

LEARNING ABOUT AND LEARNING FROM STUDENTS: TWO TEACHERS'
CONSTRUCTIONS OF STUDENTS' MATHEMATICAL MEANINGS THROUGH
STUDENT-TEACHER INTERACTIONS

by

BIYAO LIANG

(Under the Direction of Kevin C. Moore)

ABSTRACT

Teacher knowledge of students' mathematical thinking has been a growing research area in mathematics education. Although researchers have made substantial progress in understanding students' mathematical thinking and created research programs and curricula tailored to the research findings, teacher education programs have, by and large, not been grounded in similarly extensive research on the cognitive nature of teachers' knowledge and its development. Moreover, teacher knowledge and student-teacher interaction are often treated as disparate research areas, and research on their interplay is sparse.

In this dissertation study, I addressed these research gaps by characterizing the mental processes by which teachers develop knowledge of students' mathematical thinking as they interact with students and reflect on students' mathematical thinking. Specifically, I focused on how teachers construct knowledge of students' mathematical thinking including how this construction, in turn, supports the teachers' personal growth in mathematical knowledge.

Adopting and integrating radical constructivism, Piagetian learning theories, and communication theories, I conceptualized the cognitive mechanisms by which teachers construct

knowledge of students' mathematics through social interactions with students. My theorization combined the notions of first- and second-order models, decentering, assimilation, perturbation, accommodation. I expanded the constructivist teaching experiment methodology to a methodology called *teaching experiment*². I included two high school teachers and supported each of them in engaging in cycles of interactions with individual or paired student(s) and reflection on those students' thinking. Each cycle included (a) a planning session, (b) a teaching session, and (c) a reflecting session in the form of video-stimulated recall interview. One teacher worked with her students in the context of linear programming, and another teacher worked in the context of geometric dilation.

My analysis identified seven key mental processes involved in the teachers' constructions of students' mathematics: re-presenting, contraindicating, explaining, inferring, modifying, comparing, and restructuring. The specification of these mental processes contributes to the current theorization of teacher knowledge by operationalizing an anti-deficit, constructive, and dynamic perspective on teacher knowledge and knowing. The findings also inform teacher educators about how to advance teachers' sensitivity to student thinking and supporting teachers in continually learning from their students in their own teaching.

INDEX WORDS: Teacher Knowledge of Student Thinking, Content Knowledge, Student-Teacher Interaction, Mathematical Thinking, Decentering, Mathematics Teacher Education, Professional Development, Learning Through Teaching, Radical Constructivism, Piaget, Linear Programming, Geometric Dilation

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DEDICATION

*To my husband Xuelin
who is the most decentering person I have ever met
and to all students
who have ever felt different*

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CHAPTER 1

SUBJECTIVITY STATEMENT

The whole purpose of education is to turn mirrors into windows.

By Sydney J. Harris

My passion for researching teacher knowledge of student thinking and student-teacher interactions originated from my experience as a mathematics learner and pre-service teacher in China. At school age, I often struggled with adjusting my strategies to solve mathematical problems with variations. I was aware that some of my classmates excelled at solving those problems, and I had been very curious about their thinking. However, due to the nature of school instruction being teacher-centered, I lacked opportunities to have access to my peers' thinking in mathematics classrooms; the thinking underlying their problem-solving had been a mystery to me at the time.

Entering into college as a pre-service teacher, I received training that exclusively focused on lecturing. To me, teaching was a solo activity where a teacher presents their personal mathematical knowledge coherently and rigorously to their students. To practice this activity, teacher education and professional development programs often included activities such as public exemplary lessons (Huang et al., 2011; Huang & Li, 2009) and mock teaching competitions (Li & Li, 2009). During my participation in these activities, I maintained an assumption that there exists a good way to teach a topic that teachers should strive to emulate in action, and a teacher can work toward this good way of teaching by sharing and discussing different instructional approaches and procedures. As a winner of a teaching competition, I had never questioned this assumption. This reminds me of the Chinese word 教学 (jiào xué), with the

literal meaning of the first character being teaching and that of the second being learning. However, most Chinese people (including myself) often use these combined characters to *only* mean teaching and we seldom interpret it as incorporating two components. Reflecting on this now, student learning is overlooked in Chinese mathematics education because “teaching and learning research” only implies “teaching research” to many people.

My view of teaching started to change as I studied in the United States. I remember the first time I was present at a class in the U.S., I felt like Alice stumbling into a new world, where student-teacher interaction was a central activity in the classroom. At the time, I was extremely anxious about talking to my classmates and responding to instructors’ questions, and I had to request to present my final project at my professor’s office to avoid interactions publicly. Gradually, and as I entered graduate school, I became adjusted to this classroom culture and comfortable with interacting with classmates, instructors, and pre- and in-service teachers in a variety of settings. I benefited from interacting with these intellectual individuals, learning to care about and listen to different perspectives, and meanwhile reflecting on my own thinking. Moreover, I was exposed to the radical constructivist epistemology that served as a grounding theory for me to reflect on my past educational experience and teaching philosophy. Teaching mathematics is not about enculturing students into the mathematical realities of teachers or mathematicians; it is about teachers and students mutually listening, interpreting, and reflecting on each other’s mathematical realities and learning from each other. There are no winners in teaching, and the best teaching is the one that tailors towards and draws on particular students’ thinking so that it triggers meaningful and accessible conversations with them.

With such an assumption in mind, I had opportunities to engage in multiple research projects aimed at characterizing students’ diverse ways of thinking that differed from my own

(i.e., an NSF CAREER grant *Advancing Secondary Mathematics Teachers' Quantitative Reasoning* with Kevin Moore and an NSF REAL project *Generalization Across Multiple Mathematical Areas* with Amy Ellis). I have been inspired by my research participants in each of our conversations, and my experiences in these settings have helped solidify my belief that every individual's knowledge and learning processes are worth being valued and respected.

Motivated by my past experience, I develop a research program that is at the intersections of mathematical cognition, social interactions, and teacher education. I am committed to understanding individuals' ongoing constructions of mathematical realities through social interactions with others. I started this journey with this dissertation study in which I investigated two in-service teachers' learning of students' mathematical thinking through student-teacher interactions and in what ways such learning was beneficial to the teachers' mathematical and professional growth.

CHAPTER 2

MOTIVATION AND PROBLEM STATEMENT

Can we not account for both science's existence and its success in terms of evolution from the community's state of knowledge at any given time? Does it really help to imagine that there is some full, objective, true account of nature and that the proper measure of scientific achievement is the extent to which it brings us closer to that ultimate goal?

By Thomas Kuhn, *The Structure of Scientific Revolutions*

In this chapter, I describe my thinking about philosophy, science, and education that motivates my interests in teachers' learning of student thinking. Then, I review a history of student-centered instruction and identify the research gaps about supporting teachers' enactment of student-centered instruction. Against this backdrop I discuss my research questions and rationale of my approach to the current study. Finally, I provide an overview of the structure of the dissertation and notes on its completion in the midst of the COVID-19 pandemic.

Philosophy, Scientific Inquiry, and Education

In this section, I discuss views of mathematics and mathematics education inspired by the radical constructivist epistemology, including the importance of understanding others' mathematical thinking. Then, I discuss ideas from *The Structure of Scientific Revolutions* by Thomas Kuhn and how they inform a scientific inquiry into someone else's thinking.

Nobody Knows the "Truth"

In history, the development in mathematics itself has challenged the notion of mathematics as a body of truths about the real world (Goldin, 1990). To the present day, there has been a debate among mathematicians, philosophers, and educators on whether mathematics

is *created* or *discovered* by human beings. Individuals who believe that mathematics is discovered assert that mathematics is constituted by a body of truths existing independently on human activity and thinking. Despite its independent existence, some believe that it is possible for humans to obtain this external reality through reasoning and logical inference (e.g., rationalism). Knowledge is considered to be objective, and every question has “a single, correct answer that is knowable with absolute certainty” (Gottlieb, 2007, p. 6). In contrast, individuals who believe that mathematics is created assert that mathematics is a result of human construction. Radical constructivism is one of the epistemologies that is committed to such an assumption. Radical constructivists believe that mathematics is necessarily constructed because no individual has direct access to an objective reality and, relatedly, no individual can compare their personal knowledge to some true representation of the real world (Goldin, 1990; Steffe & Thompson, 2000b; von Glasersfeld, 1995). An individual’s knowledge is constructed through their experience and ascribing meanings to that experience in personal ways. Likewise, an individual cannot claim that they know what another person knows because each individual lives within the constraints of their own experience, and knowledge of another’s experience must be viewed through the experience of making subjective interpretations of another’s actions.

In the past 30 years, this view of mathematics has advanced the field of mathematics education by promoting mathematics teaching and learning as a humane, inclusive, and social activity. Educators have called for a belief that each individual is capable of constructing mathematics in personal and rational ways and that each individual’s mathematical construction is legitimate and worth being heard, encouraged, and respected (Hackenberg, 2010; Steffe & Thompson, 2000b). Mathematics is not *between* teachers and students in a classroom as described by the didactical triangle as interpreted by Straesser (2007), who conceptualized

teaching as a diffusion of a priori knowledge from teachers to learners (i.e., a teacher-centered, didactic model of teaching). Rather, the existence of classroom mathematics has to *emerge from* students' and teachers' engagement with a given problem situation and active constructions of each other's knowledge through socialization (Schoenfeld, 2012). As Resnick (1989) framed it, "we may do well to conceive of mathematics education less as an instructional process (in the traditional sense of teaching specific, well-defined skills or items of knowledge) than as a socialization process" (p. 58). In this regard, mutual communication serves a critical avenue for discussion and reflection on the viability of different views of mathematics and consequently, emergence of new mathematical constructions in individuals involved in the conversation (Gash, 2020; see also *pedagogy as conversation* in Smith, 2003).

More generally, radical constructivism helps sustain harmony and cooperation in our daily life by inviting conversations on diverse perspectives and guiding us to be respectful for viewpoints that differ from our own, rather than thinking that we ourselves own the "truth" and arguing against alternative perspectives and opinions that are "not true" or "less true".

Towards a Systematic Way of Learning About and Learning from Others

Given that understanding and learning from others is important, how do we work to understand and learn from others? The essence of scientific evolution informs one answer to this question. Looking into how natural scientists attempt to know about the nature, Kuhn (1962/2012) argued that the evolution of science is not an "evolution-of-what-we-do-know" but an "evolution-towards-what-we-wish-to-know." The goal of scientific inquiry is not about discovering an illusory objective truth but about constructing and reconstructing hypotheses about the nature through continual interactions with the environment and the objects under investigation (Kuhn, 1962/2012). In doing so, scientists collectively engage in iterative and

convoluted explorations where they constantly generate hypotheses to explain phenomena, confirm or refute or revise those hypotheses with additional evidence, and construct new theories when prior theories cannot account for new observations. Conclusions that scientists make have to be results of their perception, contemplation, interpretation, and analysis (Kuhn, 1962/2012).

This scientific mindset has influenced research in education, including the endeavor of understanding individuals' mathematical thinking¹. When the subject being studied is not the natural environment or objects but human beings like teachers and students, researchers engage in interpreting these individuals' behavior and continually test and refine their interpretations during interactions with them. Likewise, regarding the roles of teachers in teaching, teachers can continually *learn about* their students by recursively formulating, testing, and reconstructing hypotheses about the students' mathematical knowledge in their daily interactions with students. Mathematics teaching is thus viewed as a scientific investigation into students' mathematical realities (Steffe & Thompson, 2000b; A. G. Thompson & Thompson, 1996). Meanwhile, teachers may *learn from* their students during this inquiry (Ball, 1993; Schifter, 1998; Simon, 2000a; T. J. Smith, 2003; Steffe, 1990; Steffe & Thompson, 2000b). Namely, they may reconstruct and reorganize their own mathematical realities by reflecting on and comparing aspects of the students' thinking and the teachers' own thinking. The Book of Rites (also known as *Lǐ Jì*) has a phrase called 教学相长 (*jiào xué xiāng zhǎng*), meaning to teach is to learn; through teaching students, teachers can become aware of the limitations in their own knowledge and improve their knowledge, and therefore teaching is also a process of learning. The beauty of

¹ Kuhn has questioned about the extension of his theory to social science because of some fundamental differences of these two fields (Stanford Encyclopedia of Philosophy, 2018). What I discuss in this paragraph comes from my own interpretations and demonstrates how I see some aspects of Kuhn's thinking as being applicable in education. In fact, his work has already influenced the development of social science to certain extent, because of his permission of a liberal conception of what science is and rejection of rules as determining scientific outcomes.

learning and teaching mathematics lies in the very process of each individual mutually inferring each other's mathematical realities and reconstructing their own mathematical realities based on those inferences. Numerous researchers have advocated that it is teachers as well as students who learn and it is students as well as teachers who teach (T. J. Smith, 2003; Steffe & Thompson, 2000b).

Then, if we maintain this view on mathematics teaching as a process of mutual learning and of scientific inquiry, a critical question for mathematics teacher educators becomes: How can we operationalize and promote this mindset in our work with teachers? Especially, how can we systematically support teachers in putting this scientific mindset into practice? And if a teacher does enact teaching consistent with this approach, what are the ways of reasoning involved in the teacher's mind? A few researchers have partially answered the latter question by doing self-studies, acting as a teacher-researcher and analyzing the thinking and decision-making involved in their efforts to understand students' mathematical thinking (Hackenberg, 2010; Schifter, 2001; Simon, 1995; A. G. Thompson & Thompson, 1996; Tzur, 1999). More research is needed to reveal patterns of in-service teachers' reasoning with student thinking, and the present dissertation seeks to contribute to this end.

Calls for Student-Centered Instruction

The present study is also situated in a larger movement of *student-centered instruction* in education, a teaching approach characterized as centering the roles of learners (also known by, among others, *learner-centered approach* and *child-centered pedagogy*). Characteristics of student-centered instruction overlap with many other instruction types referred to by a variety of names such as “reform-based teaching,” “ambitious instruction” (Lampert et al., 2013), “inquiry-oriented instruction” (Kuster et al., 2018), or “discourse-oriented teaching” (Williams & Baxter,

1996). The history of student-centered instruction can be dated back to a Chinese philosopher and educator, Confucius, in the East, and a Greek philosopher, Socrates, in the West, between the 5th and 4th centuries B.C. (Shah, 2020). The Analects of Confucius was written by Confucius's followers and contained a large collection of sayings, ideas, and conversations by Confucius and his colleagues and students. One of the well-known ideas from the Analects of Confucius is called 因材施教 (yīn cái shī jiào), which stresses teachers' attention to individual differences during teaching, including differences in personalities, cognitive capabilities, and interests.

Over two millennia passed before the twentieth century theories of several scholars such as John Dewey, Jean Piaget, and Lev Vygotsky influenced further conceptualizations and enactment of student-centered instruction in various forms. For example, John Dewey advocated for an educational structure that balances between content delivering and learners' personal interests and experiences (Dewey, 1938/1954). He is one of the proponents of experiential education, which informs the contemporary notion of inquiry-based learning (or problem-based learning), where learners engage in authentic experience of speculating, questioning, observing, and testing hypotheses to solve problems relevant to their personal interests, while teachers providing appropriate and necessary guidance. Dewey (1902/1911) also advocated psychological investigation into learners' experiential reality during their learning of subject matter. Later, Jean Piaget (e.g., Piaget, 1977, 2001; Piaget & Inhelder, 1967) and Lev Vygotsky (1978) formulated theories about individuals' learning processes by taking complementary lenses and focuses, providing theoretical foundations for Dewey's proposal of learning through experience.

Influenced by these scholars, mathematics education researchers have conceptualized student-centered mathematics instruction (Ball, 1993; Carpenter et al., 1996; Chung & Walsh,

2000; Hiebert et al., 1997) and constructivist perspectives on mathematics teaching (Confrey, 1990; Simon, 1995; Steffe & D'Ambrosio, 1995; Steffe & Wiegel, 1992; P. W. Thompson, 2002). As Steffe and Wiegel (1992) stated, “the most basic responsibilities of constructivist teachers is to learn the mathematical knowledge of their students and how to harmonize their teaching methods with the nature of that mathematical knowledge” (p. 460-461). In contrast to conventional mathematics teaching that casts the mathematics of teachers as the power of their classrooms, this perspective transforms the classrooms by shifting the focus to students’ ways of operating on mathematics, and the primary tasks of the teachers are to learn about those ways of operating and use them to create instructional goals and interactive mathematical communication. A constructivist teacher must come not only to expect, but also to seek, new mathematical ways of thinking in their teaching and let themselves to be surprised by their students (Schifter, 1998).

Related to these researchers’ suggestions, mathematics educators and writers of mathematics education policy documents continue to highlight the importance of teachers eliciting and using student thinking in teaching practices. Authors of the *Principles to Actions: Ensuring Mathematical Success to All* (National Council of Teachers of Mathematics [NCTM], 2014b) have announced eight Mathematics Teaching Practices, four of which are relevant to making instructional actions sensitive to student thinking (i.e., facilitate meaningful mathematical discourse, pose purposeful questions, support productive struggle in learning mathematics, and elicit and use evidence of student thinking). Authors of the *Standards for Preparing Teachers of Mathematics* (Association of Mathematics Teacher Educators [AMTE], 2017) have emphasized the importance of teachers having foundational understanding of

students' mathematical knowledge and being committed to developing such understandings through ongoing teaching practices.

Teachers' ability to communicate with students such that students' various ways of thinking are acknowledged, encouraged, and celebrated is also a necessary condition for broadening participation in STEM and providing more equitable learning opportunities to empower students in mathematics classrooms (NCTM, 2014a). The recent position statement of *TODOS: Mathematics for All* (2020) by Zavala and colleagues has promoted the essential action of *eliminating deficit views of mathematics learning*. Specifically, educators must “rethink what counts as valid mathematical knowledge” and “expand our understanding of what it means to be good at mathematics, make space for alternative ways of knowing and doing mathematics based in the community, and acknowledge the brilliance, both in mathematics and beyond” (p. 8). Relatedly, the equity statements of SIGMAA on RUME has argued for the importance of expanding these pervasive perspectives to the research communities, including being critical of language and perspectives in making claims about participants' learning and “not focusing on identifying their misconceptions and deficiencies in understandings” (SIGMAA on RUME, 2018, p. 2).

Challenges in Teacher Education and Research Gaps

Despite the importance of teaching in ways sensitive to students' mathematical thinking, researchers and mathematics teacher educators have provided fruitful evidence that it was nontrivial for teachers to develop knowledge, skills or dispositions necessary to enact student-centered instruction (e.g., Jacobs et al., 2010; E. Johnson & Larsen, 2012; Maher & Davis, 1990; Speer & Wagner, 2009; A. G. Thompson & Thompson, 1996). Researchers have suggested that, in mathematical teaching and learning, there is a discrepancy between the mathematical

understandings teachers intend to teach and the mathematical understandings students develop (Bauersfeld, 1980; Lew et al., 2016; P. W. Thompson, 2013; P. W. Thompson & Thompson, 1994; Yoon, 2019). This miscommunication between teachers and students has created a fundamental dysfunction in mathematics education (P. W. Thompson, 2013), which highlights the need for mathematics teachers to systematically bring forth the mathematics of students and make instructional decisions sensitive to it (Steffe & Thompson, 2000b; Teuscher et al., 2016).

In the past 20 years, researchers have made efforts to gain insights into this phenomenon, resulting in a growing body of studies and research programs on teachers' discourse moves, professional noticing, listening, modeling, decentering, teacher knowledge of student thinking, and their relationships. One of the common and important findings is that teachers' lack of knowledge of student thinking constrains them from understanding and acting upon critical moments of student thinking (e.g., Bray, 2011; Charalambous et al., 2012; Hill et al., 2008; Speer & Wagner, 2009). Complicating the matter, increase in teaching experience itself does not necessarily lead to teachers' increased ability to respond to students' mathematical thinking (Jacobs et al., 2010).

To address this challenge, teacher educators have developed a variety of professional development (PD) programs aimed at supporting teachers' learning about student thinking. These include PD programs at different grade levels and in various forms (e.g., lesson study, video club, student interviews), such as Cognitively Guided Instruction (CGI) (Carpenter et al., 1989, 1996; Carpenter & Fennema, 1992; Fennema et al., 1996; Franke et al., 2001; Steinberg et al., 2004), Mathematics Classroom Situations (Even & Markovits, 1993; Markovits & Even, 1999a, 1999b), Developing Mathematical Ideas (DMI) (Schifter, 1998, 2001), Integrating Mathematics Assessment (IMA) (Gearhart & Saxe, 2004; Kazemi & Stipek, 2001; Rhine, 1998),

Integrating Mathematics and Pedagogy (IMAP) (Philipp et al., 2002, 2007; Philipp, 2008), Problem-Solving Cycle (Borko et al., 2011, 2015, 2017; Koellner et al., 2007), Project Pathways (Carlson et al., 2007; P. G. Clark et al., 2008; Teuscher et al., 2016), Studying Teachers' Evolving Perspectives (STEP) (Jacobs et al., 2010; M. G. Sherin et al., 2011a), Learning Trajectories Based Instruction (LTBI) (Clements & Sarama, 2009; Paola Sztajn et al., 2012; P. H. Wilson et al., 2013, 2014, 2015, 2017), Iterative Model Building (IMB) (Lee, 2013; Norton et al., 2011; Weiland et al., 2014), Inquiry-Oriented Instruction (E. Johnson, 2013; Kuster et al., 2018; Rasmussen et al., 2020; J. F. Wagner et al., 2007), Ongoing Assessment Project (OGAP) (Supovitz et al., 2018), and Student-Adaptive Pedagogy (Hodkowski, 2018; Tzur et al., 2018). These PD opportunities have helped increase teachers' ability to adapt their teaching goals and activities based on student conceptions, the quality of instruction and nature of student-teacher interactions, and student achievement.

Although these PD programs have been grounded in rich research knowledge about students' mathematical cognition, they have, by and large, not been grounded in similarly extensive theories about the nature of teachers' knowledge of student thinking and its development (Doerr & Lesh, 2003). Most of their focuses have been on the outcomes of the PD programs rather than the teachers' developmental processes themselves. Some of those outcomes are captured within the contexts of the PD programs and research, while some are characterized as sustainable outcomes through longitudinal studies. While one may be convinced by the effectiveness of these PD programs, they are left wondering: How does teacher change occur or what supports the sustainability of teachers' growth in knowledge of student thinking? In other words, what is going on in the teachers' minds during their participation in these PD activities that could explain their growth? A researcher's focus on *explaining* teachers' growth likely

necessitates fundamentally different research questions and methodologies than a focus on *providing evidence for* teachers' growth. Moving forward, the field of teacher education is in need of research that goes beyond characterizing teacher change to reveal cognitive mechanisms of teachers' mathematical and professional learning associated with their reflection on student thinking. This attempt will guide teacher educators and researchers to design and implement teacher education activities to support teachers' learning of student thinking.

Research Goals, Questions, and Rationales

Addressing the aforementioned research gap, the goal of this dissertation is to study teachers' mathematical development, including development in knowledge of students' mathematical thinking, in the context of student-teacher interactions. The general question driving this study is: *What are the mental processes involved in teachers' constructions of knowledge of students' mathematical thinking in the context of student-teacher interactions, and what are the mathematical consequences of such constructions?* In the following, I unpack this research focus in terms of two types of interplay.

Interplay Between Teacher Knowledge and Student-Teacher Interaction

Teacher knowledge and student-teacher interactions have been perennial areas of focus in mathematics education. However, these areas are often treated separately, and research on their interplay is relatively thin. Prior researchers have either characterized teachers' knowledge of students' mathematics independent of student-teacher interactions (e.g., Hill et al., 2008; Jacobson et al., 2018) or investigated student-teacher interactions without providing a fine-grained analysis of the cognitive structure of teachers' knowledge of students (e.g., Brendefur & Frykholm, 2000; Wood, 1998). The stagnancy of research on their interplay is partly due to a lack of methodology that simultaneously allows teachers to engage in sustained interactions with

students and allows researchers to closely observe, hypothesize, and document the teachers' ongoing mathematical constructions. Thus, in addition to the research knowledge generated, the present dissertation contributes an emergent methodology that enables the connection between social interaction and knowledge development.

One approach pertaining to the goal of this methodology is to study teachers' mathematical development through promoting their development in the context of student-teacher interactions. Instead of disseminating to teachers research knowledge on student's mathematical thinking (cf. Sztajn et al., 2012), researchers can provide teachers with opportunities to engage in sustained interactions with individual students. Meanwhile, the researchers focus on fostering the teachers' interpretations of students' mathematical knowledge and reflection on their knowledge. This approach is an operationalization of researchers' calls for learning-through-teaching (Franke et al., 2001; Leikin & Zazkis, 2010; M. G. Sherin, 2002; Steinberg et al., 2004; A. G. Thompson & Thompson, 1996; Wood et al., 1991) and positioning teachers as researchers (Potari, 2000; A. G. Thompson & Thompson, 1996) and reflective practitioners (Schön, 1983).

Relatedly, examining teachers' ongoing mathematical development related to student-teacher interactions necessitates a non-possessive or non-evaluative stance on teacher knowledge. As mentioned above, some researchers have claimed that teachers' lack of knowledge of students constrain them from enacting teaching practices sensitive to student thinking (e.g., Bray, 2011; Charalambous et al., 2012; Hill et al., 2008; Speer & Wagner, 2009). While it is a reasonable expectation for teachers to obtain substantial knowledge of students, a researcher who focuses on evaluating the extent teachers possess such knowledge may lose the opportunity to advance the teachers' knowledge as they interact with students. Moreover, there is

no objective benchmark for “sufficiency” or “lack-ness” of an individual’s knowledge; a teacher’s actions during student-teacher interactions are constrained to their past experience and disposition, knowledge, and skills *available* to them in-the-moment, as opposed to what they *lack*. I consider that characterizing the knowledge available to a teacher provides more explanatory power with regard to why the teacher do what they do and how teacher educators can improve what they do (see P. W. Thompson, 2013).

Interplay Between Mathematical Knowledge of Students and Mathematical Knowledge

Maintaining these assumptions, I adopt a dynamic and adaptive views on teacher knowledge (Fennema & Franke, 1992), aiming to capture the constructive processes associated with teachers’ learning about students. The dynamic view assumes that teachers’ mathematical knowledge of students develops and grows through their ongoing interactions with students. By “adaptive,” I assume teachers’ development in mathematical knowledge of students is built upon their mathematical knowledge and vice versa. As Ball (1993) pointed out, teaching mathematics is about “perceiving the mathematics through the mind of the learner while perceiving the mind of the learner through the mathematics” (p. 159). Steffe (1990) made a similar point:

If teachers could only accept the premise that the mathematical knowledge of their students is also valid, then the necessary adaptations of teachers when teaching mathematics would be in the direction of the mathematical knowledge of their students as well as in the direction of their own mathematical knowledge. In other words, the mathematical knowledge of the students as seen by their teachers would become part of the knowledge of the teachers. This happy state of affairs could only improve mathematical communication in the classroom, especially in those cases where the teachers emphasize the activity of their students in learning mathematics. (p. 184)

Both researchers’ emphasis on the bifocal perspective on mathematics teaching implies a dynamic interplay between a teacher’s personal mathematics and their interpretations of students’ mathematics. On one hand, teachers’ sense-making of student thinking requires their enactment of existing mathematical knowledge; on the other hand, “analysis of student thinking

can, itself, become a powerful site for teachers' further mathematical development" (Schifter, 1998, p. 57).

Research Questions and Significance

Considering that a teacher's development in mathematics and in students' mathematics may concurrently occur, I break down the aforementioned research question into the following sub-questions:

1. What mathematical knowledge does a teacher construct *prior to* their interactions with students?
2. What knowledge of students' mathematics does a teacher construct *during* interactions with students and reflection on students' mathematics?
3. How does a teacher's existing mathematical knowledge afford their constructions of students' mathematics?
4. How are such constructions, in turn, generative to a teacher's mathematical knowledge?

Since Shulman's (1986) proposal of the notion of pedagogical content knowledge, the last three decades have seen a growing number of theories and research about the structure and nature of teacher knowledge, among which teachers' content knowledge and knowledge of students are critical components (Ball et al., 2008; Fennema & Franke, 1992; Harel, 2008a; Silverman & Thompson, 2008; Tallman, 2015). Research on teacher knowledge has also went beyond the dichotomy of these two types of knowledge to characterize the *interactions* between them. There has been research on how teachers' content knowledge influences their constructions of knowledge of students (Baldinger, 2019; A. G. Thompson & Thompson, 1996; P. W. Thompson & Thompson, 1994) and research on how teachers' constructions of knowledge

of students contribute to their growth in content knowledge (Carpenter et al., 1996; Philipp et al., 2002; Philipp, 2008; M. G. Sherin, 2002; P. H. Wilson et al., 2013, 2014). However, few researchers have investigated this bidirectional relationship in a single context. There has also been a lack of research that reveals the mental processes or modes of reasoning underlying these interactions (as opposed to characterizing teacher knowledge at two different states)².

Moreover, such interactions are expected to be implicated and sustained in the context of student-teacher interactions, where a teacher is afforded opportunities to engage in cycles of testing and refinement of their knowledge of students and reflection on their mathematical knowledge. Untangling the mental processes involved in these activities is imperative for designing future teacher education and professional development programs. A specification of these mental processes will provide teacher educators with a framework to design educational opportunities that foster these mental processes and support teachers' continual learning about and learning from their students through student-teacher interactions and reflection on student thinking.

An Overview of the Dissertation

This dissertation study is grounded in the radical constructivist epistemology in combination with Piagetian learning theories and theories of social interaction or human communication. The central tenet of radical constructivism justifies the need for teachers to engage in a scientific inquiry into students' mathematical meanings and their development.

² I view this distinction also as a distinction between *knowing* and *knowledge* (Harel, 2013) or a distinction between *process* and *outcome*. *Knowledge* is the accumulated product of knowing. Examples of focusing on knowledge include describing content of knowledge, evaluating knowledge, and identified features of knowledge. *Knowing* is the process of generating knowledge. A researcher's interference of an individual's processes of *knowing* requires the researcher to first gain insights into the substance of the individual's *knowledge* (see the notion of *meaning* in Chapter 4) at different states, and then identify the mental processes that enables the emergence of that knowledge.

Piagetian learning theories provide foundational constructs to explain and characterize the mental processes involved in teachers' learning about and learning from students. Theories of social interactions shed light on the phenomenon of intersubjectivity and characterize how teachers and students can achieve a state of intersubjectivity as the teachers construct and modify knowledge of students during ongoing student-teacher interactions.

In order to promote and study teachers' development in knowledge of student thinking, this dissertation used a two-tier teaching experiment. Two high school teachers, Ms. K and Ms. N, each conducted a teaching experiment to investigate and promote individual or paired student(s)' mathematical learning regarding different mathematical concepts. I conducted another layer of teaching experiment to investigate and promote Ms. K and Ms. N's mathematical development, including their development in mathematical knowledge of their students. The analysis offered insights into diverse mental processes involved in Ms. K and Ms. N's reasoning with mathematics and students' mathematics.

The rest of the dissertation is broken down into seven chapters. Chapter 3 provides a review of existing research programs and literature around teachers' understanding of students' mathematical thinking. Chapter 4 discusses the theoretical constructs and analytical framework of the study, including assimilation, perturbation, accommodation, first- and second-order models, and a framework of four types of decentering that incorporates these aforementioned constructs. Chapter 5 presents the methodology of the study. In particular, I propose and justify a methodology called teaching experiment² as an extension of the constructivist teaching experiment. Chapter 6 and Chapter 7 describes the results and analysis of two cases. Chapter 6 describes Ms. N and her student Taylor's mathematical activities in the context of linear programming. Chapter 7 describes Ms. N and her students Carlee and Heidi's activities in the

context of geometric dilation. Chapter 8 discusses the conclusions and implications of the dissertation, including the common themes identified across the two teacher cases. Finally, Chapter 9 addresses the limitations and future research agenda in light of the findings of the dissertation.

Notes on the COVID-19 Pandemic

This dissertation is completed in the midst of the COVID-19 pandemic. Data collection was suspended due to school closure under this special circumstance; three fourths of total number of intended sessions were completed before the suspension occurred. To complement the incompleteness of data collection and guarantee the contribution of the dissertation, the literature review in Chapter 3 is conducted in a more comprehensive manner. It branches out into and connects areas that are beyond the scope of but motivating the current study, including literature on teacher knowledge, teacher noticing, listening, and discourse moves and patterns. Another novel contribution lies in the proposal of an analytical framework of teacher decentering by combining theoretical constructs in the areas of cognition, social interactions, and teacher education. In addition, as mentioned above, this dissertation contributes to an emerging methodology for advancing and studying teachers' development in mathematics and students' mathematics, which has undergone several iterations and corresponding refinements. To conclude, the significance of the dissertation lies as much in its potential for enlightening future research as it does in the empirical findings themselves. I look forward to continuing this line of inquiry in wake of the pandemic.

CHAPTER 3

LITERATURE REVIEW

If I have seen further, it is by standing upon the shoulders of giants.

By Sir Isaac Newton

In this chapter, I review literature on teachers’ understanding of students’ mathematical thinking. To this point, I have used “understanding” generically to include research on the *acts* of understanding and the *products* of understanding (see also *knowing* versus *knowledge* in Harel, 2013). Figure 1 captures their relationships that an act of understanding generates a product of understanding and the product, in turn, supports future acts of understanding. For example, we can consider that a teacher’s act of responding to a student’s thinking (e.g., asking an extending question) is driven by their existing knowledge of mathematics and students’ mathematics, which is captured by the “supports” arrow. Then, after the student responds, the teacher may enact the act of interpreting and constructs further understanding of the student’s thinking, which is captured by the “generates” arrow.

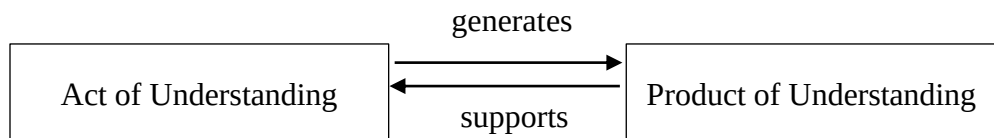


Figure 1. Relationships between act of understanding and product of understanding.

Based on the distinction made here, I divide the literature into two major categories. I consider research on teacher knowledge of student thinking as addressing the product of understanding, and I review different conceptualizations of this type of knowledge in the literature. Then, I review literature on the different acts of understanding, including (a) discourse

moves and patterns, (b) listening, (c) noticing, (d) decentering, and (e) other modes of reasoning with student thinking. As described below, some literature identifies the nature, taxonomies, and features of each act while some other literature discussed how these acts are generative to or conditioned by teachers' knowledge of student thinking.

Teacher Knowledge of Students' Mathematical Thinking

Teacher knowledge of student's mathematical thinking represents an area of research focused on the products of teachers' understanding of students' mathematics. In the mid-1980s, Shulman (1986) introduced the notion of *pedagogical content knowledge* (PCK) to address the "missing paradigm" in educational research and policies—namely, he challenged the disconnection between content and pedagogy, especially the exclusive emphasis on general pedagogy in terms of teaching skills and actions and the absence of focus on content. Shulman (1986) argued that teacher education and evaluation should "pay as much attention to the content aspects of teaching as...to the elements of teaching process" (p. 8) and that pedagogy should be conceptualized *in terms of* specific subject matter. According to Shulman (1986), PCK is "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p. 9). In particular, PCK includes comprehensive "ways of representing and formulating the subject that make it comprehensible to others" (e.g., forms of representation of ideas, analogies, illustrations, examples, explanations, and demonstration) and understandings of "what makes the learning of specific topics easy or difficult" (e.g., preconceptions and misconceptions that students bring with them to the lesson) (Shulman, 1986, pp. 9–10). While some researchers interpreted Shulman's notion of PCK as merely an amalgam of content and pedagogy (e.g., Petrou & Goulding, 2011), Silverman and Thompson (2008) and Tallman (2015) interpreted that it is the transformed content knowledge that stands on its own and that has

pedagogical power in teaching. Tallman (2015) argued that Shulman's PCK is "a *form* of content knowledge that *facilitates* effective pedagogy" and "effective pedagogy cannot exist independently of content knowledge, but is instead fashioned by the way one knows the content of his or her discipline" (Tallman, 2015, p. 17). Therefore, the implications of PCK on teaching is not enacting content knowledge together with general pedagogical knowledge and skills, but about making pedagogical decisions grounded in the teachers' content understandings.

In the past three decades, mathematics education researchers have built upon, expanded, and critiqued Shulman's (1986, 1987) conceptualization of teacher knowledge to propose a variety of theories of teacher knowledge specific to mathematics teaching. In the following, I summarize four influential perspectives in this pursuit, including how teacher knowledge of student thinking is considered within these perspectives.

Fennema, Franke, and Colleagues: Teachers' Knowledge Developing in Context

Shulman (1987) argued teachers need to "transform the content knowledge [they] possess into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by students" (p. 15). Fennema and Franke's (1992) teacher knowledge framework specified this process of knowledge transformation by capturing the *interactive* nature of teacher knowledge. In particular, the center triangle in their framework (see Figure 2 for their illustration of the framework) indicates that it is the specific context of teaching that defines the interconnected relationships between different domains of teacher knowledge and beliefs; the context is where these different components come into play interactively and govern a teacher's teaching actions (Fennema & Franke, 1992).

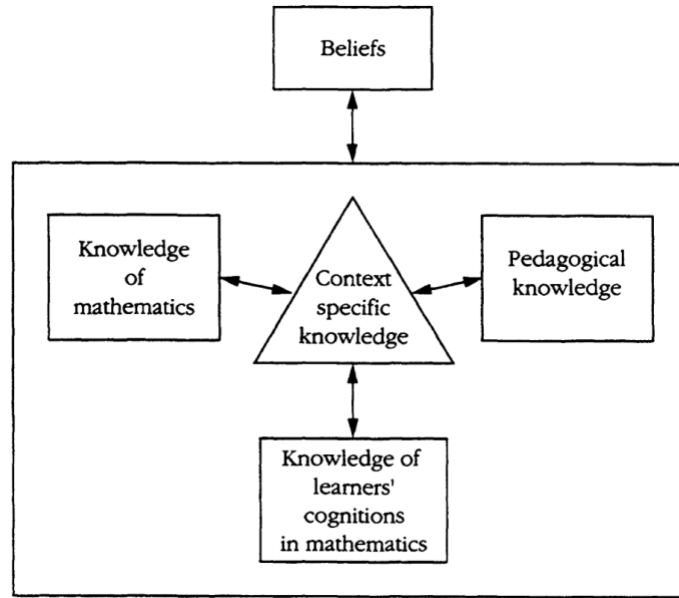


Figure 2. Teachers' Knowledge: Developing in Context (Fennema & Franke, 1992, p. 162).

Another critical feature of teacher knowledge in their framework is *dynamic*, which accounts for the possibility that teachers' knowledge and their coordination of knowledge continue to grow and change in response to students and contexts. Shulman (1987) has argued that teachers not only apply and transform their knowledge during teaching but also construct and modify their knowledge *from* teaching, claiming that "the teacher achieves new comprehension [through the acts of teaching], both of the purposes and of the subjects to be taught, and also of the students and of the processes of pedagogy themselves" (p. 19). Drawing upon Shulman's idea, Fennema and Franke (1992) conceptualized that teacher knowledge continues to grow through sustained interactions with mathematics, curriculum, and students, and through professional experiences, stating:

Knowledge acquisition is a process; the crucial aspect is not what knowledge exists, but rather that the process continues. The challenge then is to understanding knowledge as it grows and changes and to discover what experiences contribute to this growth and change. (p. 161)

Compatible with their theory of teacher knowledge, Fennema and Franke's conceptualization of teacher knowledge of students also emphasized the dynamic processes of student learning in particular contexts. They defined *knowledge of learners' cognitions in mathematics* as:

knowledge of how students think and learn and, in particular, how this occurs within specific mathematics content. This includes knowledge of how students acquire the knowledge of the mathematics content being addressed, as well as understanding the processes the students will use and the difficulties and successes that are likely to occur (p. 162)

Because teacher knowledge can only be understood in the context in which it works and develops, Fennema and Franke (1992) acknowledged that it is challenging to develop a methodology that can encompass the interrelationships between different domains of knowledge and how their roles change in the moment of teaching. Consequently, although the framework is theoretically promising, there has been lack of research on continual growth of—and—interplay between—different components of teacher knowledge.

Sociocultural and Situated Cognition Perspectives

Another group of researchers have also theorized the situated aspect of teacher knowledge (Adler, 1998; Barwell, 2013; Hodgen, 2011; Putnam & Borko, 2000). What differentiates their work from Fennema and Franke's is their explicit epistemological basis in sociocultural or situated cognition theories (Brown et al., 1989; John-Steiner & Mahn, 1996; Lave & Wenger, 1991; Wertsch, 1985; Woolley & Jarvis, 2007). While views of situatedness may vary between these theories, one view of situatedness is popular for theorizing teacher knowledge. This view of situatedness maintains that there is *no* separation between an individual's knowledge and the activity or the environment in which the knowledge is constructed, and that the individual, the activity, and the environment mutually constitute each other (Lave & Wenger, 1991). A situated perspective on teacher knowledge assumes that

“[t]eacher knowledge is not simply applied in the context of teaching mathematics but is rather situated within the complex and social world of mathematics classrooms” (Hodgen, 2011, p. 27).

As Putnam and Borko (2000) described:

This professional knowledge is developed in context, stored together with characteristic features of the classrooms and activities, organized around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations. (p. 13)

This perspective recasts the relationship between teacher knowledge and teaching contexts and materials—teacher knowledge is not possessed by a teacher or applied during teaching but is distributed across the teacher, the teacher’s students and colleagues, lesson materials, and various artifacts and structures within the learning community. Hence, teacher knowledge is also social because it has to be supported by the social communities a teacher participates in and the social relationships the teacher is involved in. These social interactions determine what the teacher learns and how learning takes place (Putnam & Borko, 2000).

Barwell’s (2013) conceptualization of teacher knowledge is an application of the social dimension of teacher knowledge. Critical to Barwell’s (2013) approach is the notion of *knowledge display*—“an utterance constructed as a statement of knowledge on the part of the speaker (and oriented to as such by the other participants)” (p. 602). Teachers’ knowledge displays in the forms of the teachers’ talks are socially-organized and co-produced in the moment of student-teacher interactions and are collectively-held in a classroom.

Under the assumptions of a situated cognition perspective on teacher knowledge, teachers’ knowledge of student thinking is located within the teachers’ participation in conversations with students. This knowledge is not a property of a teacher but of the entire activity that the teacher and their students are engaging in. A teacher becoming more knowledgeable about their students is evident by their enhanced ability to make sense of students’ language and use students’ language to convey mathematical ideas in ways sensible to

the students (Adler, 1998; Barwell, 2013). However, Norton and D'Ambrosio (2008) cautioned that learning of mathematical objects and operations is in fact more than learning mathematics as a language or a practice, and thus it is dangerous to reduce mathematical learning to learning to talk or act in activity.

Ball and Colleagues: Domains of Mathematical Knowledge for Teaching

In contrast to Shulman who created his categories of teacher knowledge as a predetermined framework that guided his study on teachers' knowledge growth, Ball et al.'s (2008) framework emerged from their analysis of mathematics teachers' teaching practices, and was thus called a "practice-based theory". In particular, they used a bottom-up approach to seek to characterize what mathematical teaching itself demands a teacher to know. They inquired:

What do teachers need to do in teaching mathematics—by virtue of being responsible for the teaching and learning of content—and how does this work demand mathematical reasoning, insight, understanding, and skill? (Ball et al., 2008, p. 395)

That the theoretical account for teacher knowledge is based on examination of teachers' practices allowed Ball and her colleagues to reveal "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p. 395). They thus termed the system of knowledge as "mathematical knowledge *for* teaching" (often known as the MKT framework).

One critical component of Ball et al.'s MKT framework is *knowledge of content and students* (KCS), which referred to "knowledge of common student conceptions and misconceptions about particular mathematical content...[for example] familiarity with common errors and deciding which of several errors students are mostly likely to make" (p. 401).

Regarding novel student conceptions, teachers need *specialized content knowledge* (SCK) to identify source of student difficulties and "size[e] up the nature of students' errors" (Ball et al., 2008, p. 401). In comparison with the previous two perspectives on teacher knowledge, this definition of KCS seems to imply a possessive view of teacher knowledge, a focus on teacher

knowledge that is already constructed in prior experiences and that enables teachers to enact certain actions in their teaching (e.g., anticipate what students will think and do, anticipate what students will find confusing, interesting, motivating, easy, or hard). This perspective has led to work on measuring teacher knowledge of students (Hill et al., 2008; Hill & Chin, 2018). Hill and colleagues' MKT instrument has significantly improved the traditional approach to assessing teacher knowledge (including knowledge of students). As Tallman (2016) commented,

the assessment...is a vast improvement over the previously used proxy variables for mathematical knowledge (e.g., degrees earned, mathematics courses taken) that were unable to tease out the nuances in teachers' mathematical understandings and were insensitive to capturing the specific ways in which teachers must know their subject. (p. 30-31)

Constructivist Perspectives

Alternatively, some researchers have proposed teacher knowledge frameworks grounded in constructivist epistemologies (Harel, 2008a; Silverman & Thompson, 2008; Tallman, 2015), believing that teacher knowledge is constructive and individually-held. In Harel's (2008a) Teachers' Knowledge Base (TKB) framework, he conceptualized *teacher knowledge of mathematics* as entailing both mathematical ways of understanding—i.e., products of teachers' mental acts—and mathematical ways of thinking—i.e., teachers' habitual ways of reasoning with mathematics. Accordingly, *teacher knowledge of student learning* refers to “a teacher's view of the process of mathematics learning and [their] understanding of the cognitive and epistemological issues involved in learning particular mathematical ways of understanding and ways of thinking” (Harel, 2008a, p. 902). Harel (2008a) further explained that this type of knowledge entails the teachers' understanding of what perturbations their students are experiencing that may lead to their modification or construction of knowledge (i.e., intellectual needs), whether the students are motivated to engage in a problem to pursue a solution (i.e.,

psychological needs), and how the students ways of understanding and ways of thinking are constructed, internalized, organized, and retained in their minds.

Echoing Shulman's idea of PCK as transformed knowledge, Silverman and Thompson's (2008) Framework for Mathematical Knowledge of Teaching sought to unpack the ways by which teachers "extend and connect their mathematical and pedagogical understandings to create a 'new' knowledge" (p. 502). Silverman and Thompson listed four types of understandings that are pedagogically powerful: (a) "models of the variety of ways students *may* understand the content," (b) "an image of how someone else might come to think of the mathematical idea in a similar way" (see also *hypothetical learning trajectories*, Clements & Sarama, 2004; Simon, 1995; Simon & Tzur, 2004; and *zone of potential construction*, Norton & D'Ambrosio, 2008; Steffe, 1991), (c) "an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea" (see also *didactic objects and didactic models*, P. W. Thompson, 2002) and (d) "an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas" (p. 508). Each type of understanding is centered around students' mathematical understandings and their development and requires a teacher to be reflectively aware of how mathematical ideas are developed and connected either from their own perspective or from their students' perspectives (Silverman & Thompson, 2008).

Contributing to Silverman and Thompson's (2008) framework, Tallman (2015) suggested that "the mathematical knowledge required for effective teaching involves more than powerful understandings of mathematical ideas; it involves an *awareness* of the mental actions and operations that constitute these understandings" (p. 597, italics added). By paying attention to factors that conditioned a teacher's enactment of content knowledge during teaching, Tallman

found that such enactment is fashioned by the teacher's awareness of their own mental actions and operations that comprise their content knowledge. It is the self-aware mental processes that are enactable and support teachers' implementation of a variety of instructional activities, including attending to students' thinking and constructing models of epistemic ways of understanding (see Figure 3 for Tallman's illustration of his framework). In other words, teachers' construction of knowledge of student thinking is contingent upon their consciously aware mathematical operations, because otherwise they will not be able to discern how students' ways of operating are distinct from the teachers' own.

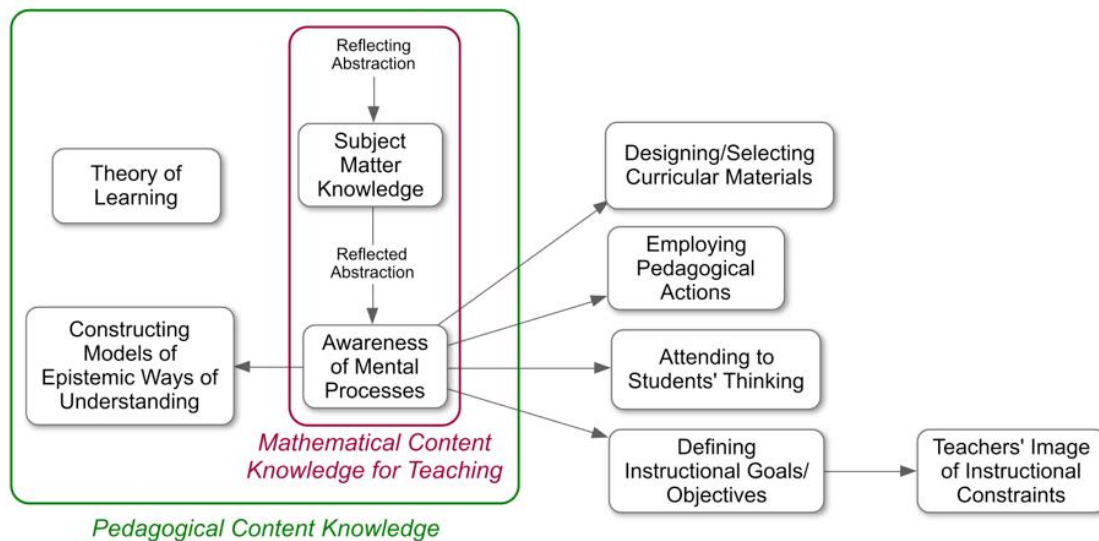


Figure 3. Tallman's framework for pedagogical content knowledge (2015, p. 599).

Constituent Elements of Teacher Knowledge of Student Thinking

As discussed above, teacher knowledge of student thinking is a critical component of a variety of teacher knowledge frameworks across diverse theoretical perspectives. Numerous researchers have further unpacked elements constituting this knowledge. For example, Hill and colleagues (2004, 2008) have constructed and validated an MKT assessment that justified KCS

as a distinctive category of teacher knowledge. According to Hill et al. (2008), four aspects of KCS were considered in the instrument:

- **common student errors:** identifying and providing explanations for errors, having a sense for what errors arise with what content, etc.
- **students' understanding of content:** interpreting student productions as sufficient to show understanding, deciding which student productions indicate better understanding, etc.
- **student developmental sequences:** identifying the problem types, topics, or mathematical activities that are easier/more difficult at particular ages, knowing what students typically learn "first," having a sense for what third graders might be able to do, etc.
- **common student computational strategies:** being familiar with landmark numbers, fact.

Marks (1990) created a similar list that specified five facets of this type of knowledge:

- **students' typical understanding:** what student know and understand.
- **students' common errors:** what students misunderstand.
- **students' learning processes:** e.g., students must understand concepts rather than learn by rote, abstract mathematical concepts and operations should be connected to experience with concrete objects.
- **what is hard and what is easy for students**
- **particular students' understanding:** e.g., making hypothesis about particular students' thinking to explain their observable actions.

Confrey, Wilson, and colleagues have framed KCS through the lens of learning trajectories (P. H. Wilson et al., 2014). They reconceptualized KCS as “knowledge about how learners progress from less to more sophisticated ways of thinking, the cognitive steps that support such development, and misconceptions often found along this progression” (P. H. Wilson et al., 2014, p. 155). They added that teachers’ construction of research-based learning trajectories of students can support teachers’ reorganization of their KCS constructed in previous instructional experiences and reflective teaching practices.

Even and Tirosh used the distinction between *knowing that* and *knowing why* to conceptualize teacher knowledge about students (Even & Tirosh, 1995; Tirosh, 2000), stating:

“Knowing that” in this context refers to research-based and experienced-based knowledge about students’ common conceptions and ways of thinking in the subject-matter. “Knowing why” refers to general knowledge about possible sources of these conceptions, and also to the understanding of the sources of a specific student’s reaction in a specific case. (Even & Tirosh, 1995, p. 17)

Even and Tirosh’s conceptualization of knowing-why involves two aspects. One type of knowing-why involves a teacher explaining student actions by relating it to other common (mis)conceptions (e.g., “they forgot first to invert and only then to multiply” in Tirosh, 2000, p. 10). Another type of knowing-why involves a teacher thinking as a student does and making sense of the reasoning and rationality behind the student’s actions (e.g., “a student thinks the $y = x$ line has a slope of 1 and the y -axis has a slope of 2 because the $y = x$ line is a 45-degree angle and the y -axis is 90 degrees” in Even & Tirosh, 1995, p. 14).

Researchers have also introduced the notion of *interpretative knowledge* (IK), one form of SCK that enables teachers to comprehend students’ atypical solutions and strategies (Di Martino et al., 2019; Jakobsen et al., 2014; Mellone et al., 2020; Ribeiro et al., 2016). Di Martino et al. (2019) defined interpretative knowledge as:

a deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge, starting from their own reasoning and productions, no matter how non-standard or incorrect they might be. IK completes the knowledge of typical errors or solution strategies, with the knowledge of possible source for typical or atypical error and the knowledge of possible use of errors. (p. 3)

Compatible with Ball et al.'s (2008) SCK, interpretative knowledge is a type of mathematical knowledge that allows teachers to interpret students' work and actions, particularly novel reasoning and strategies that are beyond their common content knowledge, and to use them as opportunities for learning and inquiry in mathematics teaching (Di Martino et al., 2019; Jakobsen et al., 2014).

Summary and Research Gaps

In summary, teacher knowledge of students' mathematics (therein after referred to as KoSM) has been conceptualized through different theoretical lenses. Although they differ in significant ways, these frameworks (including empirical work guided by these frameworks) illuminate some common understandings about the nature of this type of knowledge.

First, it is commonly accepted that teachers' development in KoSM is a dynamic process and that it can be enriched and improved over time (see Shulman, Ball and colleagues, and Fennema and Franke's conceptualizations).

Second, driven by extant theorizations, researchers have provided ample evidence that teachers' increases in KoSM are productive in many aspects. They can improve students' mathematical achievement (e.g., Carpenter et al., 1989; Carpenter & Fennema, 1992; Hill & Chin, 2018; C. Jacobson & Lehrer, 2000), enable teachers to enact instructional actions sensitive to student thinking (e.g., Bray, 2011; Cengiz et al., 2011; Charalambous et al., 2012; Hill & Chin, 2018; C. Jacobson & Lehrer, 2000; E. Johnson & Larsen, 2012; Munter & Correnti, 2016; Speer & Wagner, 2009; Yoon, 2019), and enrich teachers' personal mathematical knowledge

(e.g., Carpenter et al., 1996; Philipp et al., 2002; Philipp, 2008; M. G. Sherin, 2002; P. H. Wilson et al., 2013, 2014).

Third and related to the last aspect, it is commonly accepted that teachers' KoSM is closely related to their content knowledge. However, researchers' conceptualizations of their relationships differ. Since Shulman's (1987) and Ball et al.'s (2008) proposal of domains of teacher knowledge, most researchers have conceptualized KoSM as a distinctive category and attempted to understand its nature and consequences (e.g., Hill et al., 2008; E. Johnson & Larsen, 2012; Speer & Wagner, 2009). Gradually, there has been increasing attention to the interactive relationships between domains of knowledge. To date, it is commonly accepted that teachers' constructions of KoSM not only are contingent upon but also engender reorganizations in teachers' content knowledge (e.g., P. H. Wilson et al., 2013, 2014). Moreover, a few researchers have considered KoSM as a subset of content knowledge, assuming that teacher's growth in KoSM automatically implies their growth in mathematical knowledge (e.g., Philipp, 2008; Philipp et al., 2002). Therefore, there seems to be a spectrum regarding researchers' views of KoSM's relationships to content knowledge, from no relationship, to a mutual-influence relationship, to an intersecting relationship, and to a containing relationship (Figure 4). The end of this spectrum represents a teacher's true internalization of KoSM and treating student ways of operating mathematics as part of their own ways of operating. In Chapter 4, I justify my conceptualization that is consistent with this last perspective.

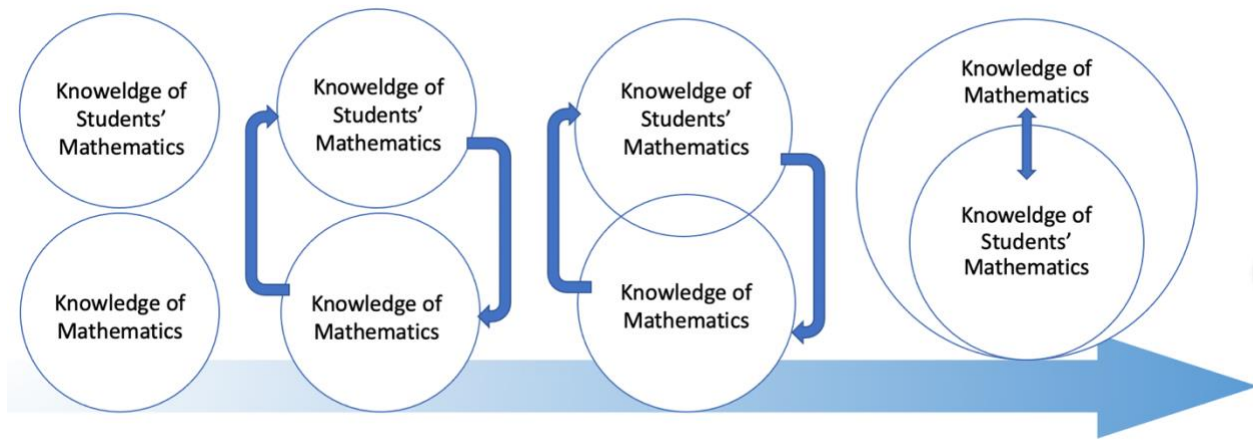


Figure 4. Relationships between knowledge of mathematics and knowledge of students' mathematical thinking as a spectrum.

Despite the fruitful evidence that teachers' constructions of KoSM are generative to their personal mathematical knowledge, there has been lack of research about how such learning occurs, including what kinds of mathematical reasoning or understandings enable or condition teachers' constructions and internalizations of KoSM. Insights into these reasoning processes will allow researchers to operationalize a dynamic or adaptive perspective on teacher knowledge, both in terms of how a teacher's KoSM develops on its own and how it is consequential to the development of content knowledge. This dissertation is focused on conceptualizing and unpacking these processes.

Another binary approach to framing KoSM is a deficit view on student thinking (Bray, 2011; Hill et al., 2008; Son, 2013; Tirosh, 2000) versus an anti-deficit view (Carpenter et al., 1996; Harel, 2008a; J. Hunt & Tzur, 2017; Silverman & Thompson, 2008). Researchers or teachers taking a deficit view of student thinking often frame student thinking as misconceptions or errors when their thinking leads to non-normative mathematical solutions from the researchers or teachers' perspectives. In other cases, they make claims about the absence of knowledge, for example, claiming a student lacks certain knowledge. What they actually mean is that the

observed student does not possess the knowledge that an observer possesses. Alternatively, an educator taking an anti-deficit view of student thinking believes students' mathematics is rational and legitimate, focusing on what students can do and building upon their thinking instead of fixing it. Some researchers also seem to adopt a combination of both perspectives, considering students' solutions as erroneous but emphasizing the importance of understanding students' reasoning and rationale underlying their solutions (Even & Tirosh, 1995; Marks, 1990). Therefore, an individual's views of student thinking can also be considered as a spectrum (Figure 5). An important question for mathematics teacher educator is that: How can teacher educators support teachers shifting from recognizing and evaluating non-normative student thinking to understanding students' rationality, and to truly appreciating and valuing students' reasoning and learning from the students (i.e., thinking towards an anti-deficit view of student thinking)? As mentioned in the previous paragraph, identifying the complex and effortful reasoning processes entailed in teachers' learning about student thinking may be critical to answer this question.



Figure 5. Deficit- and anti-deficit views of student thinking as a spectrum.

Last but not the least, I observe researchers drawing on different types of evidence to make claims about KoSM. For example, some researchers considered teachers' sentiments about students mathematical ability or general comments about student thinking as evidence of their increased KoSM —e.g., “I thought this could end up being too challenging for them, but it wasn't” (Wilkie, 2016, p. 262). In such a case, a reader is left wondering: What were the mathematical aspects of the problem that made the teacher think it could be challenging for the students? What were the mathematical understandings the teacher inferred from the students that

made them think it was not challenging? By contrast, some researchers specified these mathematical understandings when making claims about teachers' KoSM. In Chapter 4, I discuss my perspective of teacher knowledge in alignment with the latter approach to KoSM.

Discourse Moves, Listening, and Noticing

In this section, I synthesize literature on teacher moves, patterns of student-teacher conversations, listening, and teacher noticing. Each area has shed light on the important role of student thinking in teaching and learning.

Discourse Moves and Patterns

In addition to research on KoST, there has been rich research on communicative acts that support (or constrain) teachers' construction of such knowledge. Jacobs and Empson (2016) defined *responsive teaching* as "a type of teaching in which teachers' instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction in response to children's content-specific thinking, instead of being determined in advance" (p. 185). Numerous researchers have used discourse analysis to analyze how teachers talk and act in ways responsive or non-responsive to students. These researchers primarily focus on identifying characteristics of teachers' discourse moves or discourse patterns evident in student-teacher interactions. Different than teacher noticing and listening (discussed later), teachers' discourse moves sensitive to student thinking are observable actions performed by the teachers for eliciting and responding to student thinking. Researchers typically foreground their discussions on how teachers act rather than what drive the teachers to act in particular ways (e.g., what teachers know, hear, attend to, and interpret). In the following, I synthesize two groups of literature in this area: (a) modes and patterns of student-teacher interactions, and (b) teacher moves.

Modes and Patterns of Student-Teacher Interactions

Researchers have characterized modes and patterns of student-teacher interactions and of communication in general (e.g., Brendefur & Frykholm, 2000; Cazden, 2001; Knuth & Peressini, 2001; Mortimer & Scott, 2003; Truxaw & Defranco, 2008; Voigt, 1995; Wegerif, 2008; Wood, 1998). Common discourse patterns in classrooms include IRE, IRF, and IRFRF, where “I” stands for initiation (normally through teacher questioning), “R” stands for response (from the student), “E” stands for evaluation (by the teacher), and “F” stands for feedback (by the teacher) (Cazden, 2001; Mortimer & Scott, 2003). In addition, Wood (1998) distinguished a *funneling* pattern—where a teacher asks a series of guiding questions to lead a student to a correct answer—from a *focusing* pattern—where a teacher orients their questions to a particular aspect of student thinking.

Brendefur and Frykholm (2000) defined four communicative approaches in mathematics classrooms: uni-directional communication, contributive communication, reflective communication, and instructive communication. In *uni-directional communication*, teachers take the authority in the classroom and “tend to dominate discussions by lecturing, asking closed questions, and allowing few opportunities for students to communicate their strategies, ideas, and thinking” (Brendefur & Frykholm, 2000, p. 126). *Contributive communication* refers to communication among students and between teacher and students that is limited to assisting each other and sharing or presenting ideas, “often with little or no deep thought” (p. 126). In contrast, *reflective communication* refers to students and teachers using some ideas in prior conversations for deeper investigations and explorations through reflection. This notion of reflection aligns with Cobb et al. (1997) and von Glasersfeld’s (1991) conceptualizations of reflection that requires students and teachers re-presenting a chunk of their experience (e.g., what a student has

previously done or said) and discuss that experience explicitly in the discourse. The last type of communication, *instructive communication*, is drew upon Steffe and D'Ambrosio (1995)'s suggestions on conceiving communication as playing a role in bringing forth, sustaining, encouraging, and modifying the mathematics of students. In this regard, communication is not only interactive and reflective but also instructional and goal-oriented, implying that teachers not only need to understand students' thought processes and their limitations and strengths, but also began to shape instructional actions sensitive to student thinking (Brendefur & Frykholm, 2000).

Teacher Moves

Other researchers have focused on identifying certain actions of teachers as they interact with students. There has been an ongoing debate on whether teachers should *tell* when leading discussions. As Baxter and Williams (2010) asserted, “teachers are faced with what we called the dilemma of telling: how to facilitate students coming to certain understandings, without directly telling them what they need to know or to do” (p. 8). Baxter and Williams illustrated how middle school mathematics teachers successfully managed this dilemma by using *social scaffolding* and *analytic scaffolding* (Baxter & Williams, 2010; Williams & Baxter, 1996). An alternative way to resolve this dilemma is to propose a reformulation of telling as *initiating*—which is a set of teaching actions that introduce new ideas into a classroom conversation for the purpose of stimulating students' mathematical constructions—followed by *eliciting*—which is “an action intended to ascertain how students interpret the ideas introduced by the teacher” (Lobato et al., 2005, p. 111).

In addition to initiating and eliciting, researchers have identified many other questioning and responding moves sensitive to students' mathematical thinking. These moves include, among others: *inviting* (or *eliciting*, or *initiating*) (Fraivillig et al., 1999; Franke et al., 2015;

Shaughnessy & Boerst, 2018; Staples, 2007), *revoicing* (Eckert & Nilsson, 2017; Ellis, Özgür, et al., 2019; Forman et al., 1998; Forman & Ansell, 2002; Herbel-Eisenmann et al., 2009; O'Connor & Michaels, 1993; Sleep & Boerst, 2012), *supporting* (or *follow-up*, or *facilitating*, or *probing*) (Fraivillig et al., 1999; Franke et al., 2015), *extending* (Cengiz et al., 2011; Fraivillig et al., 1999), *conferring* and *nudging* (Munson, 2018, 2019), and *focusing* (Drageset, 2014; Herbel-Eisenmann & Breyfogle, 2005; Wood, 1998). Teachers' enactment of these teacher moves indicates they value student' contribution to a discussion and attempt to build upon what they interpret to be students' mathematical thinking. In contrast, teacher moves of *redirecting* (Drageset, 2014; Lineback, 2015), *progressing* (Drageset, 2014), and *funneling* (Herbel-Eisenmann & Breyfogle, 2005; Wood, 1998) entail more authority taken by the teacher rather than the students.

Researchers have also proposed frameworks that include a combination of teacher moves for analyzing classroom discourse through a certain lens (e.g., Cengiz et al., 2011; Conner et al., 2014; Ellis et al., 2019; Fraivillig et al., 1999; Franke et al., 2009; Krussel et al., 2004; Leikin & Dinur, 2007; Son, 2013; Staples, 2007; van Es et al., 2014) or guiding teachers to enact practices sensitive to student thinking (e.g., Herbel-Eisenmann et al., 2013; Hufferd-ackles et al., 2004; Leatham et al., 2015; Stein et al., 2008; Van Zoest et al., 2017). For instance, Ellis's (2019) Teacher Moves for Supporting Reasoning (TMSSR) framework places the eliciting, responding, facilitating, and extending moves on a continuum according to their potential for supporting student reasoning. Leatham et al.'s (2015) framework called Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOSTs) is a tool that guides teachers to analyze the mathematical and pedagogical potential of student thinking during instruction.

While these teacher moves are distinct, several researchers have pointed out that there is no strict boundary between productive and unproductive moves. For instance, redirecting is not necessarily unproductive in certain situations, depending on when and how teachers enact the move (Lineback, 2015). Revoicing may be commonly used to invite students' participation, but it may also seem controlling, manipulative, and authoritative to some students (Herbel-Eisenmann et al., 2009). Therefore, in addition to identifying forms and features of teachers' communicative acts, it is also important to account for teachers' intentions, orientations, and interpretations of a particular action, as well as the practical rationality they operate on (Herbst & Chazan, 2003; Lobato et al., 2005; Stockero et al., 2020). Moreover, it is crucial to reveal teachers' in-the-moment reasoning with students' mathematics and personal mathematics that governs their instructional actions (Teuscher et al., 2016).

Listening

Another lens through which researchers study teaching practices sensitive to student thinking is teacher listening. Listening is initially introduced as a technique of interviewing in qualitative research, and then Jere Confrey conceptualized listening as a feature of constructivist teaching, and Brent Davis used listening as a construct for analyzing mathematics teaching.

Piaget, Ginsburg, and Seidman: Listening as a Methodology

In Piagetian clinical interviews, listening as opposed to talking is of utmost important. Ginsburg (1997) made it clear that when conducting a clinical interview:

The child should do most of the talking, not you. Of course, you have to be active, asking penetrating questions. But you should spend most of your time listening to what the child has to say and observing. If you are talking more than the child, something is wrong. (p. 148)

To Ginsburg, listening is framed as a refrain from talking and telling, and it provides interviewees space to articulate their thought deeply and concretely. In his book, *Interviewing as*

Qualitative Research, Seidman (1991/2006) adopts a similar stance and discusses three levels on which an interviewer must listen. On the first level, an interviewer must “concentrate on the substance to make sure that they understand [the interviewee’s talking] and to assess whether what they are hearing is as detailed and complete as they would like it to be” (p. 78). On the second level, an interviewer must elicit the interviewee’s inner voice as opposed to outer or public voice. In doing so, the interviewer seeks for information that is more specific, elucidative, and elaborated than general comments or descriptions targeted to public audience (e.g., “I am confused about whether 6 is an even or odd number because 6 is 3 groups of 2” versus “even and odd numbers are challenging concepts”). On the third level, an interviewer must remain aware of the interviewing process while listening, including monitoring time, coverage, and interviewees’ energy level and nonverbal cues. As Seidman stated, “interviewers must listen hard to assess the progress of the interview and to stay alert for cues about how to move the interview forward as necessary” (p. 79).

Confrey: Close Listening

Adapting from Piagetian clinical interviews, Confrey (1993, 1998) defined *close listening* as an approach to examining student thinking deeply. Close listening demands a teacher to devote considerable amounts of time to converse with students and test and refine their interpretations of students’ thinking, including students’ experienced perturbations in a given problem context (Confrey, 1993, 1998). A listening teacher not only applies their sophisticated content knowledge and knowledge of students to explain student thinking but also has a propensity to “use student methods as opportunities to examine, strengthen, and revise their own understanding of mathematics” (Confrey, 1993, p. 301). Confrey (1998) further introduced the

distinction between *voice* and *perspective* to describe this dynamic relationship between one's own perspective and hypothetical models of student voice in the context of listening:

One heuristic is to view voice as embedded in perspective and try to disengage it, then reverse the relationship by viewing one's own perspective in light of the student's voice. As one learns to be more and more aware of this dialectic, one can come to see how frequently what is labeled as students' inadequacy is really the result of our own inflexibility in considering alternative perspectives. (p. 110)

Confrey's discussion illuminates the bidirectional interactions between a teacher's content knowledge and their knowledge of students during student-teacher interactions and interpretation. More importantly, she pointed out that a teacher's deficit view on student thinking is in fact a constraint experienced by the teacher. Close listening affords teachers an opportunity to (a) have access to student voice and conceive the rationality within a student's framework and (b) reconstruct their own perspective by reflecting on their own unexamined assumptions or aspects in light of student voice.

Davis: Three Modes of Listening

Davis (1997) offered a framework for examining mathematics teaching that includes three modes of listening: evaluative listening, interpretive listening, and hermeneutic listening. *Evaluative listening* is characterized by teachers' listening to students' thinking with the intent of "[assessing] the correctness of student responses" (Davis, 1997, p. 365; see also *assessment monitoring* in Chi et al., 2004 and *directive listening* in Empson & Jacobs, 2008). Although the teachers are attending to the verbal utterances of students, they are primarily operating within their own framework and seeking for evidence (in the form of students' observable actions) that either confirm or contra-indicate the teachers' own understandings. The teacher may also actively try to "elicit the expected response even when it is inconsistent with the child's understanding" (Empson & Jacobs, 2008, p. 268). This mode of listening overlaps with

Seidman's first and third level of listening in that listening is for the purposes of judging the students' contributions against a pre-determined answer, standard, or goal.

In contrast, teachers who are oriented to *interpretive listening* are endeavoring to accessing (as opposed to assessing) "the sense [students] were making" (Davis, 1997, p. 365), similar to Seidman's (2006) second level of listening, Chi et al.'s (2004) *diagnosis monitoring*, and Empson and Jacobs' (2008) *observational listening*. Wallach and Even (2005) further unpacked four types of interpretive listening, including (a) *over-hearing* (i.e., over-generalizing what students have said), (b) *compatible-hearing* (i.e., interpretations can explain student solutions), (c) *under-hearing* (i.e., hearing something said by the students but missing some parts), and (d) *non-hearing* (i.e., ignoring a whole part of work of students).

Although an interpretive listening teacher conveys a genuine interest in the substance of student thinking, the teacher may not intend to respond to the student by supporting or extending their thinking (Empson & Jacobs, 2008). When teachers engage in *hermeneutic listening*, the teachers interact with their students in a "negotiated and participatory" manner (Davis, 1997, p. 369). Hermeneutic listening involves teachers revising instructional activities in response to students' mathematical thinking and being open to interrogate and transform their existing mathematical understandings (see also Empson and Jacobs' (2008) *responsive listening*, T. J. Smith's (2003) *attentive listening*, and Yackel et al.'s (2003) *generative listening*). Furthermore, because hermeneutic listening teachers are full participant in classroom conversations, mathematical authority is shared by the students and teachers (Davis, 1997; T. J. Smith, 2003). Hermeneutic listening thus also promotes a view of teaching as an emotional and relational practice (T. J. Smith, 2003). T. J. Smith suggested that the pedagogy of mathematic should be

conceptualized as reciprocal conversation, and that hermeneutic (or attentive) listening helps sustain caring, receptive, and empathic student-teacher conversations. She stated:

the term *conversation* suggests an educative relationship, or way of being with learners that implies negotiation and consensus, reciprocity, opportunities for the creation and connection of knowledge, and above all attentive listening. (p. 497)

Noticing

Noticing is another research area that has received rapid attention in the past two decades. Building on Goodwin's (1994) *professional vision* and Mason's (2002) *discipline of noticing*, M. G. Sherin et al. (2011b) defined *teacher noticing* as "the processes through which teachers manage the 'blooming, buzzing confusion of sensory data' with which they are faced, that is, the ongoing information with which they are presented during instruction" (p. 5). The introduction of the construct of teacher noticing has significantly advanced the field of education by linking areas of teacher knowledge and teacher actions (Blömeke et al., 2015; Kaiser et al., 2017; M. G. Sherin et al., 2011b). On one hand, teacher noticing research focuses on teachers' in-the-moment perception, interpretation, and decision making as opposed to the product of teachers' decision making in form of instructional actions. On the other hand, teacher noticing differs from teacher knowledge in that "[t]he word *noticing* names a process rather than a static category of knowledge...the focus is on how, at a fine-grained level, the teacher interacts with the classroom world rather than solely on a teacher's reasoning" (M. G. Sherin et al., 2011b, p. 5).

Overview of Research on Teacher Noticing

Across diverse conceptualizations of aspects of teacher noticing in the literature, three key activities are considered (see also *three conceptualizations of teacher noticing* in Jacobs & Spangler, 2017, Figure 6):

- **attending:** selectively attending to particular events in an instructional setting; what teacher see, perceive, and draw attention to; a process of filtering sense data (Star et al., 2011; Star & Strickland, 2008).
- **interpreting:** making sense of events in an instructional setting in addition to attending (Goodwin, 1994; M. G. Sherin, 2007; M. G. Sherin & Van Es, 2009; van Es & Sherin, 2002).
- **deciding:** deciding how to respond (as opposed to enact the response) in addition to attending and deciding (Jacobs et al., 2010; K. R. Leatham et al., 2015; Monson et al., 2020).

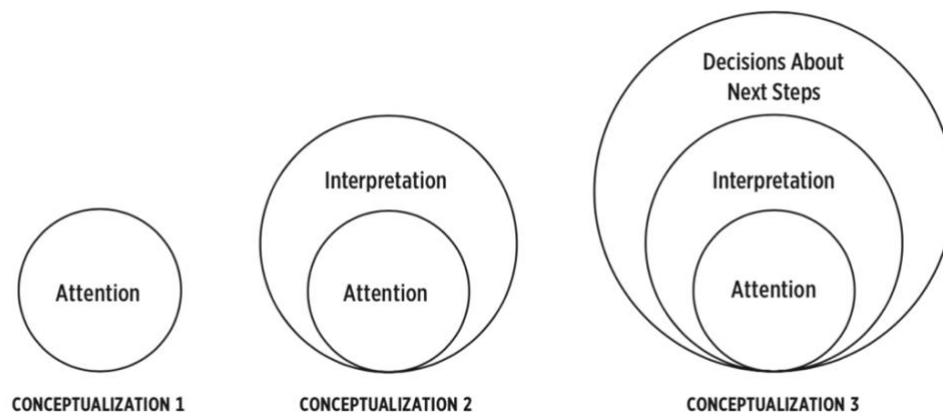


Figure 6. Three conceptualizations of teacher noticing (Jacobs & Spangler, 2017, p. 773).

Recently, van Es and M. G. Sherin (2021) have expanded the conceptualizations of noticing to include *shaping*—that is—“constructing interactions, in the midst of noticing, to gain access to additional information that further supports their noticing” (p. 773).

Adopting these constructs, mathematics education researchers have conducted fruitful research on *what* teachers notice (e.g., whom and what classroom features) and *how* teachers notice (e.g., stance, depth, specificity) in a variety of educational contexts. These contexts include in-the-moment and retrospective analysis of classroom instruction (e.g., Scherer &

Steinbring, 2007; van Es & Sherin, 2002), analysis of students' written work (e.g., Dick, 2017; Lee, 2020b; Monson et al., 2018; Shin, 2020), technological-rich environment (e.g., Ng & Chan, 2020; R. Smith et al., 2017), lesson study (e.g., Amador et al., 2016; Amador & Carter, 2018; Lee, 2019), and clinical interviews or teaching experiment (e.g., Amador, 2016, 2020; Fernandes, 2012; Lee, 2020a; Monson et al., 2018; Schack et al., 2013).

Regarding *what* teachers notice, researchers have focused on teacher noticing of (a) students versus teachers (e.g., Mitchell & Marin, 2015; M. G. Sherin & Van Es, 2009; Stockero, 2008, 2014; van Es & Sherin, 2008), (b) pedagogy (e.g., task use, representation use, teacher moves) (e.g., Dreher & Kuntze, 2015; Kersting, 2008; Lee, 2020b; McDuffie et al., 2014; Roller, 2016; Star & Strickland, 2008; Stockero, 2008; van Es, 2011; van Es et al., 2017; van Es & Sherin, 2008), (c) student thinking (e.g., Amador, 2016; Jacobs et al., 2010; Krupa et al., 2017; Lee, 2018), (d) classroom management or climate or environment (e.g., Amador, 2017; Mitchell & Marin, 2015; M. G. Sherin & Van Es, 2009; Star et al., 2011; Star & Strickland, 2008; Stockero, 2008; van Es, 2011; van Es & Sherin, 2008), (e) student-teacher or student-student communication (e.g., Scherer & Steinbring, 2007; Star & Strickland, 2008), (f) equitable issues (e.g., power, participation, bilingualism) (e.g., Dominguez & Adams, 2013; Fernandes, 2012; Hand, 2012; McDuffie et al., 2014; Turner et al., 2012; Wager, 2014), and (g) curricula (e.g., Amador et al., 2017). Mitchell and Marin (2015) reported that teachers in their study were not used to noticing students' mathematical thinking, with only (on average) 19% of the instances teachers identified were related to students' mathematical thinking compared to 42% related to pedagogy.

Regarding *how* teacher notice, I focus my following discussion on how teachers notice students' mathematical thinking due to its relevance to the current dissertation.

Teacher Noticing of Students' Mathematical Thinking

Available literature conveys four major findings regarding teacher noticing of student thinking. First, researchers have found that some teachers make *specific, elaborated, and in-depth* comments about students' mathematical thinking while others only make *general* comments or restate student's ideas (e.g., Kersting, 2008; Schack et al., 2013; Stockero, 2014; Teuscher et al., 2017; van Es, 2011; van Es & Sherin, 2008). Additionally, Lee (2018) proposed a Framework for Evaluating Quality of Teachers' Noticing Expertise, in which two major criteria are *descriptiveness* and *generality* of a teachers statement about student thinking (see Figure 7).

Main Aspects	Sub-criteria	Description	References
Descriptiveness	Thoroughness	This criterion is about whether a teacher has provided enough background information (context and tasks).	(Norton, et al., 2011)
	Evidence-based	This criterion is about whether a teacher has included explicit evidence that supports the model from details of the student's work in a segment of data selected by a teacher.	(Norton, et al., 2011)
	Generalizable statements	This criterion is about (1) whether a teacher has provided statements that indicate a generalizable feature of the student's thinking, rather than just a description of a student's solution to a specific problem and also (2) whether the generalizable statements are applicable to describing other students' thinking.	(Cobb & Steffe, 1983; IMBP, 2010)
	Faithfulness	This criterion is about (1) whether generalizable statements are faithful to a description of a student's work from a selected part of the data and (2) whether they fit with evidence from all data sources.	(Steffe & Thompson, 2000)
Generality	Predictability	This criterion is about whether others can use the model to predict how that student would respond to a similar question or task.	(Norton & McCloskey, 2008)
	Research-based knowledge	This criterion is about whether a model explicitly depends on existing theoretical constructs, such as counting schemes, developed by other researchers.	(Cobb & Steffe, 1983; Steffe & Thompson, 2000)
	Awareness of constraints in making a generalizable statement	This criterion is about whether a teacher recognizes what kinds of additional information he/she needs in order to create a more viable model.	(Cobb & Steffe, 1983)

Figure 7. Lee's (2018) Framework for Evaluating the Quality of Teachers' Focused Noticing Expertise.

Second, some teachers take an *evaluate* stance when noticing students' mathematical thinking while others take a *descriptive* or *interpretive* stance (e.g., Mitchell & Marin, 2015; M. G. Sherin & Van Es, 2009; Shin, 2019; Stockero, 2008; van Es, 2011; van Es & Sherin, 2008). Teachers may also make *root interpretations* in an attempt to "identify possible reasons behind

the student mathematical thinking or what the student's thinking might mean in relationship to their understanding of the mathematics" (Teuscher et al., 2017, p. 39). Additional stances include *confronting*, *restructuring*, *theorizing*, *explaining*, *overgeneralizing*, and *comparing* (Shin, 2019; Stockero, 2008).

Third, some researchers have focused on ways teachers use evidence to support their claims about student thinking, including the extent of evidence (e.g., lack of, limited, robust), source of evidence (e.g., transcript, student work), and whether or not teachers connect evidence to their interpretations (e.g., Amador, 2020; Ivars et al., 2019; Krupa et al., 2017; Lee, 2020b; Sánchez-Matamoros et al., 2019; Stockero, 2008; van Es, 2011). Lee (2020) reported that about 83% of the PSTs in her study gave limited or no evidence for their interpretations.

Fourth, some researchers have studied teachers' ability to attend, interpret, and decide to respond (e.g., Krupa et al., 2017; Lee & Cross Francis, 2018; Monson et al., 2018; Sánchez-Matamoros et al., 2019; Schack et al., 2013; Simpson & Haltiwanger, 2017; Teuscher et al., 2017). A common finding is that deciding to respond is the most demanding when compared to attending and interpreting (Jacobs et al., 2010; Krupa et al., 2017; Sánchez-Matamoros et al., 2019; Schack et al., 2013). Moreover, these researchers found that teachers' ability to decide to respond does not necessarily increase with more teaching experience but requires more focused coursework and professional development experiences.

Summary and Research Gaps

In summary, research on teacher discourse aims at identifying features of teacher moves and patterns of student-teacher communication. Research on listening focuses on manners in which teachers listen to their students, including the purposes of their listening and

characteristics of what is heard. Research on teacher noticing characterizes properties of teachers' attention, interpretation, and subsequent actions centered around student thinking.

Although these lenses and the related constructs differ in significant ways, they overlap with each other to certain extent. For example, *hermeneutic listening* involves a teacher listening for the purposes of engendering changes in students' thinking and adjusting instructional moves sensitive to students' thinking, which is similar to *deciding to respond* in the noticing framework or noticing at the *extended* level. As another example, Crespo (2000) studied teacher listening in the context of interpreting students' mathematics letters exchanged with their teachers. The author found that teacher's *interpretive listening* shifted from attending to correctness and making quick and conclusive comments to making thoughtful and tentative comments. These findings are parallel to van Es's (2011) characterization of noticing to be *evaluative* versus *interpretive* and *general* versus *specific*. In fact, there is no clear distinction between the constructs of noticing and listening. More precisely, their distinction is not easy to be operationalized. To put it another way, given an instance of a teacher interacting with a student, can a researcher differentiate teacher noticing from teacher listening? From my point of view, these are two highly parallel lenses for analyzing student-teacher interactions rather than two distinct practices.

There are also some connections between the work on listening and discourse. *Interpretive listening* is likely accompanied by *eliciting* and *revoicing* moves, and *hermeneutic listening* likely requires uses of *supporting* and *extending* moves. A teacher engaging in *hermeneutic listening* likely communicates with students in a manner aligned with *instructive communication*, and *interpreting listening* is associated with *contributive* and *reflective communication*. Mortimer and Scott's (2003) four classes of communicative approach contrast

an *interactive* approach with a *non-interactive* approach and contrast a *dialogic* approach with an *authoritative* approach. Relatedly, as a teacher shifts from orienting to *evaluative listening* to *interpretive* and *hermeneutic listening*, the teacher demonstrates decreasing level of authority and increasing amounts of interactions with students. To conclude, the lens of listening captures the teacher's orientation to student thinking, and the lens of discourse describes the resulting observable behaviors and communication patterns when a teacher engages in a certain listening mode. A researcher's inference of a teacher's listening orientation is likely based on the discourse features of a student-teacher conversation.

Conditions and Affordances

Empirical research adopting these lenses and constructs has provided evidence that teachers' enactment of these practices are supported and constrained by teachers' content knowledge and knowledge of students (e.g., Bray, 2011; Courtney, 2010; Dick, 2017; Dreher & Kuntze, 2015; E. Johnson & Larsen, 2012; Şen Zeytun et al., 2010; Speer & Wagner, 2009; A. G. Thompson & Thompson, 1996; P. W. Thompson & Thompson, 1994). Moreover, this body of work has illuminated a list of affordances of teachers' enactment of listening, noticing, and teacher moves sensitive to student thinking (informed by Empson and Jacobs [2008]):

- improving children's conceptual understandings and mathematical achievement (e.g., Carpenter et al., 1989; Fennema et al., 1996; Richards, 1996; Saxe et al., 1999; Villaseñor & Kepner, 2020; S. M. Wilson & Berne, 1999)
- providing a means of formative assessment (e.g., Jacob et al., 2017; Lee, 2013; Sleep & Boerst, 2012; Steinberg et al., 2004; Supovitz et al., 2018; Sztajn et al., 2012; Weiland et al., 2014)

- engendering pedagogical reasoning and decisions related to student thinking (e.g., selection of tasks, response to students) (e.g., Choppin, 2011; Doerr, 2006b; Dyer & Sherin, 2016; Fennema et al., 1996; Kazemi & Franke, 2004; Lee & Cross Francis, 2018; P. H. Wilson et al., 2015)
- restructuring mathematical authority from centering on the teacher to enabling student agency (e.g., Engle et al., 2014; Ng et al., 2020; D. Wagner & Herbel-Eisenmann, 2014)
- supporting teachers' (continual) pursuit, testing, and generation of knowledge of students' mathematics (e.g., Doerr, 2006a; Doerr & English, 2006; Franke et al., 2001; Jenkins, 2010; Kazemi & Franke, 2004; Seymour & Lehrer, 2006; Steinberg et al., 2004; Tirosh, 2000; Wilkie, 2016)
- increasing teachers' mathematical knowledge (e.g., Doerr & English, 2006; Philipp et al., 2007; Schifter, 1998; Silver et al., 2007)

Research Gaps

Despite the productivity of these frameworks and constructs, B. L. Sherin and Star (2011) acknowledge that noticing research only focuses on the emergent attributes (e.g., what people notice about, the level of specificity of what they notice) but not the underlying machinery of noticing. As they clarified:

When we say that teachers are “attending to pedagogy” in their comments, we are saying only what their comments are *about*, from a researcher’s point of view, not what they were perceiving...they do not, in any direct way, tell us anything about the underlying noticing machinery that produced those emergent features. (p. 76)

From my point of view, one way to understand the underlying noticing machinery is to understand characteristic reasoning processes involved in teacher noticing. For example, one may ask: How does a teacher operate differently when engaging in elaborated noticing than

general noticing? Why is deciding-to-respond nontrivial and effortful? What mental processes are required for and what factors condition teachers to deciding to respond?

Similarly, researchers investigating teacher listening only make claims about teachers' orientation to listen to student thinking based on patterns and characteristics of student-teacher interactions, and those who study discourse moves only focus on features of teacher actions. One may wonder: What is the content and nature of the knowledge that a teacher constructs when engaging in interpretive listening, and what knowledge transformation and modification does a teacher experience in hermeneutic listening? Given that evaluative listening is prevalent in student-teacher interactions (Chi et al., 2004; Crespo, 2000; Even & Tirosh, 1995; Heid et al., 1998; Nicol, 1998; Wallach & Even, 2005), one may wonder: How can teacher educators support teachers in shifting from evaluative to interpretive and hermeneutic listening? Related to noticing, how can teacher educators support teachers in transitioning from attending to interpreting to deciding, and how can teacher educators support teachers in enacting elaborated, in-depth noticing?

These questions motivate the present dissertation that aims to understand teachers' "underlying machinery," that is, the mental actions and operations necessary for a teacher to notice, listen to, and act upon students' mathematical thinking. In Chapter 8, I discuss how my findings may inform teacher education efforts in promoting teacher change in noticing, listening, and acting upon student thinking.

Decentering

Another area of research is called teacher decentering, which is focused on mental activities in relation to teachers' ability to think as students. This construct originated from Piaget's work on children *decentration* (cf. *centration*, *egocentrism*) starting from the mid 1900s.

In the late 1900s, scholars in child psychology and education picked up this construct to study children development in cognition and socialization, K-16 students' learning of mathematics and science, and student-student interactions in K-16 classrooms. A few researchers have also extended this construct to examine teacher decentering, attempting to understand how teachers decenter from their personal knowledge to engage with student thinking.

In this section, I categorized extant literature on decentering into four areas: (a) individuals' decentering associated with development of logico-mathematical operations, (b) individuals' decentering associated with their constructions of perspectives of others, (c) individual's decentering associated with social interactions, and (d) teacher decentering.

Development of Logico-Mathematical Operations

Piaget used the notion of *egocentrism* to describe children's centration on the self and lack of ability to separate themselves from the environment or other objects. It is non-trivial for an infant to model properties of other objects until they can decenter from the self and conceive of the objects existing independently (Ginsburg & Oppen, 1988; Piaget, 1947/2001, 1954/2013). Before children's achievement of object permanence, they believe an object does not exist until it is perceptually available. It is common to see children covering their own eyes when they intend to hide; they assume that they are invisible to others because they are invisible to themselves (Russell et al., 2012).

Summarizing the role of decentration in children development, Piaget stated:

Starting from a state of centration on a self uncognizant of itself and in which the subjective and objective are inextricably intermingled, the progressive decentration of the subject leads to a twofold movement, of externalization, tending to physical objectivity, and internalization tending to logico-mathematical coherence (Piaget, 1971, as cited in von Glasersfeld, 1982, p. 621).

As evidence of children's development towards logico-mathematical coherence, children at the concrete operational stage do not center their attention on one particular aspect of a situation;

rather, they can attend to and coordinate several aspects simultaneously (i.e., *decentering attention*) (Ginsburg & Oppen, 1988). By contrast, centration is “a cognitive fixation on the immediate goals of isolated personal activity, a focus on one’s own point of view and on what is imposed by perception” (Montangero & Maurice-Naville, 1997, p. 96). It is only through decentering that an individual shifts the locus of attention from their own physical actions and perception to logico-mathematical structure. For example, children can conceive of two bottles of liquid as having equal amount when one bottle is taller and narrower than the other. They realize that the amount-ness of the liquid preserves regardless of perceptual differences in height and width of the two containers. The child’s ability to preserve the quantitative structure of a situation (i.e., the amount-ness of water) is an evidence of achieving physical objectivity and logico-mathematical coherence. As Montangero and Maurice-N described, “decentering is the process whereby one frees oneself from initial egocentrism” (p. 96).

Another evidence of logico-mathematical coherence is related to individuals’ ability to decenter from particular states by disassociating their mathematical actions and results of those actions. Results of actions are the products generated by enactment of actions (e.g., creating a drawing, writing an equation). Actions are the underlying operations that enable that enactment; Piaget explained it as the “how” or I consider is as the “why” of an observable action. Piaget (1962) argued that one’s awareness of their action is nontrivial.

[A]wareness at first is limited to the results of actions and only later extends to the “how,” i.e., the operation itself...one understands that a subject whose perspective is determined by his action has no reason for becoming aware of anything except its results.

Researchers have provided ample evidence that many students struggle to disassociate their mental actions and results of those actions (Bednarz & Garnier, 1993; Liang & Moore, 2020; Moore et al., 2019). It is even difficult for some undergraduate students to coordinate their

mental actions and results in order to re-present, anticipate, and transform complex mathematical activities (Liang & Moore, 2020; Moore et al., 2019).

Then, Piaget (1962) continued to say, “decentering, on the other hand, i.e., shifting one’s focus and comparing one action with other possible ones, particularly with the actions of other people, leads to an awareness of “how” and to true operations.” Here, he made an important point that decentering triggers one’s shift from attending to results to attending to actions. This shift is achieved by the individual engaging in observing the results of others’ actions, comparing those results with their own (and being able to discern differences), and inquiring into the different actions that may cause those results. It is this understanding of the actions of others that leads to the individual’s increasing awareness of their own actions. In Chapter 6 and Chapter 7 of this dissertation, I discuss multiple instances of how an individual’s decentering and comparison of mental actions contribute to their increasing awareness of aspects of their own mental actions.

Constructions of Others: Thinking about Others’ Thinking

After differentiating self from the environment and other objects, a child is poised to construct others. Children shift from assuming others think like them to being aware of occasions where others may not think like them (von Glasersfeld, 1995). For example, a child, who is engaging in the Three Mountains Task (Figure 8) and viewing the mountains at Position A, may assume that another individual, who is viewing the mountains from Position C, can see the same objects as what themselves see (Piaget & Inhelder, 1948/1967). The child reveals “a type of spatial imagination which is likewise not yet decentrated as regards changes of positions, but remains centered on a position (or previously experienced positions) corresponding with [their] present viewpoint” (Piaget & Inhelder, 1948/1967, p. 217). In other words, the child is unable to decenter from the viewer-centered reference frame and consider the perspective of another

individual. Through decentering, the child frees themselves from centering on their own point of view and perception, so that they can insert “one’s own point of view in a set of possible points of view” and “the self in a universe of which it is no longer the center” (Montangero & Maurice-Naville, 1997, p. 96).

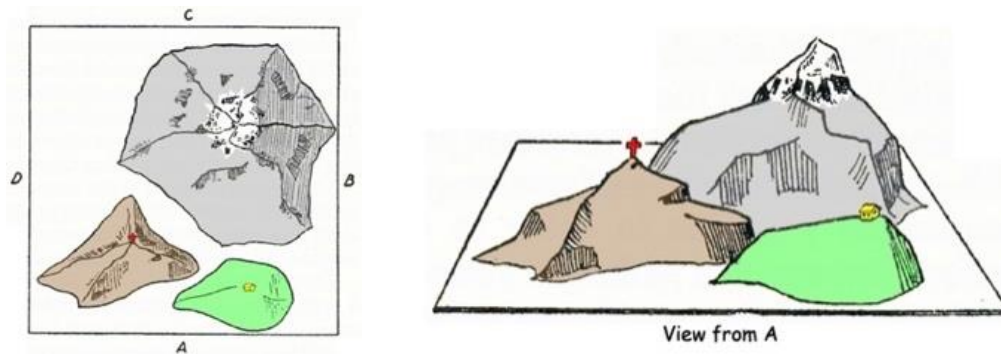


Figure 8. The Three Mountains Task (Piaget & Inhelder, 1948/1967, p. 211).

Child psychologists have also examined children’s reasoning about someone else’s thinking that differs from their own thinking or from other individuals’ thinking (e.g., Grueneisen et al., 2015; Hogrefe et al., 1986; Perner & Wimmer, 1985; Wimmer & Perner, 1983). Instead of adopting the construct of decentering, they distinguished a child’s *first-order belief* (i.e., a child’s belief about Person A’s thinking about a situation) from their *second-order belief* (i.e., a child’s belief about Person A’s thinking about Person B’s thought) and *higher-order belief* (i.e., a child’s belief about Person A’s thinking about Person B’s thinking about Person A). A well-known experiment about children’s second-order belief is conducted by Perner and Wimmer (1985) in the context of Ice-Cream Story:

This is a story about John and Mary who live in this village. This morning John and Mary are together in the park. In the park there is also an ice-cream man in his van.

Episode 1: Mary would like to buy an ice cream but she has left her money at home. So she is very sad. “Don’t be sad,” says the ice-cream man, “you can fetch your money and buy some ice cream later. I’ll be here in the park all afternoon.” “Oh good,” says Mary, “I’ll be back in the afternoon to buy some ice cream. I’ll make sure I won’t forget my money then.”

Episode 2: So Mary goes home. She lives in this house. She goes inside the house. Now John is on his own in the park. To his surprise he sees the ice-cream man leaving the park in his van. “Where are you going?” asks John. The ice-cream man says “I’m going to drive my van to the church. There is no one in the park to buy ice cream; so perhaps I can sell some outside the church.”

Episode 3: The ice-cream man drives over to the church. On his way he passes Mary’s house. Mary is looking out of the window and spots the van. “Where are you going?” she asks. “I’m going to the church. I’ll be able to sell more ice cream there,” answers the man. “It’s a good thing I saw you,” says Mary. Now John doesn’t know that Mary talked to the ice-cream man. He doesn’t know that!

Episode 4: Now John has to go home. After lunch he is doing his homework. He can’t do one of the tasks. So he goes over to Mary’s house to ask for help. Mary’s mother answers the door. “Is Mary in?” asks John. “Oh,” says Mary’s mother, “She just left. She said she was going to get an ice cream.” (p. 441)

A researcher would ask a child, “Where does John think Mary had gone?” The child is expected to have access to the information that the van is at the church (a zero-order belief) and that both John and Mary know this information as well (a first-order belief). What complicates the story is the fact that John is not aware that Mary and him share the same knowledge, and therefore the child needs to decenter from John’s, Mary’s, and their own perspectives in order to understand that John thinks Mary thinks the van is in the park (a second-order belief). A child’s ability to decenter in this situation requires them to coordinate multiple perspectives, including accounting for the absence of knowledge about Mary of John. Several psychology experiments have confirmed that children’s sophisticated understanding of second-order epistemic states develops during age 5-8 (Perner & Wimmer, 1985; Wimmer & Perner, 1983). It is remarkable that this is also the age when children transition from preoperational to concrete operational thinking, a benchmark for development of logical thought. Being able to re-present and reason with similar, different, and competing knowledge of different individuals is crucial for children’s social interactions and corporation with peers (Grueneisen et al., 2015; Perner & Wimmer, 1985).

Social Interactions

Social interaction is another activity that requires decentration. Piaget (1926/1959) has studied the critical role of decentration in children's development of speech and socialization. Young children commonly demonstrate egocentric thought by speaking only to themselves or of themselves, or speaking with the assumption that others can understand them in the same way as they do. When children gradually develop the ability to decenter (up until the age of about 7 or 8), they are able to simultaneously take into account what they want to express, their listeners' needs, and how the listeners may interpret their talking.

Drawing on this aspect of decentration, researchers have investigated the role of decentration in student-student interactions, including its productivity in promoting subject learning (Hoyles, 1985; A. Moon et al., 2017; Rasmussen et al., 2020). Hoyles (1985) found that pupil-pupil interactions facilitated pupils' mathematical growth by demanding them to (a) articulate their own thought processes and make those processes available to others, (b) frame thoughts in language such that meanings are conveyed to and acceptable by others, and (c) actively incorporate others' thinking into their own thinking through listening. As Hoyles (1985) concluded, decentration is "facilitated by the differing points of view amongst participants in a discussion and this on its own may stimulate [cognitive] progression" (p. 207). Echoing Hoyles' point, Rasmussen et al. (2020) and A. Moon et al. (2017) provided additional evidence that, as students engaged with peers' thinking through decentering, they were able to (re)engaged with their own thinking, which contributed to their conceptual growth in light of others' thinking. Furthermore, students' engagement with peers' thinking impacted the ways they participate in classroom discourse through argumentation and justification (A. Moon et al., 2017; Rasmussen et al., 2020). For example, student decentering can manifest itself in discursive features such as

the use of rebuttals and counter-arguments (A. Moon et al., 2017). When students engage in reciprocal construction of understandings with each other in a conversation, including the contradicting understandings aroused, they are poised to incorporate others' perspectives or argue against perspectives that they want to challenge or reject.

In summary, at early ages, children do not view objects, events, or other individuals in the environment as existing independently or having their own properties; they tend to only perceive every entity as relating to themselves. Gradually, they become capable of decentering their attention and realize that objects hold some invariant properties that are independent on human perception and actions. They also come to realize that different individuals may hold different perspectives regarding the same object, and they attempt to construct the perspectives of others. This realization plays a critical role in children's increasing tendency and ability to socialize with others, which in turn, promotes their own cognitive development.

Teacher Decentering

Dating back to 1994, Alba Thompson and Patrick Thompson described a student, Ann, and her teacher, Bill, miscommunicating with each other due to their holding different mathematical meanings of the concept of speed (P. W. Thompson & Thompson, 1994). The authors called for teacher education to focus on supporting teachers to develop their mathematical knowledge for teaching through:

sustained and reflective work with students and with mathematical ideas—comparing their attempts to influence students' thinking with disinterested analyses of what those students actually learn—and reflecting both on what they intended and what (they understand) they achieved” (A. G. Thompson & Thompson, 1996, p. 22)

Although the authors did not use the construct of decentering in their discussion, their findings shed light on the importance of teacher decentering and supporting teacher decentering.

Teachers' constructions of and reflection on students' mathematical thinking necessitate teachers to set aside their own thinking and put themselves in the shoes of the students.

Extending Piaget's work on decentration, Confrey, Steffe, Thompson, and colleagues have conceptualized teacher decentering during constructivist teaching (Confrey, 1990; Hackenberg, 2005, 2010; Steffe & Thompson, 2000a). They defined *teacher decentering* as a mental action of teachers setting aside their own ways of operating temporarily in order to understand the perspective of their students. "Setting aside" does not imply that teachers abandon or lose sight of their own mathematical reality (and in fact, they cannot) (Hackenberg, 2005). Instead, decentering is a disposition of orienting to the students' ways of operating rather than personal ways of operating. Decentering requires teachers to intentionally "make space to see realities that may differ significantly from their own" (Hackenberg, 2005, p. 47) and expecting "to be surprised (in the sense of delighted, stimulated, and challenged) by how students think" (Hackenberg, 2005, p. 49). A decentering teacher does not assume that their students would share same mathematical thinking with the teacher themselves; instead, the teacher actively constructs interpretations of the students' thinking and attempts to think as how the students think during their interactions with students or reflection on the students' mathematical activities.

Moreover, Confrey (1990) argued that:

Decentering, the ability to see a situation as perceived by another human being, is attempted with the assumption that the constructions of others, especially those held most firmly, have integrity and sensibility within another's framework. (p. 108)

Engaging in decentering to understand students' mathematical thinking requires the teacher to truly believe that students' mathematical experiences and their consequential understandings are legitimate, rational, and valuable, even though they are potentially distinct from that of adults' (Steffe & Thompson, 2000b). As Arcavi and Isoda (2007) summarized these two aforementioned aspects of teacher decentering:

As teachers, sometimes we may not only need to set aside our own knowledge and ways of knowing, but we also have to develop a capacity to ‘read through’ idiosyncratic ways of expression and learn to appreciate their underlying logic and potential. (p. 113)

Teacher Decentering as a Construct for Analyzing Student-Teacher Interactions

Carlson and her colleagues have conducted their pioneer work on mathematics teacher decentering by operationalizing this construct to analyze and explain authentic student-teacher interactions (Baş-Ader & Carlson, 2021; Carlson et al., 2007; Marfai et al., 2011; Teuscher et al., 2016). Teuscher et al. (2016) conceptualized and illustrated two types of student-teacher interactions that involved decentering and non-decentering teachers respectively. They approached teacher decentering by inferring from a teacher’s observable actions whether the teacher was intentional in hypothesizing students’ mathematical thinking and reflecting on aspects of students’ thinking. Baş-Ader and Carlson (2021) went beyond the binary of decentering and not-decentering to specify four levels of teacher decentering, each characterized by corresponding mental actions and observable behaviors (Figure 9). Teachers demonstrating the highest level of decentering appeared to make sense of students’ thinking and then build on and respect the rationality of their thinking. Mills et al. (2019) applied an earlier version of framework (Bas Ader & Carlson, 2018) to examine levels of mathematics tutors’ decentering actions in the moment of tutoring students.

Table 1. A framework that characterizes a continuum of graduate student instructors' decentering actions.

Mental Actions	Decentering Levels	Description of the Behaviors
- Does not reflect on aspects of the interaction that are contributed by the student - Assumes that the student's thinking is identical to her own - Uses first-order model during interaction to organize and guide her actions	Level 0: No Interest Shows no interest in the student's thinking, but shows interest in the student's answer. Makes no attempt to make sense of the student's thinking, but takes actions to get the student to say the correct answer	- Asks question to elicit the student's answer - Listens to the student's answer - Does not pose question aimed at revealing the student's thinking - Poses question that focuses on a procedure [e.g., <i>What do you do next?</i>] - Poses question to have the student echo key phrase or complete steps to get the correct answer - Poses question to reveal the student's thinking [e.g., <i>Why? What does that mean? What does that term represent?</i>], but does not attempt to understand the student's thinking - Guides the student toward her way of thinking - Poses question for the purpose of getting the student to adopt her way of thinking or has the student echo key phrase - Gives explanation aimed at getting the student to adopt her way of thinking - Evaluates how the student's response compares to her way of thinking or approach - Poses question to reveal the student's thinking and attempts to adopt the student's perspective - Does not use the student's thinking for the purpose of advancing the student's current way of thinking - Does not pose question aimed at prompting the student to reflect on or advance her current way of thinking - Redirects the conversation toward her way of thinking or approach
- Does not reflect on aspects of the interaction that are contributed by the student - Recognizes that the student's thinking is different than her own but does not intentionally analyze the student's mental actions - Uses first-order model during interaction to organize and guide her actions	Level 1: Interest Shows interest in the student's thinking, but makes no attempt to make sense of the student's thinking. Attempts to move the student to her way of thinking without trying to understand or build on the expressed thinking and perspective of the student	
- Reflects on aspects of the interaction that are contributed by the student - Recognizes that the student's thinking is different than her own and attempts to analyze the student's mental actions - Builds the second-order model of the student's thinking - Uses first-order model during interaction to organize and guide her actions	Level 2: Make Sense Makes an effort to make sense of the student's thinking and perspective but does not use this knowledge in communication	
- Reflects on aspects of the interaction that are contributed by the student - Assumes that the student has unique ways of thinking - Builds and continually adjusts the second-order model of the student's thinking - Operates from the second-order model of the student's thinking to make decisions on how to act	Level 3: Use Makes sense of the student's thinking and perspective and makes general moves to use the student's thinking when interacting with the student	- Prompts the student to explain the student's way of thinking - Poses question to gain insights into the student's way of thinking - Leverages the student's expressed thinking to advance the student's understanding of key ideas in the lesson - Poses question or gives explanation that is attentive to the student's thinking and aimed at advancing the student's understanding of the key idea - Poses question or gives explanation to support the student in making connections between different viable ways of thinking of the key idea

(Continued)

Table 1. (Continued).

- Reflects on aspects of the interaction that are contributed by the student - Assumes that the student has unique ways of thinking - Builds and continually adjusts the second-order model of both the student's thinking and how the student might be interpreting her utterance - Operates from the second-order model of both the student's thinking and how the student might be interpreting her utterance	Level 4: Use and Adjust Constructs an image of the student's thinking and perspective and then adjusts her action to take into account both the student's thinking and how the student might be interpreting her utterance	- Poses question to reveal and understand the student's thinking - Follows up on the student's response in order to perturb the student in a way that extends the student's current way of thinking - Poses question informed by the student's current way of thinking and meaning of the key idea - Gives explanation informed by the student's current way of thinking and meaning of the key idea

Figure 9. Framework for analyzing student-teacher interactions in terms of the teacher's mental actions and observable behaviors (Baş-Ader & Carlson, 2021, pp. 8–9).

Productivities and Conditions of Teacher Decentering

Teacher decentering is productive for mathematics teaching and learning in many aspects. Researchers have argued that teacher decentering can be an important avenue for generating teachers' mathematical knowledge for teaching (Silverman & Thompson, 2008; Walters, 2017). According to Silverman and Thompson's (2008) Framework for Mathematical Knowledge of Teaching (introduced earlier in this chapter), a teacher's constructions of MKT necessitate the teacher to put themselves in the place of a student and examine the operations the

student needs and the constraints the student operates under (i.e., decentering). Adopting Silverman and Thompson's (2008) MKT framework, Walters (2017) characterized how three teachers shifted their viewpoint to that of their students. Specifically, decentering is evidenced by the teachers being able to solve problems in ways similar to their students and anticipate students' solutions and conceptual difficulty. Walters' findings suggested that decentering, as opposed to egocentrism, not only requires reflective thought (see also Silverman & Thompson, 2008; Steffe & Thompson, 2000a), but is also generative to a teacher's personal mathematical growth (see also Liang, 2020). Adopting a point of view of students and reflecting on students' ways of operating serve as an important avenue for teachers to construct novel ways of thinking.

In turn, a teacher's ability to decenter is contingent upon sophisticated content knowledge (Silverman & Thompson, 2008). Without such knowledge, it would be difficult for the teacher to conceptualize meaningful ideas that they intend the students to construct, to understand what students have constructed, and to anticipate what students would be able to construct and potential perturbations they would experience. Together, I conjecture that teacher's engagement with decentering necessarily implies interactions between the teacher's personal knowledge and knowledge of students.

Decentering is also productive for teachers' learning to listen to student thinking. Teachers' capacities and orientations to listen to student thinking (especially for interpretive and hermeneutic listening) necessitate teachers to keep away from their own perspectives and pursue understandings of what they hear from students. Arcavi and Isoda (2007) implemented a teacher workshops where teachers exercised decentering through reading and understanding historical texts that introduced historical approaches to mathematical ideas. Their preliminary findings suggested that such experience supported teachers' orientation to interpretive listening to student

thinking. However, this experience also fostered value judgment of certain ways of thinking (i.e., reinforcing evaluative listening) because the participating teachers realized that some modern approaches to mathematics were more valuable than historical approaches: If one approach is so powerful and efficient, why do we bother to consider alternative perspectives? Arcavi and Isoda (2007) concluded that although their intervention design was faithful to the rationale of decentering, it was insufficient to train teachers to be good listeners given the complexity of listening practices. In particular, teachers need more opportunities and time to become truly appreciate alternative perspectives, including historical mathematics and students' mathematics, and to engage in authentic social interactions with students to practice listening and shape their own knowledge along the way.

Summary and Research Gaps

Current research knowledge about teacher decentering can be summarized as the following:

- A decentering teacher truly believes students' thinking is rational and likely distinct from the teacher's own thinking, and therefore they value the opportunity to set aside their personal thinking to engage with students' thinking (Confrey, 1990; Hackenberg, 2005; Steffe & Thompson, 2000b).
- Teacher decentering impacts the nature of student-teacher interactions (Baş-Ader & Carlson, 2021; Teuscher et al., 2016).
- Teacher decentering is contingent upon teachers' coordination of content knowledge and knowledge of students and conscious awareness of this knowledge (Silverman & Thompson, 2008; Tallman, 2015).

- Teacher decentering generates new content knowledge and knowledge of students (Liang, 2020; Silverman & Thompson, 2008; Walters, 2017).

Although decentering is introduced as a cognitive construct, extant literature on teacher decentering has been focused on its observable manifestations in student-teacher interactions and its relationship with teacher knowledge, few researchers have aimed to unpack the mathematical reasoning or mental processes underlying teacher decentering. The notion of decentering has been mostly used as a disposition, a label for a certain process, or a feature of teacher actions (as the notions of noticing and listening are used), but the characteristic mental actions of decentering, including the very processes by which teachers construct knowledge of students and subsequently enrich their personal mathematical knowledge, remain uninvestigated. A researcher's attempt to infer and generalize these mental processes requires the researcher to first conduct a fine-grained analysis of the mathematical understandings teachers develop during their interactions with students and reflection on students' mathematical understandings, and then infer the processes by which these understandings come into being. The current dissertation accomplishes this novel attempt by incorporating decentering, student-teacher interactions, and teacher knowledge development in a single study.

Other Modes of Reasoning with Student Thinking

Addressing the aforementioned research gaps, some researchers do document different modes of reasoning related to teachers' constructions of students' mathematics, such as anticipating (or predicting), evaluating, describing (or restating), explaining, comparing, and restructuring. In the following, I elaborate on these reasoning types and then discuss how the current dissertation contributes to this body of work.

Anticipating (or Predicting)

Anticipating student thinking involves a teacher making predictions about possible solution strategies, common student errors or misconceptions, and developmental pathways (Bergqvist, 2005; Carpenter et al., 1988, 1989; Nathan & Koedinger, 2000; Norton et al., 2011; Şen Zeytun et al., 2010; P. H. Wilson et al., 2015). Some researchers assessed teachers' ability to anticipate based on the extent their anticipation matched with students' actual performance. For example, Bergqvist (2005) found that teachers tended to underestimate their students' reasoning levels and believe only a small group of students could use higher level reasoning in mathematics. As another example, Norton et al.'s (2011) asked a group of prospective teachers to watch videos of individual student solving a mathematical task and then predict how the student would respond. The authors used a rubric to assess these teachers' prediction ability based on *accuracy* of predication—in terms of whether their predication matched with the student's actual response—and on *detail* of predication—in terms of the extent they used evidence to make predictions. In Nathan and Koedinger's (2000) study, they asked a group of teachers to ordered 12 arithmetic and algebra problems according to their predicted problem-solving difficulty for students. They found that teachers predicted that story problems and word-equation problems would be more difficult than symbol-equation problems to students, whereas students' performance suggested the opposite from the researchers' perspective.

Evaluating

Evaluating involves a teacher assessing a student's statement based on their own knowledge (An & Wu, 2012; Rasmussen et al., 2020; Schifter, 2001)³. On a superficial level,

³ I note that Rasmussen et al.'s (2020) study is in fact about *students'* engagement with peer students' thinking. I include the citation due to its potential of being generalized to the context of *teachers'* engagement with students' thinking. Namely, we can consider an interpreting student in Rasmussen's study as taking the role a teacher.

evaluating only requires a teacher to determine the correctness of a student's solution (An & Wu, 2012; Didis et al., 2016; Rasmussen et al., 2020; Wallach & Even, 2005). This can be evident by a teacher claiming a student's solution, or method, or way of thinking, being right or wrong, good or weak, or conveying agreement or disagreement to the student's utterances. On a deeper level, evaluating requires a teacher to assess the validity of a student's thinking, including reasoning about why the student's thinking leads to (or does not lead to) a mathematically correct solution and whether it is generalizable to other mathematical situations (Schifter, 2001).

Explaining

Explaining student thinking is the most prevalent modes of reasoning in extant literature. In the following, I discuss three forms of explaining: (a) explaining as interpreting or inferring, (b) explaining as modeling or cogitating, and (c) explaining as diagnosing.

Explaining as Interpreting or Inferring

Didis et al. (2016) defined explaining as follows:

Prospective teachers recognize the mathematically significant details of students' thoughts/actions reflected in students' work. Explaining includes prospective teachers' explicating the meaning assigned to students' thoughts/actions. Prospective teachers explain the details of students' thoughts/actions in a variety of ways such as making inferences about students' mathematical thinking, providing various possible explanations about their ways of thinking, or evidence to support their explanations. (p. 360)

This definition points to one form of explaining that is similar to other researchers' use of *interpreting* (Baldinger, 2019; E. Jacobson et al., 2018; Rasmussen et al., 2020; Sleep & Boerst, 2012) or *inferring* (P. H. Wilson et al., 2011, 2013), which requires a teacher to explain why a student thinks the way they think by making inferences about students' reasoning underlying their mathematical activities. Schifter (2001) framed this form of explaining as seeking students' mathematical logic to decide what and how to teach, saying, teachers need to:

appreciate [their] students' sense-making ability and looks for it even when their logic has gone awry...they must feel grounded in their capacity for mathematical reasoning. And in order for teachers to help, they must find the firm ground in their students' logic on which to build new ideas. (p. 128)

Some researchers have looked into features of teachers' assertions when they interpret student thinking (Baldinger, 2019; Norton et al., 2011; Sleep & Boerst, 2012). These researchers focus on whether or not a teacher uses evidence to support their assertions and whether the teacher's assertion is warranted by evidence they provide from a researcher's perspective. For example, Baldinger (2019) found that in some occasions, a teacher provided evidence not corresponding to the evidence available in the students' activities (called *evidence-evidence discrepancy*), and in some other occasions, they provided evidence that contradicted their assertion (called *assertion-evidence discrepancy*). In Sleep and Boerst's (2012) study, assertion-evidence discrepancy also occurred when some teachers' assertions were overreaching and beyond what can be inferred from the given evidence, and some teachers provided evidence not specific enough to support their assertions.

Another group of researchers have studied teachers' interpreting activities by relating them to research constructs of students' mathematical operations or strategies regarding a particular topical area (E. Jacobson et al., 2018; Son, 2013; Son & Crespo, 2009; Son & Sinclair, 2010). E. Jacobson et al. (2018) pointed out that it is necessary to gain insights into the qualitative and mathematical differences in teachers' interpretations of student thinking because their reasoning varies a lot even if their evaluations appeared to be the same.

Explaining as Modeling or Cogitating

Another form of explaining is *modeling* (Hallagan, 2006; Yoon, 2019), or what Hodkowski (2018) called *cogitation*, which is an enhanced and stable version of interpreting or inferring. Cogitation means a teacher's interpretation of a student's thinking has become stable

enough for providing hypothetical explanations for the student's observed behavior. In doing so, the teacher can explain a student's actions by articulating a collection of previous instances to support their inferences about student thinking. The teacher is able to cogitate about the student's thinking as separated from the teacher's own thinking and able to contemplate the mathematical reasoning of the student. Modeling and cogitating are closely related to and necessitate teacher decentering.

Explaining as Diagnosing

The third form of explaining is what I call *diagnosing*, which involves a teacher locating the source of a student's difficulties or inferring why the student thinks in particular ways (Rasmussen et al., 2020; Schifter, 2001). The teacher goes beyond understanding the student's thinking to think deeper about the fundamental conceptual issues the student might be faced with. In the quote mentioned above, Didis et al. (2016) described this form of explaining as "providing various possible explanations about their ways of thinking" (p. 360). This is a non-trivial task, and it likely requires a teacher to put their hypotheses into testing through sustain interactions with a student and then iteratively engage in other forms of explaining and modes of reasoning before identifying the fundamental conceptual issues of the student.

I note that the inferential feature is key to each of these three forms of explaining. Researchers have contrasted it with *describing* or *restating*, which only involves a teacher providing assertions close to what a student has said or done without reasoning through the mathematical reasoning as the student does (Baldinger, 2019; Didis et al., 2016; Wallach & Even, 2005; P. H. Wilson et al., 2011). P. H. Wilson et al. have used Figure 10 to illustrate this distinction between describing and inferring.

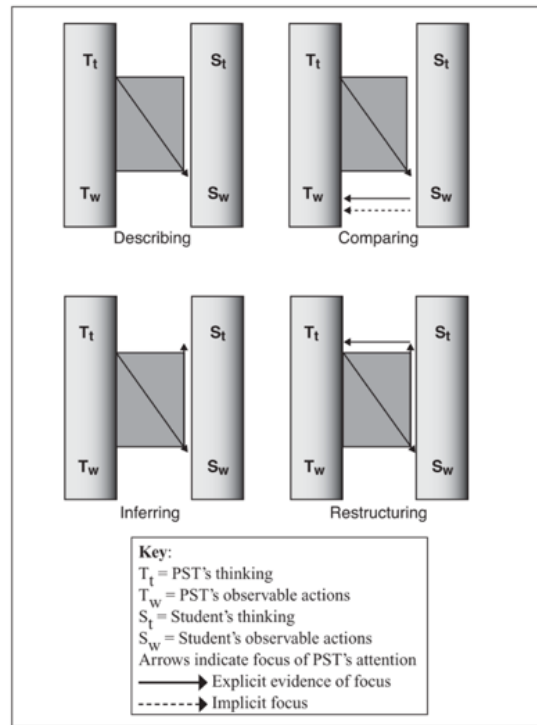


Figure 10. Wilson et al.'s (2011) models to illustrate describing, comparing, inferring, and restructuring (p. 52).

Comparing

Comparing refers to a teacher searching for similarities and differences between their own thought and/or observable actions and those of a student (Baldinger, 2019; Didis et al., 2016; Hodkowski, 2018; Sapti et al., 2019; P. H. Wilson et al., 2011, 2013). Hodkowski (2018) termed this mode of reasoning as *juxtaposition* and Rasmussen et al. (2020) termed it as *connecting*. P. H. Wilson et al. (2011) used comparing to only refer to comparison between observable actions (see Figure 10), and they distinguished *implicit comparison* from *explicit comparison*, with the latter requiring a teacher to articulate both observable actions explicitly and putting them side-by-side while the former refers to a teacher not making their own actions explicit. Sapti et al. (2019) further made a distinction between *comparing work*—meaning a teacher comparing the products of student thinking and the products of their own thinking—

and—*comparing knowledge*—meaning a teacher comparing the products of student thinking to their own knowledge.

Comparing can also occur in the context of making distinction “between one student’s mathematical reasoning and another’s mathematical reasoning, and between possible inferences of mathematical understanding within one student” (Hodkowski, 2018, p. 113). Hodkowski argued that engaging in these types of comparing indicates the teacher can entirely decenter from their own mathematics and operates as the students do.

Restructuring

The last mode of reasoning is called *restructuring*, through which teachers’ constructions of students’ mathematical thinking engender changes in the teachers’ own knowledge. Wilson et al. (2011, 2013) conceptualized restructuring as a teacher incorporating understandings of students into their own knowledge system and experiencing reorganization in their technological pedagogical content knowledge (TPACK). They stated, “models of students’ thinking become integrated into their own ways of thinking about mathematics, technology, and pedagogy and relationships among these three components” (p. 58). A teacher’s TPACK enables the teacher to make pedagogical decisions sensitive to students’ mathematical understandings and technological capacities.

Summary and Research Gaps

Anticipating, explaining, comparing, and restructuring represent four major ways in which teachers reason with student thinking, illustrating the complexity of this activity. What’s more, each of these reasoning processes are underlain by or contribute to interactions among different domains of teacher knowledge. Moving forward, I intend to contribute to this body of work by reconceptualizing teachers’ modes of reasoning with students’ mathematics in terms of

knowledge interactions and development. Before describing my theoretical perspective and research design, I discuss what I conceive to be research gaps in this group of literature.

Features of Discourse Practices Versus Mental Actions and Operations

Some but not all of the aforementioned researchers operationalized their constructs in terms of properties of teachers' discourse practices. For example, Wilson et al. (2011) categorized the following teacher's statement as an evidence of the teacher *comparing* a student's behavior and the teacher's own behavior:

This was similar to our approach in that we separated them [icons representing graduation rates at universities in North Carolina] and stacked them also. They [the students] also recognized that there was an outside influence on graduation rate. They differed from us in that they did not graph the graduation rate versus anything. They also only investigated student-faculty ratio instead of looking at various variable[s] as we did. (p. 56)

A reader is left wondering: Can the teacher infer how the student's thinking might differ from their own thinking in order to explain the behavioral differences? If yes, should we instead consider it as an evidence of inferring? If no, how significant it is to distinguish comparing from describing? Therefore, I argue that defining teachers' modes of reasoning with student thinking in terms of forms of their verbal statements may not account for times when a describing teacher can compare, or a comparing or describing teacher can infer. To address this gap, instead of identifying a teacher's describing-, inferring, or comparing-type statements, an alternative approach is to infer the mental operations available to the observed teacher and the mental operations the teacher ascribes to the students. Gaining insights into nuances of a teacher's mathematical operations necessitates a cognitive approach to teacher thinking, which requires a researcher to engage in sustained interactions with the teacher and conduct fine-grained analysis of the teacher's verbal and non-verbal actions to generate a stable, holistic profile of the mathematical understandings a teacher holds. And then, the researcher can proceed to infer the

reasoning processes that have enabled the teachers' development of these understandings. I adopt this approach in the present study, hoping to further illuminate the distinctions between the different reasoning modes identified in the literature.

Relatedly, this approach will reveal what kinds of knowledge a teacher is enacting when engaging in each mode of reasoning. For example, comparing and restructuring may commonly involve a teacher operating on both of their personal mathematical knowledge and knowledge of students. A teacher's anticipation may be only based on their personal mathematical knowledge (e.g., anticipating possible student solutions to a given problem, Nathan & Koedinger, 2000), while in some other cases, it may be based on their knowledge of students. Therefore, although these reasoning modes were identified to be distinct in the literature, it is possible that two modes of reasoning are underlain by a common or similar mental process, or one mode of reasoning entails different mental processes. Characterizing teacher's knowledge interactions related to their reasoning with student thinking will clarify features of each reasoning mode, including the mental processes it entails and its mathematical consequences.

An Evaluative Versus a Descriptive Stance on Teacher Reasoning

I observe that some researchers have made claims about teachers' reasoning of students' mathematics with a set criterion. For example, Son (2013) evaluated teachers' ability to interpret student thinking using her own (the researcher's) interpretations as a reference (e.g., 10.5% teachers misdiagnosed student Sally's errors). Some researchers have evaluated the accuracy of teachers' predication of student thinking based on students' performance or responses regarding the same problems (Bergqvist, 2005; Carpenter et al., 1988, 1989; Nathan & Koedinger, 2000; Norton et al., 2011). These researchers have taken what amounts to be an evaluative rather than descriptive stance when making claims about teacher reasoning. As a result, the teachers'

mathematical rationality underlying their interpreting and anticipating remained unexamined. Accounting for teachers' mathematical rationality is critical for explaining what they say and do. Achieving this goal requires researchers to be committed to a perspective on mathematical knowledge in terms of mathematical operations and use corresponding research methods to gain insights into these operations. It also requires the researchers to distinguish the mental operations of the students from those of the teacher and from those of the researcher, rather than conflating these perspectives when making claims. In Chapter 4, I discuss my conceptualization of mathematical knowledge as mental actions and operations and how I differentiate different perspectives in the current study.

Attention to Restructuring

Compared to literature on anticipating, explaining, and comparing, research on restructuring is sparse. As previously mentioned, there has been literature on how teachers' teaching practices sensitive to student thinking, along with their constructed knowledge of student thinking, supported reorganization in teachers' mathematical knowledge (e.g., Carpenter et al., 1996; Doerr & English, 2006; Philipp, 2008; Philipp et al., 2002, 2007; Schifter, 1998; M. G. Sherin, 2002; Silver et al., 2007; P. H. Wilson et al., 2014). Although this body of work implies restructuring *having occurred*, it does not reveal the specifics of how teachers coordinate and reorganize their mathematical knowledge during restructuring. Deeper insights into teacher restructuring can afford researchers understandings of the mathematical learning teachers experience as they reflect on students' mathematics. A researcher's attention to restructuring also indicates their attempt to operationalize a constructive and dynamic perspectives of teacher knowledge, as opposed to setting up a prior standard for what teachers should know or have already known. However, it is important to note that restructuring is a non-trivial mode of

reasoning, which likely explains why the amount of relevant literature is limited. In the present dissertation, I aim to unfold the types of reasoning necessary for a teacher to restructure. I also discuss (in Chapter 4) my conceptualizations of teacher knowing, teacher decentering, and student-teacher interactions that account for the process of restructuring and its mathematical consequence.

Summary

This chapter discusses literature on teacher knowledge of students' mathematical thinking, listening, discourse moves, noticing, decentering, and modes of reasoning with students' mathematics. A common research gap identified from this group of literature is the lack of attention to teachers' reasoning with students' mathematical thinking at a cognitive level. This research gap is partly due to a lack of theoretical work that enables such inquiry. In the next chapter, I discuss the theoretical perspectives, constructs, and frameworks that enable me to address this research gap in the present dissertation. My theorizations are committed to an anti-deficit, descriptive, and constructive stance on both student thinking and teacher knowledge. They also necessitate a corresponding methodology (discussed in Chapter 5) that allows researchers to ground their claims about teacher knowledge of students in terms of mathematical actions and operations.

CHAPTER 4

THEORETICAL PERSPECTIVES AND FRAMEWORKS

In a sense, we are recommending that teacher education turn Cobb and Steffe's (1983) idea of "Researcher as teacher and model builder" into "Teacher as researcher and model builder." We should add that, in our experience, the level of support teachers need to do this far exceeds what most teacher enhancement and teacher education programs provide. (A. G. Thompson & Thompson, 1996, p. 22)

To address the aforementioned critiques and research gaps, I propose the following theoretical frameworks that guide my inquiry in the present study. I first discuss the call for a modeling perspective on teacher knowledge development in light of a modeling approach to student thinking. Then I discuss a conceptualization of teacher knowledge of students' mathematical thinking as third-order model. By connecting theories of human communication, intersubjectivity, and Piagetian theories, I provide a theorization of student-teacher interaction as it relates to teachers' knowledge construction and enactment. Finally, I combine the aforementioned constructs to propose an analytical framework of teacher decentering.

Modeling Teachers' Modeling

According to the epistemology of radical constructivism (von Glasersfeld, 1995), knowledge is actively constructed by a knower through interactions with the environment. Knowledge is not a representation of an objective truth; rather, it functions and organizes viably within a knower's experience and is idiosyncratic to the knower. Each individual has no access to another individual's knowledge, and they can only construct hypothetical models of another individual's knowledge that viably explain their observation of the observed individual's

behaviors (Steffe & Thompson, 2000b). As Doerr and Lesh (2003) defined, models are systems of interpretation.

In the past 40 years, researchers have been adopting a modeling approach to students' mathematical cognition, resulting in fruitful findings about student's mathematical learning in various topical areas and contexts. Representative work is conducted by Leslie Steffe, Patrick Thompson, Paul Cobb, Anderson Norton, Amy Hackenberg, Amy Ellis, Kevin Moore, Heather Johnson, and colleagues. These scholars have also researched and developed K-12 curriculum materials tailored to their research findings. Extending this body of work, Doerr, Lesh, and Lehrer have called for *a modeling perspective on teacher development* (Doerr & Lesh, 2003; Lesh & Lehrer, 2003). As they argued:

Just as children, a central point for teachers' knowledge is that teaching mathematics is much more about seeing and interpreting the tasks of teaching than it is about doing them. Indeed a distinguishing characteristic of excellent teaching is reflected in the richness of the ways in which the teachers sees and interprets her practice not just in the actions that she takes. It is precisely a teacher's interpretations of a situation that influence when and why as well as what it is that the teacher does. The essence of the development of teachers' knowledge, therefore, is in the creation and continued refinement of sophisticated models or ways of interpreting the situations of teaching, learning and problem solving. (Doerr & Lesh, 2003, pp. 125–126)

There has been a number of teacher education and professional development programs aiming to provide teachers with opportunities to observe and reflect on examples of good teaching practices and develop knowledge of pragmatic skills, procedures, guidelines, and rules necessary for effective teaching. Here, Doerr and Lesh are proposing an alternative and complementary approach that teachers need to develop insights into what the observed teachers are thinking that drives their instructional actions (e.g., purposes, rationales, alternatives considered, elements attended, and meanings attached to the attended elements). Foregrounding such a focus affords researchers theoretical foundations for working with teachers and supporting teachers in their

efforts to see and ascribe meanings to a situation, rather than taking particular actions in a situation (Doerr & Lesh, 2003).

Doerr and Lesh also made it clear that teachers' models encompass more than understandings of mathematics and of students' mathematics. These models are:

interpretative and explanatory frameworks that teachers use to see children's thinking, to respond in ways that further that thinking along multiple dimensions, to differentiate the nuances of particular contexts and situations, to see principles and more generalized understandings that cut across contexts and situations, and to support the continual revision of their own interpretations in light of evidence from experiences. (Doerr & Lesh, 2003, p. 131)

A teachers' models not only include understandings of students' mathematics—but also understandings of the nature of those models and ways of fostering the development of those models—such that they are shareable and reusable—and that they serve some educational purposes in multiple contexts and situations (Doerr & Lesh, 2003). This conceptualization echoes Silverman and Thompson's (2008) idea that teachers need to transform their mathematical understandings into understandings that do powerful pedagogical work and empower student learning through informed instructional decisions.

Given that a teacher's modeling activity is multidimensional as described by Doerr and Lesh, the current dissertation is centered on particular aspects, that is, I focus on teachers' models of students' mathematical thinking and how teachers' models are tested, revised, generalized, and reused during their continual interactions with students.

However, researchers have noted that it is challenging for teachers to model students' mathematics (McCloskey, 2012; Norton & McCloskey, 2008; A. G. Thompson & Thompson, 1996; P. W. Thompson & Thompson, 1994). Kastberg (2014) also noted that teachers may not be aware of themselves as model builders even if they have constructed them. Instead of continuing to characterize these difficulties and document factors that constrain teachers'

modeling, in the current study, I start with a possibility. Namely, I choose to work with teachers who I identify to be capable and have the disposition of engaging in modeling students' mathematics, and I focus on modeling the mental processes that support their modeling activities. Reframing my research questions through a modeling perspective, I ask: *What kinds of mental processes are critical to teachers' constructions, reflection, and refinements of their models of students' mathematics, and in what ways are these activities consequential to the teachers' mathematical growth?*

Teacher Knowledge of Students' Mathematics as Third-Order Model

As discussed in the previous chapter, prior researchers have conceptualized teacher knowledge of students' mathematics in various ways (Ball et al., 2008; Fennema & Franke, 1992; Harel, 2008a; Shulman, 1986; Silverman & Thompson, 2008). I define teacher knowledge of students' mathematics in ways compatible with the radical constructivist and the modeling perspectives. In the following, I elaborate on my conceptualization around four aspects: (a) models are idiosyncratic to the observer, (b) models are mental actions and operations, (c) model building is iterative and continual, and (d) relationships between modeling, decentering, and mathematical caring relations.

Models are Idiosyncratic to the Observer

Radical constructivists clearly distinguish the living experience of the knower themselves from the observer who is trying to understand the knower's knowledge and knowing process. Steffe et al. (1983) introduced the notions of first- and second-order models, and Wilson et al. (2011) define third-order models to make such distinction. In this subsection, I discuss each type of model, their relationships, and their implications on mathematics education and research.

First- and Second-order Models

First-order models (FOMs) are “models the observed subject constructs to order, comprehend, and control [their] experience” (Steffe et al., 1983, p. xvi). Because no one can have access to another individual’s FOMs, what we call “teacher knowledge of students’ mathematics” is at best the teachers’ personal interpretations and inferences of the students’ conceptual operations. Steffe and colleagues defined such interpretations as *second-order models* (SOMs)—“[the models] observers may construct of the subject’s knowledge in order to explain their observations (i.e., their experience) of the subject’s states and activities” (Steffe et al., 1983, p. xvi). A teacher engaging in second-order modeling inquires, “What might this student be thinking so that their actions make sense from my perspective?” The teacher “puts [themselves] into the position of the student and attempts to examine the operations that [the teacher] would need and the constraints [they] would have to operate under in order to (logically) behave as the student did” (P. W. Thompson, 1982, p. 161). However, there is no way for the teachers, the observers, to confirm if their interpretations match what the students, the subjects, are “actually” thinking. As Cobb and Steffe (1983) explained:

we should not forget that a model is no more than a plausible explanation of children’s constructive activities. One can never claim a correspondence between the model and children’s inaccessible mathematical realities. Although a model can be viable, it can never be verified. (p. 92-93)

Steffe and Wiegel (1996) made a similar point:

We make no assumption that our schemes of action and operation represent the students’ mathematical knowledge in the sense of forming a match. Rather, our models of students’ mathematical knowledge consist of constellations of conceptual constructs we as researchers find useful when interacting with students. (p. 496)

Relationships Between First- and Second-order Models

To be clear, a teacher’s SOMs are also mathematical knowledge *comprising* the teacher’s FOMs. As Steffe (1990) stated:

If teachers could only accept the premise that the mathematical knowledge of their students is also valid, then the necessary adaptations of teachers when teaching mathematics would be in the direction of the *mathematical knowledge of their students* as well as in the direction of their own mathematical knowledge. In other words, the mathematical knowledge of the students as seen by their teachers would become part of the knowledge of the teachers. (p. 184)

I provide three arguments to support Steffe's statement. First, a teacher's consideration of SOMs as legitimate mathematics requires them to internalize their SOMs as an integral part of their own mathematical knowledge system. Moreover, a teacher's exploration of students' mathematics may enable the teacher to identify mathematical understandings that are more productive than others, and these productive understandings (originally constituting the teacher's SOMs) will become understandings they intend to promote during their instruction, that is, FOMs. Second, when constructing SOMs, a teacher necessarily reasons through a student's mathematics on their own, which is in fact a process of constructing FOMs. Describing a student's mathematics is unavoidably an expression of the teacher's own ways of operating mathematics (Steffe, 1991). Third, SOMs can only be constructed in terms of a teacher's FOMs. A teacher's ability to construct SOMs of another has to be occasioned by the teacher's ways and means of operating (Steffe & Thompson, 2000b). Steffe (1995) also noted that all observers operate with their FOMs and that "second-order models are understood as springing from the conceptual operations that are available to the observer, along with their modifications in the context of interacting [with the observed person]" (p. 496).

Said succinctly, a teacher's FOMs contain mathematical knowledge constructed both through reflection on student thinking and from other experiences. Regardless of sources of constructions, these are mathematical ways of operating comprising the teacher's FOMs. Like FOMs, SOMs are elements of an individual's knowledge, but they are elements they attribute to

the FOMs of another individual; “second-order” refers to an observer conceiving the knowledge as from the perspective of the observed.

Third-order Models

Considering the role of a researcher who is observing a teacher interacting with a student, Wilson et al. (2011) define *third-order models* (TOMs) to refer to the researcher’s SOMs of the observed teacher’s SOMs of the student’s mathematics. Such models consist of the researcher’s inferences about how the teacher may interpret the student’s mathematics that can explain the language and actions involved in student-teacher interactions (and any other researcher-teacher interactions). This notion of TOMs aligns with Simon and Tzur’s (1999) *accounts of teachers’ practice*; that is, explaining teachers’ perspectives from researchers’ perspectives. Again, TOMs are merely models constrained by a researcher’s FOMs, and thus they are elements of the researcher’s FOMs. Because a teacher’s SOMs are parts of their FOMs (mentioned earlier), a researcher’s TOMs are also elements of their SOMs of the teacher’s knowledge; “third-order” only indicates that the associated ways of operating have their origin in the student’s activity, and they are constructed through the eyes of the teacher. Finally, these TOMs are also elements of the researcher’s SOMs of the student’s knowledge because the researcher’s constructions of TOMs have to be constrained by what the researcher perceives as the student’s activity. It is possible that the researcher’s TOMs of the student are not compatible with the researcher’s SOMs of the same student, but they still constitute subsets of those SOMs.

From this point forward, when I speak of teacher knowledge of students’ mathematical thinking, I refer to TOMs that are *models of* teacher knowledge of students’ mathematical thinking constructed from a researcher’s perspective, which likely are not identical to an observed teacher’s SOMs of their student or the observed student’s FOMs. Figure 11 illustrates

the components of a researcher's FOMs and their relationships, and Figure 12 illustrates these different models and their relationships in the context of student-teacher interaction (with a researcher as an observer of such interaction).

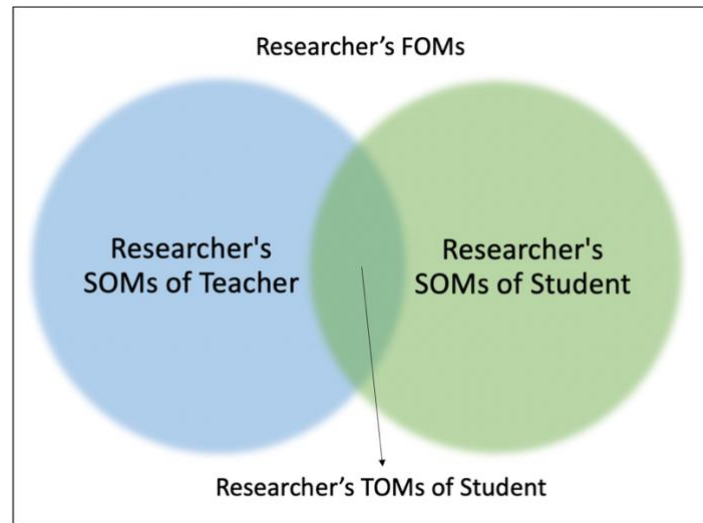


Figure 11. Components of a researcher's TOMs and their relationships.

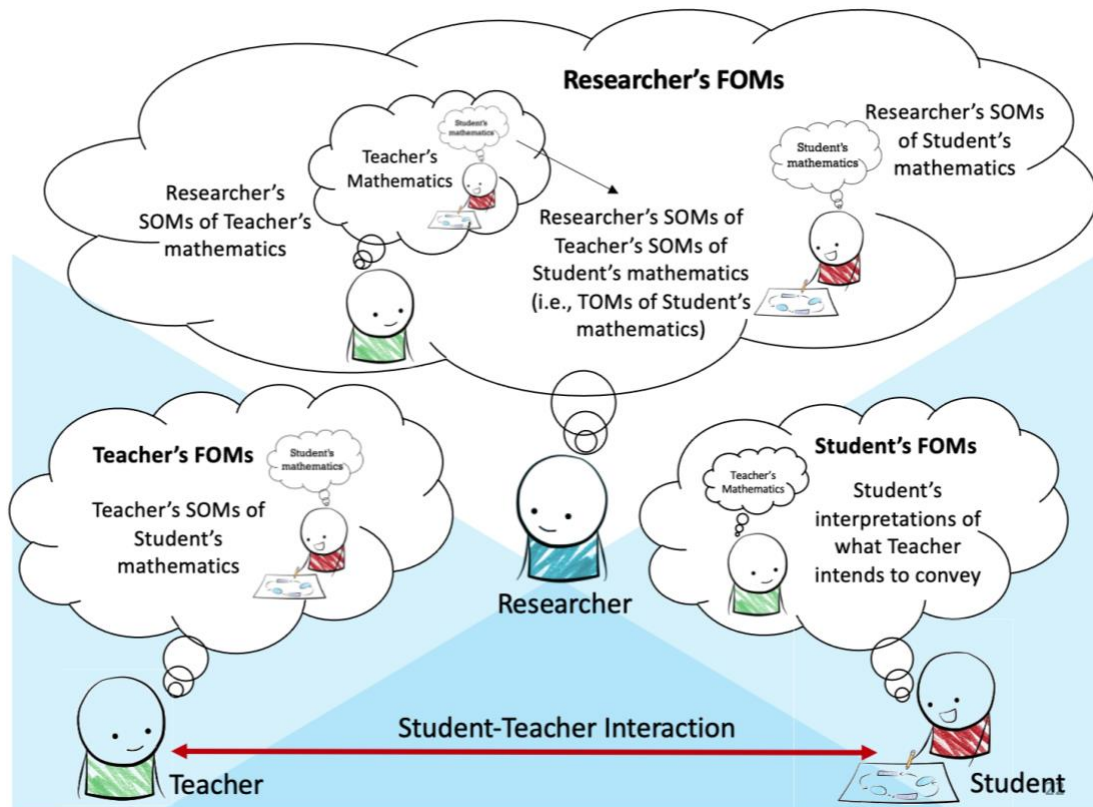


Figure 12. Types of models involved in student-teacher interactions.

Implications

The distinction between first-, second-, and third-order models have important implications on mathematics education and research. As I mentioned earlier, it is likely that a teacher's SOMs are not identical to the observed student's FOMs and that a researcher's TOMs are not identical to the teacher's SOMs of the student. Moreover, an individual can never confirm whether their SOMs of another are true representations of the other individual's FOMs. However, because each individual's mathematical realities are constructed and organized based on their personal experience in logical and rational ways, educators and researchers should be committed to the construction of SOMs as a way to grant rationality and value to other's mathematical realities—despite impossible direct accessibility to them (Confrey, 1990; Steffe & Thompson, 2000b; Ulrich et al., 2014). Second-order modeling is an important avenue through which teachers learn about their students' mathematical realities and researchers learn about teachers' and students' mathematical realities.

Models are Mental Actions and Operations

A teacher's SOMs of students can be multidimensional and constructed through various avenues. For example, a teacher may construct research-based SOMs about productive and unproductive ways of thinking (e.g., *generalized, epistemic subject type* SOMs, see Tzur, 2014; Ulrich et al., 2014) from a content course or by reading research articles; a teacher may construct SOMs of how a student has interpreted their instructional actions to understand how their actions are consequential to student learning (Yoon, 2019); more broadly, a teacher may construct SOMs about how students learn among students by taking into account non-cognitive factors such as cultures, classroom norms, and artefacts. In the current dissertation, and consistent with my

conceptualizations of SOMs in the previous section, I limit the notion of SOMs as a collection of conceptual operations comprising an individual's inferences about another individual's mathematics, and these inferences are primarily constructed in the context of social interactions and reflection upon those interactions.

Piagetian notion of *schemes* and Thompson's (2013) notion of *meanings* provide the foundations of the framing of first-, second-, and third-order models. A scheme or a meaning is not a procedure or an action in itself, but it is the underlying cognitive structure that governs an individual's assimilation and actions and that is reusable, transferable and generalizable to analogous circumstances (Piaget & Inhelder, 1968; P. W. Thompson, 2013). As I have mentioned in Chapter 2, extant literature has suggested that teachers tend to interpret student thinking in an evaluative manner or at the level of describing (e.g., only focusing on observable actions). Ironically, the researchers who made those claims often do not focus on the mathematical actions or operations (or lack thereof) constituting the teacher's knowledge of students. Here, by framing teacher knowledge in terms of mathematical meanings, I maintain an assumption that teachers' or students' mathematical knowledge is a result of their coordinating mental actions and operations in logical and rational ways. Each individual's meaning system is organized in certain ways so that one meaning implies, contains, or precedes another meaning (Piaget, 1976).

A teacher's FOMs, including their SOMs of students' mathematics, consists of a collection of mathematical meanings that govern the teacher's mathematical perception and activities, anticipation of the results of those activities, and interpretation and anticipation of students' mathematical activities (i.e., *assimilation*). When the perceived results of these activities are unanticipated and the teacher conceives such a conflict, the teacher is experiencing

a state of *perturbation* (or *disequilibrium*). Piaget asserted that each organism, including human beings, tend towards an internal equilibrium in the face of perturbations set up by the environment—namely, they tend to “organize structures into coherent and stable pattern” (Ginsburg & Oppen, 1988, p. 23). When the teacher (consciously or unconsciously) modifies or reorganizes their meanings to eliminate perturbations, *accommodation* has occurred. As von Glasersfeld (1997) explained, “that cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium” (p. 68). A researcher’s attention to a teacher’s SOMs and their development as a result of interacting with students or reflecting on students’ mathematical activities necessarily involves them making concrete claims about the teacher’s assimilating and accommodating activities in these contexts.

A focus on teachers’ second-order modeling in terms of their constructions and reorganization of meanings enables an anti-deficit point of view of teacher knowledge. As Doerr and Lesh (2003) described:

From a modeling perspective, we want to shift the emphasis from repairing the deficiencies in teachers’ mathematical knowledge to using what teachers already know to express, test, revise and refine their content knowledge and to extend that knowledge to more powerful forms for classroom teaching. (p. 130)

A researcher who characterizes a teacher’s SOMs pays explicit attention to and makes descriptive statements about the substance of teacher’s SOMs and how the teacher builds and revise the meanings of their SOMs through assimilation and accommodation. The researcher moves away from evaluative statements such as “a teacher is found *lacking* knowledge of a student” or “a teacher is found to have *misunderstanding* of a student’s knowledge.” It is the meanings (in the forms of mental actions and operations) available and viable to the teacher that constitute the teacher’s FOMs and SOMs.

Model Building is Iterative and Continual

It is worth highlighting that any SOM does not come into being effortlessly. In the context of student-teacher interactions, a teacher's construction of SOMs likely requires sustained interactions with a student in order to discern and stabilize characteristic ways of student thinking. A teacher's modeling of a student's mathematics is an iterative and cyclical process where the teacher continually develops, tests, and refines their SOMs as they observe additional student actions over time. As Doerr and Lesh argued:

teachers' models are generally developed in situations where the available information is often partial or incomplete and where the consequences of actions are not always immediate and local but sometimes remote and global. This suggests that a key perspective for a view of any particular model or system of interpretation needs to hold in the foreground that the model is likely to change as additional information and new evidence leads to revisions. (p. 132)

When engaging in second-order modeling, a teacher is putting themselves in a constraint by the student and is open to be perturbed and surprised by novel student actions. When perturbations do occur, the teacher desires to continue to converse with the student and elicit their thinking in order to reformulate their SOMs substantially and reconcile their experienced perturbations. This iterative process of second-order modeling enables a teacher to develop an *emerging-type* SOMs (i.e., working hypotheses about the thinking of a particular student in a particular context) into an *elaborated-type* SOMs (i.e., characteristic ways of thinking that are predictable and explanatory for recurring patterns of multiple students' behaviors) as they work with an increasing number of students in an increasing number of contexts (Tzur, 2014). Furthermore, a teacher may be able to work towards developmental SOMs that include possible courses of development from one understanding to another (Paul Cobb & Steffe, 1983). Accordingly, a researcher who is studying a teacher's second-order modeling also engages in a recursive cycle of formulating, testing, and

modifying their TOMs of the students and SOMs of the teacher based on ongoing conversations with the teacher.

In Chapter 5, I discuss how the design of the current study provides opportunities that address the variability and growth of a teacher's SOMs (which in fact are my development in TOMs), including how those SOMs become more explicit, stable, predictable, and general⁴ longitudinally during the course of the study.

Modeling, Decentering, and Mathematical Caring Relations

A teacher's constructions of SOMs of students' mathematics necessitate the teacher to interact with students reflectively in a conscious attempt to decenter from their personal ways of operating and engage with the students' ways of operating (Baş-Ader & Carlson, 2021; Teuscher et al., 2016). In contrast, a teacher who is constrained to their FOMs operates entirely from their own perspectives and assumes students understand their utterances as intended (i.e., not decentering) (Teuscher et al., 2016). As Hodkowski (2018) explained:

[W]ith FOMs, the observer's interpretations are essentially attributing mathematical knowing to the learner being observed through only the lens of the mathematics used by the observer herself. The observed learner may be considered to have or not to have a concept, but such an attribution, with the frequent reference to "not having" as "misconception," is likely rooted in the observer's contrasting of the other person's mathematics through assimilating it into one's own FOM. (p. 14)

And because this is an assimilation process, it is often implicit in the observer's mind in that the observer is not consciously aware that they are assimilating. The observer does not actively discard elements outside their FOM, they simply do not perceive it when they do not decenter (see the metaphor of a card-sorting machine by von Glasersfeld [1995]).

⁴ Steffe and Thompson (2000b) clarified that the generalizability of a SOM is not about the extent it represents thinking of the population from which a sample was drawn to build the SOM. It is about an observer's hypothesis of one's mental functioning is useful in "interpreting the mathematical activity of students other than those in the experiment" and in "organizing and guiding [their] experience of students doing mathematics" (p. 300).

Considering a researcher who is attempting to understand a teacher's SOMs of a student, the researcher also needs to decenter. In particular, the researcher does not only need to decenter from their FOMs to constructs SOMs of the teacher's or the student's mathematics, but also decenter from their SOMs of the student to give space for constructing TOMs of the student. These constructions require the researcher to interact with the teacher in a conscious attempt to reveal the teacher's SOMs and promote the teacher's reflection on those SOMs. Analogous to what Hodkowski (2018) said, a researcher's claim about what knowledge of student a teacher lacks or misunderstands only implies a discrepancy between—the researcher's personal SOMs of a student—and the SOMs the researcher attributing to the observed teacher.

To construct SOMs and decenter also means to care. Researchers have conceptualized teaching as a relational practice that capitalizes the role of student-teacher relationships (or teacher-teacher educator relationships) in students and teachers' learning. This body of work has generated constructs such as *caring* (Hackenberg, 2005, 2010; McCloskey, 2012; Nicol et al., 2010; Philipp et al., 2002; P Sztajn, 2008), *empathizing* and *respecting* (D'Ambrosio & Kastberg, 2012; Kastberg et al., 2019, 2017; Kitchen, 2005), and *trusting* (Kastberg et al., 2009). Hackenberg and colleagues introduced the notion of *mathematical caring relations* (MCRs, for short) to conceptualize the dynamics of emotion, decentering, and second-order modeling in the context of student-teacher interactions (Hackenberg, 2005, 2010; McCloskey, 2012). MCRs are caring relations aimed toward students' mathematical learning, meaning a teacher going beyond just deeply care about the students in general to care mathematically for the students with attempts to sustain opportunities for students' mathematical learning (McCloskey, 2012). Hackenberg (2005) explained that, "caring is not simply a feeling or a virtue of a teacher...caring is an orientation to create and participate in social interactions with certain cognitive and

affective qualities” (p. 46). P.W. Thompson (2013) also claimed that participants involved in a conversation must care to understand other participants; teachers are special participants in a conversation, and they should “manage a conversation so that students *do* have power and trust, but it is teachers who allow and nurture those opportunities” (P. W. Thompson, 2013, p. 66). A teacher’s practice of building SOMs and decentering likely facilitates students’ sense of being cared for as they feel that “they are being listened to, that their ideas are valued, and perhaps, that they are understood” (Hackenberg, 2005, p. 47).

Human Communication and Student-Teacher Interaction

There has been critiques about the constructivist and Piagetian perspectives on teacher knowledge that they do not account for its contexts of construction and enactment. For example, Barwell (2013) argued that Piaget’s representational view of mind ignores the social dimension of human cognition. Lerman (1996) critiqued that radical constructivism takes an incoherent stance in regard to the idea of intersubjectivity. Radical constructivists have argued against these critiques, claiming that accounting for the dynamics of social interactions and dialogues has been a natural application of radical constructivism (Confrey, 1995; Steffe & Thompson, 2000a; P. W. Thompson, 2000, 2013, 2014; von Glasersfeld, 1995). As Steffe and Thompson (2000a) argued, “interaction enters radical constructivism at its very core” (p. 192). In fact, (radical) constructivism has not only influenced the field of education, but also further research in communication and media science, social science, and psychotherapy, in which social interaction is one of the major objects of study (Barnes & Možina, 2020; Palmaru, 2016; Puljić & Puljić, 2020). In this section, I elaborate on a radical constructivist perspective on human communication and intersubjectivity and propose my conceptualization of student-teacher interactions with attention to the teacher’s construction and enactment of knowledge.

Intersubjectivity in Human Communication

Researchers have used the notion of second-order modeling and Piagetian constructs of assimilation and accommodation to characterize the cognitive mechanism of human communication. P. W. Thompson (2013) characterized person-to-person communication as involving each individual reciprocally assimilating the utterances of the other and constructing SOMs of the other's meanings until (ideally) no accommodations are necessary for successful assimilation. Steffe and Thompson (2000a) described this reflexivity as *intersubjective construction of knowledge in social interaction*. From a radical constructivist perspective, intersubjectivity does not refer to agreement of same or similar meanings but non-conflicting mutual interpretations at the moment (P. W. Thompson, 2000, 2013, 2014). As Thompson (2013) clarified:

a conversation being in a state of intersubjectivity has no implication for whether the participants' meanings align. Rather, the nature of a conversation that is in a state of intersubjectivity is that neither participant has any reason to believe that he has misunderstood the other. (p. 65)

Meanings are always something that individuals abstract from their experiences. Although it may seem to be compatible with the abstraction of others or may continue to adapt to be compatible with the others' in the course of social interactions, no observers can transcend their own experiential world to determine sameness between individuals' meanings (von Glasersfeld, 1995, p. 137). Two individuals being able to sustain a conversation and speak of *taken-as-shared* meanings (Paul Cobb, 1989) does not imply that they do think identically but only that they do not discern evidence that one misunderstands the conversation; they "may have in fact completely misunderstood each other" (P. W. Thompson, 2013, p. 65). This is also what non Glasersfeld (1995) called *second-order viability*:

If, then, the other does what we predicted, we may say that the piece of knowledge was found to be viable not only in our own sphere of actions but also in that of the other. This

bestows a second order of viability to the knowledge and the reasoning we assumed the other to have and act on. (p. 120)

These notions of intersubjectivity, second-order viability, and conveyance of meaning explain how miscommunication occurs among students and teachers. Teachers and students can sustain a conversation even if they hold completely different mathematical meanings mentally (e.g., Bauersfeld, 1980; E. Johnson & Larsen, 2012; Lew et al., 2016; P. W. Thompson, 2013; P. W. Thompson & Thompson, 1994; Yoon, 2019). One source of the miscommunication is that students and/or teachers are unaware of or do not speak of the mathematical meanings constituting their mathematical activities, and they only attend to mathematical procedures or outcomes (Lew et al., 2016). On some occasions, meanings are so capsulated in their use of language and procedures to the extent that they are not aware of the underlying meanings (P. W. Thompson, 2013; P. W. Thompson & Thompson, 1994). Consequently, neither can the teachers nor the students detect the different meanings held by each other because they do enact similar actions and produce similar outcomes. On some other occasions, teachers may fail to discern the discrepancy between the meanings their students construct and their own meanings and primarily assimilate students' actions into the teachers' personal meanings (Bauersfeld, 1980; E. Johnson & Larsen, 2012).

These examples of miscommunication point to the significance of students and teachers speaking of and attending to meanings, including constructing SOMs of others' meanings during student-teacher interactions (P. W. Thompson, 2013). Egocentric thought does not support construction of intersubjective knowledge (Steffe & Thompson, 2000a). It is the teachers that should take the responsibility of supporting and sustaining collaborative and meaningful conversations with students by promoting constructions of meanings and monitoring evolution of meanings.

A Conceptualization of Student-Teacher Interaction

Building on these theories of human communication, I conceptualize a model of student-teacher interaction that captures the dynamics of individuals' development in meanings (Figure 13). This model includes components of teacher activity (denoted in yellow boxes) and student activity (denoted in orange boxes). I distinguish a student or a teacher's actions from their knowledge. I acknowledge that an individual's actions are functions of their knowledge, goals, and affect (e.g., motivation, beliefs, values, attitudes, and emotion) (Schoenfeld, 1998). In this model, I only focus on the knowledge-action relationships, although affect is unavoidably intertwined with knowledge and actions during student-teacher interactions. In terms of the knowledge component, and compatible with my aforementioned theorization, I limit my focus on a teacher's FOMs (including their intended mathematical meanings during the interactions) and SOMs. I caution the reader that I am not assuming SOMs are a priori knowledge available to all teachers. In fact, a teacher's construction of SOMs is a nontrivial achievement and is likely fashioned by the nature of their FOMs (Silverman & Thompson, 2008; Tallman, 2015).

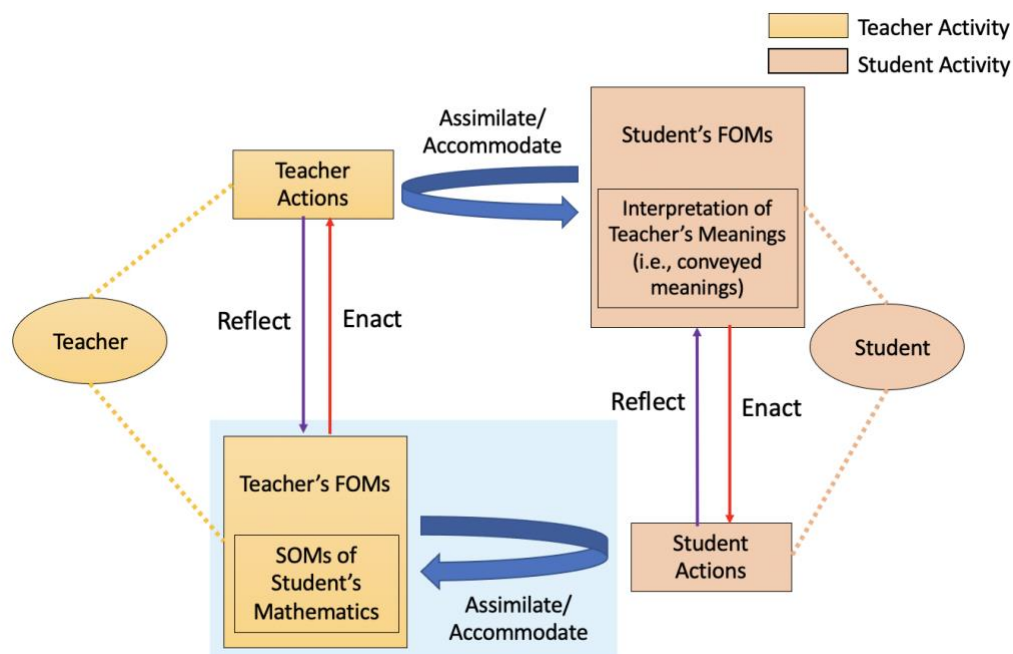


Figure 13. A model of student-teacher interaction.

When a teacher engages in interactions with a student, they either assimilate the student's actions into their FOMs (including SOMs) or accommodate their FOMs to account for novel student actions (see the blue arrow at the bottom of Figure 13). Then, the teacher enacts their FOMs to formulate a response to the student in action (see the "Enact" arrow on the left). The teacher's actions may or may not be sensitive to their SOMs. Subsequently, the student assimilates or accommodates the teacher's actions (see the blue arrow at the top) and responds to the teacher in action, which again triggers assimilation or accommodation in the teacher. The teacher may also refine their SOMs by reflecting on their teaching actions to become aware of how their ways of interacting are consequential to the student's learning or how the student has interpreted their actions (see the "Reflect" arrow on the left). This interaction cycle continues as the student and the teacher reciprocally and continually assimilate the language and observable actions of the other or construct SOMs. Ideally, both of the teacher and the student are engaging in listening and interpreting each other's meanings, monitoring and anticipating each other's responses, and correspondingly adjusting SOMs of each other (P. W. Thompson, 2013).

While this model depicts a mechanism of ongoing student-teacher interactions, some processes also apply to a teacher's retrospective analysis of such interactions. For example, a teacher can confirm or update their SOMs by re-assimilating and re-accommodating students' actions as they watch a video of their interactions with the students. They can also reflect on their actions and consider alternative actions they should have made based on deeper insights into student thinking. Other than student-teacher interactions, this model may also apply to person-to-person communication in broader contexts and situations (e.g., a teacher conversating with another teacher during professional development, a student conversating with their peers in small

group, a psychotherapist conversating with a client), although the content and goals may vary in those conversations.

Four Types of Teacher Decentering

In the present study, I focus my research questions and analysis on the processes highlighted in the blue rectangular box in Figure 13. Specifically, I am interested in what SOMs a teacher constructs, how these constructions perturb the teacher's FOMs (including previously constructed SOMs), and how such perturbation engenders accommodations in the teacher's FOMs. Answering these "how" allows me to theorize the reasoning involved in teacher decentering. Piaget has never analyzed or unpacked the mental actions and operations of decentering in detail. His discussion was often intermingled with other cognitive processes such as adaption and coordination of actions that form logical operations. Similarly, his followers (e.g., Steffe and Thompson) often discussed decentering for other purposes instead of explaining it in and of itself. As what follows, I distinguish between four types of decentering based on the different mental processes involved (see verbal descriptions in Table 1 and a visual illustration in Figure 14).

Table 1. A framework of teacher decentering

Decentering Type	Descriptions
Type 0: No Decentering	- occurs when a teacher gives lectures to students or applies their FOMs to solve a mathematical problem - occurs when a teacher conceives of a student's thinking to be similar to their own thinking without any perturbations - involves a teacher entirely operating on FOMs
Type 1: An Emergent Sense of Decentering	- occurs when a teacher experiences perturbation by a student's actions but cannot infer the student's thinking to resolve the perturbation - involves a teacher entirely operating on their FOMs

Type 2: Decentering Through Assimilation	- occurs when a teacher recognizes student thinking as being similar to some former SOMs (of the same student or of other students) - involves a teacher constructing in-the-moment SOMs via assimilation (may be accompanied by modest modifications in FOMs)
Type 3: Decentering Through Accommodation	- occurs when a teacher is perturbed by a student's actions and infers the student's thinking to remove perturbations - involves a teacher constructing in-the-moment SOMs via accommodation - the genesis of SOMs and modifications in FOMs
Type 4: Reflective Decentering	- occurs when a teacher reflects on how student thinking is different, similar, or related to their own thinking - involves a teacher simultaneously operating on their SOMs and FOMs and reorganizing their FOMs - requires a teacher's consciousness of the mental actions and operations constituting their SOMs and FOMs

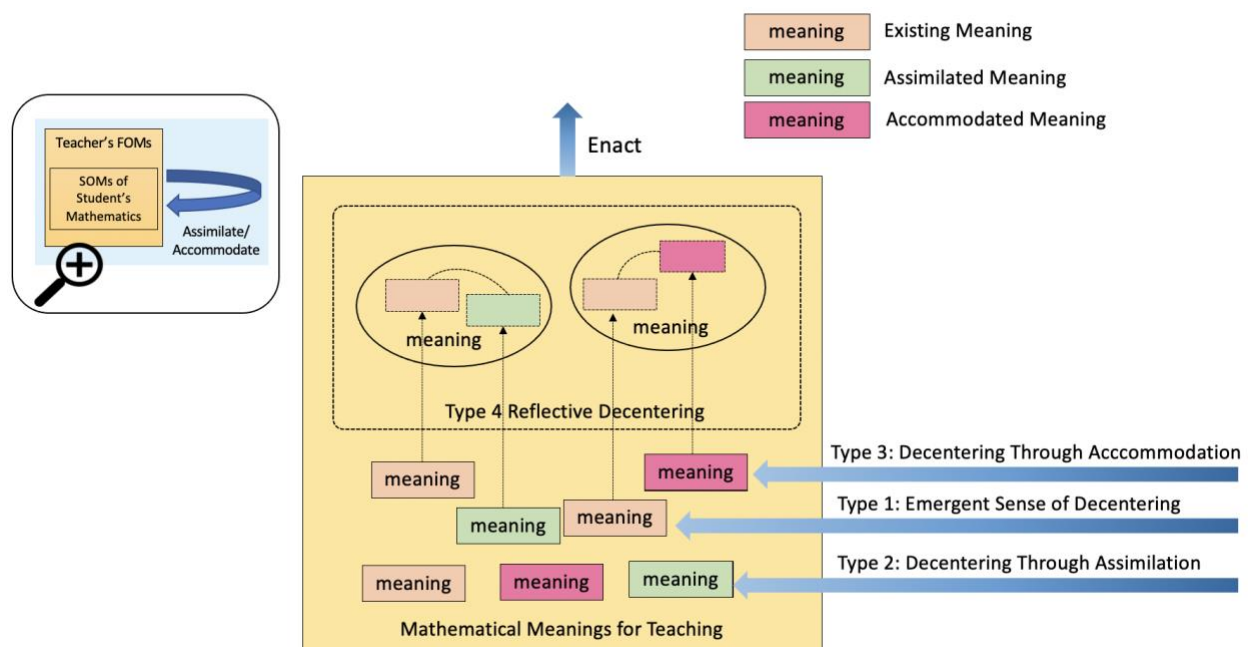


Figure 14. Mental processes underlying teachers' constructions of students' mathematics.

Note. Meanings in orange boxes are meanings existing prior to interactions with students; meanings in green boxes are existing meanings activated for assimilation; meanings in red boxes are meanings added or modified through accommodation.

Type 0: No Decentering

Type 0 Decentering involves a teacher entirely operating on their FOMs. It is called “Type 0” because it does not require the teacher to set aside their own thinking or construct SOMs of others—that is—not decentering. For example, Type 0 Decentering may occur when a teacher lectures their FOMs to their students without any student-teacher interactions. It may also occur when a teacher applies their FOMs to solve a mathematical problem on their own, including applying their internalized SOMs to solve a problem in ways resembling a student’s reasoning. Type 0 Decentering may also occur in the context of student-teacher interactions. In this case, a teacher primarily assimilates a student’s actions into their FOMs (*not* including SOMs of other students). The teacher recognizes the student’s actions by projecting their intended meanings onto the student. Namely, the teacher considers the student thinks in ways consistent with their own and does not abstract novel features from the student’s actions as they experience the student’s activity. Because there is no perturbation occurs in the teacher, the teacher does not make any modifications or reorganizations in their FOMs.

Although Type 0 Decentering is common during student-teacher interactions, it may generate miscommunication between a teacher and their students. For example, a teacher may interpret a student’s understanding to be consistent to their own or consistent to the student’s prior understanding, but in fact, the student thinks in fundamentally different ways or has experienced fundamental shift in their understanding. The teacher is not sensitive to the discrepancy between the student’s current ways of operating and the teacher’s personal ways of

operating or the student's previous ways of operating. This insensitivity on the part of the teacher may be partially due to the teacher's unawareness of the mental actions and operations constituting their FOMs and SOMs. The aforementioned phenomena of student-teacher miscommunication prevailing in mathematics teaching (e.g., Bauersfeld, 1980; E. Johnson & Larsen, 2012; Lew et al., 2016; P. W. Thompson, 2013; P. W. Thompson & Thompson, 1994; Yoon, 2019) can be characterized as a result of Type 0 Decentering.

Type 1 Decentering: An Emergent Sense of Decentering

There are times when a teacher's FOMs do not allow them to assimilate a student's actions. Type 1 Decentering captures the process by which a teacher encounters perturbation by a student's actions but cannot infer the student's mental actions to remove that perturbation (Figure 15). To enact Type 1 Decentering, a teacher may start with operating on the material aspects of the student's actions and results of those actions (e.g., movement, drawing produced, equations written) and abstracting properties of the student's behavior. And then, the teacher represents these actions and results, compares them to those governed by other ways of thinking, and infers that the current student does not think in ways compatible with the teacher's own or other students'. However, the teacher has not yet constructed a SOM that enables them to understand the reasoning behind the student's activities.

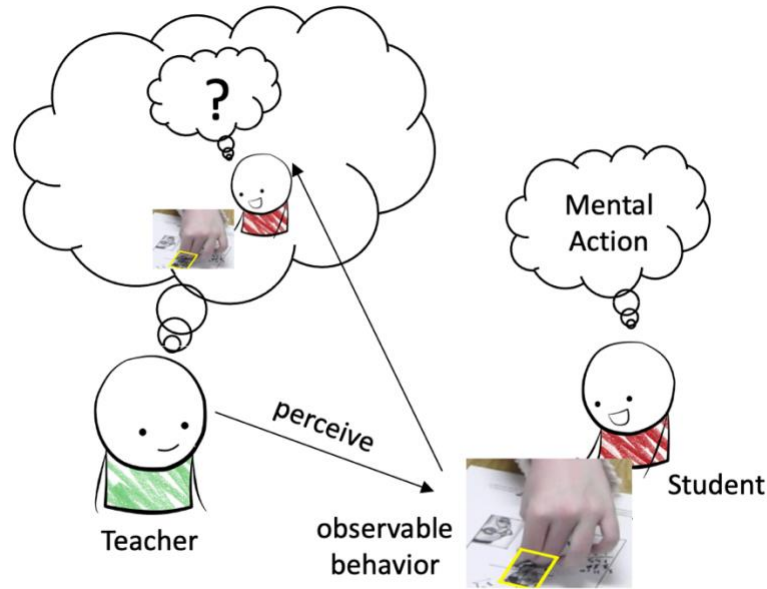


Figure 15. An illustration of Type 1 Decentering.

Just as how perturbation is critical for a student's accommodation and mathematical learning (see also *productive struggle*; Hiebert & Grouws, 2007; Warshauer, 2015; Warshauer et al., 2019; *cognitive dissonance*; Festinger, 1957; *cognitive incongruity*; Hatano, 1988), teacher's experienced perturbation as a result of reflecting on student's mathematical activities likely triggers the teacher's mathematical learning (Steffe & D'Ambrosio, 1995; Steffe & Wiegel, 1996). In a sense, such perturbation may motivate the teacher to engage in further purposeful interactions with the student to construct a SOM and furthermore, force the teacher to make a distinction between their FOMs and SOMs, which is why I call this type of decentering as an emergent sense of decentering.

Type 2 Decentering: Decentering Through Assimilation

Type 2 Decentering involves a teacher constructing SOMs of a student through assimilation of a student's activities. The teacher conceives the student's actions to be explained by their SOMs constructed in prior experiences of teaching and learning. These SOMs consist of

characteristic ways of thinking of other students or ways of thinking of the same student in prior interactions. In the latter case, the teacher's prior SOMs of this student have become stable and explanatory for the same student's current actions. To put it another way, the student's subsequent actions help confirm and stabilize the teacher's previous SOMs.

Different than Type 1 Decentering, Type 2 Decentering requires the teacher to make in-the-moment inferences that are explanatory for the student's behavior, although these inferences have already constituted their preexisting SOMs.

Different than Type 0 Decentering, Type 2 Decentering requires a teacher to be aware that the observed student's thinking *differs* from the teacher's own. Piaget (1962) have tried to make it clear that, cognitive egocentrism "stems from a lack of differentiation between one's own point of view and the other possible ones, and not at all from an individualism that precedes relations with others." As I described above, one form of Type 0 Decentering is an example of this indifferentiation or egocentrism. It is not that the teacher is not involved in listening to a student, but that the teacher cannot discern differences between the student's thinking and the teacher's own. Egocentrism is not "a deliberate choice in favor of one's own point of view" (Montangero & Maurice-Naville, 1997, p. 101) but due to a lack of sensitivity to differences. By contrast, Type 2 Decentering requires a distinction among multiple perspectives, and the occurrence of assimilation is accompanied by such conscious awareness.

Type 3 Decentering: Decentering Through Accommodation

Type 3 Decentering involves a teacher constructing SOMs through accommodation. When a teacher observes novel student actions and is perturbed by those actions, the teacher is able to actively revise their existing meanings or constructs new meanings to account for the novel features of student actions. This achievement likely requires the teacher to repeatedly

engage with similar student behaviors, abstract patterns in those behaviors, and finally infer the mental actions that govern those behaviors. These activities also imply that the teacher is able to differentiate the student's mental actions from the results of those actions (in the forms of behavior) and project these actions (coordinated with their results) to a re-presentational level (Figure 16). Ideally, and when this construction gradually becomes stable (through Type 2 Decentering), the teacher can mentally run through the student's (physical and mental) activities even if the student is not present. Due to the complexity of this activity, a researcher's sensitivity to a teacher's Type 3 Decentering likely relies on retrospective conversation with the teacher; only observing the teacher's behavior in the moment of student-teacher interactions may not be insufficient to infer whether Type 3 Decentering has occurred.

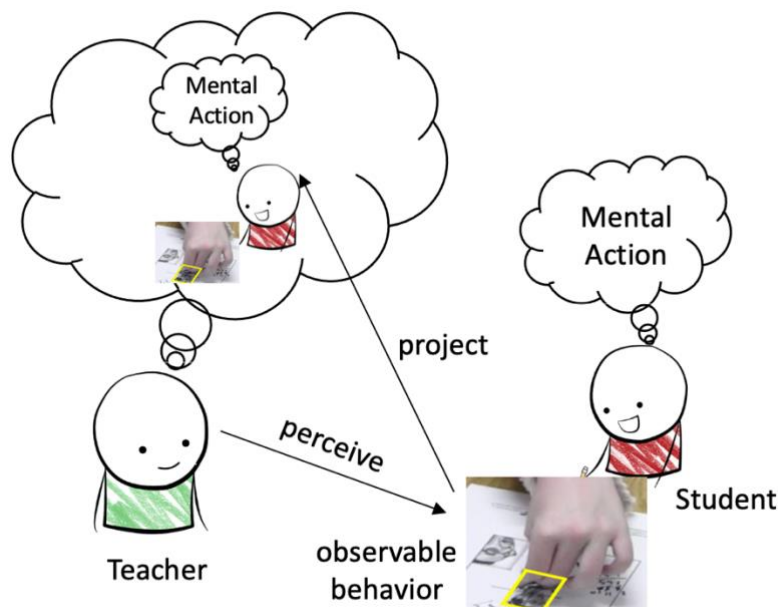


Figure 16. An illustration of Type 3 Decentering.

While Type 2 Decentering describes how a teacher's existing FOMs enables their second-order modeling, Type 3 Decentering (and Type 4 Decentering, as I explain next) accounts for how second-order modeling is generative to a teacher's FOMs as the teacher

continually refines and reorganizes their meanings. This is how decentering can be a source for teachers' reflective abstraction and mathematical learning (Steffe & Thompson, 2000a).

Type 4 Decentering: Reflective Decentering

Since a teacher engaging in Type 3 Decentering is constructing SOMs in-the-moment, they may not be fully aware of the relationships between their FOMs and SOMs or between multiple SOMs. This leads to Type 4 Decentering that involves a teacher simultaneously operating on their SOMs and FOMs or multiple SOMs and reflecting on how they are related, similar, or different (Figure 17). As such, the teacher puts themselves in cognitive constraints by their students, and questions and clarifies their existing FOMs and SOMs, including previous hypotheses about the student's thinking and the mathematical assumptions that they have been taking as given (see Hackenberg, 2010). Echoing Confrey's (1991) comment, (Type 4) decentering makes a teacher's FOMs "transformed, elucidated, and enriched" (p. 129).

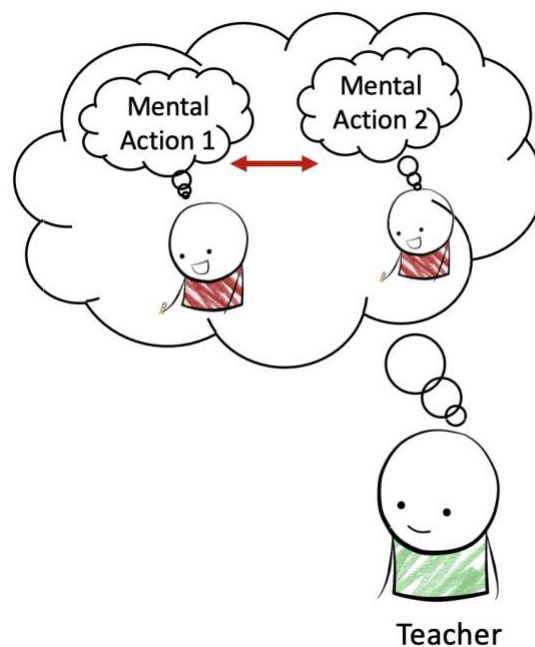


Figure 17. An illustration of Type 4 Decentering.

A necessary condition for Type 4 Decentering is a teacher's *conscious awareness* (Piaget, 2001a; Tallman, 2015) of the mental actions constituting their mathematical meanings (as opposed to the behavioral outcomes by applying those meanings). With such awareness, the teacher is able to treat their internalized actions as objects of operations and coordinate those mental actions at the level of re-presentation (Montangero & Maurice-Naville, 1997; Tallman, 2015). A teacher's conscious awareness of the mental actions constituting their FOMs provides *intellectual need* (Harel, 2008a, 2008b, 2013) for the teachers to model students' mental actions and reorganize the teachers' own mental actions. Harel (2013) defined one category of intellectual need to be the *need for causality*—that is—“one's desire to *explain*, to determine a cause of a phenomenon” (p. 126). If a teacher is not consciously aware of the mental actions constituting their FOMs, or they are only aware of the consequential physical actions and behaviors governed by those FOMs, they likely do not have the desire to explain the students' behaviors by inferring their mental actions; instead, they may focus on modeling the students' behaviors and attempt to engender changes in those behaviors. Furthermore, they may not be committed to or able to discern how their SOMs are conceptually different or related to their FOMs either. This desire to identify relationship between mental actions also relate to another category of intellectual need, the *need for structure*. The need for structure is the need to reorganize the knowledge one has learned into a logical structure. Knowledge about behaviors is harder to be structured logically than knowledge about mental actions and operations. This is because one mental action can govern a collection of appeared-to-be-different behaviors, and one behavior can be underlain by varying mental actions. To conclude, conscious awareness serves as the basis for a teacher's reflection on and comparison between multiple mathematical meanings during Type 4 Decentering.

Type 4 Decentering has three important outcomes. First, the teacher experiences reconstructions and reorganizations in their FOMs such that they are more coherent to a network of related mental actions and operations. Montangero and Maurice-Naville (1997) assert that “the process of coordination of actions and operations is the most important instrument of decentering ... and this coordination is both individual and social” (p. 96). The following quote of Piaget unpacked the meaning of “individual and social:”

All logical thought is socialized because it implies the possibility of communication between individuals. But such interpersonal exchange proceeds through correspondences, reunions, intersections, and reciprocities, i.e., through operations. Thus there is identity between intra individual operations and the inter individual operations which constitute *co-operation* in the proper and quasi-etymological sense of the word. Actions, whether individual or interpersonal, are in essence co-ordinated and organized by the operational structures which are spontaneously constructed in the course of mental development. (Piaget, 1962)

This process of “co-operation” speaks to the very process (and outcome) of Type 4 Decentering. In addition, with a more coherent and organized knowledge in mind, a teacher will be more capable of supporting students’ constructions of robust and coherent mathematical meanings in their classroom.

Second, Type 4 Decentering is the process through which a teacher’s SOMs become enactable and consequential for their interactions with students (see Silverman & Thompson, 2008; Tallman, 2015). A teacher’s attempt to situate a learning opportunity based on what a student already knows requires not only construction of SOMs of the student’s knowledge but also ability to identify the qualitative differences between those SOMs and the teacher’s FOMs (Hodkowski, 2018). Then, the teacher can hypothesize potential construction the student can make (see *zone of potential construction*; Norton & D’Ambrosio, 2008; Steffe & D’Ambrosio, 1995; *didactic model*; P. W. Thompson, 2002; *hypothetical learning trajectory*; Simon, 1995;

Simon & Tzur, 2004) and then act in ways supportive for the student's development towards the goal understandings within the student's framework.

Third, Type 4 Decentering likely supports a teacher's shift from engaging in *responsive* and *intuitive interactions* to engaging in *analytical interactions* with students (Steffe & Ulrich, 2014). During responsive and intuitive interactions:

teacher/researchers are agents of interaction and they strive to harmonize themselves with the students with whom they are working to the extent that they "lose" themselves in their interactions. They make no intentional distinctions between their knowledge and the students' knowledge, and, experientially, everything is the students' knowledge as they strive to feel at one with them. In essence, they become the students and attempt to think as they do (Thompson, 1982, 1991; van Manen, 1991). (Steffe & Ulrich, 2014, p. 105)

This type of interactions can be achieved when a teacher engages in Type 2 or Type 3

Decentering spontaneously. The teacher is acting as a *first-order observer* and focusing on constructing SOMs in the moment of the interactions (Steffe & Wiegel, 1996). As Hackenberg (2010) described, such insights into students' mathematical thinking "are not often clear at the time and not always traceable" (p. 267). In comparison, analytical interactions involves "deliberate conjecturing about mathematical ways of operating that model the mathematical activity of the student" (Hackenberg, 2010, p. 267). This deliberateness refers to a teacher constructing SOMs in a conscious attempt to engender accommodation in students' ways of operating. The teacher serves as a *second-order observer* (i.e., operating as external to a interaction where the teacher becomes an observer of their circumstance as an observer) and "become aware of how they interact and of the consequences of interacting in a particular way" (Steffe & Ulrich, 2014, p. 106). Doing so necessitates the teacher to explicitly and simultaneously takes into account their SOMs and FOMs during their interactions with students (Steffe & Wiegel, 1996), that is, Type 4 Decentering.

Remarks

This framework of teacher decentering characterizes how teacher decentering may operate in different ways at a mental level. The mental processes involved in setting aside one's FOMs to construct SOMs of another can be more complex and varied than how it has been described in the literature. Specifically, a teacher's efforts to decenter can vary depending on how the teacher constructs or reconstructs their FOMs and SOMs through assimilation and accommodation. Shedding light on the constructive and dynamic nature of teacher decentering, this framework suggests that decentering is in essence a mental process of a teacher actively and systematically applying their mathematical meanings to discern aspects of students' mathematics and meanwhile modifying and reorganizing the teacher's own mathematical meanings. In particular, Type 1 and Type 2 Decentering describe how a teacher's FOMs (including previously constructed SOMs) form the basis for student-teacher interactions, and Type 3 and Type 4 Decentering characterize how their FOMs are in turn modified by those interactions. Steffe and Tzur (1994) have described a similar bidirectional mechanism in the context of children's mathematical interactions.

Second, I note that the distinctions between the different types of decentering are not to be interpreted as hierarchical stages. Indeed, Type 1 Decentering is likely required for Type 3 Decentering, and Type 3 Decentering has to precede Type 4 Decentering as it provides the source materials that teachers operate on in Type 4. In this regard, they are somewhat hierarchical. However, I clarify that the numbering of these decentering types does not indicate a hierarchy in a way that Type 0, Type 1 or Type 2 Decentering is lower-level reasoning or less productive than Type 3 or Type 4 Decentering. Type 1 Decentering is foundational to Type 3 and Type 4. Type 2 Decentering likely occur after Type 3 (and/or Type 4). A teacher's FOMs

can become richer and more coherent as the teacher repeatedly engages in Type 3 and Type 4 Decentering, and ultimately, they are engaging in Type 2 Decentering. We can imagine a veteran teacher or an experienced researcher engaging in Type 2 Decentering during most of their interactions with students because they have constructed a substantial knowledge base of student meanings that allows them to assimilate student activities without perturbation. This framing is coherent to my aforementioned conceptualization that SOMs are integral parts of FOMs. At this point, because the teacher primarily operates on their FOMs and enact Type 2 Decentering, they may be more prepared to engage in analytical interactions with students and Type 4 Decentering in-the-moment.

Third, what type of decentering comes into play is wholly determined by the nature of a teacher's FOMs. Two teachers who are in face of what a researcher perceives to be similar student actions may engage in different types of decentering depending on the teacher's personal ways of organizing and refining their mathematical meanings in response to those observed actions. For example, in P. W. Thompson and Thompson's (1994) study, the teacher Bill's knowledge of speed was dominated by the calculations of division, and he could not discern that his student Ann, who appeared to be able to enact similar calculations, was in fact holding a different meaning. In this case, Bill was not decentering, but the researchers were engaging in Type 3 Decentering regarding Ann's activity.

Summary

This chapter addresses three key domains of theories relevant to the present study. First, in light of a modeling perspective on teacher knowledge development, I conceptualize teacher knowledge of students' mathematics as a researcher's TOMs (i.e., a researcher's SOMs of teacher's SOMs of students). A teacher's SOMs of a student's mathematics are hypothetical

conceptual operations that explain that student's language and actions, and the teacher cannot verify if their SOMs are true representations of the student's knowledge, but they are merely viable within the teacher's experiences of working with the student. Piagetian notions of assimilation, perturbation, and accommodation provide the same grounding for understanding teachers' construction of SOMs and FOMs. This construction process is effortful, iterative, and continual, allowing the teacher's SOMs to become explicit, stable, and generalizable. Second, I conceptualize how student-teacher interactions provide a site for teacher's knowledge development, where a teacher and their students engage in reciprocal assimilation and accommodation and continually construct their SOMs and FOMs. Third, I propose a framework of teacher decentering that outlines four decentering types based on teachers' ways of operating on their FOMs and SOMs.

In the next chapter, I discuss the research methodology that enables me to gain access to these four types of decentering. The methodology is conceptualized and described as an extension of the constructivist teaching experiment, which simultaneously provides the participating teacher with opportunities to interact with students and enables the researcher to model and document the teacher's development in FOMs and SOMs.

CHAPTER 5

METHODOLOGY

The clinical interview, which is based on key values, especially an ethic respect for the individual's construction of reality, leads to an interesting and distinctive view of education that stresses thinking and children's examination of it. (Ginsburg, 1997)

As mentioned in Chapter 1, teachers' modeling of students' mathematical thinking in relation to on-going student-teacher interactions has been a comparatively stagnant area in part due to methodological challenges. In this chapter, I first review existing research methods for investigating teachers' understandings of students' mathematics. Incorporating a subset of these methods and expanding the constructivist teaching experiment methodology, I propose and describe a novel methodology called *teaching experiment*². Finally, I describe my implementation of this methodology in the present study, including the participants, research site, and data collection and analysis methods.

A Review of Forms of Inquiry in Literature

Researchers have used both quantitative and qualitative methods to investigate teachers' understandings of students' mathematics. Different artefacts have been used in qualitative interviews and quantitative instruments. These artifacts are in the forms of (a) mathematical problems, (b) student work samples, (c) transcripts, (d) hypothetical student responses or scenarios, (e) videos of students working with mathematical tasks, (f) videos of a teacher's own interactions with students, (g) videos of other teachers' teaching (see literature examples in Table 2).

Table 2. Types of artefacts used to access teachers' understandings of students' mathematics

Artefact	Example Literature
Mathematical problems	(Bergqvist, 2005; Empson & Junk, 2004; Nathan & Koedinger, 2000; Şen Zeytun et al., 2010; Tirosh, 2000; Wilkie, 2016)
Student work samples	(Baldinger, 2019; Didis et al., 2016; Driscoll & Moyer, 2001; Franke et al., 2001; Hallagan, 2006; Jacobs et al., 2010; Jenkins, 2010; Kazemi & Franke, 2004; Sapti et al., 2019; P. H. Wilson et al., 2013)
Transcripts	(Dreher & Kuntze, 2015; Schack et al., 2013)
Hypothetical student responses and scenarios	(An & Wu, 2012; Bergqvist, 2005; Bray, 2011; Carpenter et al., 1989; Empson & Junk, 2004; Even & Tirosh, 1995; Hill et al., 2008; Hill & Chin, 2018; E. Jacobson et al., 2018; McEwan & Bull, 1991; Son, 2013; Tirosh, 2000)
Student videos	(Carpenter et al., 1988; Didis et al., 2016; Norton et al., 2011; Walters, 2017; Yoon, 2019)
Videos of a teacher's own interactions with students	(M. G. Sherin et al., 2009; M. G. Sherin & Dyer, 2017; M. G. Sherin & van Es, 2005; van Es & Sherin, 2008; P. H. Wilson et al., 2013)
Videos of other teachers' teaching	(Jacobs et al., 2010; P. H. Wilson et al., 2011)

Among the qualitative studies, research data is collected in a variety of contexts, including (a) survey or questionnaire, (b) researcher-teacher interviews, (c) teacher-student interviews or teaching experiment, (d) written reflection, (e) lesson planning, (f) classroom instruction, (g) post-instruction reflection or interviews, and (h) conversations occurred in teacher education courses or professional development (see literature examples in Table 3). Some researchers have used a combination of data sources to triangulate their data. For example, some researchers inferred teachers' knowledge of students' mathematics by analyzing teachers' in-the-moment interactions with students and drew on retrospective teacher interviews to support their inferences (E. Johnson & Larsen, 2012; Seymour & Lehrer, 2006; Speer & Wagner, 2009).

Table 3. Types of data sources for accessing teachers' understandings of students' mathematics

Data Source	Example Literature
Survey or questionnaire	(Dreher & Kuntze, 2015; Son, 2013; Tirosh, 2000; Wilkie, 2016)
Researcher-teacher interview	(Baldinger, 2019; Bergqvist, 2005; Bray, 2011; Carpenter et al., 1988; Didis et al., 2016; Empson & Junk, 2004; Even & Tirosh, 1995; Hallagan, 2006; E. Jacobson et al., 2018; Sapti et al., 2019; Şen Zeytun et al., 2010; Walters, 2017; Yoon, 2019)
Student-teacher interview or teaching experiment	(Fernandes, 2012; Heid et al., 1998; Hunting, 1997; Jacobs et al., 2006; Jenkins, 2010; Lee, 2018, 2021; Moyer & Milewicz, 2002; Norton & McCloskey, 2008; Weiland et al., 2014; Wickstrom & Langrall, 2020; P. H. Wilson et al., 2013; Wright et al., 2006)
Written reflection	(Crespo, 2000; Jenkins, 2010; Wickstrom & Langrall, 2020; P. H. Wilson et al., 2013)
Lesson planning	(Çelik & Güzel, 2017)
Classroom instruction	(Bray, 2011; Çelik & Güzel, 2017; Hodkowski, 2018; E. Johnson & Larsen, 2012; Seymour & Lehrer, 2006; M. G. Sherin, 2002; Speer & Wagner, 2009)
Post-instruction reflection or interview	(Hodkowski, 2018; E. Johnson & Larsen, 2012; Kazemi & Franke, 2004; Seymour & Lehrer, 2006; M. G. Sherin, 2002; Speer & Wagner, 2009; Steinberg et al., 2004; Wickstrom & Langrall, 2020; Wilkie, 2016)
Teacher education courses or professional development	(M. G. Sherin, 2002; Tirosh, 2000; P. H. Wilson et al., 2011, 2014)

Relevant to the present study are the methods of student-teacher interviews and researcher-teacher interviews independent of the participating teachers' classroom instruction. During the researcher-teacher interviews, I primarily used the video episodes of the participating teachers' own interviews with individual students. In the following, I describe the research design in detail and justify my choices of these methods in relation to the research goals and questions.

Constructivist Teaching Experiment Methodology

Researchers have been using the constructivist *teaching experiment* methodology (Steffe & Thompson, 2000b) to form inferences about students' mental actions and operations through on-going design-based interactions with individual students. Participants of a teacher experiment include (a) a teacher-researcher, (b) one, two, or a small group of students, and (c) one or two witness-researchers/observers. A teaching experiment often include three phases: (a) exploratory teaching (also called pre-teaching experiment sessions), (b) teaching experiment sessions, and (c) post teaching experiment session (Figure 18).

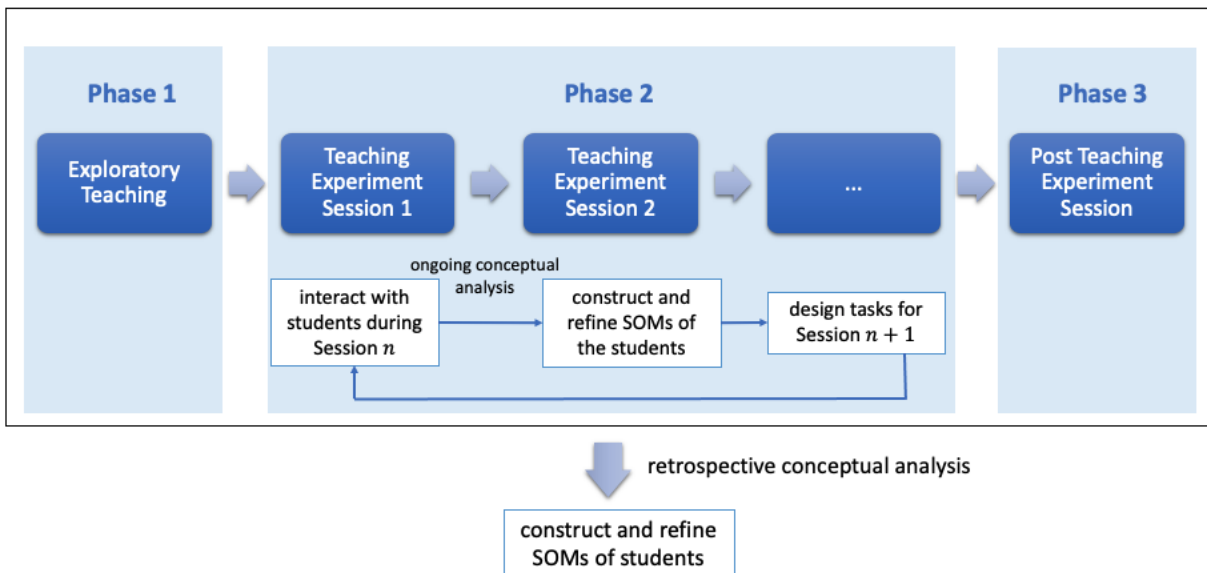


Figure 18. The processes of a teaching experiment.

Exploratory teaching consists of a series of semi-structured, task-based clinical interviews (Clement, 2000; Ginsburg, 1997; Goldin, 2000; Hunting, 1997) where the teacher-researcher attempts to “become thoroughly acquainted at an experiential level with students’ ways and means of operating in whatever domain of mathematical concepts and operations are of interest” (Steffe & Thompson, 2000b, p. 275) without judgement or evaluation. The researcher makes no attempt to influence students’ thinking, although the interactions themselves

unavoidably affect students' thinking in certain ways; rather, the researcher attempts to generate a profile of the mental actions and operations available to the students. Another goal of exploratory teaching is to make essential distinctions in students' ways and means of operating that constitute the global research hypotheses of a teaching experiment. These research hypotheses also help clarify the overall general intentions of the teaching experiment and guide the selection of students who will participate in Phase 2 of the experiment.

During each teaching session in Phase 2, the teacher-researcher engages in sustained interactions with the students in order to generate and test ongoing local hypotheses of the students' mathematics. Right after each session, the teacher-researcher discusses with the witness-researchers to formulate and confirm those hypotheses and hypothesize potential constructions the students can make in the following sessions (called *ongoing conceptual analysis*; Steffe & Thompson, 2000; Thompson, 2008). And then, the researchers design tasks for the next teaching session for the purpose of testing and stabilizing these hypotheses. It is these iterative hypothesis-testing-and-generating processes that makes a teaching experiment an "experiment." Meanwhile, the teacher-researcher shifts from intuitive interactions for constructing initial SOMs of the students to analytical interactions. The teacher-researcher's interactions and task design gradually become goal-oriented with the purpose of engendering accommodation in the students' FOMs, but those goals and hypotheses are always subject to change based on the students' responses. The teacher-researcher's increasing experiences with the students' mathematics and increasing awareness of the mathematical consequences of their interactions makes a teaching experiment also a "teaching" activity.

Transitioning to Phase 3, the teacher-researcher conducts a post teaching experiment session, often in the form of semi-structured, task-based clinical interview. The goals of this

session are to put the stable models into testing with new problem situations and understand the limitations and affordances of those models. The tasks are predicted to require minor accommodation in the students based on the researchers' SOMs. If a student is observed to experience perturbation, the researcher would interact with the students with no attempt to engender accommodation in the student's thinking. Some researchers may use same or similar set of tasks of the exploratory teaching session but with different or more open-ended prompts. This allows the researchers to compare the student's ways of operating prior to and after the teaching experiment. Some researchers may also design tasks related to other mathematical concepts in order understand the consequences of the student's ways of thinking. This will help generate future research hypotheses and research questions.

Upon completion of the three phases, the researchers iteratively watch the entire set of data and review transcripts stemming from the data to reconstruct SOMs of the students (called *retrospective conceptual analysis*; Steffe & Thompson, 2000; Thompson, 2008). These iterative retrospective analysis efforts involve the researcher constructing hypothetical mental actions and operations that viably explain the students' activities. The researchers also continually search the data for instances that the models cannot account for, and they modify their models or attempt to explain developmental shifts in the students' meanings. Products of these analyses include characterizations of distinctive mathematical meanings within and across students, affordances and limitations of those meanings, and developmental trajectories of those meanings.

Teaching Experiment² Methodology

In the current study, I expand the teaching experiment methodology to a methodology called *teaching experiment²*. Although a teaching experiment² resembles some features of a *multitiered teaching experiment* (Lesh & Kelly, 2000), I create the term "teaching experiment²"

to emphasize its root in Steffe and Thompson’s (2000b) notion of *teaching experiment* that is consistent with a constructivist perspective, and it includes a teaching experiment at two levels. The Tier-1 teaching experiment involves a teacher acting as the teacher-researcher, interacting with individual or paired student(s), and constructing, testing, and revising SOMs of the student(s) in-the-moment. The researcher primarily acts as a witness during their interactions. After each teaching session, the researcher and the teacher watch and analyze the teaching episode, formulate hypotheses of the student(s)’ thinking, and design tasks that incorporate those hypotheses. These procedures align with a constructivist teaching experiment other than the teacher being in the role of a teacher-researcher and the researcher being an observer.

However, it is important to note that teachers often lack experience of conducting a constructivist teaching experiment. In particular, they likely do not perceive their interactions with students as a scientific investigation into students’ reasoning. This necessitates the Tier-2 teaching experiment, where the researcher supports the teacher’s examination into student thinking by engendering the teacher’s learning of the student’s mathematics (the “teaching” aspect). As mentioned in Chapter 4, such learning can be achieved through the four types of decentering, and the researcher needs to ask purposeful questions to engender accommodation in the teacher’s mathematics. Meanwhile, the researcher infers and documents the teacher’s mathematical thinking by continually testing their hypotheses of the teacher’s thinking in the moment of their interactions with the teacher and retrospectively (the “experiment” aspect). Table 4 summarizes the roles of student, teacher, and researcher in the teaching experiment² methodology.

Table 4. Roles of student, teacher, and researcher in the teaching experiment² methodology

Student	Teacher	Researcher

Tier-1 Teaching Experiment	Role: subject of study	Role: teacher-researcher Meaning of teaching: engaging with student thinking and engendering accommodation in student thinking Meaning of experiment: constructing and testing hypotheses of student thinking	Role: witness-researcher
Tier-2 Teaching Experiment		Role: subject of study Role: teacher-researcher Meaning of teaching: engaging with teacher's thinking about student's thinking and engendering teacher's Type 1, 2, 3, 4 Decentering Meaning of experiment: constructing and testing hypotheses of teacher thinking	

To operationalize this two-tier teaching experiment into concrete research activities, I break down the experimental procedures into two phases. Phase 1 includes a series of pre-teaching sessions, and Phase 2 includes cycles of planning-teaching-reflecting activities. Figure 19 illustrates these two phases, and Figure 20 illustrates the model-building activities involved in each session. I discuss each session in detail in the following.

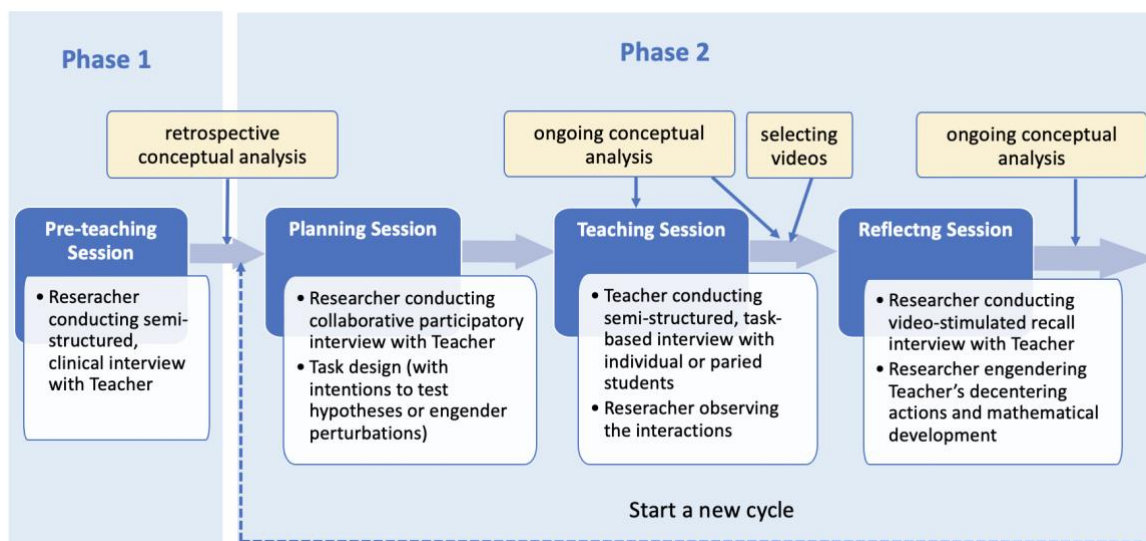


Figure 19. The teaching experiment² methodology.

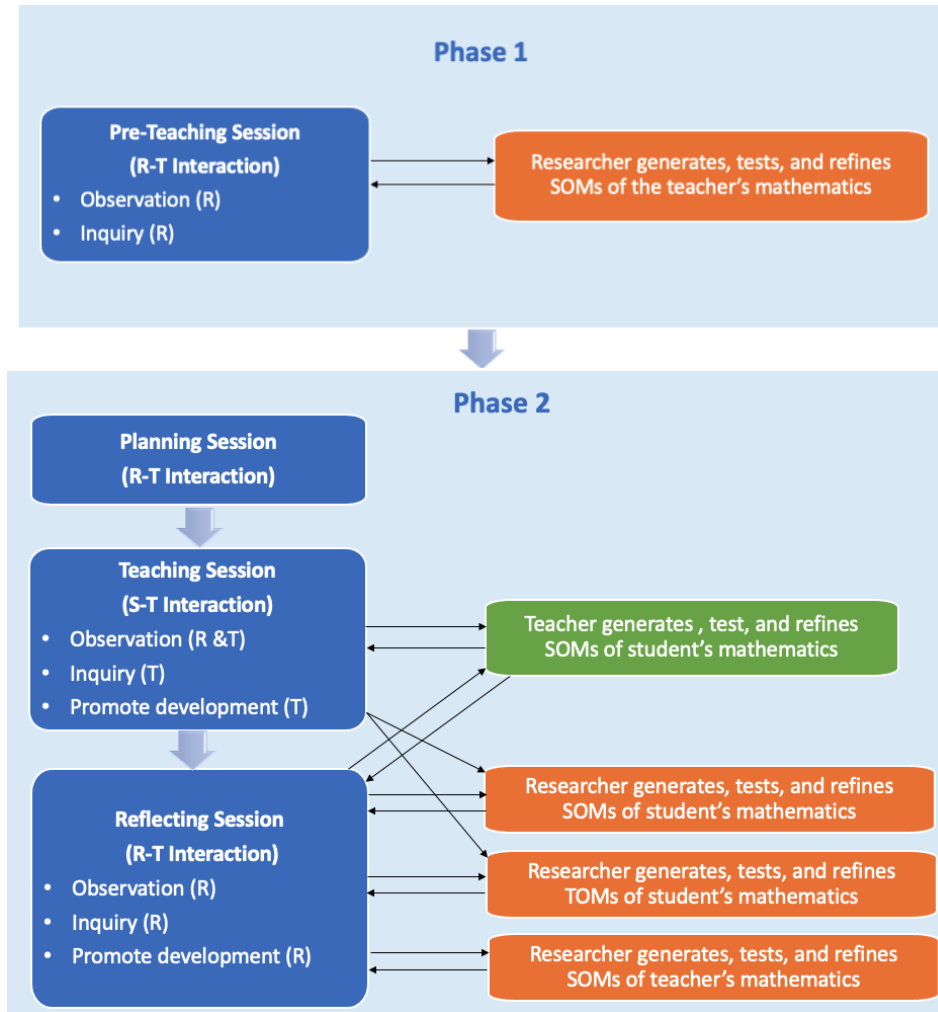


Figure 20. Participants' roles and the model-building activities involved in the teaching experiment² methodology.

Note. R refers to Researcher's activities and T refers to Teacher's activities. Green boxes indicate Teacher's modeling activities; orange boxes indicated Researcher's modeling activities.

Phase 1: Pre-teaching session

The research goal of the Phase 1 pre-teaching sessions is to formulate SOMs of the teacher's mathematical meanings *prior to* their interactions with students in Phase 2. In each pre-teaching session, the researcher conducts a semi-structured, clinical interview with the teacher on a chosen mathematical topic. The researcher asks the teacher to discuss their lesson goals,

instructional materials, the rationale of the tasks used, and experiences of working with students. The researcher also designs additional mathematical tasks with the purpose of eliciting the teacher's mathematical meanings but without providing systematic interventions that engender changes in their meanings. The researcher primarily observes the teacher's actions and inquire into the teacher's thinking. These researcher-teacher interactions enable the researcher to construct initial SOMs of the teacher's mathematics, and these SOMs are reference models with which the researcher infers the teacher's mathematical development in Phase 2. Moreover, these interactions enable the researcher to familiarize themselves with the mathematical content and construct fundamental mental actions and operations necessary for the researcher's assimilation and accommodation of the teacher's and students' activities in Phase 2.

Phase 2: Planning Session

An essential practice of conducting a teaching experiment is designing mathematical tasks that enable the teacher-researcher to build and test their hypotheses about student thinking. Designing and choosing mathematical tasks are also parts of a teacher's everyday teaching practice. Doerr and Lesh (2003) have stressed the importance of implementing professional development activities in practice-based situations:

In almost all cases, we have found it necessary to create contexts in which teachers can express their current ways of thinking. Often, this involves putting teachers in practice-based situations where they need to interpret or assess students' thinking, to plan appropriate sequences of instructional activities in light of students' current understandings, or to select, modify or extend tasks based on the results of student model eliciting activities. All three of these situations (assessing, planning, and selecting) are deeply embedded in teachers' everyday practice. (p. 136)

In a planning session, the researcher focuses on supporting the teacher in designing, choosing, or modifying mathematical tasks that enable their scientific inquiry into students' mathematical thinking. However, the researcher does not dictate the task design decision, but to ensure the teacher comes up with a task that serves the purpose of testing certain hypotheses of

student thinking in light of their mathematical activities in the previous teaching session. In a teacher's daily teaching practice, they often design tasks in line with a lesson goal based on their FOMs, and the researcher's role in the present methodology is fostering the teacher's decision-making such that their SOMs, as opposed FOMs, are foregrounded and taken into account.

To achieve this goal, the researcher and the teacher engage in a *collaborative participatory interview* (Patton, 2015). Different than other interview methods where the interviewer (or the researcher) possesses the power of inquiring and requesting information from the interviewee, a collaborative participatory interview involves both participants establishing a mutually respectful inquiry relationship—that is, they can ask questions to each other and make decisions collaboratively. Specific to the context of a teaching experiment², the conversation often starts with a brief summary or discussion on the SOMs of the students as discussed in the previous reflecting session. The participants center their conversations around the following major questions: (a) What is available in each student's thinking? (b) How is the students' thinking different or similar when compared to that in the previous sessions? (c) What are the affordances of the students' thinking? and (d) What are the limitation of the students' thinking?

And then, the researcher and the teacher work together to design tasks based on their SOMs of the students. They first discuss if there are any hypotheses worth testing in the next session. If yes, they design a task that enables them to test those hypotheses. This often involve them varying the contexts, representations, or numerical values of a previous task in order to understand how the students assimilate or accommodate those changes. If the researcher and the teacher are convinced that their SOMs of the students are relatively stable, they can hypothesize potential meanings the students can construct in the next session. Then, they design a task for the purpose of generating perturbation and promoting accommodation in the students' thinking. As

Confrey (1990) described, a constructivist teacher “must form an adequate model of the students’ ways of viewing an idea and [they] then must assist the student in restructuring those views to be more adequate from the students’ and from the teacher’ perspective” (p. 109).

Upon creating the tasks, the researcher and the teacher outline eliciting, supporting, and extending questions around the tasks and anticipate the students’ responses and difficulties. The researcher should inform the teacher that they can vary the tasks or questions in the moment of their interactions based on the students’ responses, rather than strictly following the protocol.

Phases 2: Teaching Session

A task-based interview is a method of interacting with one or a small group of individuals within a task environments in order to infer the individual(s)’ reasoning and knowledge structures (Goldin, 2000). Numerous researchers have promoted the use of task-based interview (Fernandes, 2012) or teaching experiment (Norton & McCloskey, 2008) in mathematics teacher preparation, also referred to as clinical interview (Hunting, 1997; Lee, 2018, 2021; P. H. Wilson et al., 2013), problem solving interview (Jacobs et al., 2006), diagnostic interview (Moyer & Milewicz, 2002; Schack et al., 2013), or formative assessment interview (Weiland et al., 2014). Although these forms of interviews may slightly differ in their goals or implementation, they commonly involve a teacher (as the interviewer) interacting with individual or paired students (as the interviewee(s)) and inferring the student(s)’ mathematical thinking in-the-moment and retrospectively (e.g., through watching the interview episodes or reading interview transcripts). This interviewing experience affords the teacher a focus on students’ mathematical thinking in a context free of instructional constraints, distractions, and management issues in a classroom (Amador, 2016; Lee, 2018, 2021). As a result, they have opportunities to engage in sustained mathematical conversations with individual students and gain deep insights into the students’

thinking and its development. Furthermore, removing from the classroom context affords the researcher a fine-grained analysis of the teacher's model-building processes and the consequential mathematical development.

The teaching session of the present methodology adopts a similar method. In each teaching session, the teacher, the researcher, and the students are present at the same time and in the same space. At the teacher level, the teacher engages in interactions with individual or paired students and generates, tests, and refines SOMs of the students' mathematics in-the-moment. At the researcher level, the researcher takes the role of an observer during the student-teacher interactions and generates, tests, and refines SOMs and TOMs of the student's mathematics that can explain the students and the teachers' actions respectively. In order to keep the conversation as natural as possible, the researcher avoids providing intervention or interrupting the conversations and primarily observes the student-teacher interactions.

The *reality principle* of a model-eliciting activity states that “[t]he situations which we ask teachers to interpret must be in the context of their actual practice” (Doerr & Lesh, 2003, p. 133). The teaching session follows this principle by providing teachers with opportunities to interact with their own students authentically. In addition, it generates a collection of artefacts that enable the teachers' retrospective analysis of their own students' thinking, including video episodes of their own interactions with students and the authentic student work produced from those interactions. Beisiegel et al.'s (2018) study suggested that engaging teachers to watch videos of own instruction is more likely to produce meaningful discussions than watching videos of unknown teachers. In the following, I discuss how these artefacts are used in the reflecting session.

Phase 2: Reflecting Session

Many researchers have emphasized the role of reflection in improving teachers' mathematics teaching and learning, leading to the frequent use of terms such as *reflective practitioner* (Schön, 1983), *reflective teaching* (Zeichner & Liston, 1996), *reflective practices* (Barth-Cohen et al., 2018; Karsenty & Arcavi, 2017), *reflective stance* (Stockero, 2008), and *reflection* or *self-reflection* (Courtney, 2010; Hollingsworth & Clarke, 2017; Scherer & Steinbring, 2007; Tzur, 2001) in teacher education literature. It is noteworthy that researchers' conceptualizations of these terms have differed in significant ways and that an educator can reflect on a variety of aspects of their teaching through a variety of means. In conceptualizing the present methodology, I define reflection at both behavioral and mental levels.

Reflection Through Stimulated-Recall

At the behavioral level, reflection refers to an activity in which a teacher becomes an observer of their own interactions with students and analyzes the students' mathematical thinking. This reflection activity is supported by the method of *(video-)stimulated recall interviews* (Bloom, 1953; Calderhead, 1981b; Conners, 1978; Tuckwell, 1980b, 1980a). A stimulated recall interview used in the context of teacher education consists of:

replaying a videotape of a teaching episode to enable the viewer (usually the teacher of the episode) to recollect and report on [their] thoughts and decisions during the teaching episode. (C. M. Clark & Peterson, 1986, p. 259)

Originally, video-stimulated recall was used by researchers to gain access to teachers' thought processes and decision-making during teaching (Bloom, 1953; Calderhead, 1981a, 1981b; C. M. Clark & Peterson, 1986). Later on, researchers suggested that teachers' analysis of videos also supported their development of reflective stance of their teaching experiences (e.g., Herbst & Chazan, 2003; Hollingsworth & Clarke, 2017; Jacobs et al., 2006; Osmanoglu, 2016; Scherer & Steinbring, 2007; Sherin & van Es, 2009; Star & Strickland, 2008; Stockero, 2008; van Es &

Sherin, 2008; Walkoe et al., 2019), including reflecting on aspects of student thinking. Watching a video provides teachers with opportunities to re-experience the interactions as if it is in-the-moment. As Bloom (1953) described, a subject is “enabled to relive an original situation with vividness and accuracy” (p. 161). Such re-experience enables teachers to pay attention to something that was overlooked or unattended in the moment of interactions, such as nuances of student thinking. Furthermore, a video provides the source materials, i.e., the stimuli, for the teachers’ reflection and triggers conversations around the content of the video (see also *didactic object* in Thompson, 2002). A careful selection of videos can focus teachers’ discussion on specific instances of student thinking as opposed to other events or aspects of the interactions. As a result, teachers are more likely to ground their discussion on evidence of student thinking and engage in deeper levels of reasoning with student thinking (e.g., interpreting and restructuring as opposed to describing) (Stockero, 2008).

Cark and Peterson (1986) described that variations in the use of videos in stimulated-recall interviews include:

replaying only researcher-selected portions of the recording versus replaying the complete tape; researchers asking prespecified questions each time the tape is stopped versus soliciting open-ended commentary from the teacher; and researcher control of when to stop the tape versus teacher control or shared control. (p. 259)

Regarding the methodology pertaining to the present study, the researcher selects portions of video clips and asks purposeful questions to elicit the participating teacher’s interpretations of student thinking. The researcher takes most control of when to play and stop the video clip to move the conversation forward, but the teacher can request replaying of the videos when needed. Although it may be more helpful to ask the teacher to select the video clips (as opposed to the researcher) in order to understand what the teacher draws attention to and finds interesting, I use the alternative in implementing this methodology due to several reasons. First, the researcher can

select a small set of video clips that can potentially generate focused and in-depth conversations about student thinking. This is particularly important and helpful when the reflecting session is within 1 to 1.5 hours. It also reduces the teacher's total participation time since they do not need to spend extra hours to watch and analyze the entire teaching episode; in fact, each participant, often an in-service teacher, has already devoted significant amounts of time in the study by going through the pre-teaching sessions and cycles of planning, teaching, and reflecting sessions. Second, a teaching experiment necessarily involves intervention, and focusing teachers' attention to certain high-quality clips that include important aspects of student thinking is necessary to engage the teacher in second-order modeling (see high-quality attributes of instances of student thinking in Sherin et al., 2009; Van Zoest et al., 2017). I note that it is always a trade-off between providing intervention and investigating spontaneous and naturalistic thinking, and future researchers can test both options to better understand the limitations and affordances of each.

Reflection Through Re-presentation

At the mental level, reflection is a process of taking ones' actions and thought as objects of thought (Dewey, 1910/1997; Montangero & Maurice-Naville, 1997; Piaget, 2001a; von Glasersfeld, 1995). Von Glasersfeld described reflection as a process that allows an individual to "step out of the stream of direct experience, to re-present a chunk of it, and to look at it as though it were direct experience, while remaining aware of the fact that it is not" (p. 90). Regarding the construction of SOMs specifically, reflection demands a teacher to mentally re-present the students' actions and reason with those actions by coordinating them with the teacher's own actions, the same students' previous actions, or other students' actions. As mentioned in Chapter 4, these mental activities can be achieved through Type 3 Decentering (i.e., constructing SOMs through re-presenting students' actions) and Type 4 Decentering (i.e., re-presenting, comparing

and coordinating aspects of SOMs and FOMs). Before enacting Type 3 and Type 4 Decentering, the teacher likely also needs to enact Type 1 Decentering in order to re-present the students' observable actions and compare those actions to the teacher's own actions.

Because these are cognitively demanding thought processes, they may be difficult to be enacted in the moment of student-teacher interactions, especially to novice teachers. The reflecting session provides the teacher an opportunity to reconstruct and reflect on their SOMs and FOMs. It also offers a space for the researcher to promote these reconstructions and conscious awareness of those constructions. Pragmatically, the researcher focuses their questioning on three aspects. First, the researcher focuses on stimulating the teacher's retrospective construction of students' mathematics by asking: "What do you think the student is thinking at this point?" or "How do you interpret the students' activities?" (i.e., promoting Type 1 and Type 3 Decentering) Second, the researcher focuses on fostering the teacher's coordination of SOMs and FOMs and conscious awareness of that coordination by asking: "How is the student's thinking different or related to your own thinking?" "How is the student's thinking different or related to their previous thinking?" or "How is Student A's thinking different or similar to Student B's thinking?" (i.e., promoting Type 4 Decentering) Third, the researcher asks purposeful questions to confirm, reject, or modify their TOMs inferred from the teaching session. For example, a teacher's behavior during student-teacher interactions may appear to be insensitive to student thinking, but this behavior itself may not fully indicate a lack of construction of SOMs of student's thinking. Providing an opportunity for the teacher to articulate their reasoning affords the researcher a more precise explanation for the teacher's behavior. In all of these questioning practices, the researcher presses for detail and specificity (Kazemi & Franke, 2004) to create fine-grained models of the teacher's thinking.

Doerr and Lesh (2003) summarized that the activities that drive the development of a teacher's models include (a) "perturbing existing models or eliciting new ways of thinking" (p. 136), (b) testing, revising, and refining their models and resolving mismatches within the models, and (c) "sustaining the model across a broad set of contexts" (p. 136). To promote such development, I include cycles of planning-teaching-reflecting as opposed to a single cycle in this methodology. Variations in mathematical problem contexts allow teachers to continually test, revise, and refine their SOMs of student thinking, leading to increasingly stable SOMs that support the teacher's assimilation of student activities (see the *multiple contexts principle* of a model-eliciting activity in Doerr & Lesh, 2003). Collectively, these extensive researcher-teacher interactions allow the researcher to promote and characterize the teacher's mathematical development, especially how second-order modeling is generative to such development.

The Corroboration-Through-Triangulation Principle

This methodology follows the *corroboration-through-triangulation principle* of a multitiered teaching experiment that requires a systematic juxtaposition and corroboration of several perspectives (Lesh & Kelly, 2000). For example, there exists a between-observer mismatch between the researcher's SOMs of the students and the teacher's SOMs of the same students. The reflecting sessions provide the researcher with an opportunity to recognize and model such mismatches through communication with the teacher.

There also exists between-role mismatches. For example, the researcher engages in triangulation of their SOMs of the teacher's mathematics related to the pre-teaching sessions (as an interviewer), teaching sessions (as an observer), and reflecting sessions (as a teacher-researcher). Meanwhile, the teacher triangulates their SOMs of the students by shifting from casting in a role of a teacher-researcher (or interviewer) in the teaching sessions to an observer

(or interviewee) in the reflecting sessions. The teacher's shift in roles comes with a shift in perspectives on the student's activities; viewing the student's activities as a true observer by obtaining an after-the-fact view of a videotape encourages the teacher's adaptation of their SOMs that were constructed in-the-moment. More importantly, removing from the role of an interviewer allows the teacher to reconstruct and restructure their FOMs (including SOMs) through decentering. Overall, the teacher acts as a research subject, research collaborator, teacher-researcher, model builder, and mathematics learner throughout the teaching experiment².

Different Than A Teacher Development Experiment

The teaching experiment² methodology answers Simon's (2000b) call for researchers creating situations to observe and document teachers' developmental processes in order to inform teacher education that fosters such development. Similar to Simon's (2000b) *teacher development experiment* (TDE), the present methodology aims to generate and modify models of teachers' development through ongoing interactions with the teachers and to promote that development through purposeful questioning and design. However, it differs from a TDE in that the present methodology is primarily concerned with a teacher's mathematical development, and in a TDE, the researcher simultaneously focuses on the observed teachers' pedagogical development and mathematical development (Simon, 2000b; Simon & Tzur, 1999). Due to this expanded focus, a TDE requires multiple sites during its implementation, which include whole-class instruction occurring in mathematics education courses and individual case studies in the context of interviews and classroom teaching. By contrast, the teaching experiment² methodology is limited to a setting of one-on-one or one-on-two interactions. Besides, the use of video-stimulated recall interviews is not part of a TDE but a critical component of the teaching experiment² methodology.

Participants and Research Site

This study included two rounds of data collection. In each round, I worked in partnership with one high school mathematics teacher and one or two of their students. Both teachers are chosen from the same school on a voluntary basis. The school is a top rated public high school located in the southeastern United States. Different from a private school setting, I assumed that students and teachers from a public school were more likely to represent a typical population in the area. Also, the proximity of the school was considered, as I needed to visit the school on a regular basis for extensive amount of time. At the time of the study, the high school served grades 9-12 and had an enrollment of approximately 1375 students with a student-teacher ratio of 17 to 1 (11 were mathematics teachers). Approximately 75.8% of the student body are White, 9.4% Hispanic, 7.7% Asian, 3.8% Black, and 2.9% are mixed ethnicity.

Each teacher selected their student participants from a voluntary pool of their present and previously taught students. The only two selection criteria were: (a) the students had not received instruction on the chosen topic of the study at the beginning of their participation, and (b) the students were articulate in expressing their thinking. The former criterion enabled me to characterize the students' learning as a result of their interactions with their teacher during the study and minimize the influence of classroom instruction. Table 5 and Table 6 summarize the background information of each participant.

Table 5. Backgrounds of teacher participants

	Degree	Teaching Experience	Courses Taught
Ms. K (female, Asian American)	B.S. in Secondary Mathematics Education	1.5 yrs	Geometry Support (10th grade), Geometry (10th grade), Accelerated Geometry (9/10th grade), Algebra I (9th grade), Algebra II Support (11th grade)

Ms. N (female, Asian American)	B.S. in Secondary Mathematics Education, B.S. in Mathematics, MED in Mathematics Education (in progress)	0.5 yrs	Geometry (9/10/11th grade), Geometry Support (9/10/11th grade)

Table 6. Backgrounds of student participants

	Grade	Math Courses Taken	Teacher
Taylor	11th	Algebra 1 Support, Geometry, Algebra 2	Ms. K
Carlee	10th	Geometry, Algebra 1	Ms. N
Heidi	10th	Geometry, Algebra 1	Ms. N

Ms. K and Taylor

In the Round 1 data collection, I worked with Ms. K, who was in her second year of full-time teaching, and her previous student, Taylor. I chose Ms. K because I was familiar with her teacher education backgrounds and experiences. During her time in the teacher education program, I observed that she was proficient at articulating her own thinking, and she showed herself attentive to student thinking and open to learning new ways of thinking.

Taylor was a runner athlete, and he sometimes participated in a session right after his training, soaked with sweat. Ms. K and Taylor established a good relationship and sometimes joked about each other during a session. Ms. K characterized Taylor as an average, interesting kid. He was able to reason with mathematical ideas creatively, but he did not like to recall or generalize procedures. For most of the time, Taylor was engaging and persistent in solving problems, and his thinking was often unpredictable, making the story very interesting. However, when Taylor experienced perturbations, he got frustrated; when he obtained an answer, he was

not motivated about reasoning with the “why.” In both cases, he likes to say “I don’t know” to get away from answering Ms. K’s questions or thinking deeply.

Ms. N, Carlee, and Heidi

Ms. N. participated in Round 2 data collection. She volunteered to participate in the study due to her personal interest in educational research and noticing my Round 1 data collection in Ms. K’s classroom. Two high school students, Carlee and Heidi, were recruited from Ms. N’s on-level Geometry class, and they were close friends. Both Carlee and Heidi worked well with each other. Carlee was very expressive and confident in her thinking, and she usually spoke first when Ms. N posed a question. Although Heidi was generally more quiet and less confident in her solutions than Carlee, she was able to engage with Carlee’s thinking, asked questions, and expressed her own thinking when she disagreed. As discussed in Chapter 7, Heidi was able to convey more sophisticated understandings than Carlee as the study progressed.

Data Collection

The data collection procedures followed the teaching experiment² methodology described above. Both Ms. K and Ms. N went through Phase 1 and Phase 2 of the methodology, and their students participated in the teaching sessions of Phase 2. All study sessions were conducted independently and separately from each teacher’s teaching and each student’s learning at the school.

Research Timeline

Table 7 illustrates the timeline of data collection and duration of each session. Ms. K completed six pre-teaching sessions during Summer 2019 and six teaching cycles over the course of the Fall 2019 semester, producing 26 sessions and 26 hours’ data in total. One to two sessions occurred weekly, and the third and the fifth cycles contained one additional reflecting session.

Ms. N completed five pre-teaching sessions and two teaching cycles during the Spring 2020 semester, producing 10 sessions and 14 hours' data in total. The study discontinued due to suspension of school instruction and the COVID-19 pandemic starting in March 2020.

Table 7. Data collection timeline

Participant	Phase	Session/Cycle	Date	Duration (hrs.)
Ms. K	Phase 1: Pre-Teaching	1	6/5/2019	0.7
		2	6/10/2019	1.1
		3	6/12/2019	1.0
		4	6/14/2019	1.0
		5	6/19/2019	1.2
		6	6/20/2019	1.2
Ms. K and Taylor	Phase 2: Teaching Cycle	1	9/4/2019, 9/9/2019, 9/12/2019	0.8+1.5+1.3=3.6
		2	9/16/2019, 9/20/2019, 9/25/2019	1.4+0.8+1.1=3.3
		3	9/26/2019, 9/30/2019, 10/3/2019, 10/9/2019	0.7+1.0+1.0+1.2=3.9
		4	10/11/2019, 10/24/2019, 10/28/2019	0.8+0.9+1.0=2.7
		5	11/5/2019, 11/6/2019, 11/11/2019, 11/13/2019	1.0+0.9+1.0+0.9=3.8
		6	11/20/2019, 11/21/2019, 11/25/2019	0.5+1.1+1.2=2.8
Ms. N	Phase 1: Pre-Teaching	1	1/29/2020	1.0
		2	1/31/2020	1.4
		3	2/3/2020	1.4
		4	2/5/2020	1.4
		5	2/7/2020	1.6
Ms. N, Carlee, and Heidi	Phase 2: Teaching Cycle	1	2/10/2020, 2/12/2020	1.4+1.7=3.1
		2	2/19/2020, 2/24/2020, 3/4/2020	1.0+1.5+1.6=4.1
Total			36	40.2

Data Sources

The data collection efforts produced multiple data sources for analysis, including (a) video recordings, tablet screen recordings, and transcripts, (b) research protocols and task

artefacts, (c) students' and teachers' written work, researcher notes, and research journal. Table 8 summarizes the data types generated from the pre-teaching, planning, teaching, and reflecting session, respectively.

Table 8. Data sources

	Documents	Video Recordings
Pre-teaching Session	teacher work, teacher instructional materials, researcher notes, research journal	tablet screen recordings, session video recordings
Planning Session	teaching session protocol, task handouts, GeoGebra applets, research journal	session video recordings
Teaching Sessions	student work, teacher notes, researcher notes, research journal	in-the-moment tagged videos by the researcher (Round 1 only), session video recordings (and transcripts)
Reflection Session	teacher work, researcher notes, research journal	session video recordings (and transcripts)

Video Recordings and Transcripts

Pre-teaching Session. Figure 21 illustrates the equipment set-up of the pre-teaching session. As shown in the figure, the teacher and I sat by a table, and a witness researcher sat between us but a little bit far away from the table. Both cameras are stationary, with one being a paper-focused view (focusing on the teacher's work) and the other being a face-focused view (focusing on the teacher's face). A microphone is connected to the paper-focused camera in order to obtain high-quality audio. In some sessions, the task is presented on a tablet device, which was connected to a computer for screen-recording.

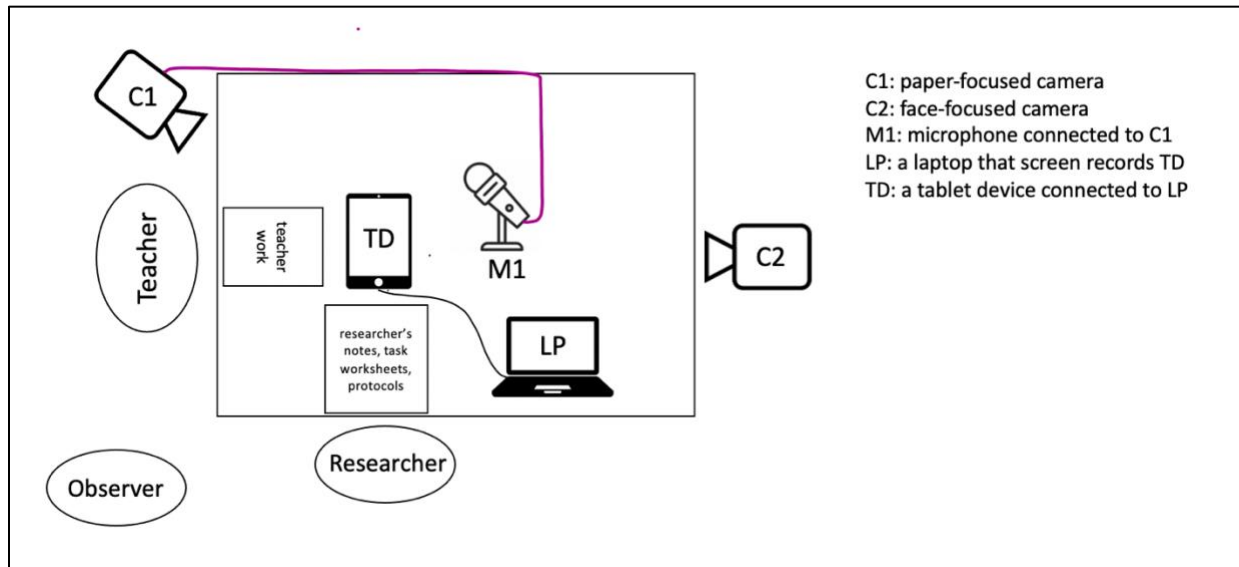


Figure 21. Equipment set-up in a pre-teaching session.

Planning and Reflecting Sessions. The planning and reflecting sessions have a similar set-up (see Figure 22). The only difference was that in a planning session, the participating teacher and I worked on our own laptop, with a working document shared online, and in a reflecting session, we positioned one laptop in the middle and watched the teaching episode together.

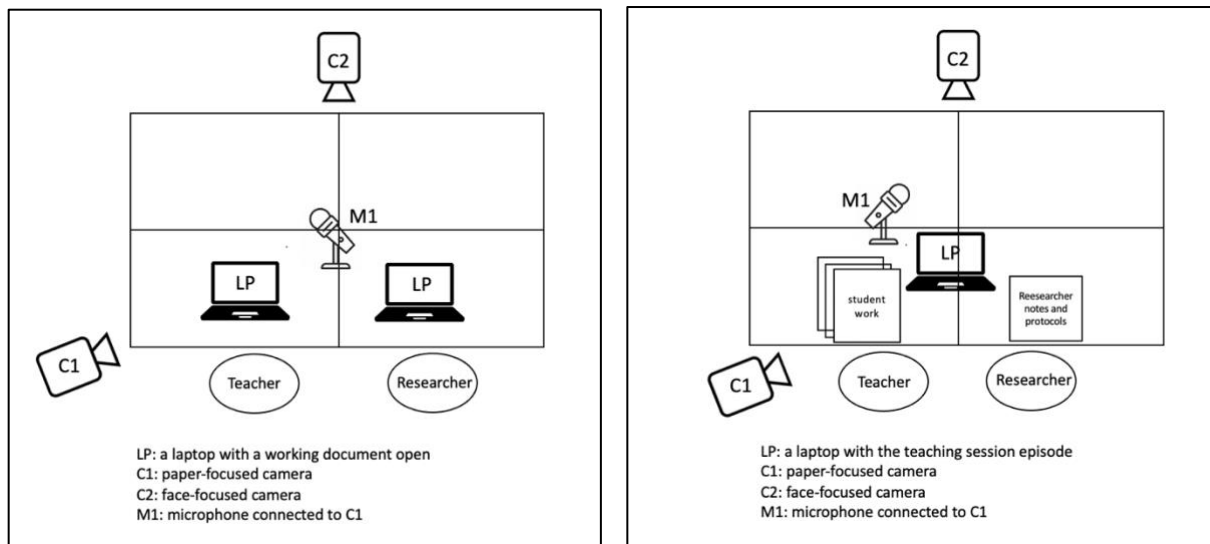
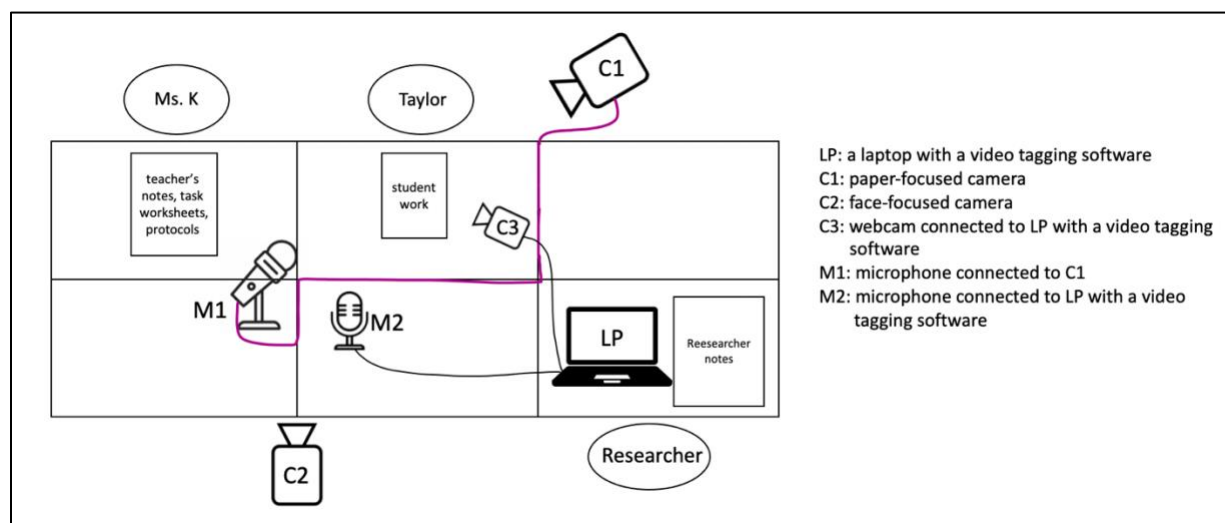


Figure 22. Equipment set-up in (left) a planning session and (right) a reflecting session.

Teaching Sessions. Figure 23 illustrates the room configuration in Ms. K and Ms. K's classroom. The face-focused camera is focused on the students and teacher's faces, and the paper-focused camera is focused on the students' work. I ensured that the paper-focused camera captured the students' activities (e.g., gestures, drawing, and writing) clearly since it produced the video for use in the reflecting session. There were two major differences in the set-up between Ms. K and Ms. N's classroom. First, because Ms. N worked with paired students as opposed to one, the seating was arranged differently than Ms. K's in order for Ms. N to have a better access to both students' activities. Second, with Ms. K, I used a live video-tagging (online) platform to tag interesting moments and write memos as the teaching video is recorded (see the equipment LP, M2, C3 in Figure 23, above). This set-up was removed in Round 2 data collection since it was difficult to capture two students' activities at the same time. In addition, tagging moments and writing memos in-the-moment became more difficult and less productive when the conversations involved frequent turn-taking among multiple individuals. Therefore, I decided to instead only focus on listening to the conversations and following the participants' thinking without tagging videos.



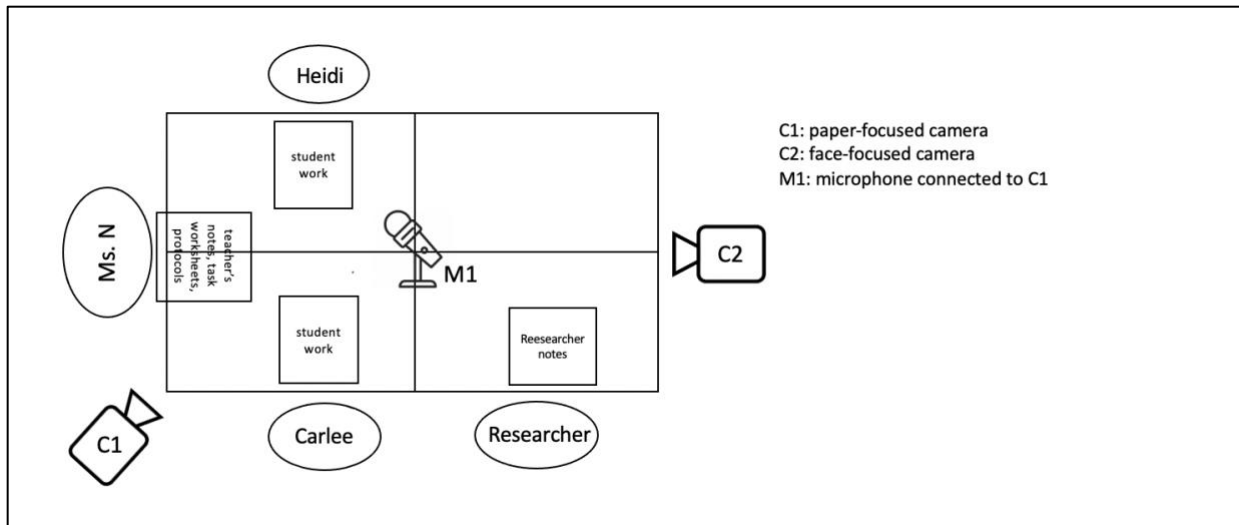


Figure 23. Equipment set-up in a teaching session in (above) Ms. K's room and (below) Ms. N's room.

Transcripts. I used automated transcript services to transcribe the teacher-researcher dialogue during the reflecting sessions, with approximately 80% accuracy. Then, instead of manually adjusting all of the transcripts by listening to all audios, I first identified critical instances after watching and analyzing the video episodes multiple times (see methods of analysis in the Data Analysis section). During the video watching, I frequently referred to the automated transcripts to assist my analysis of the conversations. I also conducted color-coding on those transcripts to organize themes emerge from the data analysis. After iterative transcripts and videos viewing and analysis, I located the relevant parts to the critical instances in the transcripts, manually edited the inaccurate transcription, and it was these edited portions of the transcripts that constituted the Excerpts in this dissertation.

Protocols and Task Artefacts

One group of documents collected from the data collection included the research protocols and task artefacts prepared prior to the implementation of each session. In the following, I describe how the protocol and task materials of each session were generated.

Pre-teaching Sessions. In preparation of the pre-teaching sessions, I chose a mathematical topic from a list of topics covered in all of the courses taught by each teacher (see a later discussion on the rationale of topic selection). Then, I conducted a conceptual analysis of a collection of related concepts, reviewed the relevant instructional materials offered by each teacher, and selected additional tasks from curriculum resources, online resources, and literature. The first two pre-teaching sessions were focused on asking each teacher to discuss their instructional materials of the chosen topic, and the rest of the sessions were focused on tasks chosen by me. Before each session, I created a research protocol to list the questions centered on the tasks in order to elicit the teacher's understanding of the content and construct SOMs of the teacher's content knowledge.

Planning Sessions. In each planning session, each participating teacher and I collaboratively generated tasks and guiding questions for use in the teaching session. All the thinking involved in task design is typed and documented electronically during the planning session (see Appendix A for an example). We often started the session by determining the goal of the upcoming teaching session. As mentioned in the previously, the goal could be testing a certain hypothesis of a student's thinking or perturbing the student in order engender accommodation in the student's thinking. Then we listed the key features of a task that would enable us to achieve the goal. Finally, we selected a problem context that incorporated those features. Due to time limitations, the planning session often ended without the worksheet or applet fully designed, but we ensured that the major task ideas and structured were agreed upon. After the planning session, I completed the formatting of worksheets and other task artefacts.

Reflecting Sessions. To prepare for the reflecting session, I watched the teaching episode and reviewed my tagged moments and memos to select video clips in which (a) the participating

teacher acted in ways responsive (or not responsive) to the students' thinking from my perspective (coded as "teacher move" in Figure 24), and (b) the students articulated their thinking that I conceived to be interesting or worth discussing (coded as "student thinking" in Figure 24). I wrote memos for each selected video clip to describe my interpretations, but they were not shown to the participating teacher during the session. In a separate document, I created an interview protocol where I listed potential questions to ask regarding each video clip or each group of clips (see Appendix A for a sample protocol).

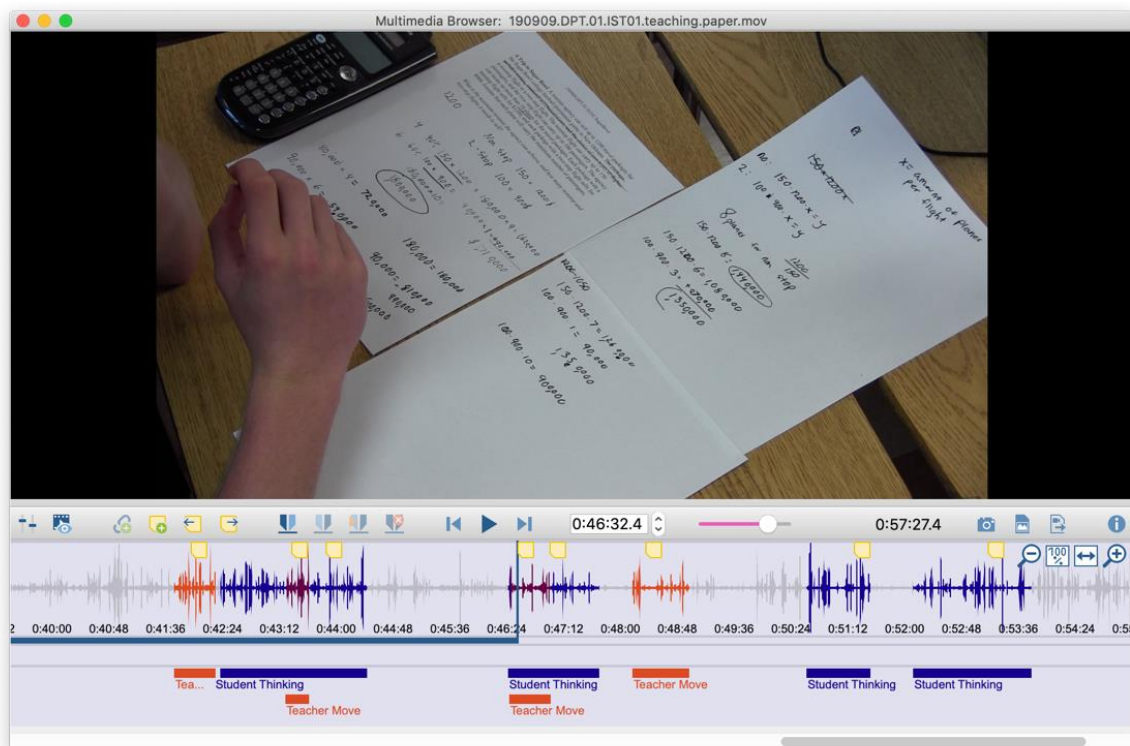


Figure 24. A computer screenshot during a reflection session.

Participants' Written Work, Researcher Notes and Research Journal

Upon completion of each session, I collected and digitalized all participants' work and researcher notes. I also kept research journal to write summaries of the session, including interesting observation. I also documented (a) ongoing hypotheses of the participants' thinking (i.e., SOMs of the students, SOMs of the teacher, and TOMs) by reviewing the session episode,

(b) hypotheses to be tested in future sessions, and (c) task design ideas or follow-up questions to task in future sessions.

Task Design

The sessions with Ms. K and Taylor were focused on linear programming. The sessions with Ms. N were focused on geometric dilation and similarity. In the following, I describe each topic of focus, explain the rationality of choosing the topic, and the tasks used.

Ms. K and Taylor: Linear Programming

A linear programming problem is a common type of optimization problem that provides students with opportunities to apply mathematical knowledge to model authentic real-world situations. Secondary curricula often introduce linear programming in two-dimensions that involves finding the maximum or minimum value of a two-variable linear function under a domain restricted by a system of two-variable inequalities. Undergraduate mathematics curricula expand this type of problems to n -dimensions, which requires more complicated algorithms (e.g., numerical optimization methods) to locate the optimal solutions efficiently.

The present study is situated in the context of secondary mathematics. In this context, understanding and solving linear programming problems requires students to draw on a web of mathematical ideas, including constructing and graphing equalities and inequalities, reasoning with multi-variations of quantities, representing quantitative relationships (both inequality and equality relationships) algebraically and graphically, and optimizing objective functions involving three variables (one variable as output and two as inputs). An objective function in the context of two-dimensional linear programming is a two-variable function in the form of $z = f(x, y)$, where $f(x, y) = ax + by$. Given extant literature on students' difficulty with coordinating two co-varying quantities in various contexts (Carlson et al., 2002; Castillo-Garsow

et al., 2013; H. L. Johnson, 2015; Moore et al., 2019), I conjecture it is also challenging for students to simultaneously coordinate three quantities and anticipate the variation of value z in relation to the covariation of x and y , given the coefficients a and b . It is also cognitively demanding to optimize an objective function because it requires students to go beyond describing quantities' multi-variation qualitatively to locate the optimal output of the objective function under certain constraints of x and y .

Extant secondary curricula often simplify the treatment of this topic by introducing the *vertices method*, that is, an optimal solution of an objective function can be obtained by evaluating the objective function at all corner points of a feasible region and comparing their outputs (e.g., Larson et al., 2001; Long & Ashley, 2008; Ryan et al., 1993; Senk et al., 1993). An underlying assumption of this approach is the *corner point principle*, which states that if there is a unique optimal solution to a linear programming problem, it must lie at a vertex of the feasible region. Most secondary curricula omit the proof of this principle and, instead, formulate steps for a solution using the vertices method (see Figure 25 for an example). One deficiency of this treatment is that it does not afford students understandings of why the corner point principle or the procedures work. For example, a student may be left wondering: Why is it sufficient to only attend to the corner points in the feasible region but not the others?

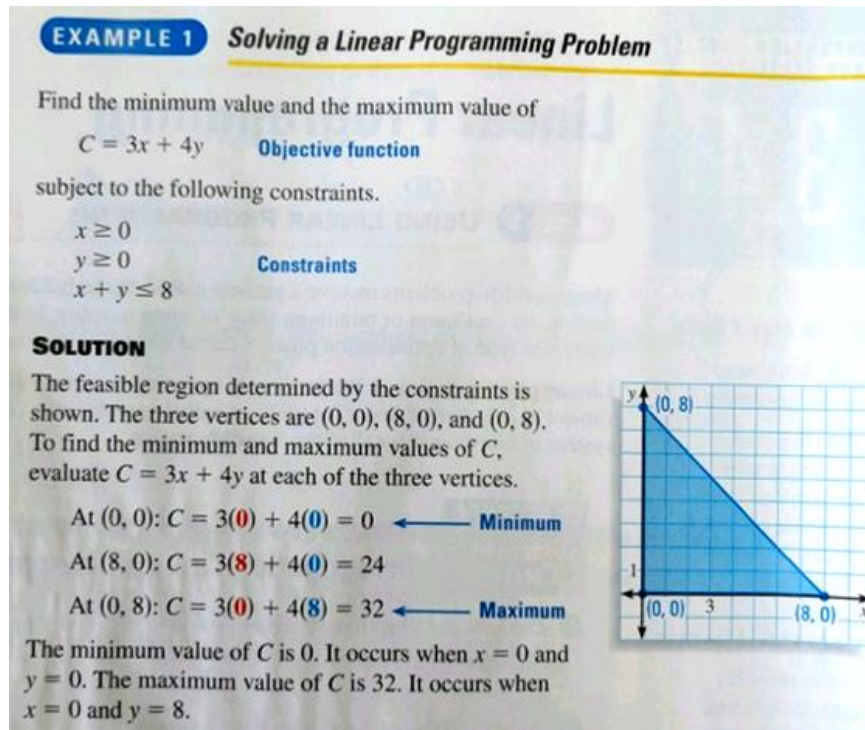


Figure 25. Curriculum examples of the vertices method (Larson et al., 2001, p. 164).

Another approach introduced in textbooks and literature is parameterizing the value z and graphing a family of parallel lines (or level curves) of $y = -\frac{a}{b}x + \frac{z}{b}$, where $\frac{z}{b}$ is the y -intercept of the line (see Figure 26) (e.g., Edwards et al., 2015; Fey et al., 2009; K. Moon, 2019; Ryan et al., 1993; Senk et al., 1993; Shama & Dreyfus, 1994). Each line represents a collection of x - y pairs that satisfy the relationship for a fixed z value. Hence, the optimization problem is transformed into finding the x - y pair(s) that allow(s) the level curve to have a maximum or minimum y -intercept (and thus, a maximum or minimum z). A third approach is visualizing the feasible region and the objective function in a three-dimensional coordinate system and describing the optimal solution as existing on the x - y coordinate pair that generates the maximum (or minimum) vertical projection within the feasible region (see Figure 27) (e.g., Brzezinski, 2017).

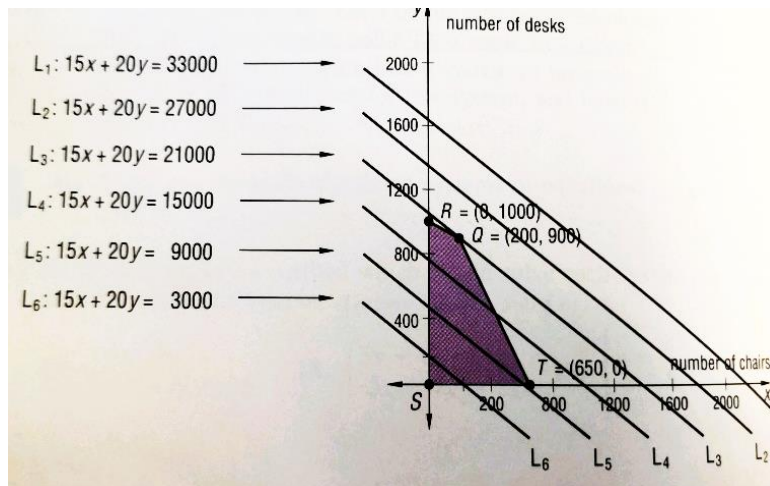


Figure 26. Curriculum examples of the level curve method (Senk et al., 1993, p. 288).

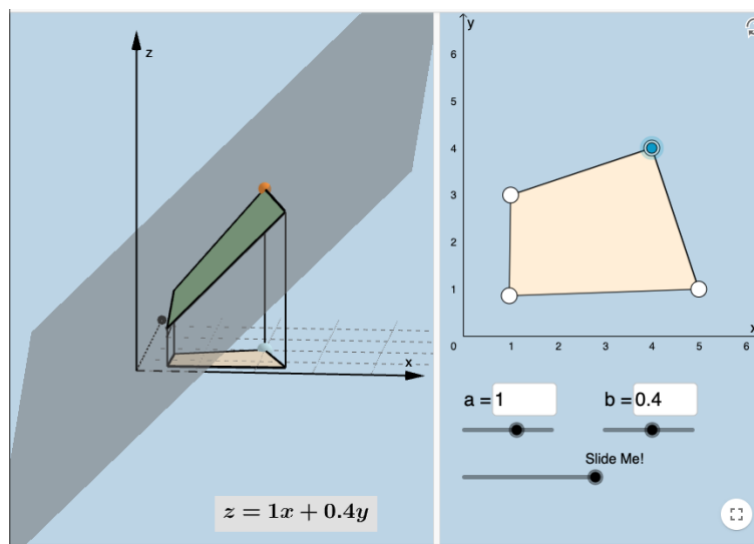


Figure 27. Curriculum examples of the 3D graph method (Brzezinski, 2017).

In general, linear programming is a relatively challenging problem type, but it can be approached through a variety of means. I conjectured that this type of problems could potentially trigger rich and creative thinking as Taylor attempted to make sense of the complex problem situation, coordinate multiple constraints of the situation, and optimize the output of the objective function. I noticed from Ms. K's instructional materials that she adopted the vertices method in her classroom teaching. I conjectured that working with Taylor, who has never received instruction on this topic, likely allowed Ms. K to gain insights into Taylor's

spontaneous thinking and reflect on the assumptions that she had taken as given in her teaching. Due to the novelty of Taylor's thinking from Ms. K's perspective, Ms. K was more likely to decenter from her personal knowledge and construct SOMs of Taylor. Each of these aspects made linear programming an appropriate topic for the current study.

What's more, despite the merits, richness, and complexity of this topic, linear programming has received a general lack of attention from the mathematics education community. I anticipated that choosing this topic would motivate future research agenda about studying students' and teachers' mathematical thinking of linear programming. More importantly, choosing a novel topic allowed me, as a researcher, to remain inquisitive throughout the study. I intentionally put constraints on myself in order to genuinely engage in making sense of Taylor and Ms. K's thinking. This disposition allowed me to act as a true observer when observing Ms. K's interactions with Taylor, without imposing my SOMs onto Ms. K's thinking. It also allowed me to give Ms. K authority to discuss Taylor's thinking during the reflecting session. Table 9 lists the ideas and tasks addressed in each pre-teaching sessions (see the tasks in Appendix B). Regarding the tasks used in the teaching cycle, since the task design decisions were sensitive to Ms. K's SOMs of Taylor, I leave the discussion on those tasks in Chapter 6.

Table 9. Summary of the pre-teaching sessions with Ms. K

Session	Ideas Addressed
Session 1	- prerequisite understandings for learning linear programming - goal understandings of linear programming
Session 2	- tasks used in class and homework - mathematical solutions to Ms. K's class activity: A Trip to Sugar Bowl - o graphing inequalities as graphing lines and shading regions

- Session 3
 - mathematical solutions to Ms. K's class activity (continued)
 - o graphing inequalities as graphing lines and shading regions
 - o understandings of objective functions from a covariational perspective
 - o justification of the vertices method and the Corner Point Principle
 - Session 4
 - two meanings for graphing inequalities in the context of Fancy's Café
 - graphing coordinate pairs with parameters (e.g., $(x, y) = (4 + n, 3 + k)$ with n and k restricted to certain intervals) in Cartesian coordinate systems and polar coordinate systems and its connections to graphing inequalities
 - Session 5
 - graphing coordinate pairs with parameters (e.g., $(x, y) = (4 + n, 3 + k)$ with n and k restricted to certain intervals) in Cartesian coordinate systems and polar coordinate systems and its connections to graphing inequalities (cont.)
 - comparison of meanings for graphing points, equations, and inequalities
 - a third meaning for graphing inequalities in the context of Fancy's Café
 - Session 6
 - a third meaning for graphing inequalities (cont.)
 - o justifications for the vertices methods in the context of Advertising Election:
 - Why is it sufficient to only check the corner points?
 - Why should the optimal solutions occur at the corner points?
-

Ms. N, Carlee, and Heidi: Geometric Dilation and Similarity

When I worked with Ms. N, I chose the topics of geometric dilation and similarity for several reasons. First, Ms. N was in her second semester of teaching at the school at the time of the study, and she only had experience of teaching geometry. Transformation is one of the key mathematical concepts in geometry, so I chose these topics from a list of topics covered in her geometry courses. Second, similar to linear programming, dilation and similarity are concepts closely related to real-life applications (e.g., shadows, graphic design, architecture, photocopy). Different than linear programming, a teacher-researcher could also have access to students' understandings of these concepts independent of real-world situations. By minimizing the influence of contextual differences on the students' conception of these concepts, a teacher-researcher could gain insights into nuances of students' mental actions and operations regarding

the geometric relationships involved. Third, these topical areas offered a context to engage students in physical activities involving measurement tools (e.g., rulers). This enabled the researcher and the teacher to model whether the students' mathematical operations were dominated by perceptual features of shapes and features of physical actions or the quantitative structures of the situations (e.g., relationships between distances in the context of dilation). Given that dilation and similarity are multifaced concepts, I anticipated that these topics would afford Ms. N rich opportunities to gain access to, tease apart, and discern multiple ways of student thinking. Table 10 summarizes a list of ideas and tasks addressed in each pre-teaching session (see the tasks in the Appendix C). I leave the discussion on the tasks used in the teaching cycle in Chapter 7.

Table 10. Summary of the pre-teaching sessions with Ms. N

Session	Ideas Addressed
Session 1	- Instructional materials and lesson goals of geometric dilation and trigonometric ratios
Session 2	- Instructional materials and lesson goals of similar triangles - Sorting Rectangle Task (Seago et al., 2013) ○ A static perspective on similarity: similarity is conceptualized in discrete terms as a numeric relationship between 2 figures. ○ Between-figure ratios, within figure ratios ○ Relating between-figure ratios to amounts of change and linear relationships
Session 3	- Similar Rectangle Problem (Seago et al., 2013) ○ A <i>transformation-based perspective</i> on similarity: similarity is conceptualized as enlarging or reducing figures proportionally to create a class of similar figures ○ ratios, proportionality, scale factor ○ line of dilation - Growing Triangles Task: similarity in dynamic and continuous contexts - Similarity Slope Graph (National Council of Supervisors of Mathematics [NCSM], n.d.) ○ Line of dilation

- Relating between-figure ratios to amounts of change and linear relationships

Session 4 - Floodlight Shadow Task (Mathematics Assessment Project [MAP], 2015a) modeling geometry problem

Session 5 - Rolling Cups Problem (MAP, 2015b): a modeling geometry problem

Data Analysis

The goal of the experiment was to formulate and test hypotheses of each teacher's SOMs and FOMs including the development of these models over the course of the experiment. The data collection methods facilitated my constructions of three types of models (Figure 12): (a) SOMs of the student's mathematical meanings, (b) SOMs of the teacher's mathematical meanings, and (c) TOMs of the student's mathematical meanings. I constructed these models through ongoing and retrospective conceptual analysis (Steffe & Thompson, 2000b; P. W. Thompson, 2008).

Ongoing conceptual analysis

The *ongoing conceptual analysis* (OCA) involved me generating and refining tentative hypotheses of the three types of models *during* each session and *between* consecutive sessions. Regarding the pre-teaching sessions, I formulated hypotheses of each teacher's FOMs during my interactions with the teacher. Right after each session, I watched the teaching episode to confirm, revise, or reject those hypotheses in order to explain the teacher's observable actions. Such analysis also informed the task design for the upcoming pre-teaching session.

During each teaching session, I focused on constructing SOMs of the students' mathematics based on the students' observable actions. At times, I also constructed TOMs of the students' mathematics based on the teacher's responses to the students' actions in-the-moment. After each teaching session, I watched the teaching episode to confirm, revise, or reject these

models and selected video clips for use in the reflection session (selection criteria were discussed previously). During each reflecting session, I continued to test and refine my TOMs in the moment of my interactions with the teacher. When I inferred from the conversations that the teacher's SOMs were different than my SOMs of the student, I asked purposeful questions to confirm this discrepancy. I was also open to revise my SOMs of the student based on my interactions with the teacher. Additionally, I focused on constructing SOMs of the teacher's mathematics to characterize the accommodation occurred as the teacher reflected upon her SOMs and FOMs during our interactions.

Retrospective conceptual analysis

The *retrospective conceptual analysis* (RCA) occurred after each phase of data collection. For RCA regarding the Phase 1 (pre-teaching sessions) data, I focused on generating SOMs of the teacher's mathematics. I recursively watched the videos and created detailed analytical notes, summaries of the teacher's actions, and interpretive memos. This iterative retrospective analysis efforts involved constructing hypothetical mental actions that viably explained the teacher's observable and audible behaviors. I continually searched the data for instances that the models could not account for, and I modified the models or attempted to explain developmental shifts in the teacher's meanings. These analysis approaches applied to both of the data involving Ms. K and Ms. N.

RCA for the Phase 2 data of Ms. K was focused on the reflecting sessions and included four rounds of analysis. During my first pass of the data, I focused on writing memos and summaries to document my inferences of the teacher's mathematical meanings, including her SOMs of the students' meanings. I was also attentive to the modes of reasoning identified in prior literature (e.g., inferring, comparing, explaining, and restructuring) and noted them in the

memos. I continually generated and reorganized codes to capture characteristics of my inferences. My second pass of the data focused on iteratively testing and refining my inferences and reassigning codes to each coded segment. I particularly focused on assigning codes to the segments that occurred before the codes were generated. This process also allowed me to reorganize the codes by identifying their relationships and distinctions. During my third pass of the data, I reviewed the instances of each code to operationalize and stabilize the coding system.

Table 11 summarizes descriptions of the top-level codes that were generated from the three rounds of analysis, and Figure 28 is a screenshot of the coding system from the analysis software. These codes were placed into three general categories: (a) researcher's TOMs, (b) researcher's SOMs of Ms. K, and (c) researcher's SOMs of Taylor. I note that there are several sets of codes omitted in Table 11 due to my focus on Ms. K's *mathematical* constructions. For instance, I excluded codes about Ms. K's teacher moves, her rationale and goals for asking a specific question, her beliefs, her general discussion on other students in her classroom, and her reflection on her classroom teaching.

Table 11. Coding system

	Codes	Descriptions
Researcher's TOMs (of the student)	Student Thinking	When a teacher infers a student's mathematical meanings
	Comparison (within/across students)	When a teacher compares: - a same student's meanings evident in previous sessions and those in the current session - two students' meanings The result of the comparison can be concluding consistencies or inconsistencies of those meanings.
	Comparison (student vs. teacher)	When a teacher (either explicitly or implicitly) compares: - their own mathematical meanings and a student's meanings. An implicit version of is that the teacher questions about the validity of a student's argument with the teacher's own knowledge - their intended questions and the student's interpretations of those questions - their expectation of a student's responses and their

interpretations of the student's actual responses The result of the comparison can be concluding consistencies or discrepancies.		
Researcher's SOMs (of the teacher)	Explanation	When a teacher provides hypothetical explanations for a student's actions; these explanations can be: - mathematical: - o when the teacher identifies contraindications: e.g., if the student had meanings XYZ, they would not say ABC; if the student said ABC, they were not thinking XYZ - o when the teacher draws on prior inferences to explain their current inference of the student's meanings - o locate source of difficulties: the student thinks XYZ because they understand the other concept as JKL - non-mathematical: e.g., the student's activity being result-oriented or computation-based, the student's school experience, the features of the problem contexts There are also cases where the teacher suggests she cannot explain the student's actions.
	Implication	When a teacher considers the mathematical implications of a student's meanings, i.e., if a student thinks in this way, what would happen in other situations? Such a discussion can be manifested in the following ways: - conjecturing the student's responses to other problem contexts or other probing questions - envisioning the student's meaning as a foundation for constructing other mathematical meanings - discussing the generalizability of the student's meanings: Do they work in all situations? What kinds of situations may fail them?
	Refinement	When the teacher's SOMs are modified due to continued interactions with the student and additional evidence of student thinking; such refinement can be in terms of the student's mathematical meanings, the implications of their meanings, or the explanations of their meanings.
Researcher's SOMs (of the students)	Existing Math	When the teacher discusses their mathematical meanings that were evident in the pre-teaching sessions
	New Math	When the teacher reconstructs their mathematical knowledge, and these meanings are not evident in the pre-teaching sessions Such discussion can be manifested in one of the following ways or a combination of them: - constructing new meanings by abstracting novel student actions - building connections between the student's meanings and personal meanings - reconstructing personal meanings in ways compatible with certain aspects of the student's mathematics
	Discrepancy	When the teacher's SOMs are different than the researcher's SOMs of the students
	Refinement	When the researcher refines her SOMs of students as a result of

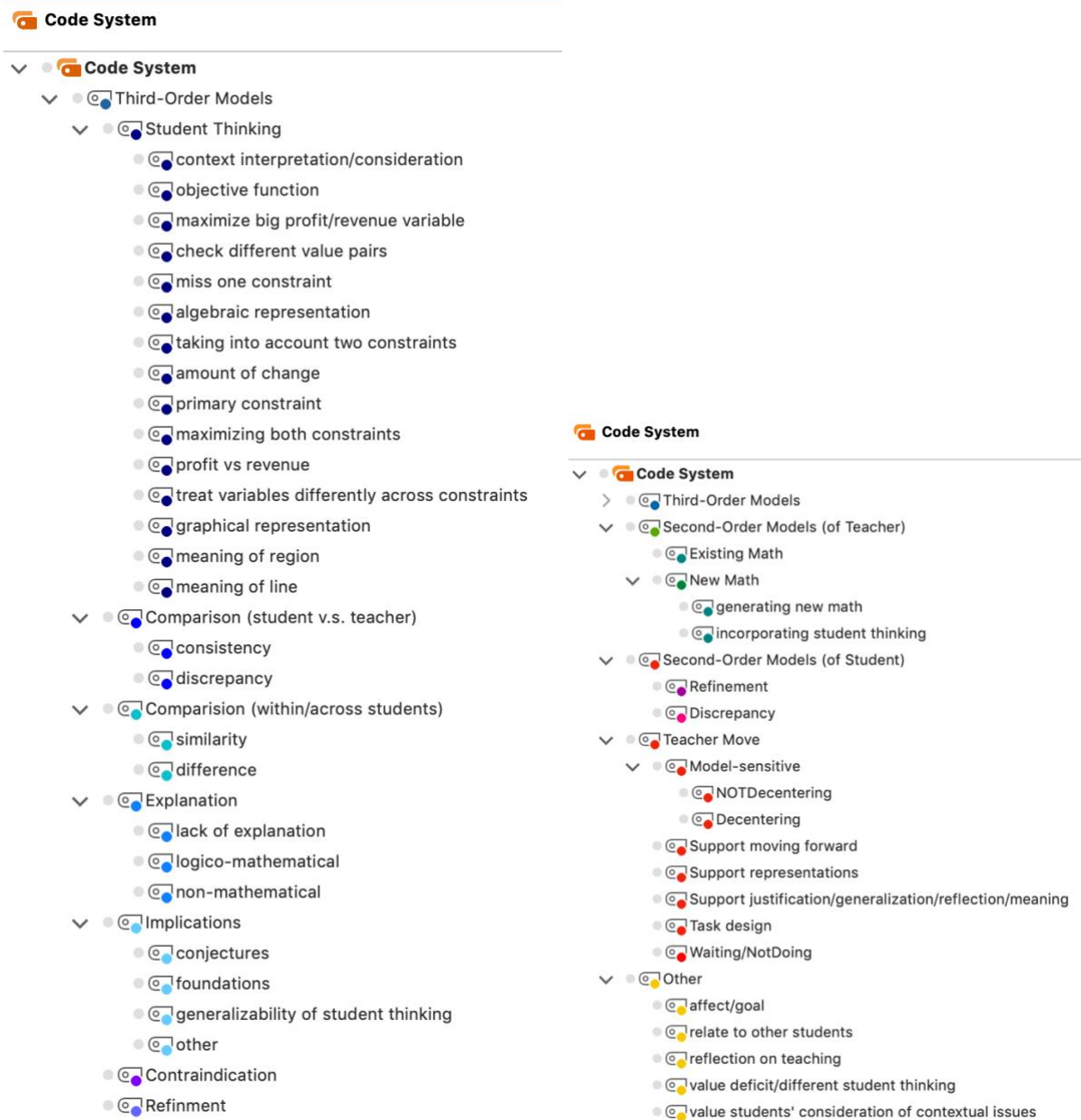


Figure 28. Screenshot of coding system.

These three passes of data analysis allowed me to select and organize a collection of critical instances for the fourth round of analysis. In this final round of analysis, my iterative analysis efforts included operationalizing the four types of decentering described in Table 1. To

infer Ms. K's Type 1, Type 2, and Type 3 Decentering, I compared my TOMs of the students' mathematics with my SOMs of the teacher's mathematics, both inferred from my interactions with Ms. K. I note that my SOMs of Ms. K's mathematics not only included the meanings I inferred from her Phase-1 activity but also those related to her interpretation of Taylor's *former* activity. Consistent with my aforementioned conceptualization, Ms. K's previously constructed SOMs will become part of her FOMs, which she operated on to enact future assimilation and accommodation. If the ways of operating constituting my TOMs of Taylor's mathematics and my SOMs of Ms. K's mathematics were consistent, I expected Type 2 Decentering to occur, hypothesizing that Ms. K would be able to assimilate Taylor's activity with her existing meanings; if they were inconsistent, I expected Type 1 or Type 3 Decentering to occur, hypothesizing that Ms. K would be at least perturbed by Taylor's activity. As I continued my analysis, I searched for Ms. K's observable actions that could confirm or reject these hypotheses. If Ms. K could articulate her inferences of Taylor's ways of thinking, I hypothesized Type 3 Decentering had occurred.

I inferred Type 4 Decentering had occurred when I observed Ms. K articulating her comparisons of multiple ways of thinking. One evidence could be her abstracting common or distinct characteristics of different ways of thinking, including her personal ways of thinking, Taylor's current ways of thinking, and Taylor's previous ways of thinking. In occasions where she identified two ways of thinking to be related, further accommodation might occur in the form of injecting coherence into a system of meanings. In occasions where she identified two ways of thinking to be different, further accommodation might occur as she brought certain aspects of her thinking into conscious awareness; these could be aspects that she took as given or were backgrounded prior to her interactions with Taylor.

These analyses efforts generated seven key mental processes involved in Ms. K's construction and enactment of SOMs: re-presenting (Type 0), contraindicating (Type 1 Decentering), explaining (Type 2 Decentering), inferring (Type 3 Decentering), modifying (Type 3 Decentering), comparing (Type 4 Decentering), and restructuring (Type 4 Decentering)⁵. In terms of the RCA of the data of Ms. N, I focused on stabilizing these constructs. Instead of going through the four rounds of analysis, I only repeated the procedures of the fourth round of analysis and coded the data in terms of the seven mental processes. This analysis helped clarify features and aspects of each mental process and decentering type and confirm their distinctions.

Summary

This chapter describes the research methodology of this investigation into two high school mathematics teachers' construction of SOMs through social interactions with students and reflection on student thinking. A methodology called teaching experiment² was proposed as an expansion of the constructivist teaching experiment methodology. This methodology includes a series of pre-teaching sessions for the researcher to model each teachers' mathematics prior to their interactions with students and cycles of planning, teaching, and reflecting sessions that support the teacher's modeling of students' mathematics. The implementation of this methodology, including ongoing and retrospective conceptual analysis of the data enabled the researcher to characterize each teacher's development in SOMs and FOMs and infer the mental processes and decentering types that could explain such development.

The next two chapters present the findings emerged from data analysis of the reflecting sessions with Ms. K and Ms. N. The findings are structured chronologically by teaching cycle. The tasks and task design rationale regarding each teaching cycle are also described.

⁵ These findings are discussed in more detail in Chapter 8 since they constitute the major findings of the study.

CHAPTER 6

THE STORY OF MS. K AND TAYLOR

I already knew, like I knew he was a smart kid, but oh my gosh, he's such a smart kid.

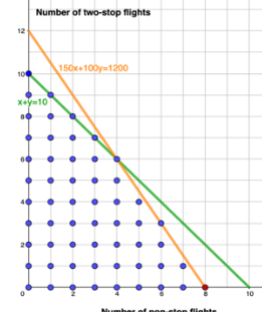
By Ms. K

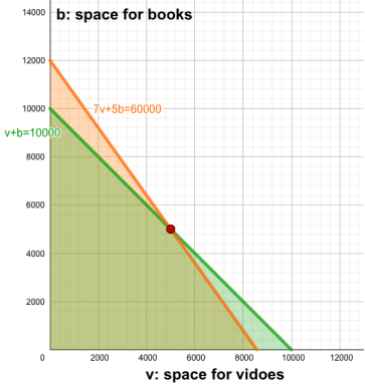
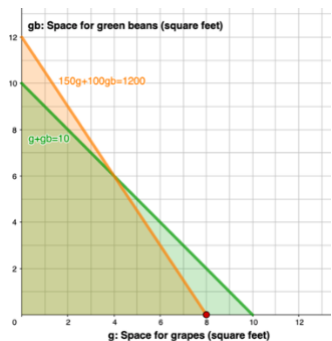
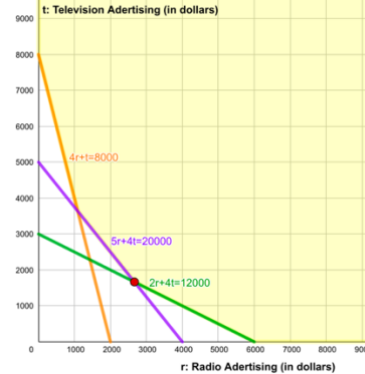
In this chapter, I provide a description of Taylor's activities through the eyes of Ms. K. I first describe the tasks used in the six teaching cycles and then discuss the selected excerpts where Ms. K discussed her interpretations of Taylor's activities and reflection on her personal mathematics. It is my interpretations of Ms. K's discussion that constitute my TOMs of Taylor's mathematics and SOMs of Ms. K's mathematics.

Tasks

The six teaching cycles involve four linear programming tasks: (1) A Trip to Sugar Bowl (ATSB), (2) Arranging Store Space (ASS), (3) Gardening Problem (GP), and (4) Advertising Election (AE). Table 12 includes the problem statement of each task and a solution with the conventional graphical approach. Since the task design decisions were sensitive to Ms. K's SOMs of Taylor within each cycle, I interweave the discussion on the task features with my description of Taylor's activity.

Table 12. Four linear programming tasks

A Trip to Sugar Bowl (ATSB). A tourism agency can sell up to 1200 travel packages for the Sugar Bowl college football postseason game in New Orleans. The package includes airfare, weekend accommodations, and the choice of two types of flights: a nonstop flight or a two-stop flight. The nonstop flight can carry up to 150 passengers, and the two-stop flight can carry up to 100 passengers. The agency can locate no more than 10 planes for the travel packages. Each package with a nonstop flight sells for \$1200, and each package with a two-stop flight sells for \$900. Assume that each plane will carry the maximum number of passengers. <i>What is the maximum revenue the agency can achieve, and how many nonstop and two-stop flights it needs to sell?</i>	x = number of non-stop flights y = number of two-stop flights Constraints: $150x + 100y \leq 1200$ $x + y \leq 10$ Objective function: $P(x, y) = 150 \times 1200x + 100 \times 900y$ Optimal solution: $P(8, 0) = 1440000$ 

Arranging Store Space (ASS). Suppose that a new store plans to lease selling space in a mall. The leased space would be divided into two sections, one for books and the other for music and videos. - The store owners can lease up to 10,000 square feet of floor space. - Furnishings for the two kinds of selling space cost \$5 per square foot for books and \$7 per square foot for music and videos. The store has a budget of at most \$60,000 to spend for furnishings. - Each square foot of book-selling space will generate an average of \$75 per month in sales. Each square foot of music and video-selling space will generate an average of \$95 per month in sales. <i>How would you help the store owners to decide how to allocate space to the two kinds of items, books or music/video to maximize monthly sales? What is the maximum sales?</i>	v= space for videos b= space for books Constraints: $5v + 7b \leq 60000$ $v + b \leq 10000$ Objective function: Revenue: $R(v, b) = 75b + 95v$ Profit: $P(v, b) = (75 - 5)b + (95 - 7)v$ Optimal solution: $R(5000, 5000) = 850000$ $P(5000, 5000) = 790000$  The graph shows the feasible region for the ASS problem. The x-axis is labeled 'v: space for videos' and the y-axis is labeled 'b: space for books'. Both axes range from 0 to 12,000. Two lines are plotted: a green line representing the budget constraint $5v + 7b = 60,000$ and a red line representing the space constraint $v + b = 10,000$. The feasible region is the area below both lines in the first quadrant. The optimal solution is marked with a red dot at the intersection of the two lines, which is at $(5000, 5000)$.
Gardening Problem (GP). Suppose there is a community garden that will be growing and selling two different types of plants, grapes and green beans. - The garden is 10 square feet. Each plant takes up one square foot. - Each grape vine produces 150 grapes. Each green bean plant produces 100 green beans. There is only space to hold 1200 pieces of produce. - Each grape will sell for 12 cents. Each green bean will sell for 9 cents. <i>How would you help the community to decide how to allocate space to the two kinds of plants, grapes or green beans to maximize sales? What is the maximum sales?</i>	g= space for grapes gb= space for green beans Constraints: $150g + 100gb \leq 1200$ $g + gb \leq 10$ Objective function: $P(g, gb) = 150 \times 12g + 100 \times 9gb$ Optimal solution: $P(8, 0) = 14400$  The graph shows the feasible region for the GP problem. The x-axis is labeled 'g: Space for grapes (square feet)' and the y-axis is labeled 'gb: Space for green beans (square feet)'. Both axes range from 0 to 12. Two lines are plotted: a green line representing the space constraint $g + gb = 10$ and an orange line representing the produce constraint $150g + 100gb = 1200$. The feasible region is the area below both lines in the first quadrant. The optimal solution is marked with a red dot at the intersection of the two lines, which is at $(8, 0)$.
Advertising Election (AE). When candidates for political office plan their campaigns, they have choices to make about spending for advertising. Suppose that advisers told one candidate that radio and television ads might reach different audiences in the following ways: - For every dollar spent on radio advertising, the candidate's message will reach 2 Republican voters. For every dollar spent on television advertising, the candidate's message will reach 4 Republican voters. The candidate has a goal to reach at least 12,000 Republican voters. - For every dollar spent on radio advertising, the candidate's message will reach 5 Democratic voters. For every dollar spent on television advertising, the candidate's message will reach 4 Democratic voters. The candidate has a goal to reach at least 20,000 Democratic voters. - For every dollar spent on radio advertising, the candidate's message will reach 4 Independent voters. For every dollar spent on television advertising, the candidate's message will reach 1 Independent voter. The candidate has a goal to reach at least 8,000 Independent voters. <i>How much money should be given to radio and television to minimize total advertising expense while still meeting all three campaign goals? What is the expense?</i>	r=dollars spent on radio t=dollars spent on TV Constraints: $5r + 4t \geq 20,000$; $2r + 4t \geq 12,000$ $4r + t \geq 8000$ Objective function: $P = r + t$ Optimal solution occurs at $(r, t) =$ $(2666.67, 1666.67)$ and $P = 2666.67 + 1666.67 = 4333.34$  The graph shows the feasible region for the AE problem. The x-axis is labeled 'r: Radio Advertising (in dollars)' and the y-axis is labeled 't: Television Advertising (in dollars)'. Both axes range from 0 to 9,000. Three lines are plotted: a green line representing the Republican voters goal $4r + t = 8000$, a purple line representing the Democratic voters goal $5r + 4t = 20000$, and a red line representing the Independent voters goal $2r + 4t = 12000$. The feasible region is the area above all three lines in the first quadrant. The optimal solution is marked with a red dot at the intersection of the green and purple lines, which is at $(2666.67, 1666.67)$.

Cycle 1: A Trip to Sugar Bowl

We started the first teaching cycle with A Trip to Sugar Bowl (hereinafter referred to as ATSB), which was a task Ms. K used in her classroom instruction.

Taylor's Initial Attempt

Taylor initially did not consider the constraint of 1200 total packages, and he reasoned that because nonstop flight was more expensive, he would like to allocate 10 nonstop flights and 0 two-stop flight, leading to a total revenue of \$1,800,000. Taylor also calculated other value pairs (e.g., (9,1), (4,6), (1,9), (0,10)) to confirm his solution (see Taylor's work in Figure 29 and Ms. K's interpretation of his activity in Excerpt 1).

$150 \times 1200 = 180,000$ $100 \times 900 = 90,000$ $180,000 \times 10 = 1,800,000$	$180,000 \times 4 = 720,000$ $90,000 \times 6 = 540,000$ $1,260,000$	$180,000 = 180,000$ $90,000 = 810,000$ $990,000$

Figure 29. Taylor's initial attempt on ATSB (the total revenue of each possible solution is circled).

Excerpt 1—Memo 26

- 1 Ms. K: He's like, if you want to make the most money, you would just use
- 2 nonstop flights because he's seeing the fact that this is a big
- 3 revenue, so this is the most revenue, get as much of that as you can.
- 4 Right, which I mean like down here [pointing to Figure 29, left] he
- 5 multiplies by 10 to show that.

Taylor's Second Attempt

As the conversation continued, Ms. K drew Taylor's attention to the constraint of 1200 total packages. Immediately, Taylor reasoned that he would allocate all the 1200 people to nonstop flights; he calculated 1200 divided by 150 and concluded that 8 nonstop flights and 0 two-stop flight should be the solution (Figure 30, left). Taylor also tested many other value pairs (e.g., (6,3), (7,1), (0, 10)) to confirm his solution (Figure 30, middle and right). Excerpt 2 illustrates Ms. K's discussion on Taylor's reasoning regarding the 6-3 combination (Figure 30, middle).

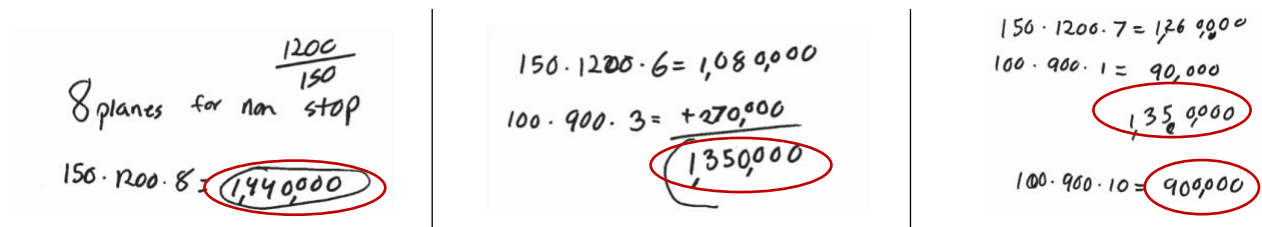


Figure 30. Taylor's second attempt on ATSB by considering the constraint of 1200 total packages (the total revenue of each possible solution is circled).

Excerpt 2—Memo 44

- 1 Ms. K: I'm going to do six of these [six nonstop flights], then that means I
- 2 have...900 people. So...1200 minus 900 divided by 100. Because a
- 3 hundred people can fit on each of these [two-stop] planes. So there are 300
- 4 people, I can take three planes full of people.

From Ms. K's perspective, Taylor's activities included him (a) finding the number of passengers carried by a certain number of nonstop flights, (b) determining the number of passengers needed to be fit into the two-stop flights when maximizing the total number of passengers to be 1200, (c) determining the number of two-stop flights by dividing the number of passengers by the number of people that each two-stop flight could take, and (d) calculating the total revenue corresponding to each flight combination and comparing the results. Ms. K commented that this was a nontrivial achievement, and she discussed how Taylor's reasoning with a single constraint was foundational for his success in coordinating two constraints (Excerpt 3).

Excerpt 3—Memo 42, 43

- 1 Ms. K: The way I can maximize revenue is still the most amount of nonstop flights
- 2 because that's still the biggest number. So what's the most amount of
- 3 nonstop flights I can have? Basically, how can I fill up all 1200 people with
- 4 nonstop flights? (...) I mean I'm speaking from like if I was having to work
- 5 through a problem that had four or five different constraints, like I feel like
- 6 at first it gets, it's a little daunting when you have to take into account like
- 7 this and this and this and this, but the fact that he could already have done
- 8 all of this [pointing to Figure 29] and then very easily just slip this one piece
- 9 in [pointing to Figure 30]. Because, and also right here [pointing to Figure
- 10 30], he's already figured, he already kind of has concrete in his mind the idea
- 11 that the thing that's going to maximize his revenue is having the most

12 amount of nonstop planes possible. Not doing some sort of combination, but
13 just maximize your nonstops (...) So going into this, he's like, okay, I know
14 how to maximize revenue. Maximize nonstop flights. Now how many
15 nonstop flights can I have at most?

Ms. K considered that Taylor's initial exploration with one constraint was consequential and beneficial for his coordination of two constraints. In Taylor's first attempt where he only took into account the 10-total-flight constraint, he reasoned that maximizing the number of nonstops to be 10 would maximize the total revenue. Then, when considering the additional constraint of 1200 people, he reasoned: "how many nonstop flights can I have at most" when having 1200 people in total (Line 14-15)? It is noteworthy that Ms. K did not conceive Taylor's reasoning with only one constraint as deficient; during their interactions, I did not observe her hastened to intervene either. Rather, she engaged with modeling Taylor's mathematics, and as suggested in this excerpt, her previous SOMs became explanatory for Taylor's current activity. Specifically, Ms. K identified a coherent aspect of Taylor's reasoning in these two instances (i.e., maximizing the number of nonstop flights as many as possible). Her ability to conceive of Taylor's reasoning with the 1200-total-package constraint as being coherent to his reasoning with the 10-total-flight constraint was an evidence of Type 2 Decentering having occurred.

Supporting Symbolic Representations

Observing that Taylor relied on calculations to solve the problem, Ms. K attempted to support him in generalizing his calculation sequence with variables, but she was unsuccessful. During the reflecting session, I asked Ms. K why she was persistent in supporting Taylor's generalization of his reasoning, she responded that his method was inefficient when dealing with situations involving many possible solutions (Excerpt 4).

Excerpt 4—Memo 50

1 Ms. K: Because I mean he said to me and I knew that's what his plan was – I can
2 just work out every single pair. It's like, okay, well is there something

- 3 where you don't have to do every single pair? Because what if we had given
 4 him a situation where like the obvious choice wasn't just to give the most
 5 to nonstop. Then, do you just work out every pair and see what happens?
 6 What happens when you give him a lot of pair options?

At this point, I hypothesized that Ms. K might think differently than Taylor, I asked her to discuss her own thinking. She said that the number of two-stop could be generalized as

$\frac{1200-150x}{100}$, where x represents the number of nonstop (Figure 31, left). Then, the total revenue of nonstop could be represented by $150 \cdot 1200 \cdot x$ (Figure 31, middle), and the total revenue of two-stop could be represented by $100 \cdot 900 \cdot \frac{1200-150x}{100}$ (Figure 31, right). However, when I prompted, she could not explain how this equation might be helpful to find the optimal solution.

The image shows three handwritten mathematical expressions in blue ink, separated by vertical lines. The leftmost expression is $\left\{ \frac{1200 - (x \cdot 150)}{100} \right\}$. The middle expression consists of two parts: $no: 150 \cdot 1200 \cdot x = y$ at the top, and $x = \text{number of flights in nonstop}$ and $y = \text{Total revenue in nonstop}$ below it. The rightmost expression shows two equations: $100 \cdot 900 \cdot \left(\frac{1200 - 150x}{100} \right) = z$ and $90000 \left(\frac{1200 - 150x}{100} \right) = z$.

Figure 31. Ms. K's generalization of Taylor's reasoning.

Justification of the Optimal Solution

Another theme of Ms. K's conversation with Taylor centered on why the optimal solution occurred at (8,0). Ms. K illustrated her justification for why the 6-3 combination did not produce a higher revenue than the 8-0 combination (Figure 32). She reasoned that reducing two nonstop flights would cause a loss of \$360,000 (each nonstop produces \$18,000 revenue) and that gaining three two-stop flights only compensated for \$270,000 (each two-stop produces \$90,000 revenue). Therefore, the 6-3 combination is less optimal than 8-0 even if it included nine total flights as opposed to eight.

$$\begin{array}{l}
 n = \text{non stop} \\
 t = \text{two-stop}
 \end{array}$$

$$\begin{array}{rcl}
 150 \cdot 1200 \cdot n & & \\
 \hline
 180,000 & 360,000 & \\
 150 \cdot 1200 \cdot 6 & &
 \end{array}$$

$$\begin{array}{rcl}
 100 \cdot 900 \cdot t & & \\
 \hline
 90,000 & - 270,000 & \\
 \hline
 & 90,000 &
 \end{array}$$

Figure 32. Ms. K's justification for why it is not worth losing 2 nonstop flights and gaining 3 two-stop flights.

In contrast to Ms. K's reasoning, Taylor's justification of why (8,0) was the optimal solution was constrained to the numerical results of his calculations. He referred back to the series of calculations he carried out in Figure 29 and Figure 30 to show that the solution of (8,0) was indeed the most optimal when compared to the other value pairs. Ms. K considered Taylor's activity as a contraindication that the idea of amounts of change was available to him at the time (Excerpt 5).

Excerpt 5—Memo 405, 339

- 1 Ms. K: I mean, part of that comes from like, maybe not super knowing the, not
- 2 formally understanding amounts of change. And so like we kind of talked
- 3 about and he said, something along the lines of less money, like a two-stop
- 4 to give you less money. And then I asked, well, what if I give you an extra
- 5 two-stop? You know? And so going back to what we just did of like
- 6 [pointing to Figure 32], is there ever a point where you feel like you're
- 7 making up that deficit of a loss of a nonstop by gaining two or more two
- 8 stops? Because you're taking more people but the amount of change is still
- 9 not worth that change and amount that I just said. Like this hasn't been like,
- 10 because he hasn't really explored this idea [pointing to Figure 32]. He can't
- 11 say without any kind of doubt like yes, this is it....He's not being able to
- 12 really like concrete, like he can't concretely say if I lose one nonstop flight,
- 13 I am sacrificing \$180,000. He knows that it's a more than I'm going to gain
- 14 from a two stop, but he doesn't know how much more. He doesn't know if
- 15 it's ever worth it.

Here, Ms. K was primarily operating on her own thinking and considered Taylor's observable actions as contraindications of him operating in ways consistent with hers. This aligns with Type 1 Decentering, meaning Ms. K experienced perturbation from Taylor's activity, and she seemed

to remain perturbed because she could not infer the mental actions or fundamental conceptual issues that could explain his activity. From my point of view, that Ms. K's perturbation persisted was understandable because Taylor himself was likely not aware of the conceptual meanings underlying his procedures either; to Taylor, (8,0) was the optimal solution only because the numerical results showed it.

Cycle 2: A Trip to Sugar Bowl (follow-up)

In the second teaching cycle, Ms. K continued to support Taylor reflecting on his calculation sequence and generalize his reasoning with variables with respect to ATSB.

Supporting Symbolic Representations

With much support, Taylor was able to relate the number of nonstop x and number of two-stop y (Figure 33). He initially wrote $\frac{150x-1200}{100}$, and when Ms. K asked him to explain what it represented, Taylor plugged in $x = 5$ and found that y was a negative number, leading to his doubt in his answer. Then, he adjusted his expression to be $\frac{1200-150x}{100}$, but he still struggled to explain the meaning of the expression in itself beyond assigning values to the unknown x and comparing the results. Excerpt 6 illustrates Ms. K's discussion on Taylor's understanding of the expression.

$$\frac{150x - 1200}{100}$$

$$\frac{1200 - 150x}{100} = y$$

$y = \text{planes } 4 \text{ 2-stop}$
 $x = \text{planes } 4 \text{ non-stop}$

Figure 33. Taylor's equation for relating number of nonstop and number of two-stop.

Excerpt 6—Memo 345

- 1 Ms. K: They're [the expressions] just, like an avenue for him. This is just a means
2 to an end. The way that you answered the questions is by getting an answer
3 and then explaining that answer. Not so much just, okay, I can use this to
4 explain my answer in it. Just this, in his mind he's like, I can't just tell you
5 about what this is without knowing a specific value that it could represent.
6 Because I can talk to you about a specific value. It's hard to talk about an
7 expression or how the different pieces fit together.

Ms. K inferred from Taylor's activity that that he was unable to operate on the algebraic expression in itself. Instead, his meaning of the expression was constrained to calculating the numerical values from the expression. Similar to Excerpt 5, this is an instance of Type 1 Decentering.

Justification of the Optimal Solution

Another goal of this teaching session was to support Taylor's justification for the optimal solution with amounts of change. Taylor argued that the optimal solution must occur at (8,0) because it was not worth allocating any two-stop flights, since each two-stop is "cheaper" than each nonstop. During the reflecting session, Ms. K questioned about the validity of this justification (Excerpt 7).

Excerpt 7—Memo 359

- 1 Ms. K: I haven't really used to this thinking like, well one is cheaper. That's it.
2 This one is worth less and because it's worth less, that's it, you are
3 multiplying by a smaller number. I think he's missing like if it's a one to
4 one, yes, it's not worth it, but what if it's a two to one or three to one? What
5 happens when it's not one to one and he won't go past a one-to-one
6 thought...It's like right. I lost one, but I gained three planes. Why is it still
7 not making up for that?

Ms. K inferred that Taylor considered the optimal solution should occur when maximizing the number of nonstop flights because nonstop flights were cheaper. However, she conceived a limitation in Taylor's thinking. That is, Taylor's argument was only valid for justifying why it was not worth decreasing *one* nonstop and gaining *one* two-stop, but not necessarily other

situations such as decreasing *two* nonstop and gaining *three* two-stop (i.e., comparing the 8-0 combination to the 6-3 combination). I considered this was an evidence of Type 4 Decentering because Ms. K was simultaneously operating on her SOMs and FOMs and draw on her own knowledge to evaluate the validity of Taylor's reasoning.

Ms. K's awareness of the discrepancies between her personal understandings and Taylor's understandings played a critical role in designing tasks that might engender perturbation in Taylor. That was when she decided to use the ASS task in Cycle 3, and she anticipated that Taylor's argument would fail in that problem situation.

Cycle 3: Arranging Store Space

Reflecting on Taylor's activity regarding ATSB, Ms. K argued that his reasoning might not be generalizable to other linear programming problems because maximizing the more expensive item did not always lead to the most optimal solution. When planning the following teaching session, we therefore designed the Arranging Store Space (ASS) problem (adapted from Fey et al., 2009), where the optimal solution occurred at (5000, 5000), anticipating that Taylor would be perturbed if he continued to maximize one of the variables.

Initial Construction of SOMs

Figure 34 illustrates Taylor's solution to ASS. Ms. K described Taylor's reasoning that he allocated all the 10,000-square-foot space for videos because video produced more sales. However, he realized that the corresponding furnishing expenses would be \$70,000 (Figure 34, left), which was over the \$60,000 budget, and therefore he desired to adjust his solution. Excerpt 8 illustrates Ms. K's summary of how Taylor proceeded to solve the problem.

$7 \times 10,000 = 70,000$	$5 \times 1,000 = 5,000 \text{ cost for B}$ $7 \times 9,000 = 63,000 \text{ cost for V}$ $5 \times 2,000 = 10,000$ $7 \times 8,000 = 56,000$	$5 \times 3,000 = 15,000$ $7 \times 7,000 = 49,000$ $5 \times 4,000 = 20,000$ $7 \times 6,000 = 42,000$ $5 \times 5,000 = 25,000$ $7 \times 5,000 = 35,000$	$15,000$ $+ 49,000$ $64,000$ $62,000$ $60,000$

Figure 34. Taylor's solution to ASS in the third teaching cycle.

Excerpt 8—Memo 104

- 1 Ms. K: So, what's really interesting about how he solves this problem is he picks
 2 one constraint, and he manipulates that one and then finds the place where
 3 this one [the budget constraint] works. So what he's doing is he's working
 4 with the 10,000 [space] constraint and he's going to say, okay, everything
 5 I do is going to fall within the 10,000 constraint until I find, adding up to,
 6 so like nine [thousand] plus one [thousand] and eight [thousand] plus two
 7 [thousand]. And I'm going to work until I find—until those give me an
 8 output that fall within this constraint [the budget constraint].

In this excerpt, Ms. K described Taylor's thinking that he first selected a series of value pairs that satisfy the 10000-square-foot constraint (Figure 34, middle, right), including (space for books, space for videos) = (1000, 9000), (2000, 8000), (3000, 7000), (4000, 6000), (5000, 5000). Then, Taylor calculated the corresponding furnishing expense for each combination and compared the result with the \$60,000 budget. Eventually, he concluded that (5000, 5000) was the best solution because it is "the only place where you're spending all of your money and using up all of your space."

Comparing ASS and ATSB

Engaging with Taylor's reasoning, Ms. K realized that Taylor was not perturbed in ways she anticipated in the planning session (Excerpt 9).

Excerpt 9—Memo 137

1 Ms. K: We want something that wasn't on the edge so that he couldn't just go, well,
2 this one's worth more and let me give it all, buy things and be done. Yeah.
3 Like he had to do a little more, but we also thought that this would push
4 him to graph or some other, some other way of thinking. But to me it seems
5 like he did the same thing. You know, like it didn't push him to a new
6 solution method. It just pushed him to adjust his solution. And he did, he
7 did it successfully, he did it efficiently.

It was noteworthy that Ms. K conceived Taylor's thinking in the current situation was inherently coherent to that in ATSB. Later, she added that, "they are not like the exact same process, but the thinking is still the same." As I promoted, Ms. K further unpacked what she conceived to be the "same" in Taylor's thinking (Excerpt 11).

Excerpt 10—Memo 110

1 Ms. K: He chooses one constraint to make his like primary constraint. I guess it's
2 a way, it's kind of how I think of it. So, like in this one [ATSB] it was his
3 people, the amount of people on the plane. That's the one he's
4 manipulating. And then based on that one he works until he figures out the
5 best use of the other constraint. I don't know if that makes sense. Like in
6 this one [ASS], his primary constraint was space. So, he only manipulated
7 these values.

In this excerpt, Ms. K pointed out one similarity that Taylor chose one constraint to start with in both contexts (e.g., amounts of people on the plane in ATSB, amounts of space in ASS). Ms. K called this constraint as the a "primary" constraint. In Excerpt 11, she continued to describe how Taylor "manipulated" the primary constraint and "figured out the best use of the other constraint" in these two contexts.

Excerpt 11—Memo 138

1 Ms. K: So he went into this thinking, okay, video is worth more. I'm going to make
2 the most money of the square feet for videos so I should give them all of
3 these square feet that I have, which if you like parallel that with this
4 thinking you went nonstop giving me the most money, let me give nonstops
5 all of the possible planes that I have. And here [the ATSB context] that
6 worked. I could, I could give them eight. I can put all of my people to the
7 nonstop flights and it will make me the most money. But here [the ASS
8 context] I can't give all of my square feet to video because it goes over
9 budget. And that's another one of the nuances where like this one [ASS], it

10 didn't work, not because it wasn't going to maximize value, but because it
11 didn't fall within that second constraint, which was different here [ATSB].
12 It did, so he didn't have to really like push that.

From Line 1-5, Ms. K described that Taylor always started with allocating all the resources to something that was worth more under the “primary” constraint. Regarding ATSB, he allocated all of the 1,200 people to nonstop flights; regarding ASS, he allocated all of the 10,000-square-foot spaces for videos. Then, from Line 5-12, although Ms. K seemed to point out the differences in Taylor’s activity in these two contexts, she was trying to claim that these differences were not indications of Taylor’s thinking being inconsistent but due to the mathematical nature of the problems—when maximizing the value of one quantity under one constraint, the other constraint was simultaneously satisfied in ATSB, but it was not in ASS. She later added that if Taylor obtained a number greater than 10 (instead of 8) in ATSB, he would have tested other value pairs until both constraints were satisfied as he did in ASS.

I argue that this instance was an evidence of Type 3 Decentering. Recall that in the planning session of Cycle 3 (and Cycle 1 and Cycle 2), Ms. K conjectured that Taylor’s reasoning in ATSB was limited to particular situations and anticipated him to be perturbed by ASS. After her engagement with Taylor’s activities in ASS, she modified her SOMs constructed in the ATSB context. She now realized that Taylor’s reasoning could be generalized to other situations such as ASS. Ms. K’s modification in her SOMs suggested she had enacted Type 3 Decentering. I also argue that this was an evidence of Type 4 Decentering. Although Taylor’s observable actions in ASS appeared to be different than those in ATSB, Ms. K was able to abstract the coherent understandings underlying Taylor’s activities. I inferred this was a result of her simultaneously re-presenting and operating on the mental actions constituting her SOMs constructed in the two problem contexts, which suggested her enactment of Type 4 Decentering.

Comparing the Graphical and Algebraic Approaches

As the conversation continued, Ms. K discussed Taylor's algebraic approach in comparison to the conventional graphical approach. She first produced the graph shown in Figure 35 (left) and described the graphical method as requiring graphing each inequality and identifying corner points of the overlapping region. Then, she produced Figure 35 (right) and discussed Taylor's method in the same graphical context as a contrast. Excerpt 12 illustrates Ms. K's discussion on these two graphs.

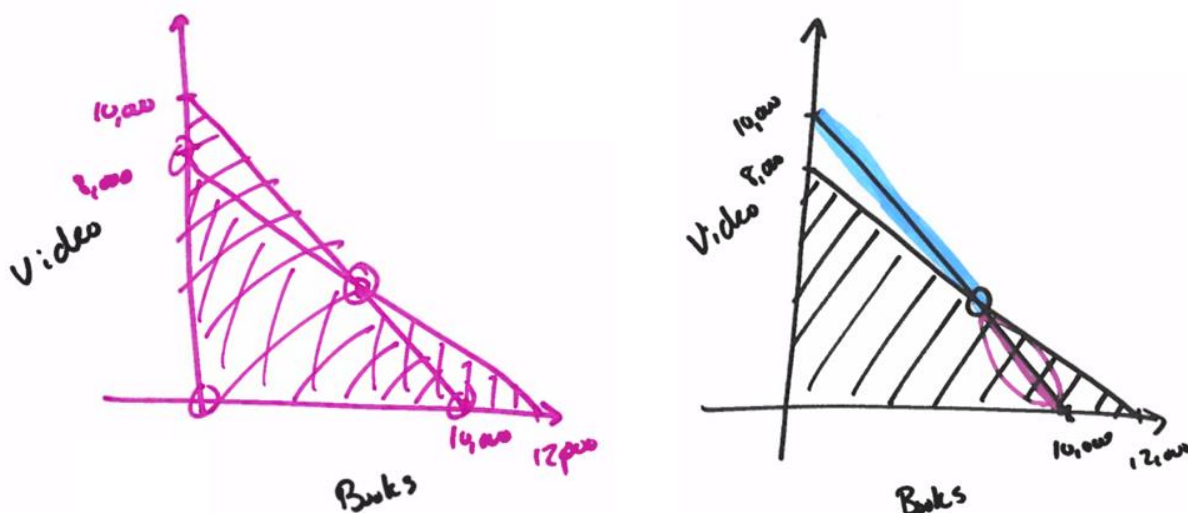


Figure 35. Ms. K's comparison of the vertices method and Taylor's method regarding ASS.

Excerpt 12—Memo 106, 107, 108

- 1 Ms. K: What he's doing is—Okay, I know I want to maximize space. So what
- 2 combinations of books and video will give me 10,000 and that's, just this
- 3 line [referring to the black segment connecting $(0, 10,000)$ and $(10,000, 0)$
- 4 in Figure 35, right]. So, he doesn't even consider the shaded part
- 5 underneath it (...) It's definitely efficient because you are not considering
- 6 all of these extra values for the 10,000. You are looking at like, honestly,
- 7 you are not considering both of them as inequalities.
- 8 (Transitioning to another reflecting session and recalling what she had
- 9 said)
- 10 He thought to himself, okay, I'm going to make the most money off video,
- 11 so let me test every possible combination of video and book that's going to
- 12 maximize space, and I'm just going to test those down [tapping her finger
- 13 along the blue segment in Figure 35, right] until I maximize budget without

14 going over budget. And so he's basically just working along this line right
 15 here [*tracing along the blue segment*]. This line right here until he gets to
 16 this point that you stopped [*pointing to the point circled in Figure 35, right*].
 17 So these values never get tested [*pointing to the segment circled*], neither
 18 of these [*pointing to the shaded region*]. But this is his like range [*tracing*
 19 *along the blue segment*]. So his like test range is just here [*shading the blue*
 20 *segment*].

Recall that in Cycle 1 and Cycle 2, Ms. K perceived more limitations in Taylor's thinking than its affordances. Here, by re-presenting Taylor's algebraic method on a graph, Ms. K realized that Taylor's method was much more efficient than the graphical method because he only considered a small subset of feasible points as opposed to all of them. Although Taylor's algebraic method and Ms. K's graphical method involved different procedures, Ms. K was able to abstract the common mathematical properties underlying them. At this point, her SOMs of Taylor no longer constituted separate meanings in her mind; rather, she injected coherence into her system of meanings by reconstructing her FOMs (i.e., her meanings for the inequality graph in particular). Consequently, Ms. K was able to assimilate Taylor's ways of operating with her own meanings for graphs and coordinate systems and assimilate the graph through the eyes of Taylor. Therefore, I argue that Ms. K had experienced accommodation through Type 4 Decentering.

A Disagreement

Although I agreed with Ms. K's interpretations, I was concerned about the generalizability of Taylor's thinking. Based on my SOMs of Taylor, I wondered: If he started with the 60,000-budget constraint instead of the 10,000-space constraint, would he conclude $(60,000/7, 0)$ to be the optimal solution because that was when he spent all budget to furnish the space for videos? Ms. K disagreed on my concern, believing that Taylor would be able to get a correct answer even if he started with a different constraint (Excerpt 13).

Excerpt 13—Memo 141

1 Ms. K:

2 I think that his method can't work if it is bizarre. And because it's so
3 centered in reasoning, because it's not like this is a method where you pick
4 a constraint and manipulate the other one, and something where they both
5 work. You're still—that's not the whole point of this. Does that make, you
6 know what I'm saying? It's not just I'm going to choose a constraint and
7 work with it until they maximize the other one, and we are done. (...) He is
8 taking into account—the nuances of each problem are still being
9 considered. (...) So I feel like because it's not just a cut and dry, like his
10 process is not just choosing a constraint, then testing values. His process is
11 doing that with understanding the nuances of the context and he take all of
that into consideration, then yeah, I do think this is a way that would work.

Ms. K anticipated that Taylor's reasoning was more than a procedure of choosing one constraint and maximizing the other constraint. She inferred from his activity that his reasoning was tied to the contexts, quantities' relationships, and value testing. Thus, what the procedure we abstracted from observing Taylor's activities was merely an artefact of Taylor's reasoning. Those differences were only perceived from an observer's perspective. She later added that, "he's not thinking about the fact that we're going to sit there, watch his process...he's going in thinking they need me to give them the right answer." She continued to say, "I feel like his thinking is...more sophisticated than that. Like just because his choice of constraint [i.e., which constraint he considered as a "primary" constraint] may not be backed by some sort of like mathematical reasoning, the choices that he makes beyond that are." Together, these instances suggested the affordance of engaging in decentering. That is, Ms. K positioned Taylor as a rational and capable thinker, believing that Taylor's reasoning was more comprehensive, flexible, and organic than a "cut and dry" procedure. And this belief likely came from her constructing SOMs of Taylor's activities in multiple problem contexts and reflecting upon the coherency and affordances of Taylor's thinking.

Cycle 4: Gardening Problem

Being aware of this disagreement, we decided to design a teaching session to test our hypotheses. In doing so, we designed the Gardening Problem (GP) to incorporate features of

both of the ASS and ATSB contexts, attempting to see how Taylor would assimilate the problem. Specifically, we used the gardening context to include the quantity of space and put the 10-square-feet space constraint as the first bullet point, hoping to see if he would choose this constraint as the primary constraint as he did in ASS. Meanwhile, we imbedded numerical values similar to ATSB in this new context, conjecturing if Taylor chose the 10-square-feet space constraint as the “primary” constraint, he would obtain (4,6) as the solution, which would be inconsistent with the normative solution (8,0).

Consistencies Between ASS (New Solution) and GP

Based on Taylor’s activity, Ms. K interpreted that he assimilated the problem as being similar to ATSB. He chose the 1200 total produce as the “primary” constraint and obtained the normative solution (8,0) (Figure 36).

$$\begin{array}{l}
 g: 150 \text{ total } g \text{ per vine, sell for 12 cents} \\
 gb: 100 \text{ total } gb \text{ per plant, sell for 9 cents} \\
 \\
 \frac{1200}{150} = 8 \text{ plants of } g \\
 150 \cdot 12 = \textcircled{1800} \quad 150 \cdot 8 \cdot 12 = \textcircled{14400}
 \end{array}$$

Figure 36. Taylor’s solution to GP.

When Ms. K presented to Taylor his work on ASS and asked him to compare, Taylor said he forgot what he did and would like to start over. Then, he decided to use all the \$60,000 budget to furnish space for videos (see the top equation in Figure 37).

$$\frac{60,000}{7} = 8,571 \text{ sq ft of v}$$

$$95(8,571) - 60,000 = 754,245$$

$$\frac{95(8,571)}{95(8,571)}$$

Figure 37. Taylor's solution to ASS in the fourth teaching cycle.

In the following excerpt, Ms. K described how Taylor's reasoning in GP and ASS was consistent. Specifically, she interpreted that the 1200-produce constraint in GP was parallel to the 60,000-budget constraint in ASS.

Excerpt 14—Memo 203

- 1 Ms. K: [In GP], to figure out how many plants you would need, you would take
- 2 how much produce you could hold [1200 produces] and divide how much
- 3 produce each plant gives you [150 grapes] to see how much square footage
- 4 you would have. [In the ASS context] You take how much money you have
- 5 to spend [\$60,000 budget] divided by how much each square foot cost [\$7
- 6 per square foot for video].

Inconsistencies Between ASS (Old Solution) and GP/ATSB

However, after Taylor calculated the profit⁶ (see the bottom equation in Figure 37), he realized that the numerical result was less than what he obtained in the third teaching cycle. Ms. K prompted him to explain the discrepancy, and he said it was because he did not maximize the space to be 10,000 square feet in the current solution (8571, 0). Ms. K was perturbed by his

⁶ The ASS problem asks for a "maximum sales." Taylor interpreted it as a maximum profit, although our intention was asking for a maximum revenue. In the rest of the discussion, we discussed his activity from his perspective instead of ours.

explanation because she observed marked inconsistencies in Taylor's actions. Excerpt 15 illustrates Ms. K's articulation of her perturbation.

Excerpt 15—Memo 210

- 1 Ms. K: And his thinking is like well I didn't maximize both things, like he didn't
- 2 choose like this thing [i.e., the constraint of 10,000 square feet]. But he
- 3 didn't have to do that for the other ones that he was convinced that they
- 4 were right. So, I don't know what makes it his head what makes it different.

By "the other ones," Ms. K was likely referring to ATSB and GP since Taylor had only worked on three problems at the time. Ms. K was perturbed by her observation that Taylor did not maximize both constraints in ATSB and GP, whose solution was (8,0), but he claimed he had to maximize both constraints in ASS in order to obtain the optimal solution. To respond, Ms. K asked Taylor to explain why he had to maximize both constraints in ASS but not the other two situations. Taylor responded that because ATSB and GP were about maximizing revenue, but ASS was about maximizing profit, saying "we need to spend money before we make money."

Excerpt 16 illustrates that Ms. K remained perturbed by Taylor's argument.

Excerpt 16—Memo 197, 198, 199

- 1 Ms. K: He got caught up on like, in his mind the difference between these three
- 2 scenarios. The reason that this one [ASS] works differently, is solely
- 3 because of the context. Because you have to sell space, like you have to buy
- 4 space and then turn around and sell it. (...) I don't know why that makes it
- 5 different. (...) Whereas this one [GP], you are only to gain money. Like
- 6 there's not, he does have a cost here. There's not something being more
- 7 cost effective, like there it is, but it's not as obvious. Whereas here like you
- 8 have to think about being cost-effective.
- 9 Biyao: So, do you think that's something that, you just couldn't make sense
- 10 of?
- 11 Ms. K: He was really confusing to me. Like it was really hard for me to work
- 12 through that.

Ms. K was perturbed by Taylor's explanation because Taylor's argument was only sensitive to the contextual difference of the problem situations. Taylor considered that the reason why ASS required maximizing both constraints while GP (and ATSB) did not was that ASS involved

spending money to furnish space and gaining money through selling videos and books while GP (and ATSB) only involved earning money. Ms. K continued to say that Taylor used the contextual difference as a “scapegoat” for not having a mathematical reason for using different solution methods. She added that:

Excerpt 17—Memo 212

- 1 Ms. K: He’s just like, that’s just in the back of his head, like, oh, cuz, that’s the
- 2 most apparent difference between the two, that’s why. Not because it’s a
- 3 different problem or anything like that. It’s literally just well, that’s the
- 4 difference in these scenarios.

As Ms. K reflected further and attempted to remove her perturbation, she provided an explanation of Taylor’s activity (Excerpt 18).

Excerpt 18—Memo 216

- 1 Ms. K: It’s not even that, in his mind, he’s thinking I need to maximize both things.
- 2 Originally, it’s just he tries something, realizes that doesn’t work, so he
- 3 keeps going. Like in the very first time he worked through this one [ASS],
- 4 it was he tried allocating space a certain way who was over budget, and so
- 5 he changed it. And he stopped when he was not over budget. Whereas in
- 6 this case or in the other two [ATSB and GP], it’s not that he was over, he
- 7 was under. And so, I think it’s also the fact that he understands that you can
- 8 be under one of the constraints, like that is a possible solution set
- 9 underneath it. But over is not. Like I have to make sure I’m working in a
- 10 solution set that is not over budget.

Ms. K explained that Taylor’s initial activity was not oriented by the goal of maximizing one or two variables. His ultimate goals were to maximize the value of the more worthwhile quantity while remaining under the two constraints: If two constraints were satisfied at the beginning, he stopped; if one of them was not satisfied, he needed to adjust his solutions until both are satisfied. By re-presenting her previous SOMs of Taylor’s activities, Ms. K realized that nothing was incoherent from Taylor’s perspective, because how many constraints he needed to maximize was not part of his thinking. Hence, what Ms. K perceived as Taylor’s “struggle” of providing a valid justification was more of a constraint she herself experienced. The difference in

maximizing one versus two constraints was merely a feature of Ms. K's SOMs of Taylor's thinking, but this difference was not necessarily aware by Taylor. This realization allowed Ms. K to shift from considering that Taylor's argument was not mathematically valid to considering that Taylor's "struggle" was understandable. This realization was also a result of Ms. K persistently decentering from her FOMs and reconstructing her SOMs to be explanatory for Taylor's activity (i.e., Type 3 Decentering).

Affordances and Limitations of Taylor's Thinking

Reflecting on Taylor's activity in these three cycles, Ms. K was more confirmed that his reasoning was organic and sensitive to specific contexts. The positive aspects of his method were that it was very sensemaking-based and efficient, and she appreciated Taylor's ability to engage in making sense of the situations rather than applying a procedure. Ms. K also started to see some limitations in Taylor's thinking based on his activity in the current cycle⁷. First, from an educator's perspective, some aspects of Taylor's thinking were inconsistent across situations, meaning that his thinking was hard to be generalized to all situations or might not be accessible to all students. Second, because Taylor thinking was sensitive to the features of the context, it was hard for someone else to anticipate his thinking. For example, one may wonder: If the difference between profit and revenue mattered to Taylor, what if we gave him a context that does not involve money?

Cycle 5: Advertising Election

To continue to test the generalizability of Taylor's thinking, we used the Advertising Election (AE) problem that involved three constraints. This task was adapted from the Core-Plus

⁷ Unfortunately, I did not have Ms. K's quotes of this conversation because I forgot to hit the Record button of the cameras. The two bullet points are my summaries of the conversations.

Curriculum (Fey et al., 2009), and Ms. K worked on the task in one of the pre-teaching sessions. Ms. K anticipated Taylor to be challenged because his prior ways of thinking required him to coordinate all constraints algebraically, which would be more difficult in the case of three constraints. Ms. K also considered AE to be a context motivating the graphical approach because Taylor's algebraic approach would not be as efficient as before when dealing with three constraints.

Initial Construction of SOMs: Two Different Assumptions

After reading through the problem statement, Taylor reasoned that because the goal was to minimize expenses, he would spend money on the more cost-effective advertising method with respect to *each* party (see Figure 38 for Taylor's work and Excerpt 19 for Ms. K's summary of his thinking).

$$\begin{array}{ll}
 \$-R = 2rv & 12,000rv - \text{goal} \\
 \$-Tv = 4rv & \sqrt[4]{12,000} = \$3,000 \\
 \\
 \$-R = 5dv & 20,000 - \text{goal} \quad \sqrt[5]{20,000} \\
 \$-Tv = 4dv & \$4,000 \\
 \\
 \$-r = 4iv & 8,000iv - \text{goal} \quad \sqrt[4]{8,000} \\
 \$-tv = 1iv & \$2,000 \\
 \\
 & R = 6,000 \\
 & Tv = 3,000 \\
 & \$9,000
 \end{array}$$

Figure 38. Taylor's solution to AE in Cycle 5.

Note. “\$-R” refers to money spent on radio, “\$-Tv” refers to money spent on TV, “rv” refers to republican vote, “dv” refers to democratic vote, and “iv” refers to independent vote.

Excerpt 19—Memo 375

1 Ms. K: The goal is maximizing votes while minimizing cost. He thought, all right,
2 if I need 12,000 [Republican] votes and TV gives me double the amount
3 that radio does and I'm going to use—I'm going to try to get all of these
4 votes from TV, and I'm going to get all of these [Democratic] votes from
5 radio, and all of these [Independent] votes from radio. So, he divided his
6 goal votes by whatever was going to give him more out of the two and then
7 said, okay, I'm just going to give that much money to TV or radio in that
8 sense. And then he figured out the total first and then he went back and
9 figured out how much to give each by saying, okay, these amounts
10 [*pointing to the "\$4,000" and "\$2,000" in Figure 38*] came from giving
11 all of my money to radio. So that's 6,000 [for radio], which leaves 3,000
12 [for TV].

Taylor's approach to optimizing the advertising plan involved him choosing the more efficient advertising method to achieve the goal votes for each party. For example, since each dollar spent on TV produces 2 more Republican votes than radio, he would spend \$3,000 on TV to achieve the 12,000 goal votes for Republican. By contrast, radio generates more votes than TV for Democratic and Independent, and thus he would spend \$4,000 and \$2,000 on radio to achieve the goal votes for these two parties respectively. In Excerpt 20, Ms. K discussed how Taylor's understanding was inconsistent with his understandings in the prior contexts.

Excerpt 20—Memo 377

1 Ms. K: I think he's not thinking about them in tandem. He's thinking about them
2 like one at a time. He's thinking, okay, with this group of—I think this is
3 the Republicans. If I'm trying to do 12,000 people, it's cheaper for me to
4 do all of the money that I would spend on the TV versus radio because I'm
5 reaching more people for the same amount with TV. And then he did that
6 same thinking process with the Democrats, and then with the Independent.
7 And what he's missing here that we've seen him do all the other times is
8 being able to like balance all of the different constraints in his head.
9 Whereas here, I feel like he is thinking about them very individually. And
10 he's not thinking about like, okay, well if I'm spending \$3,000 up here and
11 then \$4,000 right here and then \$2,000 right here, I'm not thinking about
12 how I can like mix these and try to spend less money.

Ms. K observed that in the prior contexts, Taylor was able to coordinate multiple constraints at the same time and determine value pairs satisfying those constraints. However, in AE, Taylor treated each constraint individually; he was imagining spending different amounts of money on

TV or radio with respect to different campaigns. This is an instance of Type 4 Decentering because Ms. K was able to re-present Taylor's thinking in different contexts and discern nuances in his thinking. Later, Ms. K also compared Taylor's understanding with her personal understanding regarding this problem (Excerpt 21).

Excerpt 21—Memo 378

- 1 Ms. K: The way that the problem is set up, if you're spending \$6,000 on a radio,
- 2 you're, you're reaching two and five and four at the same time [*pointing to*
- 3 "*2rv*," "*5dv*," and "*4iv*" in Figure 38]. Right? But he's not seeing that.
- 4 He's seeing it as like I'm going to give this, this TV, the 3000 [*pointing to*
- 5 "*\$3,000*"], and these two [*pointing to* "*\$4,000*" and "*\$2,000*"] radios, the
- 6 6,000, but he doesn't realize that he's still reaching these two [*pointing to*
- 7 "*\$=R=2rv*" "*\$=Tv=4rv*"] cause' he did give money to radio. It's still
- 8 getting put into that.

Ms. K conceived an inconsistency between her FOMs and SOMs of Taylor. She imagined every dollar spent on TV (or radio) as contributing to votes in each campaign at the same time (Line 2), while Taylor thought every dollar spent on TV (or radio) only gained votes for a specific campaign (Line 6-7). From Ms. K's perspective, the candidate would achieve more votes than what they need by spending \$6,000 on radio and \$3,000 on TV. However, Taylor's belief that (\$6,000, \$3,000) was the most optimal solution persisted in his activity, which Ms. K considered as a contraindication that they shared the same assumptions of the context. This is an instance of Type 4 Decentering in that Ms. K identified the contradictory assumptions made by herself and by Taylor. This comparison facilitated Ms. K to foreground the assumption in her thinking, as it was likely implicit before her interactions with Taylor. In Excerpt 22, she discussed how modeling Taylor's thinking informed her future teaching where she would ensure her assumption was explicit to her students.

Excerpt 22—Memo 379, 380

- 1 Ms. K: Like that's something that if I was teaching this, I would make explicit. I
- 2 wouldn't have them like, like that's not something that I would work

3 towards. That's one of the things that I would make sure every student
 4 understood before working on the problem. Does that make sense?
 5 Biyao: So, could you, could you be more specific about what—
 6 Ms. K: the understanding that like they are not individual pieces, and if you tell me
 7 that you're going to spend \$50 on radio, then that means you're spending
 8 \$50 across the board and you're going to get \$50 worth of votes through
 9 radio for each of the parties. I don't think that was ever apparent to him (...)
 10 Because if he realized it, then he would see that he's, he's going way over
 11 his goals and which is obviously gonna go over.

Ms. K's comment suggested how decentering, in particular Type 4 Decentering, was generative to a teacher's pedagogical decisions. As she pointed out, she would make sure her students had consistent interpretation of the context before introducing the graphical method. Generally speaking, some aspects of a teacher's FOMs might be implicit or unaware, but decentering served as an avenue for a teacher to reflect upon their FOMs and bring aspects of their thinking into the forefront so that they know how to better communicate with their students.

Confirmation of SOMs

As the conversation continued, Ms. K observed additional evidence that supported her inference. She asked Taylor what would happen if the candidate spent \$2 on radio. Taylor considered the total money spent on radio to be \$6, which produced 4, 10, and 8 votes for each campaign. He then determined the amount of money needed to be spent on TV for each campaign: \$4997.5, \$2999, \$7992, and calculated the total expenses to be \$15999.5 ($= (\$4997.5 + \$2999 + \$7992) + \$6$) (see Ms. K's discussion in Excerpt 23 and Taylor's work in Figure 39). Towards the end of this exchange, Taylor was more convinced by his initial solution because the total expense resulting from this solution was greater than \$9,000.

$$\begin{array}{l}
 \$6 = \begin{array}{l} 4v \\ 10v \\ 8v \end{array} \\
 4\sqrt{19,990} = 4,997.5 \\
 4\sqrt{11,996} = 2,999 \\
 1\sqrt{7,992} = 7,992 \\
 \begin{array}{r}
 15,999.5 - 8v \\
 + \quad 6 - 5 \\
 \hline
 15,997.5
 \end{array}
 \end{array}$$

Figure 39. Taylor's work in response to Ms. K's question about spending \$2 on radio.

Excerpt 23—Memo 267, 268

- 1 Ms. K: So he went in not understanding the idea that if I give \$2 to radio, it means
- 2 everyone gets \$2 put into the radio. And so like this, for example, I give
- 3 you \$2 for radio, that does not become \$6, that is still just \$2. Because he
- 4 was thinking is like two here, two here, two here, not \$2 that gives like,
- 5 these should be \$2 ends up with four votes and 10 votes and eight votes.
- 6 But in his mind, it was six because he's going two, two and two (...)
- 7 because he's treating each television differently, television money. This is
- 8 like television money for Republicans, television money for Democrats,
- 9 television money for independence. So the money I was spending on
- 10 television is all of these added up together, not this much [*pointing to*
- 11 "*\$4997.5*"] or this much [*pointing to* "*\$2999*"]. It's, I spent this much
- 12 [*pointing to* "*\$15999.5*"] on television to meet all of the goals because they
- 13 are three different types of televisions. And I spent six dollars because I
- 14 gave radio \$2. (...) He never carried the two or the, he never carried the
- 15 money for radio throughout.

This additional observation allowed Ms. K to confirm her prior SOMs of Taylor's thinking. Her utterances (Line 6-15) suggested she was able to run through Taylor's reasoning with the assumption held by Taylor. In other words, Ms. K was able to assimilate Taylor's current activities with her previous SOMs without perturbation, which was an example of Type 2 Decentering.

As the teaching session proceeded, Ms. K asked Taylor to remove the last constraint, conjecturing that removing one constraint might allow Taylor to assimilate the problem in ways

consistent with that in the previous problem contexts. Taylor responded that the solution would be \$3000 on TV and \$4000 on radio because we did not need to spend the \$2000 on radio for Independent. Excerpt 24 illustrates Ms. K's comment on his activity.

Excerpt 24—Memo 396

- 1 Ms. K: It was to see how he would work when working with just the two variables
- 2 and if you can balance them better in his head because he's good at
- 3 balancing two constraints in his head. Nah, when he did that I was so upset.
- 4 Oh my God. Because I was like, honestly my first thought was like, no, that
- 5 makes sense. Not like mathema— Is it correct? No. But does it make sense
- 6 with how you're thinking about this problem? It does. So like of course
- 7 that's what you're going to do and like I don't have a good way to tell you
- 8 that you can't do that.

Although Taylor's response was inconsistent with Ms. K's anticipation, it again stabilized Ms.

K's SOMs that Taylor ultimately treated each constraint individually but not imagining the value pairs (r, tv) being applied to each constraint simultaneously.

Re-presentation of SOMs

Following up on this exchange and assuming that only two constraints were considered, I asked Ms. K to create a solution as Taylor would do if he did interpret the context in ways consistent with hers. Figure 40 illustrates Ms. K's solution, and she summarized her work in Excerpt 25.

Handwritten work showing calculations for radio and TV constraints. The work includes several arithmetic problems and a final boxed solution.

Top left notes: 2 radio, 4 TV, 12,000 goal; 5 radio, 4 TV, 20,000 goal.

Top right calculations:

$$\begin{array}{r} 2 \cdot 2000 = 4000 \\ 4 \cdot 2000 = 8000 \\ \hline 12,000 \end{array} \quad \begin{array}{r} 5 \cdot 2000 = 10,000 \\ 4 \cdot 2000 = 8,000 \\ \hline 18,000 \end{array}$$

Middle right calculations:

$$\begin{array}{r} 2 \cdot 4000 = 8,000 \\ 4 \cdot 1000 = 4000 \\ \hline 12,000 \end{array} \quad \begin{array}{r} 5 \cdot 4000 = 20,000 \\ 4 \cdot 1000 = 4000 \\ \hline 24,000 \end{array}$$

Bottom right calculations:

$$\begin{array}{r} 2 \cdot 3,000 = 6000 \\ 4 \cdot 1500 = 6000 \\ \hline 12,000 \end{array} \quad \begin{array}{r} 5 \cdot 3,000 = 15,000 \\ 4 \cdot 1500 = 6000 \\ \hline 21,000 \end{array}$$

Bottom left calculations (boxed):

$$\begin{array}{r} 5 \cdot 4600 = 23,000 \\ 4 \cdot 3,000 = 12,000 \\ \hline 35,000 \end{array} \quad \begin{array}{r} 2 \cdot 16,000 = 32,000 \\ 4 \cdot 3,000 = 12,000 \\ \hline 44,000 \end{array}$$

$$\begin{array}{r}
 12 - 5.6 = 6.4 \\
 2 \frac{1}{5} = 0.4 \\
 20
 \end{array}$$

$$\begin{array}{r}
 2 \cdot 3200 = 6400 \\
 4 \cdot 1400 = 5600 \\
 \hline
 6400 + 5600 = 12,000
 \end{array}$$

$$\begin{array}{r}
 5 \cdot 3200 = 16,000 \\
 4 \cdot 1400 = 5,600 \\
 \hline
 16,000 - 5,600 = 10,400
 \end{array}$$

$$\begin{array}{r}
 2 \cdot 2,900 = 5,800 \\
 4 \cdot 1,600 = 6,400 \\
 \hline
 5,800 + 6,400 = 12,200
 \end{array}$$

$$\begin{array}{r}
 5 \cdot 2,900 = 14,500 \\
 4 \cdot 1,550 = 6,200 \\
 \hline
 14,500 - 6,200 = 8,300
 \end{array}$$

Figure 40. Ms. K's re-presentation of Taylor's algebraic approach in the context of AE.

Excerpt 25—Memo 247

- 1 Ms. K: I think the plan was we wanted to see if, well we told him to take out a
- 2 constraint and if we did that, would he try to work with his normal method.
- 3 So, this was like me trying to work through in his way of picking two
- 4 numbers and then like applying it to one constraint. Like using one
- 5 constraint to pick another and then applying it to the second constraint to
- 6 see if it works. And so, we started with three thousand three thousand [see
- 7 *Figure 40, top, left*]. We realized it didn't work cause' we went over so that
- 8 I just did 2000, 2000 [see *Figure 40, top, right*] cause then it still gave me
- 9 12,000 here, but it was still too much so that I adjusted lower and so on.
- 10 And then right here [see *Figure 40, bottom*] we trying to get in close to then
- 11 you start fine tuning it a little bit. So instead of jumping by thousand, started
- 12 jumping by 500 and then my 10 and so on.

Here, Ms. K was picking up on Taylor's way of reasoning to coordinate the two constraints regarding Republican and Democratic. Her solutions resembled some features of Taylor's approach, such as coming up with a value pair satisfying a "primary" constraint (i.e., making sure $2x + 4y = 12,000$) and considering if the other constraint was also satisfied. She also adjusted the value pair so that the total number of Democratic votes got closer and closer to 20,000. However, I also observed that her reasoning seemed not to be systematic. For example, she switched from using the Democratic constraint as a "primary" constraint (see her work highlighted in the red box; spending \$1,600 on radio and \$3000 on TV to achieve 20000 goal

votes) to using the Republican constraint (see her work in other places). Also, she tested the 3200-1400 combination (see her work highlighted in the green box) right after 3000-1500, from which I inferred she did not conclude from the previous three set of calculations that increasing money spent on radio from 3000 would get away from the 20,000 goal votes for Democratic (as opposed to getting closer)⁸.

These observations were not to claim that Ms. K's SOMs of Taylor were not identical to Taylor's mathematics or to claim that Ms. K's ability to solve a problem with this approach was less sophisticated than Taylor's. Rather, I claim that a researcher can support a teacher's mathematical learning through promoting the teacher's re-presentational activity associated with their SOMs. As Ms. K admitted at the end of the study, working through ideas in ways consistent with Taylor's thinking was challenging but eye-opening to her (Excerpt 26).

Excerpt 26—Memo 333

- 1 Ms. K: Oh my gosh. [Taylor] has so many crazy misunderstandings...the times
- 2 where I had to work a problem, the way he would work it, it would always
- 3 really get me. I had to like try to solve this as if I was doing it his way.
- 4 Okay. And then finishing this graph as if he was doing it [*referring to her*
- 5 *work shown in Figure 42, right*]. Those were really, really eye-opening for
- 6 me, like, oh my gosh.

Ms. K's Attempt to Intervene

As the teaching session continued, Ms. K tried to intervene by asking Taylor to write equations corresponding to the three constraints and identify value pairs satisfying those equations (Figure 41). She hypothesized that reasoning with these equations and their associated value pairs might allow Taylor to realize that one value pair should apply to three equations at

⁸ I note that this might be because Ms. K's calculation of the 2000-2000 combination was incorrect. The total votes for Democratic should be 18,000 instead of 21,000. This might influence her anticipation of how the total votes would change as the value pair changed.

the same time but not separately. However, she did not achieve her goal despite this attempt (see her reflection in Excerpt 27).

$$\begin{array}{l} 12,000 = 4x + 2y \\ 20,000 = 4x + 8y \\ 8,000 = 8x + 4y \end{array} \quad \begin{array}{l} 3,000.0 \\ 2,000.3000 \\ 0.6,000 \\ 500,5,000 \end{array}$$

Figure 41. Taylor's equations related to AE.

Excerpt 27—Memo 249

- 1 Ms. K: I really was just saying if he could list numbers like choosing, if you already
- 2 have 3000 per radio that you know that television's going to have zero and
- 3 maybe that was going to cause him to see that that doesn't meet all three of
- 4 my constraints cause' I have to reach the goal all the way through and he
- 5 still hasn't gone there. He never really did get there. Yeah. So maybe if I
- 6 was making him list all of the different things, he would see that just
- 7 because it satisfies one constraint doesn't mean it satisfies all of them.
- 8 Which again is so weird because like...um, cause' that's not an issue he
- 9 had in the other tasks. Like he understood the idea that like I could pick
- 10 something that's gonna satisfy one of these constraints and not necessarily
- 11 satisfy the other one and I can adjust to know when it fixes both. But he
- 12 never really got there here because he was treating these as like these
- 13 variables so differently.

Again, like Excerpt 20, Ms. K was referring to Taylor's activities in the previous sessions, where he was able to coordinate two variables so that they satisfied two constraints simultaneously (Line 9-11). Drawing on this SOM, Ms. K anticipated that Taylor might be able to assimilate these equations in ways similar to what he did in the prior contexts (Line 1-4, Line 6-7).

However, Taylor's responses did not match Ms. K's anticipation. Ms. K's failure to engender accommodation in Taylor's thinking in turn confirmed her SOMs that Taylor ultimately held a different assumption of the problem context (Line 12-13). Towards the end of this exchange, Ms.

K remained perturbed by not being able to support him in generalizing his prior understanding to the current situation.

Promoting a Graphical Representation

Towards the end of the teaching session, Ms. K decided to shift the conversation by asking Taylor to graph all the possible radio-tv combinations. Taylor created a graph shown in Figure 42, where he related *goal votes* for each campaign (on the horizontal axis) and *total expenses* for the corresponding campaign (on the vertical axis). Although Ms. K realized that Taylor was relating two quantities different than what she expected, she decided to build upon Taylor's thinking and support him graphing a vertical line, as opposed to discrete points, to represent all possible total expenses with respect to each goal vote, but Taylor did not respond as intended, and the session ended.

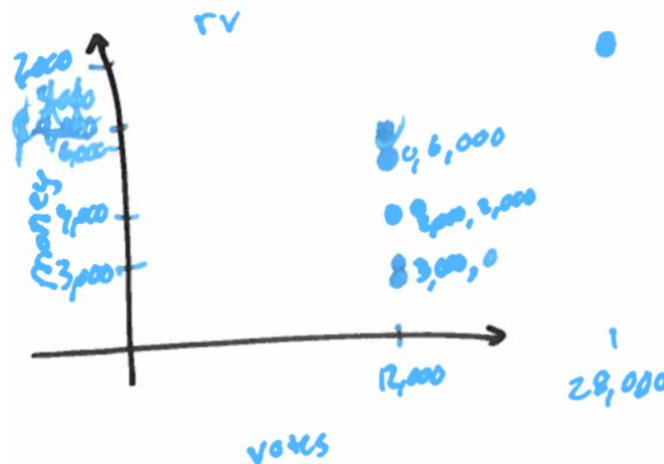


Figure 42. Taylor's votes-money graph.

I asked Ms. K to show how she would create a similar graph. She drew the graph shown in Figure 43, where she identified the range of expense regarding each campaign. For example, in order to gain 8000 Independent voters, the advertising expense ranged from \$2000, when the candidate only spent money on radio, to \$8000, when the candidate only spent money on TV.

Other advertising plans led to expenses between \$2000 and \$8000. Ms. K also claimed that the optimal solution should occur within the highlighted segments (see the black segments in Figure 43), saying “the amount of money you give to radio and the amount of money you give to television needs to be, the total amount for that, should be between four and five thousand dollars.” And values outside this range did not contain solutions that satisfied the three goal votes constraints at the same time.

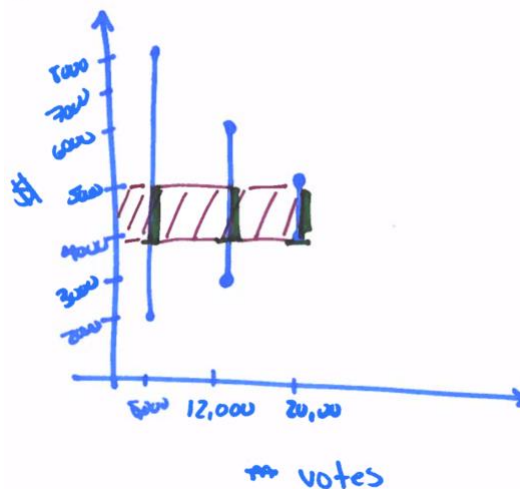


Figure 43. Ms. K's votes-money graph.

Then, Ms. K claimed that it would be possible to determine the exact solution with this graph since the number of value combinations were limited within a certain range (Excerpt 28).

Excerpt 28—Memo 263

- 1 Ms. K: Let's say that he was thinking about it this way, you could from, I mean
- 2 from there, like that really limits the amount of value pairs you have to test
- 3 because now you're just testing between four and five thousand dollars and
- 4 he's good at testing value pairs.

However, as she thought further, she realized that it was almost impossible to come up with an exact solution given the range of total expenses of radio and TV, saying “so it's more like you have to be able to pick numbers within this range, but then also the numbers that only fall on these lines, which is just a lot”. Ms. K realized that the expenses of radio and of TV not only

needed to add up to a given total that is within the range of 4000 and 5000, which included infinitely many possibilities, but should also determine a value pair (r, t) such that $r + t =$ *a selected value of total expenses*, $5r + 4t = 20000$, $2r + 4t = 12000$, $4r + t = 8000$ were simultaneously satisfied. Regardless, I consider this as an example of constructing SOMs being generative to Ms. K's mathematical development. Ms. K had not thought about the relationship between votes and money before she worked with Taylor, and she was persistent in thinking through the mathematics inspired by Taylor's reasoning.

Cycle 6: Advertising Election (follow-up)

Before starting this cycle, we learned that Taylor had learned the linear programming lesson at school⁹. We decided to focus this last teaching session on supporting his reflection on the graphical method he learned and the algebraic method he previously used. We also planned to engage him in a conversation about graphing inequalities to gain insights to the meanings he constructed from the lesson.

Resolving Advertising Election

Ms. K started the session by asking Taylor to rework on the AE problem. As illustrated in Figure 44 (top), he defined the two variables x and y , wrote three inequalities to represent the constraints, and converted each inequality into a slope-intercept form. Then, he graphed the three inequalities on a Cartesian coordinate system and circled two corner points of the shaded region (Figure 44, bottom left). To speed up his problem-solving process, we told him the accurate coordinate pairs of those two corner points to be approximately $(2667, 1667)$ and $(1091, 3637)$. Then, he calculated the total expenses to be \$13,002 $(= 2667 \times 3 + 1667 \times 3)$ and

⁹ Taylor did not receive the instruction from Ms. K but from another teacher.

\$14,184 ($= 1091 \times 3 + 3637 \times 3$), assuming that the candidate spent \$2667 or \$1091 on radio and \$1667 or \$3637 on TV for each campaign (Figure 44, bottom right). Eventually, he determined the optimal solution to be $(x, y) = (2666, 1666)$ since it corresponded to a lower cost.

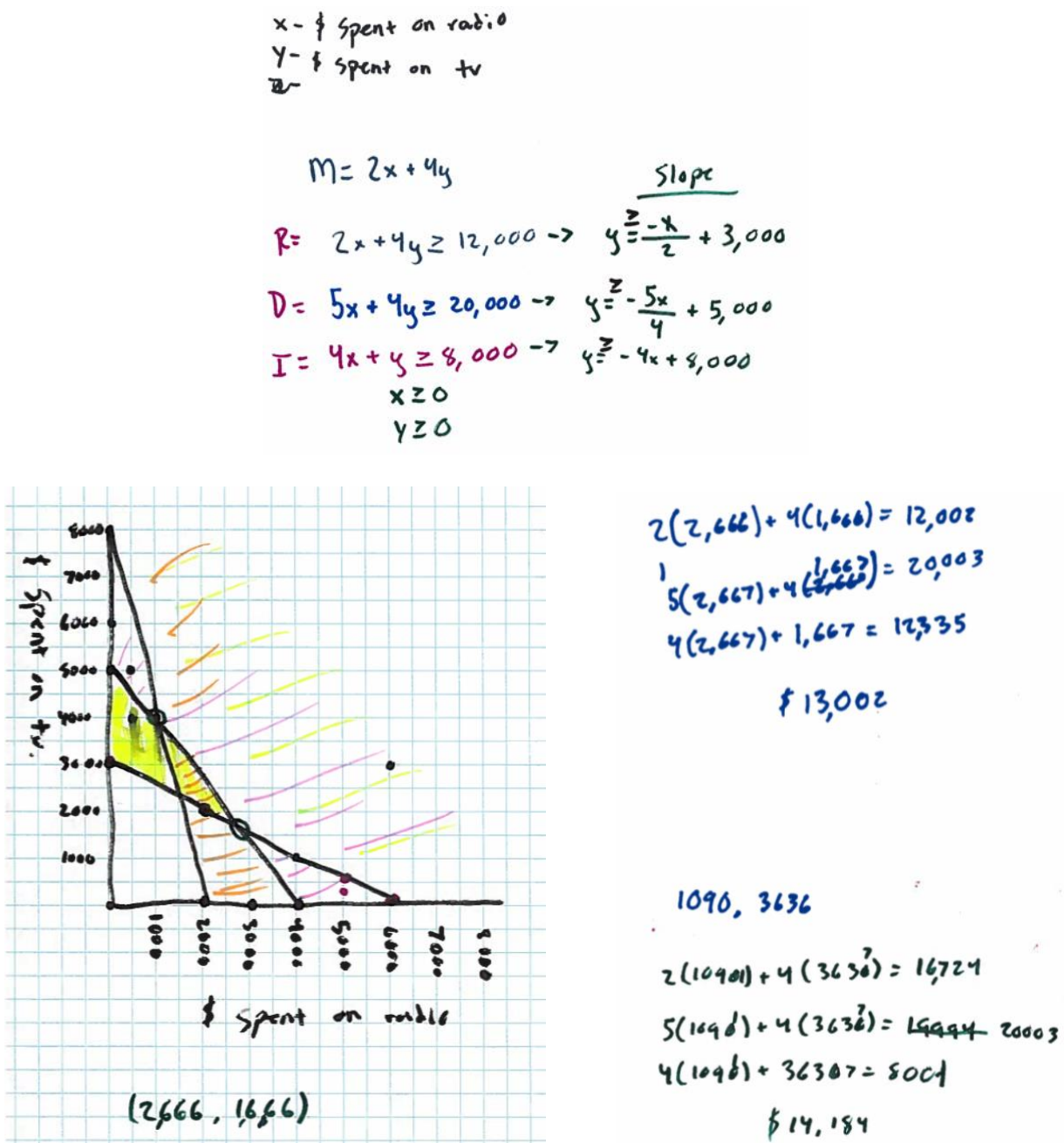


Figure 44. Taylor's solution to AE in Cycle 6.

Meanings of Lines and Shaded Region

I also draw the reader's attention to the six equations shown in Figure 44 (bottom right), where Taylor evaluated each of the three inequalities with the two coordinate pairs to determine whether the goal votes were reached. Ms. K was surprised by his actions and interpreted Taylor's activity as a contraindication of him understanding what a graphed line represented (Excerpt 29).

Excerpt 29—Memo 312

- 1 Ms. K: He doesn't know what a line is! [laughing] You can hear me like smirking
- 2 when I'm talking to him. Like what do you mean you have to test the
- 3 points? It's on the line! It's like that's what it means to be on the line! I
- 4 think this goes back to the fact that it's taught so procedurally. He's not
- 5 taught to think about it as like, all right, these are all of the points that will
- 6 satisfy the equality of this equation and everything above it or below it
- 7 satisfies the inequality part of it. He's just taught like, okay, graph it and I
- 8 find these places, and like you're not thinking about what you're doing as
- 9 you do it. And so, pushing him on those, he's like, I guess I can check to
- 10 make sure they work. It's like, no, you don't have to do that because that's
- 11 the whole point of graphing. Like that's why you graph it, cause' you know
- 12 they will work.

From Ms. K's perspective, it was unnecessary to do those calculations because the fact that the points were chosen from the lines bounding the shaded region automatically implied that their coordinate pairs satisfied the associated inequalities (Line 1-7, Line 10-12). Taylor's actions thus generated perturbation in Ms. K because Ms. K failed to assimilate his actions with her FOMs. Therefore, she claimed that Taylor's actions were contraindications that he understood "what it means to be on the line," which was an evidence of Type 1 Decentering.

Attempting to continue to understand Taylor's meaning of his inequality graph, Ms. K prompted him several times to discuss the meaning of the shaded region by asking, for example, "what does it mean for a region to be shaded by all three of them?" Taylor responded that it meant that "wherever those points meet, you plugged in those numbers into the formula" and that

“you need to know the points to choose from.” Excerpt 30 and Excerpt 31 illustrate Ms. K’s interpretation of his understanding being procedural.

Excerpt 30—Memo 303

- 1 Ms. K: Like it’s very procedural. It’s like you do this thing and then you do this
- 2 thing and then you do this thing. Um, without a lot of context behind why
- 3 you do.
- 4 Biyao: So, could you say more about what you meant by—like—what you are
- 5 imagining?
- 6 Ms. K: He’s thinking like, okay, I’m going to graph it and then I’ll have a feasible
- 7 region, and with the feasible region, I’ll have like my boundary points, and
- 8 I’ll plug those boundary points into the objective function, and that’ll give
- 9 me my whatever my solution is.

Excerpt 31—Memo 320

- 1 Ms. K: I think that ties back to the procedural thing like this. Like the idea that
- 2 okay, you do this step to be able to do this next step...It is just an
- 3 understanding that like, okay, I have to shade so that I can find the point so
- 4 that I can plug them in and so that I can get my answer.

Ms. K interpreted that Taylor’s understanding of his inequality graph was constrained to the sequence of physical actions of shading regions, locating the corner points, and evaluating the objective function. As an additional evidence of this procedural meaning, Taylor was perturbed by the shaded region in the current context being not a polygon, saying “in class, there was a shape that was shaded by all three of them”. When discussing this instance, Ms. K conjectured that Taylor’s teacher likely emphasized the process of identifying a polygon in her instruction (Excerpt 32).

Excerpt 32—Memo 346

- 1 Ms. K: And I think in class, they really do focus a lot on like the feasible region
- 2 being a very distinct shape, like some sort of polygon. And so I think that’s
- 3 what he was looking for because that’s what he’s used to.
- 4 Biyao: Do you think that’s an important thing to emphasize?

5 Ms. K: No. And that's why I went back and asked him, what does that shape
 6 represent or what does that, what does that place where you, like all three
 7 parts are shaded in something like that. To show him like, okay, that's the
 8 feasible solutions, it's not the shape. It's the fact that all three constraints
 9 are shading the same area of the graph, which means that this is the section
 10 where they have the same possible solutions.

This excerpt represented Type 4 Decentering because Ms. K was contrasting two meanings of an inequality graph. One meaning was associated with her SOM that Taylor conceived of the shaded region as a shape and as a result of carrying out a sequence of actions, and the other meaning was associated with her FOM that a shaded region represented all the feasible solutions.

Meanings of a Point

Ms. K also observed that Taylor struggled to describe the meaning of the coordinate point (1000, 4000)¹⁰ on his graph (Figure 44, bottom left). Specifically, he was confused about whether the 1000 represented the total money spent on radio for the three campaigns (i.e., the \$1000 should be split into three parts) or represented the money spent on radio for each campaign (i.e., the total money spent on radio was not \$1000 but \$3000). Ms. K explained his confusion as coming from his assumption of the context (Excerpt 33).

Excerpt 33—Memo 309

1 Ms. K: Like I understand where he's coming from in the context of like, well, like
 2 Republicans and Democrats and Independents aren't gonna be using like
 3 the same advertising company...like it makes sense to treat them
 4 differently. And this graph isn't doing that and so like what that \$1,000,
 5 you don't... I don't know. It's harder to understand it in the context of
 6 the...Because like you would want to say like, oh, if that's the total spent
 7 and then like these three numbers, like you're get like Republican has so
 8 much and Democrats so much and Independent so much. But that's not the
 9 case. It's that each one is spending \$1,000 on campaign advertising. But

¹⁰ Before we provided Taylor with the accurate coordinate values, he estimated the point (1091, 3637) to be (1000, 4000).

10 contextually that just kind of falls apart because why would they all be
11 doing the same thing?

Ms. K considered Taylor's assumption was contextually reasonable because candidates usually advertised in different channels for different campaigns rather than spending money on one channel to gain votes on different campaigns. However, a 2-dimensional graph was not compatible with such an assumption because one coordinate value (e.g., \$1000) should have a consistent meaning with respect to the three graphed lines rather than being split into three values or representing the value of three different variables. It is also noteworthy that Ms. K did not take a deficit view on Taylor's struggle; rather, she considered Taylor's confusion as being reasonable because the way he interpreted the context was not viable with a conventional meaning for a point on a Cartesian coordinate system. I claim that Ms. K's reflection here entailed Type 4 Decentering by which she simultaneously operated on two contradicting meanings constituting Taylor's and her own thinking. This instance was also an evidence of Type 2 Decentering in that Ms. K's prior SOMs became explanatory for Taylor's current activity. Taylor's activity did not cause perturbation in Ms. K; instead, her SOMs became stabilized, and she could apply them to assimilate Taylor's activity.

Summary

In this chapter, I described Ms. K's reflection on Taylor's activities during the six teaching cycles. I provided evidence of Ms. K enacting Type 1 Decentering as she was perturbed by Taylor's actions (e.g., Excerpt 5, Excerpt 6, Excerpt 29), Type 2 Decentering as she applied her SOMs to assimilate Taylor's activities (e.g., Excerpt 3, Excerpt 23), Type 3 Decentering as she constructed her initial SOMs (e.g., Excerpt 1, Excerpt 8, Excerpt 19) or revised her existing SOMs (e.g., Excerpt 11, Excerpt 18), and Type 4 Decentering as she compared her SOMs and FOMs (e.g., Excerpt 7, Excerpt 21, Excerpt 33) or compared multiple SOMs of Taylor (e.g.,

Excerpt 11, Excerpt 20). Ms. K's journey of decentering represented a scientific inquiry process, where she iteratively constructed, confirmed, and rejected hypotheses about Taylor's thinking. During this journey, some aspects of Ms. K's SOMs became more and more stable, but meanwhile, new perturbations occurred, and some of the perturbations persisted towards the end of the experiment.

The next chapter (Chapter 7) presents the story of Ms. N, Carlee, and Heidi in the context of geometric dilation. In Chapter 8, I provide a broader discussion on the cross-cutting themes of these two stories.

CHAPTER 7

THE STORY OF MS. N, CARLEE, AND HEIDI

You know how the math education world is trying to move away from the word “misconception.” Because to the student is still a conception, it’s not a misconception.

By Ms. N

In this chapter, I describe Carlee and Heidi’s mathematical activities through the eyes of Ms. N. The experiment contained two completed teaching cycles and a set of tasks planned for the third teaching cycle. The first teaching cycle was focused on the concept of dilation in relation to two real-world contexts. The second teaching cycle examined the students’ meanings for image, preimage, center of dilation, and their relationships. The third teaching cycle was aimed to support students’ understanding of dilation in terms of distances, including distinguish dilating distances from dilating areas. In each of the following subsections, I first give an overview of the task progression in each cycle and then discuss Ms. N’s interpretations of Carlee and Heidi’s activities with respect to each task. It is my interpretations of Ms. N’s discussion that constitute my TOMs of the two students’ mathematics and SOMs of Ms. N’s mathematics.

Cycle 1: Photocopy Faux Pas and Triangle Shadow

The first teaching cycle addressed two major tasks: Photocopy Faux Pas and Triangle Shadow. Towards the end of the teaching session, Ms. N also engaged the students in a discussion on the mathematical connections they conceived from these two tasks.

Task 1: Photocopy Faux Pas

Photocopy Faux Pas (Figure 46) was adapted from the Geometry curriculum by the Mathematics Vision Project (MVP, 2017).

Burnell has a new job at a copy center helping people use the photocopy machines. Burnell thinks he knows everything about making photocopies, and so he didn't complete his assignment to read the training manual.

Mr. and Mrs. Donahue are making a scrapbook for Mr. Donahue's grandfather's 75th birthday party, and they want to enlarge a sketch of their grandfather which was drawn when he was in World War II. They have purchased some very expensive scrapbook paper, and they would like this image to be centered on the page. Because they are unfamiliar with the process of enlarging an image, they have come to Burnell for help.

"We would like to make a copy of this image that is twice as big, and centered in the middle of this very expensive scrapbook paper," Mrs. Donahue says. "Can you help us with that?"

"Certainly," says Burnell. "Glad to be of service."

Burnell taped the original image in the middle of a white piece of paper, placed it on the glass of the photocopy machine, inserted the expensive scrapbook paper into the paper tray, and set the enlargement feature at 200%.

In a moment, this image was produced.

"You've ruined our expensive paper," cried Mrs. Donahue.

"Much of the image is off the paper instead of being centered."

"And this image is more than twice as big," Mr. Donahue complained. "One fourth of grandpa's picture is taking up as much space as the original."

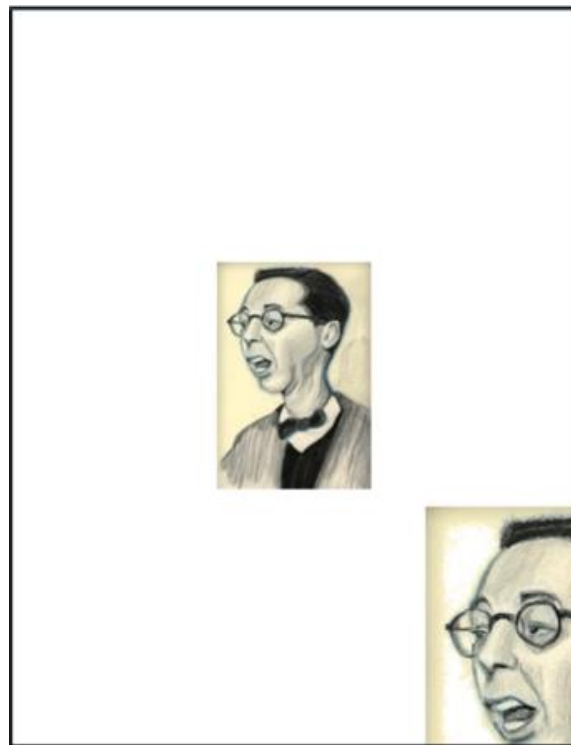


Figure 45. Burnell's photocopy with the original picture taped at the center of the paper.

Figure 46. Photocopy Faux Pas.

Ms. N's Solution

Ms. N used this task to teach dilation in her classroom instruction. Specifically, she focused her students on the following four questions: (a) Judging from the way Burnell taped the picture in the middle of the paper, where does Burnell think the center of dilation is? (b) Judging from the way the image was enlarged, where is the *actual* center of dilation? How do you know? (c) Where should Burnell have placed the original image if he wanted the final image to be centered on the paper? (d) Mr. Donahue complained that the copy was four times bigger than the original. What do you think? Did Burnell double the image or quadruple it? What evidence would you use to support your claim?

Figure 47 illustrates Ms. N's solution to this task during the first pre-teaching session. By extending two segments that connect two pairs of corresponding points on the preimage (the original photo) and the image (the photocopy), Ms. N determined the intersection of the two segments at the upper left vertex of the paper and claimed that it was the actual center of dilation (see the point pointed by the little blue arrow in Figure 47). In contrast to Ms. N's thinking, Burnell thought the center of dilation was at the center of the paper, and therefore he placed the preimage in the middle, hoping that it would be dilated around the center. Regarding question (c), Ms. N claimed that Burnell should have placed the original image at the upper left corner of the paper so that its upper left vertex coincides with that of the paper. However, Ms. N realized from her interactions with Carlee and Heidi that she was wrong because this placement would not make the image to be at the center of the paper. Regarding question (d), Ms. N claimed that the copy was a double of the preimage because the length and the width of the image were both twice as long as those of the preimage.

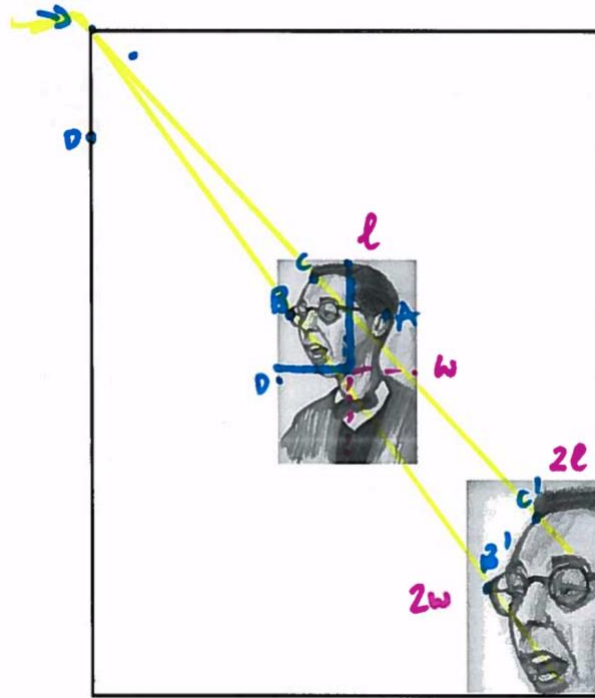


Figure 47. Ms. N's solution to Photocopy Faux Pas.

Moving the Resulting Image

During the planning session, Ms. N and I decided to modify Photocopy Faux Pas to be more open-ended, and thus we removed the four prompts she presented in her classroom instruction. After the Carlee and Heidi read through the scenario, Ms. N asked them, “how would you help Burnell to fix this problem if he had a chance to reproduce a copy?” Carlee suggested that Burnell should move the image to the center. Ms. N was surprised by Carlee's response and asked several follow-up questions to confirm that Carlee indeed wanted to move the image. In Excerpt 34, Ms. N discussed her interpretation of this instance.

Excerpt 34—Memo 42

- 1 Ms. N: I was very surprised. I was like haha interesting! Cuz she said, I guess like
- 2 she was thinking it's the result that's wrong so you have to fix the result,
- 3 whereas like for me is like—again back to the idea of you have to think about
- 4 what's causing it to be that way and fix the source of the problem, but she's
- 5 thinking well the result is what's wrong so you fix the result because she's
- 6 saying you have to fix the picture that end up in the corner.

Based on Ms. N's interpretation, Carlee's rationale was that because the location of the image was wrong, Burnell needed to fix the image as opposed to the preimage (Line 2, Line 5). By contrast, from Ms. N's point of view, in order to change the resulting image, Burnell needed to relocate the preimage in order to affect the image. As she further explained, "you can't really move something that is a result of something else. You have to move that thing that's causing it to be this way...I'm thinking move the original because the original is what's affecting where the result goes." I consider this was an instance of Type 4 Decentering because Ms. K was able to put two ways of understandings side by side and identify their differences.

Changing the Enlargement Percentage

As the conversation proceeded, Ms. N decided not to intervene Carlee's thinking and instead turned to Heidi for an alternative suggestion. Heidi proposed that Burnell should reduce the enlargement percentage from 200% to 100% in order to center the image. Excerpt 35 and Excerpt 36 illustrate Ms. N's interpretations of Heidi's reasoning.

Excerpt 35—Memo 29

- 1 Ms. N: I think she thought that way because the picture went off the paper. So
- 2 maybe if you make it smaller, the whole image would be contained in the
- 3 paper.
- 4 Biyao: Say more, like—
- 5 Ms. N: For example, if you make it like—I think she just thinks that if it's the
- 6 pictures all the way out here [*pointing to the image at the lower right corner*
- 7 *of Figure 45*] that it needs to move closer to here [*moving her hands to the*
- 8 *center of the paper*], so if you enlarge it less, then it will be in the paper.

Excerpt 36—Memo 37

- 1 Ms. N: She referred back to the original itself, and she said, one fourth, like this is
- 2 one fourth of the original [*pointing to the image at the lower right corner*
- 3 *of Figure 45*], and you want it to be half or twice as much. So, it's up 200.
- 4 You would just do half of that because maybe she's thinking 200% gives
- 5 you a picture that quadruple from the original. So, if you want half of that,
- 6 then you will take half of 200%, which gives you 100.

In Excerpt 35, Ms. N interpreted that Heidi thought reducing the enlargement percentage would allow Burnell to relocate the image to be closer to the center of the paper. And more precisely, as she discussed in Excerpt 36, because 200% produced an image that was a quadruple of the preimage (from Heidi's perspective), reducing that enlargement scale by half should generate an image that was a double of the original, which was what Burnell intended. Together, these two excerpts were examples of Type 3 Decentering by which Ms. N could make sense of Heidi's arguments and constructed SOMs of her thinking.

However, Ms. N argued that although reducing the enlargement scale might allow the image to be fully contained in the paper, it would not be a doubling of the preimage; from her perspective, scaling the preimage by 100% meant the picture was "not really enlarging at all." This was an example of Type 4 Decentering since Ms. N was operating on her FOM to consider one aspect of her SOM was not valid. She added that it was really hard for her to "resist the urge of stepping in and say, what does it mean to enlarge something by 100%" in the moment of the interactions and that she should have asked this question to Heidi to gain more insights into her thinking. I claim that Ms. N's comparison between her FOM and SOM motivated her inquiry into Heidi's thinking, suggesting she had experienced perturbation as a result of this comparison.

Building on Heidi's idea of changing the enlargement scale, Carlee alternatively suggested that the enlargement scale should be associated with the percentage of the preimage with respect to the entire paper. Excerpt 37 illustrates Ms. N discussion on her reasoning.

Excerpt 37—Memo 32, 35

- 1 Ms. N: And Carlee was saying, well, you have to know how much this [*pointing to*
- 2 *the original picture at the center of Figure 45*], the original— how much
- 3 that original takes up in terms of the whole paper, which is confusing to me.
- 4 Because let's say like this original takes up is gonna estimate— one, two,
- 5 three, one, two, three [*counting how many original pictures can be fit into*
- 6 *the whole paper; there were three rows with three in each row*], maybe like

7 one tenth? One ninth or one tenth of the piece of paper. How is that gonna
 8 help you? You know? I'm just confused. 'Cause like, one tenth is like 10%.
 9 So, I don't know if she was suggesting that it should be 10%
 10 Biyao: What will be the other possibility?
 11 Ms. N: Could she maybe think about if this original picture is one tenth, then you
 12 need maybe to make it 10 times larger?
 13 Biyao: Ten times as big as the original?
 14 Ms. N: Yeah, to fill up the whole thing.
 15 Biyao: What do you think about that? Is it going to work?
 16 Ms. N: No. [laughing] It goes back to the idea of what they were saying here with
 17 the sections, like the original takes up one tenth of the whole piece of paper.
 18 Yes. But going from the original to the whole piece of paper. You're not
 19 dilating at 10 times larger because the dimension is not getting 10 times
 20 larger, like the width and the length is not getting 10 times larger. It is only
 21 getting like three-ish times larger. (...) We're trying to get them to think
 22 that the dimension is what's changing, not the area. Whereas like Carlee
 23 was saying if the original takes up 10% of the whole thing then you need to
 24 multiply it by 10 maybe to get, to fill up the whole entire page. And she
 25 was saying area, not dimension.

Reflecting on Carlee's activity, Ms. N initially was perturbed and could not make sense of her rationale. As I prompted, she inferred that Carlee might aim to fulfil the paper by a scale of 10 instead of doubling the preimage. Furthermore, she claimed that Carlee was likely attending to the area of the picture but not its dimension and that Carlee had constructed a meaning of dilation in terms of iterating areas as opposed to scaling lengths. Ms. N's engagement in Type 3 Decentering here not only involved her making sense of Carlee's argument but also involved her locating the fundamental conceptual issue that might cause Carlee to assimilate the problem in a particular way. In the next subsection, I will provide more instances of Ms. N's discussion on both students' conception of dilation in terms of areas versus length.

Immediately after this exchange and reflecting on Carlee's interpretation of the problem, Ms. N desired to reread the scenario to ensure Burnell's goal was to double the image instead of fulfilling the paper (Excerpt 38).

Excerpt 38—Memo 47

- 1 Ms. N: “They have purchased some very expensive scrapbook paper, and they
2 would like this image to be centered on the page. Because they are—” Wait,
3 what? Centered on the page? You can’t center on the page if you if the
4 center of dilation is right there [*pointing to the upper left corner of the*
5 *paper, see Figure 45*]. Then you have to move it. [*pausing*] If you want it to
6 be centered on the page, you can’t, the center dilation is right there,
7 obviously [*pointing to the upper left corner of the paper, see Figure 45*], but
8 how? So, you would have to move this not to the corner but just— I just
9 don’t realize that! (...) If you don’t move this all the way up here just like
10 maybe somewhere right there [*see gestures in Figure 48*]. So, like the
11 images somewhere like that [*tracing along the yellow rectangle annotated*
12 *in Figure 48*]. Does that make sense? Like the original over here, not all the
13 way to the corner? But that adds another layer to the problem.
- 14 Biyao: Yeah, you may start here. Oh, so the answer is going to be different now.
- 15 Ms. N: But then it’s like, that’s hard. This is Introduction to Dilation. They can’t
16 do that! You know what I mean? Oh, my gosh, I just I didn’t even like, I
17 didn’t realize that, reading it back (...) Oh, they said it again [*turning to the*
18 *problem statement*], “We would like to make a copy of this image that is
19 twice as big and center in the middle.” You can’t center in the middle!

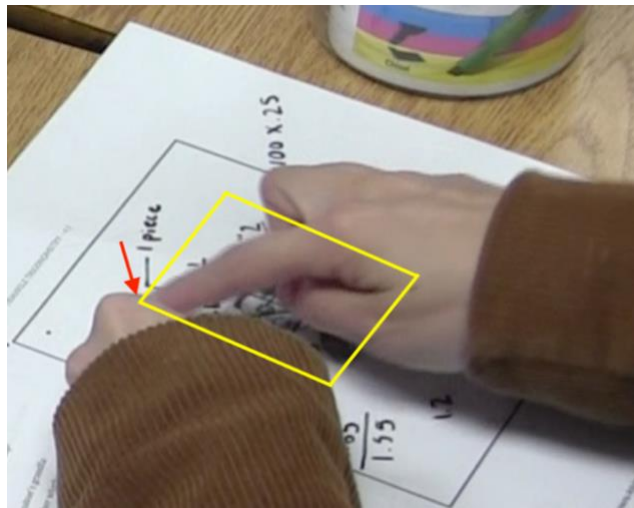


Figure 48. Ms. N’s gesture for showing the correct placement of the preimage.

Ms. N realized that asking students to producing a copy of the preimage such that it was not only twice as big as the preimage but also located at the center of the paper was inappropriate for the introductory lesson on dilation (Line 15-17). Ms. N’s activity associated with this excerpt was characterized as Type 0 Decentering since she was primarily drawing on her FOMs to resolve the task. However, I note that her construction of SOMs of the two students’ thinking, including

the students' novel responses to the task, played a critical role in motivating Ms. N's reengagement in the task. In particular, both Heidi and Carlee intended to move the location of the image towards the center of the paper but they each conveyed interesting thinking regarding the enlargement scale of the image. It was only after Ms. N's experience of these two students' thinking that she came to realize the coordination of the size and the location of the image might be nontrivial to many students. Again, this suggested how decentering and second-order modeling could be consequential to a teacher's mathematical thinking.

Quadrupled or Doubled? Area or Length?

To continue to test the hypothesis that the students conceived of dilation in terms of areas instead of length, Ms. N asked the students to determine whether the image was a quadruple or a double of the preimage and to justify their thinking. Both students agreed that the original photo was quadrupled by the copy machine. Excerpt 39 illustrates Ms. N interpretation of Carlee's justification.

Excerpt 39—Memo 51

- 1 Ms. N: She said that's— that length [*pointing to the width of the image, see Figure*
- 2 *45*] equals to that length [*pointing to the width of the preimage, see Figure*
- 3 *45*]. And that length [*pointing to the height of the image*] or that [*pointing*
- 4 *to the height of the preimage*], they are equal to each other. But this is only
- 5 one fourth of the picture [*pointing to the image*], so one fourth of the
- 6 picture, the dimension of one fourth of the picture is equal to the dimensions
- 7 of a whole, the whole original.

Carlee thought the image was a quadruple of the preimage because their dimensions were identical, but the image only captured one fourth of the original picture. Ms. N contrasted Carlee's understanding with her own understanding of the situation.

Excerpt 40—Memo 51

- 1 Ms. N: I thought they were going to get to the idea of well, this section right here,
- 2 the one, one quarter section has the length or width of 1.65 [*referring to the*
- 3 *dimension of the width of a quarter section of the original photo, see Figure*

4 49]. And this one quarter has the length or width of 3.2 [referring to the
5 *dimension of the width of the image, see Figure 49*], which is twice as much.
6 So I like that's what I think of. When I asked them to write this down, I
7 thought they were going to make the connection– well, one quarter over
8 here is 1.65 and one quarter over here is three two, so the dimension is
9 enlarging by two.

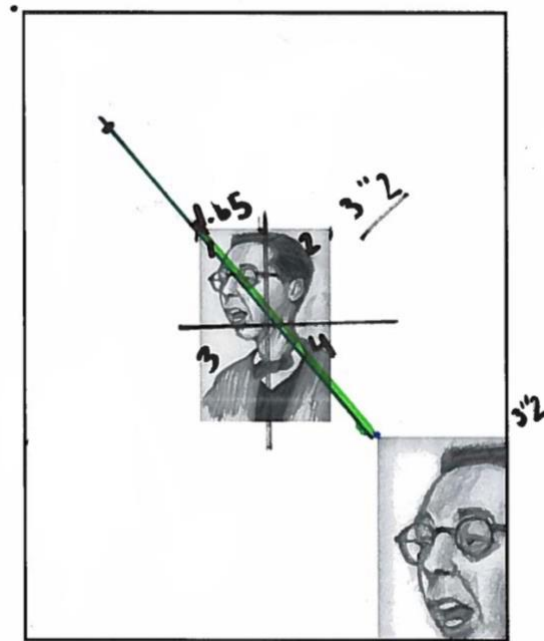


Figure 49. Ms. N's reference to the dimension of the preimage and image (drawing created by Carlee and Heidi during the teaching session).

Different than Carlee (and Heidi) who seemed to compare the dimensions of the *full* preimage to the dimensions of *a quarter* of the full image, Ms. N compared their corresponding dimensions. Namely, the 3.2-inch width of a portion of the full image should be compared to half of the width of the full preimage, which was measured as approximately 1.65 inches.

In order to perturb the students' thinking, Ms. N drew two images, both of which captured one fourth of the original picture, but one had a width labeled as "2" (Figure 50, left) and one had a width labeled as "5" (Figure 50, right). Ms. N explained that with respect to the one labeled with a "2", it was an enlargement of the original picture but smaller than the current photocopy, and the one labeled with a "5" was bigger than the original and the photocopy.

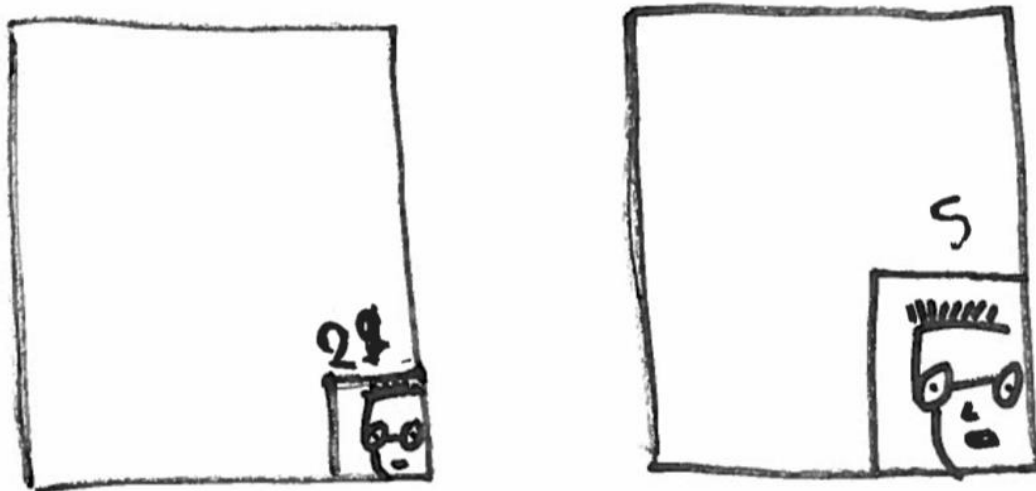


Figure 50. Ms. N's drawings of one quarter of the original picture with different dimensions.

When asked if these two pictures were both quadruples of the original, Carlee said they were not because the dimensions of these two pictures were not identical to the original and that Figure 50 (left) was smaller than the original while Figure 50 (right) was bigger. Heidi disagreed by saying that both drawings were bigger than the original because 2 and 5 were both bigger than 1.65. Excerpt 41 illustrates Ms. N's comparison of the two students' thinking.

Excerpt 41—Memo 57

- 1 Ms. N: I think it was Carlee who compared the whole dimension to that two,
- 2 whereas Heidi was starting to get the idea that you know, that's one
- 3 quarter of the pictures. You have to compare the quarter to the quarter
- 4 here. (...) Because she [Carlee] was comparing the 3.2 to 2 whereas Heidi
- 5 is comparing the 1.65 to the 2.

Based on the different responses from the two students, Ms. N interpreted that Heidi was able to compare the dimension of the corresponding length while Carlee did not. This instance suggested how Ms. N's SOMs were consequential to her actions of interacting with students, especially making intentional actions that could reveal nuances in the students' thinking. In a latter activity (see Excerpt 49), this discrepancy between Heidi and Carlee's thinking became more apparent to Ms. N.

In concluding her reflection on Carlee and Heidi's activities with respect to this task, Ms. N commented on how the two students' thinking was novel to her (Excerpt 42).

Excerpt 42—Memo 52

- 1 Ms. N: I never thought about like this. I never stopped and like— for me is like-
- 2 even like because this is my first-year teaching geometry and I had to pretty
- 3 much reteach myself the whole thing. And even then, like I never thought
- 4 about area. When I think about— I never thought area when I think about
- 5 dilation. I would think about dimensions. And for me like that's just such
- 6 an automatic thing that I never— I just take it as a given, like two plus two
- 7 is four, kind of thing.

Ms. N's reflection suggested that understanding geometric dilation in terms of areas was novel to her. She had been taking it as a given that dilation was about scaling dimensions or lengths instead of areas. Ms. N further explained where this discrepancy between her understanding and the students' understanding might come from. (Excerpt 43).

Excerpt 43—Memo 53, 54

- 1 Ms. N: I think it goes back to the language because if you say, "something is twice
- 2 as big as the other." What does that mean for something to be twice as big?
- 3 It could mean area, you know, or like it could mean— like it could
- 4 mean...like the dimensions are twice as big. Like what makes something
- 5 twice as big as something else? I think it's just that language is very
- 6 confusing. (...) I think when we say twice as big like I think it's not clear
- 7 to... twice as big like what does that mean for something to be twice as big?
- 8 (...) So, I think, again, like this just helps me so much with implementing
- 9 this lesson in the future, just like really stop and make sure students know
- 10 what does it mean for it to be twice as big? Like, what is he really wanting?
- 11 I might want them to maybe talk about like, can you draw me two things
- 12 where one is twice as big as the other? You know, what was that mean?

Here, Ms. N reflected that there had been a lack of reference to the measurable attribute when someone said "something is twice as big as the other." As she said, it could mean areas, and it could mean dimensions (Line 3-4). I argue that Ms. N's ability to distinguish these two meanings of dilation suggested a modification in her FOMs through Type 4 Decentering. That is, before Ms. N's interactions with these two students, she had taken it as a given that dilation was about

dilating dimensions or lengths, meaning this meaning was backgrounded in her thinking; after the interactions, this meaning had become foregrounded in her thinking, and therefore she considered that allowing students to clarify their meaning of dilation, including supporting their attention to the length of a shape was critical for their construction of dilation (Line 8-12).

Task 2: Triangle Shadow

In the second activity called Triangle Shadow, Ms. N turned off the light of the classroom and worked with the two students to produce a shadow of a triangle cutout on the whiteboard with a flashlight. As shown in Figure 51, Heidi was holding a triangle cutout, and Carlee was holding a flashlight to project a shadow of the triangle on the whiteboard. Ms. N started with asking them to move the triangle and the light source to observe how the shadow changed. Carlee and Heidi observed that when the triangle got closer to the light, the shadow became bigger, and that when it got away from the light, the shadow got smaller. Ms. N asked them to explain why this happened, and Excerpt 44 illustrates Ms. N's interpretation of Carlee's response.

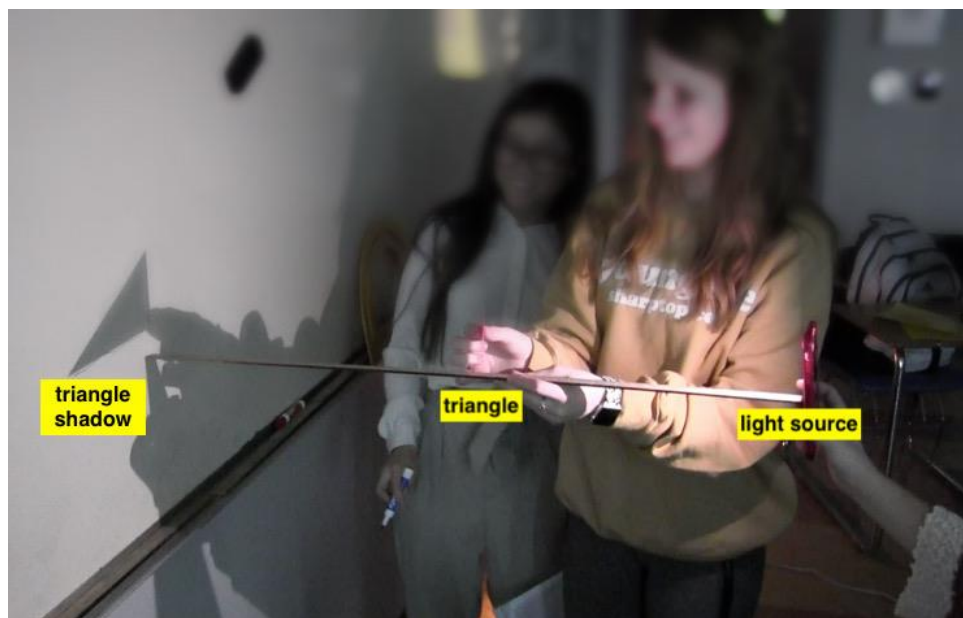


Figure 51. The Triangle Shadow activity (without the ruler in their initial exploration).

Excerpt 44—Memo 62

- 1 Ms. N: She's saying the light is emitting some sort of like— it's emitting some sort
2 of light rays. And when the triangle is closer to the flashlight, it's blocking
3 more path, like the traveling path of the ray. That's why the shadow appears
4 much bigger. Whereas if it's far away, then it's not taking up as much of
5 the travel path of the ray. It's blocking less of the light ray from traveling.
6 I don't know if that's the reason, but I could see that making sense, because
7 you can imagine like, let's say, a flashlight has 100 rays coming off of it.
8 And if you're closer to it, then you're covering more of those 100 rays,
9 rather than being far away. Like if you're like, imagine if your hand is right
10 on top of these [*see gestures shown in Figure 52, left*], then is blocking all
11 100 rays. And if you're a little further away [*see gestures shown in Figure*
12 *52, right*], that is blocking maybe like 98 of the rays, the rays are like,
13 coming out like that, because the ray is not traveling straight across. I think
14 it's more like it's going out like that. And then so like when you're further
15 away, you're only blocking the ones that's going straight rather than like
16 the one that's going in different directions.



Figure 52. Ms. N was demonstrating Callie's explanation.

Ms. N was modeling Carlee's explanation for why getting closer to the light source produced a bigger shadow. Carlee reasoned that when the triangle moved closer to the light source, it blocked more light rays than when it moved away from the light source, and therefore it produced a bigger shadow. Although Ms. N was unsure if that was really the reason, but she was able to fully make sense of Carlee's logic (Line 6-7), suggesting she had constructed a SOM of Carlee's thinking through Type 3 Decentering.

As the conversation continued, Ms. N asked the students where they would locate the triangle such that the shadow was twice as big as the triangle cutout. In response, they put the

triangle cutout in the middle between the light source and the whiteboard (see Figure 51), and Ms. N drew the outlines the shadow on the whiteboard. Comparing the shadow outline and the triangle cutout in her hand, Heidi commented that the shadow looked much bigger than a doubling. Ms. N said that she was confused by Heidi's comment during the teaching session, but now she was able to make sense of it because both students ultimately attended to area instead of side length (Excerpt 45).

Excerpt 45—Memo 61

- 1 Ms. N: Now looking back at it, I see it, but like when I was in the moment, I was
- 2 like of course it is two times as big, like just look at the size. I think like
- 3 looking at it now, because I was so focused on the dimensions, and now
- 4 that we've talked more about the idea of area that I just realize, Oh yeah, I
- 5 can see what you mean.

This excerpt suggested Type 2 Decentering had occurred because Ms. N was able to assimilate Heidi's utterance with her previously constructed SOM.

Then, Carlee and Heidi had a discussion on the relationships between the dimensions of the triangle shadow and those of the triangle cutout (see their work in Figure 53). Carlee claimed that from the cutout (on the right) to its shadow (on the left), each side length was doubled and added by .75 inches. She added that the measurement of 10 inches should be 9.75 inches in order for that relationship to hold. Excerpt 46 illustrates Ms. N's comment on this thinking.

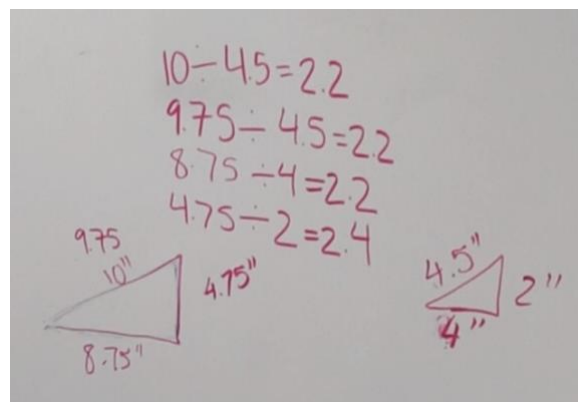


Figure 53. Measurements of the triangle cutout (right) and its shadow (left).

Excerpt 46—Memo 64

- 1 Ms. N: So, they understand the idea here is that you have to be consistent. So, like
2 going from 4 to 8.75. If you're multiplying by two and adding 0.75 more,
3 then the other sides have to be the same, which is true, but they're not— but
4 this is strictly in terms of multiplication. What they are doing is sort of like
5 half multiplication and then half additive [*laughing*].

Ms. N acknowledged the students' understanding that the relationship should be consistent across different pairs of corresponding sides. However, she expressed surprise that these students conceived of dilation in terms of multiplication and addition. This reminded her about the examples she used in her classroom teaching when teaching the concept of similarity (see Figure 54 for an example), where the numerical values “set them up to think about multiplication.” In Excerpt 47, Ms. N discussed what kind of examples should be used to support student learning of similarity and dilation.

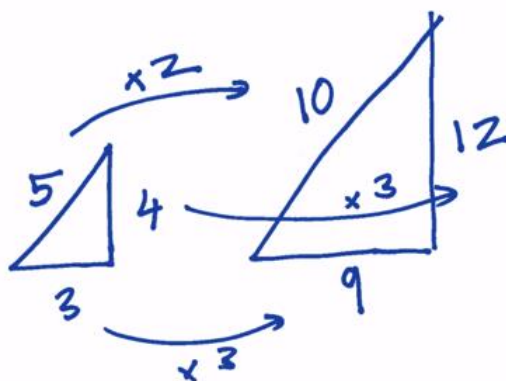


Figure 54. An example Ms. N used in her classroom teaching.

Excerpt 47—Memo 65

- 1 Ms. N: If I give students something like this [*pointing to Figure 53*], I want them
2 to see that everything has to be multiplied by the same scale factor. But I
3 think I never thought about giving them an example where it's like they
4 have to stop and think, like, are these similar? You know, and what makes
5 them similar?

Ms. N said that she should have selected numerical values that could provide students with opportunities to consider what relationships are actually preserved among similar geometric

shapes. Because the example shown in Figure 53 allowed multiple ways of thinking, it enabled students' discussion of similarity in terms of multiplicative relationship as opposed to other types of relationship, while the one shown in Figure 54 might not reveal diverse perspectives or motivate a similar discussion. Ms. N's reflection suggested the productivity of constructing SOMs. Instead of using mathematical examples that only reflected her FOMs, she became attentive to make pedagogical decisions based on her SOMs. Specifically, she considered to design mathematical tasks that would bring forth diverse student thinking, engender perturbations in student thinking, and support students' reflection on multiple perspectives.

Task 3: Relating the Two Tasks

Upon completion of the discussion on the two triangles on the whiteboard, Ms. N asked Carlee and Heidi to produce a drawing on a paper to illustrate the Triangle Shadow activity. They drew a diagram shown in Figure 55 and emphasized that the triangle cutout (the triangle in the middle of the drawing) had equal distance from the light source and the shadow (or the whiteboard) and that the shadow was twice as big as the cutout.

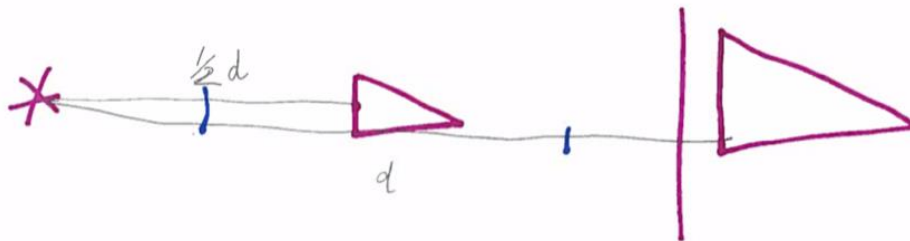


Figure 55. Carlee and Heidi's drawing of the Triangle Shadow activity.

Attempting to support the students relating this activity with their activity in the context of Burnell's photocopy, Ms. N asked them to determine the "light source" of the copy machine such that the image was a quadruple of the preimage. Carlee drew a dot on the upper left corner of the paper (see Figure 56), but Heidi suggested that it should be closer to the preimage. Heidi

referred to Figure 55 to show that the triangle cutout needed to be in half-way between the light source and its original location in order for the shadow to be four times as big as the cutout (see the marker of “ $1/2d$ ” in Figure 55). She further described that the left blue tick mark represented one fourth, the middle triangle represented one half, and the right blue tick mark represented three fourths of the total distance from the shadow to the light source.

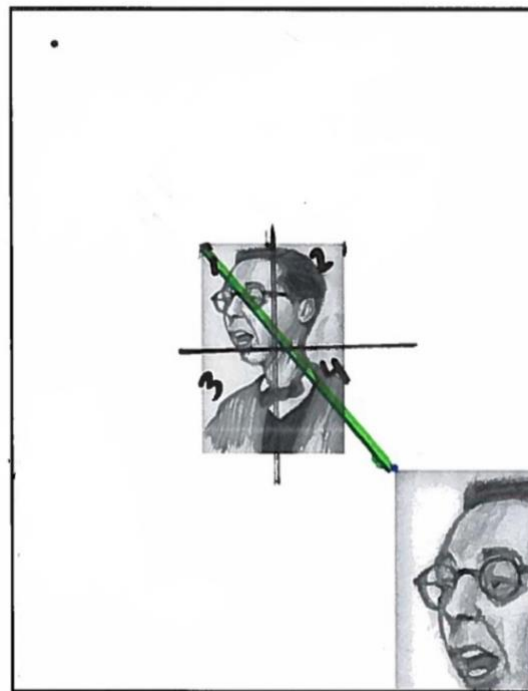


Figure 56. Carlee’s label of the “light source” of the copy machine on the upper left corner and Heidi’s reference to the distance between the preimage and the image (the green segment).

Following up on Heidi’s discussion, Ms. N asked the students where they perceived the one-fourth in Figure 56. Excerpt 48 illustrates the two students’ conversations.

Excerpt 48—Memo 26

- 1 Carlee: Really, we can just compare the image to the distance as a whole, because
- 2 we can see that this is a whole distance [see her gesture in Figure 57, left]
- 3 and this is a whole image [see her gesture in Figure 57, right], so we have
- 4 to relate the distance to the picture basically.
- 5 Ms. N: [Turning to Heidi] Do you have anything to add here?

6 Heidi: Well, like maybe like measure this [*pointing to the green segment in Figure*
7 *56 that connected two corresponding vertices of the preimage and the*
8 *image*] so like the image [*pointing to the middle triangle in Figure 55*] is
9 like halfway, so measure this from here to here [*pointing to the green*
10 *segment in Figure 56*], that's like three fourths of the way till the light
11 [*pointing to the distance between the left blue tick mark and the shadow*
12 *triangle in Figure 55*]. So, went another fourth. Somewhere like here
13 [*pointing to somewhere near the upper left corner of the preimage*], but this
14 area [*pointing to the green segment in Figure 56*] is three fourths of the total
15 length.

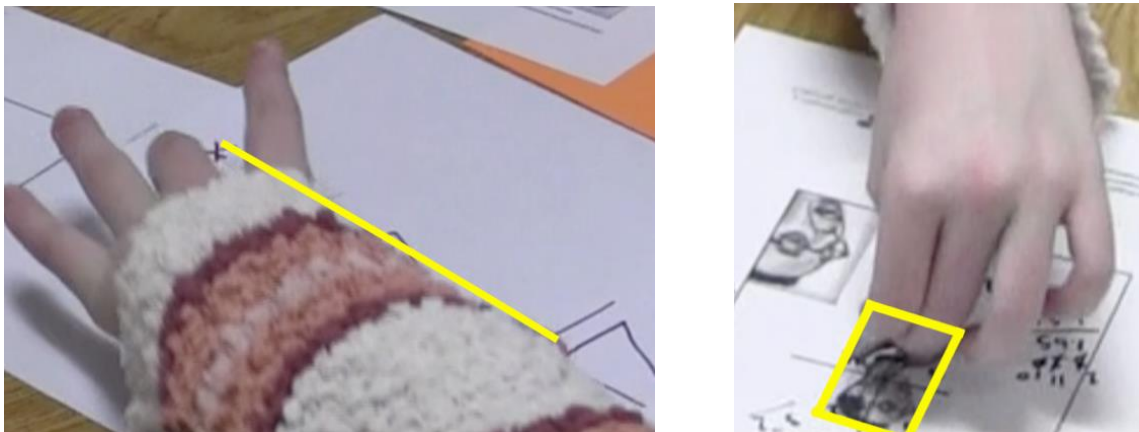


Figure 57. Carlee's gestures for relating the "whole" distance to the "whole" image.

Watching this video episode, Ms. N expressed confusion about Carlee's response, talking to herself, "What is she trying to say? I am so confused. [*pausing*] Were she may be thinking about the dimension? I was confused because she kept doing like [*mimicking Carlee's gesture shown in Figure 57, right*], she kept talking about like, she keeps doing this with her finger [*mimicking Carlee's gesture shown in Figure 57, right*]." I characterized Ms. N's wondering as an indication of Type 1 Decentering. She was perturbed by Carlee's utterances and gestures and could not infer the underlying reasoning that could explain Carlee's actions. Later, Ms. N admitted that she did not pay attention to Carlee's statement in-the-moment because she was very excited about the connections that Heidi was able to make. As she watched the episode again and again, Ms. N started to understand Carlee's activity (Excerpt 49).

Excerpt 49—Memo 67

- 1 Ms. N: Because in the light source activity, we were talking about distance at one
2 dimensional. The distance from one thing to another. And she was going
3 back to here [*pointing to the preimage at the center of Figure 56*], and she
4 was saying the whole picture, but the whole picture is not one dimensional,
5 is two dimensional.
- 6 Biyao: But that whole picture is parallel to?
- 7 Ms. N: The whole distance...Ahh! If you take one quarter of that, it's getting
8 smaller. Yeah, I see it. Because like, the way that we want students to think
9 about is this is not the whole [*pointing to the preimage at the center of*
10 *Figure 56*]. The whole is the distance from the center to the image. That's
11 what Heidi got. But then Carlee was thinking the whole as in the picture
12 itself. Hmm.
- 13 Biyao: And you have one fourth of a whole picture.
- 14 Ms. N: Yeah. It's like, in this case, we want students to see that, that image, the
15 original [*pointing to the preimage at the center of Figure 56*], is the same
16 as the cutout triangle. And the photocopy [*pointing to the image at the*
17 *lower right corner of Figure 56*] is the shadow. And the center of dilation
18 is the light source. But she was saying this "a whole" in a completely
19 different way where—
- 20 Biyao: Complete the sentence.
- 21 Ms. N: What she just took it as the whole is the original and like the photocopy was
22 one fourth of the original. [*pausing*] Wow.

Ms. N eventually removed her perturbation through Type 3 Decentering by understanding that Carlee was conceiving the "whole" distance from the shadow and the light source as being equivalent to the "whole" photo itself (Line 11-12), and one fourth of that "whole" distance was equivalent to one fourth of the original photo (Line 21-22). By contrast, Ms. N associated the triangle cutout with the original picture (i.e., the preimage), the triangle shadow with the photocopy (i.e., the image), and the flashlight with the light source of the copy machine (i.e., the center of dilation); from her perspective, the "whole" was consistently referred to the distance between the image and the center of dilation in both contexts. Ms. N's utterances suggested she

was consciously aware of these differences (Line 8-11, Line 14-22), meaning she had engaged in Type 4 Decentering.

As the teaching session proceeded, Ms. N continued to ask the two students to work together to locate the “light source” of the copy machine. I note that this move was not sensitive to the different understandings held by these two students because Ms. N did not discern that difference at the time. In response, Carlee seemed to pick up on Heidi’s comment about a distance being three fourths of a total distance. Carlee suggested that they should extend the green segment by making three more copies (gesturing along the segments PA , AB , and BC annotated in Figure 58, left) and that the center of dilation would be outside the paper (see Point C in Figure 58, left). To re-present Carlee’s activity, Ms. N produced the drawing shown in Figure 14 (right) and described that from her perspective, the scale factor was actually four thirds because CP' was four thirds of CP .

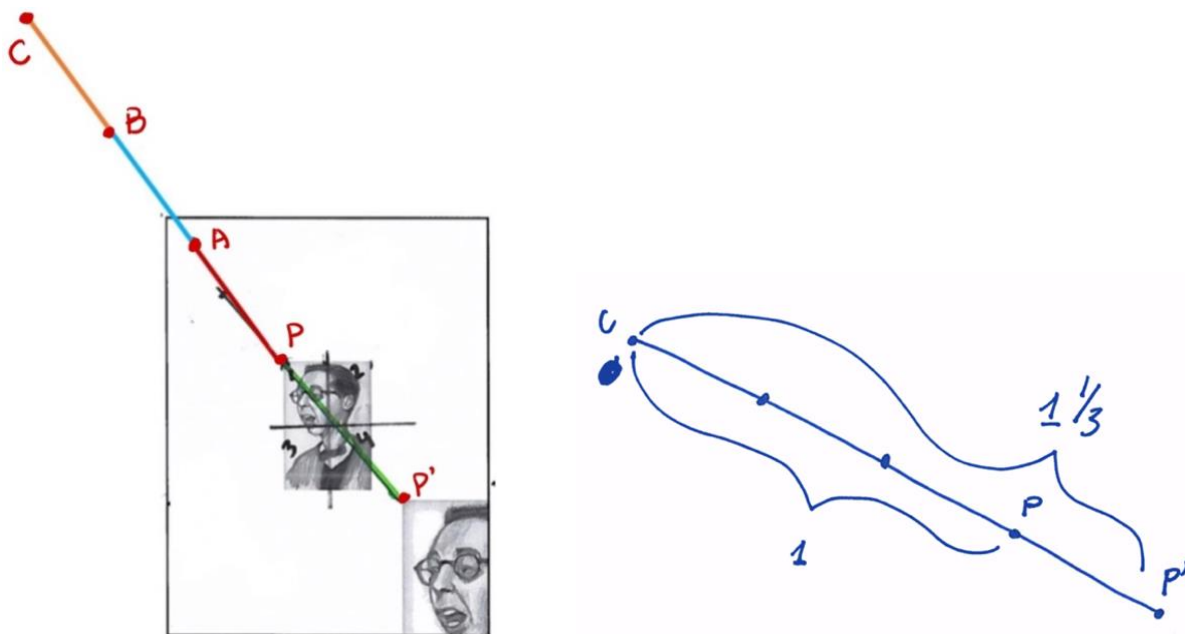


Figure 58. (left) An illustration of Carlee’s activity and (right) Ms. N’s re-presentation of Carlee’s activity.

Note. C corresponds to the center of dilation; P corresponds to the upper left vertex of the preimage, and P' corresponds to the upper left vertex of the image.

Alternatively, Heidi reiterated that, in order for the scale factor to be 4, the green segment PP' needed to be three fourths of the distance between the light source and the image P' (see her previous quotes in Excerpt 48, Line 6-15). Therefore, to locate the light source, they needed to extend the green segment by “another fourth.” Because Carlee was holding the ruler and pen at the time, Heidi had to give her instructions to draw the center of dilation. She asked Carlee to draw a point at the middle of the red segment AP (see Point C in Figure 59, left, where AP equals PP'). Ms. N created another drawing to re-present Heidi’s thinking (Figure 59, right, with blue ink) and claimed that the actual scale factor was three as opposed to four.

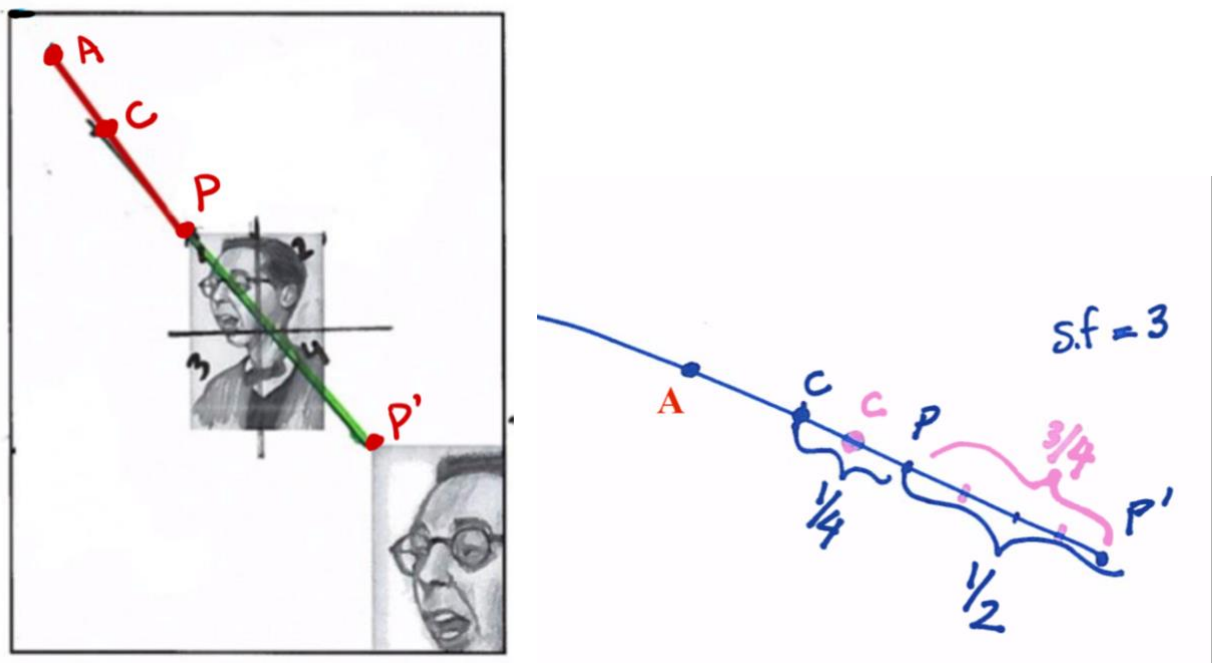


Figure 59. (left) An illustration of Heidi’s activity and (right) Ms. N’s re-presentation of Heidi’s (in blue) and her own thinking (in pink).

Towards the end of the reflecting session, I asked Ms. N to show how she would locate a center of dilation such that the scale factor was four. Excerpt 50 illustrates her reasoning.

- 1 Ms. N: So, this would represent that to be three fourths [*drawing the bracket and*
2 *writing “3/4” in Figure 59, right*]. And so, you need one fourth more
3 [*drawing the partitions between P and P’ in Figure 59, right*]. So, you need
4 to figure out how much is one fourth and then add one fourth more [*drawing*
5 *the pink dot and labeling it as “C” in Figure 59, right*] (...) Like [Heidi]
6 knew that this [*pointing to the segment PP’ in Figure 59, right*] has to be
7 three fourths, and one fourth more, but then somewhere maybe it was
8 because like she didn’t have the ruler maybe. And that’s why she just takes
9 this as like, a half. Like two fourths, not three fourths.

Ms. N’s activity involved her partitioning PP' into four equal sections and iterating one fourth of PP' to locate the Point C (in pink) (Line 1-5). Ms. N perceived her activity to be consistent with Heidi’s goal of adding “one fourth” onto a “three fourths” (Line 6-7). However, neither of them was aware of that the result of their activity did not align with the relationship they claimed to represent. From a researcher’s perspective, Heidi’s activity implied that CP (with “ C ” being the Point C in blue) was one fourth of “the whole” AP' . And Ms. N conceived of CP (with “ C ” being the Point C in pink) as one fourth of the “the whole” PP' . To accurately locate a Point C such that CP was one fourth of CP' and that PP' was three fourths of CP' , they should have partitioned PP' into three equal sections and added one section onto PP' to locate the center of dilation (see Figure 60).

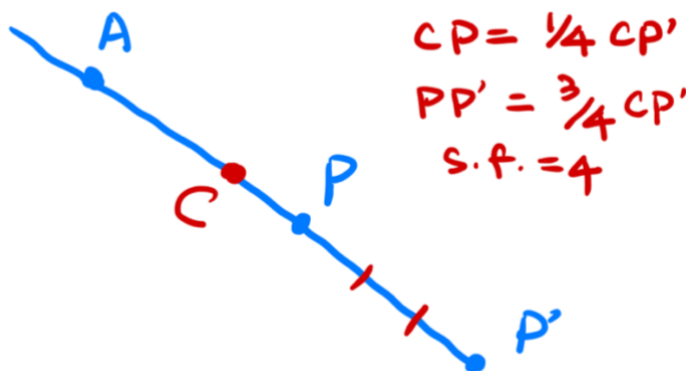


Figure 60. An illustration of the accurate location of center of dilation C .

Cycle 2: Center of Dilation, Preimage, and Image

In summary, Ms. N and my experience in Cycle 1 enabled us to realize that it was nontrivial for students to attend to and operate on distances and coordinated those operations with their physical actions of partitioning. As we planned the second teaching session, we intended to continue to investigate the students' understandings of the relationship between center of dilation, preimage, and image, especially the relationships of the distances between them. As a side note, Ms. N informed me that by the time the students participated in the teaching session, they should have taken some lessons on dilation in her class. Therefore, we designed three tasks as extensions of her classroom instruction: (a) Finding the Image, (b) Finding the Preimage, and (c) Finding the Center of Dilation (Part A). In each task, we provided the students with two of the three components (i.e., center, preimage, and image) and asked them to determine the missing element. Regarding the first task, we asked the students to find the image when the preimage and the center of dilation were given, and we hypothesized that they would not experience difficulty because it was a conventional type of task addressed in school instruction. In the second and the third tasks, we asked the students to determine the preimage and the center of dilation respectively when the other two elements were given. We conjectured that these tasks would challenge the students due to their novelty as each of them required them to reverse their operations and coordinate their physical actions, the results of those actions, and the quantitative structures they intended to re-present. We intentionally designed each task to be a pure geometric construction task independent of real-world contexts, so that we could gain insights into the nuances of their coordination of actions without influences from contexts.

Task 1: Finding the Image

With the first set of tasks, we asked the students to draw the image of a given preimage with different scale factors.

Dilating a Rectangle with a Scale Factor of -1

The first task asked the students to draw an image of a given rectangle $ABCD$, with the center of dilation located within the rectangle and the scale factor being 2.5 (Figure 61). Figure 62 illustrates the two students' work on this task, and Excerpt 51 illustrates Ms. N's comparison of their activities.

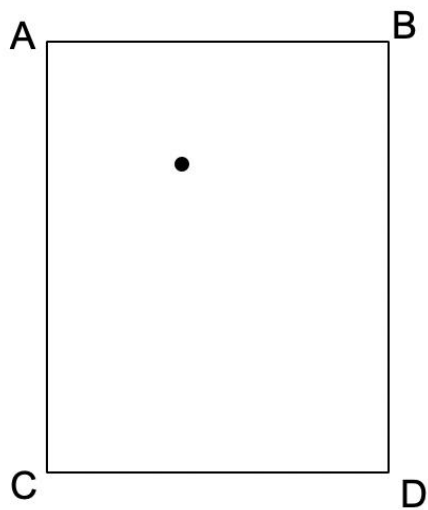


Figure 61. Task 1: Find the image of the preimage $ABCD$ with a scale factor of 2.5.

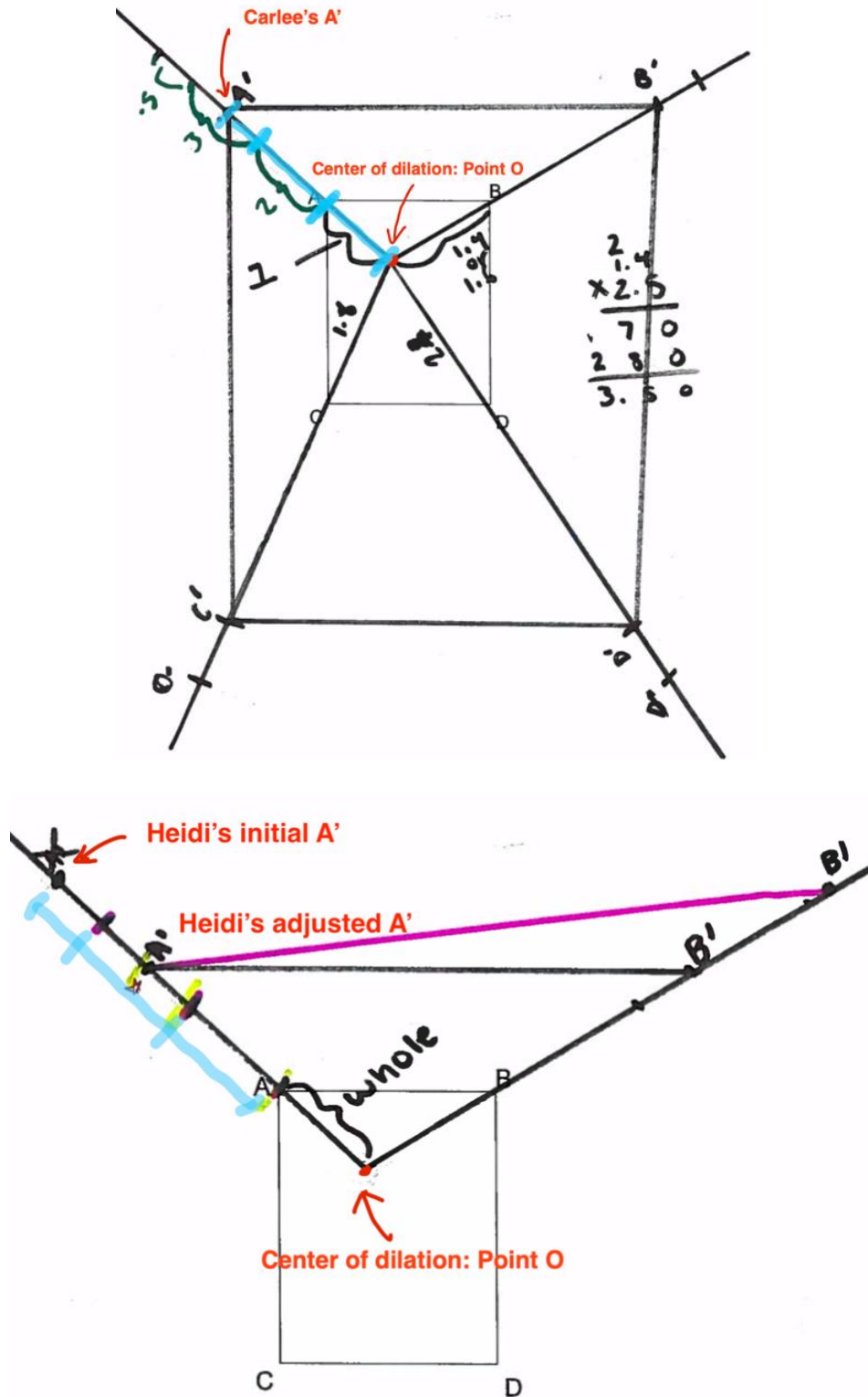


Figure 62. (above) Carlee and (below) Heidi's drawing of image $A'B'C'D'$ (Scale Factor=2.5).

Note. Point O and the light blue segments are annotated for the reader.

- 1 Ms. N: I guess [Heidi] is thinking about dilating out from the preimage rather than
 2 the center of dilation, whereas [Carlee] is thinking dilation from the center
 3 of dilation. So that's like [Carlee] started at the center of dilation, and
 4 [Heidi] started at the preimage.
- 5 Biyao: Okay, so how does that cause them to think about the relationship— Like
 6 here is 2.5, right? How do they see the 2.5 maybe a little bit differently?
- 7 Ms. N: I think, for me, if we're dilating by 2.5 scale factor in itself, it's referring to
 8 something's getting 2.5 times bigger, whereas this one, if we start from here
 9 [*pointing to Point A in Figure 62, below*], for me, that just sounds— just
 10 makes it seem like it's 2.5 times more. I don't know if that makes sense.
 11 This one [*pointing to Heidi's initial A' in Figure 62, below*] is you're
 12 adding 2.5 times this length on to it whereas this one [*pointing to Heidi's*
 13 *adjusted A' in Figure 62, below*] you are multiplying the whole by 2.5 (...)
 14 It makes sense to me that you start from the center of dilation, because you
 15 are enlarging the one that you have, like the length that you have, so you
 16 are taking this original [*putting two fingers at Point O*] and you are
 17 enlarging it out [*moving the left hand finger from Point O to A'*]. So you
 18 are not going to start where it ends [*putting the right hand finger at Point*
 19 *A*], you are going to start with where it starts [*moving the right hand finger*
 20 *back to Point O*] and enlarging it from there.
- 21 Biyao: So, I want to get another distance that is 2.5—
- 22 Ms. N: Times the original. Because like I just imagine looking at a computer. You
 23 have a line on a computer, and you are trying to make it longer. You keep
 24 one end, usually, and you extend the other end out or in. So that's what I
 25 think of when I think about this. I imagine using my mouse click and then
 26 clicking on point [*pointing to Point O*] and dragging it out more [*moving*
 27 *her finger from Point O to A'*]. And I'm keeping the center, like the
 28 endpoint at the center of dilation.

In this conversation, Ms. N articulated contrasting differences between Carlee and Heidi's understanding of dilation, which was an indication of Type 4 Decentering. Heidi imagined dilating from the preimage, and the scale factor of 2.5 meant that the distance from the image to the preimage AA' was 2.5 times the distance from the preimage to the center AO . Ms. N interpreted her thinking as implying that the distance from the image to the center $A'O$ was 2.5 times *more* than AO . In contrast, Carlee conceived of dilation as starting from the center of dilation, and the scale factor of 2.5 referred to the multiplicative relationship between $A'O$ and AO (i.e., $A'O$ was 2.5 times as long as AO). In addition, Ms. N considered Carlee's image of

dilation to be consistent with her own, especially the aspect of sustaining one endpoint when dilating a segment around that point. She discussed this image of dilation in the context of a computer environment where a displayed segment could be stretched out when one endpoint of the segment was fixed (Line 22-28).

Based on my observation of Ms. N's interactions with the students, I conjectured that her ability to model and discern these differences in the two students' thinking was a result of her persistent efforts in asking them to specify which distance was 2.5 of which distance and to use tick marks to clearly show 2.5 on their drawings. In the reflecting session, I asked her about the relationality of these moves in order to gain insights into the extent that they were intentional from her perspective. Excerpt 52 illustrates her response.

Excerpt 52—Memo 97

- 1 Ms. N: I just wanted them to see why is wrong. I want them to see why is the other
- 2 one wrong. (...) I see where [Heidi] is coming from— I see where she's
- 3 saying she would go 2.5 from A. It's just I think— part of me is like it's right
- 4 in a way. It's right within the conception that she has.
- 5 Biyao: Something is 2.5 of something.
- 6 Ms. N: Right. Right. But her conception doesn't fit into [Carlee]'s conception
- 7 because [Carlee]'s conception starts at the center. And so, I think it's hard.
- 8 Like my goal was trying to show them like why it is not that [*pointing to*
- 9 *Heidi's initial A' in Figure 62, below*]. Because it's the scale factor at 3.5.
- 10 But that's only if you understand that it starts— everything starts at the center
- 11 of dilation. Heidi doesn't seem to understand that. And again, it showed up
- 12 in her quiz. She doesn't understand that we are talking about distance and
- 13 comparing distance between— distance is usually we always refer back to
- 14 the center of dilation. Everything is in relation to that. Yeah, you're right,
- 15 like something is still 2.5. So that's why like, I mean, I've talked with [an
- 16 instructor at her university] about this before, and how she said like to a
- 17 student, a misconception is a conception and like, if I can't get them out of
- 18 that conception like that thinking, then I— no matter how much like
- 19 counterexample, like no matter like— doing, taking steps like these, I don't
- 20 think I could shake them out of that if they already bought into it.

Ms. N initial intention of asking both students to visualize the 2.5 was to surface the differences in their thinking, hoping that Heidi would fix her “misconception” (Line 1-2, Line 8-11).

However, she realized that her goal was not achieved; although Heidi could temporarily adjust her actions during her interactions with Carlee, Ms. N observed that Heidi's image of dilating from the preimage persisted in her class work (e.g., assignments and quizzes) (Line 11-14). Reflecting on why her questioning could not engender perturbation and stable accommodation in Heidi's thinking, Ms. N realized that dilating from the preimage was ultimately viable within Heidi's thinking. It was not that Heidi could not see one distance being 2.5 of another distance; it was that she saw 2.5 in a different way than Carlee and Ms. N herself (Line 2-4, Line 15) Ms. N's reference to the quote "misconception is a conception" indicated that she truly believed in the rationality and viability of Heidi's thinking. Ms. N also came to understand that it was her that experienced constraints by the students rather than the other way around. It was her that felt challenging to intervene Heidi's thinking, while from Heidi's perspective, her conception was viable in multiple problem contexts. This is an evidence of Ms. N's Type 4 Decentering as she was able to differentiate the Heidi's perspective from her own perspective.

Building on this conversation with Ms. N, I asked her if she had any conjecture about where Heidi's thinking might be from. Excerpt 53 illustrates her responses.

Excerpt 53—Memo 98

- 1 Ms. N: I wonder if it's like going back to the idea of like, multiplication. Like I
- 2 wonder if she's struggling because like, she doesn't understand the concept
- 3 of multiplying. (...) For example, let's do two times three [*producing the*
- 4 *drawing in Figure 63*]. So, two times three, technically means you start out
- 5 with a group of two. So, if you have two times three ideas, you start out
- 6 with a group of two. And then you're multiplying that group three times,
- 7 but technically in terms of adding, you're only adding it two, like you
- 8 adding two more groups of it. (...) Whereas this one, like, if we were to
- 9 apply Heidi thinking to this situation, what it would mean for her is, you
- 10 will start out with a group two, and you add three more, because that three
- 11 means three more. Not three times the original.

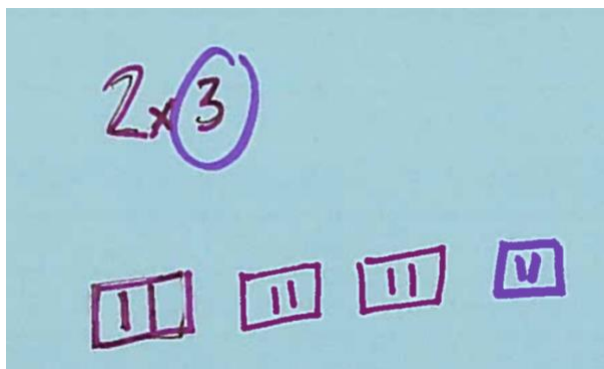


Figure 63. Ms. N's representation of 2 multiplied by 3 as three groups of 2.

Ms. N hypothesized that Heidi's struggle with sustaining the center of dilation was associated with her understanding of multiplication. From Ms. N's perspective, multiplying 2 by 3 meant adding two groups of two instead of adding three groups of two (Line 6-8). In the case of Heidi, if she did understand 2 multiplied by 3 as adding three copies of two, it would not be surprised that she considered a scale factor of 2.5 as adding 2.5 copies of a given distance. This instance suggested another form of Type 3 Decentering, by which Ms. N explained Heidi's understanding of dilation by inferring her understanding of multiplication.

Dilating a Rectangle with a Scale Factor of -1

As the teaching session proceeded, and in order to gain more insights into the students' meanings for dilation, Ms. N asked the two students to create another image of the same rectangle $ABCD$ with a scale factor of -1 . Carlee and Heidi's work are shown in Figure 64. Carlee created the image $A'B'C'D'$ by interpreting the effect of scale factor of -1 as flipping the directed distance from the center of dilation to the preimage to an opposite orientation. For example, the directed distance $\overrightarrow{OA'}$ should have the same magnitude as $\overrightarrow{OA}$ but point towards the opposite direction (see Carlee's drawing in Figure 64, left). Heidi initially had no idea how to start, and she decided to follow Carlee's approach. She determined A' and B' in ways consistent

with Carlee, but she forgot to go to the opposite direction to locate C' and D' , resulting in an image that was a trapezoid rather than a rectangle (see Heidi's drawing in Figure 64, right).

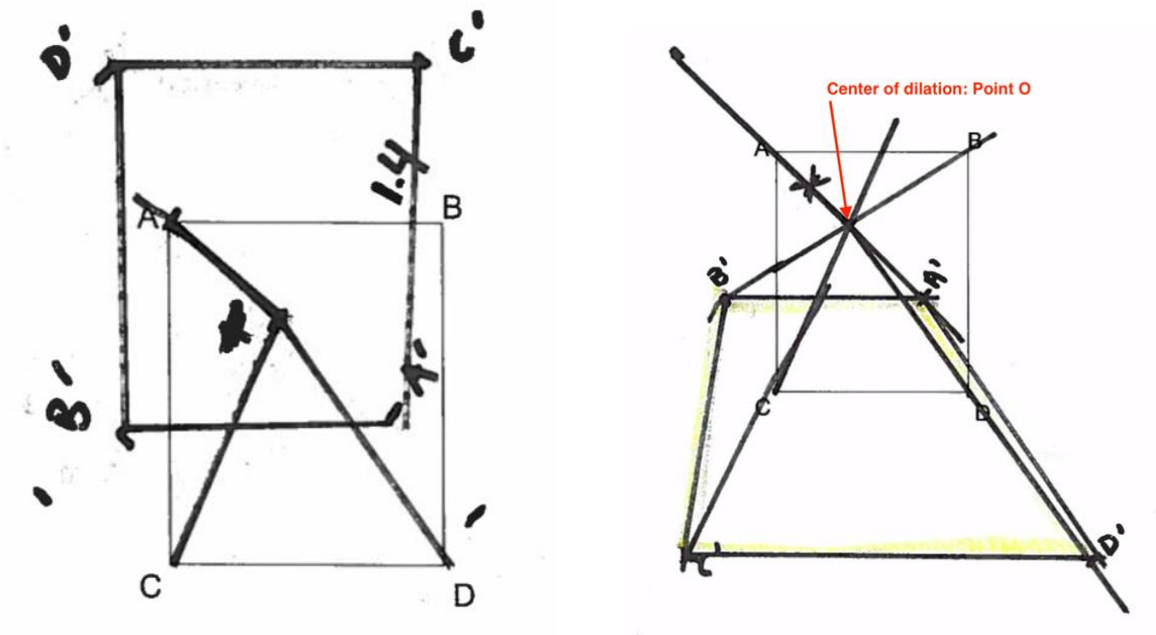


Figure 64. (left) Carlee and (right) Heidi's drawing of the image $A'B'C'D'$ (Scale Factor = -1).

Then, Ms. N asked the two students to consider Point D' on Heidi's drawing and to determine the actual scale factor if it was not negative one. Both students initially responded that the scale factor was 1. Then, Carlee put a ruler on the diagram (see Figure 65), and immediately, both students said the scale factor was 2. Excerpt 54 illustrates Ms. N discussion on this instance.

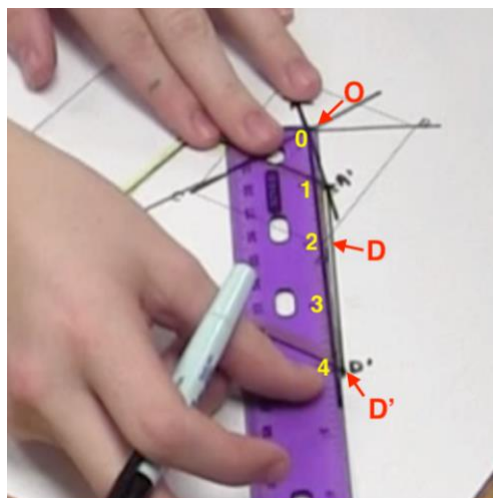


Figure 65. Carlee put a ruler on Heidi's diagram.

Excerpt 54—Memo 100

- 1 Ms. N: Again like, I think going back to this idea she's still stuck on the mindset
- 2 of I need to dilate from the preimage not the center. Because just like the D
- 3 she dilated, she thought it was one because this [*pointing to Segment DD' in Figure 64, right*] is one more than that one [*pointing to Segment OD, also see her gesture in Figure 64, right*]. So again, like she's stuck on that
- 4 mindset that I have to dilate from the preimage out.
- 5
- 6
- 7 Biyao: So why does the ruler help?
- 8 Ms. N: I think like they just saw it as like this is two inches or something like that
- 9 [*pointing to Segment OD*] and this is four inches [*pointing to Segment OD'*].
- 10 (...) I think to see like length of things help. To see the length that this is
- 11 two [*pointing to Segment OD*] and this length is four [*pointing to Segment OD'*]
- 12 whereas like we don't see that the distance between D and the center
- 13 is that long [*pointing to Segment OD*] and distance from here to here is that
- 14 long [*pointing to Segment DD'*].
- 15 Biyao: Yeah. It makes me wonder why it is difficult for them to— like without
- 16 seeing the values, why is it so difficult for them to see two?
- 17 Ms. N: Yeah, I agree. I don't know either. Maybe it's because like without the
- 18 ruler, if you want to get twice the length of this [*pointing to Segment OD*],
- 19 you are measuring how long this is [*see her gesture in Figure 66, left*] and
- 20 you are adding on to it [*see her gesture in Figure 66, middle*]. So you are
- 21 adding like— I measured this [*see her gesture in Figure 66, left*] and add on
- 22 one more length that this was one [*see her gesture in Figure 66, middle*].
- 23 (...) because like without a ruler you can't— because like I don't know what
- 24 double this length will look like [*see her gesture in Figure 66, left*] without
- 25 copying it out [*see her gesture in Figure 66, middle*]. Does that make sense?
- 26 Because like double this I would have to kind of like— well double this will
- 27 be like [*keeping her middle finger at Point O and moving her thumb away from the middle finger; also see Figure 66, right*], but I wouldn't sure. You
- 28 know what I mean? I have to copy this [*see her gesture in Figure 66, left, and then Figure 66, middle*]. And then—
- 29
- 30
- 32 Biyao: Move it over.
- 33 Ms. N: Right.

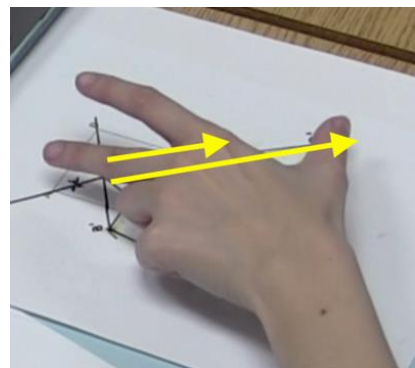
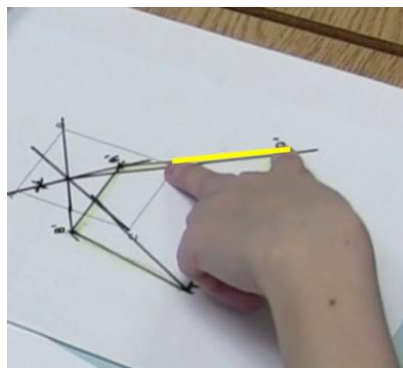
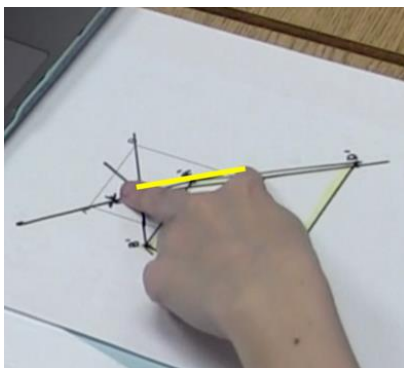


Figure 66. Ms. N's gestures of an image of (left and middle) copying a distance versus (right) stretching a distance.

Ms. N considered the students' initial response of the scale factor being 1 to be an additional evidence that they were thinking of dilation additively (Line 1-6). This was an example of her SOM becoming stable and explanatory for the students' actions through Type 2 Decentering. Furthermore, Ms. N explained that the numerical values displayed on the ruler allowed the students to perceive the accumulated distance of OD' to be 4 inches and the distance of OD to be 2 inches, which supported them perceiving a scale factor of 2 (Line 8-14). When the numerical values were unavailable, their thinking was foregrounded by their physical actions of iterating a distance and conceiving DD' as *one* additional copy of OD , and the idea that the accumulated distance OD' was *twice* as long as OD was less apparent (Line 17-30). I interpreted that Ms. N had modified her SOMs through Type 3 Decentering by making an additional inference to explain the students' activities. Recall that, in Excerpt 53 (and Figure 63), she inferred that the students' conception of dilation in terms of dilating from the preimage was associated with their conception of multiplication. Here, she was making an alternative explanation that the students' difficulty with conceiving the multiplicative relationship between distances might be partially due to their thinking being dominated by the iterating actions they carried out and relied on operating on concrete measures of the distances to construct that relationship.

Classroom Instruction Informed by the Study

In the upcoming planning session, Ms. N shared with me a task she designed for her classroom instruction, where she incorporated the aforementioned reflection about students' meanings for dilation (Figure 67). In this task, she provided with students a center of dilation C , a preimage segment AB , and its image $A'B'$. She asked her students to choose the true statements

that captured the relationships between AB and $A'B'$ (Statement I), $A'A'$ and CA' (Statement II), and CB and CB' (Statement III). It is noteworthy that she specified in the task that the segments were “not drawn to scale”, and Excerpt 55 illustrates her discussion on her rationale.

Use the following diagram to answer Question 1.

NOTE: not drawn to scale

1) The scale factor is $\frac{3}{4}$ and the center of dilation is C. CIRCLE ALL true statements.

I. $\overline{AB}$ is equal to $\frac{3}{4}$ the distance of $\overline{A'B'}$.

II. $\overline{A'A}$ is equal to $\frac{1}{3}$ the distance of $\overline{CA'}$.

III. $\overline{CB}$ is $\frac{4}{3}$ times the distance of $\overline{CB'}$.

Figure 67. A task Ms. N designed for classroom instruction.

Excerpt 55—Memo 88

- 1 Ms. N: You know why I did this and like some of them kind of like, why, well, I
- 2 would have gotten that right if it was drawn to scale. And I was like, no,
- 3 you are right. And I was like actually, even people who has never done
- 4 dilation can get that question right if it was drawn to scale. You know what
- 5 I mean? It's like, well, yeah, like duh you are going to get— like if it was
- 6 drawn to scale then it is a super easy question. Because it's not ask— like
- 7 require you to think about the meaning of dilation at all. (...) You are just
- 8 eyeballing it, you are not really deeply thinking about the concepts like the
- 9 three fourths, like, why is it? Like, why is this one third, or why is this four-
- 10 thirds? You know? That doesn't match up with the scale factor at all.

Ms. N explained that asking students to reason with segments that were not drawn to scale could facilitate their understanding of dilation and the quantitative relationships involved. When these segments were drawn to scale, students did not have the intellectual need to draw on knowledge

of dilation or attend to the scale factor to solve the problem, and they might only rely on direct measurement and its numerical results to choose the true statements (Line 6-10). I note that Ms. N designed this task completely on her own. It was exciting to see that her second-order modeling activities during the study was consequential to her decision-making in her classroom instruction. It was through modeling Carlee and Heidi's thinking that Ms. N realized it was nontrivial for students to operate on the relationship between displayed segments or distances when their measures were unavailable. Therefore, she intentionally designed tasks that would engender perturbation in the students' thinking and require students to reason with distances' relationships. Again, this was an evidence that Ms. N's SOMs became pedagogically powerful in her teaching.

Task 2: Finding the Preimage

With the second task, Ms. N asked the students to draw a preimage of the image $A'B'C'D'$, with the a center of dilation being within the image and a scale factor being 2.5 (Figure 68).

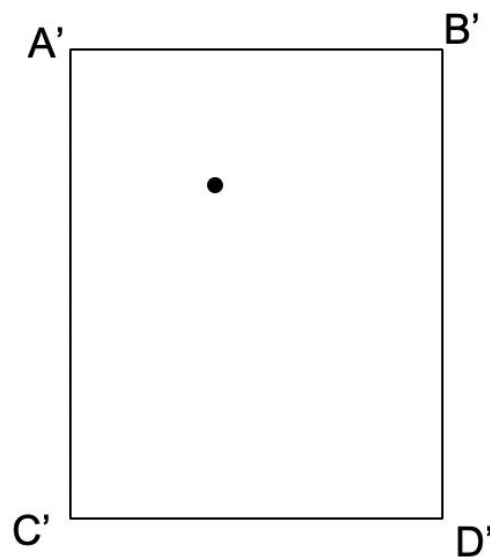


Figure 68. Task 2: Find the preimage of the image $A'B'C'D'$ with a sclae factor of 2.5.

Carlee and Heidi's Initial Solutions

Carlee interpreted this task as a “backward” of the previous task, and she produced a preimage shown in Figure 69 (left). Excerpt 56 illustrates Ms. N’s interpretation of her activity.

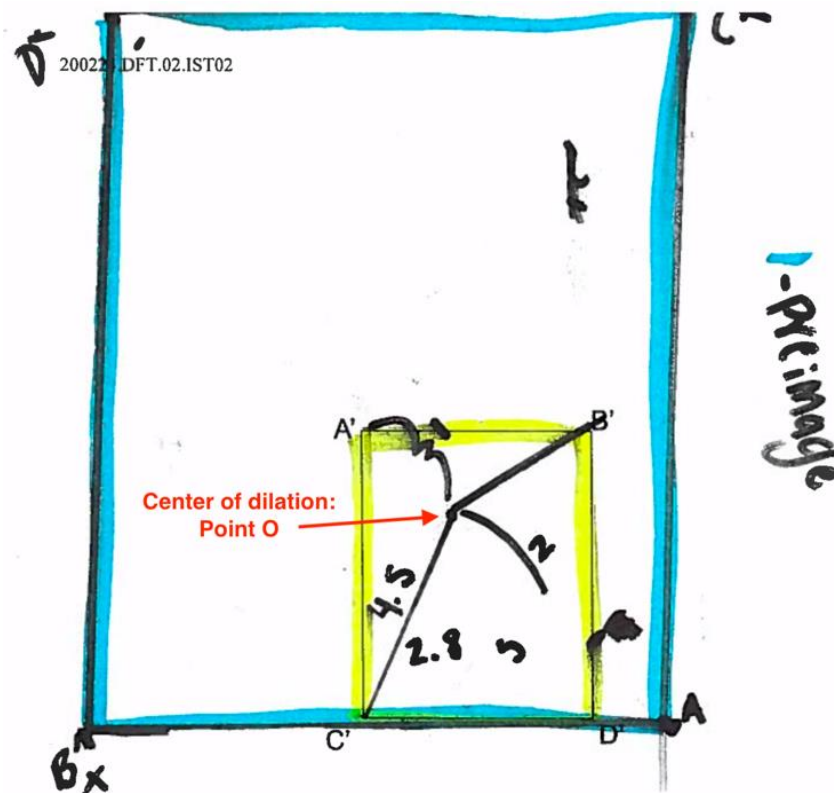


Figure 69. Carlee’s drawing of the preimage $ABCD$ with a scale factor of 2.5 (Point O was annotated for the reader).

Excerpt 56—Memo 102

- 1 Ms. N: Yeah, I think she said this scale factor is 2.5, so you have to go backwards
- 2 2.5 from the center of dilation the other direction. She was thinking more
- 3 like negative 2.5. ‘Cause I guess like backwards, for her like— since we are
- 4 on backwards to find the preimage, backwards also means like the other
- 5 direction. (...) So, like, instead of going this way [*moving her finger from*
- 6 *the center of dilation to Point A’ in Figure 69*] you are going the other way
- 7 [*moving her finger from the center of dilation to Point A*]. (...) I think she
- 8 took it literally like you need to go backward.
- 9 Biyao: Yeah, literally.
- 10 Ms. N: Yeah, but what we are thinking is more like, you need to like, thinking
- 11 about it backwards. Like, here’s the image, where is the preimage.

Ms. N interpreted that Carlee was making an association between two meanings of “backward”:
 (a) finding the preimage given the image was a “backward” of finding the image given the preimage, and (b) dilating the directed distance $\overrightarrow{OA'}$ by going towards an opposite direction. Drawing on her FOM, Ms. N considered that Carlee was actually dilating the rectangle $A'B'C'D'$ by a scale factor of -2.5.

In comparison, Heidi created a drawing shown in Figure 70. Upon completion of her drawing, Heidi tested her solution with a ruler. She realized that the scale factor was 2 instead of 2.5, but she was unsure how to adjust her diagram. To explain why Heidi could anticipate the preimage to be smaller than the image but could not draw the preimage with an accurate scale factor, Ms. N inferred that Heidi likely generalized her experience in Task 1, saying, “I think she was thinking well before we also had 2.5 [*referring to the Figure 62, above*]. But the preimage was between the image and the center of dilation, so why should it be different here.”

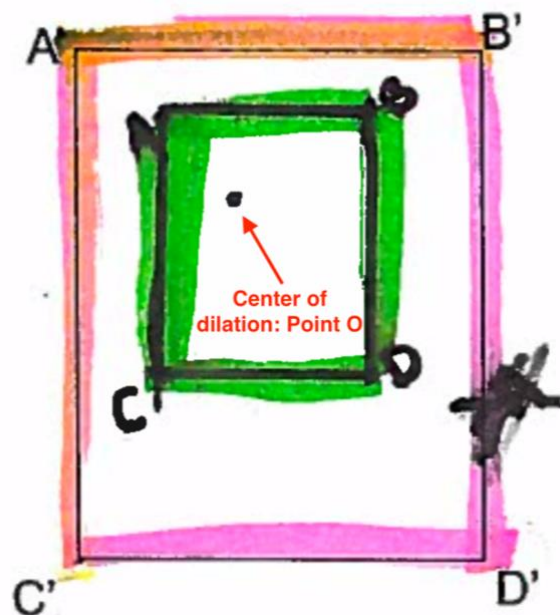


Figure 70. Heidi's drawing of the preimage $ABCD$ with a scale factor of 2.5 (Point O was annotated for the reader).

A Revision

As the teaching session proceeded, Ms. N attempted to intervene and support the students reconstructing a preimage. She put the two drawings (Figure 69 and Figure 70) side by side and described that she saw one preimage bigger than the image $A'B'C'D'$ and one smaller. Then, she asked the students, “what does it mean by a scale factor of 2.5?” Carlee immediately responded that it meant the picture became larger and that Heidi’s drawing was correct. Heidi admitted that her picture was actually a scale factor of 2, using her fingers to show AO and AA' was equal. Then, Ms. N asked, “how could you modify your work to be a scale factor of 2.5?” Both students proposed that they needed to shrink the rectangle $ABCD$ that Heidi drew. To gain more insights into their thinking, Ms. N offered them a new worksheet and asked them to redraw a Point D as a preimage of the Point D' . Carlee measured OD' to be 2 inches, and she divided 2 by 2.5 to obtain 0.8. She asked Heidi to draw the Point D at 0.8 inches away from the center. Carlee also annotated where she perceived 2.5 by specifying that OD' , which was measured as 2 inches (drawing the arrow in Figure 71), consisted of two 0.8 inches and one 0.4 inches (see “1,” “2,” and “.5” labeled in Figure 71).

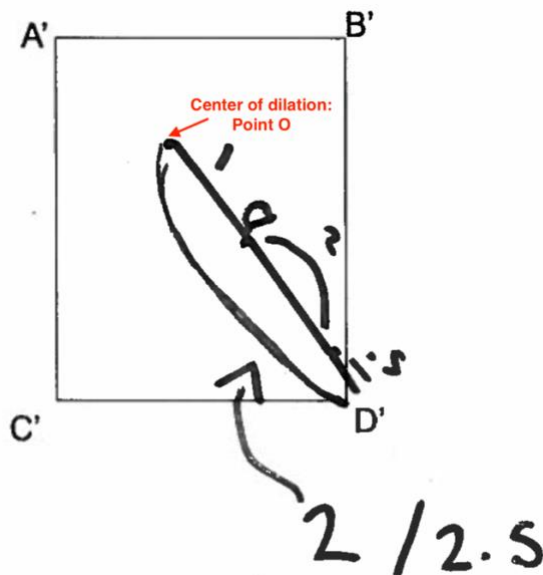


Figure 71. Carlee and Heidi's rework on Task 2 (Point *O* is labeled for the reader).

When asked to explain why she used division, Carlee said that “because we were looking for the preimage not the image.” Ms. N hypothesized that Carlee might consider division as a “backward” or inverse operation of multiplication. In the previous task, the students used multiplication to locate the image when the preimage was given, so in the current task, they used division to find the preimage when the image was given. Ms. N admitted that she was really unsure about why Carlee used division, saying “I don’t know what makes [Carlee] think to divide. I should have asked that. But I don’t know why she divided— but I don’t think she was thinking in terms of like, the same way that I did.” Inferring that Ms. N conceptualized the situation differently, I asked her to explain her own thinking.

Ms. N's Solution

In Excerpt 57, Ms. N described how she would solve the task.

Excerpt 57—Memo 110

- 1 Ms. N: Like if this is the center of dilation [*drawing the center of dilation in Figure*
- 2 72]. And this was the [image] [*drawing the Point A' in Figure 72*]. Two
- 3 times would mean— like for me, I didn’t think about in terms of division
- 4 [*pointing to “2/2.5” in Figure 71*]. I like very visually did this in my head,
- 5 but I think 2.5 means, there’s two, two wholes and a half. Like a whole
- 6 [*drawing the left bracket and labeling it as “W” in Figure 72*], another
- 7 whole [*drawing the middle bracket and labeling it as “W”*], and then a half
- 8 [*drawing the right bracket and labeling it as “1/2”*]. So that half would fit
- 9 into this whole length, one, two, three, four, five times [*drawing the four*
- 10 *tick marks between the “Center” and “A’”*]. So, it will be one fifth of that.
- 11 That’s what I was thinking in my head. I wasn’t thinking, take two and then
- 12 divide it by 2.5.
- 13 Biyao: Okay. Okay. And how would you find— I think they talked about the point
- 14 eight or something. What does that refer to?
- 15 Ms. N: It’s just the actual distance. Because the whole thing is [*pointing to Segment*
- 16 *OD’ in Figure 71*— I think they said it was two inches. So, this was like,
- 17 point eight of an inch [*pointing to Segment OD in Figure 71*], point eight
- 18 of an inch [*pointing to the segment labeled as “2” in Figure 71*] and point
- 19 four of an inch [*pointing to the segment labeled as “.5” in Figure 71*].
- 20 Biyao: And how does that connect to here [*pointing to Figure 72*]?

- 21 Ms. N: So, this will be point eight [pointing to one of the segments labeled as “ W ”
22 in Figure 72]. This will be point four [pointing to the segment labeled as
23 “ $1/5$ ” in Figure 72].
24 Biyao: Okay, and then you have five—
25 Ms. N: Point four. Right. Yeah. And you have two and a half point eight that fits
26 into the whole length.

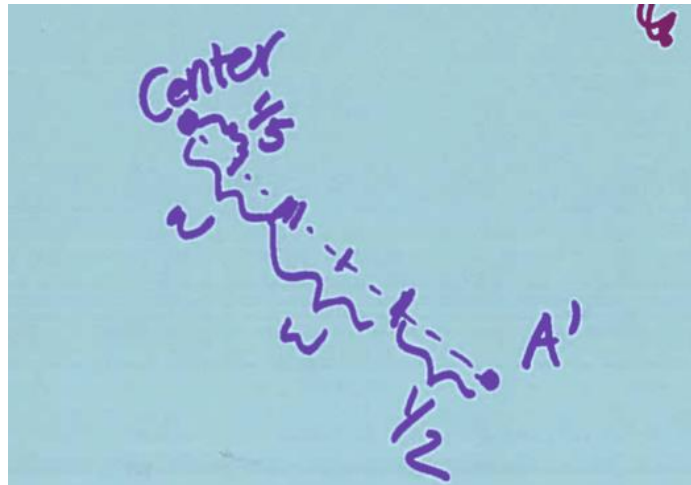


Figure 72. Ms. N’s illustration of her personal solution to Task 2.

Ms. N’s reasoning involved her imagining the Segment OA'^{11} (with a measure of 2 inches) being partitioned into 5 sections with each section having a measure of 0.4 inches. She did not seem to make a connection to Carlee’s calculation of $2/2.5$ (Line 3, Line 11-12), although she knew that both ways of understanding led to the same result (i.e., Point D is .8 inches away from the center).

To promote Ms. N to make that connection, I asked her if she could find a way to make sense of Carlee’s calculation, given that she did produce a consistent result. Pausing for a moment, Ms. N recalled the N-M-P multiplication model she learned in her pre-service teacher education program (Excerpt 58).

Excerpt 58—Memo 116

¹¹ The Point A' in Ms. N’s diagram corresponded to the Point D' in Heidi’s diagram. For some reason, Ms. N used a different letter as a generic name for an image point.

- 1 Ms. N: 2.5 is representing the group number, number of groups [writing
2 "2.5" above "m" in Figure 73], and the product amount is two
3 [writing 2 above "P"]. So, you are really finding how many are in
4 one group when you're dividing.
5 Biyao: Which is?
6 Ms. N: Which is point eight. Yeah, point eight inches in one group, and there
7 are 2.5 groups.
8 Biyao: So how does this point eight in one group relate to this context of
9 dilation? What does that mean?
10 Ms. N: The point eight is the whole. (...) 2.5 would mean you are taking the
11 whole and enlarging it by 2.5. And so like, the whole in this case will
12 be the one group, like how many in one group. Like the number of
13 groups in here will be the scale factor [writing "s.f." under "m"]. This
14 will be like how long is like one whole [pointing to "n"]. And then this
15 will be whatever [pointing to "P"].
16 Biyao: When you say that "how long is one whole", could you speak of the—
17 Ms. N: Oh, okay, like center to preimage [writing "center of p.i" under "n"].
18 And then this will be the center to image [writing "center of image"
under "P"].
19 Biyao: That's very cool. Have you made that connection before?
20 Ms. N: No. I have not. I have not.

The diagram shows the equation $2/2.5 = 0.8$ at the top. Below the numbers, the variables are defined: n is labeled "# in 1 group" and "Center of p.i"; x is the multiplication symbol; m is labeled "# of groups" and "s.f."; $=$ is the equals sign; and P is labeled "center of image".

Figure 73. Ms. N's reconstruction of $2/2.5 = 0.8$ using the N-M-P model.

Ms. N's activity suggested she had reconstructed a meaning for Carlee's division, $2/2.5 = 0.8$.

Using the N-M-P model, Ms. N considered the number of units in one group, N , to be associated with the distance from the center to the preimage (measured as 0.8 inches), the number of groups, M , to be associated with the scale factor of 2.5, the product amount, P , to be associated

with the distance from the center to the image (measured as 2 inches). I argued that this was a novel mathematical construction through Type 4 Decentering. Ms. N experienced reorganization in her mathematical meanings by injecting coherence between her meanings for geometric dilation and meanings for multiplication (and division). Although I did not get a chance to ask Ms. N to relate this discussion to her drawing shown in Figure 72, I conjectured making such a connection would not be a difficult construction to her. As evidenced in Excerpt 57 (Line 25-26), she did conceive of her diagram as representing 2.5 groups of 0.8 inches making “a whole” 2 inches. What was missing at the time was why a division operation made sense regarding these three values, and now she should be able to conceive that relationship.

Task 3: Finding the Center of Dilation (Part A)

The third task asked the students to determine the center of dilation given one point of a preimage, its image, and a scale factor of $\frac{1}{4}$ (Figure 74). Both students’ work is presented in Figure 75.

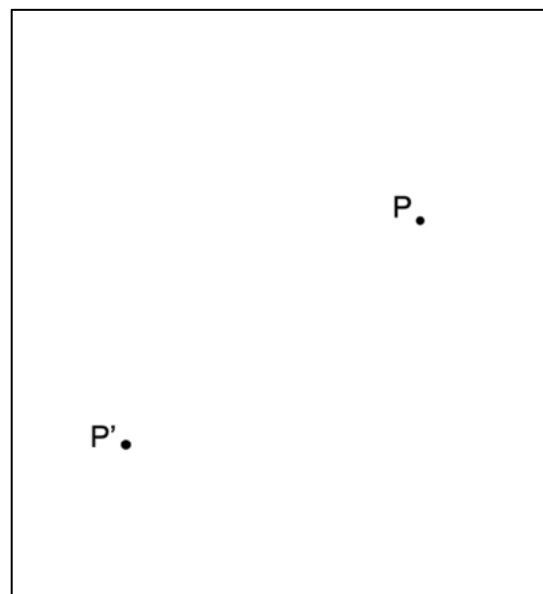


Figure 74. Task 3: Finding the center of dilation (rectangular frame indicates the boundary of the worksheet).

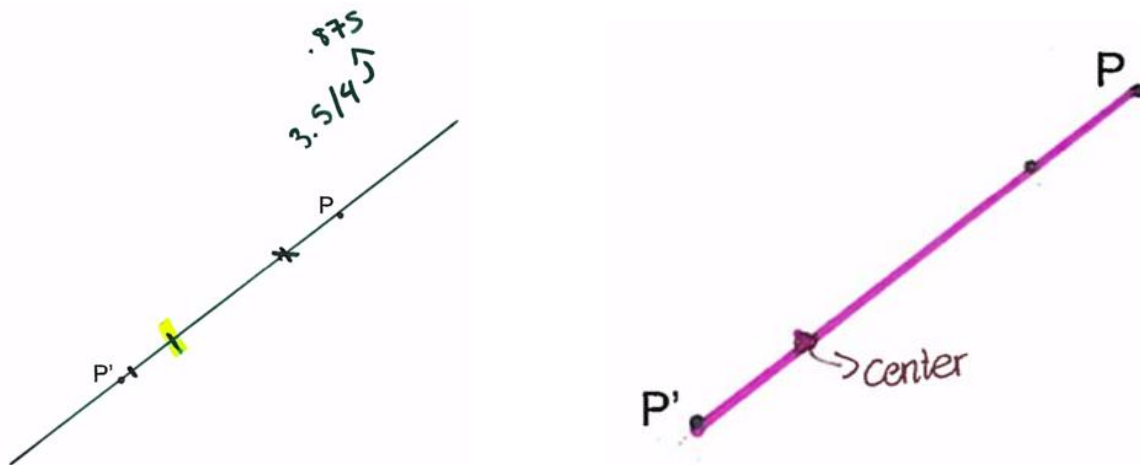


Figure 75. (left) Carlee and (right) Heidi's solution to Task 3.

Ms. N interpreted that both students considered PP' —which Carlee measured to be 3.5 inches—as one whole—and located the center of dilation in between P and P' such that the distance from the center to P' is one fourth of the distance of PP' . For example, Carlee calculated $3.5/4 = 0.875$ and determined the center of dilation at 0.875 inches from Point P' (see the yellow tick mark in Figure 75, left). Excerpt 59 illustrates Ms. N's discussion on the students' activities.

Excerpt 59—Memo 117

- 1 Ms. N: I think there was some incorrect understanding in there because like if
- 2 they said that's the center, then that would make the distance from the
- 3 center to P' shorter than the distance from the center to P , which is what
- 4 you want, if you're dilating by a scale factor of one—like smaller than one.
- 5 So, you want a distance from center to image to be shorter than distance
- 6 from center to preimage. And it is. But this is not the scale – And this is
- 7 the thing like, I think like we have the same issue over here, because
- 8 [Carlee] was trying to do it. When we asked her to find the preimage by a
- 9 scale factor of 2.5. She did negative 2.5. And this is a negative scale factor
- 10 [pointing to Figure 75, right]. But they didn't see that.
- 11 Biyao: Yeah, and it's not even negative one fourth. It's like negative—
- 12 Ms. N: Yeah, negative one third. Yeah, negative one third.

Ms. N drew attention to two aspects of the students' actions. First, their actions suggested they conceived of a scale factor of one fourth as implying the distance from the center to P' being

smaller than that from the center to P (Line 1-6). Second, their solution did not correspond to a scale factor of $1/4$ but a scale factor of $-1/3$ (Line 12). This reminded her about Carlee's prior activity where she conflated a negative and positive scale factor (see Excerpt 56 and Figure 69, left).

Ms. N's Solution

Then, I asked Ms. N to illustrate how she would solve the task (Excerpt 60).

Excerpt 60—Memo 112

- 1 Ms. N: So I would think about it as this is one fourth of this whole thing [*writing*
2 *"1/4" and drawing the bracket of CP in Figure 76, left*]. Yeah, but since I
3 only have this [*pointing to the Segment PP'*] then that means the center to
4 P [*drawing the bracket above "4/3"*] is going to be four thirds time the
5 distance of P' to P [*partitioning $P'P$ into three equal parts and writing 4/3*
6 *of $P'P$*]. So, I would have measured this [*pointing to the Segment $P'P$ in*
7 *Figure 75, right*], take a third of it and then add it on to P' [*see gesture in*
8 *Figure 76, right*]. I would have measured how long this was [*pointing to*
9 *the Segment $P'P$ again*]. Take a third of that, and then add on one third
10 more [*see gesture in Figure 76, right*].
- 11 Biyao: Why one third?
- 12 Ms. N: Because like I said [*referring back to Figure 76, left*], like the whole C to
13 P is four third time P' to P , which means C to P' is one third of P' to P .
- 14 Biyao: And where does this come from [*pointing to the "4/3"*]?
- 15 Ms. N: This four-thirds? It stands for if you're taking the distance from P to P' and
16 divide it into three parts [*pointing to the tick marks in between P' and P in*
17 *Figure 76, left*], then the distance from C to P would, it will have four of
18 those one thirds. Like the one third, one third right here [*pointing to the far*
19 *right section of CP*]. So there's four of them. One, two, three, four, one
20 third [*pointing to each equal section of CP one by one from the right to the*
21 *left*].

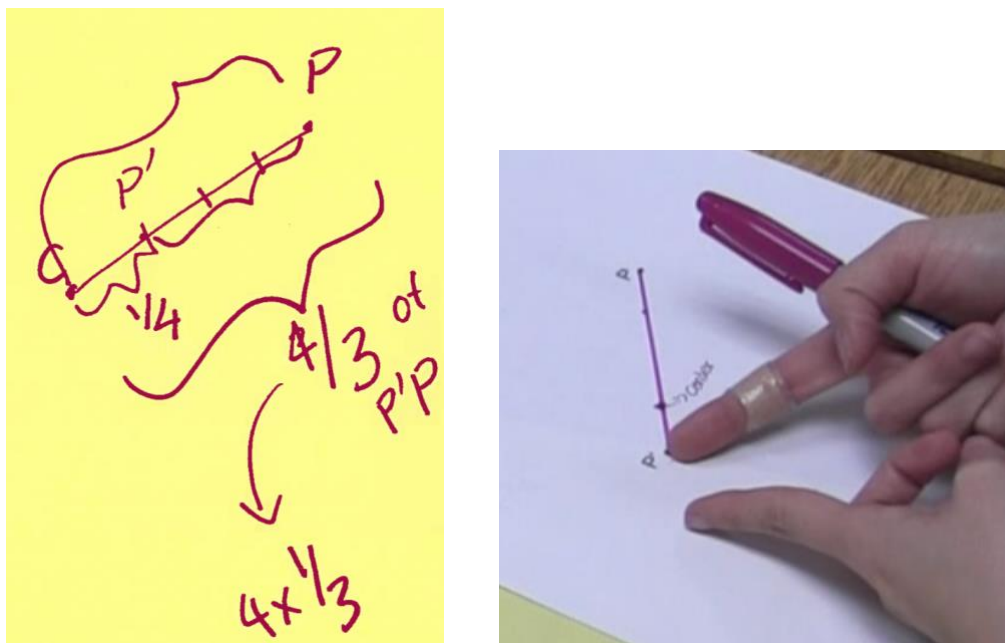


Figure 76. (left) Ms. N's solution to Task 3 and (right) her gesture showing adding one third of $P'P$ on to P' .

Ms. N conceived of a scale factor of $1/4$ as implying that CP' was $1/4$ of CP (Line 1-2). This relationship further implied that there should be four copies of one third of $P'P$ fit into CP , making CP to be $4/3$ of $P'P$ (Line 3-5, Line 15-21). Therefore, in order to locate the center of dilation C , Ms. N partitioned $P'P$ into three equal parts and iterate one part on to Point P' (Line 6-10). I considered Ms. N's reasoning entailing sophisticated fractional reasoning and ability to coordinate, partition, and iterate units. Compared to Carlee and Heidi whose activities were foregrounded by operating on available measures and segments (see Figure 65 and Figure 75), Ms. N's activities suggested her actions, especially the actions of partitioning $P'P$ and iterating one third of $P'P$, were governed by the quantitative structures and relationships she anticipated to re-present. She could coordinate her physical actions, mental actions, and the perceptual results of those actions such that they entailed consistent relationships between distances.

Ms. N's Attempt to Intervene

In response to the students' explanation related to Figure 75, Ms. N drew a Point P and a point labeled as “center” in Figure 77 (left) and asked them to find the image of Point P with a scale factor of $1/4$. She discussed her rationale for this move in Excerpt 61.

Excerpt 61—Memo 122

- 1 Ms. N: I just wanted them to see if they change their mind if I give them the
2 center and the preimage. Because this is like going backwards
3 [pointing to Figure 75, right], whereas this one is going forward
4 [pointing to Figure 77, left]. So, I was thinking, would they change
5 their mind? Because that's what I've been doing with a lot of these.
6 It's like showing where their thinking contradicts a previous problem.

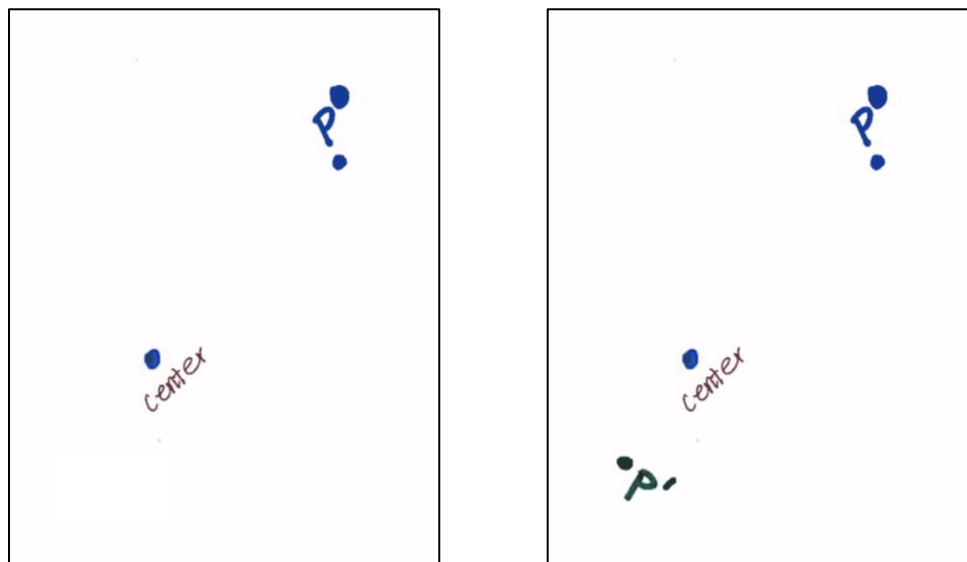


Figure 77. (left) A follow-up task and (right) Carlee and Heidi's solution.

In making this move, Ms. N attempted to engender perturbation in the students' thinking. Specifically, she believed that the students would be able to appropriately find an image given a center and preimage since it was “going forward”, and she conjectured that engaging in this follow-up task might allow them to realize that the relative position of the three points in their current solution were inconsistent with that in their solution to Task 3.

In response to this new task, Carlee and Heidi measured CP to be 2.5 inches and then located Point P' such that CP' was $1/4$ of 2.5 inches (see Figure 77, right). In Excerpt 62, Ms. K described what she learned from her observation.

Excerpt 62—Memo 117, 119

- 1 Ms. N: I'm very confident that they could have done a scale factor of one fourth
2 if we had given them this first [*pointing to Figure 77*]. And that brings
3 back the idea because you know how the math education world is trying
4 to move away from the word "misconception." Because to the student is
5 still a conception, it's not a misconception. And this shows that students,
6 whatever that conception they have, they will try to make everything fit
7 that conception. 'Cause like, this is the conception they had [*pointing to*
8 *Figure 75, right*], and they were trying to be consistent [*pointing to*
9 *Figure 77, right*]. So, like, what misconception is weird because like
10 there's still some consistency like, there's still like, patterns in it like and
11 it makes sense, you know, within that conception and make sense,
12 because they were trying to fit it to this idea [*pointing to Figure 75,*
13 *right*]. (...) It was really interesting. I really liked this activity where I
14 could see like, wow, that's so crazy that like students will make
15 everything fit their conception (...) and then when I asked them what is
16 the whole they still said P' to P and it's like it's not consistent with
17 they're doing because like— Like I said, I think if we had done this the
18 opposite direction, this first [*pointing to Figure 77, right*] and this second
19 [*pointing to Figure 75, right*], they would have be able to find P'
20 correctly. (...) It's just like for this one [*pointing to Figure 77, right*]
21 they— I think they try too much to make this fit [*pointing to Figure 75,*
22 *right*].

Conceptualizing Ms. N's activities from the lens of decentering, her Type 4 Decentering in this instance was evidenced by her comparing her SOMs and FOMs and comparing her SOMs of the students in different contexts, each of which generated perturbations in Ms. N. First, Ms. N was perturbed by the students being unable to solve a conventional problem type appropriately (Line 1-2, Line 17-20). The students' actions were contraindications of her SOMs constructed in her previous experience of working with the students. That is, she anticipated the students to be able to construct an appropriate relationship between CP' and CP when the preimage P and the center of dilation C were given, but the students' actions suggested they could not. Second, what was

more surprising to Ms. N was that the students conceived of their activities in the two related tasks to be consistent. From the students' perspective, they considered *PP'* as “the whole” of $\frac{1}{4}$ in both situations (Line 15-17), and they enacted similar actions and produced perceptually similar results (e.g., measured the length of a given distance, divided that measure by 4, and located a point). By contrast, from Ms. N's perspective, the students did not treat *PP'* as “the whole” of $\frac{1}{4}$ in the second task since *CP* was the distance that they divided by 4. Such comparison allowed Ms. N to realize that the inconsistency or perturbation was only conceived from her perspective but not Carlee or Heidi's, and thus what she considered to be a “misconception” was in fact a viable and coherent conception to the students (Line 9-13).

Ms. N's Final Reflection

Towards the end of the reflecting session, Ms. N summarized her take-aways by reflecting on her observations holistically (Excerpt 63).

Excerpt 63—Memo 120, 121

- 1 Ms. N: Yeah, the fluency, like that requires so much conceptual understanding of
- 2 being able to go back and forth, like going— start from center, image, scale
- 3 factor, and going back to preimage. Like, I think giving a preimage, a
- 4 center and a scale factor, like they're so used to that, they're so used to
- 5 getting the result, like calculate this. Just like the same thing with— if we
- 6 do two plus three equals five like they know two plus three is equal to five,
- 7 but if you try to go backwards with two plus what equals to five, then they
- 8 have trouble of like, thinking backwards. 'Cause like I think the students
- 9 are so used to like, perform, like given something, like perform this action
- 10 and not think about like, the underlying meaning of it. So, like, going
- 11 backward is just hard for them. (...) And I think that's exactly what
- 12 happened here. Like and again like I'm making an assumption here
- 13 because I am giving the benefit of the doubt that like, if we are giving them
- 14 the center and the preimage, they would have been able to do the— find the
- 15 images. But when you're giving them the preimage and the image, find the
- 16 center, is hard. And like they're so used to finding the center, giving two
- 17 sets of points, and then you just draw lines and whatever they intersect is
- 18 the center. They're not used to pay attention to the relationship between
- 19 the distance.

Reflecting on the students' activities, Ms. N became aware of the importance of reversibility. She considered that a student's ability to reverse their actions of dilation (e.g., finding the image given a center and a preimage versus finding the center given a preimage and image or finding the preimage given a center or image) was an indication of them having constructed quantitative relationships between distances in the context of dilation (Line 8-10, Line 18-19). Carlee and Heidi's struggle with reversing their actions, including coordinating the results of measurement with actions of partitioning and iterating was a contraindication of such construction; from Ms. N's perspective, their actions did not preserve the relationships they intended to represent. At the end of this excerpt (Line 16-19), Ms. N also referred to a common class activity where the students were asked to locate a center of dilation given a preimage $ABCD$ and image $A'B'C'D'$ (Figure 78). She became realized that a student's ability to locate the center by intersecting lines that connect corresponding points did not necessitate their construction of the relationship between distances. In other words, a teacher should not take for granted that students have constructed such a relationship if they could enact this activity to determine the center of dilation.

I claim that Ms. N's decentering actions (Type 4 Decentering in particular) afforded reorganization in her FOMs (including SOMs), through which she could make explicit differentiation of the quantitative relationships involved in dilation from the physical actions that produce a certain result (e.g., locating a center of dilation, an image, or a preimage). Given her observation that students' thinking foregrounded by physical actions were limited in multiple contexts, she foregrounded reversible operations on distances and their relationships and considered them as critical markers for conceptual understandings of geometric dilation (Line 1-3).

Figure $A'B'C'D'$ is a dilation of figure $ABCD$. Determine the center of dilation.

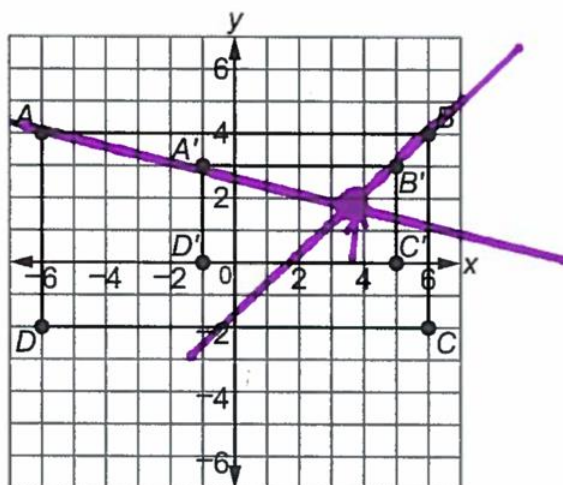


Figure 78. A task example of determining a center of dilation given a preimage $ABCD$ and an image $A'B'C'D'$.

Cycle 3: Properties of Dilation

In Cycle 3, Ms. N and I only completed a planning session, and the experiment was suspended. In the following, I describe three tasks we designed, (a) Dilation Statement (b) Finding the Center of Dilation (Part B), and (c) Area Versus Side Length. Regarding each task, I describe the problem statement, design rationale, and implementation protocol. Although we designed all three tasks in the planning session of Cycle 3, we anticipated the students to struggle and expected the tasks to be covered by two or three more cycles.

Task 1: Dilation Statements

In this task, we intended to present students with two drawings (Figure 79) and ask them to choose *all* the true statements (from two sets of statements shown in Figure 80) with respect to *each* drawing. The two drawings were designed based on Task 3 of Cycle 2. Figure 79 (left) was consistent with the students' solution (see Figure 75) and Figure 79 (right) was consistent with Ms. N's solution (Figure 76). Based on our SOMs of the Carlee and Heidi, they were

comfortable with operating on distances and their relationships when the preimage, image, and center of dilation were given. Therefore, we intended to use this task to engage them in reasoning with the relationships involved and hopefully differentiating the two situations.

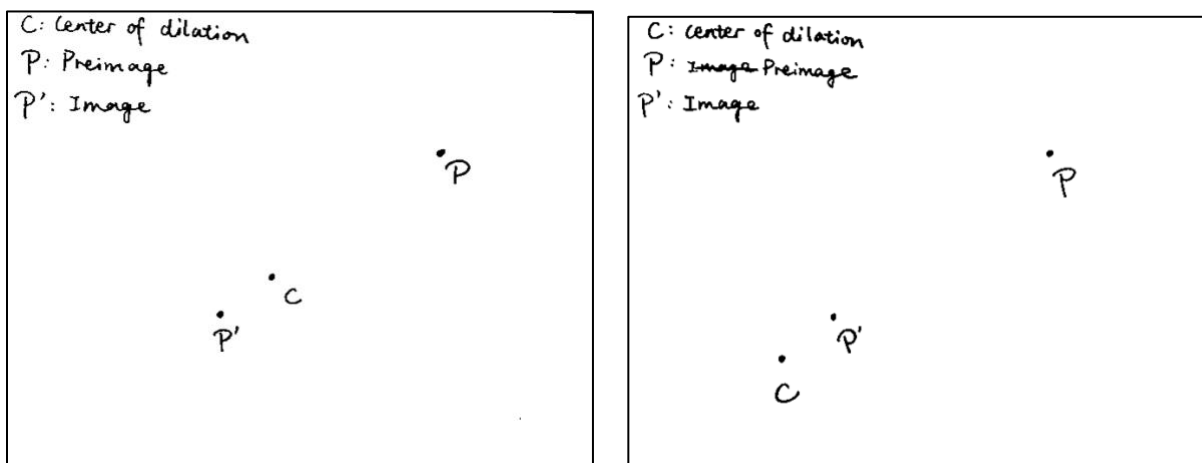


Figure 79. Task 1: Two drawings.

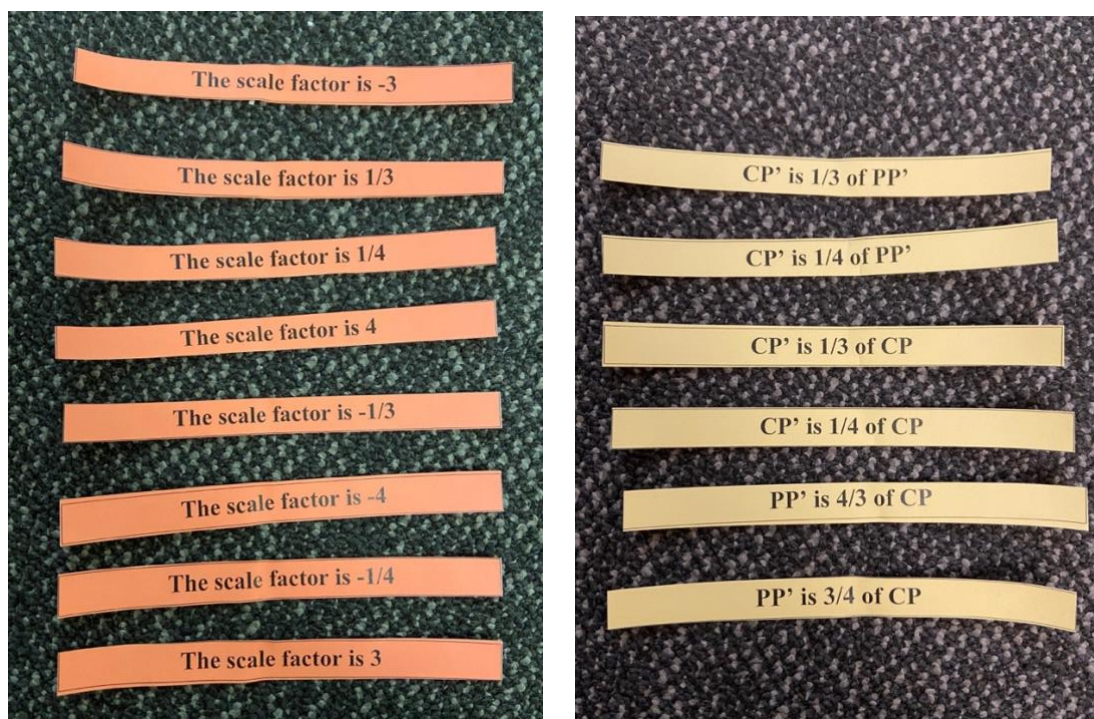


Figure 80. Task 1: two sets of statements.

To better achieve our goals, we decided not to allow the students to use rulers but instead provide them rubber bands and Wikki Stix (Figure 81). The students could choose whichever

they preferred. Both materials are easy to be bended or cut, which afforded the students' actions of partitioning, comparing, and iterating. The students could also use markers to draw tick marks on them to indicate partitions without actually cutting them into pieces. Since no measures are available on these materials, the students likely had to operate on the distances displayed on a rubber band or a Wikki Stix and their relationships (e.g., a section of a Wikki Stix in relation to another section of the same rubber band) in order to choose the appropriate statements.



Figure 81. Rubber bands and Wikki Stix.

To implement the task, Ms. N would first ask the students to choose the true statement about scale factor for each drawing. Then, Ms. N would ask the students to choose the true statements about distances' relationships. They do not need to categorize all of the statements, and they can use one statement for both drawings if appropriate. With respect to each task, Ms. N would also prompt them to explain how they determine the scale factor and describe in detail any physical actions they carry out and any mathematical relationships they conceive. She would also offer them extra copies of the two drawings if they would like to draw on the paper. Lastly, Ms. N would come up with some hypothetical scale factors (assuming that the two drawings were not drawn to scale) and ask the students to anticipate and justify some distance relationships without physically carrying out the activity. For example, regarding Figure 79 (left), she would ask, "If

the scale factor was $-1/11$, what relationships between CP , CP' and PP' could you describe?”

This activity would allow Ms. N to gain insights into the extent the students could reason with the relationships between distances with a given scale factor.

Task 2: Finding the Center of Dilation (Part B)

Task 2 is also adapted from Task 3 in Cycle 2 (see Figure 74). We changed the position of the two points to make it appear to be a novel task. Then, Ms. N would ask the students: “Given a Point P and its image Point P' , could you find the center of dilation that would best fit a scale factor of $1/5$?” Depending on the students’ progress on the task, Ms. N might ask them to consider a scale factor of 5 , -5 , or $-1/5$. The goal of this task was to understand if the students’ engagement with the previous task, including their experiences of operating with distances in relation to multiple scale factors, was helpful for them to reverse their actions to determine a center of dilation when the preimage and image were given.

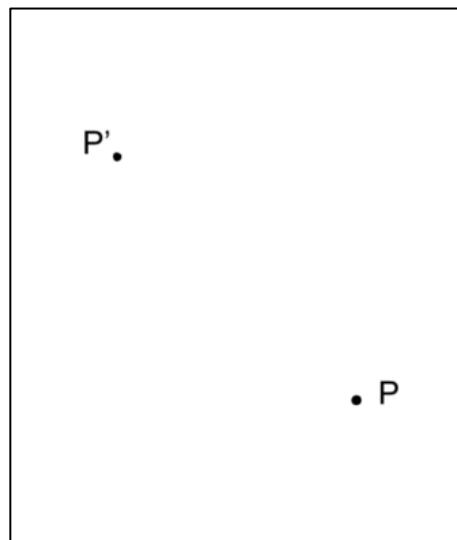
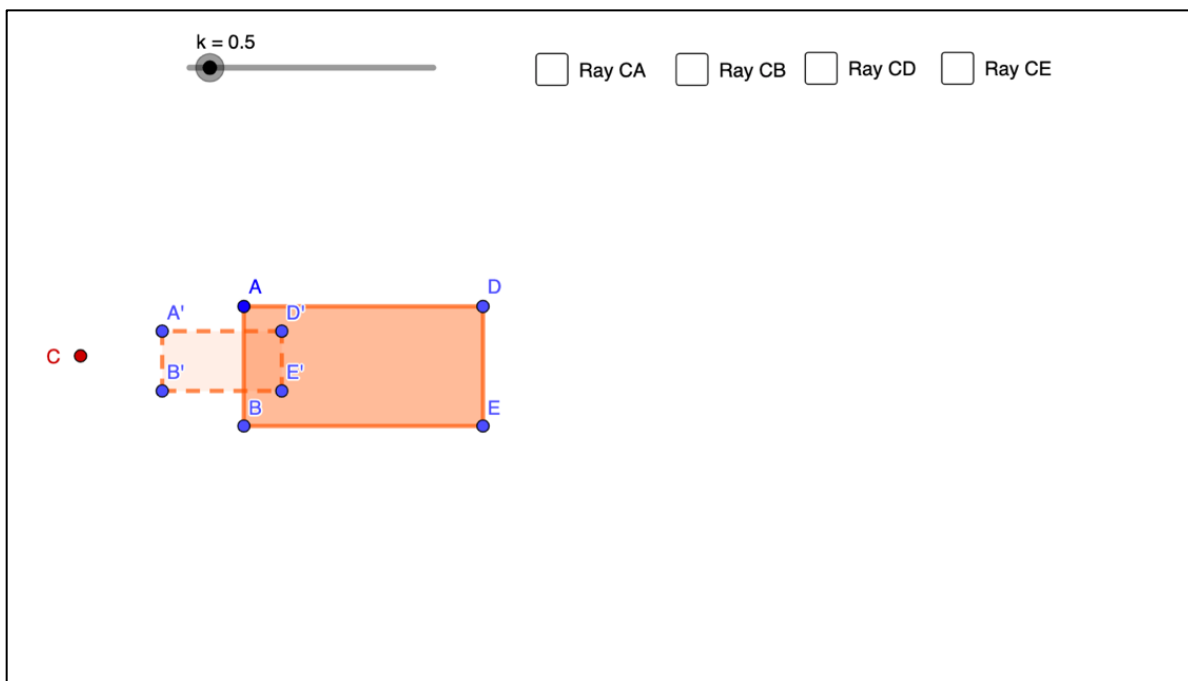


Figure 82. Task 2 of the third teaching cycle.

Task 3: Some Dynamic Situations

This task was intended to support the students’ construction of relationships between areas and relationships between lengths in the context of dilating a rectangle. Each student would

be offered a tablet device presenting a GeoGebra activity (see <https://www.geogebra.org/m/vuwy2cu8#material/bsqntnqn>). In the first activity, Ms. N would ask the students to set up the scale factor to be 0.5 (Figure 83, above) and determine the relationship between the areas of the two rectangles without any measurement available. They would also be asked to provide a justification for their statements. Then, Ms. N would ask them to determine the relationship between the perimeter of these two rectangles. Ms. N would repeat the same sequence of questions after asking them to set up a scale factor greater than 1 (e.g., 2.1, see Figure 83, below). Depending on the students' progress, Ms. N might also ask a more challenging question: "If I would like to scale the preimage so that the area of its image is tripled, what the scale factor should be? Explain your reasoning." The overall goals of this activity were to examine whether and to what extent the students could differentiate dilating areas from dilating length and how the relationships between areas or between length were associated with a scale factor.



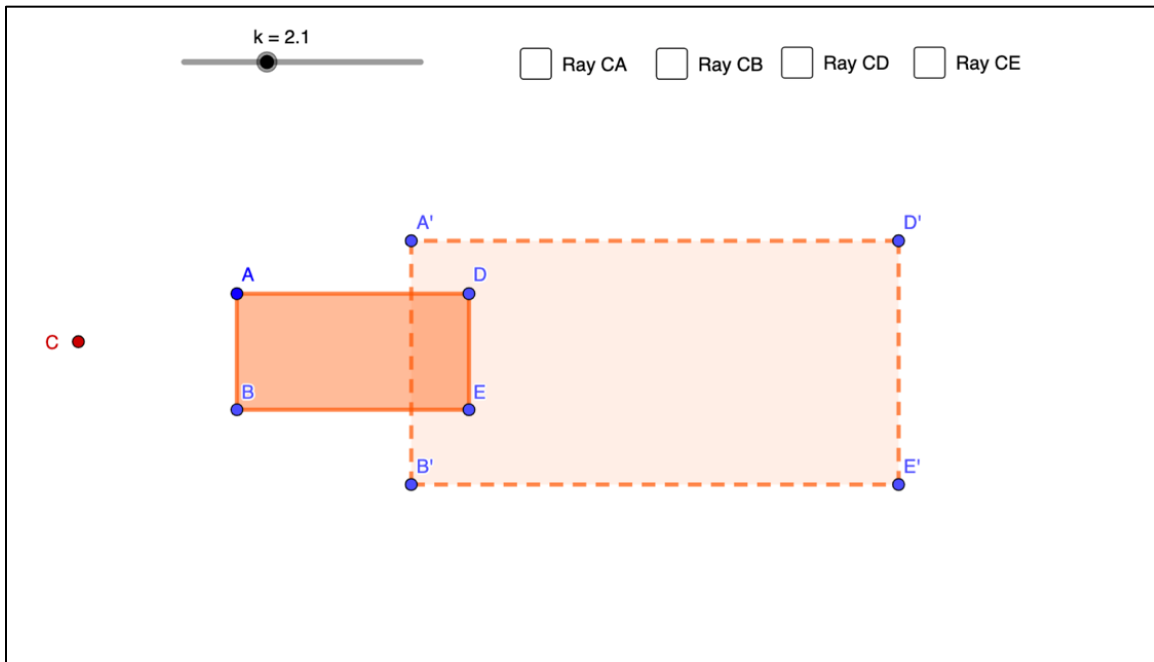


Figure 83. Screenshots of the GeoGebra activity when $k = 0.5$ and $k = 2.1$.

In terms of the second activity in the same context, Ms. N would ask the students to move the slider to vary the scale factor k from 0 to 6 and generally describe (a) how the image $A'B'C'D'$ changes, including its variation in location and size and how the distance AA' change. This is for the students to get acquainted with the dynamic situations and observe some quantities' variations. Then, Ms. N would ask the students to describe the relationship between AA' and AC and consider how this relationship relates to the scale factor. If the students struggled, they could check “Ray CA ” to show the ray CA as an assistance. The goal of this activity was to understand whether and to what extent the students could transfer their understanding of distances' relationships from the previous tasks to the current dynamic situation and generalize those relationship across infinitely many cases.

The third activity within this context required the students to ask them to predict the image's movement and change in size (if any) with respect to each of the following scenarios: (a) moving the preimage so that Point A is moving towards the Center C and (b) moving the Center

C away from Point A . Ms. N would start with asking the students to make prediction and verify their prediction without moving the objects on the applet. If they struggle, Ms. N would let them play with the applet, observe, and explain why the changes make sense. Both scenarios involved varying the distance CA , and we intended to understand whether and how the students would anticipate the variation in the distance of AA' (or CA') when the scale factor was fixed. Again, this activity aimed to enhance and examine the students' understanding of relationship between distances in the context of dilation.

A final question Ms. N would ask was that: "Is it possible to move the Center C such that the relationships between CA , AA' , and CA' were preserved? If so, how?" One answer to this question is that one can move Point C along a circle path with Point A as the center of the circle. The intention of this question was to understand whether and to what extent the students could draw on the relationships involved in this situation to control or anticipate the dynamic variations of the objects involved. In particular, this activity required the students to coordinate multiple relationships between distances and the scale factor, including the conditions for their variations and invariance in a dynamic context.

Summary

In this chapter, I described Ms. N's reflection on Carlee and Heidi's mathematical activities in the context of geometric dilation. Their conversations addressed several key ideas in secondary geometry, including preimage, image, center of dilation, enlargement, measurement, and fractional reasoning. Similar to Ms. K, Ms. N's activities suggested her engagement with Type 1 Decentering (see the discussion below Figure 57), Type 2 Decentering (e.g., Excerpt 45 and Excerpt 54), and Type 3 Decentering manifested in the forms of inferring the students' students' mental actions (e.g., Excerpt 35, Excerpt 36, Excerpt 37, Excerpt 39, Excerpt 44,

Excerpt 46, Excerpt 49), inferring the students' understanding of other mathematical concepts to explain the current activity (e.g., Excerpt 53), and modifying existing SOMs (e.g., Excerpt 54). Ms. N's constructions of SOMs also facilitated her engagement with Type 4 Decentering to reorganize her FOMs, through which she learned that (a) dilation is about dilating length instead of areas (e.g., Excerpt 37 and Excerpt 43), (b) coordination of size and location in the context of dilation might be nontrivial (e.g., Excerpt 38), (c) dilation could be reconceptualized in terms of division (e.g., Excerpt 58), and (d) conceptual understandings of dilation should be foregrounded by quantitative relationships between distances instead of physical actions of partitioning, iterating, and measuring (e.g., Excerpt 54 and Excerpt 63). This story of Ms. N, Carlee, and Heidi not only provided vivid examples of how a teacher systematically learned about and learned from her students but also interacted with students and designed mathematical tasks in an intentional manner to engender perturbations in the students' thinking. The students' activities also implied potential research agenda about studying students' meanings for geometric dilation, including their ability to reason with quantitative relationships and coordinate physical and mental actions.

In the next chapter, I discuss a framework that specifies the types of mental processes involved in each type of decentering, hoping to further illuminate the specific reasoning involved in teacher decentering. I illustrate each mental process with data from Ms. N and Ms. K.

CHAPTER 8

DISCUSSION AND IMPLICATIONS

What radical constructivism may suggest to educators is this: The art of teaching has little to do with the traffic of knowledge, its fundamental purpose must be to foster the art of learning.

By Von Glasersfeld

In this chapter, I discuss seven key mental processes identified from the analysis of Ms. K and Ms. N's decentering activities. I also discuss how these mental processes correspond to the four types of teacher decentering mentioned in Chapter 4 (Table 1) and propose a revised framework of teacher decentering. I conclude this chapter by summarizing the findings of this dissertation and discussing their implications on teacher education.

Seven Key Mental Processes

Figure 84 illustrates seven mental processes emerged from Ms. K and Ms. N's constructions of students' mathematical thinking: (a) contraindicating, (b) inferring, (c) explaining, (d) modifying, (e) comparing, (f) restructuring, and (g) re-presenting. This model includes three compartments: (a) a student's observable actions perceived by the teacher (a current action S_{A1} and a subsequent action S_{A2}), (b) a teacher's FOMs that consist of the teacher's mathematical meanings constructed *prior to* a particular exchange with the student (T_{M0}) and the teacher's mathematical meanings constructed *after* a particular exchange with the student (T_{M1}), and (c) the teacher's observable actions governed by their FOMs (T_{A0} and T_{A1}).

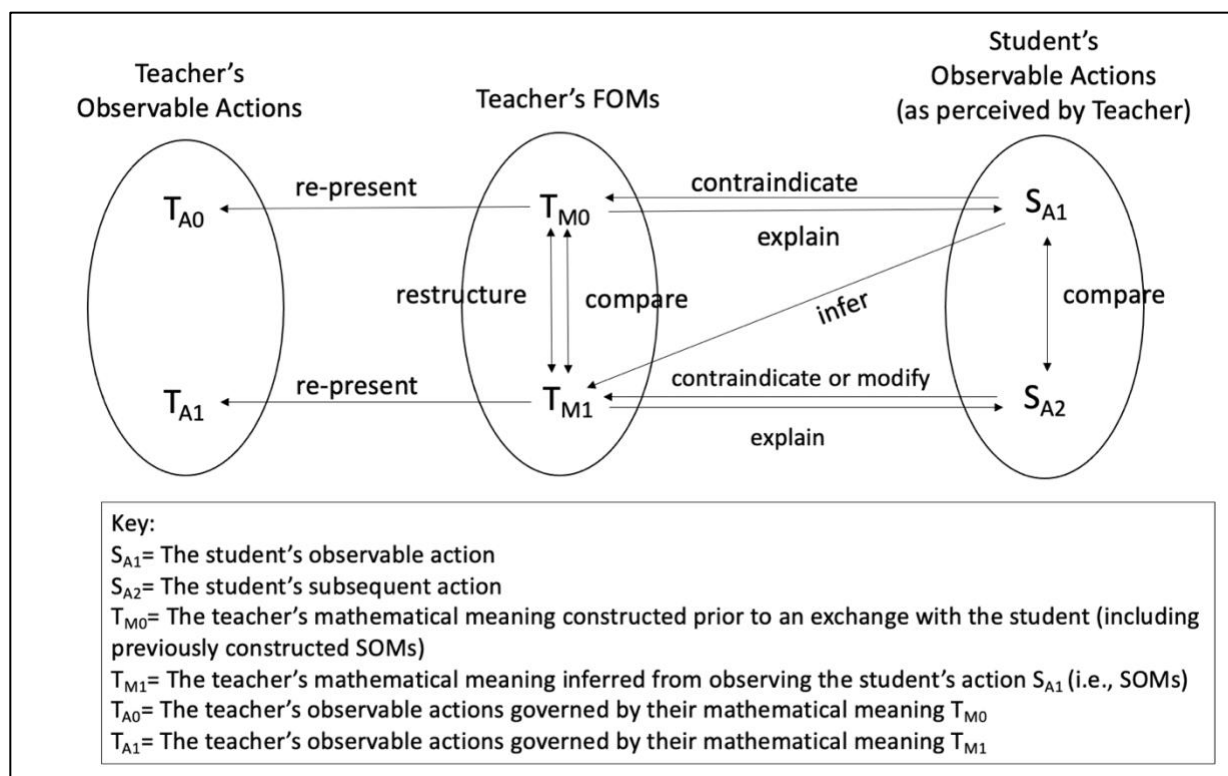


Figure 84. Seven mental processes involved in a teacher's constructions of students' mathematical meanings.

Contraindicating

Contraindicating refers to the process by which a teacher conceives of the student's observable actions as contraindications of their FOMs (see the arrow from S_{A1} to T_{M0} and the arrow from S_{A2} to T_{M1}). These FOMs include (a) the teacher's personal mathematical meanings constructed from prior school experiences, teacher education courses, professional development programs, and other life experience, (b) their SOMs of students mathematical meanings constructed from prior interactions with students, teacher education courses, professional development programs, tutoring, reading research articles on student thinking, and other life experience, and (c) their SOMs of the current student related to previous problem contexts. When enacting the mental process of contraindicating, the teacher primarily operates on their

existing mathematical meanings, ultimately seeking for evidence that the student shares similar understandings. Any observable action that is not evidence of such sharing is a contraindication that the student holds a meaning compatible with that of the teacher, of other students, or of the current student's prior activity.

One observable manifestation of this mental process is a teacher expressing confusion or surprises about a student's thinking. These moments of perturbation implies that the teacher's existing FOMs do not allow them to assimilate the student's activity. For example, Ms. N was very perturbed by Carlee's statement that the whole original photo was parallel to the whole distance between the light source and the triangle shadow (see Excerpt 48 and related discussion), and she could not infer Carlee's meaning until later.

Another observable manifestation of contraindicating is a teacher claiming a student lacks certain understandings or does not think in a certain way. The teacher is constrained to providing a claim of absence and cannot infer the mathematical meanings of the student. The teacher may notice that the student is doing something different, and they are convinced that the student is thinking differently, but they cannot describe the mathematical operations underlying the student's actions. For example, Ms. K explained that Taylor relied on carrying out a series of numerical calculations to confirm his solution to ATSB (see Figure 29 and Figure 30) because the idea of amounts of change was unavailable to him (see Excerpt 5). With respect to AE, Taylor evaluated the three inequalities at a corner point in order to confirm the solution indeed satisfied the three constraints, which Ms. K considered as a contraindication that he had constructed a quantitative meaning of a line in a Cartesian coordinate system (see Excerpt 29). Regarding the case of Ms. N, she was perturbed by the two students not being able to appropriately identify an image given a preimage, center of dilation, and scale factor (see the

task and student solution in Figure 77). The student's activity contraindicated Ms. N's SOMs that Carlee and Heidi had constructed the relationship between distance from center to preimage and distance from center to image (see Excerpt 62). As another example, Ms. N considered that the students' difficulty with reversibility (e.g., identifying a preimage given an image) was a contraindication that they had constructed a quantitative relationship between distances (see Excerpt 63).

Inferring

In other occasions, contraindicating is accompanied by *inferring*. Namely, a teacher not only hypothesizes that the student thinks differently but can also infer what the student thinks. Inferring is the genesis of the teacher's SOMs based on their observation of the student's actions (see the arrow from S_{A1} to T_{M1}). The teacher decenters from their personal mathematical meanings and hypothesizes the student's meanings such that if they imagined themselves holding those meanings, they would act in ways consistent with the student. As an example, Ms. K not only considered Taylor did not interpret the AE situation in a way consistent with hers (see Excerpt 21), but also described how she understood Taylor to have interpreted that situation (see Excerpt 19). Related to an example mentioned above, Ms. N initially was very confused by Carlee's comments in Excerpt 48, but after repeated watching of her activity and listening to her utterances, Ms. N was able to resolve her perturbation and understood that Carlee thought the "whole image" or "whole photo" in the Photocopy Faux Pas context was parallel to the "whole distance" from the light source to the image in the Triangle Shadow context (see Ms. N's inference in Excerpt 49). As another example, Ms. N was able to make sense of the students' argument about the photocopy being a quadruple of the original photo, although she herself considered it to be a double (see Excerpt 39).

Ms. N's activity also conveyed another form of inferring that involved her conjecturing about the students' understandings of other mathematical concepts that might explain the students' current activity. For example, in Excerpt 53 (and Figure 63), she referred to Heidi's understanding of multiplication and considered that as the source of her difficulty with understanding dilation in terms of dilating from the center as opposed to from the preimage.

Explaining

Explaining is a process by which a teacher's existing SOMs are explanatory for a student's actions. That is, the teacher can assimilate the student's actions into their existing SOMs without perturbation or accommodation. One form of explaining involves a teacher inferring from the student's actions that the student shares similar or identical meanings of other students (see the arrow from S_{A1} to T_{M0}). These meanings are part of the teacher's SOMs constructed prior to their interactions with the current student. Another form of explaining characterizes the process by which a teacher's SOMs constructed in a prior exchange with the current student become explanatory for the same student's *subsequent* actions (see the arrow from S_{A2} to T_{M1}). In this case, the student's actions can be considered as additional evidence that confirms or stabilizes the teacher's SOMs of the student's thinking. For example, Taylor's response to Ms. K's question about spending \$2 on radio confirmed Ms. K's inference that Taylor interpreted the AE context in a way different than hers (see Figure 39 and Excerpt 23).

As another example, in the teaching session, Ms. N was perturbed by Heidi's comment that the triangle shadow on the whiteboard appeared to be more than twice as big as the triangle cutout. After reflecting on Heidi's thinking in the reflecting session, including constructing a SOM that Heidi was primarily operating with areas instead of length, Ms. N was able to apply this SOM to assimilate Heidi's statement (see Excerpt 45). In another instance, Ms. N considered

Heidi's thinking that the scale factor of D' with respect to D was 1 instead of 2 (see Figure 64, right) to be an additional evidence that she conceived of dilation from the preimage as opposed to from the center (see Excerpt 54, Line 1-6).

Modifying

In cases where a student's subsequent action causes perturbation in the teacher, the teacher may *modify* their SOMs to account for the student's actions (see the arrow from S_{A2} to T_{M1}). One form of modifying involves a teacher replacing the elements constituting their prior SOMs of the student due to additional evidence. For example, Ms. K anticipated that the ASS situation would perturb Taylor given her SOMs of his thinking in ATSB (see Excerpt 9, Line 1-3). However, Taylor adjusted his reasoning by accommodating to the ASS situation, leading to Ms. K's modification of her SOMs. She realized that Taylor was able to coordinate two constraints at the same time beyond just maximizing the more "expensive" item (see Excerpt 9, Line 4-7, Excerpt 13). Although Ms. K might not be aware that her SOMs of Taylor had changed, from an observer's perspective, I noticed her shifting from considering Taylor's way of thinking in Cycle 1 to be limited to considering his way of thinking to be valuable and efficient in Cycle 2. Therefore, I argue that Ms. K's new model has replaced the old model.

Modifying may also occur as a teacher provides an alternative viable explanation for a student's actions due to additional evidence. For instance, Heidi had consistent difficulty with understanding dilation in terms of dilating from the center. She conceived of a scale factor as being associated with the multiplicative relationship between distance from a center to a preimage and distance from the preimage to its image (as opposed to distance from the center to the image) (see Figure 62 and Excerpt 51). Ms. N initially conjectured that this difficulty might be due to her conception of multiplication (see Excerpt 20, Figure 63). Then, after she reflected

on Heidi activity associated with Figure 65, she inferred that her thinking might be dominated by the physical action of iterating one copy of the distance from center to preimage instead of by in image of sketching a length with one fixed end point (see Excerpt 54, Line 15-33). Instead of rejecting the previous explanation, Ms. N provided an alternative explanation. Both inferences were made to explain Heidi's conceptualization of dilation in terms of dilating from a preimage.

Comparing

Comparing is the mental process by which a teacher positions two (observable or mental) actions against each other and abstracts similarities or differences in those actions. In the following, I differentiate four types of comparing evident in the data.

Comparing Observable Actions

One form of comparing refers to a teacher constructing similarities or differences between two observable actions (see the double arrow between S_{A1} and S_{A2} , with the former occurring prior to the latter). For example, Ms. K observed that Taylor maximized both constraints in ASS but not in ATSB or GP (see Excerpt 15). Then, Ms. N continued to interact with Taylor with an attempt to understand what he perceived as differences in those contexts. Generally, this instance suggested that comparison between observable actions might motivate a teacher to construct SOMs to explain the result of comparison. On some occasions, the teacher may be able to construct an explanation based on the student's response. On some other occasions, this attempt may not be successful. For example, Taylor conceived the differences to be differences between contextual features of the situations (i.e., profit versus revenue), but Ms. N remained perturbed by his statement towards the end of their conversation (see Excerpt 16 and Excerpt 17).

Comparing Mental Actions

Another form of comparing refers to a teacher simultaneously operating on two mental actions and abstracting common or distinct features of them (see the double arrow between T_{M0} to T_{M1}). The major difference between comparing and contraindicating is that contraindicating only involves a teacher inferring a student does not think in a particular way, but *not* requiring the teacher to infer what meanings the student holds. Recall an aforementioned example where Ms. K interpreted Taylor's justification of his solution to be constrained to carrying out a series of calculations to test the total revenues corresponding to different value pairs. Ms. K primarily attended to Taylor's behavior and inferred Taylor did not know amounts of change (see Excerpt 5). In contrast, comparing requires defining two kinds of meanings and putting them against each other. The following are examples of comparing between mental actions.

Mental Actions of the Same Student. One type of comparing mental actions involves a teacher comparing two mental actions of the same student at different times (with T_{M0} indicating a former mental action and T_{M1} indicating a latter mental action, both of which constituting the teacher's SOMs). For example, Ms. K described Taylor's thinking in ATSB to be coherent to his thinking in ASS (see Excerpt 10 and Excerpt 11). As another example, when describing the difference between Taylor's activity in ASS and AE, Ms. K simultaneously re-presented her SOMs constructed in ASS and those in AE and argued that Taylor could mentally coordinate multiple constraints in ASS but not in AE (see Excerpt 20).

Mental Actions of Different Students. A second type of comparing mental actions involves a teacher comparing the mental actions of different students (with T_{M0} indicating an SOM of one student and T_{M1} indicating an SOM of another student). For example, Ms. N inferred from Carlee and Heidi's responses to Figure 50 that Heidi was attending to the

dimension of the corresponding sides of the two figures while Carlee was primarily operating on the “whole” length of both figures, where the “whole” did not correspond (see Excerpt 41). As an additional example, Ms. N interpreted that Carlee conceived of “dilating from the center” while Heidi conceived of “dilating from the preimage” (see Excerpt 51).

Own Mental Actions and Those of Students. The third form of comparing mental actions involves a teacher comparing the teacher’s personal meaning to meanings of their SOMs (with T_{M0} indicating the teacher’s own mental action and T_{M1} indicating the mental action of the student). For example, in ATSB, Taylor argued that because nonstop was more expensive, he would maximize the number of nonstops in order to achieve the highest revenue. Ms. K interpreted that his argument was valid only when considering one-unit amount of change (i.e., the revenue would decrease when decreasing *one* nonstop and increasing *one* two-stop) (see Excerpt 7). In comparison, Ms. K’s own justification involved considering an increase of *three* two-stops and a decrease of *two* nonstops (see Figure 32). As a result of such comparison, she argued that Taylor’s justification was insufficient because he did not account for other feasible solutions regarding the situation. In this example, Ms. K’s comparing activity involved her operating on her FOM to evaluate the validity of the way of reasoning constituting her SOM.

In the case of Ms. N, she inferred that the Carlee and Heidi conceived of dilation in terms of dilating area, which was contrasted with Ms. N’s conception of dilation in terms of dilating dimension (see Excerpt 37). Relatedly, Carlee made an interesting comment about what she conceived to be the similarity between the Photocopy Fax Pas and Triangle Shadow. That is, the “whole” image of the original picture was parallel to the “whole” distance from the light source to the shadow (see her gestures in Figure 57). This understanding was contrasted by Ms. N’s conception that the distance from the light source to the shadow should be parallel to the distance

from the light source of copy machine to the photocopy (see Excerpt 49). As an additional example, Ms. N inferred Carlee conflated two meaning of “backward”: “backward” in terms of finding an image given a preimage and “backward” in terms of dilating with a negative scale factor; the latter perspective was only perceived by Carlee but not Ms. K (see Excerpt 56).

Restructuring

When a teacher simultaneously operates on their SOMs and their personal mental actions through comparing, they are poised to *restructuring* their FOMs. Restructuring describes a teacher modifying or reorganizing their own mathematical knowledge as a result of constructing and reflecting upon SOMs (see the double arrow between T_{M1} and T_{M0} with T_{M0} indicating a mental action of the teacher’s personal mathematical knowledge and T_{M1} indicating a mental action constituting the teacher’s SOMs).

One form of restructuring is accompanied by a teacher’s comparison between their own mental actions and those of a student. As a teacher constructs SOMs of the student and compares the students’ mental actions with the teacher’s own, the teacher becomes more consciously aware of the mental actions constituting their own thinking. Before such construction, these mental actions were backgrounded in the teacher’s mind. In other words, it is through inferring the mental actions of the student that the teacher surfaces the corresponding mental actions of their own thinking. This form of restructuring often occurs when T_{M1} entails problematic meanings. I define “problematic” not in terms of meanings that lead to nonnormative solutions or incoherent mathematical meanings from a researcher’s perspective, but in terms of meanings that are incompatible with the teacher’s personal meanings from the teacher’s perspective. For example, in the context of AE, Taylor assumed that the candidate advertised in three radio and three TV channels while Ms. K assumed the candidates advertised in one radio and one TV channels (see

Excerpt 21). Consequently, Ms. K realized that a coordinate pair in a 2-dimensional graph only made sense with her assumption but not Taylor's (see Excerpt 33). Understanding the nonconventional interpretation of Taylor allowed Ms. K to clarify this aspect of her thinking. This is an example of restructuring because the assumption that Ms. K had taken as given and had been in the background of her thinking now became foregrounded. I also observed similar evidence in the case of Ms. N. Through working with Carlee and Heidi on the Photocopy Faux Pas and Triangle Shadow tasks, Ms. N realized that both students conveyed meanings of dilation in terms of dilating areas. Reflecting on those instances, Ms. N became aware that she had taken it as a given that when speaking of a scale factor, she was actually referring to the relationship between the measurable attributes of two distances as opposed to two areas. Constructing the alternative perspective of reasoning with areas allowed Ms. N to become more consciously aware of her own thinking that was grounded in reasoning with distances (see Excerpt 43). This form of restructuring through foregrounding aspects of an individual's FOMs explains how decentering, the mental process of constructing another's perspective, promotes the individual's own knowledge growth.

Another form of restructuring involves the teacher injecting coherence into two mental actions beyond discerning their differences or similarities (via comparing). This often occurs when the mental actions of the teacher's SOMs entail what the teacher conceives to be propitious meanings. For example, Ms. K re-presented Taylor's reasoning with a graphical representation and realized that his method was efficient in that he did not consider all feasible points to generate an optimal solution (see Excerpt 12). That Ms. K genuinely valued Taylor's understanding and considered it as a viable, rationale, and alternative way of understanding was an indication that she had restructured her FOMs by incorporating novel understandings. An

important benchmark for Ms. K's restructuring was her ability to conceive both ways of understanding to be coherent; that is, she was able to assimilate Taylor's ways of operating with her meanings for the graph and vice versa. Together, I underscore that these two forms of restructuring are nontrivial mental processes because they require a teacher to first conceive *contraindication*, and then *infer* the student's meanings to resolve perturbation, and then *compare* those meanings with other meanings, and finally *restructure*.

Re-presenting

Re-presenting refers to the process by which a teacher's FOMs determine their observable actions (see the arrow from T_{M0} to T_{A0} and that from T_{M1} to T_{A1}). One form of re-presenting involves the teacher reasoning with a certain mathematical situation independent on their SOMs. This is when the teacher discusses what they conceived to be the normative solution of a task. In the context of ATSB, Ms. K's justification for why the optimal solution occurred at (8, 0) instead of (6, 3) was governed by her meanings for quantities' amounts of change in that context (see Figure 32). Ms. N also demonstrated how she would find the preimage $ABCD$ given $A'B'C'D'$ (see Excerpt 57 and Figure 72) and how she would find the center of dilation C given P' and P (see Excerpt 60 and Figure 76).

A second form of re-presenting involves a teacher re-presenting some aspects of their SOMs in combination with their own meanings. For example, through interaction with Taylor, Ms. K considered the relationship between votes and money as an additional viable relationship regarding the AE situation, implying her mental structure of the AE situation had been restructured to include a new pair of relationship. Then, she reconstructed the votes-money graph based on her personal understandings of the quantities' relationships (Figure 43). Given that Carlee and Heidi both considered the scale factor of the photocopy to be 4, Ms. N demonstrated

how she would locate the corresponding center of dilation in line with that scale factor (see Excerpt 50 and Figure 59, right).

A third form of re-presenting occurs when a teacher re-presenting the mental actions of their SOMs to reason with a problem in ways that resemble features of a student's activities. Ms. K's application of Taylor's algebraic method to solve the AE problem was an example of this form of re-presenting (see Excerpt 25 and Figure 40).

I note that the differences between these three forms of re-presenting are nuanced. In essence, they all require a teacher to reconstruct their understandings of a problem situation as if they were a student. Their differences only lie in the extent that those understandings have their origin in an observed student's activities. I also note that other mental processes, including explaining, modifying, comparing, and restructuring, also entail re-presentational activities, but those activities were centered around the students' activities evident in the teaching sessions. The term "re-representing" I use as the seventh mental process refers to a particular type of re-presentational activity that requires a teacher to enact their FOMs to reason with a problem situation in novel ways. I speak of this novelty only from a researcher's perspective in the context of their constructing SOMs of the teacher's mathematics.

Methodologically, a teacher's re-presenting activity could be captured in the pre-teaching sessions, but because the teaching cycles involved ongoing task design and the teacher might make a statement that implied their operations on their own thinking during the reflecting sessions, the researcher might prompt the teacher to re-present their thinking during the reflecting sessions in order to gain further access to their FOMs, especially when the researcher infer novel ways of thinking that was not evident in the previous sessions. A researcher's prompting associated with the second form of re-presenting can also provide the researcher with

insights into how the teacher's understanding of a mathematical situation is restructured in light of their SOMs. Finally, and related to the third form of re-presenting, a teacher's re-presentation of the mental operations constituting their SOMs in novel contexts can provide researchers with further insights into the content and nature of their SOMs and the extent that the SOMs are stable.

To this point, I have defined seven key mental processes and illustrated each with instances of Ms. N and Ms. K's reasoning. Table 13 summarizes the aforementioned descriptions of each mental process.

Table 13. A summary of the seven mental processes

Mental Process	Descriptions
contraindicating	A student's observable actions (as conceived by the teacher) contraindicates the teacher's FOMs (including the teacher's personal mathematical meanings and previously constructed SOMs). The teacher is experiencing perturbation by the student's actions, but they cannot infer the student's rationale.
inferring	A teacher hypothesizes the student's mental actions in order to explain the students' observable actions.
explaining	A teacher applies their FOMs to explain a student's observable actions. This mental process, in turn, helps stabilize a teacher's SOMs when the teacher repeatedly observes student actions (in different contexts) that confirm their SOMs.
modifying	A teacher revises their SOMs to account for novel features of a student's observable actions. Modifying is a means to resolving perturbation or contraindication.
comparing	A teacher abstracts common or distinct features of multiple mental actions or observable actions. The comparison can be between a student and their own, between two students, or between a student's prior and subsequent actions.
restructuring	A teacher reorganizes their FOMs by foregrounding/backgrounding aspects of personal mathematical meanings or by injecting coherence between their personal mathematical meanings and their SOMs.

re-presenting A teacher re-presents their FOMs (including internalized SOMs) to reason with a mathematical situation.

A Framework of Teacher Decentering

Framing these mental processes in terms of decentering, I use Table 14 to illustrate the alignment of the seven mental processes with the Type 0, 1, 2, 3, 4 Decentering.

Table 14. A revised framework of teacher decentering

Decentering Types	Mental Process	Cognitive Activities	Observable Manifestation (researcher prompts italicized)
Type 0: No Decentering	re-presenting	- entirely operates on FOMs, including stabilized and internalized SOMs, to assimilate a mathematical situation	- demonstrates what they believe to be a normative solution of a task (<i>Could you show me how you would solve this problem?</i>) - resolves a task by incorporating aspects of their SOMs (<i>Could you solve this problem as if you were [student name]?</i>)
Type 1: An Emergent Sense of Decentering	contra-indicating	- experiences unresolved perturbation resulting from observing a student's action - entirely operates on FOMs (including internalized SOMs)	- expresses confusion or surprises by a student action (<i>Could you describe what the student was thinking?</i>) - claims that certain meanings are not available to the student (<i>Could you explain why the student might think in this way?</i>)
Type 2: Decentering Through Assimilation	explaining	- entirely operates on previously constructed SOMs to assimilate a student's observable action - is likely accompanied by functional accommodation	- discusses how a student's action is similar to other students' or similar to the same student's previous action
Type 3: Decentering Through Accommodation	inferring	- abstracts characteristics of a student's observable actions and infers mental actions of the student - the genesis of SOMs - reconciles perturbation through accommodation	- describes a student's mathematical reasoning as if they themselves are reasoning with the mathematical situation (<i>Could you describe what the student was thinking?</i>) - infers a student's understanding of another mathematical idea that can explain the understanding of the current idea

	modifying	- abstracts characteristics of a student's observable actions and modifies mental actions constituting an existing SOM - reconciles perturbation through accommodation	- provides an alternative explanation for the student's actions either by replacing a previous SOM or adding an additional hypothesis (<i>Could you describe what the student was thinking?</i>)
Type 4: Reflective Decentering	comparing	- abstracts common or distinct characteristics of multiple mental actions - simultaneously operates on multiple FOMs (including internalized SOMs) - requires conscious awareness of mental actions and operations	- discusses how two ways of thinking are similar or different (<i>How is this student's understanding different or similar to your own understanding/their previous understanding/other students' understanding?</i>)
	restructuring	- simultaneously operates on multiple FOMs (including internalized SOMs) and reorganizes FOMs - accommodates through bringing aspects of thinking into conscious awareness or injecting coherence into a system of meanings	- discusses an understanding that has been taken as given (<i>What have you learned from the student?</i>) - discusses how two ways of thinking are coherent (<i>Do you think the student's thinking is compatible with how you think, and why? Why do you think the student's method always works?</i>)

Re-presenting aligns with Type 0 Decentering because it does not require in-the-moment construction of student thinking, although it may involve the teacher re-presenting aspects of their previously constructed SOMs. The teacher entirely operates on their FOMs, and there is no major learning or accommodation occurring from this process.

Contraindicating is representative of Type 1 Decentering because it describes a teacher experiencing perturbation by a student's observable actions, but the teacher cannot resolve such perturbation by inferring the mental actions of the student. The teacher only abstracts features of the student's actions and infer that the student does not think in ways compatible with the teacher's FOMs.

Explaining describes a teacher's enactment of FOMs to assimilate a student's action, which aligns with Type 2 Decentering. It is not Type 0 Decentering because it does require the

teacher's construction of the student's meanings; it is not Type 2 or Type 3 Decentering because it does not entail major accommodation. The teacher only experiences modest accommodation through applying their FOMs to make sense of an additional student action. Importantly, a teacher's ability to draw on their constructed SOMs to explain a student's subsequent action is a landmark of their SOMs becoming stable. Thus, decentering in the form of explaining plays a critical role in second-order modeling since it builds upon inferring (Type 3 Decentering) and is foundational for restructuring (Type 4 Decentering) and re-presenting (Type 0 Decentering).

Inferring and modifying are the constructive processes of SOMs. When the teacher observes novel student actions and experiences perturbation, they construct novel SOMs or revise existing SOMs to resolve the perturbation. These two mental processes align with Type 3 Decentering because they require the teacher to accommodate by adding or modifying meanings in their FOMs.

Type 4 Decentering requires a teacher to abstract features of multiple mental actions, which is compatible with comparing and restructuring. Comparing involves a teacher simultaneously operating on multiple mental actions and discerning differences or similarities in them. Although comparing necessarily entails abstraction of actions or operations, it may not stimulate major accommodation if the teacher does not experience coordination or reorganization in their meanings; the teacher still treats two meanings as entailing separate mental actions. Restructuring characterizes a teacher experiencing coordination or reorganization in their meanings. One benchmark of restructuring is a teacher bringing certain aspects of their mental actions into conscious awareness. An alternative benchmark is a teacher injecting coherence in a collection of mental actions and becoming aware of how two ways of operating are coherent and connected, regardless of the differences in their observable entailments. Both comparing and

restructuring describe how second-order modeling is generative and consequential to a teacher's mathematical development. The difference between these two mental processes is nuanced as it is only tied to the extent a teacher accommodates; in fact, comparing can be considered as a minor version of restructuring.

I iterate that I call these distinctions to be decentering *types* instead of decentering *levels* to not to imply that these different types of decentering or mental processes are hierarchical. First, I note that the latter types of decentering do not necessarily require the enactment of the former types. For instance, a teacher does not need to engage in re-presenting (Type 0) or contraindicating (Type 1) in order to engage in explaining (Type 2). However, a teacher likely needs to first engage in contraindicating (Type 1) in order to engage in inferring or modifying (Type 3). And they may also need to engage in Type 2 or Type 3 before engaging in comparing or restructuring (Type 4), since those mental processes provide them with the source material to operate on during Type 4 Decentering.

Second, characterizing a mental process as a decentering type with lower numbers is not a criticism of them or suggesting they are trivial or unimportant. For instance, contraindicating is foundational to and can engender subsequent decentering actions such as inferring and modifying. A teacher with rich SOMs may mostly engage in Type 2 Decentering to assimilate a student's actions without perturbation, as they likely have gone through Type 1, Type 3, and Type 4 to construct and stabilize those SOMs before their interactions with the current student. In other words, the teacher's existing SOMs have already been explanatory for many students' mathematical activities. Although Type 0 Decentering (or re-presenting) itself does not involve any decentering action, but some re-presenting actions are built upon prior decentering actions such as inferring, comparing, and restructuring. For instance, a teacher's ability to re-present a

student's ways of operating in a novel context is an evidence that the teacher has internalized the student's ways of operating and used it to govern their own mathematical activities. This conceptualization is consistent with Piaget's description of the relationship between centration and decentration. That is, the results of teacher decentering can provide the source material a teacher operates on during their non-decentering activities, and the teacher ultimately treats the students' ways of operating as their own ways of operating.

Remarks

In closing this section, I highlight three goals I accomplish by proposing this framework of teacher decentering (see Figure 84, Table 13, and Table 14). First, this framework describes the dynamic mechanisms by which a teacher constructs, modifies, and enacts their FOMs and SOMs in the context of reflecting on students' mathematical thinking. The abbreviation of each component (e.g., S_{A1} , T_{M0}) in Figure 84 is a generic label of a particular type of meanings or actions instead of a specific meaning or action. As the student-teacher interaction proceeds, T_{M1} or S_{A2} in the current exchange will become T_{M0} or S_{A1} in the next exchange. T_{M0} does not only include what a teacher considers as normative mathematical meanings, but also SOMs constructed in their prior experience, and T_{M0} will continually include new mental actions as the teacher's SOMs are developed and internalized through mental operations. Therefore, the chronological order of the presence of each meaning, action, and context matters to the evolution of the teachers' mathematical constructions. I note that this order is a result of the teacher's personal construction during the reflection session, which may not necessarily align with what an observer perceives in the teaching session in-the-moment.

I also highlight that the development of this model is an example of adopting an adaptive and dynamic perspective to investigate teacher knowing. The model illustrates that a teacher's

decentering actions entail a series of constructive processes. Importantly, engaging with student thinking can be generative to a teacher's mathematical growth. In particular, this growth is achieved through three key mental processes: (a) contraindicating produces perturbation in the teacher; (b) inferring and modifying removes perturbation; (c) comparing and restructuring engender reorganizations in FOMs.

Second, I strive to operationalize each mental process such that their distinctions are sensitive to a teacher's mental actions and operations. In doing so, I differentiate between a teacher's conceived observable actions of a student, the teacher's FOMs (including inferred mental actions of the student), and the teacher's own observable actions, and each mental process describes a distinct type of interaction within or between two components. In making claims about Ms. K and Ms. N's activities, I sought for indications and contraindications of each mental process. For instance, making a claim of contraindicating requires a contraindication of the teacher being able to infer the student's meanings. Similarly, claiming that a teacher was comparing two observable actions requires a contraindication of them being able to infer the mental actions underlying those observable behaviors.

However, a researcher's attempt to operationalize these differences does not imply that there always exists a one-on-one relationship between a mental process and a particular instance. One reason is that some mental processes are interrelated or may occur simultaneously. For instance, explaining and modifying may be accompanied by comparing between two observable actions; a teacher may first compare a current action with a prior action at an observable level and then become aware that whether their previous SOM is explanatory to the current action or not, thereby engaging in explaining or modifying. Besides, a teacher's ability to conceive of two mental actions to be coherent through restructuring may imply that the teacher has enacted

explaining to assimilate the student's observable actions. As such, restructuring is only an aftereffect of explaining. These nuances are not necessarily distinguishable within discourse practices, but a researcher's analysis is necessarily constrained by inferring the teacher's reasoning based on their utterances.

Another reason is that these seven mental processes are distinguished based on an assumption that a researcher can discern whether a teacher is operating on mental or observable actions. However, a student's activity necessarily has both mental and physical entailments, which may not be distinguishable in discourse practices either. A teacher's reference to an observable action does not necessarily imply that the teacher is not operating on the mental aspect associated with that action and vice versa. A researcher can try their best to tease out these differences in their interactions with the teacher, but it is not a guarantee that the researcher is able to avoid any misinterpretations. This limitation relates to a previous point I have made that a researcher's TOMs are not considered to be true representations of a teacher's SOMs.

Third, and related to this limitation, the implications of this framework are more than labeling or categorizing a teacher's ways of operating with clear criteria. Despite the inextricable links between these mental processes and the difficulty to discuss them in isolation, I discuss them separately in an effort to demonstrate their existence and to elucidate their distinctions. The productivity of this framework lies in providing researchers an analytical tool to conduct a fine-grained analysis of a teacher's mathematical constructions in the context of student-teacher interactions. Such analysis is underlain by a descriptive as opposed to an evaluative stance to inquiring teachers' mathematical knowing and knowledge. It affords researchers an understanding of the complex, interwoven mathematical actions and operations involved in a teacher's decentering and second-order modeling activities. This collection of mental processes

also constitutes researchers' modeling practices when researching individual's mathematical cognition, which are often left implicit in the literature or unaware by the researchers. In the following sections, I continue to discuss the implications of this framework on teacher education and the multiple ways researchers can advance the field with the framework.

Conclusions

I conclude this dissertation by summarizing the empirical findings of the study and highlighting the corresponding methodological and theoretical approaches that support the emergence of these findings.

Empirical Findings

The major finding of this dissertation is a framework of teacher decentering that characterizes seven mental processes associated with teacher's constructions of students' mathematics. Contributing to the common definition of decentering, that is, "setting aside one's own thinking and constructing perspectives of another," this framework provides a theoretical conceptualization that specifies the mathematical reasoning entailed in decentering. Scholars in the area of teacher noticing have also called for research that "unlock[s] the black box of noticing and offer[s] us insights into the underlying mechanisms of productive teacher noticing" (Choy & Dindyal, 2019, p. 484; B. L. Sherin & Star, 2011). My work fills in this gap by revealing the mathematical reasoning entailed by teacher noticing.

In particular, the complexity of the involved reasoning processes can be seen by the different kinds of interactions between and within a teacher's own mathematical meanings (i.e., FOMs) and their inferred mathematical meanings of students (i.e., SOMs). The evolution of a teacher's mathematical meanings is achieved by their continual constructions, modifications, and reorganizations of their SOMs and FOMs. These reasoning processes also vary by the types of

scheme activity a teacher engages in, ranging from assimilation, perturbation, to accommodation. By interweaving these different lenses and constructs to theorize teacher decentering, I provide a descriptive and explanatory account for teachers' capacities and orientations to decenter.

I iterate that these processes do not simply emerge from my first-order thinking; rather, they emerge from my modeling of the concrete mathematical activities of two in-service teachers and their students. Despite the complexity of the reasoning processes involved in teacher decentering, it is promising and exciting to see those different ways Ms. K and Ms. N decentered. This suggests that although decentering is nontrivial and effortful, teachers are capable of decentering when appropriate environment and support are given.

In addition to the insights into the nature of teacher decentering, the findings reveal several productive consequences of teacher decentering. First, teacher decentering advances teachers' mathematical development. In a non-decentering conversation, students are constrained by their teacher's actions, and it is the students who try to comprehend the teacher's actions in order to construct knowledge. In this case, the teacher primarily operates on their FOMs and they do not experience accommodation in their FOMs. By contrast, during a decentering conversation, teachers are constrained by their students' actions, and in resolving these perturbations to achieve equilibrium, the teachers accommodate their mathematical knowledge. Prior researchers have provided evidence that teachers' mathematical knowledge increases as they construct knowledge of student thinking (e.g., Carpenter et al., 1996; Philipp, 2008; Philipp et al., 2002; Sherin, 2002; Wilson et al., 2013, 2014). The present dissertation contributes to this body of work by specifying the mechanism of how teaching becomes learning (e.g., through inferring, modifying, comparing, and restructuring).

Second, teacher decentering has pedagogical potential and power. Prior conceptualizations of teacher knowledge have explained how teachers' attention to and constructions of student thinking play critical roles in transforming their personal understandings to pedagogically powerful understandings (Shulman, 1987; Silverman & Thompson, 2008). Based on the proposed framework of teacher decentering, this transformation is necessarily achieved through Type 4 Decentering. In particular, teachers need to simultaneously operate on their FOMs and SOMs to discern consistencies and inconsistencies in those meanings. Through foregrounding aspects of their FOMs, the teachers also become more aware of the intended meanings in light of the alternative meanings students hold. These constructions afford the teachers' realization of the necessity of clarifying certain assumptions that they have taken as given, of bringing forth diverse perspectives among students, and of designing tasks to engender perturbations in ways tailored to particular students. They also come to realize some limitations of their pedagogical moves in that there are occasions where the students are not perturbed by an instructional action, but the teacher anticipates that it does. It is through reflecting on and comparing one's own anticipation and the perceived consequence of an action that the teacher modifies their SOMs (through Type 3 Decentering) and adjusts their instructional actions correspondingly.

Relatedly, research on teacher noticing has provided fruitful evidence that *deciding* (how to respond) is non-trivial to teachers when compared to attending and interpreting (Jacobs et al., 2010; Krupa et al., 2017; Sánchez-Matamoros et al., 2019; Schack et al., 2013). Kastberg (2014) also noted that teachers may not genuinely draw on their models to make instructional decisions even if they have constructed them. The findings of the present dissertation explain why this is the case. That is, teachers need to enact a series of effortful decentering actions, including

inferring, comparing, and restructuring, before making a pedagogical move sensitive to students' thinking; Type 2 and Type 3 Decentering are likely insufficient. Because those reasoning processes likely do not occur spontaneously without necessary support, it is not surprising that deciding is reported to be the most difficult action by numerous studies focused on assessing teachers' noticing capability.

Third, teacher decentering engenders teachers' continual learning about and learning from students. This dissertation provides vivid examples of how teachers learned to make more specific claims that account for students' activities in specific contexts as opposed to making overgeneralized claims based on a single action of students. They became aware that their predications often failed at the beginning because their existing FOMs (including SOMs) were not explanatory for students' actions in other contexts. As Ms. K argued, Taylor's thinking is so complex, organic, and unpredictable. Gradually, the teachers learned to develop more and more stable and explanatory models of student thinking by continually constructing, rejecting, and refining their hypotheses about student thinking. During this journey, the teachers not only learned *about* their students but also learned *from* the students through ongoing interactions with their students.

Fourth, teacher decentering affects teachers' professional dispositions. Through engaging in decentering actions, teachers construct first-hand experience of how students think. This experience is often associated with surprises, confusion, perturbation, enlightenment, and frustration. These compound feelings have led to dispositional change. To illustrate, I present Excerpt 64 where Ms. K discussed her feelings and take-aways as a final reflection on her participation in the study.

Excerpt 64

- 1 Ms. K: It was really, really cool getting to like work with [Taylor] ‘cause like I’ve
2 taught him for two years and I never got to have like this much in-depth
3 conversations with them. So that was really, really neat. I already knew,
4 like I knew he was a smart kid, but oh my gosh, he’s such a smart kid. I
5 picked him because I’m going to be honest, like I picked him because I
6 thought he was like average, and I was wrong, like he showed me up,
7 which I am proud of him. Like he has such— I think that a lot of our
8 students like lack problem solving skills, like basic reasoning and problem
9 solving because they haven’t been taught to use them and like he has that
10 and I just didn’t expect it, which is super, super great. (...) It definitely
11 makes me take a step back and think about, being more, you said
12 something about like, um, there’s a word for being able to like adopt a
13 mindset of another student.
- 14 Biyao: Decenter?
- 15 Ms. K: Yes. Trying to do that more I think because like I do think that’s
16 something that I can do. I just don’t ever try. I don’t try ‘cause I never am
17 trying to, like I’m not trying to get myself to understand their thinking.
18 I’m trying to get them to understand my thinking. And so that’s one of
19 those things I was like, that’s an easy switch that I can make because I
20 already know how to do it. I just don’t think about doing it often.
- 21 Biyao: So, what does this attention to the student’s thinking afford you?
- 22 Ms. K: I think it helps me give them stronger understandings because then I can
23 understand where they’re coming from. It helps me like scaffold. It helps
24 me like fill in holes that they might have things like that versus just being
25 like, no, this is how you do it. I mean I’m going to brag, but I don’t think
26 that I like just go in and like force them. Like I do think that I have the
27 ability to like talk to my students and understand where they’re coming
28 from and then reroute them.

Ms. K acknowledged that she underestimated Taylor’s mathematical ability before her participation in the study (Line 1-10). Now, she positioned Taylor as a knowledgeable and capable individual who had sophisticated mathematical reasoning skills. During the teaching cycles, Ms. K repeatedly expressed surprises about Taylor’s ability to solve and make sense of a novel problem with nonstandard approaches. This suggests that a teacher’s engagement in authentic in-depth student-teacher conversations—including decentering—may influence the teachers’ beliefs in students’ mathematical competences (see Kazemi & Franke, 2004)—and enable them to truly appreciate and value the students’ ability to reason with mathematics (see Stockero et al., 2020).

Decentering also enabled the teachers to build mathematical caring relations with their students (Hackenberg, 2005, 2010). For example, Ms. K commented that she felt the need to decenter more in her future teaching (Line 15-20), suggesting she became more attentive to and cared about her students mathematically.

Methodological Approaches

In addition to these empirical findings, I draw the reader's attention to the methodological contributions of the study. The design of the teaching experiment² afforded teachers meaningful and authentic experiences of engaging in in-depth conversations with students and reflection on the students' mathematical thinking. Moreover, as I mentioned earlier, this is not only a teaching experiment that *studied* teacher decentering (the "experiment" aspect), but also a teaching experiment that *promoted* teacher decentering in the meantime (the "teaching" aspect). Therefore, the researcher's support during the experiment should not be downplayed. In the following, I highlight the intervention I provided during the experiment as well as the challenges and affordances identified from my implementation of the methodology.

Pre-Teaching Sessions

Although the data of the pre-teaching sessions are not reported in this dissertation, my SOMs of the teachers' mathematics played critical roles in many aspects of the study. As I mentioned earlier, I intentionally chose topical areas that were overlooked by mathematics education research. The pre-teaching sessions allowed me to formulate foundational ways of operating and have access to the instructional materials and practices relevant to those topics. When transitioning to the teaching cycles, these understandings constituted necessary schemes for me to assimilate the teachers' and the students' mathematical activity. During the reflecting sessions, both teachers frequently referred to ideas discussed in the pre-teaching sessions as they

reflected on the students' thinking. Those schemes also allowed me to hypothesize the accommodation and perturbation the teachers might experience during their interactions with students. Finally, my interactions with the teachers during the pre-teaching sessions allowed me to establish relationships with them, which was essential for our later interactions as those interactions required the teachers to exchange ideas with me and articulate their thinking as much as possible. To conclude, the pre-teaching session is an indispensable component of the methodology.

Planning Sessions

A planning session involved collaboration between the researcher and the participating teacher with the goal of designing tasks and protocols for the following teaching session. However, this collaboration was more challenging than expected. I summarize these challenges in terms of two dilemmas. One dilemma lies in identifying my role in this session. On one hand, the teachers positioned me as an expert in conducting the study. They were aware that this was my dissertation study and desired to be "helpful". Consequently, they sometimes tended to look for my inputs regarding task design. On the other hand, I tended to wait for their contribution and gain insights into their decision-making, including the extent their decisions were sensitive to their SOMs. This dilemma made it hard for us to put down a concrete protocol within a one-hour conversation.

Relatedly, it was challenging for me to decide how much intervention and contribution I should provide. On one hand, I intended to give the teachers autonomy when designing a task. On the other hand, I considered it necessary to offer guidance and suggestions, especially orienting the teachers to design open-ended tasks and prepare questions that might be helpful for constructing, testing, and refining their SOMs. This goal was not necessarily shared with the

teachers at the beginning of the study. For example, Ms. K tended to “fix” Taylor’s understanding based on her SOMs constructed in a prior teaching session, while I tended to test and confirm those SOMs before providing specific intervention. This discrepancy might come from Ms. K holding a teacher-mindset while me holding a researcher-mindset. These different dispositions influenced our choice of tasks and goals during the planning sessions. It took us a few cycles to navigate the water, and as the study proceeded, we gradually adjusted mutually.

I conjecture that most teachers would need certain intervention to learn about how to design tasks during a teaching experiment. Therefore, it is reasonable and necessary for researchers to offer some guidance and suggestions especially at the first several cycles. As the study proceeds, researchers can decide if they want to provide less intervention. This will allow them to gain more insights into how the teacher’s disposition and decision-making change as they develop their SOMs. Future researchers who implement this methodology need to find a trade-off based on their research goals and research progress.

Another issue is about analyzing the planning sessions. Because the nature of the planning sessions was collaborative, the conversation was often too scattered to analyze. It was hard to tease apart which decision was coming from whom. This is why I did not discuss any instances of the planning sessions and only discussed the final products (and rationale) we agreed upon. Future researchers can consider how to structure and analyze a planning session so that it can provide useful information about the teacher’s thinking. In particular, if the research goal is to understand teachers’ pedagogical reasoning in response to their SOMs, the planning session may need to be the focus of analysis, and the researcher may need to interact with the teacher in more intentional way to reveal their decision-making.

Teaching Sessions

The goals of the teaching sessions are for the teachers to engage in authentic interactions with students and model their thinking. The researcher primarily serves as an observer and does not provide much intervention. During its implementation, I intentionally asked Ms. K to work with an individual student and Ms. N to work with a pair of students in order to compare the affordances and limitations of each approach. Including two students made the interactions more interactive and allowed the researcher to gain insights into the extent that the teacher was sensitive to the nuances of the two students' thinking. The nature of the student-teacher interactions was also different. When working with two students, the teacher often needed to ensure both students could have chance to respond to the same question. The tendency to orient the questions to another student might stop the teacher from asking follow-up questions to the previous student. By contrast, working with an individual student allowed the teacher to construct a comprehensive model of a single student's thinking without the interruption of another student. However, I also noticed moments of Taylor being disengaged because Ms. K asked many prompting questions. Future researchers can continue to monitor the affordances and limitations of either approach, hoping to generate more insights into how the number of participants may influence a teaching experiment, including the generated models, emotions, communication patterns, and power dynamics in an experiment.

Reflecting Sessions

The reflecting sessions adopted the method of video-stimulated recall interview. This method has been widely used in educational research, and different researchers used it in different contexts for different purposes. In the current study, a video-stimulated recall interview involved the teacher and I discussing a third person's mathematical thinking. In this context, I

felt unnatural to simply act as an interviewer who only requested information without contributing. When the teachers shared their interpretations of a student's thinking, I could sense that they were also looking for my affirmations and inputs and treating me as a colleague instead of an interviewer. Researchers who conduct a video-stimulated recall interview in this context may need to act and respond in a slightly different way than a traditional clinical interview. However, there is a limitation when a researcher contributes their insights or revoices the teacher's utterances. When the researcher overinterprets the teacher's thinking by imposing ideas of the researcher's own SOMs, the teacher often conveys agreement. In that case, the researcher will not be sensitive to the discrepancy between their SOMs and the teacher's because they appear to be in a state of intersubjectivity. This issue suggests that researchers should avoid overinterpretation and always interact with the teacher with a conscious attempt to test their TOMs.

Another challenge of conducting a reflecting session lies in making decisions on what instances to include in the 1 to 1.5 hours' time period. As I mentioned in the Chapter 5, my questioning focused on revealing the teacher's SOMs, including their rationality of particular moves. My analysis suggested that the questions focused on teacher moves did not reveal useful information about their SOMs. For example, when I asked: "Why did you ask this question," I expected either of the following two kinds of responses: (a) with respect to the instances where I considered the teacher moves were sensitive to the student's thinking, I expected the teacher to discuss her SOMs and explained how her questioning was sensitive to her models; (b) with respect to the instances where I considered the teacher moves were *not* sensitive to the student's thinking, I expected the teacher to explain her rationale. In fact, with respect to either type of instances, the teacher's spontaneous responses were mostly not mathematical. For example, she

would say “because I want to move the conversation forward” or “I don’t have a particular answer in my mind when asking that question, I just want to ask and see what happen.” One explanation is that the questions I posed seemed not to imply a conceptual or mathematical response. Another explanation is that in-the-moment decisions are hard to be articulated in the context of retrospective analysis. Researchers who focus on understanding teachers’ in-the-moment decision-making likely needs additional methods to have access to that kind of thinking.

To maximize the amount of useful data generated from a reflection session under certain time constraint, I suggest that future implementation should focus on asking conceptual questions. The decentering framework I propose offers guidelines of how to ask these questions. For instance, researchers should ask teachers to describe what the students might be thinking and should be attentive to whether the teacher experiences perturbation in their attempt to explain the student’s actions. This will allow the researchers to infer mental processes such as contraindicating (Type 1), explaining (Type 2), and modifying (Type 3). Meanwhile, the researchers should elicit the teachers to articulate student thinking with specificity and detail. Especially when the teachers only describe the physical actions of the students, the researchers should ask for inferences or explanations, prompting the teachers to enact inferring (Type 3) to discuss mental actions beyond observable behavior. Researchers can also ask questions that support comparing (Type 4) or re-presenting (Type 0). Example questions corresponding to each mental process are included in Table 14. It is also productive to focus on a smaller number of instances but have a deeper discussion about those instances instead of covering a wide range of instances in a reflecting session.

In addition to the issues related to the implementation of a reflecting session, I acknowledge that the opportunity to conversate with the teachers to analyze student thinking

retrospectively affords the researcher's testing and refinement of their TOMs. Characterizing teacher decentering only based on in-the-moment observation of a teacher's observable actions may not fully account for the teachers' rationality or goals of those actions. A researcher's efforts to test and revise their TOMs by interacting with the teacher are essential to strengthen their claims about teachers' decentering actions.

Theoretical Approaches

I also want to conclude about how teacher knowledge and knowing were approached and analyzed in the study. Specifically, this dissertation is committed to operationalize an anti-deficit, dynamic, and adaptive perspectives on teacher knowledge and knowing to inform teacher education and research.

A dynamic (or sustainable) perspective assumes teacher knowledge is not fixed but develops through teachers' ongoing teaching practices. The research design of this study included repetition of teaching cycles that offered teachers opportunities to engage in cycles of planning-teaching-reflecting to model students' mathematics. The research goal was not to evaluate or describe the teacher's existing knowledge but to capture the evolution of teacher knowledge during the course of ongoing student-teacher interactions. Another aspect that aligns with this perspective is the focus on characterizing the mental processes of knowing. The seven mental processes depicted the mechanisms by which teacher knowledge comes into being, and therefore, they are not features of knowledge states but are types of reasoning that can engender sustainable growth in teacher knowledge. When these reasoning modes become habitual, teachers can continually update their knowledge in response to their experiences.

An adaptive (or constructive) perspective assumes teacher knowledge develops by building on existing knowledge. This assumption relates to a distinction between the *application*

and *construction perspectives* on teacher knowledge in my prior work (Liang, 2019). An application perspective assumes teachers' observable behaviors are driven by existing knowledge. A researcher adopting this perspective explains a teacher's actions by inferring what knowledge the teacher possesses. By contrast, a researcher adopting a construction perspective focuses on what knowledge a teacher constructs in-the-moment. In reviewing extant literature on teacher knowledge, there are more literature that adopts the application perspective than the construction perspective, and only a few discuss their perspectives explicitly. The proposed framework of teacher decentering conveys both perspectives. Both Type 1 and Type 2 Decentering involve a teacher operating on their existing mathematical meanings, and a researcher's sensitivity to these two types of decentering requires the researcher to build SOMs of those existing meanings. In comparison, Type 3 and Type 4 Decentering capture the constructive processes entailed by the emergence of new knowledge. The associated mental processes, including modifying, comparing, and restructuring, shed light on how teacher knowledge develops through reflecting upon prior knowledge. This perspective explains the productivity of conceptualizing FOMs as containing SOMs (see the last model in Figure 4) as it allows the possibility of SOMs and FOMs to evolve into a coherent system.

The adaptive perspective on teacher knowledge also necessitates a view of mathematical knowledge in terms of mental actions and operations. The term "knowledge" may refer to a wide range of ideas an individual knows (e.g., a teacher knows their students learn better with visuals), but mathematical meanings, SOMs, and FOMs as I use in this dissertation must refer to mental actions and operations. Mental actions and operations constitute the cognitive structure that governs an individual's mathematical activities. They are not properties of thinking, but the content of the thinking itself. In realizing this perspective during my interactions with the

participating teachers, I always pressed for speaking with meanings. Accordingly, I speak with meanings when making claims about the teachers' decentering actions in this dissertation. It is only through fine-grained analysis of mathematical meanings and constructions of holistic SOMs of the teachers that I can infer the diverse mental processes that engender those meanings.

Last but not the least, I argue for an anti-deficit perspective on teacher knowledge and knowing. None of the seven mental processes are framed to be deficit processes. They are all mental processes critical for teachers' sense-making of student thinking. I do not characterize a teacher as lacking certain knowledge of students or misinterpret a student's thinking; instead, I decenter from my SOMs of the students to model what the teachers understand about the students and how the teachers revisit, revise, or reorganize their meanings. Therefore, an anti-deficit perspective necessitates the adaptive perspective mentioned above. I also note that using an anti-deficit discourse to discuss teacher knowledge and knowing does not mean that a researcher does not attend to the limitations of teacher knowledge and knowing (see Adiredja, 2019; Adiredja & Louie, 2020). As seen in my descriptions of results, I repeatedly referred to the perturbations the teachers experienced and how their existing SOMs could not account for the students' mathematical actions. These perturbations were in fact fundamental to motivate subsequent actions of decentering, and therefore they are worth being valued as well.

In addition to my commitment to an anti-deficit perspective, decentering also served as an avenue for teachers to shift towards an anti-deficit perspective on student thinking. For example, Ms. K shifted from conceiving Taylor's activities to be inconsistent to realizing that the inconsistency was only perceived from her own perspective but not Taylor's. Ms. N was also explicit that a misconception is a conception to a student, acknowledging the inherent coherence of student thinking despite the challenge of disturbing that equilibrium from her perspective.

I note that these perspectives are not specific to particular taxonomies or frameworks of teacher knowledge in extant literature. A researcher may or may not be committed to these perspectives when they operationalize specific constructs of teacher knowledge to explain teacher actions. Summarizing these perspectives adopted in the present dissertation is to illuminate their impacts on how I make claims about teacher knowledge and knowing, including the choices of methodological and analytical approaches that enable those claims. I call for future researchers to also explain their perspectives beyond describing the teacher knowledge frameworks and constructs in use. While the relevant constructs do play essential roles in guiding a study, it is equally important to discuss how those constructs are operationalized in our work with teachers and analysis of teacher thinking.

Implications on Teacher Education

In closing this chapter, I discuss how my findings inform teacher educators of designing and structuring teacher education courses and professional development programs.

Design of Mathematics Content Courses

A mathematics content course is a common type of course offered to prospective mathematics teachers. The major goal of a content course is to support prospective teachers' constructions of productive mathematical meanings that reflect extant research on students' mathematical cognition (i.e., research-based SOMs) (therein after referred to as G1). A typical structure of a content course is engaging teachers in mathematical activities and discussions as if they were students. Teachers' participation in these activities serve as the experiential basis to reflect upon their preexisting FOMs. In supporting this reflection, an instructor focuses on perturbing the teachers' unproductive meanings and engendering construction of generative meanings that are consistent with the research-based SOMs. This construction lays the

foundation for subsequent learning, including learning about and learning from students' mathematics (described below) and considering pedagogy of teaching the relevant content.

In light of the theoretical and empirical work of this dissertation, I propose three additional goals of a content course. Specifically, I suggest infusing students' mathematics into content courses given the powerful role of SOMs in teachers' mathematical development.

Supporting Teachers' Constructions of Second-Order Models

First, I suggest teacher educators provide teachers with rich opportunities to construct SOMs of students' mathematics (therein after referred to as G2). As implied by this dissertation, teachers' SOMs develop through rich mathematical conversations with students and in-depth retrospective analysis of the students' thinking. Teacher educators can use authentic student videos or student work and ask teachers to interpret particular students' thinking. During this process, teacher educators should support teachers making inferences about student thinking beyond describing student behavior and support them speaking with meanings when making an inference. Teacher educators can also ask teachers to conduct clinical interviews with individual students to model and describe their thinking with specificity and detail. As suggested in the present study, both participating teachers appreciated the opportunities to converse with individual students, as these opportunities enabled them to build fine-grained models of student thinking, which might not be accessible during whole class instruction.

Supporting Teachers' Conscious Awareness of Mathematical Meanings

Additionally, I suggest that teacher educators focus on increasing teachers' conscious awareness of the mental actions constituting their FOMs (G3). I note that teachers' mathematical activities associated with G1 and G2 do not necessarily imply that they are fully consciously aware of the meanings they construct or operate on. Teacher educators can engender teachers'

conscious awareness of their FOMs and SOMs through multiple avenues. First, teacher educators can present teachers with curricula examples and prompt them to critically reflect on how the curricula texts may hinder or support students' constructions of the meanings compatible with their recently constructed FOMs (associated with G1). Second, building on G2, teacher educators can ask teachers to compare their SOMs with their FOMs to discern consistencies and inconsistencies in those meanings (i.e., Type 4 Decentering). This allows the teachers to become more aware of their own mental actions in light of their SOMs of the students' mental actions. Humans are rationalizers and tend to assimilate. Student actions trigger perturbations in teachers' knowledge system. Observing and reflecting upon students' actions allow teachers to be skeptical about what they know and not take their FOMs as given. For example, Ms. N had been operating on length when reasoning with geometric dilation, but this operation was not brought into the forefront until she considered an alternative meaning of dilation (e.g., dilating areas). To support teachers' distinction between different ways of thinking, teacher educators can also introduce to teachers relevant research constructs or distinctions (e.g., direction of change versus amounts of change; Carlson et al., 2002; P. W. Thompson & Carlson, 2017; static graphical shape thinking versus emergent graphical shape thinking; Moore, 2021; speed as a rate versus speed as a length; P. W. Thompson, 1994; P. W. Thompson & Thompson, 1994).

To conclude, through iteratively discerning and contrasting mathematical ideas conveyed from these different sources, the teachers are able to clarify and solidify the mathematical meanings constituting their FOMs and SOMs. The teachers also become reflectively aware of how ideas are developed and connected both from their own perspectives and from students' perspectives so that these perspectives are internally coherent and pedagogically actionable.

Fostering Modes of Reasoning with Students' Mathematics

Third, I suggest that teacher educators foster modes of reasoning with students' mathematics that enable teachers' continual generation of and reflection upon personal and students' mathematics (G4). There is a Chinese saying that “授人以鱼不如授人以渔,” whose English translation is “It's better to teach someone to fish than giving them a fish.” Telling teachers about research-based SOMs is “giving them a fish.” This is an important avenue to construct SOMs because, as I repeatedly emphasize, such constructions are effortful and time-consuming, and teachers likely do not have time to analyze every student's thinking deeply. In addition to that, teacher educators should also teach teachers how to construct SOMs from their own interactions with students. Instead of directly disseminating to teachers research knowledge about student thinking, teacher educators should support teachers coming to know about students' mathematics. The seven mental processes identified from this study provide guidelines of how teacher educators may support sustainability of teachers' growth in knowledge of student thinking. For example, teacher educators can design concrete tasks, activities, materials, and educational opportunities to foster each mental process in systematic ways. They can prompt teachers to enact specific reasoning process through strategic questioning as they analyze student work or videos. For example, one can ask, “How does Student A's thinking different than Student B's thinking?” to support comparing. This kind of questions can guide teachers who might otherwise not be inclined to ask themselves those questions to engage in a targeted mental process. Such intervention can first be provided in the context of retrospective analysis of student thinking, and then teachers can practice them during in-the-moment interactions with individual students or a small group of students. It may gradually improve teachers' sensitivity to student thinking and ability to act responsively during instructional situations. In general, the goal is to

nurture habitual and systematic ways of engaging with students' mathematics and personal mathematics so that teachers' SOMs and FOMs can continually grow.

Reflection on Teaching Contests and Exemplary Lessons

Recall that in the Introduction chapter, I mentioned that teaching contests and exemplary lessons are common teacher professional activities in China. Although these activities are structured in a variety of ways in different countries, and scholars have different opinions about their goals, one common cultural value conveyed by these activities is that mathematics teaching can be competed and compared. According to Li and Li (2009), features of a prize-winning lesson (also as one form of exemplary lesson) include: (a) depth and coherency of mathematical content conveyed by a teacher's "clear identification and handling of both important and difficult points" of a topic, "careful and intensive study of the textbook," and appropriate sequencing of problems and problem variations, (b) student engagement and involvement evidenced by "frequent and various interactions between the teacher and students, in addition to students' own efforts either individually or in groups," and (c) use of multiple instructional strategies such as differentiation and generalization from specific examples. Feature (a) suggests that a lesson is evaluated in term of the teacher's first-order conceptualization of the content, including how mathematical ideas are progressed and connected. Feature (b) and Feature (c) suggest that a lesson is evaluated in terms of teacher actions. These features are consistent with my personal experience of participating in teaching contests and being a participating student in numerous exemplary lessons by my former teachers at schools.

I notice that although student engagement and participation are considered to be a key feature of a good lesson, the detail of student thinking and the teacher's responsiveness to that thinking are not being evaluated or considered. Due to this lack of focus on student thinking,

when the video tapes of prize-winning and exemplary lessons are disseminated or studied during lesson studies, other fellow teachers likely only learn from content delivery and instructional actions but not how students think. Often times, teachers simply mimic the entire instructional sequence in their own classrooms because the lesson is considered to be a “good” lesson. Wong (2009) suggested that teachers should reflect deeper on certain aspects of an exemplary lesson beyond imitating the lesson. This leads to an important question: What should a teacher learn from a prize-winning or exemplary lesson? As implied by the present dissertation, perhaps an alternative focus can be on the students’ mathematical thinking and the student-teacher interactions that elicit and draw upon that thinking. Teacher educators should support teachers building professional communities where teacher can engage in in-depth conversations around second-order thinking in addition to first-order thinking and instructional actions. I also wonder: How often do mathematics teacher educators or researchers provide educational or teaching recommendations solely based on first-order knowledge as opposed to second-order knowledge, and what are the consequences of this orientation?

Any form of teaching competition requires quantification of a teacher’s knowledge and skills. It implicitly reinforces an evaluative stance of teacher knowledge and knowing and instilling in teachers that teaching skills and teacher knowledge are comparable among individuals and that their professional growth is achieved in comparison with others. This assumption contradicts a constructivist view of learning and teaching that learning is personal and teaching is a personal journey of exploring students’ mathematics. I am not claiming that teachers should not be expected to possess and construct a large number of SOMs, but I argue that this construction process is so personal and cannot be forced. As seen in the two teacher cases of the present study, each teacher’s constructions of students’ mathematics entailed

iterative and effortful processes, and only the students' actions that were perceived, internalized, and sustained in the teacher's mind might generate pedagogical power in their classrooms. Such constructions likely cannot be achieved by simply observing someone else's lesson. Therefore, it is imperative for teacher educators to support teachers learning from their own practices, from others' practices, and from their own students. The present dissertation offers examples and guidelines of how such support can be provided.

CHAPTER 9

LIMITATIONS AND FUTURE WORK

The more I learn, the more I realize how much I don't know.

By Albert Einstein

In this chapter, I outline the future research agenda in light of this dissertation. One area of focus is to develop a generalizable account of how individuals learn from others through social interactions, including studying decentering in the context of student-teacher interactions, student-student interactions, and teacher professional development. These efforts will also contribute to the refinement of the teaching experiment² methodology and the proposed framework of decentering. The second area of focus is to formulate the endeavor of second-order modeling, including clarifying the types of inquires involved in second-order modeling and what a SOM encompasses. This formulation will inform researchers and teacher educators about how to design curriculum materials to support teachers and future researchers in modeling individuals' thinking scientifically. The third area is concerned with the connection between teacher decentering and pedagogical reasoning. Finally, I propose debate as a professional development activity to foster teacher decentering.

Teacher Decentering and Student-Teacher Interactions

One major line of research is to continue to investigate teacher decentering in the context of student-teacher interactions. I break down the potential research agenda into four directions.

Broadening the Contexts

One way of broadening is to work with a broader student and teacher population. In the current study, I only worked with two teachers, both of whom were early career teachers with

strong mathematical backgrounds. They also came from the same teacher education program, which offered sufficient training that emphasized attention to student thinking in mathematics teaching. Future researchers can work with teachers with more diverse experience and beliefs including veteran teachers and teachers who are less oriented to student thinking. Comparison of the decentering actions of different teachers will afford insights into the factors that condition or support teacher decentering. It will also be interesting to compare elementary teachers' decentering actions to secondary teachers' decentering actions. Elementary teachers work with students of younger age and on more basic mathematical ideas, so I conjecture that it might be more challenging for them to decenter than secondary teachers. Besides, elementary children are likely less articulated than secondary students, and I am curious if that may constrain teacher from decentering.

Another way of broadening is by working on other topical areas in mathematics and STEM. In this study, I only focused on the topics of linear programming and geometric dilation. Future researchers can focus on other topics to understand how the framework of teacher decentering (including the seven mental processes) may or may not apply to broader STEM contexts and improve the framework. Besides, I mentioned that I intentionally chose areas that I was unfamiliar with and that had received little attention in literature. This allowed me to genuinely engage with the teachers' and the students' thinking without imposing my first-order thinking during interactions with them. Moving forward, I am open to work on areas that I possess more research-based SOMs and reflect on how my ways of operating might differ. Such investigation will afford an understanding of whether and how a researcher's choice of mathematical topics is critical to the researcher's implementation of this methodology.

I also call for future researchers to work with teachers in more diverse educational contexts, for example, comparing teacher decentering in a paper-and-pencil environment and that in a technology-rich environment. Within a technology-rich environment, student thinking is often embodied in and conveyed by gestures and physical actions (e.g., pointing, dragging), and I wonder: How teacher decentering may manifest in different ways when students' utterances are replaced by frequent use to physical actions? Moreover, mathematical learning in emerging technological contexts (e.g., 3D printing, computing, gaming, dynamic geometry software, graphing calculator) is likely novel to many teachers. I conjecture that decentering may not only be an important avenue for teachers to access student thinking but also provide teachers with experiential basis of building content knowledge associated the use of a particular technological tool.

Finally, I wonder how the nature of student-teacher interactions and teacher decentering might differ in classroom settings when compared to laboratory settings. Teachers likely consider themselves as taking more roles in a classroom than in a research study, and classroom teaching requires in-the-moment interpreting and responding. I anticipate that second-order modeling in a classroom might be more challenging to teachers. Future research can investigate the extent the framework of teacher decentering may apply to situations that require teachers to work with larger group of students. Teacher educators can also consider how to support teachers transferring their decentering actions from a small-scale setting to a classroom setting and from retrospective analysis to in-the-moment teaching? If classroom second-order modeling is a demanding task, perhaps a teacher can offload some responsibilities to the students by supporting student decentering (see a more detailed discussion in a latter section)?

Methodology Testing and Refinement

In Chapter 8, I have mentioned that future researchers can test and refine the teaching experiment² methodology by varying the extent that they provide intervention in the planning sessions or by working with different number of student participants in the teaching sessions. In addition, future researchers (including myself) can test two alternative decisions. First, in preparing the reflecting session, all video clips were chosen by me instead of the teacher, and therefore I did not have insights into what instances the teacher considered to be critical or interesting. In this mplementation, each instance was considered as a didactic object (P. W. Thompson, 2002) that triggered the teachers' mathematical discussion, and those instances might not be what the teachers would pay attention to spontaneously. I decide not to ask the teachers to watch and select video clips on their own because I tried not to increase the teachers' participation hours. If possible, future researchers can adopt the alternative approach to understand its affordances and limitations. Another way to gain insights into the critical moments that catch the teachers' attention is to conduct a debrief session immediately after the teaching session. Ms. K was exhausted after every teaching session and full-day teaching, and thus we did not do the debrief. Ms. N was willing to do a debrief because she desired to discuss her observation when her memory of the teaching session was fresh. However, she only participated in two cycles, so the differences of these two approaches were not apparent. Future work can continue to compare the affordances and limitations of either approach.

Improving the Theorization of Teacher Decentering with Piagetian Constructs

In this study, I only studied the mental processes entailed by teacher decentering and the productivities and mathematical consequences of teacher decentering. I have not studied the conditions of teacher decentering; that is, what are the factors that constrain teacher decentering?

In my theorization, I have emphasized that teachers' conscious awareness of the mental actions constituting their FOMs is critical for them to decenter, especially for enacting Type 3 and Type 4 Decentering to accommodate their FOMs. This leads to a broader research question for future researchers: How does the nature of a teacher's content knowledge (or FOMs) enable or condition their accommodation? How do the SOMs of a teacher who frequently enacts Type 3 and Type 4 Decentering differ from a teacher who primarily enacts Type 0 and Type 1 Decentering? What enables a teacher's inferences about student thinking and sensitivity to nuances of student thinking?

In addition, future researchers may consider incorporating Piagetian notions of pseudo-empirical abstraction, reflecting abstraction, and reflected abstraction to make distinctions between the four types of decentering. Insights into how these constructs can be operationalized in analyzing teacher decentering (i.e., identifying indications and contraindications in terms of these types of abstraction) may help further clarify the four types of decentering as I introduce in this dissertation.

Design-Based Research

When the framework of teacher decentering is stabilized and generalized, researchers can conduct design-based research to develop, test, and refine curriculum materials for teacher education courses and professional development programs. Researchers can conduct interventionist studies to identify ways of supporting teachers' development in their ability to decenter, including supporting their enactment of all kinds of mental processes in different educational contexts.

Student Decentering and Student-Student Interactions

Future research can also extend the research line of teacher decentering by studying student decentering in the context of student-student interactions (e.g., Paul Cobb et al., 1992; Hoyles, 1985; A. Moon et al., 2017; Rasmussen et al., 2020). When working with Carlee and Heidi, I observed some listening and interpreting among their interactions. I conjecture that student-student interactions, including students' reflection on each other's thinking, would be generative to students' mathematical learning. Students have their own language and unique ways of communicating with each other. It may be beneficial for them to learn through listening to and reflecting on their peers' thinking. They may also be more engaging with understanding peers' thinking when compared to understanding adults' thinking due to both cognitive and non-cognitive factors. Therefore, it will be interesting to explore: How do students decenter from their personal mathematical understandings and understand the perspectives of their peers? How do this engagement and the subsequent reflection on multiple understandings contribute to the students' mathematical development? How may student decentering differ from teacher decentering? Some students may be more willing to share their thinking with their peers in small group than with their teachers during whole-class discussion. Insights into the learning processes involved in student-student interactions will inform teachers of how to support those interactions and create an inclusive learning environment.

Autistic students are often labeled as having difficulties with communicating and socializing. Research has shown that peer interactions are beneficial to autistic students' learning (DiSalvo & Oswald, 2002; Kamps et al., 2002). However, existing research has primarily focused on how socially competent peers model and reinforce social behavior for autistic peers. Future research can investigate the nature of peer-mediated instruction in the context of

mathematical learning and how learning occurs (e.g., Tan & Alant, 2018). Research is also needed to understand how teachers can better support non-autistic peers to engage in productive mathematical conversations with autistic students. These research focuses directly answer the recent call for anti-deficit and neurodiversity perspectives on the mathematical capabilities of students with disabilities (J. H. Hunt et al., 2019; Lambert et al., 2018; Lambert, 2018; Lambert et al., 2020; Lambert & Tan, 2017; Tan et al., 2019; Yeh et al., 2020).

Teacher Decentering and Teacher-Teacher Interactions

The third line of research is concerned with teacher decentering in the context of teacher-teacher interactions, including interactions in professional development programs. Future researchers can focus on unfolding the mechanism of teachers' mathematical and professional learning through interacting with colleagues. Relevant findings will inform designs of teacher education and professional development programs for supporting teachers in engaging in productive conversations in professional contexts.

Towards a Generalizable Account of Learning About and From Others

Drawing from findings of the aforementioned research, researchers can formulate general learning theories about how individual learns from others mathematically through social interactions. They can also test and refine the theories in contexts outside education and clarify the affordances and limitations of the theories in varied contexts. This attempt includes extending extant research on teacher decentering to the field of social psychology, where researchers can study how decentering manifests in romantic relationships, psychotherapy treatment, legal aid services, courtroom debate, and other social situations. Researchers who are interested to investigate these phenomena should also continue to develop, implement, and refine methodologies that simultaneously provide individuals with opportunities to engage with others'

thinking and that allows researchers to observe, hypothesize, and document the individuals' ongoing understanding of others and their actions in response to that understanding.

It will also be interesting to investigate the applicability of the theories of decentering in international contexts. For example, what are the limitations and affordances of these theories in explaining Chinese students' and teachers' subject learning? Understanding what happens at the confluence of culture, philosophy, and education will allow researchers to expand the theories to account for broader phenomena in broader contexts.

Formulating the Endeavor of Second-Order Modeling

Although this dissertation is a study about teachers' second-order modeling activities, it also implies patterns and features of researchers' endeavor of second-order modeling. Moving forward, I call for researchers to formulate a generalizable account of the endeavor of second-order modeling. The proposed framework of teacher decentering in this dissertation specifies the mental processes related to second-order modeling, but I iterate that the emergence of these processes was dependent on intensive support of the researcher who had experience of modeling student thinking. The two participating teachers likely were not aware of the different kinds of mental processes they engaged in because they were merely processes modeled from a researcher's perspective. To make the endeavor of second-order modeling more accessible to teachers, teacher educators, and other researchers, it is imperative to explicate the different modes of inquiry a researcher engages in during second-order modeling. A researcher can ask: What do I ask myself when modeling and analyzing student thinking? To partially answer this question, I list seven aspects in the following (see also Figure 85).

- **Doing What:** focusing on the student's observable behavior and identifying regularities in the student's behavior. It involves inquiring "What is the student

doing?” “Does what the student doing contraindicate what I will do?” (see also Type 1 Decentering)

- **Thinking What:** inferring the students’ mental actions to construct SOMs (see also Type 2 and Type 3 Decentering). It involves inquiring “What might the student be thinking that drives them to perform a certain behavior?”
- **Why:** hypothesizing other related mental actions that can explain a particular behavior or way of thinking SOMs (see also Type 2 and Type 3 Decentering). It involves inquiring “Why does the student think in this way?”
- **What’s Different, What’s Similar:** comparing multiple ways of thinking to identify differences and similarities (see also Type 4 Decentering). It involves inquiring “How is the student’s thinking compared to how I think?” “How is the student’s thinking compared to how other students think?” “How is the student’s thinking compared to how the same student previously thinks?”
- **Then What:** considering implications or mathematical consequences of a way of thinking, including conjecturing the perturbation the student may experience in other situations or how the student may generalize their thinking to other situations. It involves inquiring “If the student thinks in this way, what is going to happen in situation X?”
- **What’s Next:** hypothesizing the next mental action or operation that the student might be able to construct and designing a situation that may engender that mental action. It involves inquiring “Given what the student is thinking now, what progress can they make?” “What tasks may be able to support that thinking?”

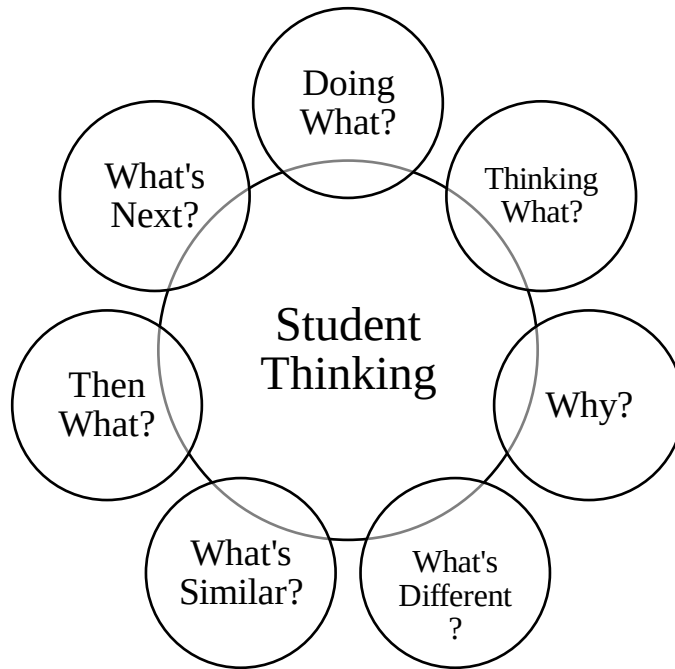


Figure 85. Types of inquiries in relation to second-order modeling.

Looking across these forms of inquiry, one may realize that the constructions of stable and explanatory second-order models is more than inferring what a student is thinking (i.e., Thinking What). It is a systematic and scientific exploratory process that requires intentional decision-making and specific modes of reasoning. Each of these forms of inquiry are enacted during the course of a wide range of research activities, including research design (e.g., ongoing task design in response to evolving SOMs), implementation (e.g., enacting discourse moves in the moment of interactions with the participants), data analysis (e.g., making claims about the participants' thinking when reporting research outputs), and generating future research agenda of investigating student thinking in relation to a particular idea or reasoning process.

Teasing apart these different dimensions of second-order modeling also motivates a question about what we mean by an SOM. The research community may need to reach a consensus in terms of what an SOM encompasses: Do we define all these aspects as constituting components of an SOM, or an SOM only includes *thinking what* (and maybe also *thinking why*)?

Furthermore, in this study, I limit the notion of SOMs to only include hypothetical mental actions and operations of individual students. A broader conceptualization of SOMs may also include models of how students learn among students or models, how student thinking develops (e.g., learning trajectories), and how students interpret a teacher's instructional actions (e.g., constructing SOMs of the teacher's mathematics; Yoon, 2019). Each of these types of SOMs are worth studying, including their nature, constituent components, and roles in influencing teaching and learning.

Together, these research efforts in formulating the notions of second-order modeling and SOMs will have practical implications in doctoral education and teacher education. Researchers, educators, and mentors can draw on the relevant findings to develop curriculum materials and research guides to support graduate students, future researchers, and teacher leaders to conduct teaching experiment and construct SOMs. Teacher educators can use these materials to support future teachers to become teacher-researcher in their classrooms. In the meantime, researchers will become more aware of the aspects constituting their second-order modeling practices and thus be able to make more careful and rigorous claims about individuals' thinking. For example, when disseminating their results, they should be explicit about their observed evidence (i.e., indications and contraindications), their inferences, their ongoing hypotheses, their wonderings, their decision making, and their first-order thinking, and differentiate rather than conflate these aspects.

Linking Teacher Decentering and Pedagogical Reasoning

I have briefly mentioned in my results that teacher decentering is pedagogically powerful. Both Ms. N and Ms. K's SOMs influenced their task design, goal structures, and teacher moves. Those tasks, goals, and instructional actions in turn supported their constructions of SOMs. I

have not analyzed the bidirectional influence between teacher decentering and pedagogical reasoning in detail in the present study. Future research can explore how teachers' decentering actions may influence or be influenced by their pedagogical decisions and goals.

Many researchers have unpacked the reasoning and decision-making associated with their second-order modeling activities during a teaching experiment (Ellis, Fonger, et al., 2019; Hackenberg, 2010; A. G. Thompson & Thompson, 1996; Tzur, 1999). For example, Ellis and colleagues (2019) explained how learning trajectories (as one form of SOMs), task choice, and pedagogical supports worked together in the design, implementation, and analysis of a teaching experiment. More research is needed regarding teacher decentering in the context of teaching experiment². Researchers can study how teachers make pedagogical decisions in response to their SOMs and how these decisions generate productive outcomes in student learning. And as I mentioned earlier, Type 4 Decentering beyond Type 3 Decentering is likely critical to teachers' transformation of their SOMs into pedagogical decisions. Therefore, one way to understand the "how" is to further identify the types of reasoning aligned with Type 4 Decentering. Relevant findings about the linkage between teacher decentering and pedagogical reasoning will inform design of coherent content and pedagogy courses in teacher education programs.

Debate as a Teacher Processional Activity

I also consider promoting debate as a professional activity in teacher education. Debate is a formal discussion on a particular topic in which opposing arguments are put forward. In our daily life, there are different forms of debates such as court debates, presidential debates, and parliamentary debates. Many universities have student debate associations, and their members regularly participate in competitive debates at the local, national, and international level. To conduct a debate, a group of participants can be divided into three groups: (a) one affirmative

team consisting of 3 to 4 members who support a resolution, (b) one opposing team consisting of 3 to 4 members who oppose a resolution, (c) one judge team (also the audience) who judge the quality of the evidence and arguments of both teams, and (d) a debate moderator who ensures the debate event moves at pace. Before the debate starts, the judges vote for one standpoint of the resolution based on their own opinion. Then, speakers from both teams will take turn to present their arguments (first speaker from the affirmative team, first speaker from the opposing team, and so on). When the debate is finished, the judges vote for another time, and the winner of the debate is the team that gains votes (as opposed to losing votes) after the debate. Before the result is announced and after the debate is finished, the audience can have a debrief and discussion, asking questions and contributing additional arguments and opinions.

Preparing and participating in a debate requires an individual to hypothesize others' standpoints and arguments, which is in essence a decentering activity. Teacher education programs can organize debate events to engage teachers in discussing some prevalent educational topics, policies, and practices. This kind of activity will offer pre- and in-service teachers opportunities to not only enhance their ability to decenter but also think critically about central issues in mathematics and mathematics education. There is recent publication titled *Debates in Mathematics Education* (Ineson & Povey, 2021), which includes leading scholars' discussion on contemporary debates in mathematics education. Teacher educators can design debate resolutions based on this resource. Additionally, I provide some sample debate resolutions in the following:

- Do you think mathematics is discovered or invented?
- Someone claimed that $2+2=4$ is cultural but not objective, do you agree or disagree?

- If you can press a magic button such that knowledge can be instantly shared among all human brains, will you press it or not?
- Some researchers suggest that cross multiplication should be removed from mathematics curriculum, do you agree or disagree?
- Should calculus ideas be introduced to elementary students?

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APPENDICES

Appendix A: Sample Protocols

Planning Session (Cycle 5, Ms. K)

1. In the last session, we interpreted that the student's decision on whether he needed to maximize both constraints was related to the contextual differences in the problems. What does this imply about his thinking? What are the mathematical consequences of it?

- How does this affect a context that doesn't involve money
- He applies the implications of the contexts to his solutions and his methods to make sure they make sense.
- The solution method is subjective, making it difficult to generalize or anticipate.
- It makes it harder for the student to be consistent in his solution methods.
- It makes it harder to know what constraints to keep track of in certain scenarios.
- Lacks metacognition
 - He uses the contextual difference as a scapegoat for not having a mathematical reasoning for using different solution methods or making adjustments to his solution methods.

2. Thinking of the student's method of solving linear programming in general, what are the advantages and limitations of it?

- Advantages:
 - o It's efficient
 - o Not a lot of work for the sake of doing work
 - o Limits the amounts of points to be tested in the objective function because he does not consider all the possible solutions.
- Limitations:
 - o It's not accessible for all students (doesn't have many entry points for students)
 - o More difficult for students who lack reasoning skills
 - o The adjustments that are made between problems are intuitive (students must be flexible in their solution)

3. Why is a graphical approach counterintuitive to students? Hypothesis?

- Graphing is not a default way to find solutions
- Students feel more comfortable with an algebraic approach to solving problems
- Lacking foundational knowledge of functions

4. What are the mathematical understandings we can focus on supporting in the next session:

5. Task idea?

- 3 constraints

- Election Task

6. Hypotheses of how he will think and of his difficulties?

- Struggle with writing the constraints
- Generalizing his method into an equation

7. Probing/supporting questions to ask:

- How many possible combinations that satisfy each/one specific constraint?
- What are all the possible combinations that satisfy one of the constraints?
- How can we represent ALL the possible solutions?
- If you spend \$2 on radio, how much do you have to spend to make sure you are reaching at least 2000.

Reflecting Session (Cycle 3, Ms. K)

Supplies

- ☐ My computer
- ☐ Two cameras and two tripods
- ☐ Markers, pens, highlighters, White papers
- ☐ Protocol
- ☐ Student work

[06:20-06:57] Could you summarize his thinking in this clip? At the beginning, when he said “that’s way too much money,” what do you think he meant? And then, you drew his attention that that was money he brings in, why do you think that was important or helpful?

[07:22-07:46] What do you think he was confused about after testing out the first case? You asked him to try out something, why do you think that would be helpful?

[08:26-08:57] How did he determine “it was too much” or not?

[09:46-10:19] Why did you ask him to clarify how he chose the numbers?

[13:23-13:56] How did he decide 5000,5000 is the best solution? And maybe ask her to give a summary of the student’s entire method.

[15:24-15:51] [19:42-20:25] Why do you think he did division to find the total income?

[21:11-22:45] What was this conversation about? How do you interpret his response? At the end, you mentioned the cost of video, what were you thinking?

[24:50-25:37] What were you thinking about the other task? [Before this, the student don’t feel good about his division equations any more, but struggled to move forward]

[24:50—25:37] What did you learn from this conversation?

[27:58-28:35][skip][30:04-30:22] I think here you shifted the conversation away from the percentage. Was that intentional? What was your goal here?

[31:22-33:07] What was he trying to do here? Was that different than how you would solve it?

[36:32-37:11] Why was he convinced that he got the maximum? You asked a question to try to challenge him, what was that? [Un until here, ask her to talk through the student's thinking about finding the max income]

[39:04-39:23] What were you trying to challenge him?

[45:08-45:44] How do you interpret his response to our question? [I asked him about the 4999-5001 combination]

[Ask her to summarize the student's reasoning about the formula using the student's work]
[prepare for two printed graphs of both problems]

Now let's take some time to reflect on this student's thinking in the two problems. Could you summarize how he did each problem?

Do you think this student's ways of thinking in these two problems are consistent?

- If so, could you summarize his method in a general way? Like how he would reason with a collection of similar problems?
- If not, could you compare them and how they are different?
 - o What will be a way of thinking parallel to the A Trip to Sugar Bowl in this second situation?
 - o What will be a way of thinking parallel to this second situation in the A Trip to Sugar Bowl situation?

Do you remember in the planning session, why we decided to pick this task? Why did we think this task may be helpful for advancing his thinking or challenging him? Based on his reasoning here, what do you learn? Why do you think this situation does not challenge him at all?

Let's take some time to reflect the traditional method and the student's method. Could you compare these two methods and talk about how they relate and how they differ? What are the limitations and advantages of each?

- In what situations does his method work?
- In what situations does his method not work?

Appendix B: Pre-Teaching Session Tasks with Ms. K

A Trip to Sugar Bowl

A tourism agency can sell up to 1200 travel packages for the Sugar Bowl college football postseason game in New Orleans. The package includes airfare, weekend accommodations, and the choice of two types of flights: a nonstop flight or a two-stop flight. The nonstop flight can carry up to 150 passengers, and the two-stop flight can carry up to 100 passengers. The agency can locate no more than 10 planes for the travel packages. Each package with a nonstop flight sells for \$1200, and each package with a two-stop flight sells for \$900. Assume that each plane will carry the maximum number of passengers.

Fancy's Café

Fancy purchased a selling space in a new shopping mall to build a Café. In addition to the Café section, she also plans to include a book section so that her customers can enjoy reading books while drinking coffee. Available space, budget, and books lead to the following constraints:

- The store takes up to 10,000 square feet of floor space.
- Fancy has a budget of at most \$60,000 to spend for furnishings. Furnishing for the two kinds of space cost \$4 per square foot for books and \$7 per square foot for the Café section.
- To guarantee enough space for the Café section, the book section should be at most 1.5 times the space for the Café.

Graph all the possible ways of allocating space to the book section and the Café section under the constraints.

GeoGebra Applet:

<https://www.geogebra.org/m/j7zhnmja#material/vuzfqvpm>

<https://www.geogebra.org/m/j7zhnmja#material/m6r3guzb>

Note. This task is adapted from Fey et al. (2009, p. 137). Core-Plus Mathematics, Course 3, Unit 2.

Advertising Election

The 2020 United States presidential election is scheduled for November 3, 2020. Advertising is an important means to gain voters. When candidates plan their campaigns, they have choices to make about spending for advertising. Suppose that advisers told one candidate that radio and television ads might reach different audiences in the following ways:

- Every dollar spent on radio advertising will assure that the candidate's message will reach 5 Democratic voters, 2 Republican voters, and 4 Independent voters.
- Every dollar spent on television advertising will assure that the candidate's message will reach 4 Democratic voters, 4 Republican voters, and 1 Independent voter.

The candidate's goals are to reach at least 20,000 Democratic voters, 12,000 Republican voters, and 8,000 Independent voters with a minimum total advertising expense. What is the advertising plan that can minimize total advertising expense? What is the expense?

GeoGebra Applet:

<https://www.geogebra.org/m/j7zhnmja#material/jappy9y2>

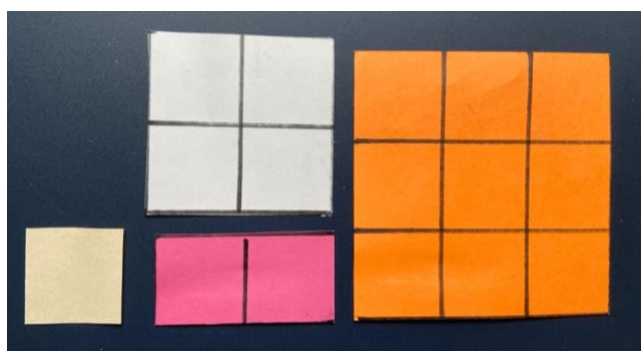
Note. This task is adapted from Fey et al. (2009, p. 147). Core-Plus Mathematics, Course 3, Unit 2.

Appendix C: Pre-Teaching Session Tasks with Ms. N

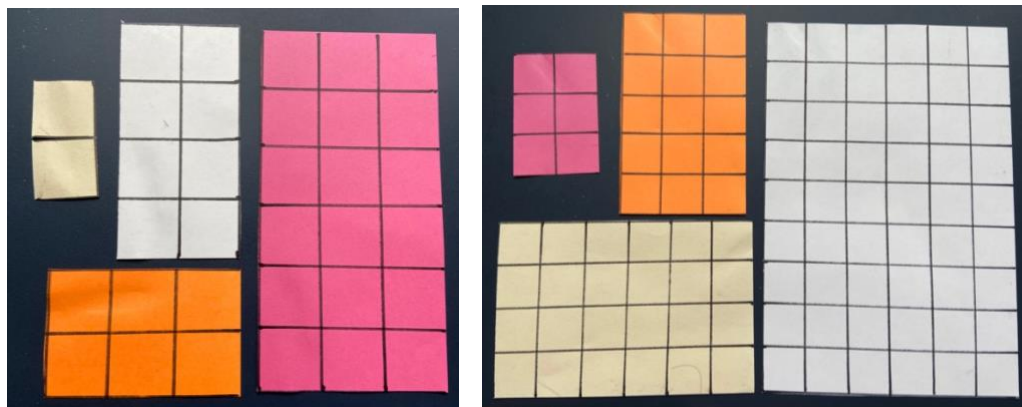
Sorting Rectangle Task (Seago et al., 2013)

Part A. Inside each bag of four rectangles, there is one rectangle that is either not an enlargement or a reduction of the other rectangles. Identify the rectangle that doesn't belong in each bag. Explain why.

Bag A



Bag B (left) and Bag C (right)



Part 2. Evaluate Victoria and Makayla's arguments about Bag B.

Victoria:

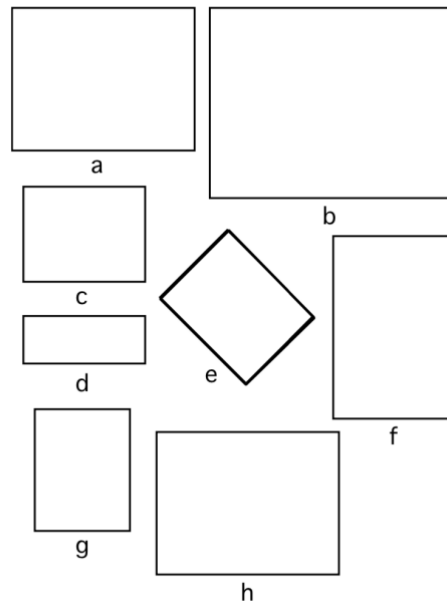
3x2 doesn't belong because notice the other numbers: 2x1, 1 is half of 2; 6x3, 3 is half of 6; 4x2, 2 is half of 4. But 2 is not half of 3.

Makayla:

3x2 doesn't belong because from a 1x2 to a 2x4 you are adding 1 to the width and 2 to the length. It's the same from a 2x4 to a 3x6.

Similar Rectangle Task (Seago et al., 2013)

Which rectangles are similar to rectangle a? Explain the method you used to decide.



Growing Triangles

Watch each of the following videos, which animation creates a series of similar triangles?

Triangle A: <https://youtu.be/bYvQDRjJMzg>

Triangle B <https://youtu.be/1157ric4h-M>

Triangle C: <https://youtu.be/W9YVwqE7QxI>

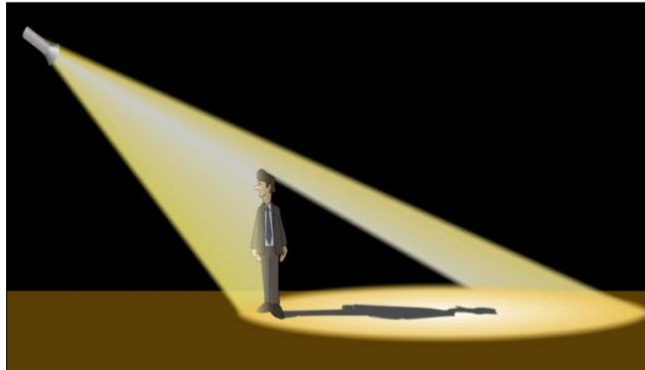
Triangle D: <https://youtu.be/sr7Oitva3iY>

Similarity Slope Graph (National Council of Supervisors of Mathematics [NCSM], n.d.)

1. Create four similar rectangles. List the dimensions for each rectangle.
2. Plot the dimensions on a coordinate graph by aligning the bottom left vertex of each rectangle with the origin (0,0), and orienting the rectangles in the same direction, thus nesting the rectangles.
3. Write the equation for the line that passes through the origin and the upper right vertices of each rectangle. Draw the line.

Floodlight Shadow Problem (MAP, 2015a)

Part A: Eliot is standing underneath a floodlight and walking away from the floodlight. Describe the relationship between Eliot's distance from the floodlight and the length of his shadow.



Part B:

Eliot is playing football.

He is 6 feet tall.

He stands exactly halfway between two floodlights.

The floodlights are 12 yards high and 50 yards apart. The floodlights give Eliot two shadows, falling in opposite directions.

What is the total length of Eliot's shadows?

How will the total length of Eliot's shadows change if he walks in a straight line towards one of the floodlights?

Rolling Cups Problem (MAP, 2015b)

1. Here are 6 cups/containers. Could you conjecture which of these objects will roll the largest circle? Why?
2. What are the factors that may affect the roll radius? Why?
3. How would you use your mathematical knowledge to test, refine, or justify your conjectures?



Appendix D: IRB Form



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Athens, Georgia 30602
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IRB@uga.edu
<http://research.uga.edu/hso/irb>

Human Research Protection Program

APPROVAL

August 29, 2019

Dear [Kevin Moore](#):

On 8/29/2019, the IRB reviewed the following submission:

Title of Study:	Teachers' Construction of Knowledge of Students' Mathematics
Investigator:	Kevin Moore
Co-Investigator:	Biyao Liang
IRB ID:	PROJECT00000317
Funding:	National Science Foundation
Grant ID:	FP00000377
Review Category:	Expedited 6, 7(a)

Materials Reviewed: Recruitment letters; Parent/Guardian Consent Form; Teacher Consent Form; Student Assent Form; Oconee County Schools Authorization Letter; Sample Questions for Reflection Session; Sample Task for Planning Session.

The IRB approved the protocol on 8/29/2019. Please submit a Progress Report by 8/28/2022.

Study activities are only approved in Oconee County Schools. When authorization is received from Clarke County, upload it via a modification.

To document consent, use the consent documents that were approved and stamped by the IRB. Go to the Documents tab, Final column, to download them.

Please close this study when all human subject research activities and data analysis of identifiable information is complete.

In conducting this study, you are required to follow the requirements listed in the Investigator Manual (HRP-103).

Sincerely,

William Westbrook, IRB Analyst
Institutional Review Board
University of Georgia