ORDER OF OPERATIONS: PLEASE EXCUSE MY DEAR AUNT SALLY AS HER RULE IS DECEIVING

by

JOSHUA EVAN NADDOR

(Under the Direction of Kevin C. Moore)

ABSTRACT

Order of Operations is a topic in mathematics learned around the fourth grade. Researchers have indicated students incorporate a multitude of unique approaches to evaluating expressions. The unique approaches indicate the variety of perceptions students have for the order of operations. Researchers have also recommended mathematics teachers provide a refresher on how to apply the order of operations in various math classes. This thesis presents a brief history of the conventional order of operations, research on why students approach expressions differently, and data from a survey gauging participants' understanding of order of operations.

INDEX WORDS: Order of Operations, Mathematical Conventions, PEMDAS, Quantitative Reasoning

ORDER OF OPERATIONS: PLEASE EXCUSE MY DEAR AUNT SALLY AS HER RULE IS DECEIVING

by

JOSHUA EVAN NADDOR

BS, The University of Georgia, 2017

BS, The University of Georgia, 2017

A Thesis Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

MASTER OF ARTS

ATHENS, GEORGIA

2020

ORDER OF OPERATIONS: PLEASE EXCUSE MY DEAR AUNT SALLY AS HER RULE IS DECEIVING

by

JOSHUA EVAN NADDOR

Major Professor: Kevin C. Moore
Committee: Kelly Edenfield
Cameron Byerley

Electronic Version Approved:

Ron Walcott
Interim Dean of the Graduate School
The University of Georgia
August 2020

ACKNOWLEDGEMENTS

First, I would like to give special thanks to my advisor, Dr. Kevin Moore. Without his unwavering patience and guidance, I would not have completed this degree. Additionally, I would like to thank Ms. Katrina Neidlinger for helping me navigate the various intricacies of UGA's graduate school. I would like to thank all of my friends and family who have helped me throughout my academic career. Their love, support, and encouragement were some of the major motivating factors to help me through this program. I would also like to thank the following professors who helped me throughout my academic career: Robert Rumely, Mark Werner, Abhyuday Mandal, Maurice Hendon, Kevin Moore, Margaret Morgan, Paul Pollack, Kelly Edenfield, Giorgis Petridis, and Amy Ellis. Not only did these wonderful people teach me various interesting topics and concepts, but their guidance and support helped me become the mathematics educator I am today. Additionally, I would like to thank each of my committee members for their support and expertise. Special thanks to Pat McAfee Inc. for providing me with copious mental vacations, all Athens, GA Chick-Fil-A locations for excellent food and phenomenal customer service, George Zheng and Zack Stokes for assistance with R, my parents their unconditional support, and my cat Champ for putting up with my stress cuddles. Lastly, I would like to thank everyone who assisted me by taking the survey for this thesis and sharing the survey amongst their friends and family.

TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Order of operations is a commonly taught topic in mathematics classrooms, with students typically learning order of operations somewhere between fourth grade and seventh grade. This set of rules provides a convention for determining the precedence for mathematical operations. Whether it be in an expression, an equation, an inequality, or another mathematical statement, these conventional rules can support individuals in interpreting expressions, equations, etc. in consistent and compatible ways. An order of operations convention also assists educators in helping students develop a thorough understanding of mathematics, as it would be cumbersome (and nearly impossible) to follow a different order of operations for each student.

When students learn order of operations, they are commonly given a mnemonic device to assist in understanding and remembering the conventions. In the United States, the mnemonic device used is PEMDAS: Parenthesis, Exponents, Multiplication, Division, Addition, and lastly Subtraction. This mnemonic device can be slightly confusing as it does not demonstrate the conventional order of operations in terms of its logical structure. The conventional order of operations is the following:

1. Parenthesis
2. Exponents
3. Multiplication or Division. The leftmost operation occurs first.
4. Addition or Subtraction. The leftmost operation occurs first.

When presented with multiplication and division, the operation that appears first is what is evaluated first. For example, $8 \div 2 * 4$ is evaluated as $(8 \div 2) * 4 = 4 * 4; 4 * 4 = 16$, -as division appears before multiplication. To alleviate the memorization of this mnemonic device, students are commonly told to "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally". Other parts of the world use similar mnemonic devices. Parts of Canada and New Zealand use BEDMAS. The United Kingdom, Pakistan, India, Bangladesh, Australia, and South Africa use BODMAS (R. Purbrick, personal communication, May 21 2020). The B stands for Brackets, the E stands of Exponents, and the O stands for Operations. The last four letters in this mnemonic device stand for the same words as PEMDAS. Additionally, the United Kingdom has also used BIDMAS instead of BODMAS. The difference between these two is the I stands for Indices in the former.

Although the aforementioned mnemonic devices can be useful for memorization purposes, I have seen an overreliance on them in my experience as an educator. Mnemonic devices like these can imply that multiplication should *always* happen before division, and the example above illustrates that is not always the case if following convention. This overreliance and the rise of ambiguous mathematical expressions on social media are the main inspirations for this thesis. Additionally, my experiences have also suggested people's overreliance on a mnemonic device like PEMDAS stems from not understanding order of operations as a convention; that is, in terms of an individual constructing a concept in relation to an image of the practices of some perceived community (Moore et al, 2019). I expand upon this in my literature review section.

Background and Rationale

Throughout the past decade, expressions similar to $8 \div 2(2 + 2)$ have appeared in some form or fashion on social media posts. I repeatedly found the comments and responses to such posts interesting, which invariably included a variety of different values. With respect to

$8 \div 2(2 + 2)$, the majority of commenters stated the value of the expression is either 16 or 1. The following procedures show how each of these values are commonly obtained:

$8 \div 2(2 + 2)$	$8 \div 2(2 + 2)$
$8 \div 2(4)$	$8 \div 2(4)$
$4(4)$	$8 \div 8$
16	1

It is important to note that these values can be obtained in different ways:

$8 \div 2(2 + 2)$	$8 \div 2(2 + 2)$
$4(2 + 2)$	$8 \div (4 + 4)$
$4(4)$	$8 \div 8$
16	1

While the comment sections on social media are not full of experts in mathematics, I was particularly interested in seeing the reasoning behind why they selected their value. The majority of the time, those responding in the comments appealed to the mnemonic devices they learned in school. Additionally, I found it interesting how many of these people referred to order of operations by the mnemonic device. It seemed as if they simply believed that was the name of a mathematics concept, and thus unquestionable and absolute in correctness, as opposed to some convention for expression evaluation. The nature of these responses motivated me to gain additional insights into individuals' use of order of operations. Another major motivating factor for this thesis was a desire to learn more about student thinking. If educators can understand why students think the way they do, it can provide benefits in future teaching.

Research on order of operations has highlighted many different facets of student understanding. Kirshner (1989) argued "that the visual structure of the notation system is highly correlated with the semantic categories underlying a propositional" (p. 275). Although there is a unique conventional order of operations, students could hold a similar set of rules regarding the visual syntax. "In visual terms, the same subrule means that diagonal juxtaposition has higher precedence than horizontal or vertical juxtaposition, which have higher precedence than wide spacing" (Kirshner, 1989, p. 276). Other researchers have emphasized students might execute an operation that they perceive is easier: "All the years I was taught that you first multiply. It's easier this way" (Linchevski & Livneh, 1999, p.180). A technique I often used while tutoring is using more than enough parenthesis to make the order in which operations should be evaluated more obvious. However, research from Gunnarsson, Sönnerhed, and Hernell (2015) showed that a superfluous use of brackets did not help students with order of operations. "The data do not seem to support the use of brackets to detach the middle number (b) from the first operation (±) in $a \pm b \times c$ type of expressions" (p. 91).

This thesis includes four additional chapters. Chapter 2 will be a concise literature review discussing the history of the conventional order of operations, algebraic structure, and the use of superfluous parentheses. In Chapter 3, the methods for my thesis will be discussed. This will include discussing the design of the survey I used to gain insights into individuals' evaluation of various expressions as it relates to order of operations. In Chapter 4, the results from the survey will be analyzed. In Chapter 5 I close with an overall discussion of the study including a summary of the findings, potential implications of the study, and avenues of future work.

CHAPTER 2:

LITERATURE REVIEW

History of Order of Operations

While there is not a definitive answer on when the conventional order of operations was adopted in mathematics, aspects of the conventional order of operations have been in use by mathematicians for centuries. Similar to how Newton worked with fluxions and Leibniz worked with infinitesimals, mathematicians of the 16th century used different symbols for the same mathematical operations (Miller, 2017). Although these mathematicians used different symbols for identical mathematical operations, the conventions they used were similar. It is hypothesized that this similar use of conventions is due to the distributive property: "distributive property implies a natural hierarchy in which multiplication is more powerful than addition, and makes it desirable to be able to write polynomials with as few parentheses as possible" (Peterson, 2000). Important to note, experts believe that facets of the conventional order of operations were around before the establishment of algebraic notation. While a convention similar to "multiplication before addition" did not exist until the beginning of algebraic notation, it is believed that this convention already existed in the verbal and geometric modes that preceded algebraic notation (Peterson, 2000).

Other conventions like left-to-right evaluation and equal precedence for multiplication and division appear to have been established within the past 150 years. Miller (2017) synthesized various mathematical journals from 1892 to 1929.

> In 1892 in *Mental Arithmetic*, M. A. Bailey advises avoiding expressions containing both ÷ and ×.
>
> In 1898 in *Text-Book of Algebra* by G. E. Fisher and I. J. Schwatt, a÷b×b is interpreted as (a÷b)×b.
>
> In 1907 in *High School Algebra*, Elementary Course by Slaught and Lennes, it is recommended that multiplications in any order be performed first, then divisions as they occur from left to right.
>
> In 1910 in *First Course of Algebra* by Hawkes, Luby, and Touton, the authors write that ÷ and × should be taken in the order in which they occur.
>
> In 1912, *First Year Algebra* by Webster Wells and Walter W. Hart has: "Indicated operations are to be performed in the following order: first, all multiplications and divisions in their order from left to right; then all additions and subtractions from left to right."
>
> In 1913, *Second Course in Algebra* by Webster Wells and Walter W. Hart has: "*Order of operations*. In a sequence of the fundamental operations on numbers, it is agreed that operations under radical signs or within symbols of grouping shall be performed before all others; that, otherwise, all multiplications and divisions shall be performed first, proceeding from left to right, and afterwards all additions and subtractions, proceeding again from left to right."
>
> In 1917, "The Report of the Committee on the Teaching of Arithmetic in Public Schools," *Mathematical Gazette 8*, p. 238, recommended the use of brackets to avoid ambiguity in such cases.
>
> In *A History of Mathematical Notations* (1928-1929) Florian Cajori writes (vol. 1, page 274), "If an arithmetical or algebraical term contains ÷ and ×, there is at present no agreement as to which sign shall be used first." (Miller, 2017).

As early as 1892, mathematicians found it advisable to avoid expressions containing both multiplication and division. As Cajori noted “there is at present no agreement as to which sign shall be used first” (Cajori, 1928-1929 p. 274, as cited in Miller, 2017).

While it is not something students will think of when learning the conventional order of operations, the need for an agreed upon order of operations is also likely due to the incorporation of computers in society (Pappanastos, Hall, & Honan, 2002). Although people may interpret an expression like 8 ÷ 2(2 + 2) differently to obtain different values, a computer will interpret this expression to obtain one unique value based on how it is programmed. A class of 60 calculus students was given the following problem:

$$Solve\ 2x/3y - 1\ if x = 9\ and\ y =\ 2$$

Out of the 60 students, 58 evaluated the expression to have a value of 2. However, Javascript, WolframAlpha, and Mathics all yield 11 as the value of the expression (Knill, 2014). As computers will interpret an expression uniquely, it is paramount those using the computers have a strong understanding on the conventional order of operations. Most importantly, individuals should understand order of operations as a convention so that they can anticipate computers being programed to interpret expressions in particular ways. A simple mistake based on human interpretation can have disastrous consequences: “given the computer-driven world in which we live, the implications of this erroneous understanding could have a potentially devastating impact on businesses whose employees are required to use spreadsheet programs” (Pappanastos et al., 2002, p.81).

Order of Operations as a Convention

As described in Chapter 1, a convention can be described as an individual constructing a concept in relation to an image of the practices of some perceived community. Conventions are important to mathematics; two notable examples are order of operations and notational systems. Speaking to conventions, Moore and colleagues (2019) defined them as follows:

> A convention is a personal construct that an individual has externalized as if it is a property of some community. We claim that a person has constructed a convention when that person has in mind a concept, a community of individuals, and some representational

practice that he perceives as a choice in that community among a variety of equally valid choices. (p. 181).

Moore and colleagues emphasized that a convention is not an objective object that exists independent of the person who conceives it; a practice may be perceived as a convention to one person and as an absolute rule by another person. Hence, what a student perceives as a convention might not be consistent with what another person (or teacher) perceives as a convention. Or, what a person or teacher perceives as a convention might instead be perceived as a rule by a student. As yet another option, a student might choose to follow some convention that differs from an alternative convention, while understanding the equal validity of each choice. For instance, a person could choose to negotiate and establish a different conventional order of operations within some community they interact in. If the adopted order of operations was the following, it would still be a convention as long as that person understands they could choose another order as an equally valid option.

1. Parentheses
2. Exponents
3. Addition and Subtraction. The rightmost operation is evaluated first.
4. Multiplication and Division. The leftmost operation is evaluated first.

Under this different order of operations, expressions and equations would be evaluated differently than the convention typically taught in schooling. Throughout the remainder of this thesis, I will use *conventional order of operations* to refer to my personal convention for the order of operations, which I take to be consistent with that taught in schooling and often referred to as PEMDAS. I will use *unconventional order of operations* to refer to when I infer students to use some order of operations that differs from the conventional order of operations. I acknowledge that there are two drawbacks to this choice of terminology. First, an unconventional order of operations may actually be understood as a conventional order of operations to the student. Second, defining a set of operations as conventional without situating it with respect to a particular individual's cognitive activity somewhat contradicts Moore's definition of convention. But, in order to aid the reader, I use conventional order of operations as described above. I will reference various research throughout the remainder of this chapter. It is possible that the students mentioned in said research hold the same conventional order of operations that I hold, but it is not necessarily required or clearly identified by the authors.

Algebraic Structure

Researchers have suggested that students' difficulties with the algebraic structure of expressions and equations are partially due to their lack of understanding of structural notions in arithmetic. To research this phenomenon, Linchevski and Livneh (1999) interviewed 53 sixth graders. As a tool to measure student understanding, they created an assessment with a series of expressions and equations. When simplifying expressions, they identified that students tend to group like terms. However, this process can be misleading. When students were asked to solve the following equation, researchers made an unexpected discovery. Students evaluated $4 + n - 2 + 5 = 11 + 3 + 5$ as $4 + n - 7 = 19$. This tendency of ignoring the preceding operation is known as *detachment* (Linchevski & Livneh, 1999). Detachment occurs when students disregard a mathematical operation and apply it later. In the example above, students detached the subtraction sign from the 2; they evaluated the expression as $4 + n - (2 + 5)$. If the conventional order of operations is followed, $4 + n - 2 + 5 = 11 + 3 + 5$ becomes $4 + n + 3 = 19, 7 + n = 19$, and lastly $n = 12$. By detaching the 2 from the subtraction sign, $4 + n - 2 + 5 = 11 + 3 + 5$ becomes $4 + n - 7 = 19$, $-3 + n = 19$, and lastly $n = 22$. Another equation students followed detachment on was $115 - n + 9 = 61$. Namely, some of the participants evaluated the expression as $115 - 9 + n = 61$ (Linchevski & Livneh, 1999). This detachment yields $n = -45$ while following the conventional order of operations yields $n = 63$. Detachment did not occur only in examples where variables were present; when the researchers asked the participants to *quickly* evaluate the expression $50 - 10 + 10 + 10$, several responded with 20. 20 is obtained by evaluating the expression as $50 - (10 + 10 + 10)$. If detachment did not occur in this expression, students would obtain 60. It is important to note that the use of the word *quickly* could be the reason for this detachment. Instead of approaching the expression as they would if the word *quickly* was not present, students might have tried to use an arithmetic trick to *quickly* solve the expression.

Table 1. Linchevski's and Livneh's (1999) Table I (p.178).

Choice of first operation (percentage)

	Israel	Canada	Is + Ca	Israel	Canada	Is + Ca
Choice of 1st operation	(1) $5 + 6 \times 10 =?$			(2) $17 - 3 \times 5 =?$		
multiplication first	33	44	38	52	55	53
addition/subtraction first	67	56	62	48	45	47
Choice of 1st operation	(3) $8 \times (5 + 7) =?$					
brackets first	100	100	100			
Choice of 1st operation	(4) $27 - 5 + 3 =?$			(5) $24 \div 3 \times 2 =?$		
subtract./div. first	78	61	70	67	56	62
addition/ mult. first	22	39	30	33	44	38

Reviewing the results in Table 1, the first expression resulted in more than half of the students choosing to add before they multiplied. "One of the students who added $5 + 6$ before multiplying did it while citing the rule "multiply before adding" correctly in his own words.

When asked to explain, he justified his action by saying, ‘I have to know what to multiply with!’” (Linchevski & Livneh, 1999, p.178). The authors noted that although the majority of students followed the conventional order of operations on the last two expressions, there were several students that did not. They hypothesized the reasoning for using addition before subtraction and multiplication before division is due to mnemonic devices such as BODMAS. As mentioned earlier, mnemonic devices such as these can be misleading as they do not account for the fact that multiplication and division have the same order of precedence. The same is true with addition and subtraction. Another explanation the authors provided was when given the opportunity to choose between addition and subtraction, students may choose what is more convenient.

It is important to note that after each student completed the expressions, the researchers presented solutions that used different approaches to the order of operations (Figure 1). Several students who initially calculated expressions in an unconventional order altered their approach to use the conventional order. However, some students did not change their approach: “Nevertheless, there were five students who remained steadfastly sequential, claiming that ‘. . . you have to go from left to right’” (Linchevski & Livneh, 1999, p. 179). Additionally, many students who applied an unconventional order recalled how they were taught. One student claimed they were taught to always multiply first as it is easier that way.

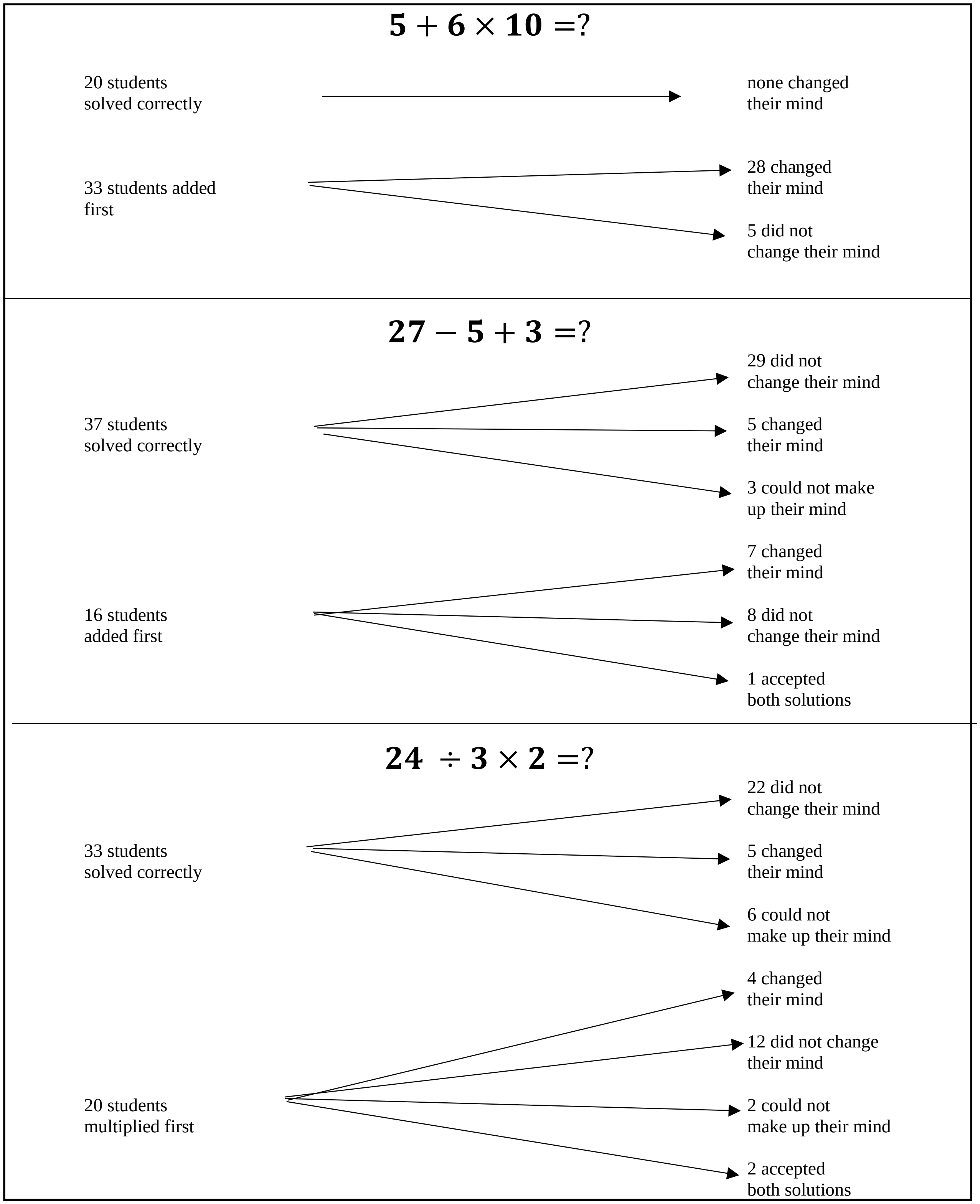

$5 + 6 \times 10 = ?$
20 students solved correctly
none changed their mind
33 students added first
28 changed their mind
5 did not change their mind
$27 - 5 + 3 = ?$
37 students solved correctly
29 did not change their mind
5 changed their mind
3 could not make up their mind
16 students added first
7 changed their mind
8 did not change their mind
1 accepted both solutions
$24 \div 3 \times 2 = ?$
33 students solved correctly
22 did not change their mind
5 changed their mind
6 could not make up their mind
20 students multiplied first
4 changed their mind
12 did not change their mind
2 could not make up their mind
2 accepted both solutions

Figure 1. Linchevski's and Livneh's (1999) *Figure 1* Responses to challenge (frequency): Israel + Canada. (p.180).

The results of Linchevski's and Livneh's findings showed only 6 of the 53 students did not use detachment and 14 of the 53 used the conventional order of operations. The authors' analysis suggests that some of the unconventional student responses made might stem from the structure of the expression and biasing number combinations. When comparing the student responses to the expressions $217 - 17 + 69$ and $267 - 30 + 30$, the authors noted that while these expressions have the same structure of operations, students may interpret the mathematical structure of each expression differently. Other research also echoes these claims (e.g., Bell et al., 1981; Fischbein et al., 1985; Linchevski and Vinner, 1990, as cited in Linchevski & Livneh, 1999). Bell et al. (1981) illustrated that when children were presented with problems with the same mathematical structure, they used different operations to solve the problem. Their different operations were dependent on the specific numerical data given. However, the authors did not present any ideas on how educators can combat this unconventional use of the order of operations. As students already hold a plethora of differing ideas on the topic, Linchevski & Livneh agreed that more research is needed on the area.

While it is not as closely related to the algebraic structure as detachment and unconventional approaches to order of operations, the visual syntax of expressions is another explanation for how students evaluate expressions. Tables 2 and 3 demonstrate how the conventional order of operations can be represented by visual characteristics:

Tables 2 and *3*. Kirshner's (1989) Table 1: *Syntactic Convention Using Hierarchy of Operation Levels* and Table 2: *Alternative Characterization of Operation Level*. (p. 276).

Table 1
Syntactic Convention Using Hierarchy of Operation Levels

Level 1	Addition	Subtraction
Level 2	Multiplication	Division
Level 3	Exponentiation	Finding roots

(Level 3 is said to be higher than Level 2, which is higher than Level 1.)

Syntactic Convention

1. Higher level operations have precedence over lower level operations.
2. In case of an equality of levels, the left-most operation has precedence.

Table 2
Alternative Characterization of Operation Level

Level	Visual characteristics	Examples
1	Wide spacing	$a + b$, $a - b$
2	Horizontal or vertical juxtaposition	ab, $\frac{a}{b}$
3	Diagonal juxtaposition	a^b, $\sqrt[a]{b}$

More examples of equations displayed by these visual characteristics are $2+3=5$, $71-49=22$, $12(8)=96$, $\frac{52}{13}=4$, $3^5=243$, and $\sqrt[6]{64}=2$. Addition and subtraction tend to have the largest amount of spacing between numbers. Multiplication and division tend to have very little spacing and exponentiation has even less. Kirshner's research demonstrated how students would perform various expressions without the use of spacing. Motivated by the potential ambiguity of standard notation, Figure 4 illustrates notations he devised for his study:

Table 4. Kirshner's (1989) Table 3: *Nonce Notations*. (p.277)

Table 3
Nonce Notations

Standard	Unspaced nonce	Spaced nonce
$a + b$	$a\mathrm{A}b$	$a \quad \mathrm{A} \quad b$
$a - b$	$a\mathrm{S}b$	$a \quad \mathrm{S} \quad b$
ab	$a\mathrm{M}b$	$a \ \mathrm{M} \ b$
$\frac{a}{b}$	$a\mathrm{D}b$	$a \ \mathrm{D} \ b$
a^b	$a\mathrm{E}b$	$a\mathrm{E}b$
$\sqrt[a]{b}$	$a\mathrm{R}b$	$a\mathrm{R}b$

Rewriting several of the examples above in unspaced nonce yields $2A3 = 5, 12\ M8 = 96$, and $3E5 = 243$. Rewriting several of the examples above in spaced nonce yields $71\ \ S\ \ 49 = 22$, $52\ D\ 13 = 4$, and $6R64 = 2$. Kirshner compared the performance of how students *performed* on expressions using the unspaced nonce and the spaced nonce. However, he did not mention how students would perform if the operations switched their level of spacing. For example if the spaced nonce was written as $a\ M\ b, a\ D\ b, a\ E\ b, a\ R\ b, aAb$, and aSb as opposed to that displayed in Figure 4, how would students approach expressions like these? After conducting his study, Kirshner claimed “The unspaced form of the notation proved more difficult than the spaced form. $F(1,369)=14.6$, $p < 0.001$” (Kirshner, 1989, p. 281). While the majority of students in this study were able to evaluate expressions similar to $1 + 3x^2$ for $x = 2$, students had more difficulty transferring this skill to the unspaced nonce notation compared to the spaced nonce notation. Kirshner concluded that for some students, the surface features of ordinary notation provide a necessary cue to successful syntactic decision.

Superfluous Parentheses

A common technique to avoid ambiguity in expressions is to use sets of parentheses. While expressions similar to $12 \div 2 * 3$ have a unique value if the conventional order of operations is followed, the conventional order is not always used. If division is applied first, the expression has a value of 18. If multiplication is applied first, the expression simplifies to 2. As mentioned in the 1917 *Mathematical Gazette 8*, the use of brackets can avoid such ambiguity. If parentheses are applied to the aforementioned expression, it would be clear which operation should occur first. If the expression was $(12 \div 2) * 3$, division would be applied first, and the result would be 18. If the expression was $12 \div (2 * 3)$, multiplication would be applied first, and the result would be 2. This practice of including additional parentheses has been used in classrooms for decades. In my experience as a mathematics educator, this practice has merits in aiding students understanding the conventional order of operations.

However, research by Gunnarsson, Sönnerhed, and Hernell (2015) shows this practice might not have any additional benefits. These three researchers attempted to provide an answer to Linchevski's and Livneh's (1999) work on the pedagogical approaches educators can take to reduce uses of detachment or a use of an unconventional order of operations. Their study consisted of 169 students aged 12 and 13 in Sweden. These students were split into two groups. Both groups completed a pretest, but the brief instruction each group received was different. One of the groups was exposed to emphasizing brackets during their brief instruction and the other group was not.

> The teaching intervention, for both groups, started with a claim that if there are different operations in one expression they are supposed to be conducted in the following order: first brackets, then multiplication and division and last addition and subtraction. Then four examples with necessary brackets were worked out (Gunnarsson, Sönnerhed, and Hernell, 2015, p. 96)

The bracket group was exposed to a teaching intervention where examples were worked out *with* brackets, and the control group was exposed to a teaching intervention where the examples were worked out *without* any brackets. After the instruction, both groups completed a posttest. Each group took the same test. Table 5 shows the expressions each test consisted of.

Table 5. Gunnarsson's, Sönnerhed's, and Hernell's (2015) Table 1. (p. 95).

Pretest item no	Expression	Posttest item no	Expression
F1	(3+5)·2	E1	(7−2)·3
F2	2·7+3	E2	2·5+4
F3	3+5·2	E3	8−3·2
F4	4·(5−3)	E4	4·(5−3)
F5	8−3·2	E5	2·7+3
F6	3+4·2	E6	3+4·2
F7	(7−1)·2	E7	(7−1)·2
F8	2·5+4	E8	2+5 3
F9	(3+7)·2	E9	(3+7)·2
F10	2+3·2	E10	2+3·2
F11	4+7·3	E11	4+7·3
F12	2·(3+6)	E12	2·(3+6)
F13	4·(4−2)	E13	4·(4−2)
F14	3+5·2	E14	3+5·2
F15	4·4−3	E15	4·4−3
F16	8−3·2	E16	8−3·2

The researchers showed that there was a statistically significant difference between the score of each group's pretest and posttest. However, there was not a statistically significant difference between each group's posttest score. It is possible that the different types of instruction each group received could be the reason the scores on the posttest were higher, but this increase in score could also be due to the fact that students tend to do better on assessments if they have already seen the material. While it was not statistically significant, the researchers found the control group to perform better than the group who received instruction on the incorporation of superfluous brackets. The authors concluded their study by acknowledging this lack of statistical significance between the two groups could be due to the brevity of the instruction, but as a response to assisting students with improperly using detachment and applying an nonnormative set of conventions, the use of superfluous brackets do not benefit student understanding of the conventional order of operations. (Gunnarsson, Sönnerhed, and Hernell 2015).

Other Relevant Research

Various researchers have identified that students have a variety of different beliefs in regard to understanding of the conventional order of operations (Pappanastos, Hall, Honan, 2002; Kirshner, 1989; Linchevski & Livneh 1999; Gunnarsson, Sönnerhed, and Hernell, 2015; Lee and Messner, 2000; Zazkis and Rouleau, 2017). Some researchers have recommended temporary fixes to assist students in constructing conventional order of operations:

> We suggest that instructors implement a refresher course on the order of operations convention, which is as follows:
>
> 1. Symbols first (brackets, braces, parentheses, etc.), starting with the innermost ones
> 2. Exponents
> 3. Negation
> 4. Multiplication and division from left to right
> 5. Addition and subtraction from left to right. (Pappanastos, Hall, Honan, 2002, p. 84)

Other temporary solutions consist of rewriting mnemonic devices. Instead of writing the mnemonic device as $PEMDAS$, it could be written as $PE\frac{M}{D}\frac{A}{S}$ or $PE(MD)(AS)$ (Zazkis and Rouleau, 2017). While multiple researchers have proposed temporary solutions to aid in student understanding of this convention, everyone agrees more research is needed in order to ensure students garner a robust understanding of the conventional order of operations.

CHAPTER 3

METHODS

In this chapter I describe the methods driving my investigation of the participants' understanding of order of operations. I start by explaining the construction of the survey. Next, I describe the setting and participants of those who completed my survey. Lastly, I describe the process and techniques I used to analyze and interpret the data.

Survey Construction

As the previous chapters have illustrated, there are many unique conceptions students can hold in regard to the order of operations. Relatedly, there are numerous forms of expressions within which I could investigate students' order of operations conceptions. I chose to narrow my focus of these forms into several categories. The survey for this thesis originally consisted of twenty expressions that were classified into one of the following categories:

1. A unique ordering of multiplication, division, addition, and subtraction
2. Purposely ambiguous
3. Repeated division
4. Negative squaring

These four categories were inspired by a combination of literature and my personal experiences. The first category was heavily influenced by the work of Linchevski & Livneh (1999). The expressions in this category were designed to see if participants would apply principles of detachment in solving expressions. The second category consisted of expressions similar to

8 ÷ 2 (2 + 2). These expressions are purposely ambiguous due to the conventional order. If an expression contains both multiplication and division, the expression is evaluated left-to-right. However, as expressions like these involve an implied multiplication through the distributive property, participants might apply multiplication before division. In my experience as a mathematics educator, students are eager to distribute terms, and they often distribute terms before applying other operations. Additionally, due to mnemonic devices like PEMDAS, participants might tend apply multiplication before division due to the simple fact that the M comes before the D in the mnemonic device (Zazkis and Rouleau, 2017). The third category was created due to division being an operation that students face difficulty with. I believed repeated division would allow for the most variety in responses compared to repeated subtraction. Repeated addition and repeated multiplication were not considered as both of those operations are commutative ($a + b = b + a$ and $a * b = b * a$). The fourth category was inspired by Lee and Messner (2000). These researchers noted that expressions such as -2^4 can be interpreted as $-(2^4) = -16$ or $(-2^4) = 16$. In addition to these four categories, several of the original twenty expressions were repeated versions of other expressions, except I replaced a number with a variable such as x or y (Moore, 2020). The expression 8 ÷ 2 (2 + 2) appears as the fifth expression. Shortly after, participants are asked to evaluate 8 ÷ 2 (2 + x). I was curious to see if participants would use the same approach on each expression. After these twenty expressions, participants would need to answer five short answer questions related to topics covered by expressions. These questions were included to allow participants to explain their logic in more detail. They were useful in gauging how participants approached several of the expressions. An example of one of the free response questions:

> When using his calculator to evaluate 8 ÷ 2 (2 + 2), Tommy obtained 1 as the answer.
> When using her calculator to evaluate 8 ÷ 2 (2 + 2), Gina obtained 16 as the answer.

> Please provide as many potential explanations as you can think of for why Tommy and Gina obtained different answers when they both used calculators to simplify the expression?

I then decided to shorten the length of the survey by removing several expressions. The majority of the original twenty expressions belonged to the first category. To help balance out how many of each category appeared, I removed several expressions from the first category. Additionally, these expressions resulted in possible responses that were not integers. I did not want participants to be too caught up on the unpleasantness of the numbers, so those expressions were removed. The fourth category was remade into a free response question. To obtain a deeper insight into each participant's logic, I also updated the survey to ask participants to explain how they arrived at their solution immediately after solving specific expressions. The complete survey as it appeared online is located in Appendix A. The figure on the next page contains the finalized questions on the survey:

1. What is the value of 18 ÷ 2 * 3 - 5 + 7?

2. What is the value of 48 ÷ 2 (9 + 3)?

3. What is the value of 16 ÷ 8 ÷ 4 ÷ 2?

4. What is the value of 18 + 2 ÷ 3 * 5 - 7?

5. **What is the value of 8 ÷ 2 (2 + 2)?**

6. What is the value of 72 ÷ 12 ÷ 4 ÷ 2?

7. **What is the value of 8 ÷ 2 (2 + *x*)?**

8. **What is the value of 3 + 6 * x^2 ÷ *x* + (-4)?**

9. **What is the value of 36 + 14 * 6 - 3 ÷ 2?**

10. What is the value of 16 ÷ *x* ÷ 4 ÷ 2?

11. **What is the value of 72 ÷ 12 * 3 - 4 + 2?**

12. **What is the value of 18 ÷ 2 * 3 - 5 + *x*?**

13. **What is the value of 36 + 14 * *y* - 3 ÷ 2?**

Key:
Bold: Questions contained "Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason."
Question #: Unique ordering of Multiplication, Division, Addition, and Subtraction
Question #: Purposely ambiguous
Question #: Repeated Division
Question #: Variables used in addition to one of the aforementioned

Figure 2. The expressions that appeared on the final version of the survey.

There are seven questions that are categorized as a unique ordering of multiplication, division, subtraction, and addition; three questions categorized as purposely ambiguous; and three questions categorized as repeated division. Six questions contain variables in addition to one of the aforementioned categories. I found it important to emphasize the unique ordering of the operations compared to the other categories as this is the category likely with the most variety for different responses. (Linchevski & Livneh, 1999; Pappanastos, Hall, and Honan, 2002; Zazkis and Rouleau 2017). As this category was the most frequently used and what I deemed to be the most important, questions meeting this category were selected to be the first and last question. This is also the only category that repeats over consecutive questions. However, these consecutive questions do not follow the same exact permutation of operations, hopefully allowing for different responses. I spaced the other categories out throughout the entirety of the survey making it so there was at least one question of a different category before a repeat.

Survey Setting

It is important to note that this study was completed during a very unique time in history: The Covid-19 pandemic and quarantine. Due to this, the setting for this study was somewhat untraditional. After the survey was constructed, it was distributed through word of mouth and social media with the use of Qualtrics. The study concluded with a total of 70 participants with varying demographics and backgrounds in mathematics. Participants selected a category that included their age.

Table 6. Number of Participants for each Age category.

Age	**18-20**	**21-29**	**30-39**	**40-49**	**50-59**	**60 or older**
Number of Participants	1	39	5	5	7	13

Additionally, participants selected a category that contained their highest completed degree.

Table 7. Number of Participants for each Degree category.

Highest Completed Degree	**Less than a High School Degree**	**High School Degree or Equivalent**	**Associate's Degree**	**Bachelor's Degree**	**Graduate Degree**
Number of Participants	8	3	2	32	25

Participants' professions included to a variety of different fields. These participants were located throughout the globe. The majority of participants resided in Georgia, but eight participants were international. Each participant was assigned a randomly generated name. Due to the nature of the survey and its implementation, I do not claim the data to be a representative sample. Rather, the data and all inferences should be perceived as foundational groundwork for the design and implementation of the survey, as well as for developing tentative hypotheses for consideration during future work with more strategically designed and representative samples.

Data Analysis

The following terms will be used consistently throughout the remainder of this thesis. This is how they should be interpreted:

Order of Operations	PEMDAS	PEMDSA	PEDMAS	PEDMSA
First	Parentheses	Parentheses	Parentheses	Parentheses
Second	Exponents	Exponents	Exponents	Exponents
Third	Multiplication	Multiplication	Division	Division
Fourth	Division	Division	Multiplication	Multiplication
Fifth	Addition	Subtraction	Addition	Subtraction
Sixth	Subtraction	Addition	Subtraction	Addition

Figure 3. Explanation of various forms of PE(MD)(AS).

Before the survey was distributed, I approached each of the expressions using multiple different orders of operations. I simplified each expression using the conventional order, PEMDAS, PEMDSA, PEDMAS, PEDMSA, Left-to-right, and other orders I perceived participants might use. For example, I evaluated $18 \div 2 * 3 - 5 + 7$ following all 4 approaches listed in Figure 3.

Order of Operations	PEMDAS	PEMDSA	PEDMAS	PEDMSA
First	$18 \div 2 * 3 - 5 + 7$	$18 \div 2 * 3 - 5 + 7$	$18 \div 2 * 3 - 5 + 7$	$18 \div 2 * 3 - 5 + 7$
Second	$18 \div 2 * 3 - 5 + 7$	$18 \div 2 * 3 - 5 + 7$	$18 \div 2 * 3 - 5 + 7$	$18 \div 2 * 3 - 5 + 7$
Third	$18 \div 6 - 5 + 7$	$18 \div 6 - 5 + 7$	$9 * 3 - 5 + 7$	$9 * 3 - 5 + 7$
Fourth	$3 - 5 + 7$	$3 - 5 + 7$	$27 - 5 + 7$	$27 - 5 + 7$
Fifth	$3 - 12$	$-2 + 7$	$27 - 12$	$22 + 7$
Sixth	-9	5	15	29

Figure 4. Approaches to $18 \div 2 * 3 - 5 + 7$.

Several of the approaches overlapped as the PEDMSA approach yields the same value as evaluating Left-to-Right. I evaluated each expression on the survey similarly to Figure 4 using a minimum of four different approaches to see what possible responses might be.

Participants' responses to the expressions in the survey resulted with a multitude of different values. Instead of focusing on the numeric value itself, I compared the values with the multiple approaches I applied beforehand. Each response fell into one of the following categories:

Category	Description
Conventional	This answer was obtained by using the conventional order of operations.
PEMDAS	This answer was obtained by using the PEMDAS approach.
PEDMAS/PEMDSA	This answer was obtained by using the PEDMAS or the PEMDSA approach.
Left-to-Right	This answer was obtained by evaluating the expression left to right.
Grouping	This answer was obtained by grouping like terms together.
Solved	The participant attempted to solve for the variable.
Unsure	The participant was unsure of how to approach the expression.
Other	This answer was obtained by none of the methods above.

Figure 5. Table of categories for approaches.

Before responses were placed in the *Other* category, I read their explanations on why the participant chose to approach the problem the way they did. This allowed several responses to be placed into different categories. If the explanation did not adequately explain the participant's reasoning or the expression did not have an explanation, these responses were placed in the *Other* category. Additionally, several responses were perceived as typos. I consulted with a fellow researcher on how responses like these should be categorized. After each response was categorized, I observed the processes each participant used. I was curious if participants would use the same process on similar problems. If they would not, I hoped to gain insight on why they changed their approach. Additionally, an exploratory data analysis was conducted on multiple variables for each participant.

CHAPTER 4

RESULTS

In this chapter, I present the results of this study. I start by listing the number of approaches participants implemented throughout the thirteen expressions. Next, I describe the approaches participants used in order to evaluate each expression on the survey. The order in which I present the results is split by the three categories defined in the previous chapter: (1) Unique Ordering of Multiplication, Division, Addition, and Subtraction, (2) Purposely Ambiguous, and (3) Repeated Division. I then highlight the explanations and reasoning of a select few participants in order to illustrate various themes in the data. The results for each expression are located in the Appendix B.

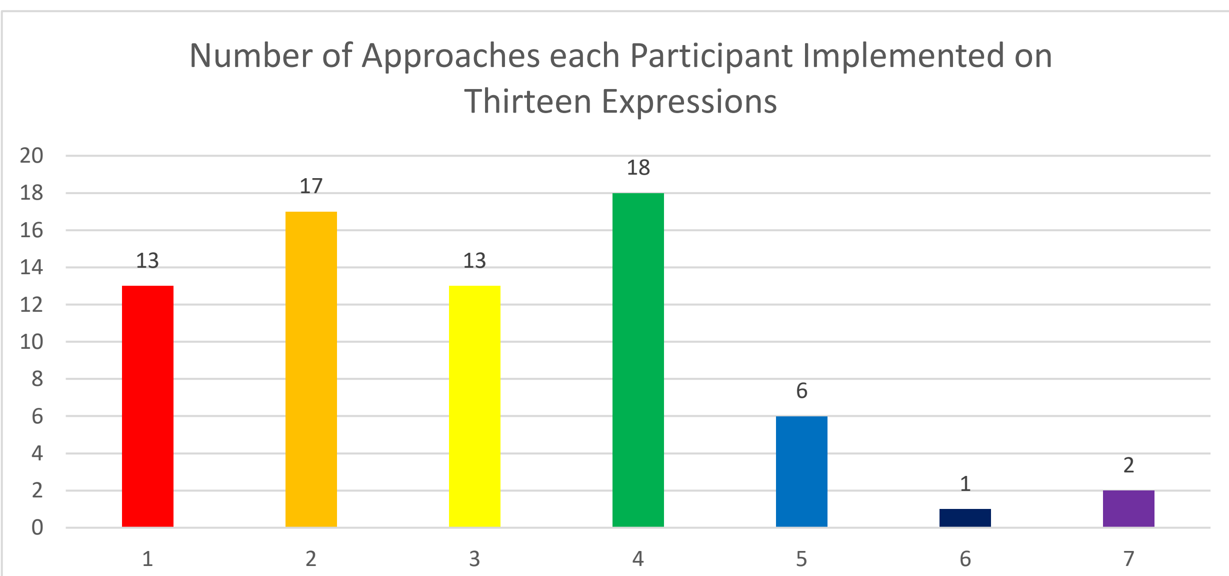


Figure 6. Number of Approaches each Participant Implemented on Thirteen Expressions.

Figure 6 illustrates how many approaches each participant implemented on the survey. Thirteen participants implemented the same approach on each expression. Those thirteen participants implemented the conventional approach on each expression. One participant did not follow the conventional approach on any of the expressions. However, this participant implemented 5 different approaches throughout the survey. I hypothesized that the less the conventional approach was used, the number of unique approaches used would increase. The aforementioned participant might be an outlier in this relationship as the two participants who followed 7 different approaches used the conventional approach once and twice respectively.

Unique Ordering of Multiplication, Division, Addition, and Subtraction

As mentioned in Chapter 3, I hypothesized that the category with the most variability in responses would be questions belonging to the category Unique Ordering of Multiplication, Division, Addition, and Subtraction. The data suggests this is true as expressions in this category averaged 12.66 unique responses per expression. The following table shows the approaches participants used in order to evaluate each expression belonging to this category.

Table 8. Approaches to a Unique Ordering of Multiplication, Division, Addition, and Subtraction.

Expression	Conventional	PEMDAS	PEMDSA/ PEDMAS	Left-to-Right	Other	Solved	Unsure
1. 18 ÷ 2 * 3 - 5 + 7	56	3	6	0	4	0	1
4. 18 + 2 ÷ 3 * 5 – 7	38	8	0	1	20	0	3
8. 3 + 6 * x^2 ÷ x + (-4)	47	0	0	4	4	10	5
9. 36 + 14 * 6 - 3 ÷ 2	56	0	0	3	9	0	2
11. 72 ÷ 12 * 3 - 4 + 2	50	3	10	0	5	0	2
12. 18 ÷ 2 * 3 - 5 + x	44	12	0	0	5	8	1
13. 36 + 14 * y - 3 ÷ 2	47	0	0	7	0	8	8

Twenty-five participants followed the same approach on all seven expressions, fourteen participants implemented two approaches in their evaluations, eight participants used three different approaches, and the remaining twenty-three participants used four or more different approaches. The first expression resulted in eight unique responses, far more than I anticipated. This will be a theme throughout the remainder of these results. Expression 4 yielded *twenty* unique responses; the most out of any expression. I believe the increase in unique responses can be explained by the division of coprime numbers (two numbers are considered coprime if their greatest common divisor is 1). As participants were told to refrain from using a calculator, applying division could result in this surplus of unique responses. Results from the eighth expression surprised me as I perceived this to be the most difficult expression on the survey due to the presence of x^2. As this was the expression I perceived to be the most difficult, I hypothesized this expression would result in the most unique responses. However, there were only eight unique responses to this expression, far less than the number of responses expression 4 yielded. Multiple participants explained their response to expression 9 was due to mnemonic devices.

Kate: BODMAS was really drilled into our heads at junior school. as soon as I see division or multiplication in a string I automatically do those first. to then do 120-(3/2) I find it easier to convert 120 into a fraction ie [sic] 240/2 then (240-3)/2

Participants evaluated expression 11 to obtain nine unique responses, expression 12 to obtain twelve unique responses, and expression 13 to obtain seven unique responses. I did not expect this last expression to have the fewest number of unique responses out of all of the expressions in this category. Not only was this an expression with a variable, but it was an expression with a variable that had not been used until this point. I hypothesized expressions with variables would have more unique responses than those that did not. While this was true for the average for each of the aforementioned, it was not true for every expression. The reasoning for most of the responses to expression 13 were very similar to Matthew's response.

Matthew: Multiplication and division first: 14 * y = 14y and 3/2 = 1.5. Then addition/subtraction: 36 -1.5 = 34.5. Add on the 14y to yield = 14y + 34.5.

Towards the end of these expressions, there was an increase in responses belonging to the unsure category. I believe this is because the inclusion of variables confused many of the participants. It might also have been the result of survey fatigue.

Comparing approaches participants implemented on similar expressions provides intriguing results. Expressions 1, 4, and 12 made use of the same numbers. Considering how similar these expressions were to each other, I find it interesting how many participants varied in their approaches. Thirty-three of the seventy participants did not use the same approach throughout these three expressions and thirteen of the seventy participants used a different approach for each of the expressions. Comparing responses from $18 \div 2 * 3 - 5 + 7$ and $18 + 2 \div 3 * 5 - 7$ showed an increase in responses following the mnemonic device PEMDAS. The former had three participants follow PEMDAS while the latter had eight participants implement this device. I did not expect to see an increase in responses belonging to the PEMDAS category as the survey progressed as the results from Gunnarsson, Sönnerhed, and Hernell (2015) showed students did not score significantly lower on the posttest compared to the pretest. Additionally, I assumed the participants of my survey would try to reevaluate the expression if the value they obtained was not a "common fraction". Expression 12, $18 \div 2 * 3 - 5 + x$, and expression 1, $18 \div 2 * 3 - 5 + 7$, had similar results to each other. Most of the participants who changed the approach they used in these two expressions tried to solve for the variable. This was something I did not expect to occur. I did not expect so many of the participants to set expressions with variable equal to zero and solve for the unknown variable. I perceive this to be the reason there were differences in approaches to the two expressions. The only other expressions in this category that were similar were $36 + 14 * 6 - 3 \div 2$ and $36 + 14 * y - 3 \div 2$. Similar to the expressions mentioned before these, I perceive this shift in ideology to be due to many of the participants attempting to solve for the variable instead of writing the expression in a different form. The last noteworthy trend for this category is the evaluation approach Left-to-Right was used more in expressions that involve variables compared to those that do not.

The following figures show the approaches taken to expressions in this category. Figure 11 contains all expressions, Figure 12 only contains expressions without variables, and Figure 13 only contains expressions with variables.

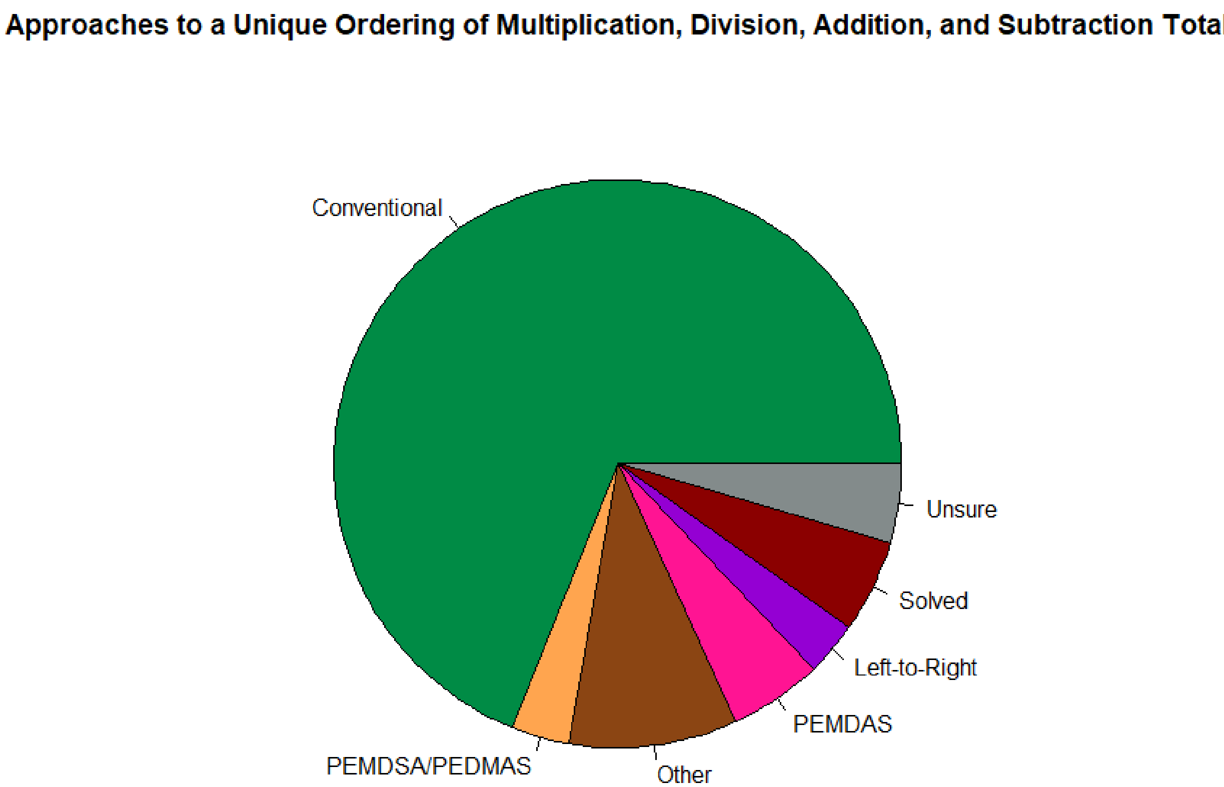


Figure 7. Approaches to a Unique Ordering of Multiplication, Division, Addition, and Subtraction Total.

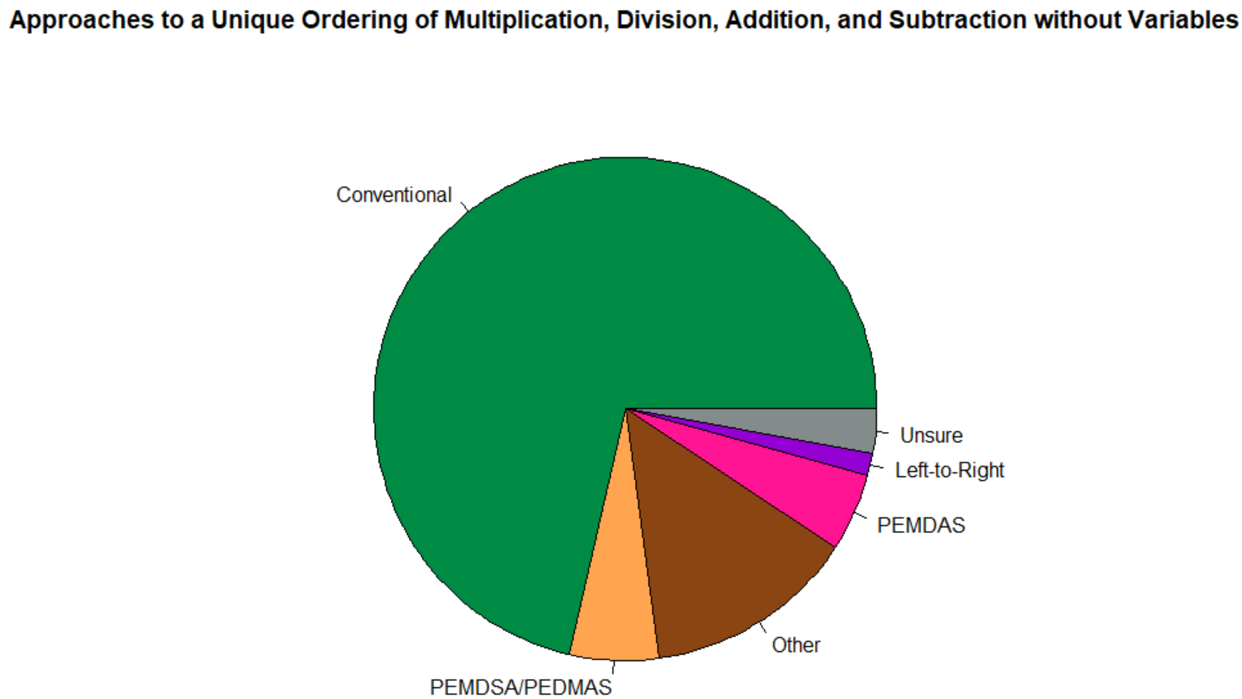


Figure 8. Approaches to a Unique Ordering of Multiplication, Division, Addition, and Subtraction without Variables.

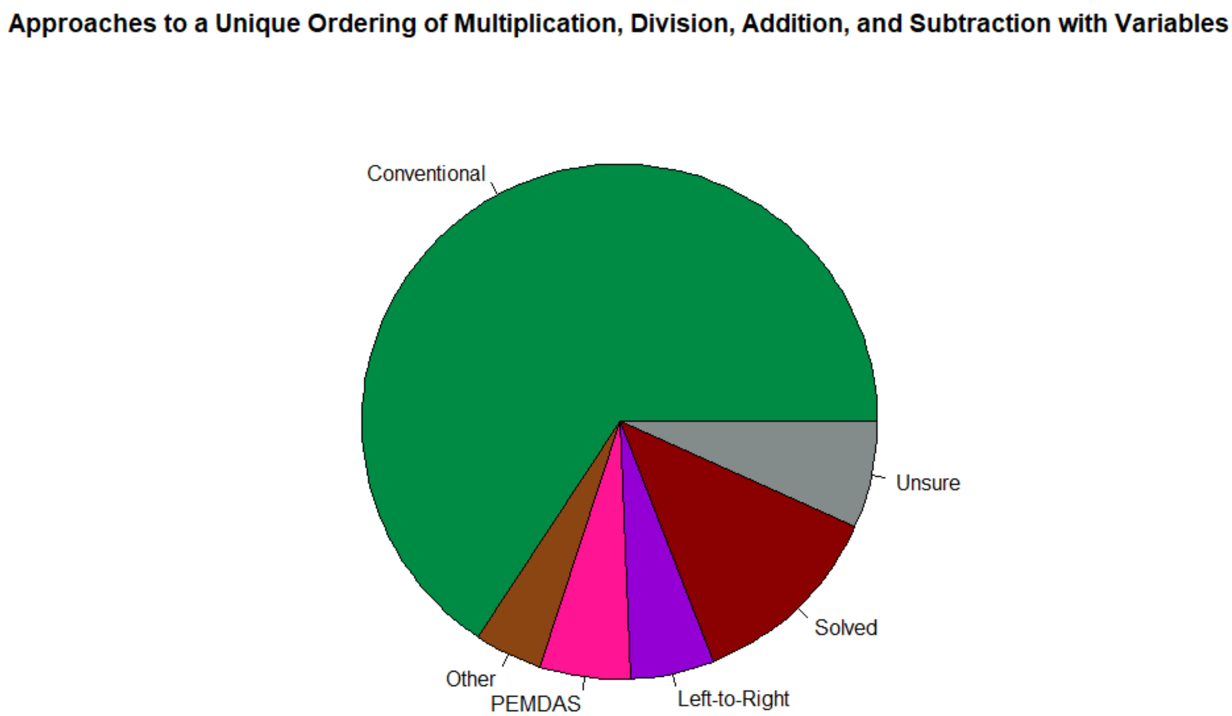


Figure 9. Approaches to a Unique Ordering of Multiplication, Division, Addition, and Subtraction with Variables.

Purposely Ambiguous

Table 9. Approaches to Purposely Ambiguous expressions.

Expression	Conventional	PEMDAS	Other	Solved	Unsure
2. 48 ÷ 2 (9 + 3)	31	34	4	0	1
5. 8 ÷ 2 (2 + 2)	36	33	0	0	1
7. 8 ÷ 2 (2 + *x*)	27	15	13	12	3

I predicted that the category with the most responses generated by using PEMDAS would be expressions that were purposely ambiguous. The data shows that this prediction was accurate. Expressions in this category averaged 9 unique responses. Expressions in this category were those that had equal weight amongst all approaches. Additionally, $48 \div 2(9 + 3)$ was the only expression that the conventional approach was not the most common. Nineteen of the participants used the term PEMDAS in their explanation for how they arrived at their response for $8 \div 2(2 + 2)$. It is important to note that the person who was unsure for both expressions 2 and 5 chose to respond with the value one obtains from applying the conventional order of operations and the value one obtains from applying PEMDAS. As multiple responses were submitted, this was categorized as unsure.

Morgan of Toronto recalled BODMAS for her response of 1 for expression 5.

Morgan: For order of operations I use BEDMAS which is the Canadian notion for Pemda, [*sic*] as such it requires opening the brackets first and performing the operation that results from opening the brackets which in this case is the 2(4). After that simple division results in the answer.

I was surprised by Morgan's reasoning as the mnemonic device BEDMAS contains division before multiplication. However, Morgan's response demonstrates that she chose to multiply before dividing. Jarrett also responded with 1 to this expression. However, his reasoning for his response showed he would have responded with 16.

Jarrett: I evaluate expressions in parentheses first, then any coefficients, then multiplication and division going left to right, then addition and subtraction going left to right.

Another participant, Kirk, had this to say in regard to the aforementioned expression:

Kirk: In my humble opinion, this expression is poorly written because the author doesn't explicitly demonstrate the intended order of operation. It would not surprise me if it was meant as 8 ÷ (2 * (2 + 2)), but the author lacked the due care when writing a division in a single line - on a blackboard the (2+2) term could be in the denominator. That said, strictly speaking, you solve multiplications and divisions from left to right, so it's (8 ÷ 2)(2 + 2) = 4*4.

I would agree with what Kirk said. This expression *is poorly written*! The variety in responses show how different people interpret expressions like these. Other participants recalled the conventional order of operations in their reasoning, but their responses were those obtained by using PEMDAS. Randy’s response to the fifth expression was 1, but 1 was the value that participants would determine if they followed PEMDAS. Randy showed that Multiplication and Division have the same priority. Randy’s reasoning for his response was the following:

Randy: PEMDAS: Parentheses, Exponents, Multiplication/Division, Addition/Subtraction.

As was a theme with expressions from the previous category, expressions with variables included tended to yield more unique responses than those that did not. The number of participants who followed the conventional approach was not as close to those who used the PEMDAS approach on this expression. Some of this drop-off is explained by twelve participants attempting to solve this expression. However, there was a considerable increase in participants applying a method I did not foresee. Several participants had noteworthy reasoning for this expression.

Duke: I chose this approach to be consistent with how I did the last problem, but now I'm annoyed at how these problems are written with ambiguity. I did 8/2=4 and then 4(2+x)=8+4x.

Frank: I multiplied out the parenthesis first, leaving me with 8 ÷ 4 + 2x. I then subtracted 2x from the equation giving me 8 ÷ 4 = -2x. I then divided each side by -2 giving me (8 ÷ 4) ÷ 2 = -x which reduced down to 2 ÷ 2 = -x further to 1 = -x and then x = -1

Bill: Similarly to before, without the multiplication sign between the 2 and the parenthesis, I assumed that everything after the division sign was the denominator.

I found it interesting how Bill interpreted the omission of the multiplication sign as remaining terms in the expression become the denominator.

Five participants applied PEMDAS on expression 2 and the conventional approach on expression 5. I expected several participants to change their approach in this way. However, two participants applied the conventional approach on expression 2 and PEMDAS on expression 5. This shift in perception is not something I expected to see as Gunnarsson, Sönnerhed, and Hernell (2015) showed students did not score significantly lower on the posttest compared to the pretest. Additionally, this seems different than what Linchevski and Livneh saw. This is not to necessarily say that these two participants regressed but merely to demonstrate it was unexpected. Thirty-three participants implemented a different approach on expressions 5 and 7. Although these questions were not immediately following each other, I did not expect to see nearly half of the sample change their perception on expressions that were so similar. However, this illustrates that what I perceived was similar does not necessarily mean the participants viewed it the same way. Some of this shift is explained by the presence of a variable as a noticeable number of participants attempted to solve for the unknown variable. However, this does not explain all of the shifting that occurred. Two participants followed the conventional approach on expression 5 and the PEMDAS approach on expression 7. Similarly, two participants followed the PEMDAS approach on expression 5 and the conventional approach on expression 7. Thirty-four participants used the same approach on all three of these expressions, thirty-two participants implemented two unique approaches, and four participants followed a different approach for each of the three expressions.

The following figures show the approaches taken to expressions in this category. Figure 10 contains all expressions, Figure 11 only contains expressions without variables, and Figure 12 only contains expressions with variables.

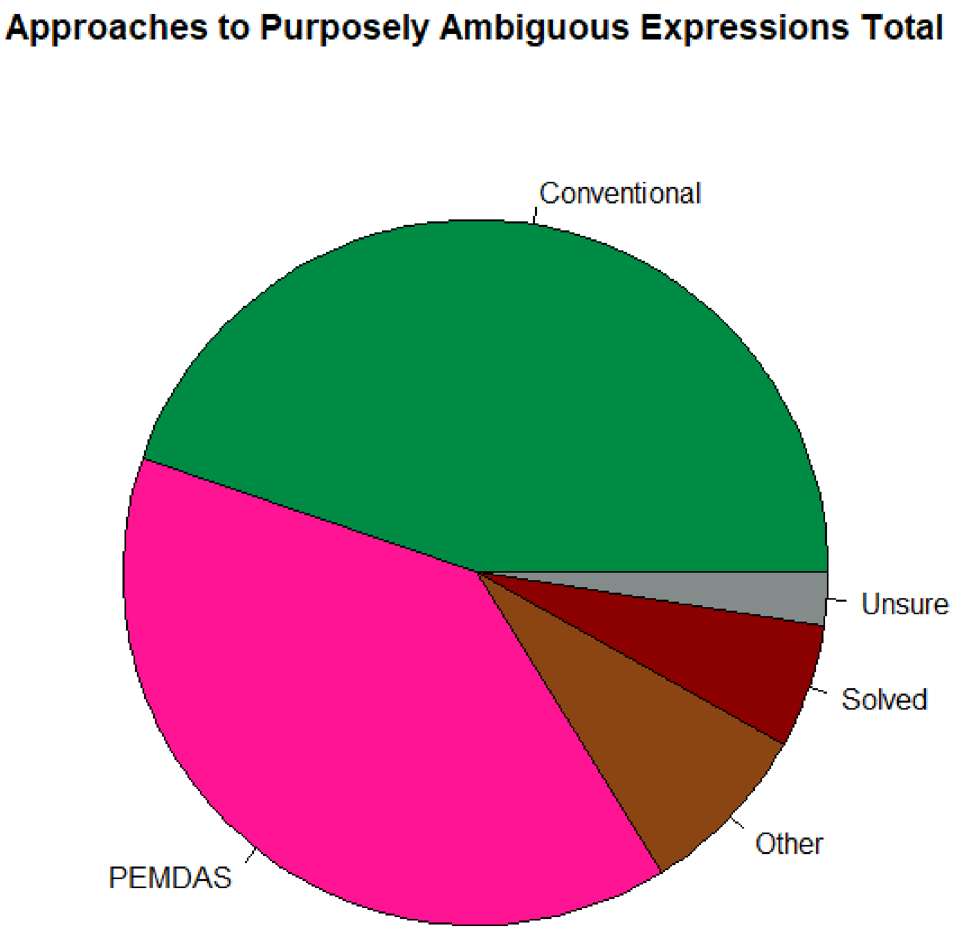


Figure 10. Approaches to Purposely Ambiguous Expressions Total.

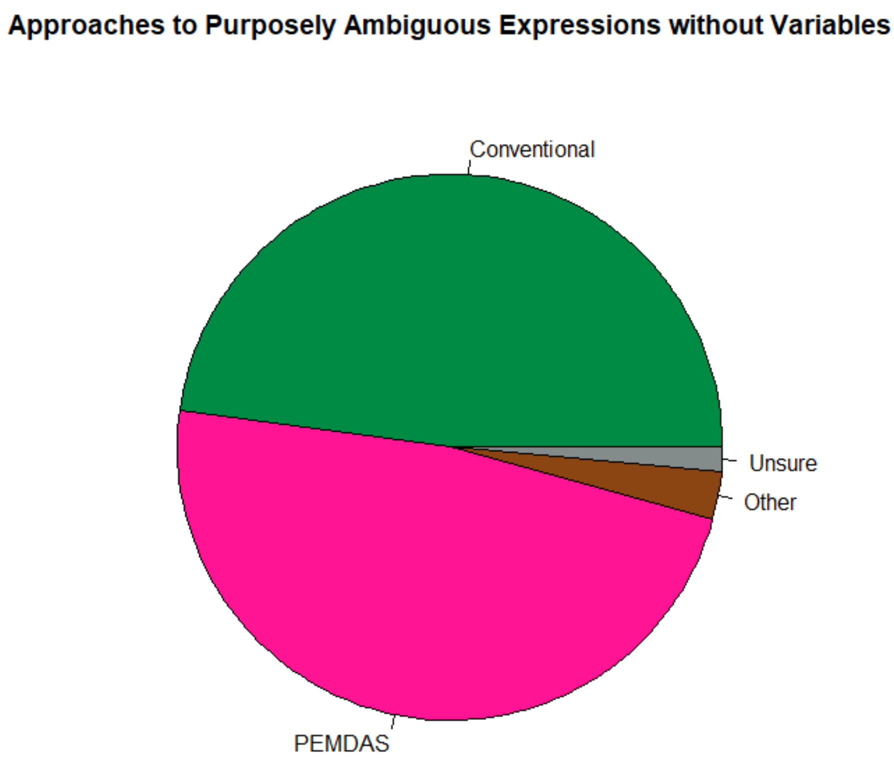


Figure 11. Approaches to Purposely Ambiguous Expressions without Variables.

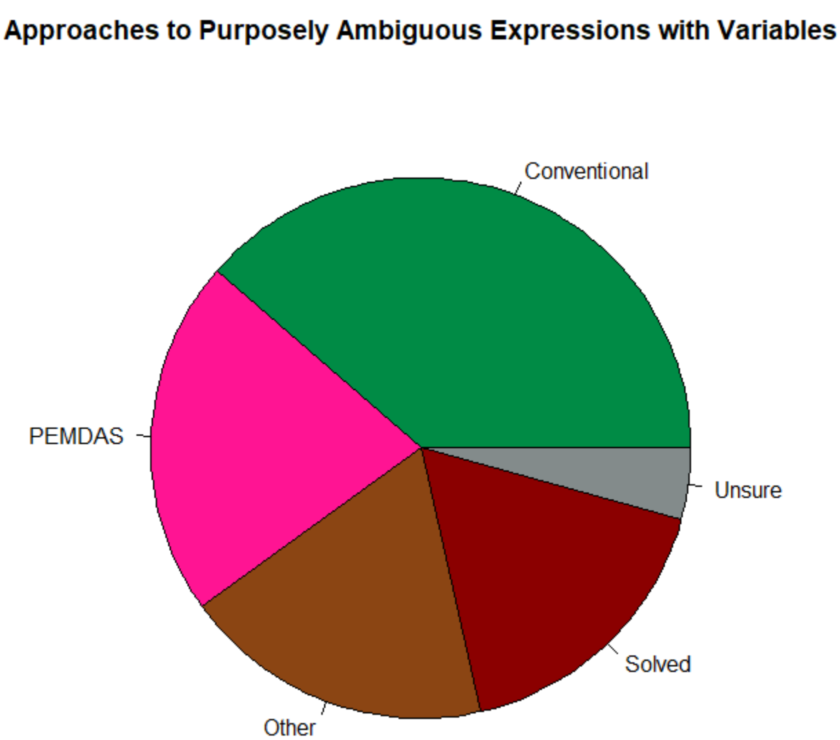


Figure 12. Approaches to Purposely Ambiguous Expressions with Variables.

Repeated Division

Expression	Conventional	Grouping	Other	Solved	Unsure
3. 16 ÷ 8 ÷ 4 ÷ 2	60	8	2	0	0
6. 72 ÷ 12 ÷ 4 ÷ 2	60	5	5	0	0
10. 16 ÷ *x* ÷ 4 ÷ 2	37	4	14	7	8

Table 10. Approaches to Repeated Division expressions.

This last category had the lowest variation in the number of unique responses. The expression in this category averaged seven unique responses. This was also the only category where sixty or more participants implemented the conventional approach in their reasoning. Expression 3, $16 \div 8 \div 4 \div 2$, yielded 4 unique responses: $0.25, 1, 2, and\ 2.5$. If participants used the conventional approach, they obtained 0.25. Participants who chose to implement some form of grouping obtained 1 as they treated $16 \div 8 \div 4 \div 2$ as $(16 \div 8) \div (4 \div 2)$. The latter expression simplifies to $(2) \div (2) = 1$.

Three participants used a grouping technique on both expressions 3 and 6, and one of those three grouped all three expressions in this category. This was also the only category where participants used a grouping technique. Thirty-five participants followed the conventional approach on all three expressions in this category.

The following figures show the approaches taken to expressions in this category. The top figure contains all expressions, the bottom-left only contains expressions without variables, and the bottom-right only contains expressions with variables.

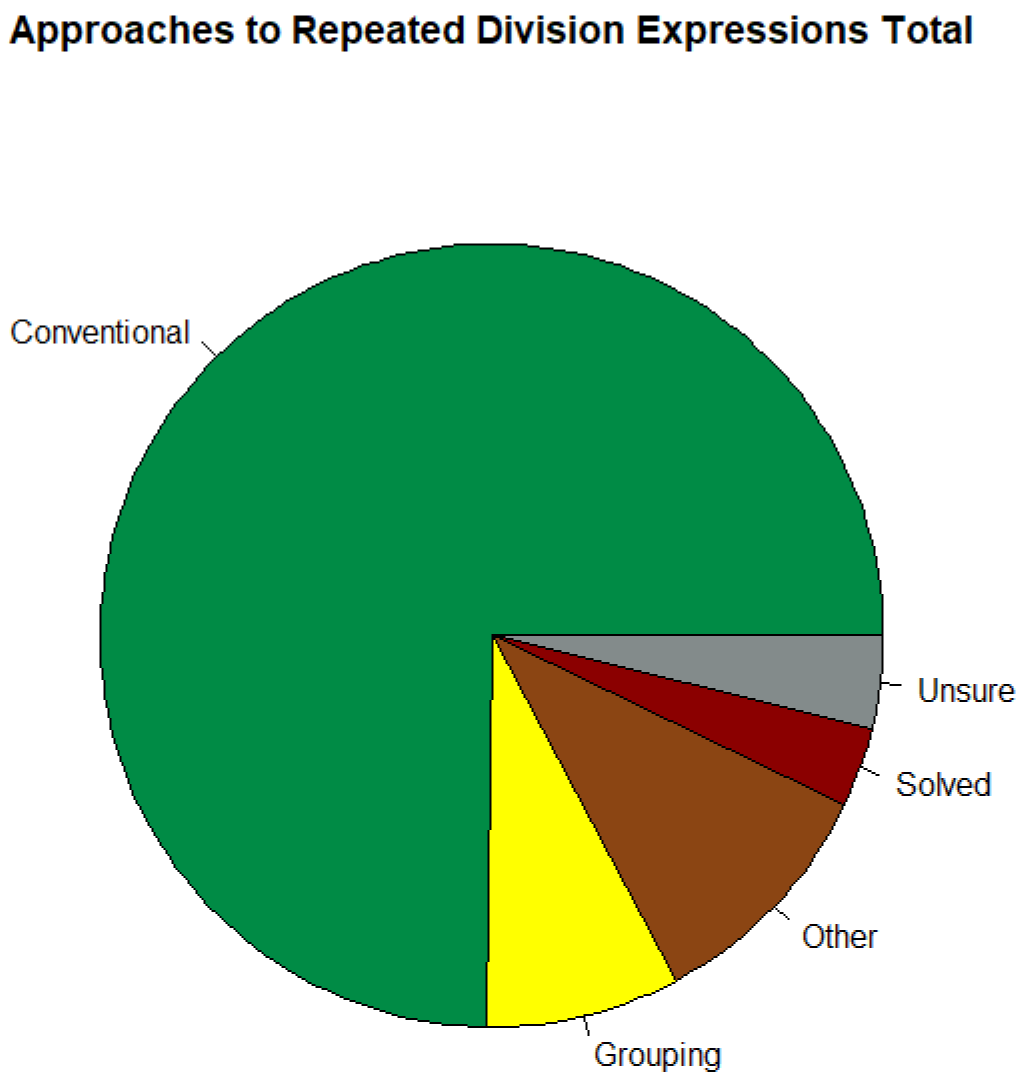


Figure 13. Approaches to Repeated Division Expressions Total.

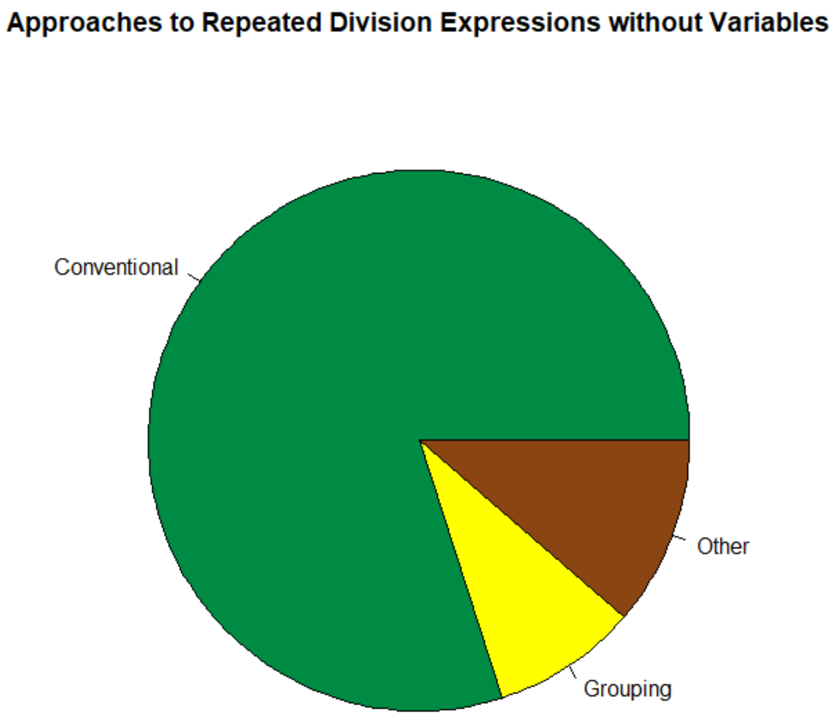


Figure 14. Approaches to Repeated Division Expressions without Variables.

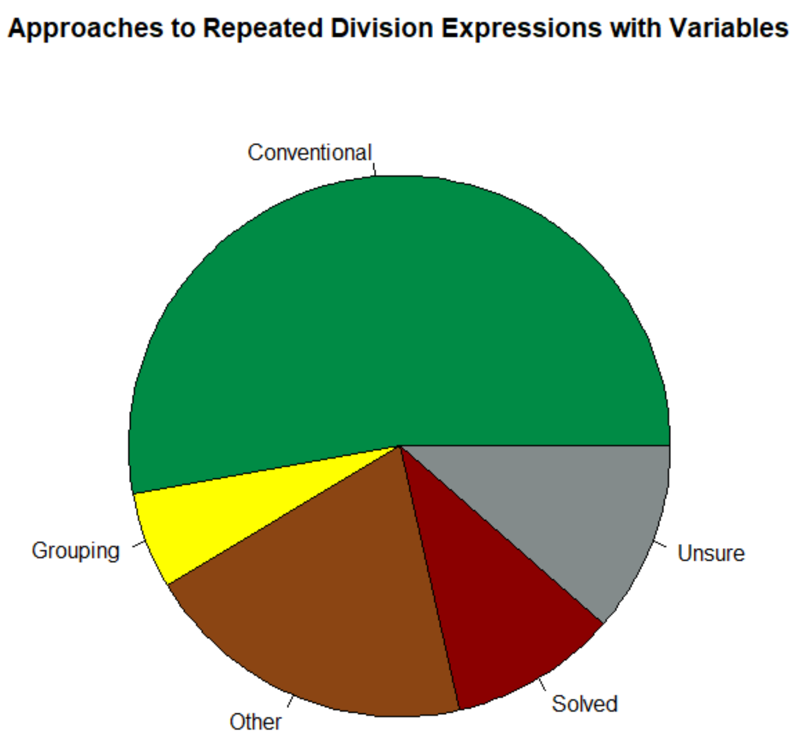


Figure 15. Approaches to Repeated Division Expressions with Variables.

Participant Highlights

This section provides a brief overview of the reasoning of several of the participants. The participants mentioned are Molly, Duke, Fred, and Velma. These participants were randomly selected from a list of several participants who provided detailed reasoning for their answers. Additionally, several participants were selected due to their differing demographics compared to several of other selected participants. While more than half of the participants were in the 21-29 category for age, significantly more than half of useable explanations were from this category.

Molly

Molly belonged to the 60 or older age demographic. She holds a graduate degree and the last time she took a math class was 1999. Molly used the conventional approach on expressions 1, 6, and 9, the PEMDAS approach on expressions 2 and 5, PEMDSA on expression 11, grouping on expression 3, and an approach that qualifies as other on expression 4. Molly attempted to solve for the unknown variable on expressions 7, 8, 10, 12, and 13. After responding with 2, the PEMDAS approach, on expression 5, Molly had this to say:

Molly: Parenthesis always come first, multiplication and division before addition and substraction [*sic*] ...I think!!

When evaluating expression 8, Molly attempted to solve for x. She had this to say afterwards:

Molly: Because I forgot how to do Algebra that I took in 1971.

Given how Molly took Algebra almost half a century ago, it is understandable that Molly claims to have forgotten several aspects of Algebra. Molly's responses show she is a firm believer in evaluating the inside of parentheses before anything else.

Molly was consistent with her approaches on purposely ambiguous expressions; she implemented PEMDAS on both expressions without variables. Molly also used the conventional approach on two of the unique ordering expressions that did not have variables. Additionally, she attempted to solve for the unknown variable on every expression a variable was present. However, that is where the consistency in her approaches ends. Molly followed a different approach on each of the repeated division expressions (grouping, conventional, and solving). Unfortunately, there is no explicit reasoning why she did this. My assumption would be due to Molly's claim that she has forgotten several aspects of algebra.

Duke

Duke belonged to the 21-29 age demographic. He holds a graduate degree and he took a math class this spring. Duke implemented the conventional approach on twelve of the thirteen expressions. The lone expressions he did not use the approach was expression 2. On this expression, Duke evaluated and obtained 2, the value used by following PEMDAS. Duke was one of the few participants who applied PEMDAS on expression 2 but applied the conventional approach on expression 5. If participants were given the ability to change previous responses, it would be interesting to see if Duke reevaluated expression 2 using the conventional approach. Duke was one of several participants who cited PEMDAS as their reasoning for their response while the response they entered suggests they used the conventional approach. After responding to expression 9, Duke had this to say:

Duke: I suppose standard PEMDAS is guiding my decisions. I did 14*6 is 84, and then 36+84 is 120 - 3/2 is 118.5

When asked to explain the difference between the mnemonic device PEMDAS and PE(MD)(AS), Duke demonstrated an understanding of the desired difference between the two:

Duke: Most people read PEMDAS as do all things with parenthesis, exponents, multiplication, division, addition, and subtraction in that order. The second tries to clarify that multiplication/division and addition/subtraction should be done together because they are the same "type" of operation. Ie. [*sic*] Whether I am multiplying by 1/4 or diviiding [*sic*] by 4, it should not change how I approach the problem.

Fred

Fred also belongs to the 21-29 category. He holds a bachelor’s degree and the last math class he took was in 2014. Like Duke before him, Fred followed the conventional approach on twelve of the thirteen expressions. Unlike Duke however, Fred used an unexpected other approach on expression 10. His response was $2x$. While it is possible he meant to type $2/x$, the value participants that used the conventional approach responded with, there was no explanation for how he arrived at his value. Like Duke, Fred demonstrated a knowledge of the conventional order of operations when he cited PEMDAS in his response:

Fred: I added the values inside the parentheses first, because PEMDAS. Then I worked left to right in order dividing 8 by 2 then multipling [*sic*] that result by 4 (the result of the equasion [*sic*] inside parentheses).

There were several instances where Fred was not confident in his responses, but he sufficiently explained how he arrived at the value. Fred responded with $8 + 4x$, the value participants that used the conventional approach responded with, to expression 7. His reasoning for his response was the following:

Fred: I straight up guessed. I mean, I kinda [*sic*] combined PEMDAS and FOIL and just "FOILed" the result of 8÷2 with (2+x).

Fred cited multiple mnemonic devices in his reasoning. He used these devices sometimes in their intended manner in these circumstances. However, it is difficult to know if Fred always uses these devices in the aforementioned manner.

Fred also demonstrated an understanding between the mnemonic devices PEMDAS and PE(MD)(AS):

Fred: I've never seen the second version. However, I would assume the parentheses indicate the enclosed operations be done simultaneously as opposed [*sic*] to doing multiplying before dividing.

Fred's response to the last free response question was also enlightening.

> When using his calculator to evaluate 8 ÷ 2 (2 + 2), Tommy obtained 1 as the answer. When using her calculator to evaluate 8 ÷ 2 (2 + 2), Gina obtained 16 as the answer. Please provide as many potential explanations as you can think of for why Tommy and Gina obtained different answers when they both used calculators to simplify the expression?

The following picture demonstrates the conundrum Tommy and Gina experienced:

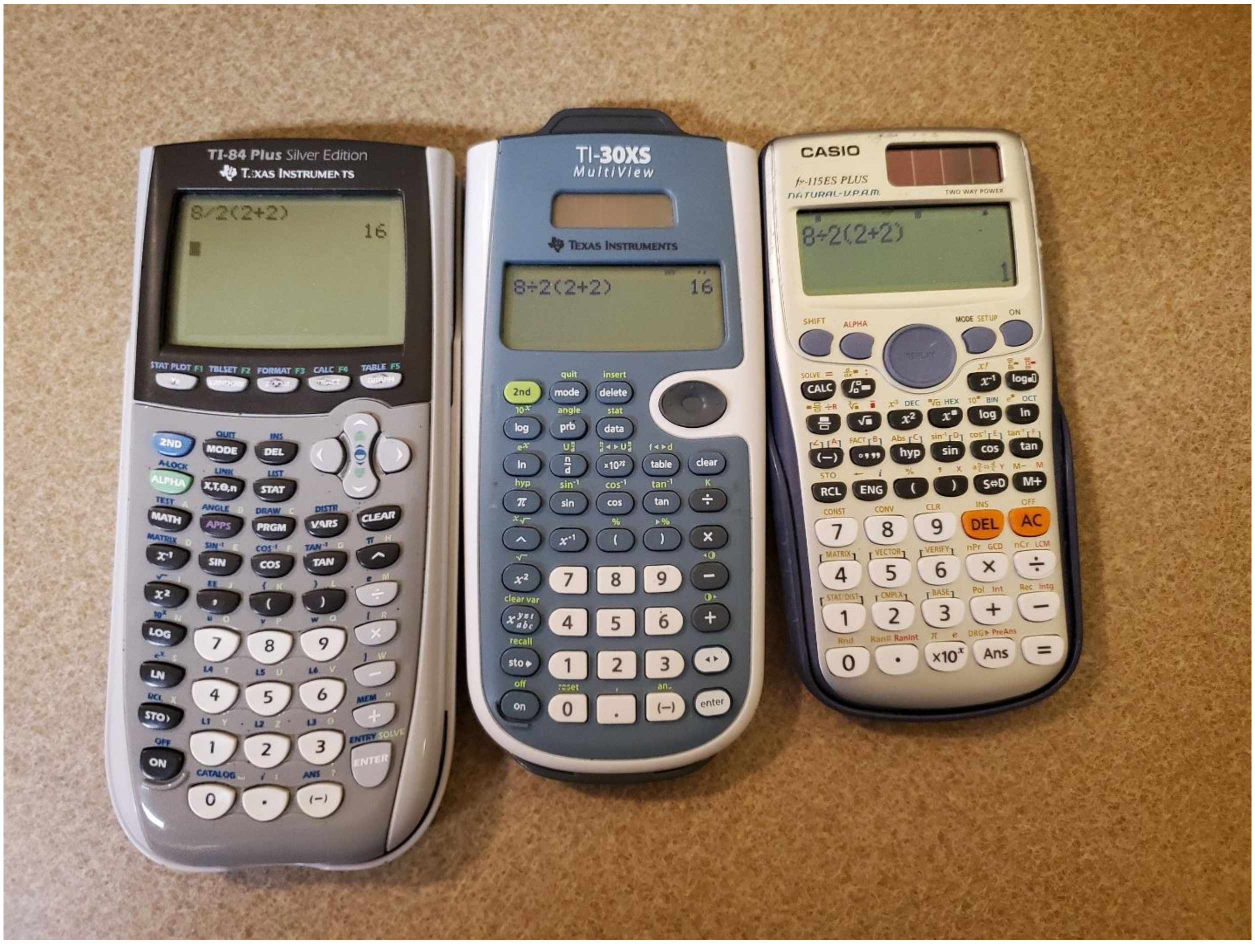


Figure 16. Calculators with Identical Inputs Result in Different Outputs

Fred: Tommy multiplied first, then divided 8 by the result. Functionally, 8 ÷ (2 (2 + 2)). Gina did things in the proper order, first adding the numbers in parentheses 2 + 2, then dividing 8 by 2, then multiplying the result by 4 (2 + 2).

Fred, like many other participants, stated Gina "did things in the proper order" implying he believes that there is a proper way to approach expressions like these. I found it interesting how Fred emphasized that it was Tommy's use of the calculator that resulted in his answer of 1. Fred, like most of the participants, did not mention the programming of the calculator. One can infer Fred believes this error is due to Tommy and not due to how the calculator was programmed.

Velma

The last participant I highlight in this section is Velma. Velma belongs to the 40-49 category. Velma holds a graduate degree and the last time she took a math class was in 2001. Like Duke and Fred, Velma followed the conventional approach on twelve of the thirteen expressions. The expression she did not use this approach on was expression 7. She responded with a response that demonstrated she followed PEMDAS:

Velma: The inside of the parentheses could not be simplified I did the distributive property then I express my answer as a fraction and reduced it

Velma was also one of the few participants that recited PEMDAS in their reasoning while responding with a value obtained by using the conventional approach. Several of Velma's reasoning responses were simply "Pemdas". Additionally, Velma provided another technique to alleviate with the memorization of PEMDAS. While most students in the United States learn PEMDAS as Please Excuse My Dear Aunt Sally, Velma suggested "Please Excuse My Drunk Ass Sister."

CHAPTER 5

DISCUSSION

In this chapter, I summarize and explain the participants' responses on the survey. First, I describe my inferences on the participant's responses. Next, I describe what I perceive to the main takeaways of the data. Lastly, I describe limitations of the study and propose areas that need further exploration.

Summary of Participants' Responses

The participants of my survey had far more unique responses than I anticipated. One can infer that even with what I perceived to be common approaches, there are still many more unique approaches one can take to evaluate an expression. Most participants used the same approaches for similar expressions. However, several participants implemented multiple approaches throughout the survey. Figure 6 from Chapter 4 illustrates this.

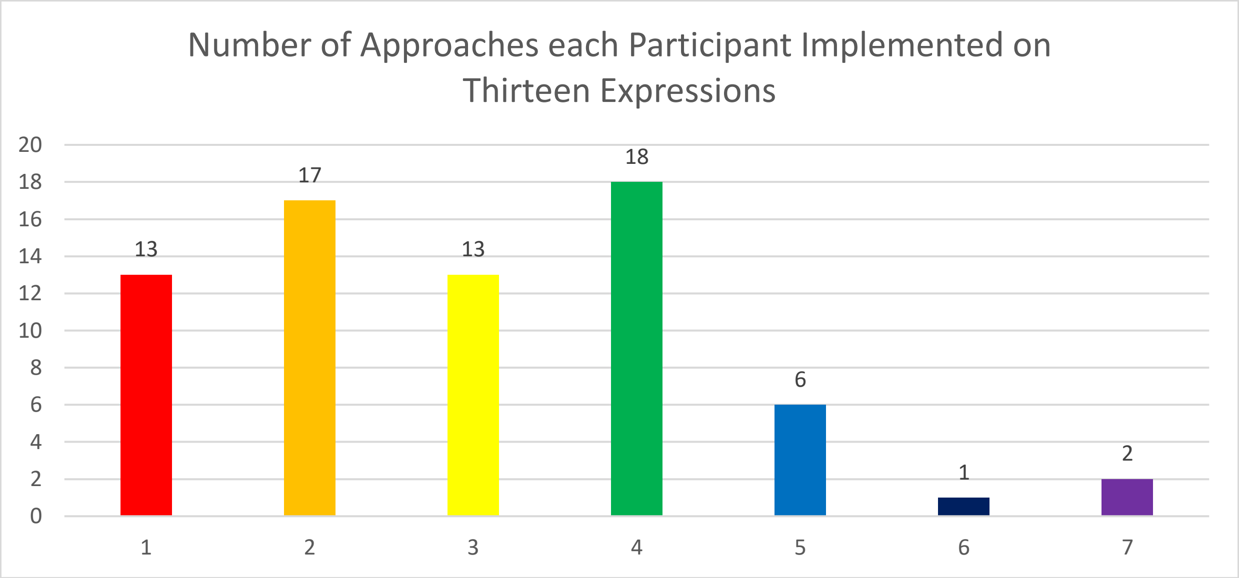


Figure 6. Number of Approaches each Participant Implemented on Thirteen Expressions.

Two participants followed seven different approaches on the survey; the largest number of unique approaches used. Every participant that used one unique approach on each of the thirteen expressions used the conventional approach, and all but one of the participants used the conventional approach at least once. Additionally, there seems to be a correlation between frequency of implementation of the conventional approach and the number of unique approaches implemented throughout the survey. Several participants noticed how similar expressions could be approached with a different method. The majority of participants that changed their approach used the conventional approach the second time around, but several participants implemented the conventional approach followed by an unconventional approach. Additionally, several participants implemented the conventional approach on expressions that were a unique ordering of multiplication, division, addition, and subtraction, but they implemented a different approach on repeated division expressions. This theme was consistent with several participants: they implemented different approaches on the three categories of expressions, but they used the same approach on each expression in the category.

As referenced in the last chapter, expression 4, $18 + 2 \div 3 * 5 - 7$, had *twenty* unique responses. The expression with the second most unique responses that did not have a variable was expression 9, $36 + 14 * 6 - 3 \div 2$. Expression 9 yielded twelve unique responses. Like expression 4, the majority of responses to expression 9 were not integers. I hypothesize that this large number of unique responses on expressions like these is due to how many of the responses were not integers. On expressions without variables with a majority of non-integer responses, there were sixteen unique responses on average. On expressions without variables with a majority of integer responses, there were eight unique responses on average. Only seven of the responses to expression 4 were integers. Seven responses to expression 9 were integers as well. While some of this variation in unique responses is explained by rounding (a response of 14.4 was treated the same as 14.33, the response obtained from using the conventional approach but a response of 14.5 was considered different than 14.33), this does not account for all of the variation. As I alluded to earlier, I believe the increase in the number of unique responses is due to the non-integer values obtained. I hypothesize that if participants were allowed to use a calculator, this increase would not be as significant. However, figure 16 demonstrates why the use of a calculator is not ideal.

Main Takeaways

The results from this data showed multiple pieces of information. I did not expect to see such a significant shift in ideology when participants were introduced to expressions with variables. Not only did I not foresee the noteworthy number of participants who chose to solve for the variable, but the increase in different approaches was also surprising. Expressions with variables averaged nearly eleven unique responses while those without variables averaged slightly less than nine responses. The sample size was not large enough to identify this difference as statistically significant, but it is still interesting nevertheless.

As participants had more unique approaches on expressions involving variables compared to those that did not, it causes me to ponder about how each participant thinks about variables. Based off of the data, it seems that a substantial number of participants always believe a variable is something that needs to be solved and it has no other purpose. As several of the participants applied the conventional order of operations to expressions with variables, it appears they start off with the conventional approach but attempt to solve for the unknown variable when they have "combined like terms." When asked if Order of Operations still applied to expressions with variables, Eddie had this to say:

Eddie: Order of Operations is still relevant, but often you may need to combine like terms via addition/subtraction before you can find the answer

When asked the same question, Patrick responded with the following:

Patrick: yes, you have to solve for those variables in the same order of operations

While this sample is not necessarily representative of how the average student thinks about variables, responses like Eddie's and Patrick's show there are people that believe that expressions with variables need to be treated differently than those that do not.

Additionally, expressions that are purposely ambiguous such as $8 \div 2(2 + 2)$ and $48 \div 2(9 + 3)$ do not have any real benefit to testing student knowledge of various facets of mathematics. As Miller (2017) referenced, various mathematical journals from 1892-1929 recommend avoiding expressions that use both multiplication and division due to the ambiguity. In addition to the techniques mentioned in the aforementioned journals, there are other techniques educators can take to prevent this ambiguity. As Kirk referenced in the previous chapter, "In my humble opinion, this expression is poorly written because the author doesn't explicitly demonstrate the intended order of operation." This ambiguity is essentially the mathematical equivalent of asking a student to spell the word "weather" and providing it in a sentence with "I don't know whether the weather will improve" (Jean and Kruse 2003).

Expressions like these are designed to result in multiple meritorious responses. Both of the aforementioned expressions appeared in the survey. Both expressions yielded more than thirty participants implementing the conventional approach and more than thirty participants implementing the PEMDAS approach. Additionally, no other expression had two approaches with more than fifteen participants each. When a question has this much divide in its responses, it is necessary to reevaluate the true purpose of the question. While expressions like these can cause controversy on social media, their classroom applications are limited.

As many researchers (Zazkis and Rouleau, 2017; Linchevski & Livneh, 1999) suggested, mnemonic devices such as PEMDAS can have unintended effects on students understanding. These devices suggest multiplication must *always* come before division and addition must *always* come before subtraction. Several participants agreed with this mindset. When asked the last free response question (listed on page 45), Cyrus had this to say:

Cyrus: Tommy (the correct answer) multiplied the 2*4 prior to dividing it into 8. Gina divided 8 by 2 first then multiplied that by 4. Multiplication should be done before division.

Although division occurred before multiplication in the operation, Cyrus stated “Multiplication should be done before division.”

Instructional Implications

Expressions such as $8 \div 2(2 + 2)$ should be avoided in instruction due to their purposeful ambiguity. While students may need to evaluate an ambiguous expression in a future math course or their career, these ambiguous expressions should not be emphasized when students learn the conventional order of operations for the first time. Furthermore, a greater emphasis needs to be placed on teaching order of operations in a manner that treats it as a convention. Reiterating Moore’s (2019) definition, what a student perceives as a convention might not be consistent with what another person (or teacher) perceives as a convention. Or, what a person or teacher perceives as a convention might instead be perceived as a rule by a student. This greater emphasis would allow students to garner a more robust understanding of variables, equations, etc. In turn, this would allow them to identify different mathematical objects. Moreover, a greater emphasis on treating order of operations as a convention would reduce unintended learning. As referenced in chapter 2, there are many different techniques to aid in student understanding of the conventional order of operations. As mnemonic devices such as PEMDAS and BODMAS can often have unintended learning, restructuring these mnemonic devices would be beneficial for future teaching. As Zazkis and Rouleau suggested, implementing the devices as $PE\frac{M}{D}\frac{A}{S}$ or $PE(MD)(AS)$ could allow student to understand that multiplication does not always come before division and addition does not always come before subtraction. Additionally, the omission of the mnemonic devices would also aid in this unintended learning. Instead of providing students with the mnemonic device, educators could provide students with the list of the conventional order.

1. Parenthesis
2. Exponents
3. Multiplication or Division. The leftmost operation occurs first.
4. Addition or Subtraction. The leftmost operation occurs first.

While some students might struggle in gaining an understanding on the conventional order of operations due to the lack of a mnemonic device, the omission of a mnemonic device prevents the unintended learning that currently occurs due to its presence. Please Excuse My Dear Aunt Sally has been in use as a tool for alleviating memorization with this conventional order, but it is apparent our Dear Aunt Sally has been deceiving students.

Limitations and Further Research

Lastly, the research from this thesis and the data itself helped change my perspective on this topic. I originally was interested why a large number of people would get the "wrong answer" when responding to expressions similar to the aforementioned. The research gathered and the data gave me a sense of clarity. This is not necessarily a "problem" that needs fixing. The different approaches participants implemented in the survey showed that although participants responded to expressions differently, each participants' approach has its own merit and should not be simply disregarded. I alluded to student understanding of variables earlier in this chapter. Future research needs to occur to understand how students think of variables. As participants were informed the survey consisted of expressions and not equations, I believe this research should focus on how students think of variables in expressions and not equations. Additionally, research comparing how students perceive expressions and equations is necessary. It is quite possible the participants who attempted to solve for the unknown variable viewed expressions and equations as synonyms and not different mathematical objects. Future research on the aforementioned will assist in determining if any instructional change is necessary. Additionally, this research gave me a new understanding on conventions. What a student perceives as a convention is not necessarily perceives as a convention by a teacher, and what a teacher perceives as a convention might be perceived as an absolute rule by a student. The results also helped me gain a stronger understanding on how students will approach expressions like this in the future. These new understandings on conventions and practice will help me become a better educator.

This thesis was completed during a very unique and challenging time in all of our lives. Due to this, there was an abundance of limitations. Participants did not always provide explicit reasoning for their logic. This caused several inferences to be made. Additionally, I was unable to truly understand why participants think the way they do as I did not wish a large number of participants to spend several hours completing this survey. A study of this sort would have done better with interviews. This would have provided me with additional information on why the participants think the way they do. I would not have felt as worried about participants spending copious time on this survey if it was completed through an interview. Additionally, there is future research necessary on many facets of order of operations. While expressions that are purposely ambiguous are not a good measuring-stick for student understanding of order of operations, there will be copious instances where students will need to not have an erroneous understanding of the order of operations. “Given the computer-driven world in which we live, the implications of this erroneous understanding could have a potentially devastating impact on businesses whose employees are required to use spreadsheet programs” (Pappanastos, Hall, Honan, 2002, p.81). An example of one of these potentially devastating impacts on businesses occurred in 2003. A bridge was built across the Rhine River between Laufenburg, Germany and Laufenburg, Switzerland. This bridge was built simultaneously from each side. Both countries used a different reference point for sea level. The difference in reference points was known and a calculation occurred to counter this difference in reference points. Unfortunately, the calculation was doubled instead of eliminated due to improper computations. This resulted in additional time, additional money, and certainly additional frustration (Lewis, 2015).

As there seemed to be noticeably more unique responses on expressions with variables compared to expressions that did not have variables, it is apparent more research is necessary on how students interact with expressions with variables present. As previously mentioned, it will be interesting to see how students interact with expressions compared to how they interact with equations. Additionally, further research is needed to truly understand why expressions with non-integer responses yielded far more unique responses compared to those that yielded integer responses. A stronger understanding of students' number sense is desired.

References

Bell, A., Swan, M. and Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics* 12, 399–420.

Gunnarsson, R., Sönnerhed, W. W., & Hernell, B. (2015). Does it help to use mathematically superfluous brackets when teaching the rules for the order of operations? *Educational Studies in Mathematics, 92*(1), 91-105. doi:10.1007/s10649-015-9667-2

Jean, A. (Producer), Kruse, N. (Director). (2003) *The Simpsons: I'm spelling as fast as I can* [Television Broadcast]. Los Angeles, CA: Twentieth Television Inc.

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education, 20*(3), 274-287. doi:10.2307/749516

Knill, O. (2014, April 9). *Ambiguous pemdas*. http://people.math.harvard.edu/~knill//pedagogy/ambiguity/index.html

Lee, M. A., & Messner, S. J. (2000). Analysis of concatenations and order of operations in written mathematics. *School Science and Mathematics*, *100*(4), 173-180. doi:10.1111/j.1949-8594.2000.tb17254.x

Lewis, H. (2015). Math mistakes that make the news. *PRIMUS*, *25*(2), 181-192. doi:10.1080/10511970.2014.928657

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40,* 173-196.

Miller, J. (2017, June 2). *Earliest uses of grouping symbols*. http://jeff560.tripod.com/grouping.html

Miller, J. (2017, June 2) *Earliest uses of symbols of operation*. http://jeff560.tripod.com/operation.html

Moore, K. C., Silverman, J., Paoletti, T., Liss, D., & Musgrave, S. (2019). Conventions, habits, and U.S. teachers' meanings for graphs. *The Journal of Mathematical Behavior, 53*, 179-195. doi:10.1016/j.jmathb.2018.08.002

Pappanastos, E., Hall, M. A., & Honan, A. S. (2002). Order of operations: Do business students understand the correct order? *Journal of Education for Business, 78*(2), 81-84. doi:10.1080/08832320209599702

Peterson, D. (2000, November 22). *History of the order of operations*. http://mathforum.org/library/drmath/view/52582.html

Zazkis, R., & Rouleau, A. (2017). Order of operations: On convention and met-before acronyms. *Educational Studies in Mathematics, 97*(2), 143-162. doi:10.1007/s10649-017-9789-9

Appendix A

UNIVERSITY OF GEORGIA

INFORMED CONSENT FORM

Advancing Students' Quantitative Reasoning

STATEMENT

We are asking you to participate in a research study titled "Advancing Students' Quantitative Reasoning" conducted by Kevin C. Moore from the Department of Mathematics and Science Education at the University of Georgia (542-3211) and funded by the National Science Foundation (DRL-1350342). Your participation is voluntary. You can refuse to participate or stop taking part without giving any reason, and without penalty or loss of benefits to which you would otherwise entitled. Any educational grades, class standing, or professional standing will not be affected whether you choose to participate or not participate. Please read the following information carefully, and please ask the researcher if there is anything that is not clear or if you need more information. Please keep a copy of this narrative for your records.

PI

Kevin C. Moore; Department of Mathematics & Science Education; kvcmoore@uga.edu; 706.542.3211

PURPOSE

The reason for this study is to explore understand peoples' quantitative reasoning in the context of secondary mathematics ideas. We are particularly interested all individual's reasoning, and hence our request for your participation.

PROCEDURES

If you volunteer to take part in this study, you will be asked to do the following things:

1. Complete a survey of mathematics problem and allow your responses to be recorded. There are approximately 10-15 problems, and we estimate the survey will take at most 15- to 30-minutes. During the survey, you will be asked to solve mathematical problems involving secondary mathematics topics.

You can end the survey at any point in order to terminate your participation in the study.

BENEFITS

The benefit for you is that you may develop a deeper understanding of quantitative reasoning and mathematics by answering open-ended questions. In addition, your participation in this research may lead to improvements in curriculum design, which will subsequently improve the quality of education for students, and hopefully their teaching practices. This research may lead to changes in how mathematics and mathematics education courses are structured and conducted.

DISCOMFORTS & RISKS

Aside from any initial discomfort of working math problems in an online environment, no risks or discomforts are anticipated.

INCENTIVES FOR PARTICIPATION

You will not be financially compensated for your efforts and time.

CONFIDENTIALITY

No information that identifies you will be shared with others without your written consent, unless otherwise required by law. You will be assigned a pseudonym, and this pseudonym will be used to identify your assignments and in all interview transcripts. The code key which will be used to link your pseudonym to your real name will be kept indefinitely so that the researchers can recruit you to participate in future research on this topic. Furthermore, any publication from the study will use pseudonyms. Any data gathered will be stored in a locked office.

TAKING PART IS VOLUNTARY

Your involvement in the study is voluntary, and you may choose to terminate your participation at any time without penalty or loss of benefits to which you are otherwise entitled. If you decide to withdraw from the study, the information that can be identified as yours will be kept as part of the study and may continue to be analyzed, unless you make a written request to remove, return, or destroy the information.

CONFIDENTIALITY

No information that identifies you will be shared with others without your written consent, unless otherwise required by law. You will be assigned a pseudonym, and this pseudonym will be used to identify your assignments and in all interview transcripts. The code key which will be used to link your pseudonym to your real name will be kept indefinitely so that the researchers can recruit you to participate in future research on this topic. Furthermore, any publication from the study will use pseudonyms. Any data gathered will be stored in a locked office

RESEARCH SUBJECT'S CONSENT TO PARTICIPATE IN THE RESEARCH:

By clicking the "Continue" button on the screen, you acknowledge that you have read this information and agree to participate in the research to complete a survey of mathematics problems, with the knowledge that you are free to withdraw your participation at any time without penalty.

Please answer the following questions about your demographics:

Which category below includes your age?

17 or younger

18-20

21-29

30-39

40-49

50-59

60 or older

If you are currently employed, what is your profession? If you are not currently employed, please leave this field blank.

What is the highest level of school you have completed or the highest degree you have received?

Less than high school degree
High school degree or equivalent (e.g., GED)
Some college but no degree
Associate degree
Bachelor degree
Graduate degree

Which of the following math classes have you taken? Select all that apply.

Algebra
Algebra 2
Trigonometry
Precalculus
Calculus
Advanced Calculus
Number Theory
Abstract Algebra
Other undergraduate/graduate level math courses

When was the last time you took a mathematics course? (approximate year)

Where are you located? (city, state)

The following pages will consist of various expressions. Please evaluate each expression and enter your answer in an integer, fraction, or decimal form. For repeating decimals, please enter three decimal places.

Examples:
Integer: 42, 10, -5
Fraction: 10/3, 4/7, -3/5
Decimal: 1.435, -0.667, 0.250

After each expression will be several free response questions. Please answer each question in complete sentences.

Note: The symbol " * " will be used for multiplication.
Example: 2 * 3 = 6

PLEASE DO NOT USE A CALCULATOR

PLEASE DO NOT USE A CALCULATOR

1. What is the value of 18 ÷ 2 * 3 - 5 + 7?

PLEASE DO NOT USE A CALCULATOR

2. What is the value of 48 ÷ 2 (9 + 3)?

PLEASE DO NOT USE A CALCULATOR

3. What is the value of 16 ÷ 8 ÷ 4 ÷ 2?

PLEASE DO NOT USE A CALCULATOR

4. What is the value of 18 + 2 ÷ 3 * 5 - 7?

PLEASE DO NOT USE A CALCULATOR

5. What is the value of 8 ÷ 2 (2 + 2)?

Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason.

PLEASE DO NOT USE A CALCULATOR

6. What is the value of 72 ÷ 12 ÷ 4 ÷ 2?

PLEASE DO NOT USE A CALCULATOR

7. What is the value of 8 ÷ 2 (2 + *x*)?

Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason.

PLEASE DO NOT USE A CALCULATOR

8. What is the value of $3 + 6 * x^2 \div x + (-4)$?

Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason.

PLEASE DO NOT USE A CALCULATOR

9. What is the value of $36 + 14 * 6 - 3 \div 2$?

Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason.

PLEASE DO NOT USE A CALCULATOR

10. What is the value of $16 \div x \div 4 \div 2$?

PLEASE DO NOT USE A CALCULATOR

11. What is the value of 72 ÷ 12 * 3 - 4 + 2?

Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason.

PLEASE DO NOT USE A CALCULATOR

12. What is the value of 18 ÷ 2 * 3 - 5 + *x*?

Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason.

PLEASE DO NOT USE A CALCULATOR

13. What is the value of 36 + 14 * *y* - 3 ÷ 2?

Why did you choose to take the approach you did? Please answer in a few sentences describing your approach, both in what you did and why you made that choice if you have a clear reason.

Please place parenthesis around the expression 16 ÷ 8 ÷ 4 ÷ 2 to demonstrate the order in which you approached this expression.

When Arnold simplified 18 ÷ 2 * 3 - 5 + 7, he obtained an answer of 5. How do you think Arnold obtained 5?

Do you agree or disagree with his strategy?

Do the rules of Order of Operations still apply to expressions with variables involved? If not, how should these be approached?

If sets of parentheses are placed around the expression 72 ÷ 12 ÷ 4 ÷ 2 to make it 72 ÷ 12 ÷ (4 ÷ 2), would that change the answer? If not, explain.

What, if anything, is the difference between the following:

$(-6)^2$, -6^2, (6^2), and (-6^2)

Consider 18 ÷ 2 * 3 - 5 + 7 and 18 ÷ 2 * (3 - 5) + 7. Do these have the same value? Explain.

In your own words, explain the difference between the following two mnemonic devices:

PEMDAS PE(MD)(AS)

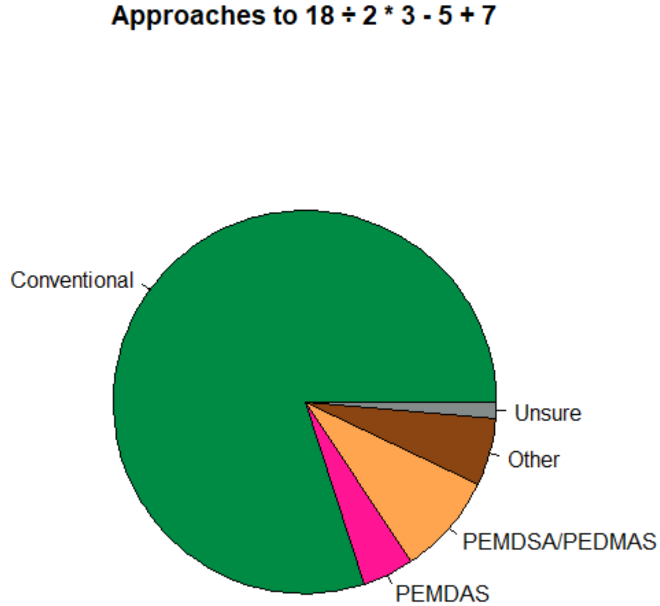
If you are unfamiliar with these, please write so.

When using his calculator to evaluate 8 ÷ 2 (2 + 2), Tommy obtained 1 as the answer. When using her calculator to evaluate 8 ÷ 2 (2 + 2), Gina obtained 16 as the answer. Please provide as many potential explanations as you can think of for why Tommy and Gina obtained different answers when they both used calculators to simplify the expression?

Powered by Qualtrics

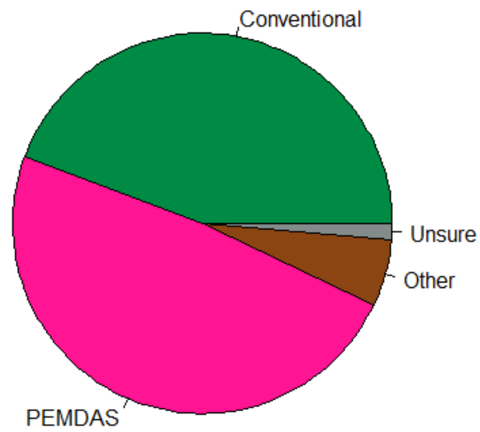
Appendix B

Expression 1: 18 ÷ 2 * 3 - 5 + 7
Expression Category: Unique ordering of Multiplication, Division, Addition, and Subtraction



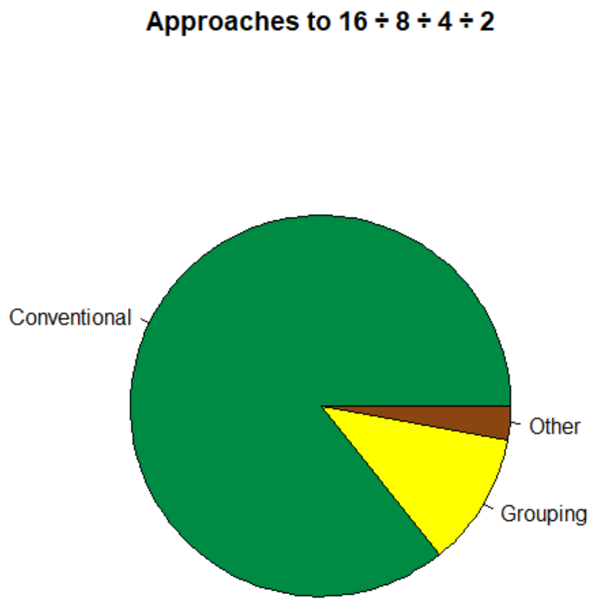
Approach	Conventional	PEMDAS	PEMDSA/PEDMAS	Other	Unsure
Number of Participants	56	3	6	4	1

Expression 2: 48 ÷ 2 (9 + 3)
Expression Category: Purposely Ambiguous
Approaches to 48 ÷ 2 (9 + 3)



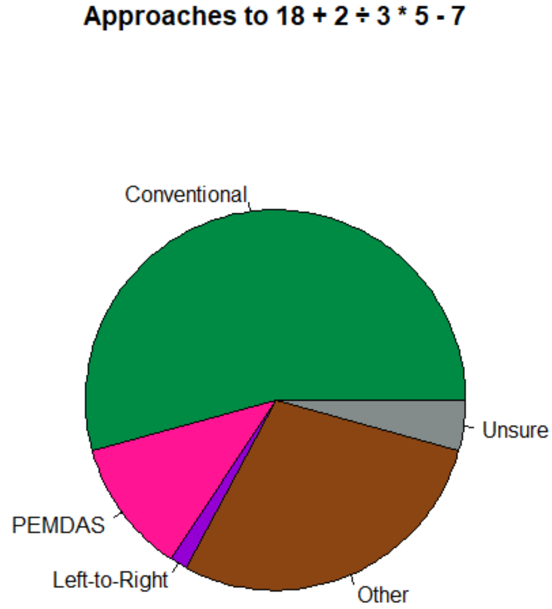
Approach	Conventional	PEMDAS	Other	Unsure
Number of Participants	31	34	4	1

Expression 3:
Expression Category: Repeated Division



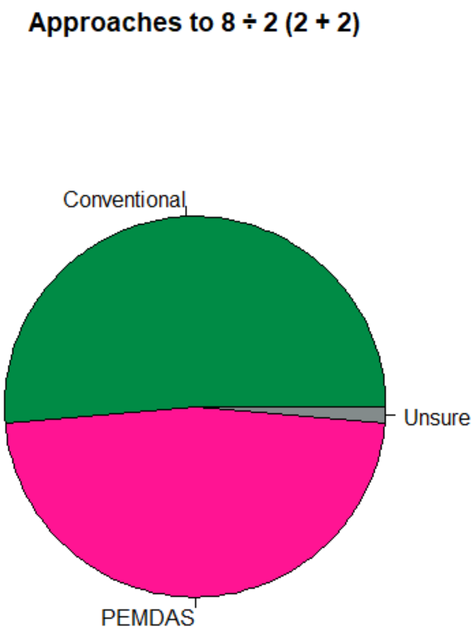
Approach	Conventional	Grouping	Other
Number of Participants	60	8	2

Expression 4: 18 + 2 ÷ 3 * 5 – 7
Expression Category: Unique ordering of Multiplication, Division, Addition, and Subtraction



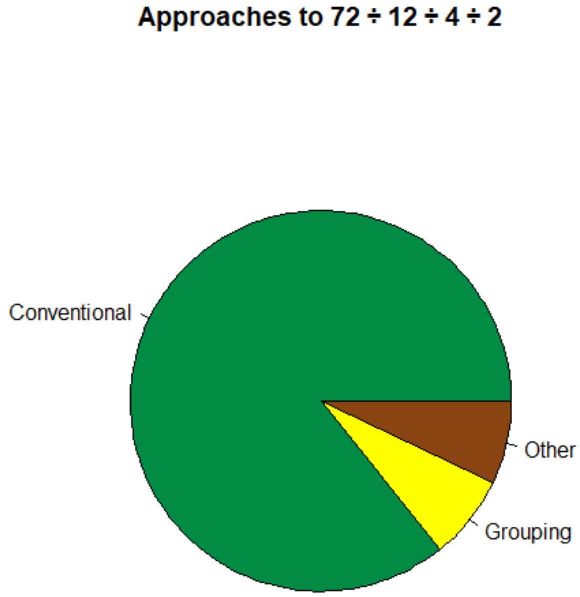
Approach	Conventional	PEMDAS	Left-to-Right	Other	Unsure
Number of Participants	38	8	1	20	3

Expression 5: 8 ÷ 2 (2 + 2)
Expression Category: Purposely Ambiguous



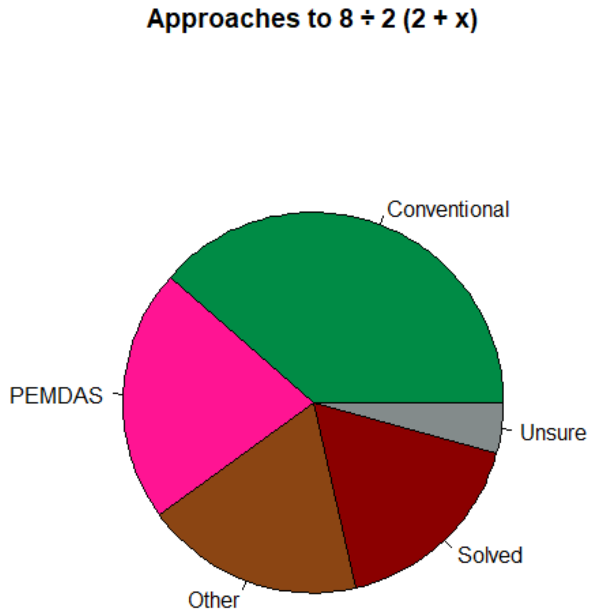
Approach	Conventional	PEMDAS	Unsure
Number of Participants	36	33	1

Expression 6: 72 ÷ 12 ÷ 4 ÷ 2
Expression Category: Repeated Division



Approach	Conventional	Grouping	Other
Number of Participants	60	5	5

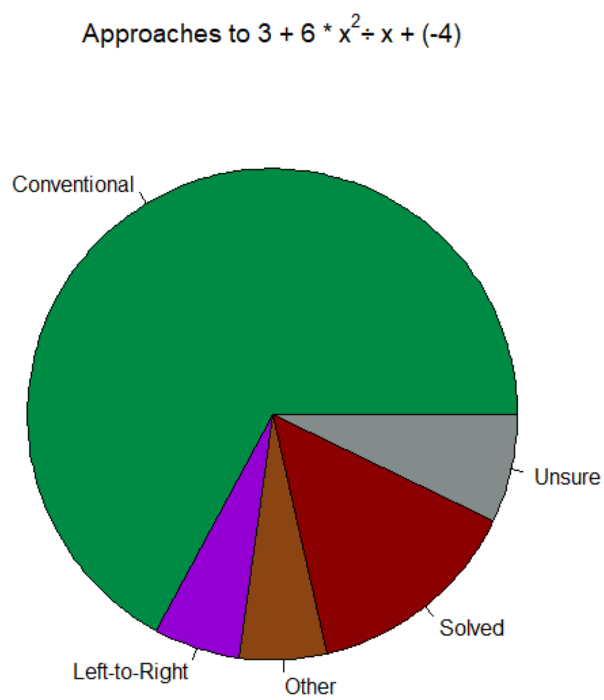
Expression 7: 8 ÷ 2 (2 + *x*)
Expression Category: Purposely Ambiguous with Variable Inclusion



Approach	Conventional	PEMDAS	Other	Solved	Unsure
Number of Participants	27	15	13	12	3

Expression 8: 3 + 6 * x^2 ÷ x + (-4)

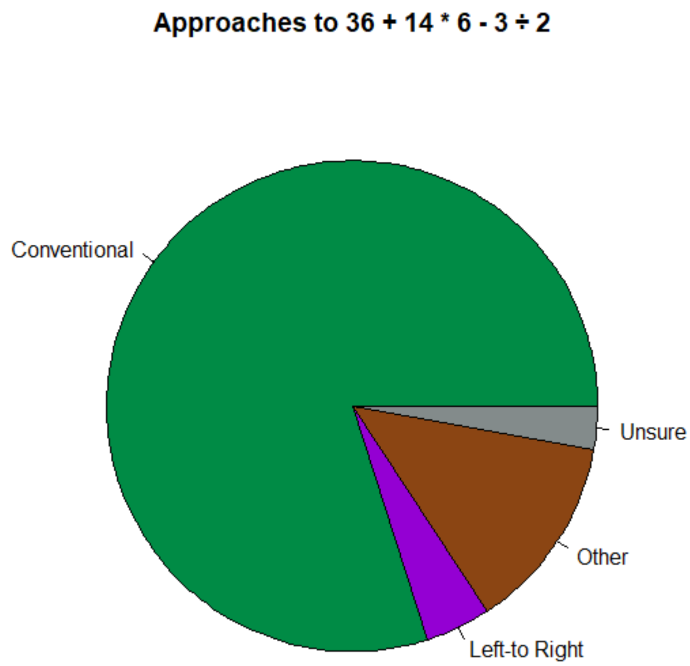
Expression Category: Unique ordering of Multiplication, Division, Addition, and Subtraction with Variable Inclusion



Approach	Conventional	Left-to-Right	Other	Solved	Unsure
Number of Participants	47	4	4	10	5

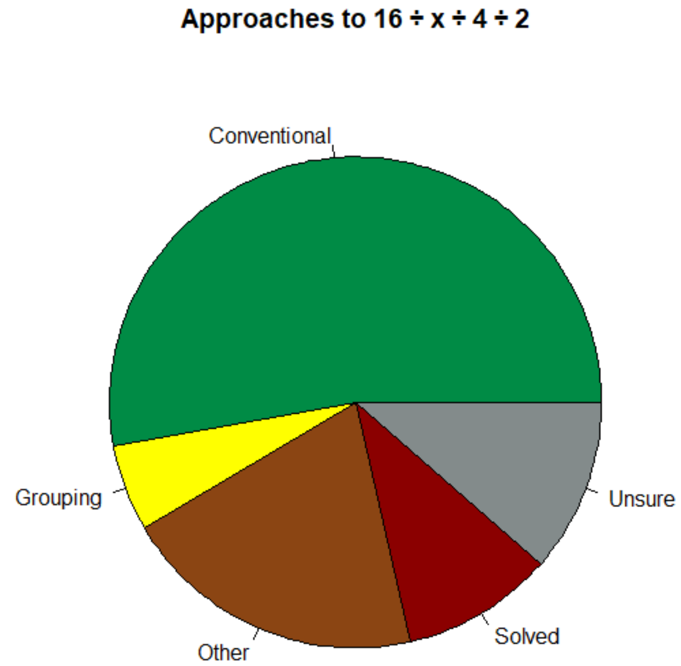
Expression 9: 36 + 14 * 6 - 3 ÷ 2

Expression Category: : Unique ordering of Multiplication, Division, Addition, and Subtraction



Approach	Conventional	Left-to-Right	Other	Unsure
Number of Participants	56	3	9	2

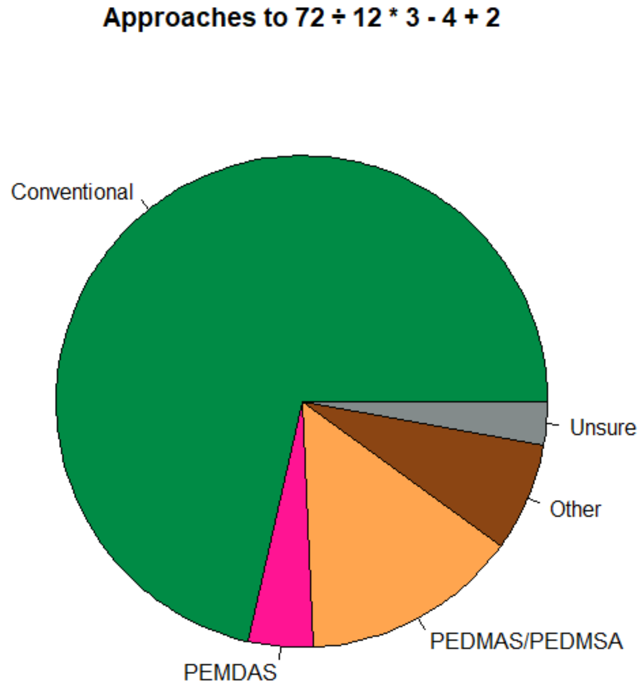
Expression 10: 16 ÷ *x* ÷ 4 ÷ 2
Expression Category: Repeated Division with Variable Inclusion



Approach	Conventional	Grouping	Other	Solved	Unsure
Number of Participants	37	4	14	7	8

Expression 11: 72 ÷ 12 * 3 - 4 + 2

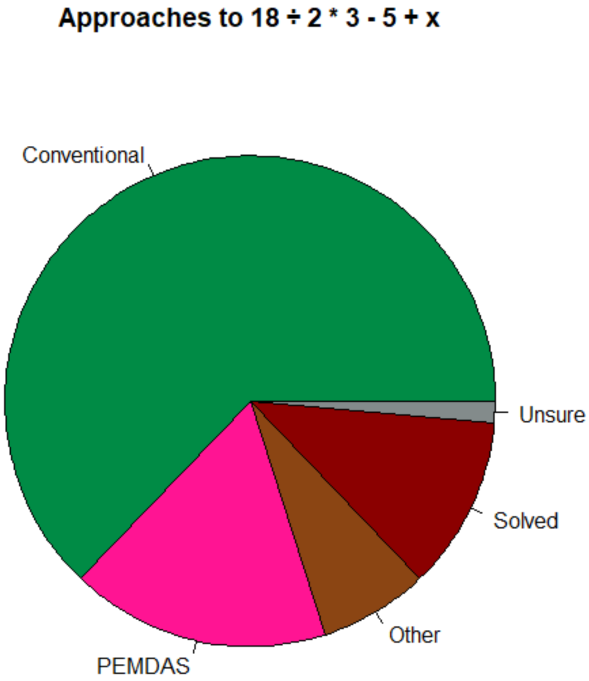
Expression Category: Unique ordering of Multiplication, Division, Addition, and Subtraction with Variable Inclusion



Approach	Conventional	PEMDAS	PEDMAS/PEDMSA	Other	Unsure
Number of Participants	50	3	10	5	2

Expression 12: 18 ÷ 2 * 3 - 5 + *x*

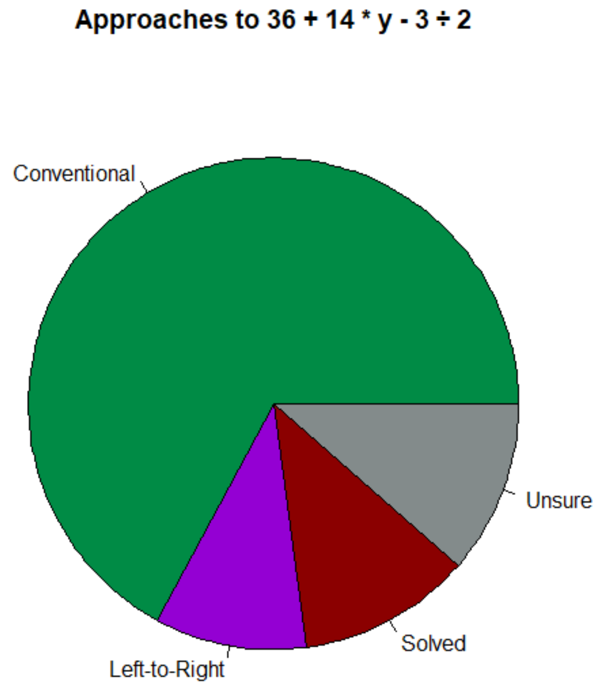
Expression Category: Unique ordering of Multiplication, Division, Addition, and Subtraction with Variable Inclusion



Approach	Conventional	PEMDAS	Other	Solved	Unsure
Number of Participants	44	12	5	8	1

Expression 13: 36 + 14 * *y* - 3 ÷ 2

Expression Category: Unique ordering of Multiplication, Division, Addition, and Subtraction with Variable Inclusion



Approach	Conventional	Left-to-Right	Solved	Unsure
Number of Participants	47	7	8	8