Applications of Alexander ideals to knotted surfaces in 4-space

by

Jason Joseph

(Under the direction of David Gay)

Abstract

In this thesis we give two applications of Alexander ideals to knotted surfaces in S^4. First we prove that the Alexander ideal induces a homomorphism from the 0-concordance monoid $\mathscr{C}_0$ of oriented surface knots in S^4 to the ideal class monoid of $\mathbb{Z}[t^{\pm 1}]$. Consequently, any surface knot with nonprincipal Alexander ideal is not 0-slice and in fact, not invertible in $\mathscr{C}_0$. This proves that the submonoid of 2-knots is not a group and reproves the existence of infinitely many linearly independent 0-concordance classes.

The second application is to regular homotopies of 2-knots in S^4, and is joint work with Michael Klug, Benjamin Ruppik, and Hannah Schwartz. Analogous to classical unknotting number, we define the Casson-Whitney number of a 2-knot as the minimal number of Whitney moves during any regular homotopy to the unknot, and prove that if K_1 and K_2 each have nontrivial determinant, then the Casson-Whitney number of $K_1 \# K_2$ is at least 2. A corollary is that the Casson-Whitney number is not equal to the stabilization number, the minimal number of 1-handle stabilizations needed to produce an unknotted surface. We also prove a strong version of nonadditivity for both the Casson-Whitney number and the stabilization number.

Index words: Low-dimensional topology, 4-manifolds, knotted surfaces, knot groups, Alexander module, concordance, unknotting number, regular homotopy, ribbon 2-knots, twist-spun knots

Applications of Alexander ideals to knotted surfaces in 4-space

by

Jason Joseph

B.A., Bates College, 2010

M.A., University of Georgia, 2018

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

August, 2020

Applications of Alexander ideals to knotted surfaces in 4-space

by

Jason Joseph

Approved:

Major Professor: David Gay

Committee: Gordana Matic
Arunima Ray
Michael Usher

Electronic Version Approved:

Ron Walcott
Interim Dean of the Graduate School
The University of Georgia
August 2020

Acknowledgements

I would like to acknowledge my family for believing in and supporting me throughout this period of learning and growth. I would also like to acknowledge my advisor, David Gay, for his advice and wisdom and for always having a great attitude and making math fun. Also thanks to Benjamin Ruppik and Hannah Schwartz for making some of the figures. Part of this research was funded by NSF DMS 1664567, "FRG: Collaborative Research: Trisections – New Directions in Low-Dimensional Topology." I would also like to thank the Max Planck Institute for Mathematics in Bonn, Germany, where I conducted portions of this research and was fortunate enough to spend the last year of this PhD program.

Contents

Chapter 1

Introduction

The goal of this thesis is to study closed surfaces smoothly embedded in S^4. The main tools we will utilize are knot groups and the Alexander module, which are classical invariants but contain much information. The ***Alexander ideal*** $\Delta(K)$ of a knotted surface K is the first elementary ideal of its Alexander module. It is well known that this ideal may fail to be principal, in which case K 'has no Alexander polynomial'. The first recorded instance of this was Example 12 of *A Quick Trip Through Knot Theory* by Ralph Fox [Fox62]; later this 2-knot was identified with the 2-twist-spun trefoil of Zeeman. Hence many authors define the Alexander polynomial of a knotted surface to be a generator of the smallest principal ideal which contains the Alexander ideal, thereby obtaining a polynomial at the expense of throwing away information. One goal of this thesis is to promote the study of the Alexander ideal as is. Indeed, the ideal class monoid of a PID is trivial, so Theorem 1.1 would be completely missed if one only considered principal ideals. The impetus for studying these comes from the ribbon obstruction for 2-knots: any ribbon 2-knot has a Wirtinger presentation of deficiency 1, and all such knot groups have principal Alexander ideals. What we show here is that having a nonprincipal ideal is in fact a 0-sliceness obstruction, which unlike the ribbon obstruction generalizes to any genus.

A ***2-knot*** in S^4 is a smooth embedding of a 2-sphere into the 4-sphere, considered up to isotopy. Kervaire proved that all 2-knots in S^4 are slice (concordant to the unknot) [Ker65], so it is natural to seek restricted forms of concordance. Paul Melvin introduced the notion of 0-concordance in his thesis [Mel77]. Two 2-knots are 0***-concordant*** if there is a concordance between them such that each regular level set consists of only spheres. Melvin proved that 0-concordant 2-knots have diffeomorphic Gluck twists in 1977, but it was unknown until recently if any 2-knots in S^4 were not 0-slice, i.e. 0-concordant to the unknot. This is Problem 1.105a on the Kirby problem list [Kir97].

Question (Melvin)**.** *Is every 2-knot 0-slice?*

Sunukjian showed in [Sun15] that all genus g surface knots in S^4 are concordant and extended the notion of 0-concordance to higher genus surfaces. The set of all oriented surface knots in S^4 modulo 0-concordance forms a commutative monoid under connected sum, which we denote $\mathscr{C}_0$. The set of all 2-knots modulo 0-concordance is an important submonoid of $\mathscr{C}_0$, which we denote $\mathscr{K}_0$.

We produce a 0-concordance obstruction by determining exactly how the Alexander ideal can change during such a concordance. Namely, any surface knot which is 0-slice must have a principal ideal. This is analogous to the Fox-Milnor theorem, which says that if a classical knot is slice, its Alexander polynomial must factor as $f(t)f(t^{-1})$. More generally, the Alexander ideals of 0-concordant surface knots are equivalent in the ideal class monoid of $\mathbb{Z}[t^{\pm 1}]$. This comes from a useful factoring of 0-concordances into two opposing ribbon concordances, and the key lemma for Theorem 1.1 which shows that during a ribbon concordance the Alexander ideal changes by multiplication by a principal ideal. The ***ideal class monoid*** of an integral domain R, denoted $\mathcal{I}(R)$, is a quotient of the monoid of nonzero ideals of R under multiplication, by the equivalence relation $I \sim J$ if there exist nonzero $x, y \in R$ such that $(x)I = (y)J$. The Alexander ideal of a connected sum is the product of the individual ideals, so the operations on these monoids are compatible.

Theorem 1.1. *The Alexander ideal induces a homomorphism* $\Delta : \mathscr{C}_0 \to \mathcal{I}(\mathbb{Z}[t^{\pm 1}])$.

By definition, 0-concordant surface knots have the same genus; still we say a surface knot K is *0-slice* if it is 0-concordant to the unknotted surface of the same genus, and *invertible* if there exists a surface knot J so that $K \# J$ is 0-slice. Since the ideal class monoid of $\mathbb{Z}[t^{\pm 1}]$ has no inverses, any surface knot with a nonprincipal Alexander ideal is not invertible in $\mathscr{C}_0$.

Corollary 1.2. *If a surface knot* K *has a nonprincipal Alexander ideal, then it has no inverse in* $\mathscr{C}_0$*, i.e. for all surface knots* J*,* $K \# J$ *is not 0-slice.*

As we have already observed, there exist 2-knots with nonprincipal Alexander ideals; this proves that $\mathscr{K}_0$ is not a group. Next we analyze the effect of twist-spinning on the Alexander ideal to prove that infinitely many twist-spins of the same classical knot have nonprincipal ideals, as long as the knot's determinant is not a unit. The n-twist-spin of a classical knot K in S^3 is denoted $\tau^n K$.

Theorem 1.3. *If* K *is a classical knot such that* $|\Delta_K(-1)| \neq 1$*, then there exist infinitely many* $n \in \mathbb{Z}$ *such that* $\Delta(\tau^n K)$ *is not principal. In particular, if* n *is even and* $\Delta(\tau^n K)$ *is principal, then* $\Delta_K(t)$ *has a root* z *such that* $z^n = 1$.

In the special case of 2-twist-spins of 2-bridge knots, the determinant is an invariant of 0-concordance.

Corollary 1.4. *Any 2-twist-spin of a 2-bridge knot is not invertible in* $\mathscr{C}_0$*. If* K *and* J *are 2-bridge knots and their 2-twist-spins are 0-concordant, then* $|\Delta_K(-1)| = |\Delta_J(-1)|$.

The structure of the ideal class monoid is in general quite complicated, although in the case of ideals which admit a prime factorization we understand the situation completely. In particular, the maximal ideals of $\mathbb{Z}[t^{\pm 1}]$ independently generate a free commutative submonoid of the ideal class monoid of infinite rank. If K is a 2-bridge knot with prime determinant $p = |\Delta_K(-1)|$, then the Alexander ideal of its 2-twist-spin is the maximal ideal

$(p, t+1)$. Hence any collection of 2-twist-spun 2-bridge knots with pairwise distinct, prime determinants yields a basis for a free commutative submonoid of $\mathscr{K}_0$.

Theorem 1.5. *$\mathscr{K}_0$ contains a submonoid isomorphic to $\mathbb{N}^\infty$.*

These techniques provide a very different answer to the 0-concordance problem than was provided recently by several authors [Sun19], [DM19]. However, all of these approaches utilize the factoring of a 0-concordance into two ribbon concordances. This was first made explicit by Sunukjian in [Sun19], and is also essentially pointed out in [Kir97].

Sunukjian used this and techniques from Heegaard Floer homology, applied to Seifert 3-manifolds of 2-knots, to produce a 0-concordance obstruction, proving the existence of infinitely many 0-concordance classes [Sun19]. This line of attack was strengthened by Dai and Miller using spin rational homology cobordism, and enabled them to find infinitely many linearly independent 0-concordance classes [DM19]. It is interesting to note that these approaches are independent of ours, in that each works to obstruct 0-concordance in situations where the other fails (see Remark 4.17). Furthermore, while Dai and Miller had already established the existence of infinitely many linearly independent 0-concordance classes, Theorem 1.1 provides the only known method to obstruct invertibility in the 0-concordance monoid. These are also the only techniques known to generalize to higher genus surfaces.

Understanding the image of Δ gives us information about $\mathscr{C}_0$. We determine its image by characterizing the ideals which occur as Alexander ideals of knotted surfaces. Kinoshita proved that any polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$ satisfying $f(1) = \pm 1$ is the Alexander polynomial of a ribbon 2-knot [Kin61]. For an ideal $I \subseteq \mathbb{Z}[t^{\pm 1}]$ and $a = \pm 1$, let $I|_a$ denote the nonnegative generator of the ideal $\{f(a) : f(t) \in I\} \subseteq \mathbb{Z}$. We prove the following generalization of Kinoshita's theorem to arbitrary genus.

Theorem 1.6. *An ideal I of $\mathbb{Z}[t^{\pm 1}]$ is the Alexander ideal of a surface knot if and only if*

$I|_1 = 1$.

The last chapter of this thesis concerns regular homotopies of 2-knots in S^4, and is joint work with Michael Klug, Benjamin Ruppik, and Hannah Schwartz [JKRS20]. Smale proved that all 2-knots are regularly homotopic [Sma58]. Additionally, any generic regular homotopy can be decomposed into a finite sequence of finger and Whitney moves. Given a 2-knot K, we consider the minimal number of Whitney moves (or equivalently finger moves) in any regular homotopy from K to the unknot. This is an invariant of K, which we call the ***Casson-Whitney number*** of K, denoted $u_{\text{cw}}(K)$.

There is another notion of an unknotting number for 2-knots, namely the minimal number of 1-handle stabilizations needed to produce an unknotted surface. We call this the ***stabilization number*** of a 2-knot, denoted $u_{\text{st}}(K)$. The goal of [JKRS20] is to study the relationship between these unknotting numbers. In this thesis we will present several theorems from that paper which draw on algebraic techniques. The main theorem of this chapter is that the unknotting numbers are not equal.

Theorem 1.7. *There are infinitely many 2-knots K for which $u_{\text{st}}(K) = 1$ and $u_{\text{cw}}(K) = 2$.*

Miyazaki found examples of 2-knots K_1 and K_2 for which $u_{\text{st}}(K_1) = u_{\text{st}}(K_2) = 1 = u_{\text{st}}(K_1 \# K_2)$ [Miy86]. In these same examples, and more generally whenever the determinants of K_1 and K_2 are nontrivial, we prove that $u_{\text{cw}}(K_1 \# K_2)$ is at least 2.

Theorem 1.8. *Let K_1, K_2 be 2-knots with determinants $\Delta(K_i)|_{-1} \neq 1$. Then*

$$u_{\text{cw}}(K_1 \# K_2) \geq 2.$$

The previous theorem suggests the possibility that the Casson-Whitney number of a connected sum is at least as large as the number of summands. We conclude by proving, on the contrary, a strong form of nonadditivity for both the stabilization number and the Casson-Whitney number.

Theorem 1.9. *Let $\upsilon = u_{\mathrm{st}}$ or u_{cw}. For any $c, n \in \mathbb{N}$, there exist 2-knots $K_1, \ldots, K_n$ with*

$$\upsilon(K_i) = c \text{ for all } i, \text{ and}$$

$$c \leq \upsilon(K_1 \# \cdots \# K_n) \leq 2c$$

Chapter 2

Background

2.1 Knotted surfaces in 4-space

Throughout we will consider smoothly embedded, closed, connected surfaces $K : \Sigma_g \hookrightarrow S^4$, considered up to smooth isotopy. We will assume Σ is oriented except when noted otherwise. When $g = 0$, K is called a *2-knot*. An embedded surface K is ***unknotted*** if K bounds a smoothly embedded handlebody. Up to isotopy, there is only one unknotted surface for each genus, which we denote $\mathcal{U}_g$, with $\mathcal{U} = \mathcal{U}_0$ the unknotted sphere. An ***unlink*** is an embedding $L : \sqcup S^2 \hookrightarrow S^4$ such that each component of L bounds a 3-ball, disjoint from all the other 3-balls.

The first examples of 2-knots are due to Artin in [Art25]. Artin described a way to start with a classical knot K, i.e. a smoothly embedded circle in the 3-sphere, and produce a 2-knot $Spin(K)$ via a process called *spinning*. This was generalized to twist-spinning and eventually deform-spinning, by Fox, Litherland, Zeeman, *et al* [Zee65], [Lit79]. These techniques provide many examples of knotted spheres and tori, although they all 'come from' classical knots and are a small subset of all knotted surfaces. Indeed, enumerating knotted surfaces is a difficult task and even a rough classification seems very difficult.

The situation with classical knots is much simpler. Whitten proved that prime knots whose complements have isomorphic fundamental groups must have homeomorphic complements [Whit85], and Gordon and Luecke proved that knots with homeomorphic complements are isotopic [GL89]. Indeed, classical knots admit prime factorizations, a topic which is very much open for knotted surfaces. In both contexts, however, the ***knot group***, which is the fundamental group of the exterior of the knotted manifold, plays a central role in determining many of its interesting properties.

We begin by introducing ribbon surface knots and twist-spun knots, which will be the main examples considered in this thesis. Then we review some important properties of knot groups and introduce the Alexander module, a powerful invariant which comes from the knot group. The ***Alexander ideal*** is the first elementary ideal of this module and will be our main tool in most of the results herein.

2.1.1 Ribbon surface knots

A particularly simple and easy to visualize class of knotted surfaces is the class of ribbon surfaces.

Definition 2.1. A ***ribbon presentation*** of a knotted surface $K : \Sigma_g \hookrightarrow S^4$ is obtained from an unlink L of n components by fusing along $n-1+g$ 3-dimensional 1-handles embedded in S^4, whose $S^0 \times D^2$ boundary components are embedded in L and equal the intersection of the 1-handles with L, in such a way as to produce an orientable surface of genus g. Any surface knot K admitting such a presentation is called ***ribbon***.

This definition is analogous to ribbon 1-knots, which are obtained from an unlink of n components by fusing along $n-1$ bands (2-dimensional 1-handles), and bound obvious ribbon-immersed disks in S^3 with only ribbon singularities. Indeed, a ribbon knotted surface bounds a ribbon-immersed handlebody in S^4 with only 'ribbon' self-intersections: disks

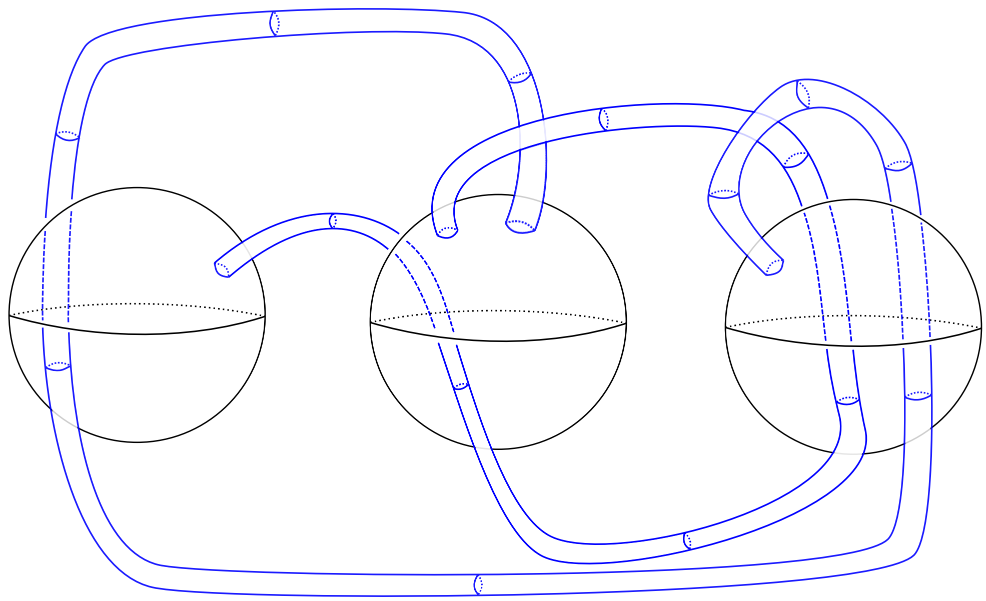

Figure 2.1: A ribbon presentation for a ribbon 2-knot.

embedded in S^4 whose preimages in the handlebody consist of the disjoint union of a properly embedded disk and a disk embedded in the interior of the handlebody.

These surfaces are convenient for computations involving knot groups, as meridians to the unknots generate the knot group and each fusion tube gives a Wirtinger relation. Conversely, given a Wirtinger presentation which abelianizes to $\mathbb{Z}$, one can always create a ribbon surface with unknots corresponding to generators and fusion bands corresponding to relations which has exactly this presentation for its knot group.

2.1.2 Twist-spun knots

The first examples of knotted spheres are due to Artin [Art25]. Note that S^4 admits an open book decomposition with binding the unknotted S^2, which we call S: $S^4 \setminus S$ is fibered by 3-ball pages $B_\theta, \theta \in S^1$. Explicitly, start with $B^3 \times S^1$ and form the quotient

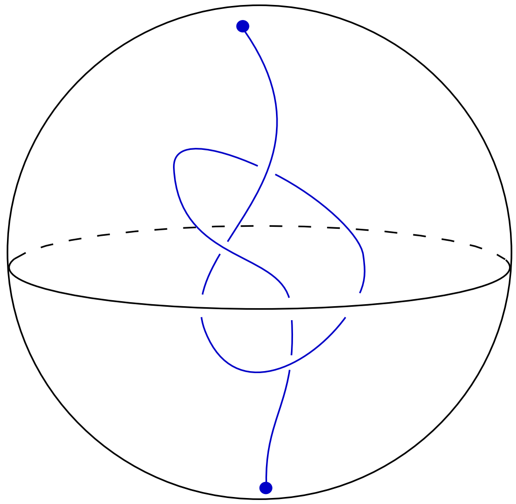

Figure 2.2: A (3-ball, knotted arc) pair used in the spinning construction.

space $B^3 \times S^1/(p, 0) \sim (p, \theta)$ for all $p \in \partial B^3, \theta \in S^1$. This space is diffeomorphic to S^4, the images of $B^3 \times \{\theta\}$ are the B_θ, and the common boundary sphere ∂B_θ is the binding S.

Starting with a classical knot k, Artin described a way to 'spin' a (3-ball, knotted arc) pair through this decomposition, producing a knotted sphere [Art25]. Given $k : S^1 \hookrightarrow S^3$, remove a small 3-ball neighborhood B of a point on k to obtain $k^\circ : I \hookrightarrow S^3 \setminus B$. Identify $S^3 \setminus B$ with B^3 so that ∂k° is exactly the north and south pole of the unit ball $B^3 \subseteq \mathbb{R}^3$.

Definition 2.2 (Artin)**.** The ***spin*** of k, denoted $\mathit{Spin}(k)$, is the image of $k^\circ \times S^1$ in S^4, thought of as the above quotient space.

Artin proved that the group of $\mathit{Spin}(k)$ is isomorphic to the group of k, showing that every 1-knot group is also a 2-knot group. It can be calculated from the complement of the knotted arc inside of B^3, as in Figure 2.2.

Twist-spinning was introduced by Zeeman in [Zee65] as a generalization of the spinning construction (this was further generalized by Litherland to deform spun knots, but in this thesis we will not consider these). Let $\rho_\theta : B^3 \to B^3$ be the self-diffeomorphism which rotates B^3 about the vertical axis through an angle of θ radians. Fix $n \in \mathbb{Z}$ and consider the quotient space $B^3 \times S^1/(p, 0) \sim (\rho_{n\theta}(p), \theta)$ for all $p \in \partial B^3, \theta \in S^1$.

Definition 2.3 (Zeeman)**.** The n***-twist-spin*** of k, denoted $\tau^n k$, is the image of $k^\circ \times S^1$ in S^4, thought of as the above quotient space.

By definition $\tau^0 k \cong Spin(k)$. Zeeman proved that if the number of twists n is not 0, the resulting 2-knot is fibered by the n-fold cyclic branched cover of k. Thus $\tau^{\pm 1} k = \mathcal{U}$ for any k, as this 2-knot is fibered by 3-balls. By the homotopy exact sequence, the commutator subgroup of $\pi(\tau^n k)$ is the fundamental group of the fiber, the n-fold cyclic branched cover of k, for $n \neq 0$. Twist-spun knots provide a large generalization of spun knots which are fairly well understood and display many interesting properties. Cochran proved that any twist-spun knot $\tau^n k$ for k nontrivial and $n \neq 0, \pm 1$ is not ribbon [Coc83], in contrast to spun knots, which are always ribbon.

2.2 Knot groups

The knot group of a knotted surface is the fundamental group of its complement. This is a powerful invariant for studying knotted surfaces. The spinning construction of Artin shows that all 1-knot groups are also 2-knot groups, but there are many interesting group theoretic properties that 2-knot groups can have which 1-knot groups never do.

Definition 2.4. Let K be a surface knot. A **meridian** of K is an element of πK which can be represented by a simple closed curve $\gamma : S^1 \hookrightarrow S^4 \setminus K$ bounding a disk in S^4 that transversely intersects K in a single point.

The set of positively oriented meridians of a knotted surface forms a conjugacy class of its group. That is, x is a meridian of K if and only if $w^{-1} x w$ is as well, for any $w \in \pi K$.

The meridians of a surface knot K generate πK, so any meridian normally generates it, i.e. $\pi K/\langle\langle x\rangle\rangle \cong 1$.

Definition 2.5. Let K be a surface knot. The minimal number of meridians of K which generate πK is called the ***meridional rank*** of K.

Knot groups admit special presentations called Wirtinger presentations which are generated by meridians. These can be calculated from a projection to $\mathbb{R}^3$ in a manner analogous to the Wirtinger algorithm for classical knots, or can be obtained from the motion picture method, by performing the Wirtinger algorithm on the central cross-section of the normal form described in [Kaw96], then adding relations for each band which identify the meridians on either end of it.

Definition 2.6. Let G be the group of a surface knot. A **Wirtinger presentation** of G is a presentation of G of the form $\langle x_1, \dots, x_n | r_1, \dots r_m\rangle$, where each $r_i = x_{i_1}^{-1} w_i^{-1} x_{i_2} w_i$, for some $w_i \in G, i_j \in \{1, \dots, n\}$.

Although not all 2-knot groups are 3-knot groups, every knot group of an orientable knotted manifold $M^n \hookrightarrow S^{n+2}$ is the group of some knotted surface. Indeed, in [Kam01] it is shown that all these knotted manifold knot groups admit Wirtinger presentations. Conversely, any Wirtinger presentation is realizable as the knot group of a knotted surface, by first starting with an unlink of 2-spheres whose meridians will be $x_1, \dots, x_n$, and then performing m 1-handle stabilizations along tubes which link these spheres to carry out the relations.

If $K : \Sigma_g \hookrightarrow S^4$ is any embedded, closed, oriented surface, then by Alexander duality $H_1(S^4 \setminus K; \mathbb{Z}) \cong H^2(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$, so all knot groups abelianize to the integers. A simple but useful fact is that knot groups are semidirects products of their commutator subgroups with their abelianizations. Let K be a knotted surface with group πK. The abelianization short exact sequence

$$1 \to (\pi K)' \to \pi K \to \mathbb{Z} \to 1$$

is easily seen to be split exact, as sending a generator of $\mathbb{Z}$ back to a meridian of K commutes with the abelianization. Thus, $\pi K \cong (\pi K)' \rtimes \mathbb{Z}$, where the semidirect product structure is induced by conjugation by a meridian. Understanding this action is critical to understanding the knot group: for example, any two fibered 1-knots of the same genus g have a free group of rank $2g$ as the commutator subgroup of their knot group, but prime 1-knots are determined by their groups and so must have different semidirect product structures.

It is worth noting that the preimage in πK of $1 \in \mathbb{Z}$ has in general many distinct conjugacy classes, only one of which is the conjugacy class of positively oriented meridians. For example, Suciu showed that there are infinitely many ribbon 2-knots, all with the group of the trefoil, no two of which have the same meridians [Suc85]. In fact he shows there is no automorphism of the trefoil group carrying the meridians of one of these 2-knots to another, so the semidirect product structures must be distinct. This semidirect product structure guarantees that any $g \in \pi K$ may be written as $x^n w$, where x is a meridian of K (or indeed any element which abelianizes to 1), $n \in \mathbb{Z}$, and $w \in (\pi K)'$.

Example 2.7 (Twist-spun knots). The knot group of $\tau^n K$ is obtained as a quotient of πK by making the n^{th} power of a meridian lie in the center of the group. To see why, consider two meridians x and y of K, where x is drawn near the north pole as in Figure 2.3. Isotoping y through the twist-spin, keeping the basepoint fixed, we see that it gets caught n times around the axis of twisting. Thus $x^{-n} y x^n = y$ in the group of the twist-spin, or equivalently x^n commutes with y. Repeating this argument with x and a generating set of meridians we see that x^n is in the center of the group of the n-twist-spin of K.

Let $\langle x_0, \ldots, x_m | r_1, \ldots, r_m \rangle$ be a Wirtinger presentation for πK (such a presentation can be obtained from any diagram for K). Then $\langle x_0, \ldots, x_m | r_1, \ldots, r_m, [x_0^n, x_1], \ldots, [x_0^n, x_m] \rangle$ is

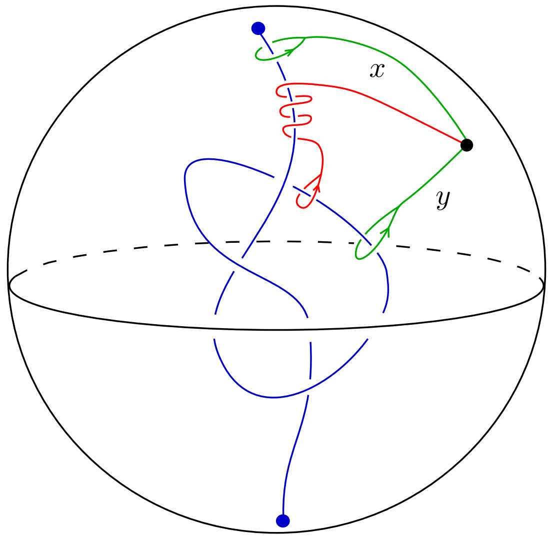

Figure 2.3: An illustration of the knot group calculation for the 3-twist-spun trefoil. After isotoping the meridian y around the twist-spin, we see it is isotopic rel basepoint to the red meridian $x^{-3}yx^3$.

a Wirtinger presentation for $\pi(\tau^n K)$. Sometimes it will be convenient to use the equivalent presentation $\langle x_0, \ldots, x_m | r_1, \ldots, r_m, x_0^n x_1^{-n}, \ldots, x_0^n x_m^{-n} \rangle$. These are equivalent because all the meridians of πK are conjugate (any two elements which are conjugate and in the center of a group must be equal, and conversely if $x_0^n = x_i^n$, then $x_0^n x_i = x_i^{n+1} = x_i x_0^n$). In Theorem 1.3 we work out the ideals of these 2-knots explicitly.

2.3 The Alexander module

As noted in Section 2.2, the group πK of any knotted surface K abelianizes to $\mathbb{Z}$. Thus, as in the case of classical knots, it makes sense to study the commutator subgroup $C := (\pi K)'$ of the knot group, together with the action of conjugation by a meridian. The covering space corresponding to the commutator subgroup of the knot group is called the ***infinite cyclic cover*** $\widetilde{X}$, or universal abelian cover, of the knot complement. It admits a $\mathbb{Z}[t^{\pm 1}]$ module structure, induced by the action of conjugation by a meridian of K. This is called the ***Alexander module*** of K. To make this precise, let K be a knotted surface and let x be a meridian. For any element $\alpha \in H_1(\widetilde{X}, \mathbb{Z})$, $t \cdot \alpha$ is defined as $[x^{-1}ax]$, where a is any lift of α to C and brackets denote equivalence class in C/C' (which is canonically isomorphic to $H_1(\widetilde{X}; \mathbb{Z})$ as a group). This does not depend on the choice of a because if $[a'] = \alpha$ as well, then $a' = aa''$ for $a'' \in C'$, so $[x^{-1}a'x] = [x^{-1}aa''x] = [x^{-1}ax] + [x^{-1}a''x] = [x^{-1}ax]$ (note that $[x^{-1}a''x] = 0$ because $x^{-1}a''x \in C'$, as C' is a characteristic, hence normal subgroup of πK). Lastly, if y is any other meridian of K, then $y = cx$ for some $c \in C$, so $[y^{-1}ay] = [x^{-1}c^{-1}acx] = -[x^{-1}cx] + [x^{-1}ax] + [x^{-1}cx] = [x^{-1}ax]$, so this action is well defined.

Our main method of computing a presentation matrix for the Alexander module will be Fox's free differential calculus, which we now describe. For more details see [CF63], [Fox62], [Fox53]. Any group homomorphism $G \to H$ has a unique extension to a ring homomorphism between the group rings $\mathbb{Z}G \to \mathbb{Z}H$. When G is a knot group, expressed as a presentation $\langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$, the two homomorphisms we will consider are the quotient map $F \to G$ which defines G from a free group F on n generators, and the abelianization map $G \to \langle t \rangle \cong \mathbb{Z}$. The benefit of this last homomorphism is that all knot groups abelianize to $\mathbb{Z} \cong H_1(S^4 \setminus \Sigma_g)$, so we have a well defined universe in which to compare, and since $\mathbb{Z}$ is abelian, its group ring is commutative, so one can define determinants and elementary ideals. Of course, commutative rings are far better understood than noncommutative ones, and the ring $\mathbb{Z}[t^{\pm 1}]$ has some additional nice properties which will be exploited in Chapter 4.

A *derivative* is a linear mapping $D : \mathbb{Z}F \to \mathbb{Z}F$ which obeys a Leibniz rule. On elements of F, this takes the form $D(g_1 g_2) = Dg_1 + g_1 Dg_2$, and then one extends linearly to define D on all of $\mathbb{Z}F$. We are concerned with the case that F is a free group, generated by $x_1, \dots, x_n$. Then for each free generator x_j there is a unique derivative $\partial/\partial x_j$ satisfying $\partial x_i / \partial x_j = \delta_{ij}$.

Note that $\partial 1/\partial x = 0$ and $\partial x^{-1}/\partial x = -x^{-1}$.

The *Alexander matrix* $\mathbf{A}$ corresponding to a knot group presentation $P = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ has entries the images of the r_i under the composition

$$\mathbb{Z}F \xrightarrow{\partial/\partial x_j} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\mathfrak{a}} \mathbb{Z}\langle t \rangle$$

where $F \xrightarrow{\gamma} F/R \cong G$ is the canonical homomorphism defining G from P and $G \xrightarrow{\mathfrak{a}} \langle t \rangle$ is the abelianization. So $A = (a_{ij})$, where $a_{ij} = \mathfrak{a}\gamma(\partial r_i/\partial x_j)$. It is a presentation matrix for the Alexander module. Two matrices are considered equivalent if one is obtained from the other by a sequence of row and column operations, adding a row of zeroes, or stabilization: $\mathbf{A} \to \begin{pmatrix} \mathbf{A} & \vec{0} \\ \vec{0} & 1 \end{pmatrix}$. Different presentations of the same group give rise to equivalent Alexander matrices.

Starting with P as before, $\mathbf{A}$ will be an $(m \times n)$-matrix. The k^{th} ***elementary ideal*** ε_k of $\mathbf{A}$ is the ideal of $\mathbb{Z}\langle t \rangle$ generated by the determinants of the square $(n-k)$-minors of $\mathbf{A}$. When $n - k \leq 0$, $\varepsilon_k = (1)$. The ***Alexander ideal*** is the first elementary ideal. Equivalent matrices define the same chain of elementary ideals, so these are invariants of the oriented knot K. Since t encodes an orientation, these are really invariants of the pair $(\pi K, \varepsilon)$, where ε is an orientation of K.

Recall that P is a *Wirtinger presentation* if all generators x_i abelianize to t, the generator of $H_1(S^4 \setminus K)$, and all relations are of the form $x_i = w x_j w^{-1}$, where $i, j \in \{1, \ldots, n\}$ and w is a word in the x_i. A nice consequence of all generators abelianizing to t is that the sum across any row of the Alexander matrix is zero, so we can always replace one column with a column of zeroes when working with Wirtinger presentations (Theorem 8.3.7 [CF63]). When K is a classical knot or a ribbon n-knot, it has a Wirtinger presentation with $m+1$ generators and m relations. After replacing one column with zeroes, we see that there is only one $(m \times m)$-minor with a nonzero determinant, so these knots always have principal

Alexander ideals. In this case a generator of the ideal is called the *Alexander polynomial of K*.

Example 2.8. To illustrate these techniques we use the 2-twist-spun trefoil as an example, with its standard Wirtinger presentation $\langle x, y | xyxy^{-1}x^{-1}y^{-1}, x^2yx^{-2}y^{-1}\rangle$.

$$\begin{pmatrix} \frac{\partial r_1}{\partial x} & \frac{\partial r_1}{\partial y} \\ \frac{\partial r_2}{\partial x} & \frac{\partial r_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + xy - xyxy^{-1}x^{-1} & x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1} \\ 1 + x - x^2yx^{-1} - x^2yx^{-2} & x^2 - x^2yx^{-2}y^{-1} \end{pmatrix}$$

$$\xrightarrow{a\gamma} \begin{pmatrix} 1 + t^2 - t & t - t^2 - 1 \\ 1 + t - t^2 - t & t^2 - 1 \end{pmatrix} \sim \begin{pmatrix} t^2 - t + 1 & 0 \\ 1 - t^2 & 0 \end{pmatrix}$$

Since there are 2 columns, the Alexander ideal is generated by the (1×1)-minors, and $\Delta(K) = (t^2 - t + 1, t^2 - 1)$. One can check this is equal to $(3, t + 1)$, which makes it clear that the quotient $\mathbb{Z}[t^{\pm 1}]/(3, t + 1) \cong \mathbb{Z}_3$, so $\Delta(K)$ is maximal. $\mathbb{Z}[t^{\pm 1}]$ is a regular ring of Krull dimension 2, so every maximal ideal is minimally generated by 2 elements. Thus the 2-twist-spun trefoil is not ribbon.

Example 2.9 (Twist-spun 2-bridge knots). Generalizing the previous example, we show that if k is a 2-bridge knot then the Alexander ideal of $\tau^n k$ is $(\Delta_k(t), t^n - 1)$. Since k is 2-bridge, πK has a Wirtinger presentation $\langle x, y | r\rangle$, and $\pi(\tau^n K)$ is then presented by $\langle x, y | r, [x^n, y]\rangle \cong \langle x, y | r, x^n y^{-n}\rangle$. The Alexander ideal, as computed from these presentations, is $(\Delta_k(t), t^n - 1) = (\Delta_k(t), \frac{t^n - 1}{t - 1})$, where $\Delta_k(t)$ is the Alexander polynomial of k.

Alexander ideals are multiplicative under connected sum, which is of critical importance to Theorem 1.1.

Proposition 2.10. *Let K and J be surface knots. Then $\Delta(K\#J) = \Delta(K)\Delta(J)$.*

Proof. Let $\langle x_1, \dots, x_n | r_1, \dots, r_k\rangle$ and $\langle y_1, \dots, y_m | s_1, \dots, s_l\rangle$ be Wirtinger presentations for πK and πJ. Then $\langle x_1, \dots, x_n, y_1, \dots, y_m | x_1 y_1^{-1}, r_1, \dots, r_k, s_1 \dots, s_l\rangle$ is a Wirtinger presentation for $K\#J$. Let $a_{ij} = \mathfrak{a}\gamma\left(\frac{\partial r_i}{\partial x_j}\right)$ and $b_{ij} = \mathfrak{a}\gamma\left(\frac{\partial s_i}{\partial y_j}\right)$. Then the Alexander matrix for K is $A = (a_{ij})$ and the matrix for J is $B = (b_{ij})$, so the matrix for $K\#J$ is:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ a_{11} & a_{12} & \dots & a_{1n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & b_{11} & b_{12} & \dots & b_{1m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & b_{l1} & b_{l2} & \dots & b_{lm} \end{pmatrix}$$

As usual, we replace a column by zero, the sum of all columns. It is convenient to replace the $(n+1)^{\text{st}}$ column with zeroes, resulting in

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ a_{11} & a_{12} & \dots & a_{1n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & b_{12} & \dots & b_{1m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & b_{l2} & \dots & b_{lm} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & a_{12} & \dots & a_{1n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{k2} & \dots & a_{kn} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & b_{12} & \dots & b_{1m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & b_{l2} & \dots & b_{lm} \end{pmatrix}$$

$$\sim \begin{pmatrix} a_{12} & \dots & a_{1n} & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k2} & \dots & a_{kn} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & b_{12} & \dots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & b_{l2} & \dots & b_{lm} \end{pmatrix}$$

In the last step we used the inverse of the stabilization move. To simplify things further, we delete the column of zeroes and remember to take the square minors which use all of the columns, i.e. the minors of size $n + m - 2$.

$$\begin{pmatrix} a_{12} & \dots & a_{1n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k2} & \dots & a_{kn} & 0 & \dots & 0 \\ 0 & \dots & 0 & b_{12} & \dots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & b_{l2} & \dots & b_{lm} \end{pmatrix} = \begin{pmatrix} \mathbf{A}' & \mathbf{0} \\ \mathbf{0} & \mathbf{B}' \end{pmatrix}$$

Note that $\mathbf{A}'$, $\mathbf{B}'$ are obtained from $\mathbf{A}$, $\mathbf{B}$ by deleting the first column, which may as well have been zero anyway. The claim now is that unless we choose a minor with $(n-1)$ rows from $\mathbf{A}'$ and $(m-1)$ rows from $\mathbf{B}'$, we will get a determinant of zero. Without loss of generality, suppose that we chose at least n rows from $\mathbf{A}'$. This minor is of the form $\begin{pmatrix} \mathbf{A}'' & \mathbf{0} \\ \star & \mathbf{C} \end{pmatrix}$, where both $\mathbf{A}''$ and $\mathbf{C}$ are square and the matrix $\mathbf{C}$ has a row of zeroes. The determinant of this minor is $|\mathbf{A}''| \cdot |\mathbf{C}| = |\mathbf{A}''| \cdot 0 = 0$. Therefore, the only minors with nonzero determinants are of the form $\begin{pmatrix} \mathbf{A}'' & \mathbf{0} \\ \mathbf{0} & \mathbf{B}'' \end{pmatrix}$, where $\mathbf{A}''$ and $\mathbf{B}''$ are $(n-1)$ and $(m-1)$-minors of $\mathbf{A}'$ and $\mathbf{B}'$, respectively. The determinant is $|\mathbf{A}''| \cdot |\mathbf{B}''|$. Note that $|\mathbf{A}''|$ is a generator of $\Delta(K)$ and $|\mathbf{B}''|$ is a generator of $\Delta(J)$. By choosing all possible minors of this form, we obtain a generating set for $\Delta(K\#J)$, each generator equal to the product of a generator of $\Delta(K)$ and a generator of $\Delta(J)$. Since ranging through all $\mathbf{A}''$ provides a generating set for $\Delta(K)$, and likewise with all $\mathbf{B}''$ and $\Delta(J)$, $\Delta(K\#J)$ is equal to the product of ideals $\Delta(K)\Delta(J)$. □

2.4 Regular homotopies of 2-spheres in 4-space

Part of Smale's classification of immersions of spheres in Euclidean spaces of dimensions at least 3 proves that all 2-spheres embedded in $\mathbb{R}^4$ are regularly homotopic, i.e. homotopic through immersions [Sma58]. Given two knotted spheres K_0 and K_1, embedded in S^4, any generic regular homotopy between them can be decomposed into a sequence of finger moves, Whitney moves, and isotopies. A Whitney move is a standard model for a regular homotopy which removes a pair of transverse double points of the immersion of opposite sign. The reverse homotopy is called a finger move, because a 'finger' of the surface pokes out and intersects itself, introducing two transverse double points, as in Figure 2.4. More details can be found in [FQ90], as well as Casson's lectures in [Cas86].

Definition 2.11. The local model for the regular homotopy removing a pair of double points is called a ***Whitney move***; this homotopy is supported in the regular neighborhood of a ***Whitney disk*** W. The inverse to this homotopy is called a ***finger move***, which is supported in a regular neighborhood of a ***guiding arc*** α.

Given a knotted sphere K, counting the minimal possible number of Whitney moves in a generic regular homotopy as above to the unknotted 2-sphere $\mathcal{U}$ is an invariant of K, which we call the Casson-Whitney number.

Definition 2.12. Let K be a 2-knot in S^4. The ***Casson-Whitney number of*** K**,** $u_{\mathbf{cw}}(K)$,

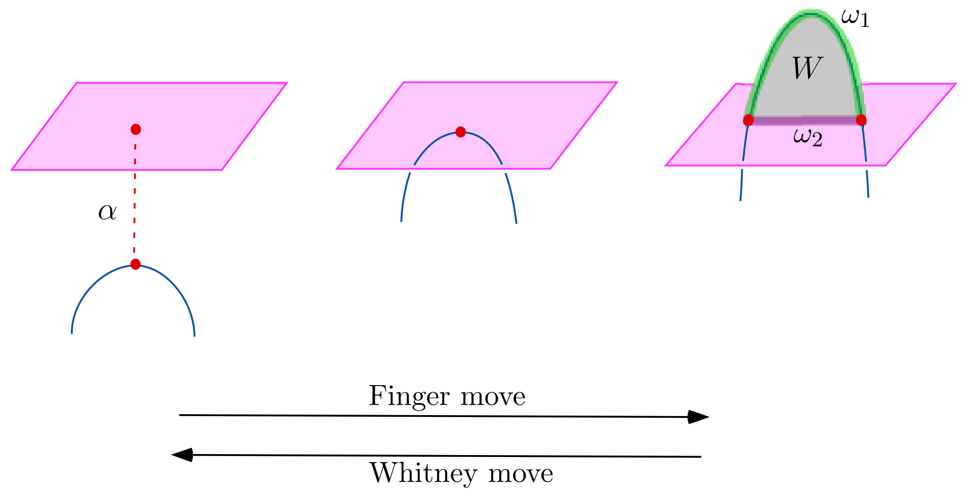

Figure 2.4: The local model of a finger move along the guiding arc α, and Whitney move along the Whitney disk W. These are 3-dimensional slices of a local 4-ball neighborhood, shown at key stages of the homotopy.

is the minimal number of Whitney moves taken over all regular homotopies from K to the unknot.

By a standard reordering argument, it is possible to perform all the finger moves first and then all the Whitney moves, without changing the numbers of these moves. The immersion in the central cross section, which we call Σ and occurs just after all the finger moves have been performed on K but before the Whitney moves, can also be thought of as the result of finger moves on the unknot $\mathcal{U}$, since $\mathcal{U}$ is obtained from Σ by a sequence of Whitney moves (see Figure 2.5). This is the situation in which the knot group provides a lower bound on $u_{\text{cw}}(K)$, but we first need to understand the effect of a finger move on the knot group. We follow Casson's notes [Cas86].

Let K be an immersed sphere in S^4, and let K' be the effect of performing a single finger move on K. We can isotope the base of the finger to lie very near where the finger move

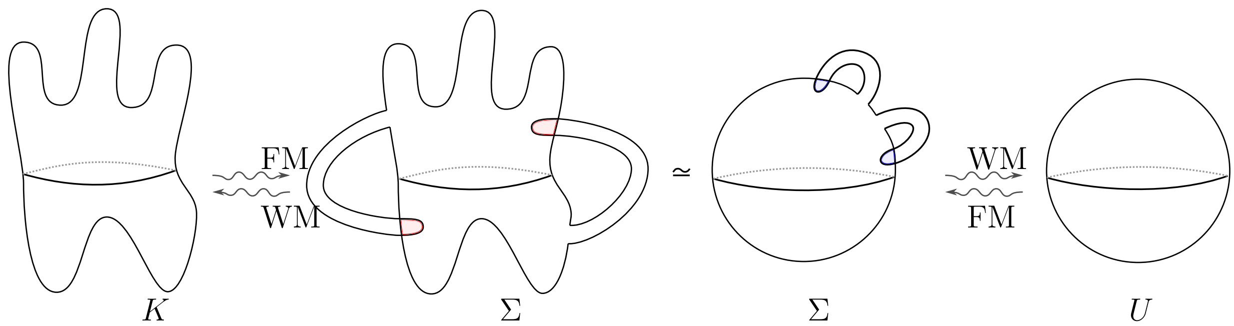

Figure 2.5: Decomposing a regular homotopy from a 2-knot K to the unknot U. The immersed sphere Σ obtained after the finger moves and before Whitney moves on K is drawn from two different perspectives (middle left and middle right) to show the knotted and standard Whitney disks (red and blue respectively).

is performed, so that the guiding arc for the finger move is really a loop $w \in \pi K$. If x is the meridian of K at the base of the finger, then the group of K' is obtained from the group of K by adding the relation $[x, x^w] = 1$, where $x^w = w^{-1}xw$. To see why, consider a small arc A from the tip of the finger to K just before the finger move is performed, as in Figure 2.6. Let W be the Whitney disk inverse to the finger move, which appears just after the finger move is performed. The fundamental groups of the complements of K, $K \cup A$, and $K' \cup W$ are isomorphic: removing an arc from a 4-manifold does not change π_1, and the complements of $K \cup A$ and $K' \cup W$ are homotopy equivalent. To obtain the group of K', we need to add the interior of W back into the complement before taking π_1. Adding in this disk will add the specified relation. Consider a regular $D^2 \times D^2$ neighborhood of one of the double points of the finger move. The two local sheets of K' can be identified with the unit disks $x^2 + y^2 \leq 1$, $z^2 + t^2 \leq 1$, intersecting at $(0,0) \times (0,0)$. Consider the Clifford torus $\partial D^2 \times \partial D^2$. This is disjoint from K' and punctures the Whitney disk W in a single point. Note that the meridian and longitude of this torus represent the meridians x and x^w of the knot group. Adding W back into the complement "unpunctures" this torus, thus adding the commutator relation $[x, x^w] = 1$ to the knot group.

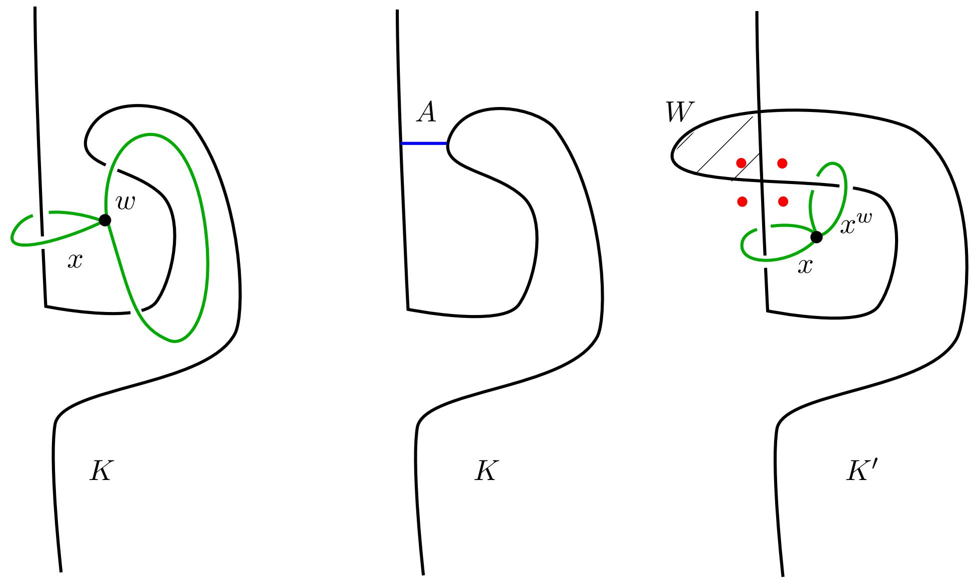

Figure 2.6: A schematic for the effect of a finger move on the knot group. The Clifford torus is pictured as four red dots.

Thus the group of Σ is obtained from the group of K by adding one such relation for each finger move. However, it is also obtained from the group of the unknot, $\mathbb{Z}$, by adding (trivial) commutator relations. Thus $\pi\Sigma \cong \mathbb{Z}$, and the finger move relations performed on K had the effect of abelianizing πK. Thus the minimal number of finger move relations which abelianize the knot group of K is a lower bound for $u_{\mathrm{cw}}(K)$. This motivates the definition below, which is defined for general n-knots because we will show in Section 5.3 that it is a lower bound for the classical unknotting number of 1-knots.

Definition 2.13. Let K be an n-knot. The minimal number of relations of the form $xy = yx$ which abelianize the knot group, where x, y are meridians of K, is called the ***algebraic Casson-Whitney number*** $a_{\mathrm{cw}}(K)$ of K.

Proposition 2.14. *If K is a 2-knot, then $a_{\mathrm{cw}}(K) \leq u_{\mathrm{cw}}(K)$.*

This is the sharpest lower bound we are aware of for the Casson-Whitney number. In Section 5.1 we will use it to prove that the Casson-Whitney number of a 2-knot is not equal to the stabilization number for infinitely many 2-knots. The stabilization number is a more well-studied invariant of a 2-knot; see e.g. [HK79], [Miy86].

Definition 2.15. Let $K : \Sigma_g \hookrightarrow S^4$ be an orientable surface knot and let $h : D^1 \times D^2 \hookrightarrow S^4$ be an embedding such that $K \cap h = \partial D^1 \times D^2$ and such that the surface $K + h := (K \setminus h) \cup D^1 \times \partial D^2$ is orientable. The operation of replacing K with $K + h$ is called a ***stabilization*** of K, and the minimal number of stabilizations necessary to produce an unknotted surface is called the ***stabilization number of*** K, denoted $u_{\mathbf{st}}(K)$.

Note that this quantity is always finite, since knotted surfaces bound Seifert 3-manifolds. Drilling out the cocores of the 2-handles in any Seifert 3-manifold results in a new surface which bounds a solid handlebody and is thus unknotted. Hence this is often called the *unknotting number* of a surface in the literature. The main goal of [JKRS20] is to compare these two notions of unknotting number, and in it the inequality $u_{\text{st}}(K) \leq u_{\text{cw}}(K) + 1$ is shown for any 2-knot K. The main difficulty in proving $u_{\text{st}} \neq u_{\text{cw}}$ is that most of the lower bounds we are aware of for u_{cw} are also lower bounds for u_{st}. The algebraic Casson-Whitney number, however, can be strictly greater than u_{st}, as we will see in Section 5.2.

Since it will appear in Chapter 5, we briefly introduce the algebraic stabilization number here. Stabilizing a knotted surface also adds a commutator relation to the knot group. Let K be a knotted surface and x a meridian of K. If the stabilization is performed along a loop $w \in \pi K$, the meridian x at the base of the tube and x^w at the opposite end will now cobound a cylinder in the complement and therefore be identified in the knot group. So, in contrast to finger move relations, which force meridians to commute, stabilization relations identify two meridians. This is considerably stronger, and this difference is how we will show $u_{\text{st}} \neq u_{\text{cw}}$.

Definition 2.16. Let K be an n-knot. The minimal number of relations of the form $x = y$, where x, y are meridians of K, which abelianize the knot group is called the ***algebraic stabilization number*** $a_{\mathrm{st}}(K)$ of K.

Another important fact we will use in Section 5.4 is that if the knot group of a knotted surface or immersed 2-knot K is $\mathbb{Z}$, there is only one stabilization, resp. finger move, which can be performed on K, up to isotopy. This was first proved for stabilizations in [HK79]; the corresponding statement for finger moves is proved in [JKRS20].

Chapter 3

Alexander ideals of knotted surfaces

In this chapter we characterize the ideals which occur as Alexander ideals of knotted surfaces and generalize the determinant of a knotted surface to the case of nonprincipal ideals.

3.1 Characterization

In 1960, Kinoshita proved that any polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$ with $f(1) = \pm 1$ is the Alexander polynomial of a ribbon 2-knot [Kin61]. Yajima strengthened this theorem by achieving the same result with 2 generator, 1 relator Wirtinger presentations [Yaj69]. In Theorem 3.1 below we provide a new proof of this fact, which improves upon Yajima's in that it solves for the Alexander polynomial of any 2 generator, 1 relator Wirtinger presentation explicitly in one general formula. This result allows us to generalize Kinoshita's theorem to a complete characterization of which ideals occur as the Alexander ideals of surface knots. If $I \subseteq \mathbb{Z}[t^{\pm 1}]$ is an ideal and $a = \pm 1$, let $I|_a$ denote the nonnegative generator of the ideal $\{f(a) : f(t) \in I\} \subseteq \mathbb{Z}$.

Theorem 1.6. *An ideal I of $\mathbb{Z}[t^{\pm 1}]$ is the Alexander ideal of a surface knot if and only if $I|_1 = 1$.*

Theorem 3.1. *If $f(t) \in \mathbb{Z}[t^{\pm 1}]$ satisfies $f(1) = \pm 1$, then there is a ribbon 2-knot K of meridional rank 2 with $\Delta(K) = (f(t))$.*

Proof. Our basic strategy is the same as Kinoshita's, but we achieve $f(t)$ as the Fox derivative of a single Wirtinger relator as opposed to the determinant of a large matrix. We will construct a Wirtinger presentation $\langle x, y | r \rangle$, where $r = xwy^{-1}w^{-1}$ for some word $w \in \langle x, y \rangle$. Any such presentation presents the knot group πK of a ribbon 2-knot K (see [Kaw96]). The Jacobian is $\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix}$.

Let r_x denote $\frac{\partial r}{\partial x}$. Since this is a Wirtinger presentation, x and y abelianize to t as before. Likewise, after abelianization the sum across the row is zero, so $\mathfrak{a}\gamma(r_y) = -\mathfrak{a}\gamma(r_x)$. The Alexander ideal of K is then generated by the abelianization of $r_x = 1 + xw_x - xwy^{-1}w^{-1}w_x$, i.e. $\Delta(K) = (1 + \mathfrak{a}\gamma(w_x)(t-1))$.

Since the Alexander polynomial is only defined up to a unit, we may assume $f(1) = 1$, so that $f(t) = 1 + g(t)(t-1)$ for some polynomial $g(t)$. We will show that w can be chosen so that $\mathfrak{a}\gamma(w_x) = g$, i.e. so that the abelianized matrix is $\begin{pmatrix} f(t) & -f(t) \end{pmatrix}$ and $\Delta(K) = (f(t))$.

For clarity, we first point out that for any $w = y^{n_1}x^{n_2} \dots y^{n_{2k-1}}x^{n_{2k}}$, $n_i \in \mathbb{Z}$, the Alexander polynomial calculated from this presentation will be

$$1 + t^{n_1}(t^{n_2} - 1) + t^{n_1+n_2+n_3}(t^{n_4} - 1) + \cdots + t^{n_1+\cdots+n_{2k-1}}(t^{n_{2k}} - 1),$$

as can be checked directly (note that $\frac{dx^n}{dx}(x-1) = x^n - 1$ for all $n \in \mathbb{Z}$). We will only need the case $n_{2i} = \pm 1$, and add the desired terms $t^{n_i}(t-1)$ and $t^{n_i}(t^{-1}-1) = -t^{n_i-1}(t-1)$ as many times as needed. In particular, if

$$f(t) = 1 + t^{m_1}(t^{m_2} - 1) + t^{m_3}(t^{m_4} - 1) + \cdots + t^{m_{2k-1}}(t^{m_{2k}} - 1),$$

then by letting $\begin{cases} n_1 = m_1 & \\ n_{2i} = m_{2i} & i \geq 1 \\ n_{2i+1} = m_{2i+1} - (m_{2i-1} + m_{2i}) & i \geq 1 \end{cases}$
we arrive at the desired form for $w = y^{n_1}x^{n_2}\dots y^{n_{2k-1}}x^{n_{2k}}$.

□

Proof of Theorem 1.6. Let I be an ideal such that $I|_1 = 1$. Since $\mathbb{Z}[t^{\pm 1}]$ is Noetherian, I admits a finite generating set $g_1(t), \dots, g_m(t)$, so by assumption $(g_1(1), \dots, g_m(1)) = (1)$. This implies there is a linear combination $f_0(t) = \sum a_i g_i(t)$, $a_i \in \mathbb{Z}$, such that $f_0(1) = 1$. Let $f_i(t) = g_i(t) - (g_i(1) - 1)f_0(t)$, $1 \leq i \leq m$. Then $I = (g_1(t), \dots, g_m(t)) = (f_0(t), g_1(t), \dots, g_m(t)) = (f_0(t), f_1(t), \dots, f_m(t))$ has a generating set such that each generator evaluates to 1 at 1.

Then, as in Theorem 3.1, we can build a relation r_i for each f_i to obtain a Wirtinger presentation $\langle x, y | r_0, \dots, r_m\rangle$ for a knot group G, from which we can construct a genus m ribbon surface knot K with $\pi K \cong G$. To do this, first construct a 2-knot K_0 with presentation $\langle x, y | r_0\rangle$, then attach a 1-handle to K_0 for each additional relation. The resulting Alexander matrix is $(m + 1) \times 2$, and the Alexander ideal is then $\Delta(K) = (\mathfrak{a}\gamma(r_{0x}), \dots, \mathfrak{a}\gamma(r_{mx})) = (f_0(t), \dots, f_m(t)) = I$.

It remains to show that if K is a surface knot, then $\Delta(K)|_1 = 1$. This will follow from the observation that we can compute $\Delta(K)|_1$ by evaluating the entries of the Alexander matrix at 1 before taking determinants of minors, and by using an especially nice Wirtinger presentation for πK.

Let P be a Wirtinger presentation for πK. Since K is connected, all of the generators $x_0, \dots, x_n$ of P are conjugate. Therefore we can rewrite the relations to obtain a presentation of the form $\langle x_0, \dots, x_n | r_1, \dots r_n, s_{n+1}, \dots, s_m\rangle$, such that $r_i = x_i w_i x_0^{-1} w_i^{-1}$, $1 \leq i \leq n$, and $s_i = x_0 w_i x_0^{-1} w_i^{-1}$, $n + 1 \leq i \leq m$. If r is any Wirtinger relation, then $\mathfrak{a}\gamma(r_{x_i})|_{t=1}$ is equal to the exponent sum of x_i in r (this is shown in one case in the proof of Theorem 3.1. The other cases are similar, but note that $\mathfrak{a}\gamma(w) = t^N$ for some N, since w is an arbitrary word in x and y. So $\mathfrak{a}\gamma(w)|_{t=1} = 1$). Therefore, when we form the Alexander matrix for K and evaluate the entries at 1, we obtain the matrix:

$$\begin{pmatrix} r_{1x_0} & \dots & r_{1x_n} \\ r_{2x_0} & \dots & r_{2x_n} \\ \vdots & \ddots & \vdots \\ r_{nx_0} & \dots & r_{nx_n} \\ s_{1x_0} & \dots & s_{1x_n} \\ \vdots & \ddots & \vdots \\ s_{mx_0} & \dots & s_{mx_n} \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 1 & 0 \\ -1 & 0 & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix}$$

Since the identity matrix is an $(n \times n)$-minor, $\Delta(K)|_1 = 1$.

□

Corollary 3.2. *Let I be an ideal of $\mathbb{Z}[t^{\pm 1}]$ such that $I|_1 = 1$. If I is minimally generated by g elements, then there is a ribbon surface knot K of genus g and meridional rank 2 with $\Delta(K) = I$.*

In particular, any maximal ideal m is minimally generated by two elements $f(t), g(t)$. The gcd of $f(1)$ and $g(1)$ is 1 if and only if m is the ideal of a surface knot, and the above construction yields a surface of genus 2 realizing this ideal. If m can be written as $(f(t), g(t))$ such that $f(1) = 1$, then m is the ideal of a knotted torus.

3.2 The determinant of a surface knot

The definition below generalizes the notion of determinant to the case of nonprincipal ideals. This will be an odd integer, which, as in the case of a single generator, follows from the fact that $\Delta(K)|_1 = 1$.

Definition 3.3. Let K be a surface knot. The *determinant* of K is $\Delta(K)|_{-1}$.

Proposition 3.4. *The determinant of a surface knot is odd.*

Proof. Let K be a surface knot. We saw in Corollary 3.2 that $\Delta(K)$ has a generating set $(f_1(t), \ldots, f_n(t))$ where each $f_i(1) = 1$. Thus $f_i(t) = (t-1)g_i(t) + 1$ for some $g_i(t)$. Therefore $f_i(-1) = -2g_i(-1) + 1$ is odd for each i, so $\Delta(K)|_{-1}$, the positive generator of $(f_1(-1), \ldots, f_n(-1))$, is odd as well. □

Proposition 3.5. *Let K be a classical knot. The determinant of $\tau^n K$ is equal to the determinant of K, $|\Delta_K(-1)|$, if n is even and 1 if n is odd.*

Proof. We will show in the proof of Theorem 1.3 that if $\langle x_1, \ldots, x_{m+1} | r_1, \ldots, r_m \rangle$ is a Wirtinger presentation for πK, then $\Delta(\tau^n K) = \sum_{j=1}^{m+1} ((t^n - 1)^{j-1}) \varepsilon_j(K)$. When n is even, $(-1)^n - 1 = 0$, so $\Delta(\tau^n K)|_{-1} = \varepsilon_1(K)|_{-1} = |\Delta_K(-1)|$, and when n is odd $(-1)^n - 1 = -2$, so $\Delta(\tau^n K)|_{-1}$ has both $\Delta_K(-1)$, which is odd, and $(-2)^m$ as generators, hence the determinant is 1. □

Remark 3.6. The previous corollary shows that this definition contains more information than evaluating the usual definition of the Alexander polynomial of a surface knot at $t = -1$. The usual definition of the j^{th} Alexander polynomial of K is to take a generator of the smallest principal ideal which contains $\varepsilon_j(K)$. When K is a 2-bridge knot with determinant $|\Delta_K(-1)| = p > 1$, the ideal of its 2-twist-spin is $(\Delta_K(-1), t+1) = (p, t+1)$, and the only principal ideal which contains it is the unit ideal (1). Its first Alexander polynomial is therefore 1, as is its determinant. With our definition, however, $(\Delta_K(-1), t+1)|_{-1} = p \neq 1$. This is the more desirable answer, since a nontrivial Fox p-coloring of K extends to one of $\tau^2 K$ in the obvious way.

Proposition 3.7. *Let K be a classical knot with a nontrivial Fox p-coloring $\phi : \pi K \to D_p$. Then ϕ factors through the quotient map $\pi K \to \pi(\tau^n K)$ if and only if n is even.*

Proof. Let $\phi : \pi K \to D_p$ be a nontrivial p-coloring and $\langle x_0, \ldots, x_m | r_1, \ldots, r_m \rangle$ a Wirtinger presentation of πK. As discussed previously, we can form $\pi(\tau^n K)$ from πK by adding the relations $x_0^n x_i = x_i x_0^n$, $1 \leq i \leq m$, or equivalently $x_0^n = x_i^n$, $1 \leq i \leq m$. The condition for ϕ to factor through the natural quotient map is for the images of these equations to be satisfied in $\phi(\pi K)$. We use the latter set of equations. When n is even, this is automatic, since Fox colorings map meridians to reflections, which are of order 2. When n is odd, the highest even power of $\phi(x_j)^n$ will vanish by the above observation, leaving $\phi(x_0) = \phi(x_i)$, $1 \leq i \leq m$. Thus the only colorings which factor through the group of an odd twist-spin were trivial to begin with. $\square$

A classical knot K admits a Fox p-coloring for a prime p if and only if p divides the determinant $|\Delta_K(-1)|$. We end this section by noting that this fact generalizes to the case of nonprincipal ideals as well. The proof is the same as the classical case: given a knot group presentation, one sets $t = -1$ in the Alexander matrix and then reduces modulo p to obtain a system of linear equations over $\mathbb{Z}_p$ whose solution set is in 1-1 correspondence with the space of Fox p-colorings. As in the classical case, the determinant (which is the generator of the first elementary ideal of the Alexander matrix evaluated at $t = -1$) is equivalent to 0 mod p if and only if the knot admits more than just the trivial constant p-colorings. This is equivalent to evaluating the Alexander module at $t = -1$ to obtain a torsion $\mathbb{Z}$-module, i.e. a finite abelian group G, then checking whether $|G| \equiv 0 \mod p$.

Proposition 3.8. *Let K be a surface knot and p prime. Then K admits a Fox p-coloring if and only if p divides $\Delta(K)|_{-1}$.*

Chapter 4

0-Concordance of knotted surfaces in 4-space

4.1 Concordance of surface knots

In this section we define the various notions of concordance which will be of interest. Let K_0 and K_1 be oriented surface knots of genus g in S^4.

Definition 4.1. A **concordance** between K_0 and K_1 is a smooth embedding $C : \Sigma_g \times I \hookrightarrow S^4 \times I$ such that $C|_{\Sigma_g \times \{i\}} = K_i$ for $i = 0, 1$, and such that projection onto the I factor is Morse.

Definition 4.2. A **ribbon concordance** $K_0 \to K_1$ is a concordance C with critical points of index 0 and 1 only.

Note that ribbon concordance is not symmetric. The historical terminology for $K_0 \to K_1$ is "K_1 is ribbon concordant to K_0", denoted $K_1 \geq K_0$. The arrow in our notation is to indicate the direction of time during the concordance. Also note that a 2-knot K is *ribbon* if and only if there is a ribbon concordance $\mathcal{U} \to K$.

Definition 4.3. A **0-concordance** between K_0 and K_1 is a concordance C such that at each regular level set, $S^4_t \cap C$ consists of a connected genus g surface and possibly some genus 0 components.

So far all of the theorems in the literature which obstruct 0-concordance between surface knots utilize the following factorization of a 0-concordance into two opposing ribbon concordances.

Proposition 4.4 (Sunukjian [Sun15])**.** *If K_0 and K_1 are 0-concordant surface knots, then there exists a surface knot J and ribbon concordances $K_0 \to J \leftarrow K_1$.*

Proof. Let $C : \Sigma_g \times I \hookrightarrow S^4 \times I$ be a 0-concordance. We can isotope C ambiently so that all index 0 and 1 critical points occur before any index 2 or 3 critical points. So C has a handle decomposition where we attach all 0-handles and 1-handles before any 2 or 3-handles. If a 1-handle was cancelled by a 2-handle, then its feet were attached to a single component of the level set in which it was attached, thereby increasing the genus of that component, hence C is not a 0-concordance. Therefore, all 1-handles are cancelled by 0-handles, and since the concordance is connected there must be the same number of 0 and 1-handles. Turning the concordance upside down, the same must be true of the 2 and 3-handles, which form a ribbon concordance in the reversed direction. □

4.1.1 Ribbon concordance

In [Gor81], Gordon proved that for a ribbon concordance $C : S^1 \times I \hookrightarrow S^3 \times I$ from K_1 to K_0, or in our notation $K_0 \to K_1$, the knot groups of K_0, C, and K_1 obey
(i) $\pi K_0 \hookrightarrow \pi C$ and (ii) $\pi K_1 \twoheadrightarrow \pi C$.

The following proposition displays the difference between ribbon concordance in the classical case with all higher dimensions, and suggests that the knot groups of K_0 and K_1 should play a fundamental role. Namely, the surjection above becomes an isomorphism, so by composing with its inverse there is a homomorphism from the group of K_0 to the group of K_1, which in many cases remains injective. Let $X_i = (S^4 \times \{i\}) \setminus \nu K_i$ and $Y = (S^4 \times I) \setminus \nu C$ (so $\pi K_i = \pi_1 X_i$, $\pi C = \pi_1 Y$).

Proposition 4.5. *A ribbon concordance $K_0 \to K_1$ induces a homomorphism $\phi : \pi K_0 \to \pi K_1$.*

Proof. We recall Gordon's proof; the only change is due to the dimension of the cobordism being one higher. Let C be a ribbon concordance $K_0 \to K_1$. Since the projection onto I is Morse, Y can be built from $X_0 \times I$ by adding handles. Every time we pass a critical point of index 0, respectively 1, of C, we get a critical point of index 1, respectively 2, in Y. From this perspective

$$Y = (X_0 \times I) \cup (1\text{-handles}) \cup (2\text{-handles})$$

In order for C to be a concordance, we must have added the same number of 0 and 1-handles. Therefore, $\pi C = \frac{\pi K_0 * \langle z_1, \ldots, z_n \rangle}{\langle\langle r_1, \ldots, r_n \rangle\rangle}$, where each r_i is of the form $z_i w_i x^{-1} w_i^{-1}$, for some meridian x of K_0 (each r_i can be chosen to be a Wirtinger relator).

Turning the cobordism upside down, we have

$$Y = (X_1 \times I) \cup (3\text{-handles}) \cup (4\text{-handles})$$

Thus the inclusion $X_1 \hookrightarrow Y$ induces an isomorphism on fundamental groups: $\pi K_1 \cong \pi C$. The inclusion $X_0 \hookrightarrow Y$ induces a homomorphism $\pi K_0 \to \pi C$, so composing with the inverse of the isomorphism yields a homomorphism $\phi : \pi K_0 \to \pi K_1$ induced by these inclusions. □

Remark 4.6. It is conjectured that the homomorphism ϕ is always injective; this is a strong form of the Kervaire conjecture. Gordon's original proof of injectivity applies whenever πK_0 is residually finite, so in this case ϕ will be injective for any ribbon concordance $K_0 \to K_1$. All

3-manifold groups are residually finite, so for classical ribbon concordance this is sufficient. Moreover, cyclic extensions of residually finite groups are residually finite, so the group of any fibered 2-knot is as well. Together these include all spun and twist-spun knots. Gordon points out that πK_0 locally indicable is also sufficient. Thus the group map can obstruct some ribbon concordances; for instance it gives an easy proof of Corollary 2.1(i) of [CSS06], which states that for p, q distinct primes, there is no ribbon concordance $\tau^2 T(2,p) \to \tau^2 T(2,q)$ (the group of $\tau^2 T(2,p)$ is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}$).

Remark 4.7. The homomorphism ϕ sends meridians of K_0 to meridians of K_1, so in fact a ribbon concordance $K_0 \to K_1$ induces a quandle homomorphism $\varphi : Q(K_0) \to Q(K_1)$, where $Q(K)$ is the fundamental quandle. This is equivalent to the diagrammatic interpretation in [CSS06], where it is shown that a coloring of K_1, i.e. a quandle homomorphism $Q(K_1) \to X$, induces a coloring of K_0. The induced coloring is the composition $Q(K_0) \xrightarrow{\varphi} Q(K_1) \to X$.

The proof of Proposition 4.5 shows that, given any presentation for πK_0, we can obtain a presentation for πK_1 by adding the same number of generators and relations. The *deficiency* of a finitely presentable group G is the maximal difference $g - r$ between the number of generators and relators, taken over all finite presentations $\langle x_1, \dots, x_g | s_1, \dots, s_r \rangle$ of G.

Corollary 4.8. *If $K_0 \to K_1$ is a ribbon concordance, then* $\operatorname{def}(\pi K_0) \leq \operatorname{def}(\pi K_1)$.

We end this section with the key lemma for Theorem 1.1. This is a generalization of the fact that ribbon 2-knots have principal Alexander ideals, because every ribbon 2-knot K is the result of a ribbon concordance $\mathcal{U} \to K$.

Key Lemma. *Let $K_0 \to K_1$ be a ribbon concordance. Then $\Delta(K_1) = (f)\Delta(K_0)$, for some $f \in \mathbb{Z}[t^{\pm 1}]$.*

Proof. By Proposition 4.5, a Wirtinger presentation for the knot group of K_1 can be obtained from a Wirtinger presentation for πK_0 by adding m generators and m relations, corresponding to the index 0 and 1 critical points of the concordance, respectively. So if $\pi K_0 \cong \langle x_1, \ldots, x_n | r_1, \ldots, r_j \rangle = P_0$ is Wirtinger, then there is a Wirtinger presentation $P_1 = \langle x_1, \ldots, x_n, z_1, \ldots, z_m | r_1, \ldots, r_j, s_1, \ldots, s_m \rangle$ for the knot group of K_1. From this presentation we compute the Alexander ideal of K_1 using Fox calculus. Below $\mathbf{A}$ is the Jacobian corresponding to P_0, which gives rise to the matrix on the right hand side for P_1. $\Delta(K_1)$ is the ideal of $\mathbb{Z}[t^{\pm 1}]$ generated by the determinants of all of the $(n+m-1)$-minors of the abelianized matrix. Since this is a Wirtinger presentation, we can replace the first column with a column of zeroes. This amounts to the realization that we may as well leave the first column out of any of our chosen minors, which leaves only the last $n+m-1$ columns.

$$\begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_j}{\partial x_1} & \cdots & \frac{\partial r_j}{\partial x_n} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_j}{\partial x_1} & \cdots & \frac{\partial r_j}{\partial x_n} \end{pmatrix} & \mathbf{0} \\ \begin{pmatrix} \frac{\partial s_1}{\partial x_1} & \cdots & \frac{\partial s_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_m}{\partial x_1} & \cdots & \frac{\partial s_m}{\partial x_n} \end{pmatrix} & \begin{pmatrix} \frac{\partial s_1}{\partial z_1} & \cdots & \frac{\partial s_1}{\partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_m}{\partial z_1} & \cdots & \frac{\partial s_m}{\partial z_m} \end{pmatrix} \end{pmatrix} \xrightarrow{\mathfrak{a}\gamma} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \bigstar & \mathbf{B} \end{pmatrix}$$

When choosing $n+m-1$ rows, we will only obtain a nonzero determinant by choosing all of the bottom m rows, i.e. all the rows of the $m \times m$ matrix $\mathbf{B}$. Otherwise, we obtain a minor of the form $\begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \bigstar & \mathbf{Y} \end{pmatrix}$, where $\mathbf{X}$ is an $(n-1) \times (n-1)$ matrix and $\mathbf{Y}$ is an $m \times m$ matrix with a row of zeroes, so the determinant of this minor is $|\mathbf{X}| \cdot |\mathbf{Y}| = |\mathbf{X}| \cdot 0 = 0$. So any minor of the right size with nonzero determinant is of the form $\begin{pmatrix} \mathbf{A}' & \mathbf{0} \\ \bigstar & \mathbf{B} \end{pmatrix}$, where $\mathbf{A}'$ is a square $(n-1)$-minor of $\mathbf{A}$. The determinant of this minor is $|\mathbf{A}'| \cdot |\mathbf{B}|$. Since the Alexander ideal of K_0 is generated by exactly the determinants of these $\mathbf{A}'$, we have that

$\Delta(K_1) = (|\mathbf{B}|)\Delta(K_0)$.

$\square$

Remark 4.9. It is interesting to compare with the recent result of Friedl and Powell on homotopy ribbon concordances of classical knots. They prove that if $K_0 \to K_1$ is a homotopy ribbon concordance of classical knots, then the Alexander polynomial of K_0 divides that of K_1 [FP19]. It is not clear that the corresponding result holds in this dimension, for a *homotopy ribbon concordance* of surface knots, i.e. a smooth concordance between K_0 and K_1 such that the group of K_1 is isomorphic to the group of the concordance exterior. The key lemma stated above does hold (by the same proof) for an in-between notion called *handle ribbon concordance*, which is defined as a smooth concordance whose exterior can be built with only 1-handles and 2-handles.

4.2 The ideal class monoid of $\mathbb{Z}[t^{\pm 1}]$

In this section we define the ideal class monoid of a ring and prove some fundamental properties in the case of $\mathbb{Z}[t^{\pm 1}]$. Let R be an integral domain. The set of nonzero ideals of R, denoted $\mathbb{I}(R)$, forms a commutative monoid under ideal multiplication. Say $I \sim J$ if there exist nonzero $x, y \in R$ such that $(x)I = (y)J$. The quotient monoid $\mathbb{I}(R)/\sim$ is called the **ideal class monoid** of R, denoted $\mathcal{I}(R)$. The identity element of this monoid is precisely the set of principal ideals of R. Hence an ideal class $[I]$ is nontrivial if and only if any representative I is not principal.

A useful characterization of this equivalence relation is that $I \sim J$ if and only if $I \cong J$ as an R-module. Therefore the minimal number of generators of an ideal I is an invariant of its ideal class. The way we will produce an infinite rank submonoid of $\mathscr{K}_0$ is by showing that any set of maximal ideals independently generates a free commutative submonoid of $\mathcal{I}(\mathbb{Z}[t^{\pm 1}])$, and then find an infinite family of 2-knots with distinct, maximal Alexander ideals. The main goal of this section is to prove the statement about the ideals in detail.

The group of units of $\mathcal{I}(R)$ is the set of ideal classes $[I]$ such that there exists a class $[J]$ so that $[IJ] = [(1)]$; i.e. so that IJ is principal. This is called the **Picard group** of R, denoted $Pic(R)$, and whenever R is a Noetherian UFD it is trivial. This will enable us to prove that any surface knot K with nonprincipal Alexander ideal is not invertible in $\mathscr{C}_0$ (where by K is *invertible* we mean that for any surface knot J, $K\#J$ is not 0-concordant to the unknotted surface of the same genus). We refer the reader to Section 20 of [Mat86] for details. In brief, the *divisor class group* $C(R)$ of a Krull ring R is trivial if and only if R is a UFD. Now let $R = \mathbb{Z}[t^{\pm 1}]$. Since R is Noetherian and integrally closed, it is Krull, and of course it is a UFD as well. Furthermore, when R is a Krull domain, $Pic(R)$ is naturally a subgroup of $C(R)$, hence is trivial for any Noetherian UFD.

Corollary 4.10. *No nontrivial ideal class of $\mathcal{I}(\mathbb{Z}[t^{\pm 1}])$ is invertible, i.e. for any nonprincipal ideal I and any nonzero ideal J, IJ is not principal.*

Now we turn our attention to the minimal number of generators of an ideal $I \subseteq \mathbb{Z}[t^{\pm 1}]$. One elementary observation is that if I is a proper ideal and $|R/I|$ is finite, I cannot be principal. Indeed, if $I = (n)$, $n \geq 2$, then $R/I \cong \mathbb{Z}_n[t^{\pm 1}]$ is infinite because it has polyomials of arbitrary degree. On the other hand, if $I = (f(t))$, where $deg(f) \geq 1$, then R/I is infinite because it has $\mathbb{Z}$ as a subring. This gives a quick test to check if an ideal I is nonprincipal (see Corollary 1.4), but to distinguish nontrivial ideal classes from each other we will need more sophisticated tools.

Note that $R = \mathbb{Z}[t^{\pm 1}]$ is a regular ring of dimension 2. This means that for any maximal ideal m, the localization (R_m, m) is a regular local ring of dimension 2, i.e. the unique maximal ideal m of R_m is minimally generated by 2 elements. In fact, the maximal ideals of $\mathbb{Z}[t^{\pm 1}]$ can be described explicitly: they are of the form $(p, f(t))$, where p is a prime integer and $f(t)$ is irreducible mod p.

Let m be a maximal ideal of R, and consider the localization R_m. If $I \subseteq R$ is an ideal, then the pushforward of I is an ideal of R_m, denoted IR_m. The minimal number of generators of IR_m is a lower bound for the minimal number of generators of I as an ideal of R, since the image of a generating set of I generates IR_m. The benefit of working in the localization is that R_m is a local ring, i.e. it has a unique maximal ideal, mR_m. Now assume (R, m) is a local ring. This allows some powerful techniques for computing lower bounds for the minimal number of generators of m^n. In this case, Nakayama's lemma implies that the minimal number of generators of m^n is equal to the minimal number of generators of m^n/m^{n+1}. Since m annihilates this R-module, it is a vector space over the field R/m, so its minimal number of generators is equal to its dimension. In general, if M is a finitely generated R-module, the **Hilbert function** $H_M(n)$ of M is:

$$H_M(n) := \dim_{R/m} m^n M/m^{n+1}M$$

The following theorem is a combination of Theorems 1.11 and 12.1 from [Eis95].

Theorem 4.11 (Hilbert)**.** *There is a polynomial $P_M(n)$, of degree $\dim(R) - 1$, which agrees with $H_M(n)$ for sufficiently large n.*

We are interested in the case $H_R(n) = \dim_{R/m} m^n/m^{n+1}$, where R is the localization of $\mathbb{Z}[t^{\pm 1}]$ at a maximal ideal m. The dimension of such an R is 2 ($= \dim(\mathbb{Z}[t^{\pm 1}])$), so $P_R(n)$ is a linear polynomial, which after some $N > 0$ agrees with $H_R(n)$. Thus the minimal number of generators of m^n/m^{n+1}, and therefore of m^n, eventually agrees with a linear polynomial. A priori the minimal number of generators of m^n as an ideal of $\mathbb{Z}[t^{\pm 1}]$ may not agree with these values, but is certainly bounded below by them.

Corollary 4.12. *If $m \subseteq \mathbb{Z}[t^{\pm 1}]$ is a maximal ideal, then the minimal number of generators of m^n grows arbitrarily large as n approaches infinity.*

Corollary 4.13. *Let $R = \mathbb{Z}[t^{\pm 1}]$. The maximal ideals of R form a basis for a free commutative submonoid of $\mathcal{I}(R)$, isomorphic to $\mathbb{N}^\infty$.*

Proof. Let $m_1, \ldots, m_n$ be any finite set of maximal ideals in R. The claim to be proved is that for any two vectors $\boldsymbol{i} = (i_1, \ldots, i_n)$, $\boldsymbol{j} = (j_1, \ldots, j_n)$, $m_1^{i_1} m_2^{i_2} \cdots m_n^{i_n} \sim m_1^{j_1} m_2^{j_2} \cdots m_n^{j_n}$ implies $\boldsymbol{i} = \boldsymbol{j}$. Suppose on the contrary that the ideals are related but $\boldsymbol{i} \neq \boldsymbol{j}$, so there exist $f, g \in \mathbb{Z}[t^{\pm 1}]$ such that

$$(f) m_1^{i_1} m_2^{i_2} \cdots m_n^{i_n} = (g) m_1^{j_1} m_2^{j_2} \cdots m_n^{j_n} \qquad (*)$$

and k so that $i_k \neq j_k$. Now localize at m_k. The equation $(*)$ pushes forward to the equation $(f) m_k^{i_k} = (g) m_k^{j_k}$ $(\dagger)$ in R_{m_k}, since all other m_α contain an element in the complement of m_k. (R_{m_k}, m_k) is a local ring of dimension 2, so by Corollary 4.12 there exists $N > 0$ such that for all distinct α, $\beta \geq N$, m_k^α and m_k^β have a different minimal number of generators. Multiply both sides of $(\dagger)$ by m^N to obtain $(f) m_k^{N+i_k} = (g) m_k^{N+j_k}$. Since the left hand side and right hand side of this equation have the same minimal number of generators as $m_k^{N+i_k}$ and $m_k^{N+j_k}$, respectively, this is a contradiction.

This proves that the maximal ideals generate a free commutative submonoid of $\mathcal{I}(R)$. Since there are infinitely many maximal ideals, this submonoid is isomorphic to $\mathbb{N}^\infty$. □

Remark 4.14. Restricting to maximal ideals may seem rather restrictive; however in terms of $\mathcal{I}(\mathbb{Z}[t^{\pm 1}])$ it is the same as restricting to ideals which admit prime factorizations. This is because every height 1 prime ideal in $\mathbb{Z}[t^{\pm 1}]$ is principal, so the only nonprincipal prime ideals are height $2 = \dim(\mathbb{Z}[t^{\pm 1}])$, hence are maximal. So, as long as an ideal admits a prime factorization, we can pin down its ideal class uniquely by looking at the multiplicities of the maximal ideals in that factorization.

Remark 4.15. There is another, in some sense easier, way to prove Corollary 4.13. One can show that, in a Noetherian domain R: if an ideal I admits a prime factorization, then that factorization is unique. Then, by a similar localization argument one quickly shows that distinct products of maximal ideals lie in different ideal classes. We included the previous argument because the minimal number of generators of an ideal, though hard to compute, gives more of a quantitative sense of how ideal classes can differ than simply resorting to uniqueness of prime factorizations. Also, our main corollary applies to surface knots with nonprincipal ideals, so by establishing that there are 2-knots whose ideals require arbitrarily many generators we are putting this requirement in some perspective. Classical knots (and ribbon 2-knots) have principal Alexander ideals for the special reason that they have deficiency 1 Wirtinger presentations, while Levine showed in [Lev78] that a 2-knot group can have any deficiency less than 1 (see also [Kan83]). Certainly a 2-knot taken 'at random' should not be expected to have a deficiency 1 knot group nor a principal Alexander ideal.

4.3 0-Concordance and Alexander ideals

In this section we prove the main theorem and applications. Recall that $\mathscr{C}_0$ denotes the monoid of oriented surface knots in S^4 modulo 0-concordance. The 0-concordance monoid of 2-knots, $\mathscr{K}_0$, is a submonoid of $\mathscr{C}_0$. A surface knot K is *0-slice* if it is 0-concordant to the unknotted surface of the same genus, and *invertible* if there exists a surface knot J so that $K\#J$ is 0-slice. Note that this is looser than the usual meaning of invertibility; indeed only a genus 0 surface has a chance at having a true inverse. As a warmup to the main theorem, we carry out an example from first principles.

Example 4.16. Let K be the 2-twist-spun trefoil. Then $\Delta(K) = (3, t{+}1)$ is maximal, as shown in Example 2.8, hence minimally generated by 2 elements. Suppose that K is 0-concordant to the unknot $\mathcal{U}$. Then there exists a 2-knot J and ribbon concordances $K \to J \leftarrow \mathcal{U}$. Since J is ribbon concordant to a ribbon knot, J is ribbon. On the other hand, by the key lemma $K \to J$ implies $\Delta(J) = (f)\Delta(K) = (f)(3, t + 1)$ for some nonzero $f \in R$.

Notice $(f)(3, t+1) \cong (3, t+1)$ as an R-module, therefore has the same minimal number of generators. Thus $\Delta(J)$ is not principal, but J was supposed to be ribbon.

Theorem 1.1. *The Alexander ideal induces a homomorphism* $\Delta : \mathscr{C}_0 \to \mathcal{I}(\mathbb{Z}[t^{\pm 1}])$.

Proof. Suppose K_0 is 0-concordant to K_1. Then by Proposition 4.4 there exists a surface knot J with ribbon concordances $K_0 \to J \leftarrow K_1$. So, by the key lemma there exist $f, g \in \mathbb{Z}[t^{\pm 1}]$ such that $(f)\Delta(K_0) = \Delta(J) = (g)\Delta(K_1)$, thus $\Delta(K_0)$ and $\Delta(K_1)$ are equivalent in $\mathcal{I}(\mathbb{Z}[t^{\pm 1}])$. As shown in Proposition 2.10, $\Delta(K\#J) = \Delta(K)\Delta(J)$, so the map $[K] \to [\Delta(K)]$ is a homomorphism. □

Since an ideal class is nontrivial if and only if it consists of nonprincipal ideals, this gives an easily computable obstruction to being 0-slice. In fact, since the group of units of $\mathcal{I}(\mathbb{Z}[t^{\pm 1}])$ is trivial (see Corollary 4.10), any surface knot with nonprincipal Alexander ideal is not invertible in $\mathscr{C}_0$.

Corollary 1.2. *If a surface knot K has a nonprincipal Alexander ideal, then it has no inverse in $\mathscr{C}_0$, i.e. for all surface knots J, $K\#J$ is not 0-slice.*

Twist-spun knots provide many examples of 2-knots with nonprincipal ideals. Together with the previous corollary, this proves that the 0-concordance monoid of 2-knots, $\mathscr{K}_0$, is not a group.

Theorem 1.3. *If K is a classical knot such that $|\Delta_K(-1)| \neq 1$, then there exist infinitely many $n \in \mathbb{Z}$ such that $\Delta(\tau^n K)$ is not principal. In particular, if n is even and $\Delta(\tau^n K)$ is principal, then $\Delta_K(t)$ has a root z such that $z^n = 1$.*

Proof. First we compute the Alexander ideal of $\tau^n K$. Let $\langle x_0, \ldots, x_m | r_1, \ldots, r_m \rangle$ be a Wirtinger presentation for πK. As discussed in Subsection 2.1.2, $\langle x_0, \ldots, x_m | r_1, \ldots, r_m, [x_0^n, x_1], \ldots, [x_0^n, x_m] \rangle$ is a Wirtinger presentation for $\pi(\tau^n K)$. The

Alexander matrix calculated from this presentation is equivalent to the following matrix, which we get by replacing the first column with zeroes as before.

$$\begin{pmatrix} 0 & a_{11} & \dots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m1} & \dots & a_{mm} \\ 0 & & & \\ \vdots & & (t^n-1)\mathbf{I_m} & \\ 0 & & & \end{pmatrix}$$

By deleting the first column altogether and remembering to take determinants of minors of size m, we arrive at the following convenient form $\begin{pmatrix} \mathbf{A} \\ (t^n-1)\mathbf{I_m} \end{pmatrix}$. Note that $|A| = \Delta_K(t)$ is the generator of $\varepsilon_1(K) = \Delta(K)$. Then the Alexander ideal is

$$\sum_{j=1}^{m+1}((t^n-1)^{j-1})\varepsilon_j(K) = (\Delta_K(t), (t^n-1)\varepsilon_2(K), \dots, (t^n-1)^{m-1}\varepsilon_m(K), (t^n-1)^m)$$

(recall that $\varepsilon_j(K)$ need not be principal and that $\varepsilon_{m+1}(K)$ is (1) by definition). What we need here is that $\Delta_K(t)$ and $(t^n-1)^m$ are in $\Delta(\tau^n K)$.

We will actually prove that such a K has infinitely many *even* twist-spins with non-principal ideal (cf Proposition 3.5). Suppose n is even and $\Delta(\tau^n K) = (f_n(t))$ is principal. Evaluating the above equation at $t = 1$ we obtain $(f_n(1)) = (\Delta_K(1)) = (1)$, and at $t = -1$, $(f_n(-1)) = (\Delta_K(-1)) \neq (1)$ by assumption. Therefore f_n has degree at least one.

Since Δ_K and $(t^n-1)^m$ are in (f_n), there exist g_n, h_n such that $\Delta_K = g_n f_n$ and $(t^n-1)^m = h_n f_n$. The second equation implies that all the roots of f_n are n^{th} roots of unity. The first equation implies that they are also roots of Δ_K. This proves the theorem.

Note that $n = 2$ always works, since $\Delta_K(\pm 1)$ is odd. If we list the primitive m_i^{th} roots of unity which are roots of Δ_K, then as long as $0 \neq n \in 2\mathbb{Z} \setminus \{km_i : k \in \mathbb{Z}\}$, we are guaranteed that $\Delta(\tau^n K)$ is nonprincipal, and thus $\tau^n K$ is not invertible in $\mathscr{C}_0$. In particular, if Δ_K has no roots of unity as roots, then all of its nonzero even twist-spins are not invertible. □

Corollary 1.4. *Any 2-twist-spin of a 2-bridge knot is not invertible in $\mathscr{C}_0$. If K and J are 2-bridge knots and their 2-twist-spins are 0-concordant, then $|\Delta_K(-1)| = |\Delta_J(-1)|$.*

Proof. Notice that $(f(t), t-a) = (f(a), t-a)$. This is because $f(t)-f(a)$ is divisible by $t-a$. When K is a 2-bridge knot, $\Delta(\tau^2 K) = (\Delta_K(t), t+1) = (\Delta_K(-1), t+1)$, so the ideal of the 2-twist-spin of K is generated by $t+1$ and the determinant of K. When $|\Delta_K(-1)| = n > 1$, the quotient $\mathbb{Z}[t^{\pm 1}]/(n, t+1) \cong \mathbb{Z}/n\mathbb{Z}$ is finite and nonzero, hence $(n, t+1)$ is not principal. The proof will be finished once we establish the following claim.

Claim: Let $n, m \geq 0$ be odd integers. If $(n, t+1) \sim (m, t+1)$, then $n = m$.

Suppose the ideals are related, then there exist f, g so that $(f)(n, t+1) = (g)(m, t+1)$. Localize by inverting the multiplicative set $\{(t+1)^k | k \geq 0\}$: in the localization, this equation becomes $(\overline{f}) = (\overline{g})$. Since $t+1$ is irreducible, there exist j, k so that in $\mathbb{Z}[t^{\pm 1}]$, $((t+1)^j f) = ((t+1)^k g)$. Multiplying the original equation by $(t+1)^j$, we see that

$$((t+1)^j f)(n, t+1) = ((t+1)^k g)(n, t+1) = ((t+1)^j g)(m, t+1)$$

Evaluating both sides of this equation at $t = 1$, we obtain:

$$(2^k \cdot g(1))(1) = (2^j \cdot g(1))(1)$$

Therefore $j = k$. Then $((t+1)^j g)(n, t+1) = ((t+1)^j g)(m, t+1)$ implies $(n, t+1) = (m, t+1)$, which implies $n = m$ (by looking at quotients, or by evaluating at $t = -1$).

□

Remark 4.17. The Stevedore knot 6_1 has determinant 9, so $\Delta(\tau^2 6_1) = (2t^2 - 5t + 2, t + 1) = (9, t + 1)$ is not principal. Since the Stevedore knot is slice, its double branched cover (which is the fiber of its 2-twist-spin) is spin rational homology cobordant to S^3. This can be seen by capping off a concordance $(S^3 \times I, C)$ between the Stevedore and the unknot with $(B^4, \text{Seifert surface})$ pairs on both sides to get a closed surface in S^4, then taking the double branched cover of S^4 over this surface. This is a spin 4-manifold, and by restricting to the relevant pieces we get a spin rational homology cobordism from the double branched cover of the Stevedore knot to S^3. Also, the double branched cover of a knot in S^3 has a unique spin structure, so this agrees with the one induced by S^4 on the fiber of the 2-twist-spin of the Stevedore. Thus this is an example where the techniques of [Sun19], [DM19] cannot obstruct 0-concordance, but Alexander ideals can. There are infinitely many slice 2-bridge knots with any given nonunit, square determinant, so all of their double branched covers share this property. Conversely, the 5-twist-spun trefoil has $\Delta(\tau^5 3_1) = (1)$, but a Seifert solid (the Poincaré homology sphere) with nonzero d-invariant [Sun19]. Dai-Miller also produce many examples where their invariant distinguishes 0-concordance but the Alexander ideal is trivial. This shows that the homomorphism Δ is not injective. We will determine its image in Subsection 4.3.1.

We turn now to identify an infinite rank submonoid of $\mathscr{K}_0$.

Theorem 1.5. *$\mathscr{K}_0$ contains a submonoid isomorphic to $\mathbb{N}^\infty$.*

Proof. In Corollary 4.13 we showed that any set of maximal ideals is linearly independent in $\mathcal{I}(\mathbb{Z}[t^{\pm 1}])$. Therefore any set of 2-knots with distinct, maximal Alexander ideals is linearly independent in $\mathscr{C}_0$, by Theorem 1.1.

Let K_p be any 2-bridge knot with prime determinant $p = \Delta_{K_p}(-1)$. Then the 2-twist-spin of K_p has maximal Alexander ideal: $\Delta(\tau^2 K_p) = (\Delta_{K_p}(-1), t + 1) = (p, t + 1)$, as in Corollary 1.4. For instance, K_p could be the 2-twist-spin of the $(2, p)$-torus knot. For any set of odd prime numbers $\{p_i\}$, the corresponding 2-knots $\tau^2 K_{p_i}$ form the basis for a linearly independent submonoid of $\mathscr{K}_0$.

□

Example 4.18. Another interesting family is the p-twist-spins of $(2,p)$-torus knots, with p an odd prime. The Alexander ideal of $\tau^p T(2,p)$ is $I_p = (\Phi_{2p}(t), \Phi_p(t))$. Note this is equal to $(2, \Phi_p(t))$, since $2 = (1+t)\Phi_{2p}(t) + (1-t)\Phi_p(t)$, and $\Phi_{2p}(t) = \Phi_p(t) - 2(t^{p-2} + \cdots + t^3 + t)$. The quotient $\mathbb{Z}[t^{\pm 1}]/I_p$ has order 2^{p-1}, so none of these 2-knots are 0-slice. When 2 is a primitive root mod p, $\Phi_p(t)$ is irreducible mod 2, so I_p is maximal. If the Artin conjecture is true, then 2 is a primitive root for infinitely many primes p, so this would give an infinite basis for another linearly independent family. It would also show that one can obtain finite fields of order 2^k for arbitrarily large k as $\mathbb{Z}[t^{\pm 1}]/\Delta(K)$ with K a 2-knot.

4.3.1 The image of Δ

Let $\mathbb{I}_K = \{I \subseteq \mathbb{Z}[t^{\pm 1}] : I|_{t=1} = 1\}$, i.e. $\mathbb{I}_K$ is the set of surface knot ideals. Let $\mathcal{I}_K = \{[I] : I \in \mathbb{I}_K\}$ be the submonoid of $\mathcal{I}(\mathbb{Z}[t^{\pm 1}])$ of classes with a representative in $\mathbb{I}_K$. This is manifestly the image of the homomorphism $\Delta : \mathscr{C}_0 \to \mathcal{I}(\mathbb{Z}[t^{\pm 1}])$.

Theorem 4.19. *The image of Δ is $\mathcal{I}_K$.*

There is another monoid one might consider. Its construction is the same as the ideal class monoid, but we restrict the equivalence relation $\sim$ to $\mathbb{I}_K$: for ideals $I, J \in \mathbb{I}_K$, $I \sim_K J$ if there exist $f(t), g(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $f(1) = 1 = g(1)$ (which is the same as requiring $(f), (g) \in \mathbb{I}_K$) and $(f)I = (g)J$. Then $\mathbb{I}_K/\sim_K$ is the monoid of interest. In fact this is isomorphic to the monoid $\mathcal{I}_K$ above. In the ideal class monoid, $I \sim J$ if there exist $f(t), g(t) \in \mathbb{Z}[t^{\pm 1}]$ so that $(f)I = (g)J$, and this is equivalent to the existence of a $\mathbb{Z}[t^{\pm 1}]$-module isomorphism $\phi : I \to J$. The less obvious direction of the equivalence is that if ϕ is such an isomorphism, then for any $f \in I$, $(\phi(f))I = (f)J$. When $I \in \mathbb{I}_K$, I contains some element f such that $f(1) = 1$, so $I \cong J$ implies $I \sim_K J$. Thus $I \sim J$ if and only if $I \sim_K J$, and $\mathcal{I}_K$ is canonically isomorphic to $\mathbb{I}_K / \sim_K$.

A harder question is determining the image when restricted to 2-knots. In Corollary 4.21 we will point out out some ideals in $\mathbb{I}_K$ which, due to a theorem of Gutiérrez, are not the Alexander ideals of any 2-knots; it seems likely that these ideal classes are also missed in the image of Δ restricted to the 2-knot monoid $\mathscr{K}_0$.

When $I \in \mathbb{I}_K$ and $f(1) = 1$, we showed how to construct a surface knot K with $\Delta(K) = I$ and a ribbon 2-knot J with $\Delta(J) = (f)$. Connect summing with a ribbon 2-knot is always a ribbon concordance, so we have a ribbon concordance $K \to K\#J$ realizing these ideals.

Question 4.20. *If $I_0, I_1 \in \mathbb{I}_K$ and $I_0 \sim I_1$, do there exist 0-concordant surface knots K_0, K_1 such that $\Delta(K_0) = I_0$, $\Delta(K_1) = I_1$?*

If there exists an ideal J such that $I_0 = (f)J$, $I_1 = (g)J$ for some f, g, then this is clearly true: by Theorem 1.6 there is a surface knot K such that $\Delta(K) = J$, and by Theorem 3.1 there are ribbon 2-knots K_f, K_g with $\Delta(K_f) = (f)$, $\Delta(K_g) = (g)$. Thus there are ribbon concordances

$$\begin{array}{ccccc} & & K\#K_f\#K_g & & \\ & \nearrow & & \nwarrow & \\ K\#K_f & & & & K\#K_g \\ & \nwarrow & & \nearrow & \\ & & K & & \end{array}$$

so $K_0 = K\#K_f$ is 0-concordant to $K_1 = K\#K_g$, and $\Delta(K_j) = I_j$.

When there exists no such ideal J, the situation is unclear; however we do not know of any ideals $I_0 \sim I_1$ for which this is the case.

4.3.2 Inversion of t

Reversing the orientation of a surface knot K amounts to changing t to t^{-1} in $H_1(S^4 \setminus K)$. This change is not detected by the Alexander polynomial of a classical knot, since these polynomials are all symmetric. For surface knots this is not the case, and in fact the ideal class of a surface knot can be distinct from that of its reverse.

Corollary 4.21. *There exist infinitely many ribbon tori in S^4 which are not 0-concordant to their reverses.*

Proof. Let $I = (f(t), p)$ be an ideal of $\mathbb{Z}[t^{\pm 1}]$ satisfying:

1) p is prime and $f(t)$ is irreducible mod p.

2) $f(t)$ is not symmetric mod p.

3) $f(1) = 1$.

Then $I = \Delta(K)$ for a ribbon torus knot K which is not 0-concordant to $-K$. Since $f(1) = 1$, $I = (f(t), p - (p-1)f(t))$ is of the right form to apply Corollary 3.2 and build a ribbon torus K with $\Delta(K) = I$ (if $f(1) \neq 1$ but $f(1)$ and p are coprime, then a genus 2 surface not 0-concordant to its reverse can be constructed). Condition 1 guarantees that I is maximal. Reversing the orientation of K has the effect of changing t to t^{-1}, so condition 2 guarantees that $\Delta(K) = (f(t), p) \neq (f(t^{-1}), p) = \Delta(-K)$, so these tori are not isotopic. In fact $(f(t^{-1}), p)$ is also maximal, since $t \to t^{-1}$ is an automorphism of $\mathbb{Z}[t^{\pm 1}]$, so K and $-K$ are not 0-concordant by Corollary 4.13.

Let p be an odd prime. Then $I_p = (2t - 1, p)$ satisfies 1-3, so this is one infinite family of examples.

□

Many more examples could be constructed along these lines. Of course, achieving the result for tori implies it for any higher genus, since adding trivial handles does not change the knot group nor the ideal. We remark that any ideal of the form $(f(t), p)$, where p is prime and $(f(t)) \neq (f(t^{-1}))$ as ideals of $\mathbb{Z}_p[t^{\pm 1}]$, is not the ideal of a 2-knot [Gut72]. Indeed, knot groups giving rise to these ideals are prototypical examples of 3-knot groups which are not 2-knot groups [Kaw96].

There are 2-knots whose ideals are not invariant under inverting t, for instance any $f(t)$ with $f(1) = 1$ and $(f(t)) \neq (f(t^{-1}))$ is the Alexander polynomial of such a 2-knot, but these ideals all represent the trivial ideal class.

Question 4.22. *Is there a 2-knot K such that the ideal class of $\Delta(K)$ is not equal to the ideal class of $\Delta(-K)$?*

No twist-spun knot has this property, which we prove now.

Proposition 4.23. *If K is a classical knot, then $\Delta(\tau^n K) = \Delta(-\tau^n K)$.*

Recall from Theorem 1.3 that if $\langle x_0, \ldots, x_m | r_1, \ldots, r_m \rangle$ is a Wirtinger presentation for πK, then the Alexander ideal of $\tau^n K$ is

$$\sum_{j=1}^{m+1} ((t^n - 1)^{j-1})\varepsilon_j(K) = (\Delta_K(t), (t^n - 1)\varepsilon_2(K), \ldots, (t^n - 1)^{m-1}\varepsilon_m(K), (t^n - 1)^m).$$

For a classical knot K, all of the elementary ideals $\varepsilon_j(K)$ are invariant under inversion of t (Theorem 9.2.3, [CF63]). Since $(t^n - 1)$ is as well, we see directly that $\Delta(\tau^n K) = \Delta(-\tau^n K)$.

4.4 0-Concordance and peripheral subgroups

In this section we show that under a mild condition on the knot group, the peripheral subgroup of a surface knot is also a 0-concordance invariant. All 2-knots have infinite cyclic peripheral subgroup, so this invariant is only useful for higher genus surfaces.

Definition 4.24. For a surface knot $K : \Sigma_g \hookrightarrow S^4$, the *peripheral subgroup* $P(K)$ is the image of $i_* : \pi_1(\partial X) \to \pi_1(X)$, where $X = \overline{S^4 \setminus \nu K}$ is the exterior and $i : \partial X \hookrightarrow X$ is the inclusion.

Note that $\nu K \cong \Sigma_g \times D_2$, so $\partial X \cong \Sigma_g \times S_1$. Therefore $P(K) \cong \mathbb{Z} \oplus G$, where the first factor is generated by a meridian of K and the second factor is some quotient of $\pi_1(\Sigma_g)$. The unknot $\mathcal{U}_g$ of genus g always has peripheral subgroup $\mathbb{Z}$. When $g = 1$, it is known that G can be $0, \mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Z} \oplus \mathbb{Z}_n$, or $\mathbb{Z} \oplus \mathbb{Z}$ [KK94].

Recall from Proposition 4.5 that a ribbon concordance $K_0 \to K_1$ induces a homomorphism $\phi : \pi K_0 \to \pi K_1$.

Lemma 4.25. *If $K_0 \to K_1$ is a ribbon concordance, then the induced homomorphism $\phi : \pi K_0 \to \pi K_1$ restricts to a surjection $P(K_0) \twoheadrightarrow P(K_1)$.*

The proof comes down to a diagrammatic method of writing a generating set for $P(K)$, which we now introduce. A similar method is outlined in [Yaj69]. Let D be a broken surface diagram for K. Write down the Wirtinger presentation for πK induced by D. Choose a basepoint region and record the meridian corresponding to that region, say x. Draw a generating system of curves for $\pi_1(\Sigma_g)$ on the surface. For each such curve γ, we can write a pushoff of γ into the exterior of K in the Wirtinger generators in a manner analogous to writing the longitude of a classical knot group. First orient gamma, then traverse the curve once, starting at the basepoint. When passing through a double curve crossing while on the undersheet, write down the generator corresponding to the oversheet, raised to the sign of the crossing. The sign of the crossing is $+1$ if the normal to the oversheet agrees with the orientation of γ, and -1 if not. After traversing the curve, multiply by x raised to the negative of the exponent sum of the word just created. If $\{\gamma_1, \ldots, \gamma_{2g}\}$ is a generating system of curves, then $\langle x, \gamma_1, \ldots, \gamma_{2g} \rangle \leq \pi K$ is the peripheral subgroup of K.

This method of calculation and the following proof were inspired by the quandle 2-cocycle invariant of [CSS06]. Indeed, the proof below is very similar to the proof of Theorem 1.2 in [CSS06], and prompts the subsequent question. In the quandle 2-cocycle calculation, one chooses a curve λ representing a homology class on a diagram of a surface knot, then

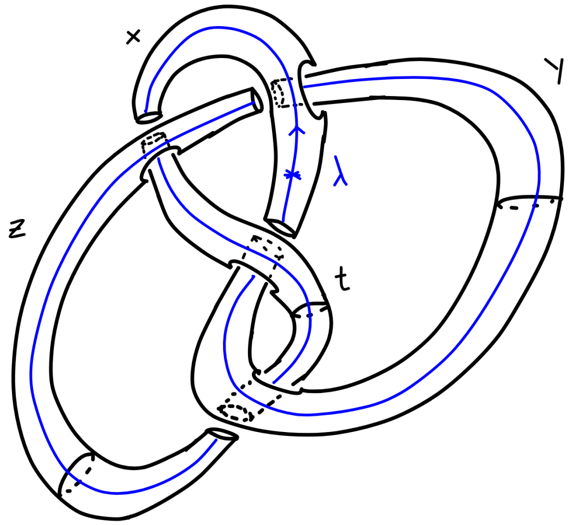

Figure 4.1: The spun torus of the figure 8 knot, represented as a tube [Sat00], with longitude $\lambda = z^{-1}yx^{-1}tx^0$.

computes a quandle cocycle calculation with respect to a fixed 2-cocycle θ at each double curve undercrossing.

Proof of Lemma 4.25. Let D_0 be a diagram for K_0. Then a diagram D_1 for K_1 is obtained from D_0 by taking a split union of D_0 with an unlink of 2-spheres, and joining them along the boundaries of some 3-dimensional 1-handles which are allowed to link the rest of the diagram. The solid 1-handles intersect the rest of the surface in disks, which can be assumed to be as small as we like, hence miss any double curve crossings.

Now, we can choose a generating system of curves for π_1 of the surface on D_0 which miss all of the double curve crossings from the 1-handles to be joined. Call this system of curves $\{\gamma_1, \dots, \gamma_{2g}\}$. Since this is a ribbon concordance, the same system of curves generates π_1 of the surface on D_1. Moreover, the image of γ_i under the homomorphism ϕ is the 'same' curve γ_i considered on D_1. Therefore every γ_i on D_1 is in the image of $\phi|_{P(K_0)}$. If x is the meridian for the basepoint region on K_0, then $\phi(x)$ is the corresponding meridian for K_1, so

$\phi(P(K_0)) = P(K_1)$. □

Question 4.26. *Let λ be a curve on a diagram for a surface knot. If the 2-cocycle invariant $\theta_\lambda(K)$ is nontrivial, must it be the case that $\lambda \neq 1 \in P(K)$?*

If the answer to this question is no, then the quandle 2-cocycle invariant could theoretically detect irreducibility of knotted surfaces when the peripheral subgroup fails (see Example 3 below). For instance, the standard way to show a knotted torus is not a 2-knot with a trivial handle attached is to show that the peripheral subgroup is bigger than $\mathbb{Z}$. It would be remarkable if the quandle invariant could be nontrivial even when the peripheral subgroup is infinite cyclic. On the other hand, if the answer is yes then this would provide an interesting link between quandle cocycle invariants and the peripheral subgroup.

Now, if $K_0 \to K_1$ is any ribbon concordance and πK_0 is residually finite or locally indicable, then ϕ is injective, as pointed out in Remark 4.6, so by the previous lemma $P(K_0) \cong P(K_1)$.

Theorem 4.27. *If K_0 is 0-concordant to K_1 and the knot groups of K_0 and K_1 are residually finite or locally indicable, then $P(K_0) \cong P(K_1)$.*

Proof. If K_0 is 0-concordant to K_1, then there exists a surface knot J and ribbon concordances $K_0 \to J \leftarrow K_1$. Each of the induced homomorphisms $\pi K_0 \to \pi J \leftarrow \pi K_1$ is injective, so $P(K_0) \cong P(J) \cong P(K_1)$. □

Since 0-concordance doesn't involve the genus of a knotted surface in any significant way, this is not surprising. We conjecture that the hypothesis on the groups may be removed.

Example 4.28. First we note some examples of surface knots whose knot groups are residually finite. The twist-spun torus knots and turned twist-spun torus knots of Boyle [Boy93] have the same groups as the twist-spun 2-knots of Zeeman. Since these 2-knots are fibered [Zee65], their groups are residually finite, so this gives a large number of knotted tori for which the peripheral subgroup is a 0-concordance invariant. Moreover, the property of being residually finite for knot groups is closed under taking connected sums: if πK and πJ are residually finite, then $\pi(K\#J)$ is residually finite as well [BE73].

In the next examples we show that peripheral subgroups and ideal classes each can distinguish some 0-concordance classes when the other cannot. We also point out that there are many ribbon knotted tori which are not 0-concordant to the unknot, in contrast with the case of 2-knots.

Example 4.29. 1. Let K be a nontrivial classical knot. Then the spun torus knot of K is a knotted torus with peripheral subgroup $\mathbb{Z}^2$ (for $K = 4_1$, see Figure 1, pictured as the "tube" of K). This is a ribbon torus knot which is not 0-concordant to the unknotted torus. Since it has a classical knot group, its Alexander ideal is principal.
2. Let K be a 2-bridge knot with nonunit determinant and let J be the 2-twist-spun torus knot of K. Boyle proved that these tori are reducible, so $P(J) \cong \mathbb{Z}$, however $\Delta(J)$ is nonprincipal by Corollary 1.4, so J is not 0-slice because its ideal class is nontrivial (more generally, attach a trivial handle to any 2-knot with nonprincipal ideal).
3. The twist-spun torus knots of the previous example can all be replaced with ribbon torus knots, by starting with the spun 2-knot of K and attaching a handle to effect the relation $y^{-2}xy^2 = x$. One can check that the longitude of this handle is trivial.

Remark 4.30. The ribbon tori of Example 3 are most likely irreducible, although this is difficult to prove. A ribbon torus has peripheral subgroup at most $\mathbb{Z}^2$, but in [Lit81] there are examples of tori with peripheral subgroup $\mathbb{Z}^3$. It seems likely that these tori are not 0-concordant to any ribbon torus.

4.5 Questions

It is interesting to consider the directed graph of ribbon concordances. The vertices are surface knots, and there is a directed edge from K_0 to K_1 whenever there is a ribbon concordance $K_0 \to K_1$. Two surface knots are 0-concordant if and only if they are in the same connected component of the ribbon concordance graph. We say K is a *root* of the graph if whenever $J \to K$, $J \cong K$ (such a K is called *minimal* in [Gor81]). A chain of ribbon concordances $\cdots \to K_{-1} \to K_0 \to K_1$ gives rise to an ascending chain of ideals $\Delta(K_1) \subseteq \Delta(K_0) \subseteq \Delta(K_{-1}) \subseteq \cdots$ by the key lemma. Since $\mathbb{Z}[t^{\pm 1}]$ is Noetherian, this chain must stabilize. This is some suggestion that the ribbon concordance graph has roots. Question 4.20 is related to the following question:

Question 4.31. *Does any connected component of the ribbon concordance graph have more than one root?*

For instance, if $\Delta(K)$ is a maximal ideal, then by the key lemma it must be true that for any ribbon concordance $J \to K$, $\Delta(J) = \Delta(K)$. Of course this does not imply that $J \cong K$, but under some additional assumption this is perhaps the case, e.g. when πK is prime. A knot group πK is *prime* if $\pi K = \pi(J \# J')$ implies πJ or $\pi J'$ is infinite cyclic, e.g. if its commutator subgroup is indecomposable as a free product. One such family of examples is the 2-twist-spins of $(2, p)$-torus knots, K_p, for p an odd prime. In this case the knot group has a simple decomposition: $\pi K_p \cong \langle x, a | xax^{-1} = a^{-1}, a^p = 1 \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}$, with commutator subgroup $\mathbb{Z}_p$.

Proposition 4.32. *Let $J \to K_p$ be a ribbon concordance. If the induced homomorphism $\phi : \pi J \to \pi K_p$ is injective, then it is an isomorphism preserving meridians.*

Proof. By Proposition 4.5, the induced map ϕ always takes meridians to meridians. Now, ϕ restricts to an injection of commutator subgroups, so $(\pi J)' \cong 1$ or $\mathbb{Z}_p$. It can't be 1, since then $\pi J \cong \mathbb{Z}$, so $\Delta(K_p) = (f)\Delta(J) = (f)$ would be principal. So $(\pi J)' \cong \mathbb{Z}_p$, and ϕ maps this isomorphically onto $(\pi K_p)'$. Therefore the image of ϕ contains $\mathbb{Z}_p$ and a meridian, but this is enough to generate πK_p, so ϕ is surjective as well as injective.

□

Note that since the group map ϕ is injective and preserves meridians, the induced quandle map φ is also an isomorphism. By Theorem 1.1 of [CSS06], J would also have a nontrivial output with the quandle 3-cocycle invariant. It seems likely this would necessarily be the same output as for K_p; this is the case when $p = 3$, since the tricoloring quandle R_3 is triply symmetric and so the output under each nontrivial coloring is the same.

Conjecture 4.33. *For p prime, the 2-twist-spin of $T(2, p)$ is a root of the ribbon concordance graph.*

Modulo the unknotting conjecture, it seems likely that the unknots $\mathcal{U}_g$ are roots as well. If there exists a ribbon concordance $K \to \mathcal{U}_g$, then $\pi K \cong \mathbb{Z}$ or is not residually finite (see Remark 4.6).

As pointed out in the introduction, 0-concordance is the smallest equivalence relation generated by ribbon concordance, which parallels the case of classical knot concordance precisely. Therefore, both cases have a natural slice-ribbon problem. Cochran produced nonribbon 2-knots in [Coc83] which are 0-null-bordant (allowing for 3-manifolds with 2 boundary components besides $S^2 \times I$), but as far as we know there are no examples of 0-slice, nonribbon 2-knots. Such a 2-knot K would have a ribbon concordance $K \to J$, where J is ribbon.

Question 4.34. *Is every 0-slice 2-knot ribbon?*

The techniques of this paper show that if a 2-knot K is invertible in $\mathscr{K}_0$, then $\Delta(K)$ must be principal.

Question 4.35. *Is any nontrivial 0-concordance class invertible?*

Chapter 5

Unknotting numbers of 2-knots in 4-space

This chapter is joint work with Michael Klug, Benjamin Ruppik, and Hannah Schwartz. In [JKRS20] we compare two unknotting operations on 2-knots, the stabilization number u_{st} and the Casson-Whitney number u_{cw}. See Section 2.4 for definitions and background. In this chapter we analyze various algebraic lower bounds for these invariants, prove that $u_{\mathrm{st}} \neq u_{\mathrm{cw}}$, and prove a strong nonadditivity theorem for both invariants.

We will use the following results from [JKRS20] without proof.

Corollary 5.1. *For any twist-spin $\tau^n k$, $u_{\mathrm{cw}}(\tau^n k) \leq u(k)$, for $u(k)$ the classical unknotting number of k.*

Theorem 5.2. *Any non-trivial twist-spin $\tau^n k$ of a 2-bridge knot k has $u_{\mathrm{cw}}(\tau^n k) = 1$.*

5.1 Algebraic lower bounds

In this section, we discuss the *algebraic Casson-Whitney number* $a_{\mathrm{cw}}(K)$ of a 2-knot K, the minimal number of meridian-commuting relations which abelianize the knot group of K (see

Definition 2.13 for the precise definition). This algebraic invariant is the sharpest lower bound we are aware of for the Casson-Whitney number u_{cw}, and in Section 5.3 we show that it is also a lower bound for the classical unknotting number. It is clear that $a_{\text{st}}(K) \leq a_{\text{cw}}(K)$, as stabilization relations identify two meridians, while finger move relations merely force them to commute. This subtle difference is used to prove Theorem 1.7, in which we give 2-knots for which $a_{\text{st}}(K) < a_{\text{cw}}(K)$, and for which this difference is realized geometrically.

The minimal number of generators of the Alexander module, called the ***Nakanishi index*** $m(K)$, is a classical lower bound for the unknotting number of 1-knots [Nak81]. In [Miy86], [MP19] it is shown that the Nakanishi index is also a lower bound for the stabilization number $u_{\text{st}}(K)$ of 2-knots. A subtler but sharper bound for the classical unknotting number is the ***Ma-Qiu index*** $a(K)$, defined as the minimal number of relations needed to abelianize the knot group [MQ06].

A similar yet sharper bound for the classical unknotting number is the algebraic stabilization number $a_{\text{st}}(K)$ (Definition 2.16), the minimal number of stabilization relations needed to abelianize the knot group, which is a natural lower bound for the stabilization number $u_{\text{st}}(K)$. This is defined in [Kan96], where it is called the *weak unknotting number*. The proof in [MQ06] actually shows that a_{st} is a lower bound for the classical unknotting number as well, since the relations they obtain identify meridians. In this section, we investigate the *algebraic Casson-Whitney number*, which is a natural lower bound for the Casson-Whitney number. By Corollary 5.1 it is also a lower bound for the classical unknotting number, but not the stabilization number $u_{\text{st}}(K)$. We summarize the previously known results regarding these invariants in the proposition below.

Proposition 5.3 (Kanenobu, Ma-Qiu, Miyazaki, Nakanishi)**.** *If k is a 1-knot, then*

$$m(k) \leq a(k) \leq a_{\text{st}}(k) \leq u(k).$$

If K is a 2-knot, then

$$m(K) \leq a(K) \leq a_{\rm st}(K) \leq u_{\rm st}(K).$$

As pointed out in [MQ06], the first inequality above is often strict: the Ma-Qiu index is positive whenever πK is not abelian, but the Alexander module and hence the Nakanishi index can be zero for nontrivial knots, e.g. Alexander polynomial one 1-knots. While $m(K), a(K)$, and $a_{\rm st}(K)$ are known to be nonadditive on certain classical knots (see the end of Section 5.2), we are unaware of any classical knots for which $a_{\rm cw}$ is nonadditive. We show in Section 5.4 that is nonadditive on certain 2-knots.

5.2 The algebraic Casson-Whitney number

Recall from Section 2.4 that each finger move on a 2-knot K adds a relation of the form $[x, y] = 1$, where x, y are meridians of K. As noted in Section 2.2, y is equal to x^w for some $w \in (\pi K)'$. Therefore, the algebraic Casson-Whitney number $a_{\rm cw}(K)$ is equal to the minimal number of elements $w_i \in (\pi K)'$ such that the relations $\{[x, x^{w_i}] = 1\}$ abelianize πK.

These finger move relations are 'weaker' than the relations induced by stabilizations, in that every finger move relation is also a stabilization relation. Recall from Definition 2.16 that $a_{\rm st}(K)$ denotes the minimal number of stabilization relations needed to abelianize the knot group; these relations are of the form $x = y$, where x and y are meridians, or equivalently $[x, w] = 1$, where $w \in (\pi K)'$ and $y = x^w$. Thus $a_{\rm st}(K)$ is the minimal number of elements $w_i \in (\pi K)'$ such the relations $\{[x, w_i] = 1\}$ abelianize πK. Although x^w is not in the commutator subgroup, $x^w = x[x, w]$, so the finger move relation $[x, x^w] = 1$ is equivalent to the stabilization relation $[x, [x, w]] = 1$, and we see that $a_{\rm st}(K) \leq a_{\rm cw}(K)$.

On the other hand, an obvious upper bound for $a_{\rm cw}(K)$ is $\mu(K) - 1$, where $\mu(K)$ is the meridional rank of K: forcing any single meridian to commute with the rest of a generating set of meridians will force that meridian into the center of the group. Since all knot groups are normally generated by any meridian, this abelianizes the group. We summarize the relationships between these invariants below, which are defined for n-knots because we will later refer to the case $n = 1$ as well as our usual case $n = 2$ (although these invariants are well defined for all $n \geq 1$ because they only depend on the knot group and the information of a meridian).

Proposition 5.4. *For any n-knot K,*

$$m(K) \leq a(K) \leq a_{\text{st}}(K) \leq a_{\text{cw}}(K) \leq \mu(K) - 1.$$

In Theorem 1.8 we show that the inequality $a_{\text{st}}(K) \leq a_{\text{cw}}(K)$ can be strict. In fact, we find infinitely many 2-knots K with $a_{\text{st}}(K) = u_{\text{st}}(K) = 1$ and $a_{\text{cw}}(K) = 2$, enabling us to prove in Theorem 1.7 that $u_{\text{st}}(K) < u_{\text{cw}}(K)$ for infinitely many 2-knots K. The last inequality may also be strict: in [BK20] there are examples of 1-knots of arbitrarily large meridional rank and unknotting number equal to one. Spinning these examples produces 2-knots of arbitrarily large meridional rank (equal to the original knot's meridional rank) and Casson-Whitney number one, by Corollary 5.1.

Proposition 5.5. *For $\alpha \in \{a, a_{\text{st}}, a_{\text{cw}}\}$ and for n-knots K_1 and K_2,*

$$\max\{\alpha(K_1), \alpha(K_2)\} \leq \alpha(K_1 \# K_2) \leq \alpha(K_1) + \alpha(K_2).$$

Proof. The proof is the same in all three cases; we follow Kanenobu in [Kan96] for $\alpha = a_{\text{st}}$. Let $g_1, \ldots, g_n$ be a minimal set of relators of the required form (depending on α) which abelianize $\pi(K_1 \# K_2)$. Let ϕ be a surjection $\phi : \pi(K_1 \# K_2) \twoheadrightarrow \pi K_1$ which sends all meridians of K_2 to a fixed meridian of K_1. Notice that $\pi K_1 / \langle\langle \phi(g_1), \ldots, \phi(g_n) \rangle\rangle \cong \mathbb{Z}$ and that each $\phi(g_i)$ is a relator of the required form for computing $\alpha(K_1)$. Therefore, $\alpha(K_1 \# K_2) \geq \alpha(K_1)$. Repeating the argument for K_2 obtains the desired result. □

As a first application of Proposition 5.4, we show that any natural number can occur as the Casson-Whitney number of a 2-knot.

Proposition 5.6. *Let $n \in \mathbb{N}$. Then there exists a 2-knot K with $u_{\mathrm{cw}}(K) = n$.*

Proof. Let J be any 2-knot with $u_{\mathrm{cw}}(J) = 1$ and $m(J) = 1$, for instance J could be any even twist-spin of a 2-bridge knot, by Theorem 5.2: 2-bridge knots have nontrivial determinants, which are preserved by even twist-spinning by Proposition 3.5. Since the Alexander module is nontrivial, it must be cyclic since the original 2-bridge knot had a cyclic module. Then letting $K = nJ$ obtains the desired result, since $m(nJ) = n \leq u_{\mathrm{cw}}(K)$ and K can be unknotted in n pairs of finger and Whitney moves by performing the optimal regular homotopy for J on each summand. □

Scharlemann proved that unknotting number one knots are prime, i.e. if K_1 and K_2 are nontrivial classical knots, then the unknotting number of $K_1 \# K_2$ is at least 2 [Sch85]. Here we prove a special case of the analogous statement for u_{cw}, which works whenever the 2-knots in question have nontrivial determinants, or equivalently whenever their knot groups admit nontrivial Fox colorings. This reproves the same special case of Scharlemann's theorem for classical knots, via the bound given by Corollary 5.1. The technical core of our proof is a Freiheitssatz for one-relator quotients of free products of cyclic groups due to Fine, Howie, and Rosenberger [FHR88].

Theorem (Fine, Howie, Rosenberger)**.** *Suppose $G = \langle a_1, \dots, a_n | a_1^{e_1}, \dots, a_n^{e_n}, R^m \rangle$, where $n \geq 2$, $m \geq 2$, $e_i = 0$ or $e_i \geq 2$ for all i, and $R(a_1, \dots, a_n)$ is a cyclically reduced word which involves all of $a_1, \dots, a_n$. Then the subgroup of G generated by $a_1, \dots, a_{n-1}$ is isomorphic to $\langle a_1, \dots, a_{n-1} | a_1^{e_1}, \dots, a_{n-1}^{e_{n-1}} \rangle$.*

Their result generalizes the more well-known Freiheitssatz for one-relator groups, a classical result in combinatorial group theory characterizing the torsion in a one-relator group. It is proved by finding explicit representations of these groups into $\mathrm{PSL}_2(\mathbb{C})$.

The determinant of a 2-knot K is defined in Section 3.2 as the nonnegative generator of the evaluation of the Alexander ideal at $t = -1$. As with classical knots this is always an odd integer, and in Section 3.2 it is shown that even twist-spinning preserves the determinant, while odd twist-spins always have determinant 1. The classical fact that a 1-knot k admits a Fox p-coloring for a prime p if and only if p divides the classical determinant $|\Delta_k(-1)|$, where $\Delta_k(t)$ is the Alexander polynomial of k, carries over without change to this definition of determinant for nonprincipal ideals.

Theorem 1.8. *Let K_1, K_2 be 2-knots with determinants $\Delta(K_i)|_{-1} \neq 1$. Then*

$$u_{\text{cw}}(K_1 \# K_2) \geq 2.$$

Proof. Let x be a meridian of $K_1 \# K_2$. The claim to be proved is that for any $w \in C := \pi(K_1 \# K_2)'$, the relation $[x, x^w] = 1$ does not abelianize $\pi(K_1 \# K_2)$, since then $u_{\text{cw}}(K_1 \# K_2) \geq a_{\text{cw}}(K_1 \# K_2) \geq 2$.

Let p_1 and p_2 be prime divisors of $\Delta(K_1)|_{-1}$ and $\Delta(K_2)|_{-1}$, respectively. Then K_i admits a Fox p_i-coloring $\phi_i : \pi K_i \twoheadrightarrow D_{p_i} \cong \mathbb{Z}_{p_i} \rtimes \mathbb{Z}_2$. Let x_i be a meridian of K_i such that $\phi_i(x_i)$ is the generator of $\mathbb{Z}_2$. Then the group of the connected sum $\pi(K_1 \# K_2) \cong \frac{\pi K_1 * \pi K_2}{\langle\langle x_1^{-1} x_2 \rangle\rangle}$ admits a surjection ϕ onto the group

$$G := \langle z, a_1, a_2 \mid z^2 = a_1^{p_1} = a_2^{p_2} = 1, z a_1 z = a_1^{-1}, z a_2 z = a_2^{-1} \rangle \cong (\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}) \rtimes \mathbb{Z}_2.$$

This is obtained by first defining $\phi_1 * \phi_2 : \pi K_1 * \pi K_2 \to D_{p_1} * D_{p_2}$ in the obvious way and then noticing that this descends to the quotients. Notice that G can be formed from $D_{p_1} * D_{p_2}$ by identifying the images of the meridians: $G \cong \frac{D_{p_1} * D_{p_2}}{\langle\langle \phi_1(x_1)^{-1} \phi_2(x_2) \rangle\rangle}$. We will show that $\frac{\pi(K_1 \# K_2)}{\langle\langle [x, x^w] \rangle\rangle}$ is not abelian by showing that its induced image $G / \langle\langle \phi([x, x^w]) \rangle\rangle$ is not abelian.

We can assume x is the meridian of amalgamation, i.e. x is the image of x_1 and x_2 in

$\pi(K_1 \# K_2)$. Notice that $\phi(x) = z$ and $\phi([x, x^w]) = [z, z^v]$, where $v = \phi(w)$ is in the commutator subgroup $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$ of G. Then $G/\langle\langle [z, z^v] \rangle\rangle$ is the image of the induced homomorphism which we would like to show is nonabelian. We will do this by showing that its commutator subgroup is nontrivial. Let $N = \langle\langle [z, z^v] \rangle\rangle$, the normal closure of $[z, z^v]$ in G. As $[z, z^v]$ is a commutator, N is contained in the commutator subgroup $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$ of G. The goal is now to show that $(\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2})/N$ is not the trivial group.

Note that $[z, z^v] = z(v^{-1}zv)z(v^{-1}zv) = (zv^{-1}zv)^2 = [z, v]^2$. It will be convenient to describe N as the normal closure in $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$ of some elements of $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$. Denote $g = [z, v]$. N is the normal closure of all elements of the form $h^{-1}g^2h$, where $h \in G$ is arbitrary. Any $h \in G$ can be written as $z^n c$, where $n = 0$ or 1 and $c \in \mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$. Then $h^{-1}g^2h = c^{-1}z^n g^2 z^n c$. Since $c^{-1}g^2c$ is already in the normal closure of g^2 in $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$, it suffices to consider $n = 1$, i.e. $h = zc$. Notice that $zg^2z = (zgz)^2 = (z[z, v]z)^2 = (v^{-1}zvz)^2 = [v, z]^2 = [z, v]^{-2} = (g^2)^{-1}$. Then $c^{-1}zg^2zc = c^{-1}g^{-2}c = (c^{-1}g^2c)^{-1}$, so in fact N is the normal closure in $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$ of just g^2. By the Freiheitssatz, $(\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2})/\langle\langle g^2 \rangle\rangle$ is nontrivial for any element $g \in \mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$.

□

Corollary 5.7. *Let k_1 and k_2 be classical knots with determinants $|\Delta_{k_i}(-1)| \neq 1$. Then $u_{\mathrm{cw}}(\tau^n k_1 \# \tau^m k_2) \geq 2$ for any even integers n, m.*

Corollary 5.8. *Let K_1 and K_2 be even twist-spins of 2-bridge knots. Then $u_{\mathrm{cw}}(K_1 \# K_2) = 2$.*

Proof. Since 2-bridge knots have nontrivial determinants, $u_{\mathrm{cw}}(K_1 \# K_2) \geq 2$ follows from Theorem 1.8. The reverse inequality follows from Theorem 5.2 and the elementary fact that $u_{\mathrm{cw}}(K_1 \# K_2) \leq u_{\mathrm{cw}}(K_1) + u_{\mathrm{cw}}(K_2)$. □

It is interesting to note that in the case of 2-bridge knots k_1, k_2, the knot group $\pi(\tau^2 k_1 \# \tau^2 k_2) \cong (\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}) \rtimes \mathbb{Z}$, where $p_i = |\Delta_{k_i}(-1)|$, and that the proof of Theorem 1.8 goes through in that setting without the further quotient to G. In fact, G arises naturally as the group of $\tau^n k_1 \# \tau^m k_2 \# \mathbb{R}P^2$, where n, m are even and $\mathbb{R}P^2$ denotes a standard projective plane.

For odd integers $p, q \in \mathbb{Z}$, let $K_{p,q}$ denote the spin of $T(2,p)\#T(2,q)$. Miyazaki proved that $u_{\text{st}}(K_{p,q}) = 1$, whenever $q = p+2, p+4$, or $p+6$, when $\gcd(p, p+6) = 1$ [Miy86]. Therefore, u_{st} fails to be additive in these cases. However, it follows from Corollary 5.8 that u_{cw} is additive in these cases, and in particular that $u_{\text{cw}}(K_{p,q}) = 2$. This proves Theorem 1.7.

Theorem 1.7. *There are infinitely many 2-knots K for which $u_{\text{st}}(K) = 1$ and $u_{\text{cw}}(K) = 2$.*

The technique used in Theorem 1.8 to prove Theorem 1.7 does not obviously extend to be able to show that the algebraic Casson-Whitney number of a 2-knot is at least 3, in the absence of another lower bound like the Nakanishi index. It would be interesting to know if the difference between u_{cw} and u_{st} is ever greater than 1.

Question 5.9. *Does there exist a 2-knot K such that $u_{\text{cw}}(K) - u_{\text{st}}(K) > 1$?*

As mentioned in Section 5.2, Theorem 1.8 is a partial result towards an 'algebraic analogue' of the theorem of Scharlemann that unknotting number one knots are prime. As prime factorizations are not proven to exist for smooth 2-knots, we can still ask if a 2-knot with Casson-Whitney number one is prime on the level of knot groups.

Question 5.10. *Are Casson-Whitney number one 2-knots K "algebraically prime", i.e. if $K = K_1 \# K_2$, then at least one of K_1 or K_2 has knot group $\mathbb{Z}$?*

5.3 Application to classical unknotting number

As noted at the start of Section 5.1, the Nakanishi index, Ma-Qiu index, and algebraic stabilization number are all previously established lower bounds for the classical unknotting number. In this section we point out that the algebraic Casson-Whitney number is also a lower bound for the classical unknotting number, which is sharper than the aforementioned invariants in many cases.

Perhaps the most interesting reason to study a_{cw} as a lower bound for the unknotting number is that the above three invariants all fail to be additive in many simple cases, such as $T(2,p)\#T(2,q)$ when p,q are coprime [KY10]. By Theorem 1.8, $a_{\text{cw}}(T(2,p)\#T(2,q)) = 2$ for all (odd) p,q. We do not know any cases where a_{cw} fails to be additive on classical knots, although it seems difficult to prove this is always the case. Still, this poses a potentially interesting avenue to study the classical unknotting number, via a lower bound which comes from four dimensional techniques.

Let k be a 1-knot. Remembering that spinning preserves the knot group (and its meridians), $a_{\text{cw}}(k) = a_{\text{cw}}(Spin(k))$. Of course, $a_{\text{cw}}(Spin(k)) \leq u_{\text{cw}}(Spin(k))$, and by Corollary 5.1, $u_{\text{cw}}(Spin(k)) \leq u(k)$. Putting these facts together, we have:

Proposition 5.11. *For any 1-knot k, $a_{\text{cw}}(k) \leq u(k)$.*

As noted in Section 5.2, this reproves a special case of Scharlemann's theorem that unknotting number one knots are prime [Sch85]. Namely, if k_1 and k_2 are classical knots with nontrivial determinants, then $u(k_1\#k_2) \geq 2$.

5.4 Strong non-additivity of u_{st} and u_{cw}

As noted in Section 5.2, Miyazaki was the first to prove that u_{st} is non-additive. For certain p,q (see section for precise description) he showed that $u_{\text{st}}(\tau(T(2,p)\#T(2,q))) = 1$. As pointed out by Kanenobu [Kan96], the Nakanishi index proves that taking iterated connected sums of $K = \tau(T(2,p)\#T(2,q))$ has $u_{\text{st}}(nK) = n$, while $u_{\text{st}}(nT(2,p)) + u_{\text{st}}(nT(2,q)) = 2n$. This shows the existence of 2-knots K_1, K_2 with $u_{\text{st}}(K_1) + u_{\text{st}}(K_2) - u_{\text{st}}(K_1\#K_2)$ arbitrarily large. In this section we investigate and prove a stronger version of non-additivity for both the stabilization and Casson-Whitney number. For notational convenience, throughout the section we use α to denote either a_{st} or a_{cw}, and υ to denote the corresponding u_{st} or u_{cw}.

Our geometric study of strong non-additivity is inspired by Kanenobu's work in [Kan96] establishing the non-additivity of a_{st}. In particular, for each $n \geq 1$, Kanenobu gave examples of 2-knots $K_1, \ldots, K_n$ with $a_{\mathrm{st}}(K_i) = 1$ and $a_{\mathrm{st}}(K_1 \# \cdots \# K_n) = 1$.

Question 5.12 (Kanenobu)**.** *Is $u_{\mathrm{st}}(K_1 \# \cdots \# K_n) = 1$ as well?*

We generalize Kanenobu's result for a_{st} and prove a corresponding result for a_{cw}. We then prove analogous results for the geometric versions u_{st} and u_{cw}, answering Kanenobu's question in the affirmative at the expense of a small correction factor. In fact, Corollary 5.17 shows that the connected sums $K_1 \# \cdots \# K_n$ in Kanenobu's original examples have both stabilization number and Casson-Whitney number at most 2.

Theorem 5.13. *Let $K_1, \ldots, K_n$ be 2-knots with $\alpha(K_i) \leq c$. Suppose that there exist meridians $x_i \in \pi K_i$ and relatively prime integers $j_i \in \mathbb{Z}$ such that each $x_i^{j_i}$ lies in the center $Z(\pi K_i)$ of the knot group of K_i. Then, $\alpha(K_1 \# \cdots \# K_n) \leq c$.*

Proof. We will prove the case $\alpha = a_{\mathrm{st}}$ and $c = 1$ in detail, then point out the changes necessary for the general result.

Since $\alpha(K_i) = 1$, there exists an element $w_i \in (\pi K_i)'$ such that $\pi K_i / \langle\langle [x_i, w_i] \rangle\rangle \cong \mathbb{Z}$. Let $K = K_1 \# \cdots \# K_n$, and let $x = x_i$ be the meridian of amalgamation. We will show that $\pi K / \langle\langle [x, w_1 w_2 \cdots w_n] \rangle\rangle \cong \mathbb{Z}$. For $m \leq n$, let

$$R_m = [x, w_1 w_2 \cdots w_m] \text{ and}$$
$$G_m = \pi(K_1 \# \cdots \# K_m) / \langle\langle R_m \rangle\rangle$$

Note that $G_1 \cong \mathbb{Z}$ by assumption; we will show that $G_m \cong G_{m-1}$, so that by induction $G_n \cong \mathbb{Z}$.

Since j_1 and $j_2 j_3 \cdots j_m$ are coprime, there exist integers s and t so that $s j_1 + t j_2 j_3 \cdots j_m = 1$. Notice that $x^{s j_1} \in Z(\pi K_1)$ and $x^{s j_1 - 1} = x^{-t j_2 \cdots j_m} \in Z(\pi(K_2 \# \cdots \# K_m))$. The relation

R_m is equivalent to $x = (w_1 \cdots w_m)^{-1} x (w_1 \cdots w_m)$. Raising both sides to the sj_1 we obtain:

$$\begin{aligned} x^{sj_1} &= (w_2 \cdots w_m)^{-1} w_1^{-1} x^{sj_1} w_1 (w_2 \cdots w_m) \\ &= (w_2 \cdots w_m)^{-1} x^{sj_1} (w_2 \cdots w_m) \\ &= (w_2 \cdots w_m)^{-1} x (w_2 \cdots w_m) x^{sj_1 - 1} \end{aligned}$$

which is equivalent to $x = (w_2 \cdots w_m)^{-1} x (w_2 \cdots w_m)$. We can repeat this procedure until we reach $x = w_m^{-1} x w_m$, or $[x, w_m] = 1$, the relation which abelianizes πK_m. Since w_m is in the commutator subgroup of πK_m, it is trivial in the abelianization. Thus

$$\begin{aligned} G_m &= \pi(K_1 \# \cdots \# K_m) / \langle\langle [x, w_1 w_2 \cdots w_m] \rangle\rangle \\ &\cong \pi(K_1 \# \cdots \# K_{m-1}) / \langle\langle [x, w_1 w_2 \cdots w_{m-1}] \rangle\rangle = G_{m-1}. \end{aligned}$$

Now, if $c > 1$, we simply repeat the previous argument c times, making a choice to group the nc assumed relations into c relations, each one the combination of one of the assumed relations from each knot group, as above.

The proof for $\alpha = a_{\mathrm{cw}}$ is similar, so we only list the changes here. When $c = 1$, each πK_i has a finger move relation $[x_i, x_i^{w_i}] = 1$ such that $\pi K_i / \langle\langle [x_i, x_i^{w_i}] \rangle\rangle \cong \mathbb{Z}$, for some $w_i \in (\pi K_i)'$. We combine these into one relation: $[x, x^{w_n w_{n-1} \cdots w_1}] = 1$, which will abelianize the group of $K_1 \# \cdots \# K_n$.

Let $v_i = w_m w_{m-1} \cdots w_i$, so e.g. $v_1 = v_2 w_1$ and choose s and t as before. The relation

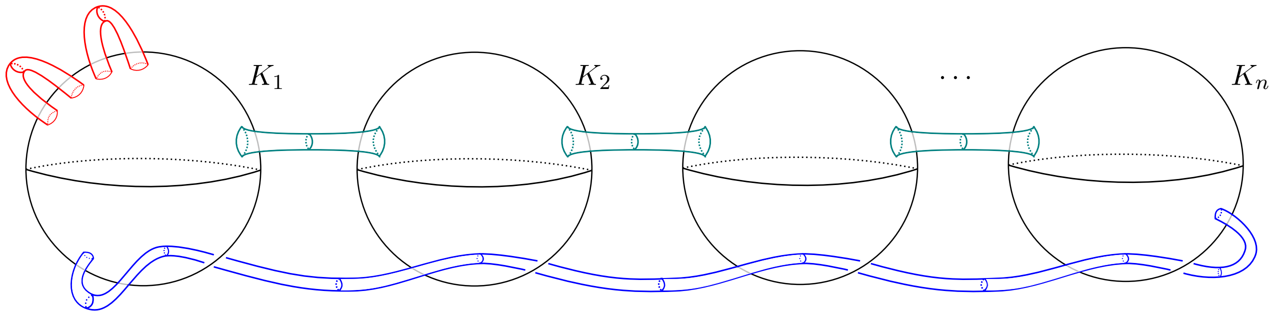

Figure 5.1: A schematic for the proof of Lemma 5.16, where $u_{\text{st}}(K_1) = 2$ and $a_{\text{st}}(K_1\#\cdots\#K_n) = 1$. The blue handle abelianizes the group of $K_1\#\cdots\#K_n$ and the trivial red handles allow us to inductively unknot each summand.

$[x, x^{v_1}] = 1$ is equivalent to $x = (x^{v_1})^{-1}xx^{v_1}$. Raising both sides to the power sj_1, we obtain

$$\begin{aligned} x^{sj_1} &= v_1^{-1}x^{-1}v_1x^{sj_1}v_1^{-1}xv_1 \\ &= v_1^{-1}x^{-1}v_2w_1x^{sj_1}w_1^{-1}v_2^{-1}xv_1 \\ &= v_1^{-1}x^{-1}v_2x^{sj_1}v_2^{-1}xv_1 \\ &= v_1^{-1}x^{-1}v_2xv_2^{-1}xv_2x^{sj_1-1}w_1 \\ &= w_1^{-1}v_2^{-1}x^{-1}v_2xv_2^{-1}xv_2x^{-1}w_1x^{sj_1} \end{aligned}$$

After canceling the x^{sj_1} terms from both sides, we can further cancel the w_1 terms to obtain $x = v_2^{-1}x^{-1}v_2xv_2^{-1}xv_2$, or $1 = [x, x^{v_2}]$. Repeating this procedure we eventually reach $1 = [x, x^{v_m}] = [x, x^{w_m}]$, the relation which abelianizes πK_m. Thus

$$\begin{aligned} G_m &= \pi(K_1\#\cdots\#K_m)/\langle\langle [x, x^{w_m w_{m-1}\cdots w_1}]\rangle\rangle \\ &\cong \pi(K_1\#\cdots\#K_{m-1})/\langle\langle [x, x^{w_{m-1}\cdots w_1}]\rangle\rangle = G_{m-1}, \end{aligned}$$

and by induction $G_n \cong \mathbb{Z}$. The adaptation to $c > 1$ is the same as in the previous case. $\square$

Remark 5.14. There are many nontrivial examples of 2-knots $K_1, \dots, K_n$ satisfying the hypotheses of Theorem 5.13. For instance, the technical condition that the j^{th} power of a meridian is central is satisfied by any j-twist-spun knot (see Example 2.7). Indeed, Kanenobu uses twist-spun knots with coprime twist indices to construct his examples of strong algebraic non-additivity in [Kan96].

Recall Proposition 5.5, which says that for a pair of 2-knots K_1, K_2, the algebraic lower bounds satisfy $\max\{\alpha(K_1), \alpha(K_2)\} \leq \alpha(K_2 \# K_2) \leq \alpha(K_1) + \alpha(K_2)$. Kanenobu used the nonadditivity result that he proved for a_{st} to prove the following theorem. We note that by Theorem 5.13, his original examples work to prove the following corollary for $\alpha = a_{\text{cw}}$ as well.

Corollary 5.15 (Kanenobu)**.** *For any positive integers $p_1, \dots, p_n$ and any integer q with $\max\{p_i\} \leq q \leq p_1 + \cdots + p_n$, there exist 2-knots $K_1, \dots, K_n$ satisfying:*

1. $a_{\text{st}}(K_i) = a_{\text{cw}}(K_i) = p_i$ *for all i, and*
2. $a_{\text{st}}(K_1 \# \cdots \# K_n) = a_{\text{cw}}(K_1 \# \cdots \# K_n) = q.$

While these examples show that the algebraic finger-Whitney index a_{cw} is non-additive on general 2-knot groups, we do not know of any classical knot groups for which this is the case. This is in contrast with the algebraic stabilization number a_{st}, which fails to be additive for classical knots by [Miy86] (see the discussion at the end of Section 5.1). Now, to extend these algebraic results on the non-additivity of a_{st} and a_{cw} to their geometric counterparts u_{st} and u_{cw}, we first relate these invariants through the following lemma.

Lemma 5.16. *Let $K = K_1 \# \dots \# K_n$. If $\upsilon(K_i) \leq c$ for each i, then $\upsilon(K) \leq c + \alpha(K)$.*

Proof. We prove only the statement for $\alpha = a_{\text{st}}$ and $\upsilon = u_{\text{st}}$ by induction on the number n of summands. Clearly, the result holds for $n = 1$. Indeed, it will be convenient for the inductive step to prove a slightly stronger statement: K can be unknotted by first stabilizing $a_{\text{st}}(K)$

times to obtain a surface F with $\pi F \cong \mathbb{Z}$, and then by stabilizing c times along guiding arcs which are necessarily trivial since πF is cyclic. This statement holds in the case $n = 1$ since the guiding arcs for the trivial stabilizations can be isotoped in the complement of F to be guiding arcs for a collection of c stabilizations that smoothly unknot $K = K_1$. So, we proceed with the inductive step, and assume that $n > 1$.

Now, since $\pi F \cong \mathbb{Z}$, the guiding arcs for the c trivial stabilizations are isotopic in the complement of F to guiding arcs for a different set of c stabilizations which unknot K_1. Therefore, the surface resulting from c trivial stabilizations of the surface F is isotopic to the surface resulting from c trivial stabilizations of a surface F' obtained from $K_2 \# \ldots \# K_n$ by the same $a_{\text{st}}(K)$ stabilizations used to abelianize πK. It follows from the proof of Proposition 5.5 that these stabilizations also abelianize $\pi(K_2 \# \cdots \# K_n)$ once πK_1 has been abelianized, and so $\pi F' \cong \mathbb{Z}$. Therefore by induction, F' is unknotted by c trivial stabilizations. □

Our first examples of the non-additivity of the stabilization and Casson-Whitney number now follow as a corollary of Theorem 5.13 and Lemma 5.16.

Corollary 5.17. *For $n \geq 1$, consider the j_i-twist-spins $K_i = \tau^{j_i} k_i$ of classical knots $k_1, \ldots, k_n$, where each k_i is either 2-bridge or has unknotting number one, with pairwise coprime twist indices $j_i \geq 2$. Then,*

$$\upsilon(K_i) = 1 \textit{ for all } i, \textit{ and}$$

$$\upsilon(K_1 \# \cdots \# K_n) \leq 2.$$

Proof. First note that by either Corollary 5.1 or Theorem 5.2 (depending on whether the knot k_i is 2-bridge or unknotting number one), $u_{\text{cw}}(K_i) = 1$ for each i. So, it just remains to show that $u_{\text{st}}(K_1 \# \cdots \# K_n) \leq 2$ and $u_{\text{cw}}(K_1 \# \cdots \# K_n) \leq 2$. This follows from the previous results of this section. In particular, as noted in Remark 5.14 above, the twist-spins K_i have $a_{\text{st}}(K_i) = 1$ as well as meridians $x_i \in \pi K_i$ such that $x_i^{j_i} \in Z(\pi K_i)$. Therefore, these knots satisfy the hypotheses of Theorem 5.13, and so $a_{\mathrm{st}}(K) = a_{\mathrm{cw}}(K) = 1$ as well. Now Lemma 5.16 applies, and we can conclude that both $u_{\mathrm{st}}(K_1 \# \cdots \# K_n), u_{\mathrm{cw}}(K_1 \# \cdots \# K_n) \leq 2$, as desired. □

Moreover, using a different family of twist-spun 2-knots, we formulate the more general non-additivity result featured in the introduction.

Theorem 1.9. *Let $\upsilon = u_{\mathrm{st}}$ or u_{cw}. For any $c, n \in \mathbb{N}$, there exist 2-knots $K_1, \ldots, K_n$ with*

$$\upsilon(K_i) = c \text{ for all } i, \text{ and}$$

$$c \leq \upsilon(K_1 \# \cdots \# K_n) \leq 2c$$

Proof. For the i^{th} prime $p_i \in \mathbb{N}$, let K_i be the connected sum of c copies of $\tau^{p_i} T(2, p_i)$, the p_i-twist-spin of the $(2, p_i)$-torus knot. Since the Alexander module of each summand $\tau^{p_i} T(2, p_i)$ is cyclic, the Nakanishi index $m(K_i)$ of the connected sum is equal to c. This matches the upper bound for υ given by Theorem 5.2, and so $\upsilon(K_i) = c$. Now, each K_i can also be thought of as a single p_i-twist-spin of the connected sum of c copies of $T(2, p_i)$. Therefore $K = K_1 \# \cdots \# K_n$ is a connected sum of twist-spun knots with coprime twist indices, and so Theorem 5.13 applies to show that $a_{\mathrm{st}}(K) = c$. Then by Lemma 5.16, $\upsilon(K) \leq 2c$. □

The proof of the next corollary follows from Corollary 5.1, Theorem 5.2, and Lemma 5.16.

Corollary 5.18. *Let $n \in \mathbb{N}$ and let $k_1, \ldots, k_n$ be 1-knots, each either 2-bridge or with unknotting number one. Let $j_1, \ldots, j_n$ be coprime integers at least 2 and let $K_i = \tau^{j_i} k_i$. Then $\upsilon(K_i) = 1$ for all i and $\upsilon(K_1 \# \cdots \# K_n) \leq 2$.*

Bibliography

[Art25] Emil Artin. Zur Isotopie zweidimensionaler Flächen im $\mathbb{R}^4$. *Abh. Math. Sem. Univ. Hamburg* 4(1) (1925), pp. 174–177.

[BK20] Sebastian Baader and Alexandra Kjuchukova. Symmetric quotients of knot groups and a filtration of the Gordian graph *Mathematical Proceedings of the Cambridge Philosophical Society* 169(1), (2020) pp. 141-148.

[BE73] James Boler and Benny Evans. The free product of residually finite groups amalgamated along retracts is residually finite. *Proc. Amer. Math. Soc.* 37 (1973), pp. 50–52.

[Boy93] Jeffrey Boyle. The turned torus knot in S^4. *J. Knot Theory Ramifications* 2.3 (1993), pp. 239–249.

[CSS06] J. Scott Carter, Masahico Saito, and Shin Satoh. Ribbon concordance of surface-knots via quandle cocycle invariants. *J. Aust. Math. Soc.* 80.1 (2006), pp. 131–147.

[Cas86] Andrew J. Casson. Three lectures on new-infinite constructions in 4-dimensional manifolds *Progr. Math.* 62 (1986) pp.201-244.

[Coc83] Tim Cochran. Ribbon knots in S^4. *J. London Math. Soc.* (2) 28.3 (1983), pp. 563–576.

[CF63] Richard H. Crowell and Ralph H. Fox. *Introduction to knot theory.* Ginn and Co., Boston, Mass. 1963, pp. X+182.

[DM19] Irving Dai and Maggie Miller. The 0-concordance monoid is infinitely generated. 2019. arXiv: 1907.07166 [math.GT].

[Eis95] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry.* Vol. 150. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[FHR88] Fine, Howie, and Rosenberger. One-relator quotients and free products of cyclics. *Proc. Amer. Math. Soc.* 102 (1988) pp. 249-254.

[FQ90] M. H. Freedman and F.Quinn *Topology of 4-manifolds* Princeton University Press, Princeton, 1990.

[Fox53] Ralph H. Fox. Free differential calculus I. Derivation in the free group ring. *Ann. of Math.* (2) 57 (1953), pp. 547–560.

[Fox62] R. H. Fox. A quick trip through knot theory. *Topology of 3-manifolds and related topics* (Proc. The Univ. of Georgia Institute, 1961). PrenticeHall, Englewood Cliffs, N.J., 1962, pp. 120–167.

[FP19] Stefan Friedl and Mark Powell. Homtopy ribbon concordance and Alexander polynomials 2019. arXiv: 1907.09031v2 [math.GT].

[Gor81] C. McA. Gordon. Ribbon concordance of knots in the 3-sphere. *Math. Ann.* 257.2 (1981), pp. 157–170.

[GL89] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *Journal of the American Mathematical Society* 2 (2) (1989) pp. 371-415.

[Gut72] M. A. Gutiérrez. On knot modules. *Invent. Math.* 17 (1972), pp. 329– 335.

[HK79] Fujitsugu Hosokawa and Akio Kawauchi. Proposals for unknotted surfaces in 4-spaces. *Osaka J. Math.* 16 (1979) pp. 233-248.

[Jos19] Jason Joseph. 0-concordance of knotted surfaces and Alexander ideals. 2019. arXiv: 1911.13112 [math.GT].

[JKRS20] Jason Joseph, Michael Klug, Benjamin Ruppik, and Hannah Schwartz. Unknotting numbers of 2-spheres in the 4-sphere. *In progress.*

[KY10] Teruhisa Kadokami and Zhiqing Yang. An integral invariant from the knot group. *Osaka J. Math.* 47 (2010) pp.965-976.

[Kam01] Seiichi Kamada Wirtinger presentations for higher dimensional manifold knots obtained from diagrams *Fundamenta Mathematicae* 168.2 (2001): 105-112.

[Kan96] Taizo Kanenobu. The weak unknotting number of a composite 2-knot. *Journal of Knot Theory and Its Ramifications* 5 (1996) pp. 161-166.

[Kan83] Taizo Kanenobu. Groups of higher-dimensional satellite knots. *J. Pure Appl. Algebra* 28.2 (1983), pp. 179–188.

[KK94] Taizo Kanenobu and Ken-ichiro Kazama. The peripheral subgroup and the second homology of the group of a knotted torus in S^4. *Osaka J. Math.* 31.4 (1994), pp. 907–921.

[Kaw96] Akio Kawauchi. *A survey of knot theory.* Translated and revised from the 1990 Japanese original by the author. Birkhäuser Verlag, Basel, 1996, pp. xxii+420.

[Ker65] Michel A. Kervaire. Les nœuds de dimensions supérieures. *Bull. Soc. Math. France* 93 (1965), pp. 225–271.

[Kin61] Shin'ichi Kinoshita. On the Alexander polynomials of 2-spheres in a 4-sphere. *Ann. of Math.* (2) 74 (1961), pp. 518–531.

[Kir97] Problems in low-dimensional topology. *Geometric topology* (Athens, GA, 1993). Ed. by Rob Kirby. Vol. 2. AMS/IP Stud. Adv. Math. Amer. Math. Soc., Providence, RI, 1997, pp. 35–473.

[Lev78] J. Levine. Some results on higher dimensional knot groups. *Knot theory (Proc. Sem., Plans-sur-Bex, 1977).* Vol. 685. Lecture Notes in Math. With an appendix by Claude Weber. Springer, Berlin, 1978, pp. 243– 273.

[Lit79] R. A. Litherland. Deforming twist-spun knots. *Trans. Amer. Math. Soc.* 250 (1979) pp. 311-331.

[Lit81] R. A. Litherland. The second homology of the group of a knotted surface. *Quart. J. Math. Oxford Ser.* (2) 32.128 (1981), pp. 425–434.

[MQ06] Jiming Ma and Ruifeng Qiu. A lower bound on unknotting number. *Chinese Ann. Math. Ser. B* 27 (2006) pp. 437-440.

[Mat86] Hideyuki Matsumura. *Commutative ring theory.* Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1986, pp. xiv+320.

[Mel77] Paul Michael Melvin. *Blowing up and down in 4-manifolds.* Thesis – University of California, Berkeley. 1977.

[MP19] Miller and Powell. Stabilization distance between surfaces. 2019. arXiv: 1908.06701 [math.GT].

[Miy86] Katura Miyazaki. On the relationship among unknotting number, knotting genus and Alexander invariant for 2-knots. *Kobe J. Math.* 3 (1986) pp. 77-85.

[Nak81] Yasutaka Nakanishi. A note on unknotting number. *Math. Sem. Notes Kobe Univ.* 9 (1981) pp. 99-108.

[Sat00] Shin Satoh. Virtual knot presentation of ribbon torus-knots. *J. Knot Theory Ramifications* 9.4 (2000), pp. 531–542.

[Sch85] Martin Scharlemann. Unknotting number one knots are prime. *Invent. Math.* 82 (1985) pp. 37-55.

[Suc85] Alexander I. Suciu. Infinitely many ribbon knots with the same fundamental group *Math. Proc. Cambridge Philos. Soc.* 98 (1985) pp. 481-492.

[Sma58] Stephen Smale. A classification of immersions of the 2-sphere. *Trans. Amer. Math. Soc.* 90 (1958) pp. 281-290.

[Sun15] Nathan S. Sunukjian. Surfaces in 4-manifolds: concordance, isotopy, and surgery. *Int. Math. Res. Not.* IMRN 17 (2015), pp. 7950–7978.

[Sun19] Nathan Sunukjian. 0-Concordance of 2-knots. 2019. arXiv: 1907.06524 [math.GT].

[Whit85] Wilbur Whitten. Knot complements and groups. *Topology* 26 (1) (1987) pp. 41-44.

[Yaj69] Takeshi Yajima. On a characterization of knot groups of some spheres in $\mathbb{R}^4$. *Osaka Math. J.* 6 (1969) pp. 435-446.

[Zee65] E. C. Zeeman. Twisting spun knots. *Trans. Amer. Math. Soc.* 115 (1965), pp. 471–495.