors α_n and k.
- If $\Phi = C_n$ and $w = s_{\alpha_i} s_{\alpha_n}$ with $i \neq n-2$ then M has factors α_{n-1} and k.
- If $\Phi = D_n$ and $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$ then M has factors $\alpha_{n-2} + \alpha_n, \alpha_{n-2}$ and k.
- If $\Phi = D_n$ and $w = s_{\alpha_{n-3}} s_{\alpha_n}$ then M has factors $\alpha_{n-2} + \alpha_{n-1}, \alpha_{n-2}$ and k.
- If $\Phi = E_6$ and $w = s_{\alpha_2} s_{\alpha_3}$ then M has factors $\alpha_4 + \alpha_5 + \alpha_6, \alpha_4 + \alpha_5, \alpha_4$ and k.
- If Φ is of type E_6, E_7, E_8 and $w = s_{\alpha_2} s_{\alpha_5}$ then M has factors $\alpha_1 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4$ and k.
- If Φ is of type E_6, E_7, E_8 and $w = s_{\alpha_3} s_{\alpha_5}$ then M has factors $\alpha_2 + \alpha_4, \alpha_4$ and k.
- If $\Phi = E_7$ and $w = s_{\alpha_2} s_{\alpha_3}$ then M has factors $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_4 + \alpha_5 + \alpha_6, \alpha_4 + \alpha_5, \alpha_4$ and k.

- If $\Phi = E_8$ and $w = s_{\alpha_2}s_{\alpha_3}$ then M has factors $\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_4+\alpha_5+\alpha_6, \alpha_4+\alpha_5, \alpha_4$ and k.

- If $\Phi = F_4$ and $w = s_{\alpha_1}s_{\alpha_3}$ then M has factors $\alpha_2+\alpha_3, \alpha_2$ and k.

If $\lambda = w\cdot 0 + p\nu \in X(T)_+$ for $w \in W$ where w is one of the reflections listed above, then

$$\mathrm{H}^2(G_1, H^0(\lambda)) \cong \mathrm{ind}_B^G(M \otimes \nu)^{(1)}.$$

The structure of these modules are listed in Appendix C.

7.2 General case

For $r > 1$, using the isomorphism in Theorem 7.0.2, we can use the B_r cohomology results, found in Theorem 5.11 and apply the induction functor to get the following theorem.

Theorem 7.2.1. *Let $p = 2$, $r > 1$ and $\lambda \in X(T)_+$. Then*

(a) If Φ is not of type A_3, B_n, C_n, D_n, F_4, or G_2, then

$$\mathrm{H}^2(G_r,\lambda) \cong \begin{cases} \mathrm{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + p\nu) \text{ with } l(w) = 0, 2, \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0, \text{ with } l(w) = 2, \\ & \quad 0 \le l < r-1, \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha, \text{ with } 0 < l < r, \alpha \in \Delta \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha \\ & \quad \text{with } 0 \le l < t < r, \ \alpha,\beta \in \Delta \\ 0 & \text{else} \end{cases}$$

(b) If $\Phi = A_3$*, then*

$$\mathrm{H}^2(G_r,\lambda) \cong \begin{cases} \operatorname{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + p\nu) \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0, \text{ with } l(w) = 2, \\ & \quad 0 \le l < r-1, \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha, \text{ with } 0 < l < r, \\ & \quad \alpha \in \Delta \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha, \\ & \quad \text{with } 0 \le l < t < r, \quad \alpha,\beta \in \Delta \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu + p^{r-1}\alpha_2 - p^l\alpha, \\ & \quad \text{with } 0 \le l < r-1, \ \alpha \in \Delta \\ H^0(\nu+\omega_1)^{(r)} \oplus H^0(\nu+\omega_3)^{(r)} & \text{if } \lambda = p^r\nu + p^{r-1}\omega_2 - p^l\alpha, \\ & \quad \text{with } 0 \le l < r-1, \quad \alpha \in \Delta \\ 0 & \text{else} \end{cases}$$

(c) If Φ is of type B_3. Then,

$$\mathrm{H}^2(G_r,\lambda) \cong \begin{cases} \mathrm{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \textit{if } \lambda = p^{r-1}(w\cdot 0 + p\nu), l(w) = 0,2 \\ H^0(\nu)^{(r)} & \textit{if } \lambda = p^r\nu + p^l w\cdot 0, \ \textit{with } l(w) = 2, \\ & \quad 0 \le l < r-1 \\ H^0(\nu)^{(r)} & \textit{if } \lambda = p^r\nu - p^t\beta - p^l\alpha, \\ & \quad \textit{with } 0 \le l < t < r, \ \alpha,\beta \in \Delta \\ H^0(\nu)^{(r)} & \lambda = p^r\nu - p^l\alpha \ \textit{with } 0 \le l < r, \ \alpha \in \Delta \\ H^0(\nu+\omega_1)^{(r)} \oplus H^0(\nu+\omega_3)^{(r)} & \lambda = p^r\nu - p^{r-1}\alpha_2 - p^l\alpha, \ \textit{with} \\ & \quad 0 \le l < r-1, \alpha \in \Delta \\ H^0(\nu)^{(r)} & \lambda = p^r\nu - p^{l+1}(\alpha_2+\alpha_3) - p^l\alpha_2 \ \textit{with} \\ & \quad 0 \le l < r-1 \\ \mathrm{ind}_B^G(M_{B_3}^{(r)} \otimes \nu)^{(r)} & \lambda = p^r\nu - p^{r-1}\alpha_2 - p^l\alpha, \ \textit{with} \\ & \quad 0 \le l < r-1, \alpha \in \Delta \\ \mathrm{ind}_B^G(M_{B_3}^{(r)} \otimes \nu)^{(r)} & \lambda = p^r\nu - p^{r-1}\alpha_3 - p^l\alpha_2 \ \textit{with} \\ & \quad 0 \le l < r-1, \\ 0 & \textit{else} \end{cases}$$

(d) If Φ is of type B_4. Then,

$$\mathrm{H}^2(G_r,\lambda) \cong \begin{cases} \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r-1)} & \text{if } \lambda = p^{r-1}(w\cdot 0 + p\nu) \text{ with } l(w) = 0,2 \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0, \text{ with } l(w) = 2, \\ & \quad 0 \le l < r-1 \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha, \\ & \quad \text{with } 0 \le l < t < r,\ \alpha,\beta \in \Delta \\ H^0(\nu)^{(r)} & \lambda = p^r\nu - p^l\alpha \text{ with } 0 \le l < r,\ \alpha \in \Delta \\ \nu^{(r)} & \lambda = p^r\nu - p^{l+1}(\alpha_3+\alpha_4) - p^l\alpha_2 \\ & \quad \text{with } 0 \le l < r-1 \\ H^0(\omega_1+\nu)^{(r)} \oplus & \lambda = p^r\nu - p^{r-1}\alpha_i - p^l\alpha \text{ with } i \in \{1,3\}, \\ \quad \mathrm{ind}_B^G(M_{B_4}\otimes\nu)^{(r)} & \quad 0 \le l < r-1,\ \alpha \in \Delta \\ \mathrm{ind}_B^G(M_{B_4}^{(r)}\otimes\nu)^{(r)} & \lambda = p^r\nu - p^{r-1}\alpha_4 - p^l\alpha_3 \\ & \quad \text{with } 0 \le l < r-1, \\ 0 & \text{else} \end{cases}$$

(e) If Φ is of type B_n. Then,

$$\mathrm{H}^2(G_r,\lambda)\cong\begin{cases}\operatorname{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r-1)} & \text{if } \lambda=p^{r-1}(w\cdot 0+p\nu) \text{ with } l(w)=0,2\\ H^0(\nu)^{(r)} & \text{if } \lambda=p^r\nu+p^l w\cdot 0,\\ & \quad \text{with } l(w)=2,\ 0\leq l<r-1\\ H^0(\nu)^{(r)} & \text{if } \lambda=p^r\nu-p^t\beta-p^l\alpha,\\ & \quad \text{with } 0\leq l<t<r,\ \alpha,\beta\in\Delta\\ H^0(\nu)^{(r)} & \lambda=p^r\nu-p^l\alpha \text{ with } 0\leq l<r,\ \alpha\in\Delta\\ H^0(\nu)^{(r)} & \lambda=p^r\nu-p^{l+1}(\alpha_{n-1}+\alpha_n)-p^l\alpha_{n-1}\\ & \quad \text{with } 0\leq l<r-1\\ \operatorname{ind}_B^G(M_{B_n}^{(r)}\otimes\nu)^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_{n-1}-p^l\alpha,\\ & \quad \text{with } 0\leq l<r-1,\ \alpha\in\Delta\\ \operatorname{ind}_B^G(M_{B_n}^{(r)}\otimes\nu)^{(r)} & \lambda=p^r\nu-p^{r-1}\alpha_n-p^l\alpha_{n-1}\\ & \quad \text{with } 0\leq l<r-1,\\ 0 & \text{else}\end{cases}$$

(f) If Φ is of type C_n. Then,

$$\mathrm{H}^2(G_r,\lambda)\cong\begin{cases}\operatorname{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r)} & \text{if } \lambda=p^{r-1}(w\cdot 0+p\nu) \text{ with } l(w)=0,2\\ H^0(\nu)^{(r)} & \text{if } \lambda=p^r\nu+p^l(w\cdot 0) \text{ with } 0\le l<r-1,\\ & \quad l(w)=2\\ H^0(\nu)^{(r)} & \text{if } \lambda=p^r\nu-p^l\alpha \text{ with } 0\le l<r, \alpha\in\Delta\\ & \quad \text{and } l\neq r-1 \text{ if } \alpha=\alpha_n\\ H^0(\nu)^{(r)} & \text{if } \lambda=p^r\nu-p^t\beta-p^l\alpha \text{ with } 0\le l<t<r\\ & \quad \alpha,\beta\in\Delta\\ H^0(\nu)^{(r)} & \lambda=p^r\nu-p^{l+1}(\alpha_{n-1}+\alpha_n)-p^l\alpha_{n-1}\\ & \quad \text{with } 0\le l<r-1\\ \operatorname{ind}_B^G(M\otimes\nu)^{(r)} & \text{if } \lambda=p^r\nu-p^{r-1}\alpha_n-p^l\alpha \text{ with}\\ & \quad 0\le l<r-1, \alpha\in\Delta\\ \operatorname{ind}_B^G(M\otimes\nu)^{(r)} & \text{if } \lambda=p^r\nu-p^{r-1}\alpha \text{ where } \alpha\in\{\alpha_{n-1},\alpha_n\}\\ \operatorname{ind}_B^G \mathrm{H}^1(B_{r-1}, M^{(-1)}\otimes\lambda_1) & \lambda=p\lambda_1, \text{ where } \lambda_1\in X_{r-1}(T)\\ \quad\oplus \operatorname{ind}_B^G \mathrm{H}^2(B_{r-1},\lambda_1) & \\ 0 & \text{else}\end{cases}$$

with M defined in Lemma 5.3.1.

(g) If Φ is of type D_4, then

$$\mathrm{H}^2(G_r,\lambda) \cong \begin{cases} \operatorname{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r)} & \text{if } \lambda = p^{r-1}(w\cdot 0+p\nu), l(w)=2,0 \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0 \text{ with } l(w)=2, \\ & \quad 0 \le l < r-1, \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha \text{ with } 0 < l < r, \\ & \quad \alpha\in\Delta, \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha \text{ with} \\ & \quad 0\le l<t<r, \alpha,\beta\in\Delta \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\gamma + p^{r-1}\alpha_2 - p^l\alpha \text{ with} \\ & \quad 0\le l<r-1, \alpha\in\Delta \\ H^0(\nu+\omega_1)^{(r)} \oplus H^0(\nu+\omega_3)^{(r)} & \text{if } \lambda = p^r\gamma + p^{r-1}\omega_2 - p^l\alpha \text{ with} \\ \quad \oplus H^0(\nu+\omega_4)^{(r)} & \quad 0\le l<r-1, \alpha\in\Delta \\ 0 & \text{else} \end{cases}$$

(h) If Φ is of type D_n, $n \ge 5$, then

$$\mathrm{H}^2(G_r,\lambda) \cong \begin{cases} \operatorname{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0 + p\nu)^{(r)} & \text{if } \lambda = p^{r-1}(w\cdot 0+p\nu), l(w)=2,0 \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu + p^l w\cdot 0, \text{ with } l(w)=2, \\ & \quad 0 \le l < r-1, \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^l\alpha, \text{ with } 0 < l < r, \\ & \quad \alpha\in\Delta \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu - p^t\beta - p^l\alpha \text{ with} \\ & \quad 0\le l<t<r, \alpha,\beta\in\Delta \\ H^0(\nu)^{(r)} & \text{if } \lambda = p^r\nu + p^{r-1}\alpha_i - p^l\alpha \text{ with} \\ & \quad 0\le l<r-1, \alpha\in\Delta,\ i\ne n-1,n \\ H^0(\nu+\omega_{n-1})^{(r)} \oplus H^0(\nu+\omega_n)^{(r)} & \text{if } \lambda = p^r\gamma + p^{r-1}\omega_{n-2} - p^l\alpha, \\ & \quad \text{with } 0\le l<r-1,\ \alpha\in\Delta \\ 0 & \text{else} \end{cases}$$

(i) If Φ is of type F_4. Then

$$\mathrm{H}^2(G_r,\lambda)\cong\begin{cases}\mathrm{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r-1)} & \textit{if } \lambda=p^{r-1}(w\cdot 0+pv) \textit{ with } l(w)=0,2\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu+p^lw\cdot 0 \textit{ with } l(w)=2,\\ & \quad 0\le l<r-1\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^l\alpha \textit{ with } 0\le l\le r-1,\\ & \quad \alpha\in\Delta\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^t\beta-p^l\alpha \textit{ with}\\ & \quad 0\le l<t<r\alpha,\beta\in\Delta\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^{l+1}(\alpha_3+\beta)-p^l\alpha_2\\ & \quad \textit{with } 0\le l<r-1,\ \beta\in\{\alpha_2,\alpha_4\}\\ \mathrm{ind}_B^G(M_{F_4}^{(r)}\otimes\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^{r-1}\alpha_2-p^l\alpha \textit{ with}\\ & \quad 0\le l<r-1,\alpha\in\Delta\\ \mathrm{ind}_B^G(M^{(r)}\otimes\nu)^{(r)} & \textit{if } \lambda=pr^r\nu-p^{r-1}\alpha_4-p^l\alpha_2\\ & \quad \textit{with } 0\le l<r-1\\ 0 & \textit{else}\end{cases}$$

(j) If Φ is of type G_2. Then,

$$\mathrm{H}^2(G_r,\lambda)\cong\begin{cases}\mathrm{ind}_B^G \mathrm{H}^2(B_1, w\cdot 0+p\nu)^{(r-1)} & \textit{if } \lambda=p^{r-1}(w\cdot 0+p\nu) \textit{ with } l(w)=0,2\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu+p^lw\cdot 0 \textit{ with } l(w)=2,\\ & \quad 0\le l<r-1\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^l\alpha \textit{ with } 0\le l<r,\ \alpha\in\Delta\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^t\beta-p^l\alpha \textit{ with}\\ & \quad 0\le l<t<r,\alpha,\beta\in\Delta\\ H^0(\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^{l+1}(\alpha_1+\alpha_2)-p^l\alpha_2\\ & \quad \textit{with } 0\le l<r-1\\ \mathrm{ind}_B^G(M_{G_2}^{(r)}\otimes\nu)^{(r)} & \textit{if } \lambda=p^r\nu-p^{r-1}\alpha_2-p^l\alpha \textit{ with}\\ & \quad 0\le l<r-1,\alpha\in\Delta\\ 0 & \textit{else}\end{cases}$$

The structure of the induced modules in the above theorems are described in Section 7.1 and Appendix C. This appendix demonstrates that $\mathrm{H}^2(G_r, H^0(\lambda))$ does have a good filtration.

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Appendix A

Cohomology Classes

Table A.1: The following table is a list of the explicit cohomology classes that show up in the U_1-cohomology

Φ	w	cohomology class
A_n	$s_{\alpha_i}s_{\alpha_{i+2}}$	$\phi_{\alpha_{i+1}} \otimes \phi_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} + \phi_{\alpha_i+\alpha_{i+1}} \otimes \phi_{\alpha_{i+1}+\alpha_{i+2}}$
	$1 \leq i \leq n-2$	$\phi_{\alpha_i} \otimes \phi_{\alpha_{i+2}}$
B_n	$s_{\alpha_i}s_{\alpha_{i+2}}$	$\phi_{\alpha_{i+1}} \otimes \phi_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} + \phi_{\alpha_i+\alpha_{i+1}} \otimes \phi_{\alpha_{i+1}+\alpha_{i+2}}$
	$1 \leq i \leq n-2$	$\phi_{\alpha_i} \otimes \phi_{\alpha_{i+2}}$
	$s_{\alpha_i}s_{\alpha_{n-1}}$	$\phi_{\alpha_i} \otimes \phi_{\alpha_{n-1}+2\alpha_n}$
	$1 \leq i \leq n-3$	$\phi_{\alpha_i} \otimes \phi_{\alpha_{n-1}}$
	e	$\phi_{\alpha_i} \otimes \phi_{\alpha_i+2\alpha_{i+1}+2\alpha_{i+2}+\ldots+2\alpha_n} + \phi_{\alpha_i+\alpha_{i+1}} \otimes \phi_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\ldots+2\alpha_n}$ $+ \ldots + \phi_{\alpha_i+\alpha_{i+1}\ldots+\alpha_{n-1}} \otimes \phi_{\alpha_i+\alpha_{i+1}\ldots+\alpha_{n-1}+2\alpha_n}$ $+\phi_{\alpha_i+\alpha_{i+1}\ldots+\alpha_{n-1}+\alpha_n} \otimes \phi_{\alpha_i+\alpha_{i+1}\ldots+\alpha_{n-1}+\alpha_n}$
C_n	$s_{\alpha_i}s_{\alpha_{i+2}}$	$\phi_{\alpha_{i+1}} \otimes \phi_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} + \phi_{\alpha_i+\alpha_{i+1}} \otimes \phi_{\alpha_{i+1}+\alpha_{i+2}}$
	$1 \leq i \leq n-3$	$\phi_{\alpha_i} \otimes \phi_{\alpha_{i+2}}$
	$s_{\alpha_{n-2}}s_{\alpha_n}$	$\phi_{\alpha_{n-2}} \otimes \phi_{\alpha_n}$
	$s_{\alpha_{n-1}}s_{\alpha_n}$	$\phi_{\alpha_{n-1}} \otimes \phi_{2\alpha_{n-1}+\alpha_n}$
	$s_{\alpha_i}s_{\alpha_n}$	$\phi_{\alpha_i} \otimes \phi_{2\alpha_{n-1}+\alpha_n}$
	$1 \leq i \leq n-3$	$\phi_{\alpha_i} \otimes \phi_{\alpha_n}$
	e	$\phi_{\alpha_n} \otimes \phi_{2\alpha_i+2\alpha_{i+1}+\ldots+2\alpha_{n-1}+\alpha_n} + \phi_{\alpha_i+\ldots+\alpha_n} \otimes \phi_{\alpha_i+\ldots+\alpha_n}$

D_n	$s_{\alpha_i}s_{\alpha_{i+2}}$	$\phi_{\alpha_{i+1}} \otimes \phi_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}} + \phi_{\alpha_i+\alpha_{i+1}} \otimes \phi_{\alpha_{i+1}+\alpha_{i+2}}$
	$1 \leq i \leq n-4$	$\phi_{\alpha_i} \otimes \phi_{\alpha_{i+2}}$
	$s_{\alpha_{n-3}}s_{\alpha_{n-1}}$	$\phi_{\alpha_n} \otimes \phi_{\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n} + \phi_{\alpha_{n-2}+\alpha_n} \otimes \phi_{\alpha_{n-3}+\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$ $+\phi_{\alpha_{n-3}+\alpha_{n-2}+\alpha_n} \otimes \phi_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$
		$\phi_{\alpha_{n-2}} \otimes \phi_{\alpha_{n-3}+\alpha_{n-2}+\alpha_{n-1}} + \phi_{\alpha_{n-3}+\alpha_{n-2}} \otimes \phi_{\alpha_{n-2}+\alpha_{n-1}}$
		$\phi_{\alpha_{n-3}} \otimes \phi_{\alpha_{n-1}}$
	$s_{\alpha_{n-3}}s_{\alpha_n}$	$\phi_{\alpha_{n-1}} \otimes \phi_{\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n} + \phi_{\alpha_{n-2}+\alpha_{n-1}} \otimes \phi_{\alpha_{n-3}+\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$ $+\phi_{\alpha_{n-3}+\alpha_{n-2}+\alpha_{n-1}} \otimes \phi_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$
		$\phi_{\alpha_{n-2}} \otimes \phi_{\alpha_{n-3}+\alpha_{n-2}+\alpha_n} + \phi_{\alpha_{n-3}+\alpha_{n-2}} \otimes \phi_{\alpha_{n-2}+\alpha_n}$
		$\phi_{\alpha_{n-3}} \otimes \phi_{\alpha_n}$
	$s_{\alpha_{n-1}}s_{\alpha_n}$	$\phi_{\alpha_i} \otimes \phi_{\alpha_i+2\alpha_{i+1}+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}+$ $\phi_{\alpha_i+\alpha_{i+1}} \otimes \phi_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$ $+\ldots+\phi_{\alpha_i+\ldots+\alpha_{n-1}} \otimes \phi_{\alpha_i+\ldots+\alpha_{n-2}+\alpha_n}, \quad 1 \leq i \leq n-3$
		$\phi_{\alpha_{n-1}} \otimes \phi_{\alpha_n}$
E_6	$s_{\alpha_1}s_{\alpha_4}$	$\phi_{\alpha_3} \otimes \phi_{\alpha_1+\alpha_3+\alpha_4} + \phi_{\alpha_1+\alpha_3} \otimes \phi_{\alpha_3+\alpha_4}$
		$\phi_{\alpha_1} \otimes \phi_{\alpha_4}$
	$s_{\alpha_4}s_{\alpha_6}$	$\phi_{\alpha_5} \otimes \phi_{\alpha_4+\alpha_5+\alpha_6} + \phi_{\alpha_4+\alpha_5} \otimes \phi_{\alpha_5+\alpha_6}$
		$\phi_{\alpha_4} \otimes \phi_{\alpha_6}$
	$s_{\alpha_2}s_{\alpha_3}$	$\phi_{\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} + \phi_{\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$ $\phi_{\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + \phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$
		$\phi_{\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_4+\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_3+\alpha_4}$
		$\phi_{\alpha_2} \otimes \phi_{\alpha_3}$

E_6	$s_{\alpha_2}s_{\alpha_5}$	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_3} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$ $\phi_{\alpha_1+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_3} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $\phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4} \otimes \phi_{\alpha_2+\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_4+\alpha_5}$
		$\phi_{\alpha_2} \otimes \phi_{\alpha_5}$
	$s_{\alpha_3}s_{\alpha_5}$	$\phi_{\alpha_2} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_3+\alpha_4} \otimes \phi_{\alpha_4+\alpha_5}$
		$\phi_{\alpha_3} \otimes \phi_{\alpha_5}$
E_7	$s_{\alpha_1}s_{\alpha_4}$	$\phi_{\alpha_3} \otimes \phi_{\alpha_1+\alpha_3+\alpha_4}$
		$\phi_{\alpha_1} \otimes \phi_{\alpha_4}$
	$s_{\alpha_4}s_{\alpha_6}$	$\phi_{\alpha_5} \otimes \phi_{\alpha_4+\alpha_5+\alpha_6} + \phi_{\alpha_4+\alpha_5} \otimes \phi_{\alpha_5+\alpha_6}$
		$\phi_{\alpha_4} \otimes \phi_{\alpha_6}$
	$s_{\alpha_5}s_{\alpha_7}$	$\phi_{\alpha_6} \otimes \phi_{\alpha_5+\alpha_6+\alpha_7} + \phi_{\alpha_5+\alpha_6} \otimes \phi_{\alpha_6+\alpha_7}$
		$\phi_{\alpha_5} \otimes \phi_{\alpha_7}$
	$s_{\alpha_2}s_{\alpha_5}$	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_3} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$ $\phi_{\alpha_1+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_3} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4} \otimes \phi_{\alpha_2+\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_4+\alpha_5}$
		$\phi_{\alpha_2} \otimes \phi_{\alpha_5}$
	$s_{\alpha_3}s_{\alpha_5}$	$\phi_{\alpha_2} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_3+\alpha_4} \otimes \phi_{\alpha_4+\alpha_5}$
		$\phi_{\alpha_3} \otimes \phi_{\alpha_5}$

		$\phi_{\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7}+\phi_{\alpha_6+\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_5+\alpha_6+\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$
E_7	$s_{\alpha_2}s_{\alpha_3}$	$\phi_{\alpha_6}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}+\phi_{\alpha_5+\alpha_6}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$ $\phi_{\alpha_4+\alpha_5+\alpha_6}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$
		$\phi_{\alpha_5}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}+\phi_{\alpha_4+\alpha_5}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}+\phi_{\alpha_2+\alpha_4+\alpha_5}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4}+\phi_{\alpha_2+\alpha_4}\otimes\phi_{\alpha_3+\alpha_4}$
		$\phi_{\alpha_2}\otimes\phi_{\alpha_3}$
	$s_{\alpha_1}s_{\alpha_4}$	$\phi_{\alpha_3}\otimes\phi_{\alpha_1+\alpha_3+\alpha_4}+\phi_{\alpha_1+\alpha_3}\otimes\phi_{\alpha_3+\alpha_4}$
		$\phi_{\alpha_1}\otimes\phi_{\alpha_4}$
	$s_{\alpha_4}s_{\alpha_6}$	$\phi_{\alpha_5}\otimes\phi_{\alpha_4+\alpha_5+\alpha_6}+\phi_{\alpha_4+\alpha_5}\otimes\phi_{\alpha_5+\alpha_6}$
		$\phi_{\alpha_4}\otimes\phi_{\alpha_6}$
	$s_{\alpha_5}s_{\alpha_7}$	$\phi_{\alpha_6}\otimes\phi_{\alpha_5+\alpha_6+\alpha_7}+\phi_{\alpha_5+\alpha_6}\otimes\phi_{\alpha_6+\alpha_7}$
		$\phi_{\alpha_5}\otimes\phi_{\alpha_7}$
E_8	$s_{\alpha_6}s_{\alpha_8}$	$\phi_{\alpha_7}\otimes\phi_{\alpha_6+\alpha_7+\alpha_8}+\phi_{\alpha_6+\alpha_7}\otimes\phi_{\alpha_7+\alpha_8}$
		$\phi_{\alpha_6}\otimes\phi_{\alpha_8}$
		$\phi_{\alpha_1}\otimes\phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5}+\phi_{\alpha_1+\alpha_3}\otimes\phi_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$ $+\phi_{\alpha_1+\alpha_3+\alpha_4}\otimes\phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}+\phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\otimes\phi_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$
	$s_{\alpha_2}s_{\alpha_5}$	$\phi_{\alpha_3}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}+\phi_{\alpha_3+\alpha_4}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}+\phi_{\alpha_2+\alpha_3+\alpha_4}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4}\otimes\phi_{\alpha_2+\alpha_4+\alpha_5}+\phi_{\alpha_2+\alpha_4}\otimes\phi_{\alpha_4+\alpha_5}$
		$\phi_{\alpha_2}\otimes\phi_{\alpha_5}$
		$\phi_{\alpha_2}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}+\phi_{\alpha_2+\alpha_4}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}+\phi_{\alpha_2+\alpha_3+\alpha_4}\otimes\phi_{\alpha_2+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5}+\phi_{\alpha_3+\alpha_4}\otimes\phi_{\alpha_4+\alpha_5}$
	$s_{\alpha_3}s_{\alpha_5}$	$\phi_{\alpha_3}\otimes\phi_{\alpha_5}$

E_8	$s_{\alpha_2}s_{\alpha_3}$	$\phi_{\alpha_8}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8}+\phi_{\alpha_7+\alpha_8}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_6+\alpha_7+\alpha_8}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_5+\alpha_6+\alpha_7+\alpha_8}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$
		$\phi_{\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7}+\phi_{\alpha_6+\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_5+\alpha_6+\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$
		$\phi_{\alpha_6}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}+\phi_{\alpha_5+\alpha_6}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$
		$\phi_{\alpha_5}\otimes\phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}+\phi_{\alpha_4+\alpha_5}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}+\phi_{\alpha_2+\alpha_4+\alpha_5}\otimes\phi_{\alpha_3+\alpha_4+\alpha_5}$
		$\phi_{\alpha_4}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4}+\phi_{\alpha_2+\alpha_4}\otimes\phi_{\alpha_3+\alpha_4}$
		$\phi_{\alpha_2}\otimes\phi_{\alpha_3}$
F_4	$s_{\alpha_1}s_{\alpha_3}$	$\phi_{\alpha_3}\otimes\phi_{\alpha_1+2\alpha_2+2\alpha_3}+\phi_{\alpha_2+\alpha_3}\otimes\phi_{\alpha_1+\alpha_2+2\alpha_3}+\phi_{\alpha_1+\alpha_2+\alpha_3}\otimes\phi_{\alpha_2+2\alpha_3}$
		$\phi_{\alpha_2}\otimes\phi_{\alpha_1+\alpha_2+\alpha_3}+\phi_{\alpha_1+\alpha_2}\otimes\phi_{\alpha_2+\alpha_3}$
		$\phi_{\alpha_1}\otimes\phi_{\alpha_3}$
	e	$\phi_{\alpha_2}\otimes\phi_{2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4}+\phi_{\alpha_1+\alpha_2}\otimes\phi_{\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4}$ $+\phi_{\alpha_1+2\alpha_2+2\alpha_3}\otimes\phi_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}+\phi_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}\otimes\phi_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}$
		$\phi_{\alpha_1}\otimes\phi_{\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4}+\phi_{\alpha_1+\alpha_2+2\alpha_3}\otimes\phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}$ $+\phi_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}\otimes\phi_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}$
		$\phi_{\alpha_1}\otimes\phi_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}+\phi_{\alpha_1+\alpha_2}\otimes\phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}+\phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$
		$\phi_{\alpha_1}\otimes\phi_{\alpha_1+2\alpha_2+2\alpha_3}+\phi_{\alpha_1+\alpha_2}\otimes\phi_{\alpha_1+\alpha_2+2\alpha_3}+\phi_{\alpha_1+\alpha_2+\alpha_3}\otimes\phi_{\alpha_1+\alpha_2+\alpha_3}$
		$\phi_{\alpha_2}\otimes\phi_{\alpha_2+2\alpha_3+2\alpha_4}+\phi_{\alpha_2+\alpha_3+\alpha_4}\otimes\phi_{\alpha_2+\alpha_3+\alpha_4}$
		$\phi_{\alpha_2}\otimes\phi_{\alpha_2+2\alpha_3}+\phi_{\alpha_2+\alpha_3}\otimes\phi_{\alpha_2+\alpha_3}$
G_2	e	$\phi_{\alpha_1}\otimes\phi_{\alpha_1+2\alpha_2}$

Remark A.0.2. In the above table, e denotes the identity element. When $w = s_{\alpha_i}s_{\alpha_j}$ when i and j are not connected and aren't separated by a single vertex, then in all of the classical types, there is a single cohomology class: $\phi_{\alpha_i} \otimes \phi_{\alpha_j}$. In the exceptional cases, there are some other classes that occur, and are demonstrated in Appendix B.

Appendix B

weights and cohomology classes in exceptional cases

Table B.1: The extra cohomology classes in $\mathrm{H}^2(U_1, k)$, explicitly written out, for the exception groups.

Φ	cohomology class	weight
E_6	$\phi_{\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_4+\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$	$(0,1,1,2,2,0)$
	$\phi_{\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} + \phi_{\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$	$(0,1,1,2,2,2)$
	$\phi_{\alpha_3} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$	$(0,1,2,2,1,0)$
	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_3} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$ $+\phi_{\alpha_1+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$	$(2,1,2,2,1,0)$
	$\phi_{\alpha_2} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_4+\alpha_5}$	$(0,2,1,2,1,0)$
E_7	$\phi_{\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_4+\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$	$(0,1,1,2,2,0,0)$
	$\phi_{\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} + \phi_{\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$	$(0,1,1,2,2,2,0)$

E_7	$\phi_{\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7}$ $+\phi_{\alpha_6+\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_5+\alpha_6+\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6+\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$	$(0,1,1,2,2,2,2)$
	$\phi_{\alpha_3} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$	$(0,1,2,2,1,0,0)$
	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_3} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$ $+\phi_{\alpha_1+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$	$(2,1,2,2,1,0,0)$
	$\phi_{\alpha_2} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_4+\alpha_5}$	$(0,2,1,2,1,0,0)$
E_8	$\phi_{\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_4+\alpha_5} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$	$(0,1,1,2,2,0,0,0)$
	$\phi_{\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} + \phi_{\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$	$(0,1,1,2,2,2,0,0)$
	$\phi_{\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7}$ $+\phi_{\alpha_6+\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_5+\alpha_6+\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6+\alpha_7} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$	$(0,1,1,2,2,2,2,0)$
	$\phi_{\alpha_3} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5}$	$(0,1,2,2,1,0,0,0)$

E_8	$\phi_{\alpha_8} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8}$ $+\phi_{\alpha_7+\alpha_8} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_6+\alpha_7+\alpha_8} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_5+\alpha_6+\alpha_7+\alpha_8} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$ $+\phi_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8} \otimes \phi_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$	$(0,1,1,2,2,2,2,2)$
	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_1+\alpha_3} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$ $+\phi_{\alpha_1+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$	$(2,1,2,2,1,0,0,0)$
	$\phi_{\alpha_2} \otimes \phi_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + \phi_{\alpha_2+\alpha_4} \otimes \phi_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$ $+\phi_{\alpha_2+\alpha_3+\alpha_4} \otimes \phi_{\alpha_2+\alpha_4+\alpha_5}$	$(0,2,1,2,1,0,0,0)$
F_4	$\phi_{\alpha_2} \otimes \phi_{\alpha_1+\alpha_2+\alpha_3} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_2+\alpha_3}$	$(1,2,1,0)$
	$\phi_{\alpha_3} \otimes \phi_{\alpha_1+2\alpha_2+2\alpha_3} + \phi_{\alpha_2+\alpha_3} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3} + \phi_{\alpha_1+\alpha_2+\alpha_3} \otimes \phi_{\alpha_2+2\alpha_3}$	$(1,2,3,0)$
	$\phi_{\alpha_2} \otimes \phi_{\alpha_2+2\alpha_3}$	$(0,2,2,0)$
	$\phi_{\alpha_2} \otimes \phi_{\alpha_2+2\alpha_3+2\alpha_4}$	$(0,2,2,2)$
	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+2\alpha_2+2\alpha_3} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3}$	$(2,2,2,0)$
	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}$	$(2,2,2,2)$
	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4} + \phi_{\alpha_1+\alpha_2+2\alpha_3} \otimes \phi_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}$	$(2,2,4,2)$
	$\phi_{\alpha_2} \otimes \phi_{2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{2\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4}$ $+\phi_{\alpha_1+2\alpha_2+2\alpha_3} \otimes \phi_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}$	$(2,4,4,2)$
G_2	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+2\alpha_2}$	$(2,4)$
	$\phi_{\alpha_1} \otimes \phi_{\alpha_1+3\alpha_2} + \phi_{\alpha_1+\alpha_2} \otimes \phi_{\alpha_1+2\alpha_2}$	$(2,3)$

Appendix C

G_r-cohomology module structure

C.1

The following lemma gives the module structure for the G_1-cohomology, with all factors listed from top to bottom.

Lemma C.1.1. *Let $p = 2$ and M be a module as above with corresponding $w \in W$. Suppose $\nu \in X(T)$ with $w \cdot 0 + p\nu \in X(T)_+$.*

(a) Suppose $w = s_\alpha s_\beta$ with $\alpha + \beta \notin \Phi^+$ and $\alpha + \beta + \gamma \in \Phi^+$. Then $\langle \nu, \alpha_i^\vee \rangle \geq 0$ for $\alpha_i \neq \alpha, \beta, \gamma$, $\langle \nu, \alpha^\vee \rangle \geq 1, \langle \nu, \gamma^\vee \rangle \geq -1, \langle \nu, \beta^\vee \rangle \geq 1$. Let $\delta \in \Delta$ be such that $\delta + \alpha \in \Phi^+$ and $\delta \neq \gamma$. Further,

(i) If $\langle \nu, \gamma^\vee \rangle = -1$, then $\operatorname{ind}_B^G(M \otimes \nu) = 0$.

(ii) If $\langle \nu, \gamma^\vee \rangle = 0$, then $\operatorname{ind}_B^G(M \otimes \nu) \cong H^0(\nu)$.

(iii) If $\langle \nu, \gamma^\vee \rangle \geq 1, \langle \nu, \delta^\vee \rangle = 0$, then $\operatorname{ind}_B^G(M \otimes \nu) \cong H^0(\nu)$.

(iv) If $\langle \nu, \gamma^\vee \rangle \geq 1, \langle \nu, \delta^\vee \rangle \geq 1$, then $\operatorname{ind}_B^G(M \otimes \nu)$ has factors $H^0(\alpha + \nu)$ and $H^0(\nu)$.

(b) Φ is of type B_n with $w = s_{\alpha_{n-2}} s_{\alpha_n}$. Then $\langle \nu, \alpha_i^\vee \rangle \geq 0$ for $1 \leq i \leq n-3$, $\langle \nu, \alpha_{n-2}^\vee \rangle \geq 1, \langle \nu, \alpha_{n-1}^\vee \rangle \geq -1, \langle \nu, \alpha_n^\vee \rangle \geq 1$. Further,

(i) If $\langle \nu, \alpha_{n-1}^\vee \rangle = -1$ and $\langle \nu, \alpha_n^\vee \rangle = 1$, then $\operatorname{ind}_B^G(M \otimes \nu) = 0$.

(ii) If $\langle \nu, \alpha_{n-1}^\vee \rangle = -1$ and $\langle \nu, \alpha_n^\vee \rangle \geq 2$, then $\operatorname{ind}_B^G(M \otimes \nu) = H^0(\alpha_{n-1} + \nu)$.

(iii) If $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 0$ and $\langle \nu, \alpha_n^\vee \rangle = 1$, then $\operatorname{ind}_B^G(M \otimes \nu) = H^0(\nu)$.

(iv) *If* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 0$ *and* $\langle \nu, \alpha_n^\vee \rangle \geq 2$, *then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-1} + \nu)$ *and* $H^0(\nu)$.

(c) Φ *is of type* B_n *with* $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$ *for* $1 \leq i \leq n-4$, $\langle \nu, \alpha_{n-3}^\vee \rangle \geq 1, \langle \nu, \alpha_{n-2}^\vee \rangle \geq -1, \langle \nu, \alpha_{n-1}^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_n^\vee \rangle \geq -1$. *Further,*

(i) *If* $\langle \nu, \alpha_n^\vee \rangle = -1$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = 0$.

(ii) *If* $\langle \nu, \alpha_n^\vee \rangle \geq 0$ *and* $\langle \nu, \alpha_{n-2}^\vee \rangle = -1$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_{n-2} + \nu)$.

(iii) *If* $\langle \nu, \alpha_n^\vee \rangle \geq 0$ *and* $\langle \nu, \alpha_{n-2}^\vee \rangle \geq 0$, *then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \nu)$ *and* $H^0(\nu)$.

(d) Φ *is of type* B_n *with* $w = s_{\alpha_j} s_{\alpha_{n-1}}$ *with* $j \neq n-3$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$ *for* $1 \leq i \leq n-2, i \neq j$, $\langle \nu, \alpha_j^\vee \rangle \geq 1, \langle \nu, \alpha_{n-1}^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_n^\vee \rangle \geq -1$. *Further,*

(i) *If* $\langle \nu, \alpha_n^\vee \rangle = -1$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_n + \nu)$.

(ii) *If* $\langle \nu, \alpha_n^\vee \rangle \geq 0$, *then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_n + \nu)$ *and* $H^0(\nu)$.

(e) Φ *is of type* C_n *with* $w = s_{\alpha_j} s_{\alpha_n}$ *with* $j \neq n-2$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$ *for* $1 \leq i \leq n-2$ *and* $i \neq j$, $\langle \nu, \alpha_j^\vee \rangle \geq 1, \langle \nu, \alpha_{n-1}^\vee \rangle \geq -1$, *and* $\langle \nu, \alpha_n^\vee \rangle \geq 1$. *Further,*

(i) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle = 0$ *and* $\langle \nu, \alpha_{n-1}^\vee \rangle = -1$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = 0$.

(ii) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_{n-1}^\vee \rangle = -1$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_{n-1} + \nu)$.

(iii) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle = 0$ *and* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 0$, *then* $\mathrm{ind}_B^G(M \otimes \nu) \cong H^0(\nu)$.

(iv) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 0$ *then* $\mathrm{ind}_B^G(N_{C_n} \otimes \nu)$ *has factors* $H^0(\alpha_{n-1} + \nu)$ *and* $H^0(\nu)$.

(f) Φ *is of type* C_n *with* $w = s_{\alpha_{n-2}} s_{\alpha_n}$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$ *for* $1 \leq i \leq n-3$, $\langle \nu, \alpha_{n-2}^\vee \rangle \geq 1, \langle \nu, \alpha_{n-1}^\vee \rangle \geq -1$, *and* $\langle \nu, \alpha_n^\vee \rangle \geq 1$. *Further,*

(i) *If* $\langle \nu, \alpha_{n-1}^\vee \rangle = -1$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = 0$.

(ii) *If* $\langle \nu, \alpha_{n-3}^\vee \rangle = 0$ *or* $\langle \nu, \alpha_{n-1}^\vee \rangle = 0$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\nu)$.

(iii) *If* $\langle \nu, \alpha_{n-3}^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 1$*, then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \nu)$ *and* $H^0(\nu)$*.*

(g) Φ *is of type* D_n *with* $w = s_{\alpha_{n-3}} s_{\alpha_{n-1}}$*. Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$ *for* $1 \leq i \leq n-4$*,* $\langle \nu, \alpha_{n-3}^\vee \rangle \geq 1, \langle \nu, \alpha_{n-2}^\vee \rangle \geq -1$*,* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 1$*, and* $\langle \nu, \alpha_n^\vee \rangle \geq 0$*. Further,*

(i) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle = -1$ *and* $\langle \nu, \alpha_n^\vee \rangle = 0$*, then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_{n-2} + \alpha_n + \nu)$*.*

(ii) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle = -1$ *and* $\langle \nu, \alpha_n^\vee \rangle \geq 1$*, then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \alpha_n + \nu)$ *and* $H^0(\alpha_{n-2} + \nu)$*.*

(iii) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle \geq 0$ *and* $\langle \nu, \alpha_n^\vee \rangle = 0$*, then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \alpha_n + \nu)$ *and* $H^0(\nu)$*.*

(iv) *If* $\langle \nu, \alpha_{n-2}^\vee \rangle \geq 0$ *and* $\langle \nu, \alpha_n^\vee \rangle \geq 1$*, then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \alpha_n + \nu), H^0(\alpha_{n-2} + \nu)$ *and* $H^0(\nu)$*.*

(h) Φ *is of type* D_n *with* $w = s_{\alpha_{n-3}} s_{\alpha_n}$*. Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$ *for* $1 \leq i \leq n-4$*,* $\langle \nu, \alpha_{n-3}^\vee \rangle \geq 1, \langle \nu, \alpha_{n-2}^\vee \rangle \geq -1$*,* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 0$*, and* $\langle \nu, \alpha_n^\vee \rangle \geq 1$*. Further,*

(i) *If* $\langle \nu, \alpha_{n-1}^\vee \rangle = -1$ *and* $\langle \nu, \alpha_n^\vee \rangle = 0$*, then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_{n-2} + \alpha_{n-1} + \nu)$*.*

(ii) *If* $\langle \nu, \alpha_{n-1}^\vee \rangle = -1$ *and* $\langle \nu, \alpha_n^\vee \rangle \geq 1$*, then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \alpha_{n-1} + \nu)$ *and* $H^0(\alpha_{n-2} + \nu)$*.*

(iii) *If* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 0$ *and* $\langle \nu, \alpha_n^\vee \rangle = 0$*, then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \alpha_{n-1} + \nu)$ *and* $H^0(\nu)$*.*

(iv) *If* $\langle \nu, \alpha_{n-1}^\vee \rangle \geq 0$ *and* $\langle \nu, \alpha_n^\vee \rangle \geq 1$*, then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_{n-2} + \alpha_{n-1} + \nu), H^0(\alpha_{n-2} + \nu)$ *and* $H^0(\nu)$*.*

(i) Φ *is of type* E_6 *with* $w = s_{\alpha_2} s_{\alpha_3}$*. Then* $\langle \nu, \alpha_1^\vee \rangle \geq 0, \langle \nu, \alpha_2^\vee \rangle \geq 1, \langle \nu, \alpha_3^\vee \rangle \geq 1$*,* $\langle \nu, \alpha_4^\vee \rangle \geq -1, \langle \nu, \alpha_5^\vee \rangle \geq 0$*, and* $\langle \nu, \alpha_6^\vee \rangle \geq 0$*. Further,*

(i) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0$ *and* $\langle \nu, \alpha_6^\vee \rangle = 0$*. Then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$*.*

(ii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0$ *and* $\langle \nu, \alpha_6^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \nu)$.

(iii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_6^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(iv) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_6^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(v) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0$ *and* $\langle \nu, \alpha_6^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\nu)$.

(vi) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0$ *and* $\langle \nu, \alpha_6^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\nu)$.

(vii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_6^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(viii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_6^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(j) Φ *is of type* E_6, E_7, E_8 *with* $w = s_{\alpha_2} s_{\alpha_5}$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$, *if* $i \in \{1, 3, 6, 7, 8\}$, $\langle \nu, \alpha_2^\vee \rangle \geq 1, \langle \nu, \alpha_4^\vee \rangle \geq -1$, *and* $\langle \nu, \alpha_5^\vee \rangle \geq 1$. *Further,*

(i) *If* $\langle \nu, \alpha_1^\vee \rangle = 0, \langle \nu, \alpha_3^\vee \rangle = 0$ *and* $\langle \nu, \alpha_4^\vee \rangle = -1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu)$.

(ii) *If* $\langle \nu, \alpha_1^\vee \rangle = 0, \langle \nu, \alpha_3^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_4^\vee \rangle = -1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(iii) *If* $\langle \nu, \alpha_1^\vee \rangle \geq 1, \langle \nu, \alpha_3^\vee \rangle = 0$ *and* $\langle \nu, \alpha_4^\vee \rangle = -1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu)$ *and* $H^0(\alpha_3 + \alpha_4 + \nu)$.

(iv) *If* $\langle \nu, \alpha_1^\vee \rangle \geq 1, \langle \nu, \alpha_3^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_4^\vee \rangle = -1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$*has factors* $H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu)$, $H^0(\alpha_3 + \alpha_4 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(v) *If* $\langle \nu, \alpha_1^\vee \rangle = 0, \langle \nu, \alpha_3^\vee \rangle = 0$ *and* $\langle \nu, \alpha_4^\vee \rangle \geq 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu)$ *and* $H^0(\nu)$.

(vi) *If* $\langle \nu, \alpha_1^\vee \rangle = 0, \langle \nu, \alpha_3^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_4^\vee \rangle \geq 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(vii) *If* $\langle \nu, \alpha_1^\vee \rangle \geq 1, \langle \nu, \alpha_3^\vee \rangle = 0$ *and* $\langle \nu, \alpha_6^\vee \rangle \geq 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu), H^0(\alpha_3 + \alpha_4 + \nu)$ *and* $H^0(\nu)$.

(viii) *If* $\langle \nu, \alpha_1^\vee \rangle \geq 1, \langle \nu, \alpha_3^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_4^\vee \rangle \geq 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_1 + \alpha_3 + \alpha_4 + \nu)$, $H^0(\alpha_3 + \alpha_4 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(k) Φ *is of type* E_6, E_7, E_8 *with* $w = s_{\alpha_3} s_{\alpha_5}$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$, *if* $i \in \{1, 2, 6, 7, 8\}$, $\langle \nu, \alpha_3^\vee \rangle \geq 1, \langle \nu, \alpha_4^\vee \rangle \geq -1$, *and* $\langle \nu, \alpha_5^\vee \rangle \geq 1$. *Further,*

(i) *If* $\langle \nu, \alpha_2^\vee \rangle = 0$ *and* $\langle \nu, \alpha_4^\vee \rangle = -1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_2 + \alpha_4 + \nu)$.

(ii) *If* $\langle \nu, \alpha_2^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_4^\vee \rangle = -1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_2 + \alpha_4 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(iii) *If* $\langle \nu, \alpha_2^\vee \rangle = 0$ *and* $\langle \nu, \alpha_4^\vee \rangle \geq 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_2 + \alpha_4 + \nu)$ *and* $H^0(\nu)$.

(iv) *If* $\langle \nu, \alpha_2^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_4^\vee \rangle \geq 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$*has factors* $H^0(\alpha_2 + \alpha_4 + \nu)$, $H^0(\alpha_4 + \nu)$, *and* $H^0(\nu)$.

(l) Φ *is of type* E_6, E_7, E_8 *with* $w = s_{\alpha_4} s_{\alpha_6}$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$ *for* $i \in \{1, 2, 3, 7, 8\}$, $\langle \nu, \alpha_4^\vee \rangle \geq 1$, $< \langle \nu, \alpha_5^\vee \rangle \geq -1$, *and* $\langle \nu, a_6^\vee \rangle \geq 1$. *Further,*

(i) *If* $\langle \nu, a_5^\vee \rangle = -1$, *then* $\mathrm{ind}_B^G(M \otimes \nu) = 0$.

(ii) *If* $\langle \nu, a_5^\vee \rangle = 0$, *then* $\mathrm{ind}_B^G(M \otimes \nu) \cong H^0(\nu)$.

(iii) *If* $\langle \nu, a_5^\vee \rangle \geq 1$ *and* $(\langle \nu, a_2^\vee \rangle = 0$ *or* $\langle \nu, a_3^\vee \rangle = 0)$, *then* $\mathrm{ind}_B^G(M \otimes \nu) \cong H^0(\nu)$.

(iv) *If* $\langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_2^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_3^\vee \rangle \geq 1$ *then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(m) Φ *is of type* E_7 *with* $w = s_{\alpha_2}s_{\alpha_3}$*. Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$*, if* $i \in \{1,5,6,7\}$*,* $\langle \nu, \alpha_2^\vee \rangle \geq 1$*,* $\langle \nu, \alpha_3^\vee \rangle \geq 1$*, and* $\langle \nu, \alpha_4^\vee \rangle \geq -1$*. Further,*

(i) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$*. Then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$*.*

(ii) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$*.*

(iii) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \nu)$*.*

(iv) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$ *and* $H^0(\alpha_4 + \nu)$*.*

(v) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$*,* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$*, and* $H^0(\alpha_4 + \alpha_5)$*.*

(vi) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\alpha_4 + \nu)$*.*

(vii) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\alpha_4 + \nu)$*.*

(viii) If $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$*,* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\alpha_4 + \nu)$*.*

(ix) If $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$ *and* $H^0(\nu)$*.*

(x) If $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\nu)$*.*

(xi) If $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$*. Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\nu)$*.*

(xii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$*has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xiii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5)$ *and* $H^0(\nu)$.

(xiv) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xv) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xvi) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_7^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu), H^0(\alpha_4 + \nu)$, *and* $H^0(\nu)$.

(n) Φ *is of type* E_8 *with* $w = s_{\alpha_2} s_{\alpha_3}$. *Then* $\langle \nu, \alpha_i^\vee \rangle \geq 0$, *if* $i \in \{1, 5, 6, 7, 8\}$, $\langle \nu, \alpha_2^\vee \rangle \geq 1$, $\langle \nu, \alpha_3^\vee \rangle \geq 1$, *and* $\langle \nu, \alpha_4^\vee \rangle \geq -1$. *Further,*

(i) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu) = H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$.

(ii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$.

(iii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$.

(iv) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$*has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \nu)$.

(v) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0$, *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(vi) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$.

(vii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \nu)$.

(viii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$, *and* $H^0(\alpha_4 + \nu)$.

(ix) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$, *and* $H^0(\alpha_4 + \alpha_5 + \nu)$.

(x) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$, *and* $H^0(\alpha_4 + \nu)$.

(xi) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle = 0,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \nu)$, *and* $H^0(\alpha_4 + \nu)$.

(xii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors*
$H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\alpha_4 + \alpha_5 + \nu)$.

(xiii) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\text{ind}_B^G(M \otimes \nu)$ *has factors*
$H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(xiv) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle =\geq 1, \langle \nu, \alpha_7^\vee \rangle = 0,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(xv) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\alpha_4 + \nu)$.

(xvi) *If* $\langle \nu, \alpha_4^\vee \rangle = -1, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu),$ *and* $H^0(\alpha_4 + \nu)$.

(xvii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$ *and* $H^0(\nu)$.

(xviii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu)$ *and* $H^0(\nu)$.

(xix) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\nu)$.

(xx) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$*has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\nu)$.

(xxi) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xxii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu)$ *and* $H^0(\nu)$.

(xxiii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle = 0$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\nu)$.

(xxiv) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle = 0,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xxv) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\nu)$.

(xxvi) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xxvii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle = 0,$ *and* $\langle \nu, \alpha_8^\vee \rangle = 0$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xxviii) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle = 0, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu)$ *and* $H^0(\nu)$.

(xxix) *If* $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle = 0, \langle \nu, \alpha_7^\vee \rangle \geq 1,$ *and* $\langle \nu, \alpha_8^\vee \rangle \geq 1$. *Then* $\mathrm{ind}_B^G(M \otimes \nu)$ *has factors* $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \nu)$ *and* $H^0(\nu)$.

(xxx) If $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle =\geq 1, \langle \nu, \alpha_7^\vee \rangle = 0$, and $\langle \nu, \alpha_8^\vee \rangle \geq 1$. Then $\text{ind}_B^G(M \otimes \nu)$ has factors $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu), H^0(\alpha_4 + \nu)$ and $H^0(\nu)$.

(xxxi) If $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1$, and $\langle \nu, \alpha_8^\vee \rangle = 0$. Then $\text{ind}_B^G(M \otimes \nu)$ has factors $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu), H^0(\alpha_4 + \nu)$ and $H^0(\nu)$.

(xxxii) If $\langle \nu, \alpha_4^\vee \rangle \geq 0, \langle \nu, \alpha_5^\vee \rangle \geq 1, \langle \nu, \alpha_6^\vee \rangle \geq 1, \langle \nu, \alpha_7^\vee \rangle \geq 1$, and $\langle \nu, \alpha_8^\vee \rangle \geq 1$. Then $\text{ind}_B^G(M \otimes \nu)$ has factors $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \nu)$, $H^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \nu), H^0(\alpha_4 + \alpha_5 + \alpha_6 + \nu), H^0(\alpha_4 + \alpha_5 + \nu), H^0(\alpha_4 + \nu)$ and $H^0(\nu)$.

(o) If Φ is of type F_4 and $w = s_{\alpha_2} s_{\alpha_4}$. Then $\langle \nu, \alpha_1^\vee \rangle \geq 0, \langle \nu, \alpha_2^\vee \rangle \geq 1, \langle \nu, \alpha_3^\vee \rangle \geq -1$, and $\langle \nu, \alpha_4^\vee \rangle \geq 1$. Further,

(i) If $\langle \nu, \alpha_3^\vee \rangle = -1$ then $\text{ind}_B^G(N_{F_4} \otimes \nu) = 0$.

(ii) If $\langle \nu, \alpha_3^\vee \rangle = 0, 1$ or $\langle \nu, \alpha_1^v ee \rangle = 0$ then $\text{ind}_B^G(N_{F_4} \otimes \nu) \cong H^0(\nu)$.

(iii) If $\langle \nu, \alpha_3^\vee \rangle \geq 2$ and $\langle \nu, \alpha_1^v ee \rangle \geq 1$ then $\text{ind}_B^G(N_{F_4} \otimes \nu)$ has factors $H^0(\alpha_2 + \nu)$ and $H^0(\nu)$.

(p) If Φ is of type F_4 and $w = s_{\alpha_1} s_{\alpha_3}$. Then $\langle \nu, \alpha_1^\vee \rangle \geq 1, \langle \nu, \alpha_2^\vee \rangle \geq -1, \langle \nu, \alpha_3^\vee \rangle \geq 1$, and $\langle \nu, \alpha_4^\vee \rangle \geq 0$. Further,

(i) If $\langle \nu, \alpha_2^\vee \rangle = -1, \langle \nu, \alpha_3^\vee \rangle = 1$ and $\langle \nu, \alpha_4^\vee \rangle = 0$ then $\text{ind}_B^G(N_{F_4} \otimes \nu) = 0$.

(ii) If $\langle \nu, \alpha_2^\vee \rangle = -1, \langle \nu, \alpha_3^\vee \rangle = 1$ and $\langle \nu, \alpha_4^\vee \rangle \geq 1$ then $\text{ind}_B^G(N_{F_4} \otimes \nu) = H^0(\alpha_2 + \alpha_3 + \nu)$.

(iii) If $\langle \nu, \alpha_2^\vee \rangle = -1, \langle \nu, \alpha_3^\vee \rangle \geq 2$ and $\langle \nu, \alpha_4^\vee \rangle = 0$ then $\text{ind}_B^G(N_{F_4} \otimes \nu) = H^0(\alpha_2 + \nu)$.

(iv) If $\langle \nu, \alpha_2^\vee \rangle = -1, \langle \nu, \alpha_3^\vee \rangle \geq 2$ and $\langle \nu, \alpha_4^\vee \rangle \geq 1$ then $\text{ind}_B^G(N_{F_4} \otimes \nu)$ has factors $H^0(\alpha_2 + \alpha_3 + \nu)$ and $H^0(\alpha_2 + \nu)$.

(v) If $\langle \nu, \alpha_5^\vee \rangle \geq 0, \langle \nu, \alpha_3^\vee \rangle = 1$ and $\langle \nu, \alpha_4^\vee \rangle = 0$ then $\text{ind}_B^G(N_{F_4} \otimes \nu) \cong H^0(\nu)$.

(vi) *If* $\langle \nu, \alpha_2^\vee \rangle \geq 0, \langle \nu, \alpha_3^\vee \rangle = 1$ *and* $\langle \nu, \alpha_4^\vee \rangle \geq 1$ *then* $\text{ind}_B^G(N_{F_4} \otimes \nu)$ *has factors* $H^0(\alpha_2 + \alpha_3 + \nu)$ *and* $H^0(\nu)$.

(vii) *If* $\langle \nu, \alpha_2^\vee \rangle \geq 0, \langle \nu, \alpha_3^\vee \rangle \geq 2$ *and* $\langle \nu, \alpha_4^\vee \rangle = 0$ *then* $\text{ind}_B^G(N_{F_4} \otimes \nu)$ *has factors* $H^0(\alpha_2 + \nu)$ *and* $H^0(\nu)$.

(viii) *If* $\langle \nu, \alpha_2^\vee \rangle \geq 0, \langle \nu, \alpha_3^\vee \rangle \geq 2$ *and* $\langle \nu, \alpha_4^\vee \rangle \geq 1$ *then* $\text{ind}_B^G(N_{F_4} \otimes \nu)$ *has factors* $H^0(\alpha_2 + \alpha_3 + \nu), H^0(\alpha_2 + \nu)$ *and* $H^0(\nu)$.

Proof. The proof is similar for all these cases, so let's just examine the general case when $w = s_\alpha s_\beta$ for any type. In particular, consider the case $w = s_{\alpha_i} s_{\alpha_{i+2}}$. Then $w \cdot 0 = \omega_{i-1} - 2\omega_i + 2\omega_{i+1} - 2\omega_{i+2} + \omega_{i+3}$. Let $\nu = \sum_{i=1}^n c_i \omega_i$. If $w \cdot 0 + 2\nu$ is dominant, then $c_j \geq 0$ for $j \notin \{i, i+1, i+2\}$, $c_i \geq 1, c_{i+1} \geq -1$ and $c_{i+2} \geq 1$. To determine the structure of the induced modules, the argument follows [BNP1, Proposition 3.4] with [BNP1, Lemma 3.3], which explains that it is necessary to determine precisely when our module factors are dominant. The module $M \otimes \nu$ has factors $\alpha_i + \nu$ and ν. Consider

$$\alpha_i + \nu = \sum_{j=1}^{i-2} c_j \omega_j + (c_{i-1} - 1)\omega_{i-1} + (c_i + 2)\omega_i + (c_{i+1} - 1)\omega_{i+1} + \sum_{j=i+2}^{n} c_j \omega_j$$

which is dominant precisely when $c_{i-1} \geq 1$ and $c_{i+1} \geq 1$. Now, let's look at the different cases. If $\left\langle \nu, \alpha_{i+1}^\vee \right\rangle = -1$, then ν and $\alpha_i + \nu$ aren't dominant and $\text{ind}_B^G(M \otimes \nu) = 0$. If $\left\langle \nu, \alpha_{i+1}^\vee \right\rangle = 0$ or $\langle \nu, a_{i-1} \rangle = 0$, then $\alpha_i + \nu$ isn't dominant, but ν is, so using [BNP1] and [Jan1, II.4.5], then $\text{ind}_B^G(\nu) = H^0(\nu)$; hence $\text{ind}_B^G(M \otimes \nu) = H^0(\nu)$. If $\left\langle \nu, \alpha_{i+1}^\vee \right\rangle \geq 1$ and $\langle \nu, a_{i-1} \rangle$ *and* ≥ 1, then both $\alpha_i + \nu$ and ν are dominant. Again using [Jan1, II.4.5] and [BNP1, 3.4] then $\text{ind}_B^G(\alpha_i + \nu) = H^0(\alpha_i + \nu)$ and $\text{ind}_B^G(\nu) = H^0(\nu)$. Hence $\text{ind}_B^G(M \otimes \nu)$ has factors $H^0(\alpha_i + \nu)$ and $H^0(\nu)$. Thus (a) follows. □