SUPPORTING STUDENTS TO MAKE MATHEMATICAL CONNECTIONS:

PROSPECTIVE SECONDARY MATHEMATICS TEACHERS LEARNING TO SPECIALIZE EVERYDAY PRACTICES

by

JONATHAN KYLE FOSTER

(Under the Direction of Denise A. Spangler)

ABSTRACT

Students participating in making mathematical connections leads to several encouraging outcomes, such as developing students' conceptual understanding and promoting their recall of mathematical procedures. However, designing this kind of instruction is difficult for many teachers, especially novice teachers. There also remains a paucity of evidence for how teachers learn to design such instruction. In this dissertation, I conducted three studies that examined how prospective secondary mathematics teachers began to leverage and specialize the everyday practices of noticing and audience design to support students to make mathematical connections.

In the first study, I examined what mathematical connections a cohort of 12 prospective secondary mathematics teachers noticed when working with secondary students in small-group instruction. Results indicated the prospective teachers attended to several kinds of connections. Further thematic analysis of the mathematical connections revealed five pedagogical considerations by prospective teachers, which led to the development of the *Pedagogical Considerations of Mathematical Connections* (PCMC) framework.

In the second study, I investigated how three secondary mathematics student teachers supported students to attend to, contribute, or provide reasoning for mathematical connections while they led whole-class discussions. In particular, I examined how the student teachers' assessment of students' expertise in the moment was evident in how they elicited, responded, facilitated, and extended students' connection-making and reasoning. From the results, I argue that student teachers' attention to students' expertise in the moment mediated how the student teachers designed and coordinated their eliciting, responding, facilitating, and extending moves to support students' participation in making connections.

In the third study, I observed how a secondary mathematics student teacher taught one lesson and then taught the same lesson again to a different class the next day. In the lesson, the student teacher intended students to make two mathematical connections through generalizing their mathematical activity. The comparison of the lesson implementations revealed that the student teacher made micro-adjustments in the second implementation to build and maintain common ground with students. By building and maintaining common ground, students were able to make the intended mathematical connections.

INDEX WORDS: Mathematical Connections, Audience Design, Common Ground, Teacher Noticing, Mathematics Teacher Education

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DEDICATION

This dissertation was possible because of the novice teachers that agreed to participate. I dedicate this dissertation to them. I also dedicate this dissertation to all the students and teachers in my life. They inspire me. I am blessed to have gotten to know and work with such amazing and talented human beings.

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CHAPTER 1

INSTRUCTION THAT FOSTERS CONNECTION-MAKING: AN INTRODUCTION AND AGENDA FOR RESEARCH

A frequently stated hallmark for conceptual understanding of mathematics is the ability to make mathematical connections. Some scholars suggest that a defining characteristic of what it means to know and do mathematics is the act of making connections (Boaler, 2002; Burton, 1998). For many education authorities and organizations in countries across the world, a goal for school mathematics is to instill in students a sense of mathematics as a coherent and well-connected discipline (e.g., Australian Curriculum, Assessment, and Reporting Authority, n.d.; Ministry of Education Singapore, 2019; National Council of Teachers of Mathematics, 2000). For example, the National Council of Teachers of Mathematics stated students should be able to "recognize and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole; and recognize and apply mathematics in contexts outside of mathematics" (NCTM, 2000, p. 64). Because mathematical connections are considered as evidence for conceptual understanding, a defining characteristic of what it means to do mathematics, and a goal for school mathematics, it is important to consider how teachers support students in making mathematical connections.

In their meta-synthesis of empirical studies, Hiebert and Grouws (2007) found that students developed conceptual understanding of mathematics if given the opportunity to explicitly attend to mathematical connections among facts, procedures, and concepts during instruction. Explicit attention to mathematical connections during instruction could include, but is not limited to,

> discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attend to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons ideas (Hiebert & Grouws, 2007, p. 383).

Unfortunately, The Third International Mathematics and Science Study (TIMSS) 1999 Video Study found that students in US classrooms experience limited opportunities to explicitly attend to mathematics connections in comparison to other developed countries (e.g., Australia, Czech Republic, Japan, Netherlands, and Switzerland) (Hiebert et al., 2003). For instance, eighth grade mathematics students experienced few opportunities to share alternative solution method (Jacobs et al., 2006). There was also a lack of evidence of the teacher or students developing a rationale, making generalizations, or using counterexamples. Few of the problems within a lesson were related in mathematically significant ways, and few problems afforded students an opportunity to make a mathematical connection within the problem. Only 1% of problems were implemented in a way that made mathematical connections explicit for students. In summary, it seems that US students experienced few opportunities to construct mathematical connections among ideas, procedures, and concepts.

The results of the TIMSS 1999 Video Study add credence to a previous call by Hiebert and Carpenter (1992) for more studies focusing on "how instruction is designed to foster connections" and to carefully assess "the connections that students make as a result" (p. 86).

Since the call by Hiebert and Carpenter, there have been some studies examining how expert mathematics teachers design instruction to foster mathematical connections (e.g., Ball, 1993; Boaler & Humphreys, 2005; Even et al., 1993; Lampert, 2001; Leinhardt & Steele, 2005). While these studies provide insight into how instruction can foster mathematical connections, they offer little insight into how teachers learn to design such instruction. This lack of insight is problematic because novice mathematics teachers typically struggle to design instruction that fosters explicit attention to mathematical connections in comparison to expert teachers (Borko & Livingston, 1989; Even et al., 1993; Leinhardt, 1989; Livingston & Borko, 1990).

Efforts to support the development of teachers' practice typically take either a top-down or bottom-up approach. The top-down approach takes the practice of expert teachers and deconstructs this practice in way that it is made accessible to other teachers, especially novice teachers. This is generally the perspective of practice-based teacher education with the emphasis on core practices for teaching (e.g., Ball & Cohen, 1999; Grossman et al., 2009; Jacobs & Spangler, 2017; McDonald et al., 2013). The five practices for orchestrating productive mathematical discussions (Smith & Stein, 2011; Stein et al., 2008) is an example of a top-down approach. Stein et al. described the intent of their five practices model (anticipating, monitoring, selecting, sequencing, and connecting) as "to make discussion facilitation something that is manageable for novices" (p. 321). The five practices are to culminate in students making mathematical connections between students' strategies to a mathematical task and highlight key mathematical ideas and practices within and across strategies.

The alternative approach, which the studies in this dissertation take, is the bottom-up approach. This approach emphasizes the beginning state of teachers' practices and seeks to foster productive and transformative change through small incremental changes to their practice.

"Incremental change does not require teachers to make sense of entirely new ideas about instruction but rather to understand small innovations in practices that they already rely upon heavily" (Star, 2016, p. 61). For example, consider the everyday practice of wait time (Rowe, 1974, 1986). Supporting teachers to incrementally increase their wait time beyond the typically 1 to 3 seconds has been found to lead to other productive changes in teachers' practice, such as asking fewer low-level questions (Tobin, 1986, 1987). In particular, this dissertation examines how everyday practices of teacher noticing and audience design can incrementally become specialized for the purposes of teaching.

Problem Statement

Problem One: Novice Teachers' Ability to Foster Mathematical Connections

The research literature depicts novice mathematics teachers as having limited ability to design and facilitate instruction that fosters students in making mathematical connections. This conclusion arose across several expert-novice studies that compared the practice of expert and novice mathematics teachers (Borko & Livingston, 1989; Even et al., 1993; Leinhardt, 1989; Livingston & Borko, 1990). Novice teachers' lesson plans often lacked any mention of mathematical connections. Novices' lessons also had fewer connections between lessons; their lessons were often presented as segmented ideas and concepts. There were few instances when novices referenced previously learned concepts or ideas. In contrast, experts' lessons were well-connected, with one lesson building on the next. For example, experts sometimes used the same representations and contexts across several lessons. Experts also reviewed previously learned concepts and ideas before proceeding on to new ideas in the lesson. In comparison to expert teachers, novice teachers were less likely to take up opportunities to support students in making connections among students' ideas or relate those ideas back to the concepts and skills identified in the objectives of the lesson. Some novice teachers argued there was no need nor time to focus on mathematical connections during lessons; other novice teachers thought that mathematical connections were important but felt it was difficult to highlight mathematical connections due to having to cover the content of the lesson, integrate unexpected student ideas, and lack of knowledge of connections among ideas.

Problem Two: Few Links Between Instructional Practices That Foster Connection-making and Student Outcomes

Few studies have fully addressed the call for research by Hiebert and Carpenter (1992) to understand "how instruction is designed to foster connections" and to carefully assess "the connections that students make as a result" (p. 86). The general consensus from research is that teachers should implement inquiry-based instructional tasks in a manner that is student-centered that supports students' mathematical understanding and practices (e.g., Ball, 1993; Boaler & Humphreys, 2005; Lampert, 2001; Sherin, 2002; Stein et al., 2008). This implementation of rich, cognitively demanding problems with explicit attention to connecting procedures, ideas, and concepts together during instruction was characteristic of high-achieving countries in the 1999 TIMSS study (Hiebert et al., 2003).

However, teachers struggle to develop these ambitious teaching practices (Franke et al., 2007). A recent investigation of the nature and quality of ninth-grade algebra instruction in five large, metropolitan school districts in the United States found that the majority of the algebra lessons followed a teacher-directed format. Students had minimal opportunities to engage in mathematical work or provide significant mathematical contributions during the lesson (Litke, 2020a). However, there was evidence that teachers attended to mathematical connections, although the quality of the instruction varied and depended on the kinds of mathematical connections. For instance, a little more than half of the lessons included at least one segment of lesson that focused on connections between representations such as graphs, tables, and contexts. On the other hand, approximately one-third of lessons included no segments where teachers and students made connections among algebraic concepts within the algebra curriculum or the span of larger mathematics curriculum. Nevertheless, there is some evidence that teachers may already have some existing instructional practices that provide students some opportunities to make mathematical connections (Litke, 2020b).

Problem Three: Need for Understanding How Teachers Learn to Foster Mathematical Connections

Studies of experts' practice provide models of instruction that seem to be productive in fostering students to make mathematical connections; studies of novices' practice identifies knowledge and skills for teaching mathematics that need to be developed in teacher education programs. What the literature does not account for yet is how teachers develop their practice to design instruction that fosters students to make mathematical connections, difficulties teachers may face in developing this practice, and how teachers overcome such difficulties with the resources available in their context (Ghousseini, 2015).

Russ et al. (2016) critiqued that the primary focus of research on teacher learning is on teachers' starting and ending states and not enough attention on the learning progression. Their critique, I argue, applies to the literature on how teachers design instruction to foster mathematical connections as well. To address the lack of attention to teachers' learning progressions, Russ et al. called for the field of teacher education to attend to how the development of teaching expertise builds on resources prospective teachers bring into their teacher education programs.

If mathematicians, educational researchers, teacher educators, and professional teaching organizations of mathematics see value in students making mathematical connections, but teachers, especially novices according to Even et al. (1993), feel uncertain or unprepared to design instruction to support students in making mathematical connections, then it falls on the former to develop novice teachers' practice in designing instruction that promotes students in making mathematical connections.

Overview and Purposes

While the literature provides examples of expert teachers' instructional support for connection-making, especially at the K-8 level (e.g., Ball, 1993; Boaler & Humphreys, 2005; Lampert, 2001), this qualitative dissertation study focused on understanding how prospective secondary teachers fostered students to make mathematical connections. Understanding how prospective teachers foster students to make mathematical connections provides mathematics teacher educators opportunities to take an incremental approach to build on their beginning practice. This dissertation contributes to the field to inform this incremental approach by jointly considering the *specialized* and *everyday* knowledge and practices (e.g., teacher-moves that support students' mathematical reasoning and audience design, respectively) prospective teachers leveraged to support students in making explicit mathematical connections.

In this section, I begin by providing a brief overview of the dissertation, and then I describe the purposes for the dissertation. The dissertation is being written in a 3-manuscript format. In the first manuscript (Chapter 2), my co-author (Lee) and I examined the kinds of mathematical connections prospective secondary mathematics teachers attended to and the pedagogical considerations they contemplated surrounding the mathematical connections. From the study, we developed the *Pedagogical Considerations of Mathematical Connections* (PCMC) framework. The PCMC framework provides a model for mathematics teacher educators to expand prospective teachers' attention to and awareness of mathematical connections. The manuscript has been revised and resubmitted for publication in *Mathematics Teacher Education and Development*.

The studies in the second and third manuscripts were a secondary data analysis of a subset of the classroom videos and lesson artifacts collected for a larger multi-year study, *Learning to Support Collective Argumentation in Secondary Mathematics Classes* (LSPAM). The goal of LSPAM was to understand how secondary learn to facilitate collective argumentation. LSPAM followed a cohort of prospective secondary mathematics education teachers in their program at a large university in the southeastern United States. Six participants from the cohort were followed into their student teaching experience. Of those 6 participants, I selected three, Melissa, Robin, and William, to include in this dissertation study. The secondary data analysis focused on the lessons these three participants taught for one unit of instruction in ninth grade mathematics classes. For context, Melissa and Robin cotaught together during their student teaching, while William cotaught with another student teacher who did not agree to participate during his student-teaching as a part of LSPAM. I excluded the other three participants in the student-teaching phase of LSPAM because studies have shown that instructional task can have a significant influence on classroom discourse (e.g., Hiebert & Wearne, 1993; Stein, Grover, & Henningsen, 1996). Because Melissa, Robin, and William co-planned lessons together, there was a lower chance for variation in instructional tasks across these participants and so the instructional tasks would be an unlikely factor contributing to the variation in classroom discourse.

In the second manuscript (Chapter 3), I argue prospective teachers' everyday practice of audience design provided opportunities for students to make mathematical connections during discussions. This attention to audience design was evidenced by the prospective teachers' decisions to make certain connections explicit during discussions and choice of teacher moves in response to students' reasoning. The method of analysis may assist mathematics teacher educators in identifying features of audience design that could be productive in building teachers' practice. The manuscript will be submitted to *Journal of Mathematics Teacher Education*.

In the third manuscript (Chapter 4), I argue that prospective teachers are able to build and maintain mutual understanding (i.e., common ground) to support students in making mathematical connections. I identified ways in which the teachers made micro-adjustments, such as through their gestures, during instructional explanations to build or maintain common ground to facilitate students in making connections. The study raises implications for studying how novice teachers learn in and from their practice. The manuscript will be submitted to *Journal of Mathematical Behavior*.

Next, I discuss the purposes for the three studies within the dissertation. Each study addressed at least one, if not more, of the purposes. Some purposes were more salient than others across the studies. However, these purposes guided the focus of the dissertation.

Purpose One: Make Sense of Early Teaching Practice from the Perspective of Novice Teachers

One purpose for the dissertation was to make sense of the early teaching practice of novice teachers from their perspective. Typically, novice teachers are depicted in a deficient manner due to their lack of teaching experience. However, novice teachers are not blank slates;

they bring multiple experiences into their early practice. Mathematics teacher educators should seek to understand how novice teachers make sense of their early teaching practice just like mathematics teachers should seek to understand how students make sense of mathematics.

The first study (Chapter 2) mostly aligns with this purpose. I sought to understand how prospective teachers made sense of their practice during moments when they attended to mathematical connections. By taking the perspective of prospective teachers, I found that they had several pedagogical considerations of mathematical connections that were either absent in the literature or described as lacking for novice teachers.

Purpose Two: Document the Early Practices of Novice Teachers that Support Students' Opportunities to Make Mathematical Connections

An additional purpose for the dissertation study was to document novice teachers' early practices that were supportive of supports' opportunities to make mathematical connections. As previously mentioned, novice teachers' practice is often depicted in the literature as lacking when it comes to supporting students in making mathematical connections (e.g., Borko & Livingston, 1989; Even et al., 1993; Leinhardt, 1989; Livingston & Borko, 1990). However, these depictions of novice teachers' practice are usually evaluated through from a top-down perspective. In other words, these studies document what lacks in novice teachers' practice in comparison to expert teachers' practice or expert models of teacher practice. There is a possibility that this top-down perspective has privileged certain practices over others.

This dissertation study documents early practices of novice teachers that were supportive of students' opportunities to make mathematical connections. Each of the three studies aligned with this purpose. In the first study (Chapter 2), I document, from the perspective of prospective teachers, what practices they thought supported or hindered their students in making mathematical connections. The second study (Chapter 3) provides documentation of how secondary mathematics teachers coordinated eliciting, responding, facilitating, and extending teaching moves to support students in attending to or providing opportunities for connection-making. Finally, the third study (Chapter 4) provides evidence of the ways a prospective mathematics teacher made micro-adjustments to her practice that built and maintained common ground with students to support their connection-making.

Purpose Three: Understand How Teachers Begin to Learn to Foster Students' Connection-making

Another purpose for the dissertation was to understand how teachers, especially novices, begin to learn to foster students in making mathematical connections. The literature suggested that in order for teachers to foster connection-making, teachers need profound mathematical understanding (Ma, 1999; Simon, 2006), mathematical knowledge for teaching (Ball et al., 2008; Rowland et al., 2005; Silverman & Thompson, 2008), or some other cognitive construct. Thus, the proposed solution to foster connection-making in classrooms is to develop teachers' mathematical knowledge. The research literature gives little attention to other explanatory constructs, other than teachers' knowledge, for understanding how teachers foster connection-making.

This dissertation study took an incremental approach to understand how teachers learn to foster connection-making. Using this incremental approach, I sought to understand how teachers specialize their everyday practices for teaching mathematics. In the first study (Chapter 2), I examined how the everyday practice of noticing emerged as prospective teachers learned to attend to and make sense of students' mathematical thinking and the implications of this noticing for fostering mathematical connections during instruction. In the second and third studies

(Chapter 3 and Chapter 4, respectively), I examined the everyday practice of audience design and how novice teachers productively leveraged audience design to support students in making mathematical connections.

	Purpose 1: Make sense of early teaching practice from the perspective of novice teachers	**Purpose 2**: Capture the early practices of novice teachers that support students' opportunities to make mathematical connections	**Purpose 3**: Understand how teachers begin to learn to foster students' connection-making
Problem 1: Novice teachers' limited ability to foster mathematical connections	Study 1	Study 1 Study 2	
Problem 2: Few Links Between Instructional Practices That Foster Connection-making and Student Outcomes		Study 2	Study 3
Problem 3: Need for understanding how teachers learn to foster mathematical connections		Study 3	Study 3

Figure 1.1. Problem-Purpose Matrix: The Alignment of the Problems and Purposes to the Studies in the Dissertation

Definitions and Background Literature

In this section, I provide definitions and background literature for the dissertation. The purpose of this section is to provide an overview of the main constructs and to situate them within the literature. The aim is not to provide a cumulative, summative, or critical review of the literature but to present definitions and clear themes in the literature. First, I describe how mathematical connections have been conceptualized in the literature and then provide my definition for mathematical connections. Next, I discuss the different conceptualizations of teacher noticing and how the literature on teachers' noticing of students' mathematical thinking suggests a relationship between teacher noticing and their ability to foster students' connection-making. I close this section with discussing audience design and common ground. I discuss the relationship between audience design and common ground and also how these constructs are similar to other constructs in other disciplines.

What are Mathematical Connections?

Mathematical connection is a term that is frequently used in mathematics education literature but is often undefined (Singletary, 2012). In this section, I review the conceptions of mathematical connections from two perspectives: as conceptualized in the mathematics education literature and from mathematics teachers. Then, I provide my definition for mathematical connections.

Conceptions of Mathematical Connections in the Literature

According to Businskas (2008) and later expanded on by Singletary (2012), there are three conceptions for mathematical connections evident in the mathematics education literature: (a) a feature of mathematics, (b) a construction of the mind, and (c) dynamic process of doing mathematics. Below, I describe each of these conceptions and provide additional examples of these conceptions evident in the research literature.

Mathematical connection: A feature of mathematics. The conception of mathematical connection as a feature of mathematics assumes that "mathematical ideas are linked by particular relationships and those connections can be identified a priori and independently of the learner" (Businskas, 2008, p. 8). Moon, Brenner, Jacob, and Okamoto (2013) provide an example of research operating under this conceptualization. Moon et al. were interested in how preservice teachers come to understand the Cartesian Connection (Knuth, 2000) in the context of conic curves. They articulated the connection as "A point is on the graph of the mathematical relation R(x, y) = 0 if and only if its coordinates satisfy R(x, y) = 0" (Moon et al., 2013, p. 204). The research question for their study was, "What are the difficulties and associated big ideas that learners encounter when making connections among representations?" (Moon et al., 2013, p. 204). Thus, Moon et al. were interested in investigating the cognitive difficulties that limited students in forming the predetermined connection they were looking for students to develop. It was not Moon et al.'s intention to understand the connection that the students were constructing during the lesson intervention on representations of conic sections. If they had, then this would indicate another conception of mathematical connections, which I discuss next.

Mathematical connection: A construction of the mind. This conception assumes that "making a mathematical connection is a process that occurs in the mind of the learner(s) and the connection is something that exists in the mind of the learner; it is a mental construction of the learner" (Businskas, 2008, pp. 12–13). For example, García-García and Dolores-Flores (2018) defined mathematical connections as "a cognitive process through which a person relates or associates two or more ideas, concepts, definitions, theorems, procedures, representations and meanings among themselves, with other disciplines or with real life" (p. 227). García-García and Dolores-Flores completed clinical interviews with high school calculus students using calculus-based tasks. They used thematic analysis (Braun & Clarke, 2006) to categorize the mathematical connections student reasoned about as a result of the tasks. One example of a connection García-García and Dolores-Flores found in their study was *the derivative is the slope of the tangent to a curve*.

Mathematical connection: Dynamic process of doing mathematics. "When making mathematical connections is conceptualized as a process, this process is often described as the byproduct of engaging in other mathematical processes, such as multiple representations, problem solving, proof, and real-world applications and mathematical modeling" (Singletary, 2012, p. 14). In other words, mathematical connections are derived from other processes. For example, in the context of problem solving, Boaler claimed it was not enough for students to know mathematical connections because "…the act of linking one mathematical area to another, in order to solve a problem, is an action that extends beyond knowledge" (2002, p. 12). Thus, it is the act of doing mathematics (e.g., considering multiple representations or problem solving) that allows for mathematical connections. While this perspective might be considered consistent with the previous one, it makes more explicit the dependency of mathematical connections on other processes of doing mathematics.

Although Orrill and Kittleson (2015) did not define their conception of mathematical connection explicitly, the perspective of mathematical connections as a dynamic process of doing mathematics was evident. Orill and Kittleson observed the connection making practices of a middle grades teacher, Donna, in the context of a professional development (PD) on proportional reasoning. During the PD, Donna used multiple approaches and representations to explain her reasoning on proportional tasks. After the PD, Donna implemented one of the tasks from the PD in her class. Orill and Kittleson observed how Donna supported her students to engage in the practices of making connections that Donna experienced in the professional development. The researchers found that Donna's practice of making connections by using multiple representations during the PD was not sufficient to translate into her teaching in supporting students in making connections. For example, even though Donna showed learning via a pre/post-test on reasoning with multiple representations to model proportional relationships from the PD, there was little evidence of her questioning students to reason with and connect multiple representations in her practice. Therefore, by limiting students' opportunities to engage with multiple respresentations, Donna did not allow her students to engage in the process of making connections.

Teachers' Conceptions of Mathematical Connections

While Businskas (2008) detailed three conceptions of mathematical connections found in the mathematics education literature, that does not necessarily mean mathematics teachers share those conceptions. Therefore, I now consider the literature on teachers' conceptions of mathematical connections. Two independent studies (Businskas, 2008; Roddy, 1992) illustrated two conceptions of mathematical connections held by secondary mathematics teachers. The studies were conducted at different sites (United States and Canada). Both studies concluded that secondary mathematics teachers conceived of mathematical connections as *applying and/or modeling mathematics* and/or r*elating to students' prior knowledge or experiences*. These are different conceptions of mathematical connections in comparison from those found in the literature above and even standards document such as the *Curriculum and Evaluation Standard of School Mathematics* (National Council of Teachers of Mathematics, 1989). For reference, NCTM (1989) defined mathematical connections as "equivalent representations and between corresponding processes in each" (p. 146). NCTM contrasted mathematical connections with modeling connections, which are connections "between problem situations that may arise in the real world or in disciplines other than mathematics and their mathematical representations" (NCTM, 1989, p. 146).

Mathematical connections: Application of or modeling with mathematics. Roddy (1992) found that several teachers understood connections as "applications" or the usefulness of mathematics to an applied problem or situation. In particular, applications positioned mathematics as useful in solving a practical problem. However, Roddy was careful to distinguish that applications did not include students' construction of a mathematical model to determine a solution. Similar to Roddy, Businskas (2008) found that when teachers spoke spontaneously about mathematical connections, they conceptualized it as real-world connections. Sometimes, the teachers expressed wanting to connect students' mathematics to their everyday lives for them to see mathematics as personally relevant. Other times, the teachers expressed wanting students to see mathematics as a meaningful tool for others (e.g., scientists or business analysts). A pilot interview where I interviewed a secondary mathematics teacher revealed consistencies between her conception of mathematical connections and the results of Roddy and Businskas.

Mathematical connections: Relating to students' prior learning or experience. Another conception of connections evident in both studies was the relationship between students' prior learning or experiences with a new mathematical concept or idea. A teacher in Roddy's (1992) study termed this as a "cognitive connection." Teachers remarked that during instruction they would pull in background information to allow students to make sense of new topics because they considered this an important piece in the teaching-learning process and useful for introducing new material. In addition, some teachers in Businskas' (2008) study explained connections as how the content they taught was related to concepts students would encounter in the future. Again, a pilot interview where I interviewed a secondary mathematics teacher revealed consistencies with this conception of mathematical connections. In sum, this conception views connections as the building up of mathematical ideas and concepts on top of previous learned concepts, ideas, and/or experiences.

Summary. Roddy (1992) and Businskas (2008) found that when teachers spoke spontaneously about mathematical connections, they conceptualized them as the application or modeling of mathematics and linking to students' prior knowledge or experiences. Mathematical connections, as defined by NCTM (1989), was not evident in teachers' conceptions even though NCTM's standards have influenced the teaching and learning of mathematics in both the United States and Canada. In addition, Roddy and Businskas recognized that teachers' understanding of mathematical connections was largely tacit and associated with their teaching practice.

Mathematical Connections as Discourse

Drawing on Gee's (2005) conception of Discourse, I define *mathematical connections* as the discursive ways or practices in which an individual or a community of learners come to make and describe a relationship between a mathematical entity and another mathematical or non-mathematical entity. I use the term entity to encompass ideas, concepts, objects, representations, procedures, or methods.

It is important to note that Gee distinguished what he termed as "little d" discourse and "Big D" Discourse. "Little d" discourse, sometimes referred to as "language-in-use" (Gee, 2005), means "connected stretches of language that make sense" (Gee, 1996, p. 127 as cited in Ryve, 2011). "Big D" Discourse, sometimes referred to as the melding of language-in-use with language-in-action (Gee, 2005), is more encompassing:

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and 'artifacts,' of thinking, feeling, believing, valuing, and acting that can be used to identify oneself as a member of a socially meaningful group

or 'social network,' or to signal (that one is playing) a socially meaningful role (Gee, 1996, p.131).

"Language-in-action" means using actions such as symbolic expressions or gestures to enact specific activities and identifies. A helpful analogy might be thinking of "language-in-use" as "talking the talk" and "language-in-action" as "walking the walk," and when these two come together then Gee would say it is Discourse.

Conceptualizing mathematical connections as Discourse affords two analytical benefits. The first benefit is what Gee termed as the "seven building tasks" of language. One of the seven building tasks is *connections*: "We use language to render certain things connected or relevant to other things, that is, to build connections or relevance" (Gee, 2005, p. 12). This allows one to answer such questions as how does language connect or disconnect things; how does language make one thing relevant to another? In the context of a mathematics classroom, the question might be how does language connect or disconnect mathematical ideas? The second analytical benefit is to understand how individuals use Discourse to signal they are playing a socially meaningful role. For example, Gee's Discourse allows me to investigate how teachers use their gestures to signal they have a meaningful role in organizing and supporting students' connection-making activities.

Mathematical Connection Framework. Singletary (2012) developed The Mathematical Connections Framework (MCF) from her observation of the instructional practices of three secondary mathematics teachers. There were five kinds of connections evident in the teachers' practice: (a) *connecting through comparison*, (b) *connecting through logical implication*, (c) *connecting methods*, (d) *connecting specifics to generalities*, and (e) *connecting to the real world*. In addition, Singletary distinguished how explicit the connection (i.e., level of connection) was during instruction: *suggested*, *provided*, and *provided-and-explained*. Table 1.1 is a summary of Singletary's MCF.

Table 1.1

Mathematical Connections Framework (Reprinted with permission from Singletary, 2012)

<table>
<tr><th rowspan="2">Kind of Connection</th><th colspan="3">Level of Connection</th></tr>
<tr><th>Suggested</th><th>Provided</th><th>Provided-and-explained</th></tr>
<tr><td>Connecting through comparison</td><td rowspan="5">A and B are somehow related or A is related to something (where the something is left unsaid).</td><td>A is similar to B.
A is the same as B.
A is not the same as B.
A or B similarly defines or describes C.</td><td>A is similar to B because of C.
A is the same as B because of C.
A is not the same as B because of C.</td></tr>
<tr><td>Connecting through logical implication</td><td>If A, then B.
If A, then B and not C.</td><td>If A, then B because of C.</td></tr>
<tr><td>Connecting methods</td><td>A or B can be used to find C.</td><td>A or B can be used to find C because of D.</td></tr>
<tr><td>Connecting specifics to generalities</td><td>A is an example of B.</td><td>A is an example of B because of C.</td></tr>
<tr><td>Connecting to the real world</td><td>A is an example of B in the real world.</td><td>A is an example of B in the real world because of C.</td></tr>
</table>

Even though Singletary did not explicitly draw on discourse analysis in her study, she focused on the teachers' discourse as a social action in relating mathematical entities (ideas, concepts, etc.). She was careful in articulating that she was studying teachers' practice of connection making, which implies that she was studying the social actions of the teachers' practice. This is consistent with the principle that discourse is action-oriented. In other words, the teachers' discursive practices were performing a social action of relating mathematical ideas together, which Gee (2005) would say is one of the building tasks of language. Thus, Singletary primarily treated mathematical connection as an interactional concern for action.

Next, I discuss noticing and the common themes apparent in the line of scholarly inquiry of teacher noticing of students' mathematical thinking. In addition, I describe the relation between teacher noticing and mathematical connections.

What is noticing?

Generally, noticing is "the act of focusing attention on and making sense of situational features in a visually complex world" (Jacobs & Spangler, 2017, p. 771). Our attention, or lack thereof, has long been of interest to psychologists and other researchers (e.g., Gibson, 1979; Goodwin, 1994; Shiffrin & Schneider, 1977; Simons, 2000). More recently, scholars have examined how individuals' attention to certain situational features related to their profession. These scholars have defined this unique attention in a number of ways: professional vision (Goodwin, 1994), disciplined perception (Stevens & Hall, 1998), and discipline of noticing (Mason, 2002). Regardless of the term, these scholars generally agree that as individuals gain expertise in a profession, they attain a more nuanced way of noticing beyond the everyday noticing of laypersons. Mathematics teacher education scholars have furthered our understanding of the disciplined ways in which mathematics teachers notice, which I discuss next.

Mathematics Teacher Noticing

Mathematics education researchers have argued that mathematics teachers develop a specialized and disciplined form of noticing unique to teaching mathematics, which is often referred to as teacher noticing or just noticing. Erickson et al.'s (1986) study is one of the first to argue that teachers' prior teaching experience was influential on teachers' noticing. For example, veteran teachers noticed a wider range of student behaviors and made more comprehensive interpretations of the student behaviors they noticed in comparison to novice teachers. However, more recent research seems to suggest that teaching experience alone does not guarantee the development of teacher noticing (Dreher & Kuntze, 2015; Jacobs et al., 2010). Regardless of teaching experience, teachers can learn to develop their noticing over time with support (e.g., Jacobs et al., 2010; Schack et al., 2013; Star & Strickland, 2008; van Es & Sherin, 2008).

A prominent theme in the teacher noticing literature, and also related to the dissertation study, is teacher noticing of students' mathematical thinking (Jacobs & Spangler, 2017). The ability to notice students' mathematical thinking is important for designing instruction that builds on students' ways of thinking, which has been linked to student achievement (e.g., Fennema et al., 1996; Jacobs et al., 2007). Furthermore, teachers can learn from their students' mathematical thinking, even after professional development supports end (Franke et al., 2001). I restrict the background literature on teacher noticing to the noticing of students' mathematical thinking. Next, I describe the various conceptualizations of teachers' noticing, prospective secondary mathematics' noticing of students' mathematical thinking, and how teacher noticing relates to supporting students in making mathematical connections.

Conceptualizations of Teacher Noticing. In their review of the mathematics teacher noticing literature, Jacobs and Spangler (2017) identified three conceptualizations of teacher noticing. The three conceptualizations of teacher noticing progressed in inclusion of the following components: (a) attention, (b) interpretation, and (c) decisions about next steps. The first conceptualization of teacher noticing is essentially the first component (i.e., teacher's attention). It seeks to understand what features of students' mathematical thinking capture teachers' attention. The second conceptualization of teacher noticing extends beyond teachers' attention to students' mathematical thinking to also include teachers' interpretations of students' mathematical thinking. Researchers with this conceptualization of teacher noticing seek to capture teachers' attention and also how they interpret specific details of students' thinking by relating them to principles of teaching and learning. The third conceptualization of teacher noticing extends even further to include teachers' decisions about how to respond with teachers' attention and interpretation. After teachers attend to and interpret students' mathematical thinking, then teachers decide how to respond to students' mathematical thinking. Teachers' reasoning for how to respond is the third and final component for teacher noticing. However, researchers with this conceptualization of teacher noticing typically assume that attention, interpretation, and decision of how to respond happen almost simultaneously.

Prospective Secondary Mathematics Teachers Noticing. In their review of prospective teacher noticing literature, Amador et al. (2021) found that several studies had mixed or neutral outcomes when it came to developing prospective teacher noticing. However, Amador et al. did not disentangle the outcomes of studies with prospective secondary teachers as participants from prospective elementary teachers. Prospective secondary teachers take more advanced mathematics courses and likely have more enriched mathematical content knowledge than prospective elementary teachers. Some studies have documented links between mathematical knowledge for teaching and teacher noticing (e.g., Dreher & Kuntze, 2015). I focus the remainder of this section on the results of teacher noticing studies with prospective secondary mathematics teachers as focus participants.

In terms of what classroom events prospective secondary mathematics attend to, studies have found mixed results. The mixed results could be explained by methodological differences. To illustrate, I discuss and compare two studies (Roller, 2016; Star & Strickland, 2008). Star and Strickland (2008) had prospective secondary teachers watch a video of classroom instruction from the TIMSS 1999 Video Study (Hiebert et al., 2003). They found prospective secondary teachers early in their teacher education program were highly attentive to classroom management and tasks and not as attentive to classroom environment, mathematical content, and communication. In comparison, Roller (2016) had prospective secondary teachers watch video of their micro-teaching to peers and found that prospective teachers attended to student-peers with a strong focus on student learning and mathematical content. Both of these studies were situated in methods courses for prospective secondary teachers. Therefore, these studies are not just a case of what prospective secondary teachers attend to when watching videos of classroom instruction, but rather were situated within methods courses that have provided different opportunities to shape prospective teachers' attention.

Several scholars have argued prospective secondary teachers' ability to notice students' mathematical thinking can be developed with supports (e.g., participating in video-clubs, clinical interviews with students, etc.). These supports include video-tagging significant classroom events (van Es & Sherin, 2002; Walkoe, 2015), participating in online discussions about student solutions to a task (Fernández et al., 2012), and participation in a curricular module that supported prospective teachers to conduct a clinical interview with a secondary student (Krupa et al., 2017). All of these supports, scholars argue, developed prospective teachers' ability to attend to and interpret students' mathematical thinking.

In my review of the literature, no studies have explicitly examined what mathematical connections prospective mathematics teacher notice, if at all, during instruction. However, the literature does provide some clues to the ability of prospective secondary teachers to notice mathematical connections during instruction. For example, Star and Strickland (2008) found that many prospective teachers did not recall a moment in the lesson when a student made a mathematical connection between two algebraic expressions. In contrast, Walkoe (2015) described a prospective teacher's participant in a video-club as follows: "We can see that even in the early tagging assignment, Heidi was able to infer aspects of the student's understanding. She explored connections the student was making" (p. 539). In a different study, a prospective teacher noticed the following in a teacher's use of two different representations to explain the sum and product of two fraction: "It [the reaction] confuses the student more instead of helping him. As a result, the understanding of a connection between multiplication and addition gets worse, since different representations are used. Addition and multiplication should be explained in the same representation" (Dreher & Kuntze, 2015, p. 104). From these excerpts in the literature, it seems that prospective secondary mathematics teachers are able to notice some connections during instruction and perhaps they also possess some perspectives or beliefs on how to support students' connection-making.

Teacher Noticing Relation to Connection-making

Explicit attention to mathematical connections during instruction promotes students' conceptual understanding of mathematics (Hiebert & Grouws, 2007). Teachers can design instruction that explicitly attends to making mathematical connections only if they notice mathematical connections among mathematical facts, procedures, and concepts in the curriculum. In addition, teachers have to be aware of and responsive to the mathematical connections among students' ideas and how they connect to the discipline of mathematics. Teachers cannot act in these ways if they do not notice.

Developing and refining teachers' attention and awareness of mathematical connections in the school mathematics curriculum and students' ideas has the potential to impact instruction. Ma's (1999) comparative study of elementary teachers' understanding of mathematics in the US and China gives an illustrative example of the importance of developing this type of awareness. She proposed that the difference between the subject matter knowledge of mathematics between a teacher and a non-teacher was that teachers drew upon knowledge that tended to make explicit connections between mathematical ideas, concepts, and procedures that otherwise remain tacit for a non-teacher. Mason (1998) similarly made the following explication of expert mathematicians and expert mathematics teachers:

> The mark of expert mathematicians is that they make problem solving and proof look easy: they are articulate with technical terms, they make the choice and use of techniques look easy, and they are aware of connections between otherwise apparently disparate topics. The mark of expert mathematics teachers is that they make exposition, explanation, task-design, and relating to students look easy. (p. 243)

Furthermore, Ma detailed a distinct difference in how the teachers described their understanding of subtraction. She stated, "…some teachers showed a definite consciousness of connections, while others did not. This difference was associated with significant differences in teachers' subject knowledge" (p. 21-22). While the direction of the association was not known (e.g., greater mathematical content knowledge leads to greater awareness of mathematical connections or vice versa), Ma claimed that developing mathematical content knowledge was the result of deliberate study. However, Mason (1998, 2002) argued developing the consciousness described by Ma is a necessity for developing a disciplined knowledge base. I suggest that both developing and refining teachers' mathematical content knowledge and awareness of their knowledge to facilitate mathematical connection-making is mutually beneficial to supporting teachers in designing instruction that fosters explicit attention to mathematical connections for students.

Having defined teacher noticing, I will now move on to discuss audience design. The following section will define audience design and common ground and their relation to each other.

What is audience design?

Researchers in several disciplines (e.g., philosophy, psychology, linguistics) have sought to understand how speakers design their speech and gestures for their communication partners (Avineri, 2021; Fischer, 2016). Several scholars have used the term audience design but with different meanings. For this dissertation, my use of the term audience design aligns with Clark and his colleagues (Clark, 1992; Clark & Carlson, 1982; Clark & Murphy, 1983). Broadly, *audience design* is when speakers design their utterances and gestures based on what they know, believe, or suppose hearers also know, believe, or may suppose. For example, if I lost my dog and I asked my friend Eric, "Did you see Beau running this way?", then I believe Eric will understand that I am directly addressing him, Eric knows that my dog's name is Beau, and Eric knows what Beau looks like. However, if I met a stranger on the street, I may design my utterance as follows: "Excuse me, sir. Did you see a small brown dog running this way recently?" For the purposes of this dissertation, I examine audience design at the level of sentence structure, word choice, gesture form, and gesture frequency.

Common ground

When designing their speech or gestures, speakers have to deal with disparities in *common ground* (Clark, 1996, 2021; Clark & Carlson, 1982; Clark & Marshall, 1981), or the shared mutual knowledge, beliefs, and understandings among speaker and addressees. Robert Stalnaker (1978) coined the term common ground but the notion of common ground can be found in similar ideas of *common knowledge* (Lewis, 1969), *mutual knowledge* or *belief* (Schiffer, 1972), and *joint knowledge* (McCarthy, 1990). Common ground is also somewhat akin to *common knowledge* (Edwards & Mercer, 1987), although common knowledge is uniquely situated within educational contexts. Edwards and Mercer were concerned with the ways in which students and their teacher come to a shared understandings or common knowledge, which may be either ritual (e.g., procedural) or principled (e.g., conceptual). While the idea of common ground appears in earlier ideas and constructs, there are different assumptions in the ways common ground, or shared understanding, develops between interlocutors. Before discussing how common ground develops between interlocutors, I first need to discuss two types of common ground.

Clark (1996) distinguished two types of common ground: communal and personal. *Communal common ground* is the assumed knowledge, beliefs, knowhow, and practices of a particular community. For example, those associated with the University of Georgia community know that the appropriate way to spell out the phrase “Go Dogs” is “Go Dawgs.” Clark was fairly inclusive of the different bases of cultural communities. The following are just some bases for cultural communities: nationality (e.g., American or Australian), occupation (e.g., teacher or nurse), hobby (e.g., golfer or violinist), language (e.g., English or Spanish speaker), religion (e.g., Christian or Muslim), cohort (e.g., baby boomer or Gen-Z), and gender (e.g., cis-gender man or trans-woman). For the purposes of dissertation, communal common ground was not an explicit focus, although I recognized its importance for a mathematics classroom community. I focused on the other type of common ground in mathematics classrooms: personal common ground.

Personal common ground is the shared knowledge between speaker and addressees resulting from prior experiences with each other or their current situation. It arises from joint perceptual experiences (e.g., teacher and her students examining number line on the board) or joint actions (e.g., teacher and students identifying the location of one-fourth on the number line). Over the course of conversation, the personal common ground between speaker and addressees accumulates via *grounding* (Clark, 1996; Clark & Schaefer, 1989). In other words, a speaker will contribute something to addressees and then they work together to reach a mutual belief that the speaker's contribution has been understood for current purposes. Speakers seek positive evidence of common ground from addressees whether through speech or gestures. Consider the following exchange between a teacher and students:

Teacher: If we have a ratio that's 1 to 3, what's the total number of parts?

Student A: Four.

Teacher: Why did Student A say there were four parts?

Student B: Because you have 3 parts and you have like 1 part, so you add them together.

In this small exchange, the teacher and students are grounding their shared understanding for the total number of parts. First, the teacher elicits evidence from students that they understand the total number of parts for a 1 to 3 ratio. Student A presented her understanding that there are four total parts. Then, the teacher sought further evidence that Student A's understanding was shared by others in the class by asking the follow up: "Why did Student A say there were four parts?" In response, Student B elaborated on his understanding for the four parts. The students' responses provide positive evidence that they share a common understanding for the total number of parts for a ratio of 1 to 3.

Implications of common ground for audience design

According to Clark (1992), "audience design revolves around the notion of common ground and cannot be accounted for without it" (p. 249). Common ground allows for interlocuters to successfully coordinate their joint actions. Several studies have examined the coordinate problems that speakers have to consider when designing their utterance for an audience and how common ground intermediates speakers' audience design (e.g., Alibali et al., 2013; Clark &

Carlson, 1982; Clark & Schaefer, 1989; Horton & Keysar, 1996; Yoon & Brown-Schmidt, 2018; Yoon & Brown-Schmidt, 2019). For the purposes of this dissertation, I narrowed my focus to examining two coordination problems within the common ground literature: disparities between common knowledge between interlocuters (e.g., expert-novice, adult-child, or teacher-student conversations) and multi-party conversations. In my review of the literature, the personal common ground among interlocuters has implications for how speaker designed their utterance in order to coordinate with addressees to engage in their proposed joint activities.

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CHAPTER 2

PROSPECTIVE TEACHERS' PEDAGOGICAL CONSIDERATIONS OF MATHEMATICAL CONNECTIONS: A FRAMEWORK TO MOTIVATE ATTENTION TO AND AWARENESS OF CONNECTIONS[1]

[1] Foster, J. K. & Lee, H. Y. Submitted to *Mathematics Teacher Education and Development*.

Abstract

Research findings and reform-oriented standards emphasise the importance of mathematical connections in support of students' conceptual development. Previous research on teachers attending to mathematical connections has tended to focus on expert teachers' practice. Complementing previous research, this study describes how a cohort of twelve prospective mathematics teachers attended to and made sense of mathematical connections that arose when working with secondary students in small-group instruction. Results indicated prospective teachers were able to attend to mathematical connections during instruction and made several pedagogical considerations around such connections. We present a framework, the *Pedagogical Considerations of Mathematical Connections* (PCMC) framework, which offers mathematics teacher educators a new model to expand prospective teachers' attention to and awareness of mathematical connections. The study contributes to the existing literature on teacher noticing by providing a new kind of theme-specific noticing (i.e., mathematical connections) and informing mathematics teacher educators of how prospective secondary teachers attend to mathematical connections.

Keywords: Attention, Awareness, Mathematical connections, Noticing, Teacher education

Introduction

Hiebert and Grouws (2007) found, from their review of literature, that when teachers and students explicitly attended to connections among mathematical facts, procedures, and concepts during instruction, students more likely acquired a conceptual understanding. Reform-based standards emphasise mathematical connections as a goal of school mathematics. For example, in the Australian mathematics curriculum, a key idea of understanding mathematics is that students will "make connections between related concepts and progressively apply the familiar to develop new ideas" (Australian Curriculum, Assessment and Reporting Authority, n.d.). In the United States (US), the National Council of Teachers of Mathematics (NCTM, 2000) emphasised instructional programs should enable students to "recognise and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole, and recognise and apply mathematics in contexts outside of mathematics" (p. 64).

In response to calls for students and teachers to explicitly attend to mathematical connections, mathematics teacher educators (MTEs) will have to contemplate how to assist teachers in supporting these goals. One approach is to examine how expert teachers explicitly attend to mathematical connections (e.g., Ball, 1993; Boaler & Humphreys, 2005; Even, et al., 1993; Lampert, 2001). Examining experts' practice has led to the development of models for instruction. For instance, the five practices of anticipating, monitoring, selecting, sequencing, and connecting is a model to support teachers in productively using students' thinking in mathematical discussions (Stein, et al., 2008). In particular, the practice of *connecting* supports teachers in explicitly attending to mathematical connections between students' strategies and how ideas within those strategies are related to important mathematical ideas. However, models such as these do not describe *how* teachers *begin* attending to mathematical connections.

Russ et al. (2016) argued, "If we are truly interested in teacher learning as a process, then we must pay more attention to the starting state and how learning progresses from this starting state to the end state" (p. 410). Teachers, even novices, bring a range of experiences. Just as teachers should pay attention to and use students' ideas to further their instruction, MTEs should attend to the ideas of prospective teachers (PSTs) and use them to develop PSTs' understandings of significant instructional practices.

While some have described how novice teachers attend to mathematical connections (e.g. Borko & Livingston, 1989; Even et al., 1993; Livingston & Borko, 1990), the focus has been on describing the novice teachers' actions, or lack thereof, from the perspective of the expert or researcher. This study takes a complementary approach by seeking to understand, from PSTs' perspective, what mathematical connections they explicitly attend to in their instruction and how they make sense of those connections. We illustrate how PSTs begin to attend to mathematical connections and suggest activities MTEs could implement to support PSTs to attend to mathematical connections.

Drawing on Mason's construct of the discipline of noticing (1998, 2002), we investigated how twelve secondary PSTs enrolled in a mathematics methods course attended to mathematical connections during their field experience. We illustrate (a) the kinds of mathematical connections the PSTs anticipated before and recalled encountering after their instruction and (b) the pedagogical considerations they made surrounding these instances while working with students.

Literature Review on Teachers' Attending to Mathematical Connections

In this review, we highlight some results from teacher noticing studies and expert-novice studies regarding teachers attending to mathematical connections during instruction. We situate our study within these two areas of research and explain how the findings from our study inform MTEs to prepare PSTs to notice mathematical connections.

Teacher Noticing

In a complex sensory world, several phenomena compete for one's attention. Scholars have argued the way professionals select what to attend to and interpret among phenomena competing for their attention is unique and distinguishable from the layperson (Goodwin, 1994; Mason, 2002; Stevens & Hall, 1998). Mathematics education researchers have identified this expertise in mathematics teachers as *teacher noticing* or simply *noticing*.

Dreher and Kuntze (2015) noted that studies on teacher noticing focus on a particular instructional feature, which they termed as *theme-specific noticing*. Two significant, but not necessarily distinct, lines of theme-specific noticing in mathematics teacher education research are noticing *students' mathematical thinking* and noticing *indicators of equitable mathematics instruction* (Jacobs & Spangler, 2017). Both bodies of literature imply teachers *can* develop their noticing skills (e.g., Jacobs, et al., 2010; Schack et al., 2013; Sherin & van Es, 2008). In line with this perspective, our study contributes to the existing literature on teacher noticing by offering a new theme-specific noticing, namely, mathematical connections, and providing MTEs with an understanding of the scope of what secondary PSTs *can* attend to in their field experiences.

Teachers Attending to Connections

Comparison studies of expert-novice teachers highlight the difficulties novices experienced when attending to mathematical connections in their lesson planning and instructional explanations. Experts' plans often included explicit connections among previous lesson discussions or used the same representations or contexts across several lessons (Even et al., 1993; Leinhardt, 1989). Furthermore, experts used unplanned opportunities to make mathematical connections during lessons (Even et al., 1993). Novices' plans did not include relationships between concepts across lessons, and they did not typically leverage unplanned opportunities to make connections with students (Even et al., 1993). In addition, experts' explanations during lessons tended to focus on critical concepts and made explicit connections across problems; whereas novices' explanations tended to focus on procedures not linked to concepts (Borko & Livingston, 1989; Livingston & Borko, 1990). Novices either expressed there was no need or time to make connections or that connection-making was essential but challenging to plan and facilitate during instruction (Even et al., 1993). Additionally, while not a traditional expert-novice study, Star and Strickland (2008) studied what first-semester secondary PSTs attended to when watching Year 8 mathematics lessons from the TIMSS 1999 Video Study (see Hiebert et al., 2003) at the beginning and end of a methods course. The research team (experts) designed a survey from what they noticed in the video and administered the survey to the PSTs. They found many PSTs did not recall a moment in the lesson when a student made a mathematical connection between two algebraic expressions.

In comparison to the expert-novice studies, different results emerged from teacher noticing studies that sought to describe what PSTs attended to rather than what they did not attend to in relation to experts. For example, Walkoe (2015) asked PSTs to identify moments in a video of "interesting student algebraic thinking" and found that some PSTs attended to connections. Walkoe described a PST's (Heidi's) early participation in a video-club as follows: "Heidi was able to infer aspects of the student's understanding. She explored connections the student was making" (p. 539). Another example comes from a study by Dreher and Kuntze (2015). They asked PSTs to read vignettes of a classroom situation in which a teacher makes a change in mathematical representations in response to a student's comment or question. In a response to one of the vignettes, a PST wrote the following about the teacher's change in representation, "It confuses the student more instead of helping him. As a result, the understanding of a connection between multiplication and addition gets worse, since different representations are used" (Dreher & Kuntze, 2015, p. 104). These studies and others (e.g., Krupa et al., 2017; Monson et al., 2020) provide evidence that suggests PSTs can attend to mathematical connections. Findings from our study contribute to this line of study not only in terms of adding more evidence that PSTs can attend to mathematical connections, but also extends the literature base by presenting a framework that captures the different kind of pedagogical considerations the PSTs in our study made around mathematical connections.

Even though reform efforts have focused on students' conceptual development in Australia and the US since the TIMSS 1999 Video Study, Star and Strickland's (2008) study suggests attending to mathematical connections during instruction may still be difficult for novices. However, other studies (e.g., Walkoe, 2015) have provided evidence to suggest PSTs can attend to connections with support. Operating on the premise that teacher noticing is a learnable practice from previous research (see Jacobs & Spangler, 2017), we pursued understanding how PSTs attended to mathematical connections from their perspective and to identify potential productive opportunities for developing PSTs' noticing of mathematical connections that MTEs may leverage, which others have yet to explicitly study.

Theoretical Orientation and Conceptual Framework

Our goals in this study were to investigate (a) the kinds of mathematical connections the PSTs anticipated before and recalled encountering after their instruction and (b) the pedagogical considerations they made surrounding these instances while working with students. In this section, we describe our theoretical orientation toward mathematical connections and explain our conceptual lens used to address (a) and (b) above, which include Mason's (1998, 2002) discipline of noticing and Singletary's (2012) Mathematical Connections Framework (MCF).

Theoretical Orientation Towards Connections

We take the constructivist orientation in viewing mathematical connections as constructions individuals make and not as ideas that live within mathematics in-and-of-itself. That is, we consider mathematical connections arise from reflectively abstracting from and reorganizing activity; they are not perceived or intuited from the world (Cobb, 1988; von Glasersfeld, 1995). Teaching actions consistent with this orientation towards connections include, but are not limited to, constructing models of students' understanding of mathematical connections (Confrey, 1990), selecting and sequencing conceptually-rich mathematical tasks that afford students opportunities to construct mathematical connections (Stein et al., 1996), and carefully initiating and eliciting ideas to promote students' engagement with mathematical connections (Lobato et al., 2005).

Modifying Singletary's (2012) definition, we define a mathematical connection as a relationship one constructs between a mathematical entity and another mathematical (or non-mathematical) entity. In other words, we view the relationship between entities as becoming one only when an individual making the connection conceives of and establishes one. In the context of this study, an individual refers to either PSTs, secondary students, or MTEs.

Mathematics Connections Framework

As our analytical framework for distinguishing different kinds of connections PSTs anticipated before and recalled encountering after their instruction, we used Singletary's (2012) Mathematical Connections Framework (MCF; see Table 2.1). *Connecting through comparison* is when an individual makes a comparison between concepts A and B; A is similar to B. *Connecting specifics to generalities* is when an individual relates a specific case A to a more generalised concept B; A is an example of B. *Connecting methods* is when an individual relates two or more methods to a solution; A or B can be used to find C. *Connecting through logical implications* is when an individual provides a connection through implication or proposition; if A, then B. Finally, *connecting to the real world* is when an individual presents an application of a mathematical concept outside the domain of mathematics; A is an example of B in the real world. Singletary developed the MCF from connections expert teachers made during instruction. We used the framework to analyse the kind of mathematical connections secondary students made (as noticed by the PSTs) or PSTs made in the field and described in online discussion board (ODB) posts.

Table 2.1

Examples of Kinds of Connections Using the MCF

Kind	Description	Examples
Connecting through comparison	A is similar to B. A is the same as B. A is not the same as B. A or B similarly defines or describes B.	Solving the equation $3^{5x+4} = 3^{x+8}$ for x is similar to solving $5x + 4 = x + 8$ for x.
Connecting specifics to generalities	A is an example of B.	The distance formula is an example of the Pythagorean theorem.
Connecting methods	A or B can be used to find C.	Completing the square or the quadratic formula can be used to find the roots of a quadratic polynomial.

Connecting through logical implication	If A, then B. If A, then B but not C.	If two linear equations have the same slope, then the lines are parallel.
Connecting to the real world	A is an example of B in the real world.	The Hohenzollern Bridge in Cologne, Germany is an example of a catenary in the real world.

Discipline of Noticing and Our Notion of Considerations

Following our aforementioned theoretical orientation, we draw upon the notions of markings and considerations to *infer* PSTs' attention to mathematical connections. According to Mason (2002), the discipline of noticing is a purposeful set of actions one takes to improve their professional practice. It involves iteratively refining one's attention and reflection in-the-moment to act in a more disciplined way. Mason distinguished this *intentional noticing*, from which professional practice develops, from that of *ordinary-noticing*, which is easily forgotten or only available when explicitly reminded. Intentional noticing is bringing specific data to mind through *marking* or *recording*. Marking is recalling and describing a salient incident to oneself or others. Recording, a higher form of noticing than marking, is to make a record of an incident within some medium to externalise thoughts and to create data to be able to re-enter the incident for analysis and reflection for future action. In our study, we draw on the notion of marking as recalling incidents salient to PSTs during their field experience.

Mason (2002) argued, "To develop your professional practice means to increase the range and to decrease the grain size of relevant things you notice, all to make informed choices as to how to act in-the-moment, how to respond to situations as they emerge" (p. xi). Relatedly, Mason (1998) conceptualised *awareness* different from *attention*. Attention is what an individual is attending to in-the-moment. Awareness is how an individual makes sense of or interprets what she attended to, either consciously or unconsciously. Teacher noticing entails what teachers attend to and how they make sense of what they attend to (Sherin, Russ, & Colestock, 2011).

Building on these notions, we use the term consideration to refer to a phenomenon related to teacher noticing.

What we refer to as *pedagogical consideration*s are the ways teachers, in our case PSTs, interpret the pedagogical implications surrounding an instance they marked. This sense making may include their interpretations or responses to the objects of their attention. We use the term consideration to note that we inferred, for some PSTs, the object of their attention as mathematical connections. In some instances, it was unclear whether PSTs were making sense of or responding to a mathematical connection, although from our perspective, we saw the instance they marked as being one. A consideration only infers the object of teachers' attention from the teacher's sense-making actions, while noticing is a direct relation to what teachers attend to (i.e., attention) and how they make sense of it in-the-moment (i.e., awareness).

Methodology and Procedures

To study the nature of PSTs' markings of mathematical connections, we examined PSTs' marking of a salient moment in the ODB or their reflections of a mathematical connection they made or observed students making. We extracted and coded each marking from PSTs' written text. First, we coded each PSTs' markings using two coding schemes: (a) whether the instance was an explicit or non-explicit connection marking and (b) kind of mathematical connection using Singletary's (2012) MCF. Whereas the MCF aided us to identify the kinds of connections PSTs marked, it did not capture PSTs' introspection of the pedagogical significance of the connection. Therefore, we used thematic analysis (Braun & Clarke, 2006) to describe how PSTs made pedagogical considerations of the mathematical connections they marked.

Participants and Context

The study occurred at a large public university in the south-eastern US. The participants, a cohort of twelve second- or third-year undergraduate PSTs were enrolled in their first semester (approximately 16 weeks) methods course of the secondary mathematics education program. Data were collected from this methods course, in which the second author was the instructor. The main objective of the course was for PSTs to make sense of and facilitate students' mathematical thinking. The course included a field experience for PSTs to engage in small-group instruction at a local secondary school (Years 9–12) once a week for approximately 90 minutes over 8 weeks. The PSTs worked with three Year 11 classes: Advanced Algebra[2], Advanced Algebra Support[3], and Accelerated Pre-Calculus. Table 2.2 lists the weeks allotted to and topics covered in each class. Each PST worked with the same students for each class. Full ethical approval was gained for this study, and all names that appear in this paper are pseudonyms.

The secondary teachers shared their lesson plans and materials for each lesson before PSTs visited the school each week. To enhance PSTs' learning from the field experience and attention to secondary students' mathematical thinking, the instructional team (the university instructor, a teaching assistant, and classroom teachers) assigned PSTs to engage in an ODB before working with students and write reflections after working with students (Fernández et al., 2012; Krupa et al., 2017; Mason, 2002; van Es & Sherin, 2002). These two assignments were data sources and are further described in the next section. PSTs also attended lab sessions with the university instructor to review the lesson plan and the mathematical content in the lesson

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[2]Advanced Algebra is equivalent to a traditional Algebra II course in the US with the addition of some statistical topics.

[3] Advanced Algebra Support was an elective class for secondary school students who needed additional assistance completing Advanced Algebra. The support class either reviewed or previewed Advanced Algebra course material.

with PSTs. These lab sessions were hour-long class meetings on the university campus a day before the field visit, designed to allow PSTs to discuss and prepare their instruction with secondary students. Lab sessions also provided PSTs the opportunity to further discuss topics posted on the ODB.

Table 2.2

Course and Topics Summary by Weeks

Field Visit Week	Class	Topics
1-3	Advanced Algebra	Operations with polynomials: Polynomial multiplication, long division and synthetic division, factoring polynomials
4-5	Advanced Algebra Support	Interval notation, characteristics of polynomial functions
6-8	Accelerated Pre-calculus	Solving trigonometric equations, law of sines, law of cosines

Data Collection

Online Discussion Board

Before each lab session, the PSTs were required to solve all the problems in the lesson and engage in an ODB to discuss the upcoming lesson (See Figure 2.1 for an example of an ODB post). The PSTs were required to post at least one comment or question and then reply to at least one post. The ODB posts were counted for completion as part of the PSTs' participation grade. There were no specific prompts for the PSTs to follow when posting. It was left open for any type of comment or question about the field visits.

During the lab session, the university instructor organised questions or comments from the ODB posts into four of the five practices of orchestrating productive mathematical discussions (Smith & Stein, 2011)—anticipating, selecting, sequencing, and connecting—to be discussed. For instance, the instructor used Sherry's post in Figure 2.1 to facilitate a discussion on the relation between the methods identified by Sherry for polynomial division as an instance to support PSTs' practice of connecting. The text produced by PSTs in the ODB posts over eight weeks served as one data source.

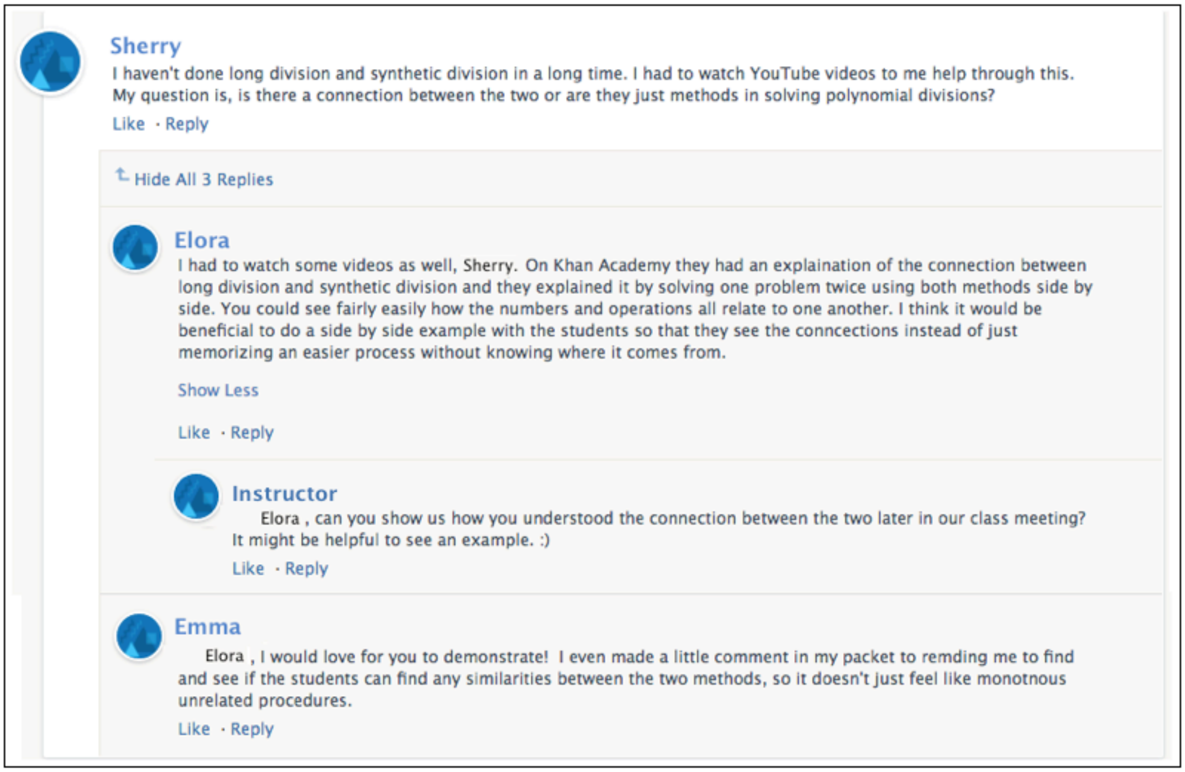

Figure 2.1. Example ODB Post.[4]

Reflections

PSTs wrote two reflections for each group of students they worked with within a class, one after their first visit and one after their last, for a total of six reflections. The instructor asked PSTs to respond to a set of focused prompts common across the weeks and a weekly prompt that changed for each reflection. See Appendix for the focused prompts and weekly prompts. The

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[4] *Note*. A post made by a PST, Sherry, on the ODB with comments by PSTs, Elora and Emma, and the methods instructor in the reply thread to the post. This post is a recreation of the original post to preserve anonymity, but the content and structure of the post and replies appear exactly as on the ODB.

reflections were graded as a course assignment. The text written in these six reflections (Weeks 1, 3, 4, 5, 6, and 8) served as the other data source.

Analysis

Phase 1: Identifying Markings of Mathematical Connections

From the ODB posts and written reflections, we identified markings of mathematical connections first by identifying *explicit* connection keywords, which either started with connect (e.g., connected, connecting, connection, disconnect etc.), referenced a relation (e.g., relate, related to, related, relating, relationship), or involved some comparison (e.g., alike, similar to, different from). As we investigated the data, we noticed there were other instances when the PSTs alluded to, from our perspective, connections but did not use such explicit keywords. Therefore, we identified markings of mathematical connections to include (a) when PSTs explicitly used connection keywords and (b) when we inferred some mathematical connection from their writing. We considered the use of explicit connection keywords to be evidence of PSTs' attention and awareness of a connection while *non-explicit* connections may or may not entail conscious awareness by the PST, but we inferred they were attending to a mathematical connection. (See Table 2.3). The unit of analysis was a collection of text written around one mathematical connection. We compiled extracted markings of mathematical connections into a spreadsheet. In either explicit or non-explicit connections, a marking had to involve a discussion of some relationship between a mathematical entity (e.g., objects, topics, concepts, and procedures) and some other mathematical or non-mathematical entity. For example, in Amanda's reflection, she wrote:

> At one point, she [the student] looked up and complained, "I don't understand something, and I don't know what it is. Something is missing, and I don't feel like I'm getting the

right answer." I assumed this was because she was forming relationships on the worksheet and not producing numerical answers. (Week 8 Reflection).

While Amanda discussed a student "forming relationships," it was unclear to us what mathematical entities were being connected and so did not count as a marking of a mathematical connection.

Phase 2: Identifying Types of Mathematical Connections Using MCF

To study what kind of mathematical connections PSTs marked, we coded each marking according to Singletary's (2012) MCF. When a PST explicitly marked a connection, but we could not infer the kind of connection due to ambiguity, we coded the kind of connection as 'undefined' to capture when a PST was attempting to relate two entities (i.e., students' previous knowledge to the ideas, procedures, concepts, or topics in a future lesson) but the nature of the relationship between the entities was left unstated. Table 2.3 lists examples of mathematical connections in conjunction with explicit and non-explicit markings.

Table 2.3

Examples of Kinds of Mathematical Connections in the Data

Kind of Connection	Explicit	Not explicit
Comparison	I asked my students if they noticed something about the area model that could be *related to* something other than drawing tedious boxes out. [A student] was the first to say how you could distribute. I asked if he could explain.	How could I have used a visual representation to show why distributing an exponent to a polynomial is incorrect?
Logical implication	[A student] said [the solution interval] was between 0 and 2π but did not make the *connection* that you cannot divide by 0. Thus, I asked, "Since we know $\sin\theta$ is between 0 and 2π, if $\sin\theta$ is equal to 0 what is $\sin\theta / \sin\theta$?"	[The students] were able to realise that a problem had no solution when the solution was greater than one, or greater than the radius of the unit circle.

Methods	My mathematical learning goal for the lesson was to create *connections* between solving trig[onometric] equations using the unit circle and using a graph…	I had the students use both distribution and the area model for each problem. Here they could try both strategies on the same problem.
Real-world	Any thoughts on how to *relate* negatives to the real world other than owing money (or something else)?	I am curious if there would be a way to provide a high-school level real-world situation using trigonometric functions.
Specifics to generality	We worked through different examples to see the *connections* between these different depictions [inequality statement, number-line, and interval notation] of inequality statements.	[Understanding sine and cosine as inverse functions] will help [students]…generalize their understanding to new situations.
Undefined	…the [curriculum] material often seemed *disconnected*.	

Note. We added the *italicised* emphasis to highlight the explicit connection keywords.

Phase 3: Development of Themes from PSTs' Pedagogical Considerations

Reviewing the data, we noticed that pedagogical considerations tended to arise about PSTs' markings. We drew upon thematic analysis (Braun & Clarke, 2006) to understand what pedagogical considerations PSTs made when marking a mathematical connection. Thematic analysis is a systematic approach to identifying and interpreting *themes* (i.e., a patterned response or meaning) in qualitative data. After extracting an initial set of themes, we coded two weeks of written reflections (third and fifth week randomly chosen). We elaborate on the final themes—students making connections, suggested practice, knowledge to draw connections, curriculum, and affect—in the Results section.

As we continued re-examining the data according to these themes and compared our codes, we refined our descriptions of each theme, consistent with the thematic analysis methodology as well as Mason's (2002) discipline of noticing. Mason stated, "Where several themes emerge, it is useful to look at how the various themes interact, and whether there really are differences between them or whether they are all manifestations of a yet more general theme"

(p. 120). For instance, there were markings originally coded as students making connections as the focus was on students. However, as the affective considerations recurred in our data, we noticed a pattern in which our PSTs were attending to affective elements, related to but distinct from, the actual activity of students making connections. Therefore, we generated a new theme and coded for instances of *affect*.

Findings

We identified a total of 282 connection markings, 246 from PSTs' written reflections and 36 from the ODB posts. First, we present the kinds of mathematical connections the PSTs explicitly or implicitly marked. Second, we present the *Pedagogical Considerations of Mathematical Connections* (PCMC) framework outlining the following five themes of pedagogical considerations: students making connections, knowledge to draw connections, suggested practice, curriculum, and affect.

Mathematical Connections

PSTs considered a variety of connections: connecting through comparison (n=167), connecting specifics to generalities (n=31), connecting through logical implication (n=28), connecting methods (n=23), and connecting to the real world (n=19). Among the 282 connection markings, 183 were markings in which PSTs used explicit language, which was evident across each kind of connection. The percentage of explicit connections was highest in connecting through logical implication (75%) and lowest in connecting methods (39%).

Looking across the weeks, certain kinds of connections appeared at different times during PSTs' field experience. For example, connecting through comparison showed prominently in Weeks 1 through 7, whereas connecting specifics to generalities occurred predominantly in

Week 8. Table 2.4 provides the frequencies of each kind of connection in the written reflections and ODB.

Table 2.4

Kinds of Mathematical Connections Marked Across Weeks

Kind of Connection	Week								Total
	1	2	3	4	5	6	7	8	
Comparison	32	4	30	39	23	5	31	3	167
Explicit	24	4	25	26	18	5	16	2	117
Not explicit	8		5	13	5		15	1	50
Logical implication	3		2	1	9		6	7	28
Explicit	3		2	1	9		3	3	21
Not explicit							3	4	7
Methods	7	2	1				11	2	23
Explicit	2	2	1				4		9
Not explicit	5						7	2	14
Real-world	4		1	4	3	1	2	4	19
Explicit	3			2	3		1		9
Not explicit	1		1	2		1	1	4	10
Specifics to generality	1		2	3	1		5	19	31
Explicit	1		2	3			2	5	13
Not explicit					1		3	14	18
Undefined	3		2	1	4		3	1	14
Explicit	3		2	1	4		3	1	14
Not explicit									
Total	50	6	38	48	40	6	58	36	282

Note: Week 2 and 6 only contain markings from the ODB posts.

To summarise, the PSTs considered all five kinds of mathematical connections identified in the MCF. The kinds of connections differed across the weeks PSTs were in the field and the explicitness of mathematical connection markings varied by kind of connection and the mathematical content involved.

Pedagogical Considerations of Mathematical Connections Framework

The PCMC framework organizes the themes from PSTs' pedagogical considerations of mathematical connections. See Table 2.5 for an overview of the framework with select examples. We elaborate on each component of this framework.

Table 2.5

Pedagogical Considerations of Mathematical Connections

Pedagogical Considerations	Description	Example PST quotes
Students making connections	Consideration of students' connection-making process	"[Student] … was the one who helped connect distribution, factoring, and the Area Model to the other two students before I even had the chance."
Suggested Practice	Consideration of practices that assisted or could assist students' connection-making	"I anticipate students having a hard time connecting the ambiguous case reasoning to these wonky SSA triangles… Thus, I think some helpful things to ask when we're are going over the ambiguous case with our students are: [lists potential questions]
Knowledge to draw connections	Considerations of knowledge (or lack of) to facilitate students' connection-making	"One question I have about trigonometric functions is how I could, as a teacher, provide a contextual problem at a high-school level."
Curriculum	Consideration of how curriculum impacts connection-making for students	"…when I left [the school] I wondered if I could have introduced the ideas and topics in a different order that would have better set the students up to engage and make connections."
Affect	Consideration of students' affective behaviours (motivations, feelings, beliefs, etc.)	"I asked [my student] why he disliked it the area model and he responded that it's too complicated for him to make the connections…"

Students Making Connections

Students making connections refers to PSTs' considerations of their students' connection-making process. These instances included when the PSTs discussed (a) students' roles in making connections, (b) connections students developed or were yet to develop, or (c) their instructional goals regarding students making mathematical connections. We present illustrative examples of each.

First, in their reflections, some PSTs marked students' roles in making connections. For example, Natalie noted a student contributed a connection to the group:

> "[A student], a lover of the Distributive Property, was the one who helped connect distribution, factoring, and the Area Model to the other two students before I even had the chance." (Week 3 Reflection).

This instance, and others like it, contrasted with moments in which PSTs described the roles *they* took to support students' connection-making (See Suggested practices). Instead, PSTs emphasised the role that a *student* had in assisting in a group in establishing a connection.

Second, PSTs discussed connections their students developed or were yet to develop. For example, Amanda reflected on a discussion with her student on how to factor trigonometric expressions in a manner similar to polynomial expressions:

> [My student] did have a significant 'ah-ha' moment concerning factoring and expanding polynomials and seemed happy with herself once she realised how she could manipulate trigonometric equations the same way. (Week 7 Reflection)

Amanda considered her student as developing a connection between factoring and solving trigonometric equations similar to her previous understanding of factoring and solving polynomial equations. Conversely, Amanda considered when students had not yet developed certain connections as well:

> I realise that the students did not understand what the axis stood for. This showed me that the students had not understood the meaning of their answers in context of the graphs, but rather the unit circle. They had not connected the two yet. (Week 7 Reflection)

There were also instances in which PSTs considered connections students developed even though these connections might not be valid from a mathematician's perspective. For example, in one of his reflections, Cory wrote:

> [A student] asked, "Well if law of sines is $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$, is the law of cosines a $\cdot \cos A =$ b $\cdot \cos B$? " I believe that she thought since sine and cosine were 'opposites' of each other that you would do the opposite operation (Week 8 Reflection).

We interpreted Cory's understanding of the student's question as a connection because of a suggested relation between sine and cosine as 'opposites.' While this connection is not mathematically valid, from our perspective, it is a connection taken from the student's perspective, as interpreted and described by Cory.

Third, PSTs considered their instructional goals regarding *students* making mathematical connections. One of Ali's reflections provides a descriptive example: "I decided that my goal for my students would be to recognise the similarities between regular equations and trigonometric equations and be able to solve trigonometric equations and have a conceptual understanding of the answer" (Week 7 Reflection).

Suggested Practices

Suggested practices were markings where PSTs considered ways in which they might assist or did assist students in making mathematical connections (e.g., a sequence of questions, a demonstration, selecting tasks, etc.). For instance, before teaching the law of sines, in the ODB, Joann posted,

> I anticipate students having a hard time connecting the ambiguous case reasoning to these wonky SSA triangles… Thus, I think some helpful things to ask when we're going over the ambiguous case with our students are: (a) Which side "swings" in SSA triangles? (b)

How does this swinging side relate to what is given? (c) From which angle do we drop *h*? (d) Why do we drop *h* from this angle as opposed to the other 2 angles? If your students happen to finish early, see if they can retry #2 on the Warm-Up, taking a second triangle into consideration, assuming that they only found one solution the first time they went through this problem (Week 7 ODB).

Joann suggested a thoughtful sequence of questions and tasks to her peers to guide students to make a connection through generalisation. Furthermore, she suggested as a closure having students revisit the warm-up activity to potentially elicit another solution strategy.

In contrast to Joanna, Emma reflected on what she *could have* done to make a more explicit connection with students. She stated, "I could have used this as an opportunity to make more explicit connections about the even and odd characteristic being directly related to the parity of the leading variables exponent as well as the end behaviour, and so on" (Week 5 Reflection).

Knowledge to Draw Connections

Knowledge to draw connections were markings when PSTs considered their knowledge (or lack thereof) of mathematical connections or (b) their knowledge (or lack thereof) in aiding students in drawing mathematical connections or demonstrating connections. Although these instances involved, to some extent, considerations of students making mathematical connections or a suggested practice, the main focus was on *their knowledge* to support students' making mathematical connections.

The ODB and reflections provided differing opportunities for PSTs' considerations of the knowledge to draw connections. The ODB provided an opportunity for PSTs to reach out to peers and the instructional team to address gaps in their knowledge whereas the written reflections provided an opportunity for PSTs to reflect on their knowledge to draw connections.

For example, in the ODB, Sherry posted about a question about the connection between long division and synthetic division for polynomials (see Figure 2.1). In her reflection, Elora revealed her misunderstandings of a connection that emerged when working with students. She noted, "My questions this time have to do with … the connection between the graph and the unit circle. I got confused at one point while trying to explain a connection between the unit circle and the graph of trig functions" (Week 7 Reflection).

Curriculum

Curriculum was markings when PSTs focused on how the curriculum materials aided or hindered opportunities for their students to make connections. These included PSTs' consideration of the (a) coherence in instructional material such as a task or worksheet, (b) sequencing of mathematical topics across or within tasks, and (c) appropriateness of the context in a task (or sequence of tasks).

Few PSTs considered the coherence of the instructional material. Joann posted how the coherence of the task impacted her thinking when solving the task: "While I was going through this worksheet, I had a really hard time making connections and keeping my thinking consistent from the first page to the last page of the packet" (Week 7 ODB). Similarly, Cory wrote in his reflection, "Another problem I came across was as the students did more practice problems, they lost the connection between multiplying polynomials and area. With the task being long, they were focused more on completion rather than conceptual understanding" (Week 1 Reflection). These examples illustrate how PSTs considered the task's features (e.g., length of the task) limited the coherence and thus distracted from of the goal of the task for making connections between concepts and procedures.

PSTs also considered the sequencing of mathematical topics across or within tasks. In a reflection, Emma stated, "…when I left [the school] I wondered if I could have introduced the ideas and topics in a different order that would have better set the students up to engage and make connections" (Week 1 Reflection). Later in the course, Emma posted, "I'm trying to determine the best way to introduce this to my students, and what sequence of problems will set them up to make the most connections. Do you guys have any suggestions?" (Week 3 ODB). Emma considered the sequencing of the problems or tasks within a lesson appeared to be important. Amanda, on the other hand, considered the sequence of topics across lessons by questioning the sequencing in which students learned concepts and how it impacted their connection-making. In a reflection, Amanda wrote, "Regarding the specific topic of coordinate planes, I wonder how it was first taught to them, how they have used graphs in the past, and what kind of connections were missing that caused this lack of foundation" (Week 5 Reflection). The lack of foundation that Amanda was referring to is the coordination of the independent and dependent variables on a graph.

Furthermore, some PSTs considered the appropriateness of a context in a task (or sequence of tasks) for fostering students' connection-making. For example, Cory wrote, "The only thing I would change about this lesson would have been… coming up with a similar question to the baseball one that would have been more relevant to my students" (Week 8 Reflection). Cory recognised that the context of the task was not meaningful to his students and so he articulated a need for attending to culturally relevant contexts (c.f., Aguirre et al., 2013).

Affect

We coded markings as *affect* when PSTs considered their students' affective behaviour, such as motivation, feelings, or beliefs, related to making mathematical connections. For example, Sherry reflected on her student's feelings about using the area model: "I asked [my student] why he disliked it, the area model, and he responded that it's too complicated for him to make the connections and felt it was much more complicated than the FOIL method" (Week 1 Reflection). In contrast, Emma considered a positive affective experience with her student: "[He] seemed truly interested when I was explaining to him why the Law of Sines doesn't work to find obtuse angles. I think this is because students really do want to make sense of the material they are learning. They like filling in the blank spaces with information that connects ideas" (Week 8 Reflection).

Table 2.6 summarises the frequencies across the five pedagogical considerations in the data. Across both data sources, PSTs most frequently considered suggested practice. In their reflections, PSTs marked several instances of students making connections. We see this focus as a natural one in that it is more likely PSTs will discuss students making connections after having worked with them, and also a celebratory shift in that such transition is not guaranteed.

Table 2.6

Frequency of Each Consideration Across Data Sources

Pedagogical Considerations	ODB	Reflections	Total
Students making connections	0	112	112
Suggested practice	24	118	142
Knowledge to draw connections	10	18	28
Curriculum	3	7	10
Affect	0	10	10
Total	37	265	302

Note. There are more pedagogical considerations than the mathematical connection instances identified, as a PST may have considered more than one pedagogical consideration per connection marking.

Discussion

The findings address two interrelated ideas: (a) kinds of mathematical connections PSTs marked and (b) the pedagogical considerations PSTs made about these mathematical connections. The PCMC framework addresses the second of these ideas. We discuss and interpret these main findings and then offer potential activities that MTEs could use to support PSTs in attending to and becoming aware of mathematical connections during instruction.

PSTs' Attention and Awareness of Mathematical Connections

PSTs in our study attended to various kinds of mathematical connections, whether those connections were constructed by them or students. The majority of the connections that we identified in the data were explicitly identified by PSTs as connections. Each kind of mathematical connection in the MCF was evident in PSTs' markings. The PSTs often attended to connecting through comparisons. The high frequency of attending to connection through comparison found in PSTs' markings is consistent with expert teachers' practice (Singletary, 2012). Overall, these findings highlight that secondary PSTs are able to attend to and explicitly identify the kinds of mathematical connections found in experienced mathematics teachers' practice. Demonstrating such ability is important because explicit attention to mathematical connections during instruction is generative for students' learning, promotes recall, and impact students' beliefs about mathematics (Hiebert & Carpenter, 1992).

The MCF (Singletary, 2012) afforded us a framework to interpret the kinds of connections PSTs attended to across particular strands of mathematics and different contexts (ODB posts and reflections). Our finding that some PSTs were aware of all five kinds of mathematical connections suggests that PSTs can develop an explicit awareness-in-discipline (Mason, 1998) of these different kinds of connections described in the MCF. Therefore, we find it an appropriate framework to introduce to PSTs to support them in reflecting on the explicit mathematical connections evident in their work with students and call for future studies that investigate the effects of such an approach.

We attribute the kinds of connections PSTs marked to (a) the tasks implemented in the field, (b) PSTs' mathematical knowledge in relation to the topics covered, and (c) the nature of the small-group discussions with secondary students during the field experience. First, mathematical tasks have considerable influence on students' opportunities to make connections (Stein et al., 1996). The tasks PSTs worked on with their students varied in potential opportunities for connection-making. For instance, some tasks focused on the execution of procedures (e.g., factoring quadratics) while others were supportive of connecting procedures with concepts (e.g., deriving the law of cosines). However, we also emphasize the importance of the PSTs' agency in building on such opportunities when implementing tasks. Recall, Joann suggested a thoughtful sequence of questions to support students in making connections as they worked through tasks. Joann's vision for implementing the task was not an explicit feature of the tasks nor an explicit goal stated in the lesson plan. She designed an attentive implementation plan using the tasks given to her by the classroom teacher that could provide opportunities for students to make connections.

Second, teachers' mathematical knowledge for teaching is considered influential in leading to opportunities for students to build mathematical connections (Hill & Charalambous, 2012). Teachers with robust mathematical knowledge for teaching are able to traverse the mathematical terrain, flexibly respond to students' mathematical thinking, and support students in making mathematical connections. Relatedly, some PSTs became explicitly aware of the gaps in their mathematical knowledge for teaching in preparation for or during their field experience.

This awareness was evident in one of Elora's reflections when she recalled getting confused in trying to explain a connection between the unit circle and the graph of trigonometric functions. Therefore, the PSTs' mathematical knowledge for teaching likely contributed to the opportunities PSTs and secondary students had to make mathematical connections and whether PSTs marked the mathematical connections.

Finally, the nature of the small-group discussions likely had an influence on the mathematical connections PSTs had an opportunity to observe. While facilitating small-group discussions, PSTs were learning how to attend to student thinking and build a math-talk community (Hufferd-Ackles et al., 2004). As PSTs and their students moved to higher levels of math-talk learning community, there were more openings to a diverse set of students' ideas and hence more opportunities to build on and connect students' mathematical ideas. For instance, Natalie marked an instance in her reflection when a student took on the role of explaining the connection between the distributive property and the area model for multiplying polynomial expressions. Giving students opportunities to contribute ideas during small-group discussions allowed PSTs and their students to make mathematical connections among ideas.

Our finding that PSTs attended to and explicitly identified mathematical connections in working with students somewhat contrasts with Star and Strickland's (2008) study, which found that few PSTs attended to a connection when watching a video of a mathematics lesson. They conjectured that PSTs may have not attended to the connection due to limited content knowledge, lack of recent experience with the particular content, or just failing to notice. We believe such contrast between the two findings may be, in some ways, attributed to the different approaches taken in the studies. From a lesson recording, Star and Strickland identified several significant features of a teacher's instruction, including making mathematical connections, and sought to determine if PSTs would identify such features. Our approach was to highlight the perspective of PSTs in identifying and marking mathematical connections that arose in their own instruction with secondary students. In other words, the data generated was from PSTs' markings. While it is important for PSTs to attend to significant mathematical connections in the discipline as identified by experts, it is also important to consider the implications of what PSTs attend to. For example, Cory recognized a student's connection-making that was not mathematically valid (i.e., If law of sines is $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$, then the law of cosines $\mathrm{a} \cdot \cos A = \mathrm{b} \cdot \cos B$ because sine and cosine are "opposites"). Even though the connection might not be valid or one that a mathematician would make, it is evident that Cory recognized an attempt by a student to make a connection between the law of sines and law of cosines. We believe both approaches are important for informing MTEs and mutually supportive for PSTs' conceptual development. PSTs should attend to mathematical connections that experts have identified as important for students' learning and should receive support from MTEs in developing the mathematical content knowledge to do so because teachers' mathematical knowledge for teaching enables teachers to successfully support students in explicitly making mathematical connections (Hill & Charalambous, 2012). MTEs also need to examine the mathematical connections PSTs attend to and how those connections may provide insight into understanding students' mathematical thinking.

Contribution of the PCMC Framework

While there are exemplars in the literature detailing expert teachers' instruction of mathematical connections and their pedagogical decisions (e.g., Ball, 1993; Boaler & Humphreys, 2005; Lampert, 2001), the field can benefit from insights on how to prepare novice teachers for such intricate work. There is a need to support MTEs' awareness to the sensitivities that PSTs need to design and facilitate discussions that explicitly attend to mathematical connections (i.e., awareness-in-counsel; Mason, 1998).

The PCMC framework contributes to the field of mathematics teacher education as a way to structure MTEs' awareness-in-counsel so they may guide and develop opportunities for PSTs to design and facilitate discussions of mathematical connections. The framework was developed from PSTs' attempts to make sense of mathematical connections within the context of teaching and learning in the field by considering their pedagogical implications. The framework outlines five pedagogical considerations emerging around PSTs' markings: (a) students' connection-making, (b) practices that assisted or may potentially assist students' connection-making, (c) their knowledge (or lack thereof) to facilitate students' connection-making, (d) curricular influences on connection-making, and (e) students' affective behaviours (e.g., motivation, feelings, beliefs, etc.) towards connection-making. These pedagogical considerations are not exhaustive; other considerations may arise in the moment. Although not exhaustive, the strength of the PCMC framework lies in the fact that it emerged from PSTs' markings. The findings revealed that PSTs *can* engage in these pedagogical considerations; and therefore, the PCMC framework can serve as a starting point for MTEs to draw upon in supporting novice teachers to design instruction that explicitly attends to mathematical connections.

In Table 2.7, we offer some activities MTEs could use in their instruction for each of the pedagogical considerations. For instance, MTEs can explicitly direct PSTs' attention and awareness (Mason, 1998) to the five pedagogical considerations for a lesson or sequence of lessons (e.g., in the planning phase or analysing a video of a lesson).

Table 2.7

Potential Activities for Developing PSTs' Attention to and Awareness of PCMC.

Pedagogical Considerations	Potential Activities for Developing PSTs' Attention and Awareness
Students making connections	When PSTs observe or watch videos of instruction, design reflection questions that prompt PSTs to attend to *students' roles* in connection-making and the different ways students appear to understand the mathematical connections. Have PSTs write instructional goals for future lessons with a focus on *students'* mathematical connections.
Suggested practices	Have PSTs solve tasks and anticipate potential connections students might make and generate ways to support students in making connections they identified. Have them reflect on how a teacher's or their support afforded or constrained students' connection-making after an observation or watching a video of instruction.
Knowledge to draw connections	Ask PSTs to reflect on their knowledge of mathematical connections in various activities such as while working on a mathematical task with their peers or watching different ways students solve the same problem.
Curriculum	Ask PSTs to examine how tasks are sequenced in curriculum materials and consider how the sequence afford or constrain students' connection-making. Ask PSTs to examine when and how connections occur in curriculum materials and learning progressions. Ask PSTs to examine if the context of the task will allow students to make meaningful connections and if so, how.
Affect	Have PSTs observe and describe students' cognitive, behavioural, and emotional engagement surrounding mathematical connections. Have PSTs share successful ways they found to engage students in making connections with peers.

Conclusion

Our study revealed that PSTs could consider various kinds of mathematical connections in their field experience. When marking connections, PSTs also recognized an assortment of pedagogical considerations: students' mathematical connection-making, suggested practices, knowledge to draw connections, curriculum, and affect (Table 2.5). These pedagogical considerations are productive beginnings for novice teachers, and we described potential activities MTEs might use to foster such considerations in their instruction (Table 2.7). Future research studies may evaluate these activities or design others to understand how an activity developed or refined teachers' pedagogical considerations for mathematical connections. Furthermore, future studies may use the PCMC framework to understand the qualitative differences between the considerations of novice and expert teachers, thus providing further direction in understanding how to develop and refine teachers' pedagogical considerations for supporting explicit attention to mathematical connections during instruction.

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CHAPTER 3

IDENTIFYING AND BUILDING UPON BEGINNING TEACHERS' PRACTICE TO FOSTER EXPLICIT MATHEMATICAL CONNECTIONS DURING DISCUSSIONS[5]

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Abstract

Students participating in mathematical discussions and making mathematical connections have been identified as key characteristics of reform-based mathematics instruction leading to several encouraging outcomes for students as identified by teachers and researchers. While the literature provides examples of expert teachers' instruction, there are limited examples of novice teachers' early practice to foster students' participation in connection-making. Understanding how novice teachers foster connection-making is supportive of an incremental approach to build upon their practice. I report on how novice teachers supported students to attend to, contribute, or provide reasoning for mathematical connections during whole-class discussions. The findings reveal that these novice teachers created opportunities for students to participate in making connections but also chose at times to contribute mathematical connections or reasoning to build upon students' thinking. I contend novice teachers' everyday practice of audience design was essential to their creating opportunities for students to attend to and participate in making connections. The analytical approach in this paper may guide teachers and teacher educators in identifying potential features of audience design to build upon in teachers' practice.

Keywords: Audience design, Classroom discourse, Mathematical connections, Reasoning

Introduction

National standards documents, curriculum guidelines, and policy statements convey that mathematical connections are an important goal in school mathematics (e.g., National Council of Teachers of Mathematics, 2000, 2014; *National curriculum in England*, n.d.; New Zealand Ministry of Education, 2007). Several empirical studies suggest that explicit attention to connections promotes students' conceptual development (Hiebert & Grouws, 2007). However, drawing explicit attention to mathematical connections is a difficult undertaking for teachers, especially novice teachers. Expert-novice studies have highlighted particular difficulties for novice teachers. For example, novices tend to focus their explanations on procedures rather than linking the procedures to key mathematical concepts (e.g., Borko & Livingston, 1989; Livingston & Borko, 1990). One means to support novices to draw explicit attention to mathematical connections, rather than solely focusing on procedures, is through selecting mathematical tasks that require students to make connections (e.g., Jacobs et al., 2006; Stein, Grover, & Henningsen, 1996). Selecting a task that requires making connections, however, does not always ensure that explicit attention to connections will occur during a lesson. The nature of the discussions surrounding the task is important. For example, the TIMSS 1999 video study found that of the problem statements that required students to make a mathematical connection, less than 1% of the discussions in U.S. classrooms made explicit references to mathematical relationships and/or mathematical reasoning (Hiebert et al., 2003). In Japan, teachers and students explicitly discussed connections in 48% of the problem statements that required students to make a connection. These findings suggest that other features, beyond the mathematical task, may also have a significant influence in drawing explicit attention to mathematical connections. For instance, Henningsen and Stein (1997) found that teacher's actions, such as pressing students to provide explanations or make meaningful connections, may support students in conceiving of and subsequently making mathematical connections.

There is also growing evidence on the importance of the teacher's role in supporting students' participation in mathematical discussions (c.f. Walshaw & Anthony, 2008). Some scholars have focused on the teacher's role in promoting students' participation in discussions about mathematical connections. For example, Turner et al. (2013) examined a teacher's role in supporting individual English learners' participation in a discussion by providing and justifying connections and found that the frequency with which individual English learners provided or justified mathematical relationships was relatively low. Similarly, Otten and Soria (2014) found that students' participation, at the class-level, was often low when justifying or making connections. In agreement with Henningsen and Stein (1997), both Turner et al. and Otten and Soria suggest that teachers' actions, in particular teachers' strategic questioning, may support students' contributions of mathematical connections and reasoning.

While the previously mentioned studies (Henningsen & Stein, 1997; Otten & Soria, 2014; Turner et al., 2013) suggested that teachers, in general, experience difficulty in supporting students to participate in making connections and explaining their reasoning, some expert-novice studies further posit that novice teachers do not leverage opportunities to make mathematical connections during discussions with students (Even et al., 1993; Leinhardt, 1989). Broadly, these opportunities are similar to what Stockero and Van Zoest (2013) called *pivotal teaching moments*. Stockero and Van Zoest characterized the types of pivotal teaching moments that a group of novice teachers encountered and their responses to them. When the novices decided to respond to a pivotal teaching moment by encouraging students to make connections, the response was likely to lead to positive student outcomes, regardless of how skillfully the novice teachers responded. This result contrasts with the earlier expert-novice studies. One explanation may be that the results in Stockero and Van Zoest's study are reflective of reform-based and practice-based teacher education movements. Their result suggests that novice teachers bring early skills and practices in getting students to attend to mathematical connections. However, Stockero and Van Zoest did not provide descriptive accounts of how the novices supported students to make mathematical connections during these pivotal teaching moments. Descriptive accounts of these early skills and practices may provide mathematics teacher educators with insights on how to build on prospective teachers' beginning practice of supporting students to attend to mathematical connections.

There are examples in the literature of expert teachers leveraging in-the-moment opportunities during instruction to support connection making (e.g., Ball, 1993; Boaler & Humphreys, 2005; Lampert, 2001; Leinhardt & Steele, 2005). These examples provide mathematics teacher educators some insights into designing learning opportunities for prospective teachers. However, these examples do not account for the development of the practice of supporting students to attend to and participate in making mathematical connections, the difficulties that prospective teachers face in learning the practice, or the resources available to prospective teachers within their context to overcome those difficulties (Ghousseini, 2015).

The studies that do focus on novice teachers often propose that novices do not take up opportunities for attending to mathematical connections because they do not have the mathematical knowledge for teaching needed to facilitate such discussions (e.g., Borko & Livingston, 1989; Even et al., 1993; Livingston & Borko, 1990). While I do not discount the specialized knowledge needed, Russ, Sherin, and Sherin (2016) offer a complementary proposition: Perhaps some of the issue lies in the everyday practice of audience design. In colloquial conversations, speakers will adjust their talk to ensure their meaning is understood by others (i.e., audience design). For example, Alice and Bob are planning to go see a movie at a theater. They selected a movie they would like to see and are trying to determine what time to go to the movie. Bob is looking at the local movie times. Alice notices Bob's watch on his wrist:

Alice: What time is it?

Bob: 7:30.

Alice: No, what time is it right now?

Bob: Oh, 5:45.

Alice requests the current time. Bob interpreted Alice's request as the next starting time for the movie. Alice modified her request to make her intention to know the current time to Bob, and Bob provided the current time to her.

Due to the unnatural work of professional classroom teaching (Ball & Forzani, 2009), novice teachers can experience difficulty in bootstrapping from the everyday practice of audience design to a specialized audience design for teaching. For example, teachers pervasively ask questions to which they already know the answer, which typically follow an interaction pattern of an initiation of a question by the teacher, then a reply from the student, and ending with an evaluation from the teacher (Mehan, 1979b). This initiation-reply-evaluation (IRE) pattern is similar to conversation between Alice and Bob when Bob misinterprets Alice's request for the current time. Russ et al. (2016) argued that researchers and teacher educators can boost or redirect teachers' everyday practices to better align with the purpose of professional teaching. Specialized practices aligned with the purpose of professional teaching include probing students' ideas or identifying key understandings in students' responses (Ball & Forzani, 2009).

According to Russ et al. (2016), novice teachers may experience two difficulties in audience design: assessing the expertise of the students or the faulty assignment of students to the categories of listeners. I focus on potential difficulties due to assessing the expertise of students in the moment. In everyday conversation, if a speaker assesses the addressee as lacking expertise, then the speaker will correct addressee's mistakes or show the addressee how to perform a task. A specialization of this everyday practice for teaching may seek to ask questions (in which the answer may not be known) to understand students' thinking or to provoke students' thinking. Conversely, if a speaker assesses the addressee as having expertise, then speakers will assume others know what they mean. For example, Singletary (2012) noticed that sometimes teachers hinted or suggested mathematical connections without explicitly stating the relationship to students. If students have the necessary knowledge or understanding to appreciate this connection, then hinting or suggesting may be sufficient in the moment. However, if students do not have the necessary background knowledge, then implying the connection may not foster understanding. It may signal that a teacher has overestimated students' expertise in recognizing the relationship. A specialization of this everyday practice for teaching may seek to probe students' ideas before making assumptions about students' meanings. Ultimately, professional teaching seeks to place students' meanings rather than the teacher's meanings as the focus.

Conceptual Framework

Mathematical connections have historically been conceptualized as relationships inherit in mathematics or the mental constructions of learners. I conceive mathematical connections from a discursive perspective. Gee regarded one task of language was to make connections: "We use language to render certain things connected or relevant to other things, that is, to build connections or relevance" (Gee, 2005, p. 12). Reframing mathematical connections from a discursive perspective allows one to answer such questions as, how does discourse connect or disconnect mathematical ideas? I define *mathematical connections* as the discursive ways or practices in which an individual or community of learners comes to make and describe relationships between a mathematical entity and another mathematical or non-mathematical entity. The term entity includes ideas, concepts, objects, representations, procedures, or methods.

The conceptual framework for this study was a concatenation of two empirically-based frameworks, Mathematical Connections Framework (Singletary, 2012) and Teacher Moves for Supporting Student Reasoning (Ellis et al., 2019), and one framework from a meta-analysis of mathematical reasoning (Jeannotte & Kieran, 2017). Next, I briefly describe each framework and highlight how the frameworks complement each other for the purpose of the study.

Mathematical Connections

Singletary (2012) developed the Mathematical Connections Framework (MCF) from her observation of secondary mathematics teachers. She found five kinds of connections evident in the teachers' instruction (see Table 3.1). One kind of connection is *connecting through comparison*. When comparing, a student or teacher may highlight similarities or differences between two entities. For example, two students could provide two competing claims: (a) $y + 6x = 11$ is the same as $y = 6x - 4$ because the two lines have the same slope and (b) $y + 6x = 11$ is not the same as $y = 6x - 4$ because the two lines have different slopes. While the former statement is not correct, it provides teachers with insight to how a student understands the meaning of slope and could provide an opportunity for a teacher to plan and facilitate a discussion on the meaning of slope. Individuals may also make connections by *connecting through logical implication*. For example, a teacher may state: "If two lines are parallel, then the two lines have the same slope but not the same y-intercept." Connections through logical implication demonstrate relationships of dependence and sometimes, but not always, resemble propositions or theorems. Another kind of connection is connecting methods. *Connecting methods* is when the community puts forth two (or more) methods that can accomplish the same goal-oriented activity. For example, a teacher may ask for two students to share how they found the y-intercept of the line $y + 6x = 11$. One student may have found the y-intercept through graphing the line and another student may have found the y-intercept by solving $y + 6(0) = 11$. A teacher could decide to lead a discussion of the relationship between these two methods and why they both lead to the same conclusion for the y-intercept. *Connecting specifics to generalities* is when an individual relates a specific case to a more general concept or rule. For example, a teacher may relate the tangent of the angle of inclination of a line to the more general concept of slope. Finally, *connecting to the real world* is when an individual relates a specific mathematical entity to a relatable concept or context in the real world. For example, Stump (1999) found that secondary mathematics teachers related slope to real world objects such as ski slopes and wheelchair ramps and also to functional situations such as constant speed in distance-time graphs. In either case, the teacher is relating the concept of slope to a relatable concept (e.g., constant speed) or objects (e.g., wheelchair ramps) in the real world.

In addition, MCF describes the levels the connections reached during instruction**.** The levels are distinguishable by how explicit the reasoning for the connection appeared in the discussion. There are three levels: suggested, provided, and provided-and-explained. A connection at the *suggested* level means the connection is left implicit. For instance, if a teacher said, “If two lines are parallel, then we know something about the slope,” then the connection is at the suggested level because the “something about the slope” is left implicit. In contrast, if the teacher said, “If two lines are parallel, then the two lines have the same slope but not the same y-intercept,” then the connection is at the *provided* level because the relationship between the two entities, parallel and slope, is made explicit. Similarly, a teacher may provide the former connection to students and ask them to prove the statement. Students may provide a variety reasons for why the statement is true. For example, students may prove by cases using an empirical proof scheme by verifying several cases of lines with the same slope or students may prove by using a deductive argument using properties of similar triangles (i.e., deductive proof scheme; Harel & Sowder, 1998). If explicit reasoning is given for the connection, then the connection is at the *provided-and-explained* level. It should not be interpreted, however, that all provided-and-explained connections fall within a proof scheme and require external truth validity. Rather, provided-and-explained connections are connections accompanied with some form of reasoning found in school mathematics and follow the shared norms of a classroom community. To further explicate reasoning found in school mathematics, I briefly discuss a conceptual model by Jeannotte and Kieran (2017) that informed my perspective on reasoning for mathematical connections.

Table 3.1

Mathematical Connections Framework (Reprinted with permission from Singletary, 2012)

Kind of Connection	Level of Connection		
	Suggested	Provided	Provided-and-explained
Connecting through comparison	A and B are somehow related. A is related to something (where the something is left unsaid).	A is similar to B. A is the same as B. A is not the same as B. A or B similarly defines or describes C.	A is similar to B because of C. A is the same as B because of C. A is not the same as B because of C.
Connecting through logical implication		If A, then B. If A, then B and not C.	If A, then B because of C.

Connecting methods	A or B can be used to find C.	A or B can be used to find C because of D.
Connecting specifics to generalities	A is an example of B.	A is an example of B because of C.
Connecting to the real world	A is an example of B in the real world.	A is an example of B in the real world because of C.

Mathematical Reasoning

Jeannotte and Kieran (2017) developed a conceptual model for mathematical reasoning for school mathematics from their meta-analysis of the literature. In their model, they distinguished between *structural* and *process* aspects of mathematical reasoning. For the purposes of this paper, I only focus on the process of mathematical reasoning. The processes of mathematical reasoning include (a) *searching for similarities or differences* (generalizing, conjecturing, identifying a pattern, comparing, and classifying), (b) *validating* (justifying, proving, and formal proving), and (c) *exemplifying* (inferring examples that assists searching for similarities or differences and validating). When I use the term "reasoning" in this paper, I intend the meaning of the process of *validating*. The process of *searching for similarities or difference* is similar to how I understand mathematical connections. For example, the subprocess of *comparing* is defined as "[inferring] a narrative about similarities and differences" (Jeannotte & Kieran, 2017, p. 11), which is consistent with *connecting through comparison* in the MCF. Furthermore, *exemplifying* is a process that is mutually supportive of searching for similarities or differences and validating and therefore, from my perspective, supportive of connection-making and reasoning.

Teacher Moves for Supporting Student Reasoning

To identify a teacher's actions during a mathematical discussion of a mathematical connection, I utilized the Teacher Moves for Supporting Student Reasoning (TMSSR) framework (see Figure 3.1; Ellis, Özgür, & Reitin, 2019). TMSSR broadly defines four types of teacher moves for supporting students' reasoning: (a) *eliciting*, (b) *responding*, (c) *facilitating*, and (d) *extending*. Eliciting moves seek to "draw out, identify, clarify, and understand students' ideas an contributions" (Ellis et al., 2019, p.118). Responding moves are a teacher's in-the-moment reaction to a student's thinking. Facilitating moves are also teachers' in-the-moment reactions to a student's thinking but with the purpose of pushing or building upon the student's thinking. Extending moves seek to foster more complex mathematical reasoning from students.

Ellis et al. developed the TMSSR framework using the conceptual model of mathematical reasoning by Jeannotte and Kieran (2017). Because of the close similarities to how I define mathematical connections and reasoning to the conceptual model of mathematical reasoning of Jeannotte and Kieran, I decided to use the TMSSR framework over other frameworks for teacher moves (e.g. Cengiz et al., 2011; Fraivillig et al., 1999; Staples, 2007). The TMSSR framework developed from teacher's actions that supported students' mathematical reasoning (searching for similarities or difference, validating, or exemplifying), which I interpret as inclusive of connection-making. Other reasons for using the TMSSR include its ability to categorize moves beyond teacher questioning (e.g. *re-presenting*), distinguish between high and low potential moves for supporting student reasoning, and not being limited to a particular mathematical domain or grade (Ellis et al., 2019).

Eliciting Student Reasoning		Responding to Student Reasoning	
Low	High	Low	High
Eliciting Answer	Eliciting Ideas	Correcting Student Error	Prompting Error Correction
Eliciting Facts or Procedures	Eliciting Understanding	Re-voicing	Re-Representing
Asking for Clarification	Pressing for Explanation	Encouraging Student Re-voicing	
Figuring Out Student Reasoning		Validating a Correct Answer	
Checking for Understanding			

	Facilitating Student Reasoning		Extending Student Reasoning	
	Low	High	Low	High
Guiding	Cueing	Providing Guidance	Encouraging Evaluation	Encouraging Reflection
Guiding	Funneling	Encouraging Multiple Solution Strategies	Pressing for Precision	Encouraging Reasoning
Guiding	Topaze Effect	Building	Topaze for Justification	Pressing for Justification
Providing	Providing Information	Providing Alternative Solution Strategies		Pressing for Generalization
Providing	Providing Procedural Explanation	Providing Conceptual Explanation		
Providing	Providing Summary Explanation			

Figure 3.1. The TMSSR framework (Reprinted by permission from Springer: Ellis et al., 2019)

In this study, I take up the belief that students' attention to and participation in making mathematical connections during discussions is important for conceptual development, and ask the question: How do (novice) teachers support students in attending to and contributing mathematical connections or providing reasoning for those connections during whole-class discussions? By focusing on explicit connections and teacher moves, I am able to capture how novice teachers supported students to attend to and participate in making connections from their in-the-moment assessment of students' understanding, a feature of audience design. An intention of this paper is to describe the ways in which three novice teachers supported students to attend to and contribute mathematical connections or to give reasoning for a mathematical connection during whole-class discussions. The intention of this paper is not to describe mathematical connections that are productive for student learning or make an argument that the mathematical connections presented here should be a focus of instruction.

Methods

A qualitative analysis based on existing frameworks (Ellis et al., 2019; Singletary, 2012) was conducted to understand how novice teachers supported students in attending to and participating in making mathematical connections across one unit of instruction. Next, I describe the participants, context, data collected and data analysis process.

Participants and Context

The participants selected come from a larger group of secondary preservice mathematics teachers in a multi-year project called *Learning to Support Productive Collective Argumentation in Secondary Mathematics Classes* (LSPAM). LSPAM followed a cohort of 18 prospective secondary teachers in their program at a large university in the southeastern United States. Six of the teachers were followed into their student teaching. From the six, three student teachers (Melissa, Robin, and William; all names are pseudonyms) were purposefully selected for this study because they taught the same 9th grade accelerated mathematics course at the same school and co-planned lessons with input from their mentor teachers. The other three student teachers were excluded because they were teaching a different course. Because Melissa, Robin, and William co-planned lessons together, there was a lower chance for variation in instructional tasks across these teachers and so the instructional tasks were considered a less likely factor contributing to any variation in classroom discourse (e.g., Hiebert & Wearne, 1993; Stein et al., 1996). For context, Melissa and Robin cotaught together during their student teaching in a model similar to the one described by Leatham and Peterson (2010). William also cotaught with another student teacher in the same model, but William's partner did not participate in this stage of the study.

The teachers taught a unit of instruction titled “Connecting Algebra and Geometry through Coordinates” (Georgia Department of Education, 2013). The state department of education provided a guide for the unit that included alignment of state standards, student learning objectives/outcomes, and resources (e.g., mathematical tasks with teacher commentary, graphic organizers, and links to web resources). Teachers could use or modify the materials in the unit guide as they saw fit. In the unit, students were to apply algebra techniques to solve problems in geometry. For example, students were to derive a formula that produced a point on a line segment that partitioned the segment into a given ratio. Table 3.2 lists an overview of the lessons.

Table 3.2

Overview of Lesson Goals in the Unit of Instruction

Lesson	Students will…
1	Determine if two lines are parallel, perpendicular, or neither using the slope criteria for parallel and perpendicular lines.
2	Derive a formula that can be used to find the point on a directed line segment between two given points that partition the segment into a given ratio (i.e., partitioning formula).
3	Derive a formula to find the distance between any two points on the coordinate plane.
4	Derive the midpoint formula.
5	Verify a given quadrilateral (e.g., rhombus, rectangle) or triangle (e.g., isosceles) on the coordinate plane by applying the distance formula, partitioning formula, and/or slope criteria. Compute the area and perimeter of a polygon given the coordinates.
6	Write the equation of a circle and graph it on the coordinate plane given the center and radius of the circle.

Data

The data collected included lesson plan materials and video recordings of classroom instruction for one unit in one class period for each participant. The data used in the analysis presented in this paper are from the six lessons in the unit. This included 8 days of video recording for one class period when Melissa was the focus teacher, 6 days of video recording for another class period when Robin was the focus teacher, and 7 days of video recordings for one class period in William's classroom.

Data Analysis

All the video recordings of the lessons were transcribed. To reduce the data to a manageable subset for further analysis, I identified episodes in which the whole class engaged in a content-related activity. An episode is an interaction that occurs in the engagement of a task or activity (e.g., checking homework, returning assessment to students, discussing a problem, etc.). A content-related activity is an episode involving a mathematical task or activity (e.g., discussing a solution to a problem or task) rather than a day-to-day operation of school (e.g., taking attendance). I further reduced the data by selecting episodes that included moments when a student or teacher first introduced a mathematical connection into the whole-class discussion, which I called connecting episodes. These connecting episodes typically included a reference of at least two mathematical entities with some description of the nature of the relationship between the entities and extended over several lines of transcripts. A simple example is a student claiming, "So, the [distance] formula is basically the Pythagorean theorem" during a discussion. This is a moment when a student introduced a new mathematical connection to the classroom discourse between the distance formula and the Pythagorean theorem. These connecting episodes formed the dataset. Because teachers' speech and instructional gestures have been found to be less frequent in similar connecting episodes later instruction (Alibali et al., 2014), I excluded connecting episodes from the dataset when a student or teacher stated a mathematical connection that had been previously discussed. This exclusion criterion was to avoid analyzing mathematical connections in the classroom discourse that were functioning as-if-shared.

Next, I coded the kind of mathematical connection and the level the connection reached in the connecting episodes using the MCF. I also coded who contributed the connection. If a student contributed both components (A and B) of the connection, (e.g., A is similar to B), then I coded the connection as provided by the student. Similarly, if a teacher contributed both components, then I coded the connection as provided by the teacher. However, if a teacher contributed one component (A) and a student contributed the other (B), then I coded the contribution of the connection as provided by both, student and teacher. In addition to coding who contributed the connection, I coded who contributed the reasoning for the connection (e.g., A is similar to B because of C), if one was provided. I then analyzed the actions and/or questions of the teacher in the connecting episodes using the TSMRR. I noted what moves, if any, the teacher took to prompt students in providing the mathematical connection or giving the reasoning for the connection.

Results

All five kinds of mathematical connections appeared in the data: *connecting through comparison* (31), *connecting through logical implications* (24), *connecting methods* (17), *connecting specifics to generalities* (25), and *connecting to the real-world* (3). Few (10 out of 100) mathematical connections were implicit (at the suggested level). The mathematical connections overall were explicit (at the provided or provided-and-explained level) during discussions (see Table 3.3). With the majority of the connections explicit during discussions, students in the classes had many opportunities to attend to mathematical connections. Furthermore, the proportion of connections at the provided-and-explained level (45 out of 100) also indicate that students had opportunities to attend to other students' or the teacher's reasoning for the mathematical connections. Making reasoning for the mathematical connection explicit is likely to support students in understanding the mathematical connection.

Table 3.3

Kind and Level of Connection Across Lessons

Level of Connection	Kind of Connection					Totals
	Comparison	Logical implication	Methods	Specifics to generalities	Real-world	
Suggested	2	-	1	6	1	10
Provided	6	16	12	10	1	45
Provided-and-explained	23	8	4	9	1	45
Totals	31	24	17	25	3	100

Extent to Which Students Participated in Making Connections

I analyzed two forms of participation in making mathematical connections: *providing mathematical connections* and *giving reasoning for mathematical connections*. Students likely participated in other ways, but I focused on these forms of participation because explicit contributions to discussions are resources for teachers to build on students' prior understandings and allow other students, and even the teacher, access to novel reasoning (Walshaw & Anthony, 2008).

Providing a Mathematical Connection

Students had a notable role in providing connections by introducing approximately half of the mathematical connections. During whole-class discussions, the teachers introduced 32 connections, students introduced 49 connections, and together students and teachers jointly provided 18 connections. A mentor teacher provided one connection as well. When considering students jointly providing connections with their teachers, then students participated in providing approximately two-thirds of the mathematical connections.

Students' participation in providing mathematical connections, however, varied across the kind of mathematical connections. For example, students frequently provided the connections when *connecting methods* or *connecting specifics to generalities*. When *connecting through comparison*, students and the teachers provided the connection with similar frequencies. When *connecting through logical implications* (if A, then B), the teachers often provided the antecedent (A) and asked students for the consequent (B). Figure 3.2 summarizes the contributions of connections by students and teachers across kinds of mathematical connections.

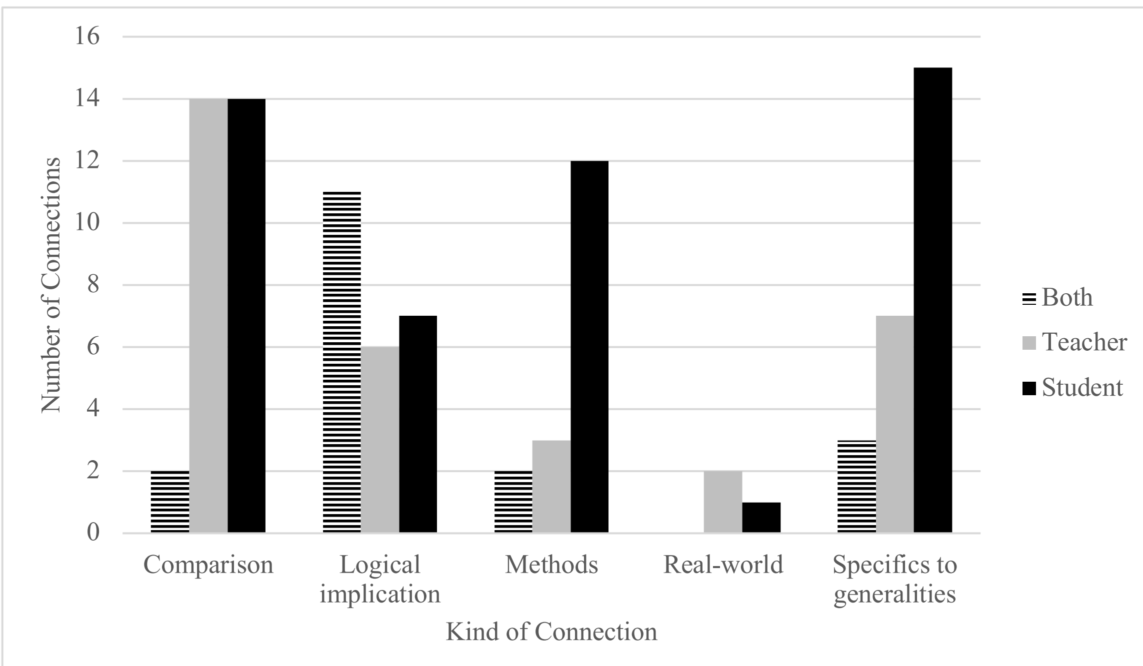

Figure 3.2. Who Contributed Connections by Each Kind of Connection

Giving Reasoning for a Mathematical Connection

Teachers and students participated in giving reasoning for connections. Recall, there were 45 (out of 100) mathematical connections during whole-class discussion at the *provided-and-explained* level. Students gave reasoning for 21 connections, teachers gave reasoning for 21 connections, and together they jointly gave reasoning for 3 connections. Thus, teachers had access to students' ways of reasoning and students had access to the teacher's reasoning for about half of the mathematical connections. The kind of mathematical connection did not seem related to whether students gave a reason for the connection. Students provided reasoning for approximately half of the *connections made through comparison* (11 out of 23), *connections made through logical implication* (4 out of 8), *connections made between methods* (2 out of 4), and *connections made from specifics to generalities* (4 out of 9). Figure 3.3 summarizes the contributions of reasonings by students and teachers across kinds of mathematical connections.

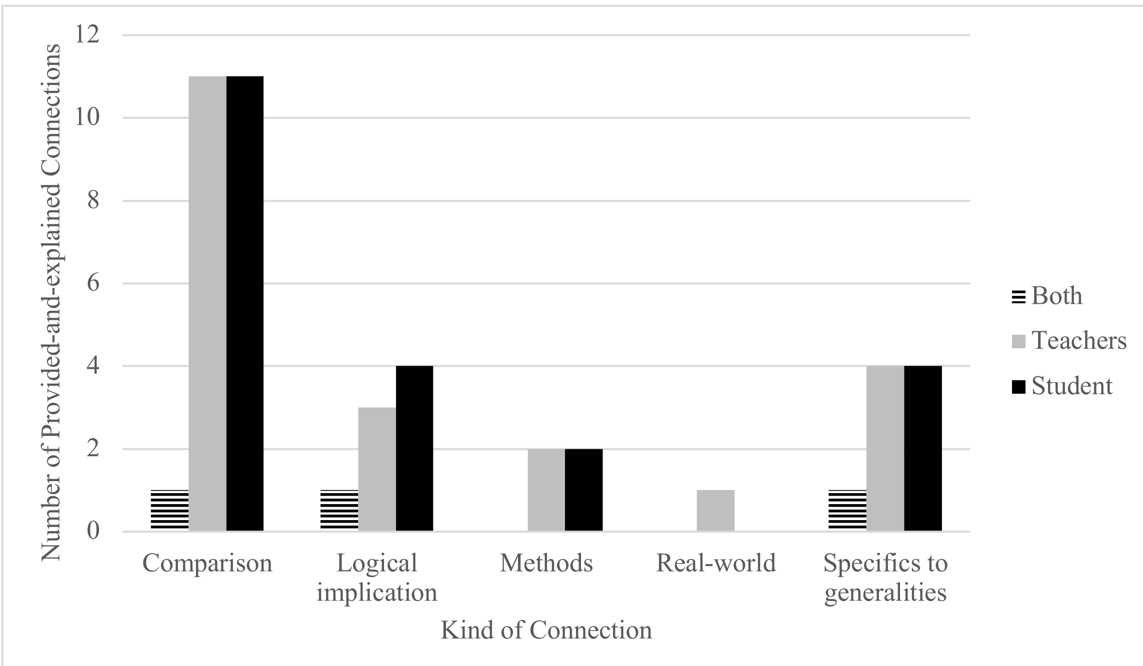

Figure 3.3. Who Contributed Reasoning for a Mathematical Connection Across Kinds of Connections

In sum, students were equal participants with teachers during episodes of connection-making in whole-class discussions. During the unit, students and teachers provided mathematical connections and reasoning for the connections. When looking at the kind of connection provided, however, there were subtle differences in who contributed connections. In particular, teachers and students were more likely to provide *connections through logical implications* together.

Also, the teachers and students were nearly equal participants in providing the reasoning for mathematical connections, regardless of the kind of connection.

Teachers' Support of Student Participation Signaling Their Assessment Students' Expertise

It is also important to understand *how* these teachers drew students' attention to and fostered participation in making mathematical connections, not just that students participated. How these teachers supported students' participation during discussions about mathematical connections signal their in-the-moment assessment of students' expertise. For example, if the teachers used an eliciting move of *eliciting understanding*, it could signal their intent to gain further insights into students' thinking to assess their understanding for making a connection. However, if teachers used a facilitating move of *providing information*, it could signal their intent to "repair" the discussion by providing necessary background knowledge for students to make a mathematical connection. Ellis et al. (2019) argued that teachers are more likely to support students' reasoning (and connection-making) when their moves were distributed across eliciting, responding, facilitating, and extending (parenthetical are the author's addition to Ellis et al.'s claim). In this study, the novice teachers used a subset of these high-potential moves to draw students' attention to and foster students' participation in making mathematical connections.

A variety of high potential eliciting, responding, facilitating, and extending moves supported students' participation of providing mathematical connections during discussions. According to Ellis et al. (2019), the use of high potential moves typically reflects discussions focused on students' ideas. Furthermore, Ellis et al. claimed that teachers were typically more successful in promoting students' reasoning (and connection-making) when they coordinated eliciting, responding, facilitating, and extending moves. I argue that this coordination of teacher moves can capture moments of teachers attending to audience design, in particular, assessing students' expertise in the moment. For example, Ellis et al. reflected on an episode in a teacher's classroom, "[The teacher] therefore relied on extending and facilitating moves in ways that combined with eliciting in order to gauge the degree to which the students made sense of the ideas she introduced" (p. 126). Gauging the degree to which students made sense of ideas is a feature of audience design (Russ et al., 2016). Next, I present connecting episodes to showcase how eliciting, responding, facilitating, or extending revealed the teachers' assessment of students' expertise in the moment.

Eliciting

Eliciting moves allow teachers to "draw out, identify, clarify, and understand students' ideas and contributions" (Ellis et al., 2019, p. 118). The teachers primarily elicited connections from students by *eliciting ideas*, which is a high potential move in the TMSSR. Episode 1 is an example of Melissa eliciting ideas from students. Prior to Episode 1, students were given a set of linear equations and asked to graph their equations on a separate piece of graphing paper. After graphing, students worked with a partner to make observations about their equations and graphs. In presenting the transcripts, I used the following conventions:

(text) denotes a description of a speaker's gestures or written inscriptions

((*text*)) denotes the author's interpretation of the speaker's speech or inaudible

{text} denotes overlapping speech between speakers

[text] denotes a word substitution to enhance clarity

[[*text*]] denotes the author's interpretation of the teacher's move using the TMSSR framework

Episode 1: What did you notice?

Melissa: Alright, so then when you matched up with your [partner], what did you notice about those two graphs or two equations? [[*Eliciting: Eliciting ideas*]]

Student 1: The lines were perpendicular.

Melissa: Lines were perpendicular [[*Responding: Re-voicing*]]. Okay, and did anyone graph the other person's equation on their graph? [[*Facilitating: Cueing*]]

Student 2: Oh, I erased it, but I did.

Melissa: You erased it? [[*Facilitating: Cueing*]]

Student 3: I didn't graph it, but we put them in front of each other and then held it up to the light.

Melissa: Okay, so you could kind of see where they intersected at. [[*Facilitating: Cueing*]]

Robin: That's a good idea.

Melissa: Did everyone see what Student 2 was talking about when she said they intersect at 90 degrees and it kind of forms four of them? [[*Eliciting: Checking for understanding*]]

Multiple: No.

Melissa: No one saw that too? [[*Eliciting: Checking for understanding*]]

Student 4: I didn't really.

Melissa: [My partner and I] graphed ours (Holds up graphing paper with both equations graphed). [[*Facilitating: Cueing*]]

In this episode, the *connection of specifics to generalities* is: The paired linear equations, $y = \frac{2}{3}x + 1$ and $y = -\frac{3}{2}x + 8$, are examples of perpendicular lines. Melissa began by eliciting ideas from students about what they noticed when comparing their equations and graphs with their partner. Student 1 made the claim that the two lines were perpendicular, which Melissa re-voiced. It is important to note that Melissa did not just elicit this mathematical connection intended from the task and move on with the discussion. She followed up her eliciting with facilitating moves of cueing to allow other students access to how Student 1 concluded the lines were perpendicular. In other words, Melissa was assessing students' expertise in the moment in recognizing the connection by cueing students' attention to the intersection of their lines.

Melissa's cueing moves were potentially informed by her prior interaction with a student during the investigation phase of the task. When Melissa and the student investigated their two linear equations, Melissa supported the student in noticing the two lines were perpendicular.

Later during the whole-class discussion, as seen in Episode 1, Melissa not only elicited ideas to support a student's contribution of the connection but also assessed whether other students recognized their pair of equations as representing perpendicular lines. She continued cueing to prompt other student-pairs to notice the intersection of their two lines. When multiple students confirmed Melissa's assessment that they had not noticed the intersection, Melissa held up the graph of her and her partner's equations to display the intersection of their two lines. Melissa inferred correctly that some student-pairs had yet to notice the intersection of their lines. Melissa's cueing brought the intersection to their attention. She did not complete a common IRE pattern with students (Mehan, 1979a). After a student claimed the lines were perpendicular, Melissa followed up with facilitating moves to draw students' attention to the graphs of the two equations. Her following up with facilitation moves makes sense if she assessed some students had yet to consider or had no knowledge of the how the lines intersected. Otherwise, it would have been reasonable, as a novice teacher, for Melissa to validate the student's claim and move on with the discussion.

Responding

Responding moves are teachers' in-the-moment reaction to students' thinking. For these teachers, responses that supported students attending to mathematical connections were often to *validate a student's answer*. According to Ellis et al. (2019), validating a student's answer confirms a student's idea by either re-voicing, re-wording, or adding information to the student's answer. Validating a student's answer was often a means by which these teachers provided an explicit mathematical connection or reasoning. For example, in Episode 2, Robin validated a student's correct answer that the product of two rational numbers that are negative (opposite) reciprocals is negative one.

Episode 2: When working with opposite reciprocals

Robin: What about opposite reciprocal? Can somebody give me an example of that? [[*Eliciting: Eliciting ideas*]]

Student 1: Two and negative one-half

Robin: Okay. (Writes '2' and '-1/2' on the board) [[*Responding: Re-voicing*]]

Robin: What happens when you multiply this? [[*Eliciting: Eliciting answer*]]

Student 2: Negative ((*inaudible*)).

Robin: Negative?

Student 2: One.

Robin: Right. [[*Responding: Validating a correct answer*]]

Robin: Alright, somebody else give me a different example of opposite reciprocal, please. [[*Eliciting: Eliciting ideas*]]

Student 3: Six-thirds and negative three-sixths.

Robin: Okay, six-thirds and negative three-sixths you said? [Writes '6/3' and '-3/6' on board] [[*Eliciting: Asking for clarification*]]

Student 3: Mm-hmm ((*affirmative*))

Robin: Thank you. If we multiply those together, that's a good example, what are we going to get? [[*Eliciting: Eliciting answer*]]

Student 3: Negative one.

Robin: Negative one. The idea here is that, I'm sure you're all familiar with this but we just wanted to review so we know what this term means, when you're working with opposite reciprocals, the product of those should always equal negative one. If you're unsure or you're picking between two equations, multiply them ((*the slopes of the two lines*)) together and if the product is not negative one, then they're probably not opposite reciprocals. [[*Responding: Validating a correct answer*]]

Robin: How do you all feel about that? Is that pretty clear to everyone? [[*Eliciting: Checking for understanding*]]

In essence, Robin provided the *connection through logical implication*: If the product of two (rational) numbers is not negative one, then they are not opposite (negative) reciprocals of each other. She provided this connection through validating a student's answer that the product of two and negative one-half is negative one. Validating a student's answer by confirming and adding information may signal that these teachers assessed some students as knowledgeable but perhaps other students had yet to make an explicit connection or were not attending to the connection. In this case, Robin elicited two examples of negative reciprocals from students and wrote the examples on the board. She then implied that the students were knowledgeable of the connection she was about to state explicitly: “I am sure you’re all familiar with this…” This statement suggests that in the moment Robin assessed that students were likely familiar with the connection she was about to state. By providing the connection explicitly, however, Robin also provided access to the logical implication definition of a negative reciprocal for students. To confirm students had understood this connection, Robin sought confirmation from students by asking, “is that pretty clear to everyone?” In cases like this one, the teacher’s validation confirms a student’s answer while also allowing the teacher to introduce an explicit connection or reason for a connection.

Facilitating

Teachers use facilitating moves to assist students to develop their reasoning through providing scaffolds (i.e., guiding) or introducing new ideas, facts, procedures, strategies, or explanations (i.e., providing) (Ellis et al., 2019). These novice teachers supported students’ participation in making a connection by *encouraging multiple solution strategies*. In Episode 3, Robin encouraged students to provide another solution strategy beyond the initial strategy of counting to find the distance between two points on a vertical line segment with one point as $x_1 = 1$ and the other as $x_2 = 9$. This is referred to as question “number one” in the episode.

Episode 3: Another way to determine distance

Robin:	So, somebody tell me what they did for part or for number one. [[*Eliciting: Eliciting a fact or procedure*]]
Student 1:	I had counted the distance.
Robin:	Okay, counted the distance. [[*Responding: Revoicing*]]
Student 1:	Which I had gotten eight.
Robin:	You got what? Eight? Okay. [[*Eliciting: Asking for clarification*]]
Student 2:	Divide by four.
Robin:	We’ll get to that in just a second. On number one, is there any other ways? Student 1 said she counted each mark. [Are] there any other ways that you all found the distance between the two points? [[*Facilitating: Encouraging multiple solution strategies*]]
Student 3:	I just counted.
Robin:	Counted, okay. [[*Responding: Re-voicing*]]

Student 2: Subtracted nine minus one.
Robin: Subtracted nine minus one, did you all hear that? So, Student 2 said he subtracted nine minus one which gave us eight. Obviously, doing either is fine, but I just want to make sure everybody is aware of that. [[*Responding: Re-voicing*]]

In this episode, the connection is: Counting from x_1 to x_2 or subtracting x_1 from x_2 can be used to find the distance between x_1 and x_2. Robin first elicited a procedure from students on how they found the distance. After Student 1 said that she counted to find the distance, Robin then encouraged other solution strategies to find the distance: "[Are] there any other ways…?" Encouraging multiple solution strategies rather than *providing alternative solution strategies* may indicate that these novices assessed students as having the expertise to develop other solution strategies rather than having to be provided an alternative. By providing alternative solution strategies, teachers *seek to inform*. According to Yoon et al. (2012), when a speaker's goal is to inform then it is not necessary for a speaker to assess the addressee's perspective. Alternatively, when a speaker *makes a request*, such as encouraging students to provide another solution strategy, then the request is likely to be successful if the speaker considered the addressee's perspective. Recall, in our initial example, Alice requested the time from Bob. She knew her request of the time would likely be successful because Bob was wearing a watch. In other words, acts of requesting require that a speaker assess the addressee as being knowledge enough to provide the requested information (Yoon et al., 2012). By encouraging multiple solution strategies, Robin likely assessed students as having the expertise to meet her request. Therefore, encouraging multiple solution strategies is indicative that Robin had monitored and assessed her students' solution strategies prior to her request.

Extending

Teachers use extending moves to foster more complex mathematical reasoning from students (Ellis et al., 2019). When initiating a mathematical connection through extending moves, these novice teachers primarily *encouraged reflection* to support connection-making. How these teachers designed their questions for students to reflect on a provided answer or explanation was indicative that these novice teachers assessed in the moment that students had the expertise to make mathematical connections through reflection. In Episode 5, students applied the Pythagorean theorem to find the length of a line segment CD. William asked a student to present her solution for the length of a line segment (see Figure 3.4). The student formed a right triangle and labeled the legs of the triangle as 6 and 8 and found the length of the hypotenuse to be 10 units. It is important to note that the problem statement only asked for students to determine the length of line segment CD. As such, William's prompt at the beginning of Episode 5 is an extension beyond the problem statement.

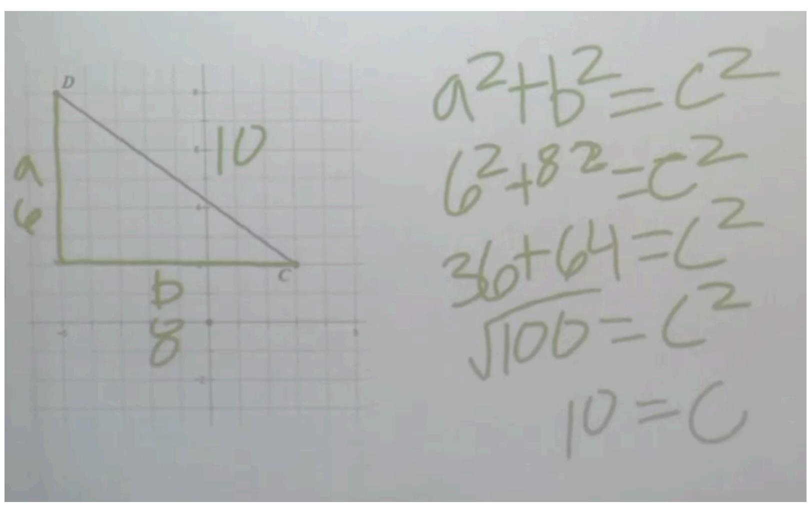

Figure 3.4. Student Solution for the Length of Line Segment CD

Episode 4: Special right triangle

William: Awesome. Did everyone get 10? Also, what kind of - do we notice this - is this a special type of right triangle? [[*Extending: Encouraging reflection*]]

Student 1: Yeah.

William:	What kind? [[*Extending: Encouraging reflection*]]
Multiple:	Right triangle.
William:	Well, yeah but more specifically. [[*Extending: Encouraging reflection*]]
Student 2:	Isn't it a 3-4-5?
William:	Perfect! So, it's a 3-4-5 but we're just times-ing each side length by 2. So that's why because it's a 6-8-10. [[*Responding: Validating a correct answer*]]
William:	So, does everyone see that connection? [[*Eliciting: Checking for understanding*]]

In this episode, the *specifics to generalities connection* is: The right triangle 6-8-10 is example of a Pythagorean triple 3-4-5 because it's "'just times-ing each side length by 2." William prompted the introduction of this connection by encouraging students to reflect on the triangle: "…is this a special type of right triangle?" William's initial question is interesting due to a lack of specificity other than "special type" of right triangle. For example, a student might have identified the right triangle as scalene or wondered if the angles in the triangle were 30, 60, and 90. Despite these alternatives and potentially other responses students could have generated, William successfully obtained from a student that the right triangle was a Pythagorean triple.

According to Yoon and Brown-Schmidt (2019), when addressing groups, a speaker designs their utterances based on the combined knowledge of the group. If the group does not share common knowledge, then the speaker will use more words in their utterance. In comparison, if the group does largely share common knowledge, then the speaker will use fewer words in their utterances. While Yoon and Brown-Schmidt did not study this phenomenon in a classroom, they hypothesized a similar result would occur. In this episode, William's brief descriptor of "special type of right triangle" does seem to suggest that William assessed students to be knowledge enough to interpret his meaning. Even when students were initially unsuccessful, William did not add more description to rebuild common ground: "Well, yeah but more specifically." A shared common ground is essential for all joint activities (Clark, 1996).

William seemed to communicate that students had the necessary background knowledge with the information on the board to decipher his question. For example, he did not add information to his requestion: "Well, yeah but which Pythagorean triple is it?" When a student did identify the triangle as a Pythagorean triple, William validated the student and provided the reasoning, "we're just times-ing each side length by 2" and thus making the connection at the level of provided-and-explained.

Discussion

The purpose of this study was to examine how novice teachers supported students in attending to and participating in making explicit mathematical connections across one unit of instruction. Explicit attention to mathematical connections is linked to promoting students' conceptual development (Hiebert & Carpenter, 1992; Hiebert & Grouws, 2007). However, teachers, especially novices, struggle to leverage opportunities to make explicit mathematical connections during discussions (Even et al., 1993; Leinhardt, 1989; Otten & Soria, 2014; Turner et al., 2013). Mathematics teacher educators need insights into how to support these teachers.

This study provides mathematics teacher educators insight into the developing practice of novice teachers to support connection-making and complements existing research of how expert teachers support explicit mathematical connections (e.g., Ball, 1993; Boaler & Humphreys, 2005; Lampert, 2001; Leinhardt & Steele, 2005). Two findings from the study merit further discussion: (a) the kinds of connections and the explicit nature of the connections evident in the discussions and (b) the evidence that novices leveraged features of audience design in their support of students' connection-making.

Explicit Nature of the Connections and Kind of Connections

A significant result of this study was the majority of the mathematical connections evident during discussions were at an explicit level. Comparatively, explicit attention to mathematical connections is often described as lacking in many US mathematics classrooms (Hiebert et al., 2003; Jacobs et al., 2006; Litke, 2020). In other words, many students in mathematics classrooms in the US have limited learning opportunities to connect mathematical objects, ideas, concepts, and procedures. The explicit nature of the connections found in the discussions led by these participating teachers is promising considering explicit connections support students' conceptual development and beliefs about mathematics (Hiebert & Carpenter, 1992; Hiebert & Grouws, 2007). However, the high frequency of mathematical connections found in the data does not mean that all the connections explicitly discussed corresponded or contributed to significant understandings for students' conceptual development. In fact, there were examples in the data where students made explicit connections that mathematicians would consider inappropriate; nevertheless, these connections provided opportunities for these teachers to address students' mathematical understandings.

An unanticipated finding was that all five kinds of mathematical connections were evident in the discussions led by the participating teachers. Singletary (2012) found that not all kinds of mathematical connections were present in discussions led by some of the experienced teachers in her study. Singletary considered possible reasons for their absence may include the teachers' beliefs about mathematics and the mathematical tasks selected by the teachers. This study did not take into account the participating teachers' beliefs about mathematics, and so, I am unable to comment on the relation between the teachers' beliefs and the kinds of mathematical connections evident in mathematical discussions. On the other hand, the mathematical tasks selected by the participating teachers were included as part of the scope of the study. It is reasonable that mathematical tasks selected by the participating teachers and how they engaged students with those tasks resulted in all five kinds of mathematical connections evident across discussions.

Generally, these participating teachers selected tasks that provided students opportunities to make comparisons and conjectures, discuss solution methods, build on previous concepts, and justify their reasoning. With these tasks, students were to communicate their mathematical ideas, which led to students or the teachers in making an explicit mathematical connection. For example, in Episode 1, students were engaged in comparing equations and their graphs together and were asked to make conjectures. As a result, students made connections through comparison, such as the similarities between equations. Students also made connections through logical implication with their conjectures. For example, if two linear equations have the same slope, then the lines are parallel. In addition, the participating teachers sometimes extended tasks in ways that offered students an opportunity to make a connection that was not apparent in the design of the task. In Episode 4, William extended a task asking students to find the length of a line segment by asking students consider what kind of Pythagorean triple was evident in a student's right triangle solution method. This provided students an opportunity to recognize the right triangle was an example of a 3-4-5 Pythagorean triple and how this recognition could have led to another solution method. In other words, William supported students in connecting specific examples to a general concept and solution methods. In summary, the teachers selected tasks that provided students opportunities to make connections and they sometimes also extended procedural tasks in ways that allowed students to make connection to previous concepts.

Audience Design as a Mediator for Teacher Moves

These novice teachers supported students' participation in making mathematical connections through their coordination of teacher moves. This coordination was possible because of their in-the-moment assessment of students' expertise, a feature of audience design. In other words, teachers' actions were mediated by their attention to audience design, specifically their in-the-moment assessment of students' expertise. Understanding teachers' in-the-moment assessment is important because how we communicate with others is guided, in part, by our knowledge or belief about what others do and do not know (Brown-Schmidt & Heller, 2018). Part of being able to communicate effectively is being able to assess what others know in the moment and adapting our utterances accordingly (i.e., audience design).

The finding that teachers' in-the-moment assessments of students' expertise are mediators for their actions is consistent with the perspective of the TMSSR framework (Ellis et al., 2019), which assumes several mediators for teachers' actions such as teachers' beliefs and goals, the mathematical task students engage in, and the classroom and mathematical norms. Building on this perspective, this study focused on how teachers' attention to audience design was evident in their coordination of their actions that supported students in attending to and participating in making mathematical connections during whole-class discussions. For example, in Episode 2, Robin explicitly stated her assessment that students were likely already familiar of the connection before she shared it with the class. Other times, the teachers' attention to audience design allowed them to support students' reasoning. In Episode 4, William encouraged students to reflect and connect a student's solution method to what they already knew about Pythagorean triples. William assessed the student's solution method on the board and his question were informative for the classroom community to make a connection to previous ideas that would be productive. Overall, the teachers' attention to students' expertise had considerable influence in how teachers support student reasoning: from when teachers support students' contributions to how teachers design and sequence their questions.

Several scholars have described teachers' questions or moves and general classroom discourse patterns in mathematics classrooms (e.g., Cengiz et al., 2011; Fraivillig et al., 1999; Lampert & Blunk, 1998; Wood, 1998). These descriptions of teacher moves and discourse patterns typically foreground and explain *what* teachers say or do. There has been little attention to how features of audience design may be mediating *what* teachers say or do and *how* teachers design their actions accordingly. In other words, how teachers' assessment or understanding of students' perspective is informing their actions. A notable exception to this is the argument by Teuscher et al. (2016) that Piaget's (1955) construct of decentering is able to explain teachers' action relative to students' thinking. They claimed that teachers' ability to decenter during student-teacher interactions has implications for whether teachers pose questions, the kinds of questions teachers ask, the quality of teachers' explanations, and the opportunities afforded to students to contribution. In some ways, the outcomes of this study agree with these implications. However, analyzing student-teacher interaction only through the construct of decentering is limiting. Decentering mediates the planning of teachers' actions. Audience design is more encompassing; it mediates the planning, production, monitoring, and repair of teachers' speech and actions. For example, an audience design perspective would provide explanatory power for when a teacher monitors and adjusts her initial question, which was based on her interpretation of a student's perspective, in order to re-engage a student that stated uncertainty in how to respond to her initial question. In other words, the teacher planned a question based on her interpretation of the student's thinking, but then she rephrased her initial question in a form that the student could engage with.

Attention to audience design is considered important to the professional work of teaching (Russ et al., 2016). This attention to a feature of audience design by the novices in this study suggests that novice teachers can leverage productive aspects of audience design into their early teaching practice. This study is supportive of Russ et al.'s (2016) assertion that teachers' everyday practices, such as audience design, may bolster teachers' specialized practices, such as supporting students' reasoning. Future research on other features of audience design and its implications for other specialized teaching practices is warranted because explicit attention to audience design by mathematics teacher educators may be constructive to incrementally build teachers' professional practice. I further discuss how mathematics teacher educators may do this in the implications section with the results from the study.

Implications

The results raise implications for mathematics teacher educators to consider how novice teachers' everyday practice of audience design could be boosted or redirected for the specialized practice of teaching, such as fostering mathematical connections during discussions. The approach taken to examine the practice of these novice teachers may provide direction for mathematics teacher educators in identifying features of audience design that could become specialized. In particular, the TMSSR framework is able to discern patterns in how teachers engage with their students in the moment (Ellis et al., 2019). To illustrate, I provide an example of a recurring pattern in the data and explain how this pattern may inform mathematics teacher educators in boosting audience design for the purposes of teaching.

The novice teachers in this study frequently checked students' understanding of a connection at the end of connecting episodes. In Episode 1, Melissa asked, "Did everyone see what Student 2 was talking about…?" In Episode 2, Robin asked, "How do you all feel about that? Is that pretty clear to everyone?" In Episode 4, William asked, "So, does everyone see that connection?" In contrast, notice how Cathy Humphreys, a highly regarded master teacher, has come to regard such questions in her instruction: "…a question like 'Does everybody understand that?' has only one correct answer – yes…I try to ask questions that seek to ferret out the complexity in mathematical ideas and explanations." Humphreys further provided alternative questions, such as, "What does and does not make sense about the explanation?" and "Why does that make sense?" I claim Humphreys is describing explicit attention to audience design in her reflection, in particular an attention to assessing students' expertise in the moment. She explained that questions like those asked by Melissa, Robin, and William offer little insight into students' understanding. In everyday conversation, speakers will monitor their addressees for understanding (Clark & Krych, 2004). Humphreys, however, seemed to notice that monitoring or assessing students' understanding is more specialized in teaching. A question like, "Does everybody understand?" is not likely to allow her to monitor students' understanding and, furthermore, the question cast understanding as an all-or-nothing transmission rather than a development of ideas to students. Humphreys' careful attention to how her questions affect students' contribution and her ability to assess students' understanding is evidence of the specialization of the everyday practice of audience design for the purposes of teaching.

While I argued Melissa, Robin, and William showed attention to audience design in supporting students' attention to and participation in making connections, I believe they could continue to specialize in how they check for students' understanding. As Russ et al. (2016)

discussed, teachers learn to boost or redirect their practice of audience design incrementally. A mathematics teacher educator working with Melissa, Robin, and William could make checking for understanding an explicit focus as a particular feature of audience design. These student teachers could analyze and reflect on their practice with support from a mathematics teacher educator to develop an understanding similar to Humphreys'. In general, a mathematics teacher educator could ask novices to critically examine transcripts or video case-studies for teacher moves using the TMSSR and consider which sequencing of moves allowed for a teacher to assess students' expertise in the moment. For example, if a teacher only uses low-potential eliciting and responding moves (i.e., similar to an Initiation-Response-Feedback pattern; Mehan, 1979a, 1979b), then he or she is provided little insight into students' understanding and is unlikely to lead students to engage in meaningful connection-making. Novices could reflect on how alternative moves, such as the high-potential moves in TMSSR, impact a teacher's ability to assess students' expertise in the moment.

Conclusion

I argue that novice teachers can foster opportunities for students to attend to, provide, or give reasons for mathematical connections during whole-class instruction. Teachers used a variety of eliciting, responding, facilitating, and extending moves to assess and build upon students' expertise, an essential skill emerging from the specialization of audience design in teaching. Examining teachers' practice as an incremental specialization of audience design for teaching, a perspective which this paper takes, is a means to understand teachers' practice and may likely be a means for teachers to reflect on and develop their practice. Teachers also have to consider the opportunities they provide for students to contribute connections or provide reasoning to the discussions. However, just providing students opportunities to participate will not lead to productive discussions. Teachers need to recognize pivotal moments that arise for connection-making and improve incrementally upon their existing practice in fostering connection-making.

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CHAPTER 4

MAKING MATHEMATICAL CONNECTIONS: BUILDING AND MAINTAINING COMMON GROUND[6]

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Abstract

Successful everyday activity with others is possible by building shared understanding (i.e., common ground). During a mathematics lesson, building common ground is a challenging but essential factor that affects the mathematics learning of a classroom community. In this paper, I analyzed how a secondary mathematics student teacher made micro-adjustments to build common ground with students to support them in making a mathematical connection. The analysis supports the assertion that teachers can begin to specialize their everyday practice of building common ground for teaching given time and experience to learn in and from their practice.

Introduction

Instructional explanations are essential to the practice of teaching. According to Leinhardt (2001), instructional explanations demonstrate, convince, structure, or convey some portion of a discipline to others; model questioning and appropriate ways such questions might be answered in a discipline; and imply valid metacognitive behaviors for working in a discipline. Instructional explanations may arise from the teacher's or students' questions or when an idea or concept is flagged as important by the teacher for future learning. Whether instructional explanations are delivered by the teacher alone or co-constructed with students, they have important implications for student learning. For instance, instructional explanations in mathematics are believed to help students learn, understand, and use procedures and concepts flexibly and in novel ways (Leinhardt, 2001).

However, research studies have found that instructional explanations are difficult for many mathematics teachers to deliver, especially novice teachers (Borko & Livingston, 1989; Even et al., 1993; Leinhardt, 1989, 2005; Livingston & Borko, 1990). In particular, novices were described as not being sensitive to the needs of their student audience. While there are instructional explanations of exceptional quality from experienced teachers in the literature (Ball, 1993; Boaler & Humphreys, 2005; Lampert, 2001; Leinhardt & Steele, 2005), teaching experience alone has not been found to guarantee high-quality instructional explanations (e.g., Thompson & Thompson, 1994). Therefore, understanding how teachers learn to provide instructional explanations is important for informing mathematics teacher education and professional development.

Little is known about how teachers learn to provide instructional explanations. Charalambous et al. (2011) offered two reasons for this gap. One reason was that much of the research on instructional explanations is situated in comparative studies between expert and novice teachers' performance. The other reason was that the literature on prospective teachers' instructional explanations predominately examine the practice at one moment in time and those studies that do examine prospective teachers providing instructional explanation across time do not track changes in the prospective teachers' ability. To address these gaps, Charalambous et al. designed a comparative study of prospective elementary teachers' instructional explanations on a simulation task prior to and after their elementary mathematics teacher preparation course sequence. They found prospective teachers improved in providing instructional explanations after their course sequence, which may suggest the practice of providing instructional explanations is learnable. Charalambous et al. noted it was an open question whether the prospective teachers' capacity to provide instructional explanations transferred into a classroom setting. Furthermore, Charalambous et al. asked the following question for future research in mathematics teacher education: "Given that beginning teachers leave teacher education programs with certain limitations in their capacity to offer quality instruction, how can we structure opportunities for them to continue growing in teaching practices that matter for student learning?" (p. 462).

Russ et al. (2016) argued teachers could continue growing in their teaching practices, such as instructional explanations, by building from or specializing their everyday knowledge and practices, such as *audience design* (H. H. Clark & Carlson, 1982; H. H. Clark & Murphy, 1983). In other words, "the teacher's specialized task of constructing an instructional explanation that communicates disciplinary content is supported by and continuous with the everyday work of designing utterances for specific audiences" (Russ et al., 2016, p. 420). As an example of audience design, consider a doctor speaking with her patient at risk for a heart attack (Isaacs &

Clark, 1987). Hearing that her patient is a medical student, the doctor may choose to consult with her patient differently than with the layperson by telling the patient he is at risk of a potential myocardial infarction (i.e., using more specific medical vocabulary). Russ et al. argued that the everyday practice of audience design and the specialized mathematical knowledge for teaching are influential in teachers' learning to provide instructional explanations.

To begin answering whether prospective teachers' attention to audience design may be a way to continue growing their instructional explanations and if the classroom could be a place for such development to take place, I completed a micro-analysis of a secondary mathematics student teachers' practice of providing instructional explanations. I focused my investigation in two ways. First, I focused on instructional explanations of mathematical connections because explicit attention to mathematical connections is generative of students' learning, promotes recall, and impacts students' beliefs about mathematics (Hiebert & Carpenter, 1992). Second, I focused the analysis on the student teacher's ability to build common ground (i.e., shared understanding) during instructional explanations because successful audience design depends on careful assessment of what is shared.

In the next sections, I provide an overview of the theoretical construct of common ground and the relevant literature on common ground for interpreting the findings of the study. Then, I discuss the methods for the study, including the background of the study, the rationale for the case, overview of the data collected, and the data analysis process. In the findings, I provide the results of the microanalysis of two illustrative data excerpts. In closing, I discuss my interpretations of how the student teacher may have learned to provide instructional explanations and offer potential implications for mathematics teacher education programs and future research.

Theoretical Framework and Relevant Literature

Language is situated in joint activity; it is used to build and create, such as making connections (H. H. Clark, 1996; Gee, 2005). Joint activities arise when an individual initiates an activity with a dominant goal and others join to achieve that goal. Successful joint activity requires coordination among participants. For instance, in conversation, speakers may need to adapt their speech and gestures to be understood by addressees, which some refer to as *audience design* (H. H. Clark & Carlson, 1982; H. H. Clark & Murphy, 1983) and others call recipient design (Sacks et al., 1974). When designing their speech or gestures, speakers have to deal with disparities in *common ground* (H. H. Clark, 1996; H. H. Clark & Carlson, 1981; H. H. Clark & Marshall, 1981), or the shared mutual knowledge, beliefs, and understandings among speaker and addressees.

Clark (1996) distinguished two types of common ground: communal and personal. For this paper, personal common ground is particularly relevant. Personal common ground is the shared knowledge between speaker and addressees resulting from prior experiences with each other or their current situation. It arises from joint perceptual experiences (e.g., teacher and her students examining number line on the board) or joint actions (e.g., teacher and students identifying the location of one-fourth on the number line). Over the course of conversation, the common ground between speaker and addressees accumulates via *grounding* (H. H. Clark, 1996; H. H. Clark & Schaefer, 1989). In other words, a speaker will contribute something to addressees and then they work together to reach a mutual belief that the speaker's contribution has been understood for current purposes. It is assumed that a speaker will seek positive evidence of common ground from addressees whether through speech or gestures. Evidence of understanding falls into one of four classes: (1) assertions, (2) presuppositions, (3) displays, or (4) exemplifications. Presuppositions are assumed to be stronger evidence of understanding than assertions, displays stronger than presuppositions, and exemplifications stronger than displays. Table 4.1 summarizes the evidence of understandings with examples from classroom discussions. Note that, although the examples are mostly positive evidence of understanding from students, the evidence could be requested by students or negative evidence (see S1 in Table 4.1).

Table 4.1

Evidence of Understandings

Class of evidence	Description	Examples
Assertions	Students may assert that they do (or do not) understand a teacher's utterance with the expectation that the teacher will accept their assertion.	T(eacher): Do you all see that connection? S(tudents): Yes. T: Does that make sense though? S1: No (*Shakes head disconfirming*).
Presuppositions	Students may presuppose they understood the teacher's intended meaning by initiating the next relevant turn.	T: So, it's going to be our x-coordinate (*Labels* $\frac{a}{a+b}(X_2 - X_1) + X_1$ *as x-coordinate on the board)*. S2: So then, like for y, you put like A over A plus B and then like $(Y_2 - Y_1) + Y_1$.
Displays	Students may display part of their understanding by responding to a teacher's proposed joint activity (e.g., responding to a question).	T: If we have a ratio that's 1 to 3…what's the total number of parts? S3: Four.
Exemplifications	Students may exemplify what they understood the teacher's (or another student's) meaning.	T: Why did S3 say there were four parts? S4: Because, well, … you have 3 parts and you have like 1 part, then you add them together.

Much of the research on audience design and common ground has taken place in clinical research settings. These studies often examine two individuals collaborating on a *referential communication task* (Krauss & Weinheimer, 1964) to identify or arrange objects in some way. A full review of audience design is beyond the scope of this paper. Brown-Schmidt and Heller (2018), Fischer (2016), and Holler and Bavelas (2017) provide comprehensive overviews of research on audience design and common ground. In this paper, I briefly reference key findings from studies that examine expert-novice pairings, multiparty conversations, and other relevant studies for interpreting classroom data.

Expert-novice Pairs

A subset of studies of audience design and common ground have examined expert-novice pairs working together on a referential communication task or some other similar task (e.g., answer patients' health problem questions). These studies have found that experts were efficient in assessing the mutual knowledge they shared with their novice addressees and made adjustments in their speech and gestures accordingly (Bromme et al., 2005; Campisi & Özyürek, 2013; Horton & Keysar, 1996; Isaacs & Clark, 1987). For instance, Isaacs and Clark (1987) examined pairs of individuals that were and were not familiar with New York City (NYC) landmarks to work together to arrange a collection of photos of New York City landmarks by talking to each other but without seeing each other's photos. Each participant had the same collection of photos but did not know their partner's familiarity with NYC. As the pair worked together, they assessed each other's expertise in identifying NYC landmarks and collaboratively developed more efficient ways to refer to the photos with the experts supplying and novice acquiring the specialized knowledge of the names of NYC landmarks. In another study, Horton (2005) found speakers will use commonality assessments to monitor and adjust for the needs of addressees. In conversation, for example, a speaker might ask, "Remember what happened last time?" The speaker's question assesses what they share in recalling a previous event. Speakers will also use more gestures when they know that the information is highly relevant to addressees (Kelly et al., 2011). Speakers may furthermore adjust their gestures (e.g., change the speed or form) when facilitating addressees' identification of specific visual features (Peeters et al., 2015).

Some studies also suggest other influential factors between expert-novice pairings such as age of the participants, time constraints of the task, and other contextual features.

Multiparty Conversation

In many ways, speakers face more complexity when designing their speech and gestures for multiple addressees at once, especially if the addressees differ in their knowledge, beliefs, and understandings. There is an emergence of audience design studies examining the unique contextual feature of multiparty conversations (Yoon & Brown-Schmidt, 2018; Yoon & Brown-Schmidt, 2019a, 2019b). In three-party conversations, Yoon and Brown-Schmidt (2018) hypothesized that speakers would design their speech for the least knowledgeable addressee – the "aim low" hypothesis, which their study confirmed. However, there was an unexpected result. They expected speakers would produce speech similar when speaking to Addressee 1 and Addressee 2 together and when speaking only to Addressee 2, where Addressee 2 was less knowledgeable than Addressee 1. However, speakers were more elaborate in their speech when addressing Addressee 2 alone. In a follow up studies, Yoon and Brown-Schmidt (2019a, 2019b) examined 4-, 5-, 6-, and 7-party conversations. With the "aim low" hypothesis, they expected speakers would design their speech for the address with the least amount of common knowledge with the speaker, even if the number of addressees increased. However, they found that when speakers had multiple addressees, the speaker designed their speech based on the combined knowledge of the group and physical context (e.g., what each addressee could see from their visual perspective). When the group of addresses shifted from majority with common ground to minority with common ground, speakers used longer and more descriptive expressions. Speakers did not design their speech for the least knowledgeable address but seemed to design their speech on the combined needs of the group. Therefore, they concluded that their "aim low" hypothesis was not supported by the data. In classroom setting, Yoon and Brown-Schmidt hypothesize that teachers would design their speech for the needs of "naïve" students, but they also question if such an approach is optimal with teachers' pedagogical goals or the students' communicative needs.

Common Ground in Mathematics Classrooms

There is a growing number of research studies investigating common ground in mathematics classrooms. Staples (2007) described several strategies (e.g., pursuing discrepancies) that an expert secondary mathematics teacher used to establish and monitor common ground during whole-class collaborative inquiry, which she organized into three themes: (a) creating a shared context, (b) maintaining continuity over time, and (c) coordinating the collective. Alibali, Nathan, and their colleagues focused their research on how teachers' gestures establish and maintain common ground during instruction (e.g., Alibali, Nathan, Boncoddo, & Pier, 2019; Alibali et al., 2013; Nathan, Alibali, & Church, 2017). For instance, to build common ground, Alibali et al. (2013) found teachers will increase their use of gestures in response to students displaying or expressing understanding inconsistent with what the teacher intended.

These studies on common ground in mathematics classrooms describe how experienced teachers build and maintain common ground in their practice. These studies, however, do not provide insight into how this practice develops. To understand how teachers' practice of establishing and maintaining common ground develops, it is also necessary to understand how this practice first emerges from their everyday practice (Russ et al., 2016). Therefore, I examined the practice of secondary mathematics student teachers because they had minimal teaching experience, and so I was likely to observe how they begin building common ground around mathematical connections during instructional explanations.

Methods

Because of the exploratory nature of the study, it was well suited to an instrumental case study design (Stake, 2003). An instrumental case study primarily foregrounds and explores an issue or phenomenon while the specific case is of secondary interest. The phenomenon of interest was how secondary mathematics student teachers built and maintained common ground in their second enactment of a lesson in comparison to their first enactment. The cases were three lessons taught twice by a pair of student teachers, Robin and Melissa (pseudonyms), in two advanced ninth-grade mathematics classes during February 2014. For the purposes of describing and illustrating this phenomenon, the focal case for this paper is a lesson led by Robin. The following describes the background of the study, the selection of the case lesson, data sources, and the data analysis process.

Background of the Study

Robin and Melissa were participants in a multi-year research project that followed a cohort of secondary mathematics teachers in their mathematics teacher education program. Prior to student teaching, the prospective teachers completed three semesters of coursework with each semester consisting of a mathematics content-focused course and a mathematics pedagogy-focused course that included a field experience component. Each semester the field experience advanced in complexity from working with one student in the first semester to small groups of students in the second semester and then to working with whole classes of students in the third semester. In the fourth and final semester, prospective teachers completed their student teaching experience. Student teachers were placed in pairs and were expected to co-teach with their mentor teacher and partner, similar to a model described by Leatham and Peterson (2010).

Robin and Melissa were partners during student teaching and co-taught two sections of an advanced ninth-grade mathematics course. They co-planned lessons for the course with another pair of student teachers from the same cohort placed with a different mentor teacher. The advanced ninth-grade mathematics course was the first course in a high school sequence designed to prepare students for advanced mathematics courses such as Advanced Placement Calculus AB/BC and Advanced Placement Statistics. The course covered topics typically found in an Algebra 1 course and half of the topics in a high school geometry course. According to the curriculum guide, the course sought to leverage algebra to deepen and extend students' understanding of geometry (Georgia Department of Education, 2013). Robin and Melissa were free to use or modify materials (e.g., mathematical tasks with teacher commentary, graphic organizers, and links to web resources) in the curriculum guide as they saw fit. They were also free to develop or use other materials available to them (e.g., mathematical tasks suggested to them by their mentor).

Selection of the Case Lesson

There were two reasons for the selection of the focal case lesson for this paper, which places the phenomenon (i.e., building and maintaining common ground) as the dominant issue in case selection. One reason for selecting the focal case lesson was Robin's core belief about the importance of effective communication to the teaching and learning of mathematics (Conner & Gomez, 2019). Teachers' beliefs are influential to their teaching practice (e.g., Bray, 2011; Cross, 2009; Diamond, 2018; Stipek et al., 2001; Thompson, 1984). Speaker's consideration of what is in the common ground with their participants is essential to effective communication

(Brown-Schmidt & Heller, 2018). I believed Robin's core belief of being effective communicator would likely provide me with opportunities to observe how she considered common ground during instructional explanations. Furthermore, I could observe how Robin made micro-adjustments to build common ground from the first enactment to the second enactment of the lesson and then reasonably assume such changes were compatible with her core mathematics teaching belief of being an effective communicator. I selected the focal case lesson because Robin was the lead teacher in facilitating the mathematical discussion in the first and second enactments of the lesson.

The second reason for the selection of this lesson was the goals of the case lesson. Robin and her partner intended for students to derive the partitioning formula (or sometimes referred to as the section formula outside the United States) through generalization. The partitioning formula results in a point, $\left(\frac{a}{a+b}(x_2 - x_1) + x_1, \frac{a}{a+b}(y_2 - y_1) + y_1\right)$, that partitions a line segment formed by points (x_1, y_1) and (x_2, y_2) into two sections with lengths that form a ratio a to b. Robin and her partner intended for students to make two mathematical connections from their generalizations (See Results). Generalizing is an essential mathematical activity, but it is difficult for many students (Lannin, 2005; Mason, 1996; Stacey, 1989). In the focal lesson, I considered Robin's and students' generalizing as a joint activity in a specific mathematical and social context. Several researchers have argued for conceiving generalizations as a form of social activity and discourse (e.g., Ellis, 2011; Jurow, 2004). Because of the potential difficulties students were likely to experience when generalizing, I reasoned that this lesson was likely to lead to Robin and students to explicitly seek evidence of understanding and require grounding their understandings in the common ground.

Data Collection

The primary data sources included video and audio-recorded lessons across one unit of instruction (approximately 7 and 9 days) for two advanced ninth grade mathematics classes, with Robin the focus teacher for one class and Melissa the other. One stationary camera was placed in the back of the classroom to capture the class during whole-class discussion and followed the focus teacher during small-group discussions. Two microphones were used to record classroom talk – one microphone was placed in the front of the classroom to capture whole-class audio, and the other was a lapel microphone worn by the focus teacher to capture her speech. For this paper, the second enactment of the focal lesson occurred one day after the first enactment.

Secondary data sources included lesson plans and materials from the unit. This included a unit-specific curriculum guide from the state department of education, lesson plans developed by the student teachers, worksheets, and some assessments.

Data Analysis

The analysis followed interpretive techniques for analyzing transcripts and video-recordings of the lessons. Generally, the analysis proceeded in five phases, but sometimes the analysis returned to previous phases as needed to confirm interpretations of the data. Next, I describe each of these phases.

Phase 1

I began by transcribing Robin's and students' speech captured by the video-recordings for each lesson. After transcribing, I iteratively watched the recording of each enactment of the lesson in 5- to 10-minute increments and identified content-related episodes during whole-class discussions. A content-related episode included activities such as discussing a solution to a mathematical task but not the day-to-day operation of school (e.g., checking attendance). Using the constant comparative method (Corbin & Strauss, 2015; Glaser & Strauss, 1967), I paired similar content-related episodes, one from the first enactment of the lesson and one from the second enactment. I created side-by-side transcripts of the two paired episodes that I enriched by describing teachers' gestures and noting representations teachers or students displayed.

Phase 2

In the second phase of analysis, I coded for evidence of understanding (see Table 4.1). I paid close attention to negative evidence of understanding or "trouble spots" similar to those identified by Alibali et al. (2013) and noted them in the side-by-side transcripts in. Trouble spots identified included: (a) students' questions seeking clarity of a teacher's or other student's meaning, (b) students' incorrect responses to a teacher's question, or (c) uncertainty or hesitancy in students' responses. These trouble spots were considered as possible indicators of a lack of common ground (i.e., shared understanding) among participants in the discussion.

Phase 3

For the next phase of analysis, I used the constant comparative method to compare the interactions in the paired content-related episodes. I wrote memos next to the side-by-side transcripts comparing the mathematical task, trouble spots evident in the episodes, content of students' and Robin's speech, and the nature of Robin's gestures in response to trouble spots. I also included in the memos events prior to the episodes that seemed to be influential for the episode (e.g., an issue during a small group interaction that became a topic of whole-class discussion). I then compared across the paired episodes to look for similarities and differences in the sequencing of the content-related episodes from the first and second enactments of the lesson. These comparisons supported my understanding of how previous interactions in episodes may have influenced future episodes within and between lessons.

Phase 4

In phase four, I returned to my memos detailing the comparisons in Robin and students' interactions between episodes and my understandings of the changes in Robin's practice that built and established common ground in the second enactment of the lesson. At this phase, I investigated the literature from clinical and field research that focused on common ground (H. H. Clark, 1996). The purpose in delaying the investigation of the literature until this phase was to remain close to the data and my interpretations. I compared the interpretations from these studies with mine in examining Robin's lessons. These comparisons sometimes sensitized me to other concepts that I had not considered and so I returned to the data in search of confirming and disconfirming evidence. I revised my analytic memos to document how my interpretations compared with those from clinical and field research.

Phase 5

In the final phase, I wrote interpretative case summaries of the focal lesson. I shared these summaries with two mathematics education researchers with considerable experience examining mathematics classroom discourse for peer examination to warrant credibility and dependability of the findings (Merriam & Grenier, 2019). Transcripts and video recordings were also provided with the case summaries. From our discussions of my interpretations, I clarified and refined my claims of Robin's practice. To appraise the transferability and credibility of the findings, transcripts for the instructional explanations are included in this paper (See Results).

Results

Several trouble spots (e.g., students expressing uncertainty) occurred during the lesson. In this paper, I focus on two trouble spots that arose in the first enactment of the lesson. I selected these trouble spots as they provided evidence that students had not yet engaged in the generalizing actions of *relating* or *extending* (Ellis, 2007), as Robin and her partner had intended, to be able to make mathematical connections. The first connection relied on the generalizing action of relating (i.e., forming an association between two or objects). It considered the relation between the ratio a to b and the fraction $\frac{a}{a+b}$ as similarly described in the "overlapping" model (M. R. Clark et al., 2003). This model posits that in the context of part-part or part-whole relationships, ratios and fractions have conceptual convergence in that students should be able to flexibly translate between representations. In context of the lesson presented here, Robin and her partner wanted students to translate between the point Z representing a partition of the line segment XY into a 1:3 ratio and XZ representing one-fourth of the line segment of XY (see Figure 4.1). In other words, they intended for students to translate from the part-part relationship described by the ratio to the part-whole relationship of the fraction: The sum of the parts in the ratio are equivalent to the unit of the fraction.

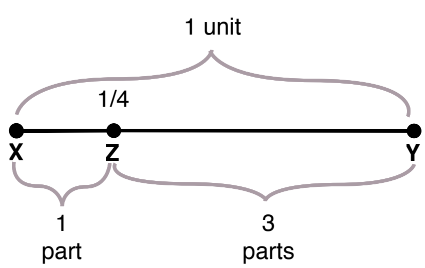

Figure 4.1. Overlapping Number Line Model Showing Part-Part and Part-Whole Relationships

Robin and her partner also intended for students to make a connection through the generalizing actions of extending (i.e., expanding a pattern or structure). They designed an activator task for students to explore partitioning a specific line segment on a number line. Then, Robin supported students in deriving a general formula to partition any line segment on a number line in a given ratio. Figure 4.2 is a recreation of the activator task. After deriving a formula for the number line, Robin asked students to further extend the structure in their general formula for the number line to partition any line segment in a coordinate plane, which Robin and Melissa referred to as the partitioning formula.

Find the distance between point X and Y.
Where would the point that is $1/4$ of the distance from X and Y (starting at X)? Mark this as Z.
What would the ratio of the $XZ:YZ$?

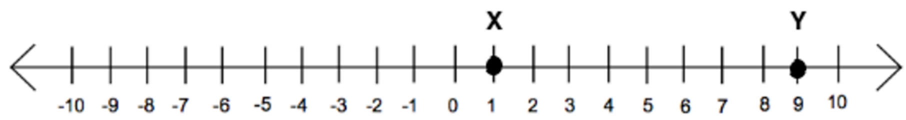

Figure 4.2. Activator Task for Partitioning a Number Line

Trouble Spot #1: The Relationship Between the Ratio $1:3$ and the Fraction $1/4$

While working with students to write a general formula for partitioning a line segment on the number line, Robin shifted students' focus to the ratio they found in the activator task. This shift appeared to be in hopes students would make a connection: the ratio 1 to 3 is related to the fraction one-fourth. Table 4.2 provides the transcript of the discussion. At the start of the instructional explanation, Robin suggested a connection between the ratio 1 to 3 and the fraction one-fourth (Lines 1.2-1.4). While students did not explicitly state any uncertainty at this point, there were several moments of silence (Lines 1.3, 1.12, and 1.15), suggesting uncertainty. Later episodes in the lesson provide further evidence that some students were uncertain about the reason for the connection. For instance, Robin's partner later worked independently with two students on understanding the relationship between the ratio 1 to 3 and the fraction one-fourth during the work period of the lesson. Also, when Robin went over a problem with a ratio of 2 to 3, some students expressed confusion as to why the ratio was related to the fraction two-fifths.

Table 4.2

Transcript of Trouble Spot #1

Line	Speech transcript	Gesture transcript
1.1	Robin: Really quick, remember our ratio 1 to 3? That's what we got to.	Underlines ratio *1:3* on the board
1.2	Robin: Can we find, or can we think about a relationship between the ratio 1 to 3	Writes *1:3* on the board
1.3	Robin: and the fraction one-fourth? (4-second pause)	Writes *1/4* on the board
1.4	Robin: Like is there another way we could write that where it could kind of relate to this ratio?	Open palm held underneath 1:3
1.5	Student 1: Can you... can you do one-third or you can do one-fourth or one dot dot four?	
1.6	Robin: Well I'm talking about - so these are two different things, right?	Open palm held underneath 1:3 and 1/4 moving back-and-forth
1.7	Robin: We have 1 to 3 and then like one four, one-fourth, right?	
1.8	Student 1: You can do percentages, can't you?	
1.9	Robin: All I'm asking is that I want this to still stay as one-fourth	Open palm held underneath 1/4
1.10	Robin: but I want to see if there's any ways we can write it	Open palm held underneath 1:3
1.11	Robin: to where we have like a 1 and a 3	Both hands are open palms moving back-and-forth
1.12	Robin: in this. (1-second pause)	Open palm held underneath 1/4
1.13	Robin: Like using plus, minus, multiplication?	Both hands are open palms moving back-and-forth
1.14	Robin: So, we have a 1 on top,	Writes *1/* on the board
1.15	Robin: what can I write on bottom using these numbers that can give you 4? (3 seconds pause)	Open palm held underneath 1:3 and then sliding across to underneath 1/4
1.16	Robin: Are you all confused by that?	
1.17	Multiple students: [crosstalk: Several students can be heard expressing confusion]	
1.18	Robin: Wait. Student 2 what'd you say?	
1.19	Student 2: 1 plus 3.	
1.20	Robin: Okay.	Writes *1+3* on the board
1.21	Robin: Is that correct?	
1.22	Student 1: So, you're asking what can give you a 4?	
1.23	Robin: Yes. Okay, we're not going to go into this quite yet, but do you all see the relationship between this?	Open palm shifting from 1/4 = 1/(1+3) to 1: 3
1.24	Robin: Okay. I know it's kind of a confusing question, but we're going to need to know this in a minute.	

Building Common Ground in Response to Trouble Spot #1

Similar to the first enactment, Robin began the second enactment of the instructional explanation by shifting students' attention to the ratio they found in the activator task before asking students to develop a general formula for partitioning a line segment on the number line in the second instructional explanation. Robin still intended students to make the same connection as in the first enactment: the ratio 1 to 3 is related to the fraction one-fourth. Table 4.3 provides the transcript of the second enactment.

To build common ground in the second enactment, Robin created a shared intellectual context (Staples, 2004, 2007). In comparing the two enactments of the instructional explanation, a small but seemingly significant difference occurred in how Robin began to build incremental common ground. In both the first and second enactments, Robin recalled a shared context: the activator task and the ratio students previously found (Lines 1.1 and 2.1-2.2). Robin's reference to the ratio found in the activator task assumed that students shared a common interpretation of the ratio as 1 to 3. In the second enactment, however, Robin followed up with a question that functioned as a strategic commonality assessment (Line 2.1-2.4). According to Horton (2005), speakers will strategically seek out evidence for commonality. Robin's question (Line 2.4) functioned as an assessment to determine if students shared a common understanding for the total number of equal partitions of the number line. Robin did not use a similar commonality assessment in the first enactment of the discussion (see Table 4.1). In fact, there was an absence of any mention of equal partitions in Robin's first enactment of the discussion. This could indicate that Robin overestimated the recognition of the equal partitions as part of the common ground with students.

Evidence to support Robin's overestimate of what was part of the common ground include Robin's adaptations of her initial question (Lines 1.4, 1.9-1.12, and 1.14-1.15) despite encountering explicit uncertainty or silence from students (Lines 1.5, 1.8, and 1.12). Notice that Robin increasingly added information, "…is there another way we could write that [one-fourth] where it could kind of relate to this ratio?" (Line 1.4), such as asking, "…what can I write [in the denominator] using these numbers [1 and 3] that can give you 4?" (Line 1.15). Her questions appear to be supporting a funneling interaction where students have to decipher Robin's intentions (Wood, 1998). In other words, Robin had to provide more information in her questions and make them more directive to get the response she intended from students. Even though Robin's thinking was evident in her questioning, students did not share a similar understanding as Robin, as they later expressed their confusion (Line 1.17).

In her second enactment of the discussion, Robin's commonality assessment elicited several different answers from students and revealed that students lacked common ground for the total number of equal partitions (Lines 2.4-2.6). Robin pursued the discrepancy in students' answers (Line 2.6) and asked a student to justify why she thought there were four equal partitions (Line 2.8). Robin's pursuit of the discrepancy was important to incrementally build common ground with students (Staples, 2007). She later leveraged this incremental common ground to support students in making a connection between the ratio 1 to 3 and the fraction one-fourth.

Robin directed students to the connection that the ratio 1 to 3 is related to the fraction one-fourth in the second enactment (Lines 2.14-2.29), which was similar to the first enactment of the lesson (Lines 1.2-1.20). However, when a student expressed uncertainty (Line 2.20 and 2.22), Robin fell back to what was previously established in building common ground (i.e., the four equal partitions). Robin pressed students to recall what they had previously determined: "You just said it, what [did] we do to get to the 4?" (Line 2.23). Notice that prior to forming her question, Robin uttered: "You just said it…" According to Horton (2005), speakers will strategically monitor "their output to ensure that what they have said will not lead to mistaken interpretations" (p. 28). If Robin did not follow up her utterance with her question, the student might have interpreted that Robin meant the previous responses of one-third or two-eighths. However, Robin followed up with her question: "…what [did] we do to get the 4?" (Line 2.23). Robin strategically monitored her utterance of "you just said it" by following up with a question that requested students to recall what they previously established when determining four total equal partitions.

After Robin's question, students returned to common ground by recognizing they summed the partitions (Line 2.24-2.28). Again, Robin expressed to students that they were "trying to find a relationship" between the ratio 1 to 3 and the fraction one-fourth (Line 2.29). A student, in response, gave a reason for the similarity between the ratio 1 to 3 and the fraction one-fourth in that they shared four equal partitions (Line 2.30). In comparison, there was no explicit reason given for the connection during the first enactment of the discussion nor elsewhere in the first lesson. Robin only stated a general procedure for mapping the ratio a to b to the fraction $a/(a+b)$ in the first enactment but, again, did not give any justification. Robin also supported the student's justification for the connection between the ratio 1 to 3 and the fraction one-fourth by building and sustaining common ground. She identified the four equal parts on the number line by drawing lines to partition the line segments into four equal parts (Lines 2.31-2.32). Robin's signaling the four equal parts focused some students' attention on the total number of equal partitions (Line 2.34). Such gestures and signaling were absent in the first enactment (see Table 4.2).

Robin's gestures and signaling are consistent with previous clinical research (Kelly et al., 2011) and field research (Alibali et al., 2019). First, Robin introduced two new representational gestures (i.e., gestures conveying semantic information closely related to the speech they accompany) in the second enactment (Lines 2.31-2.32) that did not appear in the first enactment.

According to Kelly et al. (2011), speakers will use more representational gestures with their speech if they know the information is highly relevant to their addressee. Robin clearly believed students' attention to the total number of parts was important as she made it an explicit question at the start of the discussion (Line 2.4). The first representational gesture appeared when Robin stated, "We could divide this [line segment] into like 4 parts." and then proceeded to use the open palm of her hand to show dividing the line segment into four equal parts (Line 2.31). Robin then proceeded to make her action more permanent by drawing vertical lines to show the subdivision of the line segment into four equal parts (Line 2.32), which was the second representational gesture.

Second, Robin's gestures and signaling supported a student's contribution and helped build and manage common ground among students. Alibali et al., (2019) found that teachers sometime manage common ground by repeating students' verbal contributions and then adding gestures to indicate specific referents in students' speech. Staples (2004, 2007) also found that a teacher's repetition of students' ideas built and managed common ground. Robin's repetition of the verbal utterance and accompanying gestures supported other students in deciphering the student's meaning because a student can be heard stating, "Oh, I didn't know what parts she was talking about" (Line 2.34). Robin's actions promoted a shared focus and understanding among students for the relationship between the ratio 1 to 3 and the fraction one-fourth.

Table 4.3

Transcript of Building Common Ground in the Second Enactment of the Lesson

Line	Speech transcript	Gesture transcript
2.1	Robin: Okay and before we keep going from that, let's look at, let's look at our ratio that we found.	Draws a circle around 1:3 on the board
2.2	Robin: We found a ratio of 1 to 3. If we have a ratio that's 1 to 3, 1 part	Open palm on the left hand
2.3	Robin: to 3 parts, okay.	Open palm on the right hand

2.4	Robin: Does that make sense? What, what's the total number of parts?	Both open palms held up facing each other
2.5	Multiple students: [crosstalk: Students can be heard giving answers of 3, 4, and 8.]	
2.6	Robin: Okay, what? So, I'm getting a lot of different answers.	
2.7	Student 3: 4.	
2.8	Robin: 4. Why 4?	
2.9	Student 3: Because	
2.10	Student 4: 1 plus 3	
2.11	Student 3: well, we started off like finding the distance, but then if you have 3 parts and you have like 1 part, then you add them together.	
2.12	Robin: Yeah okay, that's exactly right. So, um and then Student 4 you said…, what did you just say?	
2.13	Student 4: Oh, I just did 1 plus 3.	
2.14	Robin: Right, you're adding them. So how could I rewrite, so if I have the fraction one-fourth,	Writes *1/4* on the board
2.15	Robin: which is what we originally started with and I kind of want to write it in terms of the ratio that we found, how could I rewrite it to where it would make sense?	Writes = *1/* on the board
2.16	Student 5: 1 dot dot, wait no.	
2.17	Robin: In fraction form.	
2.18	Student 5: Oh um…	
2.19	Robin: Kind of incorporating this ratio.	Open palm held underneath 1:3
2.20	Student 5: One-third? I don't know.	
2.21	Robin: It has to, it needs to be equivalent to one fourth. You just -	
2.22	Student 5: Two-eights?	
2.23	Robin: Well, that is equivalent. You just said it, what do we do to get to the 4?	
2.24	Student 6: We added.	
2.25	Robin: You added.	
2.26	Student 5: Oh add.	
2.27	Robin: Right, so can we write this as like um… so can we write this as 1 over 1 plus 3.	Writes *1+3* on the board
2.28	Student 5: Yeah.	
2.29	Robin: Do you all see that? Okay. We're just trying to find a relationship kind of between these, okay.	
2.30	Student 5: Oh, I get it because, like one-fourth, there's like 4 parts to it.	
2.31	Robin: Yeah, exactly, exactly. So, we could like divide this into like 4 parts.	Open palm "chopping" the number line
2.32	Robin: Okay, there would be that ratio of 1 to 3, but it would be, so if I were to draw the parts like there, there, there, and there.	Draws vertical lines subdividing the line segment into four equal parts
2.33	Robin: There's 1, 2, 3, 4 parts. Do you all see that?	
2.34	Student 7: Oh, I didn't know what parts she was talking about.	

Trouble Spot #2: Interpreting and Generalizing from the Activator Task

A second trouble spot arose during a discussion when Robin asked students to make a generalization to derive a formula to partition a line segment in the coordinate plane from their initial work with a number line. The class previously created a general formula for partitioning a line segment on a number line: $Z = \frac{a}{a+b}(Y - X) + X$ where Z is the point that partitions the line segment in a given ratio *a*:*b*. Table 4.3 provides a transcript for the first discussion. Displayed on the board is the question "Can we generalize this process to create a formula that will work for partitioning any line segment?" and the expression $\frac{a}{a+b}(Y - X) + X$.

The instructional explanation started with Robin creating a coordinate plane and drawing a line segment on the board (Lines 1.41-1.44). After Robin labeled the endpoints of the line segment generally as (x_1, y_1) and (x_2, y_2), a student expressed uncertainty as to how the partitioning formula for the number line was applicable. The student expressed concern that, unlike for the number line, there were two *x*-coordinates and *y*-coordinates (Line 1.50). Robin's response suggests that she assessed the student interpreting the points *X* and *Y* in the activator task as *x*-coordinates and *y*-coordinates. She relabeled the *X* and *Y* points as X_1 and X_2 in the activator task (Line 1.52) and explained that the change in labeling the points was to reflect moving to a new context, the coordinate plane (Line 1.54).

Robin and students concluded that to generalize a formula for a point that partitions a line segment in the coordinate plane they would need to consider the x- and y-coordinates of the point. Robin suggested that students could "translate" the expression $\frac{a}{a+b}(Y - X) + X$ to the x-coordinate for the partitioning formula (Lines 1.74-1.75). Students, however, seemed uncertain how to modify the expression (Lines 1.76, 1.86, and 1.92). For example, after Robin modified the expression for the x-coordinate (Lines 1.84-1.85), students were uncertain of the reasoning for the modification (Lines 1.90-1.92). In response to this uncertainty, Robin returned to the activator task and made an analogy that the number line in the activator task was like the x-axis. The analogy supported some students in recognizing the meaning of Robin's previous relabeling of the points X and Y as X_1 and X_2. Some students, however, still did not understand Robin's meaning. For instance, later during the work period, Robin's partner worked with a student to understand the meaning of the x- and y-coordinates of the partitioning formula.

Table 4.4

Transcript of Trouble Spot #2

Line	Speech transcript	Gesture transcript
1.41	Robin: So, what do you all think about this in terms of - does anyone have any ideas of how we could make this into a formula that involves points that have two coordinates?	
1.42	Robin: So, for here	Displays the activator task on the board
1.43	Robin: you know we just had this [point] just represents one and two and three.	Circles 1 on the number line and then points at 2 and 3 on number line
1.44	Robin: But what if we have a coordinate plane like that? So, what if we have a coordinate plane	Draws coordinate plane on board
1.45	Robin: and it looks like this?	Draws line segment on the coordinate plane
1.46	Robin: And we have coordinates here and coordinates here, we'll say.	Labels ordered pairs on the endpoints of the drawn line segment as (x_1, y_1) and (x_2, y_2)
1.47	Robin: Does anyone have any ideas about how we could use this formula	Points to the partitioning formula for a line segment on a number line
1.48	Robin: for that?	Points towards the drawn line segment in the coordinate plane
1.49	Robin: Or how we could develop that formula?	
1.50	Student: If you have two different x's and two different y's how would do that?	
1.51	Robin: Well we're going (3-second pause) well that's what we're going to do. So, this was for one thing, okay?	
1.52	Robin: We could call this anything though. I mean this is Y but we could change this to X_1 and X_2 if you wanted.	Marks out X and Y and relabels as X_1 and X_2 in the activator task.
1.53	Student 1: So, you're saying the equation changes with every different problem? Like for different problems it changes? Or is it only going to change because you're going from a number line to a coordinate plane?	
1.54	Robin: We're changing it because we're going from a number line to a coordinate plane.	

Robin and students discuss that to generalize a partitioning formula for the coordinate plane that they will need to consider the x- and y-coordinates of the point that partitions the line segment. Robin writes a large empty coordinate pair on the board.

1.74	Robin: And so, if we're looking at our x's, how do you all think we could translate this	Runs left hand over $\frac{a}{a+b}(Y-X)+X$
1.75	Robin: into our x-coordiante?	Places hand under the x -coordinate position in a large empty coordinate pair on the board.
1.76	Student 1: Wait that same equation?	
1.77	Robin: This equation	Runs left hand over $\frac{a}{a+b}(Y-X)+X$
1.78	Robin: but we're trying to translate it into our x -coordinate because that's what we're going to look at right here.	Places hand under the x -coordinate position in the large empty coordinate pair on the board.
	Robin: You can think about the line.	Displays the activator task on the board
1.80	Robin: This is a straight line, right?	Moves hand left to right along the number line.
1.81	Student 2: I think that whole thing is our -	
1.82	Robin: So, this is like the x-axis, right?	Left palm face down held underneath the number line.
1.83	Student 1: Mm-hmm (affirmative)	
1.84	Robin: Okay, so what if we did change this to X_2 and we made this X_1?	Points at Y and then X in the activator task.
1.85	Robin: Let's do that.	Relabels $\frac{a}{a+b}(Y-X)+X$ as $\frac{a}{a+b}(X_2-X_1)+X_1$.
1.86	Student 1: So, it'd be $(X_2-X_1)+X_1$?	
1.87	Robin: And so, we added it to X_1, okay? Say it again.	
1.88	Student 1: $(X_2-X_1)+X_1$.	
1.89	Robin: Okay, well why?	
1.90	Student 1: That's what you have written.	
1.91	Robin: Does that make sense though?	
1.92	Student 1: No.	

Maintaining Common Ground in Response to Trouble Spot #2

To coordinate with students in generalizing a partitioning formula, Robin needed to maintain common ground with students over time (Staples, 2007). Robin managed continuity over time in the second enactment by modifying the activator task. Figure 4.3 is a recreation of the revised activator task given to students. Robin relabeled the points X and Y as X_1 and X_2, which she did in the first enactment (Line 1.85), but she did this prior to students engaging with the activator in the second enactment. In the first enactment, a student expressed a lack of continuity between the contexts of the partitioning formula for the number line in the activator task and the coordinate plane, which likely informed Robin's relabeling of the points (Line 1.50). Recall, Robin seemed to infer a student's question as interpreting the points *X* and *Y* in the activator task as *x*-coordinates and *y*-coordinates and then relabeled the points *X* and *Y* in response (Line 1.52).

Find the distance between point X_1 and X_2.
Where would the point that is $1/4$ of the distance from X_1 and X_2 (starting at X_1)? Mark this as Z.
What would the ratio of the X_1Z: X_2Z?

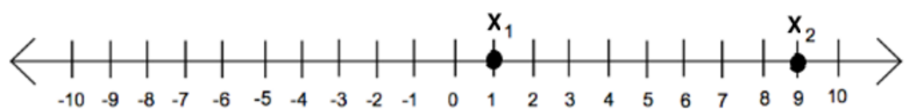

Figure 4.3. Revised Activator Task for Partitioning a Number Line

Robin's relabeling also relied on a common mathematical convention in school mathematics that the variable x represents one quantity on the horizontal axis and the variable y represents another quantity on the vertical axis. According to Clark (1996), conventions are useful solutions to coordination problems (e.g., generalizations) as they are regularities in behavior partly arbitrary (e.g., graphing independent variable on the horizontal axis) that are common in a given community (e.g., school mathematics or mathematicians). In this case, Robin attempted to coordinate with students in generalizing the partitioning formula by leveraging the graphing conventions in school mathematics. She was able to successfully coordinate with students because she inferred that students would draw on the conventional meaning for labeling the axes.

Robin's relabeling may at first seem trivial, but it provided continuity for students and supported students' generalization. Table 4.5 provides a transcript of the instructional explanation in the second enactment of the lesson. Notice that Robin begins similarly by extending the activator task into a coordinate plane (Lines 1.41-1.49 and 2.3-2.7). Subtle differences in the beginning of the instructional explanation include what is displayed on the board. In the first enactment, Robin drew a diagonal line segment in the plane with endpoints labeled (Lines 1.45-1.46). In the second enactment, Robin did not display a line segment or endpoints on the board but rather moved her hand diagonally across the coordinate plane as if indicating a diagonal line (Line 2.7). In the first enactment, Robin represented a general line segment that was static and students could repeatedly view in the first enactment of the discussion but, in the second enactment dynamically gestured and left no physical trace. Providing a permanent representation on the board provided a shared context and likely encouraged students to contribute a generalization as it was salient for all to see over a longer period of time. Robin's display on the board, however, revealed a difference in common ground caused by a lack of continuity. In the first enactment, students were initially unsure how the formula for partitioning a line segment on the number line could be generalized to the coordinate plane if given two coordinate pairs (Lines 1.50, 1.53, 1.76 and 1.92). In the second enactment, the revised task allowed students to readily discern the absence of the *y*-coordinate (Line 2.8) and hypothesize a potential expression for it (Lines 2.13-2.18 and 2.64-2.67), even though Robin did not provide a permanent representation of the general line segment or the coordinate pairs of the endpoints. It seemed to be Robin's adjustment to the task that sustained continuity over time and supported students to generalize a partitioning formula as Robin and Melissa intended.

Another difference between the first and second enactments of the discussion of the generalization of the partitioning formula is the timing of Robin's analogy between the number line context and the x-axis of the coordinate plane. In the first enactment, Robin introduced the analogy after students expressed not understanding the "translation" of the general formula for the number line to the x -coordinate for the partitioning formula (Line 1.92). In the second enactment, Robin made the analogy earlier in the discussion: "…what if we made this, this could be considered the x -axis, right?" (Line 2.2). Analogies can be useful communication strategies to ground participants' meaning (H. H. Clark & Wilkes-Gibbs, 1986; Fussell & Krauss, 1989; Iozzi & Barbieri, 2009). For example, Clark and Wilkes-Gibbs found that most participants completing a referential communication task began by using analogical descriptions for referential cards with later definite descriptions building on analogical descriptions. Furthermore, mathematics teachers' use of analogies has been documented in classroom instruction, typically following students' expressed lack of understanding (Richland et al., 2004). More specifically, teachers generally made their analogies with higher surface similarities between the source and target when students expressed lack of understanding.

Holistically, Robin took action to build common ground with students by introducing the analogy between the number line context and the x-axis of the coordinate plane earlier in the second enactment. In the first enactment, Robin introduced the analogy after students explicitly expressed evidence for a lack of common ground (Lines 1.82), which is similar to what is described in the literature (Richland et al., 2004). Knowing that other students were likely to experience a similar difficulty in the future, Robin introduced the analogy earlier in the second enactment. When there is a lack of common ground between participants in referential communication task, participants often use analogical descriptions as a means to start building common ground. Therefore, if Robin experienced a lack of common ground with students during the first enactment, then it is not surprising that Robin introduced the analogy earlier in the instructional explanation in order to maintain common ground with students.

Table 4.5

Transcript of Maintaining Common Ground in the Second Enactment of the Lesson

Line	Speech transcript	Gesture transcript
2.1	Robin: Okay so this works for this line, right.	Goes back to displaying activator task on board.
2.2	Robin: Alright, but what if we had, what if we made this, this could be considered the x-axis, right?	Pointing to number line with pen running from left to right.
2.3	Robin: Okay, so could we make this like a coordinate plane and make this the y and this the x?	Draws vertical line through 0 on the number line. Labels horizontal line as x and vertical line as y.
2.4	Student 1: Yes.	
2.5	Robin: Okay, so then what would we need to do with this?	Goes back to displaying $\frac{a}{a+b}(X_2 - X_1) + X_1)$ on the board
2.6	Robin: Like, what if we had – it works for this line,	Goes back to displaying activator task on board.
2.7	Robin: but what if we had a diagonal line [on] the axis? What do we need?	Moves hand across the drawn coordinate plane as-if indicating a diagonal line overlaying the plane
2.8	Multiple: Y, the y.	
2.9	Robin: We need our y, right.	
2.11	Student 1: Okay, so you're kind of doing it like (inaudible)	
2.11	Robin: Because this just gives us our x-coordinate, right?	Points at $\frac{a}{a+b}(X_2 - X_1) + X_1$ on the board
2.12	Robin: Say what?	
2.13	Student 1: So, you just like change the x coordinate?	
2.14	Student 2: $Y_2 - X_1$, I mean…$Y_2 - Y_1$,	
2.14	Student 1: Yeah.	
2.15	Student 1: $X_2 - Y$, I mean $X_2 - X_1$ and $Y_2 - Y_1$ and then (inaudible)	
2.16	Robin: So, you're saying we have our $X_2 - X_1$,	Points at $\frac{a}{a+b}(X_2 - X_1) + X_1$ on the board
2.17	Robin: you're saying we need a $Y_2 - Y_1$?	
2.18	Student 1: I mean it looks like that's where we're going.	
2.19	Robin: Okay, what do the rest of you guys thing?	

Another student stated she was unsure about the goal for the formula the class was trying to derive. Robin returned to the activator task to highlight how the generalized formula for the number line $Z = \frac{a}{a+b}(X_2 - X_1) + X_1$ *resulted in a point that divided the line segment into a given ratio,* $a{:}\,b$. *She explained that goal for the class was to develop a formula that would be able to partition any line segment in the coordinate plane into a given ratio.*

2.61	Robin: Okay, so if we have our first number. This number is going to represent what?	Points at $\frac{a}{a+b}(X_2 - X_1) + X_1$ on the board
2.62	Student 3: The X.	
2.63	Robin: The x-coordinate, okay. So, it's going to be our x-coordinate.	Labels $\frac{a}{a+b}(X_2 - X_1) + X_1$ as *x-coord*
2.64	Student 3: So then like for y would you put like A over A plus B and then like $(Y_2 - Y_1) + Y_1$.	
2.64	Robin:	Writes $\frac{a}{a+b}(Y_2 - Y_1) + Y_1$ and labels it as *y-coord*
2.65	Student 2: Is that, right?	
2.66	Robin: Student 1, what do you think about that? You like it?	
2.67	Student 1: I like it.	

Discussion

The purpose of this study was to investigate whether prospective teachers' attention to audience design may support their development in providing instructional explanations. Recall, Russ et al. (2016) argued that the everyday practice of audience design is a potential factor impacting teachers' development in providing instructional explanations. I also set about investigating if the classroom could be a site for prospective teachers to continue growing their practice of providing instructional explanations. Charalambous et al. (2011) found that prospective elementary teachers improved in providing instructional explanations in simulations after their methods course sequence, which implied that providing instructional explanations is learnable. However, Charalambous et al. were unsure if such learning transferred into the classroom. I discuss the results with respect to these two purposes.

Common Ground Essential for Making Mathematical Connections

Common ground is the basis from which all joint activities are made possible (H. H. Clark, 1996). One form of joint activity is building or making connections (Gee, 2005). As argued by Edwards and Mercer (1987), "a problematic aspect of education is that even well-intentioned joint action and discourse will not necessarily ensure that teachers and pupils establish a common understanding of both procedures and principles" (p. 162). The term principles, as used by Edwards and Mercer, is similar to conceptual understanding, or "an integrated and functional grasp of mathematical ideas" (Kilpatrick et al., 2001).

This study confirms the established claim that well-intentioned joint action and discourse does not necessarily ensure common understanding. In the lesson examined, Robin and her partner were well-intentioned in designing their instruction to support students in making mathematical connections through generalizing. In the analysis of the first two instructional explanations, Robin used several generalizing-promoting actions (Ellis, 2011). For example, Robin asked, "Can we find, or can we think about a relationship between the ratio 1 to 3 and the fraction one-fourth?" (Lines 1.2-1.3). This question was an instance of *encouraging relating* or prompting the formation of an association between two or more entities. However, students were not able to engage in the generalizing action of relating that Robin and her partner had intended due to a lack of common ground. There were multiple instances that revealed a lack of common ground between Robin and students, such as when students explicitly stated their confusion to Robin (Line 1.17).

Opportunely, the design of the study provided evidence that establishing and monitoring common ground are essential teaching practices that are likely to ensure joint activities of students and teachers. Robin's careful attention to building and monitoring the common ground in the second enactment of instructional explanations afforded students opportunities to generalize their mathematical activity. It became apparent that Robin built and monitored common ground through her commonality assessment, informative gestures, design of task, and timing of an analogy. These micro-adjustments in Robin's practice had significant impacts on students' abilities to make mathematical connections. Other studies have examined how teachers build and maintain common ground to support students' collaborative inquiry (Staples, 2007), students' contributions (Alibali et al., 2019), and students' attention to referents in other's speech (Alibali et al., 2013). This study contributes to the literature by documenting how a novice teacher's actions to build and maintain common ground supported students' generalizing in order to make mathematical connections.

Learning to Provide Instructional Explanations

Novices teachers often experience difficulty in providing high-quality instructional explanations (Borko & Livingston, 1989; Even et al., 1993; Leinhardt, 1989, 2005; Livingston & Borko, 1990). As expected, this study also confirmed that Robin initially experienced difficulty in providing instructional explanations that supported students in making mathematical connections. For instance, Robin over-estimated that the total equivalent partitions of the number line were in the common ground with students. With experience, however, Robin took actions to provide more effective instructional explanations in a future lesson the next day by building or maintaining common ground. In fact, Robin's actions seemed to be similar to those of expert teachers described in the literature (Alibali et al., 2013, 2019; Staples, 2007).

Finding that Robin initially struggled to consider her students' perspective when designing instructional explanations is also consistent with some findings in audience design research. For example, Horton and Keysar (1996) argued that a speaker's initial utterances are not always in consideration of common ground, but rather the speaker will monitor and adjust for the addressees' needs. In a follow up study, Horton and Gerrig (2002) found that the details of speakers' experiences need to be accounted for when making claims about the presence of audience design because "speakers may intend quite sincerely to tailor their productions for a specific audience, but lack the knowledge or resources to carry out these intentions fully" (p. 605). In other words, speakers needed to have an awareness of the need to attend to audience design under certain conditions and setting. Robin's initial experience in providing instructional explanations provided her with an awareness of the lack of common ground with her students. As a result, Robin coordinated resources (e.g., her gestures) and her understanding of students to provide more productive instructional explanations, under similar conditions, in the future.

The findings of this study also provide further credence to Charalambous et al.'s (2011) argument that research on instructional explanations needs to move beyond comparing expert and novice teacher performance and consider how teachers learn to provide productive instructional explanations. Charalambous et al. found that prospective teachers improved in providing instructional explanations by comparing their performance prior to and after their mathematics teacher education course sequence. This study examined prospective teachers learning to provide instructional explanations within their practice and on a smaller time-scale. The micro-analysis of Robin's lessons presented in this paper provides evidence that Robin made changes in providing instructional explanations from the first to the second enactments. Robin was learning to specialize the everyday practice of audience design for the purposes of teaching mathematics (Russ et al., 2016). Further micro-analysis of novice teachers' practices over time may provide insight into how teachers learn to provide instructional explanations similar to those of expert teachers.

Leveraging Multiple Lesson Enactments

In their review of research on teachers' informal learning, Kyndt et al. (2016) found that one kind of everyday professional development activity reported in the literature was teachers learning from their daily activities and practice. However, almost none of the studies in the review examined the learning outcomes related to specialized mathematics teaching practices. This study contributes to the field by carefully documenting Robin's learning outcomes through micro-analyzing her repetition of a lesson the next day. Many secondary mathematics teachers teach multiple sections of the same course in a given semester or year. This repetitive structure could be a supportive structure for teachers' everyday learning of specialized mathematics teaching practices. Some scholars have argued that the repetition of lessons is one form for improving teachers' practice (e.g., Klein, 2012). This study suggests that novice teachers can learn to develop and specialize their mathematical teaching practices from their everyday practices and learning opportunities arising from teaching a lesson more than once.

In her study of secondary mathematics teachers' everyday learning, Horn (2005) found numerous social and contextual resources supported teachers' learning. For example, teachers sometimes engaged in "teaching replays and teaching rehearsals" to recreate classroom interactions with others. According to Horn, these teaching replays "provide opportunities to analyze teaching, reflect on practice, and communicate collective standards for pedagogy" (2005, p. 228). This study was unable to account for these potential social and contextual resources that could have supported Robin's learning due to the limitations of the data. For instance, it is not known if Robin engaged in these teaching replays with her student teacher partner or mentor teacher. It is likely that Robin received feedback from her mentor teacher or partner regarding her enactments of the lesson. Analysis of this feedback and other social and contextual resources could inform us about Robin's developing practice and the conditions or resources that optimally supported her development.

Conclusion

While other studies have documented how experienced teachers establish and maintain common ground (e.g., Alibali et al., 2019; Staples, 2007), I provided an example of one student teacher's developing practice to build and maintain common round while providing instructional explanations. The data and analysis provided evidence that beginning teachers, given opportunities to work with students around a task, can start to coordinate their actions to build and maintain common ground in similar instructional explanations in the future. When the student teacher actively sought to build and maintain common ground during instructional explanations, students were often able to generalize their activity to make mathematical connections. Future micro-analysis of instructional explanations and how teachers learn to specialize their attention to audience design may provide further insights into teachers' developmental trajectory and the conditions optimal for that development.

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CHAPTER 5

CONCLUSION

I was motivated to conduct the studies within this dissertation by three challenges in supporting teachers to teach for connection-making as recommended by educational researchers and national standards and policy documents. First, teachers struggle to support students in making mathematical connections, especially novice teachers. Some teachers' lesson plans lack any attention to mathematical connections, or they do not leverage opportunities during instruction to make explicit mathematical connections among students' ideas. Second, few studies have identified instructional practices that actually foster students to make mathematical connections. Even the few instructional practices that we do know foster students' connection-making, are difficult for teacher to integrate into their daily instruction with students. Third, there is an absence in the research literature of information about how teachers develop their practice to support students to make mathematical connections. Additional research is needed to document how teachers learn to support students in making mathematical connections, the difficulties teachers face when developing this practice, and the ways teachers overcome such difficulties with the resources available in their context (Ghousseini, 2015). These three problems challenged and motivated me to design and complete the studies within the dissertation. The findings from the studies suggest several courses of action for addressing these challenges. In the chapter that follows, I first provide a summary of the findings for each of the three studies in this dissertation. Then, I interpret the findings as they relate to the purposes of the dissertation. At the end, I offer some implications for mathematics teacher education, future research, and teaching.

Summary of Findings

In this section, I summarize the findings across the three studies in the order in which they appear in their respective chapters (Chapter 2, Chapter 3, and Chapter 4). I leave the interpretations and implications of the findings for each study to later sections.

Findings of Study 1

In the first study (Chapter 2), my-coauthor (Lee) and I examined how a cohort of twelve prospective secondary mathematics teachers, in their first semester of a mathematics teacher education program, attended to and made sense of mathematical connections that arose when working with secondary students in small-group instruction. We examined prospective teachers' posts on an online discussion board prior to the field experience and the reflections they wrote after their field experiences. Specifically, we wanted to know (a) what kinds of mathematical connections the prospective teachers anticipated before and recalled encountering after their instruction and (b) what pedagogical considerations they made surrounding these instances while working with students. To analyze what kinds of mathematical connections the prospective teachers marked, we coded the texts they produced using the Mathematical Connections Framework (MCF) (Singletary, 2012). Whereas the MCF aided us to identify the kinds of connections prospective teachers marked, it did not capture prospective teachers' pedagogical considerations of the connections they marked. Therefore, we used thematic analysis (Braun & Clarke, 2006) to inductively analyze these pedagogical considerations from prospective teachers' markings of mathematical connections.

We found prospective teachers attended to all kinds of mathematical connections during their field experience: (a) connecting through comparison, (b) connecting through logical implication, (c) connecting methods, (d) connecting specifics to generalities, and (e) connecting to the real world. When marking connections, prospective teachers also recognized an assortment of pedagogical considerations, which led to the development of the Pedagogical Considerations of Mathematical Connections (PCMC) framework. The framework outlines five pedagogical considerations PSTs marked: (a) students' connection-making, (b) practices that assisted or may potentially assist students' connection-making, (c) their knowledge (or lack thereof) to facilitate students' connection-making, (d) curricular influences on connection-making, and (e) students' affective behaviors (e.g., motivation, feelings, beliefs, etc.) towards connection-making.

Findings of Study 2

In the second study (Chapter 3), I set out to examine how three secondary mathematics student teachers supported students in attending to and participating in making explicit mathematical connections. I examined the student teachers' instruction across one unit of instruction, in which they co-planned together with support from their mentor teachers. I completed a qualitative analysis of the whole-class discussions using existing frameworks: MCF (Singletary, 2012) and Teacher Moves for Supporting Student Reasoning (TMSSR) (Ellis et al., 2019). After coding for explicit connections and teacher moves, I completed an analysis of how novice teachers supported students to attend to and participate in making connections from their in-the-moment assessment of students' understanding, a feature of audience design.

The findings reveal that these novice teachers created opportunities for students to participate in making connections but also chose at times to contribute mathematical connections or reasoning to discussions themselves to build upon students' thinking. The majority of the mathematical connections evident during discussions were at an explicit level. All five kinds of mathematical connections were evident in the discussions. The novice teachers supported students' participation in making mathematical connections through their coordination of teacher moves. For instance, the teachers leveraged a strategic set of high-potential teacher moves (e.g., eliciting understanding, encouraging multiple solution strategies and encouraging reflection) that fostered students to contribute mathematical connections during discussions. This coordination of teacher moves was possible because of their in-the-moment assessment of students' expertise, a feature of audience design. In other words, teachers' actions were mediated by their attention to audience design, specifically their in-the-moment assessment of students' expertise.

Findings of Study 3

There were two aims for the third study (Chapter 4). First, I wanted to investigate whether prospective teachers' attention to audience design may support their development in providing instructional explanations. Second, I wondered if the classroom could be a site for prospective teachers to continue growing their practice of providing instructional explanations. To begin answering whether prospective teachers' attention to audience design may be a way to continue growing their instructional explanations and if the classroom could be a place for such development to take place, I completed a micro-analysis of two secondary mathematics student teachers' practice of providing instructional explanations. I focused my investigation in two ways. First, I focused on instructional explanations of mathematical connections because explicit attention to mathematical connections is generative of students' learning, promotes recall, and impacts students' beliefs about mathematics (Hiebert & Carpenter, 1992). Second, I focused the analysis on the student teachers' ability to build common ground (i.e., shared understanding) during instructional explanations because successful audience design depends on careful assessment of what is shared.

To illustrate the findings of the study, I described how one teacher made micro-adjustments to build and maintain common ground with students to support them in making mathematical connections during a focal lesson. I examined how the teacher implemented the lesson to one ninth-grade mathematics class one day and then how she implemented the lesson again with another ninth-grade mathematics class the next day. The teacher's careful attention to building and maintaining the common ground in her second enactment of the lesson afforded students opportunities to generalize their mathematical activity. It became apparent that she built and maintained common ground through her commonality assessment, informative gestures, design of task, and timing of an analogy. These micro-adjustments in her practice had significant impacts on students' abilities to make mathematical connections. In addition, the micro-analysis of the teacher's lessons provided evidence that she made changes in providing instructional explanations from the first to the second enactments of the lesson. I argue she was learning to specialize the everyday practice of audience design for the purposes of teaching mathematics (Russ et al., 2016).

Interpretation of Findings

In this section, I interpret the findings previously discussed in relation to the purposes of the dissertation. The following purposes guided the studies in this dissertation:

Purpose 1: Make sense of early teaching practice from the perspective of novice teachers
Purpose 2: Capture the early practices of novice teachers that support students' opportunities to make mathematical connections
Purpose 3: Understand how teachers begin to learn to foster students' connection-making

Interpretation of Findings Related to Purpose 1

One aim of the dissertation was to make sense of the early teaching practice of novice teachers from their perspective. Previous studies have characterized novice teachers as having limited ability to foster mathematical connections (Problem 1). For example, Star and Strickland (2008) found that, after watching a mathematics lesson, several first-semester prospective secondary mathematics teachers did not recall a moment in the lesson when a student made a mathematical connection between two algebraic expressions. They conjectured that the prospective teachers may have not attended to the connection due to limited content knowledge, lack of recent experience with the particular content, or just failing to notice. In the first study (Chapter 2), we found prospective teachers attended to and explicitly identified several mathematical connections in working with secondary students. We believe such contrast between the two findings may be, in some ways, attributed to different perspectives. After watching a video of a lesson several times, Star and Strickland identified several significant features, such as making mathematical connections, and sought to determine if PSTs would identify the same features. In other words, Star and Strickland took an expert perspective. Our approach was to highlight the perspective of PSTs in identifying and marking mathematical connections that arose when working with secondary students. In other words, the data generated was from PSTs' markings.

There are other influencing factors that potentially impacted the mathematical connections the prospective teachers attended to during their field experience. For instance, mathematical tasks have considerable influence on students' opportunities to make connections (Stein, Grover, & Henningsen, 1996). Some tasks focused on the execution of procedures (e.g., factoring quadratics) while other tasks were supportive of connecting procedures with concepts (e.g., deriving the law of cosines). Another potential factor included the implementation of tasks and students' interaction while engaging in the task (Hiebert et al., 2003). For instance, one prospective teacher designed a thoughtful sequence of questions to use with the tasks given to her by the classroom teacher to provide opportunities for students to make connections. Also, prospective teachers' mathematical knowledge for teaching likely contributed to the opportunities prospective teachers and secondary students had to make mathematical connections and whether prospective teachers marked the mathematical connections. Mathematical knowledge for teaching is instrumental in leading to opportunities for students to build mathematical connections during instruction (Hill & Charalambous, 2012). Teachers with robust mathematical knowledge for teaching are able to traverse the mathematical terrain, flexibly respond to students' mathematical thinking, and support students in making mathematical connections. However, in the case of this study, there was a rather interesting outcome. Some prospective teachers became aware of the gaps in their mathematical knowledge in preparation for or during their field experience. While this study did not measure prospective teachers' mathematical knowledge for teaching, the study did confirm that some prospective teachers were at least aware of how limitations to their knowledge hindered their ability to support students in making mathematical connections.

While it is important for prospective teachers to attend to significant mathematical connections in the discipline as identified by experts, it is also important to consider the implications of what prospective teachers attend to as well. When we studied the mathematical connections that prospective teachers attended to during their field experience, we found prospective teachers made several introspections of the pedagogical potentiality of the mathematical connections they marked. This finding led to the development of the PCMC framework, which outlines the pedagogical considerations the prospective teachers marked in relation to mathematical connections. While we can learn from studies identifying what novices can and cannot attend to in comparison to experts, we can also gain insight from studies, such as ours, that focus on what skills and competencies novices bring.

Interpretation of Findings Related to Purpose 2

The second purpose of this dissertation was to document the early practices of novice teachers that supported students' opportunities to make mathematical connections. The first study (Chapter 2) somewhat aligned to this purpose. As stated in the previous section, the early practice of teacher noticing of mathematical connections was evident in prospective secondary teachers' marking from their field experience. One of the pedagogical considerations outlined in the PCMC framework was the prospective teachers' consideration of *suggested practices:* practices that assisted or could assist student to make mathematical connections. Across both online discussion boards and written reflections, the prospective teachers frequently suggested practices in their markings. Considering pedagogical moves teachers could use to assist students' connection-making is productive for teachers to develop and refine. For instance, in the broader context of the noticing literature, Kersting, Givvin, Sotelo, and Stigler (2010) found positive correlations between students' learning gains and teachers' scores when offering suggestions to improve instruction in the videos they watched. They inferred that offering a suggestion is an indicator of developing expertise because "expert knowledge builds through deliberate practice, a process which assumes a significant effort and reflection over an extended period of time" (p. 117). Mason (2002) similarly argued, "To notice an opportunity to act requires three things: being present and sensitive in the moment, having a reason to act, and having a different act come to mind" (p. 1). While these early marking by prospective teachers seem to be supportive for their practice, I did not observe how their noticing of students' mathematical thinking and mathematical connections led to opportunities for students to make mathematical connections. The second (Chapter 3) and third (Chapter 4) studies provided a more encompassing view of this purpose, which I discuss next.

In the second study (Chapter 3), I found the prospective teachers' actions were mediated by their attention to audience design, specifically their in-the-moment assessment of students' expertise. For example, in episode 1, a prospective teacher followed up her eliciting moves with facilitating moves of cueing to support students in understanding the reason for the relation between the slopes of linear equations and how the lines of the linear equations intersect. She did not complete an IRE pattern common in classroom discourse (Mehan, 1979). Instead, her awareness of students' lack of attention to the intersection during small group exploration mediated her actions to support students in identifying and interpreting the intersection of the two linear equations during the whole-class discussion. Attention to audience design is considered important to the professional work of teaching (Russ et al., 2016). This attention to audience design by the novices in that study suggests that novice teachers can leverage productive aspects of audience design into their early teaching practice. Furthermore, this attention to particular a feature of audience design, students' expertise, suggests that it is a mediating factor for teacher moves that supported students' opportunities to make mathematical connections. The second purpose (i.e., documenting the early practices of novice teachers that supported students' opportunities to make mathematical connections) closely aligned with the second study (Chapter 3).

In addition, the third study (Chapter 4) somewhat addressed the second purpose. In that study, I found that a teacher took actions to build and maintain common ground with students to support them in making mathematical connections. She built and maintained common ground through her commonality assessment, informative gestures, design of task, and timing of an analogy. These actions provided opportunities for students to make mathematical connections and share their mathematical reasoning. This study confirms the established claim that well-intentioned joint action and discourse does not necessarily ensure common understanding (Edwards & Mercer, 1987). In the lesson examined, the pair of student teachers were well-intentioned in designing their instruction to support students in making mathematical connections through generalizing. In the analysis of the first instructional explanations, the prospective teacher used several generalizing-promoting actions (Ellis, 2011). However, students were not able to engage in the generalizing actions that the pair of student teachers had intended due to a lack of common ground. Common ground is the basis from which all joint activities are made possible (Clark, 1996). This study contributes to the literature by documenting how a novice teacher's actions built and maintained common ground to support students' generalizing in order to make mathematical connections.

In summary, the findings across all three studies captured, in some ways, the early practices of novice teachers that support students' opportunities to make mathematical connections. In particular, the everyday practices of noticing and audience design appeared to be leveraged in prospective teachers' early teaching practice. In the first study (Chapter 2), prospective teachers marked mathematical connections when working with students and considered practices that either did support or could have supported students in making mathematical connections. These considerations provide documentation of some of the teaching practices novice teachers considered to be productive in supporting students to make mathematical connections. The second study (Chapter 3) suggested that prospective teachers' attention to audience design was influential for providing opportunities to make mathematical connections. Prospective teachers' attention to audience design appeared to coordinate prospective teachers' specialized teacher moves for supporting students' mathematical reasoning: eliciting, responding, facilitating, and extending. The coordination of these teacher moves provided different opportunities for students to participate in making mathematical connections. In the third study (Chapter 4), I found that a prospective teacher leveraged everyday practices to build and maintain common ground with students in order to support them in making mathematical connections. While other studies have documented how experienced teachers establish and maintain common ground (e.g., Alibali et al., 2019; Staples, 2007), I provided examples of one student teacher's developing practice to build and maintain common ground while providing instructional explanations. Next, I address how the dissertation addressed the third purpose: understand how teachers begin learning to foster students' connection-making.

Interpretations of Findings Related to Purpose 3

The third purpose of this dissertation was to understand how teachers begin to learn to foster students' connection-making. In the third study (Chapter 4), I set out to examine whether the classroom could be a site for prospective teachers to continue growing their practice in providing instructional explanations. Novices teachers often experience difficulty in providing high-quality instructional explanations (Borko & Livingston, 1989; Even et al., 1993; Leinhardt, 1989, 2005; Livingston & Borko, 1990). As expected, this study also confirmed that a prospective teacher initially experienced difficulty in providing instructional explanations that supported students in making mathematical connections. For instance, she over-estimated that the total equivalent partitions of the number line were in the common ground with students. With experience, however, the prospective teacher took actions to provide more effective instructional explanations in future lessons by building or monitoring common ground. In fact, her actions seemed to be similar to those of expert teachers described in the literature (Alibali et al., 2013, 2019; Staples, 2007).

Finding that the prospective teacher initially struggled to consider her students' perspective when designing instructional explanations is also consistent with some findings in audience design research. For example, Horton and Keysar (1996) argued that a speaker's initial utterances are not always in consideration of common ground, but rather the speaker will monitor and adjust for the addressees' needs. In a follow up study, Horton and Gerrig (2002) found that the details of speakers' experiences need to be accounted for when making claims about the presence of audience design because "speakers may intend quite sincerely to tailor their productions for a specific audience, but lack the knowledge or resources to carry out these intentions fully" (p. 605). In other words, speakers needed to have an awareness of the need to attend to audience design under certain conditions and setting. The prospective teacher's initial experience in providing instructional explanations provided her with an awareness of the lack of common ground with her students. As a result, she coordinated resources (e.g., her gestures) and her understanding of students to provide more productive instructional explanations, under similar conditions, in her second enactment.

The findings of this study also provide further credence to Charalambous et al.'s (2011) argument that research on instructional explanations needs to move beyond comparing expert and novice teacher performance and consider how teachers learn to provide productive instructional explanations. Charalambous et al. found that prospective teachers improved in providing instructional explanations by comparing their performance prior to and after their mathematics teacher education course sequence. This study examined prospective teachers learning to provide instructional explanations of mathematical connections within their practice and on a smaller time-scale. The micro-analysis of Robin's lessons presented in this paper provides evidence that Robin made changes in providing instructional explanations from the first to the second enactments. Robin was learning to specialize the everyday practice of audience design for the purposes of teaching mathematics (Russ et al., 2016), specifically supporting students in making mathematical connections.

Implications

Thus far in this chapter, I have summarized and interpreted the findings across the three studies. Taken together, these findings suggest several implications. In this section, I discuss the implications for mathematics teacher education, future research, and teaching.

For Mathematics Teacher Education

The results across the studies raise implications for mathematics teacher educators to consider how novice teachers' everyday practice of audience design could be boosted or redirected for the specialized practice of teaching, such as fostering mathematical connections during discussions. Previous studies have examined how to boost or redirect teachers' noticing of students' mathematical thinking (e.g., Goldsmith & Seago, 2011; Krupa et al., 2017; Star & Strickland, 2008; Walkoe, 2015). However, in my search of the literature, I have found no studies that directly addressed how to boost or redirect teachers' attention to audience design. The approach taken to examine the practice of the prospective teachers in the second study (Chapter 4) may provide direction for mathematics teacher educators in identifying features of audience design that could become specialized for teaching. In particular, the TMSSR framework can help researchers discern patterns in how teachers engage with their students in the moment (Ellis et al., 2019). Teachers and mathematics teacher educators could reflect on these patterns and consider how alternative moves or sequence of moves may impact a teacher's ability to attend to audience design. Furthermore, teachers and mathematics teacher educators may consider pairing the TMSSR framework with other content related frameworks, similar to MCF, to consider how teachers' attention to audience design corresponds to students' engagement in important disciplinary practices in mathematics.

Regarding frameworks, the first study (Chapter 2) contributed the PCMC framework for mathematics teacher educators. While there are exemplars in the literature detailing expert teachers' instruction of mathematical connections and their pedagogical decisions (e.g., Ball, 1993; Boaler & Humphreys, 2005; Lampert, 2001), the field can benefit from insights on how to prepare novice teachers for such intricate work. There is a need to support mathematics teacher educators' awareness of the sensitivities prospective teachers need to design and facilitate discussions that explicitly attend to mathematical connections. Mason (1998) described this awareness as an *awareness-in-counsel*. The PCMC framework contributes to the field of mathematics teacher education as a way to structure mathematics teacher educators' awareness-in-counsel so they may guide and develop opportunities for prospective teachers to design and facilitate discussions of mathematical connections. The PCMC framework can serve as a starting point for mathematics teacher educators to draw upon in supporting novice teachers to design instruction that explicitly attends to mathematical connections.

For Future Research

Attention to audience design is considered important to the professional work of teaching (Russ et al., 2016). Future research on other features of audience design and its implications for other specialized teaching practices is warranted because explicit attention to audience design by mathematics teacher educators may be constructive to incrementally build teachers' professional practice. There has been little attention to how features of audience design may be mediating *what* teachers say or do and *how* teachers design their actions accordingly. A notable exception to this is the argument by Teuscher et al. (2016) that Piaget's (1955) construct of decentering explains teachers' actions relative to students' thinking. They claimed that teachers' ability to decenter during student-teacher interactions has implications for whether teachers pose questions, the kinds of questions teachers ask, the quality of teachers' explanations, and the opportunities afforded to students to contribute to mathematical discussions. However, I argue that analyzing student-teacher interaction only through the construct of decentering is limiting. Decentering mediates the planning of teachers' actions. Audience design is more encompassing; it mediates the planning, production, monitoring, and repair of teachers' speech and actions. I suggest that future research try to carefully consider what features of audience design are mediating the planning, production, monitoring, and repair of teachers' speech and actions. Differentiating these outcomes may prove more powerful for interpreting results as it has been for the broader research literature on audience design (c.f., Brown-Schmidt & Heller, 2018; Fischer, 2016; Holler & Bavelas, 2017).

There are also implications for research that examines how teachers learn in and from practice. In their review of research on teachers' informal learning, Kyndt et al. (2016) found that one kind of everyday professional development activity reported in the literature was teachers learning from their daily activities and practice. However, almost none of the studies in the review examined the learning outcomes related to specialized mathematics teaching practices. The third study (Chapter 4) contributes to the field by carefully documenting Robin's learning outcomes through micro-analyzing her repetition of a lesson the next day. Many secondary mathematics teachers teach multiple sections of the same course in a given semester or year. This repetitive structure could be a supportive of teachers' everyday learning of specialized mathematics teaching practices. Some scholars have argued that the repetition of lessons is one form for improving teachers' practice (e.g., Klein, 2012). This study suggests that novice teachers can learn to develop and specialize their mathematical teaching practices from their everyday practices and learning opportunities arising from teaching a lesson more than once. More research is needed to examine how mathematics teacher learn in and from their practice at an in-depth level and within a smaller timeframe.

Furthermore, there are implications for research that examines novice teachers' practice and the claims that researchers make about their practice. For example, a prospective teacher's first enactment of a lesson that she planned did not lead students to make the mathematical connections she and her partner had intended. However, she made micro-adjustments to her practice that had a significant impact on students' connection-making. If I had only one opportunity to observe the prospective teacher's enactment of the lesson, then I may have characterized her practice similar to what is found in the literature (e.g., Even et al., 1993; Leinhardt, 1989; Livingston & Borko, 1990). Fortuitously, the opportunity to observe the prospective teacher's enactment of the lesson for a second time provided me with further understandings into her developing practice. Researchers examining novice teachers' practice should seek out ways to observe novice teachers teach a lesson more than once.

For Teaching

Additionally, this dissertation also raises implications for teaching, especially for supporting students to make mathematical connections. Mathematics teachers need to continually build and monitor common ground with students if they want students to generalize their mathematical activity as a collective. Well-intentioned planned activities and discourse will not guarantee students will develop a mutual understanding of mathematical connections that is rooted in mathematical reasoning. Teachers can monitor the common ground by considering what kind of evidence of understanding they ask of students. According to Clark (1996), higher levels of evidence will provide teachers more insight into what is in the common ground with students. If teachers find that there is a lack of common ground, then they may reflect on how they could have supported students in building common ground. For instance, they may consider the features of the task or lack of informative gestures led to a lack of common ground with students. A teacher may also consider asking students what parts of discussions are confusing to them. Teachers can only regain common ground with students if they understand what is and is not mutually understood with students. This recommendation is consistent with literature that suggests that developing teachers' understanding of students' thinking can be productive in leading to changes in mathematics teachers' practice and also gains in their students' achievement (Fennema et al., 1996).

Final Thoughts

In closing, I provide some final thoughts on how this dissertation sought to position novice teachers. I believe that all teachers want to be good teachers for their students. I believe this to be especially true for eager novice teachers joining the profession. However, the research and policy on novice teachers tend to focus on what they seem to lack and overlook the many beliefs, practices, and ways of knowing that novice teachers bring into the profession. In this dissertation, I sought to bring out how novice teachers' everyday practices are an act of sense making in teaching. I was inspired to look at the everyday practices of noticing and audience design to make sense of how novice teachers supported students to make mathematical connections. I will admit that making sense of novice teachers' practice is difficult work. There were many days that I personally struggled because I felt I did not fully understand how the novice teachers' practice made sense. I had to re-examine my thoughts and assumptions about teaching mathematics and how teachers learn. However, I believe trying to understand their practice is essential to supporting teachers. I am grateful to the novice teachers in this dissertation who were willing to share their practice and sense making. They challenged me, and I hope they challenge others, to see teaching as sense making.

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APPENDIX

Reflection Prompts

Focused Prompts	What did you learn about your student(s) at the beginning? • What did he/she know at the beginning? • What was he/she struggling with at the beginning? • How do you know? What did you do/say to promote learning? • How effective was it? • How do you know? What did the student(s) know at the end of the session? • How do you know? If you had it to do over, what would you do differently (in planning/preparing or in working with your student)? Describe an interesting interaction you had with the student around a mathematical idea. • What made this interaction interesting? • What did this interaction tell you about the student's mathematical thinking, motivation, or learning preferences? What questions do you have now? • About learning? • About the mathematical topic? • About teaching?
Weekly Prompts	Week 1 From the Jacobs &Ambrose (2008), Jacobs et al. (2014), Chapin (2003) readings, which tip did you try out with your student(s)? Describe how you used it and reflect on the experience implementing it. Week 3 Some students prefer using area models and some students don't. This week reflect on the effectiveness of the area model. Describe how your student used (or did not use) the area model and how it was helpful or hindered your students' mathematical thinking. Week 4 How did you sequence your lesson? Which examples/problems did you start with? Which examples/problems did you use towards the end? Why did you make such instructional decisions? Week 5 Reflect on your anticipation activity of the 5 practices. What did you anticipate your student to do/think? Was anticipating helpful in your teaching? Why or why not? Week 6 What was the mathematical learning goal you set for this lesson? Assess whether you think your student achieved this goal or not. What makes you think so? Week 8 Based on your observations with this student, what problem or problems might you pose next? Why? Do you think your student would be motivated in solving this problem? Why or why not?