Proposing A Test For 2-D White Noise

by

Zixuan Zhao

(Under the Direction of Lynne Seymour)

Abstract

Pseudo-random number are different from the true random numbers. Testing on whether pseudo-random number generator (PRNG) is accurate is necessary because of the increasing need in the advantages of PRNG than the true ones. In this thesis, we use a new way that converts the one dimensional numbers in to two dimensional lattice, and testing whether the lattice we generate follows the white noise or not. The method we used is called the maximum pseudo-likelihood estimator (MPLE), which is from Ising model. Finally we look at the results to see where the boundaries of our method lies.

Index words: Pseudo Random Number Generator, Random Numbers, White Noise, Maximum Pseudo-likelihood Estimator, Ising Model

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Dedication

I dedicate my thesis work to my family and many friends, who have unwavering support and belief in me. Without the mental and financial support from my parents, I wouldn't have finished my graduate degree. Without the encouragement of my friends, things would became a lot harder.

Acknowledgments

First and foremost I am extremely grateful to my major advisor, Dr. Lynne Seymour for her invaluable idea, continuous support, and patience advice for me on this thesis. Her immense knowledge and encouragement have invigorate me throughout the paper.

Then I would like to express my sincere gratitude to my committee members Dr. Jaxk Reeves and Dr. Ray Bai for their insightful comments and suggestions, they also ignited my passion for statistics.

Last but not the least, I'd like to acknowledge my undergraduate professor Dr. Darci L. Kracht laiding the foundation for my mathematics path, in which I would keep moving on.

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Chapter 1

Introduction

The difference between true random number generators (TRNGs) and pseudo-random number generators (PRNGs) on generating numbers is that TRNGs use physical means from nature, and PRNGs use mathematical algorithms from computer (Haahr, 2019). Much research has been done on RNG's to find modern algorithm number sequences that generates truly random sequences. But there is still space for improvement on today's research since the results of PRNG's are still pseudo. PRNG's results can be measured, standardized, and controlled. TRNGs can not be detected since it's truly unpredictable.

A potential advantage of a PRNG is that one can reproduce the same sequence of numbers in another time by simply knowing the starting point or seed of the sequence (Lau Yu, 2018). PRNG is commonly used for its low cost, periodicity, and efficiency, which TRNGs can't match(Pandya, 2019).

To improve the PRNGs and make the number generated as close to true random numbers as possible, an efficient testing of whether they behave as white noise is necessary. There are ways to detect the "pseudo" randomness. Kenny (2005) conducted a research to generate a more advanced view of statistical tests for true random number generators. Her report was based on the NIST test suite for random numbers, which was an effect way to test test randomness during that period. An earlier stage, Professor Marsaglia (1995) developed diehard tests in seceral years in order to measure random number generators' quality. Diehard tests include a battery of statistical tests, such as the birthday spacing test, the overlapping permutation test, and many other tests which return a p-value (Florida State, 1995).
However, some early approches may have various defects. Take a very early research, conducted by Neumann (1946), as an example. The middle-square

method was applied in the study, which resulted in self-repetition for all sequences eventually. Furthermore, some conclusions may be not appropriate. For instance, Beth and Dai (1980) concluded that Kolmogorov complexity and Linear complexity are the same, but this conclusion was proved to be incorrect by Wang (1999).

What we are proposing here is a new method to test for white noise on two dimensional pseudo random numbers. We use Dr. seymour's (2001) method by first rearranging the one dimensional pseudo random numbers in a 2-D lattice, then apply maximum pseudo-likelihood estimator (MPLE) on different type of pseudo random numbers generated to see under what condition could the method be applied, and what are the results.

Chapter 2

Modeling Background

2.1 Ising model and MPLE

The Ising model was invented by the physicist Wilhelm Lenz (1920), the one-dimensional Ising model was solved by his student Ernst Ising (1925) , and two-dimensional square-lattice Ising model was analytically described by Lars Onsager (1944). It is a model of ferromagnetism in statistical mechanics, and was originally developed to describe the magnetic dipole moments of atom spins in a lattice (+1 for up or 1 for down). We use discrete variables to represent these two states, where the local structure repeats periodically in all directions, resulting each spin interact with neighbors.

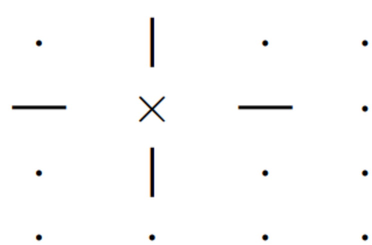

Figure 2.1: 2D Ising Model

Figure 2.1 shows a lattice in $4 * 4$ subset of Z^2, the lattice in two dimension. Consider the cross as site i in an $n * n$ lattice Λ_n, $i \in \Lambda_n$. Let j be one of the four nearest neighbours, N_i be the set of these four nearest neighbours of i, $j \in N_i$ and $N_i \subset \Lambda_n$. The spin of the atom, which is the random number taking the values -1 or $+1$ at site i is denoted by X_i. The collection of all X_i, $i \in \Lambda_n$ is denoted by X_{Λ_n}. $\beta$is a coefficient governing how strongly sites and interact. For a lattice Λ with boundary δ_Λ, the Ising model on Λ is:

$$P(X_\Lambda = x_\Lambda \mid x_{\Lambda_n}) = \frac{e^{\beta(\frac{1}{2}\sum_{(i,j)\in\Lambda}\sum_{j\in N_i} x_i x_j + \sum_{i\in\Lambda}\sum_{j\in\delta_\Lambda\cap N_i} x_i x_j)}}{Z_i}$$

Where δ_Λ is the set of lattice sites which neighbor sites in Lambda but are not contained in Lambda. Here Z_i is the normalizing constant, which sums over all configurations of Λ, and which is computationally intractable. In practice, the site i can never be on the boundary of the lattice because it must have a complete set of neighbors.
Because this distribution cannot be computed, we turn to using the single-site conditional distributions. These single-site conditional probabilities are called the local characteristics of the Markov random field (MRF):

$$P(X_i = x_i \mid x_j, j \in N_i) = \frac{e^{\beta x_i(\sum_{j\in N_i} x_j)}}{e^{\beta(\sum_{j\in N_i} x_{j[x_i=1]})} + e^{-\beta(\sum_{j\in N_i} x_{j[x_i=-1]})}}$$

The pseudo-likelihood first proposed by Besag (1975) multiplies the conditional distributions at grid sites given neighboring sites:

$$PL(\beta; X_{\Lambda_n}) = \prod_{i\in\Lambda_n} P(X_i = x_i \mid x_j, j \in N_i; \beta)$$

His idea was to treat all sites as independent, and multiple all of the local characteristics together. Use optimization to maximize in β and get the maximum pseudo-likelihood estimate (MPLE) . It gives an exponentially consistent parameter estimate called a maximum pseudo-likelihood estimate (MPLE). The we use the method in Dr. Seymour's paper (2001) to get the MPLE of the lattice, we extract each $10 * 10$ matrix side by side inside the lattice, find their pseudo-likelihood estimate, then optimal to find the maximum pseudo-likelihood estimate. ().

2.2 Hypothesis

The null and alternative hypotheses are two mutually exclusive statements about a population. A hypothesis test uses sample data to determine whether to reject the null hypothesis. The null hypothesis (H_0) is often an initial claim that is based on previous analyses or specialized knowledge. The alternative hypothesis (H_a) is what you might believe to be true or hope to prove true. (cite)
We propose the main null hypothesis to be the Ising model parameter β is 0,

meaning there are no neighboring interactions. Pseudo-likelihood estimator is likelihood estimator, which we have white noise: $H_0 : \beta = 0$. The alternative hypothesis is the Ising parameter is not 0, which is not white noise: $H_a : \beta \neq 0$ (Ising, 1925). The purpose is to test if the null hypothesis is true under different lattices. There are two tests to test if the lattice follows white noise. We apply the Anderson-Darling (AD) test for the composite hypothesis of normality and Box test for serial auto-correlation on $\hat{\beta}$ to see if the lattice is really white noise (Anderson, 2011). The null hypothesis of AD test is the data are normally distributed under mean 0 (mean for white noise). When $p \geq 0.05$ we fail to reject normality. The null hypothesis of Box test is the data have 0 serial autocorrelation. Under this test, when $p \geq 0.05$ we fail to reject that there is no significant serial autocorrelation (Box, 1970). The BOx test is also used for the auto serial correlation (ACF) plots in the later chaperter. If both tests fail to reject, we can conclude in high confidence that the MPLEs of the lattice is normal with mean 0 and have zero serial autocorrelation, which is white noise (cite).

Chapter 3

Simulation and Exploration

As discussed in the previous section, our null hypothesis is where Ising parameter $\beta = 0$, which indicates no neighboring interaction within the lattice, giving white noise. The Alternative hypothesis is that there exists a Ising parameter which is not 0. If the sample MPLEs $(\hat{\beta})$ are not significantly different from 0 and are uncorrelated, then the lattice is white noise. To test whether the MPLEs $(\hat{\beta})$ generated from sub-blocks of a given lattice meet normality, we simulate different lattices to check for Null true and Null false cases. The series of $\hat{\beta}$ generated from a lattice are the MPLEs of $10 * 10$ non-overlapping sub-lattices inside the whole lattice, for example, a $1000 * 1000$ lattice would have $(\frac{1000}{10})^2 = 10000$ of these sub-lattices and corresponding values of $\hat{\beta}$.

3.1 Null hypothesis true case

First, we generate the Null hypothesis (H_0) true case, where $p = 0.5$ for each -1 and 1 points in the lattice. A $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

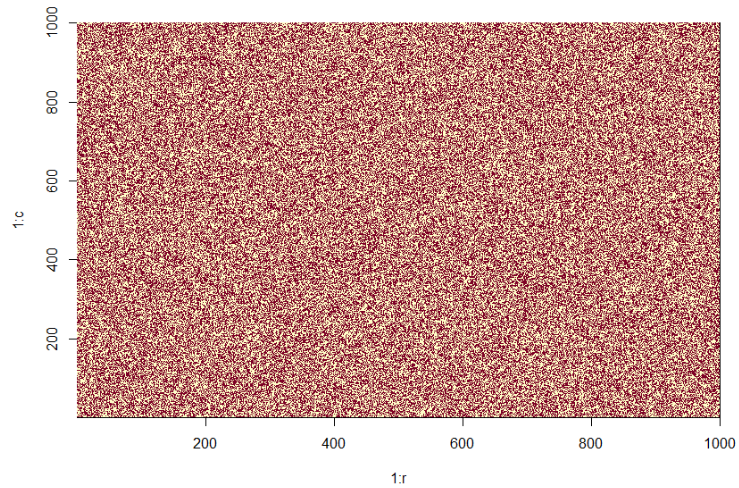

Figure 3.1: $1000 * 1000$, $p = 0.5$ lattice

Figure 3.1 looks random, it is likely white noise.

To check this, we look at the histogram to see if the distribution is approximately normal distribution with mean 0, in which the lattice is normal.

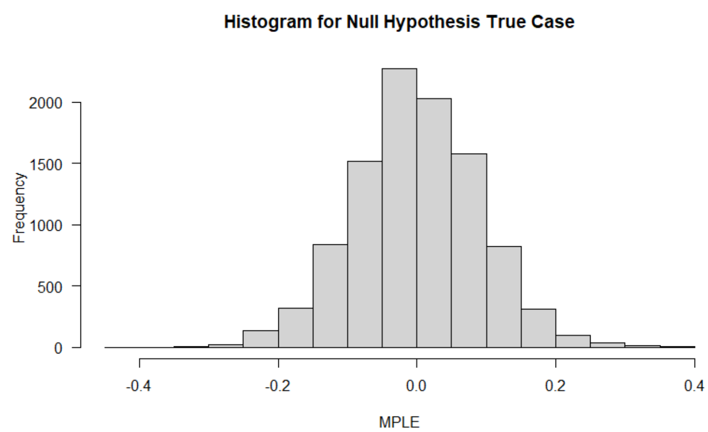

Figure 3.2: Histogram for $1000 * 1000$, $p = 0.5$ lattice

Figure 3.2 is closed to bell-shaped and symmetric close to the mean 0, we can test for normality.

An auto correlation function (ACF) plot shows serial correlation in data. Look at the ACF plot: Figure 3.3 shows the ACF, which has a spike at lag 0 and

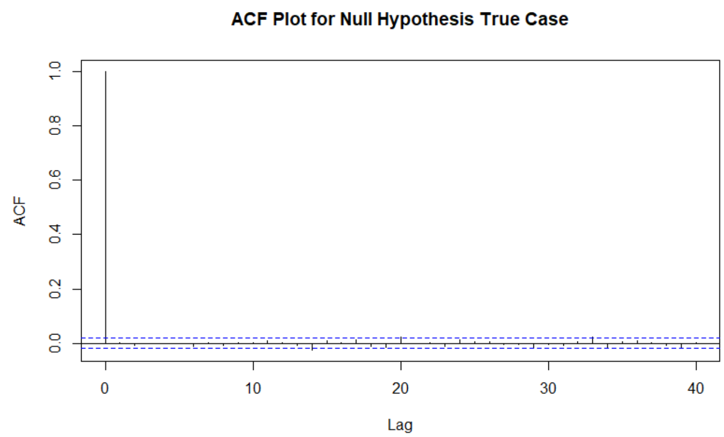

Figure 3.3: ACF of $1000 * 1000, p = 0.5$ lattice

cut-off for following lags, which are within 95% confidence interval. There is no enough statistical evidence to prove the MPLEs have serial autocorrelation. Thus, for Null hypothesis (H_0) true case, we assume the MPLEs $(\hat{\beta})$ is white noise.

3.2 Null hypothesis false cases

Then, we apply the MPLE method on Null hypothesis (H_0) false case, where there isn't a white noise case.

3.2.1 Existed Lattice

Looking at the existed lattice given by Dr. Seymour from personal communication, where $\beta = 0.3$. The $100 * 100$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

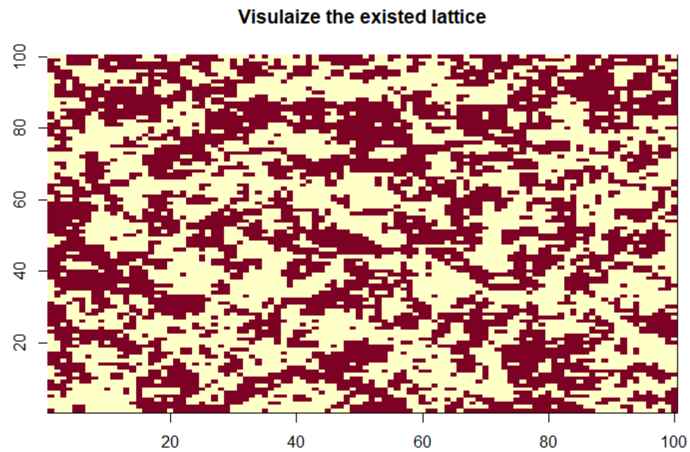

Figure 3.4: $100 * 100$, existed lattice

Figure 3.4 looks non-normal. Then we look at the histogram and ACF plots.

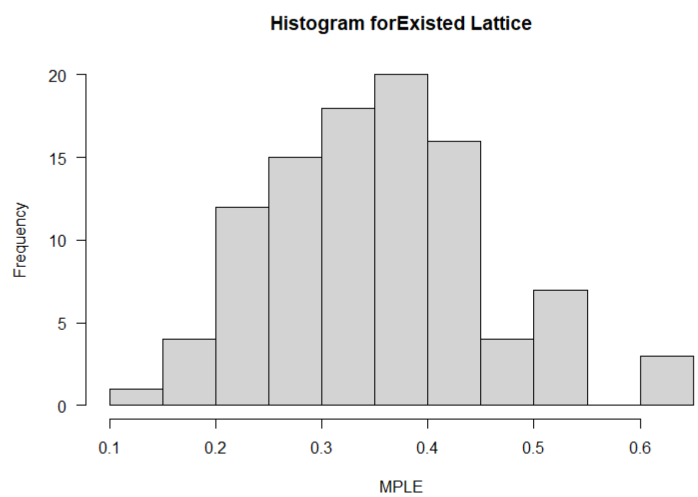

Figure 3.5: histogram for $100 * 100$, existed lattice

Figure 3.5 is not symmetric about the mean, we assume it is not normal.

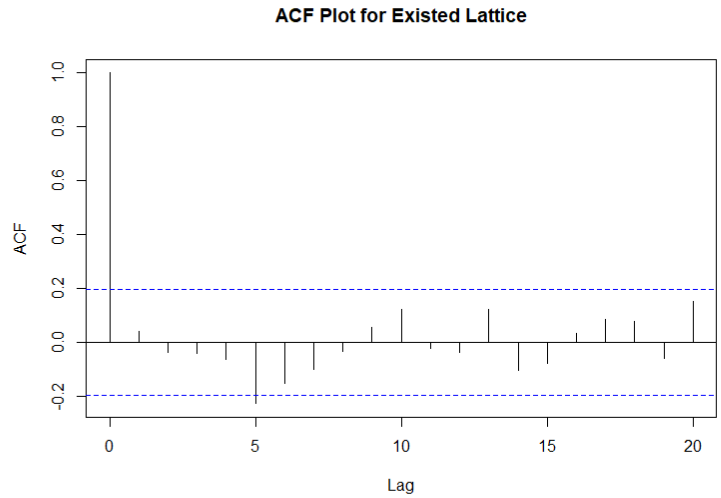

Figure 3.6: ACF of $100 * 100$, existed lattice

Figure 3.6 shows the autocorrelations plot seems to spike at lag 0 and 5, then cut-off for following lags, values for the ACF are within 95% confidence interval. The MPLEs seems to have no significant autocorrelation values. Thus, for the existed lattice, we assume the MPLEs $(\hat{\beta})$ are not white noise.

3.2.2 Replicated matrix

Then we create small $n * n$ null hypothesis true matrix and replicate several times to display them side by side in a $1000 * 1000$ lattice. Set the range of n from 20 to 500, view their results.

$n * n$=$20 * 20$ **matrix**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

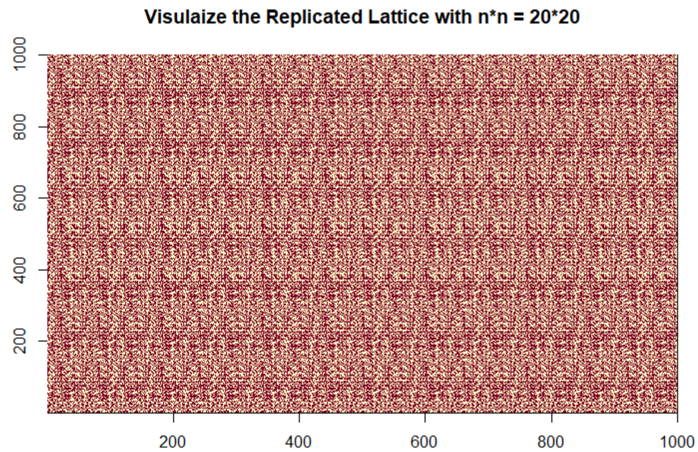

Figure 3.7: Replicated Lattice with $n * n = 20 * 20$

figure 3.7 has a clear pattern, it is not white noise. Then we look at the histogram and ACF plots.

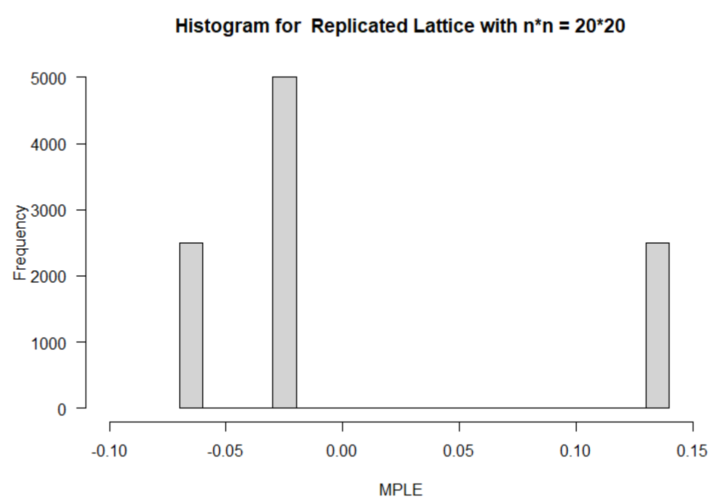

Figure 3.8: Histogram for Replicated Lattice with $n * n = 20 * 20$

Figure 3.8 does not follow the shape of normal distribution.

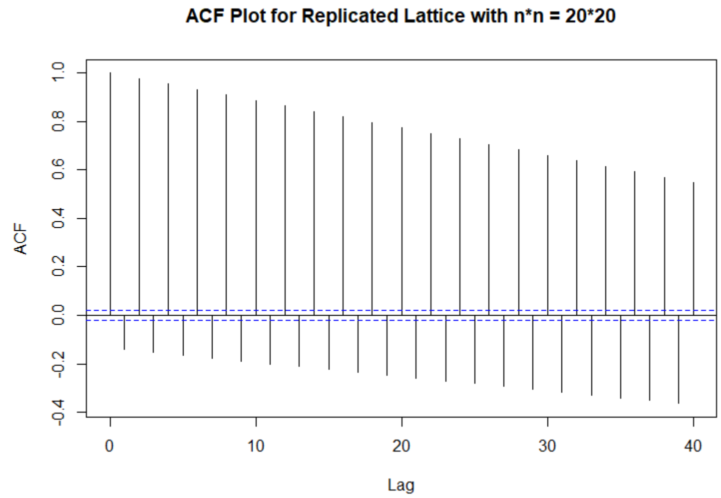

Figure 3.9: ACF of Replicated Lattice with $n * n = 20 * 20$

Figure 3.9 shows all lags of ACF are outside the 95% confidence interval in a decreasing trend. The MPLEs have clear serail autocorrelation. Thus, for the replicated Lattice with $n * n = 20 * 20$, we assume MPLEs $(\hat{\beta})$ is not white noise.

$n * n$=$25 * 25$ matrix

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

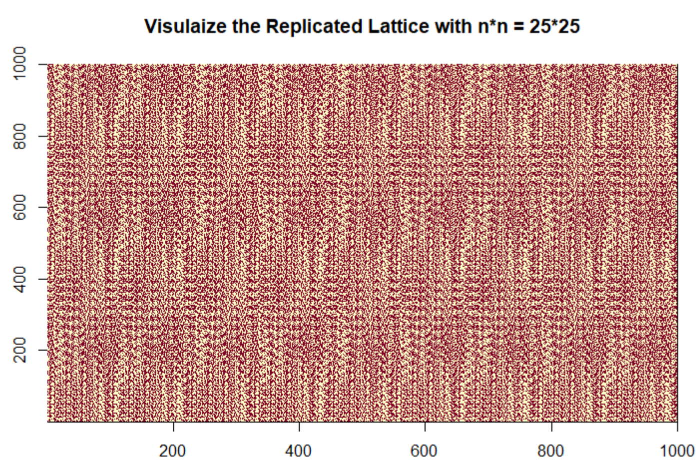

Figure 3.10: Replicated Lattice with $n * n = 25 * 25$

figure 3.10 has a clear pattern, it is not white noise. Then we look at the histogram and ACF plots.

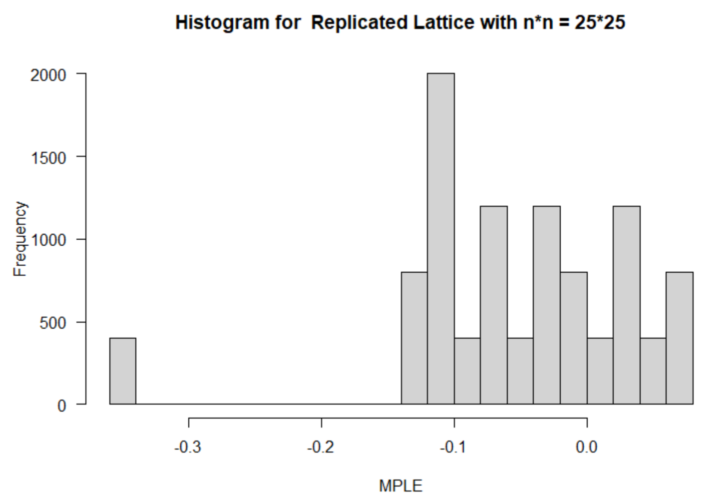

Figure 3.11: Histogram for Replicated Lattice with $n * n = 25 * 25$

figure 3.11 does not follow the shape of normal distribution.

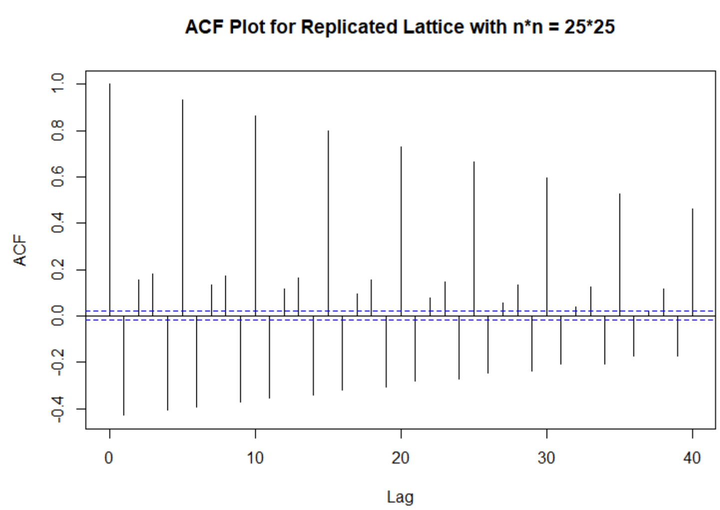

Figure 3.12: ACF of Replicated Lattice with $n * n = 25 * 25$

figure 3.12 shows ACF plot has a strong trend and periodic pattern, all lags are outside the 95% confidence interval. The MPLEs have clear serial autocorrelation. Thus, for the replicated Lattice with $n * n = 25 * 25$, we assume MPLEs $(\hat{\beta})$ is not white noise.

$n * n$=$50 * 50$ **matrix**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

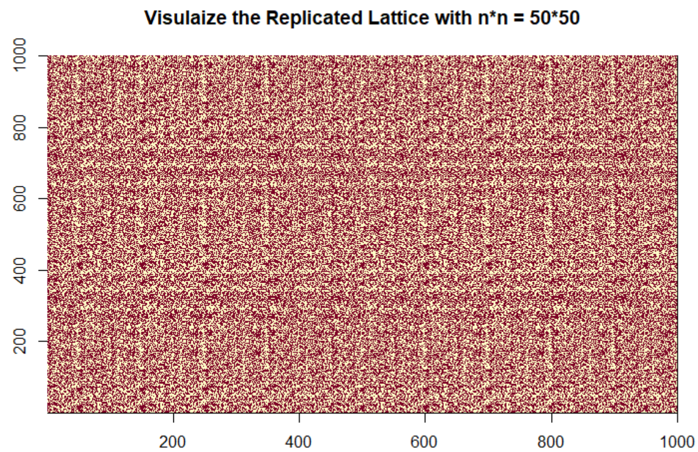

Figure 3.13: Replicated Lattice with $n * n = 50 * 50$

figure 3.13 has a pattern, it does not look like white noise. Then we look at the histogram and acf plots.

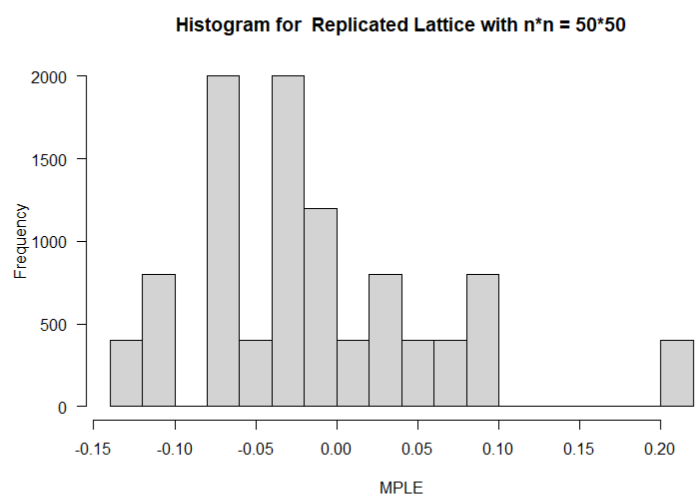

Figure 3.14: Histogram for Replicated Lattice with $n * n = 50 * 50$

figure 3.14 seems not following the normal distribution.

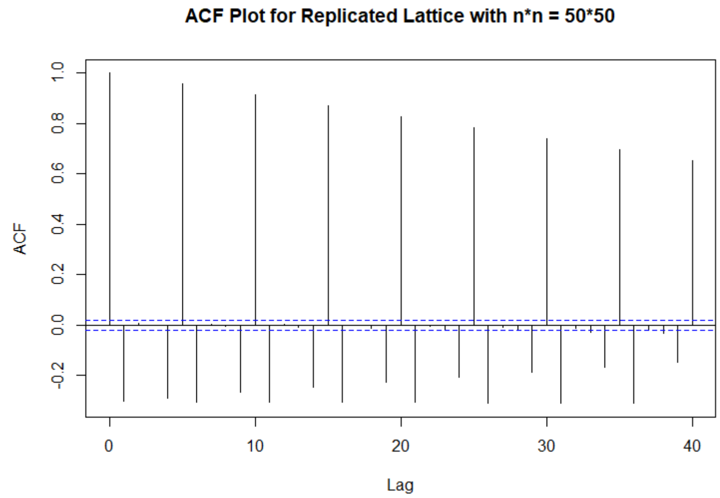

Figure 3.15: ACF of Replicated Lattice with $n * n = 50 * 50$

figure 3.15 shows ACF plot has a trend and periodic pattern. We decide the MPLEs have serial autocorrelation. Thus, for the replicated lattice with $n * n = 50 * 50$, it does not look like white noise.

$n * n$=$100 * 100$ **matrix**

The $1000 * 1000$ lattice is as shown:

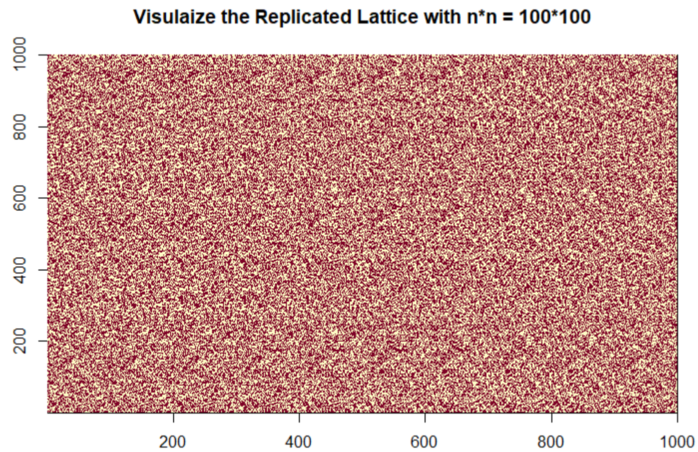

Figure 3.16: Replicated Lattice with $n * n = 100 * 100$

figure 3.16 has no clear pattern, it looks like white noise. Then we look at the histogram and ACF plots.

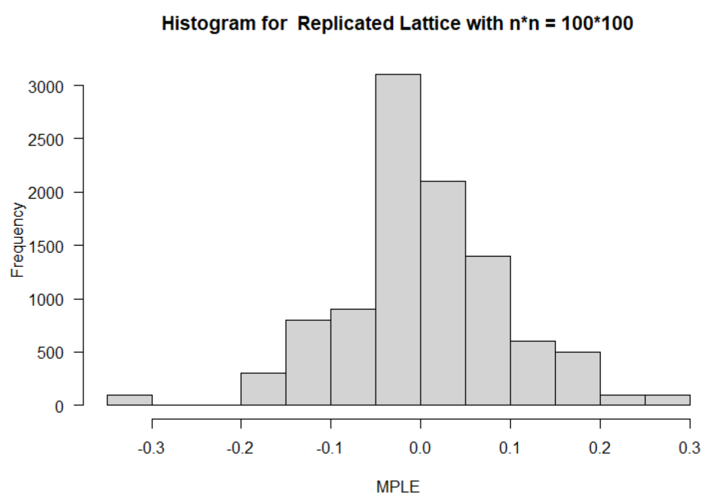

Figure 3.17: Histogram for Replicated Lattice with $n * n = 100 * 100$

figure 3.17 it does not seem to follow normal distribution.

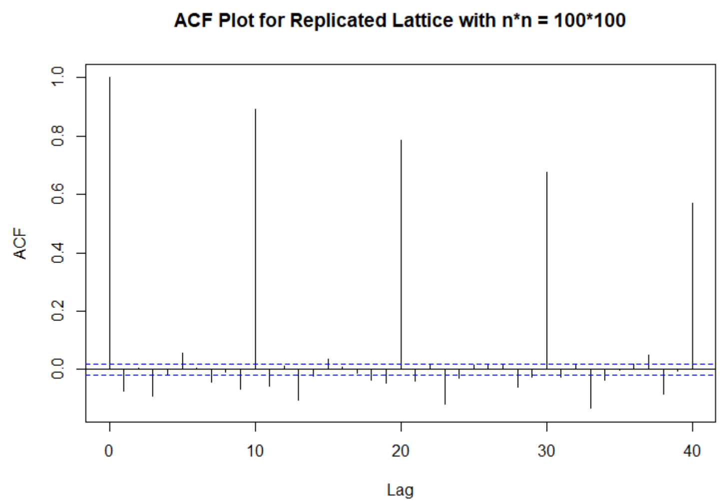

Figure 3.18: ACF of Replicated Lattice with $n * n = 100 * 100$

figure 3.18 shows ACF plot has a slowly decresing and periodic pattern with sevral spikes, the MPLEs have autocorrelation. Thus, for the replicated lattice with $n * n = 100 * 100$, it is not white noise

$n * n$=$250 * 250$ **matrix**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

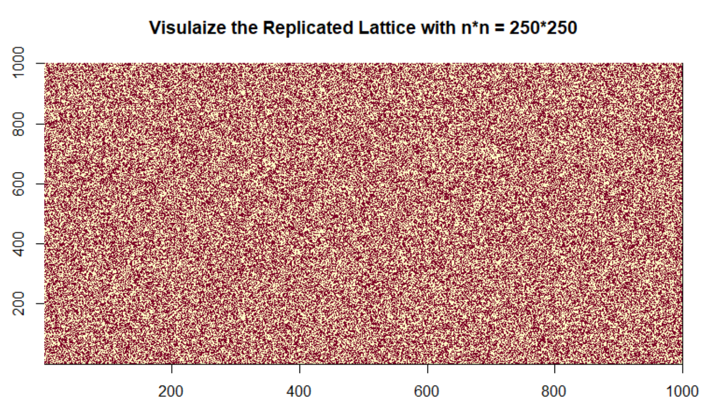

Figure 3.19: Replicated Lattice with $n * n = 250 * 250$

figure 3.19 has no pattern, it looks like white noise. Then we look at the histogram and ACF plots.

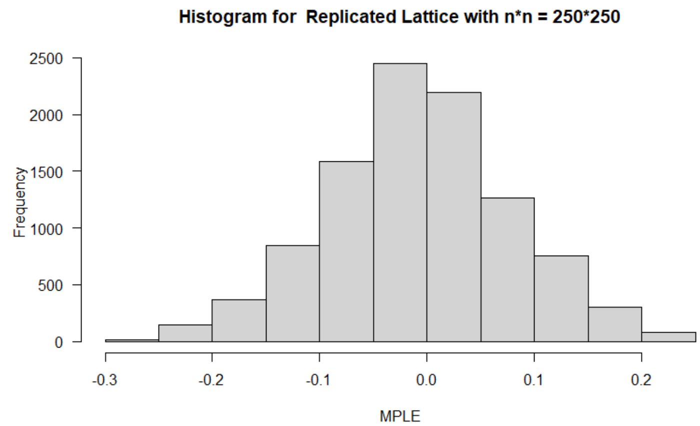

Figure 3.20: Histogram for Replicated Lattice with $n * n = 250 * 250$

figure 3.20 has a bell shape and symmetric about mean, it seems to follow normal distribution.

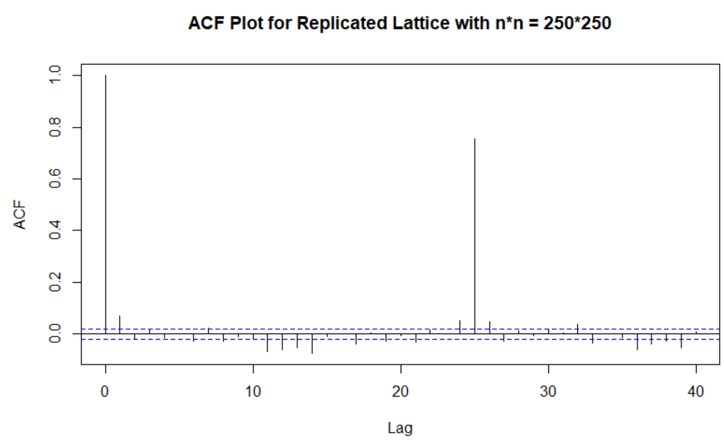

Figure 3.21: ACF of Replicated Lattice with $n * n = 250 * 250$

figure 3.21 shows ACF plot has spikes in lag 0 and 25 with a cut-off pattern. The MPLEs have serial autocorrelation. Thus, for the replicated Lattice with $n * n = 250 * 250$, we assume it is not white noise.

$n * n$=$500 * 500$ **matrix**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

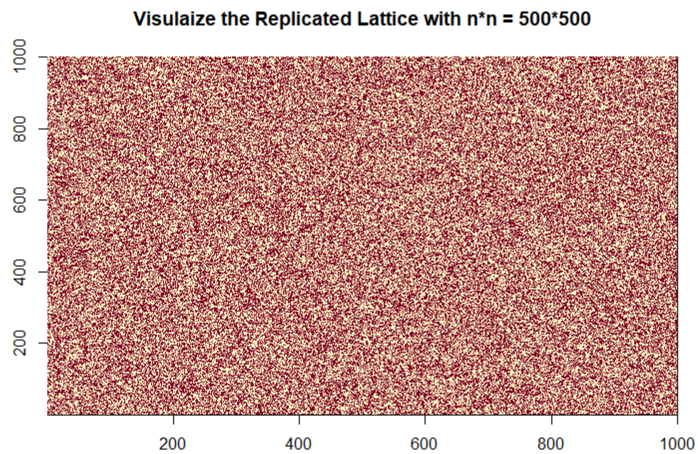

Figure 3.22: Replicated Lattice with $n * n = 500 * 500$

figure 3.22 has no pattern, it looks like white noise. Then we look at the histogram and ACF plots.

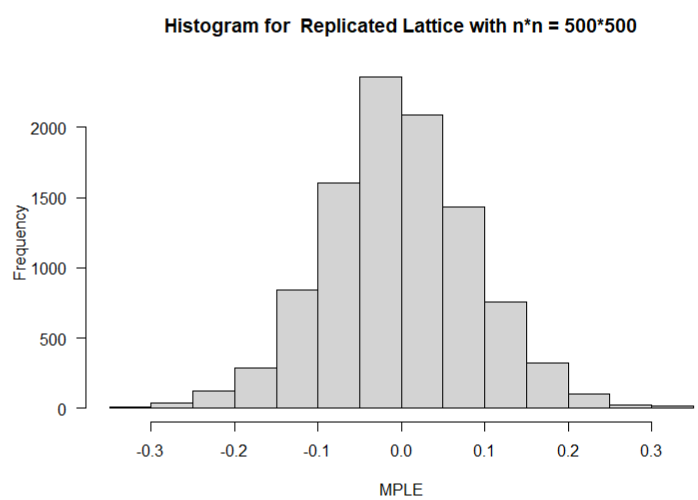

Figure 3.23: Histogram for Replicated Lattice with $n * n = 500 * 500$

figure 3.23 has a bell shape and symmetric about mean, it seems to follow normal distribution.

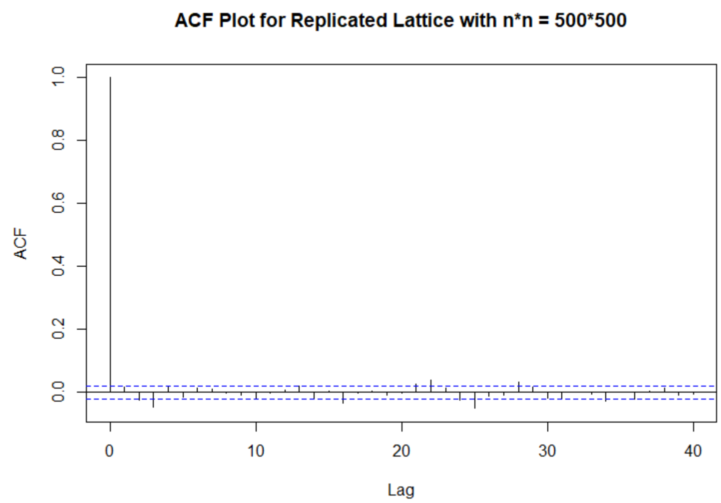

Figure 3.24: ACF of Replicated Lattice with $n * n = 500 * 500$

figure 3.24 shows ACF plot has a cut-off pattern after lag 0, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the replicated Lattice with $n * n = 500 * 500$, we assume it is white noise.

3.2.3 Blocking matrix

Within the $1000 * 1000$ null hypothesis true lattice, we set x number of $n * n$ non-overlapping blocks with all points equal to -1 inside. By visualizing the MPLEs, we can see $n * n$ size blocking squares inside the lattice. Change the number (x) and the size ($n * n$) of blocks, explore the data.

size $n * n = 5 * 5$, number $x = 100$ blocks

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

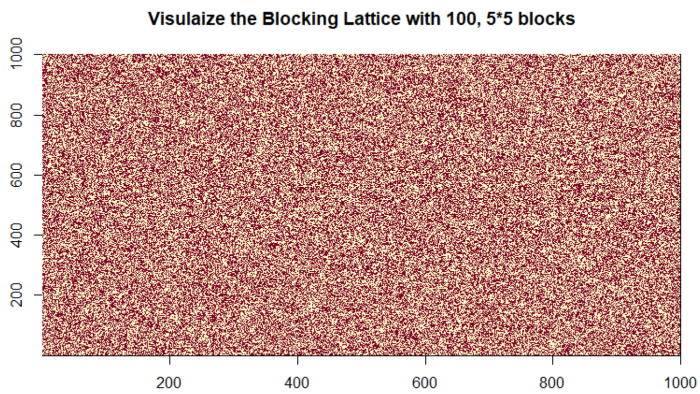

Figure 3.25: Blocking Lattice with 100, $5 * 5$ blocks

Figure 3.25 has no clear pattern, it is white noise. Then we look at the histogram and ACF plots.

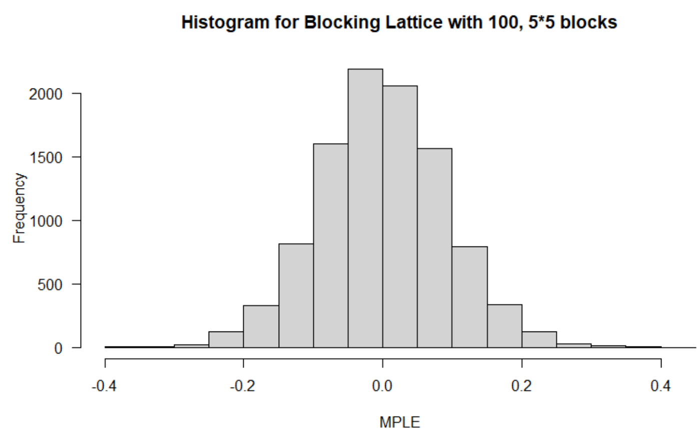

Figure 3.26: Histogram for Blocking Lattice with 100, $5 * 5$ blocks

figure 3.26 has a bell shape and symmetric about mean, it follows normal distribution.

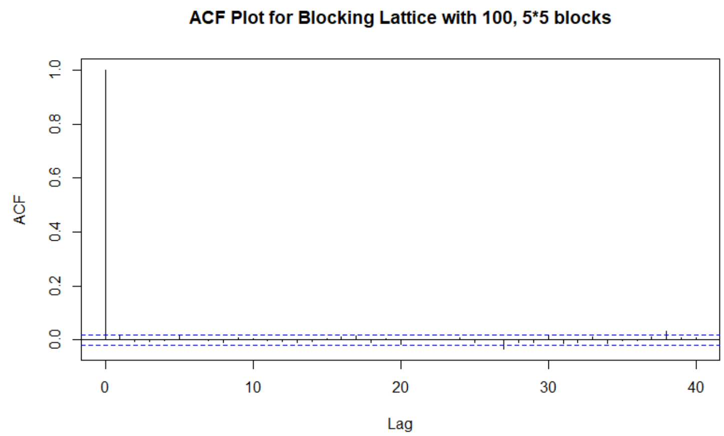

Figure 3.27: ACF of Blocking Lattice with 100, $5 * 5$ blocks

figure 3.27 shows ACF plot has a cut-off pattern after lag 0, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with 400, $5 * 5$ blocks, we assume it is white noise.

size $n * n = 5 * 5$**, number** $x = 400$ **blocks**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

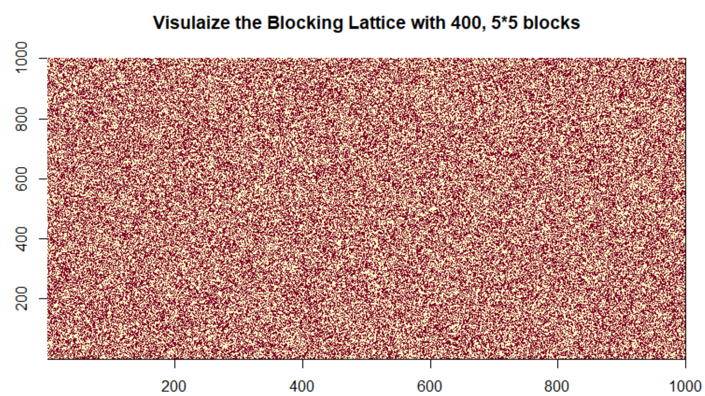

Figure 3.28: Blocking Lattice with 400, $5 * 5$ blocks

Figure 3.28 has no clear pattern, it is white noise. Then we look at the histogram and ACF plots.

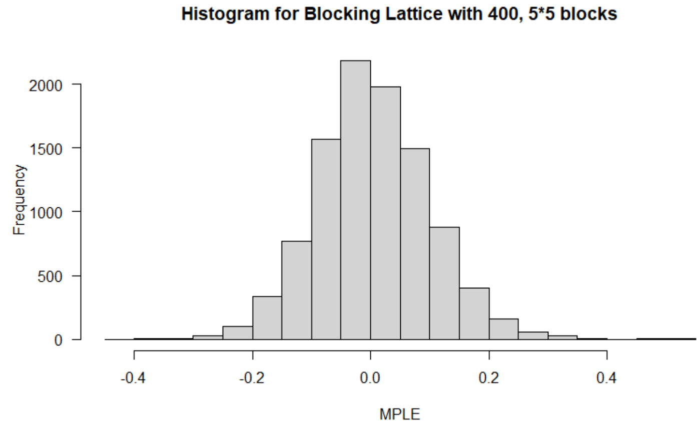

Figure 3.29: Histogram for Blocking Lattice with 400, $5 * 5$ blocks

Figure 3.29 has a bell shape and symmetric about mean, it follows normal distribution.

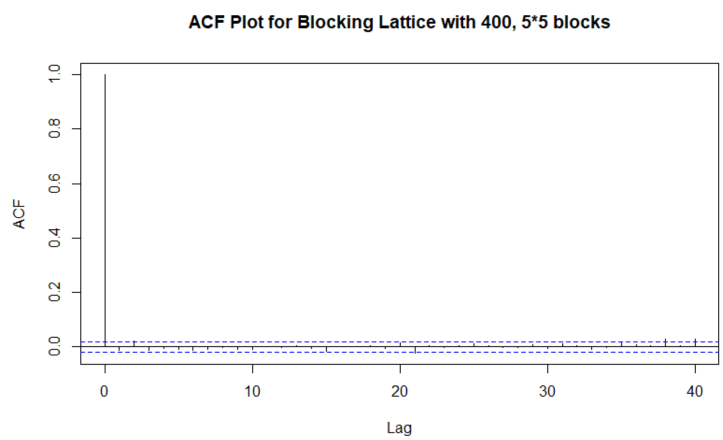

Figure 3.30: ACF of Blocking Lattice with 400, $5 * 5$ blocks

Figure 3.30 shows ACF plot has a cut-off pattern after lag 0, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with 400, $5 * 5$ blocks, we assume it is white noise.

size $n * n = 5 * 5$**, number** $x = 2500$ **blocks**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

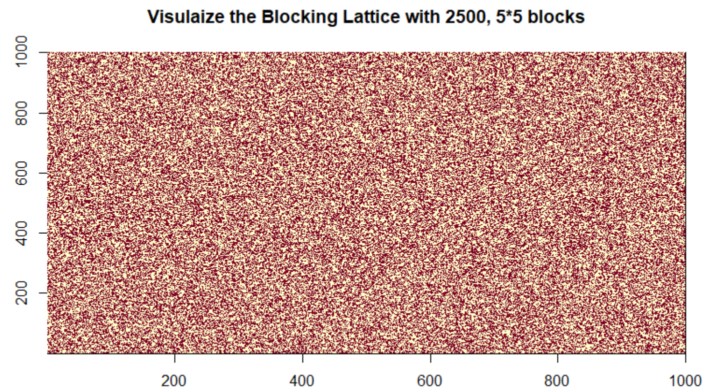

Figure 3.31: Blocking Lattice with 2500, $5 * 5$ blocks

figure 3.31 has no clear pattern, it is white noise. Then we look at the histogram and ACF plots.

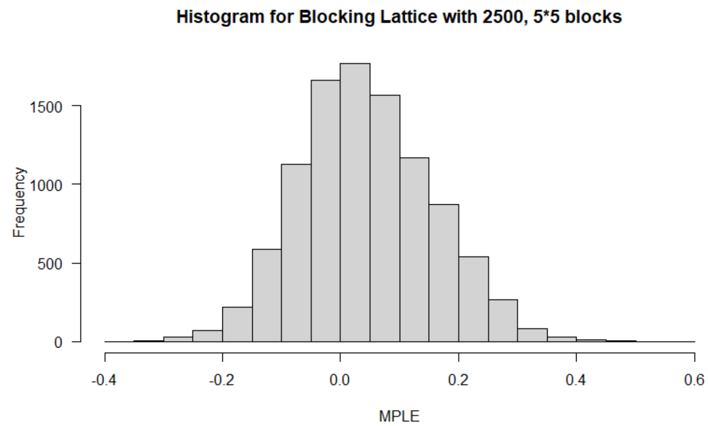

Figure 3.32: Histogram for Blocking Lattice with 2500, $5 * 5$ blocks

figure 3.32 has a mean greater than 0, it does not follow normal distribution. figure 3.33 shows ACF plot has a cut-off pattern after lag 0 and 1, most ACFs

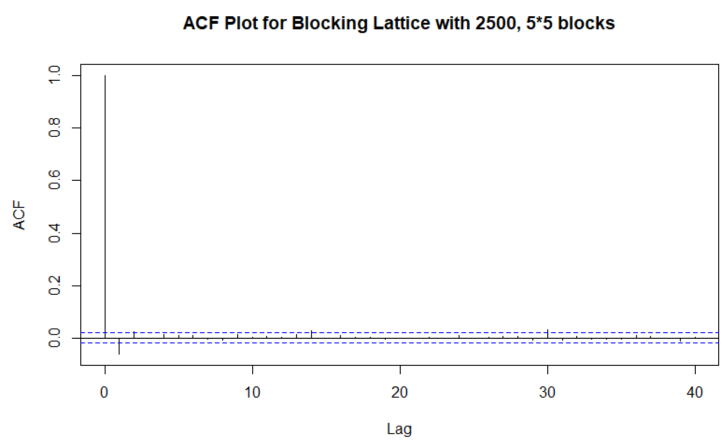

Figure 3.33: ACF of Blocking Lattice with 2500, $5 * 5$ blocks

are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with 2500, $5 * 5$ blocks, we assume it is not white noise.

size $n * n = 5 * 5$, number $x = 4900$ blocks

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

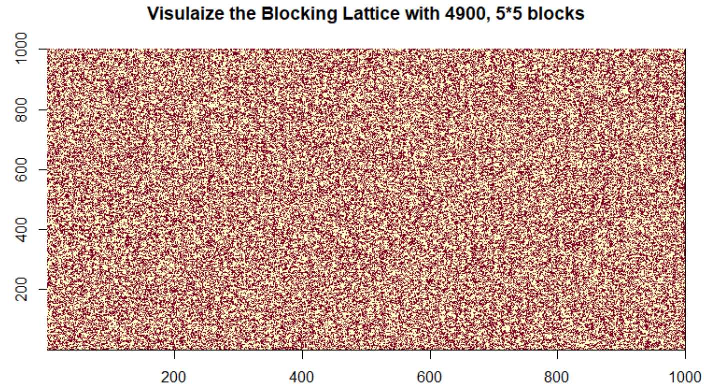

Figure 3.34: Blocking Lattice with 4900, $5 * 5$ blocks

figure 3.34 has a pattern, it is not white noise. Then we look at the histogram and ACF plots. figure 3.35has a mean greater than 0, it does not follow normal

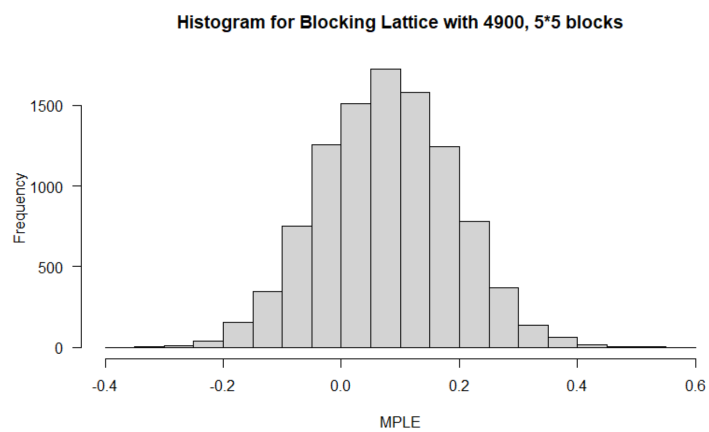

Figure 3.35: Histogram for Blocking Lattice with 4900, $5 * 5$ blocks

distribution.

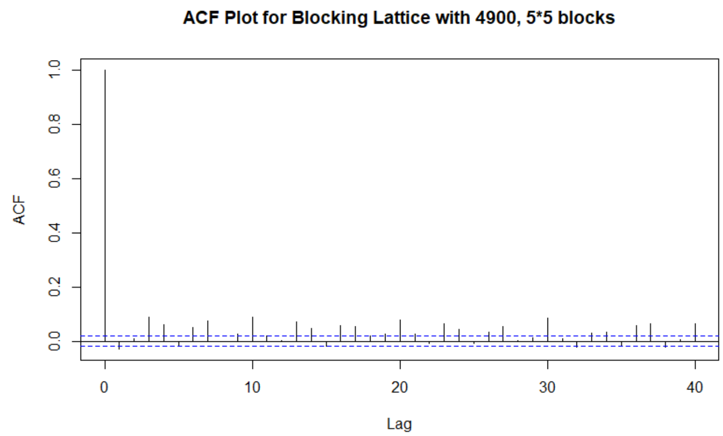

Figure 3.36: ACF of Blocking Lattice with 4900, $5 * 5$ blocks

figure 3.36 shows ACF plot has a periodic pattern after lag 0, most ACFs are outside the 95% interval. The MPLEs have serial autocorrelation. Thus, for the blocking lattice with 4900, $5 * 5$ blocks, we assume it is not white noise.

size $n * n = 5 * 5$**, number** $x = 6400$ **blocks**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

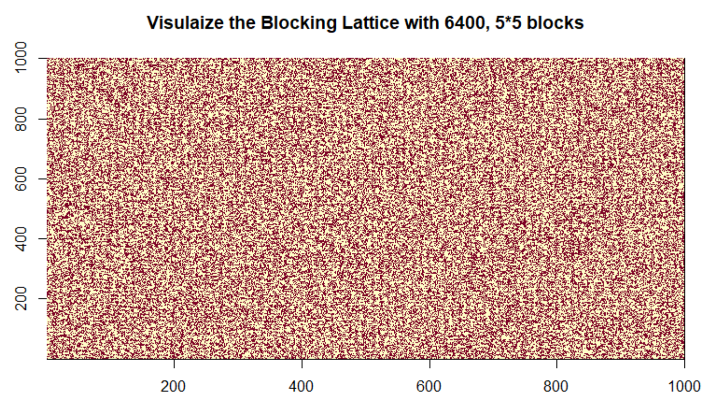

Figure 3.37: Blocking Lattice with 6400, $5 * 5$ blocks

figure 3.37 has a clear pattern, it is not white noise. Then we look at the histogram and ACF plots.

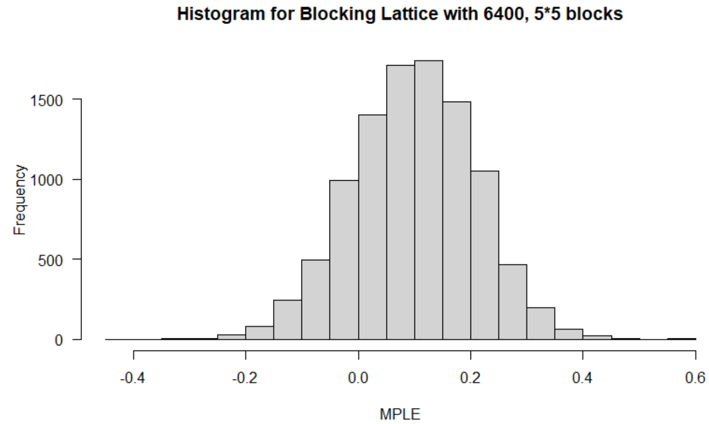

Figure 3.38: Histogram for Blocking Lattice with 6400, $5 * 5$ blocks

figure 3.38 has a mean greater than 0, it does not follow normal distribution.

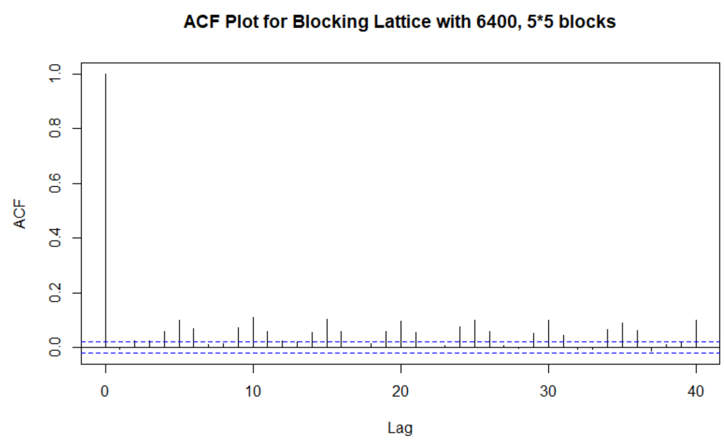

Figure 3.39: ACF of Blocking Lattice with 6400, $5 * 5$ blocks

figure 3.39 shows ACF plot has a periodic pattern after lag 0, most ACFs are outside the 95% interval. The MPLEs have serial autocorrelation. Thus, for the blocking lattice with 6400, $5 * 5$ blocks, we assume it is not white noise.

size $n * n = 10 * 10$, number $x = 25$ blocks

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

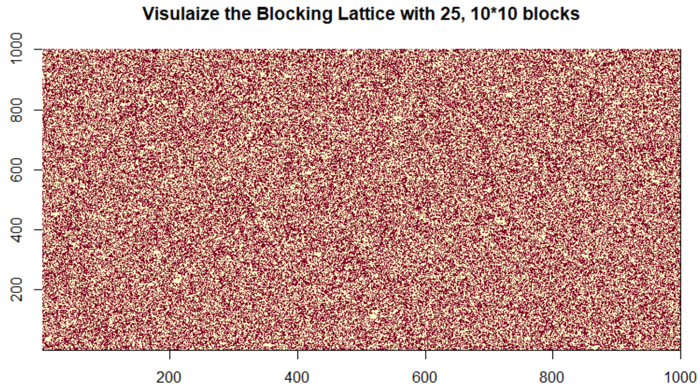

Figure 3.40: Blocking Lattice with $25, 10 * 10$ blocks

figure 3.40 has a clear pattern, it is not white noise. Then we look at the histogram and ACF plots.

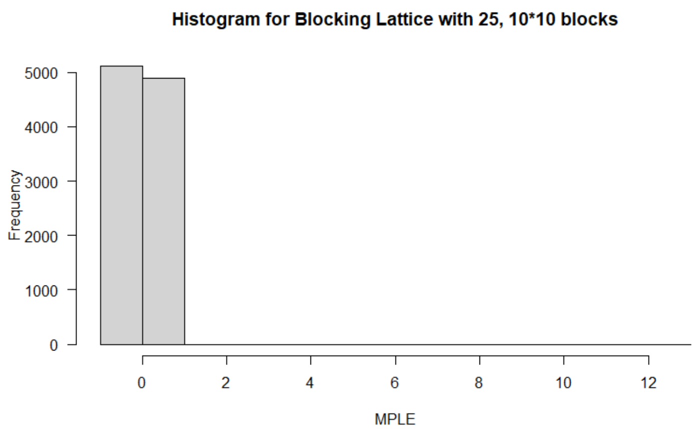

Figure 3.41: Histogram for Blocking Lattice with $25, 10 * 10$ blocks

Figure 3.41 is not normal distributed.

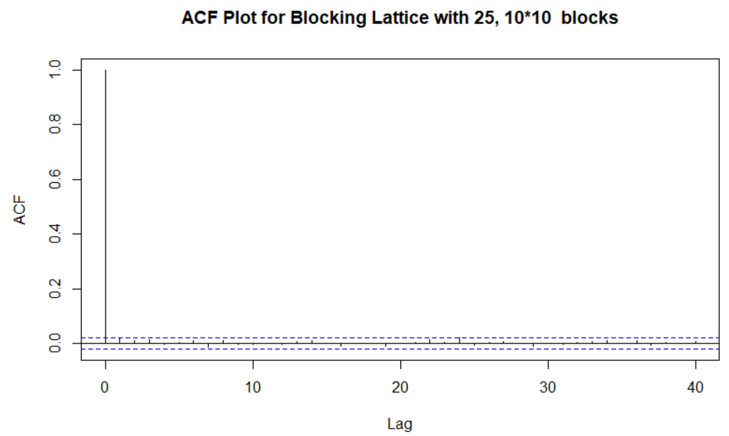

Figure 3.42: ACF of Blocking Lattice with 25, $10 * 10$blocks

Figure 3.42 shows ACF plot has a cut-off pattern after lag 0, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with 25, $5 * 5$ blocks, we assume it is not white noise.

size $n * n = 10 * 10$**, number** $x = 100$ **blocks**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

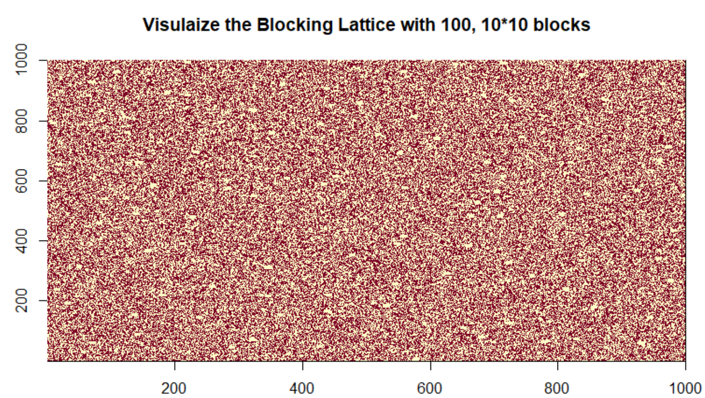

Figure 3.43: Blocking Lattice with 100, $10 * 10$ blocks

figure 3.43 has a clear pattern, it is not white noise. Then we look at the histogram and ACF plots.

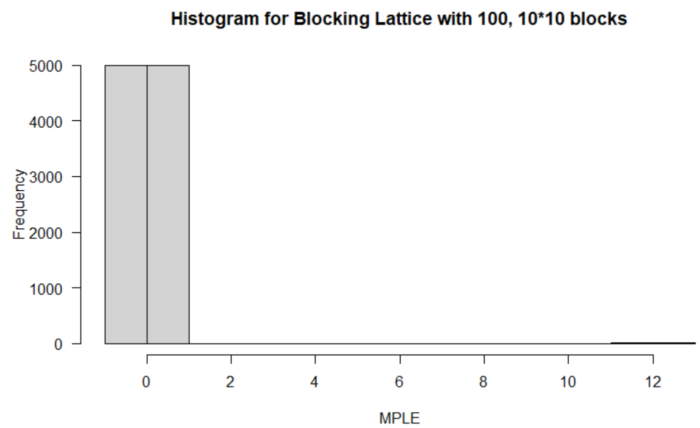

Figure 3.44: Histogram for Blocking Lattice with 100, $10 * 10$ blocks

Figure 3.44 is not normal.

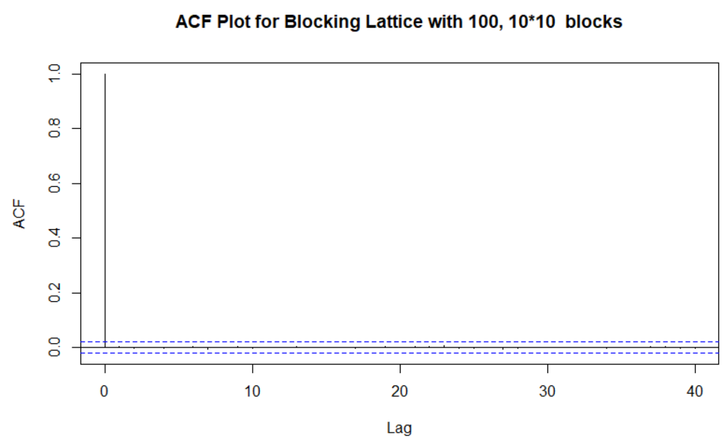

Figure 3.45: ACF of Blocking Lattice with 100, $10 * 10$ blocks

figure 3.45 shows ACF plot has a cut-off pattern after lag 0, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with 100, $10 * 10$ blocks, we assume it is not white noise.

size $n * n = 10 * 10$, number $x = 2500$ blocks

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

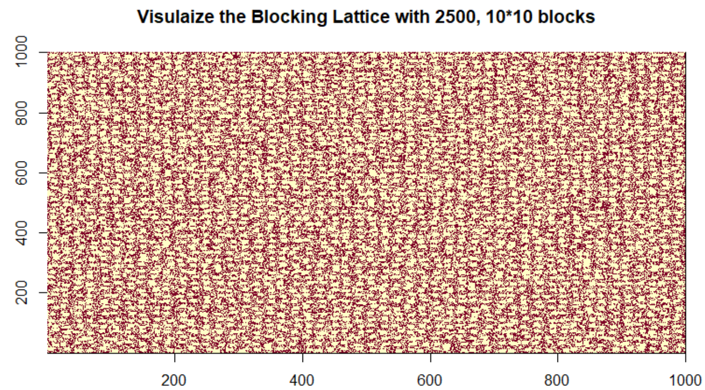

Figure 3.46: Blocking Lattice with 2500, $10 * 10$ blocks

figure 3.46 has a clear pattern, it is not white noise. Then we look at the histogram and ACF plots.

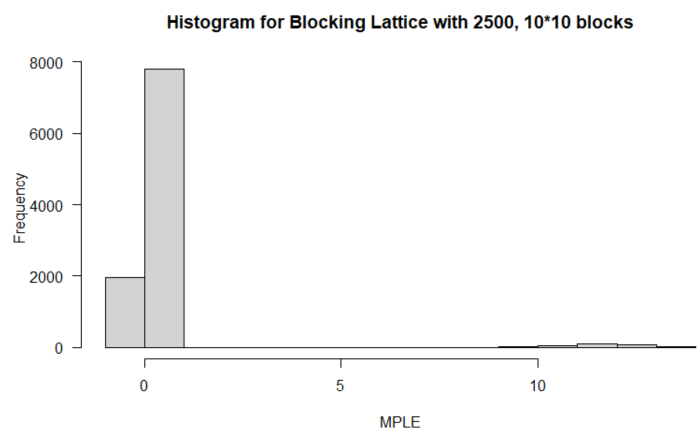

Figure 3.47: Histogram for Blocking Lattice with 2500, $10 * 10$ blocks

figure 3.47 is highly skewed, it is not normal.

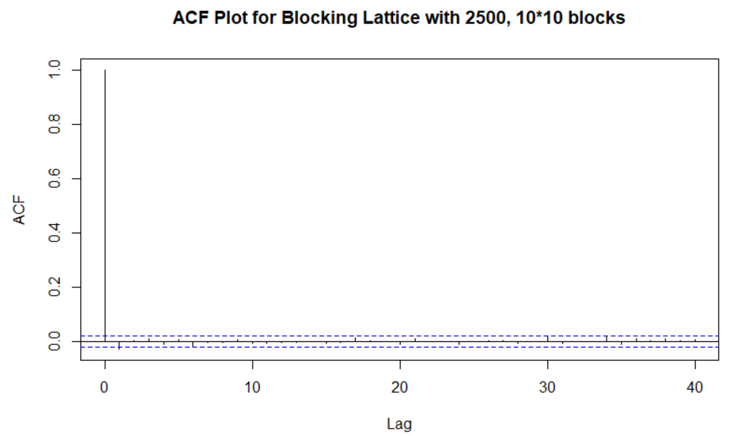

Figure 3.48: ACF of Blocking Lattice with 2500, $10 * 10$blocks

figure 3.48 shows ACF plot has a cut-off pattern after lag 0, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with 2500, $10 * 10$ blocks, we assume it is not white noise.

size $n * n = 20 * 20$**, number** $x = 1$ **blocks**

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

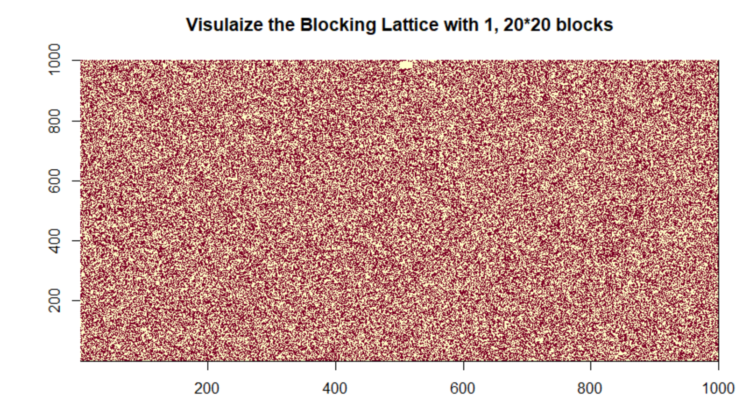

Figure 3.49: Blocking Lattice with 1, $20 * 20$ blocks

Figure 3.49 has a block on the top, it is not white noise. Then we look at the histogram and ACF plots.

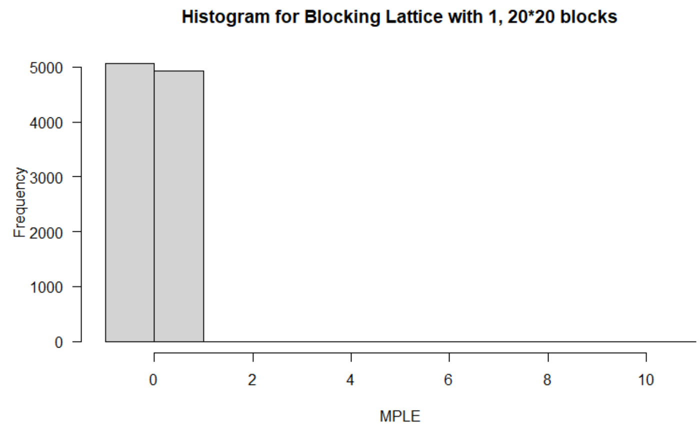

Figure 3.50: Histogram for Blocking Lattice with $1, 20 * 20$ blocks

Figure 3.50 is highly skewed, it is not normal.

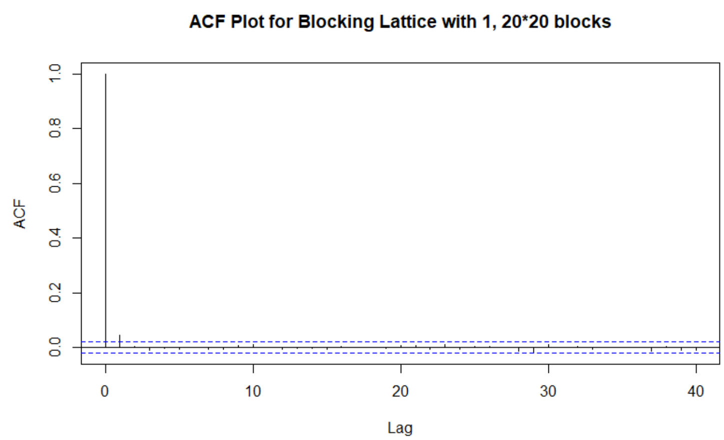

Figure 3.51: ACF of Blocking Lattice with $1, 20 * 20$ blocks

Figure 3.51 shows ACF plot has a periodic pattern after lag 0, all ACFs are outside the 95% interval. The MPLEs have serial autocorrelation. Thus, for the blocking lattice with $1, 20 * 20$ blocks, we assume it is not white noise.

size $n * n = 20 * 20$, number $x = 64$ blocks

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

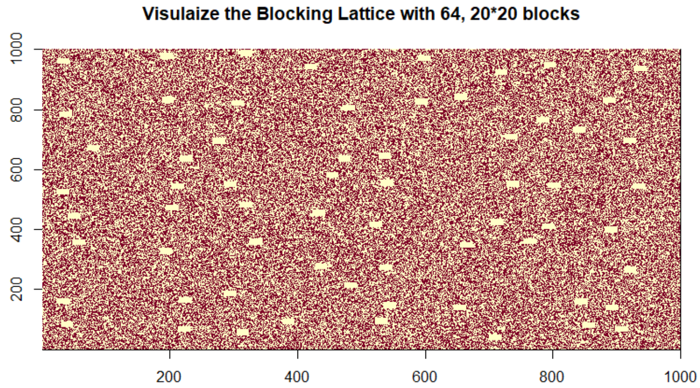

Figure 3.52: Blocking Lattice with $64, 20 * 20$ blocks

figure 3.52 is not white noise. Then we look at the histogram and ACF plots.

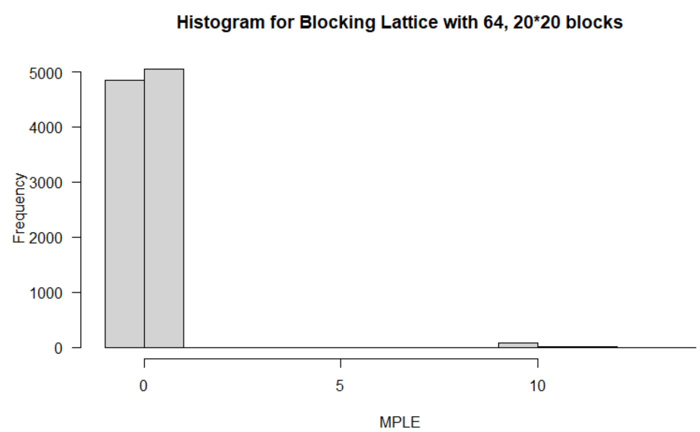

Figure 3.53: Histogram for Blocking Lattice with $64, 20 * 20$ blocks

figure 3.53 is highly skewed, it is not normal.

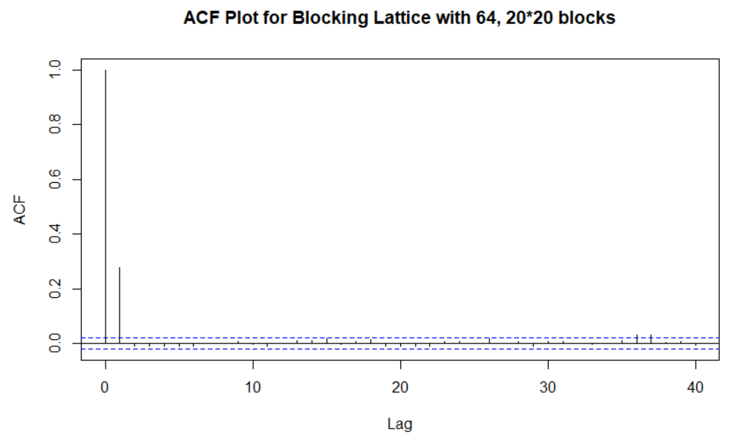

Figure 3.54: ACF of Blocking Lattice with $64, 20 * 20$ blocks

figure 3.54 shows ACF plot has a cut-off pattern after lag 1, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with $64, 20 * 20$ blocks, we assume it is not white noise.

size $n * n = 50 * 50$, number x = 1 blocks

The $1000 * 1000$ lattice is as shown, red dots are -1 points and yellow ones are 1 :

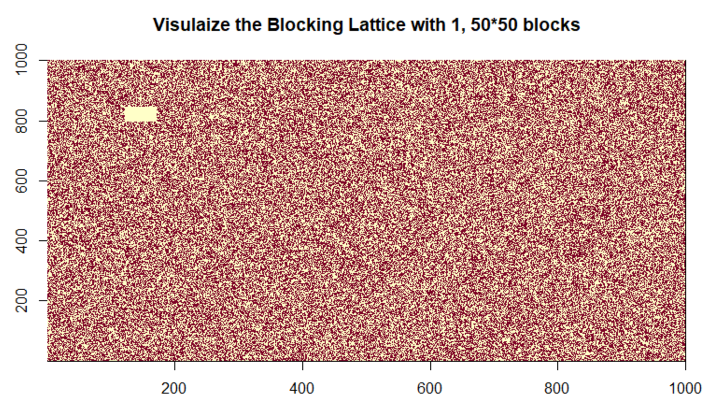

Figure 3.55: Blocking Lattice with $1, 50 * 50$ blocks

figure 3.55 has a block on the top, it is not white noise. Then we look at the histogram and ACF plots.

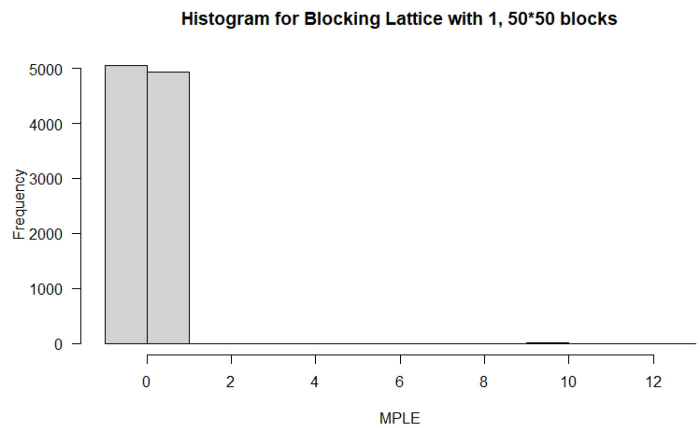

Figure 3.56: Histogram for Blocking Lattice with $1, 50 * 50$ blocks

figure 3.56 is not normal.

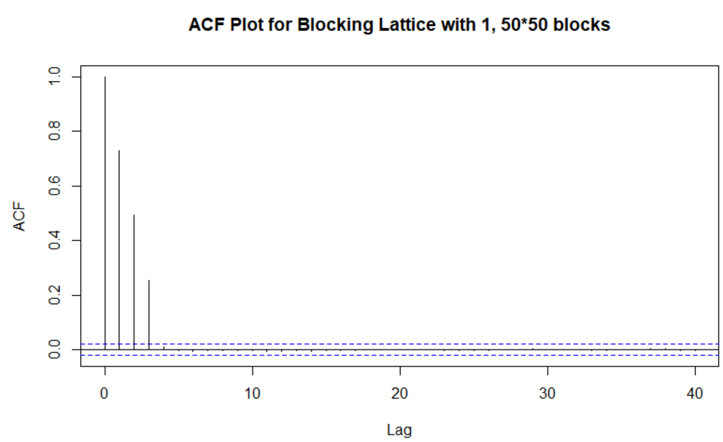

Figure 3.57: ACF of Blocking Lattice with $1, 50 * 50$ blocks

figure 3.57 shows ACF plot has a cut-off pattern after lag 4, most ACFs are within the 95% interval. The MPLEs have no serial autocorrelation. Thus, for the blocking lattice with $1, 50 * 50$ blocks, we assume it is not white noise.

Chapter 4

Results

We simulate 100 or 200 lattices for each case and see their white noise test results and the five number summary of p-values. The five-number summary describes statistical information about a dataset. Five most important sample percentiles are listed: the sample minimum, the lower quartile, the median, the upper quartile, the sample maximum. The reason we choose to do the part of the simulations 100 timesis because it takes more than 2 hours to do that for a 200 times simulation blocking matrix case like $n * n = 50 * 50$, $x = 1$, which is a time consuming test in reality. We fail to reject the 100 or 200 simulations for the two tests if more than 95% of the p-values are ≥ 0.05.

4.1 Null hypothesis true case

First we look at the null hypothesis true case:

Table 4.1: p-value Summary for Null True Case $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	95.50 %	0.9992	0.0051	0.2841	0.7936	0.5592
Box test	94.59%	0.9977	0.0037	0.2399	0.7877	0.4856

Table 4.1 shows the maximum and minimum values, the lower and upper quartiles, the median, and the percentage of $p \leq 0.05$ for normality test and serial auto-correlation test when null is true. It is obvious that more than 95% of the p-value for both AD test and Box test are larger than or equal to 0.05, indicating that $p = 0.5$ lattice is normal and have no serial auto correlation. It is obvious to conclude that it is detected as white noise under our method.

4.2 Null hypothesis false cases

Then we look at these null hypothesis false case.

4.2.1 Existed Lattice

The existed lattice does not need to be simulated 200 times, so we test on itself:

Table 4.2: p-value summary for Existed Lattice $\hat{\beta}$

	p -values
AD test	0.001
Box test	0.683

Table 4.2 shows that p-value for AD test is less than 0.05, and larger than 0.05 for the Box test, indicating the existed lattice is not normal but have no serial auto correlation. This is not white noise.

4.2.2 Replicated matrix

$n * n$=$20 * 20$ **matrix**

Then we look at the replicated matrix cases:

Table 4.3: p-value Summary for Replicated Lattice with $n * n = 20 * 20$ $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.003	0.001	0.001	0.001	0.001
Box test	1.5%	0.9613	0.000	0.000	0.000	0.000

Table 4.3 shows the maximum and minimum values, the lower and upper quartiles, the median, and the percentage of $p \leq 0.05$ for normality test and serial auto-correlation test when $n * n = 20 * 20$ for the replicated lattice. The percentage of p-value larger than or equal to 0.05 for both AD test and Box test are close to 0, indicating the lattice is not normal under mean 0 but have serial auto correlation. It is not detected as white noise.

$n * n$=$25 * 25$ **matrix**

Table 4.4: p-value Summary for Replicated Lattice with $n * n = 25 * 25$ $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	51%	0.761	0.001	0.013	0.119	0.051
Box test	9%	0.9698	0.000	0.000	0.000	0.000

Table 4.4 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are less than 95%, indicating the lattice is not normal under mean 0 and have serial auto correlation. It is not detected as white noise.

$n * n$=$50 * 50$ **matrix**

Table 4.5: p-value Summary for Replicated Lattice with $n * n = 50 * 50$ $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	54%	0.838	0.001	0.019	0.161	0.062
Box test	7%	0.908	0.000	0.000	0.000	0.000

Table 4.5 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are less than 95%, indicating the lattice is not normal under mean 0 and have serial auto correlation. It is not detected as white noise.

$n * n$=$100 * 100$ **matrix**

Table 4.6: p-value Summary for Replicated Lattice with $n * n = 100 * 100$ $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	95%	0.966	0.016	0.204	0.658	0.404
Box test	23%	0.961	0.000	0.000	0.003	0.000

Table 4.6 shows the percentage of p-value larger than or equal to 0.05 for AD test exceeds 95% and less than 95% for the Box test, indicating the lattice is tested normal with mean 0 but have serial auto correlation. It is not detected as white noise.

$n * n$=$250 * 250$ **matrix**

Table 4.7: p-value Summary for Replicated Lattice with $n * n$ = $250 * 250$ $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	96%	0.997	0.006	0.222	0.757	0.465
Box test	35%	0.866	0.000	0.000	0.013	0.003

Table 4.7 shows the percentage of p-value larger than or equal to 0.05 for AD test exceeds 95% and less than 95% for the Box test, indicating the lattice is tested normal with mean 0 but have serial auto correlation. It is not detected as white noise.

$n * n$=$500 * 500$ **matrix**

Table 4.8: p-value Summary for Replicated Lattice with $n * n$ = $500 * 500$ $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	99%	0.991	0.036	0.240	0.837	0.559
Box test	66%	0.994	0.000	0.015	0.603	0.192

Table 4.8 shows the percentage of p-value larger than or equal to 0.05 for AD test exceeds 95% and less than 95% for the Box test, indicating the lattice is tested normal with mean 0 but have serial auto correlation. It is not detected as white noise.

4.2.3 Blocking matrix

size $n * n = 5 * 5$**, number** $x = 100$ **blocks**

Finally we look at the blocking matrix cases:

Table 4.9: p-value Summary for Blocking Lattice with 100, $5 * 5$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	95%	0.999	0.009	0.264	0.726	0.469
Box test	97%	0.999	0.004	0.231	0.848	0.552

Table 4.9 shows the maximum and minimum values, the lower and upper quartiles, the median, and the percentage of $p \leq 0.05$ for normality test and serial auto-correlation test when there are 100, $n * n = 5 * 5$ blocks for the blocking lattice. The percentage of p-value larger than or equal to 0.05 for both

AD test and Box test exceeds 95%, indicating the lattice is tested normal and have no serial auto correlation. It has no differentiate than white noise under our method.

size $n * n = 5 * 5$**, number** $x = 400$ **blocks**

Table 4.10: p-value Summary for Blocking Lattice with 400, $5 * 5$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	83.10%	0.816	0.001	0.093	0.368	0.193
Box test	97.18%	0.993	0.014	0.340	0.742	0.554

Table 4.10 shows the percentage of p-value larger than or equal to 0.05 for Box test exceeds 95% and lower than 95% for the AD test, indicating the lattice is tested as not normal under mean 0, but have no serial auto correlation. It is not white noise.

size $n * n = 5 * 5$**, number** $x = 2500$ **blocks**

Table 4.11: p-value Summary for Blocking Lattice with 2500, $5 * 5$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.001	0.001	0.001	0.001	0.001
Box test	0%	0.000	0.000	0.000	0.000	0.000

Table 4.11 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are 0, indicating the lattice is not normal under mean 0 and have serial auto correlation. It is not white noise.

size $n * n = 5 * 5$**, number** $x = 4900$ **blocks**

Table 4.12: p-value Summary for Blocking Lattice with 4900, $5 * 5$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.001	0.001	0.001	0.001	0.001
Box test	0%	0.000	0.000	0.000	0.000	0.000

Table 4.12 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are 0, indicating the lattice is not normal under mean 0 and have serial auto correlation. It is not white noise.

size $n * n = 5 * 5$, number $x = 6400$ blocks

Table 4.13: p-value Summary for Blocking Lattice with 6400, $5 * 5$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.001	0.001	0.001	0.001	0.001
Box test	85.7%	0.000	0.000	0.000	0.000	0.000

Table 4.13 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are 0, indicating the lattice is tested not normal under mean 0 and have serial auto correlation. The p-values for Box test are high because the blocks are occupying most of the lattice, which can be seen in Chapter 3. It is not white noise.

size $n * n = 10 * 10$, number $x = 25$ blocks

Table 4.14: p-value Summary for Blocking Lattice with 25, $10 * 10$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	7%	0.001	0.001	0.001	0.001	0.001
Box test	99%	0.985	0.019	0.407	0.819	0.670

Table 4.14 shows the percentage of p-value larger than or equal to 0.05 for Box test exceeds 95%, which is less than 95% for AD test, indicating the lattice is not tested normal under mean 0 but have no serial auto correlation. It is not white noise.

size $n * n = 10 * 10$, number $x = 100$ blocks

Table 4.15: p-value Summary for Blocking Lattice with 100, $10 * 10$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.001	0.001	0.001	0.001	0.001
Box test	0%	0	0	0	0	0

Table 4.15 shows the percentage of p-value larger than or equal to 0.05 for both tests is equal to 0% for AD test, indicating the lattice is not normal under mean 0 but have no serial auto correlation. It is not white noise.

size $n * n = 10 * 10$**, number** $x = 2500$ **blocks**

Table 4.16: p-value Summary for Blocking Lattice with 2500, $10 * 10$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.001	0.001	0.001	0.001	0.001
Box test	0%	0.000	0.037	0.000	0.003	0.001

Table 4.16 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are 0, indicating the lattice is not normal under mean 0 and have serial auto correlation. It is not white noise.

size $n * n = 20 * 20$**, number** $x = 1$ **blocks**

Table 4.17: p-value Summary for Blocking Lattice with 1, $20 * 20$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.001	0.001	0.001	0.001	0.001
Box test	2.40%	0.220	0.000	0.000	0.001	0.000

Table 4.17 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are close to 0, indicating the lattice is not normal under mean 0 and have serial auto correlation. It isn't white noise.

size $n * n = 50 * 50$**, number** $x = 1$ **blocks**

Table 4.18: p-value Summary for Blocking Lattice with 1, $50 * 50$ Blocks $\hat{\beta}$

	% of $p \geq 0.05$	max	min	1st quartile	3rd quartile	median
AD test	0%	0.001	0.001	0.001	0.001	0.001
Box test	0%	0	0	0	0	0

Table 4.18 shows the percentage of p-value larger than or equal to 0.05 for both AD test and Box test are 0, indicating the lattice is not normal under mean 0 and have serial auto correlation. It is not white noise.

Chapter 5

Conclusion

The only possible white noise is for null hypothesis true case, the lattice is always close to white noise. For existed lattice case, the lattice have no serial auto correlation but is not normal distribution with mean 0. For replicated matrix cases, when the size of replicated matrix increase, it got closer to the normality and no serial auto correlations. When the replicated matrix $n * n$>$100 * 100$, the lattices are normal with mean 0; however, all replicated matrix cases have serial auto correlation. For blocking matrix cases, when the size of the blocks increase, holding the number of the blocks constant, the lattices preform further from white noise. Holding the size of the blocks constant, when the number of the blocks increase, the lattices preform further from white noise. We only detected white noise are when the size and number of blocks is less or equal to $n * n = 5 * 5, x = 400$. Thant means when the blocks have small size and numbers, our MPLE method may fail detecting the pseudo-randomness and treat it as white noise. This is where our method need to be improved in the future.

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