USING HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOR TO GAIN INSIGHTS INTO TEACHERS' USE OF QUESTIONING TO SUPPORT COLLECT ARGUMENTATION

by

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(Under the Direction of AnnaMarie Conner)

ABSTRACT

Collective mathematical argumentation is powerful for student learning of mathematics, and teachers' questioning strategies are pivotal in orchestrating collective argumentation. Adapting Habermas' (1998) construct of rational behavior, this study demonstrated how teachers' questioning can be framed based on this construct as a teaching method to regulate argumentative discourse. The purpose of this study was to investigate how teachers use rational questioning to organize collective argumentation with respect to Habermas' three components of rationality as a way to develop students' awareness of the rationality requirements of argumentation. The participants in this study were two beginning secondary mathematics teachers who have learned about supporting collective argumentation during their teacher education program as well as during professional development for several years. Four video-recorded lessons (two sets of two consecutive days of lessons) within a school year for each participating teacher were chosen to serve as the main data source for this study.

In this dissertation, I developed a Rational Questioning Framework from Habermas' theory of rationality and integrated it with Toulmin's (1958/2003) model for argumentation as a more powerful analytic tool to investigate teacher questioning strategies concerning both rationality requirements of argumentation and fundamental components of argumentation. Considering little consensus existed about general acceptance criteria for classroom-based argumentation, this study explored levels of truth in argumentation according to several levels of correctness of final arguments that emerged from two teachers' mathematics classrooms.

I identified various ways of using combinations of components of rational questioning to support ongoing interactions in argumentation with respect to prompting or responding to argument components, levels of truth in argumentation, and managing incorrect answers. This study suggests the necessity of using sequences of rational questions to prompt a satisfactory argument component from a teacher's perspective. The use of a privileged rationality component in rational questioning was context-dependent. Further, this study proposed a set of criteria for valid argumentative practices by considering how argumentation was gradually scaffolded in a given classroom community. The findings have important implications for theory and professional development, including teaching practice and teacher education.

INDEX WORDS: Teacher questions, Habermas' rationality, Collective argumentation, Secondary mathematics teaching

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A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2020

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August 2020

DEDICATION

This dissertation is dedicated to my dear parents for their forever love and support.

ACKNOWLEDGEMENTS

First and most obvious acknowledgement must go to my PhD advisor, Dr. AnnaMarie Conner. The words would not adequately express my gratitude and appreciation for her support, guidance, assistance, and encouragement. Every step of my PhD journey has been supported by her with patience and wisdom. In the past five years, she has provided me with numerous opportunities for research and teaching and taught me hands-on how to become a researcher and an educator. She supported me through my dissertation writing with little steps for every sentence and advice on writing research papers. Her willingness to sacrifice personal time to send dissertation to me with timely feedback and guidance will never be forgotten. Most importantly, she also taught me about life and that the journey is what is important.

Dr. Foutz, I would like to express my deepest gratitude for your support in my future research endeavors at UGA. Also, I want to thank my other committee members, Dr. Ellis and Dr. Spangler, for their helpful critiques and insightful ideas that pushed me to refine my dissertation to a quality paper.

Many thanks to all the friends and colleagues I encountered in USA during the last five years, for their support and thoughtful concern. The time we spent together along my PhD journey was one of the most unforgettable memories of my life. Last, but certainly not the least, I cherish the unconditional love and support I have received from my family.

This dissertation is based upon work supported by the NSF under Grant No. 1149436. Any opinions, findings, and conclusion or recommendations expressed in this paper are those of the author and do not necessarily reflect the views of the NSF.

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CHAPTER 1

RATIONALE

1.1 Background of the Study

In 1991, the National Council of Teachers of Mathematics (NCTM) stated that mathematics classrooms must:

- move toward becoming mathematical communities rather than a collection of individuals;
- use logic and mathematical evidence for verification rather than the teacher as the sole authority for the correct answers;
- focus on mathematical reasoning rather than memorizing;
- use conjecturing, inventing, and problem-solving rather than mechanistic answer-finding; and
- show connections between mathematics, its ideas, and its applications rather than treating it as a body of isolated concepts and procedures.

Almost a decade later, the NCTM (2000) highlighted the importance of reasoning, proving, and communicating in their *Principles and Standards for School Mathematics*. If these new standards are to be followed, then a significant change in pedagogical approaches will need to occur in U.S. mathematics classrooms so that the teacher and students rely on mathematical evidence for verification; focus on reasoning and proving; and utilize conjecturing, problem-solving, and communication. This call has been elaborated on and evolved to become more specific and concrete as presented in the most recent Common Core State Standards for Mathematics, which regarded the ability to "construct viable arguments and critique the reasoning of others" (p. 7) as one of its eight mathematical practices across grades and mathematical topics (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Internationally, standards across several countries (e.g., Australia, the United Kingdom, USA) have advocated that schools at all grade levels should highlight argumentation as one of their important instructional goals to help students better deal with the challenges in 21st-century societies (e.g., Assessment and Reporting Authority, n.d.; Department for Education, 2014; National Council of Teachers of Mathematics, 2000, 2014).

The potential benefits to introducing argumentation within the classroom have been recognized in the field of mathematics education (e.g., Andriessen, 2006; Goos, 2004; Krummheuer, 1995, 2007; Nussbaum, 2008; Singletary & Conner, 2015; Staples & Newton, 2016; Whitenack & Yackel, 2002; Wood, 1999). Researchers have claimed that a positive relationship exists between active participation in argumentation and the conceptual development of students (e.g., Krummheuer, 1995, 2007; Nussbaum, 2008; Staples & Newton, 2016; Wood, 1999). By implementing argumentation in the classroom, students have opportunities to focus on mathematical evidence and reasoning, which helps develop their understanding of mathematical concepts. As Wood (1999) stated:

> The significance of argument to conceptual understanding in mathematics is related to the development of students' thinking and reasoning that occurs during the acts of challenge and justification. That is, it is generally accepted that conceptual change and progression of thought result from the mental processes involved in the resolution of conflict in ideas. (p. 189)

Moreover, the inclusion of instruction that promotes mathematical argumentation has been positively linked to the development of students' mathematical autonomy (Yackel & Cobb, 1996), mathematical dispositions (Whitenack & Yackel, 2002) and communicative competencies (Andriessen, 2006) as well as improvements in the students' mathematics achievements (Cross, 2009). Therefore, argumentation can play a central role in promoting students' meaningful learning of mathematics.

From the teacher's perspective, argumentation as a critical approach in mathematics classrooms allows them to assess the students' thinking and development (Krummheuer, 1995, 2007), provides for new mathematical concepts and tools to emerge (Kim, 2011; Yackel, 2002), provides opportunities for students to value different forms of mathematical explanations (Wood, 1999; Yackel & Cobb, 1996), creates a public knowledge base that can be used by the class as a resource (Forman et al., 1998), aligns students with one another, and resolves students' disagreements (O'Connor & Michaels, 1993, 1996). As a long-term process, argumentation will help teachers establish a classroom context for the emergence and negotiation of norms (e.g., social norms, sociomathematical norms) and generally fosters students' development of "intellectual autonomy— aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgements as they participate in mathematical practices" (Yackel & Cobb, 1996, p. 473). In addition, some researchers (Andriessen, 2006; Hunter, 2007) believe that all teaching is in some sense an argumentative activity, and "when students collaborate in argumentation in the classroom, they are arguing to learn" (Andriessen, 2006, p. 443). Thus, argumentation can be viewed as a powerful approach to both teaching and learning mathematics.

1.2 Statement of the Problem and Purpose of the Study

Research that focuses on argumentation in the mathematics classroom has indicated that in spite of its importance for teaching and learning mathematics and national recommendations, the implementation of argumentation is not yet a common practice in most U.S. mathematics classrooms (Bieda, 2010; Boaler, 2016; Civil & Hunter, 2015; Staples et al., 2012; Stylianides, 2007; Stylianou et al., 2009). Research suggests that one of the primary impediments to developing student argumentative practices is a lack of skills on the part of teachers related to organizing argumentative discourse, and teachers are not adequately trained to identify and enhance argumentation opportunities (Asterhan & Schwarz, 2016; Sriraman & Umland, 2014). Argumentative practices differ significantly from traditional, transmission methodologies of teaching. As such, the implementation of argumentation requires sophisticated knowledge of mathematics for teaching as well as the use of a variety of teacher moves (Conner et al., 2014; Wood, 1999; Yackel, 2002). Although scaffolding argumentation is challenging for many teachers, studies have shown that, with proper support, the improvement of teachers' skills in regard to fostering productive argumentation is possible (e.g., Barkai et al., 2009; Kollar et al., 2014; Reichersdorfer et al., 2012).

As a representative of the mathematics community in the classroom, the teacher plays an essential role in creating opportunities for students' participation in argumentation, and such opportunities are critical for promoting students' growth of intellectual autonomy and social autonomy, which are major goals of mathematics education (Boavida, 2009; NCTM, 1989; Yackel & Cobb, 1996; Yackel, 2012). Teachers need to have a repertoire of supportive moves with the understanding of when one or another teacher move will be more appropriate and have more potential for promoting students' participation in authentic mathematical discourse and that different types of teacher moves serve different functions for promoting students' conceptual understanding, reasoning, arguing, justifying and so on (Cazden, 2001; Conner et al., 2014; Ellis et al., 2018; Hufferd-Ackles et al., 2004; Jacobs & Empson, 2016). However, explicit descriptions of teacher moves for creating high-quality argumentative discourse have yet to be developed in the field (Franke et al., 2007; Wagner et al., 2014), and the microanalysis of teacher moves for promoting argumentation has only partly been discussed and studied (e.g., Kim, 2011; Conner et al., 2014). Therefore, as an aid to teachers' development of such repertoires of supportive moves, the focus of this study involves the microanalysis of teachers' interactive utterances while leading argumentative discourse to provide insights into teachers' use of verbal moves to promote argumentative practices.

Studies have pointed out that a teacher's questions are a pivotal contributing factor that shapes the role that teachers play in supporting argumentative discourse and that the quality of mathematical argumentation depends on how teachers use questioning strategies to orchestrate argumentative discourse (Conner et al., 2014; Hunter, 2007; Kosko et al., 2014; Zhuang & Conner, 2018a, 2018b, in press). However, literature has consistently shown that, instead of asking questions that probe students' understanding and reasoning, teachers generally ask low-level questions that require students to recall facts, rules, and procedures, even when using reform-based curricula that include probing questions (i.e., which ask for clarification, justification, or explanation) in the teaching guides (Graesser & Person, 1994; Kosko et al., 2014; Sahin & Kulm, 2008; Zhuang & Conner, 2018a). Additionally, research has found that teachers have difficulties incorporating questioning strategies to support argumentative discourse (Kosko et al., 2014; Sahin & Kulm, 2008). More importantly, most of these studies placed more emphasis on documenting current situations or difficulties that teachers had in using questioning to regulate argumentative discourse than on developing effective ways to address some of these difficulties. Thus, it is imperative to develop a specific guiding vision of effective questioning skills involving how questions are elicited and offered and by which means the class comes to consensus (McCarthy et al., 2016). In this study, I examine the questioning strategies used by two beginning secondary mathematics teachers to handle the complexity of argumentation to develop teaching approaches to support teachers as they plan and manage argumentative discourse.

Boero (2006) proposed that Habermas' (1998) construct of *rational behavior* can become a frame to deal with the complexity of discursive practices in the intersection of three kinds of rationality: epistemic (inherent in the control of validation of statements), teleological (inherent in the strategic choice of tools to achieve the goal of the activity), and communicative (inherent in the conscious choice of suitable means to communicate understandably within a given community). Corresponding to these components, students are expected to strategically choose tools to achieve a goal (teleological rationality) on the basis of specific knowledge (epistemic rationality) and communicate in a precise way with the aim of being understood by the classroom community (communicative rationality). Douek suggested that it was beneficial to develop argumentative discourse along the three components of rationality (i.e., epistemic, teleological, and communicative) and that the teacher should support students to meet the requirements of rationality, thus dialectically forming argumentation (Boero & Planas, 2014). In order to reach such aims, Douek further proposed the idea of using *rational questioning* as a method to "organize the mathematical discussion according to the three components of rationality" (Boero & Planas, 2014, p. 210). Building on Douek's idea, in this study I advance a *Teacher Rational Questioning Framework* to classify types of rational questioning to engage student participation in argumentation with different kinds of rationality (A preliminary version of this framework can be found in Zhuang & Conner, 2018b). I define *rational questioning* as a question that contains at least one component of rationality. At times, for clarity, I use, for example, "epistemic rational question" to refer to a question containing an epistemic rationality component.

To date, while researchers have made progress in regard to illuminating knowledge, tasks, and teacher moves that promote success in generating argumentative discourse, no consensus exists in the field of mathematics education concerning the characteristics of "successful" or "productive" argumentative discourse. Researchers (e.g., Staples et al., 2017; Wagner et al., 2014) have argued that one of the challenges in teaching about supporting collective argumentation is teachers' lack of understanding of what counts as a valid argumentation. Some researchers have stated that a productive argumentation "shifts from reasoning based on intuitions to reasoning moved by logical necessity" (Prusak et al., 2012, p. 19) or proposed the concept of "epistemic argumentation" (Fielding-Wells & Makar, 2015, p. 33) that aims at seeking the epistemic truth through critical reasoning and justification. Although fostering productive or epistemic argumentation has vast potential to improve both teaching and learning of argumentation, the characteristics of productive (or quality) argumentation that attend to the classroom community are yet to be explored in proper detail in our field. In addition, research has shown that most prospective teachers have difficulties in managing incorrect answers in relation to mathematical argumentation (Shinno et al., 2018). This study responds to the call within the field (e.g., Stylianides et al., 2016) to conduct more research aimed at developing practical tools to address teachers' difficulties in supporting argumentative discourse by considering how teachers use rational questioning as a didactical tool to support different levels of truth in argumentative discourse while managing incorrect answers.

1.3 Research Questions

The main goal of this study was to investigate how teachers use rational questioning to organize collective argumentative discourse as a way to develop the students' awareness of rationality requirements of argumentation. Additionally, I investigated how rational questioning as a didactical tool can be used to scaffold different levels of truth (i.e., when incorrect answers are encountered) in argumentative discourse.

The following research questions guided the study:

1. How do beginning secondary teachers use rational questioning when guiding collective mathematical argumentation?

   (a). Which component(s) of rationality is (are) privileged in class?

   (b). What combinations of rational questions support different components of arguments in a mathematics classroom?

2. What techniques and structures of arguments do teachers use when supporting different levels of truth in argumentation? In particular, how do teachers use incorrect answers within their support of argumentation?

1.4 Significance of the Study

There is a move in mathematics education to change the IRE discourse pattern (Mehan, 1985), in which the lesson follows a teacher-dominated pattern of teacher initiation, student response, and teacher evaluation. This notion, combined with an emphasis on social constructivism, brought forth the idea that collective argumentation is central to meaningful learning of mathematics (Andriessen, 2006; Common Core Standards Initiative [CCSI], 2011;

Goos, 2004; Krummheuer, 1995, 2007; Lampert, 2001; Nussbaum, 2008; Staples & Newton, 2016; Whitenack & Yackel, 2002; Wood, 1999). Through collective argumentation, students learn that mathematics is more than a collection of rules and methods set out in textbooks. They realize that mathematics is a subject about which they should have evidence-based ideas. They also realize that they can use different methods to solve the same problem, but these methods are connected to each other through organizing concepts and theorems. Perhaps most importantly, collective argumentation provides students with opportunities to share ideas and clarify understandings, to construct convincing arguments regarding why and how things work, and to develop a language for expressing mathematical ideas (Boaler & Humphreys, 2005; National Council of Teachers of Mathematics [NCTM], 2000). Hence, the focus on argumentation in this study is strategic because facilitating collective argumentation emphasizes participation of students in mathematical discourse, which is helpful in the learning of mathematics (Koestler et al., 2013; Krummheuer, 2007; Stein et al., 2008).

Although mathematics educators have engaged in understanding various benefits of classroom discussions and their benefits may be different based on what they look for in a conversation, it has widely been agreed upon that the teacher plays a pivotal role in orchestrating argumentative discussions (Walshaw & Anthony, 2008; Yackel, 2002). Many researchers (Burns, 1985; Gall, 1970; Knott et al., 2008; Kosko et al., 2014; Martino & Maher, 1999) have recognized that teacher questioning is a key factor in the productivity of mathematical argumentation. As stated in *Principles to Actions: Ensuring Mathematics Success for All* (NCTM, 2014), effective mathematics teachers are expected to use purposeful questions to access students' conceptual understanding, prompt critical thinking, and advance students' reasoning and sense-making of mathematical ideas. Teachers' questioning has the potential to bring students into a conversation and promote participation, but the type and quality of questions can have a significant impact on the students' engagement in productive argumentation (Chen et al., 2016). However, to date, what teachers need to do to best support productive argumentative discourse has not been well characterized in the literature (Franke et al., 2007). This study contributes to the on-going efforts to develop effective questioning strategies in supporting collective argumentation.

Recently, multiple researchers (e.g., Boero & Planas, 2014; Cramer, 2015; Cramer & Knipping, 2018; Morselli & Boero, 2009, 2011; Zhuang & Conner, 2018b, in press) have taken up Habermas' theory as an analytic tool to study both students and teachers' behavior in proving and argumentative activities. An interesting aspect of this construct is that Boero et al. (2010) illustrated the possible adaptation of combining Habermas' (1998) construct and Toulmin's (1958/2003) model for argumentation to study two levels of argumentation: the meta-level, concerning the awareness of the constraints related to the three components of rational behavior in argumentation (Habermas' construct); and the content level, concerning the construction of argument components (Toulmin model). This study additionally illustrates how the integration of Habermas' construct with Toulmin's model as an analytical approach can be applied to study teachers' questioning strategies as they plan and carry out rational classroom interventions within collective mathematical argumentation.

CHAPTER 2

LITERATURE REVIEW

2.1 Overview

This chapter first provides a description of argumentation and how it has developed in mathematics education. Some researchers have argued that understanding the relationship between proof and argumentation may be essential for teaching argumentative activities in classrooms (Boero, 1999; Knipping, 2008; Pedemonte, 2007, 2008), so in order to justify the benefits of introducing argumentation as a pedagogical means for teaching and learning of proof, discussions of what the literature tells us about the relationship between proof and argumentation are included.

Various definitions of argumentation exist in the literature, and in order to undertake a classroom-based study of argumentation, it is necessary to explore how argumentation is situated in mathematics education, especially in a classroom culture. Thus, focused review of argumentation situated in a mathematics classroom will be discussed next, and it serves to set the background for me to express the perspective on argumentation that takes place in a mathematics classroom in this study.

The last section of this chapter provides a needed review of research regarding teachers' use of questioning in supporting collective argumentation, which includes the importance of teacher questioning (e.g., Burns, 1985; Martino & Maher, 1999) and teachers ways of using questioning to support argumentation (e.g., Conner et al., 2014; Sahin & Kulm, 2008). The gap in existing research on using teacher questioning to support collective argumentative activities is presented.

2.2 Argumentation

To undertake any study of argumentation, it is important to first illustrate what is meant by the term. In this section, I provide a general definition of argumentation and review how different theories of argumentation developed.

2.2.1 The Definition of Argumentation

A shared understanding exists among researchers that argumentation is a dynamic process, while arguments are static, finished objects (e.g., Boero et al., 2007; Knipping & Reid, 2015; Krummheuer, 1995). A commonly accepted definition of argumentation in general has been given by Van Eemeren et al. (1996):

> Argumentation is a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a rational judge. (p. 5)

This definition positions argumentation as a social phenomenon involving interaction and an audience. The goal of argumentation is to justify one's standpoint or to refute someone else's. Van Eemeren et al. (1996) maintained that a critical distinction exists in defining outcomes of argumentation and proving. Argumentation is supported by what the audience would regard as good reasons, in which rationality need not be conveyed solely through deductive reasoning such as proof; thus, the outcomes of argumentation cannot be certain (Van Eemeren et al., 1996).

2.2.2 The Historical Background of Argumentation

Aristotle classified three generally recognized forms of arguments: analytical, dialectical, and rhetorical, based on the different purposes they are intended to serve (Van Eemeren et al., 1996). Euclid's Elements (ca. 300 B. C. E.) started with a collection of 23 definitions, which are used to state a set of postulates and axioms. This is an application of analytical arguments (e.g., formal logic) in Aristotle's terms (Van Eemeren et al., 1996). For many centuries, Euclid's Elements was used extensively and pervasively as the principal model of argumentation for the purpose of determining the truth of mathematical statements (Durand-Guerrier et al., 2012).

In the 1960s and 1970s, Perelman and Toulmin were influential researchers in the field of argumentation. They studied argumentation through the lens of linguistic theories and asserted the importance of rationality in argumentation. However, in their research, they attributed different meanings to "rationality." Perelman and Olbrechts-Tyteca (1958) viewed argumentation as a rational activity aimed at justification, which stands alongside formal argument — it is complementary to it (as cited in Van Eemeren et al., 1996). For Perelman and Olbrechts-Tyteca (1958), argumentation is finalized less by the establishment of the validity of a statement than by its capacity to convince a listener, which means that the evaluation of argumentation depends on the audience for whom it is intended (as cited in Van Eemeren et al., 1996). In this sense, mathematical argumentation can be considered to be discursive activity aimed at convincing a listener.

Toulmin's (1958/2003) theory of argumentation is closer to Aristotelian dialectic. Like rhetoric, dialectic does not necessarily lead to true conclusions, but one who engages in dialectic should start from statements which he or she believes to be true, while one who engages in rhetoric does not need to be convinced about the truth of what he or she defends (Pedemonte, 2007). Toulmin stated that a good argument has a certain type of structure, which involves three core elements: claim (the statement, of which the speaker wishes to convince his or her audience), data (the foundations on which the argument is based), and warrants (the inference rule, which allows the data to be connected to the claim). Toulmin's model for argumentation has been employed by researchers in mathematics education to analyze classroom-based mathematical arguments (e.g., Conner et al., 2014; Krummheuer, 1995; Yackel, 2002; Zhuang & Conner, 2018b) and as an analytic framework to compare the organization of arguments (e.g., Boero et al., 2010; Pedemonte, 2007).

In summary, following the classification of arguments from Aristotle, mathematical argumentation can be analogously linked to three forms: analytic, rhetoric, and dialectic. Analytic argumentation is connected with a logically correct deduction argument, such as a proof (Krummheuer, 1995). The goal of rhetoric argumentation is to convince others, even if sometimes the reasoning provided by a speaker is insincere. In contrast, dialectic argumentation works towards mutual understanding. Argumentation can be viewed as a way of communication based on rationality and may not result in true mathematical statements. During a dialectic argumentative activity, people may hold different views but are free to express themselves based on their own rationality whilst seeking to reach mutual understanding (Durand-Guerrier et al., 2012).

2.3 The Relationship between Proof and Argumentation

Researchers in the field of mathematics education believe that understanding the relationships between proof and argumentation is essential to teaching proof in mathematics classrooms (e.g., Boero, 1999; Knipping, 2008; Pedemonte, 2007, 2008). They have considered whether proof and argumentation create obstacles for each other and which educational conditions would foster fruitful uses of either or both of the two constructs.

The relationship between proof and argumentation is complex, and it is important to distinguish *proving* as a process and *proof* as a product (Boero, 1999). In parallel, a product of *argumentation* is an *argument.* Some researchers suggested that a gap exists between proof and argumentation (e.g., Balacheff, 1999; Duval, 2007), while others argued that a possible continuity may exist between proof and argumentation (e.g., Boero et al., 2010; Douek, 1999; Garuti et al., 1996; Thurston, 1994). In this section, I provide a review for these two perspectives.

2.3.1 From a Cognitive and Logical Point of View

By referring to proving as a model of formal derivation, Duval (1991) focused on the distance between proving and argumentation:

> Deductive thinking does not work like argumentation. However, these two kinds of reasoning use very similar linguistic forms and propositional connectives. This is one of the main reasons why most of the students do not understand the requirements of mathematical proof (as cited in Douek, 1999, p. 125).

According to Duval, a structural gap exists between proof and argumentation. The structure of a proof is a deductive chain of inferences and strongly depends on the operational status of the propositions that compose it, which is independent of its content (Durand-Guerrier et al., 2012). Duval concluded that the distance between proof and argumentation is not logical, but cognitive; argumentation depends on the content of propositions whereas what is important in a proof is the way the proposition fits into the formal structure. In other words, a specific and independent learning process is necessary where deductive reasoning is concerned. According to Pedemonte (2001), the distance between these two constructs explains why most students do not understand the necessity of a mathematical proof (i.e., if there is an argumentation that justifies the statement, then the proof may be unnecessary).

2.3.2 From a Social and Epistemological Point of View

From a social and epistemological point of view, Balacheff (1999) argued that the classroom's social interaction process and behaviors run counter to the construction of a mathematical proof and fall under the definition of argumentation.

> Whereas [a] mathematical proof, in its most perfect form, is a series of structures and of forms whose progression cannot be challenged, argumentation has a non-constraining character. It leaves to the author hesitation, doubt, freedom of choice; even when it proposes rational solutions, none is guaranteed to carry the day. (Perelman, 1970, as cited in Balacheff, 1999, p. 4)

Balacheff claimed that the sources of argumentative competence are natural language, which have a profoundly different nature from those rules required by mathematical proofs. Thus, when taking into consideration of the practices of mathematicians, a fundamental opposition remains in terms of the contribution of these two activities (i.e., argumentation and proving). Balacheff suggested "Argumentation is to a conjecture what [a] mathematical proof is to a theorem" (1999, p. 6). Thus, the relationship between argumentation and proof is strictly connected to the relationship between conjecture and valid statements. According to Balacheff, such a conflict may become an epistemological obstacle when teaching a mathematical proof. Achieving the goal of a theoretical construction means becoming aware of the particular nature of mathematics, so that the particular argumentative competencies that naturally emerge in social interactions might appear inadequate (Balacheff, 1999).

2.3.3 Intersections between Proof and Argumentation

The main point that emerges from the previous discussion is whether it is possible to overcome the possible rupture and the consequent possible conflict between proving and argumentation. In response to Duval, from a mathematical point of view, Thurston (1994) wrote:

> Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas. (p. 10)

In Thurston's view, the requirements of a formal proof, as described by Duval, do not seem to fit the activities performed by many working mathematicians when they check the validity of a statement or proof. According to Thurston, social processes enhance the reliability of mathematics through important checks and balances.

For the purpose of education, some researchers stress the importance of the connections between teaching proofs and producing arguments. If we want to teach proofs, then we have to include argumentation (Knipping, 2008). In the classroom, the primary purpose of a proof is to explain why something is in the case rather than to be assured that is in the case (Hersh, 1993). Therefore, the essential mathematical activity is finding the proof, not checking after the fact that it is indeed a proof. Lakatos (1976) illustrated how mathematical proofs are co-constructed heuristically through communications between a teacher and students. At the stage of creation, proofs are advanced through heuristic methods (e.g., creating conjectures) and are subject to revisions and refinements (e.g., detecting errors or omissions) until their validity is no longer in questions by the community of the classroom (Lakatos, 1976). From this point of view, mathematical argumentation is a cyclic process of conjecturing, refuting, and refining arguments through social interactions (Lakatos, 1976). In this sense, heuristic arguments should be valued as part of argumentation activity for their function in providing insights into advancing arguments and working towards proofs.

Argumentation as a process is interfaced with proving and as a product is interfaced with mathematical proofs (Boero, 1999). Douek (1999) claimed that "proving a conjecture entails establishing a functional link with the argumentative activity needed to understand (or produce) the statement and recognizing its plausibility" (p. 135). This perspective highlights that proving itself requires an intensive argumentative activity and argumentation is important in constructing proofs. Relying on this perspective, studies have been conducted with the goal of clarifying the relationship between mathematical proofs in school mathematics and argumentation in mathematics classrooms. These studies do not deny the distinctions between proof and argumentation but are centered on argumentation as a precursor to doing mathematical proofs. This idea has been elaborated on by Garuti et al. (1996) who introduced the notion of *cognitive unity*, which highlights the continuity that potentially exists between argumentation as a process of producing a conjecture and constructing a proof.

In the context of a long-term teaching experiment involving 36 eighth grade students, Garuti et al. (1996) found that when the process of producing a conjecture showed a rich production of arguments aimed at supporting or rejecting a specific statement, it was possible to recognize an essential continuity between these arguments and the final proof. In a further investigation, Boero et al. (2007) found that during a problem-solving process, an argumentation activity was usually developed in order to produce a conjecture that can be used in the construction of a proof by organizing some of the previous arguments in a logical chain. The study from Boero et al. showed that a proof is more accessible to students if an argumentation activity is developed for the construction of a conjecture. As such, it is possible to bridge mathematical conjecturing and proving through mathematical argumentation.

Some researchers (e.g., Garuti et al., 1998; Pedemonte, 2001) have proposed that the idea of cognitive unity can be used to foresee and analyze some of the difficulties that students might have in the construction of proofs. Following the idea of cognitive unity, Pedemonte (2007) argued that the analysis of cognitive unity should concern not only the "content continuity" (the language, the mathematical theory, the drawing, the heuristic, etc.), but also "structural continuity" (deduction, abduction, and induction structures). By focusing on structural analysis between argumentation and proof in geometry and algebra, Pedemonte (2007, 2008) showed that students meet difficulties inherent in a lack of structural continuity in the geometry proof, while this issue was not the case in the algebraic proof.

The term "cognitive unity" (coined to express a hypothesis of continuity that may or may not exist) has provided a way to escape the essential irretrievable distinction proposed by researchers, such as Duval, between proof and argumentation. The main strength of the notion of cognitive unity is that it opened the way to a new approach in which the complex relationship between proof and argumentation is examined by focusing on connections, without forgetting the differences (Mariotti, 2006). While more studies are needed to understand how conjectures and proofs are related, as well as how cognitive unity can be used as a means to structure the investigation between them, the importance of the role of argumentation, which aims at connecting these two products, should not be ignored.

2.4 Argumentation in Mathematics Education

If we want to acknowledge the social dimension of argumentation, we must understand how argumentation occurs and develops in a mathematics classroom. As discussed, in section 2.2, philosophers have proposed three forms of argumentation, and these can be similarly linked to three perspectives on mathematical argumentation. In this section, these different perspectives on argumentation are discussed to frame my decision to adopt the third perspective on argumentation for this study.

In the first perspective, mathematical argumentation is viewed as an activity corresponding with proving, which involves constructing conjectures, justifying conjectures, and finally using these arguments to produce the proof (Boero, et al., 2007; Pedemonte, 2007). Thus, mathematical argumentation is a precursor of the formal proof (Boero 2010; Wood, 1999). Krummheuer (1995) defined these types of arguments as *analytic arguments* that correspond to a proof in mathematics, which means they are logically correct deductions and generate ideas by formal logic using previously established statements such as axioms, definitions, or theorems. As mentioned above, some researchers (e.g., Balacheff, 1999; Duval, 1999) have argued that a gap existed between argumentation and proof, while others (e.g., Boero, 1999; Boero et al., 2007; Pedemonte, 2001, 2007) proposed that a possibility of cognitive continuity existed between conjecturing and proving. However, researchers also found that cognitive unity is not sufficient to explain all aspects of the relationship between argumentation and proof, and a structural distance may exist from an abductive argument to a deductive proof (Pedemonte, 2007). Krummheuer (1995) criticized this perspective: “Argumentation need not be exclusively connected with formal logic as we know it from such proofs or as the subject matter of logic” (p. 235). Therefore, the perspective on mathematical argumentation as analytic (created by logical derivation inextricably connected with proving) is limited when taking into account human activities and human intuitions that are argumentative. This idea may have prevented students from learning and doing mathematics heuristically as mathematicians (Lakatos, 1976; Krummheuer, 1995; Thurston, 1994).

From a rhetorical perspective, argumentation is a discursive activity aimed at convincing listeners, and thus it suggests that in order to successfully convince others, the reasons people provide may not even convinced themselves (Van Eemeren et al., 1996). In other words, the epistemic validity of a statement may not be satisfied with an intersubjective conception of justification. In the field of mathematics education, some researchers view argumentation as a rhetorical means that can be used by an individual or a group to convince others that a statement is true or false through the use of certain modes of thought (Pedemonte, 2007; Wood, 1999). This perspective defines argumentation as a social phenomenon and lacks the property of the *ascertaining* process when an individual tries to remove his or her own doubts about the truth or falsity of a statement (Harel & Sowder, 2007; Stylianides et al., 2016).

From a dialectic perspective, researchers conceptualized argumentation as a reflexive form of communicative action (i.e., the goal is to reach mutual understanding) in which participants discuss claims (or disagreements) and support or criticize these by arguments (i.e., data, warrants, and backing) that are rationally connected to the claim (Habermas, 1984; Harel & Sowder, 2007; Krummheuer, 1995; Stylianides et al., 2016). What constitutes data, warrants, and backing is not predetermined but evolves through negotiation (Yackel, 2002). Once a community agrees that an argument is true, a new warrant or backing is established, and this can be used in further arguments to construct new knowledge (Rasmussen & Stephan, 2008; Staples & Newton, 2016).

On the basis of argumentation as a rational action aimed at a mutual understanding, some researchers view argumentation in a mathematics classroom context as a way to teach new mathematical knowledge and learn mathematics for understanding (Cobb & Bauersfeld, 1995; Forman et al., 1998; Krummheuer, 1995; Lampert, 1990; Staples & Newton, 2016; Yackel, 2002). For example, Krummheuer (1995) stated that argumentation occurs "when cooperating individuals try to adjust their intentions by verbally presenting the rational[e] of their actions" (p. 229) and defined arguments as "the intentional explication of the reasoning of a solution during its development or after it" (Krummheuer, 1995, p. 231). Following Toulmin's (1958/2003) idea of *substantial arguments*, which focus on the justificatory function of argumentation rather than inferential function of argumentation, Krummheuer classified *substantial arguments* in mathematics education as those which "expands the meaning of such propositions insofar as they soundly relate a specific case to them by actualization, modification, and/or application" (p. 235-236). According to Krummheuer, argumentation transfers from substantial to analytic when the backing for the warrant is authorized. Thus, substantial arguments do not usually have the logical rigidity of formal deductions as do analytic arguments (proofs in mathematics). Instead, a substantial argument is informative in the sense that the meaning of the premises increases or changes with the application of new cases. Some researchers (Habermas, 1984; Krummheuer, 1995; Toulmin, 1958/2003) have argued that if analytic arguments were the only acceptable form of argumentation, then the domain of rational communication would be considerably restricted, and argumentation would be irrelevant as a possible procedure of hypothetically checking claims. Thus, analytic arguments are not the only valid arguments, and with substantial arguments we see how argumentation is gradually supported. This view of argumentation aligns with framework I chose for the analysis of argumentation in mathematic classroom situations.

Argumentation involves both heuristic and deductive reasoning to work towards the truthfulness of a mathematical claim. It is a thinking process to create and advance mathematical ideas. Therefore, argumentation is not only vitally important for conceptual mathematical understanding but also serves as a potential mechanism for learning mathematics through interaction (Krummheuer, 2007).

Following a dialectic perspective on argumentation, in a mathematics classroom, argumentation can be used by an individual or a group to convince others that mathematical statements are true or false in terms of their objective and individual rationality. Yackel (2002) argued that argumentation is a useful instructional tool because the types of rationales (e.g., data, warrants, and backing) that are given for or by students are seen as powerful reasoning tools that allow the participants in a dialogue to refute, criticize, elaborate, and justify mathematical concepts and facts as well as develop an understanding of opposing perspectives as they work toward constructing a collective consensus. This definition of argumentation is also consistent with Toulmin's (1958/2003) model of argumentation, which explains that a mathematical argument can be viewed as a line of reasoning to show or explain why a mathematical claim is true. In this sense, argumentation is not only vitally important for conceptual mathematical understanding, but it can also be used as a methodological tool (i.e., the use of Toulmin's model) to demonstrate changes that take place during the collective learning of a class (Rasmussen & Stephan, 2008; Yackel, 2002).

Considering the forms of argumentation, some researchers proposed that mathematical argumentation contains a range of activities such as conjecturing, arguing, analyzing, justifying, refuting, revising, and representing mathematical ideas (Lakotos, 1976; Staples & Newton, 2016; Thurston, 1994). Similarly, Lampert (1990) maintained that argumentation in mathematics classrooms can be performed in a “zig-zag” path (p. 30) for discussing mathematical problems and results. This path begins with conjectures and involves the examination of premises through the use of disagreement and counter arguments. These views of argumentation as a cyclic process also reflected in the new standards for mathematics instruction. For example, *Principles and Standards for School Mathematics* (NCTM, 2000) demonstrated the central role of argumentation in mathematics by including it as one of eight standards of mathematical practice, which asserts that students at all grade levels should “develop and evaluate mathematical arguments” (p. 56) and are expected to create conjectures, justify claims, convince members of the classroom community and evaluate the ideas that have been created by their classmates and teacher.

2.5 Perspective on Argumentation in Mathematics Classrooms

For the purpose of this classroom-based research, a dialectic perspective on argumentation is adopted; the term *mathematical argumentation* will be used to refer to the whole cyclic process of making mathematical claims, supporting them by providing reasons and data, challenging claims, rebutting those reasons, and all other related activities aimed at constructing or responding to argument components (e.g., claims, warrants, data, rebuttal). The goal of argumentation is reaching a mutually accepted conclusion about the truth of a mathematical claim. A mathematical argument is a sequence of statements and reasons that a person puts forward in an attempt to show what he or she believes to be true. The validity of an argument is situated within a particular classroom community.

Stylianides (2007) proposed the following definition of proof in a given classroom context:

Proof is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

- It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
- It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
- It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291)

Therefore, proof is a special class of arguments that are different from empirical arguments and can be regarded as a valid argument subject to accepted mathematical statements (i.e., definitions, theorems, axioms), modes of proofs (e.g., construction of counterexamples) and appropriate communicative approaches (i.e., linguistic, physical, etc.) of the mathematical community (Stylianides, 2007). The proving process lies within the definition of argumentation and is seen as a more formal form of mathematical argumentation. Justifying, in this study, is viewed as a process that "uses accepted statements and mathematical forms of reasoning" (p. 448) to provide reasons (Staples et al., 2012). A justification can be considered as consisting of two elements: data and warrants (Toulmin, 1958/2003), and one uses these elements to establish the mathematical validity of a claim. The term "conjecturing" is intended to describe a process of "conscious guessing" to create mathematical claims and statements (Lakatos,1976), so in the stage of conjecturing, argumentation is different from formal logic, but it brings to the participants tools for arousing intuitions and for stimulating creativity (Hersh, 1993). This situation does not impair the goal of a mathematical activity, which seeks to find logical consequences through proving (Hersh, 1993; Lakatos, 1976; Thurston, 1994), but, instead, brings to this goal a world of images and conjectures. In this sense, argumentation is multimodal: some forms of argumentation are formal logic (i.e., corresponding to proving), while other forms are involved in enquiry, which focuses on conjecturing and justifying.

According to Krummheuer (1995), the term "collective argumentation" refers to when the argumentative process is constructed by two or more individuals in the classroom. Similarly, Conner et al. (2014) defined collective argumentation broadly to "include any instance where students and teachers make a mathematical claim and provide evidence to support it" (p. 404). For the purpose of the research presented here, I will focus on the social dimension of argumentation (i.e., collective argumentation) in which the teacher and students (or a small group of students) work together to construct or reject mathematical arguments.

2.6 Teacher Role in Supporting Collective Argumentation

It is widely agreed that the teacher plays a pivotal role in orchestrating mathematical argumentation in classrooms and should set up appropriate actions to promote student engagement in productive classroom-based collective argumentation (Forman et al., 1998; Hunter, 2007; Wood, 1999; Yackel, 2002). Nevertheless, teachers have difficulty when dealing with argumentation in the classroom (Conner et al., 2014; Kosko et al., 2014). One possible factor relating to this difficulty is that many teachers have limited experience with collaborative learning in mathematics (Wagner et al., 2014), and it is challenging to envision teaching in ways that one has never experienced. In addition, teaching with argumentation is pedagogically demanding. Teachers must pose questions, press for elaboration, explanation, and justification (Sahin & Kulm, 2008), establish appropriate classroom social norms and sociomathematical norms (Yackel & Cobb, 1996), respond to students' ideas in ways that develop arguments

(Lampert, 2013), choose appropriate tasks to foster understanding (Lin & Wu, 2007), and have a well-developed understanding of the practice of argumentation and modes of reasoning acceptable to the mathematics community (Stylianides, 2007). In this study, I concentrate on developing questioning strategies to support the development of the requisite pedagogies.

2.7 Teacher Questioning in Supporting Collective Argumentation

Research on teacher questioning related to collective argumentation has dealt with the importance of questioning strategies (e.g., Andriessen, 2006; Burns, 1985; Gall, 1970; Hunter, 2007) and teachers' ways of using questions in supporting argumentation (e.g., Conner et al., 2014; Franke et al., 2009; Hufferd-Ackles et al., 2004; Kosko et al., 2014; Martino & Maher, 1999; Sahin & Kulm, 2008; Wood, 1998). In this section, I provide background literature on these topics in the field of mathematics education.

2.7.1 The Importance of Questioning

It is widely accepted that teacher questioning plays an important role in teaching. Teacher questions are regarded as one of the ten major dimensions for studying the behaviors of teachers (Flanders,1970). Some researchers claimed that, in a classroom setting, the use of questioning was a critical instructional approach for teachers to convey to students the content to be learned and directions for what they were to do and how they were to do it (Cotton, 2001; Gall, 1970). Teachers' classroom questions are an important phenomenon to consider within the context of classroom interaction. They are also an essential component for developing dialogic interaction and promoting student thinking and learning in mathematics, especially in a student-centered class discussion (Burns, 1985; Gall, 1970), which may involve collective argumentation. Teacher questioning can help guide students to make public conjectures and reason with others about mathematics, justify their thinking and expand their ideas collaboratively, and open up new areas to investigate within an intellectual community (Burns, 1985; NCTM, 1991).

Researchers (e.g., Andriessen, 2006; Hunter, 2007; Kazemi & Stipek, 2001; Martino & Maher, 1999; Wood, 1999) have pointed out that it is imperative for teachers to use questioning to create a classroom context that is conducive to mathematical argumentation, justification, and reasoning. For example, Kazemi and Stipek (2001) studied four teachers in grades four and five in three low-income schools who taught the addition of fractions. They identified one dimension of classroom practice that was used to emphasize conceptual understanding and learning as "an explanation consists of a mathematical argument, not simply a procedure description" (p. 123). Their data found that the teacher questioning played an essential role in shaping classroom discourse and sustained the level of discourse and argumentation amongst the students. As Martino and Maher (1999) asserted:

> Teacher questioning can invite students to reflect on their ideas; further, it can open doors for teachers and students to become more aware of each other's ideas. The teacher who also backs away strategically when communication and reasoning flourish, allows students to play more active roles in their own and each other's learning, and thus build a classroom community that invites active participation, confidence, and further learning. (p. 75)

Thus, when questioning becomes part of the classroom norms and expectations for argumentative activities, students will feel more comfortable sharing their conjectures, justifying their thinking, explaining their thought processes, defending their solutions, and expanding their ideas rather than arriving only at correct answers. As this happens, students will experience autonomy and powerful learning, and the authority for classroom conversation shifts from the teacher to the students (Andriessen, 2006; Yackel & Cobb, 1996). Although researchers (e.g., Cobb et al., 1997; Kosko, 2016) have highlighted the role of teacher questioning in regulating argumentative discourse and its important impact on the development of mathematical autonomy on students as they participate in the practices of the classroom community, only limited studies have focused on developing effective questioning strategies in helping such autonomous learning become a reality. Therefore, it is important for the field to investigate ways of empowering teachers to strategically use questions to support students to meaningfully participate in collective argumentation in the mathematics classroom.

2.7.2 Teachers' Ways of Using Questions to Support Argumentation

According to NCTM (2014), asking questions is not enough to ensure that students can make sense of mathematics and improve their conceptual understanding; the types of questions asked is also an important factor to consider. Burns (1985) claimed that, "the types of questions the teacher was asking the student to consider were helping to create a classroom atmosphere truly conducive to developing mathematical thinking abilities (…) It's just this kind of skill with questioning that is needed to pave the way for teaching children to think mathematically" (p. 14). Burns determined that often what was lacking in mathematics classrooms was the directing of students' attention to "deciding on the reasonableness of their solutions, justifying their procedures, verbalizing their processes, [and] reflecting on their thinking" (p. 14) and identified the types of teacher questioning as the key to make these changes happen. To support argumentative discourse, Kosko et al. (2014) maintained that, "a key component of mathematical argumentation is the teacher's use of questioning strategies" (p. 459). In the field of mathematics education, many studies (e.g., Hunter, 2007; Kazemi & Stipek, 2001; Martino & Maher, 1999; Sahin & Kulm, 2008; Wood, 1998) have found that the type and quality of questions can have significant impact on students' engagement in productive argumentative activities.

Aiming to create a “math-talk learning community” (p. 81), Hufferd-Ackles et al. (2004) sought to classify questioning levels to identify changes in the teacher’s and students’ actions from only a teacher as questioner to both students and teacher as questioners. Over the course of a year, Hufferd-Ackles et al. developed a list of themes of teacher questioning by analyzing teacher actions in an urban Latino third-grade classroom. Based on the data that were collected, they categorized teacher questioning into the following four levels: level 0 is short frequency question to keep students listening and paying attention to the teacher; level 1 is question that begins to focus on student thinking; level 2 is probing and more open question; and level 3 is question that students themselves ask about their work. Hufferd-Ackles et al.’s framework provided a categorization of questioning approaches, but they did not specify which levels of questioning were instrumental in eliciting student contribution to argument components, as this was not the intent of their study.

Martino and Maher (1999) argued that teachers should be aware of the types of questions that they use in their classroom and highlighted the role of teacher questioning to establish a classroom atmosphere that is conducive to the development of mathematical argumentation. They analyzed the questions used by a third and fourth grade math teacher who taught in a school district in New Jersey as part of a 10-year longitudinal study. In this study, the teacher was working on two isomorphic problem tasks whose solutions can be represented using different methods of proof. Martino and Maher found that, in general, students were usually satisfied with trial and error methods and often believed that their solutions were sufficient for justification, even when they worked with a partner or in a small group. They also found that students did not challenge each other about the validity of arguments when they worked in pairs or groups. Hence, Martino and Maher posited that this lack of critiquing ideas or arguments among students may be due to the fact that students usually ask low levels of questions to each other. They conclude that the role of teacher questioning as an intervention is essential to encourage students to explain, to clarify, or to justify their solutions. By focusing on how teacher questions impacted student justification and reasoning, Martino and Maher suggested that a sequence of questions that facilitate justification and offer an opportunity for generalization helps students to build mathematical arguments.

Sahin and Kulm (2008) stressed that it was important for teachers to be cognizant of the types of questions that they are asking as well as their intentions for asking these questions. They analyzed questions used by a first-year teacher and an experienced teacher by using videos of five lessons for each teacher that were taken in two sixth-grade classes over a two-month period. Both teachers were teaching the same mathematical content but used different textbooks. For this study, Sahin and Kulm considered how teachers used three types of questions: probing, guiding, and factual. Probing questions were used to ask students to explain or elaborate on their thinking, to make connections with their prior knowledge, or to offer justifications or proofs for their solutions. Guiding questions were similar to leading questions and could be used to guide students work through the problem-solving process by asking for solutions, strategies or procedures, and thus scaffolding students' understanding of a concept (Sahin & Kulm, 2008). Finally, factual questions required students to recall facts or definitions as well as answers or next steps in a problem. Sahin and Kulm found that the majority of the questions posed were factual, even when using a reform-based textbook, which included probing and guiding questions in the teaching guides. They concluded that, by asking probing questions, teachers encourage students to make conjectures, construct justifications and build arguments with others, and thus create a classroom environment that effectively engages students in mathematical argumentation.

Franke et al. (2009) highlighted that teacher questioning plays an important role in orchestrating classroom discussions and suggested that it is critical for teachers to have knowledge of what types of teacher questioning will be more appropriate as they guide students toward particular answers or explanations. They articulated that, “Find[ing] the balance in the types of questions and when to ask them can make a large difference in how students continue to participate” (Franke et al., 2009, p. 381). Franke et al. also developed a coding scheme for teacher questions that focused on the questioning practices teachers used to promote student thinking and response to student explanation. They observed three elementary school classrooms in a large urban school district in Southern California, which was labeled as one of the lowest performing school districts in California. Franke et al. identified four types of teacher questioning: general questions, specific questions, probing sequences of specific questions, and leading questions. General questions were identified as those not addressing any specific part of students’ answers or explanations. In contrast, specific questions were used to clarify one aspect of students’ answers or direct students’ attention to one part of their explanations. Probing sequence of specific questions consisted of more than two related questions that enabled students to correct their incorrect or incomplete explanations or elicit further elaborations. Leading questions were used to guide students’ work towards correct answers or explanations. Franke et al. found that although teachers frequently asked students to explain their answers or to offer justifications, these follow-up questions may not always produce further explanations. Probing sequences of specific questions, according to Franke et al., served as the catalyst for the students to examine their answers or explanations, and thus provided them with opportunities to revise their incorrect answers and incomplete explanations. They argued that probing sequences of specific questions had benefits for the whole class because teachers can use this questioning strategy to make students' implicit ideas more explicit or to uncover a student's incorrect answer or reasoning. Their data also showed that a single specific question was not enough to elicit a complete explanation or justification; many specific questions, each concentrating on certain aspects of students' explanations were required. The results of this study highlighted the role of probing questions to elicit student reasoning and suggested that using sequences of probing questions may help students produce more complete and correct explanations.

O'Connor and Michaels (1996) defined *revoicing* as "a particular re-uttering (oral or written) of a student's contribution (p. 71)," which involved repetition, expansion, rephrasing, or reporting what a student says. The use of revoicing has been found to be a productive teaching move for students' mathematical argumentation (Conner et al., 2014; Forman et al., 1998; O'Connor & Michaels, 1993, 1996; Temple & Doerr, 2012). The teacher could use revoicing to reformulate student arguments to explicate student reasoning; also, revoicing is used to create alignments or oppositions in an argument (Forman et al., 1998). By positioning their arguments, students took on roles of constructors and critics in argumentation. Temple and Doerr (2012) studied teacher moves that supported students' use of symbolic notation, oral language, written language, and visual representations to construct mathematical knowledge. They found that instead of giving direct corrective feedback, teachers intentionally partially or completely repeated the student's answer as a question to signal an error or challenge the student's arguments. This questioning technique was found to enable the student as well as the rest of the class to see the error in their arguments and push the students to repair inaccuracies in their arguments or help them enhance their explanations. Temple and Doerr also found that "probing questions" (Franke et al., 2009) were particularly effective in introducing new mathematical knowledge but argued that "funneling questions" (Wood, 1998) or "leading questions" (Franke et al., 2009) supported students' use of precise mathematical language more than justification or explanation. Temple and Doerr suggested that more research was needed to focus on the balance between the use of "funneling" and "focusing" questioning strategies.

Aiming to understand why teachers infrequently asked probing questions (i.e., which ask for clarification, justification, or explanation), Kosko et al. (2014) used cartoon-based vignettes to access what questioning strategies secondary mathematics teachers perceived as beneficial when facilitating student engagement in argumentation in a high school algebra context. They examined the types of questioning teachers depicted based on the question coding schemes that were developed by Boaler and Brodie (2004) and Franke et al. (2009). Kosko et al. found that the majority of questions teachers posed were teacher statements (lecture statement), teacher silence, and generating discussion, which were in nature passively engaging students in argumentation. They suggested that this lack of the focus on students' initial explanations may be due to the fact teachers have different interpretations of mathematical argumentation from researchers. Further, following Franke et al., Kosko et al. stressed the use of a sequence of probing questions for scaffolding argumentation and suggested that more research was needed to focus on sequences of questions as well as to clarify the context in which the types of questions occur. In a recent study, by focusing on teachers' choices of probing questions in given elementary algebra LessonSketch scenarios, Kosko (2016) found that the strategic use of probing questions can create a context for richer mathematical argumentation and thus support students' mathematical autonomy. Although multiple studies have highlighted the importance of using probing questions (i.e., which ask for clarification, justification, or explanation) to invite students to share their ideas and justifications with others, it was not clear for teachers how to implement these questioning strategies and under which context they were used to support argumentation.

Wood (1998) contrasted two patterns of questioning: *funneling* and *focusing*. In funneling, the teacher asks a series of questions with the goal of leading students toward a particular solution or conclusion. These questions are often a series of "fill-in-the- blank" questions that lead students to the answer desired by the teacher. For example, "8 plus 7 is adding one more to 14, which makes ()?" is a funneling question (Wood, 1998, p. 170). In contrast, in focusing questions, the goal is to pose questions that orient the discussion to one aspect of a student's mathematical thinking. Therefore, the teacher helps students to summarize what they know, so students can solve the problem on their own. An example of this type of question would be as follows: "Ok, could you write down beside it what you did?" Focusing refers to a more open style of questioning that prompts students to explain their thinking and remain open to a mathematics problem that can be solved in many ways. Wood's finding suggested the need to concentrate on patterns of questions as well as individual instances of questioning when supporting argumentative discourse.

By analyzing argumentation episodes from secondary mathematics classrooms, Conner et al. (2014) sought to classify teacher questioning from the perspectives of teachers. They explored types of teacher questioning that directly prompted or responded to parts of arguments (i.e., data, warrant, claim) identified using Toulmin's (1958/2003) model for argumentation. They identified questions requesting a factual answer, requesting a method, requesting an idea, requesting elaboration, and requesting evaluation. This questioning framework can be used to help teachers to examine questions they ask to support argumentative discourse. However, it only includes individual questions connected to argument components, thus not accounting for sequences of questions or more general questions.

Chen et al. (2016) developed a teacher questioning framework for promoting students' cognitive responses in science classrooms. They identified four teacher roles as necessary for promoting student argumentation: dispenser, moderator, coach, and participant. Chen et al. suggested that to promote higher levels of student cognitive response in argumentative discourse, teachers should adopt multiple roles more than the dispenser role, which mainly requires short answers and engaging students in lower-level cognitive thinking. By examining a prospective teacher's questioning in supporting mathematical argumentation, Zhuang and Conner (2018a) found that Chen et al.'s questioning framework from science education transferred adequately to mathematics education and provided a perspective to support an understanding of the relationship between ownership of ideas and activities and the roles of teacher questioning in supporting mathematical argumentation. However, Zhuang and Conner found that, unlike in science classrooms, the dispenser role of questioning served important functions in mathematics classrooms, such as introducing mathematics tasks and helping students focus on important mathematical concepts they were learning. Zhuang and Conner suggested that further research needs to focus on how various roles of teacher questioning support students' engagement in mathematical argumentation and the best times and contexts to use these different roles of questioning to facilitate students toward deep mathematical conceptual understanding.

While these questioning frameworks exist, and each provides some information about how teachers' questions might relate to collective argumentation, each framework is also limited in scope to focus on individual questions or explain the correlation between types of teacher questions and subsequent student responses. Fine-grained analyses uncovering the details of teacher questioning in supporting argumentation are rare.

2.8 Summary

In this review of the literature, I synthesized the related classroom-based research that discussed the importance of questioning, different classification of questions, and the development of questioning strategies supporting collective argumentation. The field has a consensus that incorporating mathematical argumentation in classroom discourse is critical and teacher questioning plays an essential role in facilitating classroom-based mathematical argumentation. Several studies (e.g., Hufferd-Ackles et al., 2004; Sahin & Kulm, 2008; Zhuang & Conner, 2018a) classified teacher questions according to different teacher questioning schemes so as to provide insight into how different levels or types of questions were used to lead classroom discussions. For example, Sahin & Kulm (2008) found that most of teachers' questions were factual and lecture. Zhuang & Conner (2018a) found that the participant prospective teacher mainly asked direct and lecture questions (i.e., prompted lower-level cognitive thinking). However, understanding current situations or difficulties that teachers have in using questions to orchestrate classroom discourse is not enough.

A common finding across studies suggests sequences of questions are important for eliciting reasoning and understanding student arguments (e.g., a sequence of probing questions from Frank et al., 2009; patterns of focusing questions from Wood, 1998). These findings emphasized the importance of studying sequences of questions as well as individual instances of questions. Moreover, these studies to some extent may help teachers to gain some insights into the utility of questions to elicit student thinking or engage student participation in discussions, but none of these studies explained teachers' pedagogical choices to use certain types of questions and the tension that may exist between use of questioning strategies and supporting collective argumentation. Furthermore, only limited studies have targeted specific aspects (e.g., the relevance, quality or timeliness) of questions to support collective argumentation.

While overall, the research focus on teacher questioning strategies in orchestrating classroom discussion in general is growing, research attention to teachers' questioning strategies specifically used to support collective argumentation has been limited. Andriessen (2006) contended that argumentation must be "scaffolded by the environment to support a gradual appropriation of collaborative argumentation" (p. 10) and that the teacher plays a central role in facilitating development of collective argumentation in mathematics classrooms by intervening with questions to gradually support students' work toward a mutually accepted conclusion about the truth of a mathematical claim. McCarthy et al. (2016) stated that "Identifying 'good' and/or 'effective' questioning strategies is a major challenge to mathematics teachers" (p. 80). Thus, it is important not only to investigate how teachers use questions to frame argumentative discourse but also necessary to develop questioning techniques to increase teacher effectiveness and student participation in argumentation. The goal of this study was to investigate promising questioning strategies for teachers to use when aiming to support more productive collective argumentation.

Stylianides et al. (2016) called on more research in the field of mathematics education to address classroom implementations or interventions with the purpose of improving the teaching of argumentation and proof.

> The field would benefit from more research that would use theoretical ideas to design practical tools for use in the classroom and in the service or particular learning goals in different areas including argumentation and proof. (p. 336)

I therefore argue that more studies are needed in our field to analyze teachers' interventions when they employed certain types of teacher questioning to support collective mathematical argumentation as well as to investigate the effectiveness of classroom-based questioning interventions to respond to the need for developing teaching approaches to address teachers' difficulties in orchestrating argumentative discourse.

CHAPTER 3

THEORETICAL PERSPECTIVE AND ANALYTIC FRAMEWORKS

3.1 Overview

This chapter outlines the underlying theoretical perspective for this study and details the analytic frameworks developed and used. I first provide a rationale for selecting symbolic interactionism as a theoretical perspective for my study. My beliefs about mathematical knowledge and learning situated in a collective learning environment, such as collective argumentation that occurs in a mathematical classroom, and the relationship between students' mathematical learning and collective argumentation will be presented. This will be followed by a discussion of the implications of symbolic interactionism to document the development of collective mathematical learning that is situated in argumentative activities and concentrate on the role of the teacher in orchestrating these collective argumentative activities.

In the second part of this chapter, Habermas' (1998) construct of rational behavior as a useful frame to analyze didactical obstacles inherent in proving and argumentative activities is presented. I address my choice to employ Habermas' construct of rational behavior as a theoretical lens to explain the teachers' pedagogical decisions with respect to questioning that is situated in mathematic collective argumentation.

I then introduce the *Rational Teacher Questioning Framework* I developed based on Habermas' (1998) construct of rational behavior as well as the use of Toulmin's model to analyze argumentation in terms of structure. Finally, I discuss the benefits of integrating these two analytic frameworks for this study in studying teacher question moves in supporting collective argumentation.

3.2 Symbolic Interactionism

In order to make sense of classroom-based argumentation, I believe that we should take a sociological perspective on public mathematical activities. Of the various social perspectives that have been adapted in mathematics education, I drew on symbolic interactionism to guide and inform this study for three reasons. First, the symbolic interactionism perspective highlights that learning can occur in a social setting and stems from interaction, which forms the theoretical basis for my research on collective argumentation. Symbolic interactionism comes from a sociological perspective and derives from George Herbert Mead, who assumed that "knowledge is created through action and interaction" (Strauss & Corbin, 1990, p. 2). This view of knowledge stresses the need for students to work cooperatively and indicates that individual students develop their personal understanding of mathematics as they interact with the teacher and with other students. As Bauersfeld (1980) stated:

> Teaching and learning mathematics is realized through *human interaction*. It is a kind of mutual influencing, an interdependence of the action of both teacher and student on many levels. [...] the student's reconstruction of meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or peer's) sanction. (p. 35)

These remarks highlight that mathematics teaching and learning is inherently social and embedded in active participation in a communicative reasoning process (Lerman, 2001), such as collective argumentation. Therefore, collective argumentation is not only a crucial approach but also a goal for mathematics teaching and learning. Moreover, some researchers (e.g., Lave &

Wenger, 1991; Sfard, 2008) view learning as participation. Thus, in mathematical classroom situations, mathematical learning is conceived as students participating in a collective practice where the teacher and students (or a small group of students) work together to construct or reject mathematical arguments that involve explanations and justifications (Krummheuer, 2007, 2015; Staples & Newton, 2016). It is in this sense, some researchers (e.g., Krummheuer, 2007, 2015; Cramer & Knipping, 2018) claimed that learning mathematics is *argumentative* learning and it depends on students' participation in argumentative activities. Thus, a positive relationship exists between active participation in argumentation and learning (Krummheuer, 1995). Students' active involvement in a process of argumentation influences their individual learning and, conversely, their cognitive capacities influence both the course and outcome of a process of argumentation.

Second, symbolic interactionism enables the researcher to examine and trace ways of engaging in collective argumentative activities that are normative or taken-as-shared in a classroom community. One of the defining premises from symbolic interactionism theory is that people act towards things based on the meaning that those things have for them, and that these meanings are derived from social interaction and modified through interpretation (Prasad, 2005). As Blumer (1969) stated, "One has to fit one's own line of activity in some manner to the actions of others. The actions of others have to be taken into account and cannot be merely an arena for the expression of what one is disposed to do or sets out to do" (p. 8). The important idea here is that the individuals who engage in interaction have to interpret the actions of others as well as use actions to indicate to others what their own intentions are. In this sense, the teacher's actions have meaning for both the teacher and students in a classroom discussion. This notion of reflexivity implies that learning is a consequence of social interactions, along with a mutual and dialectic process between the teacher and students. The teacher and students take each other into account when they participate in the ongoing negotiation of classroom interactions and are thus at least affected by each other (Charon, 2009). Therefore, on the basis of symbolic interactionism, the teacher and students, as members of a classroom community, jointly establish classroom social norms, sociomathematical norms, and participate in collective argumentative practices, and these normative practices of a local classroom community can change over time through ongoing interaction between the teacher and students (Cobb, 2000; Cobb et al., 2001). Furthermore, symbolic interactionism regards the meaning and understanding that students construct through classroom interactions as social products that are compatible with students' interpretations (Yackel, 2001). This view of meaning implies that meanings are taken-as-shared in a classroom culture, and the development of an individual's reasoning and sense-making processes cannot be separated from his or her participation in these taken-as-shared mathematical meanings (Cobb & Bauersfeld, 1995; Yackel, 2001). Collective argumentation occurs when participants in a discussion work together in the construction of an argument, and the taken-as-shared meaning serves as the basis for mathematical communication and for students to explain and justify their thinking in the process of constructing arguments. Therefore, the adoption of the symbolic interactionism perspective allows me to regard collective argumentation as the teacher and students acting in normative or taken-as-shared ways of acting, reasoning, and arguing mathematical reality and "elaborating that reality in the course of their ongoing negotiations of mathematical meanings" (Cobb & Bauersfeld, 1995, p. 3).

Finally, symbolic interactionism is compatible with Habermas' construct (1998) of rational behavior, which has been regarded as a useful frame for analyzing individuals' didactical obstacles that are inherent in proving and argumentative activities (Boero, 2011; Boero, et al., 2010; Boero & Planas, 2014; Zhuang & Conner, 2018b). According to symbolic interactionism, *role taking* is involved in both communicating and interpreting (Blumer, 1969). Through role taking, we learn things and expect things from other people, and those people, in turn, come to know us and learn what to expect from us (Charon, 2009). Therefore, on the one hand, teachers are learners when they try to understand and interpret students' responses. On the other hand, a teacher also serves as a representative of the mathematical community who attempt to make judgements about what becomes mathematically normative in a classroom and facilitates argumentation in an accessible way, including providing relevant and appropriate data, warrants, and backings for claims that are consistent with how students construct knowledge (Yackel, 2002; Yackel & Cobb, 1996). This perspective on the role of the teacher in supporting collective argumentation highlights the significance of the teacher's own mathematical knowledge for teaching (MKT) (Ball et al., 2001), goals, beliefs, and how they work together to inspire, constrain or adjust the teacher's actions during classroom discussions. This view is consistent with Habermas' (1998) construct of rational behavior which assumes that the teacher is a "rational being" (p. 311) when participating in a communicative action toward mutual understanding, such as collective argumentation, and assumes rationality in the decision-making processes that relate to his or her knowledge, goals, and beliefs.

3.3 Habermas' Three Components of Rationality

Habermas (1998) defined a person as a rational being if a person has the ability to "give account for his orientation toward validity claims" (p. 311). Habermas proposed that this "rationality accountability" (p. 311) implies a self-relation about what the person believes, says, and does, and is interrelated with the three core structures of rationality: in the proportional structure of knowledge (knowing), in the teleological structure of action (acting), and in the communicative structure of speech (speaking). In other words, the assumption taken by Habermas is that for a rational being, discourse and reflection are integrated, and "the three rationality components — knowing, acting, and speaking — combine, that is, form a syndrome" (p. 311). Knowledge, action and speech constitute what Habermas (1998) called *epistemic*, *teleological*, and *communicative* components of rationality. In the remainder of this section, I will discuss the three components of rationality and how they can be connected to mathematical argumentative activities.

3.3.1 Epistemic Rationality (*ER*)

According to Habermas (1998), one cannot understand the meaning of an utterance without knowing the reasons for accepting it.

> In order to know something in an explicit sense, it is not, of course, sufficient merely to be familiar with facts that could be represented in true judgments. We *know* facts and have knowledge of them only when we simultaneously know why the corresponding judgments are true. The explicit 'knowing what' is bound up with 'knowing why' and oriented toward validity claims with potential justifications. (Habermas, p. 311-312)

The presumption here is that knowledge is built from "why the corresponding judgments are true" and is "intrinsically of a linguistic nature," thus it can be analyzed with the help of propositional sentences — "epistemic rationality" (Habermas, 1998, p. 311). However, Habermas argued:

> This does not mean, of course, that rational beliefs or convictions always consist of true judgments. Whoever shares views that turn out to be untrue is not *ep ipso* irrational. Someone is irrational if [he or] she puts forward [his or] her beliefs dogmatically, clinging to them although [he or] she sees that [he] she cannot justify them. In order to qualify a belief as

rational, it is sufficient that it can be held to be true on the basis of good reasons in the relevant context of justification – that is, that it can be accepted rationally. (p. 312)

The importance of this concept held by Habermas conveys his expectations of rational behavior on the epistemic side; that is, the statements or reasons that a person puts forward are explicitly an attempt to show what he or she believes to be true.

Further, Habermas (1998) was also concerned with how the factors of context influence the truthfulness of asserted epistemic rationality: “The rationality of a judgment does not imply its truth but merely its justified acceptability in a given context” (p. 312). In this sense, the truth of an argument is context dependent, which means that an argument that is accepted as valid in one community may not be validated in another and that the validity depends on what has been accepted in each community. Habermas’s perspective on the epistemic rationality of an argument is consistent with Toulmin’s idea of “dialectic arguments” which viewed that argumentation does not necessarily lead to true conclusions, but one who engages in argumentation should start from statements that he or she believes to be true (Pedemonte, 2007). This view of argumentation implies that in a mathematics classroom community, the quality of an argument is not solely dependent upon its objective mathematical truth or formal structure, but more importantly on whether the argument is based on mathematical principles and evidence that are available to members of a community of practice and are useful in extending the argument further. In summary, in mathematical argumentative practices, we can identify an epistemic dimension of rationality inherent in the desired product (acceptable arguments in context) that controls the epistemic validity of arguments and inferences that link these arguments together according to shared mathematical premises (i.e., theorems, axioms, and principles) (Boero et al., 2010).

3.3.2 Teleological Rationality (*TR*)

For Habermas (1998), all actions are intentional and goal-oriented. In this sense, each action may be understood based on an individual's conscious choice of means or tools that are used to achieve the goal of the activity. Habermas defined this process of finding "good" means that govern actions aimed at achieving success as "teleological rationality" (p. 313). However, the teleological rationality of an action does not depend on the successfulness of the action, but the strategic choice of means rests on epistemic reasoning. Habermas emphasized:

> Once again, the rationality of an action is proportionate not to whether the state actually occurring in the world as a result of the action coincides with the intended state and satisfies the corresponding conditions of success, but rather to whether the actor has achieved this result on the basis of the deliberately selected and implemented means (or, in accurately perceived circumstances, could normally have done so). (p. 313)

Built on Habermas' idea of teleological rationality, in this study I argue that mathematical argumentation is rational on the teleological side since through argumentation (which includes conjecturing, proving, modeling, finding counter examples, generalizing, etc.), students are expected to demonstrate or describe how they did something to show their capacity for choosing suitable means to reach the final claim (i.e., goal). Further, Habermas highlighted that an argumentation can be qualified as a teleological rational action even if the originally intended mathematical claim is not reached. Thus, teleological rationality not only consists of the intentional choices of means to reach the goal, but also the reflective attitude toward it, about which he stated that:

> A successful actor has acted rationally only if he [or she] (i) knows why he [or she] was successful (or why he could have realized the set goal in normal circumstances) and if (ii)

> this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that can at the same time explain its possible success. (Habermas, 1998, p. 313-314).

The first condition shows that teleological rationality about the choice of strategic means rests on epistemic rationality, and the second condition represents an enrichment of strategies in mathematical problem-solving (Boero & Planas, 2014). It implies that there are reasons for an actor to choose one tool over another, and these reasons justify an actor's ways of managing the teleological rationality. In this sense, we can identify a teleological dimension that is inherent in the conscious choice of means to obtain the desired arguments or conclusions (i.e., acceptable arguments in context) that is embedded in a goal-oriented classroom environment, such as collective mathematic argumentation (Boero et al., 2010).

3.3.3 Communicative Rationality (*CR*)

Habermas's (1998) communicative rationality is concerned with the rationality of the use of language oriented toward reaching understanding and is motivated by the attempt to create an intersubjectively shared lifeworld between speakers and listeners, which results in the possibility of referring to the same object world. He stated:

> Communicative rationality is expressed in the unifying force of speech oriented toward reaching understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (p. 315)

The "lifeworld" here refers to a "background" environment of competences, skills, attitudes, and practices representable in terms of one's cognitive horizon (Habermas, 1998). So, the use of communicative rationality as a form of social interaction serves not only to give expression to the intentions of a speaker but also to represent states of affairs and to establish interpersonal relationships with other listeners (Habermas, 1998).

When it comes to the evaluation of communicative rationality, Habermas (1998) pointed out that:

> The rationality of the use of language oriented toward reaching understanding then depends on whether the speech acts are sufficiently comprehensible and acceptable for the speaker to achieve illocutionary success with them (or for him [or her] to be able to do so in normal circumstances). (p. 315)

Here, Habermas emphasized an intentional, reflexive character that is inherent in communicative rationality. Thus, when uttering a sentence, the speaker does not merely assert his or her beliefs in the truth or validity of the sentence, but also maintains that he or she has some reasons to justify that the sentence in question is true or valid. It is in this sense that the rationality in communication rests on an internal connection between the rationality of a speech act and its corresponding justification (Habermas, 1998). Therefore, in mathematical argumentative practices, we can identify a communicative dimension that is inherent through the conscious search for means to ensure that both the communicating steps of reasoning and final claims of argumentation conform to standards in the given mathematical classroom community (Boero et al., 2010).

3.3.4 The Relationship Among Three Components of Rationality

According to Habermas (1998), the three components of rationality are inseparable, because a connection is established between the evolutionary character of knowledge (epistemic rationality), goal-oriented actions (teleological rationality) and the use of language (communicative rationality).

> Of course, the reflexive character of true judgments would not be possible if we could not represent our knowledge, that is, if we could not express it in sentences, and if we could not correct it and expand it; and this means: if we were not able to also learn from our practical dealings with a reality that resists us. To this extent, epistemic rationality is entwined with action and the use of language. (Habermas, 1998, p. 312)

Habermas' elaboration about rationality in discursive practices corresponds to what we think proving and argumentation should be in mathematics education, in which students are expected to strategically choose specific tools to achieve the goal (i.e., validity claims) on the basis of specific knowledge (mathematical rules, theorems, axioms, and principles), and communicate in a precise way with the aim of being understood by the classroom community (Boero et al., 2010; Boero & Planas, 2014). This view of how the three components of rationality are embedded in argumentative activities has important implications for this study. It is the basis of the idea of using *rational questioning* as a way to introduce the students, and lead them, to behaviors shaped by rationality requirements: *Can you tell me why? How did you figure that out? How would we write this mathematically correctly?* (Douek in Boero & Planas, 2014). Thus, it would be beneficial to identify fine-grained rationality components of teachers' questioning and thus reveal teachers' awareness of using questions to foster students' rationalization of discourse (i.e., fitting rationality requirements) when they guide argumentative practices.

3.4 Application of Habermas' Construct of Rational Behavior

In the field of mathematics education, Boero (2006) started to use Habermas' (1998) construct of rational behavior as a tool to analyze students' rational behavior in proving activities to help draw prospective teachers' attention to some aspects of students' mathematical activities, such as the teleological rationality, when they evaluate students' work. The work of Morselli and

Boero (2009) illustrated the benefits of using Habermas' construct to analyze students' argumentative approaches and their thought processes in the transition from argumentation to proofs. They pointed out that Habermas' construct can provide the researcher a comprehensive frame to analyze students' difficulties concerning proving and other mathematical discursive activities. By focusing on students' use of algebraic language in mathematical modeling and proving, Morselli and Boero (2011) emphasized that Habermas' construct of rational behavior can be a particularly useful analytic tool to describe and interpret students' mistakes as well as provide teachers with useful indications for the students' approach as related to three components of rationality. They matched Habermas' three dimensions of rational behavior with the use of algebraic language in proving and modeling. According to Morselli and Boero, epistemic rationality included *modeling requirements* (concerned coherency between the algebraic model and the model situation) and *systemic requirements* (concerned the use of algebraic language and methods); teleological rationality concerned the conscious choice and finalization of algebraic formalizations, transformations and interpretation according to the goal of the activity; and communicative rationality concerned the use of standard notations in the given community. The results showed that many student mistakes were situated in *modeling requirements* (epistemic rationality), even though the produced representations were correct (communicative rationality), and students were struggling to use these correct representations to solve the problem (teleological rationality).

In recent years, researchers have highlighted the importance of Habermas' (1998) construct of rational behavior as a tool not only to analyze students' rational behavior in proving and argumentation, but they also applied it to deal with teachers' didactical obstacles inherent in proving and argumentative activities (e.g., Boero et al., 2013, 2018; Boero & Planas, 2014;

Conner, 2017; Zhuang & Conner, 2018b, in press). For instance, Boero et al. (2013) analyzed 35 secondary prospective secondary teachers' task behaviors between different mathematical domains (analytic geometry and synthetic geometry) using Habermas' construct of rational behavior. The results showed that there was a lack of functional connections between epistemic rationality (i.e., validity of geometrical constructions) and teleological rationality (strategies referring to visual evidence); communicative rationality worked well only on analytic geometry. Boero et al.'s study contributed to the use of Habermas' construct as a tool for teachers to characterize and compare "rationalities" in different domains of mathematics activities as well as develop teachers' awareness about planning and teaching tasks according to epistemic, teleological, and communicative criteria.

Boero et al. (2010) proposed the integration of Habermas' (1998) construct of rational behavior and Toulmin's (1958/2003) model to study discursive practices related to proving and argumentative activities. According to Boero et al. (2010), combining two frameworks provided a better frame to (1) analyze the process of argumentation (i.e., content level, through Toulmin's lens); (2) plan and carry out classroom interventions aimed at promoting students' awareness of the epistemic, teleological and communicative requirements of argumentation (i.e., meta-level, through Habermas' lens). Further, Boero et al. highlighted that the meta-level argumentation is not a goal for students, but a teaching means. Adapting Habermas' construct with Toulmin's model for argumentation, I analyzed how a prospective secondary teacher used questions to support collective argumentation, which was published in Zhuang & Conner (2018b). These results suggested that the theoretical integration of two constructs can be a useful tool to frame teacher questioning strategies for promoting collective argumentation; this study resulted in a preliminary version of Rational Questioning Framework developed in this study. Habermas' lens helps to identify fine grained rationality components of teachers' questioning and also how teachers' questioning is constrained in relation to the three components of rational behavior; the teacher uses rational questioning to control the fundamental steps of argumentation as seen through Toulmin's lens.

In summary, Habermas' construct of rational behavior has drawn attention to the importance of three forms of rationality existing in discursive practices and how they are mutually dependent on each other. The use of this construct in the field of mathematics education helps illustrate the link between theoretical and applied research by examining how theoretical ideas of structures of rational behavior can be turned into practical proposals for mathematics teaching and learning aimed at promoting teachers' and students' awareness of the epistemic, teleological, and communicative requirements of argumentation.

3.5 Integrating Habermas' Rationality with Symbolic Interactionism

The integration of the symbolic interactionist perspective with Habermas' construct of rational behavior is not easy, because it requires the use of notions that are compatible with different purposes and within different frameworks. This is perhaps best summarized in Boero and Planas (2014):

> Focus on the teacher and on social interaction challenges in many ways the ideals of communication and rationality, as well as the progress of mathematics teaching and learning. The productivity of the students, in terms of effective individual learning, and that of the teacher, concerning effective creation of collective learning contexts, has to do with social aspects that intervene in broadening the reach of participation. (p. 3)

Boero and Planas (2014) claimed that Habermas' construct does not deal with social perspective in a classroom mathematical practice; it is situated from a psychological perspective that focuses on the nature of how an individual participates in discursive activities since only communicative rationality is related to social interaction. However, Boero and Planas (2014) further argued that to some extent Habermas' construct of rational behavior is compatible with social interaction by assuming that empirical knowledge and teleological actions are formed for the presumption of "acceptance" in social interaction.

In this study, I integrated the construct of rational behavior with symbolic interactionism in the analysis of classroom situations with a special focus on how teachers use questions to promote the development of students' rational behavior through argumentative activities. As discussed in this chapter, we can see that Habermas' construct of rational behavior aligns with a symbolic interactionism perspective which assumes that a purposeful teacher action in a social setting is to be considered as a key element for students' learning. The questions teachers ask and the way they ask them can influence the process of their students' engagement in argumentative discourse. It is in this sense that I argue that the integration of these two perspectives is complementary for the purpose of this study: Habermas' lens help me to identify fine-grained rationality components of teachers' questioning and how teachers' questioning is constrained in relation to the three components of rational behavior; the teacher uses rational questioning to control the fundamental steps of argumentation is traced through the symbolic interactionism perspective.

3.6 A Framework for Rational Questioning

This study builds on the literature (see Chapter 2) and shares Boero's (2006) suggestion of using Habermas' construct of rational behavior as a theoretical tool to analyze how teachers support collective argumentation. My study took the idea of Douek in Boero and Planas (2014) to use "rational questioning" (p. 210) to support collective argumentation and paid special attention to how teachers' questions are constrained in relation to the three components of rational behavior (i.e., epistemic, teleological or communicative). Douek responded to the call by Boero to use Habermas' construct of rational behavior to analyze mathematics teaching with special focus on discursive activities, such as argumentation and proving. Douek suggested that it was beneficial to develop argumentative discourse along the three components of rationality (i.e., epistemic, teleological or communicative) and that the teacher should support students to meet the requirements of rationality, thus dialectically forming argumentation (Boero & Planas, 2014). In order to reach such aims, Douek further proposed the idea of using "rational questioning" as a teaching method to promote students' awareness of epistemic, teleological and communicative requirements of argumentative practices. According to Douek, the role of rational questioning should favor student's maturation:

> From the ability to act and develop a discourse about acting towards the ability to organize strategies and express them a priori, when the situation is sufficiently mastered by supporting—in between—going back and forth from 'action' to its rationalization, accounting for validity of statements and strategies, and autonomously producing a conclusive rational discourse." (Boero & Planas, 2014, p. 211)

The idea of using rational questioning as a process of enculturating rationality discourse into the practice of argumentation corresponds to Boero et al. (2010)'s argument that rationality in argumentation must be promoted by teachers. Thus, rational questioning can be viewed as a means to develop a culture of rationality in the classroom, as discussed by Rodríguez and Rigo (2015):

> How the teacher negotiates her own rationality practices—an objective that, by way of dialogical exchange, involves the students by means of constant questions, not only about what but also about why— and how this enculturates her students in that rationality. (p. 93)

Although Douek in Boero and Planas (2014) proposed to use rational questioning as a teaching intervention to enrich the potential of collective argumentation, he did not describe the meaning of rational questioning in terms of its application and how it allows teachers to plan and manage collective argumentation. In this study, I address this gap by defining rational questioning from teachers' perspectives to engage student participation in argumentation with different kinds of rationality. In other words, my definition of rational questioning was situated in the viewpoint that teachers were rational beings who were capable of giving account of their strategic choices of didactical tools (e.g., types of rational questioning) to achieve their instructional goals (e.g., adequate epistemic reasons, efficient teleological choices, rules of communication in mathematics).

According to Douek in Boero and Planas (2014), in a Vygotskian perspective, the idea of developing the mathematical discussion along three components of rationality was under the assumption that argumentation is a means and an aim (served as mediator between students' everyday conceptions and scientific conceptualization of school mathematics), which was gradually constructed through the guidance of the teacher. Hence, through this process, the role of teachers is to create a classroom context to promote Vygotskian dialectics and develop argumentative discourse according to the three components of rationality. Following Habermas' concept of three components of discursive rationality and Douek's suggestion, I developed a *Teacher Rational Questioning Framework* (see Table 3.1) to classify rational questioning from teachers' perspective to engage student participation in different kinds of rationality. I defined *rational questioning* as a question that contains at least one component of rationality. For clarity, if a question contains an epistemic rationality component, I call it an epistemic rational questioning. Thus, if a question contains all three components of rationality, we can call it epistemic, teleological or communicative rational questioning. It is noted that a rational question could consist of various forms of rationality components when the teacher provides opportunities for students to engage in more than one dimension of rationality (e.g., justify the effectiveness of means or tools). In this sense, some rational questions include two or three components of rationality components and others may only involve one.

Unlike other teacher questioning frameworks (e.g., Boaler & Humphrey, 2005; Hufferd-Ackles et al., 2004; Sahin & Kulm, 2008), the Teacher Rational Questioning Framework (see Table 3.1) did not divide teacher questions into high- or low-level questions, open or closed questions, or categorized questions from an analysis of practice. Instead, I categorized teacher questioning into components of rationality based on Habermas' (1998) theory of rationality, which would allow teachers to plan and manage a classroom situation based on the rationality that they wish to develop, rather than focus on a high-level question, an open question or exactly what teachers should say in context. In addition, according to Boero et al. (2010), the awareness of the epistemic, teleological, and communicative requirements of rationality was inherent in expert's argumentative practices. This view of argumentation from Habermas' perspective was consistent with the field's suggested orientation toward proving and argumentation in mathematics education. In this sense, some researchers argued that the rationality requirements of argumentation must be passed over to students through the guidance of teacher by using specific educational devices (Boero et al., 2010). The Teacher Rational Questioning Framework developed in this study may serve as a starting point to help teachers introduce their students to epistemic, teleological, and communicative requirements of rationality when engaging in argumentative practice.

Table 3.1

Teacher Rational Questioning Framework

Component of Habermas' Rationality	Features	Description of Questions	Examples
Epistemic Rationality (*ER*)	Questions control the epistemic validity of arguments according to shared mathematical propositions, theorems, axioms, and principles.	• Facilitate/elicit students to reason and justify their arguments and ideas for their own benefit and for the class. • Clarify/challenge students to reason and justify their arguments and ideas when they give unclear or incorrect responses.	• Can you tell me why? • Why do you agree with his/her/their claim?
Teleological Rationality (*TR*)	Questions control the conscious choice of means/tools to obtain the desired arguments (i.e., acceptable arguments in context).	• Allow students to show or reflect on the strategic choices of means/tools that they used to achieve their arguments or ideas. • Point students toward a specific means or tools	• How did you figure that out? • Can I combine these two terms?
Communicative Rationality (*CR*)	Questions control the conscious choice of means of communication within a given community.	• Allow students to communicate or reflect on their steps of reasoning and final claims of argumentation to ensure their use of mathematical language (e.g., oral language, written language, visual representations, symbolic notation, etc.) conform to the norms of and are understandable in the given mathematic classroom community. • Guide students for correct use of mathematical terminologies, representations and phrases to form legitimate ways of reasoning.	• What we have been calling for [mathematical terminologies, representations, and phrases]? • How would we write this correctly mathematically?

3.7 Structure and Components of Arguments

An argument, as described by Toulmin (1958/2003), involves some combination of claims (statements whose validity is being established), data (support provided for the claims), warrants (statements that connect data with claims), rebuttals (statements describing circumstances under which the warrants would not be valid), qualifiers (statements describing the certainty with which a claim is made), and backings (usually unstated, dealing with the field in which the argument occurs). Toulmin conceptualized an argument as occurring with a specific structure (see Figure 4.1) in which these parts of arguments relate to one another in specific ways.

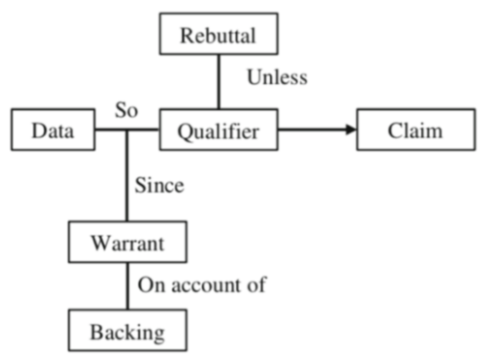

Figure 3.1. Layout or Structure of a Generic Argument (adapted from Toulmin, 1958/2003).

Toulmin's (1958/2003) description of argumentation components as well as his model have been widely used in mathematics education research as a tool for analyzing how collective argumentation developed in classroom settings (e.g., Arzarello & Sabena, 2011; Hollebrands et al., 2010; Inglis et al., 2007; Krummerheur, 1995, 2007; Rodríguez & Rigo, 2015), teachers' discursive actions around argumentation (Conner et al., 2014; Zhuang & Conner, 2018b), and other aspects of teaching mathematics, such as pedagogical arguments (Metaxas et al., 2016) or teachers' arguments (Giannakoulias et al., 2010). Following Krummheuer (1995) in using Toulmin's model of an argument, Conner (2008) focused on who contributes parts of arguments (e.g., data, warrant, claim) by including the use of color (line style) to denote the contributor(s) of components of an argument for a given argument, and they used “Teacher Support” to denote teachers’ contributions and actions that prompt or respond to parts of arguments.

In practice, arguments are often complicated and consist of multiple sets of sub-arguments to support the final claim. In addition to a single core component (i.e., claim, data, warrant), one component might serve two purposes. For instance, if a component functions as both data in one argument and as a claim in a sub-argument, I labelled it as data/claim.

In this study, I adopt Conner’s (2008) modified version of Toulmin’s (1958/2003) model of argumentation to examine how teachers used their rational questioning to lead collective argumentative discourse. Following Conner (2008), I used color and line style of the boxes representing the components of the argument: the teacher (red, solid), the students (blue, dashed), or a teacher and students co-construction (violet, dashed and dotted sequence) to signify the contributor(s) of each component of an argument (see Figure 3.2). In a classroom setting, the initial data is often the given information (e.g., task) presented to students to solve. I used a green color and a solid line to represent this information. Sometimes, parts of an argument may not be explicitly stated by the teacher or students but can be inferred from the context of the argument in the given classroom community; I labelled these implicit parts with a surrounding cloud (black). The teacher’s actions that are not components of the argument but prompted or responded to parts of the argument are represented by red ovals connected to argument components.

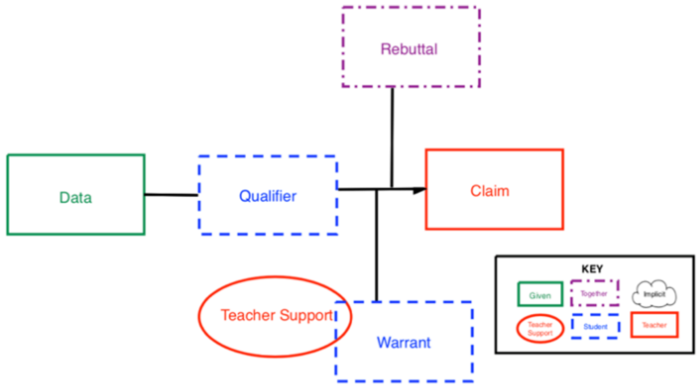

Figure 3.2. Components of an Expanded Toulmin Diagram (adapted from Conner, 2008).

Toulmin's (1958/2003) model for argumentation provided both a clear structure of an argument and a framework that enabled me to analyze the component parts of an argument and the form and roles of these elements. Investigating the processes of argumentation that occur in a mathematic classroom through Toulmin's lens does the following: First, Toulmin's model helps me to identify the elements of argument (e.g., data, warrant, claim, rebuttal) and their linkages. Second, Toulmin's model considers arguments in terms of structure, which enables me to closely examine the claim, warrant, and data, and provides me with a tool for accessing the ways of reasoning and sense-making that are being promoted. Furthermore, using "Teacher Support" to denote teachers' contributions and actions that prompt or respond to parts of arguments allows me to identify the choices that teachers are making as they support collective argumentation in mathematics classrooms.

While Toulmin's (1959/2003) model provides insight into the basic structure of an argument, in practice many researchers have observed complex arguments with components serving multiple purposes. For example, as shown in Figure 3.3, the blue box in the process of argumentation linking a green box (data) and a purple box is labelled as data/claim. It serves as a claim (in previous argument) and a data (in subsequent argument). In addition, we can notice a structural difference between the two Toulmin diagrams in Figure 3.3. The top diagram consists of a sequence of chain reasoning leading to one final claim, I call this “chain” structure of an argument. In contrast, the bottom diagram consists of two independent chains of reasoning that are parallel to each other and result in more than one final claim, I call this “parallel” structure of an argument.

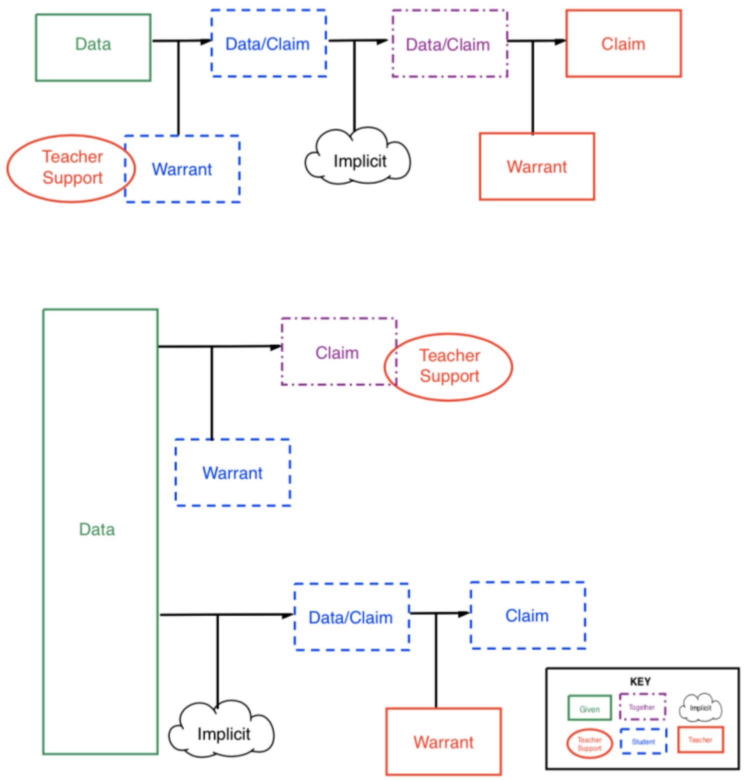

Figure 3.3. Examples of Chain Structure (top) and Parallel Structure (bottom) of Arguments.

3.8 Integration of Teacher Rational Questioning Framework and Toulmin’s Model as Analytic Tools

Following Boero et al. (2010), in this study, I considered the integration of Teacher Rational Questioning Framework built from Habermas’ (1998) construct of rational behavior with Toulmin’s (1958/2003) model for argumentation to be a more powerful analytic tool to investigate teacher questioning strategies for promoting collective argumentation. The relationship of the two analytic lenses is complementary. Habermas' construct connects the individual and the social by taking into account intentions and consciousness inherent in argumentation, but it is unable to capture the functional characteristics of products (in particular, the linkages among argument components in the process to construct final claims) and compare different structures of products (arguments). Conversely, Toulmin's model is suitable to analyze the fundamental structure of an argumentation. This integration of two constructs provides a more comprehensive tool to analyze how teachers' questioning is constrained in relation to the three components of rational behavior (Habermas' lens), and at the same time, frame the role of the teacher questioning to control the process and product of argumentation.

In summary, for this study, I employed these integrated frameworks to investigate the processes of collective argumentation that occur in mathematics classrooms and to examine how teachers used their (sequences of) rational questioning to guide the development of mathematical arguments (i.e., data, warrant, claim, rebuttal). I also focused on rational questioning within argumentative discourse that did not prompt students to construct specific components of arguments but created a context for stimulating students' gradual appropriation of the rationality requirements of collaborative argumentation as a long-term teaching intervention.

CHAPTER 4

METHODOLOGY

This chapter details the background information of this study, including research context in terms of setting and participants. A description of data analysis procedures is presented. Finally, attention is given to the validity of the qualitative nature of this study.

4.1 Background

This study was based on a 6-year longitudinal study called *Learning to Support Productive Collective Argumentation in Secondary Mathematics Classes*, focused on understanding how teachers learn to facilitate collective argumentation. The project had three phases. In the first phase, we examined a cohort of 14 secondary prospective teachers initial and changing beliefs about argumentation and what they learned about supporting argumentation from courses during their three semesters of their teacher education program. In the second phase, after the completion of coursework, we followed 6 of the 14 participants into their student teaching experiences to understand how prospective teachers supported argumentation in practice. In the third phase of the project, based on our initial analysis of their argumentative practices during student teaching, we purposefully followed 2 prospective teachers (the participants of this study) into their first 3 years of teaching to investigate how they implemented the ideas and strategies they learned in mathematics education coursework with respect to engaging students in creating and critiquing mathematical arguments (one of the eight standard mathematical practices in the Common Core State Standards for Mathematics). During the third phase, we also engaged in individualized professional development (PD).

One of the main research purposes of the third phase of this project was to investigate how teachers' ways of supporting collective argumentation changed while participating in professional development activities with respect to collective argumentation. Each teacher was video recorded during two to three consecutive days of instruction that were chosen by the teacher in collaboration with the researcher in every month of the school year, which translated into three to four times per semester. After each set of consecutive days of instruction, the research team had a semi-structured video recorded post-lesson interview with each teacher to explore the teacher's goals with respect to collective argumentation and what background factors influenced the teacher's support of collective argumentation in the classes. During the interview, the researchers also asked teachers to watch their video clips of argumentation from their classes and analyze their supportive actions for collective argumentation by introducing them to the adapted version of Toulmin's mode for argumentation (see Figure 3.2) and Teacher Support for Collective Argumentation Framework (For more details, see Conner et al., 2014).

As a research assistant, I worked on various aspects of the project. I assisted with the data collection for phase three, which included video-recording classroom observations and interviews. I worked together with our research team in creating protocols for interviews, choosing and transcribing classroom observations and interviews for analysis. Finally, I assisted in data analysis of the classroom observation transcripts and interview transcripts, such as identifying and diagramming episodes of argumentation in order to explore the structure of the individual arguments and to look for patterns in the ways teachers supported collective argumentation during classroom discussions.

4.2 Participants and Settings

The data for this study were collected as part of the third phase of the described project. These data provided a rich context for investigating how beginning secondary mathematics teachers implemented questioning strategies to support students' argumentative activities. The two beginning secondary mathematics teachers, Jill and Susan (pseudonyms), were appropriate participants for this study based on the following reasons: first, both teachers have a good understanding of mathematical argumentation because they learned about supporting argumentation from courses during their teacher education program as well as during PD. The PD involved intentional reflection on teaching through use of stimulated recall interviews (i.e., post-lesson interviews as discussed above) coupled with strategic analysis of argumentation in their classrooms and their own support for argumentation. In addition, I intentionally chose to examine their instruction during their last year of participation in the project. Their strong backgrounds of argumentation created the conditions for this study to determine the questioning strategies used by the teachers while orchestrating argumentative discourse. Second, the two teachers taught in two different schools with students from different grade level and backgrounds. Thus, this study also provides some insights into how teachers use questioning strategies to support student engagement in collective argumentation in different classroom contexts. In the following section, I provide additional information about each participant and her school context.

4.2.1 Case 1: Jill and School A

Jill was a white female and graduated with a bachelor's degree from a mathematics teacher education program at a large research university in the southeastern United States. Jill was in her third year of teaching and in her first year at her current school. She co-taught a 9th grade algebra 1 class with a special education teacher. She has a larger class size (30 students) this year than she had before. Most of the students in her class were identified as having specific learning disabilities in math and needed testing accommodations. From the first post-lesson interview in year 3, Jill thought co-teaching would help her to do more individualized activities with students and thus get the whole class engaged. In this semester, she also co-planned lessons and made quizzes and tests together with other math teachers in school. She enjoyed the co-planning and commented that “It’s nice because we have freedom within our data team, but we’re not completely alone either. So, we plan together, and we all use each other’s things but sometimes I will alter something or somebody else will alter something. But it’s very nice to have help.” (Post-lesson interview 1 in Year 3). This was a departure from the culture of her previous school. Jill was not a fan of traditional teacher-centered instruction and wanted more student-to-student interactions in her class. She liked for students to be able to work together in groups to understand the concepts and come up with the solutions to problems independently.

School A where Jill taught had a population of about 1345 students in grades 9-12 and had 12 teachers in the mathematics department. Mathematics courses were available in college preparatory algebra and trigonometry, ranging from Algebra 1 to Pre-calculus. The student demographic for this school was approximately 77% white, 9% African American, and 9% Latinx; and 54% of students were eligible for free or reduced-price lunches.

4.2.2 Case 2: Susan and School B

Susan was a white female and graduated with a master’s degree from the same mathematics teacher education program as Jill. Susan was in her second year of teaching and in her first year at her current school, where she also completed her student teaching experience. This semester Susan was teaching an accelerated Algebra II/Geometry course, and it was the first time Susan taught accelerated students. The students in her class were half gifted 9th graders and half 10th graders who wanted to do well but did not hold the gifted label. The class would spend four months on Algebra II and then three weeks on Geometry. Susan usually planned lessons by herself but would ask help from her colleagues if she had trouble with understanding content about which she commented during her first post-lesson interview of the school year, “I think I kind of like being able to be more free with what I plan, so that’s not really that hard.” In her teaching, Susan liked to provide opportunities for students to interact with each other and preferred a class structure in which students were engaged and working though problems in pairs or groups. Susan believed that students were capable of building knowledge themselves by using their past knowledge and skills. She stated:

> I am astounded by the fact that I can give them something and tell them, this might be a little bit difficult, but I think you can do it and I think you can work with your partner and kind of figure these things out for your own and then they’ll just actually do what I ask them to do. (Post-lesson interview 1 in Year 2)

School B, where Susan taught, had a population of about 1186 students in grades 9-12 and had 12 teachers in the mathematics department. Mathematics courses were available for support, regular, and accelerated students, ranging from foundations of algebra to advanced college-level calculus. The student demographic for this school was approximately 80% white, 8% Latinx, and 7% Asian; 18% of students were eligible for free or reduced-price lunches.

4.3 Data Collection

Four data sources were mainly used for this study: (1) video recordings of classroom observations, (2) transcripts of classroom observations, (3) supplemented field notes and students’ written work, (4) post-lesson interviews with transcripts.

4.3.1 Video Recordings and Transcripts of Classroom Observations

The video recordings of classroom observation and their transcripts were chosen to serve as the primary data source for this study due to the fact that these allow for an in-depth analysis of the kinds of questions and statements teachers used during the classroom discussions as well as interactions between students. To obtain a realistic picture of interactions between teacher and students, which can be described as having a practice-oriented purpose in qualitative investigation (Haverkamp &Young, 2007), I decided to observe as they take place in a natural setting rather than setting up an experimental situation in which the research design is likely to influence many aspects of the behavior involved. Although pure objective observation was nearly impossible to achieve in practice (Angrosino, 2005), my goal was to minimize any possible observer influences on the behaviour of individuals during my observation. Therefore, considering the camera effect, the first classroom observation was not analyzed. Further, considering the effect of compressed teaching schedules at the end of the school year that may have influenced the teachers' regular teaching style, the last classroom observation was not analyzed. Thus, four lessons (two sets of two consecutive days of lessons) of video recordings and their transcripts for each teacher were chosen to serve as the main data source being analyzed for this study. The data consisted of approximately 209 minutes of instruction (two long class periods and two short class periods) in total for Jill and approximately 343 minutes of instruction (four long class periods) in total for Susan.

Each lesson was videotaped using a stationary camera placed in the back corner of the room to capture teacher and students as much as possible. Because the focus of this study was teacher questioning, the main focus of the camera was the teacher. But the video recording also included students' voices to explore effects of the teacher questioning and to pinpoint the sequences of teacher questioning that directly followed students' comments within an argumentation episode. Because the study requires a fine-grained analysis of the teacher's and students' verbal utterances, transcriptions were made carefully including the time stamps for each utterance and the corresponding observation notes of non-verbal behaviors and communication. The transcripts in conjunction with video recordings helped me gain a wider perspective of the interactions within argumentative discourse.

The examination of videotaped records of lessons has become a widespread tool in research on teaching. It enables researchers to focus on microanalysis of the data to better understand various aspects of mathematics teaching and learning. For the current study, video data were particularly useful as they allowed me to identify instances of questions teachers asked as they engaged students in collective argumentation and the resulting responses, arguments, and reasoning displayed by students were captured by video.

Further, the screenshots served as an important supportive data source to capture the work of teachers (e.g., the task that was posed) and the work of students (e.g., written artifacts and representations). Videotaped records have thus served various objectives (e.g., teacher questioning, student responses) in this study and enabled me to repeat viewing of lessons to enhance data analysis and effectively meet the purpose of analysis.

4.3.2 Field Notes and Students' Written Work

As non-participant observers, the research team stayed in the back of the room controlling camera and volume when necessary and made field notes to capture information that could be missed in video-recording but might be important for understanding episodes of classroom argumentation. The field notes include the tasks or worksheets assigned to the students, descriptions of the teachers' supportive actions during argumentative practices as well as records of students' behaviors and written work (e.g., students demonstrated arguments or warrants in written forms). The field notes and students' written work as additional data sources helped me to learn more about the collective argumentation under analysis and enhanced the ability to examine the range of interactions by getting a closer look at student representations that may not be captured in the video-recordings.

4.3.3 Post-lesson Interviews

The post-lesson interviews served as a supplemental source for me to understand teachers' intentions in asking a specific question at-the-moment. Although the main purpose of post-lesson interviews was not examining teachers' intentions of asking questions, there were instances in which teachers analyzed their supportive actions for argumentation and included a rationale for their use of specific questions or a general kind of question. Table 4.1 illustrates each data type, sources, items collected and timeline of each collection. All data collection took place in Spring 2017.

Table 4.1

Data Types, Sources, Items Collected and Timeline of Collection

Data Type	Data Source	Items Collected
Videos	Video of the lessons (focus on the teacher)	4 lessons from Jill and 4 lessons from Susan
Field Notes and Students' written work	Journal from classroom observations and collection of students' written work.	Observation notes and pictures or screenshot of students' work.
Interviews	Post-lesson interviews after each unit of instruction.	2 interviews from Jill and 2 interviews from Susan

4.4 Data Analysis Procedure

As an exploratory case study, the goal of my data analysis was to investigate how teachers use questions to support students' engagement in collective argumentation. This case study was designed to gain an understanding of teacher questioning by focusing on mathematical discussions in which the teacher and students engaged in dialogue about mathematical concepts, problems, or procedures within argumentative discourse. Thus, I excluded other types of work by the teacher, such as managing behavior or collecting homework.

In this study, I drew from Cotton's (2001) definition of questions as "any sentence which has an interrogative form or function" (p. 1); in classroom settings, the teachers' questions are defined as "instructional cues or stimuli that convey to students the content elements to be learned and directions for what they are to do and how they are to do it" (p. 1). To capture teachers' moment-by-moment questions within argumentative practices, each question asked by the teacher was identified. Considering the importance of context in interpreting teacher moves (Jacobs & Spangler, 2017), an effort was made to use not only the verbal content of discourse but also non-verbal cues and other visual indicators as well as post-lesson interviews (if applicable) when interpreting types of questions the teacher was asking.

Because one of the goals of this study was to see how teachers used their question strategies to orchestrate argumentative discourse, I used Toulmin's (1958/2003) model to diagram every episode of argumentation and noted the specific component of an argument (i.e., claims, warrants, data) that each question prompted. In addition, an enumerative analysis approach was employed to reduce the subjectivity of qualitative interpretation and to quantify the numbers of rational question so as to uncover the pattern and themes that emerged from coding.

4.4.1 Identifying and Diagramming Argumentation Episodes

At the preliminary stage, I watched each video lesson multiple times and went through supplemental field notes and students' written work to become familiar with each full lesson. Then, at the first stage, each lesson was divided into multiple argumentation episodes. An argumentation episode was located by identifying the final claim of an argument and the accompanying data, warrants, and data/claims supporting the final claim the collective attempted to establish. Therefore, if there were arguments or claims that supported or refuted the initial argument, these arguments were viewed as connected with each other and included in an episode of argumentation.

As an illustration, let us consider an argumentation episode from Jill's class: during this episode, the students were learning about factoring trinomials with integer coefficients in a small group:

Given $x^2 + \_\_\_ x + 12$*, what are the possible values for the blank?*

1. Jill: **All right, what do we think?** (*Prompted claim 1*)
2. S1: It's six or nine.
3. Jill: Six or nine.
4. S2: Yup.
5. Jill: **Tell me why.** (*Prompted warrant/data 1*)
6. S1: Tell her why S2.
7. S2: Why do I have to tell her. Oh. Um, okay, so couldn't, couldn't like...
8. Jill: **Hang on. I want to hear 6 or 9 explanations first.**
9. S2: Oh gosh. Could you say the 9 explanation and I say 6 explanation?
10. Jill: **Tell me the 6 explanation.** (*Prompted warrant/data 1*)
11. S2: Okay. So, 6 times 2 is 12.
12. Jill: Yes, 6 times 2 is 12. That's true.
13. S2: Yeah, and then 6 might not work, 6 wouldn't work.
14. Jill: **Why not? Talk to me about why 6 might not work.** (*Prompted warrant/data 2*)
15. S2: Because 6 plus 2 is 8 and you have to have 12 and so because [mumbling]
16. Jill: **Hang on, hang on. You are saying things that are on the very right track. Think through it.** (*Prompted warrant/data 2*)
17. S2: Okay. 6 plus 2 is 8 but yeah do not even know where 12 like, how are you supposed to like, do you know what I am saying it's like
18. S1: You can put 8 in here. That's the point.

19. S2: Yeah.
20. S1: So we are trying to find the line, what goes on the line up here.
21. S2: Yeah.
22. S1: You can put 8 but 6 times 2 is 12 and then 6 plus 2 is 8.
23. S2: So it's 8.
24. S3: Yeah.
25. Jill: Yes. You are thinking about it in the right way. You said 8. That's okay. That's why I want you to think about it. **Now does that make sense?** (*Responded warrant 1*)
26. S2: Yeah.
27. Jill: **So does anyone come up with another number besides 8 that could go there? Anybody come up...S4, why could you do 7?** (*Prompted claim 3*)
28. S4: Oh gosh.
29. Jill: **You said you could do it. Why?** (*Prompted warrant 2*)
30. S4: 4 plus 3 is 7 and 4 times 3 is 12.
31. Jill: Very good. **Is there anything else?**

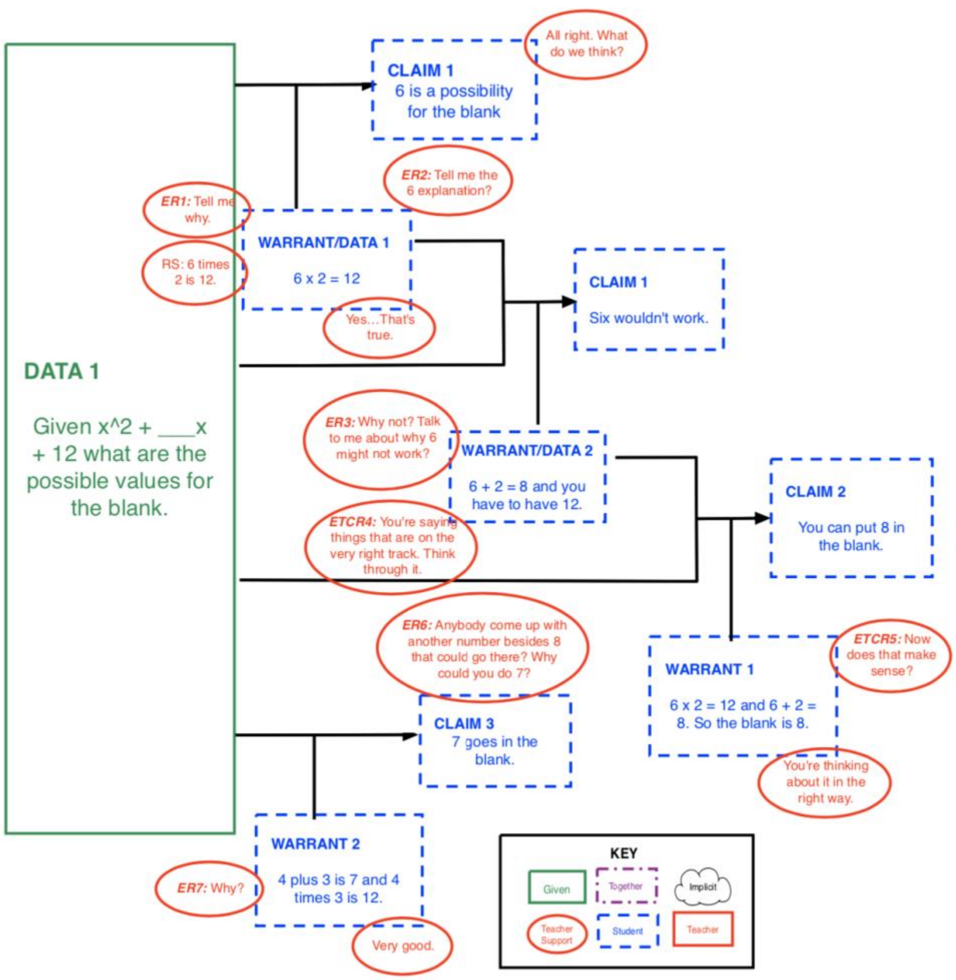

Figure 4.1. Diagram of Unit 1, 2nd Lesson, Argument 5, Small Group Discussion in Jill's Class.

After an argumentation episode was located, the adapted version of Toulmin's model (see Figure 3.2) was applied to diagram the episode of argumentation in order to help me to identify the parts of an argument that each question prompted. As shown in Figure 4.1, some questions generally prompted claims (Lines 1 and 27); some questions served to support students constructing data/warrants (Lines 5,10,14,16) or responded to warrants (Line 25); while other questions may not prompt or respond to any specific argument component in the moment (Lines 8 and 31). The expanded Toulmin (1958/2003) diagram allowed me to analyze the structure of argumentation and capture how the teacher use moment-by-moment questions to prompt or respond to different parts of an argument so as to achieve goal in the argumentation activity.

4.4.2 Identifying and Categorizing Rational Questioning

The next step was to analyze all teacher questions based on the Teacher Rational Questioning Framework (see Table 3.1). Each question was categorized as either having zero, one, two, or three components of rationality. An example of coding teacher questions for the above argumentation episode is presented in Table 4.2.

Table 4.2

Coding Teacher Questions for Unit 1, 2nd Lesson, Argument 5, Small Group Discussion in Jill's Class

Teacher Questions	Components of Rationality
What do we think? (Line 1)	Non-rational question (*N*)
Tell me why. (Line 5)	Epistemic rationality (*ER*)
Hang on. I want to hear 6 or 9 explanations first. (Line 8)	Non-rational question (*N*)
Tell me the 6 explanation. (Line 10)	Epistemic rationality (*ER*)
Why not? Talk to me about why 6 might not work. (Line 14)	Epistemic rationality (*ER*)
Hang on, hang on. You are saying things that are on the very right track. Think through it. (Line 16)	All three components of rationality (*ETCR*)
Now does that make sense? (Line 25)	All three components of rationality (*ETCR*)
Anybody come up...S4, why could you do 7?) (Line 28)	Epistemic rationality (*ER*)

You said you could do it. Why? (Line 30)	Epistemic rationality (*ER*)
Is there anything else? (Line 32)	Teleological Rationality (*TR*)

4.4.3 Identifying Levels of Truth in Argumentation Episode

The final claims of the above example were 8 and 7 could fill in the blank to enable factorization while 6 cannot. Notice that the answers provided here do not include all possible values, which was different from those argumentation episodes in which the class arrived at completely mathematically correct answers. Stylianides (2007) contended that a valid proof in a given classroom community should be acceptable from the field of mathematics. However, no consensus exists in the field about the characteristics of a quality or valid classroom-based collective argumentation. While we may agree that argumentation is critical to teaching, the lack of an analytic framework to address and assess the complexity of productive argumentation suggests that there is a lack of consensus in how argumentation may be taught and assessed. This concern of emphasizing quality of argumentation also gave rise to identifying conditions of potential "truth" of argumentation in classroom contexts. Without utilizing a pre-existing coding system, I followed the constant comparative method (Glaser & Strauss, 1967; Strauss & Corbin, 1990) through an interactive process of reviewing two teachers' argumentation episodes. I explored different levels of truth in argumentation by focusing on the final products of argumentation as represented by diagrams of arguments. These diagrams were used to explore the structure of argumentation for different truth levels. This resulted in my second research question to investigate what techniques (i.e., use of rational questioning) and structures of arguments teachers use when supporting different levels of truth in argumentation, described in Chapter 6.

4.5 Validity

To increase credibility, triangulation of data from multiple sources (video, transcript, field notes, student work, lesson interviews, etc.) was used to support the teacher questions I coded and level of truth argumentation I found. Part of the transcripts and diagrams of the argumentation episodes were verified by two research assistants on the research team. During the coding stage as well, data were coded by me first and shared with my advisor to help evaluate the process to confirm dependability. Any differences in coding were discussed until a consensus was reached regarding the most fitting code to be applied. In addition, considering the descriptive nature of this research, diagrams of argumentation episodes and codes of teacher questions are provided with detailed transcripts.

CHAPTER 5

TEACHERS' USE OF RATIONAL QUESTIONING

In this chapter, I will address my first research question:

RQ1. How do beginning secondary teachers use rational questioning when guiding collective mathematical argumentation?

(a). Which component(s) of rationality is/are privileged in class?

(b). What combinations of rational questions support different components of arguments in a mathematics classroom?

The discussion of the findings is organized into two parts. The first section (**Section 5.1**) provides an overview of argumentation episodes for each participant. The second section (**Section 5.2**) provides answers for research question one. For this part, the unit of analysis was the unit of instruction, so there were four units analyzed. Each unit addressed a different mathematical topic; they were analyzed separately to account for potential differences across topics. Findings are organized by unit of instruction.

5.1 Overview of Argumentation Episodes

The discussion of the occurrence of argumentation episodes and arguments per unit (i.e. two consecutive days of lessons) provides insight on when teachers saw opportunities to engage students in argumentative practices. Sometimes an argumentation episode included multiple arguments. Table 5.1 summarizes the frequency of argumentation episodes and arguments for the two focus participants in the study.

Table 5.1

Frequencies of Argumentation Episodes and Arguments

Teacher	Days of class	# of Argumentation episodes	# of Arguments
Jill Unit 1	2 (90 min)	23	24
Jill Unit 2	2 (118 min)	25	26
Susan Unit 1	2 (173 min)	39	43
Susan Unit 2	2 (164 min)	25	27
Total	**8 (552 min)**	**112**	**120**

As shown in Table 5.1, the total number of argumentation episodes and arguments were different for each participant teacher. This difference seems to be partly influenced by the amount of time for lessons, the content and goal of lessons, the length of individual argumentation episodes. Moreover, the number of arguments that occurred in whole class discussion and small group discussion were different between the two teachers (see Table 5.2).

Table 5.2

Numbers of Whole Class Arguments and Small Group Arguments

Teacher	# of Whole Class Arguments	# of Small Group Arguments
Jill Unit 1	15	9
Jill Unit 2	8	18
Susan Unit 1	16	27
Susan Unit 2	10	17
Total	**49**	**71**

In Jill's classroom, the amount of time spent on whole class discussions and small group discussions supporting argumentation were different across units (unit 1 more whole class arguments while unit 2 more small group arguments). Most of the whole class argumentation was directed by Jill and tended to be quite long because many of the students' responses in her class were short or incomplete and required multiple follow up questions to elicit a complete argument. In a few episodes of whole class argumentation, Jill invited students to construct two opposing arguments or multiple independent arguments and then requested that students provide justifications for why their arguments hold. In small group discussions, Jill sometimes left the conversation unresolved for students to continue work on the problem.

In the case of Susan, a fairly high number of episodes of argumentations occurred in small group discussions. Susan's class was an accelerated class and most of the students in her class appeared to be able to express their ideas and thinking clearly. In Susan's classroom, most of the time was spent on small group discussions supporting arguments. However, most of the small group argumentation occurring in her class in the teacher's presence tended to be quite short unless students were struggling in problem-solving. Due to the design of this study, I did not analyze small group discussions in the absence of the teacher. In a whole class discussion, Susan always invited students to come to the board to present their arguments. In Susan's classroom, arguments appeared more frequently in Unit 1 than Unit 2. Especially when students engaged in the Pentagon Task (Unit 2, 1st lesson), Susan spent a large amount of time in clarifying the goals and meaning of the task; thus, few arguments were observed on that day. In addition, when Susan worked with students in small group discussions, she sometimes embedded multiple tasks to make sure students were on the right track for problem-solving. In these cases, multiple arguments were presented in an argumentation episode.

5.2 Use of Rational Questioning to Support Collective Argumentation

Within each unit of instruction, episodes of argumentation were identified. After an argumentation episode was located, the adapted version of Toulmin's model (Conner, 2008) was applied to diagram every episode of argumentation to help me to identify the parts of an argument that each question prompted and how teachers used their (sequences of) questions to control the fundamental steps of argumentative practice. The next step was to analyze all teacher questions within each chosen argumentation episode to identify and categorize rational questioning based on my definition of *rational questioning*. Each question was categorized as having one, two, three or no components of rationality based on the interaction observed. Attention was given to identifying fine grained rationality components of teachers' questioning and also how teachers' questioning was constrained in relation to the three components of rational behavior through Habermas' lens; the analysis then examined how the teacher used rational questioning to manage the components of argumentation as seen through Toulmin's lens. As a result, narratives for each unit were constructed with attention to how each participant teacher used rational questioning when guiding collective mathematical argumentation. Screenshots of teachers' instructional tasks and samples of student work are displayed along with the narratives to provide details as needed.

5.2.1 Jill Unit 1 (170126 and 170127) — "Factorization"

Jill's first unit of instruction focused on factoring and expanding binomials with integer coefficients. Day 1 began with Jill guiding students to review how to find greatest common factors and how to multiply two binomials together and get the product. Jill then introduced "area model methods" to factor binomials (see Figure 5.1 as an example).

<table>
<tr><th>Factored Form</th><th>Area Model</th><th>Final Product</th></tr>
<tr><td>11.</td><td><table><tr><td>x^2</td><td>_x</td></tr><tr><td>_x</td><td>18</td></tr></table></td><td>$x^2+9x+18$</td></tr>
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Figure 5.1. An Example of Task in Jill Unit 1.

Jill's Unit 1 contained 23 argumentation episodes. Within argumentation episodes, Jill asked 148 questions, and 89% (131/148) of questions involved a rational component. Among these rational questions, the largest component was teleological rationality (67%, 88/131), followed by epistemic rationality (41%, 54/131); the portion that involved communicative rationality was smallest (24%, 32/131).

The content of Unit 1 included multiple problem-solving mathematical activities to teach students how to factor and expand binomials with integer coefficients (see Figure 5.1), and teleological (i.e., producing strategies to achieve the aim of the activity) rationality was the most common rationality component among all rational questioning. Most of Jill's teleological rational questions were strategically goal-oriented to support students with respect to filling in an area model (e.g., "*Alright now I have the inside of my area model filled out. How do I get the outside*?") and finding the greatest common factor in each row and column of the area model so as to solve the problem (e.g., "*What is the greatest common factor of the bottom row*?"). Jill also used some teleological rational questioning to encourage other students to join the discussion: "*Okay at this point we have two empty boxes. Somebody else, I want you to tell me how we find what goes into those two empty boxes that we have*." By continuing to ask other students to respond to particular students' answers, Jill developed norms that every student in the class was expected to pay attention to what other students say and be ready to share solutions. On a few occasions, Jill wanted students to use precise mathematics language to communicate their ideas and communicated this by asking them to be more specific about their solutions.

Epistemic rationality followed as the second most common component of rational questioning. Jill used most of her epistemic rational questioning to encourage her students to justify why their arguments hold (e.g., "*You are correct; it's not three, but why*?") or challenge her students to provide reasons for their arguments, especially when they gave incorrect answers (e.g., "*Why it is not 2x*?").

In this unit, Jill's goal for students was apparently less focused on communicative rationality than on teleological and epistemic rationality. Most of the communicative rational questions served to introduce a graphic representation (the area model) to help students to reason and contribute correct answers. For example, she asked, "*If this is an area model, what could I call this [points at length] and what could I call this* [*points at width*]?" Sometimes Jill asked students to rewrite a mathematical expression so that they could easily find the greatest common factor (e.g., "x_4*, how do I rewrite this one*?"). Occasionally, Jill wanted to highlight her expectations for students to use correct mathematical representations and ensure their representations could be understood in the given classroom community. An example of this type of question would be as follows: "*Have I actually finished...I need to write it in the factor form. So, tell me what to write*."

The results indicated that Jill used a variety of combined forms of rational questioning: some questions included two components (i.e., epistemic and teleological rationality components labeled as *ETR*) or three components of rationality (i.e., labeled as *ETCR*) and others only involved one component. Table 5.3 provided the numbers and percentages of rational questions and how they were associated with specific components of arguments (i.e., claims, warrants, rebuttals) as seen through an adapted version of Toulmin's model (Conner, 2008).

In addition, I noticed that often a single rational question was not enough to lead students to construct a completely satisfactory argument component from the teachers' perspective. Therefore, if a rational question was used to push students to continue to construct an argument component, for example a warrant, that students had trouble constructing or had not completed from the teachers' perspective in the beginning, I labeled this as *warrant**. For instance, in the following interactions, the reasoning S2 provided in Line 4 was labeled as *warrant** which was a supplement for her initial incomplete contribution of warrant (Line 2).

1. Jill: Why not? Talk to me about why 6 might not work.
2. S2: **Because 6 plus 2 is 8 and you have to have 12 and so because [mumbling]**
3. Jill: Hang on, hang on. You are saying things that are on the very right track. Think through it.
4. S2: **Okay. 6 plus 2 is 8 but yeah do not even know where 12 like, how are you supposed to like, do you know what I am saying it's like**.

Table 5.3

Numbers and Percentages of Combinations of Components of Rational Questioning Supporting Collective Argumentation in Jill Unit 1

Types	Numbers (Percent out of 131)	Components of Arguments Associated
Epistemic (*E*)	36 (22%)	Claims (1), Claims*(1) Data/Warrants (2) Rebuttals (1) Rebuttal/Data (1) Warrants (25), Warrants*(5)
Teleological (*T*)	59 (46%)	Claims (8), Claims*(3) Data (1) Data/Claims (18), Data/Claims*(13) Rebuttals (1) Rebuttal/Data (1), Rebuttal/Data* (1) Warrants (7), Warrants*(5)
Communicative (*C*)	3(3%)	Claims (2), Claims*(1)
ET	4(2%)	Warrants (4)
EC	4(9%)	Claims (1) Rebuttals (1), Rebuttals* (1) Warrants (1)
TC	15(9%)	Claims (4) Data/Claims (10), Data/Claims*(1)
ETC	10(9%)	Claims (1), Claims*(1) Data/Claims*(1) Data/Warrants (1) Rebuttal/Data (1) Warrants (2)

According to the expanded Toulmin diagrams, 250 argument components (except original data which are given by the teacher drawn as green boxes in diagrams) were contributed in Jill's Unit 1. Among them, 74% (186 out of 250) were contributed either only by students or by students jointly with the help of teacher. As shown in Table 5.3, 71% of Jill's rational questions in this unit consisted of a single component of rationality. A single component of epistemic rational questioning was mainly used to prompt warrants (32 out of 36) with questions like "*Why*?" and "*Why not four*?" As shown in diagrams, these questions may serve to prompt students' initial reasoning, but often students' initial responses were incomplete, incorrect or ambiguous, and they did not build a complete argument component to support their claims. Therefore, subsequent epistemic rational questioning, which served to clarify for the class the specific parts that the students had not made explicit in their original explanations, was necessary. For example, following a student's answer to an epistemic question, Jill might ask a question containing a combination of epistemic and communicative components, such as: "*So we just look at, what do you mean just look at the exponents*?" Jill used several combinations of components of epistemic rational questions to lead students to give a correct and complete explanation or construct rebuttals for their initial incorrect warrants (e.g., *ETR*, "*You are correct that it's not three, but why*?").

A single component of communicative rational questioning in Jill's Unit 1 was mainly focused on the correct use of mathematical terminology and representations (e.g., "*I need to write it in the factored form. So, tell me what to write*."). Jill used a single component of teleological rational questioning to support students contributing final claims or data/claims during the process of argumentation (43 out of 59, see Table 5.3). Additionally, these single teleological questions prompted several warrants when Jill requested students to describe how they would fill in the area model or find the greatest common factors. Combinations of teleological and communicative rational questioning were used to request students to describe steps as they filled in the area model. As an illustration, the following is a typical argumentation episode from Jill's Unit 1, 1st lesson, in which she used sequences of teleological rational questioning supplemented by epistemic and communicative rational questioning to guide students to participate in a problem-solving. It occurred at the end of the first lesson of Unit 1, in which the whole class worked on the task given in Figure 5.2.

Factored Form	Area Model	Final Product
		5x+15

Figure 5.2. Task for Unit 1, 1st Lesson, Argument 10, Whole Class Discussion in Jill's Class.

1 Jill: Okay, guys. We're about to come together now. Come on. We are looking at the computer now. S1 said she started by filling in the box. So she didn't do the outside, she started by filling in the box. **Okay, so what should I write in this box then?** (*TCR1:* Rational questioning contains teleological and communicative rational component).

2 S2: 5x [Jill wrote 5x on board].

3 Jill: **And then what should I write here?** (*TCR2*)

4 S2: 15.

5 Jill: [wrote +15 on board] Okay, **then, somebody raise you hand, what do we do next? Just one step.** (*TCR3*)

6 S1: Put 5 to the side of the box.

7 Jill: **Okay, how did you know to put 5 at the side of the box?** (*TCR4*)

8 S2: The other ones?

9 Jill: Okay, it follows the pattern of the other ones. **Why 5? Why not 3, or why not something else?** (*ER5*)

10 S2: Because any other number would be something different then what is inside the box.

11 Jill: Okay, **5 is the greatest common factor of 5x and 15, right?** (*N*)

12 S2: Okay.

13 Jill: **Right?** (*N*)

14 S2: Yeah.

15 Jill: **Then talk to me about how we filled this in?** [points above 5x] (*TCR6*)

16 S3: x.

18 Jill: **Okay, someone says x here. Why?** (*ER7*)

19 S3: Because 5 times x is 5x.

20 Jill: Here we go. So 5 times x is 5x (wrote x above 5x) and **then someone said what here?** (*CR8*)

21 S3: Plus 3.

22 S2: Plus 3. **And how do we know this is correct?** (*ER9*)

23 S4: Because 5 {inaudible}

24 Jill: 5 times 3 is 15. **Then how do we, then go here?** (*CR10*)
25 S2: 5 parentheses x+3 parentheses [Jill wrote 5(x+3) on board].
26 Jill: All right.

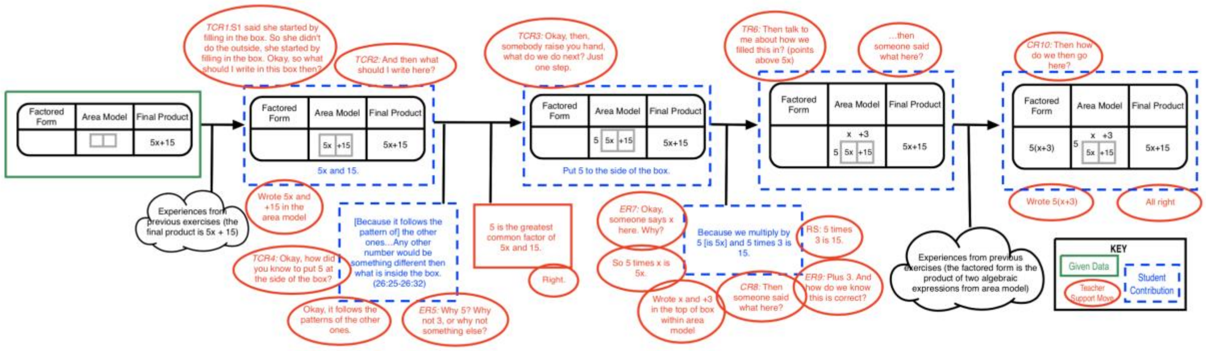


Figure 5.3. Diagram of Unit 1, 1st Lesson, Argument 10, Whole Class Discussion in Jill's Class.

As shown above, sequences of combinations of teleological and communicative rational questioning as well as epistemic rational questions were asked during this episode. Combinations of teleological and communicative rational questioning (e.g., "*And then what should I write here*?", Line 3) were used to encourage students to follow the communication rules during their process of solution; afterward, epistemic rational questions requested students to justify their steps of solution one by one (e.g., "*And how do we know this is correct*?", Line 22). Through Toulmin's (1958/2003) lens, we could see that Jill interchangeably applied combinations of teleological and communicative rational questioning as well as epistemic rational questioning (in this episode: *TCR(4)-ER-TCR-ER-CR-ER*) to control the chain structure of argumentation as students have to rely upon the previous arguments they constructed as they worked toward the final answer of the problem. In this way, Jill created contexts for students to engage in argumentation connected with epistemic, teleological, and communicative aspects of rational behavior.

5.2.2 Jill Unit 2 (170315 and 170316) — "Arithmetic and Geometric Sequences"

The goal of the Unit 2 was mainly aligned with the following standard from the *Georgia Standards for Excellence* (GSE):

> MGSE9-12.F.BF.2 Write arithmetic and geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect arithmetic sequences to linear functions and geometric sequences to exponential functions. (p. 9)

On the first day, students started with an open-ended warm up problem, discussing similarities and differences among patterns and how they could represent the patterns with a picture. Then, the class moved on to learn how to write an equation for general term of the pattern. In this unit, Jill drew special attention to directing students to explain the process of obtaining the equation.

Jill's Unit 2 contained 25 argumentation episodes. Within argumentation episodes, Jill asked 188 questions, and 93% (176/188) of questions involved at least one component of rationality. For rational questions associated with argumentation, the largest component was teleological rationality (70%, 123/176). However, in this unit, Jill used slightly more communicative rationality questioning (44%, 77/176) than epistemic rationality (42%, 74/176).

To help students make a connection between the pictured sequence and the term number, so that students could write a general formula for a geometric sequence and finally recognize geometric sequences as exponential functions, Jill often started with a sequence of teleological rational questions to lead students to arrange significant facts into meaningful patterns and then use these patterns to make conjectures. For instance, at the end of the second lesson of Unit 2, Jill posed a problem (see Figure 5.4): Write an equation to express the geometric pattern shown as Pattern 4.

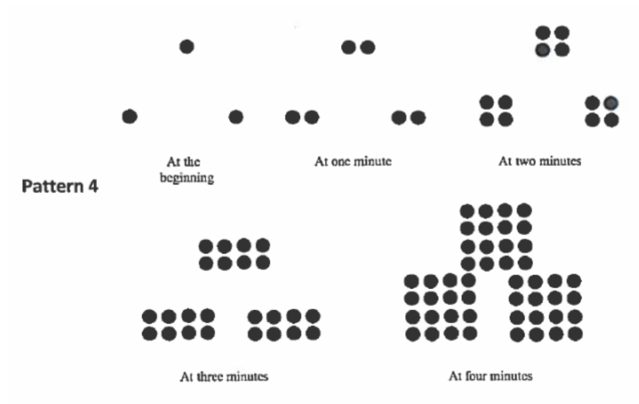


Figure 5.4. Task for Unit 2, 2nd Lesson, Argument 14, Whole Class Discussion in Jill's Class.

1 Jill: Okay, now, pattern 4. **How many are this in the (points at the 1st picture) at the beginning?** (*TR1:* Rational questioning contains teleological rational component).
2 S1: 3. [Jill writes 3 on the board]
3 Jill: **What about at one minute?** (*TR2*)
4 S2: 6. [Jill writes 6 on the board]
5 Jill: **What about at two minutes?** (*TR3*)
6 S2: Um...12. [Jill writes 12 on the board)
7 S3: 12, yeah, I think.
8 Jill: **What about at three minutes?** (*TR4*)
9 S3: 16...24.
10 S4: Multiplying by 2. [Jill writes 24 on the board]
11 Jill: **How many at four minutes?** (*TR5*)
12 S3: 48.
13 Jill: 48. [Jill writes 48 on the board]

...

A series of teleological rational questions (Lines 1, 3, 5, 8, and 11) Jill asked at the beginning of this argumentative discourse served to direct students to explore the number of dots at every minute, so the students could observe the pattern illustrated by the pictures. As illustrated by this episode, teleological rational questioning occupied a higher proportion of questions in this unit. On a few occasions, Jill wanted to introduce new concepts such as term numbers by using teleological rational questioning to strategically orient students to focus on a specific part of their activities (e.g., "*What was she plugging in*?"). In addition, several teleological rational questions were used to create time and space for students to reflect on the tools that they used to achieve their solutions (e.g., "*So, do we agree that this is the right equation? And do we see how this equation relates to this*?").

Jill put more emphasis on the use of communicative rational questions in this unit. The main reason was the content of this unit required Jill to insert terminology to enable students to use correct mathematical language to talk about the ideas under discussion. For example, Jill asked, "*And what have we been calling the term number? What have we been saying? What variable have we been using for the term number*?" Thus, Jill highlighted her expectations for students to use correct mathematical representations and ensure their representations can be understood in the given classroom community. Multiple communicative rational questions were used to follow up students' incomplete explanations. These questions served as a way to elicit or emphasize key words and phrases when students struggled to articulate or verbally express their ideas. As an illustration, see the following episode in which Jill engaged her students in discussion of similarities and differences between two patterns in a whole class discussion.

Pattern 1: 3, 4, 6, 9, 13, 18, ...

Pattern 2: 3, 7, 11, 15, 19, ...

1. Jill: Okay. So for the second one, you're just adding four. **Be more clear about what you were saying here. You're right.** (*CR1*: Rational questioning contains communicative component).
2. S6: Yeah. Adding the next number.
3. Jill: So she says from one term to the next [point at 3 and 4 in the first pattern], then from the next one to the next one [point at 4 and 6 in the first pattern], you are adding the next number. **So S5, you want to tell me what you said about that? I like what you wrote. What'd you write?** (*TCR2*: Rational questioning contains teleological and communicative component).
4. S5: The increase raises.

5. Jill: The increase raises. **So somebody else tell me what he means when he says the increase raises. Use different words than S4 used. A lot of you had this on your paper. I know you did. How can we say that in a different way? This is all correct.** (*TCR3*)
6. S3: [Inaudible] on every single topic.
7. Jill: **What is increasing by one though? Are these numbers increasing by one?** (*CR4*)
8. Multiple Students: No.
9. Jill: **What is increasing by one?** (*CR5*)
10. S3: The difference.
11. Jill: The difference is increasing by one.

This example showed that Jill used a sequence of communicative rational questions (Lines 1 to 9) to help students use precise mathematical language to communicate their ideas. By specifically asking students to express different interpretations, Jill opened the discussion to focus on the use of precise scientific terminology, keywords, and phrases to form integrated propositional statements.

Epistemic rational questions in this unit were mainly used to challenge students when their initial responses were incorrect (e.g., *Why is the equation not correct?*) or when students made changes to their initial answers (e.g., *Why did you change the 13 to a 12?*). In this way, Jill provided opportunities for students to not only uncover their incorrect strategies but also reflect on and reconsider their answers.

Table 5.4

Numbers and Percentages of Combinations of Components of Rational Question Supporting Collective Argumentation in Jill Unit 2

Types	Numbers (Percent out of 176)	Components of Arguments Associated
Epistemic (*E*)	32(18%)	Claims (1) Data (1) Data/Claims (3) Rebuttal/Data (1) Rebuttals/Claims (1) Warrants (22), Warrants*(3)
Teleological (*T*)	50(28%)	Claims (10), Claims*(2) Data (1) Data/Claims (15),Data/Claims*(11)

		Data/Warrants* (3) Warrants (5), Warrants*(3)
Communicative (*C*)	15(9%)	Claims (5), Claims*(3) Data (1), Data* (1), Data/Claims (1), Data/Claims*(1) Warrants (2), Warrants*(1)
ET	17(10%)	Data/Claims (1) Warrants (10), Warrants*(5) Warrants/Claims (1)
EC	6(3%)	Rebuttals (1) Warrants* (4) Warrants/Claims* (1)
TC	37(21%)	Claims (11), Claims*(6) Data* (1) Data/Claims (10), Data/Claims*(3) Warrants (3), Warrants*(2) Warrants/Claims* (1)
ETC	19(11%)	Claims (1), Claims*(1) Data/Claims (1), Data/Claims*(2) Data/Warrants (1) Warrants (6), Warrants* (2)

Through Toulmin's lens, I investigated how different types of rational questions supported components of arguments as summarized in Table 4. According to the expanded Toulmin diagrams, 261 argument components (except original data which are given by the teacher drawn as green boxes in diagrams) were constructed in Jill's Unit 2. Among them, 86% (225 out of 261) were contributed either only by students or by students jointly with the help of the teacher.

The use of combinations of components of rational questions in this unit served as an effective tool to follow up students' responses and foster students to view a problem from different perspectives or reflect on and reconsider answers if initial answers were inappropriate/incorrect. For example, in the following episodes Jill was working on pattern A with an individual student (see Figure 5.5).

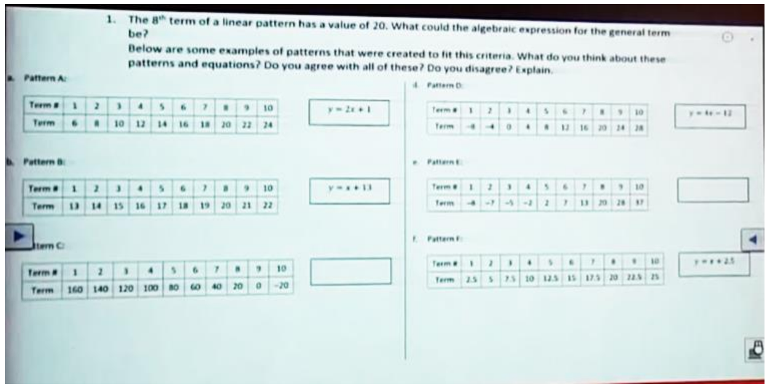


Figure 5.5. Task for Unit 2, 2nd Lesson, Argument 1, Small Group Discussion in Jill's Class.

1. Jill: Okay, so for pattern A, you made your equation y equals two x plus four, instead of what's written here which is y equals two x plus one. **Why did you change it?** (*ER1*: Rational questioning contains epistemic component).
2. S2: Because two times one is two plus four, six. Two times two is four plus four is, eight. Two times three is six plus four is, ten.
3. Jill: Okay, so you made this, and you plugged in one, and you got six. You plugged in two, and you got eight. You plugged in three, you get ten? **Okay. So, do you know of another reason why this is plus four instead of plus one? You are correct about what you just said, but do you know another reason? So what form is this?** [point at y=2x+1 on worksheet] (*ETR2*: Rational questioning contains epistemic and teleological component).
4. S2: Y slope interval [inaudible]
5. Jill: Okay, yes. **What does B represent, in any of one of these forms?** (*CR3*)
6. S2: Y-intercept.
7. Jill: Uh huh.
8. S2: That's what I did. That's how I figured it out.
9. Jill: Okay, **so, tell me how you figured it out.** (*ETR4*)
10. S2: I went like this [write down on paper] I went zero, then I went with four.
11. Jill: **Okay, so, explain this to me more clearly. What'd you get here?** [point at paper] (*ECR5*)
12. S2: [inaudible]
13. Jill: You do know, you just wrote it down. I know what you did. **But I want you to tell me what you did.** (*ECR6*)
14. S8: Because that's what x is one, then y is 6.
15. Jill: Okay, so when x is 1, y is 6. That is not the y intercept. **For the y intercept, x has to be?** (*TCR7*)
16. S2: Oh, I went back to find the Y-intercept.
17. Jill: Okay, so, this is term 1, but that's not the Y-intercept, you went backwards. **How did you get four?** (*TR8*)
18. S2: Minus two.
19. Jill: Okay. So, the pattern is plus two, you did minus two. Wonderful.

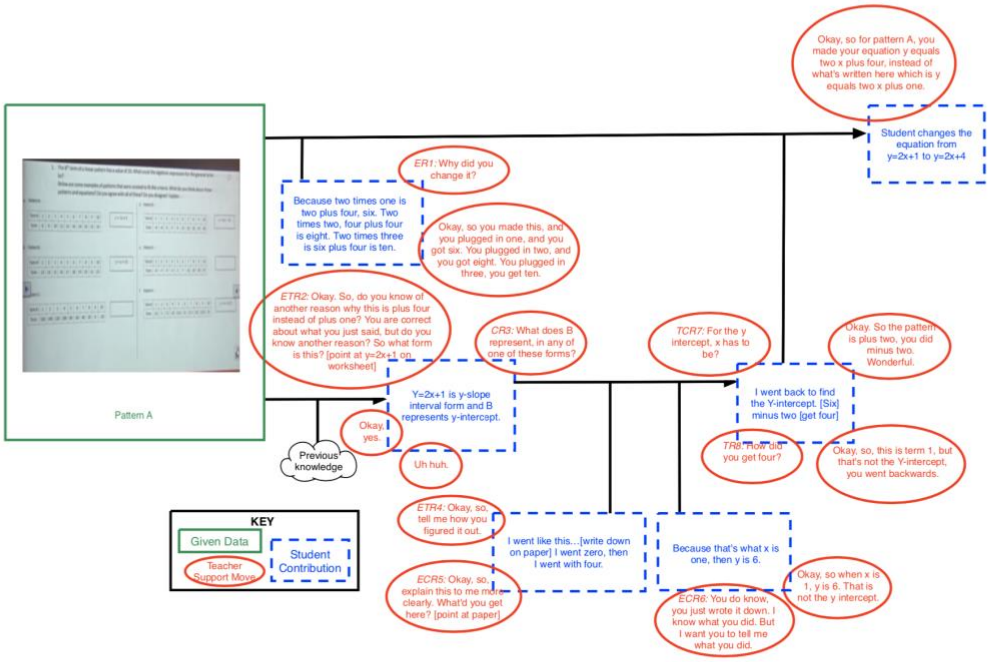


Figure 5.6. Diagram of Unit 2, 2nd Lesson, Argument 1, Small Group Discussion in Jill's Class.

In this example, Jill used a variety of combinations of components of rational questioning, such as combinations of epistemic and teleological rational questions and combinations of teleological and communicative rational questions. One way of using combinations of epistemic and teleological rational questions was to encourage students to provide additional warrants to support their claims (11 out of 17, see Table 5.4). In this context, Jill challenged S2 to look for more suitable tools to achieve their arguments more than simply checking the correctness of equation (Line 3). Thus, the teacher provided opportunities for students to view a problem from a different perspective. Sometimes, Jill also used this type of rational question to encourage students to reconstruct informal arguments they previously gave (Line 9), and thus result in a more appropriate student response (5 out of 17, see Table 5.4).

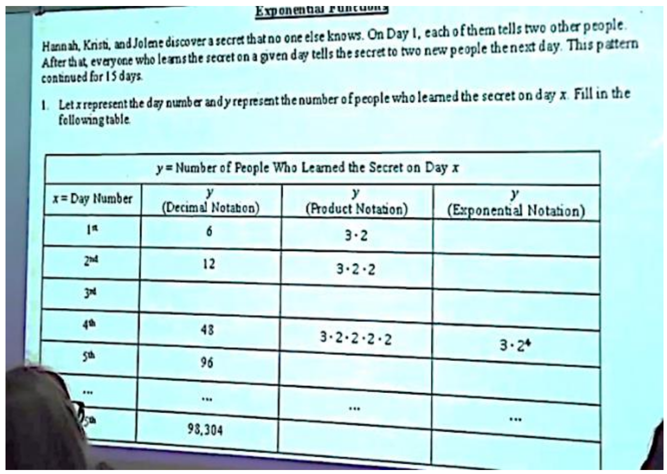
There were cases when students were unable to express their ideas in a clear way (Lines 10 to 12). When this happens, the teacher plays an important role in assisting students using language in accordance with the norms of the discipline (Temple & Doerr, 2012; Stylianides, 2019). Jill pushed S2 to use newly learned language to talk about newly learned content by asking combinations of epistemic and communicative rational questions (Lines 11 and 13). As shown in Table 5.4, most combinations of epistemic and communicative rational questions were associated with warrants that were initially not complete or were ambiguous; this served as a useful strategy to develop students' epistemic and communicative rationality as they expressed their mathematical thinking.

The goal of this unit was to connect geometric sequences to exponential functions. Jill asked multiple combinations of teleological and communicative rational questions (21%, 37 out of 176, see Table 5.4) to guide students to concentrate on pictures when investigating the patterns of a geometric sequence. For example, she asked, "*Look at the picture, don't look at that term number thing. Look at the picture. How can we get sixteen out of that?*" and "*But in your picture, I want to relate to the picture.*" These questions were particular useful when students were feeling stuck in noticing the changes in the pattern and supported students to articulate their new ideas (prompted both claims and *claims**).

Rational questions involving all three rationality components (*ETCR*) were mainly used to provide students with opportunities to make sense of other students' epistemic, teleological and communicative requirements of argumentative practices (e.g., *Wonderful. Questions*? and *All right. Are there any questions about what we just did? Yes?*). Most of these questions were associated with warrants (12 out of 19), which showed that in Jill's class students were expected to examine and evaluate the reasoning of others.

5.2.3 Susan Unit 1 (170227 and 170228) — "Exponential Functions"

The first unit of instruction observed in Susan's class focused on exponential functions. Because students in Susan's class had previously been introduced to exponential functions, Susan started the unit by engaging students in a series of tasks concentrated on interpreting exponential functions that arise in applications in terms of the context (see Figure 5.7 as an example). Then the class moved on to an activity to match given exponential equations with provided graphs. Susan spent half of the time on second day continuing the “matching activity” and then introduced partial and *n*-unit growth factor for exponential functions.



Exponential

Hannah, Kristi, and Jolene discover a secret that no one else knows. On Day 1, each of them tells two other people. After that, everyone who learns the secret on a given day tells the secret to two new people the next day. This pattern continued for 15 days.

1. Let *x* represent the day number and *y* represent the number of people who learned the secret on day *x*. Fill in the following table.

y = Number of People Who Learned the Secret on Day x			
x = Day Number	y (Decimal Notation)	y (Product Notation)	y (Exponential Notation)
1st	6	3·2	
2nd	12	3·2·2	
3rd			
4th	48	3·2·2·2·2	$3 \cdot 2^4$
5th	96		
...	...	...	...
5th	98,304		

Figure 5.7. An Example of Task in Susan Unit 1.

The total number of episodes of argumentation in Susan’s Unit 1 was 39. During Unit 1, Susan asked 213 questions when she orchestrated argumentative discourse. Among them, 89% (190/213) were categorized as rational questioning, in which 74% (140/190) contained a teleological rational component, followed by epistemic rationality (48%, 92/190); the portion that involved communicative rationality was smallest (26%, 50/190).

Most of Susan’ teleological rational questions were used to guide students to describe strategies to achieve the aim of the “match graph” activities like, “*So what’s your process for figuring out which graph goes with which one*?” or “*Okay what’s the, what’s your process for guessing and checking*?” Sometimes, Susan also used teleological rational questioning to orient students to focus on specific aspects of their strategies. For instance, when students were discussing the relationships between percent change and growth or decay factor, Susan challenged students to generate explanations based on prior knowledge and reasoning, "*But, I want to know why are we subtracting it from 1? Or why are we subtracting 1, why are we not subtracting 2 or 1.5 or 6 or? Okay, someone raise your hand.*" When students' initial responses were incorrect, instead of giving feedback to signal an error, Susan provided students opportunities to reconsider their answers by asking students to view the problem from different perspectives (e.g., "*So you just checked the Y intercept? Because C also has a Y intercept at 10.*"; "*Oh okay. Now, the question is, is that the same population as the front page*?"). On a few occasions, Susan asked students to share the teleological rationality aspects of their solutions for their own benefit or for other students: "*And now how did you pick 50 percent, because I think your partner is still confused*." In this way, Susan pushed students to learn from each other, which promoted their productive disposition toward mathematics to reach a consensus or a shared understanding in argumentation.

Epistemic rationality followed as the second most commonly observed component of rational questioning. Susan often used epistemic rational questioning to demonstrate why students' arguments were true, such as "*Okay, so give me an explanation of how you got that.*" and "*Okay, explain it.*" A certain amount of epistemic rational questioning was intertwined with teleological rational questions and was used to help students recall definition of key elements (e.g., "*Why is it exponential? It is exponential, does anybody know why*?") or focus on specific aspects of students' initial reasoning to enable problem solving (e.g., "*But the amount of people who know each day, is it increasing or decreasing?*").

In this unit, most of Susan's communicative rational questions served to ensure students wrote functions in notation that conformed to the norms in the shared mathematical community. For example, in the following conversation, Susan was addressing the correct use of mathematical notation for a student's written argument, in which the student had written $P = 3 \times 2^n$ for the answer of question: *How did you define the function G determines the number of people P who learned the secret on day n?*

1. Susan: So to find the function g that determines the number of people p who run the secret on day n. Define, hold on. Okay, so the only thing I have to say about this is the notation is a little incorrect, but that's just [inaudible] math. **What is your input in this function?** (*CR1:* Rational questioning contains communicative rational component).
2. S9: n?
3. Susan: Yes.
4. S9: The word is people [inaudible]
5. Susan: All right, I know. Brilliant question, I know. Okay, so write g of n. So remember, let's see, maybe it was Thursday or Wednesday when we were doing this.
6. S9: This p [inaudible]
7. Susan: Yeah, and it had multiple equal signs.
8. S9: Yes.
9. Susan: Yeah. Mm-hmm (affirmative)
10. S9: All right.
11. Susan: So g of n is actually the same function as p.
12. S9: Perfect.
13. Susan: So, g of n is function notation, p represents the number of people who knows so that's why we have function notation and then we also have p because it represents people who know.
14. S9: Okay.
15. Susan: Mm-hmm (affirmative), you're good. All right, excellent. Explain your reasoning to S10 too because she's not quite where you are and you want you and your partner to be on the same spot.

As shown in above example, Susan started the conversation to lead students to concentrate on the modes of representations of their arguments (Line 1) and guided students to change the original answer from $P = 3 \times 2^n$ to $P = g(n) = 3 \times 2^n$. By facilitating S9 to use requested variables and notations in functions to represent inputs and outputs, Susan not only let S9 established the truth of a claim, but in doing so, provided opportunities to deepen students' understanding of function as a mathematical concept.

Based on the adapted version of Toulmin's model (Conner, 2008), I investigated how different types of rational questions supported components of arguments as summarized in Table

5.5. Through Toulmin's lens, Susan's Unit 1 involved 291 argument components (except original data which are given by the teacher drawn as green boxes in diagrams), in which 90% (261 out of 291) components were contributed either only by students or by students jointly with the help of teacher.

Table 5.5

Numbers and Percentages of Combinations of Components of Rational Questioning Supporting Collective Argumentation in Susan Unit 1

Types	Numbers (Percent out of 190)	Components of Arguments Associated
Epistemic (*E*)	25(13%)	Warrants (25)
Teleological (*T*)	79(42%)	Claims (22), Claims*(3) Data (2) Data/Claims (28), Data/Claims*(3) Rebuttal/Claim (1) Rebuttal/Data (4) Warrants (11), Warrants*(4) Warrants/Claims (1)
Communicative (*C*)	2(1%)	Data/Claims (1) Warrants*(1)
ET	36(19%)	Data/Claims (1) Warrants (30), Warrants*(5)
EC	23(12%)	Claims (1), Claims*(1) Warrants (9), Warrants* (12)
TC	17(9%)	Claims (4), Claims*(3) Data/Claims (3), Data/Claims*(3) Warrants (2), Warrants*(2)
ETC	8(4%)	Claims (1) Rebuttal/Data (1) Warrants (3), Warrants* (2)

As shown in Table 5, based on the definition of epistemic rational questioning, it comes as no surprise that epistemic rational questioning prompted students to construct warrants. However, combinations of epistemic rationality with teleological or communicative rationality served a broader aim. These elicited more than initial warrants, including prompting students to revise initial warrants to make their explanations more complete and precise (labelled as *warrants**). For instance, in the following argumentation episode, Susan was leading a whole class conversation working on a review task about percent change and decay factor.

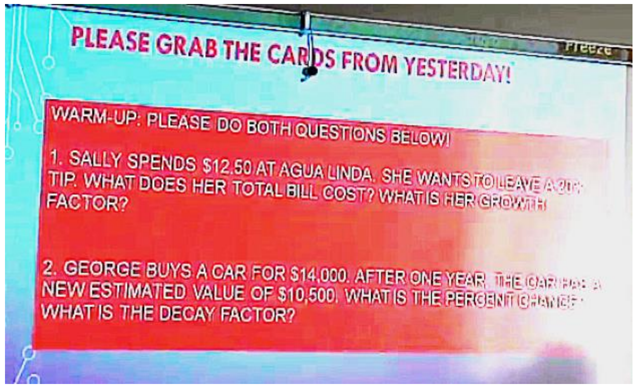


Figure 5.8. Task for Unit 1, 2nd lesson, Argument 2, Whole Class Discussion in Susan's class.

1. Susan: Okay, somebody I know you guys all have done this, so can someone come and do on the board?
2. S4: Do I have to do the work?
3. Susan: Oh, yeah.
4. S4: I kind of just like guess and check. so
5. Susan: **Go to the board and tell us what you did.** (*ETCR1:* Rational questioning contains all rational components).
6. S4: They spent you know 12.50 and they want to leave a 20 percent tip, I always find this easier for me because 10 percent, you just move the decimal here so that's just the dot and times it by two. [point at 1.250 on board]
7. Susan: That's exactly how I do it.
8. S4: So you get two dollars and 50 cents and then you just add that and you get 15 dollars. Do you want me to put that up?
9. Susan: Yeah sure, but you can't write in red because the background's red.
10. S4: Yeah okay.
11. Susan: Here, I think I've got some markers up here.
12. S4: I found a green one. [wrote on board]
13. Susan: **So our total bill is what?** (Questioning without a rational component: *N*).
14. Multiple Students: 15 dollars.
15. Susan: [15 dollars]. Okay cool. So you found the 20 percent tip would be two dollars and 50 cents, you added it, the total bills is 15 bucks. **What's her growth factor? Can anybody do that part? What did you say?** (*N*).
16. S5: One in a 20.
17. Susan: Yes okay, one and two tenths. Yes, which is also known as, what did you say S1?
18. S1: One point two.
19. Susan: One point two, so our growth factor is one point two. **Can anybody explain that? How do we know that?** (*ER2*).
20. S6: So if you multiply the original together.
21. Susan: **What did you just say S6?** (*N*).

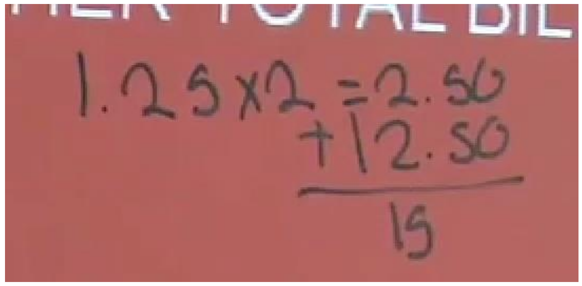


Figure 5.9. Screenshot of Students' Work from Unit 1, 2nd Lesson, Argument 2.

22. Susan: **So our total bill is what?** (Questioning without a rational component: *N*).
23. Multiple Students: 15 dollars.
24. Susan: [15 dollars]. Okay cool. So you found the 20 percent tip would be two dollars and 50 cents, you added it, the total bills is 15 bucks. **What's her growth factor? Can anybody do that part? What did you say?** (*N*).
25. S5: One in a 20.
26. Susan: Yes okay, one and two tenths. Yes, which is also known as, what did you say S1?
27. S1: One point two.
28. Susan: One point two, so our growth factor is one point two. **Can anybody explain that? How do we know that?** (*ER2*).
29. S6: So if you multiply the original together.
30. Susan: **What did you just say S6?** (*N*).
31. S6: So you multiply the initial value
32. Susan: **So you multiply the initial value which is 12 dollars and 50 cents, times one point two to get what?** (*ECR3*).
33. S6: 15.
34. Susan: 15! [wrote 12.50(1.2) =15.00 on board] Or the new amount, so this is our growth factor, it's what we multiplied our original by to get our new amount [label 1.2 as growth factor].

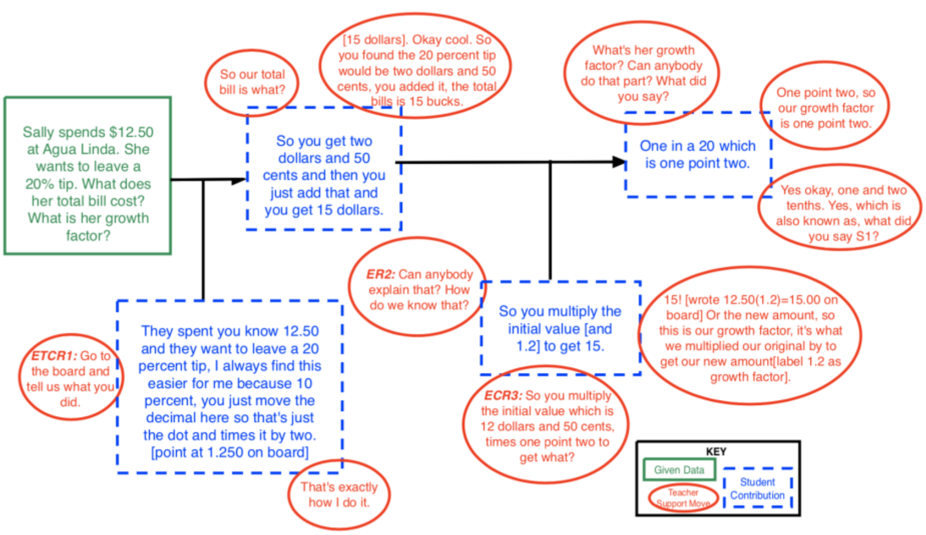


Figure 5.10. Diagram of Unit 1, 2nd Lesson, Argument 2, Whole Class Discussion in Susan's class.

Beginning with a rational question containing all three components (Line 5), Susan prompted a student-led discussion that allowed S4 to access three components of rational behavior while working on a problem-solving task. In other words, when a student came to the board, he or she was expected to strategically choose tools to achieve a goal (teleological rationality) on the basis of specific knowledge (epistemic rationality) and communicate in a precise way with the aim of being understood by the classroom community (communicative rationality). In this way, Susan made students aware of the rational requirements of argumentation. In line 28, Susan asked an epistemic rational question to allow students to justify why the growth factor was 1.2. S6 provided an initial brief answer (Line 29) that was not completely clear and beneficial for other students as listeners to make sense of her justification. At this point, a repeated epistemic rational questioning combined with a component of communicative rationality (*ECR3*, Line 32) was used to draw students' attention to the communicative dimension of their answers and thus supported students to provide a more comprehensible warrant (i.e., *warrant**) based on their initial incomplete warrant. As shown in Figure 5.10, both *ER2* and *ECR3* rational questions were associated with the second blue warrant, which provides empirical evidence that a sequence of rational questions with repeated combinations of rationality components was needed to help students to construct a satisfactory explanation.

In addition, the teleological dimension was sometimes intertwined with the epistemic dimension related to justifying the chosen strategy. As an illustration, consider the following episode of argumentation, in which Susan was working with a small group of students on the "matching activity."

1. Susan: Okay, let's see. Which ones are you, let's see. So, you have growth and decay factors already. Okay, so...how do you do percent change? Let's look at all of them (puts the cards on desk). You only have six.
2. S8: I think we have two more.
3. Susan: Oh, you have two more. **Okay. Let's just start, where do you think one of those would go?** (Questioning without a rational component: *N*).
4. S8: I don't really {inaudible}.
5. Susan: **Okay, yeah. S8 matched one, okay S8 how did you get that?** (*TR1:* Rational questioning contains teleological rational component).
6. S8: Cause it has a seventeen and {laugh} and it looked close enough.
7. Susan: **Okay, and why. So where is the seventeen in this function that allowed you to conclude that the percent change is seventeen percent?** (*ETR2:* Rational questioning contains teleological rational component).
8. S8: Seventeen hundredth (points at the card), {so}
9. Susan: {Yeah}. So this is seventeen hundredth, I like your use of math language. And so you are adding seventeen percent every time.
10. S8: So then this one would be 50 (points at the card).
11. Susan: Very good. **How come?** (*TR3*)
12. S8: Positive or negative? (points at cards)
13. Susan: I don't know you tell me, S8. **Will this one be positive or negative percent change? (points at the card)** (*TR4*)
14. S30: I don't even know what percent changes is.
15. S8: Would it be positive cause it has a one in front of it.
16. Susan: Um.
17. S8: Or does that have anything to do with it.
18. Susan: Well, let's think about this. **What does in terms of exponential, like functions. Um, what does positive versus negative mean?** (*ETR5*)
19. S8: Positive increase, negative decrease.
20. Susan: **Yeah, so positive means you're increasing it each time, so you are increasing by 50 percent or seventeen percent?** (*TR6*)
21. S8: It's increasing? (points at the card)
22. Susan: Yeah, so you went to the graph said that graph is increasing, so it's probably positive 50 percent. **And now how did you pick 50 percent, because I think your partner is still confused.** (*ETCR7*)
23. S8: Um, five hundredth, it's point five.
24. S30: Oh.
25. Susan: Cause five tenth.
26. S8: Oh yeah. So you look at the decimal.
27. Susan: Uh huh. Very good.
28. S30: So like would this one be the negative? (point at the card)
29. Susan: Excellent. **How did you come up with that?** (*TR8*)
30. S30: It does one half {inaudible} five point.
31. Susan: Yeah, very good. See so it's not as hard as you think.

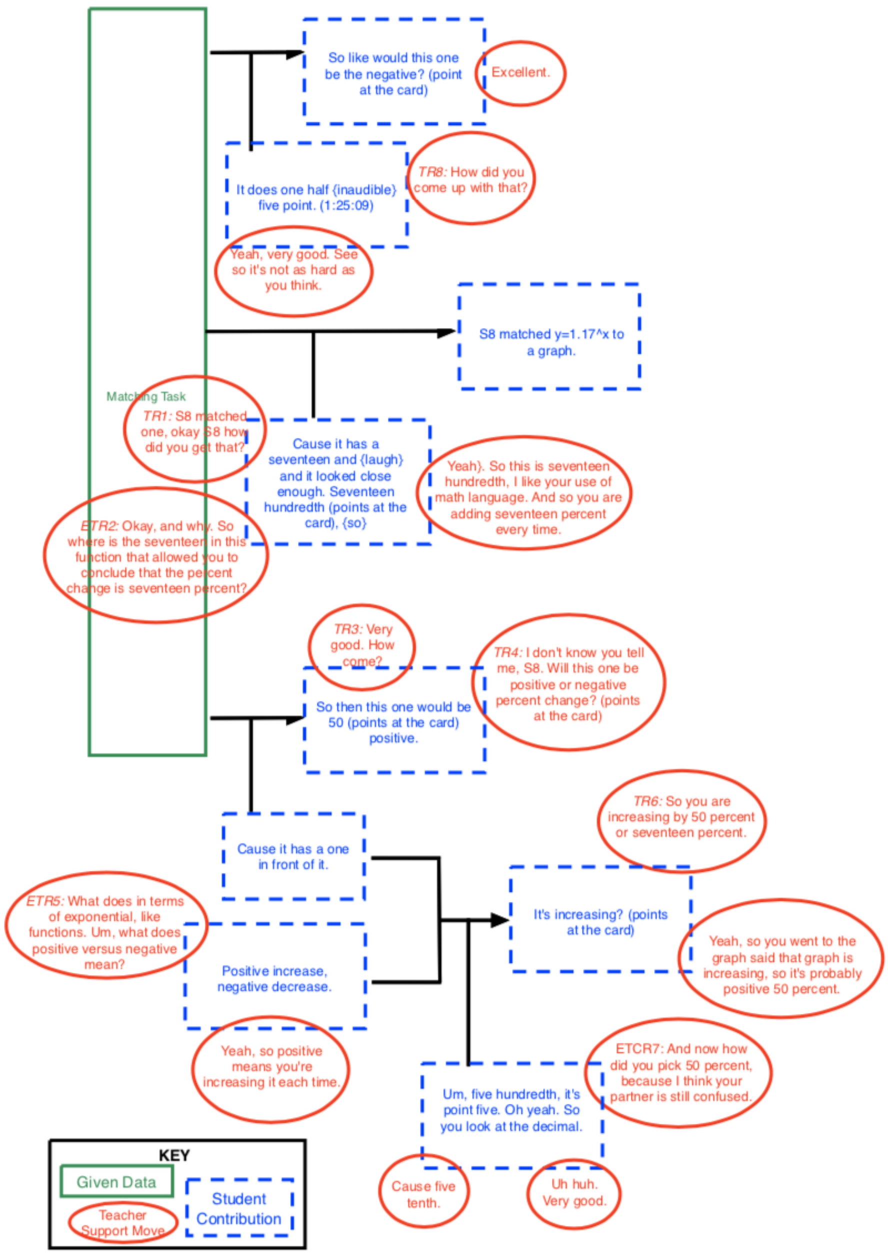


Figure 5.11. Diagram of Unit 1, 1st Lesson, Argument 20 Part (a), Small Group Discussion in Susan's class.

In this context, Susan started with a teleological rational question (Line 5, "*S8 how did you get that*?") concentrating on the strategic tools that S8 used to achieve her answer (i.e., a *pragmatic* plane of teleological dimension). S8's initial justification (Line 6, "*Cause it has a seventeen and {laugh} and it looked close enough*.") for how she matched the exponential function with provided graphs was not a completely correct mathematical statement from the perspective of mathematicians or a mathematical community. At this point, Susan pushed S8 to focus on a theoretical plane of teleological rationality (i.e., intertwined with epistemic dimension) to justify the pragmatic strategy (Line 7, "*Okay, and why. So where is the seventeen in this function that allowed you to conclude that the percent change is seventeen percent?*"), so that S8 could provide an acceptable rationale (i.e., *warrant**) based on her means to solve the problem. According to the diagram (see Figure 5.11), S8's warrant ("*Cause it has a seventeen and {laugh} and it looked close enough. Seventeen hundredth [point at the card]*) was associated with two rational questions with repeated combinations of components of teleological rationality and supplemented by epistemic rational components.

The teacher plays an important role in shaping the argumentative practices to work toward establishing a logically organized final claim on the basis of previous mathematically correct arguments. The above two argumentation episodes illustrated that it was helpful for teachers to use follow-up integrated components of rational questions to ensure students fit rationality requirements of the argumentative discourse. From Toulmin's lens, the follow-up "repeated" rational questions (in above two episodes: *ER-ECR, ER-ETR*) served to prompt a more satisfactory argument component from the teacher's perspective and provided student opportunities to reflect on and reconsider their answers.

5.2.4 Susan Unit 2 (170330 and 170331) — "Partitioning Line Segment"

On the first day of this unit, the whole class was working on an inquiry-based task (see Figure 5.12) derived from the *Georgia Standards of Excellence* (GSE) in which students were asked to formulate their own problems to solve. Susan spent a significant amount of time honoring the students' curiosity when they came up with questions that were non-mathematical in nature. Then she directed students to focus on a sequence of reasonable questions (e.g., calculating areas and perimeters of polygons) to answer using the mathematics students had learned before. Students used the rest of the class time to investigate tools to answer posed questions in small group discussions, while leaving a limited amount of time for whole class discussion at the end of the class. As a result, only seven argumentation episodes were observed on that day, which was the smallest amount observed in one class period. More than half of the episodes of argumentation occurred during small group discussions and tended to be short because students' responses tended to be short or incomplete as they were exploring different mathematical tools and were uncertain of their answers.

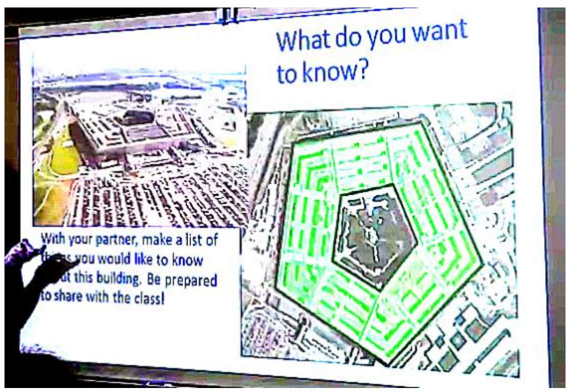
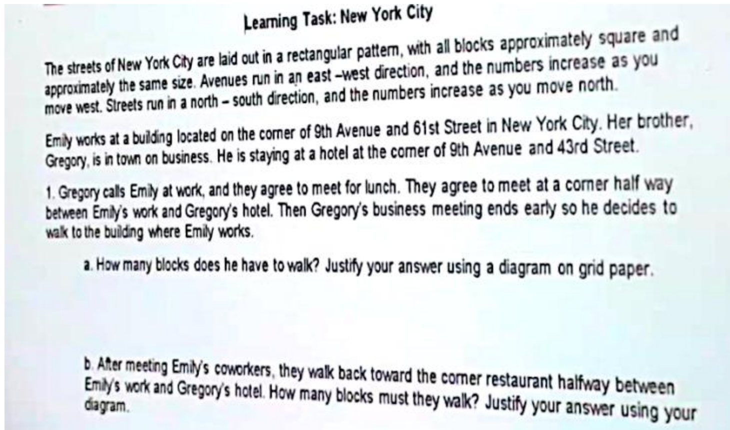


Figure 5.12. The Pentagon Task for Susan Unit 2, 1st Lesson.

The goal of the second day of Unit 2 mainly aligned with the following standard from the *Georgia Standards for Excellence* (GSE):

> MGSE9–12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio. (p. 10)

Susan created a problem-solving mathematical activity on this day, in which students were expected to address problems by using coordinates to construct the situation about the problem. For example, the New York City task (see Figure 5.13) Susan derived from the GSE was used as the beginning task on this day to provide students a guided discovery of the procedure for partitioning a segment into a given ratio.



Learning Task: New York City

The streets of New York City are laid out in a rectangular pattern, with all blocks approximately square and approximately the same size. Avenues run in an east –west direction, and the numbers increase as you move west. Streets run in a north – south direction, and the numbers increase as you move north.

Emily works at a building located on the corner of 9th Avenue and 61st Street in New York City. Her brother, Gregory, is in town on business. He is staying at a hotel at the corner of 9th Avenue and 43rd Street.

1. Gregory calls Emily at work, and they agree to meet for lunch. They agree to meet at a corner half way between Emily's work and Gregory's hotel. Then Gregory's business meeting ends early so he decides to walk to the building where Emily works.

a. How many blocks does he have to walk? Justify your answer using a diagram on grid paper.

b. After meeting Emily's coworkers, they walk back toward the corner restaurant halfway between Emily's work and Gregory's hotel. How many blocks must they walk? Justify your answer using your diagram.

Figure 5.13. The New York City Task for Susan Unit 2, 2nd Lesson.

The total number of argumentation episodes in Susan's Unit 2 was 25. Within argumentation episodes, Susan asked 134 questions, and 88% (118/134) of questions were categorized as rational questioning. Among them, the most common component was teleological rationality (89%, 105/118), followed by epistemic rationality (45%, 53/118) and communicative rationality (23%, 27/118).

In this unit, teleological rational questions were mainly used to support students to focus on the strategic choice of tools or means situated in the exploratory phase (e.g., "*How do you find the line that's perpendicular to a line*?"; "*How do you calculate the slope*?") or to deal with students' teleological obstacles when students were struggling to find a mathematical solution (e.g., "*Think about how are the slopes related in perpendicular lines*?"; "*Let's think about this point. You're talking about … What is this? What does this number represent*?"). On a few occasions, Susan created context for students to discuss their different solutions (e.g., "*Now S3 did it a different way. S3, what did you do? We have 2 minutes of class. We can do this. S3, what did you do? To get...we already know the x. Tell us how you found the y.*") and efficient teleological choices of solutions (e.g., "*Okay, so what's the much simpler way*?"). By calling on a particular student to share a different solution or discuss which solution is simpler, the class discussed what counted as a mathematically different solution, which facilitated the negotiation of sociomathematical norms (McClain & Cobb, 2001).

Epistemic rational questions in this unit were mainly used to request students to provide reasons for their teleological choices of means, such as "*Wait. 12.5 divided by four. Why did you divide by four*?" Susan's use of epistemic rational questioning controlled whether students' explanations for their choice of tools or means were understandable and supported the production of logico-deductive arguments that met the standard of mathematical theorems, axioms and principles. Sometimes, Susan posed epistemic rational questions to emphasize specific aspects of the problem or student reasoning to help students understand a concept or a solution at a micro, in-depth level. For instance, Susan prompted students to interpret "*Why are our Y values not changing* [*in this problem*]?" because the lines in this specific context were horizontal. On a few occasions, Susan asked students to explain steps of reasoning (epistemic rationality aspect) to their classmates (e.g., "*Explain to your neighbor why that is*.") for their own benefit and for other students.

Communicative rationality was subordinated to the epistemic and teleological rationality in this unit. Susan used most of her communicative rational questioning to foster student talk so they could provide more information via explicit requests (e.g., "*West, which is which way?*"). Other than that, the communicative rationality aspect often intertwined with epistemic and teleological aspects when students presented their arguments in front of the class. Students were expected to present their arguments orally and in written form (i.e., communicative rationality aspect) while they were leading argumentative discourse.

Table 5.6

Numbers and Percentages of Combinations of Components of Rational Questioning Supporting Collective Argumentation in Susan Unit 2

Types	Numbers (Percent out of 118)	Components of Arguments Associated
Epistemic (*E*)	7(6%)	Warrants (7)
Teleological (*T*)	57(48%)	Claims (15), Claims*(5) Data (2) Data/Claims (24), Data/Claims*(5) Rebuttal/Data (1) Warrants (4), Warrants*(1)
Communicative (*C*)	3(3%)	Claims* (2) Warrants (1)
ET	27(23%)	Claims (1) Warrants (23), Warrants*(3)
EC	3(3%)	Warrants (2), Warrants* (1)
TC	5(4%)	Claims*(1) Data/Claims (1), Data/Claims*(1) Rebuttal/Data (1), Rebuttal/Data* (1)
ETC	16(13%)	Claims (4), Claims* (1) Data (1) Data/Claims (3) Warrants (2)

The relationship between types of rational questioning and argument components from Susan's Unit 2 is shown in Table 5.6. Based on Toulmin diagrams, 228 argument components (except original data which are given by the teacher drawn as green boxes in diagrams) were constructed in Susan's Unit 2. Among these argument components, 87% (198 out of 228) of them were contributed either only by students or by students jointly with the help of teacher.

In this unit, Susan used a higher proportion of rational questioning including all three components of rationality, in which Susan created context and space to support students' autonomy in communicating arguments. In this way, with limited supportive actions from the teacher, students autonomously constructed claims (teleological rationality aspect) and provided warrants (epistemic rationality aspect) to support their claims with appropriate technical expressions (communicative rationality aspect). For example, the following episode showed how two students in a small group presented their method to solve the task:

Partition a segment with endpoints (2, -4) and (7, 2) into a ratio of 2:3.

1. S2: So this is six and this is five. (points to diagram drawn on paper)
2. Susan: **Wait wait wait. What? What is six and what is five?** (*CR1:* Rational questioning contains communicative rational component).
3. S2: Like there's a point here, and there's a point here. And there is six across up and down (gestures vertically). And five across up and down (gestures horizontally).
4. Susan: Oh okay I see.
5. S2: That's right?
6. Susan: Uh huh. So if you look at the coordinates 2 to 7 is 5 and then -4 to 2 is a total of 6. Ok cool.
7. S2: So um, it's a 2:3 ratio, so we said this would be like the short end of the stick. So this is easy since it's five, we just did like, two over
8. Susan: Mmhm.
9. S2: So I made that line. But this was six and that doesn't add up evenly to 2:3, so I did like what you just said, and like added it up and got 5 and then multiplied that by 1.2 to equal six. And then there's, it's like 2:3, so it will be 2.4.
10. Susan: Wow hold on. Let me (laughs) I did not do it that way. I think that should absolutely work. **So what was your final answer? This is number eight part b?** (Questioning without a rational component: *N*).
11. S2: Yes, so it's kind of like the, its um, -4, -1.6.
12. Susan: Yeah. Except for, it's not -4. Because both of your x values are positive. But yeah, I um, hmm. **Okay, so say that again? Say what you did again? Let me think about that?** (*ETCR2*)
13. S2: Um, a ratio of 2:3 like doesn't go evenly in 6.
14. Susan: Mmhm.
15. S2: So I added that up and got 5.
16. Susan: Mmhm.
17. S2: And then six divided by 5 is 5 times 1.2, is 6.
18. Susan: Okay.
19. S2: And then um, like, if you multiply 1.2 times 2 it's 2.4 and 1.2 times 3 is 3.6. And add those it equals 6. So you kind of like split it up into pieces
20. Susan: Okay. Mmhm.
21. S2: And instead of 2 times 1.2 is 2.4 so I went up 2.4. So I just did like what he did with that other - like he said he added them up and multiplied. That was, we were kind of stuck on this. And then he said to draw a right triangle. So it seemed like it worked for him.
22. Susan: Great! That's excellent.

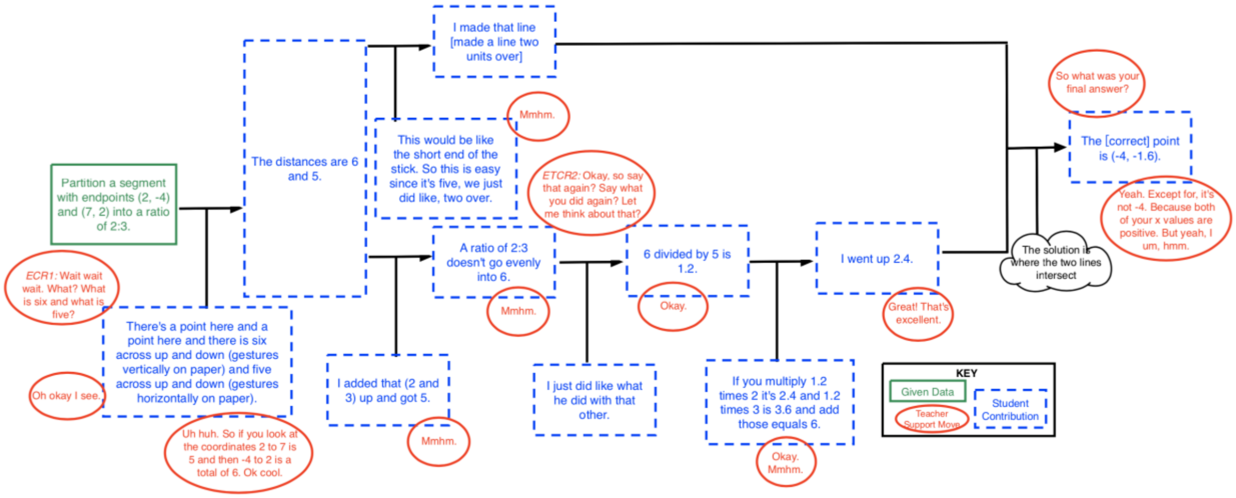
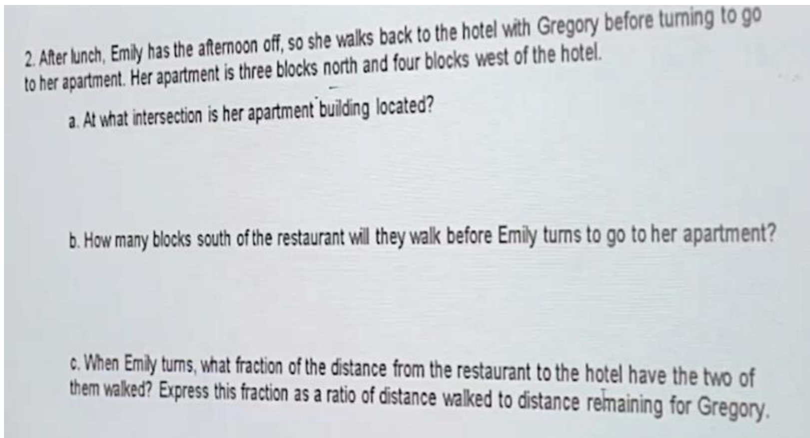


Figure 5.14. Diagram of Unit 2, 2nd Lesson, Argument 13, Small Group Discussion in Susan's Class.

In this episode, Susan only asked two rational questions (Lines 2 and 12) to open up the floor for S2 to present her thinking. As shown in Figure 13, most of the argument components in this episode involved data/claims and warrants that were constructed by S2 (i.e., blue boxes); these are accompanied primarily by responses from Susan to acknowledge the correctness of a student's statement (e.g., Mmhm). This revealed that S2 was able to draw on her own rational behavior (epistemic, teleological, and communicative rationality) while making mathematical arguments, and thus autonomously produce a rational discourse, which is a major goal of using rational questioning (Douek in Boero & Planas, 2014). Susan's role in this argumentation episode was more like a participant, in which the teacher's main tasks were to encourage students to exchange their ideas, critical thinking, and create more space for student-led negotiation (Chen et al., 2016). More importantly, in this case, the authority for classroom conversation shifted from the teacher to the students, thus fostering the development of both students' intellectual autonomy and their social autonomy, which is a major goal of mathematics education (NCTM, 1989; Yackel & Cobb, 1996).

Considering that students were still learning to lead argumentative discourse by themselves, Susan intervening in the role of coach to use rational questions to support the development of argumentation was still necessary at this stage. As an illustration, in the following episode, Susan asked one student to come to board to present his arguments on question two part c of The New York City task (see Figure 5.15).



2. After lunch, Emily has the afternoon off, so she walks back to the hotel with Gregory before turning to go to her apartment. Her apartment is three blocks north and four blocks west of the hotel.

a. At what intersection is her apartment building located?

b. How many blocks south of the restaurant will they walk before Emily turns to go to her apartment?

c. When Emily turns, what fraction of the distance from the restaurant to the hotel have the two of them walked? Express this fraction as a ratio of distance walked to distance remaining for Gregory.

Figure 5.15. The New York City Task Question Two for Susan's Unit 2, 2nd Lesson.

1. Susan: Okay class, most people seem to have gotten pretty far on the front, so I wanted to go over two part C. **So can anybody do two part C for me on the board? And you can make like a rough picture on the board as well if you want. Anybody?** (*ETCR1:* Rational questioning contains all rational components).
2. Susan: Okay, no you have to do more than just write the ratio, S30. You can do it. Yay S30. Everybody cheer on S30 who is going to come to the front of the room and do the problem. Whoo! It might be helpful. It doesn't have to be perfect but enough that your classmates can understand what you're doing. S10 [inaudible] **Okay, y'all pay attention to S30. S31, are you paying attention to S30?** (Questioning without a rational component: *N*).
3. S31: No, he hadn't started yet.
4. Susan: **Also, S30 can you tell us what you're doing when you're writing?** (*CR2*)
5. S30: I'm just numbering the line.
6. Susan: Okay.
7. S30: Okay, so the restaurant ... [draw number line on board] Okay, so the restaurant was at 9th avenue and 52nd street because-
8. Susan: 9th is on the other side of 8th.
9. S30: What?
10. Susan: 9th is on the other side of 8th.
11. S30: Okay.
12. Susan: It's confusing.

13. S30: So ... because it's halfway between her work and her ... Gregory's hotel, and his hotel was at 9th avenue and 43rd street. So it said that they ... they're walking towards his hotel, but then they had to go to her apartment which was at 12th ... yeah, it was at 13th and 46th. [Label points on graph] So they ask you what the ratio was from walking here to here when she'd have to turn. So, totally they have to walk, 1, 2, 3, 4, 5, 6, 7, 8, 9 and then she had to walk 1, 2, 3, 4, 5, 6 until she had to turn, so it'd be six to nine. Or six to three, which could reduce to two to one. So she walked like two blocks for every [inaudible]

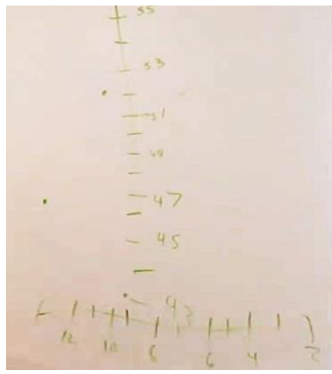


Figure 5.16. Screenshot of S30's Drawing from Unit 2, 2nd Lesson, Argument 7.

14. Susan: **Does anybody ... did anybody else get the same answer as him? Put thumbs up if you got that same answer.** (*ETCR3*)
15. Multiple students put thumbs up.
16. Susan: Okay, **let's discuss why is it six to three and not six to nine?** (*ETR4*)
17. S10: Because it's walks remaining [crosstalk]
18. Susan: Yeah, so ... it doesn't say express this as a ratio of distance walked to total distance. It says distance remaining, exactly what you said S10. Very good.

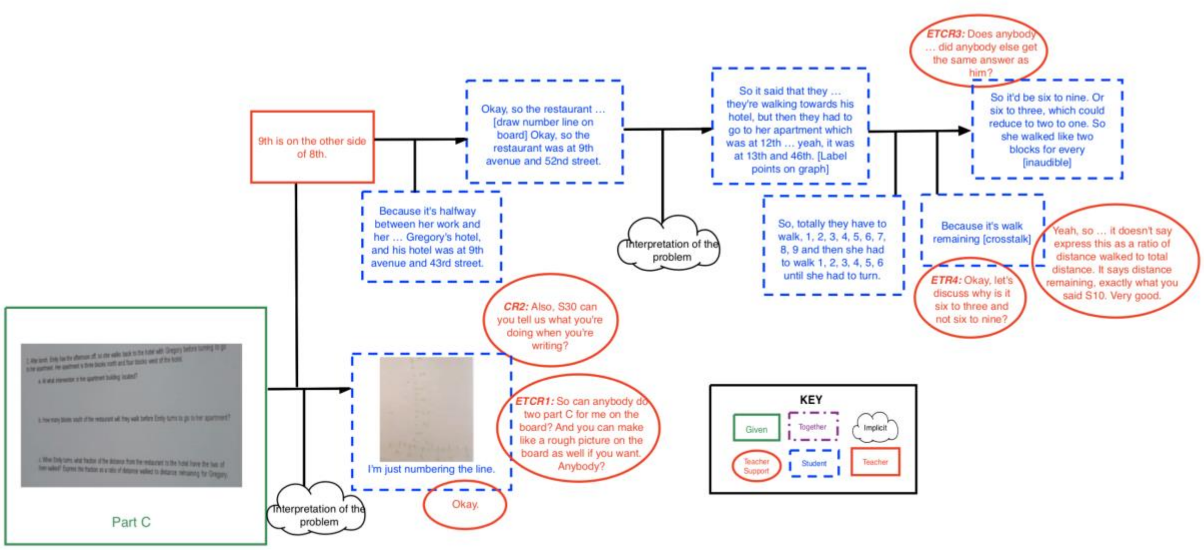


Figure 5.17. Diagram of Unit 2, 2nd Lesson, Argument 7, Whole Class Discussion in Susan's Class.

This episode began with Susan inviting S30 (Line 1) to present his arguments in front of the class. As shown in the diagram (see Figure 5.17), several argument components in this episode involved data/claims and warrants constructed by students (i.e., blue boxes) without teacher support (i.e., without attached red support), which revealed that these components were at least acceptable in the given classroom community, especially from Susan's perspective. However, there were instances in which S30 made incomplete and incorrect statements (Lines 4 and 6); at this point the teacher intervention on S30 was delicate and critical to ensure S30 used valid logical arguments appropriately sequenced to support the truth of the final answer. For instance, Susan corrected S30's representation of a number line on board (see Figure 5.16, Lines 7 to 9) and asked S30 to make his arguments more explicit to all students, "*Also, S30 can you tell us what you're doing when you're writing*?" (Line 4). Susan's statements also revealed that students who presented their arguments in front of the class were expected to not only to write words and create diagrams or drawings on board but also simultaneously to describe orally the arguments they constructed (communicative rationality aspect). In line 15, Susan also stepped in, challenging the whole class to justify (epistemic rationality aspect) their strategies (teleological rationality aspect) about why the answer is "six to three" instead of "six to nine." In this way, Susan supported the argumentative practice continuing to a deeper level and resolved students' possible difficulties in solving the problem.

5.2.5 Use of Rational Questioning Across Units

In this section, I described how two beginning secondary mathematics teachers used rational questioning to organize collective argumentative discourse in regard to Habermas' (1998) perspective on three interrelated components of rationality (i.e., epistemic, teleological and communicative). The proportions of student contributions were more than 70% in each unit, which showed both teachers offered space and context for their students to construct argument components. However, rational questions like "*Does that make sense to you*?" (*ETCR*) and "*Go to the board and tell us what you did*." (*ETCR*) sometimes did not prompt students to contribute or respond to a specific argument component in the moment, and thus, these were not associated with any specific argument component. Further, it was noticed that an argument component was often associated with multiple teacher's questions or other supportive actions. This was especially true for warrants.

The findings revealed that in both teachers' classes teleological rational questioning (75%) occupied a dominant position, followed by epistemic rationality (44%) and communicative rationality (29%) (see Table 5.7). This trend held across units, with the exception that in Jill's Unit 2, the communicative rationality dimension was slightly more prevalent than the epistemic rationality dimension.

Table 5.7

Summary of Percentages of Rationality Components

Unit	Epistemic	Teleological	Communicative
Jill Unit 1	41%	67%	24%
Jill Unit 2	42%	70%	44%
Susan Unit 1	48%	74%	26%
Susan Unit 2	45%	89%	23%
Total Average	**44%**	**74%**	**30%**

5.2.5.1 Using Combinations of Components of Rational Questioning in Support of Collective Argumentation

The results of this study showed that both beginning secondary mathematics teachers used all possible combinations of the components of rational questioning to support collective argumentation during their four lessons (two sets of two consecutive days of lessons). Table 5.8

summarizes the combinations of components of rational questioning observed, their functions with respect to prompting argument components seen through Toulmin's (1958/2003) lens, and the conditions under which they were used.

Table 5.8

Use of Combinations of Components of Rational Questioning in Supporting Collective Argumentation

Components of Rationality in the Question	Functions (Components of Toulmin's (1958/2003) model addressed)	When Used
Epistemic (*E*)	Invite students to be explicit about the warrants.	• Foster students to justify their arguments. • Elicit student's incorrect or incomplete reasoning.
Teleological (*T*)	The contribution of and linking of intermediate data/claims in support of the final claims. (Relates to their choice of tools or methods of solutions).	• Explore choices of means to solve the problem. • Guide student thinking when students were struggling in problem-solving.
Communicative (*C*)	Make claims with correct mathematical representations.	• Introduce mathematical terminologies or expressions. • Focus on presenting arguments in oral and in written form.
ET	Construct additional warrants, a more comprehensible warrant (i.e., warrants*) and rebuttals.	• Clarify or emphasize specific aspect of the problem or one part of student reasoning. • Guide students to reflect on and reconsider answers if answers are inappropriate/incorrect.
EC	Reconstruct/revise initial warrants (i.e., warrants*).	• Assist students to communicate their reasoning. • Encourage students to use appropriate mathematical language and representations.
TC	Construct valid data or data/claims to form legitimate ways of problem-solving.	• Introduce mathematical representations as tools. • Help students focus on specific aspects of the problem and

		view a problem from different perspectives.
ETC	Reflect on or construct public arguments as a whole. Not usually associated with a specific argument component.	• Provide opportunities for students to make sense of other students' arguments. • Move towards students' autonomy in communicating arguments.

Previous studies on teacher questioning have predominantly focused on describing general categories or patterns of questions that teachers asked and perceived challenges or difficulties that teachers had in using questioning to support classroom discussions. Drawing on Habermas' (1998) construct of rational behavior and Toulmin's (1953/2003) model for argumentation, this study examined teacher questioning as a way by which to develop collective mathematical argumentation along three components of rationality while seeking to engage students in actively contributing to argumentation (i.e., constructing argument components). In this sense, I viewed rational questioning as a long-term teaching intervention to support students' autonomy in producing rational argumentative discourse (i.e., fitting rationality requirements). The results of this study showed that the developed Teacher Rational Questioning Framework (see Table 4.1) captured most of the questions that teachers asked during argumentation and provided insights into how a teacher's rational questioning was related to the three components of rationality when they orchestrated argumentative discourse.

As shown from the data (see Tables 5.3 to 5.6), questions involving a single component of rationality were most commonly used to support students when generating argument components based on reasoning and prior knowledge. Rational questions with two rationality components often served a critical role in shaping argumentative discourse and moving toward valid claims (often as judged by the teacher). Despite their importance, these combinations of rational questions should not be used exclusively; instead, they were useful in following up on students' ideas or arguments. For example, a repeated epistemic rational questioning combined with a component of communicative rationality (*ECR*) was used to draw students' attention to the communicative dimension of their explanations and thus supported students to provide a more comprehensible warrant (i.e., *warrant**) in the given classroom community. In addition, questions containing all three components of rationality were observed, but these questions were infrequent. Regardless, questions containing all three components of rationality are important as they provide students with opportunities to evaluate mathematical statements made by other students (e.g., "*Does that make sense to everybody*?") or invite students to control the argumentative activities (e.g., "*Go to the board and tell us what you did*") even though these questions may not directly prompt or respond to any specific argument components in the moment.

Looking through Toulmin's (1958/2003) lens, the results showed that a single component of rational questioning was often not enough to prompt a completely satisfactory argument component from a teacher's perspective. It was observed frequently in the data that a single epistemic question (*ER*) (e.g., "*Why*?" and "*Why not*?") that asked the students to justify their arguments often resulted in incomplete, incorrect, or ambiguous responses at the beginning of argumentative practices. A follow-up epistemic rational question intertwined with teleological (*ETR*) or communicative rationality (*ECR*) was found to help the students provide a more comprehensible explanation. Thus, it is critical for teachers to go beyond one single component of rational question. These results highlight asking questions that involve combinations of components of rational questioning by considering the ongoing interactions in argumentation is necessary.

5.2.5.2 Perspectives on Privileged Components of Rationality

Although both of the participant teachers used all of the combinations of the components of rational questioning in regard to supporting collective argumentation, the results showed differences in the proportions of the questions from each component of rationality with regard to the Teacher Rational Questioning Framework (see Table 4.1), even for the same teacher when teaching a different unit of instruction. Some researchers (e.g., Martignone in Boero & Planas, 2014; Morselli & Boero, 2009, 2011) have analyzed the privileged components of rationality in students' proving processes based on Habermas' (1998) construct of rational behavior. For instance, Morselli and Boero (2009) found that the use of appropriate mathematical expressions (communicative dimension) did not guarantee that the students would achieve the goal of the activity (teleological dimension) or develop valid explicit reasoning (epistemic dimension). They stated that "detailed justification sometimes results in fallacious conclusions" (Morselli & Boero, 2009, p. 219) and suggested that teachers should ensure that the epistemic and teleological components of rationality were always in the foreground. In this section, I will discuss my view on the privileged components of rationality with respect to using rational questioning to support collective argumentation.

The results of this study indicated that teleological rational questioning was used most frequently by both teachers; on average, 74% of the teachers' rational questions contained a teleological rationality component. During multiple problem-solving mathematical activities across all four units, argumentative discourse often started with teleological rational questioning, which concentrated on the choice and use of the means or tools to achieve the goal of the activity. This method of using teleological rational questioning could be considered as *pragmatic* and was situated in the exploratory phase (i.e., constructing conjectures) of the classroom activity. In other situations, when the teleological dimension was intertwined with the epistemic and communicative dimensions, sometimes the students were asked to justify why their strategies were the most efficient ones, what counted as different solutions, or to focus on specific aspects of (other) students' mathematical thinking. This approach could be *theoretical.* These theoretical teleological rational questions contain some aspects of *focusing* questions, as defined by Wood (1998), that are used to help students summarize or reflect on the means or tools they have used, so that the students can solve the problem on their own.

Epistemic rationality followed as the second most common component of rational questioning; on average, 44% of the rational questions contained an epistemic rationality component. Epistemic rationality questions were mainly used to press students to demonstrate why their arguments held and were strongly associated with warrants (219/276, 79%) as shown in Toulmin's (1958/2003) diagrams. However, combinations of epistemic rationality with teleological or communicative rationality served a broader goal of helping the students contribute more than the initial warrants, including prompting students to revise their initial warrants to make their explanations more complete and precise (labeled as *warrants**) or leading students to construct rebuttals for their initial incorrect warrants. In doing so, the teachers provided opportunities through which the students were able to deepen their conceptual understanding of mathematical solutions.

In most of the units, the communicative rationality dimension was subordinated to the epistemic and teleological rationality dimensions; on average, 30% of the rational questioning contained a communicative rationality component. Only one lesson (Jill Unit 2, 2nd lesson) occurred in which the students needed many opportunities to explore the relationships between the geometric sequences and algebraic forms. In this lesson, a slightly higher proportion of communicative rationality questions was observed as they were used to guide the students use or explain correct graphical representations (e.g., "*Maybe explain what your picture is first, in case people can't see it and then talk about how your pictures are related*.") or develop new mathematical terminologies (e.g., "*Okay then so y would be what? What would y represent?*"). In most of the other cases, the communicative rationality questions served to help students who seemed unable to express their reasoning (epistemic rationality) or tools (teleological rationality) in a clear manner; thus these questions were strongly intertwined with epistemic and teleological rationality.

As discussed above, although in all four units, the teleological rationality component was privileged, the kinds of questions used seemed to be partly affected by the mathematical content of the activities and perhaps the structure of the tasks. As shown from Jill's Unit 2, 2nd lesson, sometimes it may be useful to emphasize a higher proportion of communicative rationality to help students focus on language or display aspects of the situation in order to enable problem-solving. More importantly, instead of using a single privileged rationality component, it may be beneficial for teachers to consider the use of different combinations of components of rational questioning to help the students develop understanding (epistemic rationality), choose and manipulate tools or methods (teleological rationality), and choose the appropriate use of mathematical language and representations (communicative rationality).

In addition, the *frequency* of the rational questions may not completely determine the development of the rational argumentative discourse. As shown from Table 5.6, the reduced number of rational questions asked in Susan's Unit 2 seemed not to impact her students' contributions to the argument components. In other words, compared to Unit 1, Susan's students contributed more argument components in Unit 2, even though she used fewer rational questions to prompt them. It should be noted that Unit 2 happened later in the school year (one month later than Unit 1). Thus, one possible reason for this difference could be that Susan may have established norms related to the students' contributions for the argument components (i.e., the rationality requirements of the argumentative practices had become normative in Susan' class). That is, contributing the appropriate components for arguments may have begun to be a taken-as-shared practice. Therefore, without or with limited supportive actions from the teacher, the students automatically made conjectures (teleological rationality) and provided justifications (epistemic rationality) to support their conjectures using the appropriate mathematical expressions (communicative rationality).

In summary, the results of this study suggest that the use of a privileged rationality component in rational questioning and the number of rational questions posed to organize collective argumentation may be related to a variety of factors, such as the content of the lesson, student responses, and taken-as-shared classroom culture. As suggested by Douek from Boero and Planas' (2014) research forum report, teachers were expected to model the components of rationality for their students at appropriate times and strike a balance between stimulating rational behavior and offering students space to doubt and suspend conclusions.

CHAPTER 6

LEVELS OF TRUTH IN COLLECTIVE ARGUMENTATION

In this chapter, I summarize the findings from the second research question:

RQ2. What techniques and structures of arguments do teachers use when supporting different levels of truth in argumentation? In particular, how do teachers use incorrect answers within their support of argumentation?

Through an iterative analysis process, four levels of truth in argumentation were identified from two participating teachers' argumentation episodes:

Table 6.1

Levels of Truth in Collective Argumentation

Truth Levels	Description
1 Mathematically Correct Arguments	Analytical mathematically correct arguments (corresponding to Toulmin's (1958/2003) description of *analytic arguments*).
2 Not Completely Mathematically Correct Arguments	Not completely mathematically correct arguments but acceptable for the purpose of instruction in-the-moment.
3 Final Arguments were Incorrect or Only Partially Correct Mathematical Statements	Students make incorrect or only partially correct mathematical arguments in-the-moment. The teacher is willing for students to persevere in attempting to make sense of the problem.
4 Incorrect Mathematical Statements or Representations	Resulted in incorrect mathematical statements or representations, include student errors that the teacher does not catch.

In this section, findings were organized by the four levels of truth inferred from the argumentation episodes examined. Findings addressed which types of rational questioning and structure of arguments appeared to be central in facilitating different levels of truth in argumentation. This microanalysis included instances in which students engaged in high-quality complex argumentation (level 1, corresponding to analytic arguments), instances of argumentation that resulted in incorrect arguments (level 4, incorrect arguments), and all levels between these. The episodes selected for inclusion exemplify noteworthy use of rational questioning strategies and structures of arguments recognized by the researcher. In addition, the microanalysis involved looking for finer details of interactions (e.g., student response, non-verbal moves) that were not represented by simply coding each question within argumentation episode. The main aim was to draw out subtle nuances, techniques and habits that teachers exhibit when supporting different levels of truth in arguments.

6.1 Level 1 — Mathematically Correct Arguments

The first level of truth in argumentation requires that mathematical arguments should be understood and accepted as truth statements in the context of the contemporary field of mathematics. This level of arguments corresponds to what Toulmin (1958/2003) called "analytic arguments," which means they are logically correct deductions and generate ideas by formal logic using previously established statements, such as axioms, definitions, or theorems. After analysis of eight lesson videos and transcripts from two participants, I identified two ways argumentation results in correct mathematical conclusions: the first was students constructing completely correct arguments (i.e., claims, warrants, etc) — level 1a of truth in argumentation. The second was students made initial incomplete, incorrect, or ambiguous arguments; and corrected them with the help of teacher or other students — level 1b of truth in argumentation.

6.1.1 Level 1a — Completely Correct Arguments

In total, 62 out of 112 argumentation episodes were 1a level of truth in argumentation across all four units. One of the techniques these two beginning teachers used to support level 1a of truth in argumentation was reflected in their use of “incorrect answers” derived from individual student or students’ group work prior to the whole class discussion. In this case, the teacher provided the incorrect solution (or asked a student to provide this initial incorrect solution) despite knowing the student had subsequently solved the problem in a correct way. The following episode illustrates how Jill used rational questioning to propose “incorrect answers” for students to help them understand more than their explanations for correct answers by drawing out why what they did was wrong at the beginning. In this argumentation episode, students were working on the following problem together in a whole class discussion:

Find the greatest common factor of $3x^2$ *and 12x.*

1. Jill: All right, let’s come together. Okay, let’s come together. I really thought you have lots of good conversations, let’s talk about as a class now. So I heard, I’m going to tell you what I heard when we were talking about this. I heard 3, I heard 3x, and I heard 3x squared. **Let’s discuss those. What do we agree with? What do we disagree with?** (Questioning without a rational component: *N*).
2. S2: 3x.
3. S3: 3x.
4. S4: 3x squared.
5. Jill: Do we agree with 3x or do we disagree?
6. S2: Yes. We agree with it.
7. Jill: **Why is not just 3 then?** (*ER1*: Rational questioning contains epistemic component).
8. S3: Because of the x’s.
9. Jill: Because there is x’s. Yes. Okay. **Is 3 a common factor between those two terms though?** (*ETR2*: Rational questioning contains both epistemic and teleological component).
10. S2: Yes.
11. Jill: It is. But it’s not the, I guess, the greatest common factor. It is 3x. **Why would it not be 3x squared? Someone tells me that.** (*ER3*)
12. S2: 3x squared would be 3x times 3, or 3 times 3 or whatever.
13. Jill: Say that again.
14. S4: Because 12 term tells you it’s just x instead of x squared.
15. Jill: **The term here only has x, and is x squared a factor of x?** (*ETR4*)
16. Multiple Students: No.
17. Jill: It’s not. Okay.

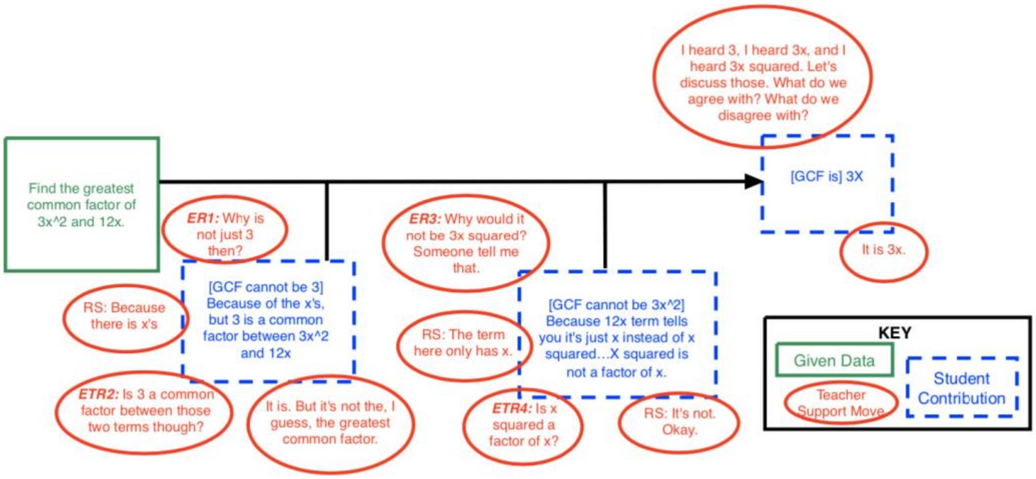
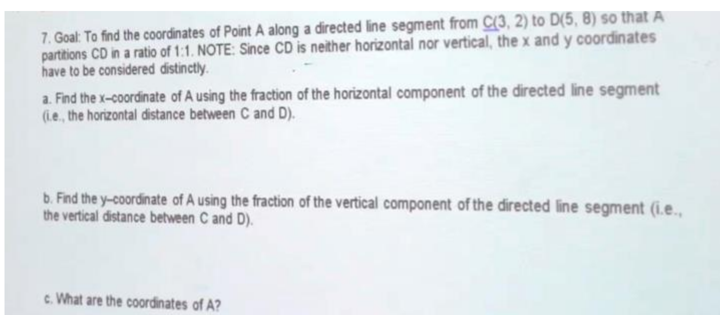


Figure 6.1. Diagram of Unit 1, 1st Lesson, Argument 7, Whole Group Discussion in Jill's Class.

Interpretation. At the beginning of this episode, Jill noticed that most students have gotten the correct answer "3x", but some originally proposed 3 or $3x^2$. Instead of giving direct feedback, Jill challenged students' arguments by asking them to provide explanations of why the incorrect answers "3" and "$3x^2$" they previously provided were incorrect (Lines 7 and 11). Here, the epistemic rational questioning such as "*Why is not just 3 then*?" (*ER1*) and "*Why would it not be 3x squared*? (*ER3*)" illustrate Jill, as a rightful participant in the discussion, created situations that enabled students to reflect on their incorrect initial mathematical thinking and reasoning. Although students' responses for these two questions (Lines 8 and 14) clarified why their initial answers were incorrect, the explanations were brief. Therefore, Jill followed up with rational questioning that contains both epistemic and teleological components (Lines 11 and 15) for more adequate justification. In this context, the combinations of epistemic and teleological rational questioning, for example, "*Is 3 a common factor between those two terms though*?" allowed students to seek connections between greatest common factor and common factor. In this way, using sequences of epistemic and combination of epistemic and teleological rational questioning (*ER-ETR-ER-ETR*), Jill facilitated her students to connect correct arguments with the incorrect data/claims they had previously constructed, and thus helped students identify errors and clarify their understandings. As a result, through the lens of Toulmin's model (see Figure 6.1), students played more active roles in the argumentation (i.e., constructed all argument components), and explicated reasons why the answers they previously provided were incorrect for their own benefit and for the class (i.e., provided more than one warrant towards final correct answer).

In other instances, the teachers created opportunities for student level 1a of truth in argumentation by selecting tasks with multiple solutions or with no right or wrong answers. These tasks are different from the kinds of tasks that involve only one single right answer or solution. In this context, students were asked to construct multiple parallel claims with appropriate justifications to support their statements. In the following whole class discussion from Susan's Unit 2, 2nd lesson, the class was working on a task that focused on partitioning a segment into a given ratio (see Figure 6.2).


7. Goal: To find the coordinates of Point A along a directed line segment from C(3, 2) to D(5, 8) so that A partitions CD in a ratio of 1:1. NOTE: Since CD is neither horizontal nor vertical, the x and y coordinates have to be considered distinctly.

a. Find the x–coordinate of A using the fraction of the horizontal component of the directed line segment (i.e., the horizontal distance between C and D).

b. Find the y–coordinate of A using the fraction of the vertical component of the directed line segment (i.e., the vertical distance between C and D).

c. What are the coordinates of A?

Figure 6.2. Task for Unit 2, 2nd Lesson, Argument 16, Whole Group Discussion in Susan's Class.

1. Susan: **Who wants to come do number seven on the board?** (*ETCR1*: Rational questioning contains all three rational component).
2. S3: I'll do it.
3. Susan: Yay, thank you. [inaudible] Okay, and don't forget to explain what you're doing ... and y'all pay attention. We're doing number seven.
4. S3: Yeah. Okay. So first, I graphed it ... and it's only in the first quadrant since they're both positive. Three, two. Five, two. [draw the picture on board] It's not in scale, by the way.
5. Susan: It's okay, we can handle it.
6. S3: So to find the X coordinate of A, which is the ... it's splitting this line here in a ratio of one to one, so basically it's splitting it in half since it's the distance between A to ... C to

... D. [Label the points] So, the distance between A to D, wherever A may be, is equal to the distance between A and C. To find the X coordinate basically I used this formula with the coordinates. X2 minus X1 squared, plus ... Okay, and what this is X2 would be where X of d, so it'd be five. So it would be five minus, and X1 would be three, so five minus three, squared. Anything is in a square root, plus Y2 minus Y1, which would be eight minus two. The same thing there. So I figured out it's two squared, plus six squared equals d. And two squared is four, six squared is ... 24 ... [36]. Oh. [crosstalk] So, it would be 40 equals [crosstalk]. Which would be 2 square root of 10, if you simplify it. So the distance here is 2 square root of 10 ... when you split it ... You know what? This is not how I did it. I just realized it.

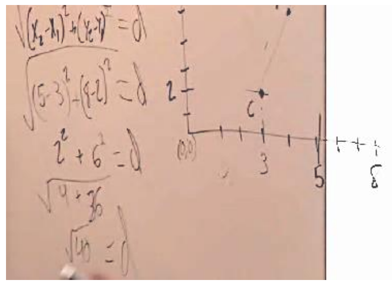


Figure 6.3. Screenshot of S3's First Drawing from Unit 2, 2nd Lesson, Argument 16.

7. Susan: (laughs) **Although what you're doing is going to work, and S6, does this look familiar?** (*TR2*: Rational questioning contains teleological component).
8. S6: Yes.
9. Susan: This is how you were initial trying to do it, and [crosstalk].
10. S3: There's a much simpler way to do this.
11. Susan: **Okay, so what's the much simpler way?** (laughs) (*ETR3*).
12. S3: So basically ...
13. Susan: You guys crack me up.
14. S3: So I find the point which is, since I know it's split in half, I find the point on the X value that is between three and five, so it's four. So that's how I know that the X coordinate's going to be four, because it's in the middle of the two.
15. Susan: Mm-hmm (affirmative).
16. S3: And then and this was really just working because it's a ratio of one to one, which is split in half, so this is why this is the simple way to do it, and this would work. The way I was doing it before would work, but just this is really easy in this problem. So then I'll find the distance between these two, which is six. Like, no. Five. Equals five. Six ... Yeah. So then, I basically draw up on here [draw on board], and that's A, and that's between the two, and that is split.
17. Susan: **Do you guys agree? Is that what you got?** (*ECTR4*).
18. S3: The point is (4, 5).
19. Susan: **The point is (4, 5)? Did you guys get that?** (Questioning without a rational component: *N*).
20. Multiple Students: For which one?
21. Susan: For part seven.
22. Multiple Students: Yeah.
23. Susan: **S3, what is the length of the line CA?** (*TR5*).

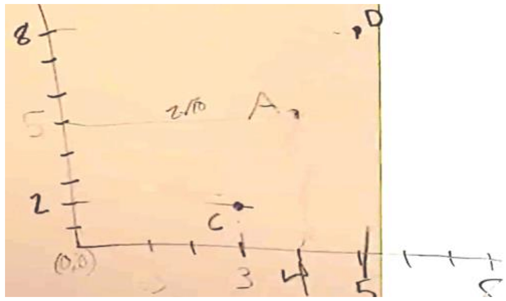


Figure 6.4. Screenshot of S3's Second Drawing from Unit 2, 2nd Lesson, Argument 16.

24. S3: CA is 10.
25. Susan: Square root of 10.
26. S3: Square root of 10.
27. Susan: Mm-hmm (affirmative).
28. S3: And then AD is 10 because we add them together ... oh, square root of ten. Sorry. Then we you add them together, you get two squared roots of ten.
29. Susan: Very good. Yes. This is the long way that S6 and S8 tried and were like, I don't really know how to find that exact distance. But this is also another way to do it. Leave it there, you're good.

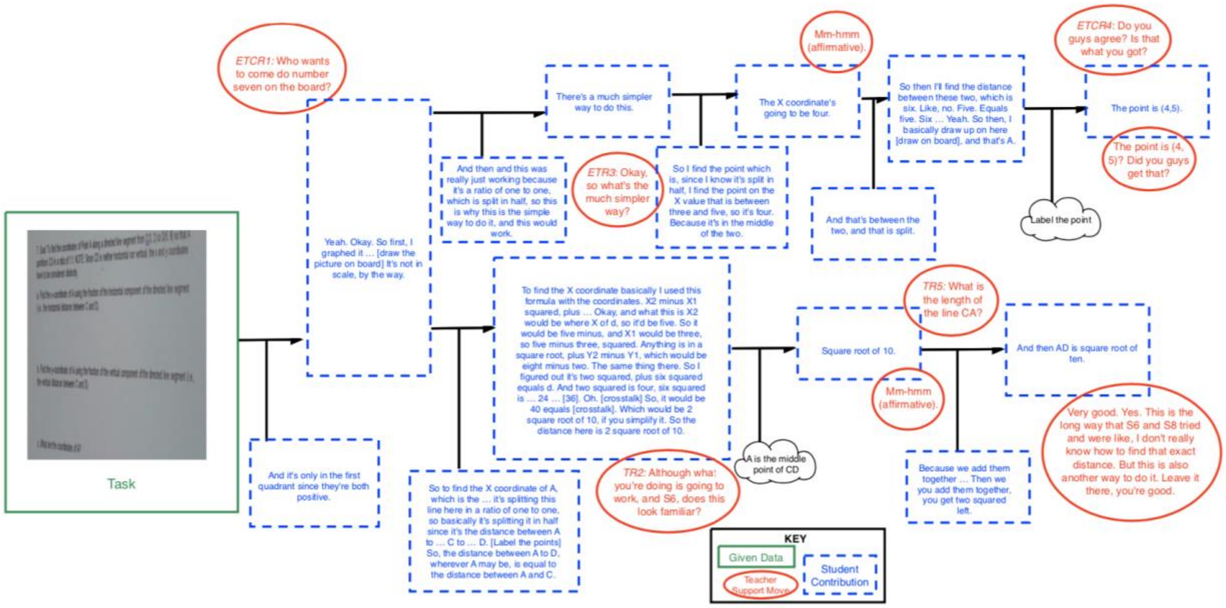


Figure 6.5. Diagram of Unit 2, 2nd Lesson, Argument 16, Whole Group Discussion in Susan's Class.

Interpretation. In this episode, Susan gave students sufficient time to generate and explore their own ideas and different solutions (Lines 4 to 16). At the beginning of the whole class discussion, Susan invited students (Line 1, "*Who wants to come do number seven on the board*?") to come to board to state problem-solving procedures, which provided opportunities for students to control the ownership of discussions and activities. In this way, Susan promoted students to be active contributors of argument components (from Toulmin's lens almost all components were contributed by students). Although the first solution S3 came up with is not the most efficient one (Line 6), Susan did not interrupt; instead she refrained from stepping in. By doing this, she provided opportunities for the student to explore a better (more efficient) way to solve the problem (Line 11).

As shown in line 18, despite students arriving at the correct final answer (4,5), Susan directed students to go back and explore the length of line segment AD, which they can get to build on the first solution. At this point, the teacher's teleological rational questioning (Line 23) served to support the development of sociomathematical norms (Yackel & Cobb, 1996) with regards to what counts as a mathematically different solutions and what counts as a mathematically efficient solution. As shown in Figure 6.5, the structure of argumentation was parallel with respect to two independent solutions that were proposed with corresponding justifications.

There were a few cases in which the teacher engaged students in level 1a of truth in argumentation by choosing a task with no right or wrong answers. As an illustration, in the following argumentation episode, Jill engaged her students in discussion of similarities and differences between the following two patterns in a whole class discussion.

Pattern 1: 3, 4, 6, 9, 13, 18, ...

Pattern 2: 3, 7, 11, 15, 19, ...

1. Jill: All right. **I would like one person to tell me, raise your hand, tell me what you found that might be similar. S2**. (Questioning without a rational component: *N*).

2. S2: They both start with a three.
3. Jill: They both start with a three. I'd call that a similarity. **Okay. Someone else, raise your hand. Did anyone else have another similarity? This could be anything. Nobody else found any similarities at all?** (*N*).
4. S3: Besides that one?
5. Jill: Besides that one. All right. Now, I want someone else to raise their hand and tell me a ... actually. **S4, tell me what's the difference that you just found out was.** (*N*).
6. S4: On the first one, it's adding, for three to go to four, you add one. And then, four to go to six, you add these.
7. Jill: Keep going.
8. S4: And then, four is ... the second one here is just adding four.
9. Jill: Okay. So for the second one, you're just adding four. **Be more clear about what you were saying here. You're right.** (*CR1*: Rational questioning contains communicative component).
10. S6: Yeah. Adding the next number.
11. Jill: So she says from one term to the next [point at 3 and 4 in the first pattern], then from the next one to the next one [point at 4 and 6 in the first pattern], you are adding the next number. **So S5, you want to tell me what you said about that? I like what you wrote. What'd you write?** (*TCR2*: Rational questioning contains teleological and communicative component).
12. S5: The increase raises.
13. Jill: The increase raises. **So somebody else tell me what he means when he says the increase raises. Use different words than S4 used. A lot of you had this on your paper. I know you did. How can we say that in a different way? This is all correct.** (*TCR3*)
14. S3: [Inaudible] on every single topic.
15. Jill: **What is increasing by one though? Are these numbers increasing by one?** (*CR4*)
16. Multiple Students: No.
17. Jill: **What is increasing by one?** (*CR5*)
18. S3: The difference.
19. Jill: The difference is increasing by one. **So S7, what did you have on your paper**? (*ETCR6*)
20. S7: You go up by one more than the last time, every time.
21. Jill: You go up by one more than the last time, every time. So the difference is what's increasing. Awesome.

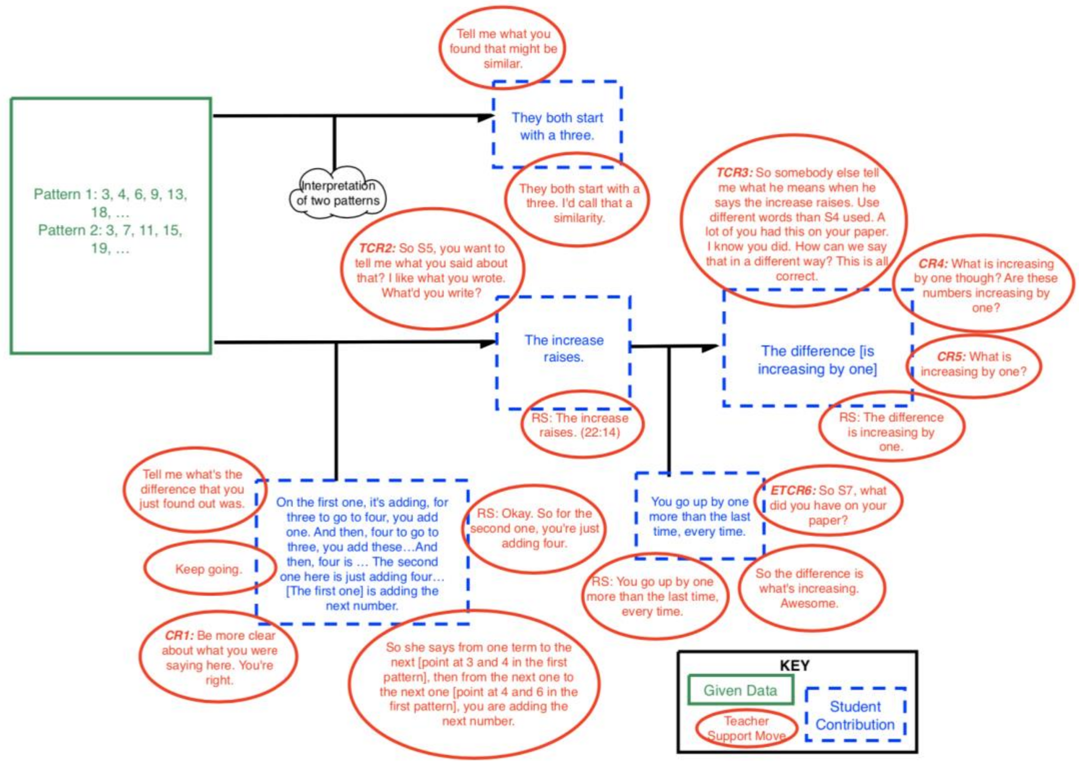


Figure 6.6. Diagram of Unit 2, 1st Lesson, Argument 4, Whole Group Discussion in Jill's Class.

Interpretation. This example shows how Jill created time and space in the discussion so the student could experience participation in level 1a of truth in argumentation to achieve more than one right answer. In this episode, Jill invited multiple students to join in an open discussion to provide justifications for why the two patterns are same or different. As shown in Figure 6.6, the structure of argumentation was parallel with respect to two independent claims that were proposed with corresponding justifications to support different conclusions. Jill used multiple sequences of communicative rational questioning (Lines 9 to 19) to help the students use precise mathematics language and appropriate mathematical representations to communicate their ideas. By specifically asking a student to "*Use different words than S4 used...How can we say that in a different way*?" (Line 13) and creating context for other students to share their thinking: "*So S7, what did you have on your paper*?" (Line 19), Jill highlighted her expectations for students to be able to learn from, and contribute to, the learning of others. At the end of this episode, the students attempted to meet these expectations, resulting in their expressed warrants and claims (Lines 18 and 20).

In sum, as illustrated from the above three episodes, two main structures were recognized when teachers supported completely correct arguments (i.e., level 1a of truth in argumentation). One way was using sequences of epistemic rational questioning or combinations of components of epistemic rational questioning (i.e., *ETR*, *ECR*) to press students to provide explicit reasons for their initial incorrect answers, juxtaposed with their rationales for correct answers. The structure of argumentation in this case was basically a chain with multiple warrants leading to one final correct claim. The second was using open-ended tasks with a focus on using teleological rational questioning to discuss different methods of solution (second episode) or using communicative rational questioning to prompt students to elaborate their different ideas and thinking that are understandable in the given mathematical classroom community (third episode). The structure of argumentation in this case was characterized by two (or more) independent correct chains of reasoning that are parallel to each other. More importantly, these two structures with associated patterns of questions illustrate how the teachers normalize valid ways of participating in level 1a of truth in argumentation in the given classroom community. For instance, in all three episodes, students contribute to all argument components as shown in Figures 6.1, 6.5 and 6.6, which suggests that in both Jill's and Susan's class students are expected to provide warrants (reasons) to justify their claims when engaging in argumentative activities.

6.1.2 Level 1b — Initial Arguments are Incomplete, Incorrect or Ambiguous

Although in these cases, correct answers resulted from the argumentation, sometimes students made incomplete, incorrect or ambiguous arguments in the process. In total, 42 out of 112 argumentation episodes were 1b level of truth in argumentation across all four units. Unlike level 1a, the structure of level 1b of truth in argumentation involves rebuttals (statements describing circumstances under which the warrants would not be valid). Through Toulmin's lens, I investigated how rebuttals were constructed building on students' initial incorrect claims. In this section, I discuss two different ways participating teachers used rebuttals within argumentative practices.

This first episode illustrates how Susan probed for the student's initial incorrect reasoning. This episode occurred on day 1 of the Unit 1 lesson, in which Susan was working with two girls in a small group on an activity that was aimed at matching given exponential equations with provided graphs. In the following conversation, they were discussing the percent change of the exponential function: $f(x) = 10 \times 1.5^x$.

1. Susan: Oh, clever. **So you think ten times one point five to the X is a percent change of fifty percent, how did you get that?** (*TR1*: Rational questioning contains teleological component).
2. S19: One point five is. Hmm- no, I don't think that's right. I take it back. I did that, too, I put that there, but then I was like, "No, I take it back."
3. Susan: **So why do you take it back?** (*ER2*)
4. S19: Because that's one and a half, not one-half. And I thought it's one half.
5. Susan: Oh. Great point. **Okay, but what happened to one-half?** (*TR3*)
6. S19: So, I guess...
7. S20: No, no, no, that makes sense, because if that one's zero then that has to... did that?
8. Susan: Okay. **Articulate what you are thinking.** (*ETCR4*)
9. S20: Okay, so it's like half of this is like negative fifty. And then one is zero, then it couldn't just be like, up another half. It would have to be like one and a half.
10. S19: So you're saying zero and a half added together
11. S20: Yeah. [crosstalk]
12. S19: Is a half.
13. Susan: **What in the half? I'm confused. Say that again?** (*ECR5*)

14. S19: Well, I thought she was taking this one and this one, like these two putting it together, because she's saying that if, because this is one and a half. So, if it's increasing by one, it's not going to change. But, if you're increasing by one and a half, then it's only going to change by half.
15. S20: Yeah. That's essentially what I'm trying to get at, but I don't; I don't know how to say it. I don't really understand it, but like it makes sense, okay.
16. Susan: I like that! I like your reasoning. You used your previous knowledge, just like you did here. So you knew a half was negative fifty percent, you knew that if it had a growth factor of one, and it really wasn't changing at all, and then you knew that if it had a growth factor of one and a half, it was increasing by fifty percent. So, now, just keep using that same knowledge.

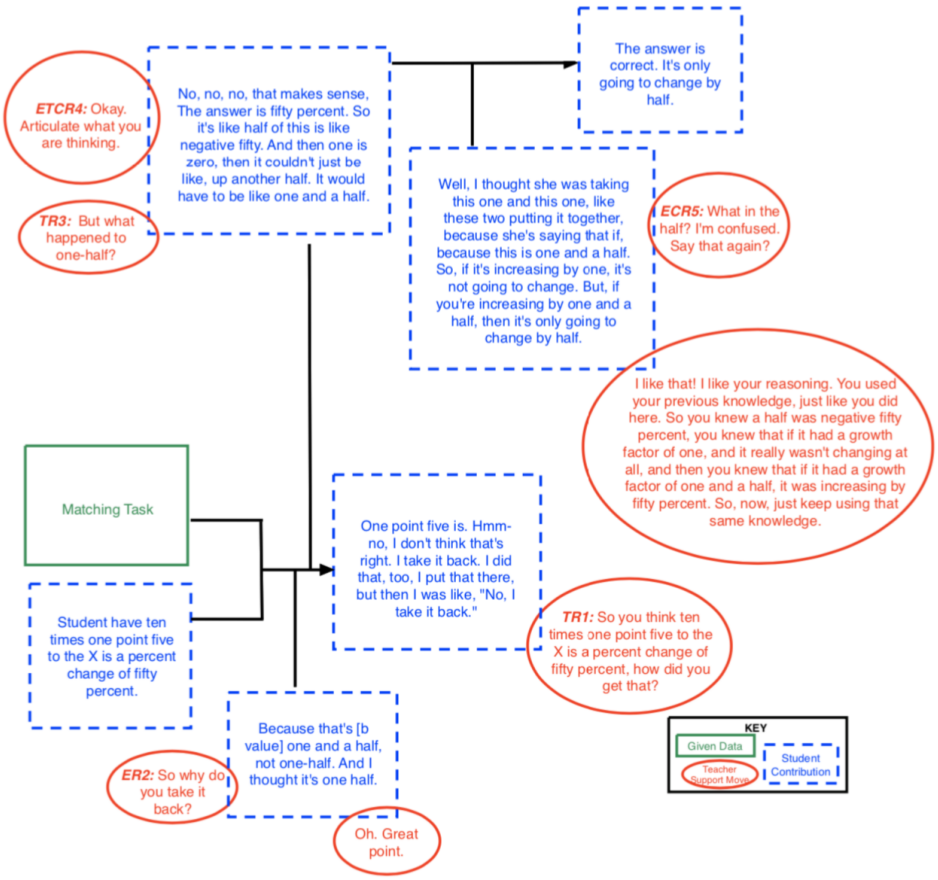


Figure 6.7. Diagram of Unit 1, 1st Lesson, Argument 14 (Part c), Small Group Discussion in Susan's Class.

Interpretation. At the beginning of this episode, S19 took back her initial correct statement that the percent change of exponential equation $f(x) = 10 \times 1.5^x$ is 50 percent. Instead of giving direct feedback, Susan posed an epistemic rational question, "*So why do you take it back*?" (Line 3) to request an elaboration to uncover the reasoning of the error made by S19

(Line 19). Using S19's reasoning as evidence, Susan directed the students to focus on a specific element of her explanation by asking a teleological rational question, "*Okay, but what happened to one-half*?" In this way, Susan challenged students' thinking and prompted the students to reflect on and reconsider their answer. However, a single teleological rational question was not enough to lead students to give a correct and complete rebuttal against their initial incorrect warrant. At this point, Susan's open-ended rational question (Line 4, "*Articulate what you are thinking*.") not only encouraged students to express their detailed thinking, but also prompted students to contribute rebuttals based on incorrect explanations, and thus, continued the argumentative discourse.

This episode illustrates Susan using epistemic rational questioning to uncover a student's incorrect reasoning and using repeated teleological rational questioning (*TR-ETCR*) to help students construct rebuttals based on an initial incorrect warrant. As a result, through the lens of Toulmin's model (see Figure 6.7), students played more active roles in reflecting on their own answers and taking a responsibility to correct themselves (i.e., construct rebuttals). The structure of this type of argumentation was usually a chain with at least one rebuttal involved. In this way, Susan created a context to foster students' development of what Yackel and Cobb (1996) called "intellectual autonomy" in mathematics, that is student should be "aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgements as they participate in mathematical practices" (p. 473), which is an essential goal of using collective argumentation in instruction.

A second episode illustrates how Jill dealt with students' incorrect answers and directly contributed a rebuttal to orchestrate level 1b of truth in argumentation. This conversation took place in a small group in which students were working on the following problem:

Given $x^2 + 8x +$ ____, what are the possible values for the blank so that the expression can be factorable form?

1. Jill: **I think you got one. What could go there?** (Questioning without a rational component: *N*).
2. S2: 12.
3. Jill: 12 could go there. **Why?** (*ER1*: Rational questioning contains epistemic component).
4. S2: Because it's on the left.
5. Jill: **That's not good enough.** (*ECR2*)
6. S3: Because two plus six equals eight and two times six equals 12.
7. S2: Yeah.
8. Jill: Thank you. Okay. So I could put 12 there.
9. S5: Who said it better?
10. Jill: What else?
11. S2: She did.
12. Jill: **What else could go there? That's correct.** (*TR3*)
13. S4: 15.
14. Jill: 15. **Why do you think 15 could go there?** (*ER4*)
15. S4: Because five plus three equals eight and five times three is 15.
16. Jill: Because five plus three is eight and five times three is 15. Good. **Anything else?** (*TR5*)
17. S6: Six.
18. Jill: Six. **Why could six go there?** (*ER6*)
19. S6: Oh. Never mind. Yeah. Six. Four times three is six. Four plus two ... four times two is eight. Four plus two is six.
20. Jill: That's true. But think about what you just said. I'll repeat what you said. You said four times two is eight and four plus two is six. Look where the blank is.
21. S7: 24.
22. Jill: **Why could 24 go there?** (*ER7*)
23. S7: Because four plus four is eight and four times four is 16.
24. Jill: I like the four plus four is eight.
25. S6: Four times four is 16.
26. Jill: But four ...
27. S7: Oh, yeah.
28. Jill: But then I could put that ... I could put 16 there. Right.
29. S7: Yeah. 16.
30. Jill: Good. Good logic.

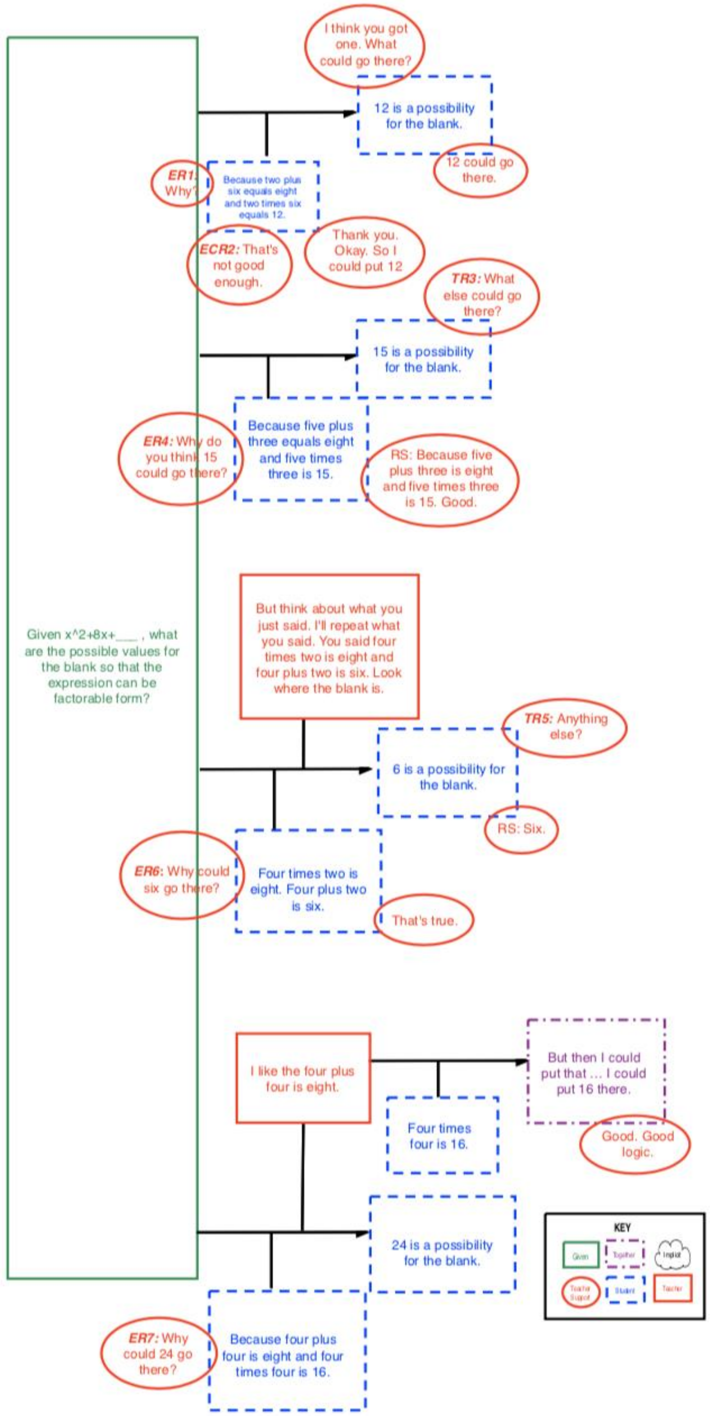


Figure 6.8. Diagram of Unit 1, 2nd Lesson, Argument 7, Small Group Discussion in Jill's Class.

Interpretation. The small group discussion started with Jill asking for the possible values for the blank so that the given expression would be factorable. Jill used a series of epistemic rational question to push her students to justify why their arguments were true (e.g., "*Why do you think 15 could go there*?", Line 14). It is important to note here that Jill never identified S6 and S7's answers (Lines 17 and 21) as being wrong or incorrect. In fact, she asked them to explicate their reasoning so that she could understand students' mathematical thinking and thus be able to provide rebuttals in response to students' incorrect explanations. The rebuttals provided by the teacher served as a new data (Line 24) to lead students to work towards correct answers. Although in this context, the teacher may change the direction of the discussion, the interventions from the teacher (i.e., contributing rebuttals) are necessary with the purpose of guiding argumentative discourse to achieve correct mathematical conclusions. The structure of this type of argumentation consisted of several parallel claims with rebuttals for incorrect claims.

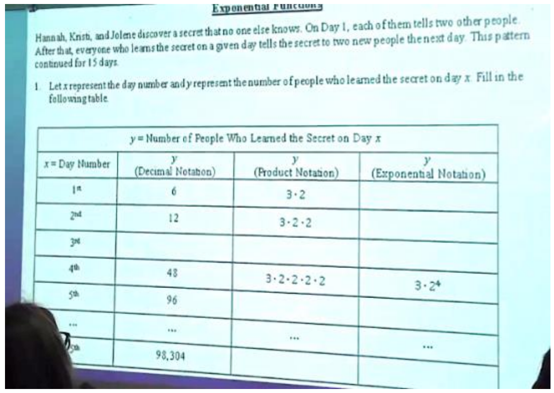
In conclusion, when students' initial arguments are incomplete, incorrect or ambiguous, as a representative of the mathematics community in the classroom (Yackel & Cobb, 1996), the teacher plays an essential role in guiding students to make sense of their mistakes and thus provide complete and correct warrants and claims. In this section, I presented two different manners of managing students' incorrect arguments by constructing rebuttals via the teacher's direct intervention, which resulted in final correct mathematical arguments (i.e., level 1b of truth in argumentation). One way is using sequences of teleological rational questioning or combinations of components of teleological rational questioning to help students construct rebuttals based on their initial incorrect warrants; the other way is the teachers themselves providing rebuttals to help students identify errors or clarify misunderstanding. Both ways highlight the importance of teachers using epistemic rational questions to elicit students' mathematical thinking, especially when their arguments are incomplete, incorrect or ambiguous (Frank et al., 2009). Moreover, as shown in two above episodes, a single epistemic or teleological rational question was not enough to recognize their warrants were incorrect or lead students to give a correct and complete rebuttal against their initial incorrect warrants; sequences of questions that concentrate on students' incorrect arguments are required. This result provides empirical evidence to support Frank and colleagues' (2009) contention that a single specific question is not enough to elicit a complete explanation or justification. The structure of level 1b argumentation included at least one rebuttal but could otherwise be structured as a chain or as multiple parallel chains.

6.2 Level 2 — Not Completely Mathematically Correct

When students engage in classroom-based argumentation, it is common for them to make mistakes and construct partially correct claims in the process of constructing conjectures and justifying conjectures, and at the moment this may result in invalid or not completely mathematically correct claims or warrants from the perspective of mathematicians or a mathematical community. In this study, I recognized the existence of instances in which students' final arguments were not completely mathematically correct, and the teachers were aware of the presence of mistakes or incomplete mathematical ideas but decided not to address it in the moment for the purpose of instruction or other reasons such as time limitation or instructional goals. The first case was students made partial or incorrect mathematical arguments (i.e., claims, warrants, etc.) when they participated in inquiry-based practices or tasks within argumentative discourse — level 2a of truth in argumentation. Another was students' final answers that did not include all possible correct arguments — level 2b of truth in argumentation.

6.2.1 Level 2a — Partial or Incorrect Mathematical Arguments

In total, I found eight argumentation episodes illustrated this level of truth in argumentation. The following argumentation episode demonstrates how Susan strategically tackled students' incorrect answers. This episode occurred on the first day of Unit 1, in which Susan was teaching the topic of exponential functions in a small group with the following task:



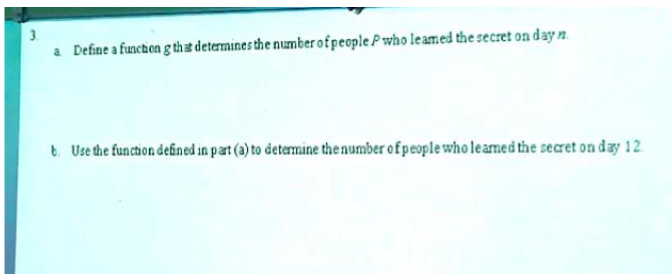


Figure 6.9. Task 1 and 3 for Unit 1, 1st Lesson, Small Group Discussion in Susan's Class.

4a. Now consider different situations, suppose that four people initially knew the secret and on day one, they each tell five people. Now apply the function you created in exercise three to reflection this change.

1. Susan: Wow, you guys are already to here? Okay, let's see. Now consider different situations, suppose that four people initially knew the secret and on day one, they each tell five people. Now apply the function you created in exercise three to reflection this change. **Okay. So, how did you come up with this function, S13?** (*TR1*: Rational questioning contains teleological component).
2. S13: I saw that four people were [crosstalk] where two were... Three were [inaudible].
3. Susan: Okay.
4. S13: And then, they [the number of people] were increasing five every day.
5. Susan: **So each of the four people would tell five people every day?** (*ECR2*)
6. S13: Right.
7. Susan: **So, how did you come up with four times five to the X versus five times four to the X?** (*TR3*)

8. S13: I saw that four was the original amount people told... and then they told five people. So that was the first day. And then you would multiply that by five again to find how many people done another day.
9. Susan: **Okay, and S14, what quantity is varying in number 4A? What's changing and what's staying the same?** (*TCR4*)
10. S14: The day is... The day number is changing.
11. Susan: All right, the day number is changing. Mm-hmm (affirmative)-
12. S14: But the number of initial people who knew is still the same.
13. Susan: Right, yeah. So, the number of initial people who knew is still the same, so they should not be affected by the day number. Right? So, is... So, this is good because you have five to the X and that's what's changing, but these four people are not being affected by what day it is. It's still just for people at the initial time.

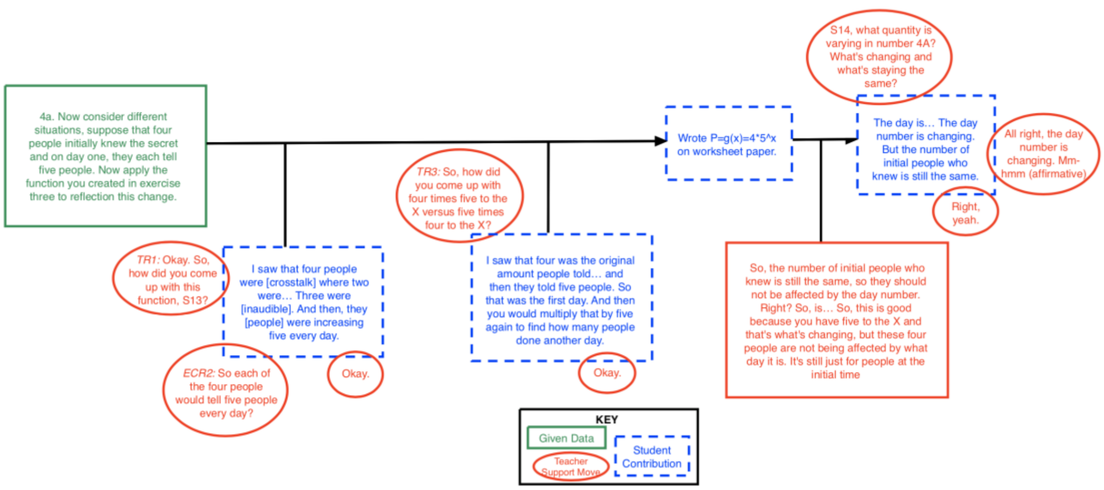


Figure 6.10. Diagram of Unit 1, 1st Lesson, Argument 6, Small Group Discussion in Susan's Class.

Interpretation. Although S13 got the correct answer $P = g(x) = 4 \times 5^x$, Susan posed a teleological rational question (Line 1) to challenge S13 to show the strategic tools he used to achieve the argument. S13, however, was unable to provide a clear interpretation in regard to exponential function $P = g(x) = 4 \times 5^x$ and made an incorrect explanation that "*And then, they [the number of people] were increasing five every day*." (Line 4). While Susan noticed S13's mistake, she did not directly point the mistake out or continually push S13 to recognize and attend to the process of correcting the error by asking S13 to clarify incorrect parts of his explanations. Instead, she redirected S13 to think about the original question in a different way by providing other possible incorrect answers, "*So, how did you come up with four times five to the X versus five times four to the X*?" (Line 7). Susan used this strategy to lead S13 to give a correct and better justification (Line 8) for his correct answer. I found both teachers used this strategy to prompt additional warrants for correct or incorrect arguments. These results supported Schleppenbach et al.'s (2007) finding that simple "why" questions may not be the only way to query students' errors. This episode showed that using teleological rational questioning to redirect students to think about the original question in a different way may be equally valuable. Moreover, the teacher proposed common incorrect answers for students (in this case "*five times four to the X*") help them build more comprehensive explanations by connecting correct and incorrect ideas and claims.

There were other cases when students engaged in an inquiry-based task, for example, in Jill's lesson when students were checking whether the equation satisfied the given pattern or not. Students only checked the first term and then concluded that the function they got was correct. Without pointing out the mistake, Jill asked students to try a second term to be sure the equation was correct. As shown in Figure 6.10, the structure of level 2a argumentation often included additional warrants for students' further explanations of their initial partial or incorrect arguments.

6.2.2 Level 2b — Not Including all Possible Correct Arguments

Sometimes when students were working on a task with multiple correct answers, they may not have been able to figure out all the answers at the same time. As an illustration, let us consider an argumentation episode from Jill's Unit 1 day two, in which students had reviewed the greatest common factor and expansion of binomials with form $(x \pm a)(x \pm b)$ on the first day of the lesson. During this episode, the students were learning about factoring trinomials with integer coefficients in a small group:

Given $x^2 + \_\_\_ x + 12$, *what are the possible values for the blank?*

1. Jill: **All right, what do we think?** (Questioning without a rational component: *N*).
2. S1: It's six or nine.
3. Jill: Six or nine.
4. S2: Yup.
5. Jill: **Tell me why.** (*ER1*: Rational questioning contains epistemic rational component).
6. S1: Tell her why S2.
7. S2: Why do I have to tell her. Oh. Um, okay, so couldn't, couldn't like...
8. Jill: Hang on. I want to hear 6 or 9 explanations first. (*N*)
9. S2: Oh gosh. Could you say the 9 explanation and I say 6 explanation?
10. Jill: **Tell me the 6 explanation.** (*ER2*)
11. S2: Okay. So, 6 times 2 is 12.
12. Jill: Yes, 6 times 2 is 12. That's true.
13. S2: Yeah, and then 6 might not work, 6 wouldn't work.
14. Jill: **Why not? Talk to me about why 6 might not work.** (*ER3*)
15. S2: Because 6 plus 2 is 8 and you have to have 12 and so because [mumbling]
16. Jill: **Hang on, hang on. You are saying things that are on the very right track. Think through it.** (*ETCR4*)
17. S2: Okay. 6 plus 2 is 8 but yeah do not even know where 12 like, how are you supposed to like, do you know what I am saying it's like
18. S1: You can put 8 in here. That's the point.
19. S2: Yeah.
20. S1: So we are trying to find the line, what goes on the line up here.
21. S2: Yeah.
22. S1: You can put 8 but 6 times 2 is 12 and then 6 plus 2 is 8.
23. S2: So it's 8.
24. S3: Yeah.
25. Jill: Yes. You are thinking about it in the right way. You said 8. That's okay. That's why I want you to think about it. **Now does that make sense?** (*ETCR5*)
26. S2: Yeah.
27. Jill: **So does anyone come up with another number besides 8 that could go there? Anybody come up...S4, why could you do 7?** (*ER6*)
28. S4: Oh gosh.
29. Jill: **You said you could do it. Why?** (*ER7*)
30. S4: 4 plus 3 is 7 and 4 times 3 is 12.
31. Jill: Very good. **Is there anything else?** (*TR8*)

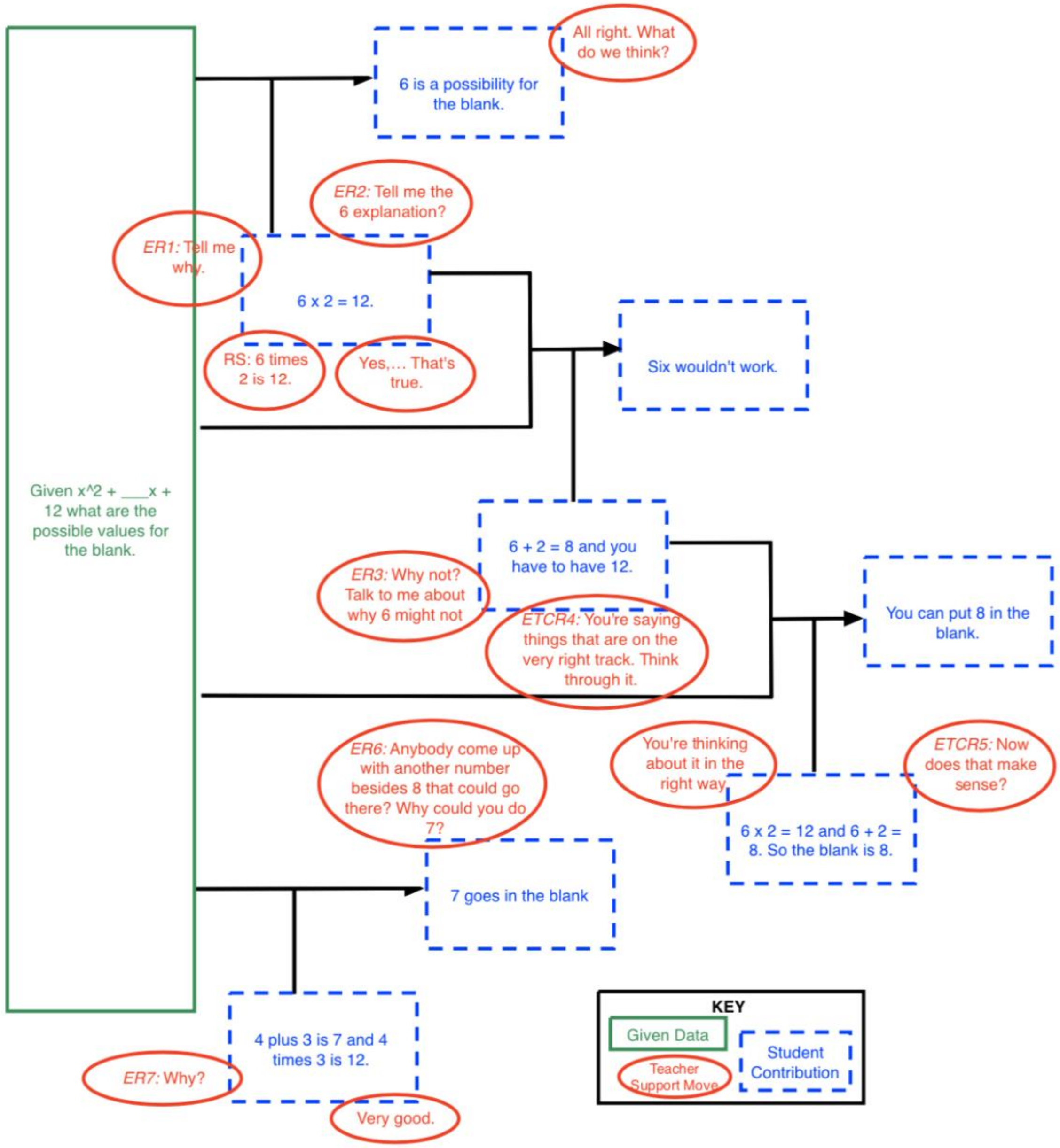


Figure 6.11. Diagram of Unit 1, 2nd Lesson, Argument 5, Small Group Discussion in Jill's Class.

Interpretation. At the beginning of this episode, both students provided incorrect answers. Without letting the student know they made an error, Jill challenged students' arguments by asking epistemic rational questions that request them to provide an explanation of incorrect answers (Lines 5, 8, and 10). These questions also illustrated that the teacher has a special role to play in trying to develop classroom social norms to address expectations for student participation in argumentative practices through ongoing negotiations. In this context, students were expected to provide warrants, reasons, or backings to justify their claims (social norms). Through the explanation of her arguments, S2 noticed that 6 was incorrect and worked towards the correct answer 8. However, she lacked the confidence to further articulate the justification in her thinking. At this point, the teacher encouraged her to explain her reasoning (Lines 16 to 17), which revealed again that students are expected to provide reasons to justify their claims (social norms). The student-student interactions (Lines 18 to 26) illustrated that pushing students to justify why their arguments hold served to support students to understand that the acceptable claims are based on mutual understanding and agreement. The question "*Now does that make sense*?" (Line 27) showed how Jill provide students with opportunities to make sense of other students' epistemic, teleological and communicative requirements of argumentative practices. It also pushed students to learn from each other, which promotes their productive disposition toward mathematics to reach a consensus or a shared understanding in argumentation.

When S4 came up with the answer 7, which was different than others, Jill called on her to explain why this could work (Lines 30 to 31), which established the expectations for students in the class to share their thinking, ideas, and solutions, even if they have answers that differ from other students' answers. At this point, the teacher's epistemic rational question served to help students understand what counts as a mathematically different solution. The final claims of this argumentation were 8 and 7 could work while 6 cannot. Notice that the answers provided here did not include all possible values. Due to the time limitation of small group discussion, Jill asked students to keep thinking of any other possible values that might exist (Line 35) at the end of this episode, but these additional values were not included in this episode. As shown in Figure

6.11, the structure of level 2a argumentation consisted of several individual parallel claims (i.e., possible correct arguments) supported with multiple warrants.

Although this argumentative discourse did not result in level 1 of truth in argumentation (i.e., analytic mathematically correct arguments), Jill created a classroom-based argumentation context involving incorrect and not all possible correct answers that support the development of social norms alongside the sociomathematical norms. According to Cobb et al. (1992), in this type of classroom, students understood they were expected to justify or explain the reasoning behind their claims, even though their arguments were incorrect or different than others. In addition, other students accepted the presence of errors and knew that they should challenge all answers and conjectures, whether they were right or wrong.

In sum, unlike level 1 truth of argumentation, level 2 truth of argumentative practices may not arrive at objective truth of mathematical claims and warrants from a mathematician's perspective. However, from the teachers' perspective, students' incompletely correct mathematical arguments were acceptable in-the-moment due to the instructional goals or other contextual factors (e.g., time limitation, grade level). Moreover, as shown in above two example argumentation episodes, teachers could strategically deal with student errors to support ongoing collective mathematical argumentation: one way was using teleological rational questions to redirect student thinking, and thus prompting student to make acceptable explanations; the other was giving students an opportunity to notice and correct the error themselves and continually figuring out other possible answers. In this way, the teacher supports students' gradual appropriation of classroom-based argumentation in a long-term teaching intervention in terms of norms (i.e., social norms and sociomathematical norms), even if parts of arguments are incomplete or mathematically incorrect in-the-moment.

6.3 Level 3 — Final Arguments were Incorrect or Only Partially Correct Mathematical Statements

Research on discourse in mathematics classrooms (e.g., Hufferd-Ackles et al., 2004; Hoffman et al., 2009; NCTM, 1991; Schleppenbach et al., 2007) has suggested that teachers are expected to strike a balance between giving students an opportunity to notice and correct errors themselves and being concerned with correcting an error immediately. There is a fine line between letting the discussion end with an error or only partially correct mathematical statements so that students could persevere in attempting to make sense of the problem and giving students direct corrective feedbacks for their errors. Through my analysis, I found several examples that demonstrate how both teachers provided opportunities for students to work through their errors but also ensured that students left the conversation on their way toward correct answers. As an illustration, see the following argumentation episode from Jill's Unit 1, 2nd lesson, in which Jill was working with an individual student to help her figure out how to factor $x^2+8x+15$.

1. Jill: Actually, I want to ask you about this one. You have number one, you have X plus two and X plus six [$(x+2)(x+6)$on student worksheet paper]. **Tell me about why this works, if it works.** [point at $x^2+8x+15$] (*ER1:* Rational questioning contains epistemic rational component).
2. S5: Because two plus six equals eight.
3. Jill: That's true.
4. S5: Then, two times... Oh, I was thinking of 12.
5. Jill: Okay, let's rethink that. **Now that it's 15, tell me how we're going to find the correct answer.** (*TR2*)
6. [Student erase the answer $(x+2)(x+6)$]
7. Jill: **If that had said 12, you would have been right. How can we think about it this time to get this?** [point at 15] (*TR3*)
8. S5: Eight.
9. Jill: Keep thinking.

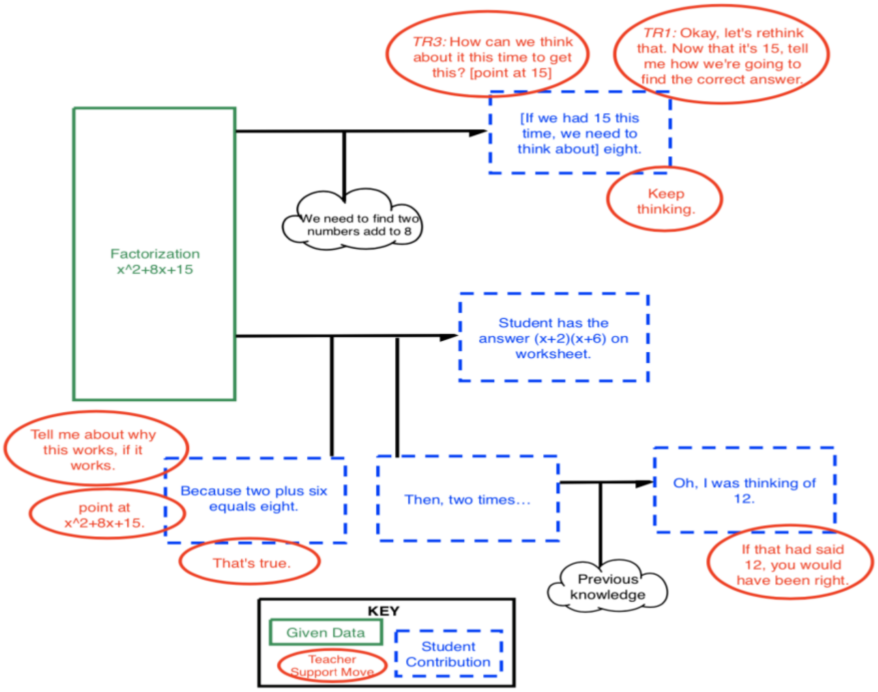


Figure 6.12. Diagram of Unit 1, 2nd Lesson, Argument 12 (Part b), Small Group Discussion in Jill's Class.

Interpretation. Jill noticed S5 got an incorrect answer, in that she wrote down $x^2+8x+15 = (x+2)(x+6)$ on her worksheet paper. Jill used an epistemic rational question to allow S5 to explicate her thinking (Line 1), and then build on S5's idea. Jill tried to lead her to formulate correct strategies. As shown in this episode, discussing a student's incorrect strategy served to help S5 to realize that $(x+2)(x+6)$worked for $x^2+8x+12$ instead of the original binomial $x^2+8x+15$. As demonstrated in this example, Jill provided students opportunities to reflect on and reconsider their answers. Jill continued challenging S5 to seek tools to find two numbers for which the product was the constant coefficient 15 by using sequences of teleological rational questions (Lines 5 and 7). S5, however, was unable to figure out the solutions in-the-moment (Line 8). Jill did not give S5 any hints or suggestions what she should do; instead, she encouraged her to keep thinking because Jill noticed that S5 knew she had to find two numbers that the sum of two numbers should go to the linear term coefficient 8 while the product went to the constant 15. At this point, I inferred that Jill was satisfied that S5 had the correct process in her mind and let her to continue work on this problem by herself.

In another example, Susan was leading a small group discussion to help students understand their answer "Ten years growth factor was 1.5" was incorrect. After several minutes of discussion, the students saw their answer was incorrect, and Susan left them to continue, saying "*So, we did it wrong. Okay, go back to the drawing board. Think about multiplication.*" Without achieving the final correct answer, Susan was comfortable that students understood their initial solution was not mathematically correct at this point and wanted to provide space and context for students to work through their errors.

In this section, I provided two examples from two participating teachers as they supported level 3 of truth in argumentation. These examples illustrated how teachers can think carefully about using student errors as learning opportunities and maintain a balance between correcting the error immediately and giving students chances to work through their error. Giving students these opportunities pushed them to seek connections among ideas and valid claims and promotes their productive disposition toward mathematics learning to make sense of a problem, a procedure, or a concept. In both examples, teachers let students explicate their incorrect reasoning and claims. Some people worry about students hearing mistakes and misconceptions, because they think that students will remember the wrong idea rather than the correct one. However, according to Bransford et al. (2000), students learn a lot when they hear wrong ideas. In addition, when students shared their initially incorrect or incomplete ideas in class, teachers gained access to the students' understanding, which they could then address through teaching, and encouraged students to consider the validity and appropriateness of their incorrect statements. Therefore, it is important for teachers to use sequences of combinations of epistemic and teleological rational questions to prompt students' thinking and consistently establish norms in class to make students not feel anxious or disheartened when they feel struggle and value different opinions and even "wrong answers." Argumentation with level 3 of truth could have multiple structures, but they always ended with incomplete correct or partially complete claim.

6.4 Level 4 — Incorrect Mathematical Statements or Representations

All teachers make mistakes and mistakes are an inevitable part of teaching. In this study, incorrect arguments were rarely observed. In total, 4 out of 112 argumentation episodes from two teachers' lessons were resulted in incorrect mathematical statements or representations, include student errors that the teacher does not catch. It is important to noted that these teachers' errors in-the-moment were not resulting from critical logical faults or completely incorrect final mathematical answers in argumentation. For example, as shown in Figure 6.13, students were working on how to factor binomials with integer coefficients. In response to students' incorrect answer that we could put *4x* and *3x* in the box, Jill made an incorrect mathematical warrant that "*There is no common factor of 3x and 10*." In fact, Jill realized her error when she watched this argumentation episode during the post-lesson interview. In the rest of this argumentation episode, Jill guided students to figure out that correct answers should be *5x* and *2x*.

<table>
<tr><th>Factored Form</th><th>Area Model</th><th>Final Product</th></tr>
<tr><td>11.</td><td><table><tr><td>x^2</td><td>_x</td></tr><tr><td>_x</td><td>10</td></tr></table></td><td>$x^2+7x+10$</td></tr>
</table>

Figure 6.13. Task for Unit 1, Unit 1, 1st Lesson, Part of Argument 11, Small Group Discussion in Jill's Class.

There were other cases in which teachers' representations on the board were mathematically incorrect. For instance, when Jill was trying to help students write an equation based on geometric patterns (as shown in Figure 6.14), she mistakenly wrote "$3 \times 3 = 9 + 1$; $3 \times 4 = 12 + 1 = 13$ and $3 \times 5 = 15 + 1 = 16$" on board.

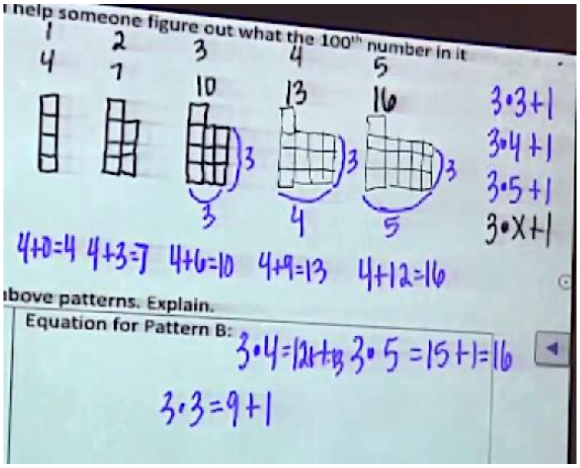


Figure 6.14. Screenshot Jill's Written from Unit 2, 1st Lesson, Argument 10 (Part b), Whole Class Discussion in Jill's Class.

Overall, the incidents observed in this study in which two teachers made mathematical errors or struggled in noticing mathematical errors (including both incorrect mathematical statements and representations) were extremely low within argumentative practices. Scaffolding mathematical argumentation practices is a highly complex activity, requiring teachers to have more than a sophisticated understanding of the subject matter (Kim, 2011) and a well-developed understanding of the practice of argumentation (Gomez et al., 2020). More research is needed to investigate what other factors affect teachers support for productive argumentative practices. Moreover, this level of argumentation, to some extent, were not illustrated by any particular structure of argumentation.

6.5 Rational Questioning and Structure in Different Levels of Truth of Argumentation

In this chapter, I explored how two teachers supported different levels of truth in argumentation by using rational questioning strategies. Through Toulmin's (1958/2003) lens, I investigated the characteristics of each level of argumentation structures emerging from two teachers' mathematics classrooms. In addition, I drew special attention to how teachers use incorrect answers they encounter or incorrect answers they construct within their support of argumentation with different levels of truth. The findings are summarized in Table 6.2.

In my review of current literature, there were no studies focused on levels of truth in classroom-based argumentation or structures of collective argumentation in secondary mathematics classrooms. This study began to address this gap by examining the complex structures of argumentation emerging in two secondary teacher's mathematics classrooms with respect to levels of truth in argumentation (see Table 6.2). The idea of different levels of truth in argumentation (except level 4) provided a description of potential valid argumentative practices that could be involved in a given classroom community when considering collective argumentation as an instructional tool for a long-term teaching intervention. This study, in addition to introducing these levels of truth in argumentation, provides insights into the structure of argumentation that can be expected for each of these levels. Thinking about these structures may help teachers plan for appropriate interventions as they facilitate argumentative practices. For instance, the construction of a rebuttal is necessary when leading an argumentation toward level 1b truth in argumentation.

As shown in Table 6.2, this study not only addressed the levels of truth in argumentation but also identified how teachers used different kinds of questions in promoting various levels of argumentation. Teacher questioning is one of the most frequently used ways of orchestrating students' reasoning and a key factor in promoting argumentation (Kosko et al., 2014). Here, the use of rational questioning was strategic, as the conceptualization of rational questioning was different from other question strategies or categories as discussed in chapter 3. Rational questioning assists teachers in working toward the goal of rationality in argumentative discourse.

Thus, relating the use of rational questioning with the levels of truth in argumentation in this study provides a conceptually grounded framework that teachers can use as they plan and develop student argumentation.

Additionally, in this study, most of the arguments exhibiting level two and level three of truth in argumentation (i.e., both levels that did not achieve completely mathematically correct arguments) were found in small group discussions. This suggested that the teachers were more comfortable letting the small group discussions end with an error or only a partially correct mathematical idea so that students could persevere in attempting to make sense of the problem. As the 2 cases presented in this qualitative study illustrates, teachers can think carefully about adapting arguments with level 2 or 3 of truth in argumentation to provide opportunities for students to work through their errors while ensuring that students leave the conversation with the correct process in mind.

Table 6.2

Summary of Findings for Research Question Two

Levels of Truth in Argumentation		Structure of Argumentation	Use of Rational Questioning	Managing Incorrect Answers
Level 1 — Mathematically Correct Arguments	Level 1a. Completely correct arguments.	***Way 1:*** Chain with multiple warrants leading to one final correct claim.	***Way 1:*** Using sequences of (combinations of components) epistemic rational questioning to press students to provide explicit reasons for their initial incorrect answers, juxtaposed with their rationales for correct answers.	The teacher observes an incorrect solution method during group work prior to the class discussion. The teacher provides the incorrect solution (or asks a student to provide this initial incorrect solution) despite knowing the student had subsequently solved the problem in a different way.
		Way 2: Two (or more) independent correct chains of reasoning that are parallel to each other.	***Way 2***: Using open-ended tasks with a focus on using teleological rational questioning to discuss different ways of solutions or using communicative rational questioning to prompt students to elaborate their different ideas and thinking that are understandable in the given mathematical classroom community.	
	Level 1b. Initial arguments were incomplete, incorrect or ambiguous.	***Way 1:*** Chain with at least one rebuttal involved.	***Way 1:*** Using epistemic rational question to elicit students' mathematical thinking and then using sequences of (combinations of components) teleological rational questioning to help students construct rebuttals based on their initial incorrect warrants.	A student provides an incorrect answer (claim or warrant). The teacher probes for the student's reasoning. The goal of the teacher is to provide an opportunity for the class to more fully understand

		Way 2: Several parallel claims with rebuttals for incorrect claims.	***Way 2:*** Using epistemic rational question to elicit students' mathematical thinking, especially when their arguments are incomplete, incorrect or ambiguous. And then the teacher provides rebuttals in response to students' incorrect explanations.	the student's solution as well as the problem.
Level 2 — Not Completely Mathematically Correct Arguments	Level 2a. Partial or incorrect mathematical arguments.	***Way 1:*** Multiple warrants leading to final correct claims.	***Way 1:*** Using teleological rational question to redirect student thinking, and thus prompt student to make acceptable explanations.	The teacher challenges students to explain why the correct answers hold by providing other possible incorrect answers.
	Level 2b. Not include all possible correct arguments.	***Way 2:*** Several parallel claims.	***Way 2:*** Using teleological rational question to give students an opportunity to notice and correct the error themselves and continually figuring out other possible answers.	
Level 3 — Final Arguments were Incorrect or Only Partially Correct Mathematical Statements		End with a single incomplete or partial correct claim.	Using epistemic rational question to elicit students' mathematical incorrect or incomplete thinking.	Using student errors for learning opportunities and keeping a balance between correcting the error immediately and giving students chances to work through their error.
Level 4 — Incorrect Mathematical Statements or Representations		No specific structure.		

CHAPTER 7

DISCUSSION AND IMPLICATIONS

This chapter provides a discussion of the findings presented in the previous two chapters (i.e., Chapter 5 and 6) and includes a discussion of the strategic ways by which teacher rational questioning can be used to support collective argumentation as observed in the practice of two beginning secondary mathematics teachers. Implications of the results, limitations of the study, and future directions for study are also presented.

7.1 Discussion of Findings

It is widely accepted that teacher questions play an essential role in shaping argumentative discourse (Walshaw & Anthony, 2008; Rodríguez & Rigo, 2015). According to Jacobs and Spangler (2017), "whether a move is good or bad is not inherent in the move but is determined by how it is enacted, for what purpose(s), and in what context" (p. 778). However, despite its prevalence and importance, fine-grained analysis uncovering the details of teacher questioning in actual classrooms is rare. In this study, I investigated aspects of teacher questioning in regard to components of rationality, development of argument components, and situational context of the conversations situated in argumentative discourse (i.e., different levels of truth in argumentation).

This study is, by nature, limited and specific to the context. However, it highlighted the role of teacher question in shaping argumentative discourse and reinforce the need for looking at teacher questioning strategies in supporting collective argumentation. This study was designed to identify teachers' questioning strategies that supported collective argumentation and found a potential benefit to using *rational questioning* as a pedagogical approach when orchestrating mathematical argumentative discourse. This study presented the adaption of Habermas' (1998) construct of rational behavior to study teacher questioning in support of collective argumentation. The findings suggest that Habermas' construct when used in conjunction with Toulmin's (1958/2003) model can be a useful analytic tool by which to conceptualize teachers' rationales for argumentative activities. According to Boero et al. (2010), teachers expect epistemic, teleological, and communicative dimensions of rational behavior to appear in students' mathematical argumentation. Thus, I considered the idea of rational questioning as a potential way by which to enrich collective argumentation through multiple perspectives. Moreover, this study illustrated how a theoretical construct from outside the field can be interpreted and adapted to offer a new and promising perspective into the study of discursive practices related to mathematical argumentation. This study is just a starting point with some possibilities of using Habermas' construct of rational behavior in mathematics education with a special focus on teacher actions to support discursive activities, such as collective argumentation. Additional study is needed to determine how this construct can be used to support other aspects of mathematics education.

7.1.1 Use of Rational Questions

It is important to note that the adapted Toulmin's (1958/2003) model (Conner, 2008) did not capture all of the rational questions that the teachers asked. For instance, many questions that contained all three components rationality were identified in episodes of argumentation but were not connected to any particular argument component. However, evidence exists from classroom dialogue and activities that the question "Does that made sense to everybody?" has the potential to provide students with opportunities to evaluate mathematical statements made by other students. Thus, I argue that it is important to identify a teacher's move when it provides students with the opportunity to engage in the rationality requirements of argumentative discourse, even though this move may not prompt or respond to any argument components in the moment. More study is needed to identify and categorize teacher moves that could support students' participation in argumentation in general and what moves appeared to be more supportive of students' contributing components of arguments. In addition, as shown from Toulmin's diagrams, the use of nonverbal aspects of communication (e.g., pointing, displaying, and gesturing) were also contributors to the development of argumentative practices. This study only analyzed part of the teachers' verbal moves (i.e., teacher questioning). Therefore, more research is needed to focus on how the emergence of teachers' nonverbal moves may also support students' participation in argumentation.

The findings of this study also indicated that not all questions in an argumentation episode were categorized as rational questioning (labelled as *N* in transcripts of argumentation episodes; for more details see Chapter 5 and 6). One type of non-rational question was used to generate data at the beginning and gather information at the end of argumentation episode. These questions often prompted data or asked students to construct results (i.e., claims). For instance, "*Can someone read me problem one*?" were often asked at the beginning of argumentation episode to give out the task information. The questions "*So, what was your final answer? This is number eight part b*?" were used to request a final claim. Another type of non-rational question was used to lead students through a method (e.g., "*Five is the greatest common factor of 5x and 15, right*?"). The students' responses in this situation were usually simply "Yes" to show their agreements with teacher's way of thinking. This type of question corresponds to what Wood (1998) called a *funneling* question, which leads students to the answer desired by the teacher and did not provide students opportunities to engage in rationality requirements of argumentative practices. This result raises questions about whether math teachers should limit these questions in order to “successfully” engage students in argumentative practices. However, according to Toulmin’s (1958/2003) diagram, the use of non-rational questions showed that, instead of getting all the information from teachers, the students were asked to provide data and final claims in the arguments, which aligned with the goal of promoting participation among the students. Therefore, these questions are beneficial to some extent even if they do not support students’ “rationalization” of discourse (i.e., awareness of rationality requirements when engaging in argumentation). Future study is needed to further investigate this question.

Further, the results indicated that although epistemic rational questioning may not always elicit correct or complete reasoning, it served as a way for teachers to negotiate social norms, including that students were expected to provide reasons to justify their claims when engaging in argumentative activities. Through leading students to work toward a specific method or foreground a particular piece of mathematics for consideration, teleological rational questioning was useful for constructing correct arguments. Further, by calling on a particular student to share a different solution, the class worked on what counted as a mathematically different solution, which facilitated the establishment of sociomathematical norms. Communicative rational questioning contributed to the development of students’ communicative competencies by asking students to make sure their representations were correct and to use appropriate mathematical terminology to communicate ideas. Questioning focused on communicative rationality also cultivated norms that students were expected to ensure their use of mathematical language and representation can be understood in the given mathematical classroom community. In addition, slight differences in teachers’ ways of using rational questioning were observed across four units of instruction (each unit has approximately one-month period gap). For instance, fewer rational questions were needed for Susan to prompt her students contribute to argument components in Unit 2. Therefore, the use of rational questioning may also serve as a way to support the emergence and negotiation of norms (e.g., social norms, sociomathematical norms) with respect to classroom-based argumentative practices in a long-term teaching intervention.

7.1.2 Significance of Different Levels of Truth in Argumentation

If teachers want to use argumentation for teaching, they have to create appropriate learning contexts and show students why and how argumentation is good for learning math (Wood, 1999). Classroom-based argumentative discourse is a form of collaborative discussion, and classroom discussions are complex and messy (Frank et al., 2007). Sometimes, the argumentation may not happen in the intended way. To facilitate collective mathematical argumentation, it is critical to understand what constitutes productive argumentative practices. However, no consensus exists in the field of mathematics education concerning the characteristics of productive argumentative discourse. Therefore, this study proposed four levels of truth in argumentation (see Table 6.1) using a set of argumentation episodes from the practice of two beginning secondary mathematics teachers.

Stylianides (2007) conceptualized proof in school mathematics as needing to meet three requirements: accepted mathematical statements, modes of proof, and appropriate communicative approaches. Stylianides argued that the validity of a proof "should be understood in the context of what is typically agreed upon nowadays in the field of mathematics" (p. 293). Although the field is still working on interpreting the relationship between proof and argumentation, a shared understanding exists among researchers that the definition of argumentation is broader than that of proof, in which proof is often described as a formal argument subject to norms (e.g., definitions, theorems, axioms) of the mathematical community (Staples et al., 2016). In this study, I conceptualize mathematical collective argumentation as a powerful instructional approach to promote students' learning of mathematics. As a long-term process, engaging students in argumentation will help teachers establish classroom contexts for the emergence and negotiation of norms (e.g., social norms, sociomathematical norms) and generally foster students' development of social and intellectual autonomy in mathematics (Yackel & Cobb, 1996). Therefore, I consider argumentation to be productive not only considering whether the arguments, modes of argumentation and the modes of argument representation are mathematically correct (Stylianides' definition of proof), but also accounting for how argumentation is gradually scaffolded in a particular classroom community. In other words, I view levels one to three of truth in argumentation (see Table 6.1) as being productive argumentative practices in a given classroom community despite levels two and three containing parts of arguments that are not completely mathematically correct from the perspective of mathematicians. Thus, from an educational perspective, the question of whether an argumentation could be productive in a classroom context concerns more than mathematically correct arguments. It may consider *norms* establishment and development as well as students' mathematical dispositions for collective argumentation. This view is supported by Krummheuer's (1995) idea of "substantial arguments" (p. 229) based on Toulmin's (1958/2003) conceptions of analytic arguments and substantial arguments, which suggests that the analytic arguments (i.e., proofs in mathematics) are not the only valid arguments, and substantial arguments which may not have the logical rigidity of formal deductions, are not taken as a weaknesses, but rather as a resource for reconstructing informal arguments.

As stated in NCTM (1991), student mistakes or incorrect conjectures can promote debates about mathematical topics and can be windows into student understanding. Teachers are expected to use student errors as learning opportunities to foster productive discussions (Schleppenbach et al., 2007); however, research has shown that most prospective teachers have difficulties in managing incorrect answers in relation to mathematical argumentation (Shinno et al., 2018). In this study, I illustrated three ways by which two beginning secondary mathematics teachers used incorrect answers to support student learning through collective argumentation, as summarized in Table 6.2. This analysis provided insights on how teachers could effectively support student argumentation, while managing or capitalizing on an incorrect answer.

The results of this study suggested that students' incorrect answers served as powerful resources by which teachers could regulate the fundamental steps of argumentation, such as constructing additional warrants (i.e., warrants*) and rebuttals. By making students' incorrect answers visible, teachers encouraged students to use their own ideas for learning and created contexts for them to be active learners as they reflected, challenged, and extended other students' thinking. In addition, research has shown that teachers might experience dilemmas related to stepping in or standing back when managing incorrect answers among students (e.g., Horn, 2008; Lampert et al., 1996). For example, conflicts can arise when students disagree, and teachers may be hesitant to step in to direct the students' discussions. However, some researchers have argued that simply avoiding "telling" (i.e., hesitant to say too much during discussions) may result in negative consequences, and it is necessary for teachers to step in and steer the focus of the disagreements toward important issues (Chazan & Ball, 1999). Data from the current study indicated that no clear evidence existed to show that the argumentative discourse from either participant teacher's class had deteriorated into oppositional or confrontational talk. In addition, no interpersonal conflicts spilled over into the intellectual content. The teachers' interventions (i.e., use of rational questioning) were delicate and necessary to make sure that argumentation continually progressed toward a productive direction, especially at the beginning stages of the school year.

7.2 Limitations

This study was clearly limited by the scope of the data, as I only described two beginning secondary mathematics teachers' classes in the context of collective argumentation (even though they taught with different content, class sizes, student populations, and school contexts). The two participating teachers have been working with this research project since they were prospective secondary teachers and learned about supporting argumentation from their courses during their teacher education program as well as during PD. Thus, the information about beginning secondary teachers using questioning strategies to support collective argumentation cannot be generalized; a random sample may have produced much different results. However, because of their backgrounds, these participants allowed for investigation of the reasonableness of the Rational Questioning Framework as well as giving rise to a framework for different levels of truth in argumentation and a description of potential ways to use incorrect answers within argumentation.

This study did not attempt to compare the frequencies and percentages of each type of rational questions between the teachers because the data set (e.g., content of lesson, class period) varied in size across units. Therefore, the findings of this study noted rational questioning strategies to support collective argumentation on the basis of the classroom contexts, content of the lessons, and student responses and reactions and did not attempt to ascertain which teacher utilized more effective questioning techniques. It was evident from the data that both teachers used a variety of combinations of components of rational questioning to invite the students to engage in argumentation and provided context for the students' conceptual development in mathematics.

Finally, because of the intent of the original data collection, the findings of this study were mainly based on data focused on the teachers' actions. Due to classroom noise and overlapping speech, the students' responses were inaudible, which, at times, affected the categorization of the teachers' questions and limited the diagrams of the whole picture related to collective argumentation. In addition, although field notes and screenshots were helpful to capture the work of the students in whole class discussions, it was difficult to interpret what they had written on their worksheets when they worked individually or in small group discussions. More data from the students' written arguments would have been helpful in understanding how collective argumentation developed during the small group discussions.

7.3 Implications

7.3.1 Implications for Research

Many researchers have used Habermas' (1998) constructs to conduct studies that centered on *students* (e.g., Cramer, 2015; Cramer & Knipping, 2018; Morselli & Boero, 2009, 2011); however, this study showed that this theoretical framework can also be used to analyze teachers' classroom actions. Specifically, this framework was useful in investigating how teachers use questioning strategies to promote collective argumentation along with the three components of rationality.

The integration of Habermas' (1998) construct with Toulmin's (1958/2003) model as an analytic tool by which to analyze teachers' questioning provides us with a more comprehensive perspective for understanding the roles of teacher questioning within collective argumentation.

The two constructs complement each other in the following sense: Habermas' lens helps us to identify fine-grained rationality components of teachers' questioning as well as how teachers' questioning is constrained in relation to the three components of rational behavior. The teachers' uses of rational questioning to manage the fundamental steps of argumentation are seen through Toulmin's lens. The results of this study provided empirical evidence to support Boero et al.'s (2010) idea of integrating Habermas' construct of rational behavior with Toulmin's model to create a powerful analytic tool to allow us to look at the development of the culture of rationality in the argumentative discourse managed by the teacher. Moreover, some researchers (e.g., Cramer & Knipping, 2018; Zhuang & Conner, in press) have integrated Habermas' construct of rational behavior with other theoretical constructs to gain a deeper understanding of argumentative practices. For instance, Cramer and Knipping (2018) integrated Krummheuer's (2007) interaction theory-based approach with Habermas' construct of rational behavior to study student participation in collective argumentation in the mathematics classroom. It may be beneficial for future studies to continue to investigate the use of Habermas theory to deal with relevant aspects of the complexity of the discursive activities, such as collective argumentation.

According to Boaler (1998), the negotiation of validation practices is important in regard to supporting teachers to construct and sustain a mathematical community in which arguments are appropriately validated. In this study, I described four levels of truth in argumentation in regard to the mathematical truth of students' final claims. The point is to capture validation of an argumentative discourse that is not just confined to the mathematical truth of students' statements but also includes the normative ways through which students engage in argumentation and the appropriate communicative competence to explicate their own arguments. Incorporating argumentative practices in the classroom is a long-term teaching intervention (Yackel & Cobb, 1996). The merit of the levels of truth argumentation framework is that it allows teachers to achieve a balance between mathematics as a discipline and the teaching and learning of mathematics through argumentation. It considers students as mathematics learners who participate in argumentation with different grade levels and emphasizes the validity of argumentation as context-dependent. In addition, it also provides a tool through which teachers can develop different requirements of argumentation aimed at promoting students' argumentative skills in school. The idea of levels of truth in argumentation provides a starting point for other researchers to define quality argumentative practices and find effective ways by which to support students in regard to implementing appropriate local acceptance criteria for argumentative practices.

7.3.2 Implications for Practice

This study's findings have implications for both teaching practice and teacher education. First, the findings provided information about how two beginning secondary mathematics teachers used questioning strategies to support collective mathematical argumentation. They implied that one specific type of rational questioning should not be emphasized more than others, but, rather, combinations of components of rational questioning should be used to allow the teachers to fulfill various pedagogical purposes and facilitate different levels of truth in argumentation in the mathematics classroom. Second, the fine-grained analysis of teacher questioning in regard to Habermas' three components of rationality may serve as a method by which to provide teachers with additional ways to think about their questions during argumentative discourse and disseminate information to teachers about how collective argumentation could be initiated and sustained. The Teacher Rational Questioning Framework (see Table 3.1) can be applied as a didactical tool beneficial for teachers when attempting to orchestrate argumentative discourse according to the three components of rationality as a way to foster students' rational behavior in argumentative activities. By purposefully using a variety of combinations of components of rational questioning, teachers can evoke student participation in argumentation and support the development of mathematical arguments. Third, the use of student errors could serve as a powerful source for teachers to support collective argumentation. This study illustrated several ways (see Table 6.2) by which teachers could support different levels of truth in student argumentation, while managing or capitalizing on an incorrect answer. The findings suggested that, by maintaining a balance between correcting the error immediately and giving the students chances to work through their errors, the teachers could help the students take responsibility for their own learning (i.e., their eagerness to figure out their mistakes), and, thus, stimulated different levels of truth in collective argumentation.

This study illustrated how two teachers used incorrect answers to support student learning through collective argumentation across one to three levels of argumentation (see Table 6.2). While these actions may be routine for experienced teachers, using these kinds of examples in teacher education and professional development may provide a more robust explanation for the use of incorrect answers in instruction. Mathematics teacher educators could incorporate these effective ways of supporting argumentation (e.g., use incorrect answers) into teacher education programs to make argumentation more approachable for prospective teachers.

7.4 Directions for Future Research

The findings from this study suggested several possible directions for future research. In this study, I used the Teacher Rational Questioning Framework (see Table 3.1) I developed from Habermas' (1998) construct of rational behavior to identify what forms of rational questioning appeared to be effective for scaffolding collective argumentation, particularly with respect to facilitating students' rational behaviors. Other questioning frameworks (e.g., Boaler & Humphrey, 2005; Conner et al., 2014) exist in the field; these precise classifications of teacher questions provide different perspectives for explaining teachers' questioning actions, and the application of a different framework in conjunction with an analysis of rational questioning may provide additional insights into the teachers' support for collective argumentation.

This study analyzed teacher questions in secondary algebra classrooms, which raises the question of whether differences exist between teacher questions in supporting argumentation across variables, such as grade levels (e.g., elementary vs. secondary), content areas (e.g., algebra vs. geometry), and classroom contexts (e.g., small group vs. whole class). Teachers' roles are to model argumentative practices and act as class resources to be drawn upon as needed (Yackel, 2002). Future research should examine what questioning techniques appeared to be effective when working with different groups of students at different grade levels.

Current literature has not reached a consensus or a shared understanding about what counts as a valid or productive collective argumentation. This study proposed the conception of truth levels in argumentation (see Table 6.1), which emphasized that the validity of argumentation is not the same as proof (see Stylianides' (2007) definition of proof) by considering the widest definition of collective argumentation — the act of individuals working together to determine the validity of a claim (Conner et al., 2014) and the use of collective argumentation as an instructional tool to gradually scaffold classroom-based argumentation as a long-term teaching intervention. Future research should continue to explore promising instruments for describing the quality of collective argumentation and how teacher questioning results in different outcomes for collective argumentation, for instance, including student responses as part of the measurement.

The results of this study suggested that the use of the privileged rationality component (teleological rationality) in rational questions was related to multiple factors, such as the content of the lessons. Moreover, Zhuang and Conner (2018b) found that teachers attempt to use different kinds of rational questioning also based on their interpretations of argumentation during course work. More research is needed to investigate teachers' intentions and associated facilitating or hindering factors when they employ certain types of teacher moves to support mathematical argumentation.

Finally, for research question two, I explored the structure of collective argumentation through Toulmin's (1958/2003) model. Krummheuer (1995) was one of the first researchers in the field to use Toulmin's diagrams to study collective argumentation in primary schools. However, the structure of the arguments he described basically involved only three core argument components (i.e., data, warrant, and claim). The results of this study have shown that collective argumentation in secondary mathematics classrooms consisted of more than a simple chain structure and were much more complex, usually involving multiple implicit warrants, rebuttals, and independent parallel arguments. For instance, as shown in Figure 5.3 and Figure 5.6, even both diagrams were categorized as a chain structure, there were some subtle differences existed between two diagrams regard to the process of argumentation. Thus, my adaptation of Toulmin's diagrams (see Figure 3.2) was not as detailed in terms of structure of collective argumentation as it might have been and was situated in a general picture (i.e., chain, parallel). Some researchers in the field (e.g., Knipping & Reid, 2015) have developed an extended argumentation model based on Toulmin's work by adding *refutation* as a new element. On the basis of this new element, Knipping and Reid (2015) proposed four types of argumentation structures: source, spiral, reservoir, and gathering structures. By adapting their framework, Cervantes-Barraza et al. (2019) conducted a teaching experiment in a fifth-grade classroom and found an argument structure that differs from the spiral structure as proposed by Knipping and Reid. More research is needed in the field to continue characterize the complex structure of collective argumentation.

In conclusion, this study highlighted the role of teacher questioning to engage students in collective argumentation that maintain a focus on students' contribution of argument components, "rationality" of discourse (i.e., three components of rationality), and levels of truth in argumentation. Teachers use appropriate questioning strategies by considering the context of the kind of instruction that is taking place in relation to the tasks (Hiebert & Wearne, 1993). Thus, the types of rational questions observed within argumentative practice may be varied across grade levels, content areas and classroom contexts. The introduction of the Rational Questioning Framework and Levels of Truth in Argumentation leads to several open questions: What should rationality requirements look like in local collective argumentative practices? How can we introduce Rational Questioning Framework in teacher education courses to help teachers identify different rationalities to support collective argumentation and in the comparison with other subject areas (e.g., science, engineering)? How does an awareness of levels of truth in argumentation impact teachers' actions with respect to questioning in argumentative practice?

REFERENCES

Andriessen, J. (2006). Arguing to learn. In K. Sawyer. (Ed). *The Cambridge handbook of the learning sciences*. (pp. 443–459). Cambridge: Cambridge University Press.

Angrosino, M. V. (2005). Recontextualizing observation: Ethnography, pedagogy, and the prospects for a progressive political agenda. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (3rd ed., pp. 729-745). Thousand Oaks, CA: Sage.

Arzarello, F., & Sabena, C. (2011). Semiotic and theoretic control in argumentation and proof activities. *Educational Studies in Mathematics, 77*(2-3), 189–206.

Asterhan, C. S., & Schwarz, B. B. (2016). Argumentation for learning: Well-trodden paths and unexplored territories. *Educational Psychologist*, *51*(2), 164 –187.

Australian Curriculum Assessment and Reporting Authority [ACARA] (n.d.) The Australian Curriculum: Mathematics. Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Balacheff, N. (1999). Is argumentation an obstacle? Invitation to a debate. *International newsletter on the teaching and learning of mathematical proof.* Retrieved from

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching,* 4th edition (pp.4 33–456). New York: Macmillan.

Barkai, R., Tabach, M., Tirosh, D., Tsamir, P., & Dreyfus, T. (2009). High school teachers' knowledge about elementary number theory proofs constructed by students. *Proceedings of PME 33, 2,* 113–120.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics*, *11*(1), 23–41.

Bieda, K. N. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. *Journal for Research in Mathematics Education, 41*(4), 351–382.

Blumer, H. (1969). *Symbolic interactionism*. Engelwood Cliffs, NJ: Prentice- Hall.

Boaler, J. (1998). Open and closed mathematics: student experiences and understandings. *Journal for Research in Mathematics Education*, *29*(1), 41–62.

Boaler, J. (2016). Designing mathematics classes to promote equity and engagement. *Journal of Mathematical Behavior*, *100*(41), 172–178.

Boaler, J., & Brodie, K. (2004). The importance of depth and breadth in the analysis of teaching: A framework for analysing teacher questions. In the *Proceedings of the 26th Meeting of the North America Chapter of the International Group for the Psychology of Mathematics Education*. Toronto, Ontario.

Boaler, J., & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning*. Portsmouth, NH: Heinemann.

Boavida, A.M. (2009). Argumentation in mathematics: the role of the teacher. In Tzckaki, M., Kaldrimidou, M. & Sakonidis, 11. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education,* Vol. I, pp. 340. Thessaloniki, Greece: PME.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof.* Retrieved from http://www.lettredelapreuve.it/Newsletter/990708Theme/990708ThemeUK.html

Boero, P. (2006). Habermas' theory of rationality as a comprehensive frame for conjecturing and proving in school. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 185–192). Prague, Czech Republic: PME.

Boero, P. (2011). Argumentation and proof: Discussing a "successful" classroom discussion. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of CERME7* (pp. 120–130). Rzeszów, Poland: ERME.

Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). Argumentation and proof: A contribution to theoretical perspectives and their classroom implementation. In M. F. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 179–205). Belo Horizonte: PME.

Boero, P., Fenaroli, G., & Guala, E. (2018). Mathematical argumentation in elementary teacher education: The key role of the cultural analysis of the content. In *Advances in mathematics education research on proof and proving* (pp. 49–67). Springer, Cham.

Boero, P., Garuti, R., & Lemut, E. (2007). Approaching theorems in grade VIII. In P. Boero (Ed.), *Theorems in school: From history epistemology and cognition to classroom practice* (pp. 249–264). Rotterdam: Sense Publishers.

Boero, P., Guala, E., & Morselli, F. (2013). Crossing the borders between mathematical domains: A contribution to frame the choice of suitable tasks in teacher education. *Proceedings of PME 37, 2*, 97–104. Kiel: IPN-PME.

Boero, P. & Planas, N. (2014). Habermas' construct of rational behavior in mathematics education: New advances and research questions. In P. Liljedahl, C. Nicol, S. Oesterle, &

D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 205–235). Vancouver, Canada: PME.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (2000). *How people learn* (Vol. 11). Washington, DC: National academy press.

Burns, M. (1985). The role of questioning. *Arithmetic Teacher*, *32*(6), 14–17.

Cazden, C. (2001). *Classroom Discourse: The language of teaching and learning* (2nd ed.). Portsmouth, NH: Heinemann.

Cervantes-Barraza, J., Cabañas-Sánchez, G., & Reid, D. (2019). Complex argumentation in elementary school. *PNA*, *13*(4), 221–246.

Charon, J. M. (2009). *Symbolic interactionism: An introduction, an interpretation, an integration*. New York City, NY: Pearson College Division.

Chen, Y. C., Hand, B., & Norton-Meier, L. (2016). Teacher roles of questioning in early elementary science classrooms: A framework promoting student cognitive complexities in argumentation. *Research in Science Education*, *47*(2), 373–405.

Civil, M., & Hunter, R. (2015). Participation of non-dominant students in argumentation in the mathematics classroom. *Intercultural Education*, *26*(4), 296–312.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, *29*(3), 573–604.

Cobb, P. (2000). The importance of a situated view of learning to the design of research and instruction. In J. Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning* (pp. 44–82), Westport, CT:Ablex.

Cobb, P., & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp.1–16). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, *28*(3), 258–277.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences*, *10*(1-2), 113–163.

Conner, A. (2008). Expanded Toulmin diagrams: A tool for investigating complex activity in classrooms. In *Proceedings of the Joint Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 32, pp. 361–368).

Conner, A. (2017). An application of Habermas' rationality to the teacher's actions: Analysis of argumentation in two classrooms. In T. Dooley & G. Gueudet (Eds.). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (CERME10, February 1 – 5, 2017). Dublin, Ireland: DCU Institute of Education and ERME.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401– 429.

Cotton, K. (2001). *Classroom questioning* (Close-Up #5). Portland, OR: Northwest Regional Educational Laboratory.

Cramer, J. (2015). Argumentation below expectation: A double-threefold Habermas explanation. In K. Krainer & N. Vondrová (Eds.), *Proceedings of CERME9* (pp. 114–120). Prague, Czech Republic: ERME.

Cramer, J. C., & Knipping, C. (2018). Participation in argumentation. In U. Gellert, C. Knipping and H. Straehler-Pohl (Eds.), *Inside the mathematics class: Sociological perspectives on participation, inclusion, and enhancement* (pp. 229–244). Cham, Switzerland: Springer Nature.

Cross, D. I. (2009). Creating optimal mathematics learning environments: Combining argumentation and writing to enhance achievement. *International Journal of Science and Mathematics Education, 7*(5), 905–930.

Department for Education (2014). National curriculum in England: Mathematics programmes of study. London: Her Majesty's Stationery Office.

Douek, N. (1999). Some remarks about argumentation and mathematical proof and their educational implications. In I. Schwank (Ed.), *European Research in Mathematics Education, I, Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Vol. 1, pp. 128–142). Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Argumentation and proof in the mathematics classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education, new ICMI study series 15* (pp. 349–367). Dordrecht, The Netherlands: Springer.

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics, 22*(3), 233–262.

Duval, R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. In P. Boero (Ed.), *Theorems in School: From History, Epistemology and Cognition to Classroom Practices* (pp. 137–161). Rotterdam, The Netherlands: Sense Publishers.

Ellis, A., Ozgur, Z., & Reiten, L. (2019). Teacher moves for supporting student reasoning. *Mathematics Education Research Journal, 31*(2), 107–132.

Ernest, P. (1991). *The philosophy of mathematics education*. New York: Falmer Press.

Fielding-Wells, J., & Makar, K. (2015). If it doesn't have an apex it's not a pyramid": Argumentation as a bridge to mathematical reasoning. *Proceedings of PME 39*, *2*, 297-304.

Flanders, N. A. (1970). *Analyzing teaching behavior*. London: Addison-Wesley.

Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and instruction, 8*(6), 527–548.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225–256). Charlotte, NC: Information Age.

Franke, M. L., Webb, N. W., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary classrooms. *Journal of Teacher Education*, *60*(4), 380–392.

Gall, M. D. (1970). The use of questions in teaching. *Review of educational research*, *40*(5), 707–721.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive Unity of Theorems and Difficulty of Proof. *Proceedings of the international group for the psychology of mathematics education PME-XXII*, vol. 2, pp. 345–352. Stellenbosch.

Garuti, R.; Boero, P.; Lemut, E. & Mariotti, M. A. (1996). Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems. *Proceedings of PME 20*, vol. 2, pp. 113–120. Valencia, Spain: PME.

Georgia standards of excellence (GSE). (2016). Mathematics standards: Accelerated GSE Algebra I/Geometry A. Retrieved from https://www.georgiastandards.org/Georgia-Standards/Frameworks/Acc-Algebra-I-Geometry-A-Standards.pdf

Georgia standards of excellence (GSE). (2016). Mathematics standards: Accelerated GSE Geometry B /Algebra II. Retrieved from https://www.georgiastandards.org/Georgia-Standards/Frameworks/Acc-Geometry-B-Algebra-II-Standards.pdf

Giannakoulias, E., Mastorides, E., Potari, D., & Zachariades, T. (2010). Studying teachers' mathematical argumentation in the context of refuting students' invalid claims. *The Journal of Mathematical Behavior*, *29*(3), 160–168.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Gomez, C. N., Park, H., Zhuang, Y., Foster, J. K., & Conner, A. (2020). Prospective Teachers' Recontextualization of Support for Collective Argumentation [Manuscript submitted for publication]. Department of Mathematics and Science Education, University of Georgia.

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education*, *35*(4), 258–291.

Graesser, A. C., & Person, N. K. (1994). Question asking during tutoring. *American Educational Research Journal*, *31*(1), 104–137.

Habermas, J. (1984). *Theory of Communicative Action, Volume One: Reason and the Rationalization of Society*. Trans. T. McCarthy. Boston: Beacon.

Habermas, J. (1998). *On the pragmatics of communication* (M. Cooke Ed.). Cambridge, MA: MIT Press.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Greenwich, CT: Information Age Publishing.

Haverkamp, B. E., & Young, R. A. (2007). Paradigms, purpose, and the role of the literature: Formulating a rationale for qualitative investigations. *The Counseling Psychologist*, *35*(2), 265–294.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*(4), 389–399.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*(2), 393–425.

Hoffman, B. L., Breyfogle, M. L., & Dressler, J. A. (2009). The Power of Incorrect Answers. *Mathematics teaching in the middle school*, *15*(4), 232–238.

Hollebrands, K. F., Conner, A., & Smith, R. C. (2010). The nature of arguments provided by college geometry students with access to technology while solving problems. *Journal for Research in Mathematics Education, 41,* 324–350.

Horn, I. S. (2008). Accountable argumentation as a participation structure to support learning through disagreement. In A. J. Schoenfeld & N. Pateman (Eds.), *A study of teaching: Multiple lenses, multiple views: Journal for Research in Mathematics Education, Monograph #14* (pp. 97–126). Reston: NCTM.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, *35*(2), 81–116.

Hunter, R. (2007). Can you convince me: Learning to use mathematical argumentation. In J. Woo, H. Lew, K. Park, & D. Seo (Eds.), *Proceedings of the 31st annual conference of the International group for the Psychology of Mathematics Education*, (Vol. 3, pp. 381–389). Seoul, Korea, The Korea Society of Educational Studies in Mathematics.

Inglis, M., Meija-Ramos, J. P., & Simpson, A. (2007). Modeling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3–21.

Ingram, D. (2010), *Habermas: Introduction and Analysis.* Ithaca, NY: Cornell University Press.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: an emerging framework of teaching moves. *ZDM*, *48*(1-2), 185-197.

Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in K–12 mathematics teaching. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp.766–792). Reston, VA: National Council of Teachers of Mathematics.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*(1), 59–80.

Kim, H.-J. (2011). *An exploratory study of teachers' use of mathematical knowledge for teaching to support mathematical argumentation in middle-grades classrooms* [Doctoral dissertation, The University of Texas at Austin]. Retrieve from http://hdl.handle.net/2152/ETD-UT-2011-12-4696

Knipping, C. (2008). A method for revealing structures of argumentations in classroom proving processes. *ZDM*: *The International Journal on Mathematics Education, 40*(3), 427–441.

Knipping, C., & Reid, D. (2015). Reconstructing argumentation structures: A perspective on proving processes in secondary mathematics classroom interactions. In *Approaches to qualitative research in mathematics education* (pp. 75–101). Springer, Dordrecht.

Knott, L., Sriraman, B., & Jacob, I. (2008). A morphology of teacher discourse in the mathematics classroom. *The Mathematics Educator*, *11*(1/2), 89–110.

Koestler, C., Felton-Koestler, M. D., Bieda, K., & Otten, S. (2013). *Connecting the NCTM process standards and the CCSSM practices*. Reston, VA: National Council of Teachers of Mathematics.

Kollar, I., Ufer, S., Reichersdorfer, E., Vogel, F., Fischer, F., & Reiss, K. (2014). Effects of collaboration scripts and heuristic worked examples on the acquisition of mathematical argumentation skills of teacher students with different levels of prior achievement. *Learning and Instruction, 32*(0), 22–36.

Kosko, K. W., Rougee, A., & Herbst, P. (2014). What actions do teachers envision when asked to facilitate mathematical argumentation in the classroom? *Mathematics Education Research Journal*, *26*(3), 459–476.

Kosko, K. W. (2016). Primary teachers’ choice of probing questions: Effects of MKT and supporting student autonomy. *International Electronic Journal of Mathematics Education*, *11*(4), 991–1012.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *Journal of Mathematical Behavior*, *26*(1), 60–82.

Krummheuer, G. (2015). Methods for reconstructing processes of argumentation and participation in primary mathematics classroom interaction. In *Approaches to qualitative research in mathematics education* (pp. 51–74). Springer, Dordrecht.

Lakatos, I. (1976). Proofs and refutations: *The logic of mathematical discovery*. New York: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29–63.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Lampert, M., Rittenhouse, P., & Grumbaugh, C. (1996). Agreeing to disagree: Developing sociable mathematical discourse. In D. R. Olson & N. Torrance (Eds.), *The handbook of education and human development: New models of learning, teaching, and schooling* (pp. 731–764). Cambridge, MA: Blackwell.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., ... & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of teacher education*, *64*(3), 226–243.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.

Lerman, S. (2001). Cultural, discursive psychology: a sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics, 46*(1-3), 84–113.

Lin, M., & Wu, C. (2007). Uses of examples in geometric conjecturing. *Proceedings of PME 31, 3,* 209–216.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In Á. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 173–204). Rotterdam, the Netherlands: Sense Publishers.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *The Journal of Mathematical Behavior*, *18*(1), 53–78.

Mehan, H. (1985). The structure of classroom discourse. In T.A. Van Dijk (Ed.), *Handbook of discourse analysis* (Vol. 3, pp.119-131). London, UK: Academic Press.

McCarthy, P., Sithole, A., McCarthy, P., Cho, J. P., & Gyan, E. (2016). Teacher questioning strategies in mathematical classroom discourse: A case study of two grade eight teachers in Tennessee, USA. *Journal of Education and Practice*, *7*(21), 80–89.

McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education*, *32*(3), 236–266.

Metaxas, N., Potari, D., & Zachariades, T. (2016). Analysis of a teacher's pedagogical arguments using Toulmin's model and argumentation schemes. *Educational Studies in Mathematics*, *93*(3), 383–397. National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Morselli, F., & Boero, P. (2009). Proving as a rational behaviour: Habermas' construct of rationality as a comprehensive frame for research on the teaching and learning of proof. *CERME6* (pp. 221–220)

Morselli, F. & Boero, P. (2011). Using Habermas' theory of rationality to gain insight into students' understanding of algebraic language. In Cai, J. & Knuth, E. (Eds.), *Early algebraization. A global dialogue from multiple perspectives* (pp. 453–481). NY: Springer.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (2014). *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards mathematics*. Washington, DC: Authors.

National Council of Teachers of Mathematics (2014). *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: Author.

Nussbaum, E. M. (2008). Collaborative discourse, argumentation, and learning: Preface and literature review. *Contemporary Educational Psychology*, *33*(3), 345–359.

O'Connor, M. C., & Michaels, S. (1993). Aligning academic task and participation status through revoicing: Analysis of a classroom discourse strategy. *Anthropology and Education Quarterly*, *24*(4), 318–335.

O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning and schooling* (pp. 63–103). Cambridge: C.U.P.

Pedemonte, B. (2001). Some cognitive aspects of the relationship between argumentation and proof in mathematics. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the international group for the psychology of mathematics education PME-25*, vol. 4, (pp. 33–40). Utrecht (Olanda).

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23–41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM: The International Journal on Mathematics Education, 40*(3), 385–400.

Prasad, P. (2005). *Crafting qualitative research: Working in the postpositivist traditions* (Ed.). Armonk, NY: M.E. Sharpe, Inc.

Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics*, *79*(1), 19–40.

Rasmussen, C., Stephan, M. (2008). A methodology for documenting collective activity. In A.E.

Kelly, R.A. Lesch & J.Y. Baek, (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 195–215), New York: Routledge.

Reichersdorfer, E., Vogel, F., Fischer, F., Kollar, I., Reiss, K., & Ufer, S. (2012). Different collaborative learning settings to foster mathematical argumentation skills. *Proceedings of PME 36, 3*, 345–352.

Rodríguez, R., Rigo, L. (2015). The cultural of rationality in secondary school: An ethnographic approach. En Beswick, K., Muir, T., y Wells, J. (Eds.). *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 89–96). Hobart, Australia: PME.

Sahin, A., & Kulm, G. (2008). Sixth grade mathematics teachers' intentions and use of probing, guiding, and factual questions. *Journal of Mathematics Teacher Education, 11*(3), 221–241.

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and US mathematics classrooms. *The elementary school journal*, *108*(2), 131–147.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Shinno, Y., Yanagimoto, T., & Uno, K. (2018). An Investigation of Prospective Elementary Teachers' Argumentation from the Perspective of Mathematical Knowledge for Teaching and Evaluating. In *Research Advances in the Mathematical Education of Pre-service Elementary Teachers* (pp. 155–169). Springer, Cham.

Singletary, L. M., & Conner, A. (2015). Focusing on mathematical arguments. *Mathematics Teacher*, *109*(2), 143–147.

Sriraman, B., & Umland, K. (2014). Argumentation in mathematics education. In *Encyclopedia of mathematics education* (pp. 46–48). Springer, Dordrecht.

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *The Journal of Mathematical Behavior*, *31*(4), 447–462.

Staples, M., & Newton, J. (2016). Teachers' contextualization of argumentation in the mathematics classroom. *Theory Into Practice*, *55*(4), 294–301.

Staples, M., Newton, J., Kosko, K., Conner, A., Cirillo, M., Bieda, K., Yopp, D., Zaslavsky, O., Hummer, J., Strachota, S., Singh, R., An, T., Going, T., & Zhuang, Y. (2017). *Using artifacts to explore conceptions of consequences of argumentation, justification, and proof* (report). PME-NA Working Group. Retrieved from: https://www.researchgate.net/publication/317267228_White_Paper_Using_Artifacts_to_Explore_Conceptions_and_Consequences_of_Argumentation_Justification_and_Proof

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park: Sage.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.

Stylianides, A., Bieda, K., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutierez, G. Leder, & P. Boero (Eds.), *2nd handbook on the psychology of mathematics education* (pp. 315–351). Rotterdam: Sense Publishers.

Stylianides, A. J. (2019). Secondary students' proof constructions in mathematics: The role of written versus oral mode of argument representation. *Review of Education*, *7*(1), 156–182.

Stylianou, D. A., Blanton, M. L., & Knuth, E. J. (2009). *Teaching and learning proof across the grades: A K-16 perspective.* New York, NY: Routledge.

Temple, C., & Doerr, H. M. (2012). Developing fluency in the mathematical register through conversation in a tenth-grade classroom. *Educational Studies in Mathematics, 81*(3), 287–306.

Toulmin, S.E. (2003). *The uses of argument* (updated ed.). New York: Cambridge University Press. (First published in 1958).

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, *30*(2), 161–177.

Van Eemeren, F. H., Grootendorst, R., Johnson, R. H., Plantin, C., & Willard, C. A. (1996). *Fundamentals of argumentation theory: A handbook of historical backgrounds and contemporary developments*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Wagner, P. A., Smith, R. C., Conner, A., Singletary, L. M., & Francisco, R. T. (2014). Using Toulmin's model to develop prospective secondary mathematics teachers' conceptions of collective argumentation. *Mathematics Teacher Educator*, *3*(1), 8–26.

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research 78*(3), 516–551.

Whitenack, J., & Yackel, E. (2002). Making mathematical arguments in the primary grades: The importance of explaining and justifying ideas. *Teaching Children Mathematics*, *8*(9), 524–528.

Wood, T. (1998). Funneling or focusing? Alternative patterns of communication in mathematics class. in H. Steinbring, M. g. Bartolini- Bussi, & A. Sierpinska (eds.), *Language and communication in the mathematics classroom* (pp. 167–178). Reston, VA: National Council of Teachers of Mathematics.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education*, *30*(2), 171–191.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*, 458–477.

Yackel, E. (2001). Explanation, Justification and argumentation in mathematics classrooms. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the international group for the psychology of mathematics education PME-25*, vol. 1, (pp. 1–9). Utrecht (Olanda).

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *The Journal of Mathematical Behavior*, *21*(4), 423–440.

Zhuang, Y. & Conner, A. (2018a, April). *Using a Framework from Science Education to Investigate Teacher Roles of Questioning in Supporting Mathematical Argumentation*

[Poster presentation]. National Council of Teachers of Mathematics Research Conference, Washington, DC.

Zhuang, Y. & Conner, A. (2018b). Analysis of teachers' questioning in supporting mathematical argumentation by integrating Habermas' rationality and Toulmin's model. In T. Hodges, G. Roy, & A. Tyminski (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1323-1330). Greenville, SC: University of South Carolina & Clemson University.

Zhuang, Y., Conner, A. (in press). Teacher Questioning Strategies in Supporting Validity of Collective Argumentation: Explanation Adapted from Habermas' Communicative Theory. *Proceedings of the 42th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Mazatlán, Sinaloa, Mexico.