DEVELOPING MIDDLE SCHOOL STUDENTS' MEANINGS FOR CONSTRUCTING GRAPHS THROUGH REASONING QUANTITATIVELY

by

HALIL IBRAHIM TASOVA

(Under the Direction of Kevin C. Moore)

ABSTRACT

In this dissertation study, I report on six middle school students' construction and interpretation of graphs and associated dynamic situations. Constructing and interpreting graphs represents a critical moment in middle school mathematics due to its opportunity to provide a powerful foundation for learning. Nevertheless, researchers have frequently reported on the challenges students experience in interpreting and making sense of graphs that ultimately affect their learning of many topics in algebra and calculus.

I explore the ways in which middle school students' graphing meanings involve quantitative and covariational reasoning. In particular, I conducted teaching experiments with each participant to engage them in activities intended to leverage quantitative and covariational reasoning with the goal of understanding and developing models of their thinking. In light of the essential role of quantitative and covariational reasoning in mathematical development, I designed tasks in order to understand opportunities for students to construct and reason with quantities' magnitudes in dynamic real-world situations. Based on my findings, I describe mental operations that constitute productive meanings for graphs, as well as those mental actions that constrain the students' graphing activity.

In my analysis, I identified various ways students organize one- and two-dimensional space to construct or make sense graphs within those spaces. I situate these different meanings in terms of representing a multiplicative object. Those meanings include representing (i) non-multiplicative object (iconic and transformed iconic translation), (ii) spatial-quantitative multiplicative object, and (iii) quantitative multiplicative object (Type 1 and Type 2). As part of my analysis of the teaching experiments, I also identify various ways of reasoning the students exhibited in dynamic situations including (i) quantitative covariational reasoning, (ii) spatial proximity reasoning, and (iii) matching the perceptual features of motion in two different spaces. Outlining those meanings and ways of reasoning can enable researchers and teachers to be more attentive to those meanings students might hold for their representational activity and to those types of reasoning students might demonstrate in dynamic situations. Based on my analysis, I also outline critical cognitive resources involved in developing a meaning for graphs as an emergent representation of two covarying quantities. The findings have important implications for research, teaching, and curriculum in terms of students' representational practices.

INDEX WORDS: Quantitative reasoning, Covariational reasoning, Graphs, Graphical meanings, Teaching experiment, Middle school students.

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DEDICATION

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CHAPTER 1

INTRODUCTION

In this introductory chapter, I present the rationale and significance of study, state the research goal and research questions that guided the study. Finally, I provide an overview of each of the subsequent chapters in this dissertation.

Statement of the Problem

Graphing is a critical aspect of understanding ideas in mathematics (Moore & Thompson, 2015, Thompson & Carlson, 2017). Moreover, constructing and interpreting graphs represents a "critical moment" in middle school mathematics for its opportunity to foster powerful learning (Dyke & White, 2004; Leinhardt et al., 1990). Graphing is also important on the upper elementary levels and middle school level of mathematics content because graphs are promising representations to support functional thinking and algebraic reasoning (Brizuela & Earnest, 2008; Kaput, 2008; Schliemann, Carraher, & Caddle, 2013).

Students experience a number of challenges in interpreting and making sense of graphs that ultimately affect their learning of many topics in algebra and calculus (e.g., Clement, 1989; Hattikudur et al., 2012; Leinhardt et al., 1990; Moore & Thompson, 2015). They tend to think about graphs as picture of situation, event phenomena, literal motion of an object, or concrete representation of an equation and function (Clement, 1989; Johnson et al., 2020; Monk, 1992; Moore & Thompson, 2015). For example, student's displayed graph might look like a hill because the situation that is being graphed involved a biker riding up and down that hill (Clement, 1989; Monk, 1992). As another example, students might draw a line graph because they notice an object in a situation travelling at a constant speed, even though distance and time are not the displayed quantities (Moore, Silverman et al., 2014; Stevens & Moore, 2016). Although each of these ways of thinking might enable students to produce correct solutions in isolated situations, these ways of thinking are not generative or flexible as it relates to students' development and learning.

There is a growing body of research (e.g., Carlson, 1998; Castillo-Garsow et al., 2013; Frank, 2017; Johnson, 2015; Johnson et al., 2017; Moore, 2016; Moore & Paoletti, 2015; Moore & Thompson, 2015; Paoletti & Moore, 2017; Paoletti et al., 2017; Ponte, 1984; Saldanha & Thompson, 1998; Stevens & Moore, 2016, 2017; Leinhardt et al., 1990; Thompson, 2011; Thompson et al., 2017) illustrating quantitative and covariational reasoning is a productive foundation for students' graphing meanings. One emergent theme in this research is the importance of constructing a multiplicative object to unite in the mind two or more quantities' attributes simultaneously (Frank, 2017, Saldanha & Thompson, 1998; Thompson & Carlson, 2017, Thompson et al., 2017). Another emergent theme is the importance of students conceiving a graph as an emergent trace of how those two quantities' values vary simultaneously in order to reason productively about the graph (see *emergent shape thinking* by Moore and Thompson, 2015). This meaning affords student the opportunity to make sense of graphs instead of conceiving of a graph as a shape qua shape, in which properties of the shape is associated with learned facts (see *static shape thinking* by Moore & Thompson, 2015).

Although there is well-established research on describing students' ability to interpret and construct graphs by (1) plotting points and scaling axes (e.g., Ponte, 1984; Wavering, 1981, 1983; Shaw et al., 1983), (2) incorporating ideas of slope and y-intercept (e.g., Hattikudur et al., 2012), (3) incorporating embodiment-based learning opportunities (e.g., Botzer & Yerushalmy, 2008; Radford et al., 2009; Nemirovsky et al., 1998), and (4) connecting with the other members of multiple representations of functions (e.g., Fonger, 2019; Knuth, 2000; Nitsch et al., 2015), little research on students' construction of graphs has focused on giving insight into the sources of (middle school) students' capacities to create a multiplicative object (Thompson et al., 2017) and construct graphs as emergent traces of this multiplicative object.

Other researchers have acknowledged the importance of understanding *situations* (e.g., some phenomena or experiential event) in interpreting and constructing graphs. For example, Janvier (1987) and Bell and Janvier (1980) reported that previous experience with the situation helped students made correct interpretations of graphs. There is research (e.g., Carlson, 1998; diSessa, Hammer et al., 1991; Kaput, 1993; Liang & Moore, 2017; Moore & Carlson, 2012; Paoletti, 2015; Stevens et al., 2015) showing the role of conceiving situation quantitatively in mathematical development. Extending this previous work, in this dissertation, I investigate how students conceive the situation as composed of quantities and their relationships as their values or magnitudes vary and how they re-present this covariational relationship between two quantities when constructing and interpreting graphs.

There are studies illustrating how middle school students reason and make sense of an invariant relationship between two quantities (Ellis, 2007a, 2011; Ellis et al., 2013, 2015; Johnson, 2012, 2015a, 2015c) In order to characterize students' meanings, the authors of these studies primarily focused on students' reasoning with (a) *numbers* in a tabular representation that are either connected to quantities (e.g., Ellis, 2007a; Ellis et al., 2013, 2015; Johnson, 2012, 2015a, Johnson, & McClintock, 2017) or disconnected from quantities (e.g., Noble et al., 2001), (b) coordinate *values* on a graph or displaying graphs on a grid paper (e.g., Ellis, 2011; Johnson, 2012, 2015a, 2015c; Johnson, & McClintock, 2017), or (c) *values* of the quantities' in a situation (e.g., Ellis, 2007a; Johnson, 2012; Johnson, & McClintock, 2017; van Reeuwijk and Wijer, 1997). Differing from those authors' focus, there is an emergent body of research (e.g., Liang & Moore, 2020; Liang et al., 2018; Tasova et al., 2019) illustrating how students reason and make sense of relationships between two quantities' *magnitudes*, independent of numerical *values*. Those characterizations have been powerful for providing empirical examples that illustrate the effective and powerful role of students' engagement in quantitative and covariational reasoning to understand the intensity of change in quantities' magnitudes. With respect to this latter focus on magnitudes, there is a need to expand the extant body of literature by demonstrating the ways in which students conceive of covariational relationships between quantities' magnitudes represented in dynamic situations and graphs. In this dissertation, I investigate how middle school students construct and represent quantitative and covariational relationships, including intensity of variation, between quantities' magnitudes.

As another motivation for the present study, traditional school curriculum does not support students conceiving a graph as an emergent trace of how two quantities' values covary simultaneously (Paoletti, Rahman et al., 2017; Thompson & Carlson, 2017). This is problematic because students are not afforded intentional opportunities to develop a meaning for graphs that researchers have illustrated to be productive for their learning. In the context of graphing, for instance, school curricula typically emphasize perceptual and indexical feature of graphs such as concavity of a curve, slantiness of a line, intersection points, locations of peaks and valleys of the curve, and name-shape pairs. These features of graphs are restricted to a particular type of representational system (e.g., conventional Cartesian plane) and do not always yield productive and generative meanings for graphs (Byerley & Thompson, 2017; Stevens & Moore, 2016, 2017; Tasova & Moore, 2018). In addition to these features of a graph being restricted in their applicability, meanings emphasizing these features can inhibit students' capacity to reason about graphs in terms of their covariation (e.g., Bell & Janvier, 1981; Carlson, 1998). Therefore, in this dissertation, I investigate the processes of students' constructing and interpreting graphs and determine the instructional conditions that foster and support productive meanings for graphs.

Also motivating the present study, the body of research emphasizing the role of quantitative and covariational reasoning in the context of graphing as an emergent trace has mostly focused on college level students, pre-service teachers, or teachers. Limited research is available that investigates how younger students (e.g., middle school students) use quantitative and covariational reasoning in constructing and interpreting graphs, especially with sensitivity to the results reported with other participant demographics. Researchers (e.g., Thompson et al., 2017) have hypothesized that adults (i.e., high school and college students, pre- and in-service teachers) have greater difficulty than younger students initially engaging in covariational reasoning to develop productive meanings for graphs, partially due to their experiences with traditional curricula and instruction (Moore & Silverman, 2015; Moore et al., 2018). This message adds to the need of understanding what ways of thinking can support younger students to develop productive meanings for graphs without being constrained to those "conventions" associated with traditional curricula and instruction. There is also a need to understand how and to what extent an instructional emphasis on certain constructs (e.g., quantitative structures, multiplicative objects, and frame of reference) in earlier grades is productive for students' abilities of construction and interpretation of graphs.

Research Questions

It is important to understand what students know including the power and utility of their quantitative and covariational reasoning at younger age levels after goal-oriented instruction.

Therefore, the purpose of this study is to explore the ways in which middle school students' graphing meanings involve quantitative and covariational reasoning. In particular, I will engage students in activities intended to leverage quantitative and covariational reasoning with the goal of understanding and developing models of their thinking. I will describe the possible mental operations that constitute productive meanings for graphs, as well as those mental actions that constrain the students' graphing activity. I address the following research questions:

1. What ways of reasoning do middle school students enact when engaged in graphing activities intended to emphasize quantitative and covariational reasoning?
2. What ways of reasoning are involved in students developing productive meanings for graphs (e.g.., emergent shape thinking)?

Graphing, quantitative reasoning, and covariational reasoning are all critical to understanding middle school mathematics. Moreover, these ideas are critical for success in STEM fields (Glazer, 2011; Gonzales, 2019; Panorkou & Germia, 2020). Thus, working with middle school students to understand their thinking and how to support these critical reasoning processes is an important step to making improvements in mathematics education.

Overview of Dissertation

In this chapter, I presented the statement of the problem, the significance of this study, research goal and research questions. In Chapter 2, I present the theoretical perspective for the study and discuss the relevant research literature on learning and understanding graphs. In Chapter 3, I discuss the methodology I used in my study, present specific methods used in the teaching experiment, and describe an overview of the task design. In Chapter 4, I provide results from the analysis of my teaching experiments exploring students' meanings of points on a plane. I describe these different meanings in terms of representing a multiplicative object in the context of graphing. I also identify various ways of reasoning the students exhibited when they engage in dynamic situations. In Chapter 5, I outline the key developmental points involved in developing meanings for graphs from the teaching experiment as I implemented an instructional sequence to support students in developing emergent shape thinking. In Chapter 6, I first summarize and discuss the findings I presented in Chapter 4 in answering my first research question. I then synthesize the findings I presented in Chapter 5 in answering my second research question. In this chapter, I use a broader lens to look across all students I included in the results in order to outline critical cognitive resources for students' constructions of productive meanings for graphs. I also describe implications these results have for research, teaching, and curriculum. I then conclude with areas for future research directions.

CHAPTER 2

THEORETICAL PERSPECTIVE AND LITERATURE REVIEW

In this chapter, I present the theoretical perspective for the study. I discuss the principles of Glasersfeld's radical constructivism (1995) and present how it guided my study. In this chapter, I also discuss the relevant research literature on learning and understanding graphs from (i) multiple representations, (ii) embodied cognition, and (iii) quantitative reasoning perspectives.

Theoretical Perspective

My theoretical perspective is grounded in my interpretations of von Glasersfeld's (1995) radical constructivism. I choose radical constructivism as my theoretical perspective not just because I want to look at *individual*s' ways of thinking, but also because the assumptions that radical constructivism afforded me to make regarding students' learning provide a productive perspective to look with. In addition to the principles described below, this is evidenced by the number of research programs on individuals' ways of thinking conducted within this perspective (e.g., Cifarelli & Sevim, 2014; Ellis, 2011; Fonger et al., 2019; Hackenberg, 2010; Norton, 2009; Tillema, 2016; Ulrich et al., 2014) as well as its fit with my desire to understand and model the mathematical realities of others.

One of the principles in radical constructivism asserts that the function of cognition serves the organization of the experiential world, rather than the discovery of ontological reality. That is, radical constructivism rejects the notion that an individual can construct an ontological reality (Glasersfeld, 1995). The assumption that "there is a pre-given objective reality" relies on a view—referred to objectivism—that has a long history in the traditional fields of cognitive science and philosophy of mind (Núñez, 1997). According to objectivism, (i) reality has a structure that is observer-independent, in turn, objective (Hardy & Taylor, 1997, p. 136) and (ii) individuals can attain knowledge that is congruent with reality's objective structure (von Glasersfeld, 1986). I believe this view has been partially challenged by radical constructivism. I use the term "partially" because radical constructivism does not deny a reality external to human experience (von Glasersfeld, 1991, 1995), but asserts that individuals do not have a way of knowing what it is objectively, so, it is unknowable to anyone in any way (von Glasersfeld, 1995, Thompson, 2000). von Glasersfeld and Cobb (1983) used the analogy between reality and a "black box" that is something "whose internal structure and functioning is forever inaccessible to the human knower" (p. 5–6). Thus, radical constructivism denies that we can understand "reality" in the sense that traditional epistemologist claims as we do not have "the God's-eye view" (von Glasersfeld, 1991, p. 6).

Specific to my study and reflecting the notion that an individual cannot construct an ontological reality; I cannot know what conceptual structures my students have associated with certain graphs. I believe that someone else's knowledge is fundamentally unknowable to me or to another person. As von Glasersfeld (1984, p. 9) stated "we can know only what we ourselves construct". The best I can do is build models of the student's mathematics (i.e. the student's mathematical realities which is not isomorphic to *my* models) that provide viable explanations for the student's words and actions. For example, as a researcher relying on radical constructivism, I can only conjecture what a graph might mean to students when they use it since I believe the graphs that they physically produced are the results of the process of their cognitive structures. Because I cannot directly observe their cognitive structures, I make *inferences* from observation by interpreting my students' written work, language, and actions including a facial expression, a gesture, a posture, or eye movement (Steffe, 1991; Steffe & Thompson, 2000; von Glasersfeld, 1995). In another words, I can try to understand (i.e., develop model of student thinking) how their conceived graph looks like by only investigating their displayed graphs along with their reasoning that comes with their displayed graph. As I come across the certain graphs again and again in a student's work, I can try to modify or reconstruct my models of the student thinking with a purpose of arriving an interpretation that fits most of the occurrences. That is, my goal in this study is to build, test, and refine models of each of my student's mathematics by relying on my interpretation of their actions and behaviors. These models of students' mathematics should not be interpreted as isomorphic to the students' thinking and reasoning. Instead, these models should be considered as models that are viable in explaining students' cognitive processes behind their actions.

Another principle in radical constructivism asserts that an individual actively constructs his/her knowledge through his/her sensory experiences, physical actions and mental operations, and reflections on these experiences and actions/operations (von Glasersfeld, 1995). "It is we who are responsible for the world we are experiencing" (von Glasersfeld, 1990). The individual engages in active participation in the construction of knowledge and through the construction of knowledge, the individual actively organizes his or her experiential reality (von Glasersfeld, 1995). From this perspective, what a radical constructivist call reality is "the domain of the relatively durable perceptual and conceptual structures which we manage to establish, use, and maintain in the flow of our actual experience" (von Glasersfeld, 1995, p. 118). A knowledge that an individual attains, therefore, cannot represent a picture of objective reality (von Glasersfeld, 1991), instead, it refers to action schemes that have proved viable in the individual's experience (von Glasersfeld, 1995).

An action scheme is a goal-directed mental activity consisting of three parts: (1) recognition of a certain situation, (2) association of a specific activity with that situation, and (3) the expectation of a certain result (von Glasersfeld, 1995, 1996). Recognition of a certain situation occurs as a result of *assimilation* as the individual recognizes certain sensory material as an instantiation of a previous experience in some way. That is, in the process of assimilation, the individual activates a scheme that has proved viable in her prior activity when she recognizes new situations (von Glasersfeld, 1995). The extent to which an individual recognizes or perceives situations relies on the individual's current cognitive constructs. That is, during the assimilation process, the individual may not be aware of other available sensory material in the new situation that does not fit into his or her existing schemes (von Glasersfeld, 1995). This might lead the individual recognize a certain situation as an instantiation of a previous experience although the new situation is quite different than the previous situations from an observer's view. For example, a 3-year-old (referring to my daughter in particular) quickly learns that a displayed video on a tablet screen can start playing when touching the play icon on the screen, and this provides the kid with the ability to play the video when desired and when the sensory materials are available. This is an example of maintaining equilibrium as it refers to a state in which the kid's cognitive structures have yielded and continue to yield expected results (von Glasersfeld, 1996).

When placed in front of what I perceive to be a computer monitor, the same kid would raise her finger and touch on the play icon on the screen that displays a paused video. Here, the assimilated situation of the scheme (i.e., presence of the play icon on the computer monitor—

which may be perceived as a tablet by the kid) triggers the activity of the scheme (i.e., touching). I, as an observer, know that the computer screen is not touchable and may say that the kid is assimilating it to her touch screen scheme to play videos on a touchable device. Note that, from the kid's perspective at that point, the monitor is a touchable item, because what the kid perceives of it is not what I would consider the features of an untouchable computer screen, but those aspects (e.g., a play icon on a screen) that fit the touching scheme to play a video on a device. Then, the specific activity associated with the recognized situation produces a result, which the individual assimilates to her expectations. That is, the individual expects that the results of the activity to be satisfying and meeting her expectations in some way (von Glasersfeld, 1995).

If the individual does not achieve her intended goal (i.e., if the result of a scheme is unexpected or if the result does not meet the person's expectations), this generates a *perturbation* (von Glasersfeld, 1995, 1996). According to radical constructivism, cognitive change and learning can happen when a goal-directed mental activity leads to perturbations, instead of producing the expected result. In the moment of perturbation, the individual feels disappointment or surprise. This may lead the individual to eliminate the perturbation, which has the potential to lead to the individual making an *accommodation* (i.e., reorganizing and constructing a scheme) that might maintain or establishes a new equilibrium (von Glasersfeld, 1995).

In my example of a 3-year-old, touching a computer screen does not yield playing the video; thus, the activity does not produce the result the kid expects. The kid's current conceptual structures are no longer viable. This generates a perturbation (e.g., the kid may be disappointed) that could set the stage for cognitive change. This perturbation may focus the kid's attention to the features of the sensory material that activated the scheme and may lead the kid to review the characteristics of the material that were not initially perceived when assimilating (von Glasersfeld, 1995). This may lead the kid to perceive some other aspects that may enable the kid to recognize what I perceive to be computer monitors as non-tablets in the future. Thus, the kid constructs new knowledge (a modest one in this case) as a result her efforts of organizing her physical and cognitive experiences by maintaining and extending equilibrium in response to a perturbation.

In my study, by engaging in experiencing their mathematical activities throughout teaching experiments, I investigate individual student's thinking to understand how they build up certain conceptual structures (Steffe & Thompson, 2000) about graphs. In my empirical research, I am primarily interested in investigating the functioning of the mind; thus, as a radical constructivist researcher, I focus on mental operations of individuals (von Glasersfeld, 1995). In my research study, I also aim to understand what ways of thinking can support students to develop productive meanings for graphs in terms of an emergent representation of a relationship between two varying quantities. So, I intend to provide students with opportunities to participate in physical and mental actions and reflect on these actions such that mathematical meanings can be constructed.

Literature Review

Researchers investigating students' graphing activities have adopted multiple perspectives including, but not limited to, graphing motion in the context of embodied learning, multiple representations, and quantitative reasoning. In this section, I first briefly review the research from different perspectives as it relates to students' construction and interpretation of graphs. Then, I provide a detailed review and synthesis of the research on graphing from a quantitative and covariational reasoning perspective. Within this synthesis, I illustrate the shape thinking construct due to its central role in my research. Finally, I reflect on studies from multiple perspectives in the light of my review of quantitative reasoning literature including that of shape thinking.

Graphing literature from Embodied Cognition perspective

Researchers adopting an embodied cognition perspective have focused on embodied mathematics activities and learning environments that support students' understanding of graphs. Researchers who provided embodiment-based learning opportunities for students have often been interested in students' understanding of graphing motion (e.g., distance-time graphs; Duijzer et al., 2019) based on the premise that physical motion plays an integral part of understanding the meaning of mathematical representations (e.g., Nemirovsky et al., 1998; Noble et al., 1995). They also believe intertwining different cognitive, physical, and sensuous (e.g., perceptual, aural, tactile) modalities and resources are the main mediating factors to help students construct mathematical meanings in the course of learning (Radford et al., 2017). These modalities include various kinds of bodily motion: seeing, touching, drawing, gesturing, the manipulation of physical and computerized artifacts (Botzer & Yerushalmy, 2008; Radford et al., 2009; Duijver et al., 2019). The most common approach among researchers when designing embodied-leaning environment has been intentionality to activating seeing and motor action at the same time (Duijzer et al., 2019).

Most of the studies from this perspective have incorporated motion sensor technology or technology called Micro-Based Laboratory (MBL) to enable students creating an *immediate* representation of motion as a graph (Duijzer et al., 2019). In their graphing activities reported in the embodied literature, students either examined their own motion—whole body (e.g., Nemirovsky et al.,1998; Radford, 2009) or part of their bodies (e.g., Botzer & Yerushalmy, 2008)—or others' motion (e.g., Ferrara, 2014; Kaput & Roschelle, 1998; Stroup, 2002; Noble et al., 2004) in order to determine a *relationship* between their kinesthetic sense of motion and graphs. I infer that researchers (e.g., Botzer & Yerushalmy, 2008; Deniz & Dulger, 2012; Stroup, 2002; Nemirovsky et al., 1998) draw students' attention to the mentioned relationship in terms of the quality/nature of the motion and perceptual features of the graphs (e.g., straight, concave down/up) or "height" of the graph. That is, the focus is on making sense of the shape of the graph in relation to the specific changes in the motion event (e.g., changing speed or direction). For reference, I refer to this relationship as a *shape-motion association*. Next, I illustrate the different versions of shape-motion associations in the embodied literature from different sub-perspectives (i.e., semiotic-cultural approach, phenomenological approach, and cognitive linguistic approach) as identified by Radford et al. (2017).

Semiotic-cultural approach

Drawing on Vygotsky's work, researchers (e.g., Andrá et al., 2015; Arzarello, 2006; Botzer & Yerushalmy, 2008) drew attention to the students' semiotic activity in teaching and learning mathematics where signs—as mediating entities of thinking (Arzarello, 2006)—including gestures and other embodied resources are explicitly used to signify meaning (Duijzer et al., 2019; Radford et al., 2017). In this perspective, a Cartesian graph is considered as "a complex mathematical sign whose objective cultural meaning was elaborated in the course of centuries" (Radford et al., 2005). Relatedly, in this perspective, learning is described through the evaluation of signs that includes a process of alignment between the subjective meaning of signs and their cultural, objective meanings (i.e., objectification of knowledge, Radford, 2006; Radford et al., 2005).

For example, Botzer and Yerushalmy (2008) presumed that a certain shape of a graph (e.g., concave down curve) has a culturally accepted meaning that conveys how motion of an object should occur. Their purpose was thus to mediate students in the transition process between their own meanings of graphs and historically constituted cultural meaning of graphs. Moreover, they reported that linking students' bodily actions with *formal* sign (e.g., representing the graphs' mathematical features through gesturing and computerized artifacts) enabled students to construct and elaborate on *the* meaning of graphs. For example, Botzer and Yerushalmy designed a mediating activity, called "the graphlets exploration activity," where students explored a relationship between the shape and the features of a motion event (i.e., a *shape-motion association*). For example, students were asked to match certain shapes (e.g., straight line, concave up/down curves, e.g.,  and ) with verbal description of motion situations (e.g., free-fall motion). Students, for instance, made association between increasingly concave downward shape and the verbal description of "increasing velocity in the positive direction." Moreover, Botzer and Yerushalmy considered the concrete action of placing the pencil as a tangent to the curve as a mediating factor to develop culturally accepted meanings of graphs.

Nemirovsky and collaborators' phenomenological approach

Different from the semiotic-cultural perspective, by focusing on the actor's emergent *experience* but not under an explicit guidance or a known goal, researches from the phenomenological perspective are interested in investigating how students' bodily activities are related to *their* understanding of functions and graphs (e.g., Nemirovsky & Ferrera, 2009; Nemirovsky et al., 1998; Nemirovsky & Monk, 2000). Nemirovsky and his collaborators do not acknowledge the dichotomies, such as signifier and signified. Nemirovsky et al. (1998) stated that "the meaning of symbols is to be found neither in the specific thoughts that they express nor in the objects to which they refer but, *in their use* [emphasis added], that is, in the practices they serve" (p. 123). For example, Nemirovsky et al. described a construct called *fusion* in order to convey their students' talking, gesturing and understanding of the graphs in ways that do not distinguish between symbols and referents. Nemirovskey et al. explained that students' description *fuses* qualities of graphs with qualities of the signified events, and they (e.g., Nemirovsky & Monk, 2000; Nemirovsky et al., 1998) presented empirical examples of students linking the shape of the graph to the phenomenon it is modeling.

To illustrate, when engaging in creating graphs generated by a motion detector that showed how a handheld button's distance from the detector varied over time, Nemirovsky et al. (1998) reported that students determined "the line on the screen becomes higher when one moves the button away from the tower and lower when one moves the button" (p. 122). Nemirovsky et al. explained a particular student's activity as follows:

> …she blended a quality of her body motion (forward) while she gesturally stressed a quality of the graphical shape (downward). What Eleanor was pointing at was not on the computer screen or in the room but was both at once; it was not in her present or in her past but was simultaneously both. (p. 148)

In an instance with another student, when asked about the student's hand motion in re-interpreting a graph that has an upward shape, Nemirovsky et al. reported that the student said, "I just moved it up really fast, but close" (p. 142). Fusion in this example is merging the quality of the graph (upness) with qualities of the student's hand motion (closeness). With this example along with additional students' descriptions of their interpretation of the motion graphs (e.g., "the closer I go, the lower it goes," p. 155), the authors pointed out that students developed a sense that their closeness to the motion detector was indexed by the "height" of the graph. Thus, students' interpretation of graphs included an indexical association between a characteristic of the motion and the shape of the graph.

Although they don't explicitly incorporate the notion of fusion, there are other researchers (Noble et al., 2004; Stroup, 2002;) who have used real-time motion and graphing motion in developing mathematical concepts (e.g., amount of change and rate of change) by making an association between the characteristics of the motion and the shape of the graph. For example, consistent with the other studies' result (e.g., Nemirovsky et al., 1998; Monk & Nemirovsky, 1994), Stroup (2002) noted that students interpret the highest point of a parabola that opens downward (e.g., a graph that shows the relationship between sunlight energy arriving as a function of month in a year) as showing both where a how much (extensive) and a how fast (intensive) quantity are the greatest. For instance, when asked "when the most sunlight energy is arriving," students pointed to the peak point whose x-coordinate is aligned with the month of June. When asked "where the sunlight energy is changing the fastest/slowest," students point to the part of the graph that goes up/down indicating where the sunlight energy changes faster/slower. Since the ideas of amount and rate seem to be confounded in students' interpretations of the graph, Stroup illustrated ways to help students to develop an intensive notion of how fast (developing a "qualitative" sense of rate) both in a how much graphical context (i.e., position as a function of time) and in a how fast graphical context (i.e., velocity as a function of time).

Stroup introduced the *Kine-Calc* program[1] that simulates a real-time motion where students create their position vs. time graphs that determine a creature's movement (see X in Figure 2.1a) on the screen. Stroup engaged students in this task with a goal of helping them to understand the essential characteristics of representing movement. Figure 2.1b shows an example

[1] Turtle Time Trials in Desmos illustrates similar ideas. See the link: https://teacher.desmos.com/activitybuilder/custom/5da9e2174769ea65a6413c93

of an eight-grade student’s response that was claimed to be a representative case illustrating the important features of understanding “qualitative calculus” in how much graphical context.

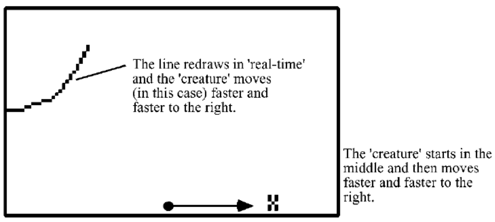


Figure 3. Pressing ENTER, a character appears and moves according to the sketched position vs. time graph.

(a)

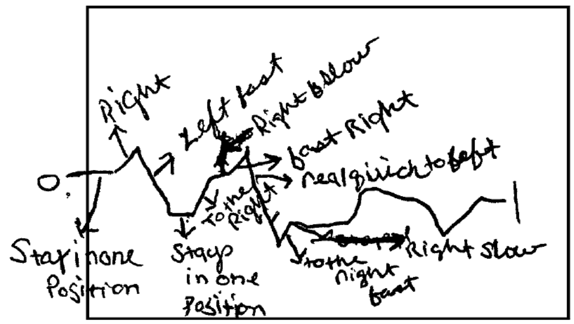


Figure 4. An eighth-grade student’s description of how the “creature” moves for a position vs. time graph.

(b)

Figure 2.1. (a) Figure 3 from Stroup (2002, p. 179), and (b) Figure 4 from Stroup (2002, p. 180).

The student associated the sections of the graph with movement of the creature. For instance, the student associated sections of the graph slanted upward/downward from left to right with movement to the right/left. No matter the length of the line segments of the graph, a “steeper” up section of the graph (labelled “to the right fast”) is associated with the movement of the creature that is faster in a positive direction by the student. Similarly, “steeper” downward is associated with faster in a negative direction. The authors claimed that the simulation environment helped students to interpret graphs in a way that the ideas of amount and rate are not confounded.

As part of the SimCalc Project in his dissertation, Stroup (1996; as cited in Stroup, 2002) developed an approach called the *delta-blocks* approach in helping elementary and middle school students to learn “mathematics of change” where they started from the integral and moved to the derivative (Kaput, 1994) by emphasizing the area under a velocity graph. The main feature of this approach, in general, was to help students to investigate how the arrangements of blocks (e.g., unifix cubes) is related to the motion of an object. Stroup’s proposed process has four steps. First, students create “walls” by arranging the blocks with constant or varying heights (see Figure 2.2a). Second, students transform these walls into graphs on paper. Then, students transition these graphs to either a calculator-based (i.e., *Baby MathWorlds*) or a computer based (i.e., SimCalc *MathWorlds*, Kaput & Roschelle, 1996, see Figure 2.2b) simulation environment. In this approach, a graph is perceived as the top edge of the arrangement of blocks. Then, students use these graphs to move an elevator simulation (see Figure 2.2b). Therefore, students explore the relationship between the motion of the elevator and the arrangement of the blocks. In this approach, the horizontal axis is perceived as the dividing line between positive and negative directions of motion (Noble et al., 2004). Using this approach, Stroup (2002) noted that students engaged rate in a qualitative way by discussing how fast the elevator moved as determined by the height of the graph.

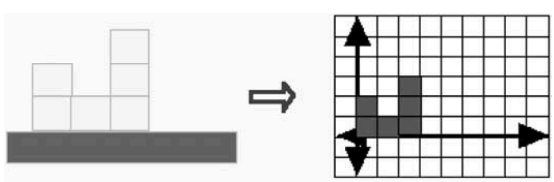


Figure 7. Side view of blocks is then transferred to graph paper.

(a)

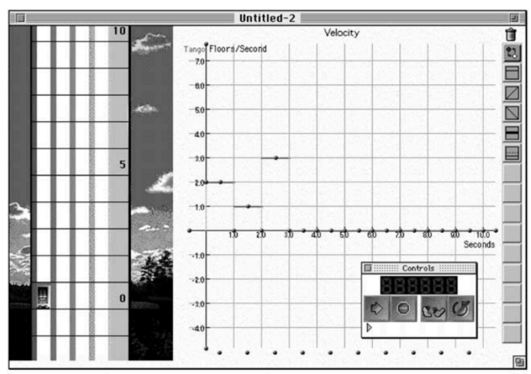


Figure 8. The same six-block pattern after being transferred to the *MathWorlds* software.

(b)

Figure 2.2. (a) Figure 7 from Stroup (2002, p. 185) and (b) Figure 8 from Stroup (2002, p. 185).

Noble et al. (2004) also used delta-blocks approach in order to help 6th grade students *learning to see* mathematical representation of motion in terms of its relation to the velocity vs. time graphs. They illustrated the progress in which students’ meanings of the graph changed over time. They reported that students initially saw the shape of the graph in MathWorlds screen as the shape of the building that the elevator travels in. In later episodes, as they developed their competence at interpreting and constructing velocity vs. time graphs, students saw the shape of the graphs as a representation of the total number of floors traveled by the elevator.

Cognitive linguistic approach inspired by Lakoff and Núñez

Lakoff and Núñez (2000) asserted that individuals conceptualize abstract concepts, via a mechanism called *conceptual metaphor*, in terms of concrete concepts. They stated that conceptual metaphor allows people to conceive one thing (e.g., numbers as abstract concepts) as if it were another (e.g., points on a number line). For example, as also pointed out by David, Roh, and Sellers (2019), using the conceptual metaphor for each value of the pair, students conceive points on a graph in a Cartesian plane as a pair of values each of which map into the physical positions as points on each axis of the plane (Lakoff & Núñez, 2000).

Emphasizing the metaphoric dimension of language, Lakoff and Núñez (2000) provided another example of the role of conceptual metaphor in making sense of graphs by using the *fictive motion* mechanism—first described by Talmy (1996; as cited in Lakoff & Núñez, 2000). In fictive motion, static objects can be conceptualized in dynamic terms, as in sentence like "The road *runs* through the woods" or "The fence *goes* up the hill" (Lakoff & Núñez, 2000, p. 38). By using fictive motion, someone can conceive a static graph of a function in the Cartesian plane as having motion tracing that graph and directionality (see also Botzer & Yerushalmy, 2008; Ferrara, 2014; Font et al., 2010). For example, someone may describe a graph as "going down", "reaching a minimum" and "going up" or may think of two lines as "*meeting* at a point" (Lakoff & Núñez, 2000, p. 38–39).

In fact, fictive motion is a manifestation of a bigger embodied (image) schema called *Source-Path-Goal schema*, which is described as "fundamental cognitive schema concerned with simple motion along trajectories" (Núñez, 2009, p. 314). Some of the researchers from embodiment literature (e.g., Font et al., 2010; Lakoff & Núñez, 2000; Malaspina & Font, 2009) highlight and confirm the importance of Source-Path-Goal schema in conceptualizing the graphs in terms of motion along a path through the metaphorical mapping of this schema. See Figure 2.3 for the elements of this schema.

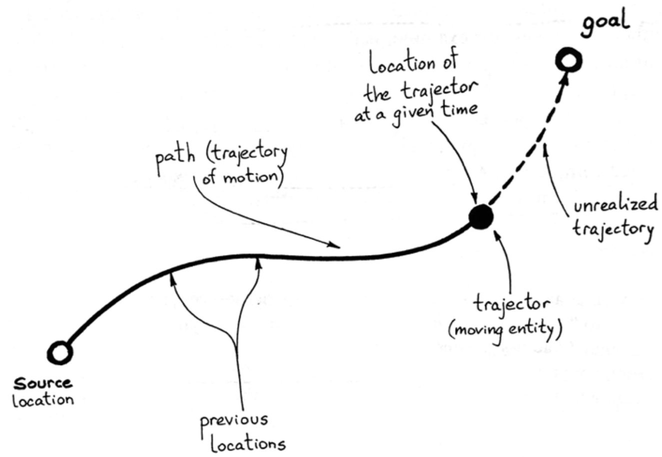


Figure 2.3. Figure 2.2 in Lakoff and Núñez (2000) illustrating a conceptualization of a linear motion using a conceptual schema in which there is a moving entity (called a trajector), a source of motion, a trajectory of motion (called a path), and a goal with an unrealized trajectory approaching that goal (p. 37).

To illustrate the elements of the source-path-goal in the context of graphing, I present the following example from Núñez (2009). Drawing on the source-path-goal schema, Núñez provided an analysis of a mathematics professor's (Guershon Harel) lecture about a mathematical proof as part of which he drew a line of the function $y = x$ on a chalkboard and gestured in front of the line. Núñez illustrated how Harel's embodied speech-thought-gesture was co-produced with the enactment of a source-path-goal scheme as follows:

> As he says "increasing," he gestures with his right hand (palm down), with a wavy upward and diagonal movement (slightly along the line $y = x$). This gesture is co-produced with the enactment of a source-path-goal schematic notion, where the source corresponds to a generic location $(x, f(x))$ with low but positive values of x and $f(x)$, the

> goal of the instantiated schema corresponds to a generic location (x, $f(x)$) with greater values for both x and $f(x)$, the trajectory is indexed by the external edge of the right hand, and the trajectory (path) is the trace left by the motion of the hand that is indexing the increasing values of the function as x gets increasingly greater values. (p. 320)

There are two different conceptualizations of graphs in this perspective: (1) one includes thinking of a graph representing a path of a continuously moving object in the plane and (2) the other includes thinking of a graph as a collection of points (as a pair of values each of which map into the physical positions on each axis of the plane). According to Lakoff and Núñez (2000), this distinction relies on the distinction between two very different conceptualization of space—*naturally continuous space* and *the-set-of-points conception of space*. This distinction was based in the difference between classical mathematics (prior to the mid-nineteenth century) and modern mathematics (after Descartes' invention of analytic geometry). They claimed that naturally continuous space is "our normal conceptualization" because we function in everyday world by using our body and brain unconsciously. On the other hand, the-Set-of-Points conception of space is consciously constructed for particular purposes (e.g., to meet certain axioms such as dimensionality and curvature). They considered the second one as a reconceptualization of the first, via conceptual metaphor (i.e., "Spaces Are Set of Points" metaphor, Lakoff & Núñez, 2000, p. 263). This reconceptualization process is called "the program of discretization" (p. 261) that is a strategy to replace naturally continuous space with infinite set of points for the sake of arithmetization, symbolization, and formalization to move from *intuitive* to *rigorous* mathematics.

Lakoff and Núñez laid out the inconsistent features of each conceptualization of space as follows:

> In one, space, lines, and planes *exist independently of* points, while in the other they *are constituted by points*. In one, properties are inherent; in the other they are assigned by

> relations and functions. In one, the entities are inherently spatial in nature; in the other, they are not. (p. 265)

For example, a circle is a spatial object independent of the points (like the path traced by a moving point) in a space in terms of the natural continuous space. Whereas a circle does not exist without considering points that constitute itself in terms of the Set-of-Points conception of space. In this conceptualization, a circle is a subset of points in a space, with particular relations to one another (i.e., a circle is a set of points in a plane that are a given distance [i.e., radius length] from a given point [i.e., center]).

Graphing literature from Multiple Representations perspective

Researchers adopting multiple representations perspective (e.g., Bixler, 2014; Martinez-Cruz, 1998; Nielsen & Bostic, 2018; Rocha, 2016; Van Dyke & White, 2004) see graphs as *a* member of multiple representations of functions (e.g., tables, equations, graphs and verbal descriptions), which have been considered equivalent in some way but not equivalent in other ways. For example, some researchers claimed that graphs are more global (universal) representations that enable students to read information about the shape and the direction of the relationship between quantities (Ainsworth et al. 2002), whereas tables (Ainsworth et al., 1998) and algebraic representations (Friedlander & Tabach, 2001) are the most effective ways to look for regularities and make generalizations. For similar reasons, Andra et al. (2013) stated that students perceive graphs "in a more holistic and condensed (synthetic) manner" (p. 17) and claimed that interpreting graphs does not require students "unpackaging (like formulas)" (p. 7) in order to understand the specific relevant elements of the representations (e.g., an exponent, a sign, or a coefficient). Looking from a somewhat different perspective, Nitsch et al. (2015) stated that graphs afford engaging in covariation (i.e., how two quantities change) whereas algebraic representations afford engaging in correspondence (i.e., for every x-value there is a y-value).

Knuth (2000) argued that multiple representations of functions are not computationally equivalent. That is, he stated that students can easily see infinitely many points in a graph; however, this is not an option in an algebraic representation; the points are given implicitly that they need to be found by computation. For example, Friedlander and Tabach (2001) stated that students can easily find the solutions of a polynomial function given its graph regardless of its degree even when they don't know an analytical way to find the solutions.

In this perspective, researchers often investigated students' ability to make a connection among multiple representations. Those researchers asserted that the integral part of conceptual and robust mathematical understanding of functions requires students to create, select, interpret, and flexibly move with ease between and within representations (Dreyfus et al.,1997; Moschkovich et al., 1993; Yershalmy & Schwartz, 1993). Note that the mentioned ability in general is named differently by researchers (i.e., *representational fluency*, Fonger, 2019; Lesh, 1999; White & Pea, 2011; Zbiek et al., 2007; *representational flexibility*. Acevedo et al., 2009; *coordination of multiple representations*, Chang et al., 2016; or *representational versatility*, Thomas, 2008).

According to the literature, there are two different factors involved in connecting multiple representations depending on the different perspectives in which students viewed functions. These two factors are distinguished as *point-wise approach* and *global approach* by Even (1998), and as *process perspective*[2] and *object perspective* by Moschkovich, et al. (1993)—those perspectives coincide respectively.

[2] Note that this process perspective is different than Thompson and Carlson's process view of a function. This process perspective is similar to Confrey's correspondence perspective.

Point-wise approach/Process perspective

"From *the process perspective*, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value" (Moschkovich et al., 1993, p. 71). Based on this perspective, Moschkovich et al. (1993) and Schoenfeld et al. (1993) identified a particular way—called Cartesian Connection—of understanding the connection between an equation and a graph. In this understanding, the main criteria for connecting both representations is as follows: "A point is on the graph of the line L if and only if its coordinates satisfy an equation of L" (Moschkovich et al., 1993, p. 73). By investigating students' abilities to use a particular aspect of the Cartesian Connection, Knuth (2000) argued that the integral part of developing the flexibility to move among representations is to understand the connection that the coordinates of any point on a graph will satisfy the equation.

Even (1998) identified a similar type of connecting the multiple representations, called a point-wise approach, where students only used specific points on a function's graph when asked to create an equation. For example, Even provided students a graph of the function $sin(x)$ and asked them to find the symbolic expressions of the functions whose graphs were given as a modified/transformed version of the original graph of $sin(x)$ that they were given (e.g., one graph was the stretched version of $sin(x)$ by a factor of two in the horizontal direction). Even reported some students checked some particular points on the graph of $sin(x)$ and on the other graph in order to figure out an equation using the same function family (i.e., sine function).

Moreover, in the process perspective, the steepness and the orientation of linear graphs was determined by the ratio of the directed line segments, which are calculated by $y_2 - y_1$ and $x_2 - x_1$for any two points on a line $(x_1,\ x_2)$ and $(y_1,\ y_2)$ (Schoenfeld et al., 1993), which was considered as conceptual understanding of the slope. However, Hattikudur et al. (2012)

challenged this understanding by reporting students' difficulties in graphing by focusing on only particular coordinate values. Given a verbal description of a situation, Hattikudur et al. investigated student's ability to construct graphs. They reported that some students constructed non-linear graphs for a situation where there is a linear relationship between quantities (i.e., saving $3 for every week) because students made arithmetic errors when calculating the coordinate values for points to plot. Hattikudur et al. pointed out that those students didn't even noticed that they draw a non-linear curve maybe because their focus of attention was on the process of calculating the specific points and increments rather than viewing the graph as a whole (which refers to the global approach). Relatedly, Even (1998) reported that students who preferred to focus on the local characteristics and specific values of the coordinates (as opposed to the students who used a global approach) were not as successful in forming an equation of a given graph if it was not simple transformation of a parent function's graph.

Researchers (e.g., Chang et al., 2016; Elia et al., 2008) noted point-wise approach (i.e., plotting points in the coordinate plane) is a very common practice provided by textbooks when creating graphs of functions. Because of this common practice, Hattikudur et al. (2012) noted that a point-wise approach on graphing is sufficient for success in school mathematics, although researchers (e.g., Even, 1998; Hattikudur et al., 2012) suggested it is important to have both perspectives (i.e., point-wise approach/process perspective and the global approach/object perspective) in understanding functions and graphs.

Global approach/Object perspective

"From the *object perspective*, a function or relation and any of its representations are thought of as entities" (Moschkovich et al., 1993, p. 71) as a whole. Someone who has this perspective might pay attention to the general behavior of a function and global aspects of the graph when making connection to an equation of the function. For example, Even (1998) provided students a graph of the function $sin(x)$ and asked them to find an equation of the function whose graph were given as a modified/transformed version of the original graph of $sin(x)$ that they were given. A student used a rule of translation when finding an equation for a graph that was horizontally stretched by a factor 1/3 compared to the given graph of $sin(x)$, "so that forces it to have a 3x inside" (p. 110). One of the graphs was the stretched version of $sin(x)$ by a factor of 2 in the horizontal direction and some students directly wrote an equation of $sin(x/2)$ for this graph because the author stated that students used a known fact about a rule of translation. Even (1998) also reported that students' global approach was not always successful as almost half of the students misused the rule in some cases (e.g., translating to the right by 1 unit meant adding +1 inside).

Knuth (2000) provided another type of example of this conception in this perspective: Recognizing that the graphs are parallel lines for the equations of the form $y = 2x + b$. In terms of more dynamic examples, researchers (e.g., Even, 1998; Rocha, 2016; Zazkis et al., 2003; Zaslavsky, 1997) often investigated students' abilities and difficulties when they were asked to observe and generalize how the global and/or perceptual features of the graphs changed as the equation changed or how a change in the graphs influenced the equation. As pointed out by Lobato and Bowers (2000), one of the factors of using this type of connection might be NCTM's (1989) standard for functions emphasizing to "analyze the effects of parameter changes on the graphs of functions" (p. 154). Moreover, by following this type of perspective, researchers and teachers (e.g., Barton, 2003; Beigie, 2014; Bixler, 2014; Chang et al., 2016; Martinez-Cruz, 1998; Moyer, 2006; Rocha, 2016; Timotheus, 2009) designed instructional materials that supported students to make connections among multiple representations of functions by incorporating dynamic geometry or mathematics technology. The typical approach of connection in those articles for practitioners included varying the coefficients of an equation of a function and examining the resulting changes in the shape of the graph[3]. For instance, they inferred that students could successfully understand the relationship between two representations if they explored how increasing the value of $|a|$ in $y = a \cdot x^2 + b$ makes the graph become skinnier.

Graphing literature from Quantitative Reasoning perspective

Quantitative and covariational reasoning plays a critical role in students' construction and interpretation of graphs. I ground this claim in a body of research (e.g., Carlson, 1998; Castillo-Garsow et al., 2013; Frank, 2017; Johnson, 2015; Johnson et al., 2017; Johnson et al., 2020; Moore, 2016; Moore & Paoletti, 2015; Moore & Thompson, 2015; Paoletti & Moore, 2017; Paoletti, Stevens, & Moore, 2017; Ponte, 1984; Saldanha & Thompson, 1998; Stevens & Moore, 2016, 2017; Leinhardt et al., 1990; Thompson, 2011) showing that students who have not engaged in reasoning covariationally encounter difficulties in developing mathematical ideas (e.g., rates of change and graphs). There is also research (e.g., Lee, Moore, & Tasova, 2019; Moore et al., 2014; Paoletti, 2015; Paoletti et al., 2018; Stevens, & Moore, 2017; Stevens, Paoletti et al., 2017; Thompson, 1994c) showing a productive shift in students' and pre-service teachers' constructing and interpretation of graphs after having developed meanings based in quantitative and covariational reasoning in relation to their graphs. In this section, I review and synthesize the research on quantitative and covariational reasoning as it relates to students' construction of graphs. First, I illustrate the central aspects and constructs of quantitative reasoning in the context of students' construction of graphs. Second, I discuss covariational reasoning and multiplicative object in the context of graphing. Then, I illustrate the shape

[3] Although it is not from a research study, Desmos's Marbleslides activity among others is an example of this approach. Here is the link: https://teacher.desmos.com/activitybuilder/custom/566b31734e38e1e21a10aac8

thinking construct—a characterization of meanings for graphs from quantitative and covariational reasoning perspective—that I drew on for my research.

Quantitative Reasoning

Thompson (2011) claimed that there are two considerations regarding students' *quantitative covariational reasoning*. One of them concerns conceiving quantities themselves and how their values or magnitudes vary in a dynamic situation. The other one is conceptualizing a multiplicative object that can be made by coupling two quantities in the mind while also maintaining the image of the dynamic situation wherein two quantities are embedded. I illustrate the former here, and the latter is addressed in the next section. In the first section, I explore *quantitative reasoning,* in general, by discussing working definitions and central aspects of quantitative reasoning. Then, in the second section, I illustrate those aspects and constructs in the context of constructing and interpreting graphs in order to explain the role of quantitative reasoning in the process of constructing and interpreting graphs.

Central aspects and constructs of quantitative reasoning. Thompson (1990) defines *quantitative reasoning* as "the analysis of a situation into a quantitative structure" where *quantitative structure* is "a network of quantities and quantitative relationships." According to Thompson, a *quantity* is a conceived attribute of an object such that it is possible to measure it by explicitly or implicitly conceiving of an appropriate unit (Thompson, 1990, 1993, 1994a, 2008). It is important to note that, for a person, carrying out the measurement process using a measurement device (e.g., a ruler) and producing a numerical value (e.g., 5 cm) might not mean that person conceives a quantity. Having the capability to make the measurement conceptually is what makes the person conceive of a quantity (Thompson, 1990; 1993; 1994a; Smith III & Thompson, 2008). For example, the distance between my home and Aderhold Hall is a quantity and I know that it is measurable (see the map in Figure 2.4). Moreover, I can compare the distance from my home to Aderhold Hall and the distance from my home to Boyd without knowing their actual measurements as I am able to anticipate a measurement process and a value that symbolizes the result. The latter is longer than the former.

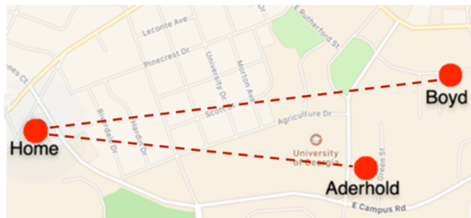


Figure 2.4. Map of my neighborhood in Athens, GA, USA.

Thompson (2011) describes "the dialectic object-attribute-quantification" as a process in which one constructs a quantity. *Quantification*, in general, is a process where one can conceive a size of an attribute of an object relative to a unit of measurement and equate the attribute's size and its measure (Thompson, 1990), but it is different than simply assigning numerical value to the attribute of an object (Thompson, 2011). For example, my height is a quantity for me because I conceptualize it as an attribute of my own body, and I know that my height is measurable (my body is the object in this case). Moreover, I can conceptualize that my height (i.e., this attribute of my body) has a unit of measure, say, 1 centimeter, such that the value or magnitude of my height (e.g., the attribute's measure) entails a proportional relationship with its unit (i.e., 1 cm). Multiplicatively comparing the amount of my height with the amounts of the unit is an example of the quantification process (Thomson, 2011). Note that a quantification process involves a unit (Thompson, 1990), which is also applicable to the case in which the quantification operation entails simply counting. Here, one considers an object to be counted for a situation and counts those objects by using a unit that is "a situationally-defined countable item" (Thompson, 1990; p. 17). For another example, average rate of change in my height with respect to elapsed time over a certain period of time is a *quantity* whereas multiplicatively comparing the amounts of change in my height with the amounts of change in time over the given period of time is *quantification* of this quantity. With this example, I also illustrated the relationship between the dialectic object-attribute-quantification and *quantitative operation*. Quantitative operation is the conception of producing a new quantity from two others (Thompson, 1990). Relatedly, a *quantitative relationship* is the conception of those three quantities (Thompson, 1990).

There is a nuanced difference between quantitative operation and quantitative relationship. While quantitative operation has to do with the *operation* in which one can create a new quantity, quantitative relationship has to do with *relating* the resultant quantity with its operands. *Operands* are two quantities that are known and being considered in operating, and *the resultant quantity* is the new quantity that becomes known—including its type and unit—as a result from operating (Thompson, 1990). In the example, average rate of change, as a new quantity, was being produced by a quantitative operation of (multiplicatively) comparing the change in two other quantities (i.e., distance and time). Thus, this quantity (i.e., rate of change) arose from operations on other quantities (not operation on numbers). Therefore, conceptualizing the image of those three quantities (i.e., average rate of change, amount of change in distance, and amount of change in time) constitutes a quantitative relationship.

As stated earlier, quantitative reasoning involves someone's reasoning about a situation by constructing quantities, quantitative operations, and relationships between conceived quantities. These mental conceptions constitute a *quantitative structure*, which Thompson (1990) defined as a network of quantitative relationships. Researchers (e.g., Carlson, 1998; Carlson et al., 2002; Lee et al., 2018; Liang et al., 2018; Moore & Carlson, 2012; Moore, Liss et al., 2013; Moore, Paoletti, & Musgrave, 2013; Moore & Silverman, 2015; Paoletti, 2015) have provided evidence that students who constructed or abstracted a quantitative structure are able to productively represent the quantities and relationships among them across different representational systems (i.e., formulas, graphs in both the canonical and non-canonical Cartesian plane, graphs in the polar coordinate plane). For example, in the box problem, Moore and Carlson (2012) showed that conceptualizing a situation as a quantitative structure (i.e., conceptualizing the quantities [box's height, base's width, and base's length] and imagining ways their values varied depending on changes in the values of the cutout's width) was helpful in creating symbolic expressions that represent quantitative relationships.

Thompson and Carlson (2017) emphasized the effective role of conceiving a quantitative structure as a foundation for students' (co)variational reasoning as "constrained variation" (p. 449). They explained that students may initially construct a quantitative structure by statically envisioning quantities (i.e., not varying) in a situation. For example, some may conceive the combination of the car's distance from City A and the car's distance from City B is the same as the distance between City A and City B, which is constant (see Figure 2.5). Then, as long as a student envisions a varying quantity in that structure, due to his or her constructed quantitative structure, the student is positioned to conceive that other quantities will also vary according to the quantitative relationships he or she has envisioned in his or her constructed quantitative structure. For example, changing the cars' distance from City A leads to a change in the distance from City B due to the quantitative structure in which the distance between City A and City B is constant and it is the combination of the car's distance from City A and the car's distance from City B. Therefore, we can say that the student's quantitative structure constrains and supports his or her variational reasoning in terms of how those quantities are varying, which is be an integral part of the ability of constructing graphs and formulas. For example, students who conceptualize this situation as a quantitative structure can represent the quantitative relationship by $x + y = c$ where x and y represents the car's varying distance from City A and City B, respectively and c represents the distance between City A and City B, which is constant.

Figure 2.5. Car situation.

Aspects of quantitative reasoning in relation to graphs. In this section, by providing empirical examples from the literature, I emphasize the role of conceiving a situation as a quantitative structure in constructing graphs.

Research on students' quantitative reasoning emphasizes the importance of conceiving situations as composed of quantities and their relationships as their values or magnitudes vary as a foundation for many mathematical concepts including rate of change (Carlson et al., 2002; Johnson, 2015; Thompson & Thompson, 1996; Thompson, 1994a; 1994b; Zaslavsky, Sela, & Leron, 2002), quadratic relationships (Ellis, 2011), exponential relationships (Castillo-Garsow, 2010; Confrey & Smith, 1994, 1995; Ellis et al., 2013, 2015; Ellis et al., 2016), and functions (Carlson, 1998; Carlson et al., 2002; Oehrtman et al., 2008; Thompson, 1994a, 1994b; Thompson & Carlson, 2017). Providing students opportunities to conceptualize quantities and their varying relationships in the situation also helps them to construct displayed graphs in a meaningful way because construction and interpretation of displayed graphs can emerge as a re-presentation of quantitative relationships (Ellis et al., 2018; Frank, 2017; Johnson et al., 2017, 2020; Liang & Moore, 2020; Moore, Paoletti, et al., 2013; Moore & Silverman, 2015; Paoletti, 2015; Paoletti & Moore, 2017; Thompson, 2011). This understanding is different than the approach of constructing a displayed graph by using some sort of memorized association or procedure (e.g., plotting points, then connecting the dots) without conceiving the graph as a representation of the relationship between quantities. Moreover, graphs themselves will be more meaningful if students relate them to a situation where students construct and/or re-present a quantitative structure.

It is important to note that when I say *a situation*, by following Moore (2016), I am not only referring to a situation of a real-life context. A situation can also be a formula, an equation, a table or even a graph. No matter the representational system, students need to engage in constructing quantitative relationships and reflect on these relationships when engaging either in a different representational system or in the same representational system. For example, students can construct quantitative relationships among quantities by interpreting a given graph in a coordinate system—the situation is a graph in this case—and can re-present those relationships when either creating a formula or constructing another displayed graph in a different coordinate system (e.g., a non-canonical Cartesian plane or a polar coordinate system). Or students can construct a quantitative structure by engaging in a formula of a relationship between two quantities (the situation is a formula in this case) and reflect on the quantitative structure when representing the quantitative relationship in a displayed graph, or vice versa.

Moore, Paoletti et al. (2013) illustrated how students engaged in quantitative covariational reasoning to make sense of graphing in the polar coordinate system (PCS) along with the connection to graphing in the Cartesian coordinate system (CCS). Overall, they reported that, by constructing a quantitative structure (i.e., conceptualizing the invariant relationship between quantities whose values covary), the students were able to display graphs to represent the relationship between quantities in different coordinate systems. For example, when representing a linear relationship, students drew a straight line on CCS whereas they drew a spiral shape in PCS. More importantly, they were able to view their displayed graphs as representing the *same* relationship even though the shape of displayed graphs perceptually looked different. Relatedly, Moore and Carlson (2012) provided evidence that conceptualizing a situation as a quantitative structure was helpful for students in creating formulas represent the quantitative relationships.

Paoletti and Moore (2017) provided findings that supports the positive role of conceiving quantitative structure in the construction of graphs by investigating students' parametric reasoning. Different than Moore et al. (2013), Paoletti and Moore (2017) explored how students construct (or understand) graphs that have the same perceptual shape in multiple ways, instead of being interested in students' construction of graphs in different coordinate systems. They reported that, by constructing a structure that entails invariant relationship between covarying quantities, their student was able to view the same graph (i.e., one graph) as producible by multiple ways by considering difference direction in which the graph is traced. Parametrically, constructed graphs were different (e.g., "$t \to (x, y)$ and $t \to (u, v)$, $0 \le t \le 2\pi$, such that $(x, y) = (t, \sin(t))$ and $(u, v) = (2\pi - t, \sin(2\pi - t)$" [p. 148]); however, the student concluded that they were the same graph perceptually (e.g., a sine graph).

Researchers have contended that students' difficulties in constructing accurate graphs and justifying curvature of graphs stem from underdeveloped images of quantities that covary. For example, Carlson (1998) reported that most talented college algebra students (i.e., Group 1, as labeled in the study) did not know how to graph a covariational relationship between two quantities as represented in a real-world situation and did not productively interpret displayed graphs in terms of covariational relationships. Carlson also showed that higher performing calculus students (i.e., Group 2) had difficulty in justifying concavity and inflection points on a graph that represented a dynamic real-world situation and had a hard time interpreting graphs representing rates of change over intervals. The author reported that, during the interview process, 74% of group 2 students constructed either a strictly concave-down graph or strictly concave-up graph in order to represent the height of the water in a spherical bottle in terms of the volume of water as the bottle was filling with water. Thus, the study revealed that most talented Calculus 2 students could not construct an accurate graph in order to represent the covariational relationship between two quantities because, as the author reported, most of them had weak understanding of the connection between how quantities are covarying in a situation and how concavity works in a graph. For example, in the spherical bottle task, Student M claimed that the entire graph should be a straight line because "as the volume comes up, the height would go up at a steady rate" (Carlson, 1998, p. 126). As another example, Student K who constructed a strictly concave down graph stated that "from here to here [referring to from beginning to the middle of the spherical part] height increases the same as the volume increases, and once you get here it [the volume] increases slower" by referring to the bottom part of spherical bottle, and "you have to put less and less [volume of water] in to get a greater height" when referring to the top part of the spherical bottle (p. 125). The reason the students continued drawing inaccurate graphs lied in their inability to conceive of the correct and appropriate covariational relationship between quantities in the situation. Thus, one source of students' difficulties in graphing is in that they did not construct an appropriate quantitative structure that informs how a quantity changes in relation to another. By only describing the directional changes in one quantity in relation to the other (as was the case for Student M), it is not fair to expect students to provide justifications for concavity and/or infer anything about the rate of change in one quantity with respect to the other quantity that is represented in a graph.

Moreover, if we review the justifications of the students' who constructed an accurate graph in the bottle task (in Carlson, 1998), we infer that most of them included an appropriate level of covariational reasoning to determine the quantitative relationships. For example, Student J, who constructed a correct graph, provided a justification considering how the height of the water in the bottle changed while imagining equal changes in volume. She told the interviewer that "if you look at it as putting the same amount of water in each time and look at how much the height would change, that is basically what I was trying to do" (p. 125). Therefore, Carlson's (1998) result, along with other researchers' results (Johnson et al., 2020; Liang & Moore, 2020; Moore, Paoletti, et al., 2013; Moore & Silverman, 2015; Paoletti et al., 2017; Stevens et al., 2015), suggests that conceiving a dynamic situation in terms of quantitative structure by knowing how to coordinate changes in quantities' value or magnitude simultaneously is foundational for constructing and interpreting graphs that represent that dynamic situations.

Covariational Reasoning

Covariational reasoning can be defined as the cognitive activities of someone who envisions how two quantities' values or magnitudes vary and envisions the ways in which they vary simultaneously (Carlson et al., 2002; Thompson, 1994a; Thompson & Carlson, 2017). Researchers have shown that covariational reasoning is essential for understanding functions as models of dynamic events (Carlson et al., 2002; Carlson et al., 2003; Carlson & Oehrtman, 2004) and for constructing and interpreting functions' graphs (as reported in aforementioned research). Even though the basic idea of covariation is accessible to elementary and middle school students (Confrey & Smith, 1994; Thompson, 1994c), it is not trivial for students to engage in covariational reasoning in order to conceive a situation in which quantities' values or magnitudes covary (Carlson, et al., 2002).

In their compendium chapter, Thompson and Carlson (2017) introduced a new variation and covariation framework by synthesizing the previous frameworks and conceptions of (co)variation produced by other researchers (Carlson et al., 2003; Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Confrey & Smith, 1994; Saldanha & Thompson, 1998; Thompson, 1994b). The new framework unifies the emphasis that was made by other researchers on (1) quantitative reasoning and the multiplicative object—which will be described in detail below—(Saldanha & Thompson, 1998; Thompson, 1994b, 2011), (2) coordinating successive values in one quantity in relation to values in another quantity discretely (Confrey & Smith, 1994), (3) coordinating the value of one quantity with changes in the other (Carlson et al., 2002), and (4) the ways in which one conceives of how quantities' values or magnitudes vary continuously (i.e., smooth and chunky; Castillo-Garsow, 2010, 2012).

Multiplicative Object

I discuss the conception of *multiplicative object* in the context of covariational reasoning, and I then provide empirical examples in order to place importance on why constructing a multiplicative object is necessary to develop productive and covariational meanings for graphs (e.g., see emergent shape thinking below).

A multiplicative object can be considered a conceptual object that is formed by uniting in the mind two or more quantities' magnitudes or values simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017). The mental operation of someone who constructs a multiplicative object is similar to the operation of someone who conceives a quarter coin as being, simultaneously, a circle *and* silver in color. In this operation, circle and silver, as two attributes of the object, have been considered, simultaneously, as one property of a quarter coin. For a dynamic example, imagine heating the quarter coin up to the melting point of silver.

As a multiplicative object, coupling the two attributes, someone could track the variation of the coin's color with the immediate and persistent awareness that, at every moment, the temperature of the coin also varies.

In the context of co-variation, Thompson (2011) represented the multiplicative object formed by uniting two quantities' variations by using the following representation: $(x_e, y_e) = (x(t_e), y(t_e))$, where $x_e = x(t_e)$ represents a variation in the values of x, where t_e represents variation in t through conceptual time over the interval $[t, t + e)$. He explained that in order for students to reason covariationally, they must unite x_e and y_e by constructing (x_e, y_e), which simultaneously represents the two. Note that the corresponding representation of this conceptual object in graphical context would be a point in a coordinate plane. I next discuss the role of conceiving a point as a multiplicative object in developing meanings for graphs.

Multiplicative Objects in the Context of Graphing. Despite the notion that graphing is critical for understanding various ideas in STEM fields (Rodriguez et al., 2019; Kaput, 2008; Leinhardt et al., 1990), students face a number of challenges (e.g., graphs as pictorial objects) in interpreting and making sense of graphs (Clement, 1989; Leinhardt et al., 1990; Moore & Thompson, 2015). Thompson and Carlson (2017) conjectured that part of these students' difficulties were grounded in being unable to conceive points on a graph as multiplicative objects, and several researchers have provided evidence to this claim (e.g., Frank, 2016, 2017; Stalvey & Vidakovic, 2015; Stevens & Moore, 2017; Thompson et al., 2017). Given this evidence, conceiving points as multiplicative objects might be an integral part of constructing productive meanings for graphs, and thus a student's construction of points should not be taken for granted.

I note that students' meanings of points on a plane can be considered as a representation of multiplicative object if students conceive a point by engaging in multiplicative operation—the operation of uniting and holding in mind two attributes of an object (i.e., quantities) simultaneously—as defined by Inhelder and Piaget (1964). Inhelder and Piaget first introduced the role of a multiplicative operation to characterize children's thinking when classifying 2-atrribute objects (e.g., objects grouped according to shape and color, as described above). They reported two different ways of children's thinking, both of which lead to normative responses when identifying a missing element in a matrix arrangement (see Figure 2.6). One is based on twofold symmetries that involve relying on perceptual configuration of the matrix arrangement and treating it as an incomplete pattern. For example, squares are symmetric over the horizontal axis of the diagram, so the blank space should include a circle. Similarly, red objects are symmetric over the vertical axis, so the blank space should include a blue circle. The other way of thinking is based on a multiplicative operation on a logical structure with reasoning about objects and coordinating two classes. For example, classifying the given three objects simultaneously in terms of shape *and* color, then identifying two elements of squares already belong to the classification of red or blue, noting the given element of circle belongs to red, and then joining circle and blue to construct the missing element.

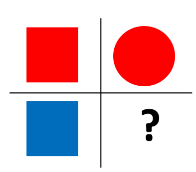


Figure 2.6. A matrix diagram designed based on the narratives of Inhelder & Piaget (1964).

Relying on these two types (e.g., perceptual features vs. reasoning about attributes), we could classify students' meanings of points as representing non-multiplicative objects or representing multiplicative objects. In addition, I note that Inhelder and Piaget (1964) illustrated that an arrangement does not have to be in a matrix form for a child to think of objects in terms of two attributes. They reported students could coordinate multiplicative classes without needing objects in a matrix form, and Tasova and Moore (2020b) illustrated that students could conceive points as representations of multiplicative objects without needing points represented in a Cartesian coordinate system. They noted that representing a multiplicative object is not restricted to plotting a point on a coordinate plane in the normative sense; it is about conceiving a point as a simultaneous representation of the two attributes of the same object.

Saldanha and Thompson (1998) illustrated that envisioning "graphs as composed of points, each of which record the simultaneous state of two quantities that covary continuously" (p. 298) is not trivial for middle school students. Frank (2016a, 2017), Stevens and Moore (2016) and Thompson et al. (2017) reported that it is also difficult for even adults (i.e., college students, pre-service teachers and mathematics teachers [especially for teachers in the U.S.]) to conceptualize a point on graph as a multiplicative object. To develop this understanding of graphs, Saldanha and Thompson (1998) stated that students needed to track two sources of information (e.g., values or magnitudes of two varying quantities that are related) simultaneously in order to construct a multiplicative object. They also drew attention to the importance of "tight coupling" two quantities so that "one variation [in one quantity] is not imagined without the other" (p. 306) in developing the multiplicative object. For example, due to his or her constructed multiplicative object (e.g., a point (x, y) on a graph), as the student imagines variation in the value or magnitude of one quantity (e.g., the values of x), he or she necessarily imagines variation in the other (e.g., the values of y) simultaneously.

Although envisioning graphs as composed of points as multiplicative object is not trivial for students, there is an emphasis on plotting points as a procedure (e.g., over and up) in school mathematics. Frank (2016b) and Thompson et al. (2017) provided arguments and evidence for the idea that conceptualizing plotting a point as a procedure (e.g., "over and up") does not help students in viewing a point on a graph or in a coordinate system as a multiplicative object that represents the value or magnitude of two quantities located on axes, simultaneously. Students usually are asked to decide (1) where to plot points (when given a coordinate pair) or (2) find the coordinates of a point (when given a coordinate system). Frank (2016b) characterizes these two scenarios "as if there is a place called (x, y) [in a coordinate system] and the student is being asked to find it" (p. 378). A student who uses "over and up" strategy could plot the point (x, y) by going over x on the horizontal axis and then up y on the vertical axis, which Thompson et al. (2017) criticized this kind of procedure by saying this is just "a recipe for locating the point" (p. 100). Thompson et al. argued that calculus students view the point $(2, f(2))$ as a value of the function, instead of the relationship between the value of the function (i.e., $f(2)$) and the value of the point (i.e., 2) for which the function was evaluated. Similarly, David et al. (2018) reported that some students treated the output of the function as the location of the coordinate point on the plane, rather than on the vertical axis (i.e., *location-thinking*). Those students—and consistent with those in Thompson et al.'s study—did not think of 2 as a measure of a magnitude located on the horizontal axis and they did not think of $f(2)$ as a measure of a magnitude located on the vertical axis in a canonical Cartesian plane. Frank (2016b) also provided similar results arguing that pre-calculus college students in her study could conceive a point in the Cartesian coordinate system as a *location* on the plane, rather than "a way of uniting two quantities' values that are represented on the axes" (p. 378).

Graphical Shape Thinking

Moore and Thompson (2015) observed three particular graphing activity patterns that entail fundamentally different mental operations regarding quantitative and covariational reasoning, and they provided three forms of the shape thinking construct to capture these patterns—static shape thinking, emergent shape thinking, and holistic shape thinking. I drew on *shape thinking* constructs for my research study, that is, I intend to use these forms to clarify important differences in students' meanings for graphs when they engage in constructing or interpreting displayed graphs. Moore and Thompson clarified that shape thinking can be used to characterize students' in-the-moment meanings for graphs, rather than a general classification of students' capabilities.

Moore and Thompson (2015) described, "[s]tatic shape thinking involves operating on a graph as an object in and of itself, essentially treating a graph as a piece of wire (graph-as-wire)" (p. 784). A student who engages in static shape thinking interprets the properties of his or her graph in terms of its straightness or concavity, its direction, and indexical associations with those properties. For example, a student engaging in static shape thinking conceives a linear graph's direction as defining its slope. A student perceives that a graph slanted downward from left to right necessarily implies the slope is negative, as actions rooted in static shape thinking are based on perceptual cues and figurative properties of shape.

A purpose of my research is to articulate the role of students' quantitative and covariational reasoning in constructing graphs. Moore and Thompson's (2015) *emergent shape thinking* provides an articulation of this role that I will build upon. They stated, "[e]mergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (p. 785). This conception involves (1) representing two inter-dependent quantities' magnitudes and/or values varying on each axis of a coordinate system, (2) forming a multiplicative object (Saldanha & Thompson, 1998; Thompson, 2011; Thompson et al., 2017) by uniting those two quantities' magnitudes or values as a single object, and (3) assimilating the process of a multiplicative object moving within the plane in ways invariant with two covarying quantities as generating a graph. The produced graph is understood as an emergent record of all instantiated moments of the simultaneous coordination of two covarying quantities so that its emergence can be anticipated when completed or presented with a given graph.

In addition to static and emergent shape thinking, Moore (in press) provided a third construct, called *holistic shape thinking*, in order to point out that a student who engages in actions that resemble static shape thinking to an observer, but to the student represent abstracted products of emergent shape thinking; their in-the-moment actions are based on convenience. For example, a student could reason the slope is positive when he or she perceives a graph that is slanted upward from left to right, but with the understanding that there is a positive linear covariational relationship between the quantities assuming the conventional Cartesian coordinate system (i.e., for each unit increase/decrease in a quantity that vary on the horizontal axis, there is a corresponding increase/decrease in the other quantity that vary on the vertical axis). The student anticipates that the graph emerged as a trace of a point that simultaneously represents these two quantities' covariation, and they can re-present the graph's emergence if necessary (e.g., if they perceive the graph to be represented in an unconventional Cartesian coordinate system, and thus requiring further examination).

Critique of the Literature from Quantitative Reasoning Perspective

In this section, I critique the studies from embodied cognition and multiple representation perspectives in the light of quantitative reasoning and emergent shape thinking. I illustrate my points in terms of (1) critique in general, (2) indexical association, and (3) potential generalizations as a result of indexical association.

Critique in general

Embodied cognition perspective is built on and necessitates a focus on the perceptual features of the graphs (e.g., shape of the graphs [e.g., "straight up"] or the "height" of the graphs [e.g., "going lower"]), which is different than a focus on quantities and their relationships. Relatedly, although there was an implicit attention to the quantities, such as "distance" between the object and the motion detector, when students use motion to enact in the situation, researchers consider distance as being spatial proximity between two objects. By referring to the several philosophers (e.g., Merleau-Ponty, 1945/1989, as cited in Nemirovsky et al., 1998), Nemirovsky et al. (1998) noted that distance "is about the impossibility of certain actions that would be possible if the objects were nearby, not a matter of meters" (p. 164). Nemirovsky et al. acknowledge that, for the students in their study, distance between the button and the tower (i.e., motion detector) was not related to meters, instead it "concerned the possibility and impossibility of creating certain graphical responses" (p. 164), which is somewhat related to my critique about indexical association that I discuss later in the next section in details.

In semiotic-cultural perspective, researchers believe that graphs convey mathematical knowledge on its own that are developed culturally over time. Thus, they mostly focus on investigating how well students' understanding of graphs align with the "culturally-accepted" mathematical relations being represented and support students to close the gap between the two. On the other hand, in quantitative reasoning perspective, researchers take the same main idea and add more nuances in our construction of that knowledge about graphs from students' perspective. Researchers from quantitative reasoning perspective is mostly interested in investigating students' conceptions of what their graphs represent, which may differ from those of researchers as well as what cultural/conventional meanings of graphs might be.

In relation to the literature on multiple representations, as also pointed by other researchers (e.g., Lobato & Bowers, 2000), my first concern is that the researchers' (or teachers' and students') focus of attention was on connecting representations and moving among the representations without an explicit consideration of the notion that tables, graphs, equations are simply different ways of representing the same relationship between two varying quantities in a situation. This is evident in their views of multiple representations being not equivalent to each other as illustrated previously. This is also evident if we consider their purpose of connecting representations. It seemed that, in some of the studies from this perspective, the purpose of connecting multiple representations was to help students to solve and/or make sense a mathematics problem (e.g., finding solutions, maxima/minima, *x* and *y*-intercepts, etc.). For example, Even (1998) supported (or expected) students to make connections between the two representations (equation and graph) to solve the following problem:

PROBLEM 1
If you substitute 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

Figure 2.7. Even's (1998, p. 106) problem.

For another example, Fonger (2019) supported students' representational fluency for the purpose of introducing equation solving. She designed activities where students were supported in constructing and interpreting graphs and tables before any symbolic manipulation. In their textbook analysis, Chang et al. (2016) showed that connecting equations and verbal descriptions to graphs were made for the purpose of visualizing the features of the function such as global/local maxima and inflection points, instead of taking students attention to relationships between covarying quantities.

I consider Kaput's (1993) call for "The urgent need for proleptic research in the representation of quantitative relationships" in order to help students to connect those multiple representations. One way to support students in this regard is to consider a "dynamic, interactive diagrammatic representation" (Kaput, 1993, p. 299) of a situation in relation to the other representations (i.e., "big three": equations, tables, and graphs). To point out "the linkage problem" in connecting students' experiences with mathematics, Kaput (1993) stated that students' mathematical experience is not connected to the experience of the "phenomena among quantities in a situation" (p. 298), in fact, he stated that students unfortunately do not usually experience the phenomena at all when engaging in some representation of it. According to him, these representations of the phenomena "serves as a surrogate, where the representation is often in the form of static, inert text, such as a word problem, perhaps augmented by static, inert graphics" (p. 298). Even though I am not interested in connecting the multiple representation of functions in my dissertation, Kaput's idea of considering the quantities and their relationships in a situation, which he called as "phenomena," and considering the multiple representations of it in relation to this phenomenon is consistent with the idea of conceiving a situation as a quantitative structure in constructing graphs as discussed earlier. If students conceive a situation forming a quantitative structure that entails quantities and their relationship, it might be easier for them to connect multiple representations of relationships and to have invariant ways of thinking across these representations. In other words, students need to determine and generalize an appropriate covariational relationship between quantities, so that they can see the same relationship in multiple representational worlds.

Indexical association

As I mentioned earlier in my review, researchers from embodied cognition perspective (e.g., Botzer & Yerushalmy 2008; Deniz & Dulger, 2012; Stroup, 2002; Nemirovsky et al. 1998; Russell et al., 2003) investigated students' attention to the relationship between the quality/nature of the motion and perceptual features of the graphs (e.g., straight, concave down/up) or "height" of the graphs. That is, students attended to make sense of the shape of the graph depending on the specific changes in the motion event (e.g., changing speed or direction). I call this relationship as shape-motion association. For example, Nemirovsky et al. (1998) talked about *fusion* as a construct describing the *simultaneous* association between the quality of graph (i.e., its visual feature, shape) and the quality of the motion. By "association," Nemirovsky and colleagues often referred to the *indexical* association between shape and the motion. They stated that the students "saw the height of the graph *indexing* [emphasis added] their distance to the tower with the handheld button" (p. 164). For example, they reported a student saying "Yeah, I remember the farther back you hold it [the button] the higher it [the graph] is" (p. 164).

In relation to multiple representation perspective, as I illustrated in the section of global approach/object perspective, researchers (or teachers) designed activities where they asked and encouraged students to determine a relationship between the shape of the graph (as a whole) and the equation by seeing a pattern between the varying coefficients of the equation and the behavior of the graph. I also count this type of activity as *indexical association*, particularly *shape-form association*, where the shape of the graph is indexed by the values of the coefficients of the equation. Said another way, values of the coefficients imply something about a particular shape or property of shape. In this type of connecting representations, students may not have a chance to see the connection between two quantities whose relationship is represented, which should be the main goal for representing relationships from my perspective. Moreover, this type of connecting representations (i.e., indexical associations) may discourage students to consider the quantitative relationship being represented because students only engage in observing the *outcomes* determined by the dynamic software where the equation and the graph are automatically linked, rather than trying to figure out where those outcomes come from.

Researchers (e.g., Frank, 2016b, 2017; Thompson et al., 2017) consider *emergent shape thinking* (Moore & Thompson, 2015) as a productive and covariational meaning for graphs from quantitative reasoning perspective. Indexical association in developing meanings for motion graphs and in connecting multiple representations does not seem compatible with the idea of emergent shape thinking. As opposed to treating a graph indexing the distance between the object and the motion detector, in emergent shape thinking, the student views a graph as a trace of a multiplicative object representing the relationship between the covarying quantities. Multiplicative object is a conceptual construct that imparts a productive meaning to *a point* on a graph; which was not taken into account seriously by the embodiment literature and multiple representation literature. Since embodiment and multiple representation researchers' focus is on the qualitative feature of the graphs (see Figure 2.1b) and global aspects of graphs, they don't attend to students' meanings of a point on a graph. However, as discussed earlier, Thompson and Carlson (2017) stated that students who do not view any point on a graph as a representation of two quantities' values or magnitudes simultaneously will likely have unproductive conceptions of graphs, as some other researchers also demonstrated in their study (e.g., Frank, 2016b, 2017; Hobson & Moore, 2017; Saldanha & Thompson, 1998; Stalvey & Vidakovic, 2015; Stevens & Moore, 2017; Stevens et al., 2017; Thompson et al., 2017).

In addition, in this particular perspective (i.e., global approach/object perspective), a graph is viewed as an object that only requires a global reading in a holistic way and doesn't require unpacking the details of the relationship being represented, such as exploring variation in variation between quantities. That means, as I illustrated earlier, aforementioned researchers from multiple representation perspective believed that graphs (compared to the other representations, e.g., tables and equations) do not allow someone to look for regularities and make generalizations beyond directional covariation between two quantities. However, in quantitative reasoning perspective, via partitioning activity—constructing equal increments of one quantity in order to investigate how the other changes in relation—(see Liang & Moore, 2020 for more details), it is possible to "unpack what a graph might want to tell us" if I use the language that those aforementioned researchers used. In fact, Tasova and Moore (2020a) developed a three-phased conception included in constructing and representing a quantitative structure by engaging in generalizing activities in the context of graphing.

In relation to the idea of indexical association, I note the idea of *immediacy* that is a common practice in both embodiment and multiple representation perspective. Due to the design of the activities in those studies from these two perspectives, students engage in constructing or manipulating graphs as an immediate response to a certain type of action (e.g., motion or change in the parameter of an equation). Aligning with the idea of fusion, in their review of research focusing on embodied learning, Duijzer et al. (2019) found that letting students use their own motion, and linking this motion *immediately* to a graph—as opposed to constructing a graph at a later moment after engaging with the motion—provided the best learning outcomes in terms of students' understanding of graphing motion. However, in quantitative reasoning perspective, there is the notion of constructing a quantitative structure first while engaging in the situation.

Then, students construct graphs by re-presenting this quantitative structure. A task design principle from embodied cognition perspective includes the idea that "actions and responses constitute each other continuously" (Nemirovsky et al., 1998; p. 127). Since the relation between motion in the situation (i.e., actions) and graph (i.e., responses to those actions) was done immediately, there may not be an opportunity for students to construct a quantitative structure while engaging in the situation and then re-present that structure later when graphing. Researchers from quantitative reasoning perspective (e.g., Carlson, 1998; Moore, Paoletti, et al., 2013; Moore & Silverman, 2015; Thompson, 2011) showed that conceiving a dynamic situation including a motion in terms of quantitative structure by knowing how to coordinate changes in quantities' value or magnitude simultaneously is foundational for constructing and interpreting graphs that represent that dynamic situations.

Potential generalizations as a result of indexical association

In this section, I draw the reader's attention to the potential generalizations that students might develop regarding the meanings of graphs as a result of indexical association (e.g., shape-motion and shape-form associations) that I discussed in the previous section. Recall that, when using Kine-Calc, referring to Figure 2.1 above, the student made association between a part of position vs. time graph that is slanted downward from left to right (i.e., shape) and an object that moved to the left in the situation (i.e., motion). I hypothesize that the meaning the student might develop about the shape of the graph (i.e., downward) in relation to the movement of the object (i.e., moving to the left) is not generalizable to graphs that show relationship between two quantities that are not position and time (e.g., distance traveled vs. time). Consider this student being asked to interpret a distance traveled vs. time graph in terms of the object's movement. It is likely that the student might interpret a graph that goes up/down in relation to an object moving to the right/left. Or if the student were asked to draw a graph of the relationship between distance traveled and time for an object that moves to the right and to the left at a constant speed, the students might draw a line that slants upward from left to right, then a line that is downward from left to right because that is what they learn as a result of shape-motion association.

This is also the case for activities that are design from multiple representation perspective. Potentially because of the teacher's focus on the indexical association (i.e., shape-form association as discussed earlier as a common way of connecting graphs and equations), Ellis and Grinstead (2008) reported a surprising generalization made by students in relation to the parameters (i.e., a, b, and c) in the general form of a quadratic function, $y = ax^2 + bx + c$. They reported that two-thirds of secondary students participated in their study conceived the parameter a in the algebraic representation of the quadratic function as a "slope" of the graph of the function. They also showed potential sources for this generalization. One of the sources that they detected was a focus on the value of the parameter a as influencing the shape of the graph in a dynamic manner. That is, in combination with teachers' language, students' explanation included a language "goes up" faster when describing how the parabola changes for larger values of a in the equation, which is the same language that students consistently used when referring to the slope in describing how linear graph goes up in relation to the value of the a in the linear equation (e.g., "it goes up by 0.25 each time" [p. 294] referring to a slope value of 0.25 in a linear function).

In order to support students to avoid making this kind of generalization, reflecting on their result and relying on Thompson's (1994) ideas regarding quantitative reasoning, Ellis and Grinstead proposed a way to shift the focus from what I call "indexical associations" to the focus on "any phenomenon described by that representation" (p. 295), which is consisted with Kaput's call for solving the linkage problem. That is, they suggested to change the focus from the symbolic representation and how changes in this representation affect the corresponding graphical representation to the quantitative relationships that are quadratic in nature.

CHAPTER 3

METHODOLOGY AND METHODS

In this chapter, I address my methodology and methods to investigate middle school students' reasoning with graphs. Recall, I organize my study around the research questions:

1. What ways of reasoning do middle school students enact when engaged in graphing activities intended to emphasize quantitative and covariational reasoning?
2. What ways of reasoning are involved in students developing productive meanings for graphs (e.g.., emergent shape thinking)?

In this chapter, I first describe the participant population and setting for my study. I then provide the explanation of the methodology and methods of data collection including a description of the underlying theoretical principles behind my methods. In this subsection, I also provide a rationale for my methods in terms of their compatibility with my theoretical perspective and research questions. I then describe the data collection procedures including the tasks and task design. Finally, I describe the data analysis techniques for the study.

Setting and Participants

The data for my study were gathered in sets of teaching experiments that occurred at a public middle school in the southeast United States. The school had an ethnically diverse population. Out of its 550 students, approximately 52.4% were African American, 29.3% were White, 10.3% were Hispanic, 2.9% were Asian American, 0.5% were American Indian. The participants for my study are sixth (age 11) and seventh-grade students (age 12) who were recruited on a volunteer basis with their parents' permission. Every student who volunteered for the study was accepted, which resulted in a sample of 4 seventh graders (Ella, Dave, Mike, and Zane) and 2 sixth graders (Melvin and Naya). All seventh-grade students were African American, and all sixth-grade students were White. Gender-preserving pseudonyms used for all students.

Why Middle School Students?

Studying middle school students' ways of thinking is important because, as stated in the problem statement, graphing is a critical aspect of understanding ideas in middle school mathematics for its opportunity to foster powerful learning (Dyke & White, 2004; Leinhardt et al., 1990) in higher-level mathematics courses. Graphing is also important in the middle school level of mathematics content because graphing is extensive representations used and promising representations to support functional thinking and algebraic reasoning (Brizuela & Earnest, 2008; Kaput, 2008; Schliemann et al., 2013). By engaging middle school students in tasks that focus on construction of graphs from emergent shape thinking perspective, middle school students have a chance to develop robust understanding of variation and covariation, which are key ideas necessary for success in calculus.

In relation to students' challenges and difficulties, I hypothesize that adults (i.e., high school and college students, pre- and in-service teachers) have greater difficulty than younger students to develop productive meanings for graphs. It might be their traditional way of learning mathematics and, in turn, their graphing habits (Moore, Silverman et al., 2019). One hypothesis is that these are learned ways of thinking that are unintended consequences and, ultimately, influence their ability to develop particular ways of thinking for graphing. It might be also that there is a period of time in students' cognitive development in which they are most open to develop the ability to conceive certain constructs (e.g., quantitative structures, multiplicative objects, and frames of reference) that are helpful in developing productive meanings for graphs. Maybe, missing the opportunity to develop those constructs makes their development more difficult later in life. For these reasons, there is a need to understand what ways of thinking can support younger students to develop productive meanings for graphs without being constrained by particular conventions and common practices, and to examine whether an instructional emphasis on covariational and quantitative reasoning in earlier grades is productive for students' later abilities of construction of graphs.

Teaching Experiment

The purpose of this study is to explore the middle school students' graphical meanings as they engage in activities designed to promote emergent shape thinking. Compatible with my research inquiry as well as the theoretical perspective (i.e., radical constructivism), I conducted a teaching experiment (Steffe & Thompson, 2000) in order to reach this goal and explore the students' mathematical progress over the course of the study.

Aligned with the notion that people are rational human beings who attempt to create viable models of reality based on their experiences, the researcher in a teaching experiment must accept that the student's mathematical reality is independent of his or her mathematical reality and fundamentally unknowable to the researcher. This brings the question of how the researcher can then understand a student's mathematical reality if it is inherently unknowable. Steffe and Thompson (2000) use the phrase *student's mathematics* to refer to the mental operations that constitute a student's mathematical reality and they use the phrase *mathematics of students* to refer to the researcher's interpretations—a model—of students' mathematics. Thus, they claim that a researcher can understand the mathematical realities of students to the extent they can construct models of student's mathematics—or mathematics of students—via finding rational grounds for the students' words and actions over the course of a teaching experiment.

It is important to recognize that these models do not directly represent the student's mathematics. These models represent the student's ways of thinking that account for the student's observable actions including gestures, facial expressions, written and oral responses (Steffe & Thompson, 2000). Thus, the researcher aims to develop models of student's mathematics that provide viable explanations of the student's words and actions when engaging in a teaching experiment. Moreover, aligned with the assumption of the idiosyncrasy of students' thinking, the researcher assumes that each model of a students' mathematics is specific to that individual at the time and in the context of that teaching experiment.

It is also important to recognize that the researcher's interpretations of the students' words and observable actions are constrained by the researcher's current ways of operating. Similarly, the researcher is constrained by the student's language and actions in a way that he or she can only create models that are viable in terms of the researcher's experiences with that student. Due to the aforementioned reasons, the researcher must routinely test his or her models for viability by questioning them in ways that enable the researcher to construct robust mathematics of students. For example, the researcher can engage the student in new problems to see if the student behaves in ways compatible with the researcher's hypothesized models (Steffe & Thompson, 2000). Hence, in this study, I engaged students in a variety of contexts across different tasks throughout the teaching experiment that enabled me to test my hypotheses for the eventual purpose of building models that are viable explanations of the students' understandings.

I designed my teaching experiment not only to test hypotheses, but also continually generate and reconstruct them (Steffe & Thompson, 2000). I began my teaching experiment with initial hypotheses to test (e.g., imagining the quantities' magnitudes on number lines enables the construction of emergent shape thinking); however, I continually generated hypotheses when I needed to formulate new situations of students thinking that I could not consider in the initial design of the teaching experiment. Steffe and Thompson (2000) emphasized that the researcher must focus on the student's mathematical activity during the teaching experiment and attempt to abandon the major hypothesis that he or she established at the beginning of the experiment. Thus, I generated other hypotheses when interacting with students in my study in order to adopt to each student's reasoning and meanings as there was a possibility that they would engage in the tasks in unexpected ways. During the teaching experiment, for example, I needed to generate new hypothesis in order to reconcile my own perturbations that were leaded by my student's actions and re-designed my models of the student's mathematics (e.g., Zane's quantitative multiplicative object, Type 1: "there is no points on a line"). This recursive feature of the teaching experiment (i.e., the continual hypothesis generating, testing, and reconstructing) allowed me to determine certain types of student thinking and mental actions that resulted in more crystallized models of student's mathematics.

In addition to constructing models of the student's current mathematics, another goal of the researcher is to understand how the student's mathematics changes and how the student develops new ways of thinking. The teaching experiment provides an opportunity for the researcher to foster student learning and development by acting as a teacher. The researcher can investigate how students modify their existing meanings as a result of their mathematical activity over the course of teaching experiments. In my study, I investigated how students developed new meanings for graphs over the course of numerous teaching sessions and attempted to understand what ways of reasoning supported students to develop productive meanings for graphs in terms of emergent shape thinking. So, my model of students' mathematics also included any modifications students made in their ways of operating. I constructed those models by engaging in the same iterative cycle previously discussed (i.e., the continual hypothesis generating, testing, and reconstructing). These models were mostly created during my retrospective conceptual analysis, which I discussed in the data analysis section in detail.

According to radical constructivism, cognitive change and learning can occur when an individual is perturbed (i.e., have unexpected results from goal-directed mental activity) because the perturbation might lead the individual to accommodate his/her action scheme to maintain or re-establish equilibrium (i.e., eliminating perturbations; von Glasersfeld, 1995). Hence, in a teaching experiment, the researcher can claim that the student has learned when the student is able to solve a problem that he or she couldn't previously solve. In relation to this idea, I seek an answer to the question of "what the situations are in which the child's schemes produce the perturbing outcomes that may it to learn" (von Glasersfeld, 1995, p. 66). This question is important and hard to answer because the perturbation occurs by the cognizing individual's *unobservable* expectations. A part of the answer to this question is *interaction*[4] that is "the most frequent cause of accommodation" (von Glasersfeld, 1995, p. 66), which is what lead me to implement teaching experiment in my study. The teaching component of a teaching experiment requires responsive and intuitive interaction (Steffe & Thompson, 2000) that allows the researcher to foster learning and cognitive change in addition to creating models of the students' mathematics. The activities in teaching experiment are designed either to test the hypothesized models of the students or to perturb students in a way that it leads them to accommodate to develop productive meanings for graphs.

[4] Learning can happen as a result of an interaction; however, the learning is still an individual accomplishment (Steffe & Thompson, 2000).

Data Collection and Task Design

I ran a set of teaching experiment sessions with Ella and Zane during the Spring 2019. Ella participated 6 teaching sessions all of which were paired with Zane. In addition to the paired teaching experiment sessions, Zane participated 10 additional individual teaching experiment sessions. Each session last for approximately one-hour. I also ran a set of teaching experiment sessions with Dave and Mike during the Spring 2019. Dave and Mike participated 4 teaching sessions as a pair. After the paired sessions, Dave and Mike also participated 7 and 6 additional individual teaching experiment sessions, respectively. The teaching sessions took place after school in a classroom at students' school, and each session last for approximately one hour with Dave and 30 minutes with Mike. Table 3.1 summarizes the teaching experiment sessions. In each session, I was the teacher-researcher who interacted with the students and seek to engender and model their ways of thinking during the teaching sessions. Other research team members (Biyao, Yufeng, and Irma) acted as witness-researchers who managed the video camera, asked follow-up questions, and provided alternative explanations of students' activities during our conversations after each session (Steffe & Thompson, 2000).

Table 3.1
Overview of the teaching experiment sessions.

Teaching Experiments (TEs)		
• Ella and Zane (7th grade) ○ 6 paired TEs ○ ~1 hour each • Zane ○ 10 individual TEs ○ ~1 hour each	• Dave and Mike (7th grade) ○ 4 paired TEs ○ ~40 minutes each • Dave ○ 7 individual TEs ○ ~1 hour each • Mike ○ 6 individual TEs ○ ~30 minutes each	• Melvin and Naya (6th grade) ○ 14 paired TEs ○ ~50 min each

I also collected data from two additional students, Melvin and Naya, who were in 6th grade from the same school district. They participated 14 teaching sessions as a pair, and each session last for approximately one hour. In each session, I was the only teacher-researcher who interacted with the students and seek to engender and model their ways of thinking during the teaching sessions. A summary of the data collected is in Table 3.2.

Table 3.2

Data summary of teaching sessions with the tasks and length.

	Ella	Zane	Dave	Mike	Melvin	Naya
S01	26-Mar	(50 min)	02-Apr	(30 min)	25-Jul	(42 min)
S02	27-Mar	(65 min)	04-Apr	(29 min)	25-Jul	(36 min)
S03	28-Mar	(59 min)	08-Apr	(40 min)	26-Jul	(49 min)
S04	03-Apr	(59 min)	11-Apr	(56 min)	26-Jul	(40 min)
S05	09-Apr	(62 min)	16-Apr (44 mi)	18-Apr (38 min)	29-Jul	(45 min)
S06	10-Apr	(60 min)	18-Apr (45 mi)	24-Apr (24 min)	29-Jul	(63 min)
S07		17-Apr (63 min)	23-Apr (67 mi)	25-Apr (37 min)	30-Jul	(62 min)
S08		22-Apr (57 min)	25-Apr (53 mi)	30-Apr (27 min)	30-Jul	(39 min)
S09		24-Apr (60 min)	30-Apr (64 mi)	02-May (31 min)	31-Jul	(68 min)
S10		01-May (54 min)	07-May (37 mi)	09-May (27 min)	31-Jul	(43 min)
S11		01-May (56 min)	09-May (36 mi)		01-Aug	(57 min)
S12		02-May (59 min)			01-Aug	(52 min)
S13		06-May (49 min)			02-Aug	(61 min)
S14		06-May (55 min)			02-Aug	(56 min)
S15		08-May (55 min)				
S16		08-May (55 min)				

Downtown Athens Task	Measurement Activity
Crow Task	Which One Task (DABT)
Skateboarder Task	Swimming Pool Task
Downtown Athens Bike Task (DABT)	Which One Task (SPT)
Matching Game Task	

Task Design and Descriptions

In each session, I provided the students a graphing activity, allowed them to investigate the activity in pairs or individually, and prompted them to discuss their thinking and ideas explicitly. I aimed to explore, understand, and support the students' thinking. The learning goals of the teaching experiment included leveraging quantitative and covariational reasoning abilities in order to support students in constructing coordinate systems and graphs and helping them view graphs as an emergent representation of two covarying quantities. I developed a sequence of tasks by considering particular design principles focused on graphing covarying quantities (e.g., Frank, 2017; Moore & Thompson, 2015; Stevens, Paoletti, et al., 2017; Thompson & Carlson, 2017). Table 3.3 below provides an overview of ideas addressed in each task during the teaching experiment. Task protocols and handouts in their final form are in Appendix A, and I summarize the design of each task in what follows.

Table 3.3

Overview of ideas addressed in each task.

Tasks	Learning Goals
	Constructing a quantitative relationship.
Downtown Athens Task Crow Task (CT)	Conceptualizing two quantities in a static situation.
Downtown Athens Bike Task (DABT)	Conceptualizing two quantities in dynamic situations.
Measurement Activity	Representing the quantities' magnitudes by directed bars on magnitude lines.
Which One? (DABT) Swimming Pool Task (SPT) Which One? (SPT)	Describing how quantities are changing in relation to each other.
	Constructing a coordinate system.
Matching Game (Part 1) Matching Game (Part 2)	Constructing a Cartesian coordinate system.
Matching Game (Part 3)	Conceiving a point in a Cartesian coordinate plane (i.e., multiplicative object).
	Constructing graphs.
CT DABT (w/ coordinate plane) SPT (w/ coordinate plane) DABT 2	Conceiving a graph as a representation of an emergent relationship between two varying quantities (i.e., emergent shape thinking).

Note: See https://www.geogebra.org/m/ayajrggf for digital version of all the tasks.

Downtown Athens Task

Downtown Athens Task (DAT) is an adapted version of the original task called "Sketching Graphs from Pictures: Particles and Paths" designed in England at the Shell Centre (Swan, 1985). DAT includes a map of Downtown Athens with seven locations highlighted and labeled: UGA Arch (hereafter Arch), Double-Barreled Cannon (hereafter Cannon), First American Bank, Georgia Theater, Wells Cargo Bank, Statue of Athena, and Starbucks (see the map in Figure 0.1, left). DAT also includes a Cartesian coordinate plane whose horizontal axis is labeled as distance from Cannon (DfC) and vertical axis is labeled as distance from Arch (DfA). Seven points are plotted without labelling in the coordinate plane to represent the seven locations' corresponding DfA and DfC (see Figure 0.1, right). I asked the participants what each of these points on the coordinate plane might represent. My goal was to observe the most spontaneous tendencies and to explore students' meanings of points in what I perceive to be a Cartesian coordinate plane. I also aimed to use this activity to get insights into if and how students conceptualize quantities in the situation. I consider this task as a prior step of the next task as I plan to explore if and how students conceive and represent quantities' magnitudes when they are not varying. Then, in the next task, I plan to have students conceive the varying quantities in the situation and determine the covariational relationship.

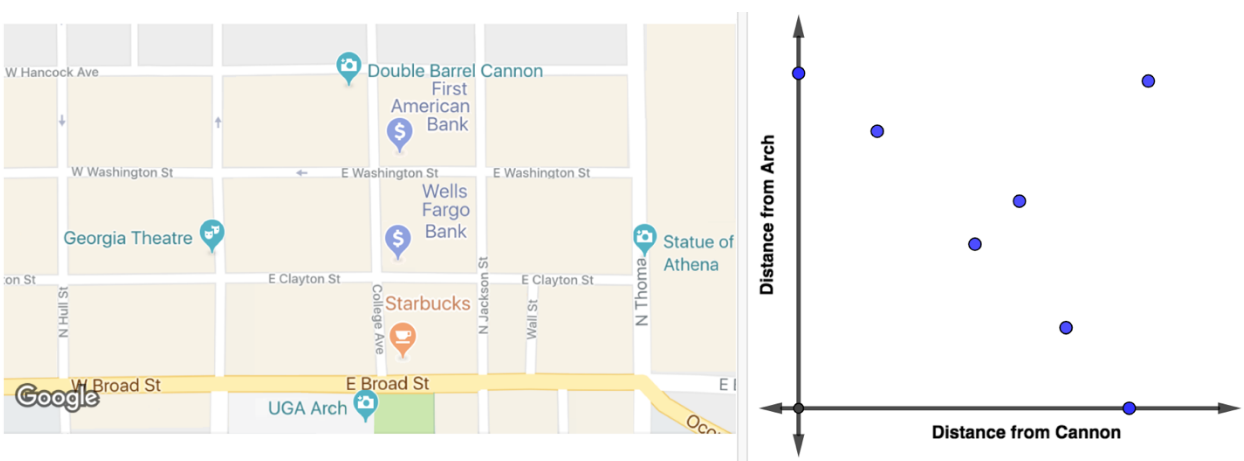


Figure 0.1. Downtown Athens Task (DAT)

Note that the distance meant in this task is the direct distances (i.e., as-the-crow-flies distance), but I ensured to explore how students interpret "distance" because there are numerous different ways of interpreting it.

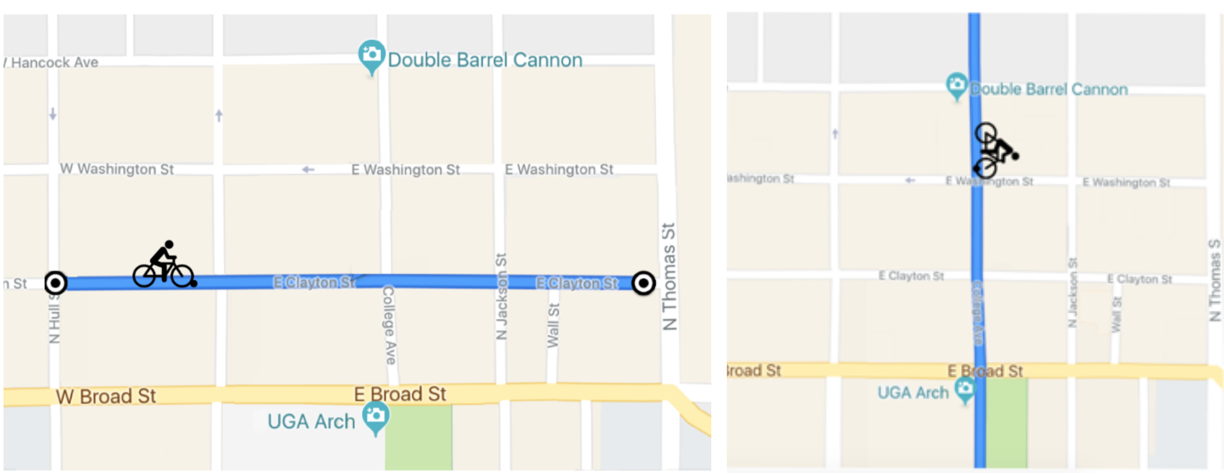
Crow Task

Crow Task's (CT) set-up is the same as DAT, but rather than a focus on landmarks, the focus was on a crow that flies in Downtown Athens. Students were able to control the crow through freely by dragging it and viewed how the corresponding black point on the coordinate plane changed (see https://youtu.be/zxv8Mk6a2Us). The questions that I asked in this task were "What is going on?" and "Why do you think the point on the coordinate system gets closer to the axis 'Distance from Arch/Cannon' as the crow flies toward to Cannon/Arch?" (see Appendix A for the task protocol). My original goal with this task was to create an environment in which reasoning based in iconic translation might lead to an experienced perturbation. In such cases, I did not plan on drawing students' attention to the varying quantities in the situation and support them in determining a covariational relationship, but rather gain insights into their experienced perturbation and spontaneous attempts to reconcile that perturbation. Students can use CT to check if their answers are correct in DAT by moving the crow on top of the highlighted places and observing where the corresponding point goes on the plane.

Downtown Athens Bike Task

Downtown Athens Bike Task (DABT) included the same map of Downtown Athens highlighting a straight road (i.e., Clayton St. or College Ave.) with two places located near the road (i.e., the Arch and Canon; see Figure 0.2) and a bike on this road. I animated the map so that the bike moved at a constant speed back and forth along the Clayton St. starting at the West side of the street in the first situation (Figure 0.2a). My purpose in this task was to explore and support students' process of conceiving the quantities' magnitudes that vary continuously and represent the varying quantities' magnitudes by students' index fingers and then by directed bars that vary on empty number lines (also called *magnitude lines*). Later, my goal was to explore and support how they would conceive the relationship between two quantities varying together in the situation. That is, this task was intended to study the strategies of the comprehension of quantities, the notion of variation and covariation, and the notion of representing quantities' magnitudes by directed bars that vary on number lines.

There are two different situations that can be used in this task. I only used the first situation (see Figure 0.2a) in DABT. By keeping the same questions with small adjustments, I planned to use the second situation (see Figure 0.2b) in case students could finish the first version too early or the first situation being too easy for students, but it did not happen in my study. In the first situation, the bike rides on Clayton St from West to East and backwards, whereas, in the second situation, the bike rides on College Avenue from North to South and backward. In the first situation, direction of change in both quantities' magnitudes is always the same (as the bike's DfA increases/decreases, DfC increases/decreases). In the second situation, I believe the quantitative structure is more sophisticated because the quantities' magnitudes vary in different directions when the bike is in between the Arch and Cannon (i.e., DfA increases as the DfC decreases).



(a) (b)

Figure 0.2. Downtown Athens Bike Task (DABT)

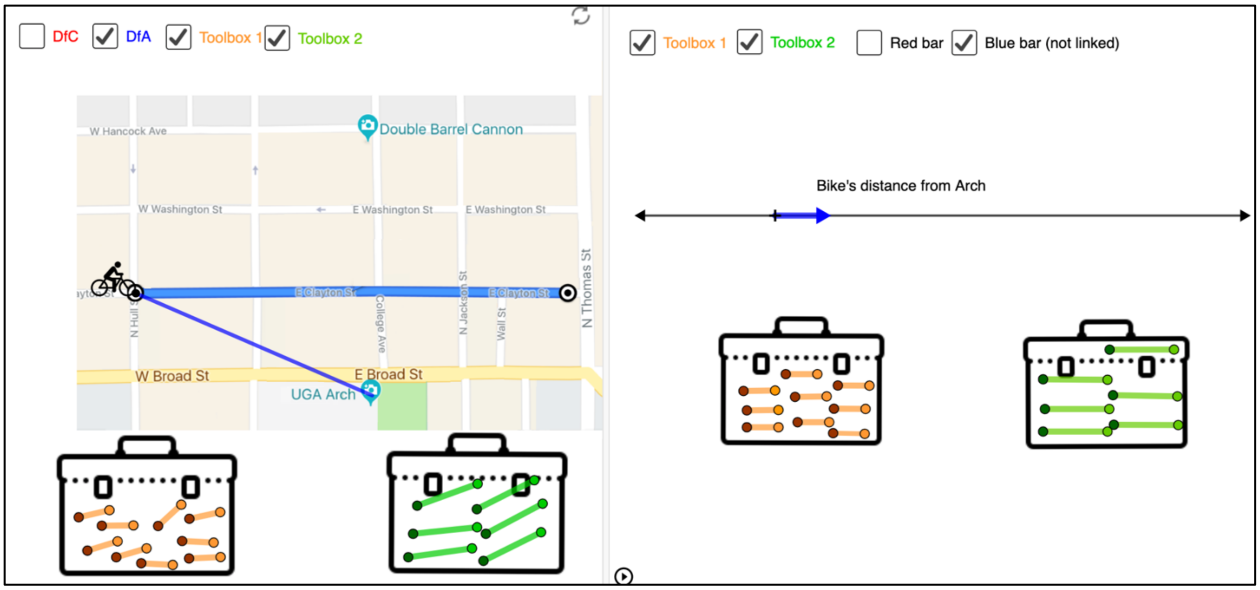
In DABT, I played the bike animation in loop and began with asking "what is going on here?" in order to let students describe situations as they saw them. When students didn't come up with quantities, I supported them by asking what the things they could consider varying and possible to measure were in the situation. When students noticed the quantities of the bike's DfA and DfC, I intended to get more insights into how they interpreted the distances because there were numerous different ways of interpreting "distance" (e.g., distance traveled, as-the-crow-flies distance). In this task, I also planned to investigate how they could conceive the variation in each quantities' magnitudes (i.e., DfA and DfC) as the bike traveled along the street back and forth. I asked them to describe the covariational relationship between DfA and DfC for the purpose of exploring their spontaneous thinking.

In the second part of DABT, I asked students to represent the varying quantities' magnitudes by using their index fingers as I believe the role of "hand" (Norton et al., 2018) in internalizing students' activity as representation of varying quantities. I directed students to place their index fingers horizontally on the table in a way that left index finger was fixed on the table and right index finger could only move in a horizontal direction. Then, I told them to move right index finger left to right so that the distance between index fingers could represent the bike's DfA (or DfC) as the bike traveled along the road. My goal in this activity was to understand if and how students could attend to the quantities' magnitudes. (i.e., the bike's DfA or DfC). I anticipated there might be students who could mimic the position of the bike as the bike travels on the map as Marfai (2017) showed that students conflated their finger with the position of the object in the situation instead of representing the variation of the attributes of the object. I repeated this finger activity by letting a pair of students working on it together to represent each quantities' variation (i.e., both DfA and DfC at the same time) by their fingers. With this activity, my goal was to understand if and how they both could work together to represent both quantities' variation at the same time.

The situation in this task was inspired by a similar task presented in Swan (1985; "Going to School"), Ponte (1984; "Journey to School"), Saldanha and Thompson (1998) and Moore and his colleagues (2016; "Going around Gainesville" and "City Travels"). Different versions of this task had been used in numerous research studies (e.g., Frank, 2017; Moore et al., 2016; Ponte, 1984, Silverman, 2005; Swan, 1982; Whitmire, 2014) as a graphing task and they often demonstrated students' difficulties during the construction of their graphs representing the relationship between two quantities (e.g., the bike's distance from City A and the bike's distance from City B). For example, Moore et al. (2016) reported that pre-service teacher had difficulties when constructing their graph due to their graphing habits (e.g., graphs should go from left to right and graphs should start on the vertical axis). I didn't plan to use this task as a graphing activity yet. I used this situation first in order to explore how students could conceive the situation quantitatively and how they could determine the relationship between quantities. In a later stage of my teaching experiment, students used the same situation in a graphing task. My conjecture was that students' mathematical activity in this early stage of the teaching experiment could help them to represent the quantitative relationship successfully later.

Measurement Activity

I designed the measurement activity to support the students in representing the quantities' magnitudes by directed bars on magnitude lines (see Figure 0.3). This task is adapted from Davydov's task (presented in Bass, 2015) for elementary students to be used to develop meanings for number lines from continuous measurement perspective without relying on numbers at first, but having numbers occupying the line as students make progress in developing conceptions of numbers. I have two general goals with the activity. First, I wanted students to know that no matter how they measure an object's attribute, the magnitude of the quantity was the same and what was changing is the value of it. Second, I wanted students to be aware that the magnitudes could be measured anytime they want as I did not want them to envision those bars as simply a visual connection between objects.



(a) (b)

Figure 0.3. Measurement Activity

The situation included the same map of Downtown Athens highlighting Clayton St. and a bike moving on this street (see Figure 0.3a). I asked students to focus on the bike's DfA in the situation and go to the other screen that included a magnitude line and a directed bar located in a *random* place on the number line (see Figure 0.3b). I asked students to change the magnitude of the bar in a way that it has the same length of the bike's DfA on the map. The situation and the bar were presented in a different screen in GeoGebra because I wanted to create a need for the measurement process. I provided tools for students to be used in measuring the magnitude of the bike's DfA and representing it on the magnitude line exactly. I provided students segments of different size (see orange and green segments in Figure 0.3) that would leave them different numerical values when used to measure the bike's DfA. Thompson et al., (2014) stated that someone who conceives of a quantity's magnitude can anticipate that the quantity's magnitude is invariant with any change in unit. So, I conjecture that students in my study could be leveraged to develop such a meaning for the quantity's magnitude when engaging in this task.

Which One Task in DABT

Which One Task (WOT) is adapted from Stevens et al. (2016) and includes the same map of Downtown Athens (Figure 0.4, left). I asked students to describe the relationship between the bike's DfA and the bike's DfC as the bike traveled along the street for the purpose of having students conceptualize the (gross) covariational relationship between two quantities. I then showed the screen that included varying bars on magnitude lines (see Figure 0.4, right). There were five bars on parallel magnitude lines. I informed the students that the blue bar represented the bike's DfA. Students could change the length of the blue bar by dragging the bike or by clicking the play button. As students moved the bike, each red bar varied in a different way in relation to the blue bar (see https://youtu.be/GZJOrStPgc0; also see Table 3.4 for the description of how the red bars vary in relation to the blue bar). I then asked the students to determine which of the four red bars, if any, accurately represented DfC as DfA varied. I hided the bike on the map in order to see if students can represent the relationship between two quantities when the situation was not visually available.

The purpose of this task was to explore if and how students could coordinate the variation of the two quantities' magnitudes at the same time and identify the gross covariational relationship between two quantities. In this task, I did not expect students to eliminate the bars by reasoning with amounts of change as it was not possible to do so in the situation with the given materials. In a later task, students had a chance to engage in identifying the covariational relationship with sensitive to the amounts of change. With this task, I also aimed to support students in constructing a multiplicative object prior to their graphing activity because Thompson et al. (2017) conjectured that conceptualizing a multiplicative object does not require a coordinate system. Moreover, Stevens et al. (2017) stated that they were able to investigate students' construction and sustaining a multiplicative object in a similar task "while minimizing the influence of the ways of thinking they have developed for graphs (e.g., iconic translations, issues of function/dependency, ways of thinking based in figurative thought)" (p. 933). Thus, I expected to gain insights into if and how students could construct and sustain a multiplicative object with respect to a situation and the varying bars.

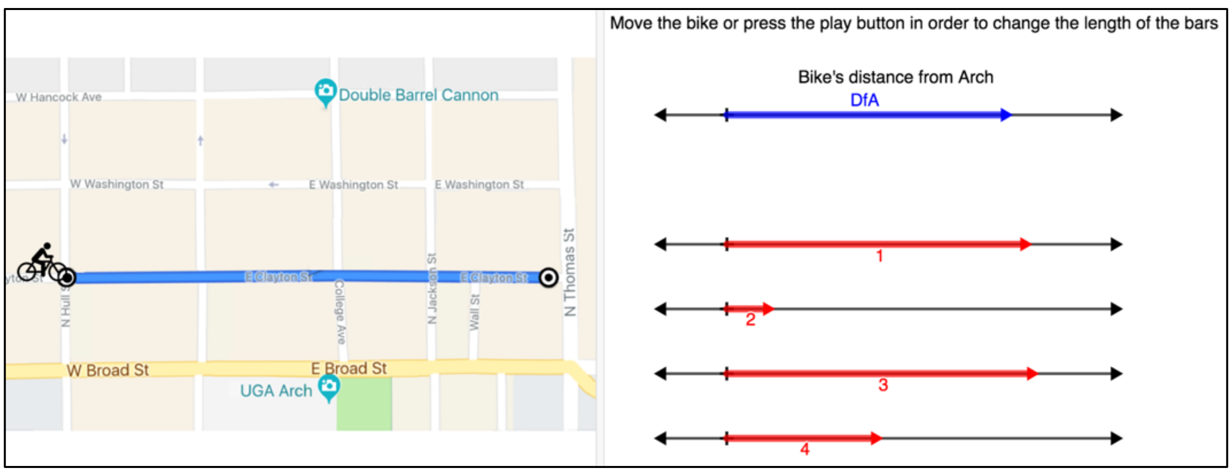


Figure 0.4. Which One Task (WOT).

Table 3.4

Information about the variation of red bars with respect to variation of blue bar.

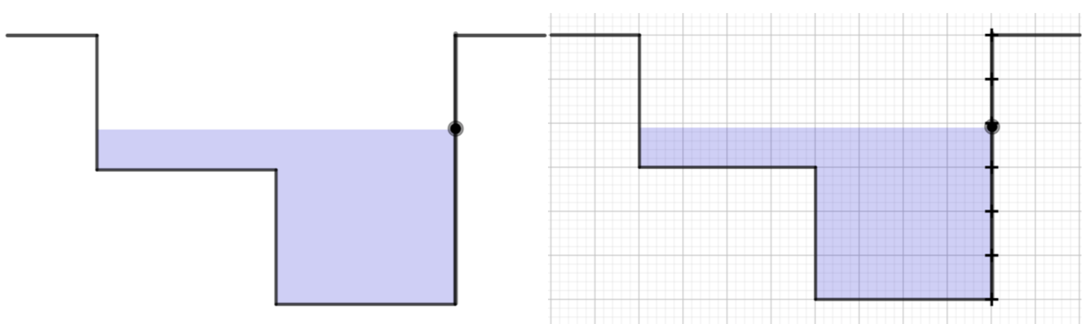
Bar	From N. Hull St to College Ave	From College Ave to N. Thomas St	From N. Thomas St to College Ave	From College Ave to N Hull St.
DfA	Decreasing	Increasing	Decreasing	Increasing
1	Decreasing	Decreasing	Increasing	Increasing
2	Increasing	Increasing	Decreasing	Decreasing
3	Decreasing	Increasing	Decreasing	Increasing
4	Increasing	Decreasing	Increasing	Decreasing

Note: #3 is the normative solution.

Finally, I asked students to sketch a rough graph showing the relationship between DfA and DfC on a Cartesian coordinate plane. The reason I intended to engage student in a graphing task was to explore to what extent students are able to assimilate their activity with bars into their graphing activity. If the students did not assimilate as such, I did not plan to further prompt students to envision varying bars on each axis representing the quantities' magnitudes at this stage of the teaching experiment.

Swimming Pool Task

Swimming Pool Task (SPT), adapted from Swan (1985), includes a dynamic diagram of a pool (see Figure 0.5a), where students can fill or drain the pool by dragging a point on a given slider. I designed the task to support students in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool.



(a) (b)

Figure 0.5. (a) A diagram of the pool (b) A diagram of the pool with grid.

I ask students to describe how AoW varied in relation to DoW as we fill the pool up. I aimed to explore how students coordinated the changes in quantities' magnitudes in the situation before they engaged in representing quantities. If students only attended to directional changes, I drew their attention to behavior of the change in both quantities beyond the directional change (e.g., amounts of change) by letting them engage in partitioning activity in the situation. I let them to have grid paper and marks on the right side of the pool (see Figure 0.5b) in order to provide them with opportunity to have equal partitioning in DoW as well as in AoW.

I then asked students to sketch a rough graph showing the relationship between AoW and DoW. With this graphing activity I planned to explore to what extent students were able to assimilate their activity in the situation and their activity with bars into their graphing activity. I did not plan to further prompt students to envision varying bars on each axis representing the quantities' magnitudes at this stage of the teaching experiment.

Which One Task in SPT

WOT in SPT includes the same pool animation (see Figure 0.6, left) and five directed bars that are placed on parallel magnitude lines (see Figure 0.6, right). The bars on the magnitude lines and the pool animation in the situation were not synced. I informed students that the blue bar represents DoW in the pool. As students moved the blue bar, each red bar varied in a different way in relation to the blue bar (see https://youtu.be/MnelghPHzWI; #3 is the normative red bar). I then asked students to determine which of the four red bars, if any, accurately represented AoW in the pool as DoW varied.

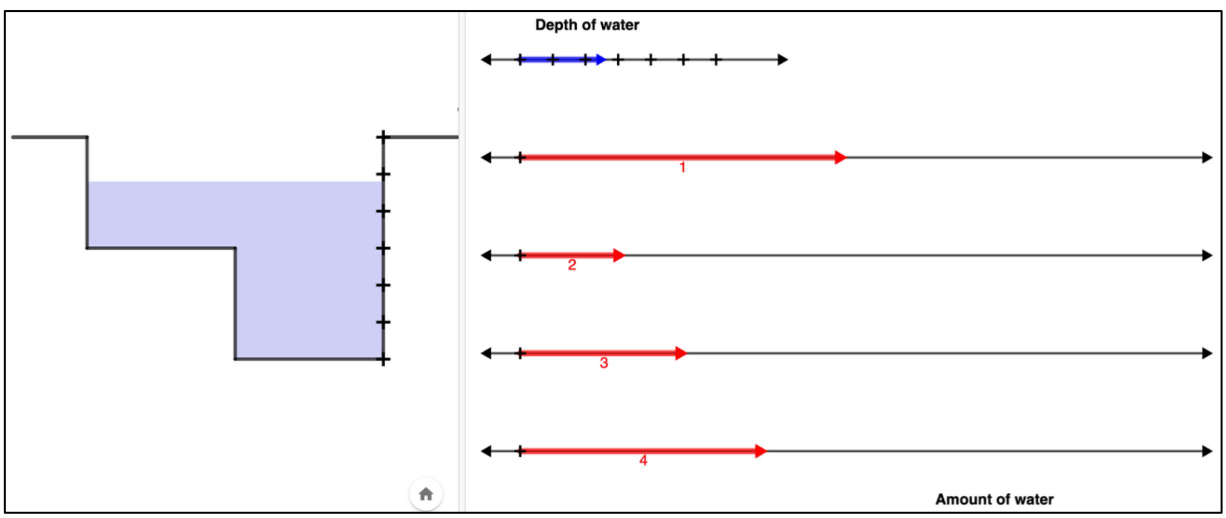


Figure 0.6. Which One Task in SPT.

Note that all of the red bars directionally varied in the same way as the normative red bar. If students would engage in only gross (directional) covariational reasoning, they would end up with all red bars satisfying the directional relationship. Thus, students should be able to engage in reasoning with amounts of change in order to find which one is the normative red bar representing AoW in the pool. In WOT, I explored if and how students engaged in what Liang and Moore (2017) called "partitioning activity" (i.e., equally partitioning one of the quantities, then investigating the corresponding change in the other quantity) in order to determine how quantities varied in relation to each other. I conjectured the partitioning activity would help students to conceptualize the invariant relationship between covarying quantities and construct a graph of this relationship accordingly.

Matching Game Task (MGT)

Matching Game Task (MGT) includes the same map of Downtown Athens highlighting a straight road (i.e., College Avenue) with two places located on the road (i.e., the Arch and the Canon; see Figure 0.7, left) and a bike riding on this road. I animated the map so that the bike traveled at a constant speed back and forth along the College Avenue starting at the North side of the road. MGT also included two bars (see Figure 0.7, right) on two parallel magnitude lines next to the situation, one represented the magnitude of the bike's DfA and the other represented the magnitude of the bike's DfC.

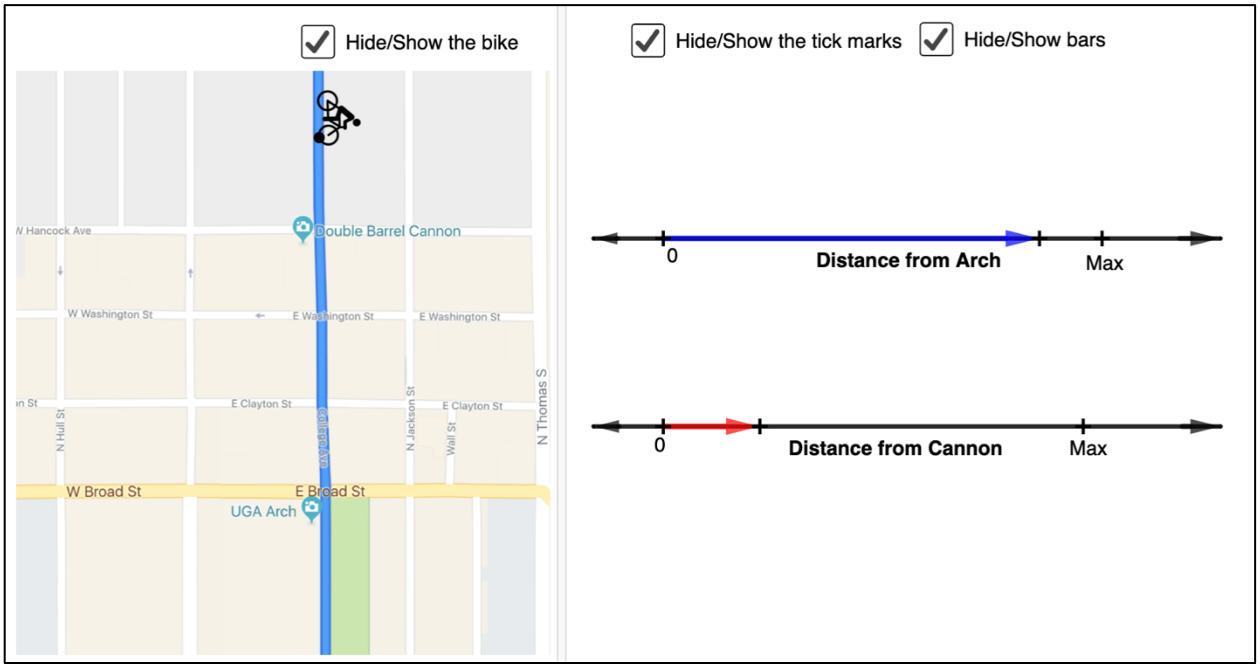


Figure 0.7. Matching Game Task (MGT).

In the first part of MGT, the bike animation and the bars on the magnitude lines were not linked. That is, when the bike moved on the map, the bars didn't move to match with the bike's DfA and DfC. I presented students with a pair of bars of a certain length and asked students where the bike would be at College Avenue for three different cases (see Figure 0.8). Students needed to conceive the quantities on the magnitude lines and re-present them in the situation to locate where the bike would be on the map. I also planned to set up a pair of bars such that their length did not match with the magnitude of DfA and DfC in the situation, that is, it was impossible to find where the bike would be. I conjectured this could help students to understand that the variation of both bars is constrained in a way that they both, respectively, represent the *same* bike's DfA and DfC.

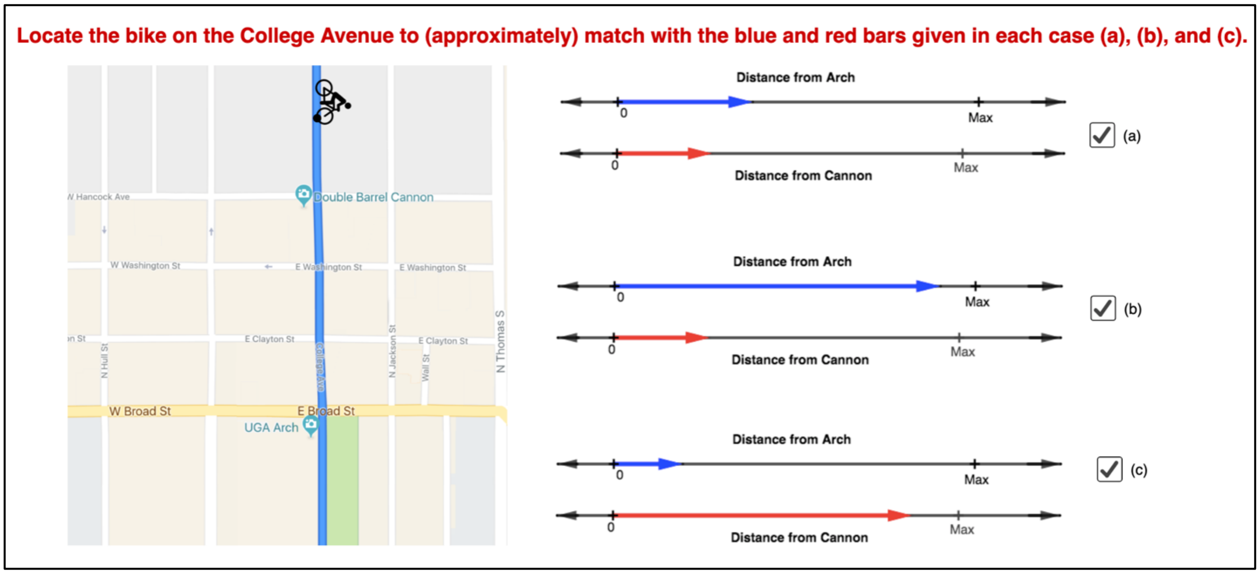


Figure 0.8. MGT Part 1

In the second part of MGT, I presented students with the situation and the two bars located next to each other (see Figure 0.9). The bike on the map and the bars on the magnitude lines were still not linked. I provided students with sliders to change the length of the blue and red bar in order to match with the bike's DfA and DfC as the bike moved on the map (see https://youtu.be/stPXDxJGuPo). I assigned a student to each bar to change their length as I played the bike animation. The goal for the students was to move in sync. I call this task as "matching game with two players." I asked them to be as much as precise in order to match where the bike was on the map. I planned to explore the extent they abstracted and coordinated quantities' magnitudes at the same time when they watched the animation as well as the other controller's movement. From my perspective, there was a constraint on how DfA and DfC vary together as the bike rides along the road. For example, the sum of the bike's DfA and DfC is constant when the bike is in between the Arch and Cannon. I repeated the same game with one player. My purpose was to explore how *one* student coordinate two quantities' magnitudes simultaneously as he or she watched the bike riding in the animation.

Figure 0.9. MGT Part 2

Finally, in the third part of MGT, the bike on the map and the bars on the magnitude lines were linked. The directed bars varied in length as students moved the bike on the map (see https://youtu.be/knrN7XCbyOY). I first hided the red and blue bars on the magnitude lines and asked students to place tick marks on the line where they thought the head of each bar could be for different locations of the bike on the map. Here, my purpose was to help students to connect the conventional use of tick marks on the number lines to the use of bars on the magnitude lines. In this way, they might be able to know that a tick mark (or a point) on a magnitude line represented the measurement of the magnitude of a quantity. This idea could also promote the idea that given *two* tick marks on each number line, students could be able to tell where the bike would be in the situation. Then, I asked the final question that necessitated why we use quantitative coordinate systems (Lee, Hamilton, & Paoletti, 2018) from my perspective. I asked students to produce *a point* that could show exactly where the bike was on the map. Previously, students used two “things” (i.e., two fingers or two tick marks) in order to play the matching game. Here, I asked them if they could come up with a new set up with the bars on the magnitude lines in a way that they could create a single point to represent the two quantities’ magnitudes. I planned to explore if and how student could see the additional representational features that was not available with a single magnitude line, including the property that a point in the coordinate plane represented two quantities’ magnitudes. With this activity, I expected to gain insights into the students’ reasoning when trying to come up with ways to get a point that represents the quantities’ magnitudes.

To solve this task, students must use two quantities’ magnitudes (i.e., DfA and DfC) and coordinate them in a new space in order to produce a point. Although I was aware students could not solve the task completely, at the very least, they had a chance to think about and try to produce *a point* that could represent two quantities’ magnitudes. If students could not produce a point in a Cartesian coordinate system simultaneously representing the two quantities’ magnitudes (i.e., DfA and DfC), I planned to support students by connecting this activity to CT that they previously engaged. Here, I intended to provide a space to students as a given structure along with the parallel magnitude lines still visually available. I expected students to use this space in order to produce a point that simultaneously represents those two quantities’ magnitudes by coordinating and projecting the two quantities’ magnitudes (e.g., dragging the two number lines and dropping them on each axis in order to produce a point).

Downtown Athens Bike Task with a Dynamic Coordinate Plane

Due to design of my instructional sequence until this point, I anticipated that students could conceptualize two quantities whose magnitudes vary in the situation, represent those quantities’ magnitudes by the varying bars on parallel magnitude lines, determine the invariant covariational relationship between those two quantities, and represent two static quantities' magnitudes by a single point on a Cartesian coordinate plane. In this version of DABT (see Figure 0.10; the point on the plane, blue and red bars were not available yet), students were provided opportunity to represent the invariant relationship between two quantities by tracing a point in a Cartesian coordinate system as two quantities vary in tandem.

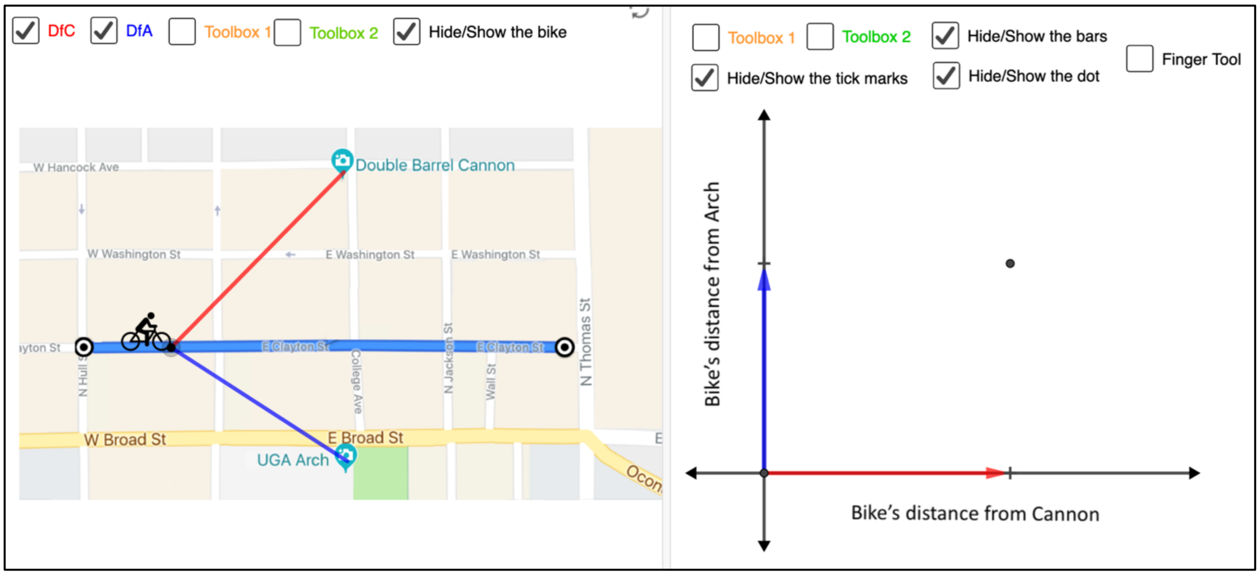


Figure 0.10. DABT with a dynamic coordinate plane

I began with reviewing the previous task asking students to represent DfA and DfC by *one single point*. Then, I asked students to construct a representation to show how the bike's DfA and DfC varied together during his or her trip in Downtown. While constructing their representations, if they didn't refer to what they were doing with the bars, I planned to ask them if there was anything that was similar to how they were thinking about the two bars in the matching game. In the tasks that was implemented earlier, students were provided opportunities to engage with those bars in order to understand how they were changing in relation to each other. Here, I asked students if and how they could use those bars to draw their graph. That is, I asked how they would use the varying bars to construct their graphs showing the relationship between two quantities. I intended to see if students could see the point and emergent trace representing the simultaneous relationship between two quantities.

Data Management

In order to best understand how students construct and interpret graphs, I videotaped the teaching sessions in order to capture the participants' exact words, gestures, and drawings. Students' written work and notes taken by research members who participated in the teaching sessions were also collected as data sources from the sessions. There were two video cameras; one filmed a wide-angle view facing the students from the front and the other filmed a focused view of the students' activities from above (see Figure 0.11). Screen recordings from a tablet device displaying animations were also be collected when used. I created transcripts and digitized students' written work for analysis.

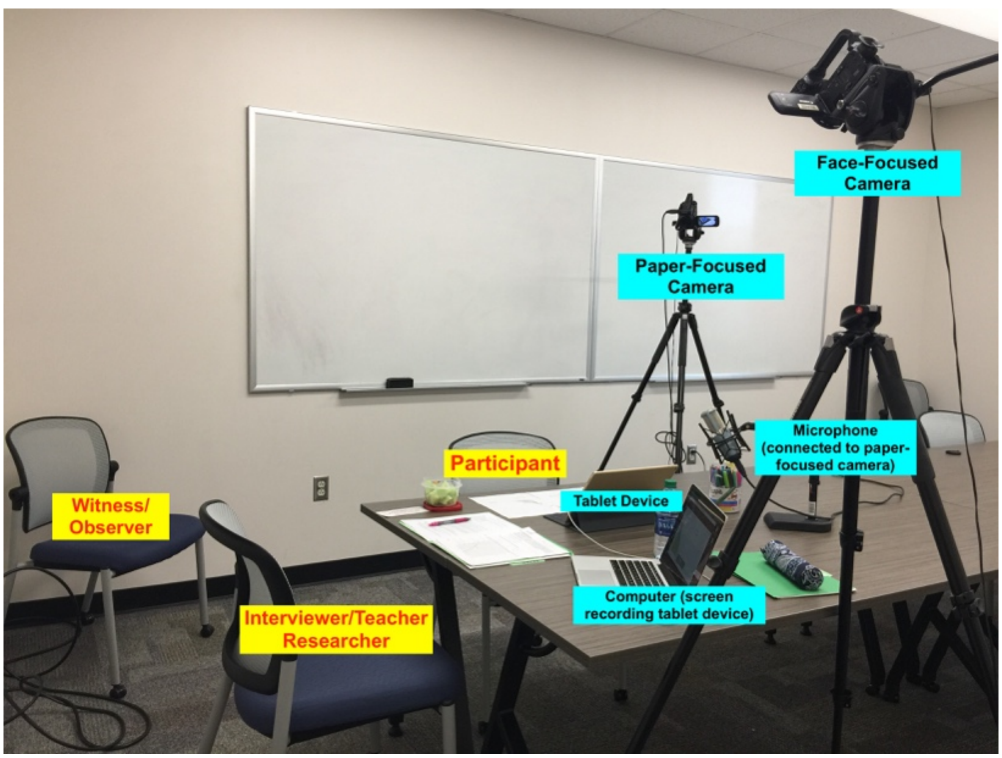


Figure 0.11. Image of the room setup[5].

[5] Thanks to Biyao for taking the photo.

Data Analysis Overview and Procedure Summary

From radical constructivism point of view, I believe I cannot enter students' head in order to investigate what conceptual structures they have associated with certain graphs as students' mathematics is inaccessible to the researchers. Therefore, as a researcher relying on radical constructivism, I can only conjecture what a graph might mean to students when they use it. In another words, I can try to understand (i.e., develop model of student thinking) how their conceived graph looks like by only investigating their displayed graphs along with their reasoning that comes with their displayed graph. As I come across the certain displayed graph again and again in a student's work, I can try to modify or reconstruct my models of the student thinking with a purpose of arriving an interpretation that fits most of the occurrences. Thus, I use a conceptual analysis (Thompson, 2008) in order to understand students' verbal explanations and actions and develop viable models of their mathematics (Steffe & Thompson, 2000).

I conducted both on-going and retrospective conceptual analyses (Steffe & Thompson, 2000; Thompson, 2008). In on-going analysis, I tested and formulated new hypotheses of student thinking throughout the teaching sessions based on the ways the students engaged in the tasks. Based on the result of my on-going analysis, I modified the subsequent session in order to further investigate students' reasoning. For example, if a student experienced perturbation when engaging in the task, I created another task for the subsequent session in order to investigate the issues that lead the student to perturb and the ways that may help him or her in reconciling his or her experienced perturbation. During my on-going analysis, I produced a set of notes about each session regarding students' engagement and ideas, which I read before my retrospective analysis.

In retrospective analysis of data, I revisited the data after completion of the teaching experiment in order to build and revise working models of students' reasoning for generative purposes (Clement, 2000). I first generated tentative models of the middle school students' mathematics by exploring their actions when engaging in graphing tasks. I planned to construct new concepts and theoretical models as I investigated how middle school students develop meanings for representing quantities in coordinate systems. It was a discovery for me rather than confirming what I expected them to think in a particular way, such as relying fully/completely on established constructs. With the aim of developing increasingly stable and viable models of students' mathematics, I tested the viability of my initial models by seeking supporting or contradictory evidence form students' moment-to-moment current ways of thinking throughout the teaching experiment. This leaded me to refine or reject certain preliminary hypotheses.

Then, I analyzed the video recordings from the teaching experiment using an open and axial coding approach (Strauss & Corbin, 1998) for convergent purposes (Clement, 2000). I made a coded analysis of the video recordings in order to provide reliable explanations of the students' activity within shape thinking framework (Moore & Thompson, 2015). The convergent analysis served me to test my hypothesis about students' mathematics (Clement, 2000) and confirm or contradict the models I developed. With the shape thinking framework, my main goal was neither merely characterizing students' thinking as static shape thinking versus emergent shape thinking nor classifying a student as static or emergent thinker. Instead, I characterized students' thinking about graphs because I want to construct a model for students' reasoning process that leads to a certain type of shape thinking. For example, I plan to form theoretical hypotheses concerning the reasoning that causes static shape thinking and the reasoning that is responsible for emergent shape thinking.

At this stage of the analysis, I conducted two passes over the video recordings, each one entailing more directed and elaborate coding of students' engagement and thinking. In the first pass, I coded each teaching experiment session for characteristics of students' interpretations and construction of graphs that emerged as they engaged in the activities. As a result, categories of students' meanings of points in coordinate plane, the axes of the coordinate plane, meanings of the segments drawn both in the situation and in the coordinate plane, the nature and extent of their reasoning emerged. Then, I tested and modified those categories outlined in Figure 4.1 in Chapter 4.

In the second pass, I attempted to connect the categories that I created and seek for potential implications of those categories. Each pass included good amount of note taking in the form of creating memos about students and the teaching experiment. I built the writing of my analysis based on those notes and codes developed in the previous stages of analysis. The process of writing my analysis also provided an opportunity for me to elaborate and refine descriptions and relations generated in previous stages.

For the purpose of my dissertation, of the six participants that went through the teaching experiment, I focused my analysis heavily on four students: Melvin, Naya, Zane and Ella. Dave and Mike are included in the analysis, but I chose not to include them in the results for the following reason. Their thinking was mostly rooted in iconic translation and, this reasoning led them struggle with the notions of magnitudes and their representation. Thus, I did not observe developmental shifts in their meanings, while they exhibited compatible meanings to the other four students in terms of iconic translation.

CHAPTER 4

RESULTS 1

In this chapter, I provide results from the analysis of my teaching experiments exploring middle school students' meanings of points on a plane. I illustrate several meanings including (i) point as an object/location and iconic translation (non-multiplicative object), (ii) point as an object/location and quantitative properties (spatial-quantitative multiplicative object), and (iii) point that represents two quantities' magnitudes (quantitative multiplicative object, Type 1 and Type 2). I also describe these different meanings in terms of representing a multiplicative object in the context of graphing. In relation these meanings, I identify various ways students organized the space including (i) conceiving the reference objects on the axis as a location, (ii) conceiving the reference objects as the entirety of an axis itself, (iii) canonical cartesian coordinate plane, and (iv) non-canonical cartesian coordinate plane. As part of my analysis of the teaching experiments, I also identify various ways of reasoning the students exhibited including (i) quantitative covariational reasoning, (ii) spatial proximity reasoning, and (iii) matching the perceptual features of motion in two different spaces. In addition to these results, I discuss the implications of such meanings and reasoning. Figure 0.1 summarizes the meanings and constructs that I illustrate in this chapter.

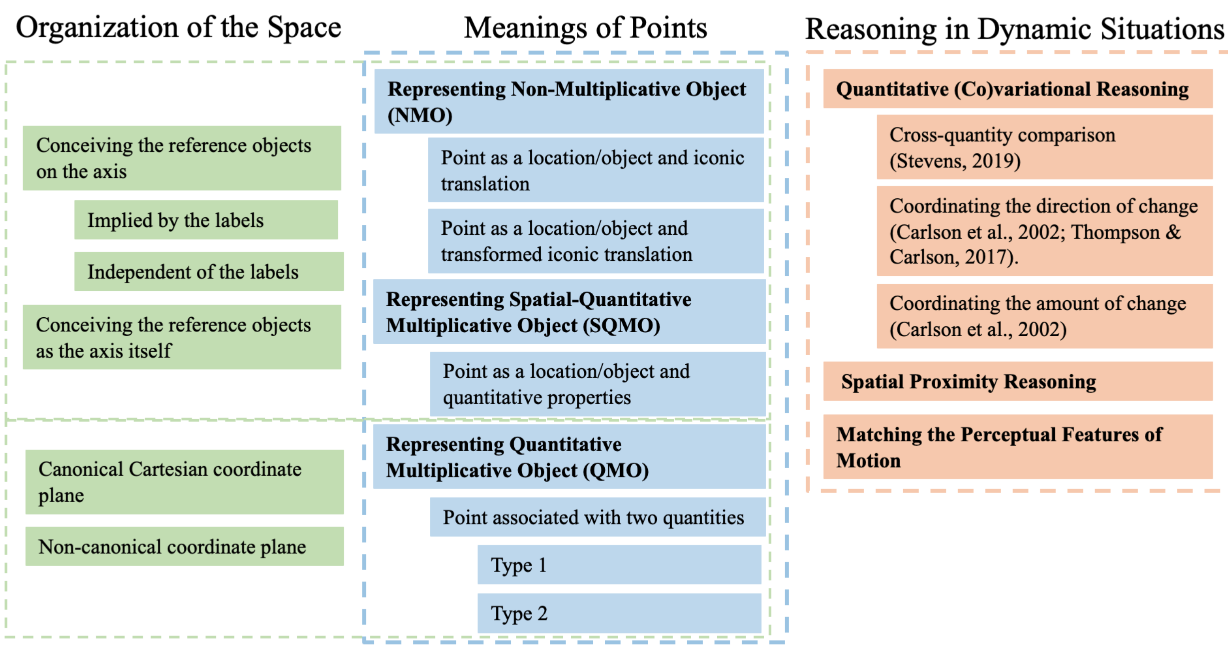


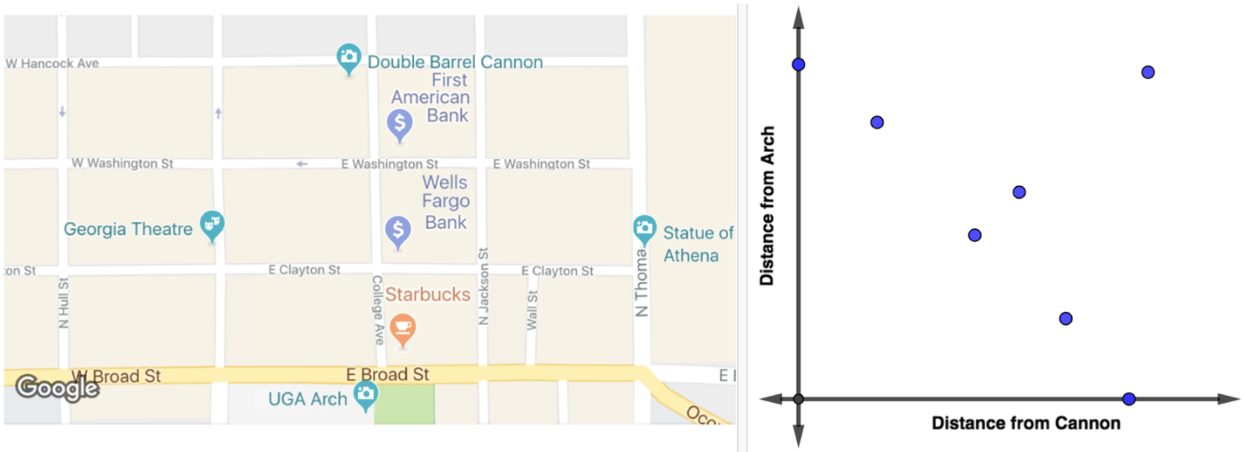
Figure 0.1. The meanings and constructs that I illustrate in this chapter.

Representing a Multiplicative Object in the Context of Graphing

In this section, I illustrate different ways students' graphing activity involved when representing quantitative relationships. I classify those meanings as representing (i) a non-multiplicative object (i.e., point as an object/location and iconic translation), (ii) a spatial-quantitative multiplicative object (i.e., point as an object/location and quantitative properties), and (iii) a quantitative multiplicative object (i.e., point that represents two quantities' magnitudes). I present empirical data from my teaching experiments to illustrate these meanings.

Representing a Non-Multiplicative Object (NMO)

Student's representing a non-multiplicative object involves envisioning points on a plane (see Figure 0.2b) as a location/object that is physically or figuratively associated with a location/object in the situation (e.g., map of Downtown Athens, see Figure 0.2a). I refer to such a meaning as representing a *non-multiplicative object* (NMO) because the point on the plane does not symbolize or unite two quantities' magnitudes or measures. Under the category of NMO, I differentiate between meanings for points on the plane in two ways: (i) the meanings that include *iconic translation*, and (ii) the meanings that include *transformed iconic translation*.



(a) (b)

Figure 0.2. (a) Downtown Athens and (b) Cartesian coordinate plane with points

Point as a location/object and iconic translation

Point as a location/object and iconic translation refers to instances in which students assimilated the points on the plane as a location/object by engaging in iconic translation. Researchers (e.g., Clement, 1989; Monk, 1992) stated that students' activity can be characterized as iconic translation when they view graphs as representing *literal* pictures of a particular situation. That is, when determining which point on the plane is associated with which place on the map, students make an association between the perceptual or spatial features of the situation and the shape of or points forming the graph (e.g., plotted points in this case, Figure 0.2b). In particular, students assimilate a point as a location/object by focusing on the position (e.g., above, below) and positional relationship of an object (e.g., a point, a line) or between objects that appear on the map. For example, a student who operates with this meaning could assimilate the point on the horizontal axis in Figure 0.2b as the physical Arch because Arch on the map (Figure 0.2a) is in the very South as well as the point on the plane.

Note that the action in developing meanings for the points is different than translating the entire picture of the situation to the graph (e.g., copy and paste). Rather, this meaning includes choosing one perceptual property of the situation (e.g., spatial orientation, relative location) and ensuring to preserve that property with the point on the plane. For example, for someone who conceived the point on the vertical axis as the physical Cannon might claim that the closest point on the plane (i.e., the point that is southeast of Cannon on the vertical axis) is the First American Bank because the First American Bank on the map is located southeast of the Cannon on the map (see Figure 0.2a). Thus, such a meaning enables one to organize and identify points based on perceived spatial properties (i.e., relative location). However, students' meanings of points that include iconic translation does not enable them to understand all of the locations of the points on the plane in relation to each location on the map one by one because they are visually different maps. For example, a student who operates with this meaning might experience a perturbation when he or she recognize that there is only one single location that is in the very East on the map whereas there are two points that are on the very East on the plane.

An illustration from Melvin and Naya's teaching experiment. As an illustration of iconic translation, I present Melvin and Naya's activity in Downtown Athens Task (DAT). DAT includes a map with seven locations highlighted and labeled (See Figure 0.2a). DAT also includes a Cartesian coordinate plane whose horizontal axis is labeled as Distance from Cannon (DfC) and vertical axis is labeled as Distance from Arch (DfA). Seven points are plotted without labelling to represent the seven locations' DfA and DfC (see Figure 0.2b). I asked students what each of these points on the plane meant to them with an intention to observe their spontaneous responses and to explore their meanings of points. Figure 0.3 illustrates Melvin and Naya's initial activity in (DAT).

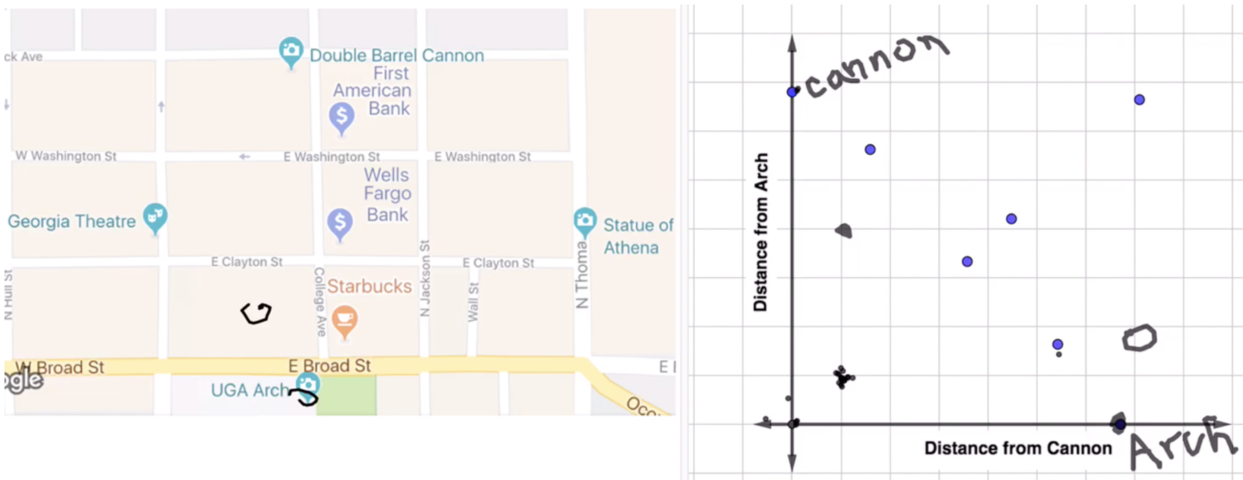


Figure 0.3. Melvin and Naya's activity in DAT (a screenshot from the tablet device).

Melvin and Naya conceived each point on the plane as a location that appears on the map by making perceptual association between the map and the graph. For example, Naya determined that the point on the horizontal axis is the physical Arch because "I see the Arch at the bottom of the map and that [*the point on the plane labeled Arch, see Figure 0.3, right*] is right there [*pointing to the point on the horizonal axis*] where I see the Arch." From this, I infer that Naya's meaning of the point included iconic translation where she made an association between the spatial attributes of the objects on the map (e.g., relative location) and points on the plane (i.e., Arch being at the bottom on the map and on the plane). After Naya and Melvin agreed that the point on the horizontal axis is the Arch, I asked them what they thought the point that is closest to the horizontal axis might represent.

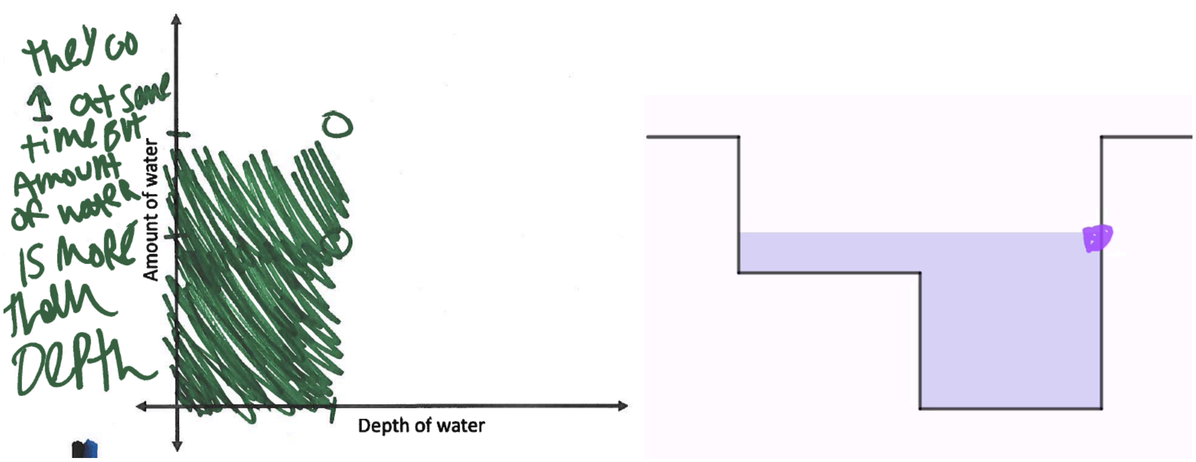
Melvin responded to my question with, "I guess it is Starbucks." But Melvin also appeared confused and unsure of his response because of an inconsistency with the spatial attributes of Starbucks (i.e., Starbucks' location relative to the Arch) within two spaces (i.e., map and plane). Specifically, Melvin conceived that Starbucks is located up and to the right of the Arch on the map whereas Starbucks is located up and to the left of the Arch on the plane. Melvin noted there is no point in the plane to match the location of Starbucks relative to Arch on the map. He said, “on the map [*pointing to the map*], the Starbucks is right there [*pointing to the plane and circling a location right next to the available point on the plane, see Figure 0.3*], not to the left.” He added:

> There is something close to the Arch [*pointing to the point on the horizontal axis labeled as Arch, see Figure 0.3, right*], a little bit up to the left [*moving the pen on the tablet screen upward from the point on the horizontal axis and to the left*], but there is nothing here [*circled a place on the map that is at the left of Starbucks, see Figure 0.3, left*] on the map. But, on the Cannon [*pointing to the point on the vertical axis*], there is something to the right [*pointing to the point that is closest to the vertical axis*], which could be the First American Bank.

I infer that his meaning of points that included iconic translation was viable because Melvin was satisfied by the Cannon’s location being on the vertical axis because there is a point on the plane that is down and to the right and similarly there is a highlighted location on the map that is down and to the right of Cannon which is the First American Bank. Drawing on the same meaning, Melvin expected to have a point on the plane for Starbucks that is located to the right of the Arch instead of to the left. Similarly, Melvin thought that in the “surroundings” of Arch on the plane, there is a point that is “up to the left, but there is nothing in the map.” He wanted an object on the map that is up and left compared to the Arch because there is a point on the plane that is up and to the left of the point on the horizontal axis, which is Arch for him. Note that this meaning of the points with iconic translation includes coordinating spatial orientation (e.g., “up and to the left”) as Melvin wanted to preserve this spatial feature of points on the plane when he moved from the map to the plane or from the plane to the map. From this activity, I infer that using a system of meanings that is mostly iconic translation created a perturbation for Melvin because the location of Starbucks didn't make sense to him in relation to the location of the Arch when comparing across the map and the plane.

An illustration from Ella's teaching experiment. For another illustration, I present Ella's graphing activity in Swimming Pool Task (SPT). I presented her a dynamic diagram of a pool (see Figure 0.4a. or https://youtu.be/jXaFMSVJ73E for a dynamic illustration), which enabled her to fill or drain the pool by dragging a point on a given slider. I asked Ella to sketch a draft of a graph that represents the relationship between amount of water (AoW) and depth of water (DoW) as the pool fills up. In this activity, Ella associated her point that she plotted on the plane (see the small circle in Figure 0.4a) with the point she plotted in the situation (see the purple dot in Figure 0.4b) and she imagined the water in place of the shaded area on the plane (see her scribbles in Figure 0.4a).

Ella began by inserting two tick marks on each axis as an indication of AoW and DoW (Figure 0.4a). She noted, as the pool is filled up, both tick marks "go up" along the axis at the same time and wanted to place the tick mark for AoW further along the vertical axis than the tick mark for DoW since she thought "they [AoW and DoW] are going up at the same time, but the amount of water is always gonna be bigger than actual depth of water."



(a) (b)

Figure 0.4. (a) Ella's first draft in SPT and (b) the swimming pool and Ella's point

Next, Ella drew a small circle on the plane to show "where those two things [*tracing her finger horizontally from the tick mark on the vertical axis to the circle in the plane, then vertically down from the circle in the plane to the tick mark on the horizontal axis*] meet here." This suggested that Ella imagined the quantities' measures along the axes and created a point on the plane where the projections of these tick marks intersected. She then shaded the rectangular area on the plane, what she called "a box" (see Figure 0.4a), to show "a bunch of water." From her description (i.e., "a box" and "a bunch of water"), I was surprised because she seemed to be making perceptual association to the pool situation in relation to what she drew on the plane. Based on this inference, I hypothesized that Ella was drawing on two different meanings here: quantitative meanings when she plotted tick marks on the axes and iconic meanings when she imagined "a box" to show "a bunch of water" on the plane after plotting her point. In order to identify which meaning was more dominant and which one she would continue to assimilate with, I asked her to explain what the point she plotted (i.e., small circle) meant for her in terms of the pool situation. I intended to see whether she related the point to the quantities (i.e., AoW and DoW) in relation to the tick marks that she placed on each axis or if she related the point to the situation perceptually.

Ella added a point in the pool situation (see her purple point in Figure 0.4b) and said, "I guess there" relating her graph perceptually to the pool situation. When asked to explain more about the point she plotted on the plane in relation to the tick marks, she said,

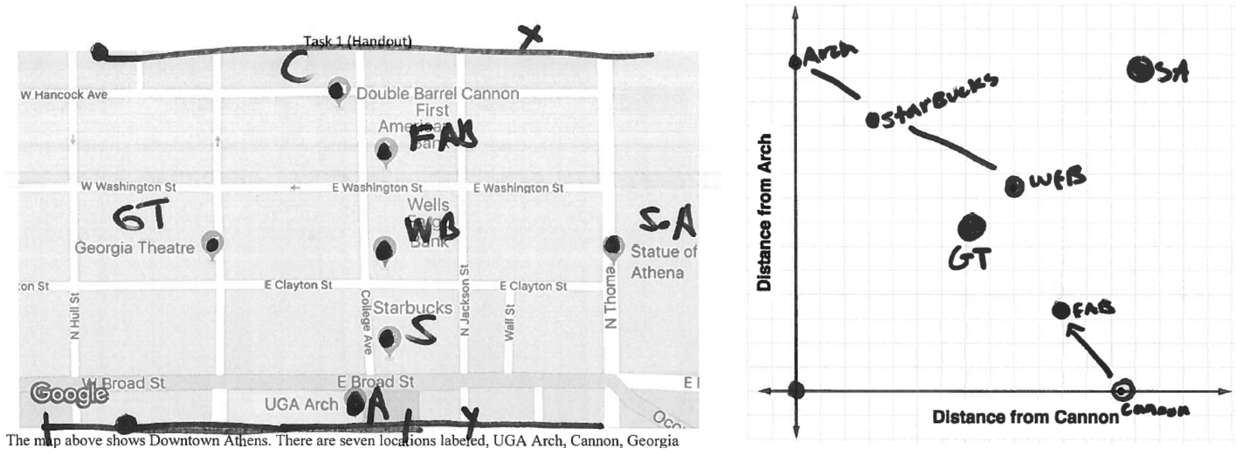
> that is just like the dot between [*tracing her finger horizontally from the tick mark on the vertical axis to the circle in the plane*] here [*tracing her finger vertically down from the circle in the plane to the tick mark on the horizontal axis*] so I can just make this box [*pointing to the shaded area in the plane*].

I infer that Ella was able to plot a point on the plane respective of the tick marks that she placed on each axis. However, Ella conceived the point as a landmark to draw "the box," which is a contraindication that her meaning of the point included the quantities' measures (i.e., AoW and DoW). Although Ella reasoned about the attributes when placing the two tick marks on the axes and used those tick marks in order to generate the point (i.e., the circle in Figure 0.4a), Ella's meanings of the point didn't include uniting the attributes of an object (i.e., AoW and DoW) in the plane; instead, Ella conceived the point in terms of a mark as a part of a procedure to set the corner for the box.

Point as a location/object and transformed iconic translation

Point as a location/object and transformed iconic translation is similar to ones above in nature because it also relies on iconic translation. However, there is a nuanced difference between iconic translation and transformed iconic translation. The previous category (i.e., iconic translation) illustrates instances in which students translate the perceptual features of the situation *as it is* to the plane (or from the plane to the situation), whereas a transformed iconic translation illustrates instances in which students translate a *transformed* version of perceptual features of the situation to the plane (or the opposite direction). The transformation can take on different forms. Some of the transformations included, but not limited to, rotating the page with the picture of a situation or the graph clockwise or counterclockwise, flipping the page vertically (i.e., reflection across a horizontal line), and possibly in combination of both.

An illustration from Naya's teaching experiment. As an example of transformed iconic translation, I present Naya's activity in Downtown Athens Task as her meaning of points included representing a location/object by making transformed iconic translations.



(a) (b)

Figure 0.5. Naya's graphing activity in DAT.

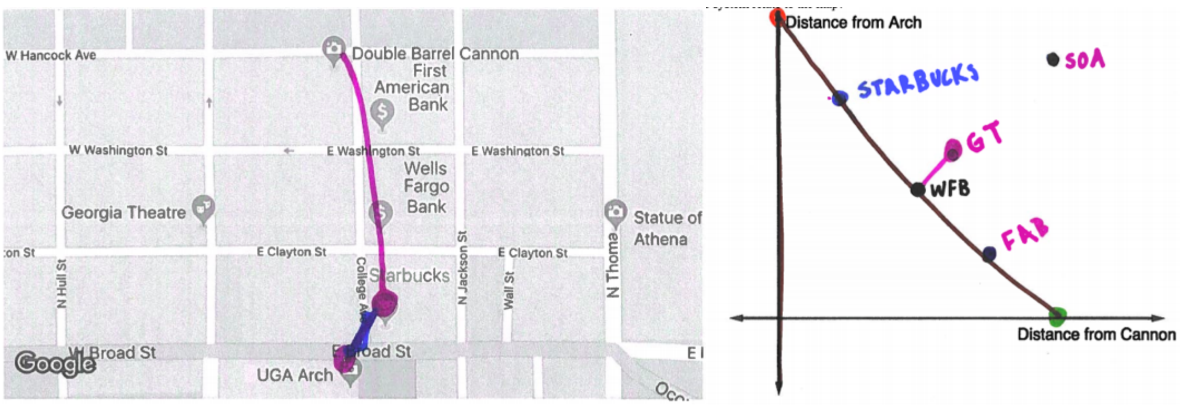
Recall that both Melvin and Naya's meanings of the points included iconic translation in DAT (see the previous section). Also recall that Melvin's meaning that included iconic translation created a perturbation for him because there was a mismatch between the spatial features of the map and the plane (i.e., Starbucks is located up and to the right of the Arch on the map, whereas Starbucks is located up and to the left of the Arch on the plane, see Figure 0.3). Melvin's argument about the location of Starbucks also created a perturbation for Naya for the same reason as Melvin. In order to reconcile this perturbation, Naya suggested making an adjustment on the plane that involved flipping the map. She said "maybe that [*pointing to the point on the horizontal axis that they labeled Arch in Figure 0.3*] is the Cannon not the Arch. … and that [*pointing to the point on the vertical axis labeled as Cannon in Figure 0.3*] would be the Arch." This change not only included simply switching the two places on the plane but also Naya imagined flipping the map vertically so that Arch is at the top and the Cannon is at the bottom on the map so that the location of Arch being on the vertical axis and Cannon being on the horizontal axis made sense in terms of her meanings included iconic translation.

With this flipped image of the map having Arch at the top and with the new organization of the plane having Arch on the vertical axis of the plane (see Naya's graph in Figure 0.4b), Naya reconciled her perturbation in terms of the location of Starbucks because Starbucks is now located to the right and down compared to the Arch both on the plane (see Figure 0.4b) and on the flipped map. She further explained:

> Once we flipped it [*referring to the map*] around, all that stuff [*pointing to the top part of the map*] is down here [*pointing to the bottom part of the map*], so this [*pointing to the horizontal axis*] is the bottom, and I considered the top kind of over here [pointing to the vertical axis] because we don't have a top [*pointing to the top part of the plane*].

This provided evidence that Naya made a perceptual association between the vertical/horizontal axis of the coordinate plane and the bottom/top of the map. In order to illustrate what she meant, she drew two parallel lines, one is at the bottom and one is at the top of the map (see Figure 0.4a). Naya imagined that the line at the top of the map is the horizontal axis of the plane (what she called the x-axis) and the line at the bottom of the map is the vertical axis of the plane (what she called the y-axis, see the labels on the map in Figure 0.4). With this image of the map, Naya conceived the highlighted locations (e.g., Georgia Theater, Starbucks, etc.) as a whole as the things "in the middle" on the map and associated those locations to the points that are "in the middle" on the plane as a whole. Moreover, she made perceptual association between how the points on the plane and the locations on the map "lined up." For example, she determined Arch, Starbucks, Wells Fargo Bank form a straight line on the map and the points labeled Arch, Starbucks, and Wells Fargo Bank also forms a straight line on the plane (see her line segments that connect these points on the plane in Figure 0.4b). From this activity, I inferred that Naya's meaning of the points included transformed iconic translation since she used the perceptual feature of the transformed map and the plane (e.g., "they lined up") in making sense of the points on the plane.

An illustration from Zane's teaching experiment. The following example is from Zane's teaching experiment in DAT and illustrates the characterization of transformed iconic translation. Zane's meanings of the points on the plane included the physical objects that appears on the map.



(a) (b)

Figure 0.6. (a) Zane's "line" on the map and (b) Zane's "line" on the plane.

Zane conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively, as implied by the labels (see orange and green dots on each axis for Arch and Cannon, respectively in Figure 0.6b). Since Arch is at the top and Cannon is at the bottom of the plane, Zane imagined rotating the map in way that Arch is at the top and Cannon is at the bottom of the map. He said, "I flipped the map [*pointing to the map in Figure 0.6*] because Arch is on this side [*pointing to the bottom of the map*] and Double Barrel Cannon is on the side [*pointing to the top of the map*]." In order to make sense the other points on the plane, Zane described:

> Like, this right here [*placing the tip of the pen on the orange point in Figure 0.6b*] from Arch, and this [*moved the pen and put it on the blue point*] probably Starbucks, this [*moved the pen and put it on the black point labeled WFB seen in Figure 0.6b*] would be like this right here [*pointing to Wells Fargo Bank on the map seen in Figure 0.6*Figure 0.6*a*], and [*moved the pen and put it on the purple point labeled FAB seen in Figure 0.6b*] the First American Bank right here [*pointing to First American Bank on the map*] would be here, to get here [*moved the pen and put it on the green point on the horizontal axis*] and it would be a straight line [*moving the pen along the path from orange point to green*

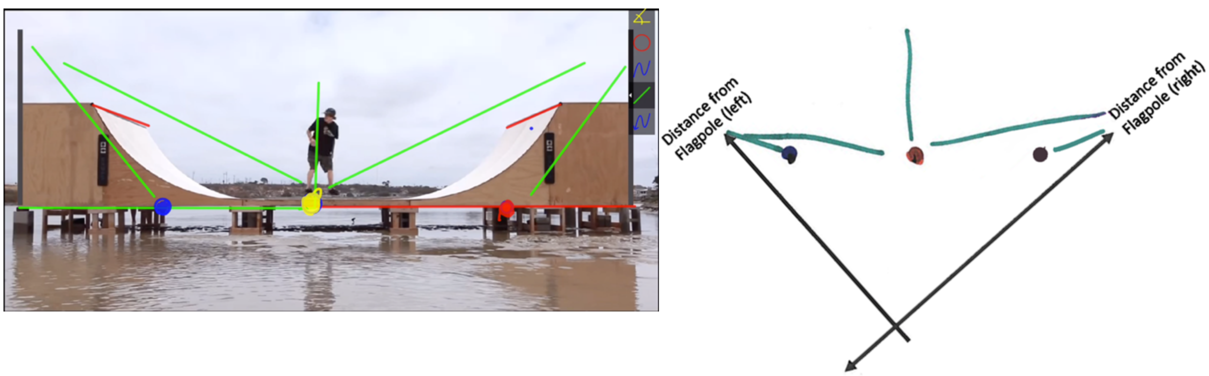
point on the plane].

He then made explicit association between the “line” that he drew on the map as seen in Figure 0.6a. and the line he drew on the plane as seen in Figure 0.6b. From this activity, I inferred that Zane’s meaning included transformed iconic translation since he rotated the map and overlaid it into the plane in order to match the line that he drew on the map with the line he drew on the plane.

An illustration from Zane’s teaching experiment. For another example, I present Zane’s graphing activity in Skateboarder Task. The situation includes a video with a skateboarder moving back and forth on a halfpipe. I also presented a Cartesian coordinate system whose horizontal axis is labeled as distance from flagpole (right) and vertical axis is labeled as distance from flagpole (left, see Figure 0.7). Zane watched the video and were asked to interpret the given points on the plane.

Figure 0.7. Skateboarder Task.

Zane first imagined the axis of the plane as the physical flagpoles. He said “This [*referring to the vertical axis*] is flagpole left and this [*referring to the horizontal axis*] is flagpole right.” When asked to explain what each of these points on the plane might represent, Zane drew three dots in the situation (see blue, yellow, and red dots that are colinear in Figure 0.8a) and he matched the points from left to right on the plane to the blue, yellow, and red dots in the situation, respectively. Zane determined that the points on the plane “make a straight line [*tracing his finger diagonally on the three points on the plane on paper*]” and his dots in the situation makes a straight line too. When asked to explain why the point in the left side on the plane is associated with the blue dot in the situation, he rotated the paper with the graph counterclockwise in a way that the points on the plane are horizontally aligned, similar to the dots he plotted in the situation (see Figure 0.8a). He then put the rotated paper next to the tablet screen where he associated each point in the graph from left to right with the dots in the situation from left to right. Note that I didn’t have evidence that Zane associated the dots in the situation with the skateboarder at the moment. Thus, Zane drew the three dots in the situation to represent three locations (without making a relation to the skateboarder) in the situation that are related the three given points on the plane.



(a) (b)

Figure 0.8. (a) Zane’s work in Skateboarder Task, and (b) Zane’s rotated graph.

When asked to explain why he rotated the paper, he said:

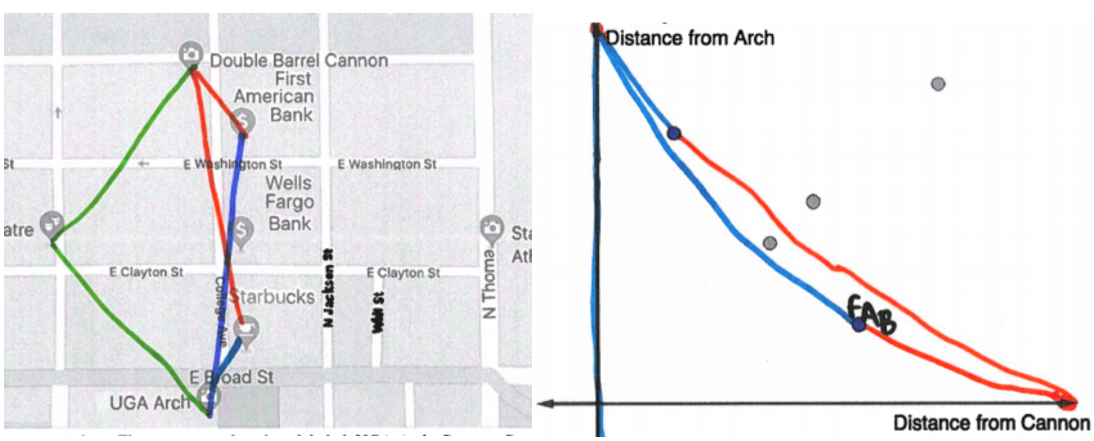
> Like, in this way [*rotating the coordinate plane on paper back to its original position*], it [*referring to the graph on paper*] doesn’t *look* [emphasis added] like this [*pointing to the situation*], but if you turn like this [*rotates the paper counterclockwise*], you see the lines [*pointing to the line segments that he drew on paper*], the green lines go like that [*tracing his finger on the green lines on both paper and in the situation on the tablet screen*].

Other than perceptually associating the three dots in the situation with the three points on the plane, Zane also made perceptual association with the green line segments that he drew both on the situation and on the plane (see Figure 0.8). For example, after rotating the plane on the paper, Zane associated the green segment that he drew *on* the skateboarder in the situation to the corresponding green segment on the plane (i.e., they are both vertical line segment after rotating the graph, see Figure 0.8). Based on Zane's activity, Zane conceived these line segments as visual connection between places in both map and plane. From this, I infer that Zane's meaning of the points included transformed iconic translation where he perceptually associated each point on the plane to a location in the situation.

Representing a Spatial-Quantitative Multiplicative Object (SQMO)

In this section, I illustrate a meaning for points as representing what I call a spatial-quantitative multiplicative object (SQMO). It is spatial and quantitative because the meaning of the points under this category includes both spatial and quantitative features of the situation. In other words, student's representing a SQMO involves envisioning points on the plane as a *location/object* by focusing on the object's *quantitative* properties and engaging in quantitative reasoning (e.g., gross comparison of two quantities' magnitudes). That is, the students' meanings of the points included determining quantitative features of an *object* in the situation (i.e., its distance from Arch and its distance from Cannon) and ensuring to preserve these quantitative properties on the plane. Moreover, students who represent a SQMO draw segments in the situation as *indication* of quantities' magnitudes (i.e., apparent magnitudes) and by making gross additive comparisons among those apparent magnitudes and be able to transform (i.e., dis-embedding and re-presenting) those magnitudes from situation to the graph (and from graph to the situation).

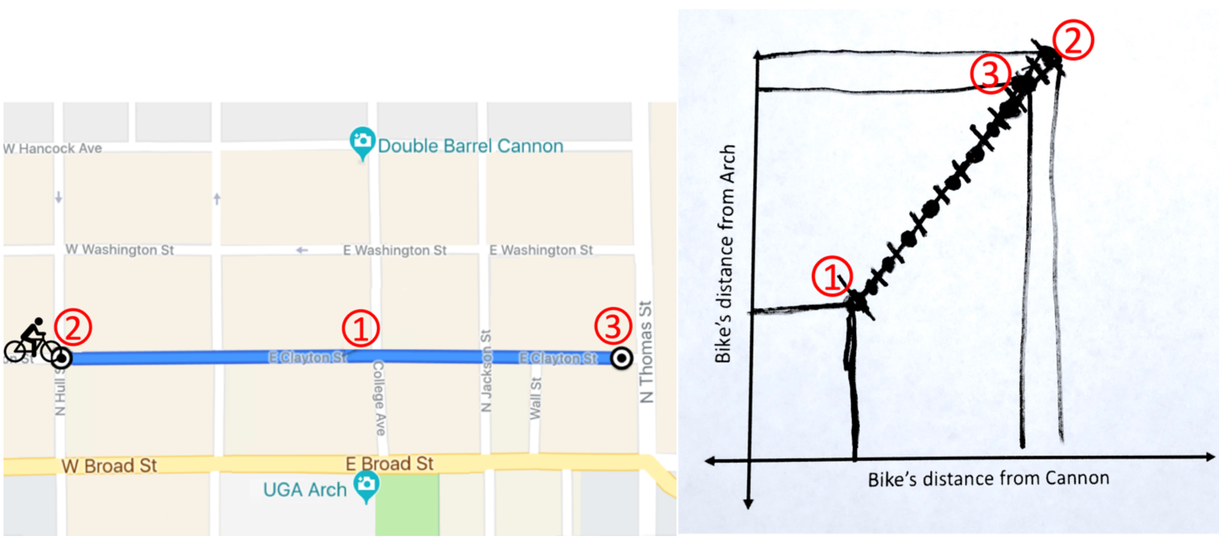
An illustration from Ella's teaching experiment. To illustrate SQMO, I present Ella's activity in DAT where she essentially formed, from my perspective, a two-center bipolar coordinate system based on gross comparisons between the two quantities' magnitudes. That is, Ella conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively, as implied by the labels (see orange dots on each axis in Figure 0.9b). Then, she made sense of the rest of the space by coordinating the radial distances between "places" on the plane and Arch and Cannon on each axis. For example, Ella labeled a point as "FAB" on the plane (see Figure 0.9b) indicating the point represents the physical First American Bank. Ella conceived the point as FAB because referring to the orange and blue line segments that she drew on the plane (Figure 0.9b), she said "the orange is shorter, and the blue is longer." Referring to the line segments on the map (see Figure 0.9a), she added, "over here, like the same thing" showing FAB is closer to Cannon and farther from Arch in the map as well as in the plane. To locate FAB in the plane, Ella represented FAB's distance from Arch as the distance from the Arch on the vertical axis and FAB's distance from Cannon as distance from the Cannon on the horizontal axis. Therefore, Ella's meaning of the points included spatial-quantitative multiplicative object (i.e., points as a location/object by coordinating quantitative features).



(a) (b)

Figure 0.9. (a) Ella's segments in the situation, (b) Ella's segments in the plane

An illustration from Melvin's teaching experiment. For another illustration of this characterization, I present Melvin's activity in Downtown Athens Bike Task (DABT). In DABT, I showed Melvin the animation where the bike rides on Clayton St. in Downtown Athens (see Figure 0.10a) at a constant speed starting from the West side of the street. The animation represents the bike's ride back and forth. I asked Melvin to draw a sketch of the relationship between the bike's distance from Arch (DfA) and the bike's distance from Cannon (DfC) as the bike moved along the road. The reader can see Melvin's graph in Figure 0.10b. Melvin conceived the vertical and horizontal axes of the plane as the psychical Cannon and Arch, respectively. Melvin then conceived each point on his graph as the psychical bike moving on its path on the plane according to the variation of its DfA and DfC that are represented by horizontal and vertical segments drawn on the plane (Figure 0.10b). The details of his activity is as follows.



(a) (b)

Figure 0.10. (a) DABT, (b) Melvin's graph (numbered labels are added for the reader).

When constructing his graph, Melvin began drawing a line upward from left to right to represent "where the bike travels." He then added tick marks and dots on his line graph "to represent like where the bike could be." For example, when the bike is at a location labeled #1 in the map (i.e., its DfA and DfC are at their minimum, see *Figure 0.10*a), he said "it [the bike] would go right here [*pointing to the tick mark on his graph near label #1 in Figure 0.10b*]" and circled a dot on that tick mark to show where the bike is on his graph (see Figure 0.10b). Then, he drew the horizontal and vertical line segments to show the bike's DfA and DfC, respectively, for this moment of the bike's travel. He said "this distance [*moving his finger up and down over the vertical line segment that he drew on the plane*] is the distance from the Cannon and this distance [*moving his finger over the horizontal line segment that he drew on the plane*] is the distance from the Arch." this suggested that Melvin conceived the length of the segments on the plane as the magnitudes of the quantities (i.e., the bike's DfA and DfC). He then added:

> And as it [*pointing to the bike on the map*] goes to either of these points [*pointing to the dots on each end side of the blue path that the bike travels on the map, see Figure 0.10a*], it [*the bike on the graph*] will be right here [*circling a dot on his graph labeled #2 in Figure 0.10b*] or right here [*circling another dot on his graph labeled #3 in Figure 0.10b and drawing the line segments from each dot to each axis of the plane, see Figure 0.10b*].

Referring to the two newly added dots on his graph, I asked him "which one refers to which locations on the map?" Melvin said:

> This dot [*pointing to the dot that is labeled #3 in Figure 0.10b*] is on the right side [*pointing to the right end side of the path on the map*] and this one [p*ointing to the dot that is labeled #2 in Figure 0.10b*] is on the left side [pointing to the left end side of the path labeled #2 on the map] … because the distance from there [*pointing to the right end side of the path labeled #3 in the map*] to there [*pointing to the Arch and Cannon on the map*] is shorter than the distance from there [*pointing to the left end side of the path labeled #2 in the map*] to there [*pointing to the Arch and Cannon on the map*].

This activity suggested that Melvin's meaning of the points included the bike by focusing on the bike's quantitative properties and engaging in quantitative reasoning (e.g., gross comparison of two quantities' magnitudes). He represented the bike's DfA and DfC as the point's (i.e., the bike for him) distance from the vertical and horizontal axis of the plane, respectively. Increasing length of the horizontal and vertical line segments on the plane indicated an increase in the bike's

DfA and DfC. The length of horizontal line segments is shorter than the length of the vertical ones for each point on the graph because Melvin determined that the bike's DfA is always less than the bike's DfC. Note that, although Melvin assimilated his graph as "where the bike travels," Melvin's graph was not the bike's path *as it is* seen in the map; it was not an iconic translation. Melvin's meaning of the points included determining quantitative features of the bike in the situation (i.e., its distance from Arch and its distance from Cannon) and ensuring to preserve these quantitative properties on the plane. Melvin drew segments as indication of quantities' magnitudes and be able to transform (i.e., dis-embedding and re-presenting) those magnitudes from situation to the graph.

Implication of Representing NMO and SQMO

In this section, I present the implications of representing NMO (i.e., points as location/object with iconic translation) and representing SQMO (i.e., points as a location/object with quantitative properties) by contrasting Zane's and Ella's activity and their reasoning in Crow Task (CT). CT's set-up is the same as DAT, but rather than a focus on landmarks, the focus is on a crow that flies in Downtown Athens (see Figure 0.11). My original goal with this task was to create an environment for students to see if the ones whose reasoning based on iconic translation could experience perturbations. Results showed that the task didn't yield a perturbation for Zane whose meanings of the points included iconic translation (i.e., NMO), but did for Ella whose meanings of the points included coordinating quantities' apparent magnitudes (i.e., SQMO).

Figure 0.11. The Crow Task.

Zane's activity in CT. While Zane was moving the crow on the map to and from the Arch and Cannon, he described the corresponding point on the plane as moving "in a straight line." Then, I moved the crow from Arch to Cannon in an upward direction in the map (see Figure 0.11 for the map and the coordinate plane) and asked Zane to explain his observations regarding the moving black dot on the plane. Zane said "[the dot] is going up ... he [*pointing to the crow in the map with the left index finger*] moves this way [*moving his index finger upward*], the dot [*pointing to the corresponding black dot on the plane*] moves that way [*moving his index finger upward*] too." I infer that Zane made a perceptual association between the nature and direction of the movement of the crow in the map (i.e., straight and up) and the nature and direction of the movement of the dot on the plane (i.e., straight and up). Zane also moved the crow in order to place it on top of Cannon in the map and, in turn, the corresponding point on the plane moved to and stopped on the vertical axis where it says "Distance from Arch." Recall that during the previous task, Zane determined that point as corresponding to the Arch (see orange dot in Figure 0.6b). It is noteworthy that this experience did not seem important to Zane; he was not perturbed by this. Zane engaged in similar experience several times (i.e., putting the crow on top of a place in the map and observing where the corresponding point located on the plane);

however, Zane could not assimilate this activity as a way to find what the points on the plane represents in terms of situation.

Ella's activity in CT. Different than Zane, Ella observed that when she moved the crow to the Cannon on the map, the corresponding point on the plane moved "farther from Cannon" (referring to the orange dot on the horizontal axis labeled "Distance from Cannon" that she claimed to be Cannon, see Figure 0.9b). That is, Ella noticed that the point that is associated with the crow moved in an opposite way than she established earlier. This was a perturbation for Ella and she explained, "it is probably farther, we have been doing wrong the whole time." She added, "So, it is basically like the opposite." Recall that Ella formed a two-center bipolar coordinate system based on gross comparisons between the two quantities' magnitudes when making sense of the point in the plane (see Figure 0.9). She conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively. Now, Ella switched the two centers (i.e., switching the location of Ach and Cannon) in order to reconcile by keeping the other properties of the system the same (e.g., the way she perceived close and far relative to the two centers). She explained, "if it [*the corresponding point on the plane*] is closer to this [*the label on the vertical axis, "Distance from Arch"*], it means it [*the crow in the map*] is farther away [*from Arch in the map*]." Later, to explain why she labeled the point as "Cannon" on the vertical axis near the label "Distance from Arch," Ella described

> Last time we did, the distance from Arch [*referring to the label on the vertical axis*] actually wasn't how close it was, but how far away it is, and, on the map [*pointing to the map*], Cannon is really far from the Arch.

Zane attended to the perceptual feature of the motion of the corresponding point on the plane when he moved the crow on the map, and thus he did not assimilate the activity as a venue to find what the points represent on the plane. In turn, he was not perturbed by the Crow task.

Ella, however, was perturbed and re-organized the space based on what she determined since she was attending to how the crow got farther from Arch and/or Cannon in the map in relation to how the corresponding point got farther from "Arch" and "Cannon" on the plane.

Note that Ella's reconciliation was merely switch (i.e., switching the location of Arch and Cannon) because she was only coping with a perturbation by doing the opposite of what she did. Ella was not able to explain why she switched other than showing the system worked after the switch (i.e., when moving the crow on top of Arch in the map, the corresponding point in plane moved near the label "Distance form Cannon" where she though where "Arch" is).

Organization of the space when representing NMO and SQMO.

In this section, I illustrate different ways students' organization of the space included when engaging in graphing tasks in teaching experiments. These organizations are directly tied to NMO and SQMO in which students assimilate points on the plane as a location/objects. I included this section as its own (separate than the other the previous two) in order to underscore how students conceive the axes of the coordinate plane regarding their goals and relative to the situation. That is, students first needed to conceive the reference objects (e.g., Arch, Cannon, flagpole, base of the pool) somewhere on the coordinate plane in order to make sense of the space and other points on the space. I identified two different conceptualizations regarding how students organized the given space: (1) conceiving the reference objects *on* the axis as a location (e.g., assimilating a point on the vertical axis as the physical Arch) and (2) conceiving the reference objects *as* the entirety of an axis itself (e.g., assimilating the horizontal axis as the base of the swimming pool).

Conceiving the physical objects *on* the axis

The first category included two sub-characterizations that refers to the same underlying reasoning; however, there is a minor difference in terms of initiation of students' activities. Students conceived the objects as a location on the axis either (a) implied by the labels of the axis (e.g., Cannon is located on the horizontal axis near label "Distance from Cannon") or (b) independent of the labels of the axis.

Conceiving objects on the axis implied by the labels. As an illustration of this characterization, I direct the reader to the example of Zane that I presented in the section of "Point as a location/object and transformed iconic translation." In DAT, recall that he began his activity by conceiving Arch on the vertical axis where it says, "Distance from Arch" (see orange dot in Figure 0.6b) and conceiving Cannon on the horizontal axis where it says "Distance from Cannon" (see green dot in Figure 0.6b). Another example is as follows: Ella said, "when it says distance from Arch [*pointing to the label on the vertical axis*], that is where the Arch is [*pointing to the same area on the vertical axis again*]." Recall that Ella switched the locations of Arch and Cannon on the plane as a result of her perturbation in CT (see the section of "Implication of Representing NMO and SQMO"). These incidents can still be considered under this characterization since her actions are still implied by the label because she established a "reverse" relationship rather than making a major accommodation. Ella said if it says "Distance from Arch" on the label that means Cannon is there.

Conceiving objects on the axis independent of the labels. This category refers to instances in which students conceived the objects (e.g., Arch and Cannon) on the axis for some reason independent of the labels of the axis. Importantly, in such cases, the location of the objects are not arbitrary points. For example, as I already illustrated in the section of "Point as a location/object and iconic translation," in DAT, Naya assimilated the point on the horizontal axis labeled as "Distance from Cannon" as the physical Ach because Arch is located at the bottom of the map, so the point at the bottom of the plane is the Arch for her (see Figure 0.3, right). Similarly, she assimilated the point on the vertical axis labeled as "Distance from Arch" as the Cannon because the Cannon is located at the top of the map. Note that where to put Arch and Cannon was not implied by the labels of the axis, as she associated the spatial features of the objects both on map and on the plane (e.g., Arch is at the bottom).

For another example, I present Ella's graphing activity in DABT after her engagement in in a task where she worked with a dynamic representation that depicted varying directed bars on two parallel number lines as representations of the bike's DfA and DfC (see Figure 0.26). I asked her to sketch a rough graph of the relationship between the bike's DfC and the bike's DfA. As seen in Ella's graph in Figure 0.27c, Ella conceived Arch and Cannon at the bottom and left side of the vertical and horizontal axis, respectively, which is different than conceiving Arch and Cannon near the labels of the coordinate system. Note that where to put Arch and Cannon was not implied by the labels of the axis, as she plotted a point as the Cannon at the very left side of the vertical axis, which is consistent with what she explored in the previous activity with parallel number line (e.g., moving to the right on the number line shows "the bike's distance is increasing" for Ella). For more details of the illustration of this characterization, I direct the reader to the section of "The implication of the way Ella and Zane assimilated the bars."

Conceiving the physical objects *as* the axis itself

This category refers to instances in which students conceived an axis in its entirety as an object (e.g., Arch or Cannon, flagpole, base of the pool). To illustrate, consider Zane's activity during CT where he assimilated the horizontal and vertical axis as the Arch and Cannon, respectively. I asked him to move the crow in the situation and observe how the corresponding point moved on the plane (see Figure 0.11). When moving the crow, Zane said:

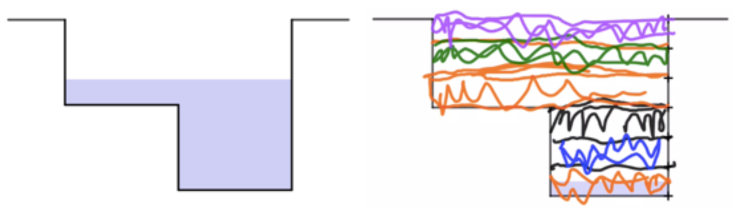
> when it [the crow] go to the Arch on the map [*dragging the crow and dropping it on the Arch in the map*], it [the corresponding point on the plane that moved according to the crow] goes by the line [*pointing to the horizontal axis labeled as Distance from Cannon, see Figure 0.11*].

For Zane, it seemed it was inconsequential where the dot "goes by the line" in CT, although each time he dragged the crow on top of Arch, the black dot went to a certain place by the horizontal axis. I do not have evidence that Zane gave attention to the placements of the dot on the axis with a reason that the dot has a certain property other than being near the axis. Thus, the location where the dot goes by the line was arbitrary for Zane and it could have been placed anywhere near the line. Zane then claimed that "this [*pointing to the horizontal axis*] is Arch." I asked what he meant by "this." He replied, "the line [*tracing the pen in the air over the horizontal axis from left to right*]." He also said, "this [*pointing to the vertical axis*] is Double Barrel Cannon" for the same reason (i.e., when moving the crow on top of Cannon in the map, the point moved by the vertical axis labeled as Distance from Arch, see Figure 0.11). His activity suggested that Zane assimilated the horizontal axis as Arch and the vertical axis as Cannon for the purpose of defining the dot's general proximity to those objects.

Representing a Quantitative Multiplicative Object (QMO)

In this section, I illustrate students' meanings for points that included a representation of a quantitative multiplicative object (QMO)—a conceptual object that is formed by uniting in the mind two or more quantities' magnitudes or values simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017). That is, students' representing a QMO included envisioning a single entity on the plane as symbolizing the two quantities' magnitudes or values simultaneously. Note that students' organization of the space does not have to be compatible with a Canonical Cartesian Plane (CCP) when representing a QMO. Students can also represent QMO by using a Non-canonical Cartesian Plane (NCP). To illustrate this meaning, I present Zane and Melvin's activity in SPT. Zane's meaning of the point included representing QMO on CCP whereas Melvin's meaning of the points included representing QMO on NCP.

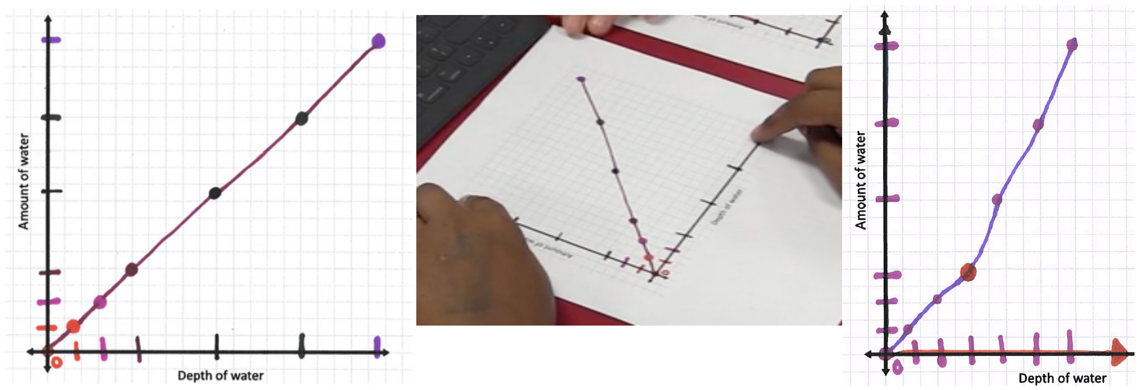
In SPT, I presented Zane a dynamic diagram of a pool (see Figure 0.12a, or https://youtu.be/jXaFMSVJ73E), where he could fill or drain the pool by dragging a point on a given slider in GeoGebra. I designed the task to support students in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool. I asked Zane to sketch a graph that shows the relationship between AoW and DoW as the pool fills up.



(a) (b)

Figure 0.12. (a) A diagram of the pool (b) illustration of Zane's partitioning activity.

An illustration from Zane's teaching experiment. Zane's meaning of the points included representing the two quantities' magnitudes simultaneously (see his graphs in Figure 0.13). Zane started his graphing activity by drawing tick marks on each axis (see Figure 0.13). Zane referred to the quantity's magnitude by drawing a line segment from the origin to the tick mark on the axis to articulate his meanings of tick marks. Moreover, Zane simulated the quantities' variation by tracing his fingers along the axis as we played the animation to fill the empty pool (Figure 0.13b). After inserting tick marks, Zane plotted points for each related tick marks correspondingly (see his color-coded points and tick marks in Figure 0.13a), then he connected those points with line segments on the plane. Figure 0.13a shows Zane's earlier graph whereas Figure 0.13c shows his final graph.



(a) (b) (c)

Figure 0.13. (a) Zane's earlier graph, (b) Zane moving his fingers on axes, and (d) Zane's final graph

To gain more insights into how he conceived his plotted points, I asked Zane to show the point on his graph representing when the pool is full. Zane first pointed to the far right and top purple tick marks on each axis (see Figure 0.13a, see also Figure 0.13b), and he then pointed to the corresponding purple point on the plane (see Figure 0.13a) by joining the two index fingers. Taken together with his description of a dot—"the dot represents both amount of water and depth of water"—his actions suggest that he could associate two tick marks (i.e., indication of quantities' magnitudes for Zane) on each axis to the corresponding point on the plane, which is an indication of representing a QMO.

An illustration from Melvin's teaching experiment. Figure 0.14 shows Melvin's first draft in SPT. Melvin drew a straight line upward from left to right because he initially determined gross covariational relationship (e.g., "The more the water, the deeper it is") between AoW and DoW in the pool. Melvin used vertical and horizontal segments on the plane to represent the quantities' magnitudes (see Figure 0.14). However, the way he organized the space was not compatible with a CCP. Note that the horizontal and vertical axis is labeled as "Depth of water” and “Amount of water,” respectively. Melvin explained that his horizontal segments represented the AoW and his vertical segments represented the DoW. Therefore, I infer that Zane conceived the direct distance between his graph and the vertical axis labeled “Amount of water” (i.e., the length of the horizontal segments) as a representation of the magnitude of AoW. Similarly, he conceived the direct distance between his graph and the horizontal axis labeled “Depth of water” (i.e., the length of the vertical segments) as a representation the magnitude of DoW. I don’t have evidence that he imagined the measurements of these lengths. Instead, he attended to an overall change in the magnitudes of AoW and DoW (see the varying size of the vertical and horizontal segments representing the magnitudes in Figure 0.14). Melvin said,

> I drew a straight line, or straight diagonal line from the origin this time because depth of water and amount of water both increase …like, say the depth of water is getting deeper [*pointing to the vertical segments on the plane and moving the pen vertically on the air from up to bottom*], it is because the amount of water is getting larger [*pointing to the horizontal segments on the plane*].

I infer that he constructed a line upward from left to right in order to show the depth of water increased as amount of water increased in the pool that were also represented by varying segments on the plane.

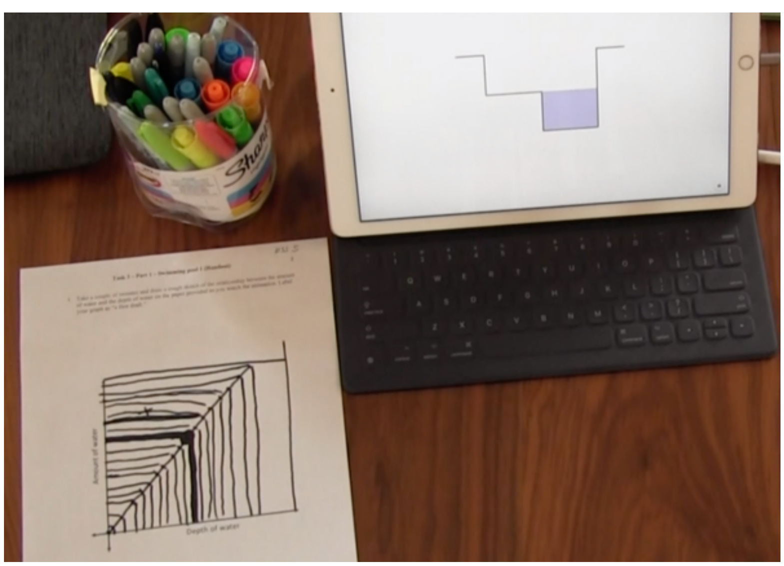


Figure 0.14. Melvin’s first draft in Swimming Pool Task.

Melvin also placed marks on his graph (see Melvin's tick marks where the vertical and horizontal segments meet on his graph in Figure 0.14). I asked Melvin to pick one of his tick marks on his graph and explain what that tick mark might represent. He highlighted the tick mark in the middle of his graph by circling a dot (see his dot in Figure 0.14) saying, "it represents the depth of the water and the amount of water." When further prompted to explain how that single dot represented both quantities, Melvin responded, "yeah, since the depth of water depends on how much water there is." This provided evidence that Melvin's meaning of the point entailed coupling the two quantities (i.e., AoW and DoW) because he was able to track one of the quantity's measure dependent on the other quantity's measure at that moment.

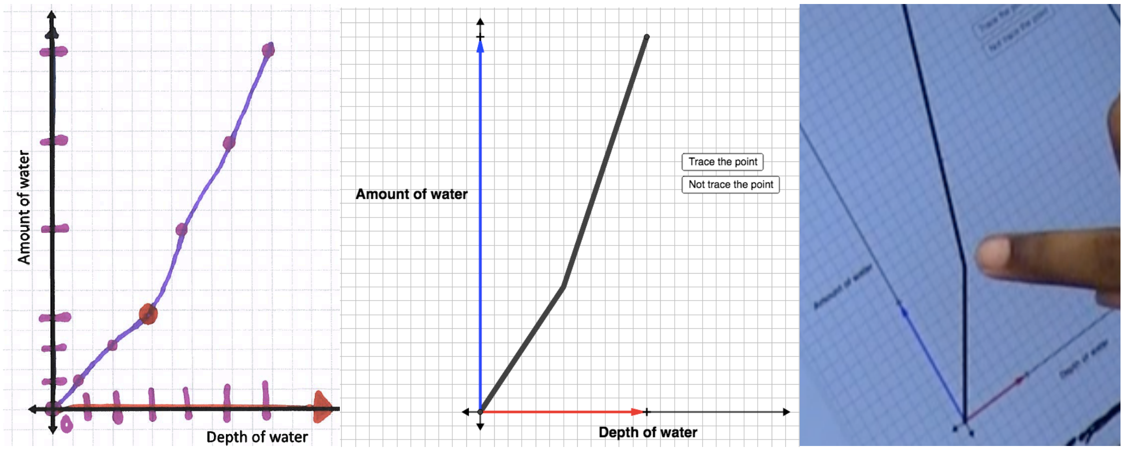
So far, I demonstrated Zane and Melvin's meanings of a single point on their graph. Next, I classify instances of representing QMO in two ways in relation to conceiving a graph (e.g., a line drawn on the plane) when students represent a relationship as two quantities vary: (1) as a path or direction of movement of a dot on the plane, and (2) as a trace consisting of infinitely many points, each of which showing the relationship of two varying quantities. I describe and illustrate those types below.

Type 1 and Type 2 QMO

Students' Type 1 QMO involves envisioning points as a circular *dot* that represents two quantities' magnitudes or values simultaneously and envisioning that points on a graph (e.g., a line) do not exist until they are physically and visually plotted. Therefore, those students conceive the graph (e.g., a line) as representing a direction of movement of a dot on a coordinate plane. For those students, a line does not have points until they are visually plotted. Students' Type 2 QMO involves being able to envision a point as an abstract object that represents two quantities' magnitudes or values simultaneously, and envision a graph (e.g., a line) as composed of points—although they are not visually plotted, each of which represent two quantities' values or magnitudes. I illustrate these meanings and their implication in the next section by providing empirical examples from the teaching experiments.

Illustration of Type 1 QMO. I illustrate Type QMO by continuing to discuss Zane's graphing activity identified above. I asked Zane whether his graph (see Figure 0.13c) showed every single moment of how the two quantities varied in the situation, Zane claimed no because one would need to stop the animation and plot an additional point in order to show the desired moment and state of the quantities. I infer that, for Zane, his line did not have points until they are visually plotted. He needed to physically plot additional points to represent moments in between two available points, even if there is a line connecting them. When questioned what the line segments that he drew in between dots meant to him, Zane responded that the line shows "where the dots go." By go, he meant a dot moving from one plotted point to the next plotted point, but not in a way that preserved an invariant relationship between those two points. In the next section, I demonstrate that such meaning for a point and a line played a role in Zane's construction and interpretation of what I perceived to be in-progress trace.

Implication of Type 1 QMO (contraindication of emergent shape thinking). Given these interpretations of Zane's meanings for points and lines, I hypothesized that he did not conceive a line consisting of infinitely many points. To test this hypothesis, I showed him an animation on an tablet device (see https://youtu.be/97EOENUM_co) and asked: "is this trace [see Figure 0.15b] showing us the relationship between depth of water and amount of water for this pool?" He replied "no" and struggled to make sense of what the animation was showing, which suggested that he did not perceive the animation as a simulation of his prior graphing activity on paper seen in Figure 0.15a.



(a) (b) (c)

Figure 0.15. (a) Zane's last draft and (b) an instance of the animation, and (c) Zane pointing to the "only" point on the trace.

Biyao asked Zane whether those dots on his paper (see Figure 0.15a) are "part of the line on the computer." Zane replied, "there is only one dot," pointing to the animating dot that produced the trace (see his gesture in Figure 0.15c). When asked "is there any other dots on this graph?" he shook his head. Moreover, he interpreted his graph (Figure 0.15a) as having more dots than the one produced in the animation (Figure 0.15b), commenting that mine is better because "mine have more dots".

Zane also claimed that he could not construct his graph in the same way as the animation did due to physical constraints of human, saying, "well, I cannot do that, because, like, can you do dots and dots [*tapping his right index finger very fast along his graph shown in Figure 0.15a*] and trace it?" This is an additional contraindication that he conceived of graphing a line as a way to represent infinitely many points.

Despite his success in being able to conceive of a point as a multiplicative object, Zane assimilated his activity as well as the animation as one dot moving along a line path instead of one dot generating infinitely many points by leaving a trace. I claim that his meaning for points and lines played a critical role in Zane's construction and constrained him from conceiving a graph as an emergent, in-progress trace (i.e., a component of emergent shape thinking).

Illustration of Type 2 QMO. I illustrate Type 2 QMO by continuing to discuss Melvin's graphing activity identified above. As an implication of Type 2 QMO, here, I also illustrate Melvin's graphing activity being suggestive of emergent shape thinking (Moore & Thompson, 2015) because his understanding of a graph included both what is made (i.e., infinitely many points that represents the corresponding magnitudes) and how it is made (i.e., covariation of coupled quantities).

Recall that Melvin's meaning of the points (i.e., tick marks) on his graph included a representation of both AoW and DoW and he was able to illustrate how each point on his graph referred to a certain instantiation of the pool animation. For example, the tick mark at the top right side of his graph referred to a moment in the animation when the pool is full (i.e., magnitudes of both AoW and DoW has their maximums). Similarly, the tick mark at the origin referred to a moment in the animation when the pool is empty (i.e., AoW and DoW are zero). Also recall that he started his graphing activity by drawing a straight line upward from left to right to represent the gross covariational relationship between AoW and DoW (i.e., as the depth of water increases, the amount of water increases too").

Since his meaning of each point plotted on his line included a representation of a moment in the pool situation in terms of both AoW and DoW, I wanted to see how Melvin could conceive the line in relation to all instantiations of the relationship between AoW and DoW as the pool fills up in the animation. To get insights into his meaning of the line independent of the physical tick marks that he added on, I told Melvin to imagine another person whose graph included only a line without tick marks or dots visually plotted on the line (see the line graph that

Melvin drew based on my description in Figure 0.16). I asked him whether that line showed every single moment of how the two quantities varied in the situation. He said, “she [*referring to the imaginary person*] gets all the, like, everything in the animation” because “the line is basically made of tick marks.” He then added “or she gets nothing because there is like no tick marks [on the line] and it [referring to the line] is not made of tick marks or anything, and she has nothing to represent.” For Melvin, a line can show every single moment of the animation under the assumption that line is made of tick marks. Melvin was also aware that if we assume the line is not made of tick marks (just a line), then the line itself shows nothing in relation to the pool situation. He said, “assuming that she meant that [*referring to the assumption that the line is made of tick marks*], I guess she would get all the, like, every exact little parts of the animation but if she was just drawing a line for a line, then I don’t know if she would.” This suggested that Melvin can hold a meaning of points and lines that is compatible with Type 2 QMO when the necessary assumption is made.

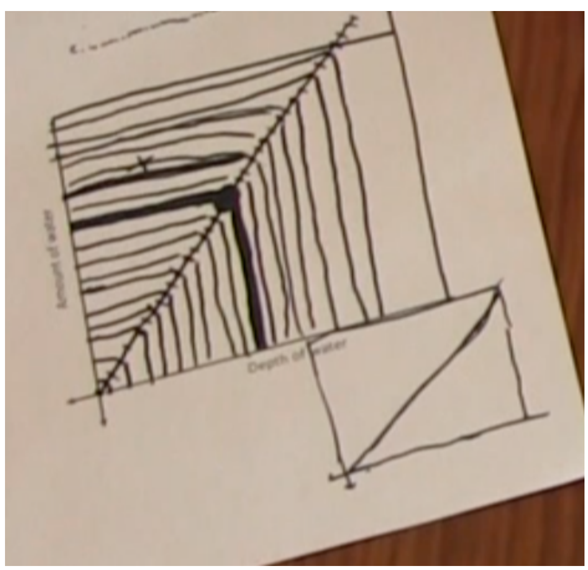


Figure 0.16. Melvin’s original graph that included both a line and visual tick marks (top left) and the imaginary person’s graph that only included a line (bottom right).

Students' Reasoning in Dynamic Situations

In this section, I present the nature of students' reasoning when they engage in dynamic situations that involve quantities varying (from my perspective). Depending on what their image of covariation or association included (i.e., quantity, proximity, and perceptual features), I classify the nature of students' reasoning as (i) quantitative covariational reasoning, (ii) spatial proximity reasoning, and (iii) matching the perceptual or spatial features of motion in two difference spaces.

Quantitative Covariational Reasoning

When students engage in a dynamic context that involve two quantities varying simultaneously, they can attend to how one quantity varies in relation to the other in tandem, which is called *covariational reasoning* (Saldanha & Thompson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). In my study, I identified that my students engaged in different types of covariational reasoning that are already characterized and illustrated by other researchers (e.g., Carlson et al., 2002; Stevens, 2019; Thompson & Carlson, 2017). I collected these instances under the category of quantitative covariational reasoning.

Cross-quantity comparison

As defined by Stevens (2019), students' quantitative covariational reasoning involves engaging in the operation of comparison between corresponding magnitudes of quantities given at a certain state (or subsequent states) of an object. This is akin to making a comparison in the columns of the table (i.e., between measure spaces) rather than trying to determine how those quantities are changing in relation to each other. To illustrate this characterization, I present Ella's activity in Which One Task (WOT) and Melvin's activity in Swimming Pool Task (SPT).

An illustration from Ella's teaching experiment. As you may see in Figure 0.17 (right), there are five directed bars that are located on magnitude lines. I informed the students that the blue bar represents the bike's DfA. As students moved the bike, each red bar vary in a different way in relation to the blue bar. I then asked Ella to determine which of the four red bars, if any, accurately represent DfC as DfA varied. In order to eliminate the bars, Ella made comparisons between the blue and the red bars in certain states of the bike (i.e., where the bike is closest and farthest to Arch and Cannon) in the situation. For example, she eliminated the red bars labeled #2 and #4 in Figure 0.17 because they were shorter than the blue bar when the bike is farthest from Arch and Cannon (see Figure 0.17 for the bike's position and the bar's length). Ella wanted to have the red bar always slightly longer than the blue bar because she established earlier that the bike's DfC is slightly bigger than its DfA during its whole trip. The other students (e.g., Naya) also reasoned in the same way. For example, Naya selected the normative red bar by saying "it always stays a little bit longer than the blue" because she determined that the bike's DfC is always longer than the bike's DfA during the bike's entire trip on the map.

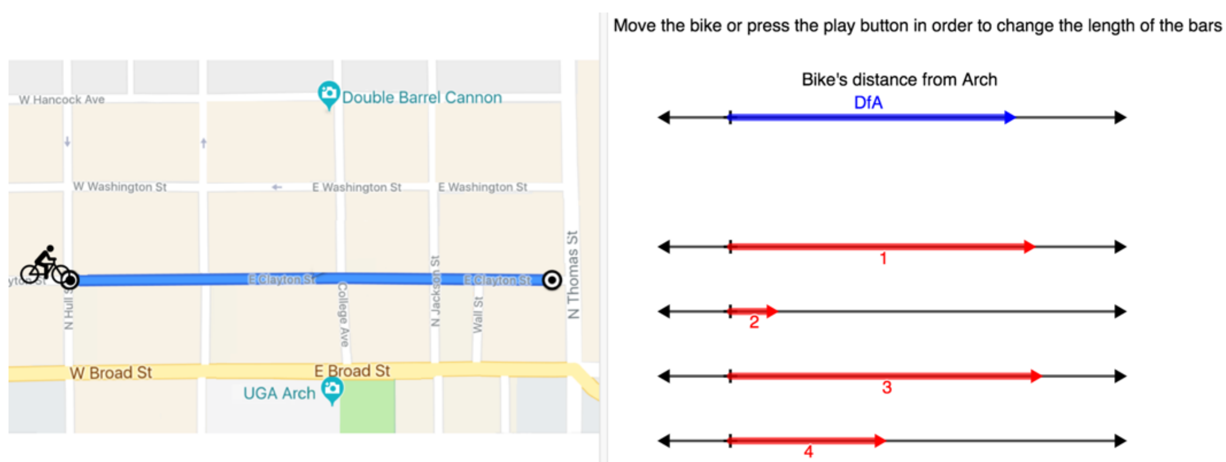
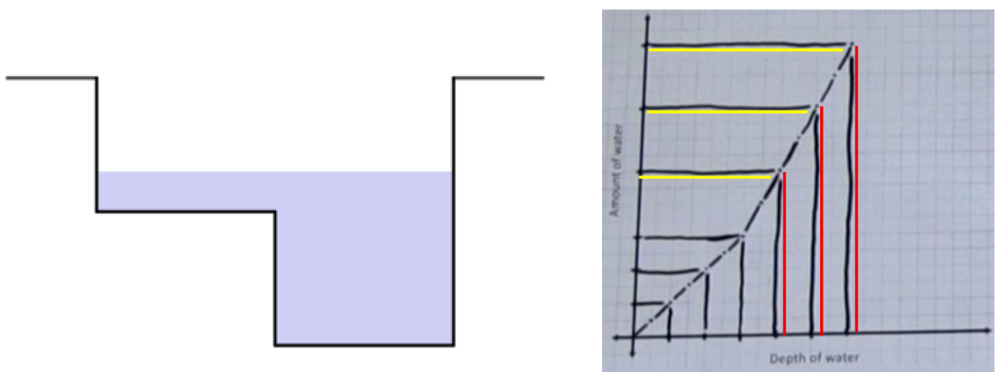


Figure 0.17. Which One Task (WOT).

An illustration from Melvin's teaching experiment. In SPT, I asked Melvin to graph the relationship between AoW and DoW on a paper with a coordinate plane. Melvin began his graphing activity by inserting tick marks on the horizontal and vertical axis to represent DoW and AoW, respectively (see Figure 0.18b). He inserted tick marks by coordinating the amount of change in AoW with changes in DoW (see the details of his graphing activity in the section of "Coordinating the amount of change"). He then plotted points on the plane according to the tick marks on each axis, and then, connected these points with line segments on the plane to represent the relationship between AoW and DoW (see Figure 0.18b). He also drew horizontal and vertical segments on the plane to represent the magnitude of DoW and AoW, respectively.



(a) (b)

Figure 0.18. (a) Swimming Pool Task and (b) Melvin's graph for the relationship between AoW and DoW (the line segments in red and yellow are added for the reader).

When describing the shape of his graph, he noticed that his graph "does not stay the same" meaning that the shape of the graph on the plane changed its direction. When asked to explain why his graph did "not stay the same", he compared the length of the segments that represented the magnitudes of DoW (see yellow segments in Figure 0.18b) with the corresponding segments that represented the magnitudes of AoW (see red segments in Figure 0.18b) for subsequent states of the pool for the higher part of his graph (i.e., the segments in yellow being shorter than the segments in red, see Figure 0.18b). He said,

> It [his graph] sort of goes this way [*tracing the pen over the higher part of his graph on the plane*] so that like these lines [*tracing the pen over the horizontal lines on the plane, see yellow segments in Figure 0.18b*] that represent the depth of water are shorter than the lines that represent the amount of water [*tracing the pen over the vertical lines on the plane, see red segments in Figure 0.18b*]. That is why sort of it goes this way [*moving his hand over his graph making an explicit turn once reached to the point on the plane where the shape of the graph changes*].

In order to explain why the shape of his graph changed its direction, Melvin engaged in cross-quantity comparisons between the magnitudes of DoW (i.e., yellow segments in Figure 0.18b) and the magnitudes of AoW (i.e., red segments in Figure 0.18b). The graph changed its direction because the yellow segments are shorter than the red segments.

Note that Melvin's quantitative covariational reasoning that included cross-quantity comparison does not identify Melvin's overall ability (or a level) to reason with covarying quantities. Melvin was also able to reason about the direction of change and the amount of change in AoW with changes in DoW when determining the relationship between the two varying quantities (see the sections of "Coordinating the direction of change" and "Coordinating the amount of change"). He could engage in different type of quantitative operation depending on his goal and the figurative material that is available for him to reason with. He inserted the tick marks on each axis by reasoning about *amount of change* in AoW with changes in DoW. In turn, he constructed his graph by relying on the tick marks (i.e., joining the two quantities as a single point on the plane and connecting these points). After this figurative material (i.e., his displayed graph with the horizontal and vertical segments on the plane) became available to him, Melvin justified the shape of the graph by comparing the length of the segments on the plane, rather than using his coordination of amount of change at the moment.

As a quick illustration from Ella in Swimming Pool Task, the reader can go to the section of "Point as a location/object and iconic translation" and read Ella's graphing activity. She determined the relationship between AoW and DoW as “they [AoW and DoW] are going up at the same time, but the amount of water is always gonna be bigger than actual depth of water.” Thus, her reasoning about the relationship between two covarying quantities included both directional change (i.e., “they [AoW and DoW] are going up at the same time”) and cross-quantity comparison (i.e., “the amount of water is always gonna be bigger than actual depth of water”).

Coordinating the direction of change

Student’s coordination of direction of change involves engaging in envisioning the values or magnitudes of the quantities increases or decreases without giving attention to the values or the size of the magnitudes changing together. This is the same as what Thompson and Carlson (2017) defined *gross coordination of values*, which is compatible with Carlson et al.’s (2002) *direction of change* (i.e., Mental Action 2 [MA2]) that is characterized as “coordinating the direction of change of one variable with changes in the other variable” (p. 357).

An illustration from Zane’s teaching experiment. To illustrate students’ coordinating the direction of change, I present Zane’s activity in WOT that I described above. Zane picked the normative red bar (labeled #3 in Figure 0.17) because he explained it was changing the direction and moving in the same way as the blue bar did in the situation. As the bike moved from left to the halfway point, he thought “because Double Barrel Cannon is lined up with Arch,” its DfC and DfA decreased together, and as the bike moved from the halfway point to the right, the bike’s DfC and DfA increased together.

An illustration from Naya’s teaching experiment. In WOT, Naya also selected the normative red bar (labeled #3) by coordinating the direction of change of the magnitude of the bike's DfC with changes in the magnitude of the bike's DfA. The following excerpt demonstrates Naya's reasoning.

> Naya: It can't be one or four [*referring to the red bars that are labeled #1 and #4*] because they are moving in their own direction, and number three turns at exactly the same time as the blue bar … it is in synced with the blue bar.
>
> TR: Uh-huh. So, you start with saying, no, I don't like one or four. Why? Let's say I moved the one here [*putting the red bar labeled #1 next to the blue bar*].
>
> Naya: Because one [referring to the #1] is going that way [*moving her right-hand thump to the right*], the blue bar is going that way [*moving her left-hand thump to the left*].
>
> TR: Why they can't go different ways?
>
> Naya: Because the Arch and the Cannon line always go the same way. That is what we practiced with our fingers [*moving her index fingers on the table*].

Later, Naya also eliminated the red bar labeled #2 because of the same reason ("it does not move in the same direction"). I infer that Naya's reasoning included envisioning the gross covariational relationship between the bike's DfA and DfC and their representation with the blue and red bars on the magnitude lines. She expected the red bar to move "in synced" with the blue bar in a way that they change direction at the same time.

An illustration from Melvin and Naya's teaching experiment. For another illustration of coordinating the direction of change, I present Melvin and Naya's activity in SPT. I presented them a dynamic diagram of a pool (see Figure 0.18a), where they could fill or drain the pool by dragging a point on a given slider. I asked them if there is a relationship between the amount of water (AoW) and the depth of water (DoW) in the pool as I played the animation. They both determined the directional change in amount of water and depth of water as the pool filled in with water. Naya said "if there is less water, there is gonna be less deep" implying that DoW decreases as AoW decrease. They also determined the relationship in the other direction. Melvin said, “the more the water, the deeper it gets” and Naya added “yep, when there is more water coming in there is more depth.” Note that they both attended to the gross variation of AoW and DoW without attending to the intensity of that variation (i.e., amount of change or rate of change) at the moment. Therefore, their graph of the conceived relationship included an increasing straight line as an implication of gross covariational reasoning (see Figure 0.19).

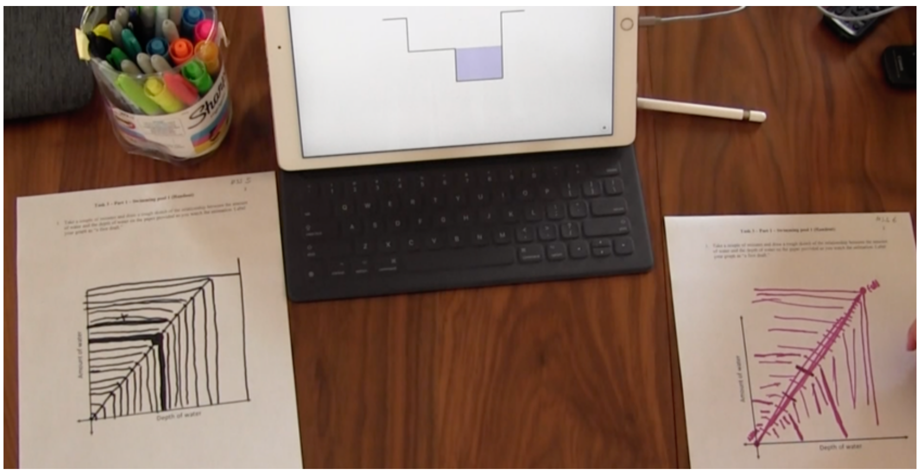


Figure 0.19. Melvin (left) and Naya’s (right) first draft in Swimming Pool Task.

When asked to represent the relationship between AoW and DoW on a given paper with a coordinate plane, both Melvin and Naya drew a straight line upward from left to right because they initially determined gross covariational relationship between AoW and DoW in the pool (“The more the water, the deeper it is”). They also used vertical and horizontal segments on the plane to represent the quantities’ magnitudes (see Figure 0.19). I don’t have any evidence that they imagined the measurements of these lengths. Instead, they attended to an overall change in the magnitudes of AoW and DoW (see the varying size of the vertical and horizontal segments). Melvin said,

> I drew a straight line, or straight diagonal line from the origin this time because depth of water and amount of water both increase …like, say the depth of water is getting deeper [*pointing to the vertical segments on the plane and moving the pen vertically on the air*

> *from up to bottom*], it is because the amount of water is getting larger [*pointing to the horizontal segments on the plane*].

Melvin constructed a line upward from left to right in order to show the depth of water increases as amount of water increases in the pool that were also represented by the varying length of the segments on the plane (see the section of "Representing a Quantitative Multiplicative Objec" for more of Melvin's graphing activity). Note that both Melvin and Naya's graphs in this task is literally what Carlson et al. (2002) described as a typical behavior (i.e., "constructing an increasing straight line" p. 357) of someone who could engage in directional change. I do not claim that constructing a line graph is always an indication of gross covariational reasoning. In this particular case, their displayed graph together with their verbal explanation provided evidence of their awareness of the direction of change of AoW while considering changes in DoW. In this sense, I claim that their graphing activity was an implication of their gross covariational reasoning (i.e., coordinating the direction of change).

Coordinating the amount of change

Students' coordination of the amount of change involves engaging in reasoning about the intensity of how a quantity increases or decreases (e.g., variation in variation) with respect to uniform change in the other quantity. This characterization involves types of reasoning that entails engaging in Mental Action 3 (MA3, i.e., "coordinating the amount of change of one variable with changes in the other variable"; Carlson et al., 2002, p. 357) when determining a covariational relationship between quantities. For example, in the context of the pool situation (see Figure 0.18a), someone who engages in MA3 could describe the relationship between amount of water (AoW) and the depth of the water (DoW) as AoW increases by the same amounts as DoW increases uniformly for the lower part of the pool. A person's mental action could involve the coordination of the relative change in magnitudes of AoW and DoW in order to recognize the relationship in this way.

An illustration from Melvin's teaching experiment. As you may see in Figure 0.20, using the given partitioning of the magnitudes of DoW into smaller intervals, Melvin considered the amount of change in the magnitude of the AoW. Melvin determined that AoW increases by the same amount as the DoW increases uniformly in the lower part of the pool. Melvin also determined that AoW increases twice as much while DoW still increases by the same amount in the higher part of the pool. The following excerpt demonstrates Melvin's reasoning.

Melvin: So, for the first three tick marks [*referring to the three tick marks on the right side of the pool from the bottom, see the labeled tick marks in Figure 0.20a*] on here [*moving the slider to fill the pool up until the water reaches to the tick mark labeled #3 in in Figure 0.20a*], the depth of water and the amount of water were like increasing and decreasing at the same rate [*moving the slider up and down to fill and drain the pool*].

TR: I will ask what you mean by rate.

Melvin: [laughs]. Hm. I mean like, there is like, between those two tick marks [*pointing to the interval between the tick mark labeled #1 and the tick mark at the bottom of the pool*], the depth of water is that much [*pointing to the same interval by placing his right index finger on the tick mark labeled #1 and his thumb on the tick mark at the bottom of the pool*], it is that deep from there to there [*jumping his right index finger from the tick mark at the bottom of the pool and the tick mark labeled #1*], and there is this much water [*pointing to the blue shaded area in the pool indicating the amount of water*]. [*Grabbing the tablet's pen*] So, that is the water [*shading the same area in black, see the black scribbles in Figure 0.20b*], and that is how deep it is [*drawing a vertical segment next to the side of the pool, see Figure 0.20b*]. And then, they [i.e., amount of water] increase by that much [*drawing the extra horizontal segments indicating AoW in Figure 0.20c*], and the same [inaudible] that much each time [*pointing to the intervals on the right side of the pool determined by the other tick marks labeled #2 and #3*].

TR: Show me when you say that much what do you mean?

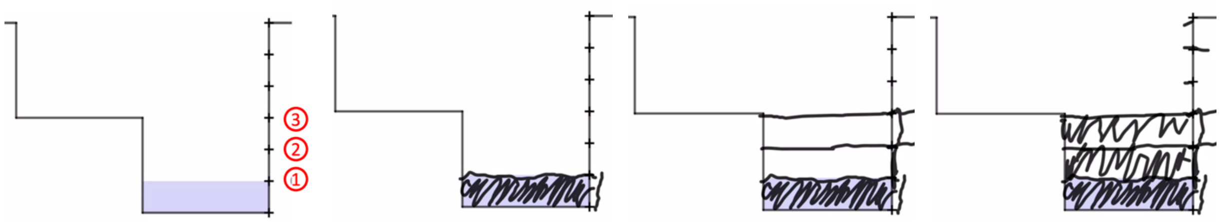
Melvin: This much [*drawing a vertical segment on the right side of the pool indicating the change in DoW from tick marks labeled #1 to #2*].

TR: This is increase in what?

Melvin: The depth. And this much [*drawing another vertical segment on the right side of the pool indicating the change in DoW from tick marks labeled #2 to #3*]. And the amount of water increases by this much each time [*shading the areas in between his horizontal segments in the pool indicating the AoW, see the black scribbles in Figure 0.20d*].

TR: Uh-hm.

Melvin: And, but after this point [*pointing to the tick mark labeled #3 in Figure 0.20a*], after here [*tracing the pen over the horizontal segment that goes through the tick mark labeled #3*], the depth keeps increasing at the same amount as it did before [*drawing additional tick marks on the side of the pool in place of the fourth, fifth, and the sixth tick marks in Figure 0.20d*], but the amount of water increases twice as much as it did before.

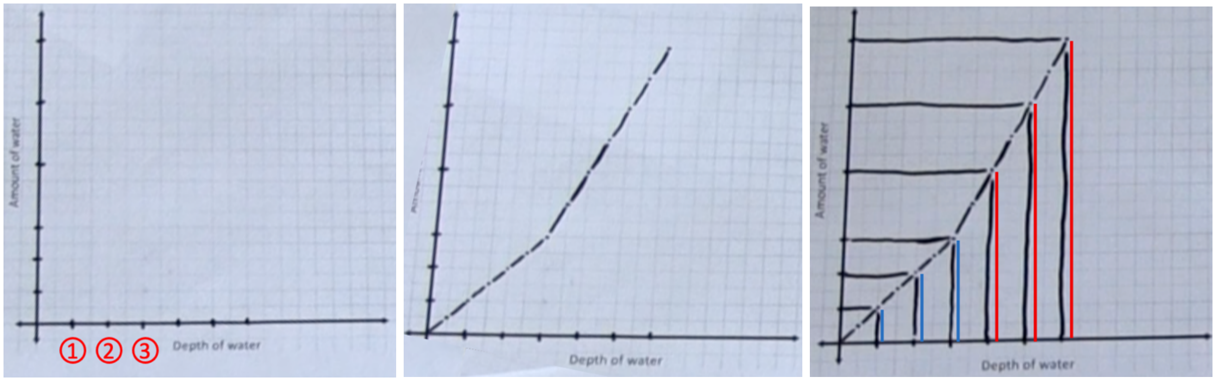


(a) (b) (c) (d)

Figure 0.20. Melvin's coordination of the amount of change in AoW with changes in DoW (numbers in (a) are inserted for the reader to label the tick marks on the side of the pool).

Although Melvin used the word "rate," from his explanation above, his reasoning included coordinating the amount of change in the magnitudes of AoW (indicated by the shaded areas) with changes in the magnitude of DoW (indicated by the vertical segments drawn in between the tick marks on the side of the pool). Thus, he determined that AoW increases by the same amount as DoW increases equally. Moreover, he determined that the amounts of change in AoW for the higher part of the pool is "twice as much" compared to the changes in AoW for the lower part of the pool while DoW "keeps increases at the same amount." Melvin multiplicatively compared the two quantities (i.e., the change in AoW for the higher part of the pool and the change in AoW for the lower part of the pool) and determined that one is two times larger than the other.

Implication of Melvin's coordination of amount of change. After Melvin determined the relationship between AoW and DoW, I asked him to sketch this relationship on a paper with a coordinate plane. Melvin began his graphing activity by inserting tick marks on the horizontal and vertical axis to represent DoW and AoW, respectively (see Figure 0.21a). Melvin's meaning of the tick marks included a representation of a quantity's magnitude on the axis as he said, "tick marks show where the end of the bars would be." His image of the tick mark involved a bar (i.e., a line segment) that is drawn on the axis from the origin to the tick mark although they do not have to be visualized on the paper. Note that Melvin drew equally distant tick marks on the horizontal axis to show DoW increased by the same amount. His tick marks on the horizontal axis (labeled with numbers in Figure 0.21a) corresponded to the tick marks on the side of the pool in the situation (see the labeled numbers in Figure 0.20a). Melvin also drew equally distant tick marks on the vertical axis; however, the distance between the two tick marks were bigger after the third tick mark because "it [amount of water] doesn't stay the same, it doesn't keep increasing by the same amount." Note that the distance between the first three tick marks on the vertical axis is 2 units (in terms of the number of grids on the paper) while it is 4 units for the other tick marks. This selection was not arbitrary for Melvin as he wanted to represent the "twice as much" idea that he established when engaging in the situation (i.e., "the amount of water increases twice as much as it did before"). Thus, the way he placed his tick marks on each axis was an implication of his coordinating the amounts of change.



(a) (b) (c)

Figure 0.21. (a) Melvin's tick marks on each axis (numbered labels in red are added for the reader), (b) Melvin's graph on the plane, and (c) Melvin's horizontal and vertical segments on the plane (colorful segments are added for the reader).

Melvin then plotted points on the plane according to the tick marks on each axis and connected these points with line segments on the plane to represent the relationship between AoW and DoW (see Figure 0.21b). He also drew vertical and horizontal segments on the plane to represent the magnitudes of AoW and DoW, respectively (see Figure 0.21c). When asked to explain why his graph contained straight lines, he explained that his graph consisted of straight lines because AoW increased by the same amount as DoW increased by the same amount each time. For example, for the lower part of the pool, he stated,

> So, it [*referring to the lower part of his graph in Figure 0.21c*] is like this [*tracing the pen over that part of the graph*] like straight from square to square up until here [*pointing to the point on the plane where the graph changes its slope*] because they are increasing at, um, the amount of water and the depth is increasing by the same amount.

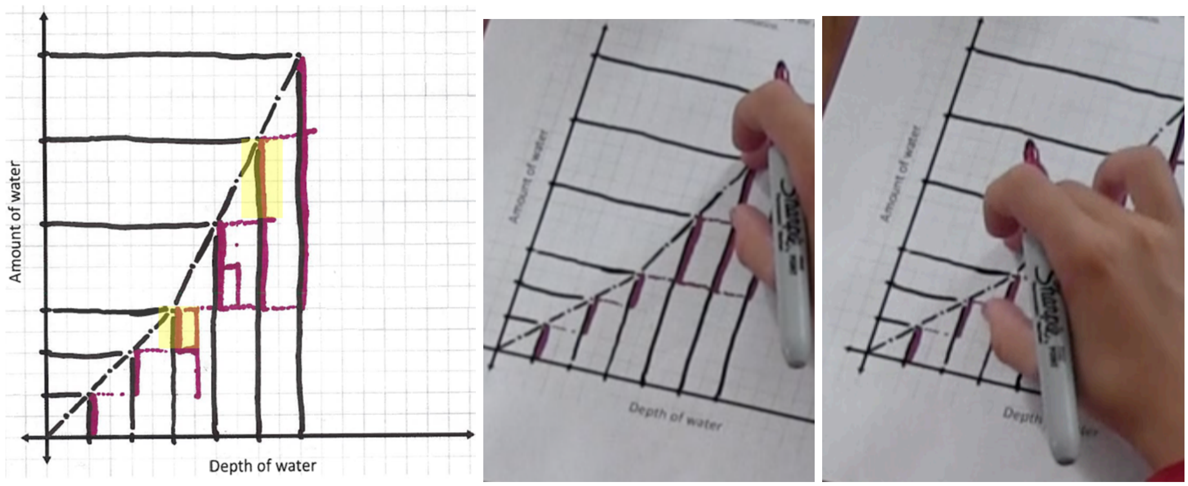
Recall that Melvin determined that the changes in AoW for the higher part of the pool are "twice as much" compared to the changes in AoW for the lower part of the pool while DoW increased by the same amount (see the excerpt above). I asked Melvin to explain why the shape of his graph changed its direction in order to see if and how Melvin could use his idea of "twice as much." His justification did not include the idea of "twice as much" at first. Instead, he *compared* the magnitude of DoW with the corresponding magnitude of AoW for each state of the pool for the higher part of his graph (i.e., the segments in yellow being shorter than the segments in red, see Figure 0.18b, see the section of "Cross-quantity comparison" for more details). Then, I explicitly asked him if his graph showed his idea of "twice as much." He first explained,

> Yeah. These lines [*pointing to the red vertical segments in Figure 0.21c*] start getting like longer, um, the length of these lines [p*ointing the same red segments on the plane in Figure 0.21c*] increase twice as much as they [*pointing to the blue vertical segments in Figure 0.21c*] were increasing before.

Melvin compared the increase in the magnitudes of AoW for the higher part of the pool (i.e., red vertical segments in Figure 0.21c) and the increase in the magnitudes of AoW for the lower part of the pool (i.e., blue vertical segments in Figure 0.21c) for the same increments of DoW. To support him in visualizing the amounts of change in AoW, I modeled highlighting the changes in AoW for the lower part of his graph (see the first three purple segments from the left in Figure 0.22a) and I asked him to show the changes in AoW for the higher part of his graph. He drew the rest of the purple segments (see the last three purple segments in Figure 0.22a) and said,

> You can see like this is [*measuring the highlighted segment on the plane, see Figure 0.22a, with his fingers, see Figure 0.22b, and he kept the same distance between the fingers and moved his hand towards the other smaller segment highlighted in Figure 0.22a*] twice as much as that [*placing his fingers over that smaller segment, see Figure 0.22c*].

Melvin was able to visually show that the change in AoW for the higher part of his graph is twice as much as the change in AoW for the lower part of his graph. He also measured the length of segments that represented amounts of change in AoW by counting the number of grids on the paper and concluded "so, it is 4 and this one is only 2." Melvin also drew the segment that is 4 units long right next to the segment that is 2 units long on the plane (see extra purple segments in Figure 0.22a) in order to show that the one is twice as much as the other. Therefore, Melvin was able to show the multiplicative comparison between the changes in AoW for the same increments of DoW in the lower and higher part of his graph.



(a) (b) (c)

Figure 0.22. (a) Melvin's final graph for Swimming Pool Task (yellow highlights are added for the reader), (b) Melvin's fingers grossly measuring the length of the longer purple segment, and (c) Melvin's fingers on the shorter purple segment.

Spatial Proximity Reasoning

In this section, I present a mode of reasoning, which I call spatial proximity reasoning, that entails (i) coordinating the (co)variation of an object's degree of proximity (i.e., closeness or nearness) to other objects (i.e., reference objects) in a dynamic situation (e.g., the bike is getting closer to or farther from Arch and Cannon), or (ii) comparing the spatial proximity of an object according to two different reference objects (e.g., the bike is farther from Cannon than it is from Arch). Note that students do not engage in coordinating the magnitudes or values of quantities, instead they coordinate the nearness between two objects without attending to the measurable attributes of these objects. For example, in the context of DABT, I observed students conceiving "the bike's distance from Arch" not as a measurable attribute of the bike, instead as the bike's degree of proximity to Arch. Indication of this type of reasoning might include having students using the linguistic adverbs "closer", "farther" when there is no evidence of students imagining quantified attributes of an object (e.g., the values or magnitudes of the bike's distance form Arch getting increase). Based on the analysis of my teaching experiments, I present evidence of spatial proximity reasoning and identify implications for students' representational activities (e.g., imagining the physical objects on the number line or on the coordinate plane) as follows.

An illustration from Ella's teaching experiment

In DABT, I showed Ella the animation where the bike rides on Clayton St. in Athens Downtown (see Figure 0.23) at a constant speed starting from the west side of the street. The animation represents the bike's ride back and forth. I asked Ella to describe how the bike's distance from Arch and Cannon changed as the bike moves along the road. Ella said, "when it is like getting in the middle [*pointing to the middle of the blue path highlighted on the map*], it is getting closer to Cannon and when it is getting over here [*pointing to the end side of the path on the map*], it is getting farther." As I wanted to see if she could describe the variation in bike's distance from Arch and Cannon in terms of how these distances increased or decreased as the bike traveled, I explicitly asked them to "talk about those distances getting decrease or increase." Ella replied in the following way:

> When it is right here [*dropped the bike in the middle of the blue path highlighted on the map, see Figure 0.23*], it is closer to the Cannon, when it is over here [*dragging the bike to the left end side of the path on the map*], it is farther, and when it is over here [*dragging the bike to the right end side of the path on the map*], it is farther. But like, right here [*moved the bike back to the middle point of the path*], well, right here, it is like, [*laughing*], so, I am like, imagining this is the number line [*referring to the blue path highlighted on the map, see Figure 0.23*]. And, over here [*pointing to the right end side of the path*], it is like, I don't know, ten, and over here [*pointing to the left end side of the road*], it is negative ten, over here [*moving the bike to the middle point of the path*], it is zero. And then, zero is like where Cannon, is like, closest to. And then if it [the bike] is just going this way [*moving the bike to the right from the middle point on the path*], it is like getting farther, and then when it is going this way [*moving the bike to the left on the*

path], it is getting farther., But, if it is like just stops [*moved the bike to the middle point again*], then it is like closest ... to Cannon.

Ella's response to my prompt (i.e., talk about bike's DfA and DfC getting decrease and increase) included imagining a number line with numbers (i.e., -10 being placed at the very left side, 0 in the middle, 10 being placed at the very right end side) overlaid onto the blue path where the bike travels on the map. When questioned "how those numbers tell you the bike is closer and farther," Ella appeared confused and unsure of her response as she said, "it kind of doesn't … I don't know." It seemed that she introduced those random numbers because I asked them to talk about something increasing and decreasing. This incident together with her language including "getting closer and farther" may suggest that Ella was engaging in spatial proximity reasoning. Moreover, Ella's activity suggested that she was not imagining the values or magnitudes of the bike's distance from Cannon when describing the bike was getting closer to and farther from Cannon.

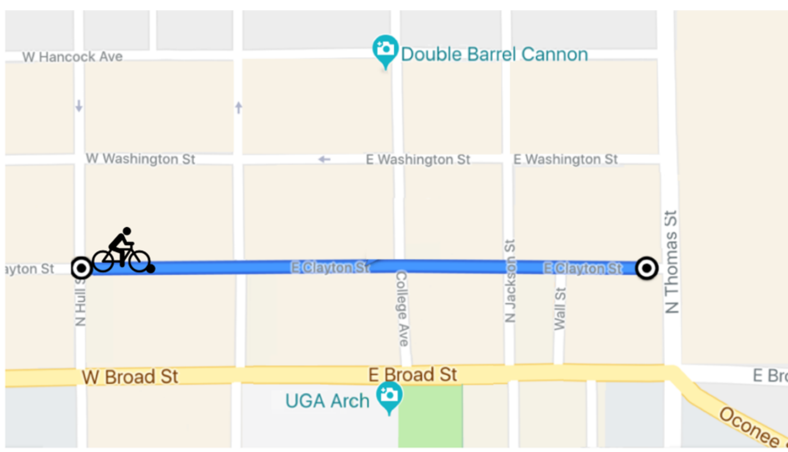


Figure 0.23. Downtown Athens Bike Task

An illustration from Zane's teaching experiment

In the first part of DABT, I showed Zane the animation where the bike rides on E. Clayton St in Downtown Athens (see Figure 0.23) at a constant speed starting from the west side of the street. I asked him to describe how the bike's DfA changed as the bike traveled on its path on the map. Zane said,

> When it [the bike on the map] is going here [*pointing to the middle of the path*] it is getting closer to Arch. Then, when it is going like here [*pointing to the right end side of the path on the map*], it gets farther away from Arch.

When asked to describe how the bike's DfC and DfA changed as the bike traveled on its path on the map. Zane said,

> Zane: I think that like when it goes in [s*liding the pen in the air from the left end side of the path to the middle*], it is getting closer to Double Barrel Cannon and Arch. So, like, when the bike is coming, they are like equal because it is coming closer into both of the places. If you go right here [*pointing to the right side of the path on the map*] it goes out to move from both of the places. So, both places' distance gets further.
>
> TR: Zane, can you show me how do you see bike's distance from Arch on the map?
>
> Zane: Like, when it [i.e., the bike] is right here [*moving the bike and dropping it at the middle of the blue path on the map*], it is closer to the Double Barrel Cannon and it is closer to the Arch. But if you go right here [*moving the bike and dropping it at the right end side of the blue path on the map*], it is farther from Cannon and it is farther from Arch.
>
> TR: Can you show me what is being far? How do you see it is far? Like, you can also draw.
>
> Zane: So, like, right here [*drawing a segment from Cannon to Arch, see Figure 0.24*], the bike goes right here [*pointing to the intersection between the bike's path and the segment he drew*]. And then if it is over here [*circling the left end side of the bike's path*], it is farther [*writing "far" near the left end side of the path*] and [*writing "far" near the right end side of the path*]. So that like, if the bike is right here [*moving the bike to the right end side of the path on the map*] it is far. But, if it is right here [*moving the bike to the intersection of his segment and the bike's path on the map*], it is closer to both of them. Its, its distances are closer to, the bike is closer to Double Barrel Cannon and UGA Arch.

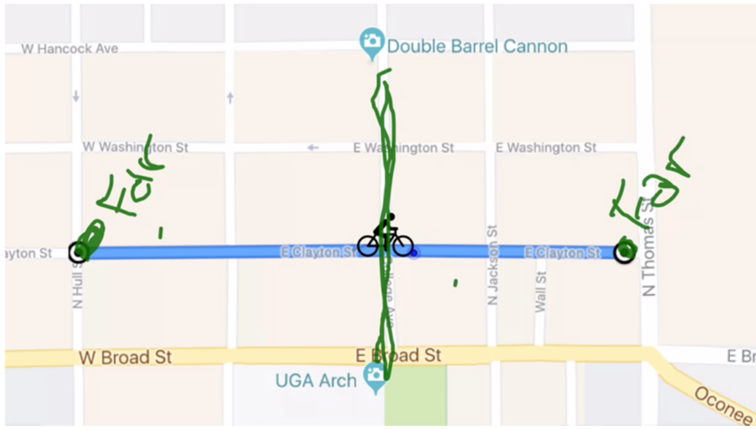


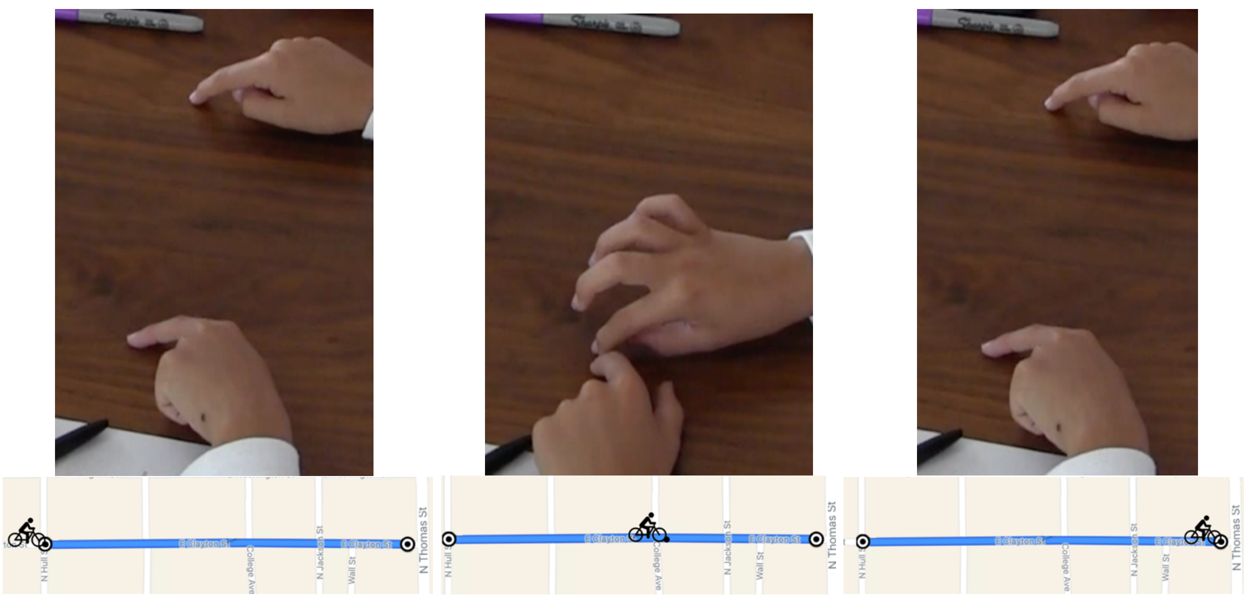
Figure 0.24. Zane's activity in DABT.

Zane's activity suggested that he coordinated the bike's proximity to both Arch and Cannon at the same time as he used the linguistic adverbs "closer" and "farther" to describe the bike's DfA and DfC. I did not have any evidence that Zane coordinated the bike's DfA and DfC as the bike's measurable attributes. I prompted him to show me how he imagined the bike's DfA, he continued to describe what he perceived to be the bike's DfA as the bike was getting closer to and farther from Arch. When I explicitly prompted him to show how he could show the bike was far from Arch by drawing on the map, he wrote "Far" on the map to show where the bike was far from Arch and Cannon (see Figure 0.24). Zane also drew a segment on the map; however, I didn't have any evidence that he conceived the segment in relation to the magnitude of the bike's DfA or DfC. He still continued to describe the bike's DfA as the bike was getting closer to or farther from Arch.

An illustration from Melvin's teaching experiment

In the first part of DABT, I asked Melvin how the bike's DfA is changing as the bike moved, he described that the bike is getting closer to and farther from Arch. As the bike started moving from the left, Melvin said "closer, closer, closer, …" until the bike reached to the middle of the path, then he said "further, further, further …" until the bike reached to the right end side of the path. In order to draw Melvin's attention to the measurable attributes of the bike (i.e., DfA) and its variation as the bike moves, I told Melvin "let's say the bike is 200 meters away from Arch" when the bike is located at the left end side of the path. Then, I asked him "if I move the bike in this way [*moving the bike from left end side of the path to the right*], what happens to this distance?" After a long pause, Melvin responded "it, it, umm, it is le, it gets closer, what did you mean?" I repeated the question by using the word "value" as follows: "What happens to this value? Like what happens to that quantity?" Then, Melvin was able to say, "it becomes less?" with a hesitated voice tone as he appeared unsure. I told Melvin "yes, it becomes less. And we can use the word decreasing." I then repeated the same task for Melvin asking him to describe how the bike's DfA was changing as the bike moves from the left end side to the right. Melvin responded, "it is getting closer." Even though I prompted him to think about the bike's distance as a quantity in terms of its value (e.g., 200 m) and asked him to imagine the variation of that value as the bike moved, he still described the variation of the bike's DfA as it is getting closer to and farther from Arch. In fact, he was able to think about the variation of a value when he was given an explicit value (i.e., 200m). However, I didn't have any evidence that he coordinated a quantity's variation in terms of its value or magnitude given the two objects on the map. From this activity, I infer that Melvin conceived the bike's DfA as a proximity between the bike and Arch, with Arch being the reference object. As the bike moved, he coordinated the bike's proximity to Arch without conceptualizing the bike's DfA as a quantity in the situation.

In the second part of DABT, I asked Melvin to focus on the bike's DfA as it travels on E. Clayton St. back and forth. I directed him to place his index fingers horizontally on the table in a way that his left index finger was fixed on the table and his right index finger could only move in a horizontal direction. Next, I told him to move his right index finger left to right so that the distance between his index fingers could represent the bike’s DfA as the bike rides along the road. Figure 0.25 shows three instantiations of how Melvin moved his fingers according to what he conceived to be the bike’s DfA. When the bike is at the left end side of the path (Figure 0.25a, bottom), Melvin positioned his fingers as seen in Figure 0.25a, top. As the bike started to move to the right, Melvin smoothly moved his right index finger to the left until both index fingers met (i.e., physical touch, see Figure 0.25b, top) when the bike was at the middle of the path (see Figure 0.25b, bottom). Then, he moved his right index finger to the right until the bike reached at the right end side of the path on the map (see Figure 0.25c).



(a) (b) (c)

Figure 0.25. Melvin’s activity in the second part of DABT.

I asked Melvin to explain how he moved his fingers. He said,

> When the bike is closest to the Arch [*pointing to the bike at middle of the path, see Figure 0.25b, bottom*] … I was as closest as I can get to this finger [*meeting his index fingers on the table, see Figure 0.25b, top*] … and when he was getting further away from Arch, … I moved my fingers away from it [*moving his right index finger to the right*].

Note that he connected his index fingers when representing what he perceived to be the bike's DfA on the table when the bike is at the middle on its path (see Figure 0.25b). I asked M "when the bike is there [*referring to the bike's location in the middle of the path as seen in Figure 0.25b, bottom*], does the distance between your fingers represents the bike's distance from Arch?" He said, "well, kind of, because when they [the index fingers] are like almost touching [*meeting his index finger on the table as seen in Figure 0.25b, top*], he [the bike] was as closest as he could get to the Arch." His activity suggested that he did not conceive the distance between his index fingers as a representation of the bike's DfA as a quantity. Instead, the separation (proximity) of two fingers represented how the bike was close to the Arch on the map for Melvin. That is, meeting fingers represented "the bike was as closest as it could get to the Arch." I claim that this was an implication of Melvin's spatial proximity reasoning. I did not have any evidence that Melvin conceived and isolated the bike's DfA as a quantity in the situation and attended to decompose it from the situation and represent its variation on the table with the distance between his index fingers. What he coordinated was the right index finger's (i.e., the bike for him) degree of proximity to the left index finger (i.e., Arch for him), with the left index finger being the reference object. Moreover, Melvin imagined the Arch in place of his left index finger and the bike in place of his right index finger on the table as he moved his fingers. This provided another evidence that Melvin was engaging in representation of his spatial proximity reasoning on the table.

Melvin also conceived the bike's distance from Cannon (DfC) in terms of the bike's proximity to the Cannon, with Cannon being the reference object. Then, I asked Melvin if there is a relationship between the bike's DfA and DfC as the bike traveled on E. Clayton St. on the map. Melvin said yes and determined an invariant relationship between what he conceived to be the bike's DfA and DfC. He compared the bike's proximity to Arch and the bike's proximity to Cannon and concluded that "the bike's distance from Cannon is a little further than the bike's distance from Arch." Note that Melvin used the word "further" rather than "bigger" when comparing the "distance from Arch" and "distance from Cannon," which suggested his reasoning entailed comparing the spatial proximities, rather than quantities. When asked to explain why, Melvin stated that "because Cannon is further away from East Clayton Street than the Arch is." I infer that the Cannon's degree of proximity to the entire street being farther than the Arch's proximity to the E. Clayton implied the bike was always farther away from Cannon (compared to Arch) as the bike travels on E. Clayton St.

Implication of Spatial Proximity and Quantitative Variational Reasoning

In this section, I illustrate and contrast the pairs from the teaching experiments (Zane vs. Ella and Melvin vs. Naya) as an implication of their spatial proximity and quantitative covariational reasoning. In both comparison cases, students who reasoned with spatial proximities abstracted the two objects (one of them is in motion) and their proximity when engaging in the situation. Thus, they conceived these physical objects in their graphing activity too. In contrast, students who reasoned with quantitative covariation conceptualized the distance between two objects as a quantity and its variation when engaging in the situation. Thus, they represented this quantity isolated and decomposed from its physical context in their graphing activity.

Zane's spatial proximity vs. Ella's quantitative variation

In this section, I illustrate and contrast Ella and Zane's activity where they were asked to interpret the varying bar on a magnitude line (see Figure 0.26, right, or https://youtu.be/6kdbDeVEF9w). I illustrate that Zane who engaged in spatial proximity reasoning assimilated what I perceived to be the representation of a quantity's magnitude on the magnitude line in a certain way that was different from Ella who engaged in quantitative covariation. Zane imagined the physical objects on the magnitude line whereas Ella represented the quantity's magnitude on the magnitude line isolated and decomposed from its physical context. In turn, their representational activity in two-dimensional space differed as an implication of their reasoning.

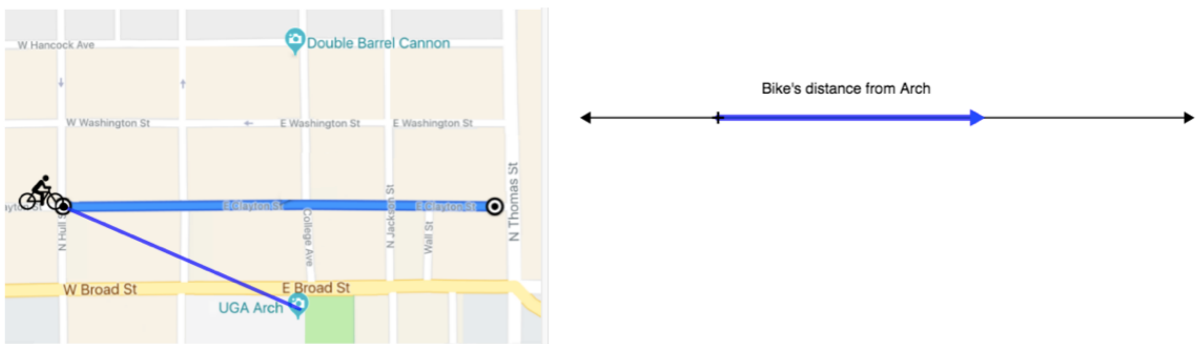


Figure 0.26. Representing the bike's distance from Arch both on the map (left) and on the magnitude line (right).

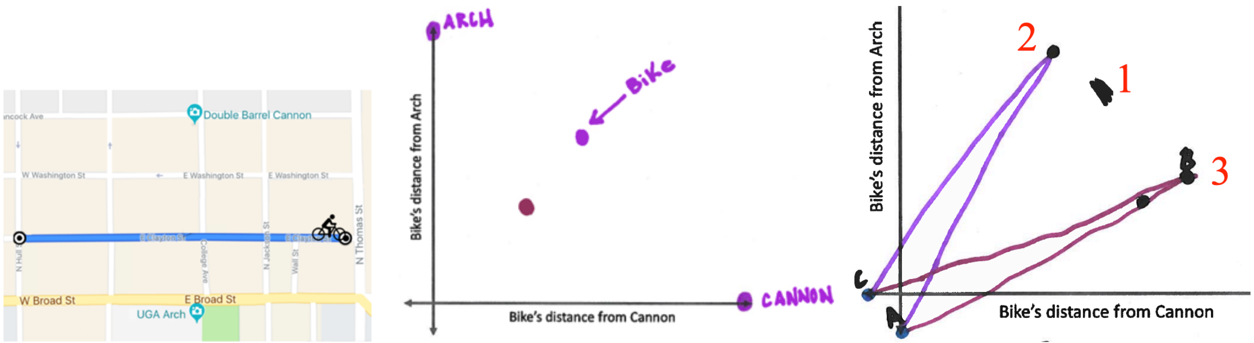
While moving the bike to the right from its position seen in Figure 0.26, I drew students' attention to the fact that the right end side of blue bar on the magnitude line was moving to the left (indicating the bike's DfA was decreasing). Then, I asked Zane and Ella to explain why that was happening. Referring to the map, Zane claimed, "the bike is getting closer to Arch," which suggested that Zane's image of variation included coordinating the variation of the bike's proximity to Arch. Then, by pointing to the blue bar on the magnitude line and tracing the pen over the line from right to left, he added it "is gonna get closer to right here [*pointing to the zero point on the magnitude line*], which is Arch." Zane conceived Arch and bike as physically placed on the left and right end side of the blue bar, respectively, and he thus perceived blue bar's change to be compatible with his spatial proximity reasoning because the bike moved closer to the Arch *along* the line, with Arch playing the role of reference object.

Differing from Zane, Ella determined that the bike's DfA is decreasing while moving the bike to the right in the map. She explained "it [*pointing to the blue bar*] is gonna get smaller because distance is smaller on the number line too." Moreover, Ella labeled the starting point as "zero", whereas Zane conceived the same point on the magnitude line as "Arch." Her activity suggested that Ella conceived the length of the blue bar on the map and on the magnitude line as a representation of the bike's DfA. I conjecture that Ella's activity was an implication of her quantitative variational reasoning (i.e., the bike's DfA is decreasing) whereas Zane's activity was an implication of his spatial proximity reasoning (i.e., the bike is getting closer to Arch).

In order to understand how Zane understood Ella's response and focus on "distance", I asked Zane to re-voice Ella. Zane's explanation included his original reasoning (i.e., imagining the Arch and the bike physically on the line). Thus, Zane assimilated Ella's answer to his meaning because they were compatible with respect to the behavior of figurative material. Since Zane's meaning included spatial proximity (i.e., nearness) between two places, it made sense to imagine Arch in place of the origin in one-dimension because the value of the bike's DfA was zero at that point. However, their meanings and reasoning have different implications when we move to two-dimensional systems, which I illustrate in the next section.

The implication of the way Ella and Zane assimilated the bars. As I illustrated above, Ella assimilated the bar on the magnitude line as representation of a quantity's magnitude. Unlike Ella, Zane assimilated the zero point on the number line as Arch and he assimilated the right end of the bar as the bike and the movement of the bar on the number line as the bike was getting closer to or farther from Arch. Here, I illustrate an implication of the way they assimilated the bar varying on the magnitude line when they were asked to represent the relationship between the bike's DfA and DfC on a given paper with a coordinate plane.

By the time I asked this question, the picture of the situation on the tablet screen was visually available to them and the bike was located at the right end side of the path on the map (see Figure 0.27a) and the animation was not playing. Consistent with his earlier activity, Zane continued to engage in his spatial proximity reasoning by conceiving the physical Arch, Cannon, and the bike on the coordinate plane. (see his labeled dots in Figure 0.27b).



(a) (b) (c)

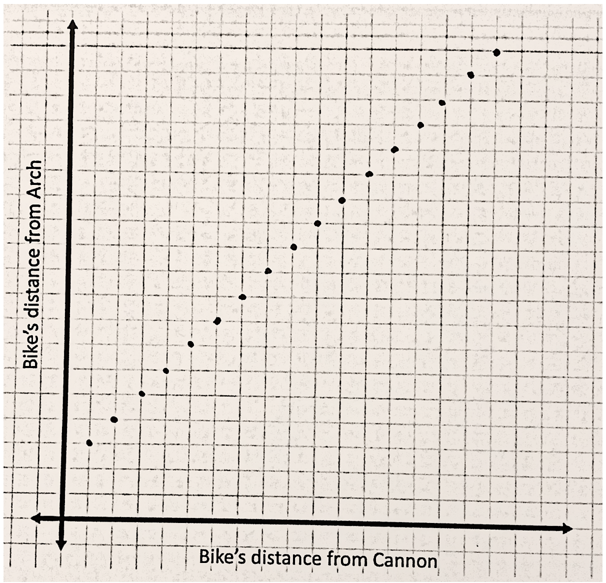
Figure 0.27. (a) Map showing the bike, (b) Zane's graph, and (c) Ella's graph

Ella re-organized the space different than her earlier actions in the teaching experiment (see Figure 0.27c vs. Figure 0.9b). In her previous graphing activity (see the section of "Representing a Spatial-Quantitative Multiplicative Object" for details), Ella was imagining Arch as a location on the vertical axis implied by the label "Distance from Arch" (see Figure 0.9b). She said, "when it says distance from Arch [*pointing to the label on the vertical axis*], that is where the Arch is [*pointing to the same area on the vertical axis again*]." Similarly, Ella conceived Cannon as a location on the horizontal axis as implied by the labels (see orange dots on each axis in Figure 0.9b). However, after her engagement in the magnitude line activity, Ella conceived Cannon at very left side of the horizontal axis[6] (see the black point labeled C in Figure 0.27c) because "farther it is here [*sweeping her finger from left to the right over the horizontal axis*] means that farther it is from Cannon." This may show that Ella's re-organization of the space was an implication of her ability to assimilate the magnitude line tool normatively. Ella still assimilated the dot she plotted on the plane as the bike (labeled B, #3 in Figure 0.27c) whose location was determined by coordinating the radial distances between the bike's DfA and DfC. Note that Ella desired to change the location of the dots (see her earlier attempts in Figure 0.27c with the numbers showing the order in which she drew) "because it [the dot labeled as B] is like farther away from Cannon than it is Arch."

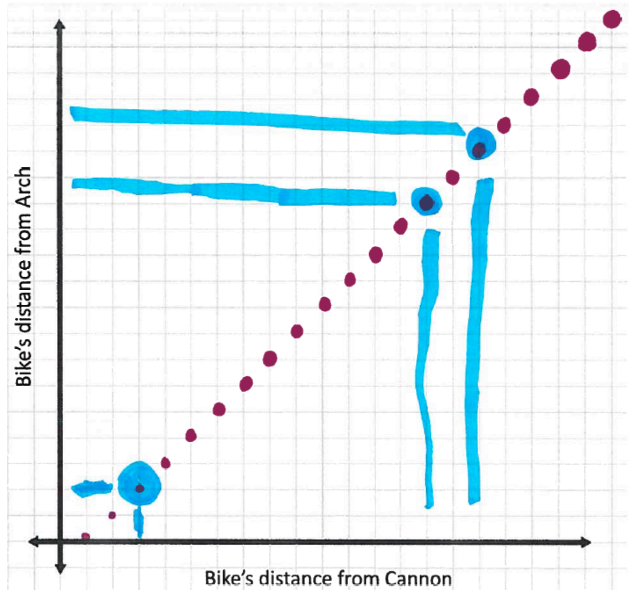
Melvin's spatial proximity vs. Naya's quantitative covariation

In DABT, before I asked them to graph the relationship between the bike's DfA and DfC, I provided Melvin and Naya an opportunity to determine the relationship in the situation. I found that Melvin's reasoning included spatial proximity whereas Naya's reasoning included quantitative covariational reasoning. Melvin determined that the bike is always closer to the Arch compared to the Cannon as the bike moved on E. Clayton St, whereas Naya determined that the bike's DfA and DfC is both increasing and decreasing together. Then, I asked them to graph the relationship between the bike's DfA and DfC as the bike traveled on E. Clayton St. on the map. Figure 0.28 shows both Melvin and Naya's first drafts. I illustrate that Melvin's graphing activity was an implication of his spatial proximity reasoning (as he brought the physical Arch, Cannon, and the bike to the plane) while Naya's activity was an implication of her quantitative covariational reasoning (as she represented the quantities' magnitudes on the plane).

[6] Note that this instance is an illustration of the characterization "Conceiving objects on the axis independent of the labels" under the section of "Organization of the space when representing NMO and SQMO."



(a)



(b)

Figure 0.28. (a) Melvin's first draft in DABT and (b) Naya's first draft in DABT.

Melvin's representation of spatial proximity reasoning. Recall that when asked to determine the relationship between the bike's DfA and DfC, Melvin compared the bike's proximity to Arch and the bike's proximity to Cannon as the bike traveled on E. Clayton St. He determined that "the bike's distance from Cannon is a little further than the bike's distance from Arch … because Cannon is further away from East Clayton Street than the Arch is." I then asked Melvin to sketch a graph that represents the relationship between the bike's DfA and DfC on a paper. Melvin conceived the entire vertical and horizontal axis of the coordinate plane as the Arch and Cannon, respectively, and he conceived his plotted points on the plane forming a path "where the bike travels" (i.e., E. Clayton St.). He said:

> I imagine this [*pointing to the vertical axis*], the y-axis as the Arch [*tracing his finger over the axis from top to bottom*] and the x-axis is the Cannon [*tracing his finger over the horizontal axis from left to right*]. And I drew East Clayton Street on the dotted line [*tracing his finger over his graph on the plane upward from left to right*]. Um, I drew it [*referring to his "dotted line"*] a little closer to the Arch than the Cannon because East Clayton Street is closer to the Arch than Cannon on the map.

Melvin represented the relationship between what he conceived to be the bike's DfA and DfC as his graphs' (the physical dots he plotted on the plane in Figure 0.28a) degree of proximity to the vertical and horizontal axis, respectively. That is, he drew his graph on the plane closer to the vertical axis compared to the horizontal axis because the bike was always closer to the Arch than Cannon on the map during its trip on E. Clayton St.

Note that Melvin's graph intersected with the vertical axis of the plane. I conjectured that this was also an implication of his spatial proximity reasoning because this was compatible with his representational activity with fingers. Recall that Melvin connected his fingers (i.e., physical touch) to represent what he perceived to be the bike's DfA when the bike was located in the middle of the path (see Figure 0.25b). Two fingers that were touched each other represented "the bike was as closest as it could get to the Arch" for Melvin, which suggested that he did not conceive the distance between his index fingers as a representation of the bike's distance from Arch as a quantity. Akin to his finger activity, Melvin's graph also reflected the same idea by intersecting his graph to the vertical axis, which is the Arch for Melvin. When the bike was located in the middle of the path (where its DfA and DfC are minimum), I asked Melvin how his graph showed this moment. He pointed to the dot that is closest to the vertical axis on his graph in Figure 0.28a. He said, "because the bike on the map is as closest as it can get to the Arch, and this dot is as closest as it can get to the y-axis." This provided an evidence that Melvin did not conceive the bike's DfA as a quantity in the situation and represented this distance as a quantity on the plane. Instead, he conceived the bike's degree of proximity to Arch (i.e., "as closest as it can get") in the situation, and he represented the same property on his graph (i.e., "as closest as it can get"). I infer that this was an implication of Melvin's spatial proximity reasoning as he conceived the bike's DfA as the bike's degree of closeness to the Arch so that "closest" moments would match in both map and graph.

Naya's representation of quantitative covariational reasoning. In DABT, before the graphing task, I asked Naya if there was a relationship between the bike's DfA and DfC as the bike traveled. She said, "when the bike gets further from the Arch and Cannon, they [the bike's DfA and DfC] increase, when it gets closer, they [the bike's DfA and DfC] decrease." Unlike Melvin's proximity reasoning, Naya attended to variation of the bike's DfA and DfC as a quantity whose measure increased and decreased as the bike traveled on the map. When asked to represent this relationship on a paper, Naya drew her first draft in Figure 0.28b. Her meaning of the points included joining two quantities' magnitudes that were represented vertically and horizontally on the plane (see the light blue bars on the plane in Figure 0.28b). Recall that Melvin imagined E. Clayton St. in place of his graph and the physical Arch and Cannon in place of the axes of the coordinate plane. After listening Melvin's explanation about his graph (see Figure 0.28a), Naya stated:

> That is not what I meant. I just imagined this [*tracing the pen on the plane vertically from the horizontal axis to the upright point on her graph and tracing the pen on the plane horizontally from the vertical axis to the same point on her graph*] being the distance. This point [*pointing to the upright dot on her graph*] being the distance between both of them and it keeps getting smaller [t*racing her finger downward from right to left over her dotted line*] and bigger [*racing her finger upward from left to right over her dotted line*].

Unlike Melvin, Naya's meanings of the graph included representing the covariational relationship between the bike's DfA and DfC on the plane. She neither imagined the physical Arch and Cannon in place of the axes nor E. Clayton St. in place of her graph. Her meaning of a point included joining the magnitudes of the bike's DfA and DfC on the plane as a single object[7]

[7] Note that Naya's meaning of the points here is also an example of quantitative multiplicative object in a non-canonical coordinate plane.

(see the light blue bars in Figure 0.28b). For example, to show the bike's DfA and DfC when the bike is at the middle of the path (i.e., when the distances at their minimum), she circled a dot on her graph (see the dot that is closed to the origin in Figure 0.28b) and drew two bars to represent the magnitudes (see the bars in light blue in Figure 0.28b). The vertical segment represented the bike's DfC, and the horizontal segment represented the bike's DfA for Naya. She also repeated the same activity for other states of the bike on the map (see the other light blue dots and segments on the plane in Figure 0.28b). Moving upward and downward direction on her graph represented both the bike's DfA and DfC was increasing and decreasing, respectively, because the length of the magnitudes was getting smaller and bigger on the plane. Therefore, I claim that although Melvin and Naya's graphs look visually similar in Figure 0.28, their meaning of the graph was different due to their reasoning about the bike's DfA and DfC (i.e., spatial proximity reasoning vs. quantitative covariational reasoning).

Matching Perceptual Features of Motion in Two Difference Spaces

In this category, students' reasoning involves engaging in associating the perceptual (or spatial) feature of the motion of the object (e.g., shape of the path or direction of the movement) in the situation and the perceptual (or spatial) feature of the motion of another, but related object on a coordinate system (i.e., a point on a magnitude line or on a Cartesian plane). Their associations are iconic (Monk, 1992) in nature as students conflate perceptual properties and shapes of the dynamic event with perceptual properties and shapes of a graph.

This type of reasoning is different than the other two types (i.e., quantitative covariational and proximity reasoning). In quantitative covariational reasoning, students coordinate the two measurable attributes of *an* object that are varying in relation to each other. For example, a student who engages in directional change (or gross covariational reasoning) may coordinate the changes in quantities' values or magnitudes as "as the depth of water increases, amount of water increases." In this instance, the object is the water, and the student coordinates its measurable attributes (i.e., the depth and amount). In spatial proximity reasoning, students coordinate the two spatial attributes of *an* object that are varying in relation to each other. For example, a student may claim "as the dot gets close to the horizontal axis, it gets farther from the vertical axis" when describing a dot's changing location on the plane. In this instance, the object is the dot, and the student coordinates its spatial attributes (i.e., the dot's proximity to both axes). However, in matching perceptual features, students coordinate perceptual features of *two* objects: one is in the situation and the other is on the coordinate plane (or on a number line). For example, when engaging in Crow Task (CT) (see Figure 0.29), a student determines that the dot moves upward on the plane as the crow moves upward on the map. In this instance, one object is the crow on the map and the other is the dot on the plane, and the student coordinates *their* spatial features (i.e., the direction that they move).

An illustration from Zane's teaching experiment

In CT (see Figure 0.29), students moved the crow on the map by dragging it without any constraints and observed how the corresponding point in the coordinate plane changed (https://youtu.be/5JgrPAuG15Y). Students who engaged in shape covariation coordinated the changing location of the object (i.e., the crow on the map) and the changing location of the black dot on the plane. Based on the data I have for now; I identified two different levels of this type of reasoning. In the lower level, students attended to the movement of the objects without paying attention to how they moved. For example, Zane said "the dot [*referring to the black point on the plane*] is moving as the crow moves [on the map]." In the higher level, students coordinated the movement of the objects paying attention to how they moved. When moved the crow from Arch to Cannon on the map (i.e., upward movement), Zane said "they [*referring to the crow on the map and the black point on the plane*] are moving in the same direction" meaning "if he moves this way [*moving his left index finger upward over the map, see Figure 0.29*], the dot moves that way too [*moving his left index finger upward over the plane*]." That is, Zane coordinated the perceptual features of the nature and direction of the movement of the crow in the map (i.e., straight and up) and the nature and direction of the movement of the dot on the plane (i.e., straight and up). Here, the student may identify a regularity in how they moved in relation to each other. So, student can make generalizations relying on figurative material (e.g., the object in the situation and the dot/graph itself on the plane).

Figure 0.29. Crow Task

Implication of Zane's matching perceptual features. In this section, I illustrate that Zane organized the space (i.e., what I perceive to be a Cartesian coordinate plane) in a certain way as a result of his engagement in matching perceptual features. Zane conceived the physical Arch and Cannon *on* and *as* the axis of the plane as an implication of his reasoning that involved matching the perceptual features of motion.

In Zane's activity in CT, he dragged the crow and dropped it on top of Arch, then as he moved the crow to the Cannon on the map, Zane determined that the black dot on the plane simultaneously moved diagonally from the right side of the horizontal axis to the up side of the vertical axis. Zane matched the perceptual feature of how the crow moved on the map and the perceptual feature of how the corresponding black dot moved on the plane. He, then, immediately concluded that the vertical axis is the Cannon because the dot on the plane "goes by the [vertical] line" as the crow approached to the Cannon on the map. Zane also claimed that the horizontal axis is the Arch because the dot went by the horizontal axis on the plane as the crow moved to the Arch on the map. For a detailed illustration of Zane's activity, the reader can go the section "Conceiving the physical objects *as* the axis itself." Therefore, I believe that Zane's matching perceptual features promoted him to conceive the horizontal and vertical axis in its entirety as Arch and Cannon, respectively.

CHAPTER 5

RESULTS 2

In this chapter, I outline the key developmental points involved in developing meanings for graphs based on my analysis of Melvin, Naya, Zane, and Ella. In each case, I begin with a brief summary of the meanings the students demonstrated across different tasks throughout the teaching experiment. Next, I illustrate these meanings and shifts in their meanings by providing examples from the teaching experiments. I then present an extended summary of the case at the end of each section.

The Case of Melvin

In this section, I present Melvin's meanings of points that he developed throughout the teaching experiment. Table 5.1 summarizes the meanings he demonstrated in Downtown Athens Task (DAT), Crow Task (CT), and Downtown Athens Bike Task (DABT) and includes the way he organized the space in solving these tasks. Melvin initially assimilated the points on the plane in relation to the physical objects that appear in the situation, and his meanings for points were based in iconic or transformed iconic translation (i.e., picture of the situation). As I implemented the instructional sequence, Melvin continued to assimilate the points on the plane in relation to the physical objects that appear in the situation, but his meanings of the points began to include quantities and their relationships (i.e., spatial-quantitative multiplicative object). Subsequently, his attention to quantities in the situation and mapping those quantities' magnitudes onto the magnitude lines supported Melvin to reorganize the space consistent with a Cartesian plane and afforded him to develop a meaning of points in terms of representing two quantities' magnitudes (i.e., quantitative multiplicative object).

Table 5.1

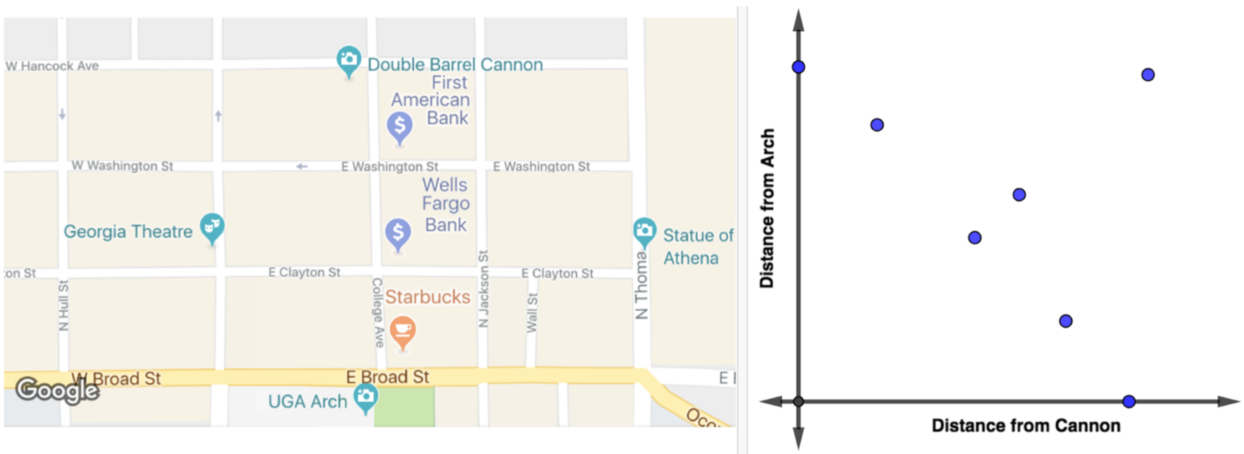
Melvin's meanings of the points and his organization of the space throughout the teaching experiment.

Tasks	Meanings of the points	Organization of the Space
Downtown Athens Task	Non-Multiplicative Object (iconic translation)	Imagining Arch and Cannon on the axis
Crow Task	Non-Multiplicative Object (transformed iconic translation)	
Downtown Athens Bike Task (1st draft)	Non-Multiplicative Object (spatial proximity)	Imagining Arch and Cannon as the axis itself
	Measurement Activity	
Downtown Athens Bike Task (2nd draft)	Spatial-Quantitative Multiplicative Object	Imagining Arch and Cannon as the axis itself
	Matching Game Task	
Crow Task	Spatial-Quantitative Multiplicative Object	Imagining Arch and Cannon on the axis
	Intervention	
Crow Task	Quantitative Multiplicative Object	Cartesian coordinate plane
	Non-Multiplicative Object (transformed iconic translation)	Imagining Arch and Cannon as the axis itself
Downtown Athens Bike Task (final draft)	Quantitative Multiplicative Object	Cartesian coordinate plane

Melvin's Activity in Downtown Athens Task

As an illustration of Melvin's initial meaning of points, I present his activity in Downtown Athens Task (DAT). DAT includes a map with seven locations highlighted and labeled (See Figure 0.2a). DAT also includes a Cartesian plane whose horizontal axis is labeled as Distance from Cannon (DfC) and vertical axis is labeled as Distance from Arch (DfA). Seven points are plotted without labelling in the coordinate system to represent the seven locations' DfA and DfC (see Figure 0.2b). I first asked Melvin what each of these points on the coordinate system might represent.

Melvin assimilated the points on the plane as a location/object by engaging in iconic translation. That is, when determining which point on the plane is associated with which place on the map, Melvin made an association between the perceptual or spatial features of the map and plotted points (Figure 0.2b). In particular, he assimilated the point as a location/object by choosing one perceptual property of the map (e.g., spatial orientation, relative location) and ensuring to preserve that property on the plane. I illustrate his meaning in detail below.



(b) (b)

Figure 0.1. (a) Downtown Athens, and (b) Cartesian plane with points

Melvin determined that the point on the horizontal axis is the physical Arch and the point on the vertical axis is the Cannon because "Arch is at the bottom on the map and Cannon is in the general top area" (Figure 0.3). This suggested that Melvin's meaning of the point included iconic translation where he made an association between the spatial features of the situation and points on the plane (i.e., Arch being at the bottom on the map and on the plane). Based on this inference, I hypothesized that Melvin would assimilate the point that is closest to the horizontal axis as Starbucks because Starbucks is the closest location to Arch on the map. To test my hypothesis, I asked him what he thought the point that is closest to the horizontal axis might represent.

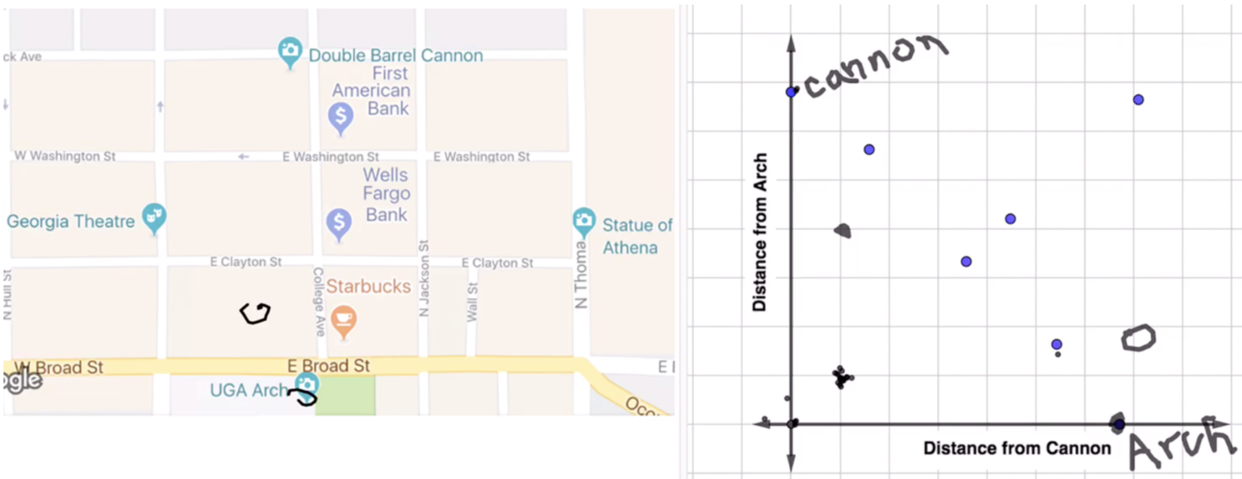


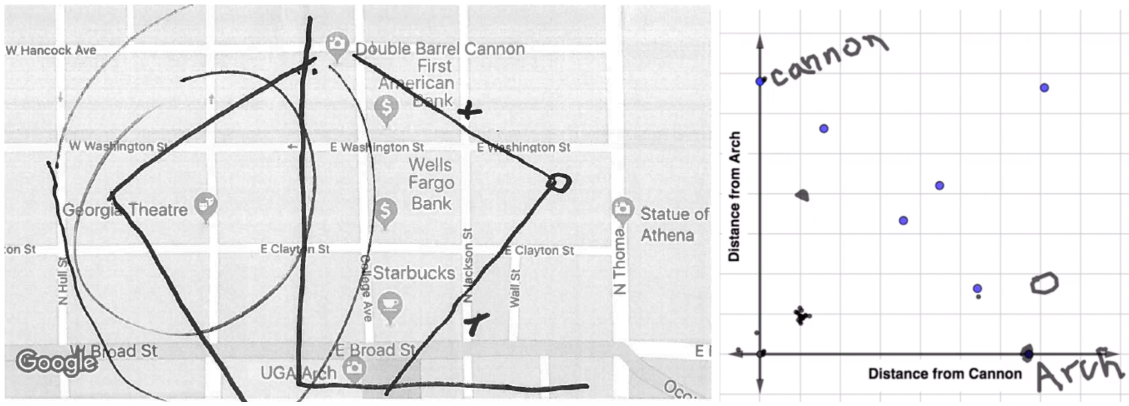
Figure 0.2. Melvin's activity in DAT (a screenshot from the tablet).

Melvin responded to my question with, "I guess it is Starbucks." But Melvin also appeared confused and unsure of his response, and he pointed out the location of Starbucks relative to Arch on the map and noted there is no such a point in the plane. He claimed, "on the map [*pointing to the map*], the Starbucks is right there [*pointing to the plane and circling a location right next to the available point on the plane, see Figure 0.3, right*], not to the left." He also added:

> There is something close to the Arch [*pointing to the point on the horizontal axis labeled as Arch, see Figure 0.3, right*], a little bit up to the left [*moving the pen on the tablet screen upward from the point on the horizontal axis and to the left*], but there is nothing here [*circled a place on the map that is at the left of Starbucks, see Figure 0.3, left*] on the map. But, on the Cannon [*pointing to the point on the vertical axis*], there is something to the right [*pointing to the point that is closest to the vertical axis*], which could be the First American Bank.

I infer that his meaning of points that included iconic translation was viable because Melvin was satisfied by the Cannon's location being on the vertical axis because there is a point on the plane that is down and to the right and similarly there is a highlighted location on the map that is down and to the right of Cannon which is the First American Bank. Drawing on the same meaning, Melvin expected to have a point on the plane for Starbucks that is located to the right of the Arch instead of to the left. Similarly, Melvin thought that in the "surroundings" of Arch on the plane, there is a point that is "up to the left, but there is nothing in the map." He wanted an object on the map that is up and left compared to the Arch because there is a point on the plane that is up and to the left of the point on the horizontal axis, which is Arch for him. Note that this meaning of the points with iconic translation includes coordinating spatial orientation (e.g., "up and to the left") as Melvin wanted to preserve this spatial feature of points on the plane when he moved from the map to the plane or from the plane to the map. From this activity, I infer that his iconic translation meaning yielded a perturbation for Melvin because the location of Starbucks didn't make sense to him in relation to the location of the Arch when comparing across the map and the plane.

As the session moved forward, Melvin attempted to reconcile this perturbation by trying to imagine a coordinate plane overlaid onto the map. He attempted to draw the coordinate plane several times on the map (see his attempts in Figure 0.3a), each of which included rotating the axis orientation in a different way so that the points on the plane could be matched with the locations on the map. Note that Arch and Cannon are diagonally aligned on the coordinate plane (see Figure 0.3b). Thus, Melvin wanted to rotate the axis to match with the Arch and Cannon on the map that are vertically aligned. None of his attempts helped him reconcile his perturbation, which led to him concluding, "I still don't get how to draw the graph on here [*referring to the map*]." Although he was not able to perceptually associate each location on the map with a point on the plane with his current meanings of the points, his actions suggest he anticipated a way to perceptually match the points on the plane with the location on the map after doing some sort of rotation of the coordinate plane. This suggested that he could potentially develop a meaning of points that include transformed iconic translation.



(a) (b)

Figure 0.3. (a) Melvin's coordinate plane overlaid onto the map in DAT and (b) Melvin's activity in DAT (a screenshot from the tablet).

Melvin's Initial Activity in Crow Task

Recall that Melvin's initial meanings in DAT included iconic translation, in which he intended to perceptually match each point on the plane with each location on the map. Such meaning created a perturbation for Melvin that he could not reconcile because of the mismatch between perceptual features of the map and the points on the plane. Next, I provided CT (as an extension of DAT) where there is a crow that flies on the map and a corresponding black point that moves on the plane according to the crow's DfA and DfC (Figure 0.29). I asked Melvin to move the crow on the map and see how he made sense of the corresponding black point's movement on the plane. Based on his activity during DAT, I conjectured that Melvin could assimilate the black dot as the crow itself. Moreover, I hypothesized he could be perturbed again by the movement of the black dot because the crow's movement and the black dot's movement do not always perceptually match. For instance, the black dot could move diagonally on the plane as the crow flies vertically on the map, see the highlights in Figure 0.29.

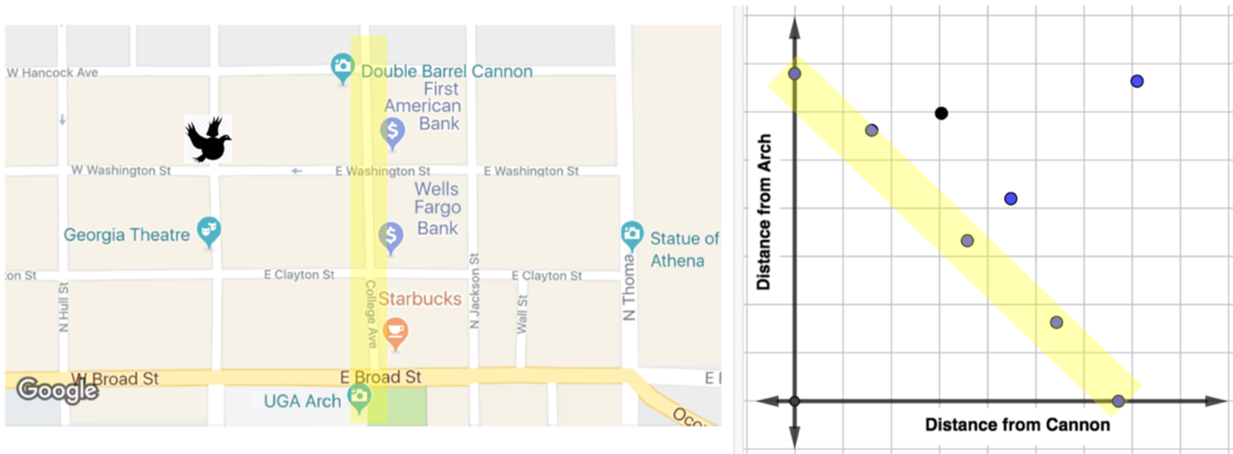


Figure 0.4. Crow Task (yellow highlights are added for the reader).

Melvin's engagement with CT created a perturbation for him as expected. Consequently, he shifted his meanings of points from iconic translation to transformed iconic translation. That is, he still conceived the points as objects/locations by using iconic translation. However, he did not translate the perceptual features of the situation *as it is* to the plane (or the opposite direction, i.e., from graph to situation); instead, Melvin translated a *transformed* version of the perceptual features of the situation to the plane (or the opposite direction). I next illustrate his activity in detail.

Recall that Melvin conceived the point on the horizontal/vertical axis as Arch/Cannon because Arch/Cannon is located at the bottom/top of the map and the point is located at the bottom/top of the plane (see Figure 0.3b). Melvin's initial activity in CT reaffirmed the point on the horizontal and vertical axis is Arch and Cannon, respectively, because

> when the bird was moved to the Cannon [*moving the crow on top of Cannon on the map*], it [the black point on the plane] is on that dot [*pointing to the point on the vertical axis, which is Cannon for him*], and when it [the crow] flies to the Arch [*moving the crow to Arch on the map*], it [*the black point on the plane*] is on that dot [*pointing* to *the point on the horizontal axis, which is Arch for him*].

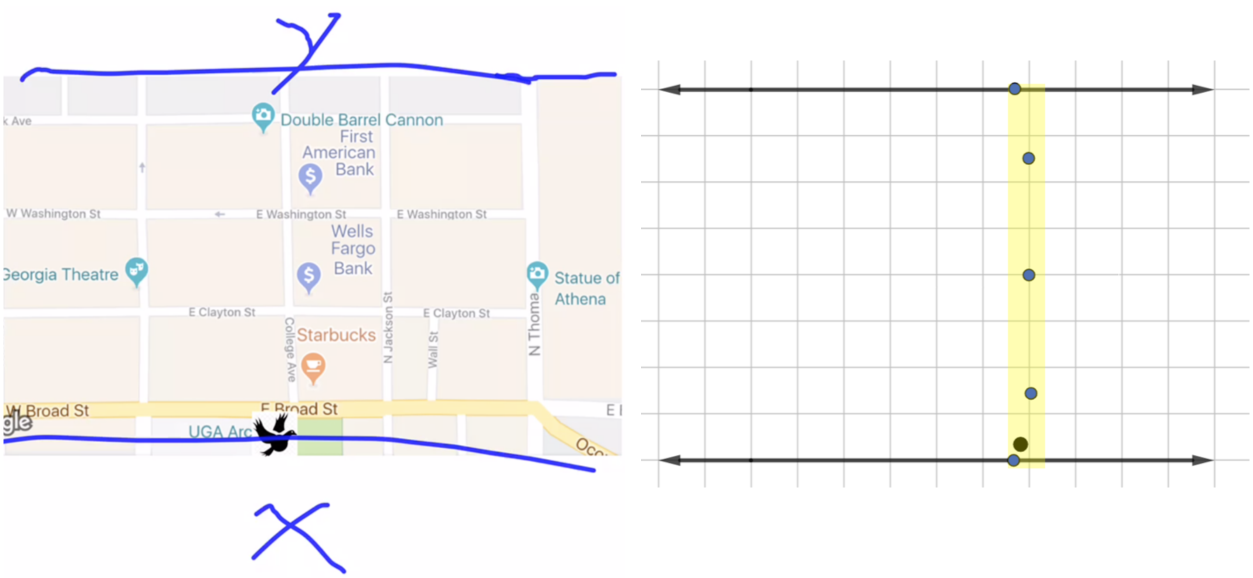
Melvin then dragged the crow and dropped it on Starbucks on the map, and in turning his attention to the plane he became surprised by the location of the corresponding black point having moved over to the point that is closest to the horizontal axis. I infer that using his meaning of the points that included iconic translation, he expected that the black point should be "to the northeast rather than northwest" of Arch. He questioned "why isn't the Starbucks like here [*pointing to the right side of point on the plane that is closest to the horizontal axis*]?" Consistent with his actions on DAT, he desired preserving the spatial property of the map (i.e., relative location of Starbucks) *as it is* when moved to the plane.

Although he could not figure out why that was happening, his engagement in CT made him believe that the point that was northwest of the Arch on the horizontal axis was Starbucks. I interpreted that he did not question the location of the black dot on the plane as it moved when the crow flied on the map because it was the task that designed that way. Instead of claiming that the black dot moved in a wrong direction according to the crow on the map, Melvin went ahead and explored the relationship between perceptual features of the motion of the crow on the map and the perceptual feature of the motion of the black point on the plane.

Melvin moved the crow from Arch to Cannon in a straight line on the map (see the highlighted path in Figure 0.29, left) and determined that the black dot on the plane moved from the *x*-axis to the *y*-axis diagonally in a straight line (see the highlighted path in Figure 0.29, right). Melvin said, "if we move it up [*moving the crow vertically from Arch to Cannon on the map back and forth several times*], it [*referring to the black dot on the plane that moved diagonally, not vertically*] won't just go directly to up." This provided evidence that Melvin associated the direction of the movement of the crow on the map (i.e., "directly up") and the direction of the movement of the black dot on the plane (i.e., "moving diagonally" not "directly up").

This perceptual difference in motion of both objects yielded Melvin to question his meanings that included iconic translation. He said, “now, I know that we can’t place this [*pointing to the coordinate plane*] on there [*pointing to the map*] and have it be exactly” because “obviously it is not like that [*tracing his finger on the axes of the plane by making an L shape on the air*] because they [*referring to the crow on the map and the black dot on the plane*] don’t move in the same way.” Because of this perceptual discrepancy, Melvin concluded he cannot draw the coordinate plane overlaying onto the map by keeping the axis orthogonal to each other (as he attempted to so, see Figure 0.3a). That is, Melvin realized that the perceptual features from the map should not be translated *as it is* to the plane because the crow on the map and the black dot on the plane moved in different directions. Melvin then concluded,

> Which means *y*-axis is on the top and *x*-axis is on the bottom [*see Figure 0.5*]. … Because if I start right here [*putting the crow on top of Arch on the map on the tablet screen, see Figure 0.5a*] and if I go directly up [*moving the crow vertically from Arch to the top of the map*], it [*referring to the crow*] goes straightly to the *y*-axis [*the horizontal line that he imagined on top of the map, see the line labeled y in Figure 0.5a*].



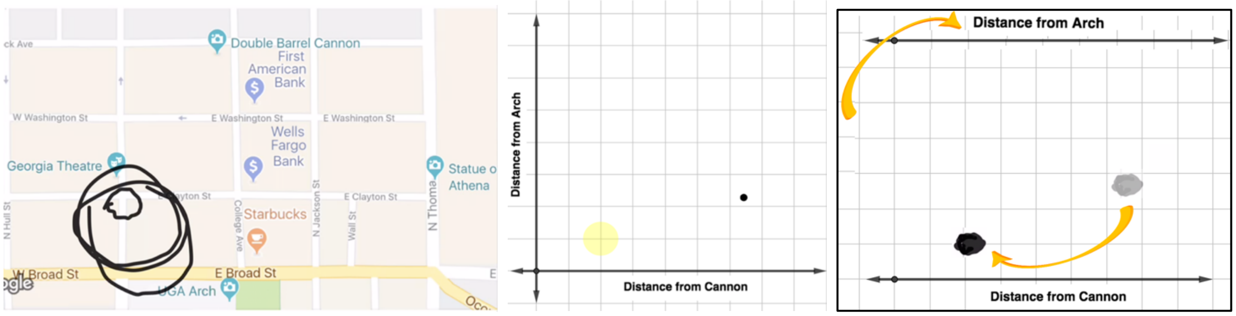
(a) (b)

Figure 0.5. (a) Drawing the axes parallel to each other overlaid onto the map and (b) re-creation of the plane for the reader to illustrate how Melvin imagined rotating the vertical axis of the plane to make it parallel to the horizonal axis on the plane.

Melvin imagined the vertical axis of the coordinate plane horizontally overlaid onto the top of the map and the horizontal axis overlaid onto the bottom of the map (see Figure 0.5a) as a way to reconcile his experienced perturbation. Melvin conceived the points that are diagonally aligned on the plane (see the highlighted points in Figure 0.29b) become vertically aligned when you imagine rotating the vertical axis and placing it horizontally at the top (see the highlighted points in Figure 0.5b). At a later moment, I hid the crow on the map and asked Melvin where the crow would be given a point on the plane (see Figure 0.6a, right). Melvin circled an area on the map for the crow's potential location (see Figure 0.6a, left). He said,

> because I just imagined turning this [*pointing to the vertical axis of the plane, see Figure 0.6, right*] up like that [*tracing the pen on the air horizontally at the top of the plane*], and then, it [*pointing to the black dot on the plane, see Figure 0.6, right*] moved over here [*sliding the pen from the black dot toward the highlighted area on the plane in Figure 0.6, right*].

Collectively, his activity provided an evidence that Melvin's meaning of the points included transformed iconic translation as he translated a *transformed* version of the perceptual features of the plane (i.e., the physical location) to the situation (see my model of his transformed iconic translation in Figure 0.6b). It made sense for Melvin to imagine the axes on the map in this way because he determined that as the point on the plane moved diagonally from the horizontal axis to the vertical axis on the plane, the crow on the map moved directly from bottom (where his x-axis is) to the top (where his y-axis is) on the map. With this image of the map that included transformed iconic translation, Melvin not only rotated the vertical axis but also mentally rotated the points on the plane in order to match with the locations on the map.



(a) (b)

Figure 0.6. (a) Melvin's designated general area for the crow's location on the map according to given black point on the plane (yellow highlight on the plane is added for the reader) and (b) my model of Melvin's transformed iconic translation.

Melvin's Activity in Downtown Athens Bike Task

Note that, in DAT and CT, I did not prompt Melvin to conceive the quantities in the situation. I first wanted to get insights into his spontaneous meanings of the point when he was given a situation and a graph together. I investigated how he could initially conceive the points on the plane before I attempt to investigate (and support) how he could conceptualize quantities in the situation before asking him to represent them as a graph. Since I identified that Melvin's initial meaning of the points included (transformed) iconic translation, I planned to take his attention to the quantities in the situation without asking him to engage in any graphing activities. I conjectured that if Melvin could conceptualize the measurable attributes of the objects on the map, he might be able to represent the quantities on the plane rather than making (transformed) iconic translation. In this section, I illustrate his activity in Downtown Athens Bike Task (DABT) where I investigated how he could conceive the quantities in the situation (e.g., the bike's distance from Arch).

DABT included the same map of Downtown Athens highlighting a straight road (i.e., Clayton St.) with two places located near the road (i.e., the Arch and the Canon; see Figure 0.7) and a bike on this road. I animate the map so that the bike moves at a constant speed back and forth along the Clayton St. starting at the West side of the street. I designed this task to explore how Melvin could conceive the situation quantitatively and how he could determine the relationship between quantities (the bike's distance from Arch [DfA] and the bike's distance from Cannon [DfC]). In particular, my purpose in this task was to explore and support Melvin's process of (i) conceiving the quantities' that vary in the situation, (ii) representing the varying quantities by his index fingers on the table, and (iii) representing the relationship between covarying quantities on a coordinate plane.

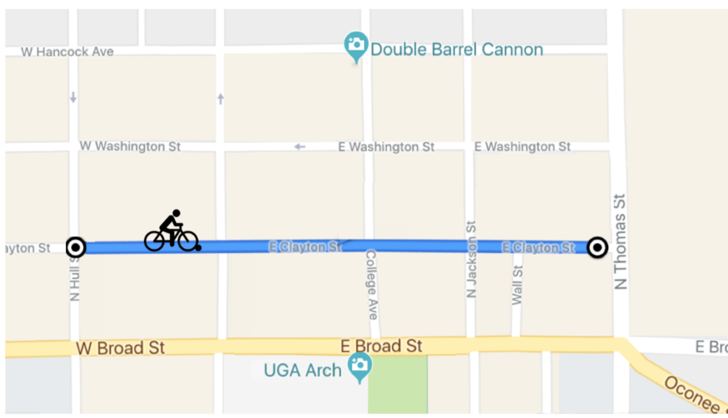


Figure 0.7. Downtown Athens Bike Task (DABT).

I illustrate that Melvin conceived the bike's DfA as the bike's *proximity* to Arch (i.e., its closeness/nearness to the Arch) without conceiving the measurable attribute of the bike and coordinated the variation of the bike's degree of proximity to Arch as the bike traveled on the map (e.g., the bike is getting closer to or farther from the Arch). Next, I illustrate how Melvin's spatial proximity reasoning influenced his graphing activity in one- and two-dimensional spaces.

Melvin's spatial proximity reasoning in DABT

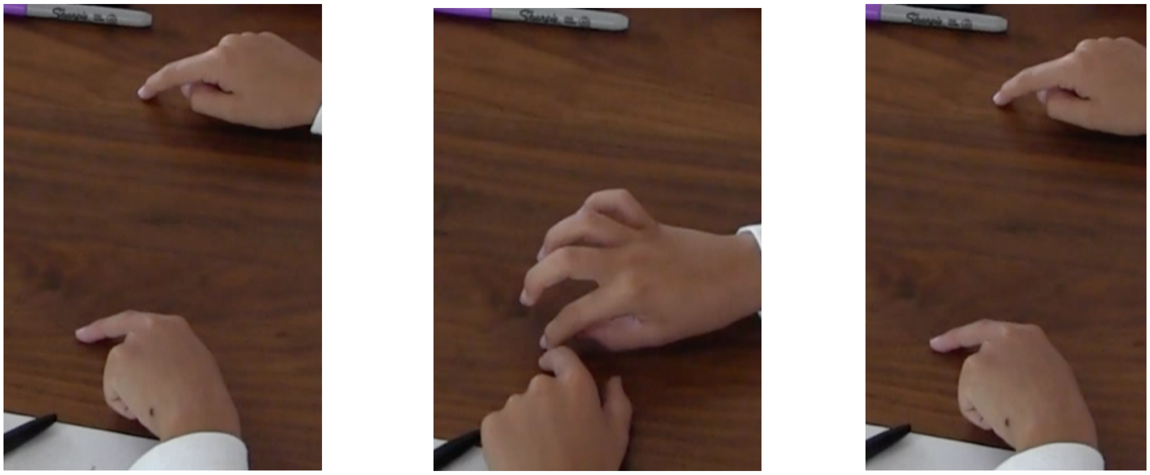
In the first part of DABT, I showed Melvin the animation where the bike rides on E. Clayton St in Athens Downtown (See Figure 0.7Figure 0.23) at a constant speed starting from the west side of the street. In order to get insights into how Melvin could conceive the dynamic situation, I first asked him to talk about things that he noticed changing as the bike moved on its path. He determined that “the direction that the bike goes” was changing in the animation. Then, I explicitly asked him how the bike's DfA is changing as the bike moved on the map. As the bike started moving from the left, Melvin said “closer, closer, closer, …” until the bike reached to the middle of the path, then he said “further, further, further …” until the bike reached to the right end side of the path. Since he seemed to be conceiving the bike’s DfA as the bike’s proximity to Arch, I decided to draw Melvin’s attention to the value of the bike’s DfA and see if he could describe the variation in terms of the measurable attributes (i.e., distances). I told Melvin “let’s say the bike is 200 meters away from Arch” when the bike is located at the left end side of the path. Then, I asked him “if I move the bike in this way [*moving the bike from left end side of the path to the right*], what happens to this distance?” After a long pause, Melvin responded “it, it, umm, it is [inaudible], it gets closer. What did you mean?” I repeated the question by using the word “value” as follows: “What happens to this value? Like what happens to that quantity?” Then, Melvin was able to say, “it becomes less?” with a questioning tone in his voice as he appeared unsure. I told Melvin “yes, it becomes less. And we can use the word decreasing.” I then repeated the same task for Melvin asking him to describe how the bike’s DfA was changing as the bike moves from the left end side to the right. Melvin responded, “it is getting closer.”

Even though I prompted him to think about the bike’s distance as a quantity in terms of its value (e.g., 200 m) and asked him to imagine the variation of that value as the bike moved, he still described the variation of the bike’s DfA as it is getting closer to and farther from Arch. In fact, he was able to think about the variation of a value when he was given an explicit value (i.e., 200m). However, I didn’t have any evidence that he coordinated a quantity’s variation in terms of its value or magnitude given the two objects on the map. From this activity, I infer that Melvin conceived the bike's DfA as a proximity between the bike and Arch, with Arch being the reference object. As the bike moved, he coordinated the bike's proximity to Arch without conceptualizing the bike's DfA as a quantity in the situation

Melvin's representation of spatial proximity reasoning in DABT

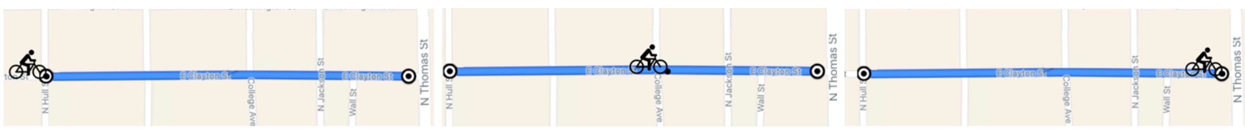
In order get more insights into his conception and representation of the bike's DfA, I decided to provide an opportunity for Melvin to represent the variation of what he conceived to be the bike's DfA. In the second part of DABT, I asked Melvin to focus on the bike's DfA as it traveled on E. Clayton St. back and forth. I directed him to place his index fingers horizontally on the table in a way that his left index finger is fixed on the table and his right index finger can only move in a horizontal direction. Then, I told him to move his right index finger left to right so that the distance between his index fingers could represent the bike's DfA as the bike rides along the road.

The reader can see three instantiations of how Melvin moved his fingers according to what he conceived to be the bike's DfA in Figure 0.25. When the bike is at the left end side of the path (Figure 0.9a,), Melvin positioned his fingers as seen in Figure 0.25a. As the bike started to move to the right, Melvin smoothly moved his right index finger to the left until both index fingers met (i.e., physical touch, see Figure 0.25b) when the bike is at the middle of the path (see Figure 0.9b). Then, he moved his right index finger to the right until the bike reach at the right end side of the path on the map (see Figure 0.25c and Figure 0.9c).



(b) (b) (c)

Figure 0.8. Melvin's activity in the second part of DABT.



(c) (b) (c)

Figure 0.9. Bike's position relative to Melvin's activity in the second part of DABT.

I asked Melvin to explain how he moved his fingers. He said,

> When the bike is closest to the Arch [*pointing to the bike at middle of the path, see Figure 0.9b*] … I was as closest as I can get to this finger [*meeting his index fingers on the table, see Figure 0.25b*] … and when he was getting further away from Arch, … I moved my fingers away from it [*moving his right index finger to the right*].

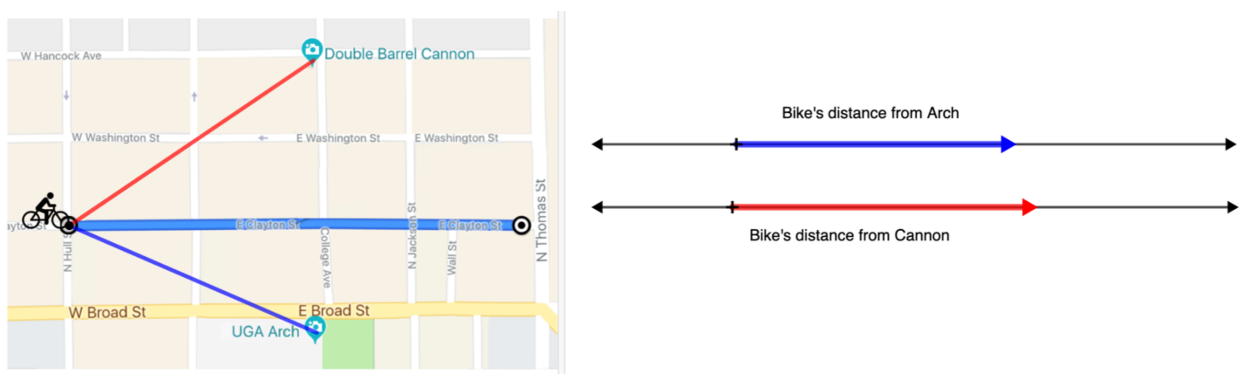
Note that he connected his index fingers when representing what he perceived to be the bike's DfA on the table when the bike is at the middle on its path (see Figure 0.25b). I asked M "when the bike is there [*referring to the bike's location in the middle of the path as seen in Figure 0.25b, bottom*], does the distance between your fingers represents the bike's distance from Arch?" He said, "well, kind of, because when they [the index fingers] are like almost touching [*meeting his index finger on the table as seen in Figure 0.25b, top*], he [the bike] was as closest as he could get to the Arch." His activity suggested that he did not conceive the distance between his index fingers as a representation of the bike's DfA as a quantity. Instead, the separation (proximity) of two fingers represented how the bike was close to the Arch on the map for Melvin. That is, meeting fingers represented "the bike was as closest as it could get to the Arch." I claim that this was an implication of Melvin's spatial proximity reasoning. I didn't have any evidence that Melvin conceived and isolated the bike's DfA as a quantity in the situation and attended to decompose it from the situation and represent its variation on the table with the distance between his index fingers. What he coordinated was the right index finger's (i.e., the bike for him) degree of proximity to the left index finger (i.e., Arch for him), with the left index finger being the reference object. Moreover, Melvin imagined the Arch in place of his left index finger and the bike in place of his right index finger on the table as he moved his fingers. This provided another evidence that Melvin was engaging in representation of his spatial proximity reasoning on the table.

Melvin's initial graphing activity in DABT (first draft)

In DABT, I planned to ask Melvin to represent the relationship between the bike's DfA and DfC on a paper with a coordinate plane. Based on his spatial proximity reasoning, I hypothesized that Melvin could bring the physical bike, Arch, and Cannon on the plane with the bike getting closer to and farther from both Arch and Cannon. In order to support Melvin to conceive the quantities in the situation instead of coordinating the proximities, I decided to provide him additional figurative material (i.e., the bars as an indication of quantities' magnitudes) that could afford him to conceive the measurable attributes of the bike in the situation.

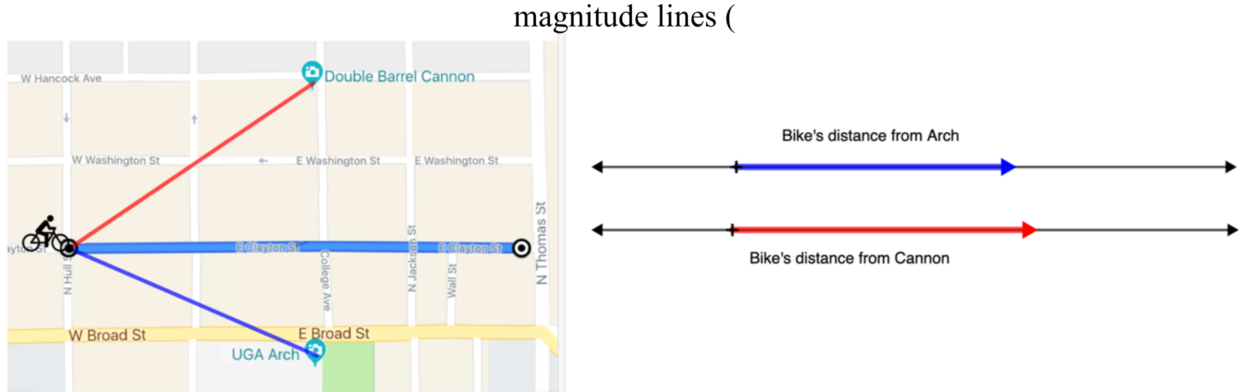
I first asked Melvin if there is a relationship between the bike's DfA and DfC as the bike traveled without proving the additional figurative material. Melvin said yes and compared the bike’s proximity to Arch and the bike’s proximity to Cannon. He said that “the bike’s distance from Cannon is a little further than the bike’s distance from Arch.” Note that Melvin used the word “further” rather than “bigger” when comparing the “distance from Arch” and “distance from Cannon,” which suggested his reasoning entailed comparing the spatial proximities, rather than quantities. When asked to explain why, Melvin stated that “because Cannon is further away from East Clayton Street than the Arch is.” For Melvin, Cannon’s degree of proximity to the entire street being farther than the Arch’s proximity to the E. Clayton implied the bike was always farther away from Cannon (compared to Arch) as the bike traveled on E. Clayton St.

Melvin’s assimilation of the bars in relation to spatial proximity. In order to draw his attention to the distances as length, I decided to show the blue and red bars on the map as (from my perspective) an indication of the magnitudes of bike’s DfA and DfC, respectively (see



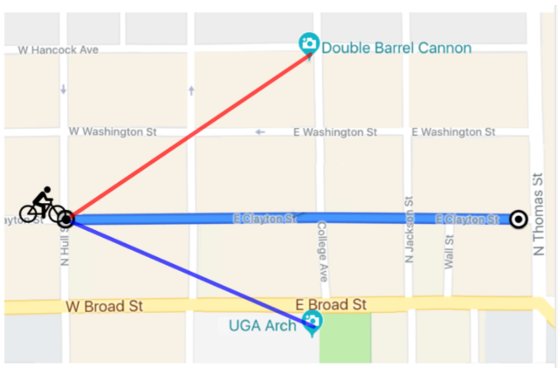
(a) (b)

Figure 0.10a). I also showed another screen where I presented a dynamic tool that could afford Melvin’s engagement with quantities’ magnitudes represented by directed bars placed on magnitude lines (



(b) (b)

Figure 0.10b). The directed bars can be varied in length as the bike moves on the map (see https://youtu.be/6kdbDeVEF9w).



(c) (b)

Figure 0.10. (a) Red and blue bars on the map and (b) red and blue bars on the magnitude lines.

I moved the bike on the map, in turn, the length of the bars on the magnitude lines varied. Based on this observation, Melvin immediately assimilated the variation of the location of the bars into his finger activity where he moved his index fingers left to right to represent the bike's DfA and DfC (see his spatial proximity reasoning illustrated in the previous section). Pointing to the bars on the magnitude lines, Melvin said, “that is what we were doing with our fingers.” He placed his right index finger in place of the right end side of the red bar and put left index finger in place of the left end side of the red bar on the magnitude line and said, “when the bike was moving, our fingers were moving, just like the lines are moving.” With this assimilation, he also imagined the physical bike moving on the magnitude lines with the Arch and Cannon being at the zero point on the magnitude lines. Referring to the right end side of the blue bar, he said “it is still supposed represent the bike, it is just the bike moving in a different way relative to the Arch [i.e., zero point on the magnitude line].” By “a different way”, he meant the bike moved to the right in its path on the map while the bike moved to the left on the magnitude line when (from my perspective) the bike’s DfA was decreasing. I infer that Melvin engaged in spatial proximity reasoning by conceiving the right end side of the blue bar as the physical bike and left end side of the blue bar as Arch on the magnitude line.

Moreover, as the bike moved on the map, Melvin described the relationship between the red and blue bar on the magnitude lines as “they turn at the same time” in that by “turn” he meant the bike “switching directions.” Note that Melvin did not describe the relationship as the bar’s length increasing and decreasing at the same time. Since he was imagining the bike on the magnitude lines, he simply determined the bike “switching directions” on both magnitude lines at the same time. His description of the relationship between the blue and red bars also included “it [*referring to the red bar on the magnitude line*] always stays a little ahead of the blue line.” This also suggested his comparison did not included the length of the bars (e.g., the red bars are longer than the blue bar), instead his comparison included imagining a bike on the magnitude lines and one of them is ahead of the other on the line. Therefore, he engaged in spatial proximity reasoning when assimilating the variation of the bars on the magnitude lines.

Melvin’s first draft in DABT. In order to investigate Melvin’s graphing activity on a coordinate plane, I asked Melvin to graph the relationship between the bike’s DfA and DfC as the bike traveled on E. Clayton Street on the map. I wondered if and how Melvin could use his image of the magnitude lines in his graphing activity on the plane. Figure 0.28 shows Melvin’s first draft. I illustrate that Melvin’s representational activity was an implication of his spatial proximity reasoning as he imagined the physical Arch, Cannon, and the bike to the plane.

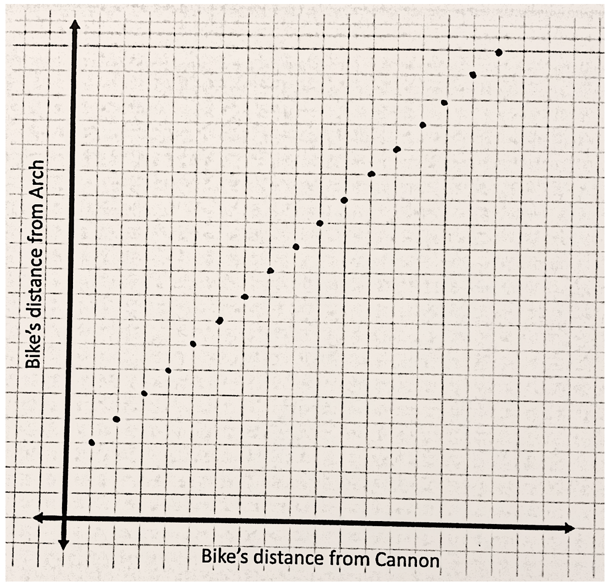


Figure 0.11. Melvin's first draft in DABT.

Melvin conceived the entire vertical and horizontal axis of the plane as the Arch and Cannon, respectively, and he conceived his plotted points on the plane forming a path "where the bike travels" (i.e., E. Clayton St.). He said:

> I imagine this [*pointing to the vertical axis*], the *y*-axis as the Arch [*sliding his finger over the axis from top to bottom*] and the *x*-axis is the Cannon [*sliding his finger over the horizontal axis from left to right*]. And I drew East Clayton Street on the dotted line [*tracing his finger over his graph on the plane upward from left to right*]. Um, I drew it [*referring to his dotted line*] a little closer to the Arch than the Cannon because East Clayton Street is closer to the Arch than Cannon on the map.

Melvin represented the relationship between what he conceived to be the bike's DfA and DfC as his graphs' (the physical dots he plotted on the plane in Figure 0.28) degree of proximity to the vertical and horizontal axis, respectively. That is, he drew his graph on the plane closer to the vertical axis compared to the horizontal axis because the bike was always closer to the Arch than Cannon on the map during its trip on E. Clayton St.

Note that Melvin's graph intersected with the vertical axis of the plane. I conjectured that this is also an implication of his spatial proximity reasoning because this was compatible with his representational activity with fingers. Recall that Melvin connected his fingers (i.e., physical touch; see Figure 0.25b) to represent what he perceived to be the bike's DfA when the bike was located in the middle of the path (see Figure 0.9b). Two fingers that were touched each other represented "the bike was as closest as it could get to the Arch" for Melvin, which suggested that he did not conceive the distance between his index fingers as a representation of the bike's distance from Arch as a quantity. Akin to his finger activity, Melvin's graph also reflected the same idea by intersecting his graph to the vertical axis, which is the Arch for Melvin. When the bike was located in the middle of the path (where its DfA and DfC are minimum), I asked Melvin how his graph showed that moment. He pointed to the dot that is closest to the vertical axis on his graph in Figure 0.28. He said, "because the bike on the map is as closest as it can get to the Arch, and this dot is as closest as it can get to the *y*-axis." This provided an evidence that Melvin did not conceive the bike's DfA as a quantity in the situation and represented this distance as a quantity on the plane. Instead, he conceived the bike's degree of proximity to Arch (i.e., "as closest as it can get") in the situation, and he represented the same property on his graph (i.e., "as closest as it can get"). I infer that this is an implication of Melvin's spatial proximity reasoning as he conceived the bike's DfA as the bike's degree of closeness to the Arch so that "closest" moments would match in both map and graph.

Melvin's Measurement Activity and His Second Draft in DABT

Recall that Melvin imagined the physical bike moving and coordinated the bike's proximity to Arch and Cannon when graphing on one- dimensional and two-dimensional coordinate systems. As an intervention, I provided Melvin with the following measurement activity to promote an understanding of the bar on the magnitude line in relation to the quantities' magnitudes, rather than imagining the bike and Arch on the magnitude line and coordinating the spatial proximity between them. In the next section, I present the features of the intervention along with my goals and Melvin's activity in the measurement task. Following the next section, I present how the intervention influenced Melvin's graphing activity in his second attempt in DABT.

Melvin's measurement activity

In this section, I present the design and instructional moves that I made in the measurement task against the backdrop of Melvin's current reasoning with spatial proximity and his meanings of the bars that included the physical bike and Arch. Note that this section only includes the description of the design of the instruction and Melvin's activity. In the next section, I illustrate his graphing activity in DABT to provide evidence that his graphical meanings slightly changed potentially because of his engagement in the measurement activity.

I presented Melvin with the same map of Downtown Athens highlighting Clayton St. and a bike on this street together with the Arch and the Canon (see Figure 0.12, left, orange and green segments were not available at the moment). I also presented him with a directed bar located in a *random* place on the magnitude line (i.e., the blue bar on the map and the blue bar on the magnitude line are not synced at the moment, see Figure 0.12, right). I asked Melvin to change the length of the blue bar in a way that it has the same length as the bike's DfA. The blue bar in the situation and the blue bar on the magnitude line were presented in different screens in GeoGebra so that they could not be physically moved for direct comparison.

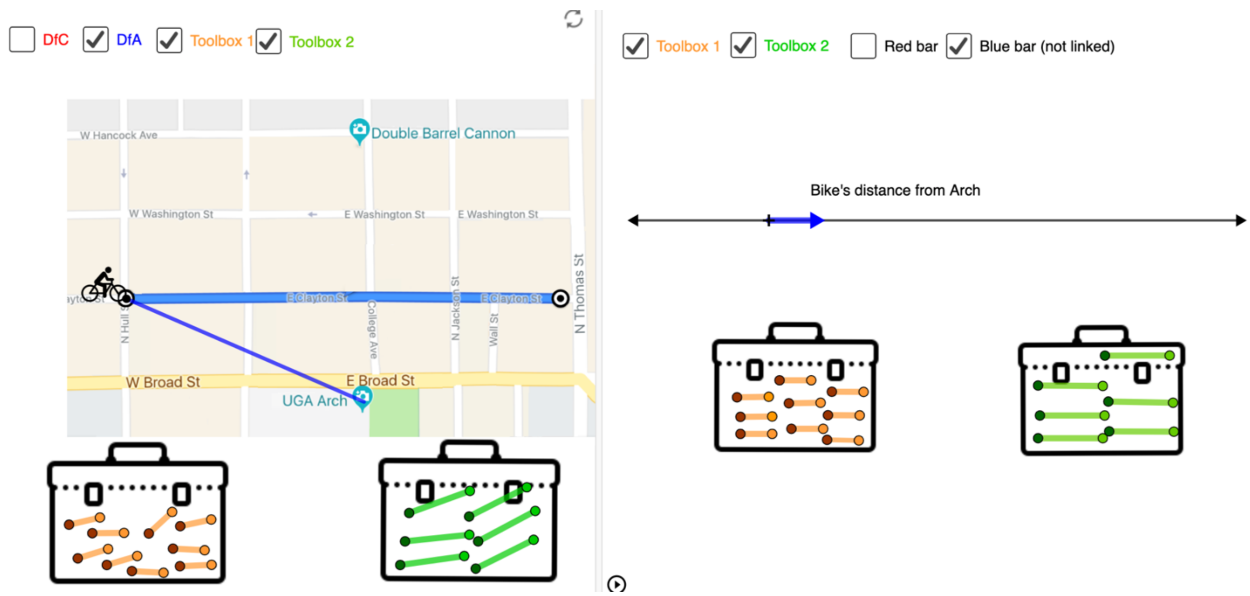


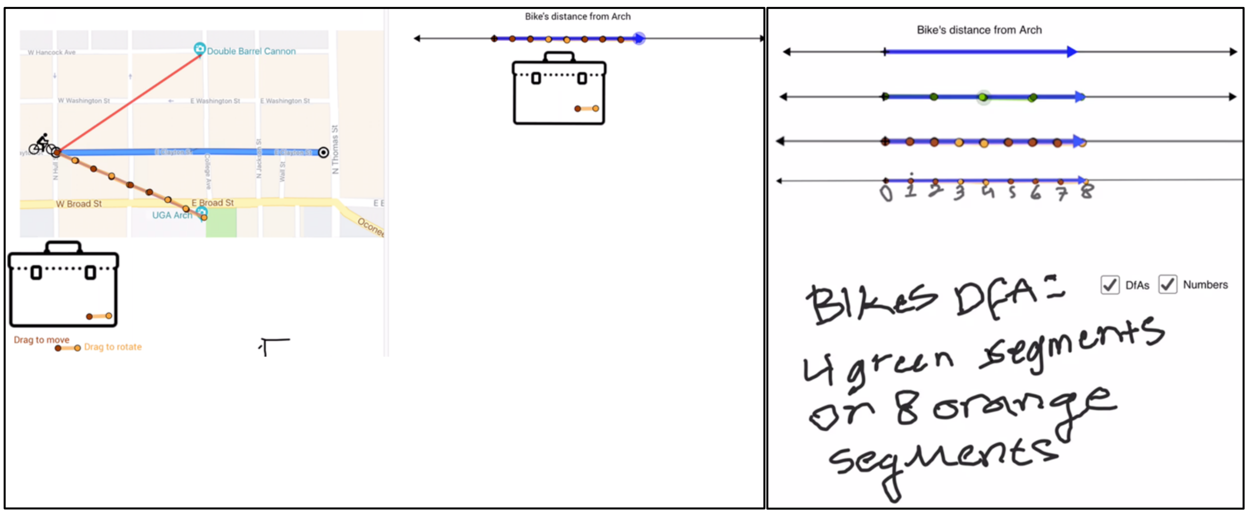
Figure 0.12. Measurement Activity.

I provided tools for Melvin to be used in measuring the magnitude of the bike's DfA and representing it on the number line exactly. I asked Melvin if and how he could use these segments in figuring out the exact length of the bar on the magnitude line to represent the magnitude of the bike's DfA when the bike was located in the left side of the road. By dragging and rotating the orange segments, Melvin measured the length of the blue bar on the map and determined the bike's DfA by counting the number of segments that he laid out on the blue bar (i.e., 8 little orange segments, see Figure 0.13a). He then carried out the same measurement activity on the magnitude line by dragging and dropping 8 orange segments from the toolbox on the magnitude line. Melvin then moved the blue bar on the magnitude line in a way that it matched with the length of 8 orange segments that he laid out on the magnitude line (see Figure 0.13a). He described his measurement activity as follows,

> [Referring to the situation], I put certain amount of little segments on the line that is from the biker to the Arch [*pointing to the blue bar on the map, see Figure 0.13a*]. And I counted how many segments I put there and [*pointing to the magnitude line, Figure 0.13a*] I put the same amount of segments in the same way up here [*pointing to the*

magnitude line]. And I put the blue line [*referring to the blue bar on the magnitude line*] right where, um, the segments ended because that is how long.

Note that Melvin used the orange and Naya used the green segments to measure the bike's DfA. I asked Malvin to compare his result with Naya's. Melvin determined that the bike's DfA is "4 green segments or 8 orange segments" (see Figure 0.13b). I conjectured that this could help Melvin to conceive the length of the blue bar as a quantity's magnitude and know (or anticipate) that the quantity's magnitude is invariant with any change in unit.



(a) (b)

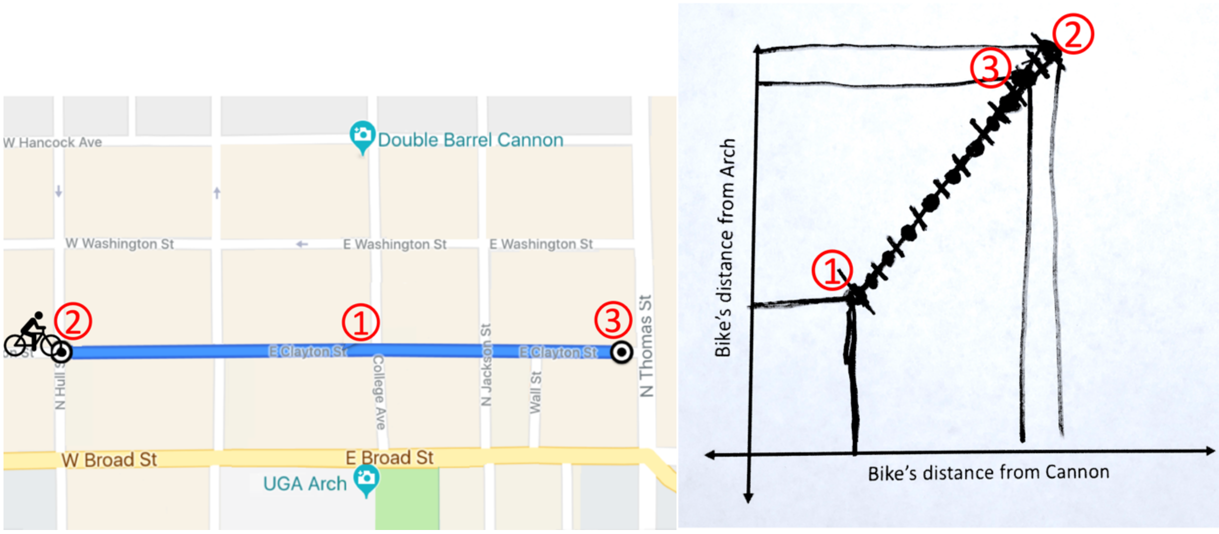
Figure 0.13. (a) Melvin measuring the bike's DfA on the map and on the magnitude line and (b) Multiple bars illustrating the result of measuring with units of different size.

After his measurement activity, I provided an opportunity for Melvin to insert numbers on the magnitude line to record his measurement process. He labeled each points on his magnitude line with numbers from zero to eight (see Figure 0.13b). I hypothesized that this could promote Melvin to develop an understanding for the magnitude line imagining the quantities' magnitudes and values without imagining the physical objects (e.g., the bike, Arch, and Cannon) on the line. I tested my hypotheses in the following activity where I asked Melvin to graph the relationship between the bike's DfA and DfC as a second draft.

Melvin's second draft in DABT

After the measurement activity, I asked Melvin to sketch his second draft to represent the relationship between the bike's DfA and DfC. In his first draft (see Figure 0.28), recall that Melvin conceived the entire vertical axis and entire horizontal axis of the coordinate plane as the Arch and Cannon, respectively, and he conceived his plotted points on the plane forming E. Clayton St. where the bike traveled. Moreover, he imagined the physical bike moving on the plane by coordinating the bike's spatial attributes (i.e., its proximity to the vertical axis, which was Arch for him, and to the horizontal axis, which was the Cannon for him) without conceiving the bike's measurable attributes. In this task, compared to his meanings and organization of the space from his first draft, I conjectured that Melvin could potentially switch from spatial proximity reasoning to quantitative covariational reasoning. That is, he could potentially conceive the bike's DfA as the bike's measurable attribute (rather than the bike's proximity to Arch) and represent this magnitude by a segment.

The reader can see Melvin's graph in Figure 0.10b, which consisted of a line upward from left to right. Melvin's meaning of the graph still included imagining the physical Arch, Cannon, and the bike; however, he coordinated the variation of the quantities' magnitudes instead of proximity between objects. Melvin conceived each point on his graph as the psychical bike moving on its path on the plane according to the variation of its DfA and DfC that were represented by the vertical and horizontal segments drawn on the plane (Figure 0.10b). I describe his activity in detail below.



(b) (b)

Figure 0.14. (a) DABT, (b) Melvin's second draft in DABT (numbered labels are added for the reader).

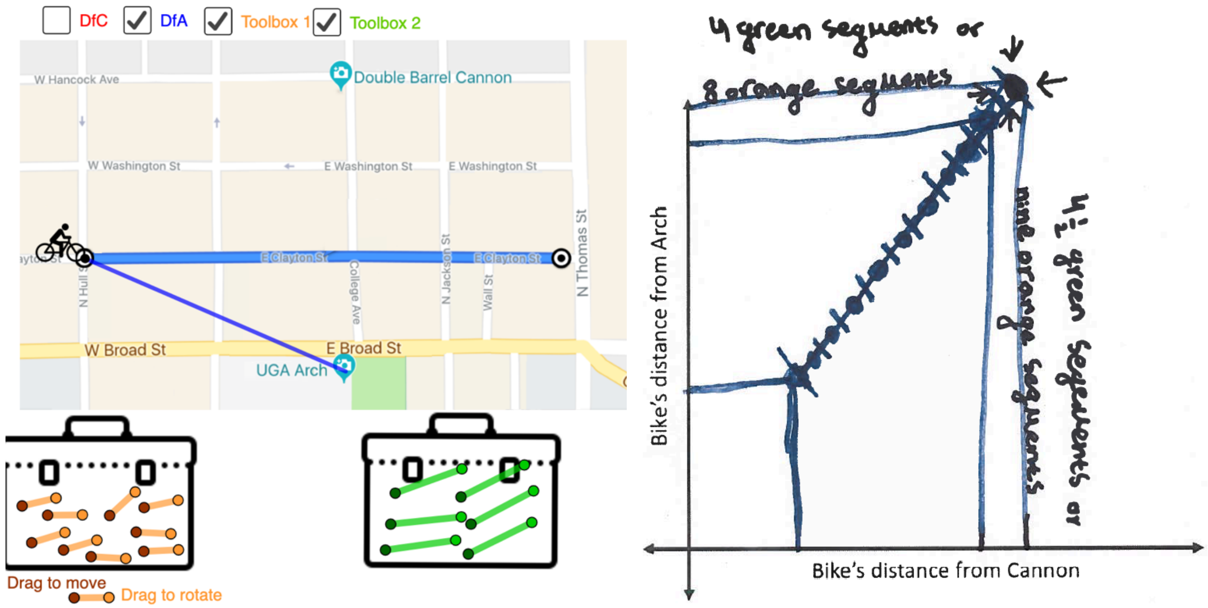
When constructing his graph, Melvin began drawing a line upward from left to right to represent "where the bike travels." He then added tick marks and dots on his line graph "to represent like where the bike could be." For example, when the bike is at a location labeled #1 in the map (i.e., its DfA and DfC are at their minimum, see *Figure 0.10*a), he said "it [the bike] would go right here [*pointing to the tick mark on his graph near label #1 in Figure 0.10b*]" and circled a dot on that tick mark to show where the bike is on his graph (see Figure 0.10b). Then, he drew the horizontal and vertical line segments to show the bike's DfA and DfC, respectively, for this moment of the bike's travel. He said "this distance [*moving his finger up and down over the vertical line segment that he drew on the plane*] is the distance from the Cannon and this distance [*moving his finger over the horizontal line segment that he drew on the plane*] is the distance from the Arch." This provided an evidence that Melvin conceived the length of the segments on the plane as the magnitudes of the quantities (i.e., the bike's DfA and DfC). He then added:

> And as it [*pointing to the bike on the map*] goes to either of these points [*pointing to the dots on each end side of the blue path that the bike travels on the map, see #2 and #3 in Figure 0.10a*], it [*the bike on the graph*] will be right here [*circling a dot on his graph labeled #2 in Figure 0.10b*] or right here [*circling another dot on his graph labeled #3 in Figure 0.10b and drawing the line segments from each dot to each axis of the plane, see Figure 0.10b*].

Referring to the two newly added dots on his graph, I asked him "which one refers to which locations on the map?" Melvin said:

> This dot [*pointing to the dot that is labeled #3 in Figure 0.10b*] is on the right side [*pointing to the right end side of the path on the map*] and this one [p*ointing to the dot that is labeled #2 in Figure 0.10b*] is on the left side [pointing to the left end side of the path labeled #2 on the map] … because the distance from there [*pointing to the right end side of the path labeled #3 in the map*] to there [*pointing to the Arch and Cannon on the map*] is shorter than the distance from there [*pointing to the left end side of the path labeled #2 in the map*] to there [*pointing to the Arch and Cannon on the map*].

He represented the bike's DfA and DfC as the point's (i.e., the bike for him) distance from the vertical and horizontal axis of the plane, respectively. Increasing length of the horizontal and vertical line segments on the plane indicated an increase in the bike's DfA and DfC. The length of horizontal line segments is shorter than the length of the vertical ones for each point on the graph because Melvin determined that the bike's DfA is always less than the bike's DfC. Note that, although Melvin assimilated his graph as "where the bike travels," Melvin's graph was not the bike's path *as it is* seen in the map; it was not an iconic translation. Melvin's meaning of the points included determining quantitative features of the bike in the situation (i.e., its DfA and its DfC) and ensuring to preserve these quantitative properties on the plane. Melvin drew segments as indication of quantities' magnitudes and be able to transform (i.e., dis-embedding and re-presenting) those magnitudes from situation to the graph.



(a) (b)

Figure 0.15. (a) Measurement activity in DABT and (b) Melvin's second draft in DABT in relation to his measurement activity

Recall that Melvin represented the magnitude of the bike's DfA as the length of the blue bar on the magnitude line in the previous task where he measured the distance by using the orange (and green) segments as a unit of measurement (see Figure 0.13). Melvin also conceived the left end side of the blue bar as a reference point on the magnitude line when measuring the bike's DfA. Here, in his graphing activity, he represented the magnitudes on the plane by using the entire axis as a reference object without conceiving the axis in relation to the magnitude lines in his measurement activity. After he constructed his graph (i.e., second draft in Figure 0.10b), I located the bike at the left end side of the path on the map (see Figure 0.15a) and asked Melvin how long was the bike's DfA when the bike was there on the map. Melvin recalled that "four green segments or eight orange segments." Then, I wanted to know how Melvin could show these numbers on his graph. Melvin pointed out the horizontal segment and labeled it as "4 green segments or 8 orange segments" (Figure 0.15b). Then, I also asked him to measure the bike's

DfC on the map. He figured that the bike's DfC was 4 and a half green-segment long or 9 orange-segment long on the map. When asked, Melvin associated the value of the bike's DfC with the length of the vertical segment he drew on the plane and labeled it as "4 ½ green segments or nine orange segments" (Figure 0.15b). This provided extra evidence that Melvin conceived the segments that he drew on the plane in relation to the measurable attributes of the bike (i.e., the values of the bike's DfA and DfC) and be able to transform these quantities' values from situation to the graph. This also provided an evidence that he didn't assimilate what I perceive to be the axis of the plane in relation to the magnitude lines as he didn't insert numbers along the axes.

Melvin's goal was to create a representation on a given space to represent how the quantities varied and his graph exactly showed the how the bike's DfA and DfC increased and decreased as the bike traveled on the map. Despite the fact that Melvin constructed a non-normative graph on the plane, his form of reasoning was productive in terms of completing the goal of the activity as he perceived it. I considered his graphing activity as a different way of graphing relationships because it still required him engage in quantitative coordination in a non-normative way although Melvin's goal included locating the bike in the space. To locate the bike in the space, Melvin represented the quantities' magnitudes in the space by committing to a reference, such as the axis itself, and where those magnitudes meet determined the location of the bike. I classified this meaning of points as spatial-quantitative multiplicative object (SQMO) in Chapter 4. This meaning of points on the space is different than conceiving a point as a quantitative multiplicative object (QMO, see in Ch. 4), which require someone to represent quantities' magnitudes on the axis and create a point by taking two orthogonal magnitudes along the axis and creating projections. In the next activity, I provided an opportunity for Melvin to develop a meaning of a point in terms of representing a QMO in a Cartesian plane.

Melvin's Activity in Matching Game Task

Recall that Melvin drew horizontal and vertical segments on the plane to represent the quantities' magnitudes (see Figure 0.15b). He did not assimilate the axes of the plane as orthogonal magnitude lines although he successfully represented a quantity' magnitude on a magnitude line (see Figure 0.13). Thus, I decided to transition to Matching Game Task (MGT, Figure 0.16) where I provided Melvin an opportunity to make connection between the parallel magnitude lines and the coordinate plane for the purpose of constructing a Cartesian coordinate system. In particular, I designed MGT in order to engage Melvin to (i) represent two quantities' magnitudes on two parallel magnitude lines and then transition to (ii) representing two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and projecting the magnitudes on the plane. I conjectured that Melvin's engagement with MGT could promote an understanding of points for Melvin as an abstract object in relation to representing two quantities' magnitudes (QMO) rather than representing a physical object that moves on the plane (SQMO).

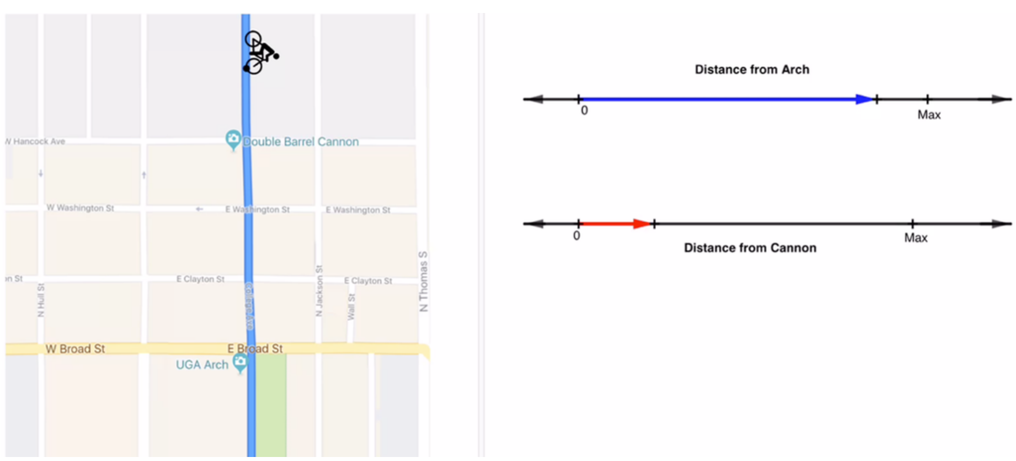
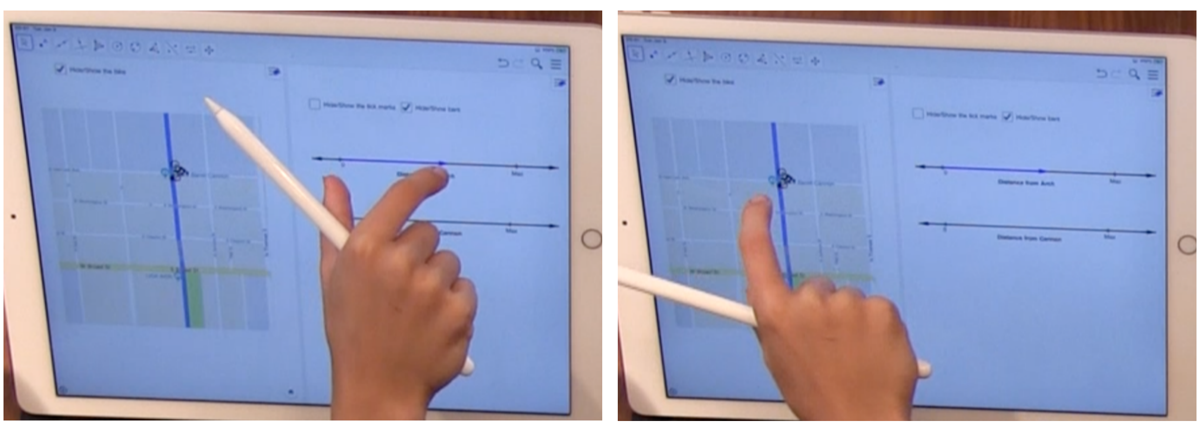


Figure 0.16. Matching Game Task (MGT).

MGT included the same map highlighting a straight road (i.e., College Avenue). Arch and Cannon are located on the road, and a bike rides along this road (Figure 0.16, left). MGT also included a dynamic tool that afforded Melvin's engagement with quantities' magnitudes represented by directed bars placed on magnitude lines (Figure 0.16, right). The directed bars can be varied in length (see https://youtu.be/knrN7XCbyOY). My goal was to promote an understanding that the length of both bars should be constrained in a way that they simultaneously represent the bike's DfA and DfC. I let Melvin to change the bike's location on the path and observe how the bars were moving on the magnitude lines. I asked Melvin what the red and blue bars might represent. He responded that the blue bar represents the bike's DfA, and the red bar represents the bike's DfC. When asked to show how he knew, he located the bike near Cannon on the map (see Figure 0.17) and said,

> Well, the bike's distance from Arch right now [*measuring the length of the blue bar on the magnitude line with his fingers, see Figure 0.17a*] is this much [*translating his fingers—by keeping the same distance between them—and placing it on the map, see Figure 0.17b*]. And it [*pointing to the bike on the map*] is right by the Cannon, so, it is not there [*pointing to the zero point on the magnitude line labeled "Distance from Cannon"*].



(a) (b)

Figure 0.17. (a) Measuring the length of the blue bar with finger and (b) translating the measurement to the situation.

I also hid the red and blue bars on the magnitude lines and asked Melvin to place tick marks on where he could think the head of each bar on the magnitude lines for different locations of the bike on the map. My purpose was to help Melvin to connect the conventional use of tick marks on the number lines to represent numbers to the use of bars on the magnitude lines. In this way, he might be able to know that a tick mark (or a point) on a magnitude line represents the measurement of a quantity's magnitude. I located the bike at the bottom of the map (see Figure 0.18, left) and Melvin successfully placed tick marks on each magnitude line accordingly (see Figure 0.18, left). When placing the tick mark for the bike's DfA, he engaged in the same measurement activity with his fingers as seen in Figure 0.17. Moreover, he stated that "tick marks show where the end of the bars would be." This suggested that Melvin's meaning of the tick marks was connected to the meaning of the bars on the magnitude line that represented the quantities' magnitudes. I conjectured that this meaning of tick marks might help Melvin to *record* the variation of the bar on the magnitude line since the varying bars do not leave trace or a mark as they move on the magnitude lines. In turn, I conjecture that this activity might help Melvin when we move to the two-dimensional space to record the relationship between two varying quantities on each axis of the plane.

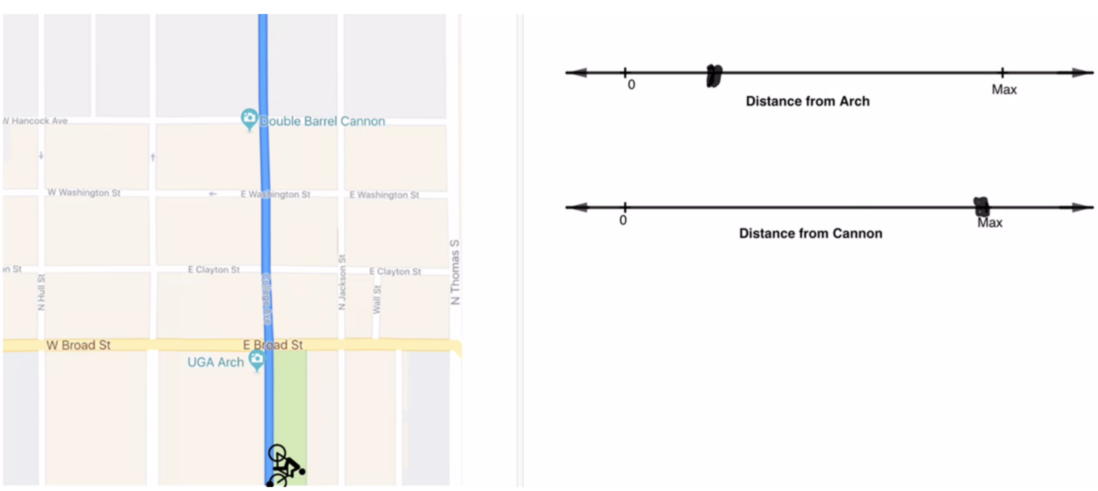


Figure 0.18. Melvin's tick marks on the magnitude lines representing the bike's DfA and DfC

With Melvin doing so successfully and indicating that he conceived of the tick marks in relation to quantities' magnitudes, I decided to transition to the next part of MGT where I asked Melvin if he knew a way to represent the bike's DfA and DfC by *a single point* instead of the two tick marks that he placed on two parallel magnitude lines. My goal was to explore if Melvin could see the additional representational features that is not available with a single number line, including the property that a point in the coordinate plane represents two quantities' magnitudes. With this activity, I expected to gain insights into Melvin's reasoning when trying to come up with ways to create a single point that represents two quantities' magnitudes.

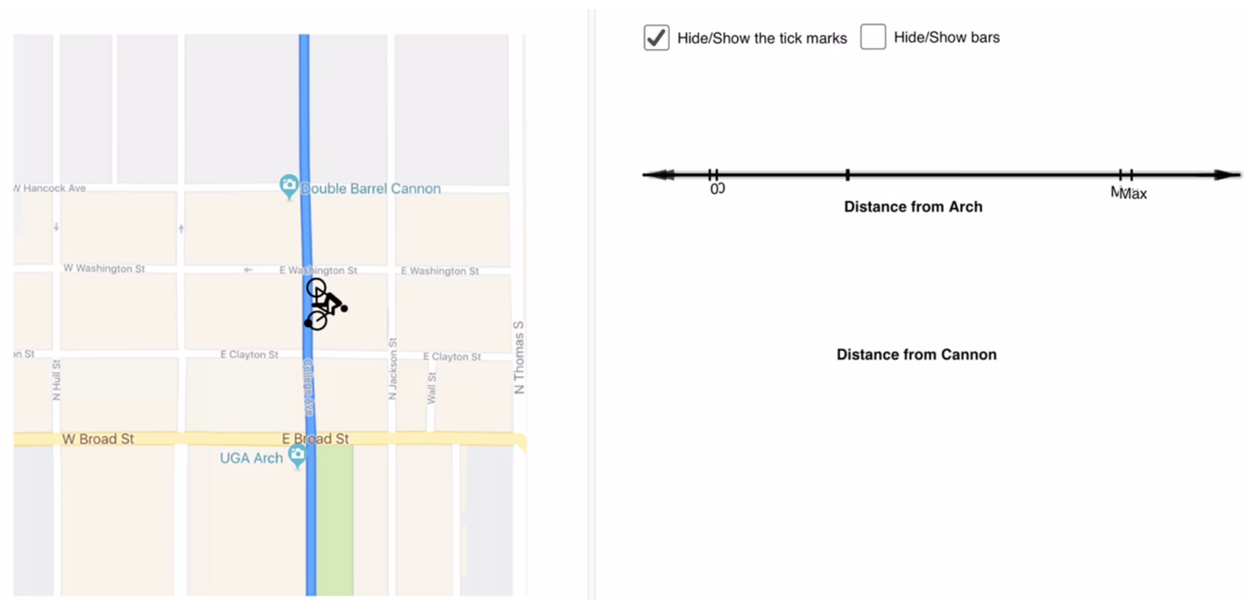


Figure 0.19. Melvin's activity seeking to create a single point

Melvin was not able to create a single point to represent two quantities' magnitudes, except for one case where the bike's DfA and DfC had the same length (see Figure 0.19). He moved one of the magnitude lines on top of each other by matching the zero and max points. He then located the bike in the middle between Arch and Cannon on the map and pointed out that the tick marks representing the bike's DfA and DfC were matched on the magnitude lines. Referring to the tick mark, he said "it represents both the distance from Arch and the Cannon because when the bike is right there [*pointing to the bike on the map, see Figure 0.19*], they are both the same distance from the Cannon and the Arch.” Melvin identified one single point and claimed that this point showed both the bike’s DfA and DfC. Putting the magnitude lines on top of each other and matching the two tick marks was a way for him to show that the single point simultaneously represented a state where the bike’s DfA and DfC were equal. He could not find a way to create a single point that simultaneously represented other states of the bike’s DfA and DfC. He needed to construct a new space that was different than the one-dimensional space in a way that he could satisfy the simultaneity for all states of the bike’s DfA and DfC. This was a need, from my perspective, why we need a two-dimensional space. I didn’t have evidence that Melvin perceived such a need at the moment as he was satisfied with his activity to create a single point that simultaneously showed the bike’s DfA and DfC for one occasion of the bike.

This suggested that I should provide other opportunities for Melvin that could afford him to structure the space in a way that he could see additional features that is not available in one-dimensional space. Thus, I decided to move to CT (see Figure 0.20) where I provided him a coordinate system as a given space. My goal was to provide Melvin with additional figurative material that might afford him to structure the space in a way that is compatible with Cartesian plane.

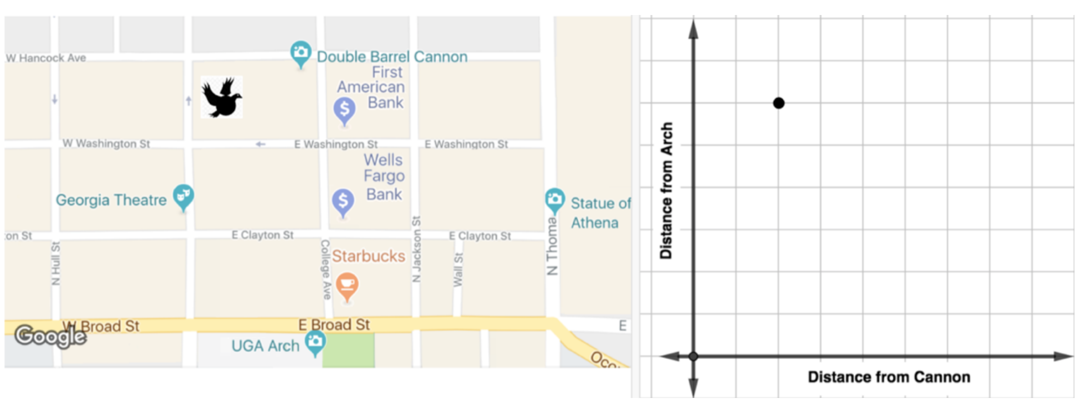


Figure 0.20. Crow Task (CT)

Melvin's Activity in Crow Task

CT included the same map in addition to a movable crow (Figure 0.20, left) and a Cartesian plane with the horizontal axis labeled as "Distance from Cannon" and vertical axis labeled as "Distance from Arch" (Figure 0.20, right). Recall that Melvin's initial meaning of the black dot included a transformed iconic translation (see the section of "Melvin's Initial Activity in Crow Task"). He conceived the black dot as the crow flying on the plane and imagined rotating the vertical axis and placing it on top of the plane in order to match the perceptual features of the map and the plane (see Figure 0.5). Because of his initial meanings in CT, I did not expect Melvin to make an immediate connection between the magnitude lines and the axes of the coordinate plane on his own. However, I conjectured that the additional figurative material provided in CT along with my prompts could afford him to make that connection. For example, I planned to ask Melvin to insert tick marks on each axis of the plane to show the crow's DfA and DfC, which might help him to conceive the axis of the plane in relation to his previous activity with the magnitude lines inserting tick marks.

Melvin was able to move the crow freely while observing how the corresponding black point on the plane moved. When asked to explain what the black point might represent on the plane, he conceived the dot as the crow, but he was not sure how the black dot moved according to how the crow flied on the map. He said, "I guess [the dot is] the crow, but I still don't understand this" meaning that he did not know how the black dot moved on the plane as the crow moved on the map. Note that, in his initial activity, he was able to explain the relationship between how two objects moved by engaging in transformed iconic translation. However, he did not make the same perceptual association here I think because the other dots on the plane were not visually available (see the other dots in Figure 0.29). In this version of the CT, I hid the other dots and there was only one single black dot that moved on the plane as the crow moved on the map (Figure 0.20). I conjecture that the figurative material that was available to him earlier (i.e., other dots) promoted an iconic translation while he was not able to make that connection here in the absence of the dots on the plane.

Melvin's meanings of tick marks on the axes of the plane in CT

Since Melvin was not able to make sense how the black point moved on the plane as the crow moved on the map and I knew he was able to reason with quantities, I wanted to draw his attention to the crow's DfA and DfC in the situation when the crow was not flying (i.e., static moment). I asked Melvin to show how he imagined the crow's DfA and DfC on the map when the crow was static on the map. He drew two segments to show the crow's DfA and DfC on the map (

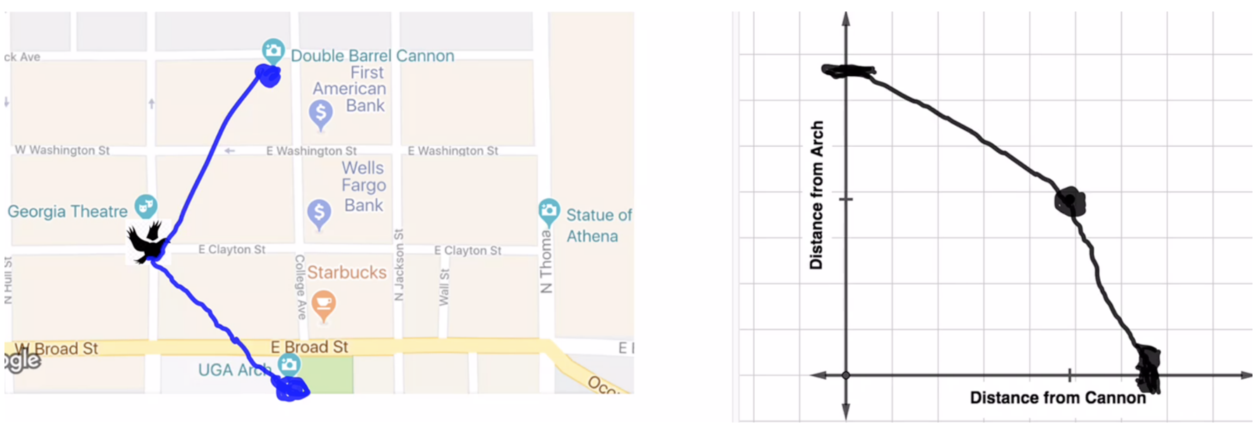


Figure 0.21, left).

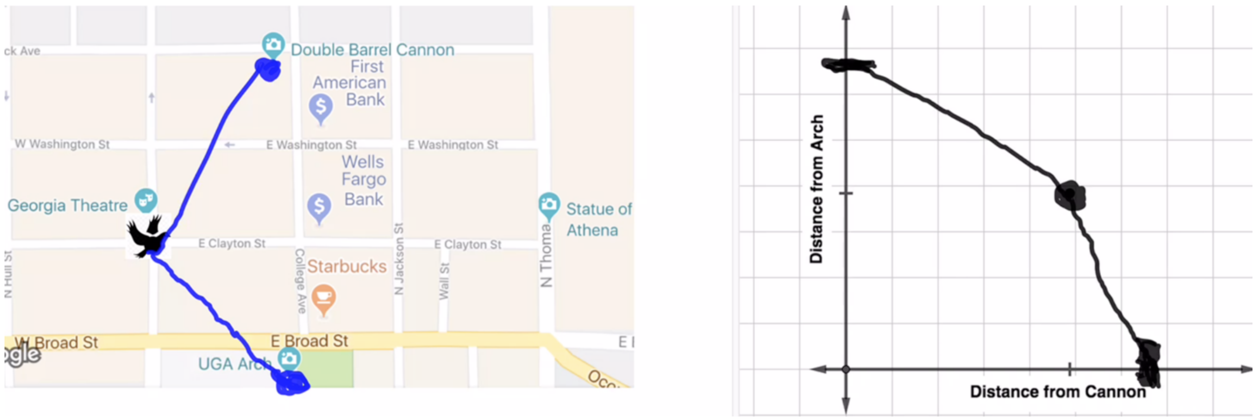


Figure 0.21. Melvin's activity in CT.

Tick marks as Arch and Cannon. Recall that, in MGT, Melvin was able to transform (i.e., dis-embed and re-present) the magnitude of the bike's DfA and DfC from situation to the magnitude lines by ensuring to preserve their length. He successfully placed tick marks on each magnitude line to represent the bike's DfA and DfC (see Figure 0.18). Thus, I thought that I could ask him to place tick marks on the axis of the plane to represent the crow's DfA and DfC. Since he never assimilated what I perceive to be the axes of the coordinate plane as the magnitude lines, I did not expect him to insert tick marks on each axis normatively in his first attempt. However, I still planned to ask this question because I thought this would be a way for me to draw his attention to the axes of the plane in later attempts. My goal was to help him to make connection between his activity of inserting tick marks on two parallel magnitude lines and inserting tick marks on the axis of the coordinate system.

To respond to my question, Melvin plotted two tick marks on each axis and drew line segments these marks to the black point (which is the crow for him) on the plane (see

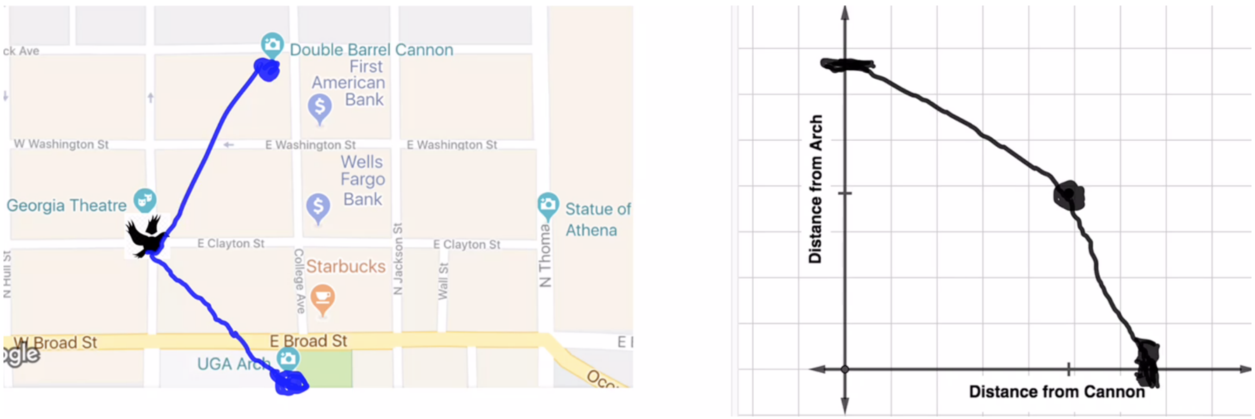


Figure 0.21, right; the computer-generated tick marks on each axis were not available at the moment). He explained,

> These tick marks are where the Cannon and Arch is, I think. So, um, and this [*pointing to the black dot on the plane*] is where the crow is. These [*sliding his index finger over the segments he drew on the plane,*
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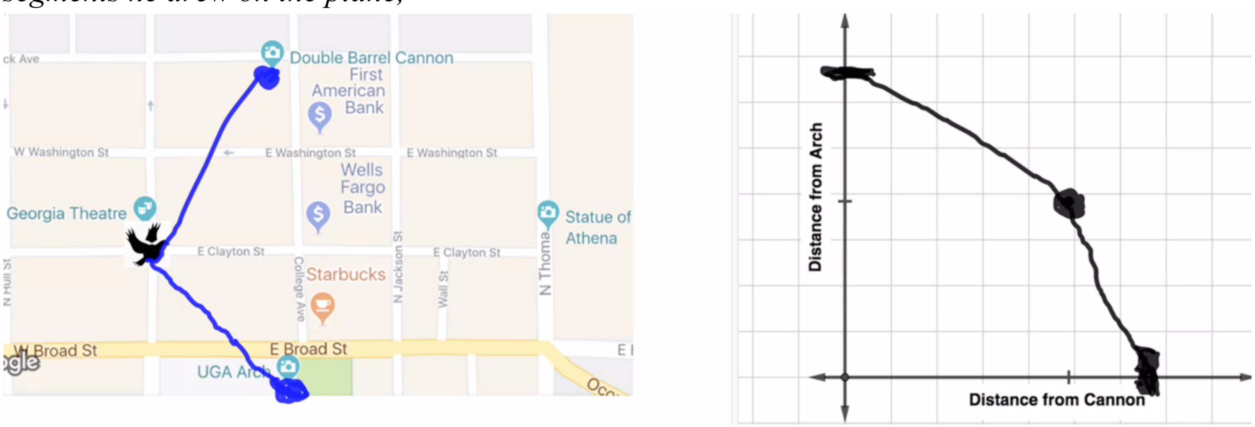
> 
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> *Figure* 0.21*, right*] are the two distance. I mean, I don't know.

Melvin plotted two tick marks on each axis where he thought where the physical Arch and Cannon were on the axes. I didn't have evidence that which of the tick marks referred to which location at that moment. To locate the crow on the plane, Melvin then represented the crow's DfA as the distance from the Arch on the axis and the crow's DfC as distance from the Cannon on the other axis. This suggested that his meaning of the point at the moment included representing a spatial-quantitative multiplicative object (SQMO). That is, he envisioned the point as a location/object by coordinating quantitative features of the object on the plane, which is the same meaning that Melvin had in his previous graphing activity in DABT (see Figure 0.10). This suggested that his representing SQMO was stable across two tasks.

Tick marks as representing the quantities' magnitudes. Then, I planned to show the computer-generated tick marks on each axis of the coordinate plane (see the additional tick marks on each axis in

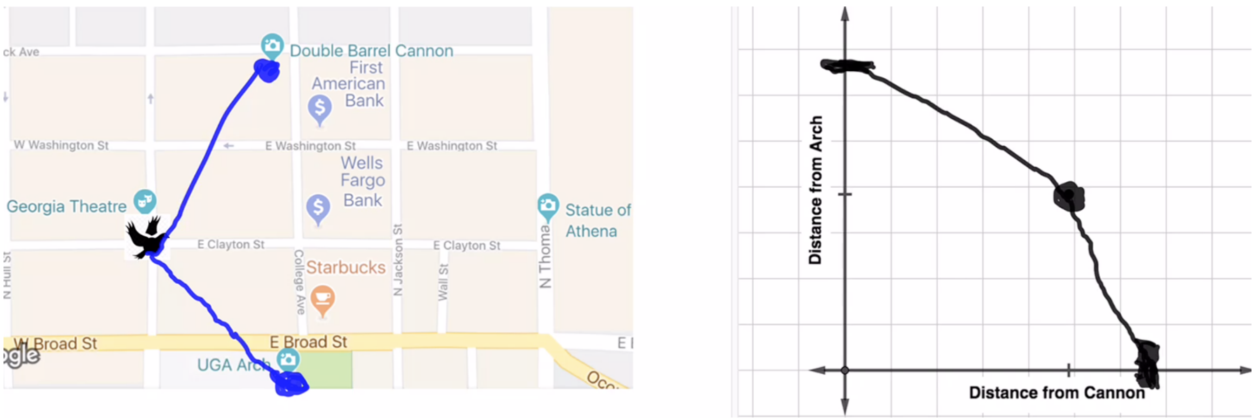


Figure 0.21, right). My goal was to provide additional figurative material for Melvin that could afford him to conceive the axis as the magnitude lines although I did not expect him to make that connection immediately. In particular, I planned to move the crow on the map, and in turn, the tick marks would move on each axis according to the crow's DfA and DFC. Since I knew Melvin could coordinate the quantities' magnitudes and representation of them on parallel magnitude lines, I hypothesized that the movements of the computer-generated tick marks could spark some ideas related to red and blue bars on the magnitude lines and make some connections to the axes of the coordinate plane.

Once I showed the computer-generated tick marks on each axis, Melvin immediately surprised by the locations of the tick marks on the axes. He questioned "why are they there" and he could not resolve this perturbation at the moment. Melvin was perturbed by the location of the computer-generated tick marks because his current meaning of the tick marks included the physical Arch and Cannon located on each axis. Since I knew Melvin had a meaning for tick marks on the parallel magnitude lines in relation to quantities' magnitudes in MGT, I decided to ask Melvin to reflect on his previous activity where he placed tick marks on the magnitude lines in MGT. I asked Melvin what the tick marks meant to him in the previous activity. He said, "end of the bar, like where the bar ended." I then asked him where these bars might be on the plane in CT. He began erasing his work on the plane in

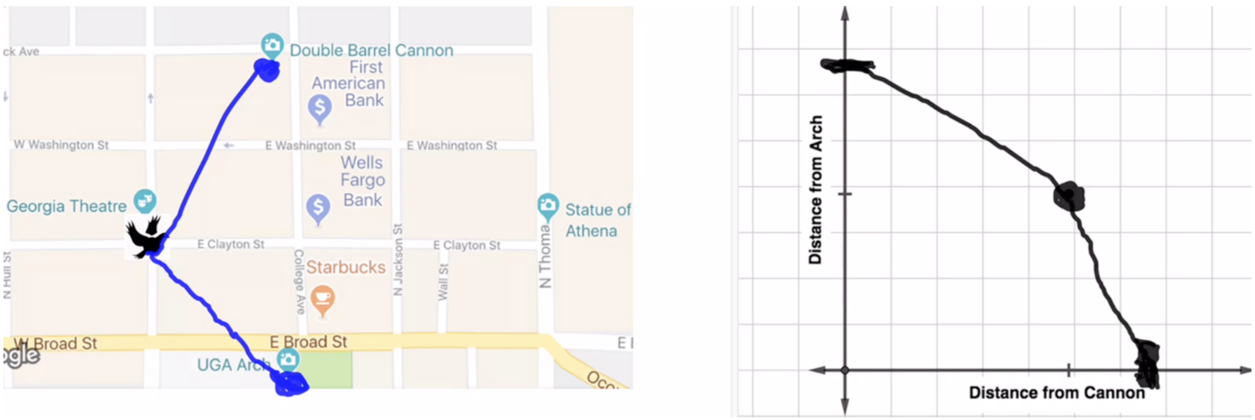
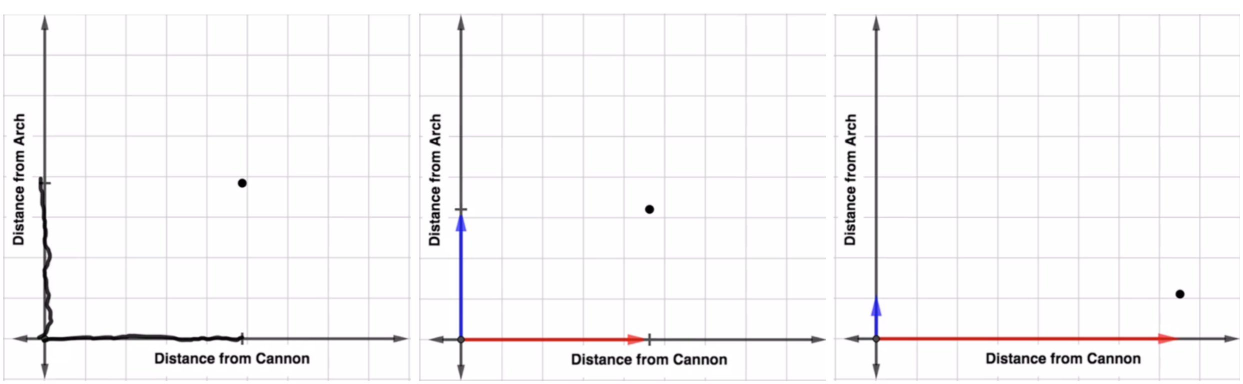


Figure 0.21 (right). Pointing to the computer-generated tick marks, he stated that

> Based on, um, the little tick marks, um, like maybe there is two bars, maybe like there is two lengths [*sliding the pen on the air over horizontal axis from origin to the tick mark and doing the same gesture for the tick mark on the vertical axis*] from the origin that go to these two [*referring to the tick marks*].



(a) (b) (c)

Figure 0.22. Melvin's tick mark on each axis representing the quantity's magnitude in CT

He then drew the horizontal and vertical bar on axes (see Figure 0.22a). I asked him to describe what these segments represented. He responded "the distances from the Cannon and Arch" although I found that he was not sure which segment showed which quantity (i.e., the crow's DfA or the crow's DfC). It was not surprising that Melvin was not able to tell which segment represented which quantity because he did not have any meanings for the label in relation to the magnitudes at the moment. Moreover, the length of two segments was almost the same (see Figure 0.22a) because the crow's DfA and DfC were almost equal to each other on the map (

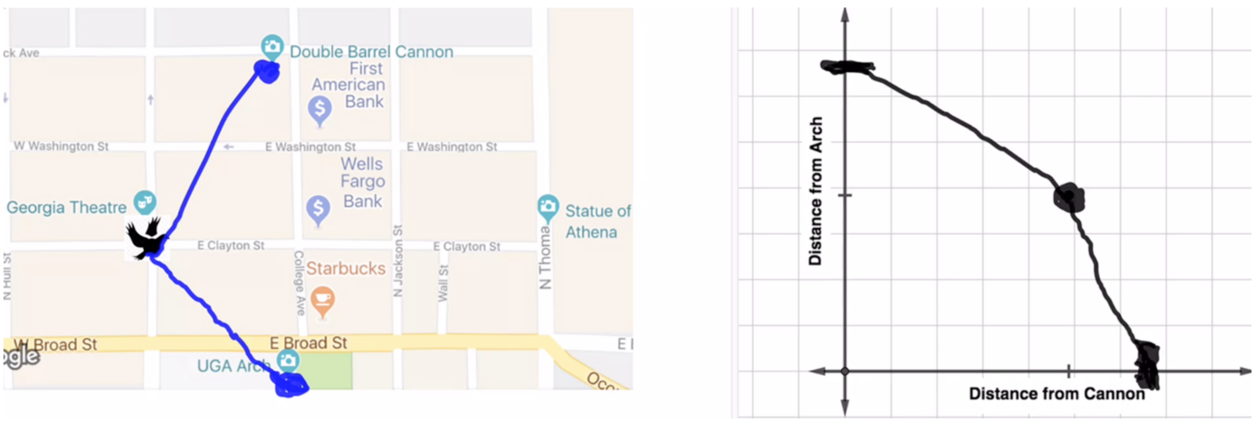


Figure 0.21, left). Thus, I hypothesized that Melvin did not have available information that could afford him to decide which segment represented which quantity on the axes.

To test my hypothesis, I decided to show the computer-generated bars on each axis and move the crow in a different location where the crow's DfA and DfC were visibly different. I showed the red and blue bars on the horizontal and vertical axis of the plane (see Figure 0.22b). Once these bars became physically available on the axes, Melvin immediately claimed that the red bar was showing the crow's DfC "because usually when it is the distance from the Cannon, it is red." For the same reason, Melvin also said that the blue bar represents the crow's DfA. I infer that Melvin reflected on the figurative meanings of the bars (i.e., its color) as he recalled that the DfC was always represented by the red bar throughout the teaching experiment. Thus, Melvin assimilated the perceptual feature of the bars (i.e., the red color) in order to make sense which bar represented which quantity.

In order to test if Melvin was also able to assimilate the quantitative features of the bars (i.e., the length of the bar) into his meaning, I asked Melvin to move the crow on the map and observe what was going on with the bars on each axis of the plane. He moved the crow right next to the Arch on the map and determined the length of the blue bar was too small (see Figure 0.22c). He said "it [the crow] is right next to the Arch [on the map] and the blue bar is really small." For Melvin, the blue bar represented the crow's DfA, and the red bar represented the crow's DfC since their length on the axes quantitatively matched with the crow's DfA and DfC on the map.

I showed the blue and red bars on the map and asked Melvin to move the crow (see Figure 0.23). As he moved the crow on the map, Melvin determined that the bars on the axes and the bars on the map "are the same." He elaborated by saying "lines are getting longer and shorter, like the lines here [*pointing to the bars in the map*] or the bars are equal to the bars like [inaudible]." This activity provided extra evidence that Melvin conceived the bars on the axes of the plane in relation to the quantities' magnitudes. Thus, I decided to investigate his meaning of a point on the plane to see if and how Melvin could join two magnitudes and generate a point that represented both.

Melvin's new meanings of points on the plane in CT

Recall that Melvin's meaning of the points so far included representing a SQMO. Now that I know Melvin was able to represent quantities' magnitudes on the axes of the plane, I planned to get insights into his meaning of the point (i.e., the black dot) on the plane. I wondered if Melvin would still conceive the point as the crow (i.e., SQMO) or as an abstract object that represents the two quantities' magnitudes (i.e., QMO). I surprisingly found that Melvin represented QMO when creating a point on the plane given the crow's location on the map whereas he represented NMO (i.e., transformed iconic translation) as well as QMO when locating the crow on the map given the point on the plane. That is, he held two different meanings (i.e., NMO and QMO) when figuring out where the crow would be on the map given a point on the plane. The details are in what follows.

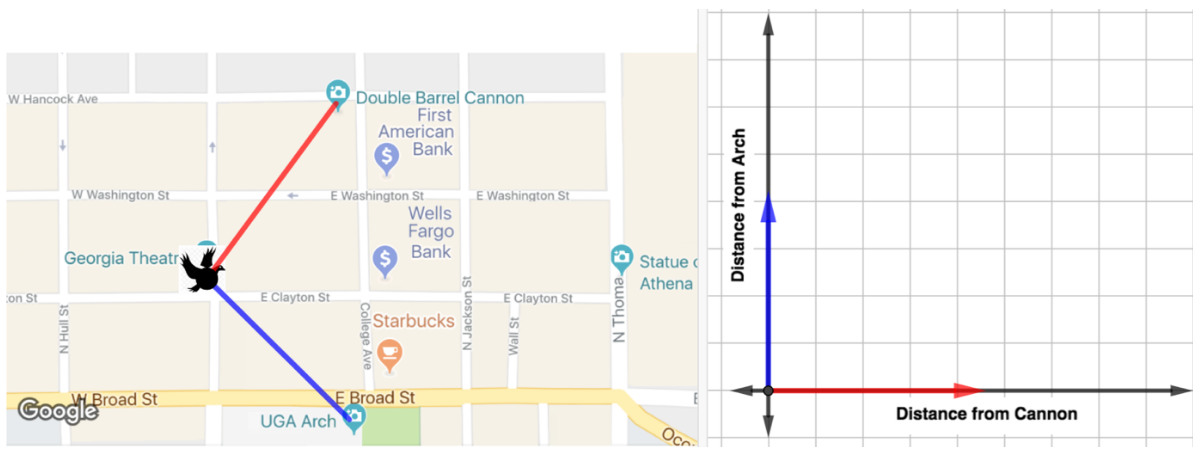
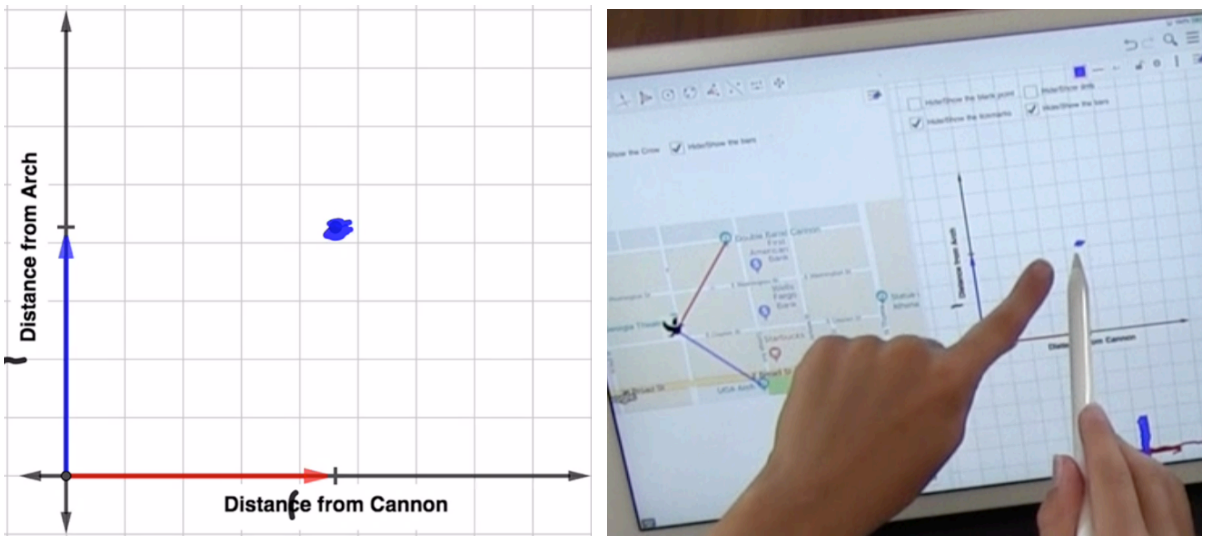


Figure 0.23. Plotting points in CT

Representing a QMO when constructing a point in CT. I hid the black dot on the plane keeping the blue and red bar on each axis visually available (see Figure 0.23, right) and asked Melvin where the black dot would be on the plane. Melvin plotted a blue point on the plane (see Figure 0.24a). When asked to explain how he knew the black dot was there, Melvin used a "multiplication table" analogy when explaining how he joined the projections of the bars on the plane. Melvin said,

> Like where these [*pointing to the tick marks where the end of the bars was on each axis*] two connect [*sliding the pen on the air vertically from the tick mark on the horizontal axis to his blue dot on the plane and then sliding the pen horizontally from the tick mark on the vertical axis to his blue dot on the plane*]. You know like a multiplication table, how, or, yeah, and like you have the two numbers [pointing to the tick marks on each axis] and then [*sliding the pen vertically from the tick mark on the horizontal axis to his*

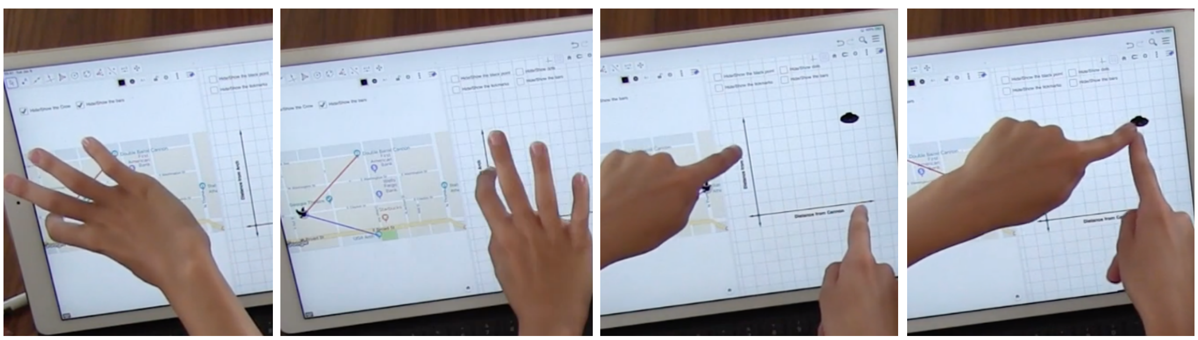
blue dot on the plane and then sliding the pen horizontally from the tick mark on the vertical axis to his blue dot on the plane] like where they meet [*simultaneously sliding the pen vertically from the tick mark on the horizontal axis to his blue dot on the plane and sliding his index finger horizontally from the tick mark on the vertical axis to his blue dot on the plane and meeting the pen and the index finger on the blue dot at the same time, see Figure 0.24b*] is like the correct answer.



(a) (b)

Figure 0.24. (a) Melvin's point in CT and (b) Melvin's physical action of joining the projection of the two magnitudes on the plane.

Melvin's activity suggested that he was able to locate a point on the plane by using the information that was given on each axis of the plane. He imagined intersecting the projections of the quantities' magnitudes that were represented on each axis of the plane. To repeat the same task with less figurative material available for Melvin to use, I changed the crow's location and hid the red and blue bars on each axis of the plane. I wanted to see if Melvin could dis-embed the crow's DfA and DfC from the situation and represent them on each axis of the plane, then, plot a point accordingly. Melvin was able to plot a point on the plane by using the same strategy. He first measured the crow's DfA and DfC on the map by using his fingers (see Figure 0.25a) and re-presented the magnitudes on the axes (see Figure 0.25b). He then located a tick mark on each axis based on the crow's DfA and DfC on the map (see his fingers pointing to his tick marks on the axes in Figure 0.25c) and joined the projections *simultaneously* on the plane (see Figure 0.25d). Note that his meaning of tick marks consistently included "where the end of the bar is," which is the representation of quantities magnitudes. Thus, I interpreted his action of joining fingers as having him intersecting the projection of the bars on the plane. Moreover, when asked to explain what the point he plotted represented, Melvin stated that "the crow's distance from the Cannon and Arch." Based on his actions and language, I infer that Melvin conceived the point on the plane as a representation of both quantities' magnitudes, which is an indication of representing QMO.



(a) (b) (c) (d)

Figure 0.25. Melvin's construction of a point on the plane in CT.

Representing a NMO when interpreting a point in CT. Melvin was able to locate a point on the plane that represented two quantities' magnitudes for given locations of the crow on the map. Although he engaged in representing a QMO on the plane when plotting a point, at some point during the session, Melvin stated that the point on the plane represented "where the crow is." From this description of the point, I was not sure if Melvin meant where the physical crow was on the plane (i.e., SQMO) or if he meant the point recorded two information (i.e., the crow's DfA and DfC indicated by the blue and red bars on the axes) so he could know "where the crow is" on the map with these two information (i.e., QMO). Thus, I decided to investigate more about his meaning of the point to see if his meaning included SQMO or QMO even though his action of locating the point on the plane (e.g., his multiplication table analogy) was suggestive of representing a QMO.

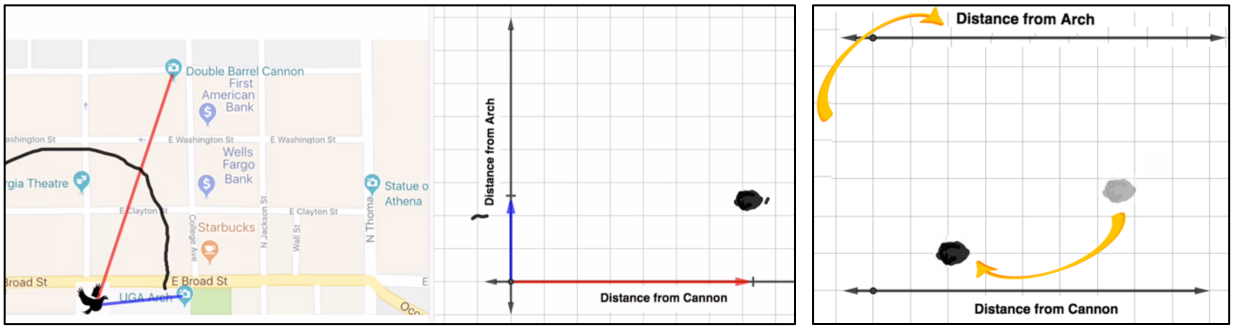
I hid the crow and bars on the map and asked Melvin where the crow would be on the map based on the given point on the plane (see the dot in Figure 0.26a on the plane; the map was empty at the moment). Melvin drew a curve to indicating a potential the area surrounded by the black curve on the map (see Figure 0.26a) and imagined the crow would be withing that area. To explain why, he said,

> This is the *y*-axis [*pointing to the vertical axis of the plane*], right. So, I turned it [*performing a rotating gesture clockwise starting from where the vertical axis was as depicted in Figure 0.26b*] and just moved this [*pointing to the black dot on the plane, see Figure 0.26a*] in mind with it. And it [the black dot on the plane] went like over here [*performing a rotating gesture clockwise starting from the black dot on the plane as depicted in Figure 0.26b*].

Melvin imagined rotating the vertical axis of the plane and placing it horizontally on top of the plane, so the point on the plane moved accordingly (see my model of his activity in Figure 0.26b). This provided an evidence that Melvin engaged in transformed iconic translation (i.e., NMO) as he translated the transformed version of the perceptual features of the graph to the situation. After he rotated the black dot on the plane (along with the rotation of the vertical axis in Figure 0.26b), the new location of the black dot spatially matched with the location of the crow on the map (i.e., they were both close to the bottom-left corner of the map/plane).

Then, I showed the computer-generated crow on the map with the red and blue bars on the map (Figure 0.26a). My goal was to potentially draw Melvin's attention to the relationship between the red and blue bar both on the map and on the plane. I thought he could associate the length of the blue and red bars on both map and on the axes of the plane (Figure 0.26a).

However, Melvin thought this was just a feedback for him as he observed that the crow appeared within his boundary in Figure 0.26a on the map.



(a) (b)

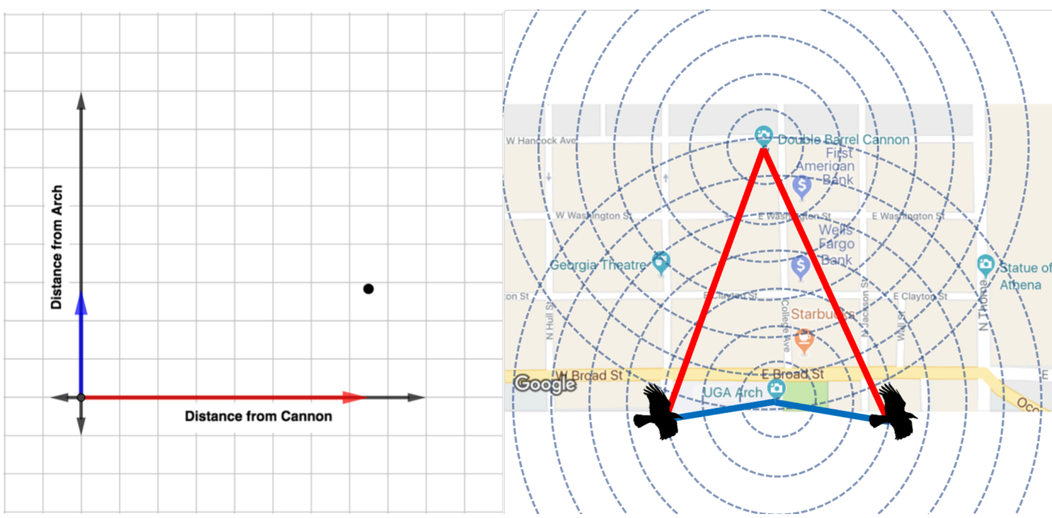
Figure 0.26. (a) Melvin locating the crow on the map and (b) My model of Melvin's transformed iconic translation.

Then, I explicitly asked Melvin if he could locate the crow on the map by using the bars on the axes of the plane. Melvin expressed that he cannot find exactly where the crow was on the map because the crow could be somewhere else on the map too. He said,

> I mean, you couldn't put, like, to exactly where it [the crow] could be because it [*pointing to the crow on the map in Figure 0.26a*] could be on the other side of the Arch [*pointing to the right side of the Arch on the map, see Figure 0.27b*] and have the same distances. So, you wouldn't be able to tell exactly where it was just from having the two bars [*referring to the red and blue bars on the axes of the plane, see Figure 0.26a*].

Melvin pointed out that there could be two different locations for the crow to be on the map if he would use the magnitudes of the crow's DfA and DfC that were given on the axes. This provided an evidence that Melvin was also able to use his quantitative meanings of the point in order to locate the crow on the map. Thus, Melvin held two different meanings for the point when locating the crow on the map given a point on the plane. I conjecture that Melvin engaged in transformed iconic translation earlier because he could not reveal enough information provided by the bars on the axes when determining a unique location for the crow on the map. So, in order to avoid this uncertainty, he chose to draw on his transformed iconic translation meaning (i.e., NMO) instead of his quantitative meaning (i.e., QMO).

I acknowledge that the task of locating the crow on the map given two quantities' magnitudes on the plane was a hard one because it requires someone to coordinate the radial quantities' magnitudes on a two-center (i.e., Arch and Cannon on the map, see Figure 0.27b) bi-polar coordinate system. Maybe because of the high cognitive demand of the task, Melvin chose to use his earlier meanings that included transformed iconic translation. I asked him which one was easier for him (i.e., the location of the black dot on the plane vs. the length of the bars on the axes) to use when locating the crow on the map. He stated that using the point on the plane would be easier to know where the crow was on the map compared to using the bars on the axes, although he could not explain why at that moment.



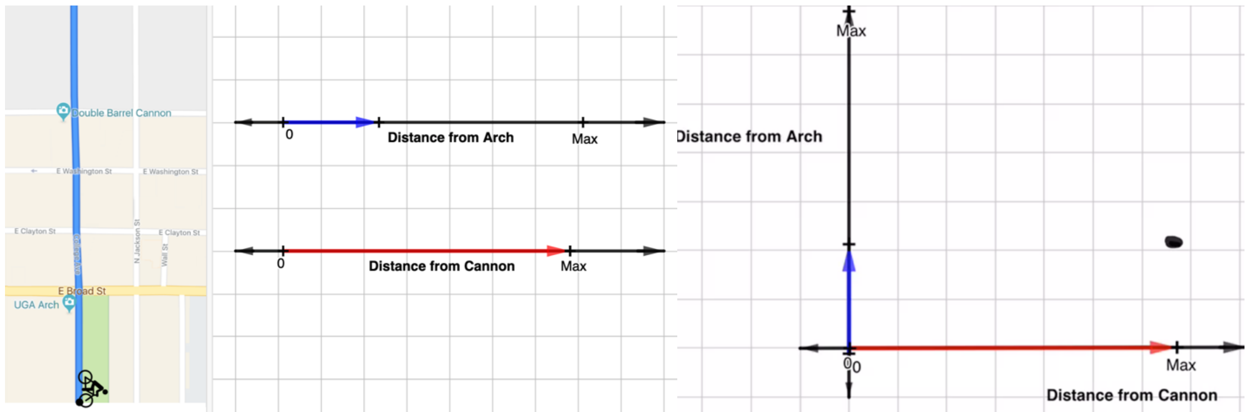
(a) (b)

Figure 0.27. (a) Cartesian plane with a point on the plane and the magnitudes on the axes and (b) A two-center bipolar coordinate system with two crows and the radial magnitudes on the space.

Melvin's Activity in Matching Game Task (Second Attempt)

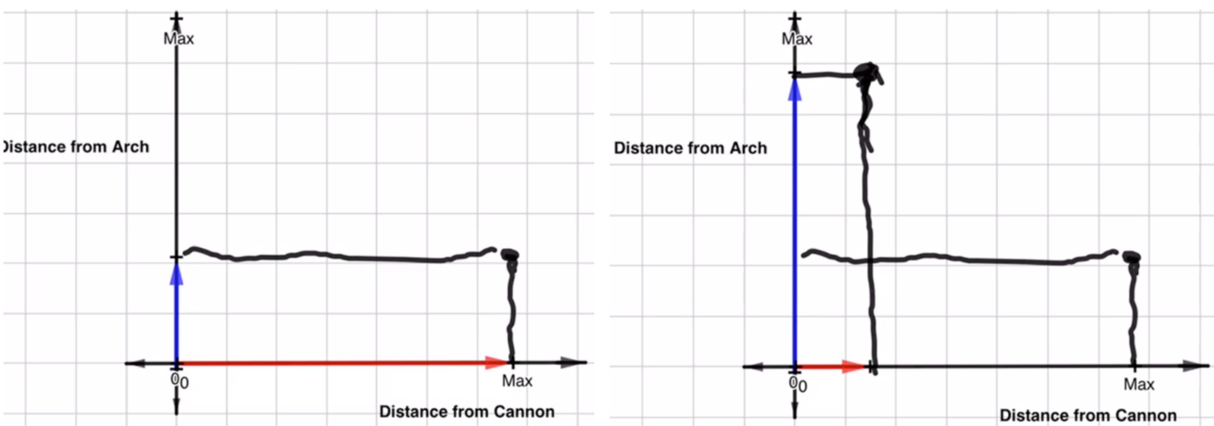
In CT, I identified that Melvin's meaning of the point included representing a QMO when constructing a point. That is, he was able to generate a single point on the plane to represent the crow's DfA and DfC given the crow on the map. I also identified that he could use his meaning of the point representing a QMO when determining the crow's location on the map given a point on the plane, although he abandoned this meaning because he could not know exactly where the crow was on the map by using the quantitative meanings.

Based on this inference, I wanted to see how Melvin could engage in the previous task (i.e., MGT) where I asked him to create a single point to represent the bike's DfA and DfC simultaneously given two parallel magnitude lines. I hypothesized that Melvin could now create a single point to show the bike's DfA and DfC because I thought he could relate to his activity of creating a single point in CT where a coordinate plane was provided as a given structure. When asked to find a way to create a single point to represent both the bike's DfA and DfC when the magnitude lines are parallel (see Figure 0.28a), Melvin said "maybe, if we make this [*pointing to the parallel magnitude lines*] a *x* and *y*-axis thing." Then, Melvin rotated one of the magnitude lines and placed it perpendicular to the other magnitude line by matching the zero points in both lines (see Figure 0.28b). He then plotted a point where the projection of the red and blue bars intersected on the plane. I asked him how he knew this point showed both the bike's DfA and DfC, he drew the vertical and horizontal segments (see Figure 0.29a) and said "it [*referring to the vertical segment*] is the bike's distance from the Arch and it [*referring to the horizontal segment*] is the bike's distance from the Arch." Melvin also successfully created a single point to represent the bike's DfA and DfC for another location of the bike (see Figure 0.29b).



(a) (b)

Figure 0.28. (a) MGT when the magnitude lines are parallel and (b) Melvin's orthogonal magnitude lines and his single point representing both quantities.



(a) (b)

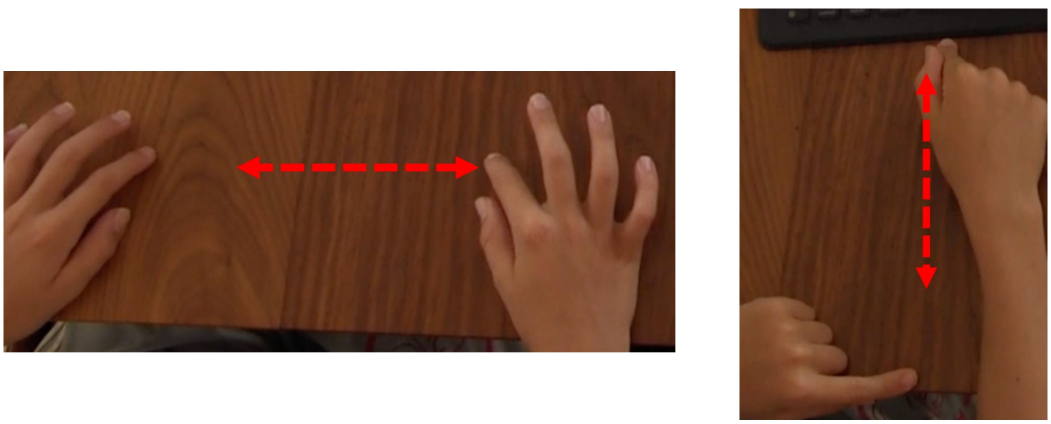
Figure 0.29. (a) Melvin's segments on the plane to show the magnitudes, and (b) Melvin's another point for another location of the bike.

Melvin's Final Draft in DABT

From his earlier activities throughout the teaching experiment, I knew that Melvin could conceptualize quantities' magnitudes in the situation, represent those quantities' magnitudes by the directed bars on magnitude lines, and represent the two static quantities' magnitudes by a single point in a Cartesian coordinate plane. In DABT, I provided Melvin an opportunity to represent the relationship between two quantities as they vary in tandem.

Melvin's finger activity in DABT

Before I asked Melvin to graph in DABT, I repeated the finger activity where I asked Melvin to represent the variation of bike's DfA by moving his index fingers. My goal was to get more insights into Melvin's conception of the variation in bike's DfA and how he could represent this variation with fingers. I directed him to place his index fingers on the table in a way that his left index finger would be fixed on the table and his right index finger could only move in a horizontal direction. Then, I told him to move his right index finger left to right so that the distance between his index fingers could represent the bike's DfA as the bike traveled along the road. He successfully moved his finger according to the bike's DfA (see Figure 0.30a).

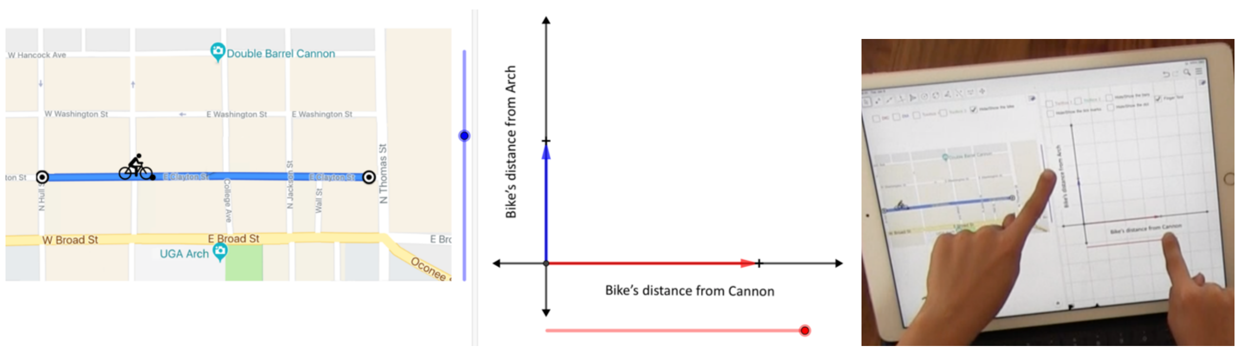


(a) (b)

Figure 0.30. (a) Moving right index finger horizontally and (b) Moving right index finger vertically (red arrows are added for the reader to indicate the path he moved his finger on).

Recall that previously he connected (i.e., physical touch) his index fingers to represent a moment when "the bike was as closest as it could get to the Arch" as an implication of his spatial proximity reasoning (see Figure 0.25b in the section of "Melvin's representation of spatial proximity reasoning in DABT"). Note that he did not physically connected his fingers this time (see the red arrow in Figure 0.30a as an indication of the finger's path on the table), which might suggest that he was coordinating the quantities' magnitudes when moving his finger, instead of proximities. In addition to his success with moving fingers in the horizontal direction, Melvin was also able to move his right index fingers in the vertical direction (see Figure 0.30b).

I also engaged Melvin in a dynamic tool where I asked him to move the sliders with his fingers to change the length of the bars on the axes of the plane in synced with the bike's DfA and DfC as I play the animation (see Figure 0.31a). One of my goals with these finger activities was to get more insights into Melvin's conception of the variation in bike's DfA and DfC and how he could represent this variation simultaneously with two fingers. I designed this finger tool activity for Melvin to engage before his graphing activity because I planned to see whether Melvin could relate his subsequent graphing activity on the plane to his finger tool activity. That way, I would be able to get more insights into his thinking and meanings when constructing or interpreting his graph on the plane.



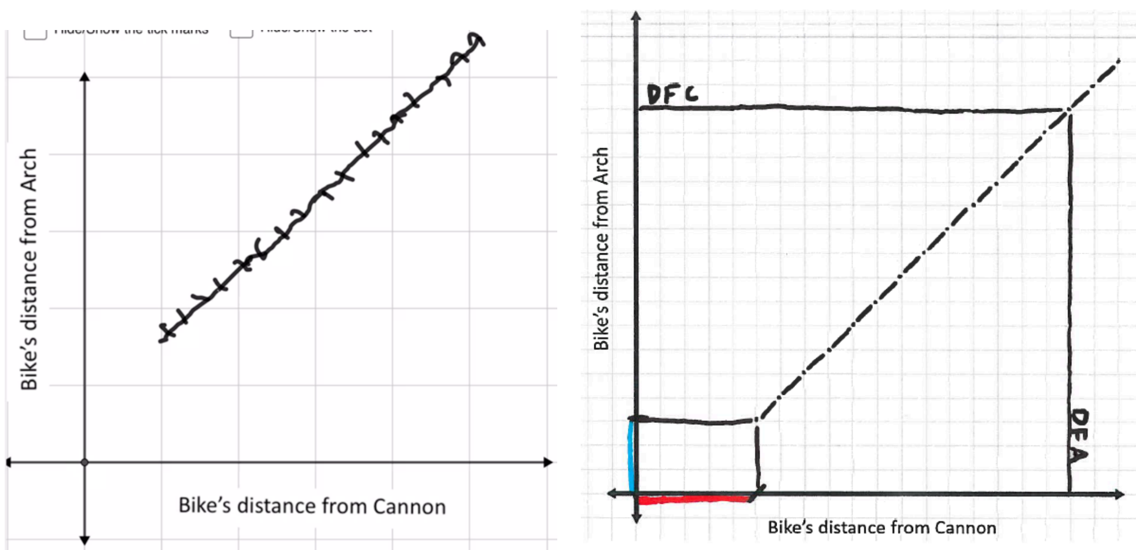
(a) (b)

Figure 0.31. (a) Finger tool activity in DABT and (b) Melvin's finger activity in DABT

In all of these finger activities, Melvin successfully moved his fingers in order to represent the (directional) covariational relationship between the bike's DfA and DfC. When simultaneously increasing and decreasing the length of the bars on the axes, Melvin also made sure to keep the length of the red bar longer than the length of the blue bar because he determined that the bike's DfC is longer than the bike's DfA as the bike traveled. This provided an evidence that Melvin was able to represent the invariant covariational relationship between two quantities on the axes of the plane as they varied in the situation.

Melvin's graphing activity in DABT

Next, I asked Melvin if he could create a representation on the plane to show how the bike's DfA and DfC changed for the whole trip, not for a single moment of the animation. Melvin began his graphing activity by drawing a straight line upward from left to right on the tablet screen (Figure 0.32a). He then added tick marks on his graph after he saw tick marks on Naya's (the other student in the session) graph. I didn't have any evidence that the tick marks had a different meaning than his line that he drew on the plane. Melvin also drew his graph on paper as a dotted and dashed straight line upward from left to right (see Figure 0.32b, the additional horizontal and vertical segments were not drawn yet).



(a) (b)

Figure 0.32. (a) Melvin's final draft in DABT on the tablet screen and (b) re-drawing his final draft on paper.

I first asked Melvin to talk about if he could relate his finger tool activity to his graph. My goal was to get insights into his meaning of the graph. In particular, I wanted to see if his image of the graph included the bars covarying on the axes of the plane. Melvin said "yes" and moved his right index fingers horizontally back and forth in the air and moved his left index finger up and down vertically in the air (see the red arrows an indication of the path for his fingers in Figure 0.33). His movement of the fingers suggested that he might be imagining the varying bars in relation to the representation that he drew on the plane.

Figure 0.33. Melvin's finger movement in the air to show how he imagined the bike's DfA and DfC varied in DABT.

To test this hypothesis, I showed the sliders and the bars on the axes in the tablet screen (see Figure 0.34a) and asked Melvin to elaborate on his finger movements. The following excerpt illustrates his activity.

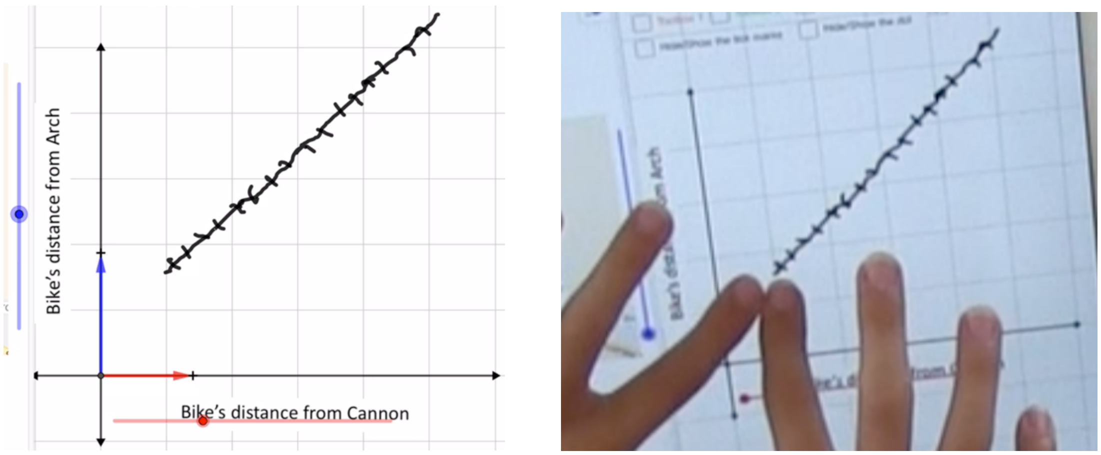
Melvin: When we moved them [*pointing to the bars on the axes in Figure 0.34a*], we moved them like [*making the same gestures in Figure 0.33 over the vertical and horizontal axis of the plane*], the two lines [*referring to the bars on the axes, see Figure 0.34a*] go like [*joining his fingers on the graph, see Figure 0.34b*] and it crosses these little tick marks [*pointing to the tick marks on his graph by sliding his finger on his graph from left to right, Figure 0.34a*], I guess.

TR: Can you do a quick illustration?

Melvin: Like, right there and there [*moving the sliders and adjusting the length of the red and blue bar on the axes as seen in Figure 0.34a*]. They are on there [*pointing to the bars on the axes*], when we move them [*moving the sliders to increase the length of the bars on the axes*], we can plot each place when we move them on a tick mark, and then they come back [*moving the sliders to decrease the length of the bars on the axes*] and then keep going on there.

TR: When the bars are there [*pointing to the bars on the axes, see Figure 0.34a*], how you are connecting the bars to your graph?

Melvin: Maybe the second tick mark [*joining the index fingers from the end of the bars on the axes to the second tick mark on his graph, similar to the gesture in Figure 0.34b*]



(a) (b)

Figure 0.34. (a) Melvin's graph on the tablet screen and the bars on the axes and (b) Melvin's index fingers joining the projection of the bars on his graph.

From his activity, I infer that Melvin imagined the projection of red and blue bars meeting on the plane along with the variation of these bars creating the graph. In order to get more insights into his meaning of the graph on paper when the additional figurative material (i.e., sliders and the bars on the axes, see Figure 0.34a) was absent on the plane, I asked Melvin to show how his graph on paper showed the bike's DfA and DfC for certain states of the bike in the situation. For example, I asked Melvin how his graph showed the moment where the bike was located at the very right side and at the middle of the path on the map. Melvin drew horizontal and vertical segments on the plane representing the bike's DfA and DfC to show these moments (see Figure 0.32b). The horizontal segment represented the bike's DfC because it was the projection of the red bar on the horizontal axis (see Figure 0.32b). Similarly, the vertical segment on the plane represented the bike's DfA because it was the projection of the blue bar on the vertical axis (see Figure 0.32b). I infer that the way he represented quantities' magnitudes was now compatible with representing QMO in a canonical Cartesian plane.

Summary of the Case of Melvin

Representing a NMO (iconic translation) in DAT

Melvin's initial meaning of the points included iconic translation as he assimilated each point on the plane as a location/object that appeared on the map. That is, Melvin made an association between the perceptual or spatial features of the map and plotted points (Figure 0.2b). In particular, he assimilated the point as a location/object by choosing one perceptual property of the map (e.g., spatial orientation, relative location) and ensuring to preserve that property on the plane.

Representing a NMO (transformed iconic translation) in CT

I transitioned to CT where Melvin was perturbed by the movement of the black dot on the plane because the crow's movement and the black dot's movement did not perceptually match. For instance, the black dot moved diagonally on the plane as the crow flied vertically on the map (see the highlights in Figure 0.29.). Consequently, he shifted his meanings of points from iconic translation to transformed iconic translation. That is, he still conceived the points as objects/locations by using iconic translation. However, he did not translate the perceptual features of the situation *as it is* to the plane (or the opposite direction, i.e., from graph to situation); instead, Melvin translated a *transformed* version of the perceptual features of the situation to the plane (or the opposite direction).

Spatial proximity reasoning in DABT

I engaged Melvin in DABT (see Figure 0.7) for the purpose of exploring how he could conceive the situation quantitatively and how he could determine the relationship between quantities (i.e., the bike's DfA and DfC). In particular, my purpose in this task was to explore and support Melvin's process of conceiving varying quantities in the situation and representing them on a coordinate plane. I found that Melvin conceived the bike's DfA as the bike's *proximity* to Arch (i.e., its closeness/nearness to the Arch) without conceiving the measurable attribute of the bike and coordinated the variation of the bike's degree of proximity to Arch as the bike traveled on the map (e.g., the bike is getting closer to or farther from the Arch). Moreover, he represented the bike's proximity to Arch in his graphing activity by imagining the physical objects (e.g., the bike, Arch, and Cannon) that were getting closer to or farther from each other on the representational system (e.g., a coordinate system or a magnitude line). For example, his representation of the relationship between the bike's DfA and DfC on the plane included imagining the bike moving on the plane with the entire axes of the plane being the physical Arch and Cannon and coordinating the bike's proximity to both Arch and Cannon on the plane as the bike traveled (see Figure 0.28).

Measurement activity in DABT

Since Melvin's reasoning were based in coordinating spatial proximities and his meanings of the bars included the physical bike and Arch, I presented Melvin with a dynamic tool that could afford Melvin's engagement with quantities' magnitudes represented by directed bars placed on magnitude lines (Figure 0.12). By using unit magnitudes, Melvin measured the length of the bar that represented the bike's DfA in the situation. Then, by using the same units, he represented the magnitude of the bike's DfA as a length of a directed bar on the magnitude line. This activity helped Melvin to conceive the bike's measurable attribute (i.e., DfA, rather than the bike's spatial proximity) in the situation and be able to re-present it by a directed bar on the magnitude line.

Representing SQMO in DABT

After the measurement activity, I asked Melvin to sketch his second draft to show the relationship between the bike's DfA and DfC. I found that his meaning of the points included representing spatial-quantitative multiplicative objects (SQMO). That is, Melvin's meaning of the graph still included imagining the physical Arch, Cannon, and the bike; however, he coordinated the variation of the quantities' magnitudes instead of proximity between objects (see Figure 0.10b). Melvin's meaning of the points included determining quantitative features of the bike in the situation (i.e., its DfA and its DfC) and ensuring to preserve these quantitative properties on the plane. Melvin drew segments as indication of quantities' magnitudes in the situation and be able to transform (i.e., dis-embedding and re-presenting) those magnitudes from situation to the graph, although he represented the magnitudes on the plane, not on the axis of the plane. In the next activity, I provided an opportunity for Melvin to develop a meaning of a point in terms of representing a quantitative multiplicative object (QMO).

Melvin's activity in MGT

Since Melvin represented the magnitudes on the plane by using the entire axis as a reference object, I engaged Melvin in Matching Game Task (MGT; Figure 0.16) where I provided him an opportunity to make connection between the parallel magnitude lines and the coordinate system for the purpose of constructing a Cartesian plane. I designed MGT in order to engage Melvin to (i) represent two quantities' magnitudes on two parallel magnitude lines and then transition to (ii) representing two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and projecting the magnitudes on the plane. Melvin successfully represented the two quantities magnitudes on two parallel magnitude lines by placing a tick mark on each magnitude line. However, he could not create *a single point* on the plane that represented the *two* magnitudes by re-organizing space. Then, I decided moved to CT providing him a coordinate system as a given space. My goal was to provide Melvin with additional figurative material that might afford him to structure the space in a way that is compatible with Cartesian plane.

Representing QMO in CT

As compatible with his earlier meaning of the points as representing SQMO in DABT, Melvin conceived the point on the plane as the crow moving on the plane according to its quantitative properties. Since he imagined the Arch and Cannon on the axis of the plane, he coordinated the black dot's (the crow for him) DfA and DfC on the plane according to the crow's DfA and DfC on the map. Then, I provided Melvin additional figurative material (i.e., tick marks and the varying bars) on each axis of the plane that afforded Melvin to assimilate the axes of the plane as magnitude lines. Consequently, Melvin developed a new meaning of a point on the plane as representing a QMO. That is, Melvin imagined the quantities' magnitudes on each axis of the plane and created a single point on the plane that simultaneously represented both quantities' magnitudes.

Representing QMO in DABT

Since Melvin was able to use the new space (i.e., Cartesian coordinate plane) in order to produce a single point that simultaneously represented two static quantities' magnitudes, I asked him to create a representation on the plane to show how the bike's DfA and DfC changed for the whole trip, not for a single moment of the animation. Melvin drew a straight line upward from left to right on the plane (see Figure 0.32b) that represented how two quantities' magnitudes varied on each axis of the plane. This provided an evidence that Melvin engaged in emergent shape thinking.

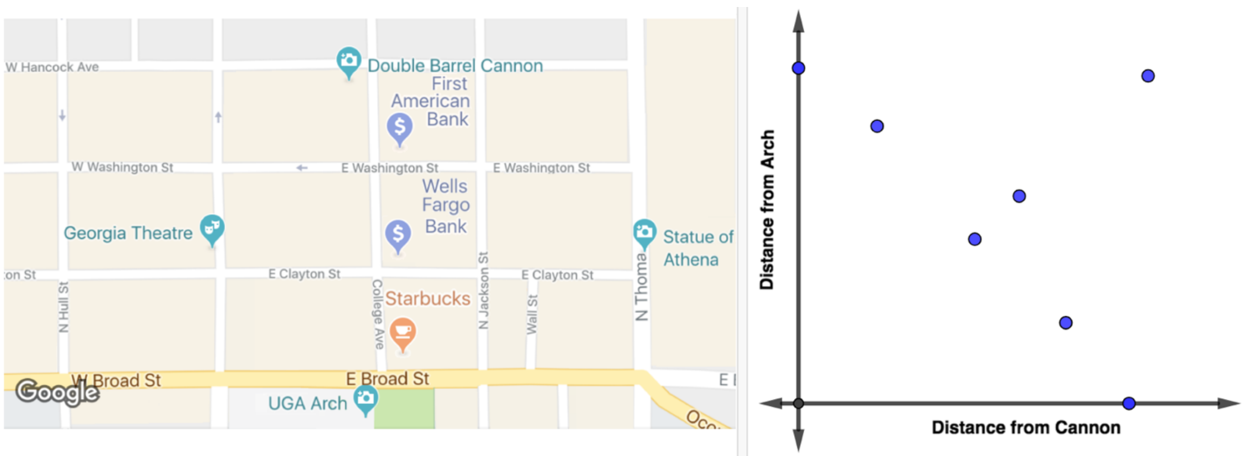
The Case of Naya

In this section, I present Naya's meanings of points that she developed throughout the teaching experiment. Table 5.2 summarizes all the meanings that she developed in Downtown Athens Task (DAT), Crow Task (CT), Downtown Athens Bike Task (DABT), and Swimming Pool Task (SPT) and summarizes the way she organized the space in solving these tasks. Naya initially assimilated the points on the plane in relation to the physical objects that appear in the situation, and her meanings for points were based in iconic or transformed iconic translation (i.e., picture of the situation). As she began to conceptualize the quantities in the situation, Naya's graphical meanings included representing two quantities in non-canonical Cartesian plane. Subsequently, her attention to quantities in the situation, mapping those quantities' magnitudes onto the magnitude lines, and assimilating the axes of the plane as magnitude lines afforded Naya to develop a meaning of points in terms of representing two quantities' magnitudes (i.e., QMO) by reorganizing the space consistent with a Cartesian plane.

Table 5.2
Naya's meanings of the points and his organization of the space throughout the teaching experiment.

Tasks	Meanings of the points	Organization of the Space
Downtown Athens Task	Non-Multiplicative Object (iconic translation and transformed iconic translation)	Imagining Arch and Cannon on the axis
Crow Task	Non-Multiplicative Object (transformed iconic translation)	
Downtown Athens Bike Task	Quantitative Multiplicative Object	A non-canonical Cartesian plane
	Matching Game Task	

Crow Task	Non-Multiplicative Object (transformed iconic translation) Spatial-Quantitative Multiplicative Object	Imagining Arch and Cannon on the axis
	Intervention	
Crow Task	Quantitative Multiplicative Object	Cartesian coordinate plane
Downtown Athens Task		
Swimming Pool Task		A non-canonical Cartesian plane Cartesian coordinate plane



(a) (b)

Figure 0.35. (a) Downtown Athens, and (b) Cartesian coordinate plane with points

Naya's Activity in Downtown Athens Task

As an illustration of Naya's initial meaning of points, I present her activity in Downtown Athens Task (DAT). DAT includes a map with seven locations highlighted and labeled (See Figure 0.35a) and a Cartesian plane whose horizontal axis is labeled as Distance from Cannon (DfC) and vertical axis is labeled as Distance from Arch (DfA). Seven points are plotted without labelling in the coordinate system to represent the seven locations' DfA and DfC (see Figure 0.35b). I asked Naya what each of these points on the coordinate system might represent with an intention to observe her spontaneous responses and to explore her meaning of points. Naya's initial meaning of the points included iconic translation as she assimilated each point on the plane as a location/object that appeared on the map. I also illustrate Naya later shifted her meaning from iconic translation to transformed iconic translation.

Naya's iconic translation in DAT

When determining which point on the plane is associated with which place on the map, Naya made an association between the perceptual features of the map and the perceptual features of plotted points (Figure 0.35b). In particular, she assimilated the point as a location/object by choosing one perceptual property of the map (e.g., spatial orientation, relative location) and ensuring to preserve that property on the plane. For example, Naya determined that the point on the horizontal axis is the physical Arch because "I see the Arch at the bottom of the map and that [*pointing to the point on the horizonal axis labeled "Arch," see Figure 0.3*] is right there [*pointing to the point on the horizonal axis*] where I see the Arch." Naya's attention to the association between the spatial features of the situation and points on the plane (i.e., Arch being at the bottom on the map and on the plane) provided evidence that Naya's meaning included iconic translation.

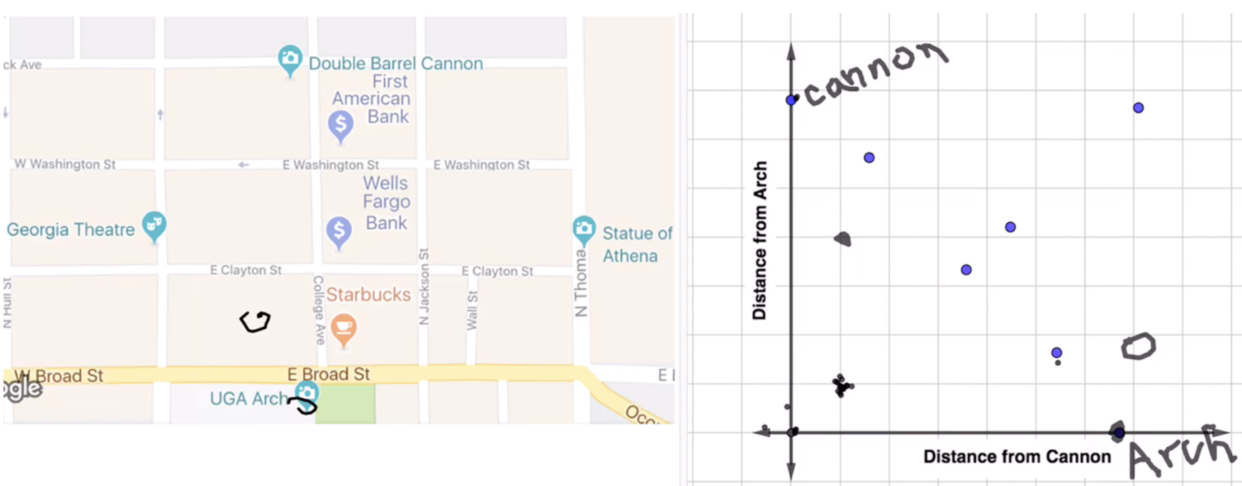


Figure 0.36. Naya's (and Melvin's) activity in DAT (a screenshot from the tablet).

Naya's transformed iconic translation in DAT

Note that Naya and Melvin were paired in the teaching experiment. I asked them what they thought the point that is closest to the horizontal axis might represent. Both Naya and Melvin thought the point represented Starbucks. This was not surprising because Starbucks is the closest location to Arch on the map. However, Melvin appeared confused and unsure of his response because he noted an inconsistency with the spatial attributes of Starbucks within two spaces. Specifically, Melvin conceived that Starbucks was located up and to the right of the Arch on the map whereas Starbucks was located up and to the left of the Arch on the plane. Melvin noted there was no point in the plane to match the location of Starbucks relative to Arch on the map. I infer that his iconic translation meaning yielded a perturbation for Melvin because the location of Starbucks didn't make sense to him in relation to the location of the Arch when comparing across the map and the plane. Since Naya's meaning of the points also included iconic translation, she was also perturbed by the inconsistency between the spatial features of Starbucks on the map and on the plane that Melvin pointed out.

In order to reconcile her perturbation regarding the location of Starbucks on the plane, Ella suggested to make an adjustment on the plane. She said,

> Maybe that [*pointing to the point on the horizontal axis that they labeled Arch in Figure 0.3*] is the Cannon not the Arch. … and that [p*ointing to the point on the vertical axis labeled as Cannon in Figure 0.3*] would be the Arch [*see the new locations of Arch and Cannon in Figure 0.3b*).

Naya conceived the point on the vertical axis as Arch and the point on the horizonal axis as Cannon. This change not only included simply switching the two places (i.e., Arch and Cannon) on the plane, Naya also imagined flipping the map vertically so that Arch was now located at the top and the Cannon was at the bottom on the map. She said,

> Say you take this [*pointing to the map, see Figure 0.37, left*] and flip it around [*performing a flipping gesture with her fingers, see* Figure 0.37, left]. It goes Arch [*pointing to Cannon on the map*] then Starbucks [*pointing to FAB on the map*], same as the map [*pointing to the coordinate plane*].

Naya made perceptual association between the flipped version of the map and the plane (i.e., transformed iconic translation). She imagined Arch on the vertical axis and Cannon on the horizontal axis because Arch was located at the top and Cannon was located at the bottom on the mentally flipped map. With the new image of the plane and map, Naya reconciled her perturbation regarding the location of Starbucks because Starbucks was now located to the right and down relative to the Arch both on the plane and on the flipped map.

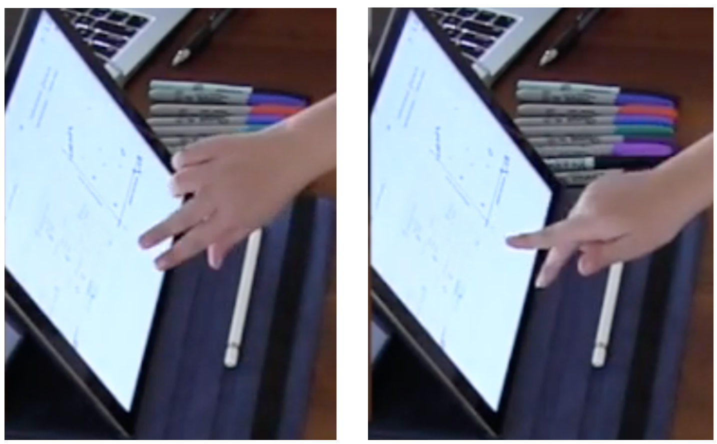


Figure 0.37. Naya's flipping gesture in DAT.

I found another piece of evidence for Naya's transformed iconic translation. When explaining her thinking about the points on the coordinate plane, Naya said,

> The top [*pointing to the top of the coordinate plane*] kind of becomes the *y*-axis [*pointing to the vertical axis*]. So, that [*pointing to the point on the vertical axis*] would be the Arch, that [*pointing to the point on the horizontal axis*] would be the Cannon, and these dots [*referring to the other points on the plane*] would line up.

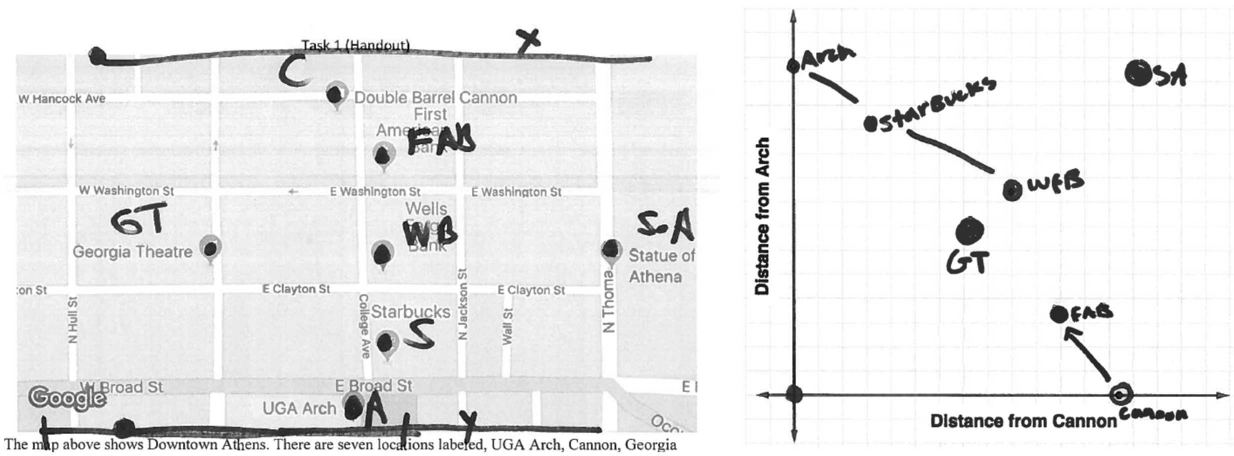
Although I infered she imagined Arch and Cannon on the vertical and horizontal axis, respectively, I could not completely understand what she meant by "the top kind of becomes the *y*-axis." In order to get more insights into her meanings, I provided the map and plane on a paper and ask Naya to elaborate what she just told me. She further explained:

> Once we flipped it [*referring to the map*] around, all that stuff [*pointing to the top part of the map*] is down here [*pointing to the bottom part of the map*], so this [*pointing to the horizontal axis of the coordinate plane*] is the bottom, and I considered the top kind of over here [*pointing to the vertical axis*] because we don't have a top [*pointing to the top part of the plane*].

After flipping the map vertically, Naya made a perceptual association between the vertical/horizontal axis and the bottom/top of the map. In order to illustrate what she meant, she drew two parallel lines, one is at the bottom and one is at the top of the map (see Figure 0.3a). The following excerpt shows how she imagined the axes on the map.

> Naya: [*pointing to the points on the plane*] [inaudible] in the middle. I think this [d*rawing a horizontal line at the bottom of the map, see Figure 0.3a*] is more like the *y*-axis, and this [*drawing another horizontal line at the top of the map, see Figure 0.3a*] is like the *x*-axis. So, you put them together. Then, you have all this stuff in the middle [*pointing to the overall locations on the map*].
>
> Melvin: So, where is the origin?
>
> Naya: When you put these two [*pointing to the left end side of each horizonal line*] together, right there [*circling a dot on each line, see Figure 0.3a*]
>
> Melvin: There is two origins?
>
> Naya: No, these two go together [*pointing to the dot on each line*].
>
> Melvin: [inaudible]
>
> Naya: No, we are imagining these two lines together. They are here [*pointing to the map*] and you put them together for the graph [*pointing to the coordinate plane*].

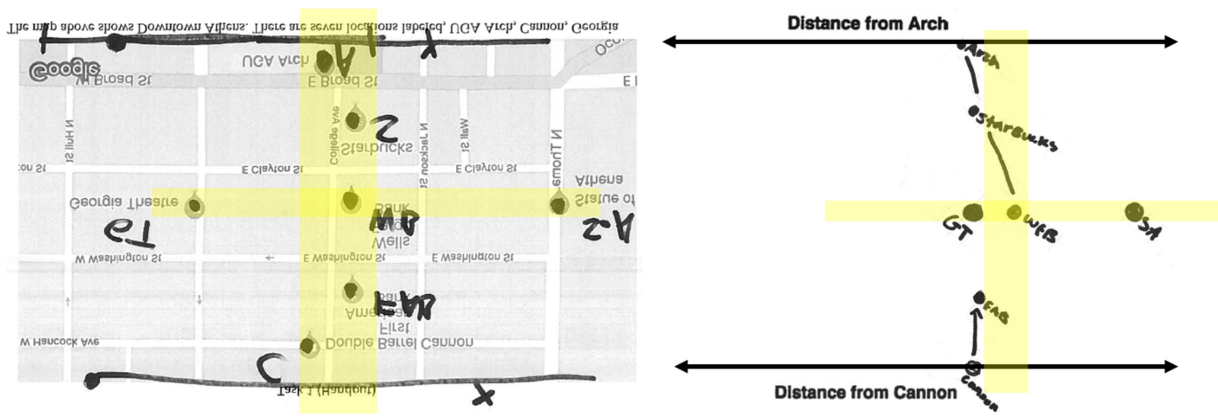
Naya imagined the horizontal axis of the plane (what she called the *x*-axis, see Figure 0.3a) at the top of the map and imagined the vertical axis of the plane (what she called the *y*-axis) at the bottom of the map. She imagined these parallel lines on the map "put them together" when relating to the axes on the coordinate plane. Thus, I infer her image of the axes on the map included a transformation in which she imagined the parallel lines intersecting when relating to the plane. With this image of the map, Naya conceived the locations on the map (e.g., Georgia Theater, Starbucks, etc.) as a whole as the things “in the middle” and associated those locations to the points that are “in the middle” on the plane. Naya conceived the axes that she drew on the map and the axis of the plane as boundary lines where the objects locate in the middle.



(b) (b)

Figure 0.38. (a) Naya’s axes overlaid onto the map in DAT and (b) Naya’s plane in DAT.

Moreover, she associated how the points on the plane and the locations on the map “lined up.” For example, she determined Arch, Starbucks, Wells Fargo Bank form a straight line on the map and the points labeled Arch, Starbucks, and Wells Fargo Bank also forms a straight line on the plane (see the line segments that connects these points on the plane in Figure 0.3b). This suggested that Naya’s meaning of the points included transformed iconic translation since she used the perceptual feature of the map and the plane (e.g., “they lined up”) in making sense of the points on the plane. In Figure 0.39b, I attempted to draw my model of Naya’s meanings regarding the points on the plane. Once Naya flipped the map (Figure 0.39a) and imagined the vertical and horizontal axis on top and bottom of the map, respectively, as you may see in Figure 0.39, the points on the plane and the locations on the map “lined up.” For example, you can see that Georgia Theater (GT), Wells Fargo Bank (WFB), and Statue of Athens (SA) are co-linear in the same order in both spaces.



(a) (b)

Figure 0.39. (a) Flipped version of the map in Figure 0.3a and (b) my model of how Naya imagined the map and points on the plane “lined up”.

Naya’s Initial Activity in Crow Task

Crow Task (CT; as an extension of DAT) included a crow that flied on the map and a corresponding black point that moved on the plane according to the crow’s distance from Arch (DfA) and the crow’s distance from Cannon (DfC; Figure 0.40). I asked Naya to move the crow on the map and explore how the corresponding black point moved on the plane. Naya continued to envision the points on the plane as a location by engaging in transformation iconic translation.

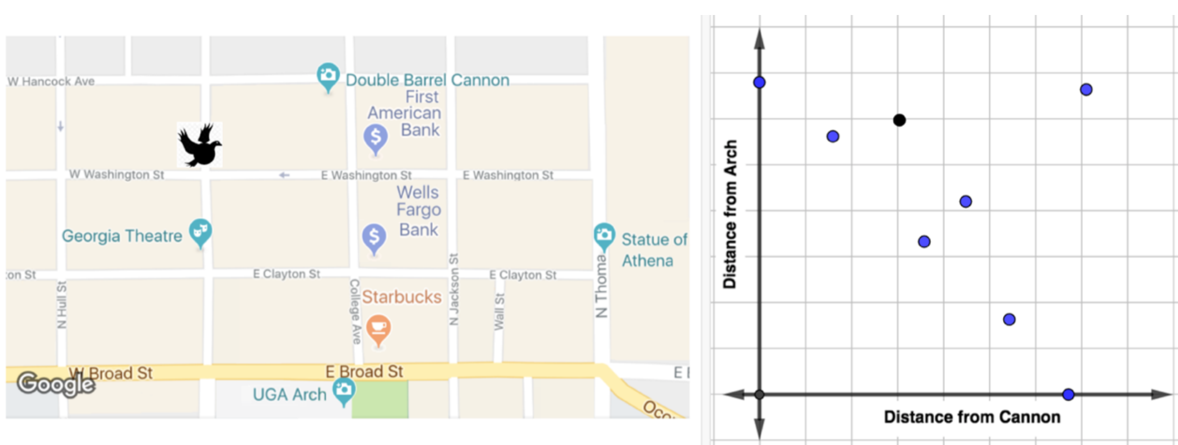


Figure 0.40. Crow Task

Naya began her activity in CT by moving the crow around Arch and Cannon on the map and observing where the black dot moved accordingly. Based on her initial observation, she immediately said "we were wrong." When the crow was on Arch on the map, she observed that the black dot was on the point that she conceived earlier as Cannon on the horizontal axis. Similarly, when the crow was on Cannon on the map, she figured that the black dot was on the point that she conceived earlier as Arch on the vertical axis. In order to reconcile her perturbation, Naya switched the location of Arch and Cannon on the axis of the plane. When I told her "now, you are thinking that Arch is on the horizontal axis and Cannon is on the vertical axis," she immediately responded "well, we *know* [emphasis added] that this [*pointing to the point on the vertical axis*] is the Cannon and this [*pointing to the point on the horizontal axis*] is the Arch." This provided an evidence that Naya's reconciliation was merely switch (i.e., switching the location of Arch and Cannon) because she was only coping with a perturbation by doing the opposite of what she did. Naya was not able to explain why she switched other than showing the system worked after the switch (i.e., when moving the crow on top of Arch on the map, the corresponding point on the plane moved to a point where she though where Arch is on the vertical axis).

Naya did not change her meaning of the points. She continued to envision the points on the plane as a location by engaging in transformation iconic translation. After switching the location of the Arch and Cannon on the plane, she also switched the axes on the map. She said "the bottoms are still the two axes. That is *y* [*pointing to the horizontal line at the top of the map that was originally labeled x. see Figure 0.3a*] and this is *x* [*pointing to the horizontal line at the bottom of the map that was originally labeled y. see Figure 0.3a*]." Since her meanings were based in iconic translation, she needed to switch the axes because she imagined Cannon on the vertical axis and Arch on the horizontal axis on the plane. In turn, the line that went through Cannon on the map should be the "*y*-axis" and the line that went through Arch on the map should be the "*x*-axis." With this new organization of the space, Naya did not need to flip the map because the perceptual features of Arch and Cannon on the map matched with the perceptual features of Arch and Cannon on the plane. That is, Cannon was at the top and Arch was at the bottom in both spaces (i.e., map and coordinate plane).

In CT, Naya's focus was on the location of Arch and Cannon so far. I wanted to draw her attention to the other locations and how the black dot moved on the plane as the crow flied on the map. I moved the crow slowly and haphazardly on the map and let Naya observe how the black dot moved on the plane. The location of the moving dot on the plane created numerous perturbations for Naya. For example, when the crow was placed at the location on the map as seen in Figure 0.41 (left), the black dot on the plane was located on the point that Naya conceived as Starbucks (see the highlighted point in Figure 0.41, right). She said, "Wait, I don't get it. The crow [*referring to the crow on the map*] is not on Starbucks when it [*referring to the black dot on the plane*] is on Starbucks [*referring to the point on the plane*]." From my perspective, this happened because the crow's DfA and DfC was the same as Starbucks' DfA and DfC on the map. Thus, the point that represented the crow's DfA and DfC (i.e., the black dot) overlapped with the point that represented Starbucks' DfA and DfC on the plane. Since Naya's meanings were based in iconic translation, she was perturbed by the location of the black dot on the plane.

Figure 0.41. Naya's activity in CT (yellow highlight and the red arrows are added for the reader).

The crow and the black dot's movement created more perturbations for Naya. For example, when the crow moved to the left on the map (see the red arrow in Figure 0.41, left, indicating the crow's path), the black dot on the plane moved diagonally to the up from left to right (see the red arrow in Figure 0.41, right, indicating the dot's path). She said,

> What! I don't get it. Why it [*referring to the crow on the map*] is going that way [*pointing to the left with her right index finger in the air*], it [*referring to the black dot on the plane*] goes up [*pointing to up with her index finger in the air*]. How come it [*referring to the black dot on the plane*] don't go further that way [*pointing to the left with her right index finger in the air*]?

From my perspective, this happened because the crow's DfA and DfC increased when the crow moved to the left as indicated by the red arrow in Figure 0.41, left. Since Naya's meaning of the points was based in iconic translation, she was perturbed by this observation because she expected that the black dot and the crow should move in the similar directions. That is, if the crow moved toward left on the map, she anticipated that the dot on the plane should move toward left too. She could not reconcile her perturbations in CT.

Naya's Activity in Downtown Athens Bike Task

Note that, in DAT and CT, I did not prompt Naya to conceive the quantities in the situation. I first wanted to get insights into her spontaneous meanings of the point when she was given a situation and a graph together. I investigated how she could initially conceive the points on the plane before I attempt to investigate (and support) how she could conceptualize quantities in the situation before asking her to represent them as a graph. Since I identified that Naya's initial meaning of the points included (transformed) iconic translation, I planned to take her attention to the quantities in the situation without asking her to engage in any graphing activities. I conjectured that if Naya could conceptualize the measurable attributes of the objects on the map, she might be able to represent the quantities on the plane rather than making (transformed) iconic translation. In this section, I illustrate her activity in Downtown Athens Bike Task (DABT) where I investigated how she could conceive the quantities in the situation (e.g., the bike's distance from Arch).

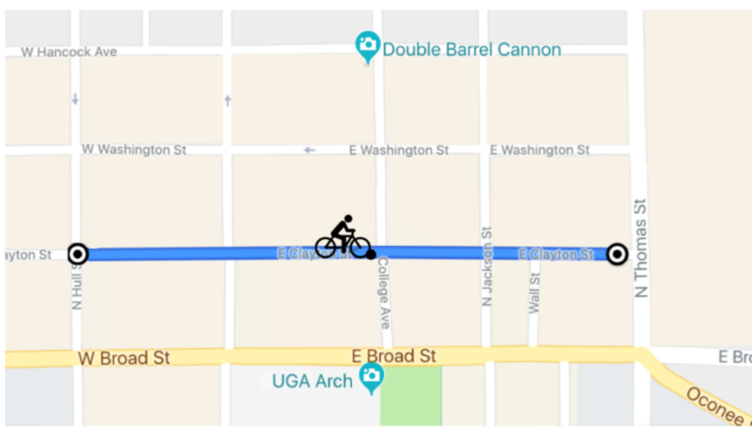


Figure 0.42. Downtown Athens Bike Task (DABT).

DABT includes the same map of Downtown Athens highlighting a straight road (i.e., Clayton St.) with two places located near the road (i.e., the Arch and the Canon; see Figure 0.42) and a bike on this road. I animate the map so that the bike moves at a constant speed back and forth along the Clayton St. starting at the West side of the street. I designed this task to explore how Naya could conceive the situation quantitatively and how she could determine the relationship between quantities (the bike's distance from Arch [DfA] and the bike's distance from Cannon [DfC]). In particular, my purpose in this task was to explore and support Naya's process of (i) conceiving the quantities' that vary in the situation, (ii) representing the varying quantities by his index fingers on the table, and (iii) representing the relationship between covarying quantities on a coordinate plane.

Naya's quantitative variational reasoning in DABT

In the first part of DABT, I showed Naya the animation where the bike rides on E. Clayton St in Athens Downtown (See Figure 0.42Figure 0.23) at a constant speed starting from the west side of the street. In order to get insights into how Naya could conceive the dynamic situation, I first asked her to talk about things that she noticed changing as the bike moved on its path. She said "directions" by sliding her index finger horizontally left to right in the air. She determined that the direction that the bike moved changed. I then prompted her to think about something that she could consider varying and possible to measure. She said, "the distance from point A [*pointing to the left end side of the blue path on the map*] to point B [*pointing to the right end side of the blue path on the map*]." Naya pointed out that the length of the blue path could be something to measure, however, she later realized "oh, yeah, it is not changing" when I asked her if that distance was varying. This conversation provided an evidence that Naya was able to conceive the measurable attributes of objects (e.g., distance between the start and finish on the blue path) in the situation although she could not determine a varying quantity in the situation yet.

Then, I explicitly asked her how the bike's DfA was changing as the bike moved on the map. I placed the bike on the left side of the path and played the animation, in turn, the bike moved to the right. Naya determined that "it [the bike's DfA] is decreasing, now [*when the bike passed the middle point of its path*] it is increasing." Naya also described the variation in the bike's DfC in the same way as the variation of the bike's DfA. This provided an evidence that Naya was able to determine the directional change in the bike's DfA as the bike moved on the map. In order to get more insights into how she conceived the bike's DfA and DfC in the situation, I asked Naya to compare the bike's DfA and DfC when the bike was located in the middle of the path (see Figure 0.42). Naya said the bike is "closer to the Arch." When asked to explain how he knew, she said, "because that [*measuring the bike's DfA with her fingers, see Figure 0.43a*] is less distance than that [*measuring the bike's DfC with her fingers, see Figure 0.43b*]." This provided an evidence that Naya was able to reason with quantities' magnitudes in the situation and make gross comparison between them determining a relationship (i.e., bike's DfA is "less than" the bike's DfC).

(a) (b) (c)

Figure 0.43. (a) Naya measuring the bike's DfA with her fingers, (b) Naya measuring the bike's DfC with her fingers, and (c) re-producing her measurement activity for the reader.

Recall that my goal in DABT was to investigate if and how Naya could conceive the quantities in the situation and (if not) support her to conceptualize the quantities (e.g.., the bike's DfA and DfC) in the situation. My analysis of her activity in DABT showed that she was able to reason with quantities' magnitudes in the situation. Next, I decided to investigate how Naya could represent the quantities' magnitudes by varying bars on parallel magnitude lines in one dimensional space before I ask her to represent the relationship on a coordinate plane.

Naya's assimilation of the bars on the magnitude lines in DABT

Recall that Naya's initial meanings of the points on the plane included (transformed) iconic translation. My goal was to support her to envision the points as a representation of two quantities' magnitudes. Before I engage her in a graphing activity in two-dimensional space, I planned to engage her in representing quantities' magnitudes in one dimensional space. I provided her with a dynamic tool that could afford Naya's engagement with quantities' magnitudes represented by directed bars placed on magnitude lines (Figure 0.44). The directed bars can be varied in length as the bike moves on the map (see https://youtu.be/6kdbDeVEF9w). I conjectured, if Naya could assimilate the bars on the magnitude lines in relation to the quantities' magnitudes, this representation could help her to develop meanings of points in terms of representing quantities in two-dimensional spaces. In order for Naya to engage in quantitative reasoning and represent the relationship between two quantities on a Cartesian plane (rather than engaging in iconic translation), she needed to conceive the axis of the plane in relation to the magnitude lines and be able to represent the quantities' magnitudes on the axis of the plane.

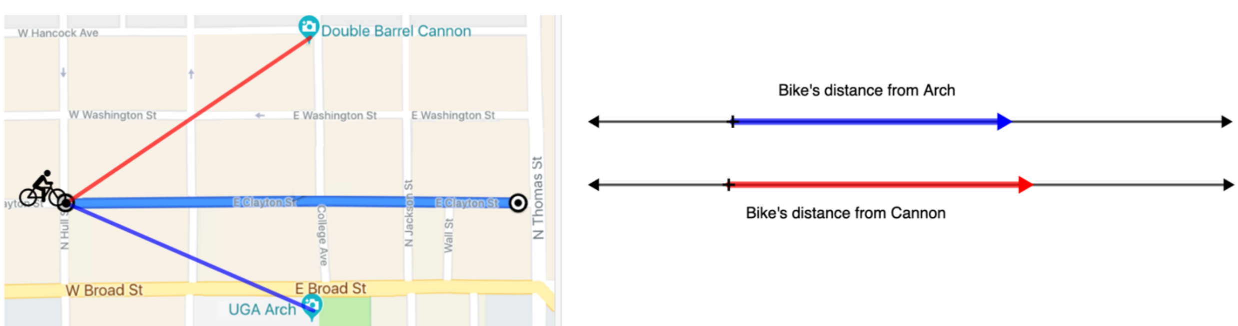


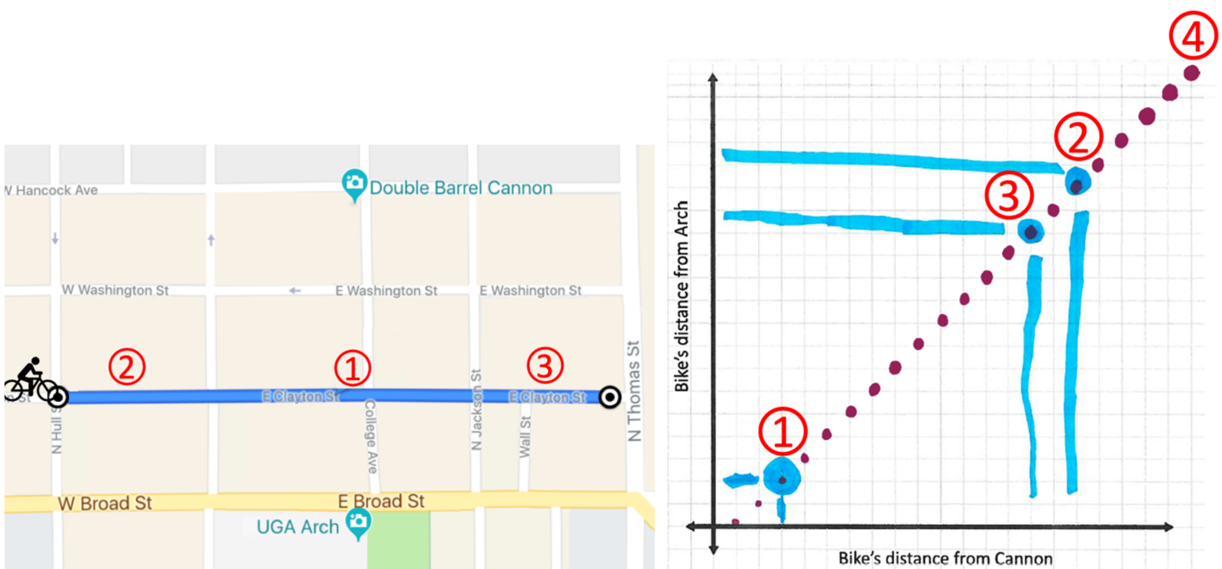
Figure 0.44. DABT and magnitude lines.

I moved the bike on the map, in turn, the length of the bars on the magnitude lines varied. I asked Naya to describe how the blue bar on the magnitude line was changing as the bike moved on the map. She said, "the length is decreasing when it [the bike] gets closer, increasing when it goes farther away." I moved the bike to the left from the middle of the path on the map and drew Naya's attention to the fact that the right end side of blue bar on the magnitude line was moving to the right (indicating the bike's DfA was increasing). Then, I asked Naya to explain why that was happening. Naya said "because the bike is still getting further away from Arch, so the line is still getting longer." From her activity, I infer Naya was able to conceive the length of the bars on the magnitude lines in relation to quantities' magnitudes. When asked to describe the relationship between the bike's DfA and DfC as the bike moved in the map, Naya was able to determine the directional covariational relationship between two quantities. She said, "when the bike [*referring to the bike on the map*] gets further from the Arch and Cannon, they [*referring to both red and blue bars on the parallel magnitude lines*] increase, when it [the bike] gets closer, they decrease." Since she successfully conceptualized the quantities' magnitudes in the situation and represented them on the magnitude lines, I decided to ask her to graph the relationship on a coordinate plane.

Naya's graphing activity in DABT (first draft)

My goal with DABT was to get Naya's attention to the quantities since she was reasoning with (transformed) iconic translation when asked to interpret a given graph in DAT. Recall that she conceived the points as a location/object and made perceptual association between the situation and the graph. She experienced perturbations in CT that she could not reconcile at the moment (e.g., she could not explain why the crow on the map and the black dot on the plane moved in different directions). Naya developed these meanings when both the situation and the graph were given, and when she was asked to interpret what she was given. Now, I have evidence that Naya conceived the quantities (i.e., the bike's DfA and DfC) in the situation and successfully represented them on two parallel magnitude lines. I wanted to see how Naya could engage in a graphing activity in two-dimensional space after her image included quantities and quantitative relationship.

Recall that Naya determined that both the bike's DfA and the bike's DfC increased and decreased together as the bike moved on the map. I asked Naya to sketch this relationship on a paper with a coordinate plane. You can see Naya's first draft in Figure 0.45b. Her meaning of the points included simultaneously representing the bike's DfA and DfC (i.e., a QMO) in a non-canonical Cartesian plane.



(a) (b)

Figure 0.45. (a) DABT and (b) Naya's first draft in DABT (the numbered labels are added for the reader).

Naya began her graphing activity by plotting points in a straight-line pattern upward from left to right (see Figure 0.45b, light blue segments were not drawn yet). As reported by Carlson et al. (2002), constructing an increasing straight line is a typical graph for someone who coordinates the directional change (i.e., gross covariation) in both quantities' magnitudes (e.g., they both increase/decrease at the same time). Thus, I hypothesized that Naya drew her graph as an implication of her gross covariational relationship between the bike's DfA and DfC being increasing and decreasing together. I asked Naya to explain what each point represented on her graph. The following excerpt illustrates how she conceived her points.

Naya: I just imagined this [*sliding the pen on the plane vertically from the horizontal axis to the point on her graph labeled #4 in Figure 0.45b and sliding the pen on the plane horizontally from the vertical axis to the same point on her graph*] being the distance. This point [*pointing to the point labeled #4 in Figure 0.45b*] being the distance between both of them and it keeps getting smaller [*sliding her finger downward from right to left over her dotted line*] and bigger [*sliding her finger upward from left to right over her dotted line*].

TR: Tell more, like, visualize what you told me. Say more.

Naya: This point, like the point right there [*pointing to the point labeled #4 in Figure 0.45b*] is all of this distance from Arch [*sliding her finger horizontally from the vertical axis to the point labeled #4 in Figure 0.45b*] and Cannon [*sliding her finger vertically from the point labeled #4 to the horizontal axis of the plane*] and it gets smaller [*sliding her finger over her graph downward from right to left*] when they are getting closer, and bigger again [*sliding her finger over her graph upward from left to right*] like kind of the lines [*pointing to the tablet screen with the magnitude lines, see Figure 0.44*].

TR: Ah. Show me those lines. Only for this point [*pointing to the point labeled #4 in Figure 0.45b*] for example.

Naya: Hmm. The longest they can be. Like, [*grabbing the tablet pen*] if we stop it [*referring to the animation on the tablet screen, see Figure 0.44*] and put the bike there [*dragging the bike on the map and putting it at the very right side of the path*], they are that [*pointing to the blue and red bars on the magnitude lines*] long.

She represented the bike's DfA on the plane as the distance between her graph and the vertical axis labeled "Distance from Arch." Similarly, she represented the bike's DfC on the plane as the distance between her graph and the horizontal axis labeled "Distance from Cannon." For example, to show the bike's DfA and DfC when the bike was located in the middle of the path (see the location labeled #1 in Figure 0.45a), she circled a dot on her graph (see the point labeled

#1 in Figure 0.45b) and drew a horizonal and vertical segment (see the light blue segments in Figure 0.45b) to represent the bike's DfA and DfC, respectively. Naya also repeated the same activity (see the other light blue dots and segments on the plane in Figure 0.45b) for other states of the bike on the map (see other labeled locations in Figure 0.45a). The vertical segments represented the bike's DfC, and the horizontal segments represented the bike's DfA. However, she later stated that the segment for the bike's DfA was supposed to be shorter than the segment for the bike's DfC because she determined the bike was closer to Arch than the Cannon. From her gestures and language, I infer that her meaning of the points included joining two quantities' magnitudes that were represented vertically and horizontally on the plane (i.e., representing QMO in a non-canonical Cartesian plane).

For Naya, moving upward and downward direction on her graph represented both the bike's DfA and DfC was increasing and decreasing, respectively, because the length of the segments on the plane was getting increase and decrease. With this new meaning of the points (i.e., QMO), while the bike moves to the left from the location labeled #1 to the location #2 on the map (see Figure 0.45a), Naya would move her finger on the graph diagonally from left to right from the point labeled #1 to the point labeled #2 (see Figure 0.45b) in order to show the bike's DfA and DfC increased.

Note that Naya made a connection between the light blue segments drawn on the plane and the red and blue bars on the magnitude lines. This provided another evidence that Naya conceived the segments on the plane as a representation of the quantities' magnitudes. This also provided an evidence that Naya was able to use her image of the varying bars on the magnitude lines when representing the relationship between quantities on a coordinate plane, although the way she represented the bars on the plane was not compatible with a canonical Cartesian coordinate plane. That is, Naya was able to mentally manipulate the bars on the magnitude lines (i.e., imagining them orthogonally on the plane while they were parallel on the magnitude lines) and represent them on the plane. This was an important cognitive ability for her as she could potentially use a similar idea when assimilating the axis of the plane in relation to magnitude lines. I hypothesize that if I provide Naya an opportunity to conceive what I perceive to be the axes of the coordinate plane in relation to the magnitude lines, Naya could represent the quantities' magnitudes on the axes rather than on the plane. Thus, I decided to transition to Matching Game Task (MGT) where I provided Naya an opportunity to make connection between the parallel magnitude lines and the coordinate plane for the purpose of constructing a Cartesian coordinate system.

Naya's Activity in Matching Game Task

Recall that Naya drew vertical and horizontal segments on the plane to represent the quantities' magnitudes in DABT (see Figure 0.45b). Naya did not assimilate the axes of the coordinate plane normatively as she did not represent the quantities' magnitudes along the axes. Thus, I designed MGT (Figure 0.46) in order to engage Naya to (i) represent two quantities' magnitudes on two parallel magnitude lines and then transition to (ii) representing two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and projecting the magnitudes on the plane. I conjectured that Naya's engagement with MGT could leverage her to structure the space in a way that is compatible with a canonical Cartesian plane.

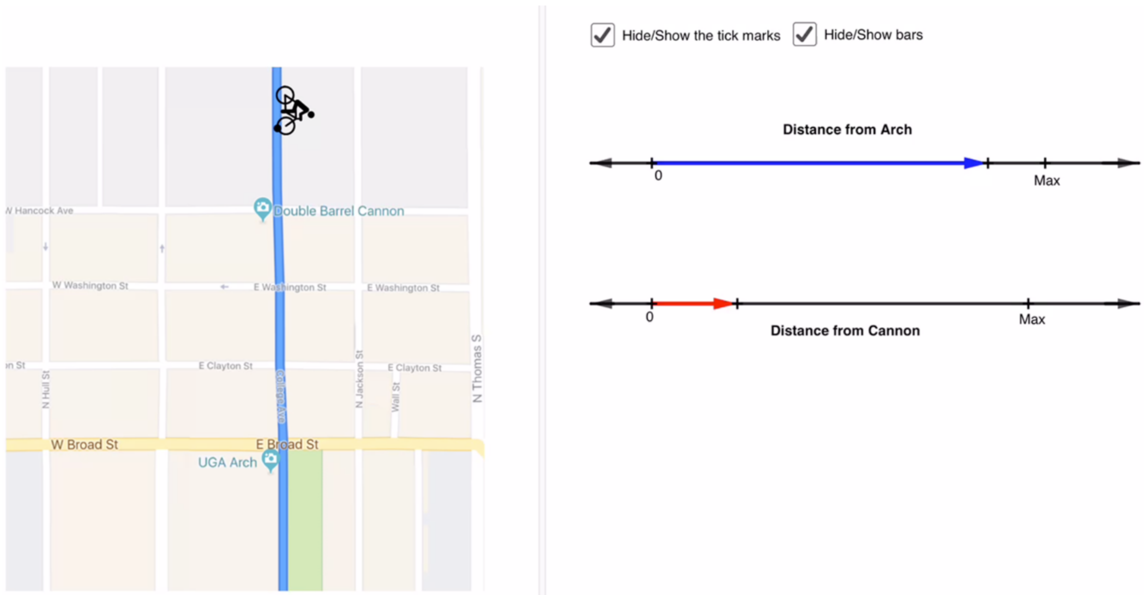


Figure 0.46. Matching Game Task (MGT).

Naya's tick marks on magnitude lines representing two quantities' magnitudes

MGT included the same map highlighting a straight road (i.e., College Avenue). Arch and Cannon were located on the road, and a bike rides along this road (Figure 0.46, left). I also presented a dynamic tool that afforded Naya's engagement with quantities' magnitudes represented by directed bars placed on magnitude lines (Figure 0.46, right). The directed bars can be varied in length (see https://youtu.be/knrN7XCbyOY). In this part of MGT, I hid the red and blue bars on the magnitude lines and asked Naya to guess what would be length of each bars on the lines according to a certain location of the bike on the map. Moreover, I asked Naya, after she guessed, to place tick marks on where she could think the head of each bar on the magnitude lines are. Based on Naya's meanings of the bars on the magnitude lines, I hypothesized this could be an easy task for Naya as she was able to represent the quantities' magnitudes—that she conceptualized in the situation—on the magnitude lines before. Here, my goal was to promote an understanding that the length of both bars should be constrained in a way that they simultaneously represent the bike's DfA and DfC. Here, I also aim to help Naya to connect the conventional use of tick marks on the number lines to the use of bars on the magnitude lines. In this way, she might be able to know that a tick mark (or a point) on a magnitude line represents the measurement of a quantity's magnitude.

When I located the bike at the bottom of the map (see Figure 0.47Figure 0.18, left), Naya successfully placed the tick marks on each magnitude line accordingly (see Figure 0.47, right). She placed a tick mark on the maximum point to represent the bike's DfC because "it [referring to the bike's DfC] is pretty much the maximum, it can't be farther away from Cannon." When placing the tick mark for the bike's DfA, she engaged in a measurement activity with his fingers as seen in Figure 0.47. She measured the bike's DfA in the situation and carried that measurement over to the magnitude lines to determine where the end of the bar might be. Moreover, she drew a segment both on the map and on the magnitude line as an indication of magnitude of the bike's DfA. Her measurement activity and her drawing a segment on the magnitude line in between the zero point and the tick mark suggested that Naya's meaning of the tick marks was connected to the meaning of the bars on the magnitude line that represented the quantities' magnitudes. I also have evidence that, at a later moment of the teaching experiment, Naya was able to reflect on the meaning of the tick mark placed on the magnitude line saying the tick mark shows "the bike's distance from Arch." When asked to explain how she knew, she drew a segment on the map between the bike and Arch and said, "that is how far the distance from the Arch is." I conjectured that this meaning of tick marks might help Naya to *record* the variation of the bar on the magnitude line since the varying bars do not leave trace or a mark as they move on the magnitude lines. In turn, I conjecture that this activity might help Naya when we move to the two-dimensional space to record the relationship between two varying quantities on each axis of the plane.

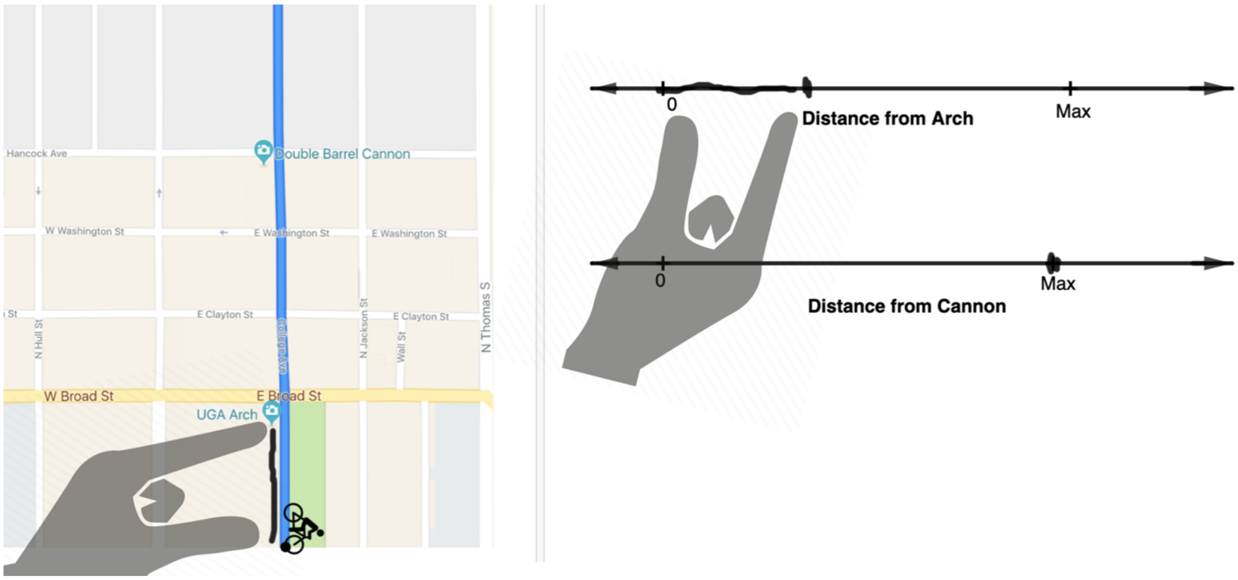


Figure 0.47. Naya's tick marks on the magnitude lines representing the bike's DfA and DfC (gesture icons are added for the reader to illustrate Naya's measurement activity).

Naya's single point on a magnitude line representing two quantities' magnitudes

Naya successfully placed tick marks on each magnitude line to represent the magnitudes of bike's DfA and DfC each time I changed the location of the bike in the situation. With Naya doing so successfully and indicating that he conceived of the tick marks in relation to quantities' magnitudes, I decided to transition to the next part of MGT where I asked Naya if he knew a way to represent the bike's DfA and DfC by *a single point* instead of the *two* tick marks that he placed on two parallel magnitude lines. I showed her the two computer-generated tick marks on each magnitude lines that were linked to the bike animation. When they moved the bike on the map, computer-generated tick marks moved accordingly on each magnitude line (see https://youtu.be/J8ASrKWtJKk). My goal was to explore if Naya could see the additional representational features that is not available with a single number line, including the property that a point on the coordinate plane simultaneously represents two quantities' magnitudes.

Recall that, in DABT, Naya was able to create a single point on a non-canonical Cartesian plane in order to represent two quantities' magnitudes (i.e., representing QMO in non-canonical Cartesian plane, see Figure 0.45b) when she was given a coordinate plane as a given structure. However, I would not be surprised if she could not create a single point given the parallel magnitude lines because there is a difference between constructing a two-dimensional space to create a point versus creating a point on a given two-dimensional space. Thus, I hypothesize that it might be hard for Naya to "invent" a way to transition to the second dimension and create a single point that would simultaneously represent two quantities' magnitudes. With this activity, I expected to gain insights into Naya's reasoning when trying to come up with ways to create a single point that represents two quantities' magnitudes.

Naya was not able to create a single point to represent two quantities' magnitudes, except for one case where the bike's DfA and DfC had the same length (see Figure 0.48). She moved one of the magnitude lines right next to the other one by matching the zero and max points. She then located the bike in the middle between Arch and Cannon on the map and placed a tick mark on one of the magnitude lines where the two computer-generated tick marks met (see Figure 0.48). Then, she moved the other magnitude line away leaving only a single tick mark on a magnitude line (see Figure 0.49). When asked to explain what that tick mark represented, Naya placed the two labels on one magnitude line and said, "distance from Arch and [*drawing a plus sign in between the two labels*] the distance from Cannon … because it [*referring to the bike on the map*] is exactly in the middle [between Arch and Cannon]." I infer that putting the magnitude lines next to each other and matching the two tick marks was a way for her to show that the single tick mark on the magnitude simultaneously represented a state where the bike's DfA and DfC were equal. She could not find a way to create a single point that simultaneously represented other states of the bike's DfA and DfC although she described her activity "this is the best I can get." She needed to construct a new space that was different than the one-dimensional space in a way that she could satisfy the simultaneity for all states of the bike's DfA and DfC.

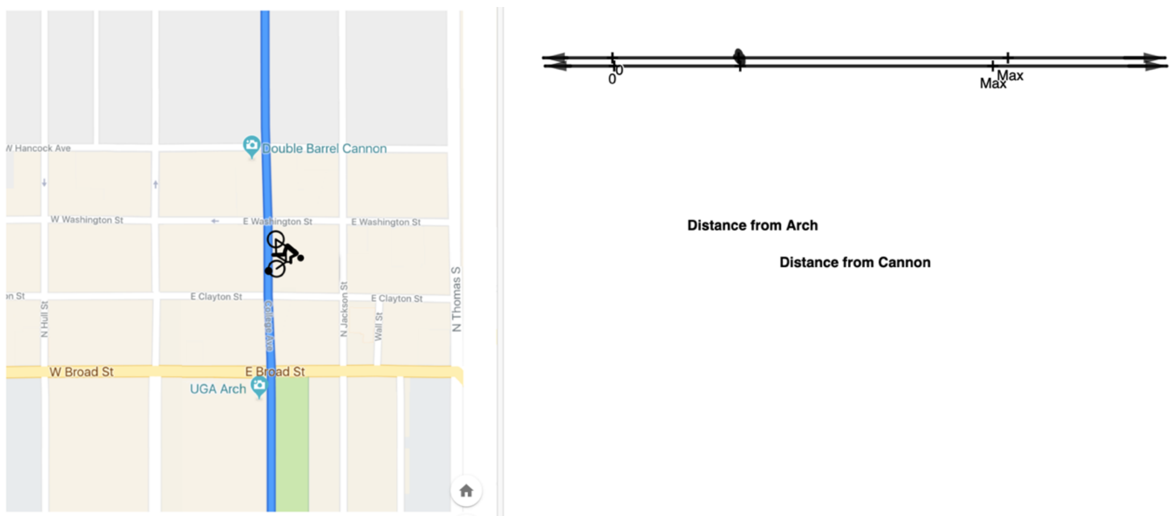


Figure 0.48. Naya's activity seeking to create a single point.

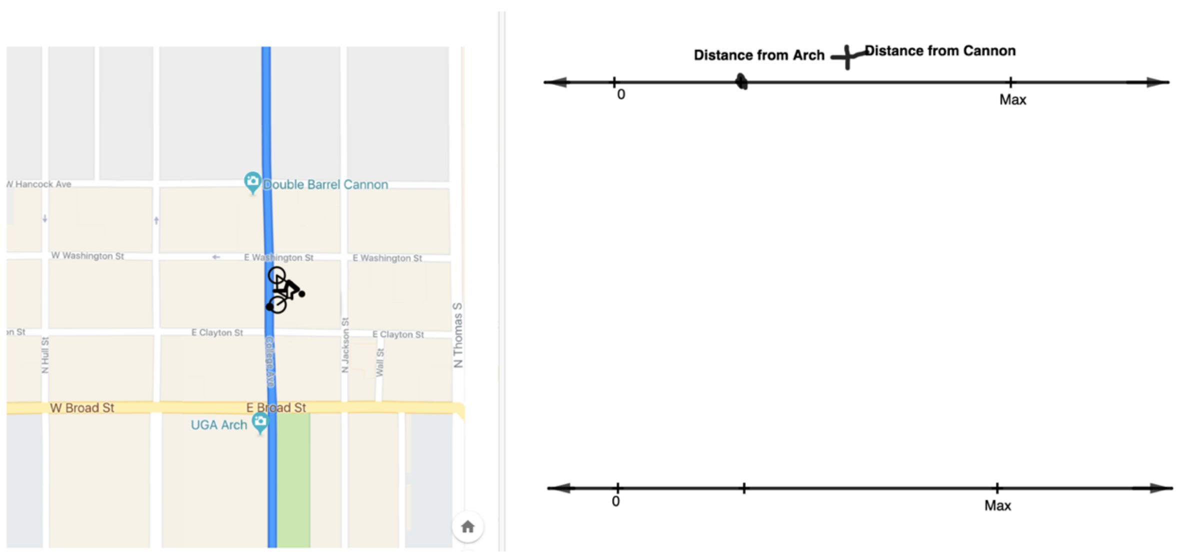


Figure 0.49. Naya's single point representing both the bike's DfA and DfC.

Naya was not able to create a single point by constructing a two-dimensional space (i.e., a canonical Cartesian plane) given two parallel magnitude lines, although she earlier constructed a single point on a given two-dimensional space where she represented QMO in non-canonical Cartesian plane in DABT. Then, I decided to transition to Crow Task (CT, see Figure 0.50) where I provided what I perceive to be a Cartesian coordinate plane as a given structure. Considering Naya's activity in MGT and her meaning for the tick marks on the magnitude lines, I hypothesized that she could potentially assimilate the axis of the plane in relation to the magnitude lines if I provide her with additional figurative material on each axis of the plane (e.g., computer-generated tick marks that could move according to the bike on the map) that could leverage her to recall her activity in MGT. Once Naya could assimilate the axis of the plane in relation to the magnitude lines, I thought she could create a single point on the plane by intersecting the projection of the magnitudes that were represented on each axis of the plane.

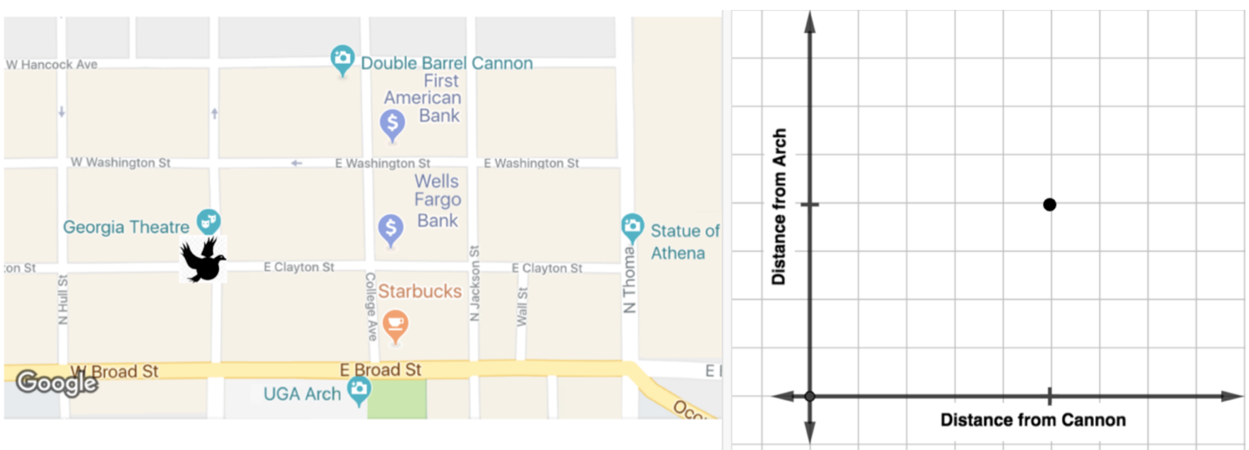


Figure 0.50. Crow Task (CT)

Naya's Activity in Crow Task

CT included the same map in addition to a movable crow (Figure 0.50Figure 0.20, left) and a Cartesian plane with the horizontal axis labeled as "Distance from Cannon" and vertical axis labeled as "Distance from Arch." The black point moved on the plane according to the crow's DfA and the crow's DfC (Figure 0.50, right, the tick marks on the axes were not available yet).

Naya's transformed iconic translation in CT

To begin her activity in CT, Naya moved the crow freely while observing how the corresponding black point on the plane moved. When I asked her to explain what the black point represented on the plane, I found that Naya used her original meaning of the points in her initial activity in CT. That is, she conceived the dot as the crow and engaged in transformed iconic translation. She said,

> This [*pointing to the top of the map*] is the *y*, this top line [*sliding the pen over the top of the map in the air horizontally from left to right*] is the *y*-axis, this bottom line [*sliding the pen over the bottom of the map in the air horizontally from left to right*] is the *x*-axis [d*rawing the lines on the map, Figure 0.51,left, and drawing the corresponding lines on top the given axis of the plane, see Figure 0.51,right*].

Figure 0.51. Naya's *x*- and *y*-axis on the map and on the plane.

She, then, "created the whole map" on the plane by putting the crow on each location on the map and labelling the corresponding location on the plane where the black dot landed. For example, Naya located the crow on top of Arch on the map (see Figure 0.51, left) and labeled a location on the horizontal axis as Arch because that was where the black dot was located on the horizontal axis (see Figure 0.51, right). She said "yeah, you can pretty much create the whole map using the crow" and, by repeating the same procedure, Naya was able to locate each location on the plane (see Figure 0.52). She further explained,

> So, when you go pretty much from top [*pointing to the bottom of the map*] to bottom [*pointing to the top of the map*] on the map [*drawing a kind of vertical segment on the map from Arch to Cannon that went through other locations, such as Starbucks, Wells Fargo Bank, and First American Bank, see Figure 0.52, left*], it [*placing the pen on Arch on the horizontal axis of the plane*] is going here [*drawing a diagonal segment starting from Arch on the horizontal axis that went though the other locations Starbucks, Wells Fargo Bank, and First American Bank in the same order, see Figure 0.52, right*], it is going diagonal on the line. Because the top of the map [s*liding the pen over the line labeled y at the top of the map*] on the graph [*transitioning the pen in the air to the plane*] is the *y*-axis [*sliding the pen over the vertical axis of the plane in the air*], the bottom is the *x*-axis [*pointing to the bottom of the plane*]. … Man, I understand this so much better now.

Figure 0.52. Naya's activity in CT

From her activity, I infer her meaning of the points included transformed iconic translation as she translated a transformed version of perceptual features of the situation to the plane (or the opposite direction). That is, Naya made a perceptual association between the horizontal/vertical axis of the coordinate plane and the bottom/top of the map. In turn, she conceived the points on the plane as a location that appeared on the map by associating their perceptual features within two spaces (i.e., forming a vertical line on the map and a diagonal line on the plane).

Naya's spatial-quantitative multiplicative object in CT

Since Naya's meaning for the points was based in transformed iconic translation in CT and because I know she could reason with quantities, I wanted to take her attention to the crow's DfA and DfC in the situation. I asked Naya to show how she imagined the crow's DfA and DfC on the map when the crow was located as seen in Figure 0.53 (left). She drew two segments: one from the Arch to the crow and the other from Cannon to the crow (Figure 0.53, left; the segments on the plane and tick marks on the axes were not drawn yet). Now, I wanted to see if and how Naya could represent these quantities' magnitudes on the plane.

Figure 0.53. Naya's activity in CT (yellow highlights are added for the reader)

Naya's tick marks on the axes representing the physical Arch and Cannon. Recall that, Naya was successfully conceptualized the quantities in the situation and represented them on the parallel magnitude lines by placing tick marks in MGT (see the section of "Naya's tick marks on magnitude lines representing two quantities' magnitudes"). She measured the bike's

DfA in the situation with her fingers and carried that measurement with the fingers over to the magnitude lines to determine where the end of the bar might be (see her gestures in Figure 0.47). Thus, she was able to transform (i.e., dis-embed and re-present) the magnitude of the bike's DfA and DfC from situation to the magnitude lines by ensuring to preserve their length. Based on this inference, I thought that I could ask Naya to place tick marks on the axis of the plane to represent the crow's DfA and DfC in CT. Since she never assimilated what I perceive to be the axes of the coordinate plane as the magnitude lines, I did not expect her to insert tick marks on each axis normatively in her first attempt. However, I still planned to ask this question because I thought this would be a way for me to take her attention to the axes of the plane in later attempts. My goal was to help her to make connection between her activity of inserting tick marks on two parallel magnitude lines and inserting tick marks on the axis of the coordinate plane.

I asked Naya to place tick marks on the axis of the plane to represent the crow's DfA and DfC that she determined in the situation. To respond to my question, Naya began plotting a tick mark on the horizontal axis, but she appeared unsure of her activity. She said "Wait, that [*pointing to her tick mark on the horizontal axis, see Figure 0.53, right, the segments on the plane were not drawn yet*] is not Arch. I don't know where the Arch is." I infer she imagined the physical Arch on the horizontal axis although she was not sure where to plot Arch on the axis. This provided an evidence that Naya didn't assimilate the axes of the plane in relation to the magnitude lines yet because she wanted to locate the Arch on the axis so that she could show the crow's DfA on the plane. I repeated the question again to emphasize that the tick marks should represent the crow's DfA and DfC. I said, "you are asked to draw tick marks to show the crow's distance from Arch and crow's distance from Cannon on the axes." She responded, "What? I don't get it. [sighs]. What are you asking?" She perturbed by my question because I was asking to insert tick marks to show the quantities' magnitudes (i.e., the crow's DfA and DfC); however, she was imagining placing a tick mark to show the physical Arch and Cannon on each axis.

I decided to let Naya do what she wanted to do originally without making emphasis on the quantities. Then, she plotted a tick mark on each axis where she thought where the physical Arch and Cannon were on the axes (see Figure 0.53

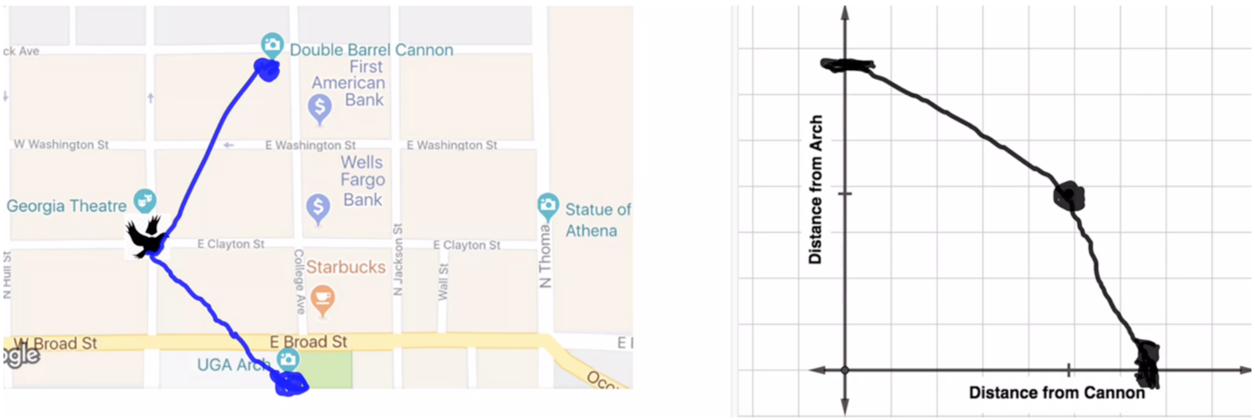


Figure 0.21, right) and drew segments from the Arch and Cannon to the black point (which is the crow for her) on the plane. After she completed her drawings, I asked Naya, "Okay, which tick mark shows the crow's distance from Cannon." To respond, she traced the pen on the segment that she drew between the black point and Cannon on the vertical axis. This suggested that Naya conceived the segments on the plane as a representation of the quantities' magnitudes (e.g., the crow's DfC and DfA). Overall, her meaning of the point included representing a SQMO. That is, she still conceived the black dot as the crow; however, she focused on the crow's quantitative properties (i.e., the crow's DfA and DfC) when making sense of the location of the black dot (i.e., the crow for her) on the plane.

Then, I let Naya to see the computer-generated tick marks on each axis of the coordinate plane (see the highlighted area on each axis for the location of the additional tick marks in Figure 0.53, right; or see the computer-generated tick marks in Figure 0.50, right). My goal was to provide additional figurative material for Naya that could afford her to conceive the axis as the magnitude lines although I did not expect her to make that connection immediately. In particular, I planned to move the crow on the map, and in turn, the tick marks would move on each axis according to the crow's DfA and DFC. Since I knew Naya could coordinate the quantities' magnitudes and representation of them on parallel magnitude lines, I hypothesized that the movements of the computer-generated tick marks could spark some ideas related to red and blue bars on the magnitude lines and make some connections to the axes of the coordinate plane.

Naya's tick marks on the axes as representing the quantities' magnitudes

Once I showed the computer-generated tick marks on each axis, Naya immediately surprised by the locations of the tick marks on the axes. She questioned "why are those there" and he could not resolve this perturbation at the moment. Naya was perturbed by the location of the computer-generated tick marks because her current meaning of the tick marks included the physical Arch and Cannon located on each axis.

Since I know Naya had a meaning for tick marks in relation to quantities' magnitudes in MGT, I decided to ask Naya to reflect on her previous activity where she placed tick marks on the magnitude lines in MGT. I asked Naya what the tick marks meant to her in the previous activity. She said, "the end of the line, distance from the bike, or the Arch." This provided evidence that she still held a meaning for a tick mark in relation to a quantity's magnitude. Then, I asked Naya to move the crow and observe how the tick marks were moving on the axes of the plane (see https://youtu.be/pQz6_i2L9Vs). As she observed the animation, she said "Oh, the tick marks are the crow's distance from Arch and the Cannon, and they move when the crow moves. I guess that makes sense now." Naya assimilated the tick mark moving on each axis of the plane as a representation of the crow's DfA and DfC.

Then, I made the red and blue bars visually available on each axis (see https://youtu.be/ya8XBAG19Wk). Naya was able to conceive the red and blue bar's lengths in relation to the crow's distance from Cannon and Arch, respectively. I asked her how she knew the red bar represented the crow's DfC and the blue bar represented the crow's DfA. To respond to my question, she moved the crow on top of Arch and showed that the length of the red bar is zero. She said, "the red bar is showing the crow's distance from Cannon, because the crow is on the Cannon and the red bar is gone." This provided an evidence that Naya conceived the quantities' magnitudes in the situation and be able to represent them by the varying bars on each axis of the plane.

Naya's quantitative multiplicative object in CT

Recall that Naya's meaning of the points in CT included representing a SQMO (Figure 0.53). Also recall that her meaning of the point in DABT included representing a QMO in a non-canonical Cartesian plane (see Figure 0.45b). Now that I know Naya was able to represent quantities' magnitudes on the axes of the coordinate plane, I planned to get insights into her meaning of the point (i.e., the black dot) on the plane to see if it would be different than representing SQMO and QMO in non-canonical Cartesian plane. My hypothesis was that she could now conceive the points in terms of representing QMO in canonical Cartesian plane.

I hid the black dot on the plane keeping the blue and red bars on each axis available (see Figure 0.54, right) because I planned to ask Naya where the black dot would be for several locations of the crow on the map. I wanted to see if and how Naya could join two magnitudes shown on each axis and generate a point on the plane that represented both. I asked Naya where the black dot would be on the plane when the crow was located in its position seen in Figure 0.54, left. Naya plotted a dot on the plane where the projection of the red and blue bars met on the plane (see Figure 0.54, right). To locate the dot on the plane, she first placed the pen at the right end of the red bar on the horizontal axis and slide the pen vertically from the end of the red bar to the location where the projection of the blue and red bars intersected (see the black arrow indicating the path for her pen in Figure 0.54, right). Naya's activity suggested that she was able to locate a point on the plane by using the figurative material that was given on each axis of the plane (i.e., the red and blue bars). She imagined intersecting the projections of the bars that were represented on each axis of the plane.

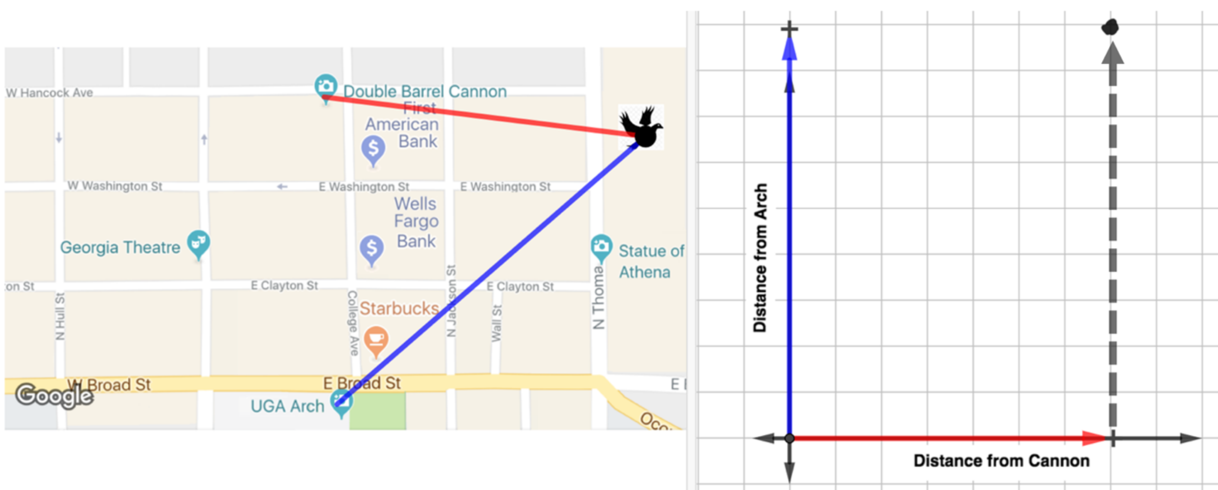


Figure 0.54. Naya's point in CT (the grey arrow on the plane is added for the reader to indicate the path for her gesture)

Since the red and blue bars in the situation and on the axes of the plane were visually available in CT, I next decided to engage Naya in a graphing activity where there was not figurative material available. To repeat the same task with less figurative material available for Naya to rely on, I transitioned to a version of DAT that included a map of Downtown Athens with no bars available and a coordinate plane with no tick marks or bars available on the axes (see Figure 0.55a). I asked Naya if she could plot a point on the plane in order to represent Wells Fargo Banks' (WFB) DfA and DfC. I wanted to see if Naya could isolate WFB's DfA and DfC in the situation, dis-embed them from the situation, and represent them on each axis of the plane, then, plot a point on the plane accordingly. I also provided extra measurement tool (i.e., a paper straight edge, see green paper in Figure 0.55b) in case she might want to use it.

(a) (b)

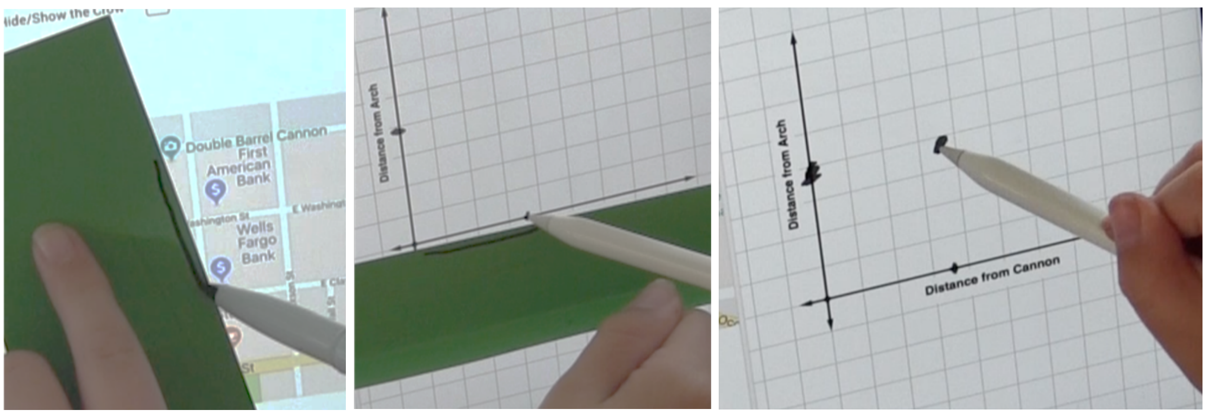
Figure 0.55. (a) A version of DAT and (b) a paper straight edge tool

Naya began her activity by placing the straight edge tool on the map (see Figure 0.56a) and drawing a segment to record the magnitude of WFB's DfC on the paper. Then, she moved the straight edge and placed it on the horizontal axis labeled "Distance from Cannon" in a way that left end side of the segment matched with the origin (see Figure 0.56b). Naya then placed a tick mark where the left end side of the segment was on the horizontal axis. In this way, Naya was able to represent WFB's DfC on the horizontal axis by a tick mark. She repeated the same activity to represent WFB's DfA on the vertical axis of the plane. Since Naya plotted two tick marks on the axis of the plane, I asked her "how about the point that I want you to plot that shows Wells Fargo's distance from Arch and Cannon?" Then, she plotted a point where the projection of the tick marks intersected on the plane (see Figure 0.56c). When asked to explain how she knew the point showed WFB's DfA and DfC on the plane, she said,

> Because we measured it, we measured the distance between Cannon [*pointing to Cannon on the map*] and Well Fargo Bank [*pointing to WFB on the map*], and then the Arch

[*pointing to Arch on the map*]. Then, put them on there [*pointing to two tick marks on the axes of the plane*], and then [*sliding her fingers in the air from both tick marks to the point on the plane*]."

From her activity, I infer Naya's meaning of the point included representing QMO in a canonical Cartesian plane. That is, she envisioned the single point on a canonical Cartesian plane as symbolizing the two quantities' magnitudes (i.e., WFB's DfA and DfC) that were represented on the axes of the plane.



(a) (b) (c)

Figure 0.56. Naya's single point representing both WFB's DfA and DfC

Naya's quantitative multiplicative object in Swimming Pool Task

In CT and DAT, I identified that Naya's meaning of the point included representing a QMO in a canonical Cartesian plane. That is, I know that she could generate a single point to represent two quantities' magnitudes when she was provided a plane as a given two-dimensional structure. Next, I decided to provide an opportunity for Naya to structure the space in a way that she could use two parallel magnitude lines to construct a two-dimensional space where she could plot her point. Thus, I transitioned to a task that was a version of MGT where I asked her to create a single point to represent two quantities' magnitudes given two parallel magnitude lines in the swimming pool situation (see Figure 0.57, left). Recall that, in MGT, Naya could not produce a single point that would simultaneously represent both the bike's DfA and the bike's

DfC given two tick marks placed on two parallel magnitude lines, except for one case where the bike's DfA and DfC had the same length (see Figure 0.48). I hypothesized that Naya could now create a single point to simultaneously represent two quantities' magnitudes by placing the magnitude lines perpendicular to each other because I thought she could relate to his activity of creating a single point in CT where a coordinate plane was provided as a given structure. Before asking Naya to create a single point given two magnitude lines in swimming pool situation, I first engaged her in Which One Task where I provided an opportunity for her to determine the relationship between two quantities (i.e., amount of water and depth of water) in the situation as they varied in tandem.

Naya's activity in Which One Task

Prior to Which One Task (WOT), Naya's conceived relationship between amount of water (AoW) and depth of water (DoW) included coordinating the direction of change in both quantities (i.e., "if there is less water, there is gonna be less deep" and "when there is more water coming in there is more depth"). By engaging her in WOT, my goal was to draw her attention to behavior of the change in both quantities beyond the directional change (e.g., amounts of change). In WOT, I presented Naya the pool animation (see Figure 0.57, left; the marks on the right side the pool was not available yet) and five directed bars that were located on parallel magnitude lines (see Figure 0.57, right; the tick marks on the magnitude line for DoW was not available yet). The bars on the magnitude lines and the pool animation in the situation were not synced (see https://youtu.be/MnelghPHzWI). I informed Naya that the blue bar represented DoW in the pool. As she moved the blue bar, each red bar varied in a different way in relation to the blue bar (see the previous link for the video to see how the red bars varied as the blue bar varied at a constant speed; #3 is the normative red bar). I then asked Naya to determine which of the four red bars, if any, accurately represented AoW in the pool as DoW varied. Note that all of the red bars directionally vary in the same way as the normative red bar. If Naya would engage in only gross (directional) covariational reasoning, she would end up with all red bars satisfying the directional relationship. Thus, Naya should be able to engage in reasoning with amounts of change in order to find which one was the normative red bar representing AoW in the pool.

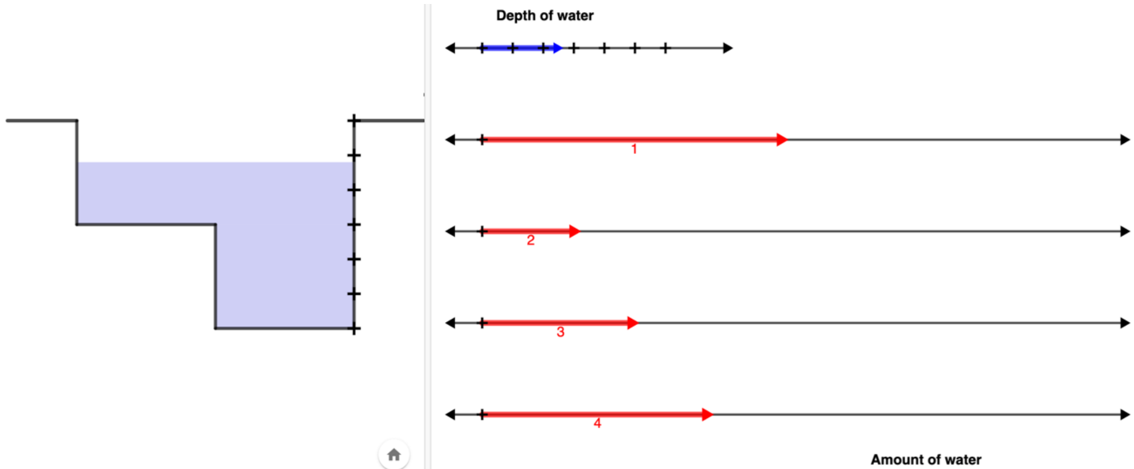


Figure 0.57. Which One Task (WOT).

Naya could not identify the normative red bar in her first attempt. She initially said, "it might be number one." Then, after Melvin claimed it was number 2, Naya said "that is possible." Then, she said "it could also be three, never mind." She then added "two and three both seemed like it could be." When investigating which one, she mentioned about she was looking at "how fast it [the red bar] is moving and how it slows down ... compared to the depth of water" but she could not elaborate on what she meant by that. Note that the blue bar moved at a constant speed (i.e., changing its position at a constant rate with respect to time) on the magnitude line when playing the animation (see https://youtu.be/MnelghPHzWI). I hypothesized that Naya compared the speed of each red bar to the speed of the blue bar and eliminated the ones that she thought did not match for her criteria. When asked to explain why she eliminated #1 and #4, she could not provide a reason and said, "I don't know."

Then, I showed Naya equally spaced marks on the right side of the pool in the situation (see Figure 0.57, left) and equally spaced tick marks on the magnitude line with the blue bar (see Figure 0.57, right). I hypothesized that this additional figurative material could afford her coordinating the amount of change in AoW in relation to equal increments of DoW in the situation and represent them on the magnitude lines. Once I showed the marks, Naya immediately said,

> That helps. [TR: How come]. Because you can tell, when you are at this [*moving the blue bar on the third tick mark from the left*] second line, or third line [*referring to the third tick mark*], it is about here [*pointing to the pool in the situation referring to the level being at the third tick mark*], and then when you start going again [*playing the animation and watching*], then that is when it speeds up [*pointing to the red bar numbered 3*], which is also when the amount of water doubles [*pointing to the higher part of the pool in the situation*], which is why it should speed up.

Using the tick marks on the magnitude line and the marks on the side of the pool for DoW, Naya was able to partition DoW equally into two parts: one for the lower part of the pool and the other for the upper part of the pool. I didn't have evidence that Naya attended to coordinate the change in AoW for each small increment of DoW in the situation (there were six equal interval). I infer that she only compared the change in AoW for the lower and upper part of the pool. She determined AoW "doubles" in the upper part of the pool compared to the change in the lower part. Then, she compared how "fast" the red bars moved for equal increment of DoW on the magnitude line with the blue bar. She determined that the third tick mark on the magnitude line for DoW was associated with the third mark on the side of the pool indicating DoW in the situation. Since she determined AoW "doubles" at the upper part of the pool, Naya expected that the red bar "should speed up" when DoW passed the third tick mark level in the situation and on the magnitude line. Thus, she decided that the red bar #3 was representing AoW in the pool because that was the only one that "speeds up" at a moment when the blue bar was at the third tick mark. I pointed out that “two [*referring to the red bar numbered 2*] is also kind of speeds up.” Naya responded “Yeah, kind of. But not at the right time.” She then added “three [*referring to the red bar numbered 3*] speeds up at the right time, but two [*referring to the red bar*] speeds up too early,” meaning the red bar numbered 2 “speeds up” before the blue bar hit the third tick mark. This provided an evidence that Naya was able to coordinate the change in AoW with the change in DoW although she used the words “speeds up” to describe the change.

In order to support Naya to coordinate the amounts of change in AoW for each small increment of DoW in the pool, I decided to transition to another version of WOT where I provided Naya with movable tick marks that could enable her to record the variation of AoW on each magnitude line. This version was the same as the previous WOT except the normative red bar was #2 (see https://youtu.be/bhYWw9aN2Ao). I asked Naya to determine which of the four red bars, if any, accurately represented AoW in the pool as DoW varied. She began her activity by placing a tick mark at the end of each red bar and the blue bar on the magnitude lines each time she moved the blue bar in equal increments (see Figure 0.58).

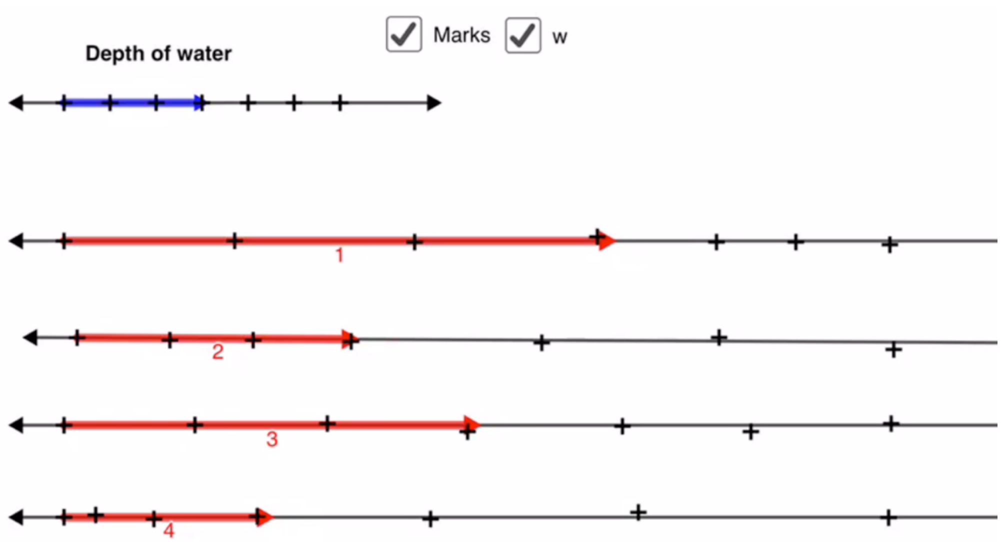


Figure 0.58. Naya’s tick marks on the magnitude lines in WOT

Once she plotted tick marks on each magnitude line, Naya noticed that the distance between the tick marks on the red bar numbered 3 were all equal. She said "it is not three [*referring to the red bar numbered 3 in Figure 0.58*]" because she believed it was for another pool that she drew on a paper (see her rectangular shaped pool in Figure 0.59). To explain why she drew a rectangular pool, she said "because the amount of water has to increase the same amount, so it has to keep the same shape [*pointing to the straight edges of the pool*]." Then Naya heard Melvin saying "I think it is two [red bar numbered 2] … because the amount of water increased by twice as much once it [*referring to the blue bar*] reaches that point [*pointing to the third tick mark on the magnitude line with the blue bar*]. Naya then agreed with Melvin saying, "yeah I think it kind of make sense." Recall that Naya earlier determined that the change in AoW for the upper part of the pool were "doubled" compared to AoW for the lower part of the pool. I thought that was why it made sense for Naya to agree on Melvin's idea that AoW increased "twice as much" in the upper part of the pool compared to the lower part of the pool for each equal increment of DoW. Thus, she selected the red bar numbered 2 indicating it represented AoW in the pool because she determined the distances between tick marks were "twice as much" as the distance between tick marks once the blue bar passed the third tick mark (see Figure 0.58).

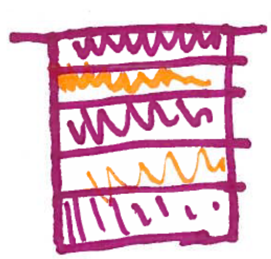


Figure 0.59. Naya's pool for the red bar numbered #3 on the magnitude line in Figure 0.58

Naya's single point in Swimming Pool Task

Since Naya determined the relationship between AoW and DoW and successfully represented them on two parallel magnitude lines, I decided to ask Naya to represent the relationship between AoW and DoW on a coordinate plane. Before asking her to create a representation for varying quantities in two-dimensional space, I want to see if she could create a single point that would represent both AoW and DoW given two parallel magnitude lines.

I presented Naya the pool animation and two parallel magnitude lines (see Figure 0.60). The bars on the magnitude lines were not synced with the pool animation. The bar's length could be changed by dragging the blue bar left to right on the magnitude line and the red bar's length would change according to the blue bar's variation preserving the quantitative relationship in the pool situation. (see https://youtu.be/DN1v277ZcdE).

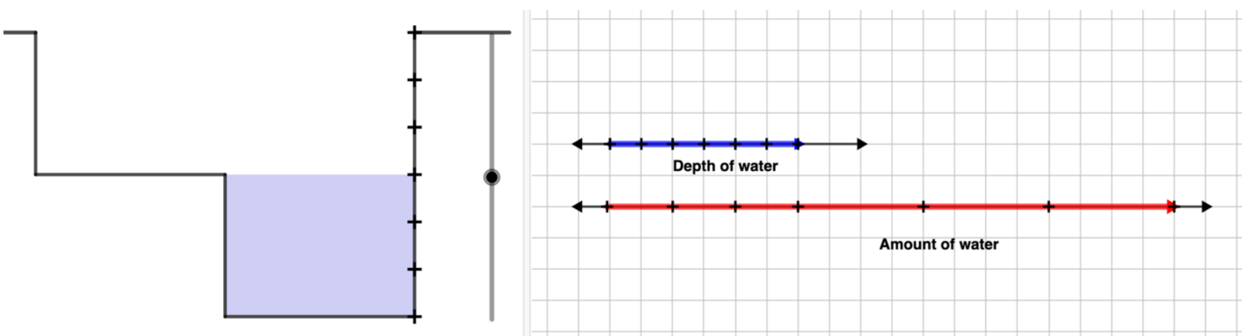


Figure 0.60. Swimming pool animation and the magnitude lines

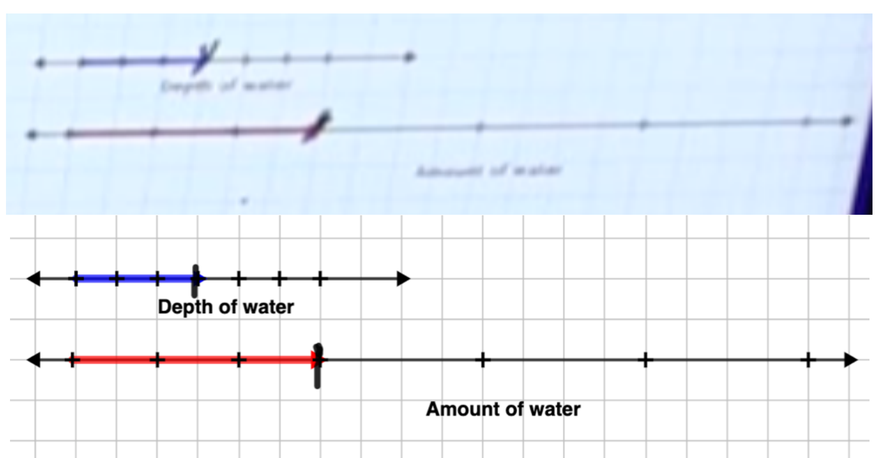
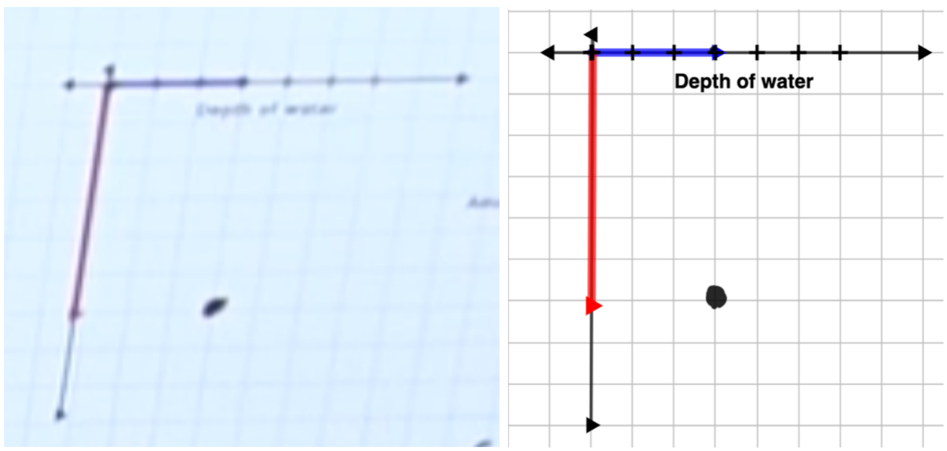


Figure 0.61. (top) Naya's tick marks on magnitude lines, (bottom) Re-producing of her work for the reader.

I first asked Naya to plot tick marks on the magnitude lines to show the amount of water (AoW) and the depth of water (DoW) in the pool when the water level was at the third tick mark on the right side of pool (see Figure 0.60, left). She moved the blue bar on the third tick mark on the magnitude line and placed a tick mark at the end of each bar on each magnitude line (see Figure 0.61). Then, I asked her to create a single point that could simultaneously show both AoW and DoW in the pool. Naya began her activity by rotating the magnitude line with the red bar downward and created the point on the plane (see Figure 0.62). This suggested that Naya's inherent organization of the space did not include the conventional use of a Cartesian plane. Naya satisfied her goal of creating a single point that simultaneously represented two quantities on the plane, which was the task she was given. Unfortunately, I could not follow up with her unconventional organization of the space because her organization was not in my attention as I was trying to fix a tech issue in Melvin's tablet that animated the pool situation. As I repeated the task for Melvin, I noted Naya changed the orientation of the magnitude line to the conventional use of Cartesian coordinate plane (see Figure 0.63). I don't know why she changed the orientation to be compatible with the conventional use of a Cartesian plane (from my perspective) as I didn't follow up with her. Nevertheless, Naya was able to rotate the magnitude lines and placed them perpendicular to each other in order to create a single point to represent both AoW and DoW on the plane (see Figure 0.63). She stated that

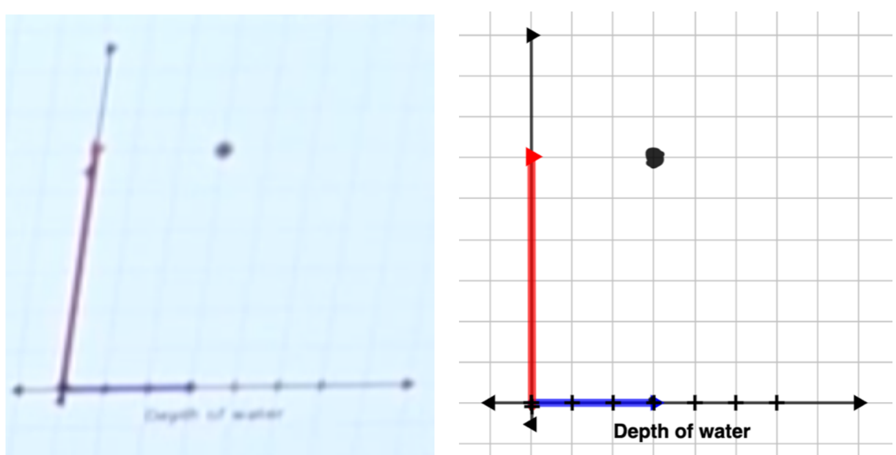
> This [*sliding the pen over the red bar on the vertical axis*] is the amount of water and that [*sliding the pen over the blue bar on the horizontal axis*] is the depth of water, so that point [*pointing to the dot she plotted on the plane*] must be both.

This data extra evidence that Naya's meaning of the point included representing both quantities' magnitudes and her organization of the space was consistent with a canonical Cartesian plane, which was consistent with her meaning in the previous activities (e.g., CT [see Figure 0.54] and a version of DAT [see Figure 0.56]).



(a) (b)

Figure 0.62. (a) Naya's single point with unconventional orientation and (b) Re-producing her work for the reader.



(a) (b)

Figure 0.63. (a) Naya's single point with conventional orientation and (b) Re-producing her work for the reader.

Naya's Graphing Activity in Swimming Pool Task

From her earlier activities throughout the teaching experiment, I knew that Naya could conceptualize quantities in the situation, represent those quantities' magnitudes by the directed bars on magnitude lines, and represent the two quantities' magnitudes by a single point in a Cartesian coordinate plane. Since Naya's meaning of the point included representing QMO in canonical Cartesian plane when the quantities were static, I now wanted to see how Naya could create a representation when both quantities varied in tandem. I illustrate that her meanings of the points in this task included both representing QMO in a non-canonical Cartesian plane (NCP) and representing QMO in a canonical Cartesian plane (CCP).

Naya's QMO in NCP in Swimming Pool Task

In this task, I asked Naya to represent the relationship between AoW and DoW on a paper with a coordinate plane as the pool animation played. Naya began her graphing activity by drawing a vertical and horizontal segment on each axis representing the magnitudes of AoW and DoW, respectively (see Figure 0.64a). She said,

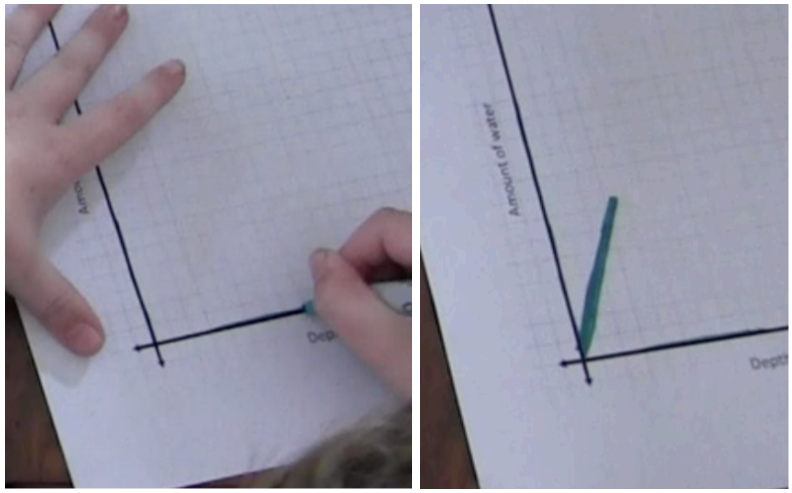
> I see these two lines [*pointing to the segments that she drew on each axis, see Figure 0.64a*] as amount [*pointing to the segment on the vertical axis that was labeled Amount of Water*] and depth [*pointing to the segment on the horizontal axis labeled Depth of Water*]. And you can raise those [*pointing to the segments on the axes*] as much as you want [*sliding the pen up over the vertical axis*], so the point could be anywhere on the graph [*moving the pen haphazardly on the plane*].

This provided an evidence that Naya initially conceived the axis of the plane as a magnitude line with the bars varying on them. Her image of the bar on each axis also included the dynamic features as she envisioned her segments "raising" on the axes. Naya was able to imagine varying bars on the axes of the plane; however, she could not imagine how the corresponding point could move on the plane. Naya claimed that the point could be anywhere on the plane as she imagined the bars moved on the axes. This idea later didn't seem reasonable to her as she said, "how do I graph like a line [*paused and sighed*] if [*paused*], oh, never mind actually" and further stated that

"I changed my mind. I was gonna say the point could be anywhere, but it can't I guess." Since she could not imagine where the point would be on the plane as she imagined varying bars on the axes, she could not draw a graph as she said, "I just don't know what to do."

Recall that Naya successfully constructed a graph on a plane in DABT to represent the relationship between the bike's DfA and DfC as they varied (see her representing QMO in non-canonical Cartesian plane in Figure 0.45b). Note that at that time, Naya was not able to conceive the axes of the plane in relation to the magnitude lines. She used the axes as reference rays to imagine the varying magnitudes on the plane. That is, she represented the bike's DfA by a horizontal segment from the vertical axis labeled "Distance from Arch." Similarly, she represented the bike's DfC by a vertical segment from the horizonal axis labeled "Distance from Cannon." In turn, Naya was able to create her graph on the plane where these two horizontal and vertical segments intersected. Here in this task when producing a representation on the plane, Naya experienced a perturbation by the dynamics of the setting because her image of a coordinate plane included varying segments on the axes, which was different than her earlier meanings. Since she could not imagine where the point would be on the plane as she imagined segments varied on the axes, she could not create her graph. Naya needed to imagine a moving point at the intersection of the projections of the segments on the axis and be able to trace that point on the plane as the magnitudes varied on the axes.

Since Naya could not produce a graph, I decided to draw her attention to only the lower part of the pool and helped her to reflect on the relationship between AoW and DoW that she determined earlier. Recall that Naya determined that AoW increased by the same amounts for equal increments of DoW in the lower part of the pool. I asked Naya to tell me how AoW and DoW were changing in the lower part of the pool. She said, "they are increasing" and began drawing a straight line on the plane upward from left to right (see Figure 0.64b) to show the relationship between AoW and DoW for the first part of the pool.

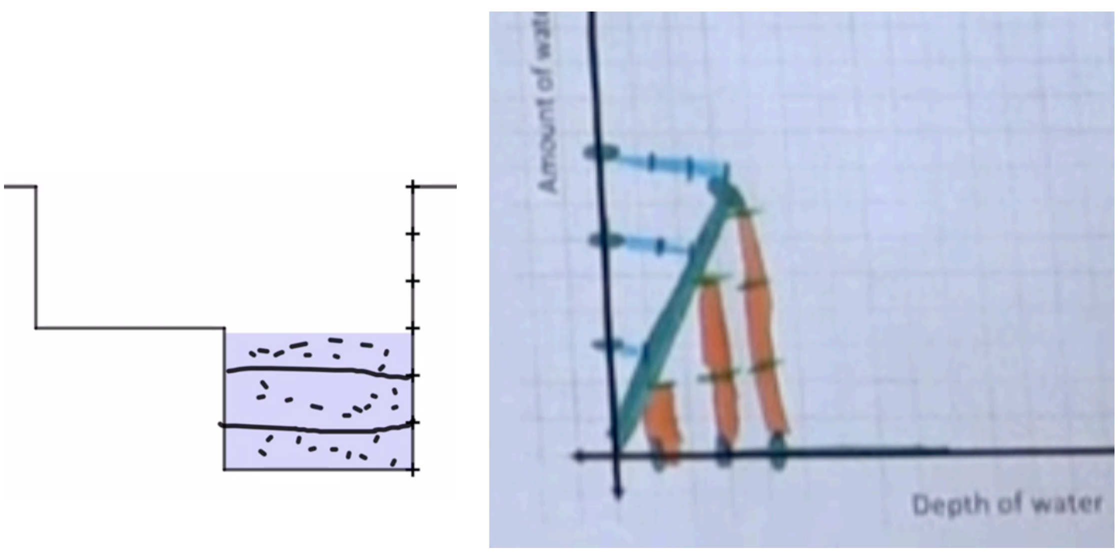


(a) (b)

Figure 0.64. (a) Naya drawing segments on each axis representing AoW and DoW and (b) Naya drawing a straight line representing the relationship between AoW and DoW.

Since Naya constructed her graph without drawing any tick marks or segments on the axes of the plane, I wanted to get insights into her meaning of the graph. I was not sure whether or not she was imagining the quantities' magnitudes on the axes when constructing her graph. Based on her previous activity drawing segments on the axes, I hypothesized that her meaning of the graph could include representing QMO in CCP (i.e., imagining the magnitudes varying on the axes although she did not visualize them). Based on her experienced perturbation (i.e., she did not know what to do), I also hypothesized that she could draw on her earlier meaning of points representing QMO in NCP (i.e., imagining the magnitudes varying on the plane) abandoning the new meaning she developed.

In order to understand Naya's meanings for the graph, I drew her attention to the relationship between AoW and DoW in the lower part of the pool and planned to investigate how she could show this relationship on her graph. I equally partitioned the quantities in the lower part of the pool and asked Naya to illustrate on her graph how AoW and DoW increased by the same amount. Naya drew three vertical segments (see orange segments in Figure 0.65b) and three horizontal segments (see light blue segments in Figure 0.65b). Then, she partitioned each tick mark equally in order to show the line segments increased by the same amount each time. She said, "you have this amount [*pointing to the shortest orange segment*] and it doubles [*pointing to the second orange segment*] and it triples [*pointing to the third orange segment*], so it gets the same amount every time." By using the partitions on the orange segment, Naya successfully illustrated the quantities' magnitude increased by "the same amount every time." In order to understand what she meant by "it", I asked her to explain which quantity she wanted to represent with the orange segments. She said "depth of water" pointing to the label of the horizontal axis (i.e., "Depth of Water" see Figure 0.65b). This data provided evidence that Naya's meaning of points included representing QMO in NCP. That is, she imagined the magnitudes of DoW (i.e., orange segments) as the distance from the axis labeled "Depth of Water" and, similarly, imagined the magnitudes of AoW (i.e., light blue segments) as the distance from the axis labeled "Amount of Water."



(a) (b)

Figure 0.65. (a) Swimming pool situation partitioned for the lower part and (b) Naya's equally partitioned segments on the plane

Naya's QMO in canonical Cartesian plane in Swimming Pool Task

Now that I understood Naya represented the magnitudes on the plane instead of on the axis, I asked both Naya and Melvin to show which segment represented which quantity. I had already known how Naya imagined the segments on the plane. I intentionally asked this question for both of them because I knew that Melvin represented the magnitudes different than Naya. Melvin represented the magnitudes on the axis and represented QMO in CCP whereas N represented QMO in NCP. I hypothesized that if I made them share their thinking, they might be able to see the differences and similarities and make adjustments if they wanted to. The following excerpt shows their conversation.

TR: Which one was showing us the amount of water and which one was showing us the depth of water?

Naya: This [*pointing to the vertical orange segment*] is the depth, this [*pointing to the horizontal segment*] is the amount.

Melvin: This [*pointing to the vertical segment*] is the amount of water and this [*pointing to the horizontal segment*] is the depth of water.

TR: You kind said different things. Tell me again.

Naya: Well, I guess, yeah, this [*pointing to the orange vertical segment*] is the amount of water and this [*pointing to light blue horizontal segment*] is the depth of water.

TR: Why you are changing?

Naya: Like. Either could be, wait, I am confused.

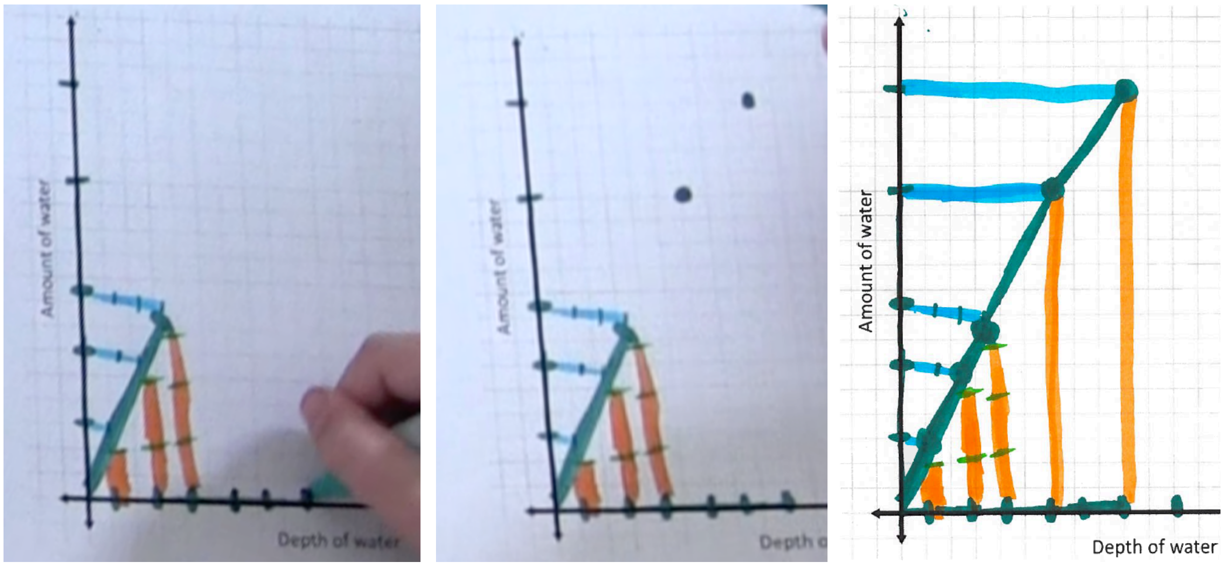
Melvin: Remember, we have to imagine the bars on here [*pointing to the axes of the plane*]

Naya: Oh, I guess he is right.

As I was running the teaching experiment, I could not find a way to perturb Naya's way of organizing the space. Her representing QMO in NCP was a valid way of representing varying quantities on two-dimensional space. It was not wrong, but uncommon. In this case, Melvin just told Naya that we have to imagine the varying bars on the axes and Naya agreed on Melvin's idea as she previously represented QMO in CCP when the quantities were static (see Figure 0.63). In order to see how she could continue her graphing activity, I asked Naya to complete her graph by sketching the relationship between AoW and DoW for the upper part of the pool. N immediately began her activity by drawing tick marks on the vertical and horizontal axis to show the variation of AoW and DoW, respectively (see Figure 0.66a). Note that she drew the tick marks in a way that the distance between the tick marks were "twice as much" as the distance between the previous tick marks on the vertical axis. She said,

> This line [*sliding the pen over the horizontal axis from left to right*] is depth of water still. This line [*sliding the pen over the vertical axis*] is amount of water still. So, when we get to this point [*sliding the pen over the vertical axis from origin to the third tick mark and pointing to the third tick mark*], the amount of water increases twice as much [*sliding the pen over the vertical axis from the third tick mark to the fourth tick mark and from the fourth tick mark to the fifth tick mark*], so that the space between the tick marks gets twice as large.

Note that she said "AoW increases" at the same time when she moved the pen upward on the vertical axis. This provided an evidence that Naya imagined the variation of AoW on the vertical axis. She then plotted points on the plane at the intersection of the projection of the tick marks on the axes (Figure 0.66b) and connected the point with line segments (Figure 0.66c). Therefore, Naya was able to create a graph to represent the relationship between two quantities (i.e., AoW and DoW in the pool) on CCP. The shape of the graph did not seem to change for the upper part of the pool compared to lower part because Naya accidentally skipped a tick mark for DoW on the horizontal axis as you may see in Figure 0.66c. She said, It [DoW] increased by the same amounts [*pointing to each interval on the horizontal axis from left to right*], spaced between them stayed equal, except for this one [*pointing to the interval between the fourth and sixth tick mark*] for some reason I just messed up a little bit.



(a) (b) (c)

Figure 0.66. (a) Naya's tick marks on the axes for the upper part of the pool, (b) Naya's points on the plane joining two tick marks, and (c) Naya's final draft in SPT.

Summary of the Case of Naya

Representing a NMO (iconic translation) in DAT

Naya's initial meaning of the points included iconic translation as she assimilated each point on the plane as a location/object that appeared on the map. That is, Naya made an association between the perceptual features of the map and the shape of the graph. In particular, she assimilated the point as a location/object by choosing one perceptual property of the map (e.g., spatial orientation, relative location) and ensuring to preserve that property on the plane. For example, Naya determined that the point on the horizontal/vertical axis was the physical Arch/Cannon because the Arch/Cannon was at the bottom/top on the map (see Figure 0.3).

Representing a NMO (transformed iconic translation) in DAT

Since Naya's meaning of the points included iconic translation, she was perturbed by the inconsistency between the spatial features of Starbucks on the map and on the plane (i.e., Starbucks was located northeast of the Arch on the map whereas Starbucks was located northwest of the Arch on the plane). Consequently, she shifted his meanings of points from iconic translation to transformed iconic translation. She still conceived the points as objects; however, she did not translate the perceptual features of the situation *as it is* to the plane; instead, Naya translated a transformed version of the picture of the situation to the plane.

Representing a NMO (transformed iconic translation) in CT

I transitioned to CT where Naya was perturbed by the movement of the black dot on the plane because the crow's movement and the black dot's movement did not perceptually match. For instance, the black dot moved diagonally to the right on the plane as the crow flied vertically to the left on the map (see Figure 0.41). She could not reconcile her perturbations in CT.

Representing QMO in non-canonical Cartesian plane in DABT

In this task, I provided an opportunity for Naya to conceptualize two varying quantities (i.e., the bike's DfA and DfC) in the situation before I asked her to graph the relationship. I found that Naya was able to determine the gross covariational relationship between the bike's DfA and DfC (i.e., DfA and DfC increased and decreased together, and the bike's DfC is "less than" the bike's DfA). In turn, her subsequent graphing activity involved Naya successfully representing the variation of the quantitates on two parallel magnitude lines (see Figure 0.44), and then representing the quantitative relationship on a non-canonical Cartesian plane (see Figure 0.45), rather than engaging in iconic translation as she did in her earlier activities.

Naya's activity in MGT

Since Naya represented the magnitudes on the plane by using the axis as her reference rays, I engaged Naya in Matching Game Task (MGT; Figure 0.46) where I provided her an opportunity to make connection between the parallel magnitude lines and the coordinate system for the purpose of constructing a Cartesian plane. I designed MGT in order to engage Naya to represent two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and projecting the magnitudes on the plane. She could not create a single point on the plane representing the two magnitudes by re-organizing space (except for one case where the bike's DfA and DfC were equal, see Figure 0.48). Then, I decided moved to CT providing her a coordinate system as a given space. My goal was to provide Naya with additional figurative material that might afford her to structure the space in a way that was compatible with a canonical Cartesian plane.

Representing a NMO (transformed iconic translation) in CT

Recall that that Naya's meaning of the black dot included a transformed iconic translation in her initial activity in CT. She conceived the black dot as the crow flying on the plane and perceptually associated the points on the plane to the locations on the map. She also imagined the vertical axis of the plane (what she called the *y*-axis) horizontally at the top of the map and imagined the horizontal axis of the plane (what she called the *x*-axis) horizontally at the bottom of the map. Naya drew on her original meaning of the points in her second engagement in CT. That is, she conceived the dot as the crow and engaged in transformed iconic translation (see Figure 0.51). Then, I provided Naya an opportunity to conceive the quantities in the situation (i.e., the crow's DfA and DfC) and decided to investigate how she could interpret the black point on the plane after she conceived the quantities in the situation.

Representing SQMO in CT

After I draw Naya's attention to the crow's DfA and DfC in the situation, I found that her meaning of the point included representing a SQMO on the plane. That is, she still conceived the black dot as the crow; however, she focused on the crow's quantitative properties (i.e., the crow's DfA and DfC) when making sense of the location of the black dot (i.e., the crow for her) on the plane. In particular, Naya's meanings of the point included determining quantitative features of the crow in the situation (i.e., its DfA and DfC) and ensuring to preserve these quantitative properties on the plane (see Figure 0.53).

Representing QMO in CT and DAT

Since Naya imagined the Arch and Cannon on the axis of the plane, she coordinated the black dot's (the crow for her) DfA and DfC on the plane according to the crow's DfA and DfC on the map. Then, I provided Naya additional figurative material (i.e., tick marks and the varying bars) on each axis of the plane that afforded Naya assimilating the axes of the plane as magnitude lines. In turn, Naya developed a new meaning of a point on the plane as representing a QMO in a canonical Cartesian plane. That is, Naya imagined the quantities' magnitudes on each axis of the plane and created a single point on the plane that simultaneously represented both quantities' magnitudes (Figure 0.54).

To repeat the same task with less figurative material available for Naya to rely on, I transitioned to a version of DAT where I asked her to represent WFB's DfA and DfC on the plane. Naya measured the WFB's DfA and DfC in the situation, dis-embed them from the situation, and re-present them on each axis of the plane, then, plot a point on the plane accordingly (Figure 0.56). This provided an evidence that Naya's meaning of the point included representing QMO in a canonical Cartesian plane.

Representing QMO in SPT

Since Naya was able to use the new space (i.e., Cartesian coordinate plane) to produce a single point that simultaneously represented two static quantities' magnitudes (AoW and DoW in the pool, see Figure 0.63), I decided to ask Naya to create a representation on the plane to show a relationship between AoW and DoW as they varied in tandem. Although she initially represented the relationship in non-canonical Cartesian plane (Figure 0.65), she later successfully created her graph to represent the relationship between AoW and DoW in a canonical Cartesian plane (Figure 0.66).

The Case of Zane

In this section, I present Zane's meanings of points that he developed throughout the teaching experiment. Table 5.3 summarizes all the meanings that he demonstrated in Downtown Athens Task (DAT), Crow Task (CT), and Downtown Athens Bike Task (DABT), and summarizes the way she organized the space in solving these tasks. Zane's initial meanings of graphs included iconic translation (i.e., picture of the situation) and representing the literal motion of the object that moves in the situation. As I implemented the instructional sequence, Zane's meanings shifted to include quantities and their relationships, including how a graph is a record of the simultaneous variation of two quantities' magnitudes. I infer Zane's actions to suggest that the experience of making parallel lines orthogonal was significant moment for him because, after this moment, he did not engage in iconic translation and/or coordinating spatial distances from the axes of the plane. Furthermore, representing quantities on the magnitude lines played a significant role in his development as he subsequently and frequently referred to the variation of quantities' magnitudes represented *on the axes* when explaining how his graphs showed certain covariational relationship on the plane.

Table 5.3

Zane's meanings of the points and his organization of the space throughout the teaching experiment.

Tasks	Meanings of the points	Organization of the Space
Downtown Athens Task	Non-Multiplicative Object (transformed iconic translation)	Imagining Arch and Cannon on the axis
Crow Task		
Downtown Athens Bike Task (1st draft)		
	Measurement Activity	
Downtown Athens Bike Task (2nd draft)	Non-Multiplicative Object (transformed iconic translation)	Imagining Arch and Cannon on the axis
	Matching Game Task	
Crow Task	Spatial-Quantitative Multiplicative Object	Imagining Arch and Cannon as the axis itself
	Intervention	
Crow Task	Quantitative Multiplicative Object	Cartesian coordinate plane
Downtown Athens Bike Task (final draft)		
Swimming Pool Task		

Zane's Activity in Downtown Athens Task

As an illustration of Zane's initial meaning of points, I present his activity in Downtown Athens Task (DAT). DAT includes a map with seven locations highlighted and labeled (see Figure 0.67) and a Cartesian plane whose horizontal axis is labeled as Distance from Cannon (DfC) and vertical axis is labeled as Distance from Arch (DfA). Seven points are plotted without labelling in the coordinate system to represent the seven locations' DfA and DfC (see Figure 0.67). I asked Zane what each of these points on the coordinate system might represent with an intention to observe his spontaneous responses and to explore his meanings of points. Zane's meanings of the points on the plane included the physical objects that appeared on the map.

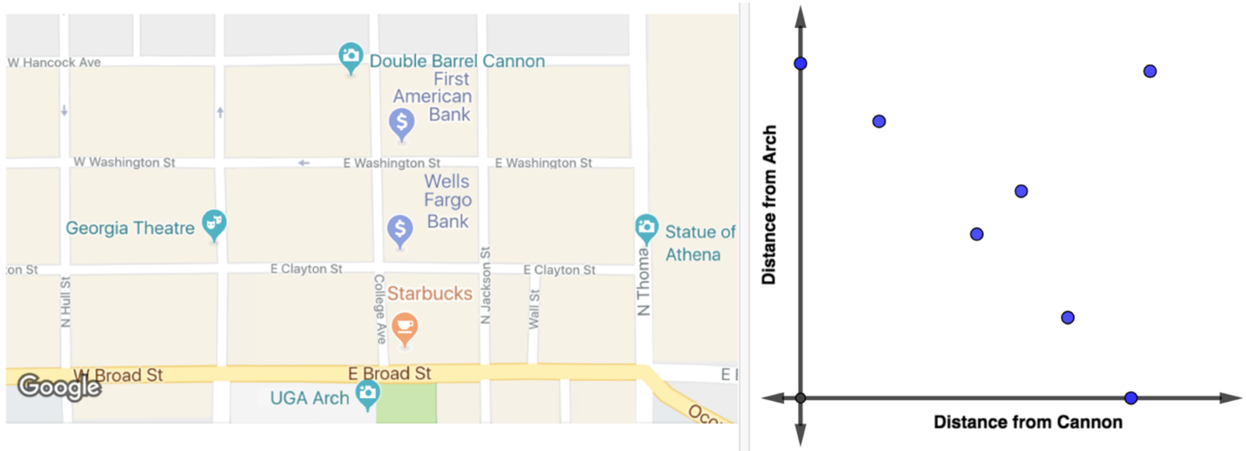


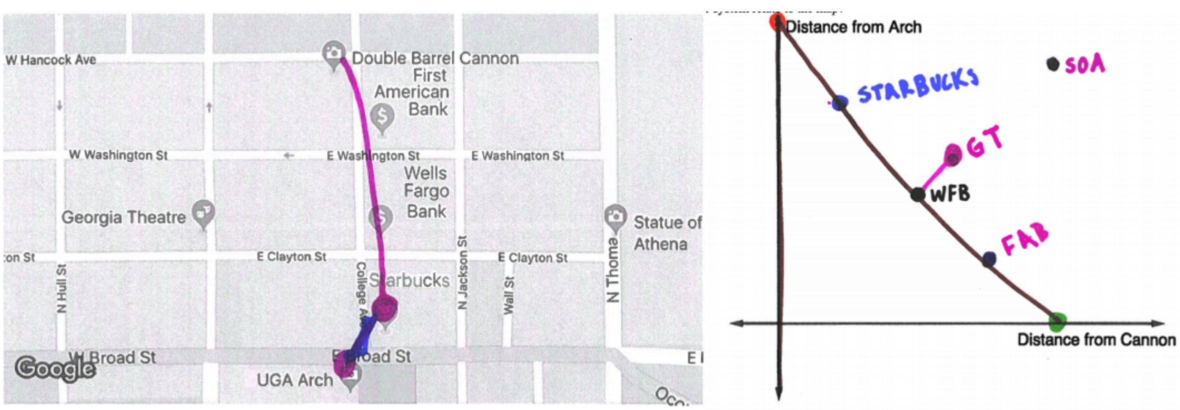
Figure 0.67. Downtown Athens Task (DAT)

Zane conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively, as implied by the labels (see orange and green dots on each axis for Arch and Cannon, respectively in Figure 0.6b). For Zane, the label "Distance from Arch" indicated where Arch was on the vertical axis. Similarly, the label "Distance from Cannon" indicated where the Cannon was on the horizontal axis for Zane. Then, he said, "I flipped the map [*pointing to the map in Figure 0.6*] because Arch is on this side [*pointing to the bottom of the map*] and Double Barrel Cannon is on the side [*pointing to the top of the map*]." Zane imagined rotating the map in way that Arch was at the top and Cannon was at the bottom of the map since Arch was at the top and Cannon was at the bottom of the plane. Zane then drew a line on the plane that went through the co-linear points on the plane starting from Arch on the vertical axis to the Cannon on the horizontal axis (Figure 0.6a). He then drew a kind line on the map that went through the locations on the plane starting from Arch to the Cannon (Figure 0.6b). Zane then made a perceptual association between the "line" that he drew on the map and the line he drew on the plane. He said,

> Like, this right here [*placing the tip of the pen on the orange point in Figure 0.6b*] from Arch, and this [*moved the pen and put it on the blue point*] probably Starbucks, this [*moved the pen and put it on the black point labeled WFB in Figure 0.6b*] would be like

> this right here [*pointing to Wells Fargo Bank on the map in Figure 0.6a*], and [*moved the pen and put it on the purple point labeled FAB in Figure 0.6b*] the First American Bank right here [*pointing to First American Bank on the map*] would be here, to get here [*moved the pen and put it on the green point on the horizontal axis*] and it would be a straight line [*moving the pen along the line that he drew from orange point to green point on the plane*].

From this activity, I inferred that Zane's meaning included transformed iconic translation since he rotated the map and overlaid it into the plane in order to match the line that he drew on the map with the line he drew on the plane. With this perceptual association, Zane was able to match each of the co-linear points on the plane with a location on the map.



(a) (b)

Figure 0.68. (a) Zane's "line" on the map and (b) Zane's "line" on the plane.

Zane's Activity in Crow Task

Recall that Zane's initial meanings in DAT included transformed iconic translation as he perceptually match each point on the plane with each location on the map. Then, I provided Crow Task (CT; as an extension of DAT, see Figure 0.11) where there was a crow that flied on the map and a corresponding black point that moved on the plane according to the crow's distance from Arch (DfA) and the crow's distance from Cannon (DfC). I asked Zane to move the crow on the map and explore how the corresponding black point moved on the plane. My original goal with this task was to create an environment for Zane if he could experience perturbations. Results showed that the task didn't yield a perturbation for Zane whose meanings of the points included iconic translation.

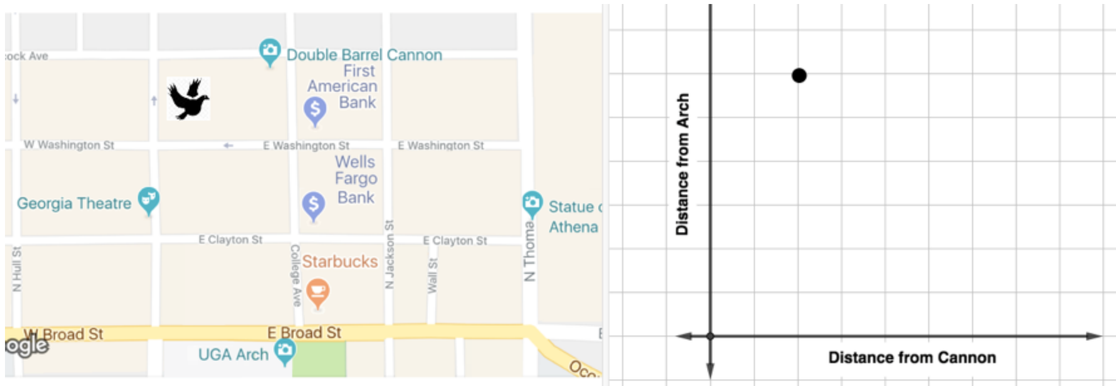


Figure 0.69. Crow Task (CT).

Based on his iconic translation meaning, I conjectured that Zane could assimilate the black dot as the crow. In turn, he could be perturbed by the movement of the black dot because when the crow was on top of Arch on the map, the corresponding black dot on the plane would be on top of the point on the horizontal axis what Zane perceived to be the Cannon (see his graphing activity in DAT in Figure 0.6b). Similarly, the black dot would get to the point on the vertical axis (what he perceived to be the Arch) when the crow was located on top of Cannon on the map.

While Zane was moving the crow on the map to and from the Arch and Cannon, he described the corresponding point on the plane as moving "in a straight line." Then, I moved the crow from Arch to Cannon in an upward direction in the map and asked Zane to explain his observations regarding the moving black dot on the plane. Zane said "[the dot] is going up ... he [*pointing to the crow on the map with the left index finger*] moves this way [*moving his index finger upward*], the dot [*pointing to the corresponding black dot on the plane*] moves that way [*moving his index finger upward*] too." I infer that Zane made an association between the direction of the movement of the crow on the map (i.e., straight and up) and the direction of the movement of the dot on the plane (i.e., straight and up). Zane also moved the crow in order to place it on top of Cannon on the map and, in turn, the corresponding point on the plane moved to and stopped on the vertical axis near label "Distance from Arch." Recall that during the previous task, Zane conceived the point on the vertical axis as the Arch (see orange dot in Figure 0.6b). It is noteworthy that this experience did not seem important to Zane; he was not perturbed by this. Zane engaged in similar experience several times (i.e., putting the crow on top of a place on the map and observing where the corresponding point located on the plane); however, Zane could not assimilate this activity as a way to find what the points on the plane represents in terms of situation. I thought since Zane attended to the perceptual feature of the motion of the corresponding point on the plane when he moved the crow on the map, he did not assimilate the activity as a venue to find what the points represent on the plane.

Zane's Activity in Downtown Athens Bike Task

Note that, in DAT and CT, I did not prompt Zane to conceive the quantities in the situation. I first wanted to get insights into his spontaneous meanings of the point when he was given a situation and a graph together. I investigated how he could initially conceive the points on the plane before I attempt to investigate (and support) how he could conceptualize quantities in the situation before asking him to represent them on the plane. Since I identified that Zane's initial meaning of the points included transformed iconic translation, I planned to take his attention to the quantities in the situation without asking him to engage in any graphing activities. I conjectured that if Zane could conceptualize the measurable attributes of the objects on the map, he might be able to represent the quantities on the plane rather than making iconic translation. In this section, I illustrate his activity in Downtown Athens Bike Task (DABT) where I investigated how he could conceive the quantities in the situation (e.g., the bike's distance from Arch).

DABT included the same map of Downtown Athens highlighting a straight road (i.e., Clayton St.) with two places located near the road (i.e., the Arch and the Canon; see Figure 0.23) and a bike on this road. I animate the map so that the bike moves at a constant speed back and forth along the Clayton St. starting at the West side of the street. I designed this task to explore how Zane could conceive the situation quantitatively and how he could determine the relationship between quantities (the bike's DfA and the bike's DfC). In particular, my purpose in this task was to explore and support Melvin's process of (i) conceiving the quantities' that vary in the situation, (ii) representing the varying quantities by his index fingers on the table, and (iii) representing the relationship between covarying quantities on a coordinate plane.

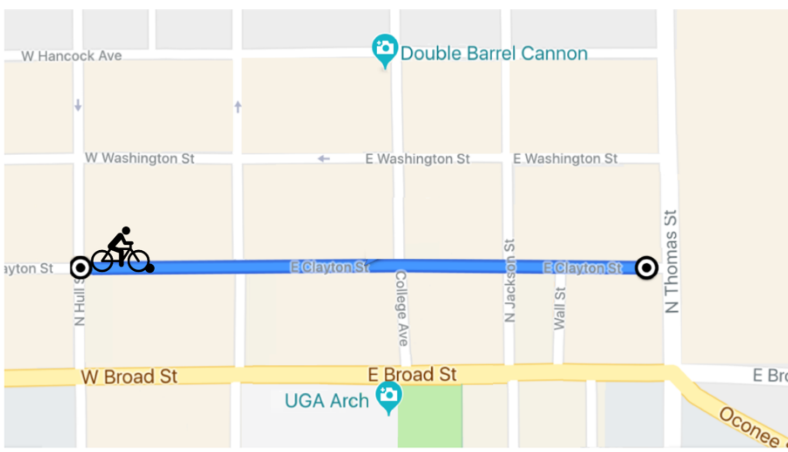


Figure 0.70. Downtown Athens Bike Task

I illustrate that Zane conceived the bike's DfA as the bike's *proximity* to Arch (i.e., its closeness/nearness to the Arch) without conceiving the measurable attribute of the bike and coordinated the variation of the bike's degree of proximity to Arch as the bike traveled on the map (e.g., the bike is getting closer to or farther from the Arch). Next, I illustrate how Zane's spatial proximity reasoning influenced his graphing activity in one- and two-dimensional spaces.

Zane's spatial proximity reasoning in DABT

In the first part of DABT, I showed Zane the animation where the bike traveled on E. Clayton St in Downtown Athens (see Figure 0.23) at a constant speed starting from the west side of the street. I asked him to describe how the bike's DfA changed as the bike traveled on its path on the map. Zane said,

> When it [the bike on the map] is going here [*pointing to the middle of the path*] it is getting closer to Arch. Then, when it is going like here [*pointing to the right end side of the path on the map*], it gets farther away from Arch.

When asked to describe how the bike's DfC and DfA changed as the bike traveled on its path on the map. Zane said,

Zane: I think that like when it goes in [s*liding the pen in the air from the left end side of the path to the middle*], it is getting closer to Double Barrel Cannon and Arch. So, like, when the bike is coming, they are like equal because it is coming closer into both of the places. If you go right here [*pointing to the right side of the path on the map*] it goes out to move from both of the places. So, both places' distance gets further.

TR: Zane, can you show me how do you see bike's distance from Arch on the map?

Zane: Like, when it [i.e., the bike] is right here [*moving the bike and dropping it at the middle of the blue path on the map*], it is closer to the Double Barrel Cannon and it is closer to the Arch. But if you go right here [*moving the bike and dropping it at the right end side of the blue path on the map*], it is farther from Cannon and it is farther from Arch.

TR: Can you show me what is being far? How do you see it is far? Like, you can also draw.

Zane: So, like, right here [*drawing a segment from Cannon to Arch, see Figure 0.24*], the bike goes right here [*pointing to the intersection between the bike's path and the segment he drew*]. And then if it is over here [*circling the left end side of the bike's path*], it is farther [*writing "far" near the left end side of the path*] and [*writing "far" near the right end side of the path*]. So that like, if the bike is right here [*moving the bike to the right end side of the path on the map*] it is far. But, if it is right here [*moving the bike to the intersection of his segment and the bike's* *path on the map*], it is closer to both of them. Its, its distances are closer to, the bike is closer to Double Barrel Cannon and UGA Arch.

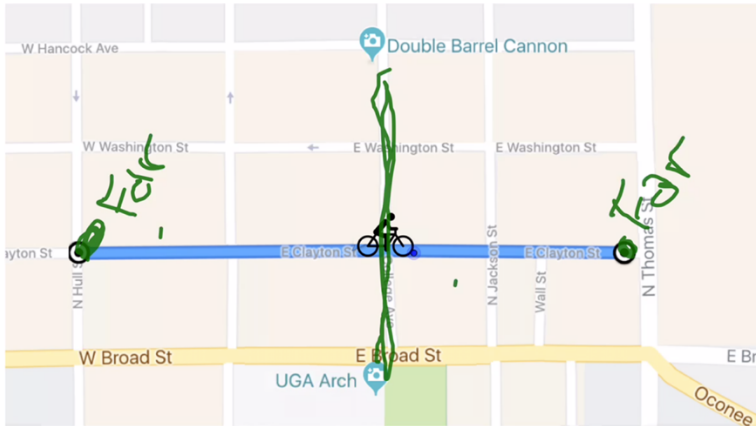


Figure 0.71. Zane's activity in DABT.

His activity suggested that Zane coordinated the bike's proximity to both Arch and Cannon at the same time as he used the linguistic adverbs "closer" and "farther" to describe the bike's DfA and DfC. I did not have any evidence that Zane coordinated the bike's DfA and DfC as the bike's measurable attributes. I prompted him to show me how he imagined the bike's DfA, he continued to describe what he perceived to be the bike's DfA as the bike was getting closer to and farther from Arch. When I explicitly prompted him to show how the bike was far from Arch by drawing on the map, he wrote "Far" on the map where the bike was far from Arch. He also drew a segment (see Figure 0.24); however, I don't have any evidence that he conceived the segment in relation to the magnitude of the bike's DfA or DfC. He still continued to describe the bike's DfA as the bike was getting closer to or farther from Arch.

Zane's representation of spatial proximity reasoning in DABT

In order get more insights into Zane's conception and representation of the bike's DfA and DfC, I provided him with a dynamic tool that could afford Zane's engagement with quantities' magnitudes represented by directed bars placed on magnitude lines (Figure 0.26). The directed bars can be varied in length as the bike moves on the map (see https://youtu.be/6kdbDeVEF9w). I conjectured that this representation could provide an opportunity for Zane to represent the quantities' magnitudes on a magnitude line in one dimension before moving into two-dimensional space.

Recall that Zane attended to the variation of bike's proximity to the Arch when engaging in the situation (i.e., "the bike is getting closer to Arch"). Similarly, when engaging with the magnitude line, Zane imagined the physical bike and Arch in place of the right and left end side of the blue bar, respectively. For example, as the bar's length got smaller on the magnitude line from my perspective, Zane assimilated it as the bike was getting closer to Arch.

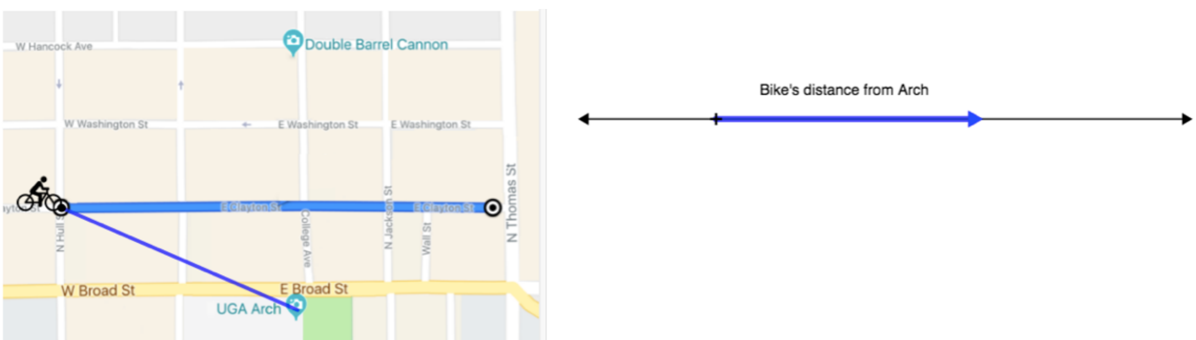


Figure 0.72. Representing the bike's distance from Arch both on the map (left) and on the magnitude line (right).

While moving the bike to the right from its position seen in Figure 0.26, I drew Zane's attention to the fact that the right end side of blue bar on the magnitude line was moving to the left (indicating the bike's DfA was decreasing). Then, I asked Zane to explain why that was happening. Referring to the map, Zane claimed, "the bike is getting closer to Arch," which suggested that Zane's image of variation included coordinating the variation of the bike's proximity to Arch. Then, by pointing to the blue bar on the magnitude line and tracing the pen over the line from right to left, he added it "is gonna get closer to right here [*pointing to the zero point on the magnitude line*], which is Arch." I infer that Zane conceived Arch and bike as physically placed on the left and right end side of the blue bar, respectively, and he thus perceived blue bar's change to be compatible with his spatial proximity reasoning because the bike moved closer to the Arch *along* the line, with Arch playing the role of reference object. Zane abstracted the two objects (Arch and bike) and their proximity when engaging in the situation. Thus, he imagined these physical objects in his representational activity too.

Differing from Zane, Ella (the other student in the teaching experiment) determined that the bike's DfA is decreasing while moving the bike to the right in the map. She explained "it [*pointing to the blue bar*] is gonna get smaller because distance is smaller on the number line too." Moreover, Ella labeled the starting point as "zero", whereas Zane conceived the same point on the magnitude line as "Arch." From this activity, I infer that Ella conceived the length of the blue bar on the map and on the magnitude line as a representation of the bike's DfA. I conjecture that Ella's activity is an implication of her quantitative variational reasoning (i.e., the bike's DfA is decreasing) whereas Zane's activity is an implication of his spatial proximity reasoning (i.e., the bike is getting closer to Arch).

In order to understand how Zane understood Ella's response and focus on "distance", I asked Zane to re-voice Ella. Zane's explanation included his original reasoning (i.e., imagining the Arch and the bike physically on the line). I then drew the numeral 0 (zero) near the origin on the number line and asked Zane how he could make sense of the number zero on the number line. The following excerpt below shows how Zane responded.

TR: Why do you think Ella put zero there [*pointing to the numeral 0 on the magnitude line*]? You are saying that Arch is there [*pointing to the zero point on the number line*]. How you are going to make sense of it? Can you make sense of it?

Zane: Do you see where the bike is right here [*pointing to the right end side of the blue bar on the magnitude line*] as I move it [*moving the bike on the map from the right end side of the path to the middle*], if it got close right here [*dropping the bike at the middle of the path on the map*], it is gonna get closer to this point [*pointing to the zero point on the magnitude line*].

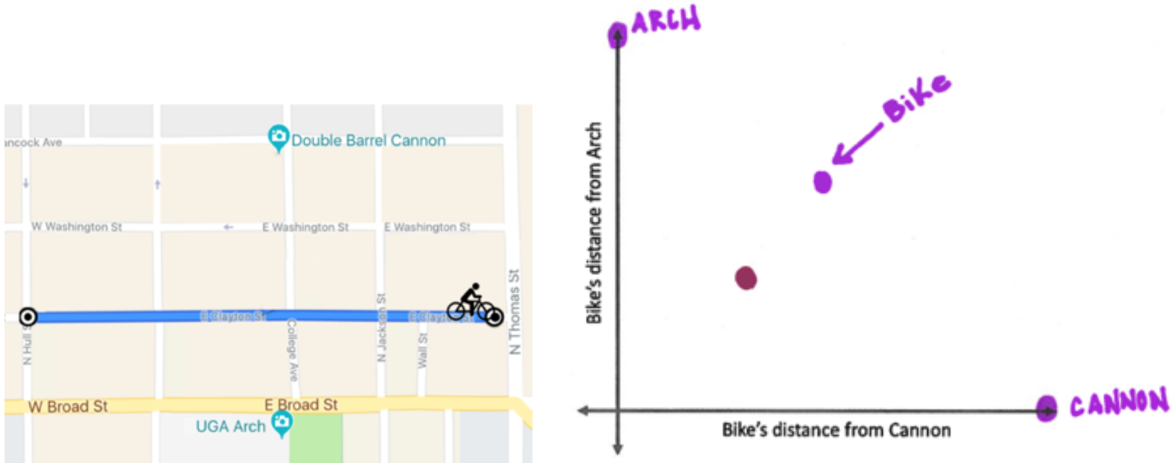
TR: How do you know the bike is getting closer to here [*pointing to the zero point on the magnitude line*]?

Zane: [inaudible]… So, every time I move it away [*moving the bike to the right on the map*], it is gonna go farther away [*pointing to the blue bar on the number line*]. But, if I move it towards [*moving the bike to the left until the bike gets to the middle of the path*], it is gonna get closer [*pointing to the bar on the magnitude line and sliding the pen to the left over the magnitude line*].

Zane assimilated Ella's answer to his meaning because they were compatible with respect to the behavior of figurative material. Since Zane's meaning included spatial proximity (i.e., nearness) between two places, it makes sense to imagine Arch in place of the origin in one-dimension because the value of the bike's DfA is zero at that point. However, their meanings and reasoning have different implications when we move to two-dimensional systems. In the next section, I illustrate how Zane's spatial proximity reasoning influenced his graphing activity in two-dimensional space.

Zane's transformed iconic translation in DABT (first draft)

In order to investigate Zane's graphing activity in two-dimensional space, I asked Zane to graph the relationship between the bike's DfA and DfC on a paper with a coordinate plane. I wondered if and how Zane could use his image of the magnitude lines in his graphing activity on the plane. You can see Zane's graph in Figure 0.27b. I illustrate that Zane's meaning of the points included transformed iconic translation as he imagined the physical Arch, Cannon, and the bike to the plane.



(b) (b)

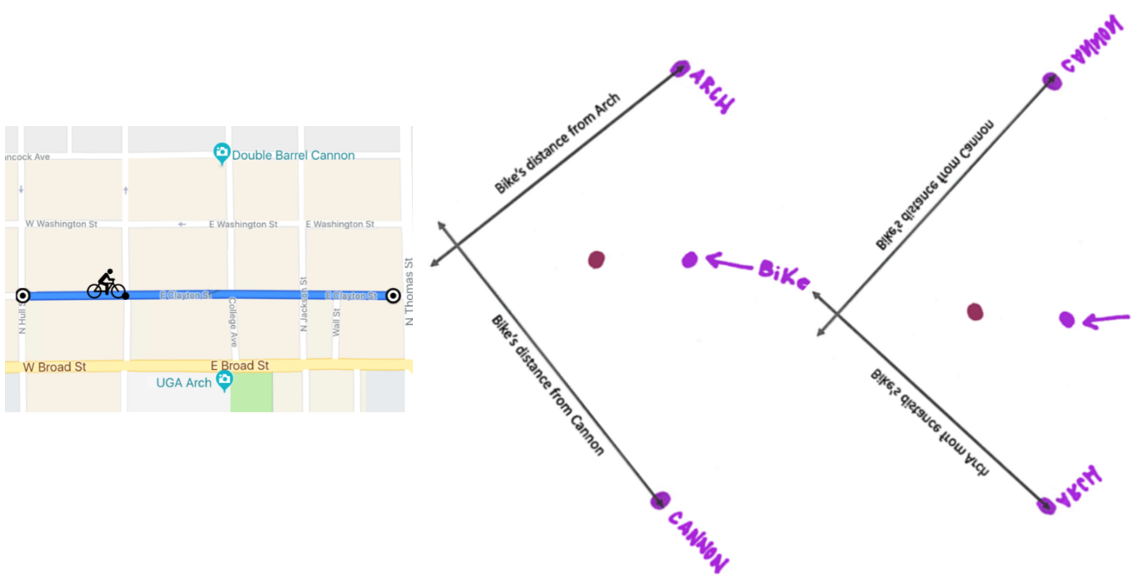
Figure 0.73. (a) The location of the bike on the map and (b) Zane's first draft in DABT.

I asked Zane to sketch a graph to represent the bike's DfC and DfA on a paper with a coordinate plane. By the time I asked this question, the picture of the situation on the tablet screen was visually available to him and the bike was located at the right end side of the path on the map (see Figure 0.27a) and the animation was paused. Zane began his graphing activity by plotting a point on the horizontal and vertical axis to indicate the physical Cannon and Arch (see Figure 0.27b). Then, he added another point and arrow with a label "Bike" on the plane. It was clear that Zane imagined the bike on the plane; however, I did not know if he was making a transformed iconic translation (as he did in DAT, see Figure 0.6) or if he was coordinating the bike's proximity to Arch and Cannon on the plane (or potentially both). In order to understand his reasoning, I changed the bike's location to the left side of the path (see Figure 0.74a) and asked Zane how his graph could show the bike's DfA and DfC when the bike was there. Zane added another point on the plane (see Figure 0.27b). When asked to explain why, he said,

> Because I do like this [*rotating the paper clockwise until Arch and Cannon were vertically aligned and the two plotted points were horizontally aligned, see Figure 0.74b*] so like, the bike [*pointing to the point he added later on the plane, the point without label in Figure 0.74b*], right here [*pointing to the bike on the map, see Figure 0.74a*]. Arch [*pointing to the Arch on the axis of the plane, see Figure 0.74b, then, he pointed to the*

Cannon on the map and paused for 3 seconds], kind of like, if we flip it like this [*flipping the rotated paper vertically, see Figure 0.74c*], Arch is right here [*pointing to Arch on the axis of the plane, see Figure 0.74c*]. This is the bike [*pointing to the non-labeled point on the plane, see Figure 0.74c*] and Cannon [*pointing to Cannon on the axis of the plane, see Figure 0.74c*].

Zane assimilated the point as the bike by making perceptual association between the map and the rotated and flipped version of the plane. He first rotated the paper with his graph clockwise in order to match with perceptual feature of the map. He rotated the paper clockwise until Arch and Cannon were vertically aligned and the two plotted points were horizontally aligned just like they were on the map (Figure 0.74). Then, he momentarily experienced a perturbation as he noticed the location of Arch and Cannon on the plane didn't match with the location of Arch and Cannon on the map. Once he viewed Arch was at the top on the rotated plane whereas Arch was at the bottom of the map, he decided to flip the paper vertically so that Arch was at the bottom on the plane (see Figure 0.74c). This provided evidence that Zane's meaning of the points included transformed iconic translation.



(a) (b) (c)

Figure 0.74. (a) The location of the bike on the map, (b) Naya's rotated graph, (c) Naya's flipped graph.

Measurement Activity

Recall that Zane imagined the physical bike, Arch, and Cannon on the plane by engaging in transformed iconic translation. Also recall that Zane imagined the bike and Arch as physically placed on the left and right end side of the blue bar, respectively, on the magnitude line. In turn, he perceived blue bar's change to be compatible with his spatial proximity reasoning as the bike moved closer to or farther from the Arch *along* the line. Thus, I decided to provide Zane with the following measurement activity to support him to conceive the quantities' magnitudes in the situation and promote an understanding of the bar on the magnitude line in relation to the quantities' magnitudes. I hypothesized that this activity could help Zane to conceive the length of the bars as a representation of the quantities' magnitudes, rather than imagining the bike and Arch on the magnitude line and coordinating the spatial proximity between them. In this section, I present the features of the intervention along with my goals and Zane's activity in the measurement task.

I presented Zane with the same map of Downtown Athens highlighting Clayton St. and a bike on this street together with the Arch and the Canon (see Figure 0.75, left, orange and green segments were not available at the moment). I also presented him with a directed bar located in a *random* place on the magnitude line (see Figure 0.75, right). The blue bar on the map and the blue bar on the magnitude line are not synced at the moment. That is, the blue bar's length on the magnitude line did not vary according to the bike's DfA on the map as the animation played. The blue bar's length on the magnitude line could be changed by dragging the right end side of the bar to the left or right. I asked Zane to change the length of the blue bar in a way that it has the same length as the bike's DfA when the bike was located at the left end of the path on the map (Figure 0.75, left). The blue bar in the situation and the blue bar on the magnitude line were presented in different screens in GeoGebra so that they could not be physically moved for direct comparison. Another reason why the situation and the bar were in different screens was about necessitating the measurement process.

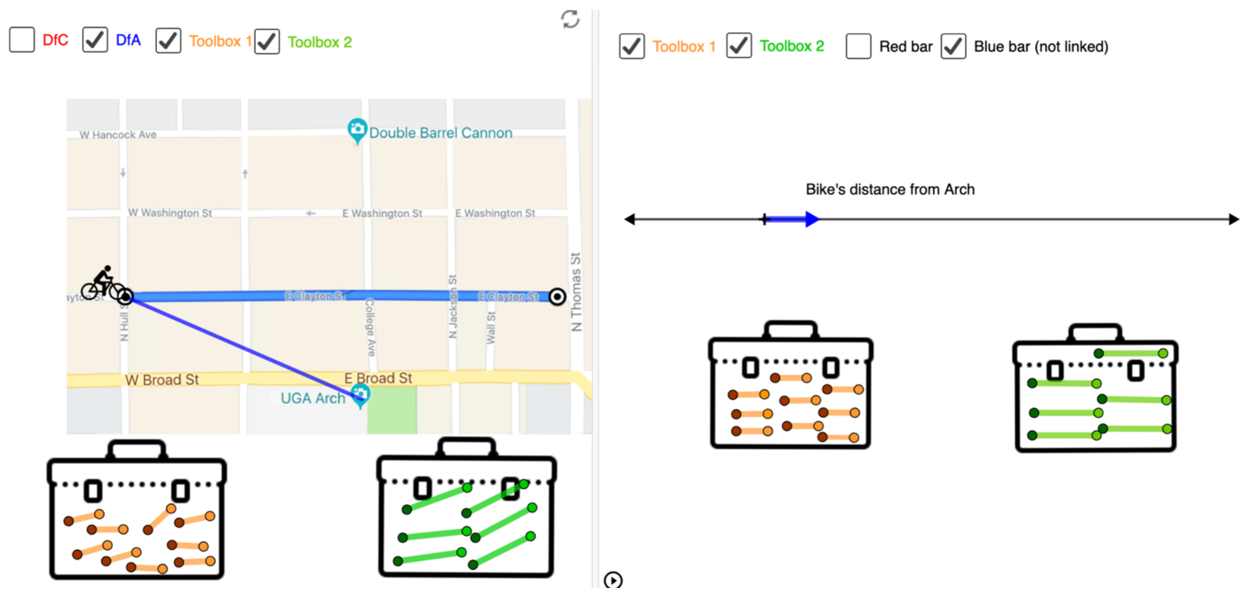
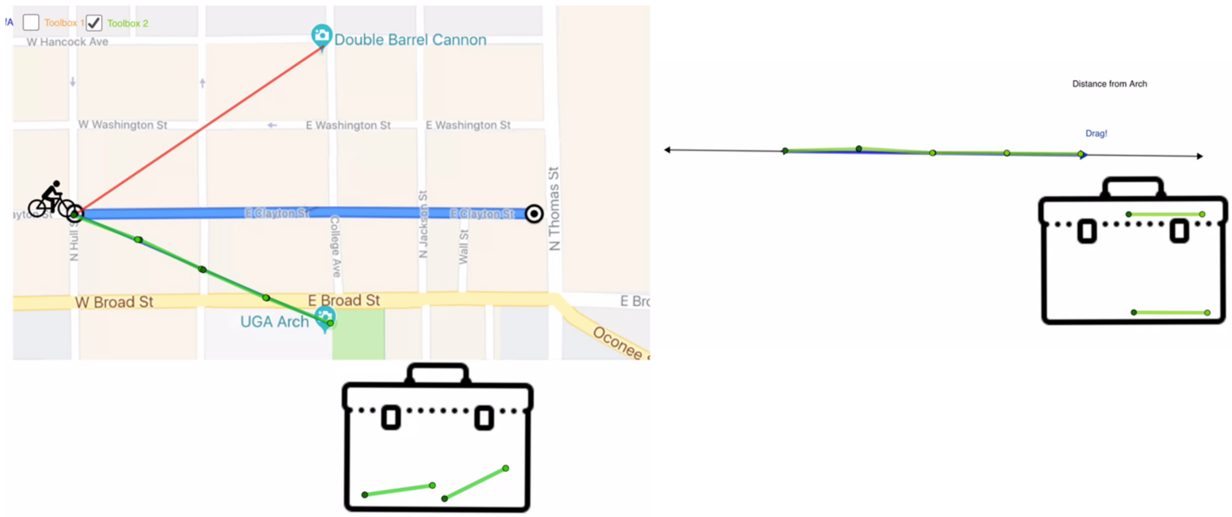


Figure 0.75. Measurement Activity.

I provided tools for Zane to be used in measuring the magnitude of the bike's DfA and representing it on the magnitude line exactly. I conjectured Zane would understand the idea of measurement and engage the concept of unit. I provided Melvin with the green and orange segments of a certain size (see Figure 0.75). I asked Melvin if and how he could use these segments in figuring out the exact length of the bar on the magnitude line to represent the magnitude of the bike's DfA when the bike is located in the left side of the road.



(a) (b)

Figure 0.76. (a) Zane measuring the bike's DfA on the map and (b) Zane measuring the bike's DfA on the magnitude line

By dragging and rotating the green segments, Zane first placed the segments on the blue bar on the map starting from the bike to the Arch (Figure 0.76a). He said, "I got four pieces to put together a straight line that matches the distance, the bike's distance from Arch. So, it is kind of like four little green lines replacing the bike's distance from Arch." Then, Zane placed the green segments on the magnitude line starting from the zero point. He said "I just did the same thing for what I did on the map. I just got four of the same green pieces to make up the bike's distance from Arch." Then, he increased the length of the blue bar until the right end side of the blue bar matched with the right end side of the last green piece on the magnitude line (see Figure 0.76b). When asked to explain how he knew that the blue bar's length on the magnitude line was exactly the same as the bike's distance from Arch on the map. He said, "because, I used four green pieces for that one [*pointing to the map*], and I used four green pieces for that one [*pointing to the magnitude line*], and they have the same green pieces, which make up the same length." From Zane's language and actions, I infer Zane was able to assimilate the length of the blue bar both in situation and on the magnitude line in relation to the bike's measurable attribute (i.e., the bike's DfA). This suggested that Zane developed a meaning for the bar as a representation of a magnitude not just a "shape" that stood between the physical Ach and Cannon on the magnitude.

Figure 0.77. Measuring the bike's DfA by using orange segments.

Then, I showed the other toolbox that had another measuring instrument (see orange segments in Figure 0.75). My goal was to bring a different unit of measure and measure the same quantity. I planned to use the "different" results of Zane's measurement process asking him what was the same and what was the different. I conjectured that Zane would be leveraged to develop a meaning for the quantity's magnitude being invariant with any change in unit when engaging in this task. Before I let Zane to use the orange segments in measuring, I asked Zane if he had any anticipation about the result if he were to measure the bike's DfA by using the orange segments. He said "I think it is gonna be more than four... Because they are shorter pieces, but the distance stays the same." Then, I let Zane to compare the length of the orange segment and the length of green segment. Zane placed the segments next to each other and figured that the length of the green segment was twice as much as the length of the orange segment (see Figure 0.77, left). After this comparison, he immediately figured that the bike's DfA is 8 orange pieces long. He said, "I put two orange pieces together to look at the length of the green one. And it makes up two pieces to make up the size of the green one. So, two times four equals eight." Then, Zane measured the blue bar on the magnitude line by using the orange segments (see Figure 0.77, right) and wrote 4 green pieces equals 8 orange pieces (see Figure 0.77, left). I asked him to explain how come he ended up with having different numbers for the same quantity, Zane responded "because the distance doesn't change, but the pieces got smaller." This provided an evidence that Zane conceived of the bar as a representation of quantity's magnitude that admitted a measurement process and conceived of quantity's magnitude as invariant with any change in unit.

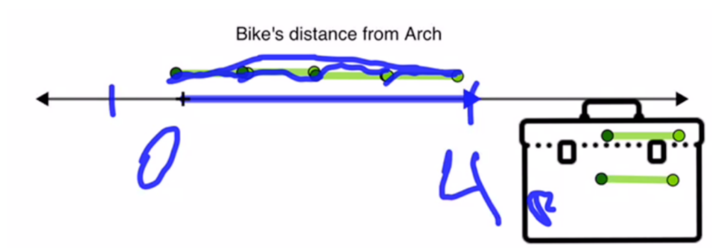


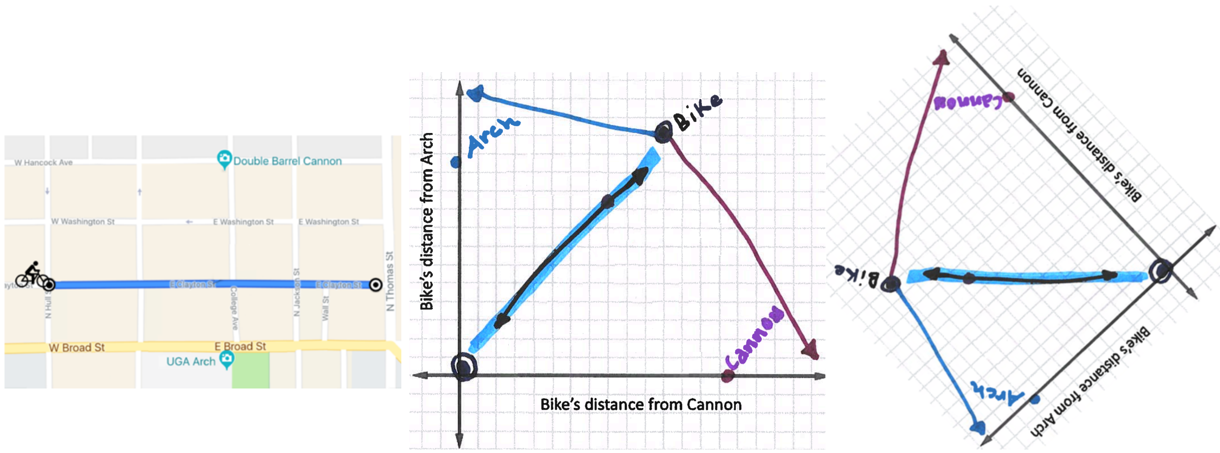
Figure 0.78. Zane putting numbers on the magnitude line.

Recall that Zane imagined the physical bike and Arch on the magnitude line previously. After his measurement activity, I provided an opportunity for Zane to insert numbers on the magnitude line to record his measurement process. I hypothesized that this could allow Zane to reflect on his new understanding for the magnitude line imagining the quantities' magnitudes and values without imagining the physical objects on the line. I asked him "if you were asked to insert a number on this number line, which number would you put there [*pointing to the right end side of the blue bar*] on this number line?" Zane said "four" because "one, two, three, four [*pointing to each segment near the magnitude line in Figure 0.78*] makes up this whole line right here [*moving the pen over the blue bar on the magnitude line from left to right*].” Zane’s activity suggested that Zane no longer imagined the physical bike at the right end side of the blue bar on the magnitude line as he successfully engaged in the measurement activity and represented the result of his measurement (quantity’s magnitude *and* value) on the magnitude line.

Zane’s Second Draft in DABT

After the measurement activity, I asked Zane to sketch his second draft to show the relationship between the bike’s DfA and DfC. I wanted to see how his graphical meanings could change given his engagement in the measurement activity in the previous task. Recall that, in his first draft, Zane’s meaning of the points included transformed iconic translation. In his second draft, Zane continued to engage in transformed iconic translation.

Zane began his graphing activity by plotting a point on the point that he labeled “Bike.” This suggested he continued to conceive the point as an object from the situation. When Zane was plotting his point on the plane, he was looking at the tablet screen where the bike was located at the left end side of the path (see Figure 0.79a). Then, he drew two segments with an arrow on the plane: one was from the bike to the end of the vertical axis where he imagined Arch (Figure 0.79b) and the other was from the bike to the end of the horizontal axis where he imagined Cannon. He then plotted another point at the origin and connected two points with a light blue segment on the plane.



(a) (b) (c)

Figure 0.79. (a) The location of the bike in DABT, (b) Zane's second draft in DABT, and (c) Zane's rotated graph.

Note that his dots on the plane resembled with the dots at each end side of the bike's path on the map (i.e., an open circle and a dot on the center). Moreover, the color for the segment on the map (i.e., light blue) was similar to the color of the path on the map. Based on these perceptual similarities, I hypothesized Zane was engaging in transformed iconic translation where he imagined the bike with its path on the plane together with Arch and Canon. I thought that, just like he did in his first draft in DABT, Zane imagined rotating and flipping the plane in a way that Arch and Cannon was at the bottom and top on the plane, respectively; so that the bike's position being on the left side of the path on map would perfectly match with the position of the bike on the plane after the rotation and flip. In order to test my hypothesis, I asked Zane to explain how he sketched his graph. He said,

> [*Rotating the paper in a way that the light blue segment became horizontal and the dot labeled "Bike" was located at the left side of the graph, see Figure 0.79c* and *Figure 0.80*] because these two points right here [*pointing to both end side of the segment on the plane with two fingers*], these are representing these over here [*pointing to the both end side of the bike's path on the plane*]. So, bike is right here [*pointing to the dot labeled "Bike" at the left end side of his segment on the plane*], and this [*pointing to the other dot at the origin of the plane*] is the other point which is right there [*pointing to the right end*

side of the bike's path on the map]. And Arch is down here [*pointing to Arch on the plane, see Figure 0.80*] and Cannon is up here [*pointing to Cannon on the plane*]. And this [*sliding his index finger over the segment on his graph on the plane left to right*] is the line it goes back and forth.

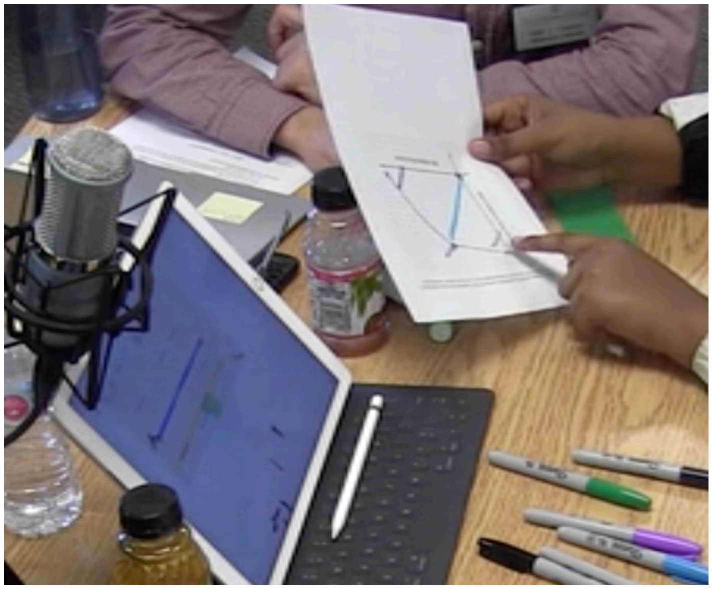


Figure 0.80. Zane pointing to Arch on his rotated graph

After rotating the paper, I infer Zane's image of the map and his image of the graph become identical. That is, after the rotation, Zane assimilated his graph on the plane as literally the map itself. His activity provided evidence that his meaning of the points included transformed iconic translation.

Note that I first hypothesized Zane would rotate the page and *then* flip it because that was what he did before in his first draft in DABT (see Figure 0.74c). Now, I found that Zane only rotated the paper (without flipping) until Arch was at the bottom and Cannon was at the top of the plane; so that bike on the plane would perfectly match with the bike on the map being located on the left end side of its path. In his first draft, Zane needed to rotate and flip because the bike on the map was located on the *right* end side of the map (see Figure 0.27a). Here, in his second draft, he only rotated the paper because the bike on the map was located on the *left* end side of the map. In both activities, Zane's goal was to find a way to perceptually match the location of the bike on the map (wherever the bike was on the map) and the location of the bike on the plane. This may suggest that his way of transformation was not a stable action across different cases. In the next activity, I provided an opportunity for Zane to develop a meaning of a point in terms of representing a quantitative multiplicative object (QMO).

Zane's Activity in Matching Game Task

Recall that, in the measuring activity, Zane successfully measured the bike's DfA in the situation and represented it on the magnitude line. His measurement activity did not seem to influence his graphing activity in DABT because I conjectured Zane did not assimilate the axes of the coordinate plane as orthogonal magnitude lines. His meaning in DABT included transformed iconic translation. I hypothesize that if I provide Zane an opportunity to conceive what I perceive to be the axes of the coordinate plane in relation to the magnitude lines, Zane could represent the quantities magnitudes on the axes (and potentially create a point that represent two quantities on the plane). Thus, I decided to transition to Matching Game Task (MGT, see Figure 0.81) where I provided Zane an opportunity to make connection between the parallel magnitude lines and the coordinate plane for the purpose of constructing a Cartesian coordinate plane. I designed MGT in order to engage Zane to (i) represent two quantities' magnitudes on two separate parallel magnitude lines and then transition to (ii) representing two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and projecting the magnitudes on the plane. I conjectured that Zane's engagement with MGT could promote an understanding of points as an abstract object in relation to representing two quantities' magnitudes (QMO) rather than representing a physical object that moves on the plane.

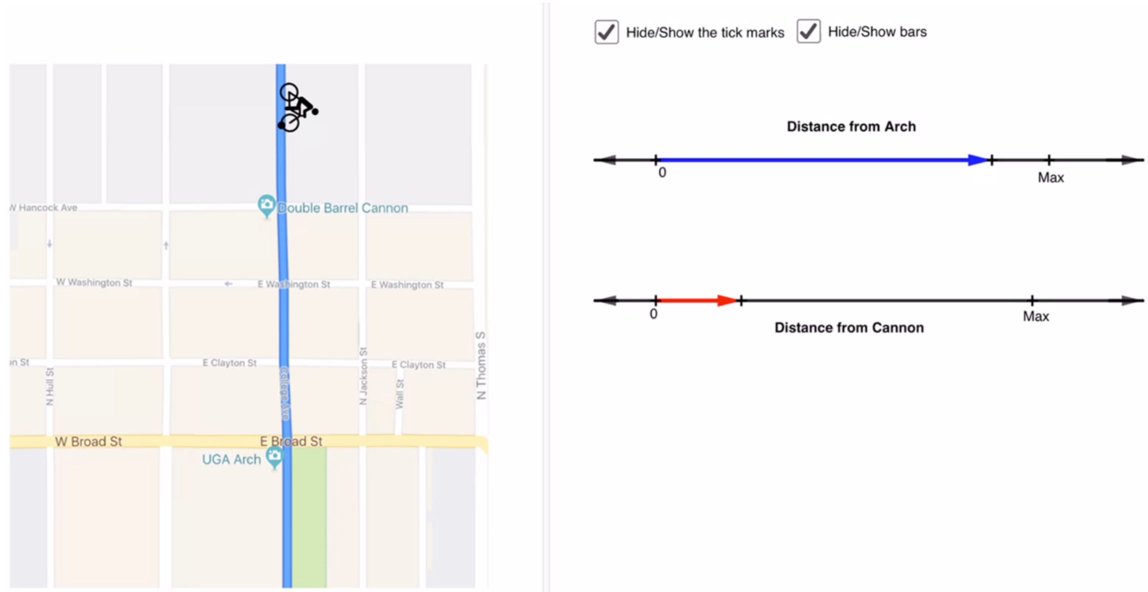


Figure 0.81. Matching Game Task (MGT)

MGT included the same map highlighting a straight road (i.e., College Avenue). Arch and Cannon are located on the road, and a bike rides along this road (Figure 0.81, left). MGT also included a dynamic tool that afforded Zane's engagement with quantities' magnitudes represented by directed bars placed on magnitude lines, Figure 0.81, right). The directed bars can be varied in length (see https://youtu.be/knrN7XCbyOY). In this task, the bars and the bike in the situation are not synced, and I asked Zane to adjust the length of the bars on the magnitude lines to match with various bike positions in the map. I hid the red and blue bars on the magnitude lines and asked Zane to place tick marks as a way to *record* those lengths on the magnitude lines before I change the bike's location to another state. Here, my purpose was to help Zane to connect the conventional use of tick marks on the number lines to the use of bars on the magnitude lines. In this way, he might be able to know that a tick mark (or a point) on a magnitude line represents the measurement of a quantity's magnitude.

I located the bike at the top area of the map (see Figure 0.82, left) and Zane successfully placed the tick marks on each magnitude line accordingly (see Figure 0.82, left). When asked to explain what the tick mark on the magnitude line represented, he said "the bike's distance from Arch." When I asked to explain how he knew that represented the bike's DfA, Zane drew a vertical segment on the map between the bike and Arch and a corresponding segment on the magnitude line between the zero point and his tick mark. This suggested that Zane's meaning of the tick marks was connected to the meaning of the bars on the magnitude line that represented the quantities' magnitudes. I conjectured that this meaning of tick marks might help Zane to *record* the variation of the bar on the magnitude line since the varying bars do not leave trace or a mark as they move on the magnitude lines. In turn, I conjecture that this activity might help Zane when we move to the two-dimensional space to record the relationship between two varying quantities on each axis of the plane.

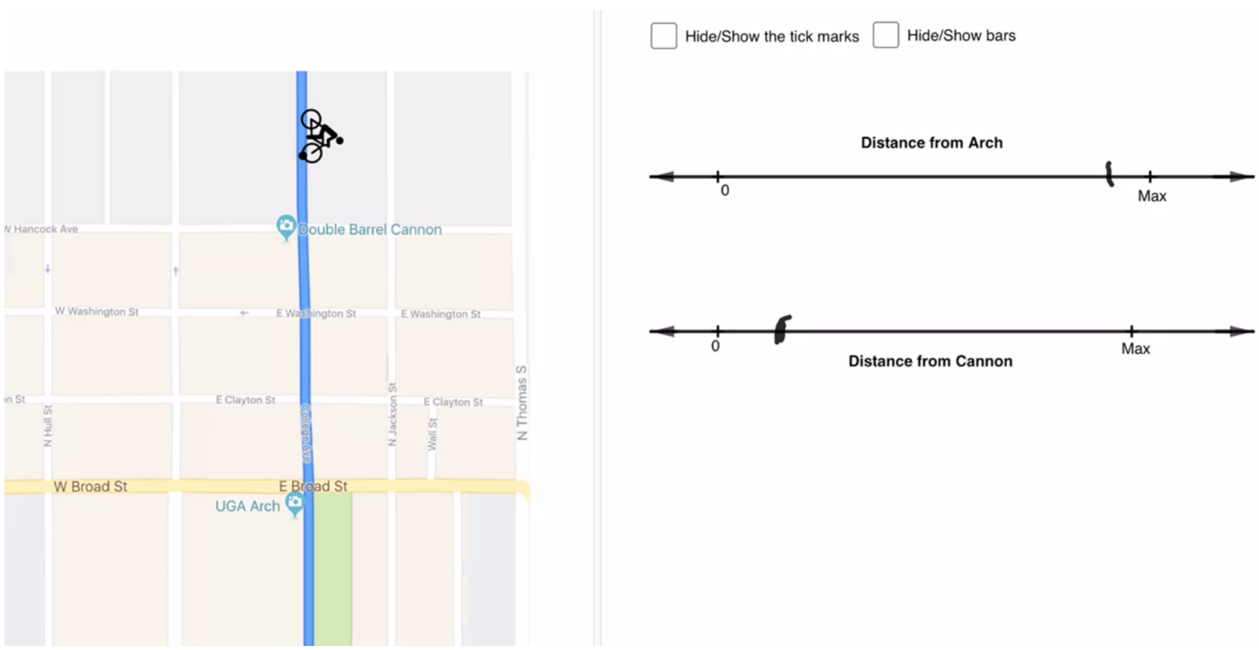


Figure 0.82. Zane's tick marks on the magnitude lines representing the bike's DfA and DfC

With Zane doing so successfully and indicating that he conceived of the magnitude bars as quantities representing the two distances from the situation, I decided to transition to the next part of MGT where I asked Zane if he knew a way to represent the bike's DfA and DfC by *one* *point* instead of the two tick marks that he placed on two parallel magnitude lines. I also informed him that he can move and rotate those lines freely on the tablet screen. With this activity, I expected to gain insights into Zane's reasoning when trying to come up with ways to create a single point that represents two quantities' magnitudes.

Zane expressed that he was unable to create such a point, although he was able to make it for one case where DfA and DfC had the same length (see his blue dot in Figure 0.83). Zane identified one single point and claimed that this point showed both the bike's DfA and DfC. From his activity, I infer that putting the magnitude lines right next to each other and matching the two tick marks was a way for him to show that the single point simultaneously represented a state where the bike's DfA and DfC were equal. He could not find a way to create a single point that simultaneously represented other states of the bike's DfA and DfC. He needed to construct a new space that was different than the one-dimensional space in a way that he could satisfy the simultaneity for all states of the bike's DfA and DfC. This was a need, from my perspective, why we need a two-dimensional space.

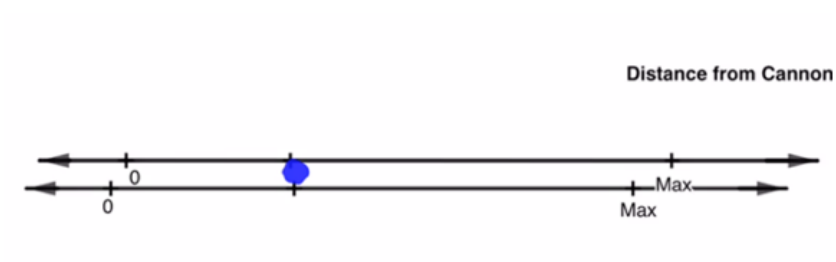


Figure 0.83. Zane's activity seeking to create a single point

This suggested that I should provide other opportunities for Zane that could afford him to structure the space in a way that he could see additional features that is not available in one-dimensional space. Thus, in order to support Zane developing a meaning for a point that simultaneously represents two quantities' magnitudes, I decided to move to CT (see Figure 0.84) where I provided him a coordinate system as a given space. My goal was to provide Zane with additional figurative material that might afford him to structure the space in a way that is compatible with Cartesian plane.

Zane's Spatial-Quantitative Multiplicative Object in CT

CT included the same map in addition to a movable crow (Figure 0.84, left). I also presented a Cartesian plane with the horizontal axis labeled as "Distance from Cannon" and vertical axis labeled as "Distance from Arch" (Figure 0.84, right). Recall that, in his initial activity in CT, Melvin's meaning of the black dot included a crow moving on the plane. Zane also made a perceptual association between the direction of the movement of the crow on the map (i.e., straight and up) and the direction of the movement of the black dot on the plane (i.e., straight and up). Also recall that Zane imagined Arch and Cannon as a location on the vertical and horizontal axis in his earlier graphing activities throughout the teaching experiment (e.g., in DAT, CT, and DABT). Because of his initial meanings in CT, I did not expect Zane to make an immediate connection between the magnitude lines and the axes of the coordinate plane on his own. However, I conjectured that the additional figurative material provided in CT along with my prompts could afford him to make that connection. For example, I planned to ask Zane to insert tick marks on each axis of the plane to show the crow's DfA and DfC, which might help him to conceive the axis of the plane in relation to his previous activity with the magnitude lines inserting tick marks.

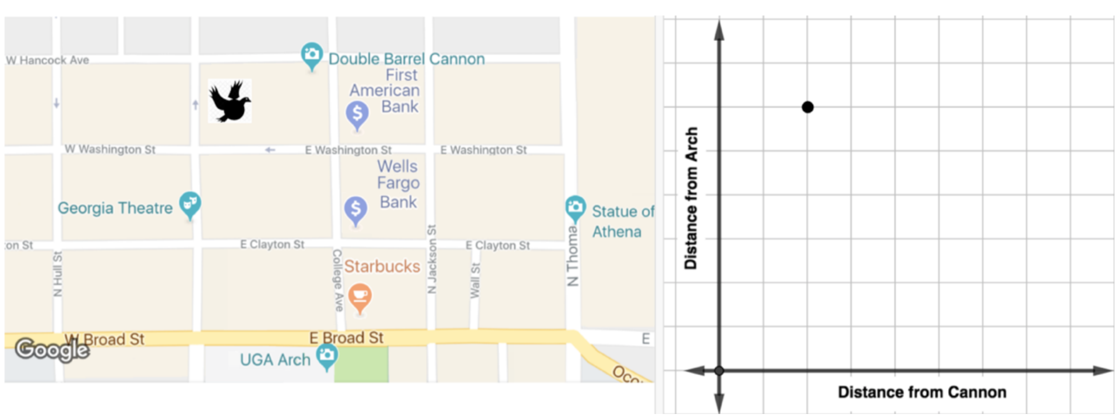
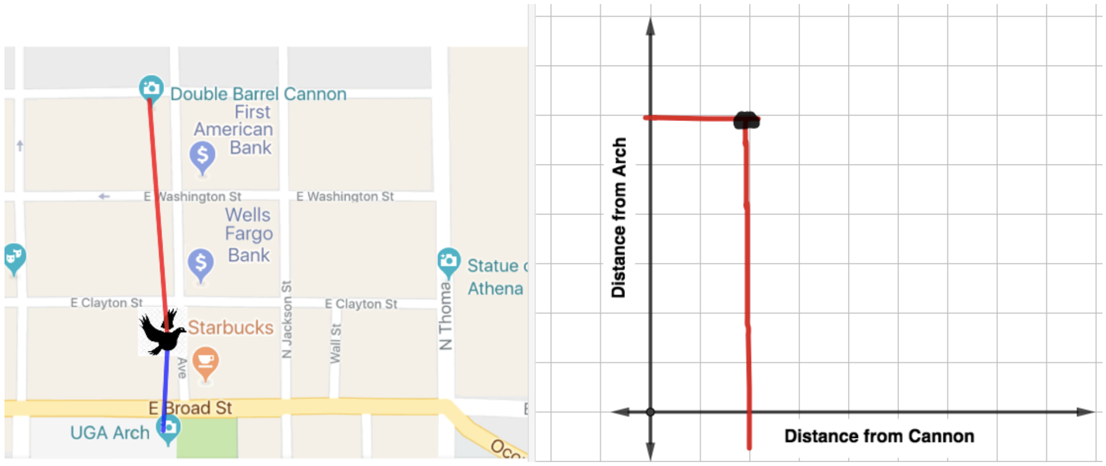


Figure 0.84. Crow Task (CT)

Zane was able to move the crow freely while observing how the corresponding black point on the plane moved. When moving the crow on top of Arch in the map, Zane observed that the black dot on the plane "goes by the line [*referring to the horizontal axis*]." Similarly, Zane moved the crow on top of Cannon in the map and observed that the black dot moved by the vertical line. The location of the dot seemed inconsequential and arbitrary for Zane other than being near the lines. Zane then claimed that "this [*pointing to the vertical axis*] is Double Barrel Cannon and this [*pointing to the horizontal axis*] is Arch." Zane assimilated the black dot as the crow by conceiving the horizontal and vertical axis itself as Arch and Cannon, respectively. This suggested that Zane's meaning of the points included representing a spatial-quantitative multiplicative object (SQMO). That is, I hypothesized that Zane made sense of the changing location of the black dot on the plane depending on the dot's distance from each axis as it related to the crow's DfA and DfC on the map.

In order to test my hypothesis, I hided the black point on the plane and asked Zane to plot a point that would represent the crow's DfA and DfC when the crow was in a place on the map as seen in Figure 0.85a. Zane began drawing a horizontal line segment starting from the vertical axis to a certain place in the plane and drew a vertical line segment from that place to the horizontal axis (see Figure 0.85b). Making connection to the blue and red bars appearing on the map (Figure 0.85a), he referred to the horizontal line segment in the plane saying, "the crow's distance form Arch is shorter" and referring to the vertical line segment in the plane, he said "the crow's distance from Cannon is longer." Then, he plotted the black dot (seen in Figure 0.85b) where these line segments intersected.



(c) (b)

Figure 0.85. Zane's SQMO in CT

To *locate* the black dot (i.e., the crow for him) on the plane, Zane represented the crow's DfA as the dot's distance from the vertical axis and the crow's DfC as the dot's distance from the horizontal axis. Note that this activity is consistent with Zane's earlier activity where he assimilated the axis of the coordinate system as Arch and Cannon. Although the position of the point he plotted is not normative in terms of a canonical Cartesian plane, this activity was valid for Zane as he was assimilating the black dot on the plane as the crow and imagining Arch in place of the vertical axis and Cannon in place of the horizontal axis itself. This provided an evidence that Zane's meaning of the point included representing SQMO. That is, Melvin's meaning of the point included an object by coordinating quantitative features of the object on the plane, which is a different meaning that Melvin had in his previous graphing activities where he engaged in transformed iconic translation.

Zane's Quantitative Multiplicative Object in CT

Since Zane conceived each axis of the plane as the physical Arch and Cannon, I transitioned the teaching experiment in an attempt to engender a reorganization that involved conceiving the axes as magnitude lines. Recall that, in MGT, Zane was able to transform (i.e., disembed and represent) the magnitude of the bike's DfA and DfC from situation to the parallel magnitude lines by ensuring to preserve their length. He successfully placed tick marks on each magnitude line to represent the bike's DfA and DfC (see Figure 0.82). Thus, I thought that I could ask him to place tick marks on the axis of the plane to represent the crow's DfA and DfC. My goal was to help him to make connection between his activity of inserting tick marks on two parallel magnitude lines and inserting tick marks on the axis of the coordinate system.

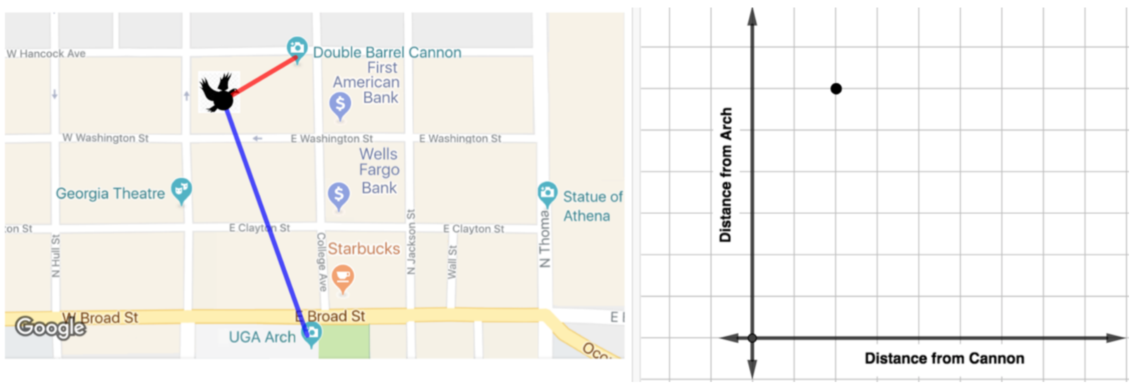


Figure 0.86. CT with the segments in the situation.

I presented the crow (with its DfA and DfC as blue and red segments, see Figure 0.86) and the black point on the plane and asked Zane to insert a tick mark on the vertical axis to show the crow's DfA. Zane first placed several tick marks on the vertical axis (Figure 0.87a). He placed a circular mark on the axis only when asked to show which one of those tick marks represented the crow's DfA. He also placed a mark on the horizontal axis when asked to show the crow's DfC.

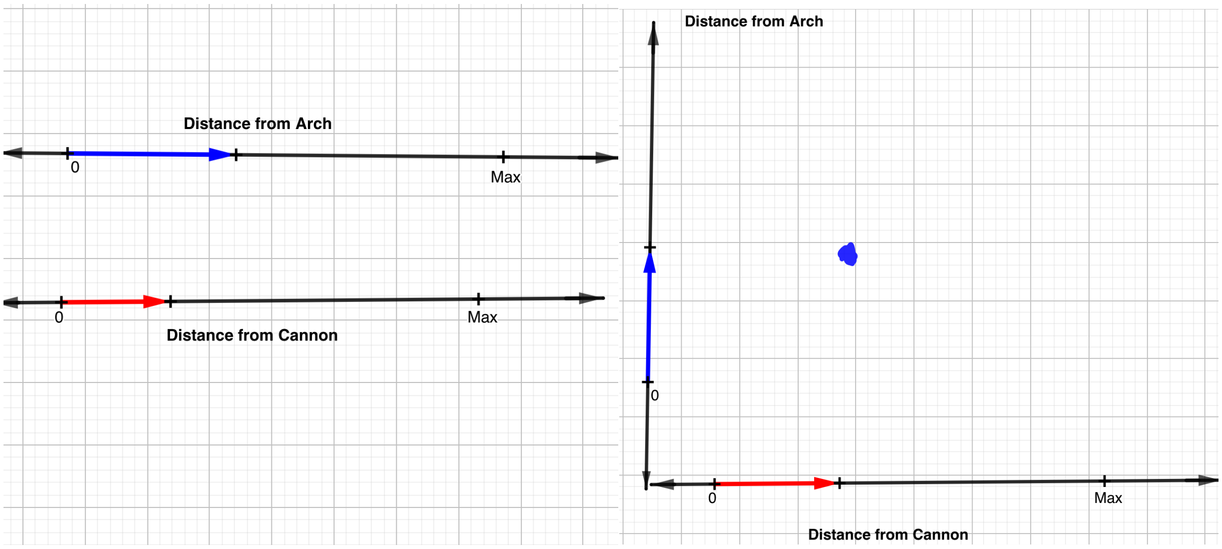
(a) (b)

Figure 0.87. Zane's activity when the black point was (a) visually available and (b) hidden.

To test if Zane place his marks on both axes by only relying on the location of the black point in the plane, I hid the black point, changed the location of the crow in the map (see Figure 0.87b), and asked Zane to place new marks on each axis to show the crow's DfA and DfC. He placed a mark on the vertical axis that is closer to the origin compared to his initial point because "the blue line [*referring to the initial magnitude of the crow's DfA, Figure 0.87a*] was longer" implying that now it is shorter. For a similar reason, Zane placed a mark on the horizontal axis to represent the increased DfC. I interpreted that Zane's activity of inserting marks on the axes was compatible with his activity of inserting tick marks on the magnitude lines in MGT. Zane coordinated and recorded the variation of the quantities on both axes according to how they changed in the situation. Then, I asked Zane to plot where the point is hidden. Zane plotted a point on the plane as seen in Figure 0.87b and his meaning of the point included "the crow's distance from Arch and Cannon." This provided an evidence that Zane's meaning of the point included representing a QMO in a Cartesian coordinate plane.

Zane's Activity in MGT (Second Attempt)

Based on my inference in CT (i.e., Zane was able to represent a QMO on Cartesian coordinate plane), I wanted to see how Zane could engage in the previous task (i.e., MGT) where I asked him to create a single point to represent the bike's DfA and DfC simultaneously given two parallel magnitude lines. I hypothesized that Zane could now create a single point to show the bike's DfA and DfC because I thought he could relate to his activity of creating a single point in CT where a coordinate plane was provided as a given structure.



(a) (b)

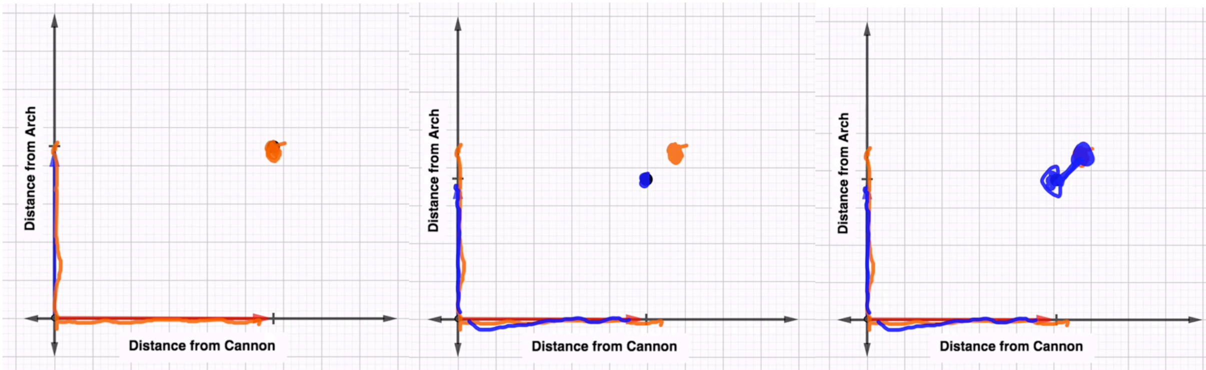
Figure 0.88. (a) Two parallel magnitude lines, and (b) Zane's orthogonal magnitude lines and Zane's single point representing both quantities.

When I directed Zane to MGT, I asked him if MGT and CT are related. Zane said yes and explained, "[*referring to CT*] because these have two distances [*pointing to the blue and red segments in the situation, see Figure 0.86, left*] and there is only one dot [*pointing to the black point in the plane, see Figure 0.86, right*]." Then, referring to the parallel magnitude lines (see Figure 0.88a), he added, "these have two distances, I only had to make the one dot." To respond the prompt, Zane then made the parallel magnitude lines orthogonal (without coinciding with the zero point on each line, Figure 0.88b) and he plotted a point accordingly in the plane (see blue dot in Figure 0.88b). From his activity, I infer that Zane created a single point on the plane as a representation of the bike's DfA and DfC (i.e., representing a QMO) by coordinating these distances in new space that was compatible with a Cartesian coordinate plane. Previously, when the magnitude lines were parallel, he was only able to create two entities to represent two things. Since Zane was able to use the new space in order to produce a single point that simultaneously represented two static quantities' magnitudes, I decided to ask him to create a representation of covarying quantities. Thus, I provided Zane another opportunity to graph the relationship between the crow's DfA and DfC as the crow flied on the map in CT.

Zane's Graphing Activity in CT

From his earlier activities throughout the teaching experiment, I knew that Zane could conceptualize quantities' magnitudes in the situation, represent those quantities' magnitudes by the directed bars on magnitude lines, and represent the two static quantities' magnitudes by a single point in a Cartesian coordinate plane. In CT, I provided Melvin an opportunity to represent the relationship between two quantities as two quantities vary in tandem.

In CT, I placed the crow at the west side of E. Clayton St. on the map and moved it to the right horizontally on E Clayton St. I and asked Zane to draw a representation showing how the crow's DfA and DfC varied. Zane drew segments on each axis and plotted corresponding points in the plane in two different colors (see orange and blue segments and points in Figure 0.89a and Figure 0.89b in the order that they were drawn) to distinguish two different states of the crow's DfA and DfC. Zane claimed that someone who came in and looked at his drawings in Figure 0.89b could understand the change of DfA and DfC "because I [he] did color code … I colored the point the same color as the distance from Arch and distance from Cannon." That is, his color-coded points showed "he [the crow] is in the different place. Like, he is, he is different distance from Cannon and Arch." This showed that Zane's meaning of a single point on a plane became stable in terms of representing QMO in a Cartesian coordinate plane.



(a) (b) (c)

Figure 0.89. (a) Orange and (b) blue segments and the dots, (c) Zane's line segment in the plane.

Note that Zane didn't draw a representation for the variation of two quantities, although he was able to represent the change in two quantities' magnitudes between two cases. That is, he only plotted two points on the plane, instead of a line graph. Two discrete points could only show the crow's DfA and DfC when the crow was at two different locations on E. Clayton St. It could not show how the crow moved between these two locations on the map. The crow could take a circular path that went through these locations and the two points on the plane could still show the crow's DfA and DfC when the crow was at these two locations on the map. Thus, by moving the crow in the map going around in a different path than on E Clayton St (but having the same start and ending point), I drew Zane's attention to the fact that there might be several ways for the crow to have these two different DfAs and DfCs. I asked him how his graph could tell that the crow flew on E Clayton St (i.e., how the graph could show the magnitudes' variations). Then, Zane drew a segment in between the blue and orange dot in the plane as seen in Figure 0.89c, although he was not able to fully reflect on what made him to draw his segment; he claimed, "I don't know."

Zane's Final Draft in DABT

To gain additional insights into his drawings of a segment, in a later session, I repeated the same activity with DABT. I asked Zane to draw a representation showing how the bike's DfA and DfC changed together during its trip. Zane drew a line segment slanted downward from right to left and circled the end points as seen in Figure 0.90.

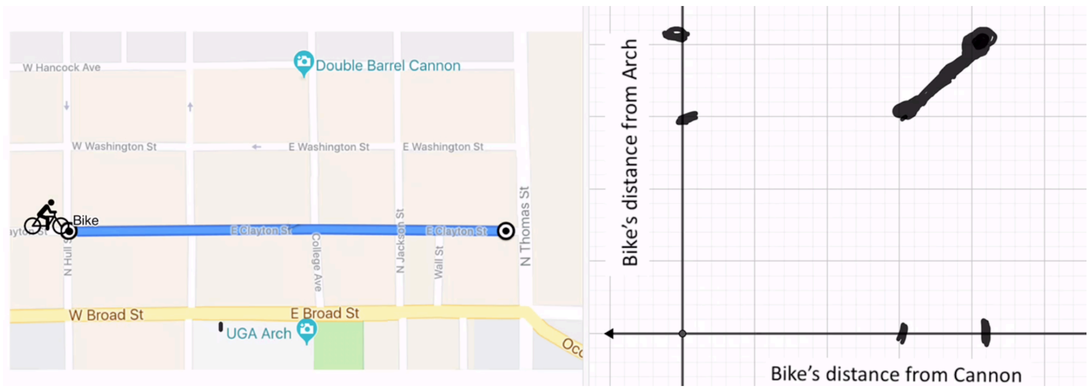


Figure 0.90. Zane's graphing activity on tablet screen.

His meaning of the graph in Figure 0.90included "the dot moving" between those end points from right to left, which implied "the bike's DfC and DfA is getting smaller … because the tick marks gets smaller [*drawing the tick marks on each axis seen in* Figure 0.90]." I asked him to elaborate on what he meant by "tick marks gets smaller" on a graph that I re-drew on a paper (see Figure 0.91). He drew line segments from origin to those tick marks representing the quantities' magnitudes on each axis. Then, he drew arrows and labels indicating the length of magnitudes decreased on each axis to represent "the bike's DfC and DfA is getting smaller" (see Figure 0.91). This provided an evidence that Zane represented QMO in a Cartesian plane.

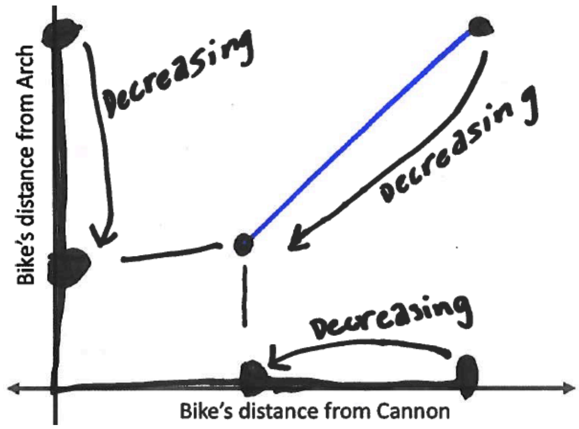
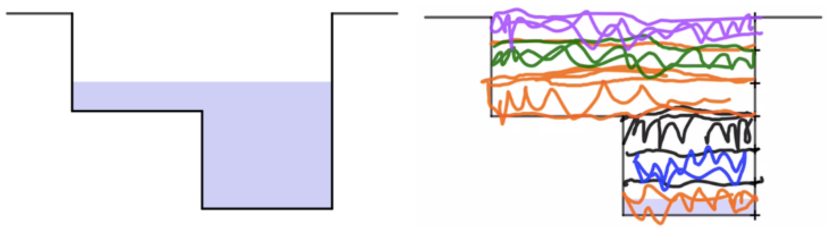


Figure 0.91. Zane's graphing activity on paper.

Zane's Graphing Activity in Swimming Pool Task

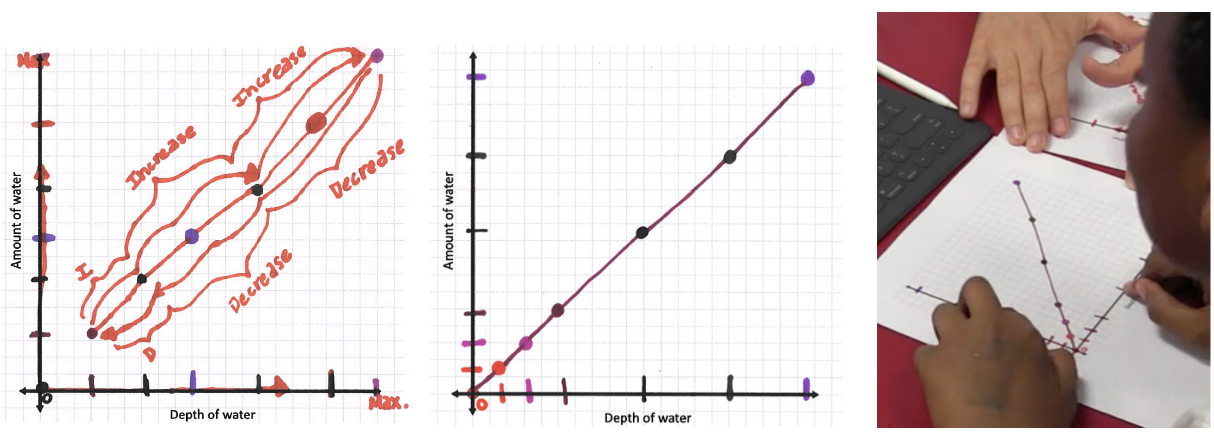
In order to see how Zane could construct a graph in a different context, I transitioned to Swimming Pool Task (SPT) where I provided Zane with a dynamic diagram of a pool (see Figure 0.92a). Zane could fill or drain the pool by dragging a point on a given slider. I designed the task to support him in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool. When determining the relationship between AoW and DoW, as you may see in Figure 0.92b, Zane used the given partitioning of the accumulated magnitudes of DoW into smaller intervals of fixed magnitudes while considering the amount of change in the accumulated magnitude of the AoW. Then, Zane recognized that AoW increases by the same amount as the DoW increases uniformly in the lower part of the pool. He said, "they all increase by one, I don't know how many, but one space [*pointing to the shaded areas on the pool diagram, as seen in Figure 0.92b*]" Zane also recognized that change in AoW was more in the top part compared to the bottom part of the pool.



(a) (b)

Figure 0.92. (a) A diagram of the pool (b) illustration of Zane's partitioning activity

I then asked Zane to sketch a graph that shows the relationship between AoW and DoW as the pool fills up. Zane started with drawing tick marks on each axis and plotting points corresponding to two related tick marks (see his color-coded points and tick marks in Figure 0.93a and Figure 0.93b), then he connected those points with line segments. He initially constructed Figure 0.93a and adjusted his graph to Figure 0.93b to represent bigger increments at the top half of the pool. He also drew arrows to show "increase" and "decrease" in both quantities (Figure 0.93a).



(b) (b) (c)

Figure 0.93. (a) Zane's first draft, (b) Zane's second draft, and (c) Zane moving his fingers on axes.

Zane's meanings for tick marks. When questioned about his tick marks, Zane referred to the quantity's magnitude by drawing a line segment from the origin to the tick mark on the axis. He also used his fingers to simulate the quantities' variation as I played the animation to fill the empty pool. He initially placed his left and right index fingers at the origin saying, “I started from zero” and then moved his left index finger up along the vertical and his right index finger to the right along the horizontal axis (Figure 0.93c). While he was moving his fingers, I inferred that he wanted to make sure both fingers hit each corresponding tick marks at the same time so as to match AoW and DoW as he perceived in the animation.

In order to determine if Zane perceived quantities’ magnitudes *in between* his tick marks, I asked him if he moved his fingers by jumping from one tick mark to another. He responded that he moved his fingers continuously and described an intermediate state:

> Because, I mean, on the thing [*pointing to the pool in Figure 0.92b*], it is not like very jumping up [*moving up his finger very fast from the bottom of the pool*]. It is really just, like, because the water can be [*pointing to the orange shaded area at the bottom of the pool, Figure 0.92b*] also a half of it too [*pointing to the water level in Figure 0.92b*].

I also played the animation backward depicting the water being drained; he moved his fingers simultaneously backwards accordingly. In summary, I infer that Zane could simultaneously coordinate both quantities’ variations on the Cartesian coordinate plane. He conceived of the distance from each finger to the origin as representing the magnitude of AoW or DoW, and he could keep track of the two quantities’ variations simultaneously and continuously, including intermediate states between tick marks.

Zane’s meanings for points. As the conversation continued, I tried to gain insights into the extent to which he coordinated those tick marks on the axes to construct his points on the graph. I asked Zane to show the point on his graph that shows the AoW and DoW when the pool is full. Zane first pointed to the far right and top purple tick marks on each axis (see Figure 0.93b, also see his gesture illustrated in Figure 0.94a), and then, he pointed to the corresponding purple point on the plane (see Figure 0.93b). His actions showed that he could associate these two tick marks on each axis to the point on the plane. Then I asked him to move his fingers correspondingly on each axis as we played the animation. The following excerpt demonstrates his activity:

TR: I am gonna take out water. You are gonna

Zane: Go down [*moving his right and left index finger to the left and down, along the horizontal and vertical axis respectively. Then, he put his finger back in their original position at the very end tick marks on each axis, see Figure 0.94a*].

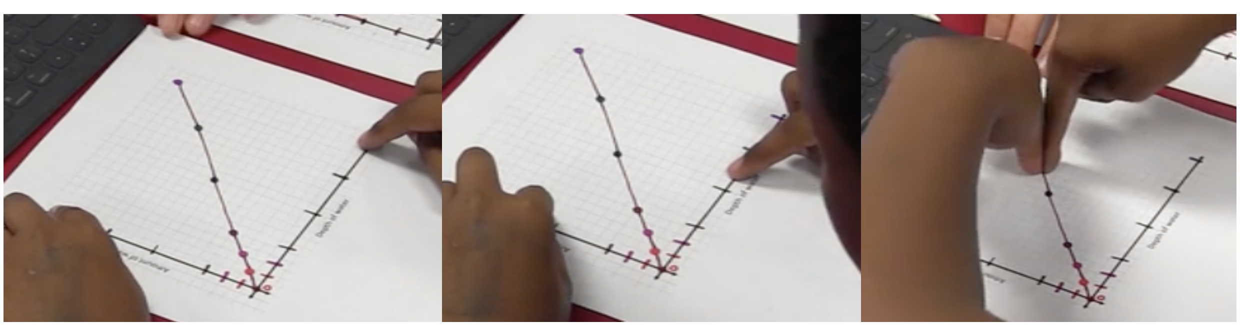
TR: Yes. But, when you do this, you gotta imagine what happens to this point [*pointing to the corresponding point*] ... when I start changing, you are gonna move your fingers and imagine what happens to the corresponding point.

Zane: [*I played the animation and Zane moved his left and right index fingers smoothly on each axis to the left and down, respectively. I stopped the animation where the water level is within the area that are shaded in green [see Figure 0.92b], and he immediately stopped moving his both fingers, see Figure 0.94b*].

TR: Okay. Where is the corresponding point?

Zane: [*He simultaneously moved his left index finger to the right horizontally and right index finger up vertically and stopped when the two fingers met; see Figure 0.94c*] like right here. [*Then he plotted a point on his graph and added two corresponding tick marks on each axis*].

I interpreted that Zane was able to conceive of a point on the plane as a QMO that simultaneously unites the two quantities' variations. Later, he described a point moving up and down along the line as representing *both* quantities' increases and decreases at the same time, saying "the dot represents both." That is, he held in mind two quantities associated with a point and imagined variation of the two quantities as the point moved.



(a) (b) (c)

Figure 0.94. (a) the location of Zane's fingers before the animation started, (b) the place where Zane stopped, and (c) Zane pointing to the corresponding point where both fingers met.

Zane's meanings for lines. When questioned why he connected the points with lines, Zane responded that the line shows "where the dots go." Additionally, he said, "it also helps to person who comes in, they will understand that the line is probably moving down and up." I infer that Zane conceived of a line on the coordinate plane as showing a path of movement of a dot in either direction. I hypothesized that this meaning for a line might be related to his meaning constructed outside a graphing context prior to the study. Therefore, I drew a straight line on a blank paper and asked him how he would define a line. He responded, "a point, hmm, something that goes and never stops." I followed up, "What is that goes and never stops?" After an eight-second pause, he said "hmm, from a start point [*placed his right index finger on the left end side of the line*] to an end point [*moves his finger to the right end side of the line*]. I inferred that Zane conceived a line as describing how an object moves from one location to another, and this was compatible with his understanding of lines in the graphical context.

Based on this inference, Biyao (the witness researcher [WR]) hypothesized that he might not conceive of a line as consisting of infinitely many points. To test this hypothesis, WR asked him, "how many points do we need to plot in order to fully describe what is going on here?" Zane added three additional points on each segment between the points that he originally plotted on his graph (see Figure 0.93b) and said "24". WR then asked him to compare two graphs, a graph with 24 dots plotted and another graph that includes a line and discuss how they were similar or different. He replied "wait, did the person who has the line also have 24 points too?" We inferred from his activity that, for him, a line with dots and a line without dots are different graphs and the dots are critical components that convey additional information to a line.

As an additional evidence, when asked whether his graph (see Figure 0.95a) shows every single moment of how the two quantities vary in the situation, Zane said no because in order to show it, you need to plot an additional point. I infer that, for Zane, lines do not have points until they are visually plotted. He needed to physically plot additional points to represent moments in between two available points, even if there is a line connecting them. That is, he did not conceive of his line as showing these extra moments. In the next section, I demonstrate that such meaning for a line played a role in his construction and interpretation of what I perceived to be in-progress trace.

Zane's Interpretations of an Emergent Trace

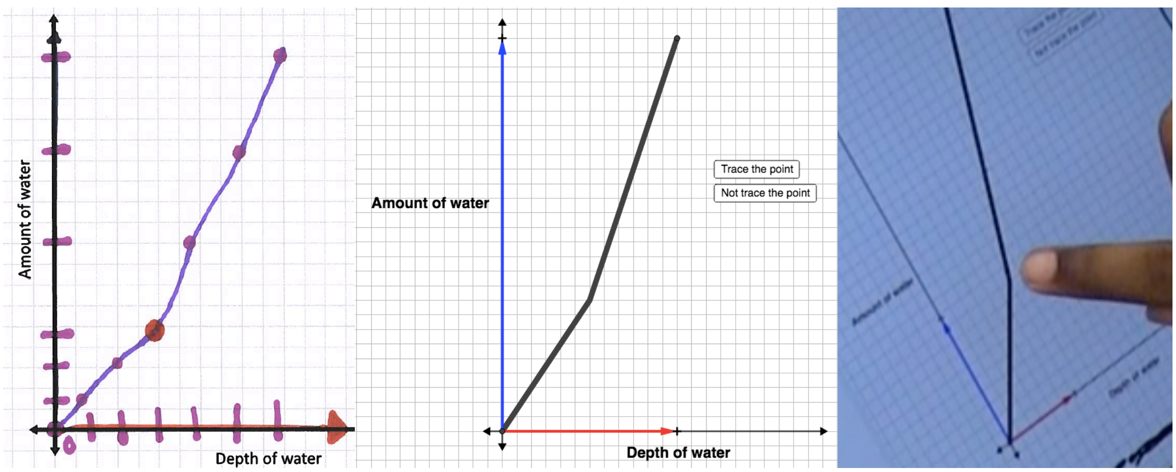
Given these interpretations of Zane's meanings for tick marks, lines, and points, I hypothesized that he likely did not interpret his prior finger activity (Figure 0.93c and Figure 0.94) as generating infinitely many coordinate points. To test this hypothesis, I showed him an animation on an tablet device (see https://youtu.be/97EOENUM_co) and asked: "is this trace [*Figure 0.95b*] showing us the relationship between depth of water and amount of water for this pool?" He replied "no" and struggled to make sense of what the animation was showing, which suggested that he did not perceive the animation as a simulation of his prior graphing activity on paper.

WR asked Zane whether those dots[8] on his paper (see Figure 0.95a) are "part of the line on the computer." Zane replied, "there is only one dot," pointing to the animating dot that produced the trace (see his gesture in Figure 0.95c). When asked "is there any other dots on this graph?" he shook his head. Moreover, he interpreted his graph (Figure 0.95a) as having more

[8] I use the word *dot* instead of *point* when we have evidence of Zane referring to a visual circular object that is plotted on the plane, but do not have explicit evidence of him holding the two quantities *in mind* (i.e., conceiving a multiplicative objective) at the moment. By using the word "dot", we are also genuine to Zane's language in this activity.

dots than the one produced in the animation (Figure 0.95b), commenting that mine is better because "mine have more dots".

Zane also claimed that he could not construct his graph in the same way as the animation did due to physical constraints of human, saying, "well, I cannot do that, because, like, can you do dots and dots [*tapping his right index finger very fast along his graph shown in Figure 0.95a*] and trace it?" This is an additional contraindication that he conceived of graphing a line as a way to represent infinitely many points.



(b) (b) (c)

Figure 0.95. (a) Zane's last draft and (b) an instance of the animation, and (c) Zane pointing to the "only" point on the trace.

Summary of the Case of Zane

Representing a NMO (transformed iconic translation) in DAT

Zane's initial meanings of graphs included transformed iconic translation (i.e., picture of the situation) and representing the literal motion of the object that moves in the situation. In DAT, Zane conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively. In order to make sense the other point on the plane, Zane imagined rotating and/or flipping the map for the purpose of associating perceptual feature of the map and the plane.

Spatial proximity reasoning in DABT

I engaged Zane in DABT (see Figure 0.23) for the purpose of exploring how he could conceive the situation quantitatively and how he could determine the relationship between quantities (i.e., the bike's DfA and DfC). Zane conceived the bike's DfA as the bike's *proximity* to Arch (i.e., its closeness/nearness to the Arch) without conceiving the measurable attribute of the bike and coordinated the variation of the bike's degree of proximity to Arch as the bike traveled on the map (e.g., the bike is getting closer to or farther from the Arch). Moreover, he represented the bike's proximity to Arch in his representational activity. He imagined the physical objects (e.g., the bike, Arch, and Cannon) that were getting closer to or farther from each other on the magnitude lines (see Figure 0.26).

Measurement activity in DABT

Since Zane's reasoning were based in coordinating spatial proximities and his meanings of the bars included the physical bike and Arch on the magnitude lines, I presented Zane with a dynamic tool that could afford Zane's engagement with quantities' magnitudes represented by directed bars placed on magnitude lines (Figure 0.76). By using unit magnitudes, Zane measured the length of the bar that represented the bike's DfA in the situation. Then, by using the same units, he represented the magnitude of the bike's DfA as a length of a directed bar on the magnitude line. This activity helped Melvin to conceive the bike's measurable attribute (i.e., DfA, rather than the bike's spatial proximity) in the situation and be able to represent it by a directed bar on the magnitude line.

Representing a NMO (transformed iconic translation) in DABT

After the measurement activity, I asked Zane to sketch his second draft to show the relationship between the bike's DfA and DfC. In his second draft, Zane continued to engage in transformed iconic translation.

Matching Game Task

His measurement activity did not seem to influence his graphing activity in DABT as Zane continued to engage in iconic translation. Recall that, in his measurement activity, Zane successfully measured the bike's DfA in the situation and represented it on the magnitude line. Thus, I designed MGT (Figure 0.81) in order to engage Zane to (i) represent two quantities' magnitudes on two parallel magnitude lines and then transition to (ii) representing two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and projecting the magnitudes on the plane. Zane successfully represented the two quantities magnitudes on two parallel number lines (by placing a tick mark on each magnitude line); however, he could not create a single point on the plane that represents the two magnitudes by re-organizing space. Then, I decided moved to the next task (i.e., CT, see Figure 0.84) and provided him a coordinate system as a given space. My goal was to provide Melvin with additional figurative material that might afford him to structure the space in a way that is compatible with Cartesian plane.

Representing SQMO in CT

In CT, Zane first conceived the black dot on the plane as the crow moving on the plane according to its quantitative properties (i.e., representing SQMO). Since he imagined the entire vertical and horizontal axis as Arch and Cannon, respectively, Zane represented the crow's DfA as the dot's distance from the vertical axis and, similarly, the crow's DfC as the dot's distance from the horizontal axis (see Figure 0.85). He determined quantitative features of the crow in the situation (i.e., its DfA and its DfC) and ensured to preserve these quantitative properties on the plane.

Representing QMO in CT

Since Zane conceived each axis of the plane as the physical Arch and Cannon, I transitioned the teaching experiment in an attempt to engender a reorganization that involved conceiving the axes as magnitude lines. I explicitly asked Zane to place tick marks on the axis of the plane to represent the crow's DfA and DfC. My goal was to help him to make connection between his activity of inserting tick marks on two parallel magnitude lines and inserting tick marks on the axis of the coordinate system. Zane successfully inserted tick marks on the axes of the plane. This additional figurative material (i.e., tick marks) on each axis of the plane afforded Melvin to assimilate the axes of the plane as magnitude lines (in relation to his previous activity in MGT). Then, he was able to create a single point on the plane that simultaneously represented both quantities' magnitudes (Figure 0.87). Thus, Melvin developed a new meaning of a point on the plane as representing a QMO.

Representing QMO in DABT

Since Zane was able to use the new space (i.e., Cartesian coordinate plane) in order to produce a single point representing two static quantities' magnitudes, I decided to ask Zane to create a representation on the plane to show how the bike's DfA and DfC changed for the whole trip, not for a single moment of the animation. Zane drew a straight line upward from left to right on the plane (see Figure 0.90) that represented how two quantities' magnitudes varied on each axis of the plane (Figure 0.91).

I infer Zane's actions to suggest that the experience of making parallel lines orthogonal was significant moment for him because, after this moment, he did not engage in iconic translation and/or coordinating spatial distances from the axes of the plane. Furthermore, representing quantities on magnitude lines played a significant role in his development as he subsequently and frequently referred to the variation of quantities' magnitudes represented on the axes when explaining how his graphs showed certain covariational relationship in the plane.

Representing QMO in SPT

In order to see how Zane could draw his graph in a different context, I asked him to graph the relationship between AoW and DoW in the swimming pool situation. Zane was able to conceive of a point on the plane as a QMO that simultaneously unites the two quantities' variations. He conceived of the distance from each tick mark to the origin as representing the magnitude of AoW or DoW, and he could keep track of the two quantities' variations simultaneously and continuously, including intermediate states between tick marks by moving his fingers on the axes accordingly. Despite his success in the finger activity and being able to conceive of a point as representing a QMO, Zane assimilated his graphing activity as well as the dynamic graph animation as one dot moving along a line path instead of one dot generating infinitely many points by leaving a trace. I claimed that his meaning for a line as describing a direction of movement (as opposed to consisting of infinitely many points) played a critical role in his construction and constrained him from conceiving a graph as an emergent, in-progress trace.

The Case of Ella

In this section, I present Ella's meanings of points that he developed throughout the teaching experiment. Table 5.4 summarizes all the meanings that she developed in Downtown Athens Task (DAT), Crow Task (CT), and Downtown Athens Bike Task (DABT), and summarizes the way she organized the space in solving these tasks. Ella initially assimilated the points in the plane in relation to the physical objects that appear in the situation, and her meanings for points were based in quantitative properties (i.e., representing SQMO). After Ella conceived the length of the bar on the magnitude line as an indication for the quantity that she conceived in the situation (i.e., the bike's DfA), she organized the space accordingly in later activities when considering two-dimensional space. I illustrate that explicit attention to quantities in the situation and mapping those quantities' magnitudes onto the magnitude lines supported Ella's re-organization of the space consistent with a Cartesian coordinate plane.

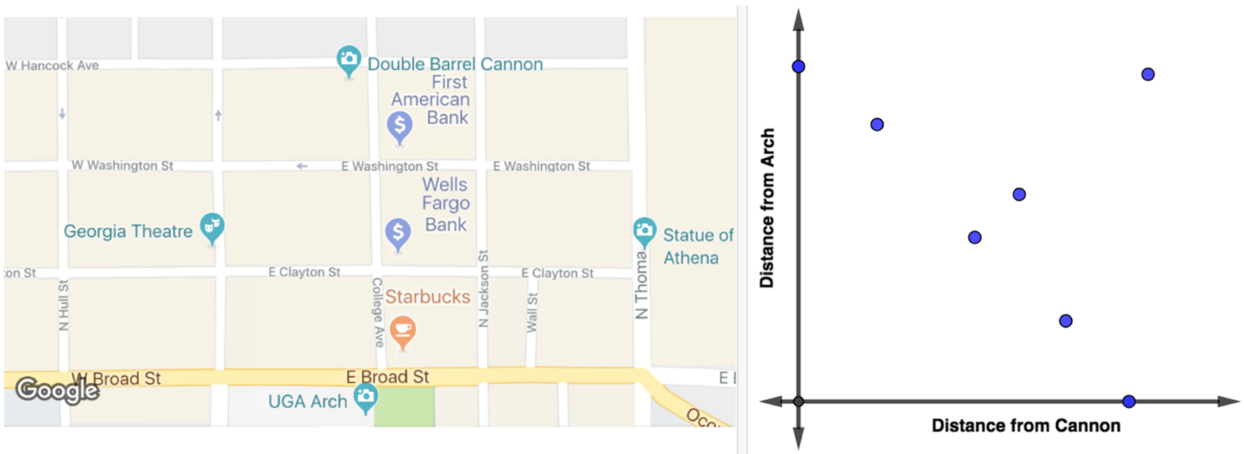
Table 5.4

Ella's meanings of the points and his organization of the space throughout the teaching experiment.

Tasks	Meanings of the points	Organization of the Space
Downtown Athens Task	Spatial-Quantitative Multiplicative Object	Imagining Arch and Cannon on the axis
Crow Task	Spatial-Quantitative Multiplicative Object	Imagining Arch and Cannon on the axis (switching the locations)
Downtown Athens Task	Non-Multiplicative Object (transformed iconic translation)	Imagining Arch and Cannon on the axis
	Intervention	
Downtown Athens Bike Task	Spatial-Quantitative Multiplicative Object	Imagining Arch and Cannon on the axis (changing the locations) Cartesian coordinate plane

Ella's SQMO in DAT

I illustrate Ella's initial meanings by using her activity during DAT. DAT includes a map with seven locations highlighted and labeled (see Figure 0.96a) and a Cartesian coordinate plane (see Figure 0.96b). Seven points are plotted without labelling in the coordinate system to represent the five locations' DfA and DfC. I asked Ella what each of these points on the coordinate plane might represent with an intention to observe her spontaneous responses and to explore her meanings of points. From Ella's activity in DAT (that illustrate below in detail), I infer she essentially formed, from my perspective, a two-center bipolar coordinate system based on gross comparisons between the two quantities' magnitudes and represented SQMO. That is, Ella assimilated points on the plane as a location/object, however, her meanings were based in focusing on object's quantitative properties.



(a) (b)

Figure 0.96. Downtown Athens Task (DAT)

She began her activity by imagining Arch and Cannon physically located on each axis as implied by the labels (see orange dots on each axis in Figure 0.97, right). Ella said, "when it says distance from Arch [*pointing to the label on the vertical axis*], that is where the Arch is [*pointing to the same area on the vertical axis again*]." Ella then made sense of the rest of the space by coordinating the radial distances between "places" on the plane and Arch and Cannon on each axis. For example, Ella labeled a point as "FAB" on the plane (see Figure 0.97, right) to indicate First American Bank, and she conceived the point as FAB based on the orange and blue line segments that she drew on the plane. She stated, "the orange is shorter, and the blue is longer… [*referring to the orange and blue line segments on the map in Figure 0.97, left*] over here, like the same thing." Ella perceived FAB is closer to Cannon and farther from Arch in the map as well as in the plane. Therefore, we infer that Ella's meanings of the points included determining quantitative features of an object in the situation (i.e., its DfA and DfC as indicated by segments) and subsequently preserving these quantitative properties via the location of a point in the plane, which is an indication of representing SQMO.

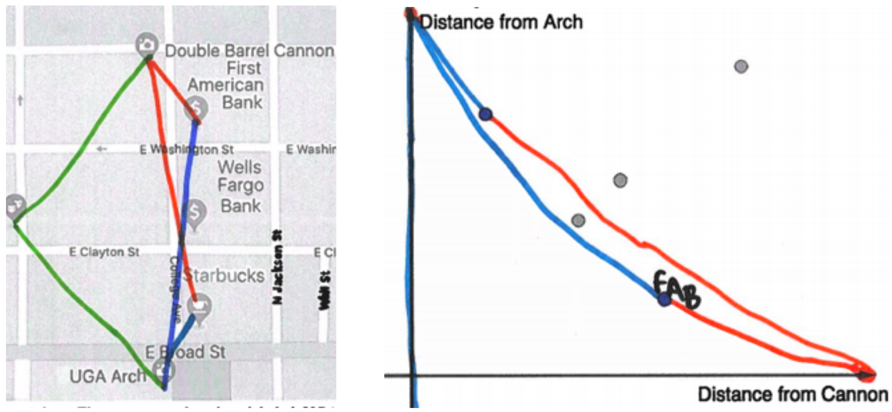


Figure 0.97. Ella's bipolar coordinate system

Ella's Activity in CT

In CT, Ella continued to envision the points on the plane as a location/location by coordinating the object's quantitative properties (i.e., representing SQMO). However, she changed how she organized the space by switching the location of Arch and Cannon on the axes. Ella observed that when she moved the crow to the Cannon on the map, the black dot on the plane moved away from a point on the horizontal axis that she claimed to be Cannon earlier, see Figure 0.97b). That is, Ella noticed that the point that was associated with the crow moved in an opposite way than she established earlier. This was a perturbation for Ella, and she explained, "it is probably farther, we have been doing wrong the whole time." She added, "So, it is basically like the opposite." Recall that Ella formed a two-center bipolar coordinate system based on gross comparisons between the two quantities' magnitudes when making sense of the point in the plane (see Figure 0.97). She conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively. Now, Ella switched the two centers (i.e., switching the location of Ach and Cannon) in order to reconcile by keeping the other properties of the system the same (e.g., the way she perceived the dot's distance from the two centers). Note that Ella's reconciliation was merely switch (i.e., switching the location of Arch and Cannon) because she was only coping with a perturbation by doing the opposite of what she did. Ella was not able to explain why she switched other than showing the system worked after the switch (i.e., when moving the crow on top of Arch in the map, the corresponding point in plane moved near the label "Distance form Cannon" where she though where "Arch" is).

Ella's NMO (transformed iconic translation) in DAT

Recall that Zane's initial meaning in DAT included iconic translation and Ella's meaning included representing SQMO. In order to see how they could make sense each other's solution, I asked to re-engage with DAT together as a pair. As Zane explained to Ella how he imagined the points as locations by making transformed iconic translation, Ella agreed with Zane and she began to assimilate the points on the plane as a location/object by engaging in transformed iconic translation. She began her activity by conceiving the points on the vertical and horizontal axis as Cannon and Arch, respectively (see the labeled C and A in Figure 0.98a, the segments and colorful points were not drawn yet). She then drew two segments on the plane: one from Arch to Statue of Athena (SoA) and the other from Cannon to SoA. She also drew the corresponding segments on the plane. She then imagined rotating the map counterclockwise (Figure 0.98b) in order to match the line segments and resulting shapes, similar to what Zane did. For example, Ella matched the shapes made by line segments (see black line segments in Figure 0.98) connecting SoA with Cannon and Arch (other segments and purple scratches in Figure 0.98 were not drawn yet by that time). By using color coding (see Figure 0.98a; the segments connecting the points were not drawn yet), Ella then associated the points on the plane with the places on the map. This provided an evidence that Ella's meaning of the point included iconic translation.

Recall that Ella initial meaning included SQMO. That is, she conceived the points as objects; however, she was coordinating the objects' quantitative properties and she was assimilating the segments as an indication of quantities' magnitudes. I did not have evidence that Ella assimilated the black segments connecting SoA with Cannon and Arch in relation to SoA's distance from Arch and Cannon. In order to get insights into her meaning of the segments, I asked Ella what the black segments represented for her. The following excerpt shows how Ella assimilated those line segments.

TR: When we were talking about this point [*pointing to the dot labeled SOA on the plane, see Figure 0.98, right*], you drew lines [*moving the pen over the black line segments on the plane from the point to a point on the vertical axis labeled C and to a point on the horizontal axis labeled A*]

Ella: Yeah.

TR: to represent what?

Ella: It, like, just, it doesn't represent anything. It is just like to show like if the lines here [*pointing to the black lines drawn on the plane from SoA to A and C*] are like this [*moving the tip of the pen horizontally and vertically over the black line segments drawn on the plane respectively*] then [*moving the pen toward the map*] if you tilted [*moving the pen over black line segments drawn in the map from SoA to Cannon and Arch in the map*] it would be like that [*repeating the last movement with the pen over line segments*].

Zane: Yeah. So, it is like that lines represents how those three places [i.e., SoA, Arch and Cannon] connect to the connection on the map.

TR: So, those lines doesn't have to do with anything about distance from Arch and distance from… [*interrupted by Ella*].

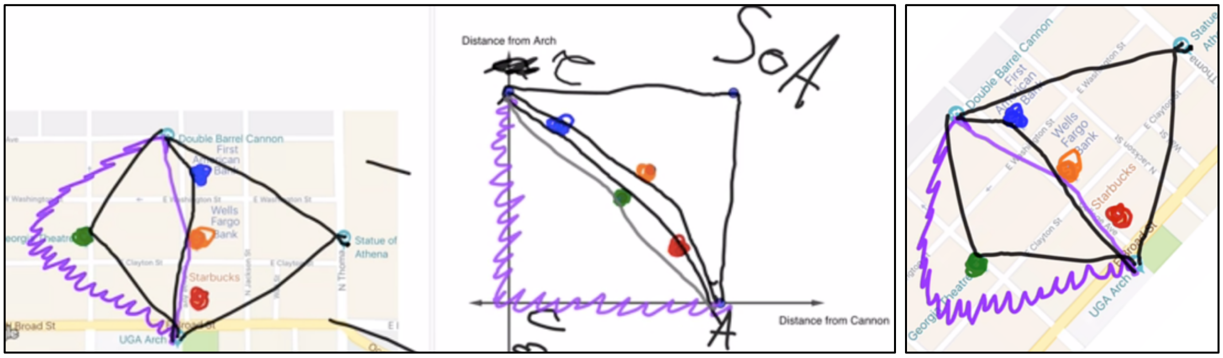
Ella: Not really. And then, if I did like a purple square thing, right here [*drawing purple scratches on both axis completing a square shape together with the black lines on the plane, see Figure 0.98a, right*]

WR: [whispering] and you just finish the purple, uh-huh.

Ella: [*continuing to draw the purple scratches on the map and completing a square-like shape on the map, see Figure 0.98a, left*] It would be like that.

TR: Mm-hm.

Ella: [inaudible] the square.



(a) (b)

Figure 0.98. (a) Ella and Zane's activity in DAT and (b) tilted map

Ella assimilated the bars as a visual connection (i.e., bars as a shape/wire) between two locations, as opposed to being representative of distances. Recall that Ella's meanings of points included quantitative features of the objects as she drew line segments indicating distances of two objects (see Figure 0.98b). On the contrary, in this instance, she assimilated the line segments as a visual connection between objects and assimilated points as objects through transformed iconic translation. With this finding, I note that the meanings of the segments and the meanings of the points were compatible so that Ella formed a system that is internally viable for her. Therefore, these meanings afforded an understanding of points in the coordinate system.

I then asked her to draw segments for other places on the map and corresponding points on the plane similar to what she did for SoA. Ella then drew segments connecting the blue, orange, and red dots in the plane with the points labeled as A and C on the axes (see Figure 0.98a, right), and she drew corresponding segments on the map connecting First American Bank, Wells Fargo Bank, and Starbucks (each of which circled with the corresponding color) with Arch and Cannon (see Figure 0.98a, left). Ella matched the resulting shapes made by the line segments for those points in the plane and places in the map. She said, "if you tilted [*rotating the map, see*

Figure 0.98b], this [*pointing to the segments drawn on the map*] is the same thing as the other ones [*pointing to the segments drawn in the plane*].” However, Ella was surprised after drawing segments connecting the green dot on the plane with Arch and Cannon on the axes (see grey segments on the plane in Figure 0.98a, right) and corresponding segments on the map (see black segments from Georgia Theater circled in green on the map to Arch and Cannon on the map, Figure 0.98a, left). She said,

> Ah, it kind of looks different now [sighs] … like, here [*referring to the map*], it is like that [*tracing a form of a zigzag on the air over the line segments that connects Georgia Theater with Arch and Cannon in the map*], over here [*referring to the plane*], it is like this [*tracing a straight line on the air over the line segments that connects green dot with Arch and Cannon on the axes*] … you know they [i.e., the green dot on the plane and Georgia Theater on the map] are like the same place. So, I probably did something wrong.

This was a moment of perturbation because Ella noticed that shapes of the segments on the plane did not match the shape of segments on the map. I interpret that Ella established a system where iconic translation provided making sense of points on the plane in relation to places on the map; and she wanted to make sense all the points in the plane within that system. Therefore, the un-matching shapes in both situation and plane yielded a perturbation for Ella in a way that she could not resolve it within that system where iconic translation and assimilating line segments in terms of shape were prominent ways of thinking.

Ella’s Activity in DABT

Since Ella represented SQMO in her initial activity in DAT, I know she was able to reason with quantities and represent them on the plane (see Figure 0.97) although Ella’s meaning of the points did not include quantities in DAT when she collaboratively worked with Zane. Thus, I planned to take her attention to the quantities in the situation. Thus, I transitioned to DABT where I present Ella with the same map of Downtown Athens highlighting a straight road (i.e., Clayton St.) with two places located near the road (i.e., the Arch and the Canon; see Figure 0.99) and a bike on this road. I animate the map so that the bike moves at a constant speed back and forth along the Clayton St. starting at the West side of the street. The overall goal with this task was to explore how Ella could conceive the situation quantitatively and how she could determine the relationship between quantities (the bike's DfA and the bike's DfC). In particular, my purpose in this task was to explore and support Ella's process of (i) conceiving the quantities' that vary in the situation, (ii) representing quantities by varying bars on the parallel magnitude lines, and (iii) representing the relationship between covarying quantities on a coordinate plane.

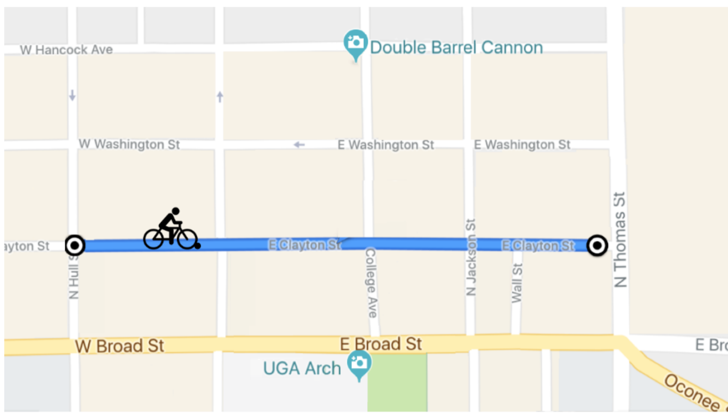


Figure 0.99. Downtown Athens Bike Task (DABT).

I found that Ella initially conceived the bike's DfA and DfC as the bike's *proximity* to Arch and Cannon (i.e., its closeness/nearness to the Arch) without conceiving the measurable attribute of the bike and coordinated the variation of the bike's degree of proximity to Arch and Cannon as the bike traveled on the map (e.g., the bike is getting closer to or farther from the Arch). Then, I provided additional figurative material (i.e., the bars on the map and on the magnitude) that afforded Ella conceiving the quantities in the situation and representing them on the magnitude line. I illustrate her activity in detail below.

Ella's spatial proximity reasoning in DABT

In DABT, I asked Ella to describe how the bike's distance from Arch and Cannon changes as the bike moves along the road. Ella said, "when it is like getting in the middle [*pointing to the middle of the blue path highlighted on the map*], it is getting closer to Cannon and when it is getting over here [*pointing to the end side of the path on the map*], it is getting farther." As I wanted to see if she could describe the variation in bike's DfA and DfC in terms of how these distances increased or decreased as the bike traveled, I explicitly asked them to "talk about those distances getting decrease or increase." Ella replied in the following way:

> When it is right here [*dropped the bike in the middle of the blue path highlighted on the map, see Figure 0.99*], it is closer to the Cannon, when it is over here [*dragging the bike to the left end side of the path on the map*], it is farther, and when it is over here [*dragging the bike to the right end side of the path on the map*], it is farther. But like, right here [*moved the bike back to the middle point of the path*], well, right here, it is like, [*laughing*], so, I am like, imagining this is the number line [*referring to the blue path highlighted on the map, see Figure 0.99*]. And over here [*pointing to the right end side of the path*], it is like, I don't know, ten, and over here [*pointing to the left end side of the road*], it is negative ten, over here [*moving the bike to the middle point of the path*], it is zero. And then, zero is like where Cannon, is like, closest to. And then if it [the bike] is just going this way [*moving the bike to the right from the middle point on the path*], it is like getting farther, and then when it is going this way [*moving the bike to the left on the path*], it is getting farther., But, if it is like just stops [*moved the bike to the middle point again*], then it is like closest ... to Cannon.

Ella's response to my prompt (i.e., talk about bike's DfA and DfC getting decrease and increase) included imagining a number line with numbers (i.e., -10 being placed at the very left side, 0 in the middle, 10 being placed at the very right end side) overlaid onto the blue path where the bike traveled on the map. When questioned "how those numbers tell you the bike is closer and farther," Ella appeared she was unsure as she said, "it kind of doesn't … I don't know." It seemed that she introduced those random numbers because I asked them to talk about something increasing and decreasing. This incident together with her language including "getting closer and farther" suggested that Ella was engaging in spatial proximity reasoning. Moreover, Ella's activity suggested that she was not imagining the values or magnitudes of the bike's distance from Cannon when describing the bike was getting closer to and farther from Cannon.

Ella's quantitative reasoning in DABT

In order get more insights into her conception and representation of the bike's DfA and DfC, I decided to provide an opportunity for Ella to engage with a dynamic tool that represented quantities' magnitudes as directed bars of varying length on parallel magnitude lines (see Figure 0.100, right). Since I know she can disembed quantities from situation and represent them on a different space that was different than the original space (see Figure 0.97), I thought she could assimilate the bars in relation to the quantities and engage in quantitative reasoning instead of spatial proximity. The directed bars can be varied in length as the bike moves on the map (see https://youtu.be/6kdbDeVEF9w). I conjectured, if Ella could assimilate the bars on the magnitude lines in relation to the quantities' magnitudes, this representation could help her to develop meanings of points in terms of representing quantities in two-dimensional spaces. In order for Ella to engage in quantitative reasoning and represent the relationship between two quantities on a Cartesian plane (rather than engaging in iconic translation, see Figure 0.98, or representing SQMO, see Figure 0.97), she needed to conceive the axis of the plane in relation to the magnitude lines and be able to represent the quantities' magnitudes on the axis of the plane.

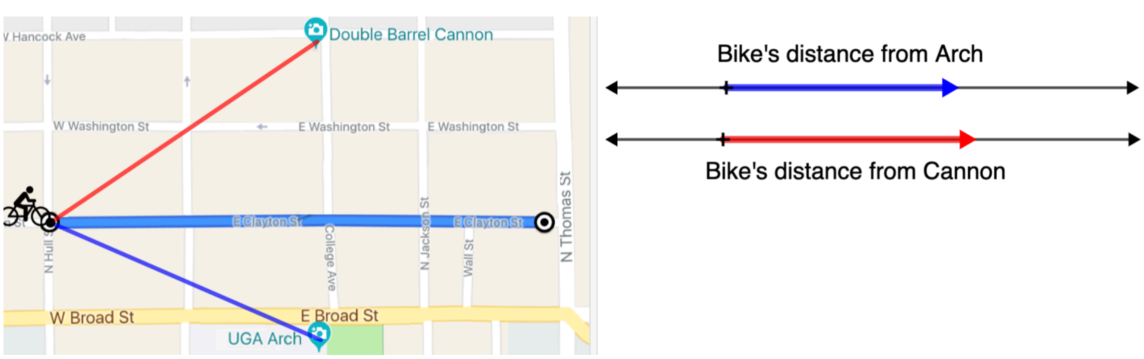
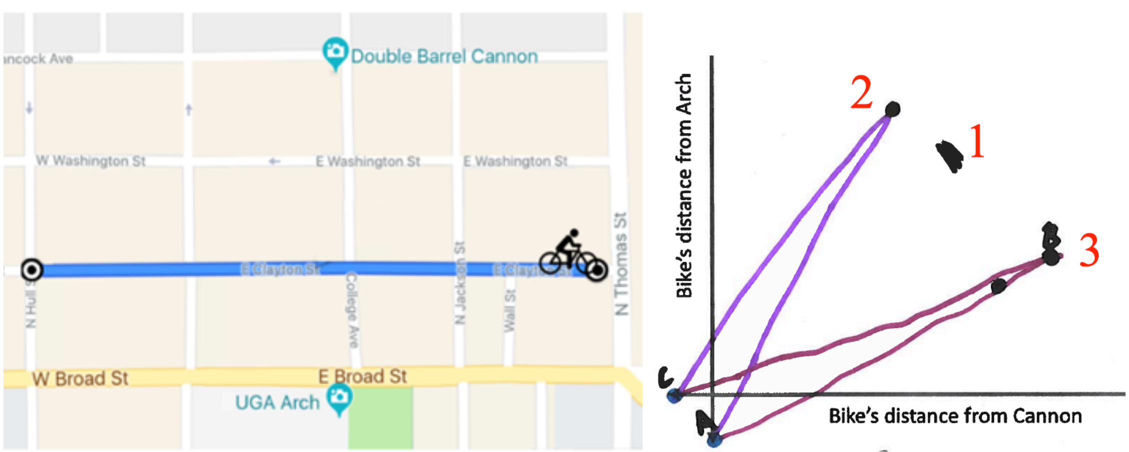


Figure 0.100. DABT with magnitude lines

I first wanted to get insights to how Ella could conceive this representation. While moving the bike from the left end side of the path to the right, I drew Ella's attention to the fact that the right end side of blue bar on the magnitude line was moving to the left (indicating the bike's DfA was decreasing from my perspective). Ella determined that the bike's DfA was decreasing while moving the bike to the right in the map. She explained "it [*pointing to the blue bar*] is gonna get smaller because distance is smaller on the number line too." Moreover, Ella labeled the starting point as "zero." From this activity, I infer that Ella conceived the length of the blue bar on the magnitude line as a representation of the bike's DfA.

Ella's SQMO in DABT

Recall that Ella assimilated the bars on the magnitude lines as representations of quantities' magnitude. Here, I illustrate an implication of the way she assimilated the bars on the magnitude lines when she was asked to represent the relationship between the bike's DfA and DfC on a given paper with a coordinate system. I illustrate that being able to assimilate the magnitude tool normatively helped Ella to re-organize the coordinate plane in a different way than she was doing before.



(a) (b)

Figure 0.101. (a) Map showing the bike's position when questioned and (b) Ella's graph.

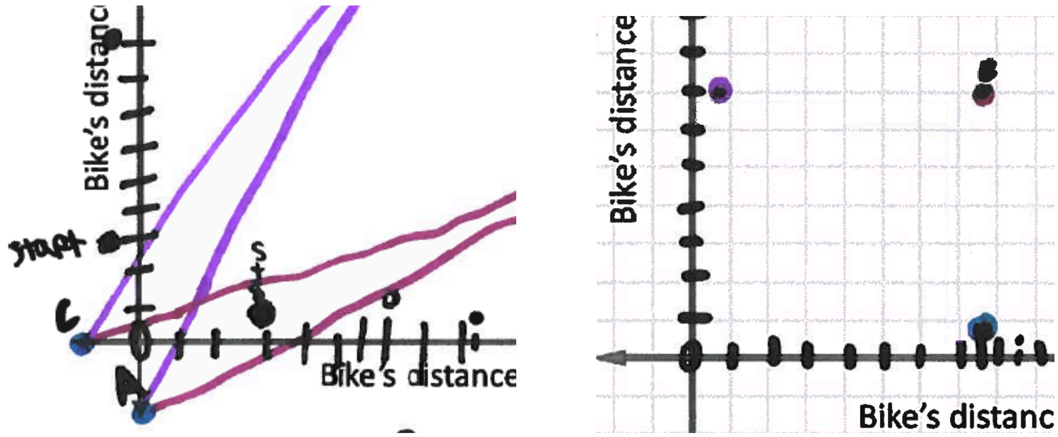
After Ella engaged with the magnitude line activity, I asked her to sketch a graph to represent the bike's DfC and DfA using a given piece of paper with a coordinate plane. At this point, the picture of the situation on the tablet screen was visually available to her and the bike was located at the right end side of the path on the map (see Figure 0.101a) and the animation was not playing. Ella re-organized the space different than her earlier actions in the teaching experiment (see Figure 0.97b vs. Figure 0.101b). In her previous graphing activity representing SQMO (Figure 0.97), Ella was imagining Arch as a location on the vertical axis implied by the label "Distance from Arch." She said, "when it says distance from Arch [*pointing to the label on the vertical axis*], that is where the Arch is [*pointing to the same area on the vertical axis again*]." Similarly, Ella conceived Cannon as a location on the horizontal axis as implied by the labels. However, after her engagement in the magnitude line activity (see Figure 0.100), Ella conceived Cannon at very left side of the horizontal axis (see the black point labeled C in Figure 0.101c) because "farther it is here [*sweeping her finger from left to the right over the horizontal axis*] means that farther it is from Cannon." This may show that Ella's re-organization of the space was an implication of her engagement with the magnitude line activity.

Note that Ella desired to change the location of the dots (see her earlier attempts in Figure 0.101b with the numbers showing the order in which she drew) "because it [the dot labeled as B] is like farther away from Cannon than it is Arch," indicated by the length of the segments she drew on the plane (i.e., the segment between the bike and Arch on the plane is slightly shorter than the segment between the bike and Cannon on the plane). From her activity, I infer Ella assimilated the dot she drew on the plane as the bike (labeled B, #3 in Figure 0.101b) whose location was determined by coordinating the radial distances between the bike's DfA and DfC, which is an indication of representing SQMO.

Ella's Final Graph in DABT

Note that Ella plotted only one point on the plane (see Figure 0.101b), although the prompt was to graph the relationship as the bike traveled. I asked her whether her graph (i.e., the dot she plotted) illustrated the relationship between the bike's DfA and DfC as I animated the bike on the tablet screen. At this point, the tablet was placed in front of Ella depicting the bike animation and the varying bars on the magnitude lines. To respond my question, she said "no" meaning that her single point on the plane did not represent the bike's whole trip. Ella then claimed, "I probably could have put a number line right here [*referring to the axes of the plane*]" to show how the bike's DfC and DfA changed as it moved. To illustrate this, she plotted tick marks on each axis on paper (see Figure 0.102a) in conjunction with tick marks plotted on the magnitude lines on the tablet screen. She added dots near certain (and somewhat arbitrary) tick marks on each axis (see black dots in Figure 0.102a) to represent certain states of bike's DfA and DfC as the bike changed its location.

During this activity, Ella did not focus on her purple line segments or the points that she drew earlier in the plane (see Figure 0.101b). She worked on the axis to represent each quantity although she did not plot points on the plane to represent them simultaneously yet. This provided an evidence that Ella assimilated the axis of the plane in relation to the magnitude lines. Note that the magnitude lines were visually available to her during her graphing activity on paper. I interpreted that the sudden shift in Ella's organization of the space occurred due to her ability to recall her previous activity with the magnitude lines from the animation on the tablet screen.



(a) (b)

Figure 0.102. (a) Ella's marks and dots on the axes, and (b) her points in the plane.

I repeated the same task with a grid paper to see if she could join those quantities on the plane. By describing "this is what I did earlier" referring to her latest activity, Ella began plotting a dot on each axis to show the bike's DfA and DfC (Figure 0.102b). Then, she plotted a point in the plane "where those two [*tracing the pen in the air from the dots on each axis to the dot on the plane horizontally and vertically, respectively*] would meet up if they have like a little line." When asked to explain what that point represented to her, Ella said, "that is where the bike is." I infer that Ella seemed to establish a way to represent two quantities in her newly organized space as a single point; although she seemed to conceive the point that she plotted on the plane as the physical location of the bike. That might suggest that she represented SQMO although her organization of the space was compatible with a Cartesian coordinate plane.

The Summary of the Case of Ella

Representing SQMO in DAT and CT

In her initial activity in DAT, Ella formed, from my perspective, a two-center bipolar coordinate system based on gross comparisons between the two quantities' magnitudes represented on the plane (see Figure 0.97). Ella initially assimilated the points on the plane in relation to the physical objects that appear in the situation, and her meanings for points were based in quantitative properties (i.e., FAB's DfA and DfC or the crow's DfA and DfC).

Representing NMO (transformed iconic translation) in DAT

In her second activity in DAT where she collaboratively worked with Zane, Ella's meaning of the points did not include quantities. She assimilated the points on the plane as a location/object by engaging in transformed iconic translation. Moreover, she didn't assimilate the segments in terms of quantities magnitudes as she attended to match the resulting shapes made by these segments on the map and on the plane. The meanings of the segments and the meanings of the points were compatible so that Ella formed a system that was internally viable for her until she experienced a perturbation.

When asked to draw segments for other places on the map and corresponding points on the plane similar to what she did for SoA, Ella noticed that shapes of the segments on the plane for a point (see grey segments on the plane in Figure 0.98a, right) did not match with the shape of segments on the map for one particular location (see segments from Georgia Theater circled in green on the map to Arch and Cannon in Figure 0.98a, left). I interpret that Ella established a system where iconic translation provided making sense of points on the plane in relation to places on the map; and she desired to make sense all the points in the plane within that system. Therefore, the un-matching shapes in both situation and plane yielded a perturbation for Ella in a way that she could not resolve it within that system where iconic translation and assimilating line segments in terms of shape were prominent ways of thinking.

Ella's activity in DABT

Then, I transitioned to DABT for the purpose of exploring and supporting Ella's process of (i) conceiving the quantities' that vary in the situation, (ii) representing quantities by varying bars on the parallel magnitude lines, and (iii) representing the relationship between covarying quantities on a coordinate plane. I found that Ella initially conceived the bike's DfA and DfC as the bike's *proximity* to Arch and Cannon (i.e., its closeness/nearness to the Arch) without conceiving the measurable attribute of the bike and coordinated the variation of the bike's degree of proximity to Arch and Cannon as the bike traveled on the map (e.g., the bike is getting closer to or farther from the Arch). However, after I provided her with additional figurative material in DABT (i.e., the bars on the map and on the magnitude, see Figure 0.100), I noted a shift in Ella's reasoning from spatial proximity to quantitative reasoning. Ella successfully conceptualized the quantities in the situation (i.e., the bike's DfA and DfC) and conceived the length of the bars on the magnitude lines as a representation of the bike's DfA and DfC.

Representing SQMO in DABT

After Ella assimilated the magnitude lines normatively, I asked her to sketch a graph to represent the bike's DfC and DfA using a given piece of paper in DABT. Ella represented a SQMO; however, she re-organized the coordinate plane in a different way than she was doing before (see Figure 0.97b vs. Figure 0.101b). She was originally imagining Arch and Cannon on the axes implied by the labels. For example, she imagined a physical Arch on the vertical axis near the label "Distance from Arch." However, Ella conceived the Arch at the bottom side of the vertical axis near origin (see Figure 0.101b) after her engagement with the magnitude line tool. Ella conceived the length of the bar on the magnitude line as an indication for the quantity that she conceived in the situation (i.e., the bike's DfA). In doing so, she conceived a constrain regarding how to represent the variation of a quantity on a magnitude line (e.g., only left and right on a horizontal line). Thus, she organized the space accordingly in DABT when considering two-dimensional space.

Representing SQMO on a Cartesian plane in DABT

Since Ella plotted only one point on the plane in her graphing activity in DABT (see Figure 0.101b), I played the bike animation on the tablet screen and asked her whether her graph (i.e., the dot she plotted) represented the relationship between the bike's DfA and DfC for the whole trip. Note that, at this point, the tablet was placed in front of Ella depicting the bike animation and the varying bars on the magnitude lines. With this figurative material available to her, she immediately assimilated the axis of the coordinate plane in relation to the parallel magnitude lines on the tablet screen. Her subsequent graphing activity showed that she successfully represented the quantities' magnitudes (i.e., the bike's DfA and DfC) on the axis of the plane. I interpreted that this sudden shift in Ella's organization of the space occurred due to her ability to recall her previous activity with the magnitude lines from the animation on the tablet screen. I also interpret her explicit attention to quantities in the situation and mapping those quantities' magnitudes onto the magnitude lines supported Ella's re-organization of the space consistent with a Cartesian plane.

Ella then was able to plot a point where the projections of the magnitudes intersected on the plane (Figure 0.102b). Ella seemed to establish a way to represent two quantities in her newly organized space as a single point; although she seemed to conceive the point that she plotted on the plane as the physical location of the bike. That might suggest that she represented SQMO although her organization of the space was compatible with a Cartesian coordinate plane.

CHAPTER 6

CONCLUSION AND DISCUSSION

In this chapter, I summarize and discuss the findings from my analysis of the students' graphing activities throughout the teaching experiment in the context of my research questions. Recall that my research questions were:

1. What ways of reasoning do middle school students enact when engaged in graphing activities intended to emphasize quantitative and covariational reasoning?
2. What ways of reasoning are involved in students developing productive meanings for graphs (e.g.., emergent shape thinking)?

I first summarize and discuss the findings I presented in Chapter 4 in answering my first research question. In Chapter 4, recall that I analyzed the students' activities in constructing graphs on a plane and I illustrated different ways students' graphing activity involved when representing quantitative relationships. I then synthesize the findings I presented in Chapter 5 in answering my second research question. In Chapter 5, recall that I illustrated each student's meanings for graphs and the development of those meanings from the teaching experiment as I implemented an instructional sequence to support students in developing emergent shape thinking. In this chapter, I used a broader lens to look across all students I included in the results in order to outline the key developmental points for students' constructions of productive meanings for graphs. After addressing the students' activities as related to the research questions, I describe implications these results have for research, teaching, and curriculum. I then conclude with areas for future research.

Representing Multiplicative Objects in the Context of Graphing

In this section, I summarize, compare, and contrast different ways students' graphing activity involved multiplicative objects when representing quantitative relationships. These examples illustrate alternative meanings of a coordinate system and coordinate points. Those meanings include representing (i) non-multiplicative object (iconic and transformed iconic translation), (ii) spatial-quantitative multiplicative object, and (iii) quantitative multiplicative object (Type 1 and Type 2). I believe outlining those meanings is important because researchers can be more attentive to those meanings students hold for their representational activity.

Non-Multiplicative Object: Iconic Translation and Transformed Iconic Translation

Student's representing a non-multiplicative object involves envisioning points as a location/object that is physically or figuratively associated with a location/object in the situation. I refer to such a meaning as representing a *non-multiplicative object* (NMO) because the point on the plane does not symbolize or unite two quantities' magnitudes or measures.

Note that there are different students' meanings that could be categorized as an NMO. A few researchers have exemplified some of those meanings. For example, Thompson et al. (2017) argued that calculus students viewed the point $(2, f(2))$ in a coordinate plane as a value of the function, instead of the relationship between the value of the function (i.e., $f(2)$) and the value (i.e., 2) for which the function was evaluated. Similarly, David et al. (2018) reported that some students treated the output of the function as the location of the coordinate point on the plane, rather than on the vertical axis (i.e., *location-thinking*). Those students—and consistent with those in Thompson et al.'s study—did not think of 2 as a measure of a magnitude located on the horizontal axis and they did not think of $f(2)$ as a measure of a magnitude located on the vertical axis in a Cartesian coordinate plane.

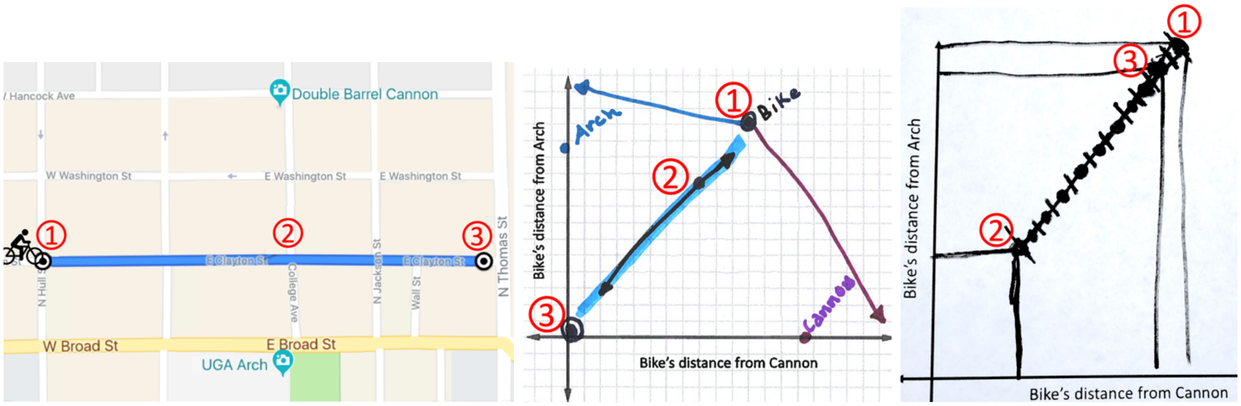
Under the category of NMO, I differentiate between meanings for points on the plane in two ways: (i) the meanings that include *iconic translation*, and (ii) the meanings that include *transformed iconic translation*. Researchers (e.g., Clement, 1989; Monk, 1992) stated that students' activity can be characterized as iconic translation when they view graphs as representing *literal* pictures of a particular situation. Although it is similar to iconic translation in nature, I introduced a new construct, called transformed iconic translation, referring to envisioning graphs representing a *transformed* version of the picture of a particular situation. Compared to iconic translation, transformed iconic translation requires students to use more cognitive recourses when creating or interpreting point on the plane, such as (mentally or physically) manipulating the picture of a situation or the shape of a graph so that they are perceptually associated with each other. The transformation can take on different forms. Some of the transformations included, but not limited to, rotating the page with the picture of a situation or the graph clockwise or counterclockwise, flipping the page vertically (i.e., reflection across a horizontal line), and possibly in combination of both.

Spatial-Quantitative Multiplicative Object

Student's representing a spatial-quantitative multiplicative object (SQMO) involves envisioning points on the plane as a location/object by focusing on the object's *quantitative* properties and engaging in quantitative reasoning (e.g., gross comparison of two quantities' magnitudes). That is, the students' meanings of points included determining quantitative features of an object in the situation (e.g., the bike's DfA and DfC) and ensuring to preserve these quantitative properties on the plane when locating the corresponding point (e.g., the physical bike for students) on the plane.

I perceive that the goal for students who conceive a point as a SQMO is locating the object in the space, which is why the primary attention is the plane rather than the axes. To locate the object on the space, students represent the quantities' magnitudes in the space by committing to a reference, such as the axis itself or a location on the axis, and where those magnitudes meet determines the location of the object (e.g., the crow, bike, or First American Bank). The meaning of this point on the space is different than students' meanings who conceive a point as NMO as they engage in (transformed) iconic translation. This is an important distinction to make because in both meanings, students envision points on the plane as an object/location. Those students who represent NMO don't coordinate quantities' magnitudes when locating the object on the plane; instead, they attend to the perceptual features of the objects in the map/plane and preserve these features when transitioned to the plane/map. For that reason, students who produce a graph by tracing a SMO will perform different actions (e.g., moving different directions on the space) than others who produce a graph by engaging in (transformed) iconic translation.

Recall that, in their initial activity in Downtown Athens Bike Task (DABT), both Zane and Melvin conceived their graphs as a part of E. Clayton St. where the bike traveled on the plane (see Figure 0.1). However, the way they drew the path, and their image of the path was different. Numbered labels in Figure 0.1 indicate how their graph on the plane match with the bike's path on the map. For Zane, the graph represented the literal path of the bike after the transformation (i.e., rotating the page). However, Melvin's graph (i.e., still E. Clayton St. for Melvin) was dictated by the quantitative relationship between the bike's DfA and DfC. That is, Melvin conceived the quantities in the situation, decomposed them from the situation, and then re-presented them on the plane. He represented *spatial*-quantitative multiplicative object because his meaning of the point included the bike, not an abstract object that simultaneously represented the bike's DfA and DfC.



(a) (b) (c)

Figure 0.1. (a) DABT, (b) Zane's transformed iconic translation, and (c) Melvin's SQMO (number labels are added for the reader).

Moreover, students who represent a SQMO draw segments in the situation as indication of quantities' magnitudes (i.e., apparent magnitudes). Students make gross additive comparisons among those apparent magnitudes to determine a relationship and are able to transform (i.e., dis-embedding and re-presenting) those magnitudes from situation to the graph (and from graph to the situation).

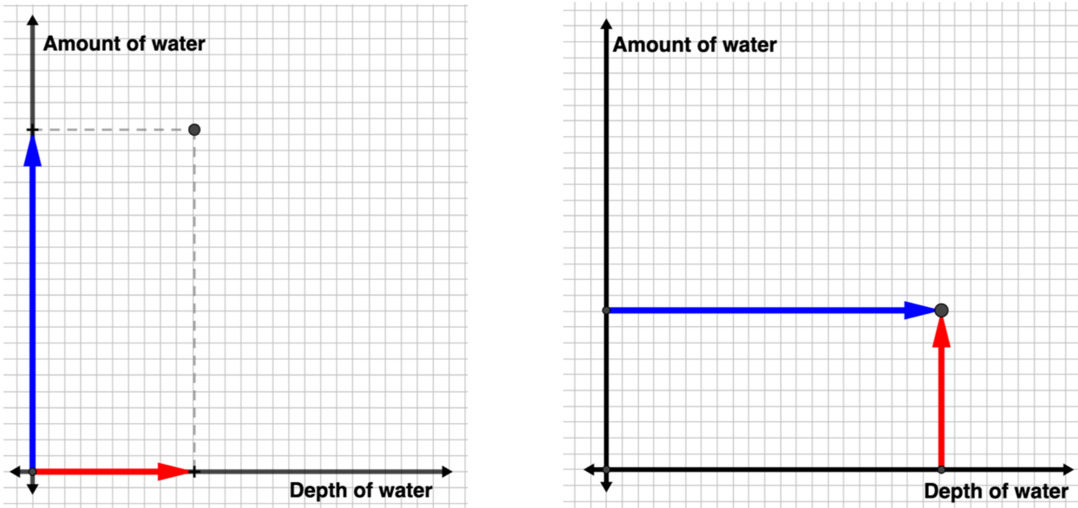
Someone may consider students' graphing activity as a non-normative graph when they represent SQMO (e.g., Melvin's graph in Figure 0.1c). I argue that their form of reasoning is productive in terms of completing the goal of the activity as they perceive it. For example, Melvin's graph (Figure 0.1c) is not wrong in terms of how it shows the relationship between the bike's DfA and DfC, which was the task he was given. His goal was to create a representation on a given space to show how the quantities varied, and his graph exactly satisfied his goal. Therefore, his activity should be considered as a different way of graphing relationships because it still required him engage in quantitative coordination in a non-normative way. Calling this type as “spatial” should not imply dismissing the role of quantitative coordination in students’ reasoning.

Note that my emphasis in this study is not on students’ use of coordinate systems (cf. Lee, Hardison, & Paoletti, 2020; Paoletti, Lee, & Hardison, 2018); instead, I categorize students’ meanings of points in terms of multiplicative objects represented in a space that may or may not be classified as any type of coordinate system that we, as researchers, know. Lee at al. (2020) distinguished two uses of coordinate systems: spatial coordinate systems and quantitative coordinate systems. They defined the spatial coordinate system “as a coordinate system mentally overlaid onto the space being represented and objects within that space being tagged with coordinates” (p. 32). The purpose of using a spatial coordinate system is to quantitatively describe the location of an object within a real-life situation (i.e., a map where a bike travels) where a coordinate system is overlaid on. Conceiving a point as SQMO should not directly imply representing the relationship on a spatial coordinate system. For example, in DABT, Melvin represented SQMO (see Figure 0.1c) by isolating the quantities in the situation, dis-embedding these quantities from the situation, and “project[ing] them onto some new space, which is different from the space in which the quantities were originally conceived” (Lee et al., 2020; p. 34). As you may see, Melvin’s activity is compatible with Lee et al.’s definition of establishing a quantitative coordinate system. As a result, I think that SQMO is represented within a quantitative coordinate system based on my understanding of the current definitions provided by Lee et al.

Quantitative Multiplicative Object

Students’ quantitative multiplicative object (QMO) involves envisioning a single entity on the plane as symbolizing the two quantities’ magnitudes or values simultaneously (Saldanha

& Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017).



(a) (b)

Figure 0.2. (a) QMO in a canonical Cartesian coordinate plane and (b) QMO in a non-canonical Cartesian coordinate plane

Note that students' organization of the space does not have to be compatible with a Canonical Cartesian Plane (CCP) when representing a QMO. Students can also represent QMO by using Non-canonical Cartesian Planes (NCPs). For example, in both Figure 0.2a and Figure 0.2b, the blue and red directed bars represent the magnitudes of amount of water (AoW) and depth of water (DoW) in the swimming pool situation. Each point in both coordinate planes represents a QMO because, in each case, the single point on the plane unites the two quantities' magnitudes simultaneously. In Figure 0.2a, the quantities' magnitudes are represented on the axes of the plane and the origin is the reference point. The point is created by taking two orthogonal magnitudes along the axis and creating projections. In turn, the space is inconsequential beyond creating the point by joining the projections when engaging in representing QMO in CCP. This characterization is compatible with quantitative coordinate system (Lee et al., 2020).

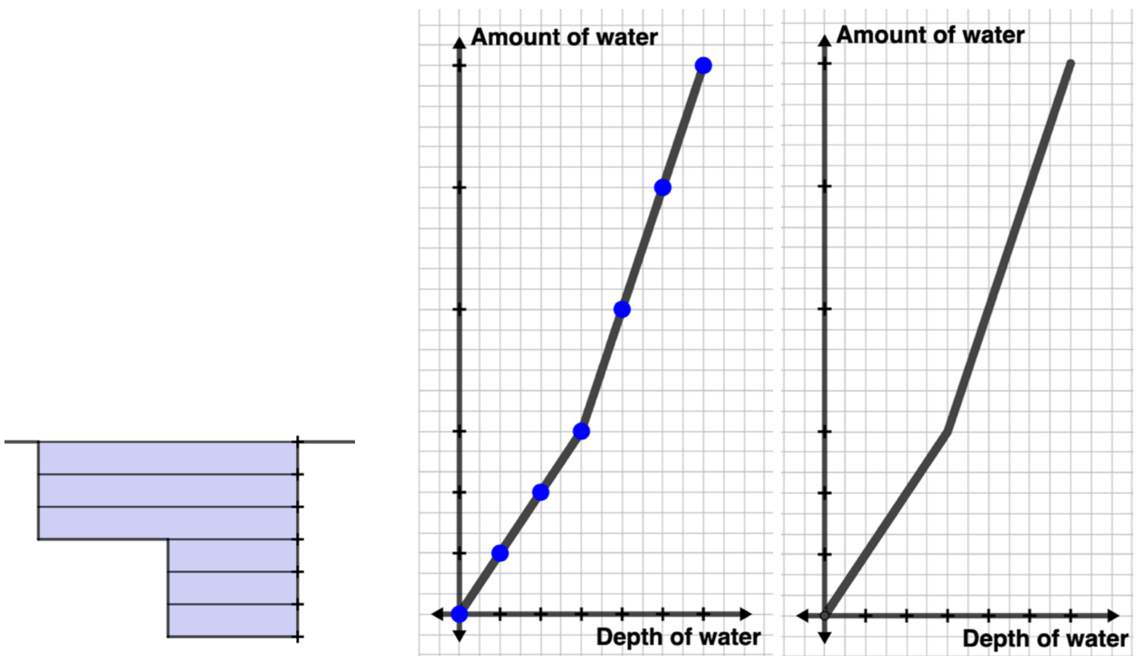
On the other hand, in Figure 0.2b, the quantities' magnitudes are represented on the plane, and the axis of the plane is the reference ray for each quantity. That is, horizontal distances from the vertical axis labeled "Amount of water" represent the magnitudes of AoW (i.e., the blue bar in Figure 0.2b), and vertical distances from the horizontal axis labeled "Depth of water" represent the magnitudes of DoW (i.e., the red bar in Figure 0.2b). The point is created where the magnitudes intersect on the plane. Students who produce a graph by tracing a QMO in NCP will perform different actions (e.g., moving different directions on the space) than others who produce a graph by tracing a QMO in CPP. In fact, those two graphs in both planes will be exactly the symmetry of each other over the line of $y = x$.

Type 1 and Type 2 QMO

I classify instances of representing QMO in two ways in relation to conceiving a graph (e.g., a line drawn on the plane) when students represent a relationship as two quantities vary: (i) as a path or direction of movement of a dot on the plane, and (i) as a trace consisting of infinitely many points, each of which showing the relationship of two varying quantities.

Students' Type 1 QMO involves envisioning points as a circular *dot* that simultaneously represents two quantities' magnitudes or values and envisioning that points on a graph (e.g., a line) do not exist until they are physically and visually plotted. Therefore, those students conceive the graph (e.g., a line) as representing a direction of movement of a dot on a coordinate plane. For example, Zane, whose meaning is characterized as Type 1, conceived the blue dots in Figure 0.3b simultaneously representing both AoW and DoW in the pool. Although there is a line that connects the blue dots, Zane conceived the entire graph only showing the six different instantiations of the pool animation as depicted in Figure 0.3a. For Zane, a line does not have points until they are visually plotted. Therefore, in order for the graph to represent AoW and

DoW on the plane in relation to another moment in the situation (other than those six), he needed to physically plot an additional dot between two available points, even if there is a line connecting them. The line only represents a path or a direction of the movement of the available dots without representing the invariant relationship between those two dots. Zane couldn't assimilate the graph in Figure 0.3c as representing the relationship between AoW and DoW because there are no visual dots on the plane. Therefore, Zane was not be able to fully engage in emergent shape thinking (Moore & Thompson, 2015) as an implication of his meanings that included Type 1 QMO.



(a) (b) (c)

Figure 0.3. (a) Partitioned swimming pool situation, (b) a graph with visually available dots, and (c) a graph without the visual dots.

Students' Type 2 QMO involves being able to envision a point as an abstract object that represents two quantities' magnitudes or values simultaneously, and envision a graph (e.g., a line) as composed of points—although they are not visually plotted, each of which represents two quantities' values or magnitudes. For example, for Melvin whose meanings included Type 2

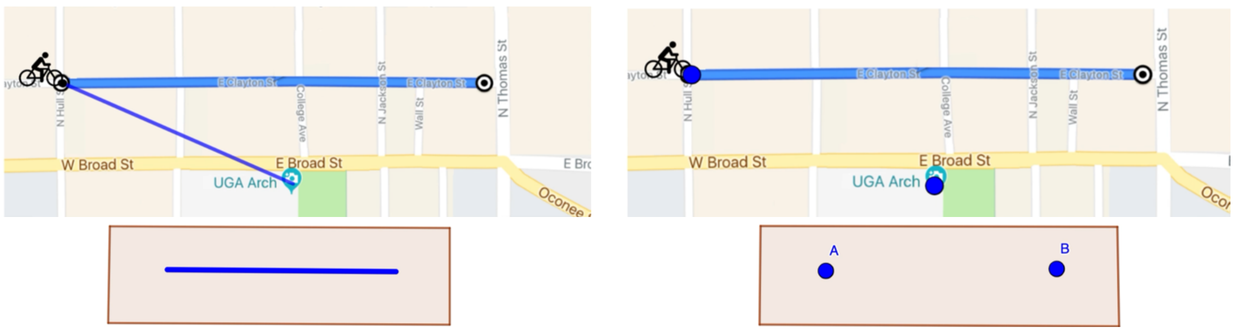
QMO, both graphs in Figure 0.3 represented a record of two covarying quantities with the result of the trace consisting of infinitely many points, each of which represents both AoW and DoW. Therefore, both graphs show every single instantiations of the pool situation because the line consist of infinitely many points. For Melvin, the visually available dots on the line (e.g., blue dots in Figure 0.3b) are used for communication purposes to talk about which part of the graph refers to which part of the animation. That is, he can visually plot additional points on the line to indicate particular AoW and DoW in relation to a particular moment in the situation. The line without the visual points (see Figure 0.3c) already shows every single moment of the situation; however, someone who comes in after the construction of graph may not immediately know which part of the graph refers to the which instantiations of the pool situation depicted in Figure 0.3a.

Students' Reasoning in Dynamic Situations

In this section, I synthesize the various ways in which students reasoned in dynamic situations. Depending on what their image of covariation or association included (i.e., quantity, proximity, and perceptual features), I classify the nature of students' reasoning as (i) quantitative covariational reasoning, (ii) spatial proximity reasoning, and (iii) matching the perceptual or spatial features of motion in two difference spaces.

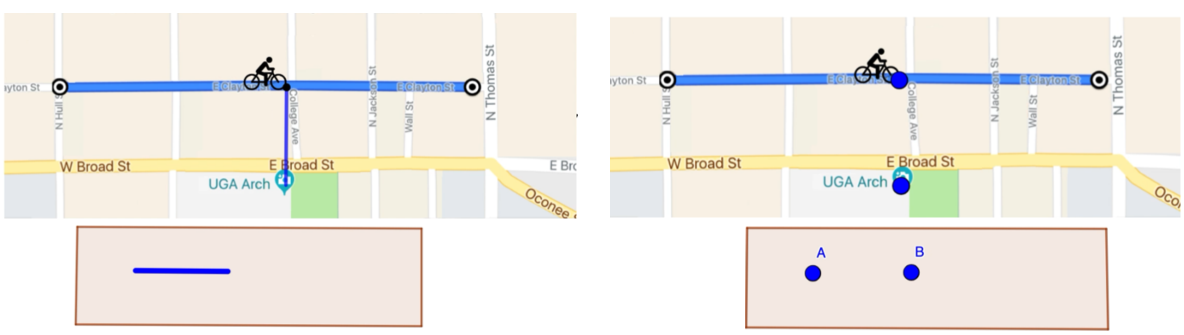
Quantitative Covariational Reasoning

In quantitative (co)variational reasoning, students coordinated the measurable attributes of objects. That is, their image of covariation included reasoning about quantities' magnitudes or values. For example, in DABT (see Figure 0.4a), students conceived the measurable attributes of the bike (e.g., its DfA) and determined the variation of the attribute as the bike moved on the map (e.g., the bike's DfA increases and decreases, see Figure 0.4a and Figure 0.5a for different instantiations of the bike). With respect to quantitative covariational reasoning, I have often observed that student's image of variation regarding the bike's DfA included the varying length of a segment (Figure 0.4a, bottom) that could be drawn in between the bike and the Arch on the map (Figure 0.4a, top) as a representation of the magnitude of the bike's DfA.



(a) (b)

Figure 0.4. (a) Representing the magnitudes of bike's DfA on the map (top) and representing the isolated magnitude (bottom), and (b) representing the proximity between the bike and on the map (top) and representing the isolated proximity (bottom). (A stands for the Arch and B stands for the bike).



(a) (b)

Figure 0.5. Same representation as in Figure 0.4 but for different instantiation of the bike on the map.

Spatial Proximity Reasoning

In spatial proximity reasoning, students coordinated an object's degree of proximity to another object, instead of coordinating the measurable attributes of an object. For example, in DABT (see Figure 0.4b), students conceived the spatial attributes of the bike (e.g., its closeness to the Arch) and determined the variation of that attribute as the bike moved on the map (e.g., the bike is getting closer to or farther from Arch, see Figure 0.4b and Figure 0.5b for different instantiations of the bike). With respect to spatial proximity reasoning, I have often observed that student's image of variation regarding the bike's proximity to Arch included two separate objects that were getting close to or farther from each other (Figure 0.4b), rather than including a varying length of a segment (cf. Figure 0.4a).

It is possible for someone to reason with spatial proximity and distance at the same time. However, I conjecture that these two types of reasoning are independent of each other. Someone can reason with distances without reasoning with spatial proximity or vice versa. What is being abstracted in spatial proximity reasoning is the two objects (e.g., Arch and Bike, see Figure 0.4b and Figure 0.5b, bottom) and their proximity when engaging in the situation (e.g., getting closer and farther). In quantitative covariational reasoning, what is being abstracted is the distance between these two objects as a quantity (see Figure 0.4a and Figure 0.5a, bottom) and its variation when engaging in the situation (e.g., increasing and decreasing). As an implication of these two types of reasoning, I often observed students' graphing activity in two-dimensional space differed although they were compatible with respect to the behavior of figurative material in one-dimensional space. For example, students who reasoned with spatial proximities imagined the physical objects in their graphing activities whereas students who reasoned with quantitative covariation represented the quantities isolated and decomposed from their physical context in their graphs.

For an example, Zane attended to the variation of bike's proximity to the Arch when engaging in the situation in DABT (i.e., "the bike is getting closer to Arch"). Thus, when engaging with the magnitude line in one-dimensional space, he conceived the physical bike and Arch in place of the right and left end side of the blue bar on the magnitude line, respectively. Differing from Zane, Ella attended to the variation of the bike's distance from Arch when engaging in the situation, in turn, she conceived the length of the blue bars both in the situation and on the magnitude line as a representation of the quantity's magnitude. As the bar's length got smaller on the magnitude line, Zane assimilated it as *the bike is getting closer to Arch* whereas Ella assimilated it as *the bike's distance from Arch is decreasing*.

Spatial proximity reasoning is not problematic in and of itself in one-dimensional space. It makes sense to imagine Arch in place of the origin in one-dimension because the value of the bike's DfA is zero at that point. Thus, Zane's and Ella's graphing activity were compatible with respect to the physical movement of the bar on the magnitude line. However, their meanings and reasoning have different implications when we move to two-dimensional systems. Zane continued to engage in his spatial proximity reasoning by conceiving the physical Arch, Cannon, and the bike on the coordinate plane whereas Ella was able to assimilate the axes of the plane in relation to the magnitude lines and, in turn, she was able to imagine the magnitudes varying on the axes of the coordinate plane. Since it has a significant impact on students' graphing activity, we, as a research or teacher, should pay attention to the types of reasoning students might demonstrate when they graph in dynamic situations.

Matching the Perceptual Features of Motion in Two Difference Spaces

The first two categorization (i.e., quantitative covariational reasoning and spatial proximity reasoning) involve students engaging in covariation of an object's measurable and spatial attributes, respectively. In matching perceptual features, students neither attended to the measurable nor spatial attributes of an object in motion. It involved the nature of students' associations between the perceptual feature of the motion of an object in a situation and the perceptual feature of the motion of another (but related) object on a representational system (e.g., a number line or a coordinate plane). For example, students determined the bike moved in a horizontal path (i.e., E. Clayton St.) on the map back and forth. Then, when asked to move their finger on a number line to represent the varying magnitude of bike's DfA, they simulated the perceptual feature of the same motion (e.g., moving their finger to the left/right on the number line when the bike moved to the left/right on the map) without considering the bike's varying attributes.

I identified the last two categories (i.e., spatial proximity reasoning and matching perceptual features) when students engaged in tasks, such as Downtown Athens Bike Task, Skateboarder Task, and the Crow Task, where the context included objects in *motion* when the quantities of interest (from my perspective) were distances (e.g., the bike's DfA as it travels in Downtown Athens, skateboarder's distance from the flagpole). I believe that spatial proximity reasoning and matching perceptual features could be a lens for researchers to have when investigating students' graphing activities in motion context.

Developing Productive Meanings for Graphs

In Chapter 5, I illustrated Melvin, Naya, Zane and Ella's meanings for graphs and the development of those meanings from the teaching experiment as I implemented an instructional sequence to support them in developing emergent shape thinking—a productive and covariational meaning for graphs from quantitative reasoning perspective (Moore & Thompson, 2015). In this section, by looking across all students' graphing activities throughout the teaching experiment, I identify five cognitive resources that were critical for developing a meaning for graphs as an emergent representation of two covarying quantities. These resources include:

1. Conceptualizing quantities and their relationship in the situation.
2. Representing the quantities magnitudes as varying bars on parallel magnitude lines.
3. Assimilating the axes of the coordinate plane in relation to the magnitude lines.
4. Creating a point on the plane by intersecting the projections of the magnitudes on the axes.
5. Conceiving a line as consist of infinitely many points.

Conceptualizing Quantities and Their Relationship in the Situation

Recall that, at the beginning of my teaching experiment, I did not explicitly prompt students to conceive the quantities in the situation. I first wanted to get insights into their spontaneous meanings of points given a situation and a graph. I wanted to investigate how they could conceive the point on the plane before I apply my instructional sequence that emphasized quantitative and covariational reasoning. The results of my study showed that, except Ella, all students initially conceived the points on the plane as a location/object by engaging in either iconic translation or transformed iconic translation in Downtown Athens Task (DAT) and Crow Task (CT). I conjecture their graphing activity was constrained by the lack of conceiving quantities and their relationship in the situation. On the other hand, in her initial activity in DAT and CT, Ella demonstrated the ability to conceptualize the quantities' magnitudes (i.e., First American Bank's DfA and DfC) and represented these quantities' magnitudes by segments drawn in the situation and on the plane. Ella was the only student among others whose initial meaning of points included quantities and their relationship. I claim her ability to conceptualize the quantities in the situation supported her representing the quantities on the plane.

Since Melvin, Naya, and Zane's initial meanings included (transformed) iconic translation, I then provided opportunities for them to conceptualize quantities in the situation before engaging them in another graphing activity. I found that, once these students conceptualized quantities and their relationship in the situation, their graphical meanings were shifted from (transformed) iconic translation to either representing SQMO or representing QMO in NCP. From these findings, I conclude that my student's conceptualization of the quantities' magnitudes (e.g., the bike's DfA and DfC) in the situation was a critical development in the teaching experiment.

This conclusion is consistent with many researchers' findings (e.g., diSessa et al., 1991; Ellis et al., 2018; Frank, 2017; Johnson et al., 2017, 2020; Liang & Moore, 2020; Moore, Paoletti et al., 2013; Paoletti, 2015; Paoletti & Moore, 2017; Saldanha & Thompson, 1998; Stevens et al., 2017; Thompson, 2011) emphasizing the importance of conceptualizing quantities and their varying relationships in the situation as they showed that construction and interpretation of graphs can emerge as a re-presentation of quantitative relationships.

Representing the Quantities Magnitudes as Varying Bars on Parallel Magnitude Lines

Once students conceptualized quantities' magnitudes in the situation, I provided opportunities for students to represent them on the magnitude lines by a length of varying bars. My goal was to provide an opportunity for them to utilize one-dimensional space to re-present the quantities' magnitudes that they isolated and dis-embedded from the situation. I found that not all students assimilated the varying bars on the magnitude lines as I intended. Both Melvin and Zane initially imagined the physical bike, Arch, and Cannon on the magnitude lines and coordinated the spatial proximities as the bike moved. In turn, their graphing activity in two-dimensional space included imagining the bike, Arch, and Cannon on the plane and coordinating the spatial proximities. In contrast, Ella and Naya assimilated the bars on the magnitude lines in relation to the quantities' magnitudes in their initial activity. In turn, their graphing activity included representing quantities' magnitudes in two-dimensional space.

I compare and contrast the way Zane and Ella assimilated bars on the magnitude lines in order to elaborate on how these meanings had different implications when representing quantities in two-dimensional space. Zane conceived the objects from the situation (i.e., Arch and the bike) on the magnitude line, and he thus conceived the varying bar as illustrating the bike was getting closer to Arch. In two-dimensional space, and via conceiving Arch and Cannon on the axes, Zane attended to the spatial proximity between objects by coordinating the bike's location on the plane. Although Zane's form of reasoning was productive in terms of completing the goal of the activity as he perceived it (i.e., showing where the bike is in the plane), this creates a problem in the normative Cartesian plane that is based on two directed distances (i.e., horizontal and vertical). On the other hand, Ella conceived the length of the bar on the magnitude line as a representation of the quantity's magnitude that she conceived in the situation (i.e., the bike's DfA). In doing so, she conceived a constrain regarding how to represent the variation of a quantity on a magnitude line (e.g., only left and right on a horizontal line). Thus, she organized the space accordingly in later activities when considering two-dimensional space. The results illustrate that explicit attention to quantities in the situation and mapping those quantities' magnitudes onto the magnitude lines is necessary for re-organizing the space consistent with a Cartesian plane.

Melvin and Zane needed to go through the measurement activity in order to develop a meaning of a varying bar on the magnitude line in relation to quantity's magnitude, rather than imagining the psychical objects (e.g., the bike, Arch, and Cannon) on the magnitude line. I reported that after their measurement activity, both Melvin and Zane demonstrated shifts in their graphical activity in two-dimensional space, from envisioning points as objects by engaging in transformed iconic translation (NMO) to envisioning points as objects by coordinating the objects' quantitative properties (SQMO). From these findings, I conclude that being able to represent the quantities magnitudes as varying bars on the magnitude lines was a critical development in the teaching experiment as it laid a potential foundation for students to construct a Cartesian coordinate plane, which I discuss in the next section.

In addition to laying a foundation for constructing a Cartesian plane, representing quantities on the magnitude lines as varying bars afforded my students to determine a relationship between two quantities beyond directional change (i.e., gross covariation). Recall that Naya's conceived relationship between AoW and DoW in the swimming pool situation included coordinating the direction of change in both quantities (i.e., "if there is less water, there is gonna be less deep" and "when there is more water coming in there is more depth"). In turn, her graphing activity included drawing a straight line upward from left to right as an implication of her reasoning with directional changes. Then, I transitioned to Which One Task (WOT) that enabled her to determine the variation in variation.

WOT included a dynamic the pool animation and five directed bars that were located on parallel magnitude lines. The bars on the magnitude lines and the pool animation in the situation were not synced and I asked Naya to determine which of the four red bars, if any, accurately represented AoW in the pool as DoW varied. Note that all of the red bars directionally varied in the same way as the normative red one. Therefore, all the red bars could be a representation of AoW in the pool because all red bars satisfied the directional relationship with the blue bar. The design of the task afforded Naya engaging in reasoning with amounts of change in order to find which one was the normative red bar representing AoW in the pool. Naya successfully engaged in what Liang and Moore (2017) called "partitioning activity" (i.e., equally partitioning one of the quantities, then investigating the corresponding change in the other quantity) in order to determine how the red bars varied in relation to the blue bar. I believe that the figurative material, such as varying bars as a representation of quantities' magnitudes, afforded her to quantitatively operate on those bars by partitioning and comparing in order to determine how quantities changed.

Assimilating the Axes of the Coordinate Plane in Relation to the Magnitude Lines

Recall that Ella initially assimilated the points on the plane in relation to the physical objects that appeared in the situation, and her meanings for points were based in quantitative properties (i.e., representing SQMO). After Ella conceived the bars on parallel magnitude lines as an indication of quantities' magnitudes that she conceived in the situation (i.e., the bike's DfA and DfC), she organized the space accordingly in later activities as she assimilated the axes of the plane as orthogonal magnitude lines. Her explicit attention to quantities in the situation and mapping those quantities' magnitudes onto the magnitude lines supported Ella's re-organization of the space consistent with a Cartesian plane.

Note that being able to represent quantities' magnitudes on parallel magnitude lines—although it is foundational—does not always imply that students can immediately assimilate the axes of the plane as orthogonal magnitude lines. This is a critical cognitive ability to have in order to construct a productive meaning for a Cartesian plane (although there are other productive meanings). In my study, all students eventually assimilated the varying bars on parallel magnitude lines as a representation of quantities magnitudes; however, except Ella, none of the students immediately assimilated the axes in relation to magnitude lines in their subsequent two-dimensional graphing activities. Melvin and Zane represented SQMO and Naya represented QMO in NCP where they imagined the quantities' magnitudes varying *on* the plane in their respective frames of reference. For example, when representing SQMO, Melvin represented the quantities' magnitudes on the plane by assimilating a point on each axis as a reference object in which he measured the radial distances from. Alternatively, Zane represented the quantities' magnitudes vertically and horizontally on the plane by assimilating the entire axes as reference objects in which he measured the distances from.

Note that Naya was the only student who represented QMO (even before the measurement activity) although her organization of space was not compatible with a CCP in DABT. Her meaning of the points included a simultaneous representation of two quantities' magnitudes (i.e., the bike's DfC and DfA) that were represented vertically and horizontally on the plane. She assimilated the vertical axis labeled "Distance from Cannon" as representing her reference ray in which she measured DfC from. Similarly, she assimilated the horizontal axis labeled "Distance from Arch" as representing her reference ray in which she measured DfA from. From my perspective, Naya constructed a quantitative coordinate system that is operationally identical to the CCP with a nuanced difference in labeling the axis. Her "Distance from Cannon" label was not a label indicating that the horizontal axis was a magnitude line. Her label for the horizontal axis indicated that everything measured from the axis represented the bike's DfC, which implicitly made the vertical axis the magnitude line from my perspective. Conventionally, we usually label the number lines in which quantities' measures are represented;

however, Naya labeled her reference rays. This does not make her coordinate system non-quantitative. For this reason, I could not create an intellectual need for Naya to change her way of organizing the space. However, I still engaged Naya in Matching Game Task (MGT) where I provided an opportunity for her to assimilate the axes in relation to magnitude lines. My goal was to support her to structure the space in a way that her graphing activity became consisted with the conventional way of graphing on a Cartesian plane.

Although Melvin, Naya, and Zane could not represent the magnitudes on the axes of the plane, they were able to imagine the quantities' magnitudes—often represented by segments/bars drawn on paper—as something that they can mentally manipulate and move. Recall that Naya was able to transfer her image of the varying bars on two parallel magnitude lines to the vertical and horizontal segments on the plane representing the quantities' magnitudes. I infer at least they showed evidence that they conceived the segments/bars as movable material that could be manipulated. I believe this is an important cognitive resource in constructing a CCP as they need to imagine those segments/bars varying orthogonally on each axis of the plane. Thus, I engaged them in MGT where I provided them an opportunity to make connection between the parallel magnitude lines and the axes of the plane for the purpose of constructing a CCP.

Creating a Point on the Plane by Intersecting the Projections of the Magnitudes

I designed MGT in order to engage students to (i) represent two quantities' magnitudes on two parallel magnitude lines and then transition to (ii) represent two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and intersecting the projections of the magnitudes on the plane. For Naya, as I mentioned earlier, I conjectured that her engagement with MGT could leverage her to structure the space in a way that is compatible with a CCP. For other students, since their meanings of points included representing a physical object (i.e., the bike; SQMO), I conjectured that their engagement with MGT could promote an understanding of points as an abstract object that simultaneously represents two quantities' magnitudes (QMO).

I found that that MGT itself was not sufficient to promote students in creating a point on a CCP. When asked to create a single point that could represent both quantities' magnitudes given two parallel magnitude lines, Melvin, Naya, and Zane proposed the same idea. They suggested moving the magnitude lines and aligning them on top of each other so that the zero and max points were matched. They placed the bike in a certain place on the map in a way that the bike's DfA and the bike's DfC were equal to each other. Then, they observed that the tick marks representing quantities' measures on both magnitude lines were perfectly matched since the bike's DfA and DfC has the same magnitude. They then claimed that this tick mark showed *both* the bike's DfA and DfC. From their activity, I infer that putting the magnitude lines on top of each other and matching the two tick marks was a way for them to show that the single point simultaneously represented a state in which the bike's DfA and DfC were equal to each other. They could not find a way to create a single point that simultaneously represented other states of the bike's DfA and DfC.

They needed to organize the space in a way that they could satisfy the simultaneity for all states of the bike's DfA and DfC. This was a need, from my perspective, why we need a two-dimensional space. I didn't have evidence that students perceived such a need at the moment as they were satisfied with their activity to create a single point that simultaneously showed the bike's DfA and DfC for one occasion of the bike. This suggested that I should provide them with other opportunities that could afford them to structure the space in a way that they could see additional features that is not available in one-dimensional space. Thus, I transitioned to Crow Task (CT) where I provided them a coordinate system as a given space. My goal was to provide them with additional figurative material that might afford them to structure the space consistent with a CCP.

I found that, in their subsequent activity in MGT after their engagement in CT, Melvin, Naya, and Zane successfully manipulated the parallel magnitude lines to create a single point on a CCP. They started with two parallel magnitude lines, rotated one of them to make them orthogonal, and plotted a point where the projection of the magnitudes intersected on the plane. This re-organization of the space was an important moment for them to be able to use the plane that they constructed in order to create a single point that simultaneously represented two quantities magnitudes. Students subsequent graphing activities in the rest of the teaching experiment suggested that the *experience* of making parallel lines orthogonal was significant moment for them because, after this moment, they did not engage in representing NMO, SQMO, or QMO in NCP (except an occasion with Naya that I discuss under the section of "Competing meanings" below). Furthermore, representing quantities on magnitude lines played a significant role in their development as they subsequently and frequently referred to the variation of quantities' magnitudes represented *on the axes* when explaining how their graphs showed certain covariational relationship on the plane.

Conceiving a Line as Consist of Infinitely Many Points

Overall, supported by these four cognitive activities I discussed above, students successfully developed a meaning for points that simultaneously represented two varying quantities' magnitudes (i.e., representing a QMO in a CCP). Recall that my goal of the teaching experiment was to support students to develop emergent shape thinking. Moore and Thompson (2015) and Frank (2017, 2018) emphasized the important role of forming a multiplicative object in students' development of emergent shape thinking. What I have found in my study is that multiplicative object is a necessary but not sufficient condition for emergent shape thinking. I found that Zane constructed a meaning for individual points as a QMO, and he could do the finger tool activity (i.e., moving index fingers on each axis) smoothly to represent continuous, simultaneous covariation of two quantities. However, he interpreted his activity of connecting points on his graph as representing a direction of movement of one point from one location to another; hence he did not imagine a segment or a line as including infinitely many points and representing every moment of the quantities' covariation. In turn, he did not perceive an emergent trace generated by a continuously moving point as a line. Thus, I claim that Zane's meanings for lines and points constrained him from developing a meaning of graphs consistent with emergent shape thinking (i.e., a graph as an emergent trace of points). On the other hand, Melvin's graphing activity was suggestive of emergent shape thinking because his understanding of a graph (i.e., a line) included both what is made (i.e., infinitely many points that represents the corresponding magnitudes) and how it is made (i.e., covariation of the coupled quantities).

Competing Meanings

In the previous section, I demonstrated the students' construction of meanings for graphs as I implemented an instructional sequence to support them developing emergent shape thinking. Overall, supported by each of the five cognitive activities, students demonstrated shifts in their graphing activity, from meanings that were based in iconic translation (i.e., representing NMO) to the meanings that were based in quantitative and covariational reasoning (i.e., representing SQMO, then, QMO). Although I designed and attempted to apply an instructional sequence to support students in developing new meanings systematically to produce a desired coherence, I found some instances where students drew on older meanings abandoning the new meanings that they developed. In this section, I discuss those moments in the context of *competing meanings* where students operate on different meanings that were compatible or incompatible in order to engage in their goal-oriented activity.

I present an instance of competing meanings from the case of Melvin. Recall that, in his final activity with CT, he was able to create a point on the plane where the projection of the quantities' magnitudes intersected in order to represent the crow's DfA and DfC (i.e., representing a QMO) given a location of the crow on the map. Then, I hid the crow on the map and asked Melvin where the crow would be based on a given point on the plane. When solving this task, Melvin surprisingly drew on his transformed iconic translation meaning (i.e., representing NMO) to locate the crow on the map. After he imagined rotating the vertical axis and placing it on top of the map, Melvin associated the spatial location of the point on the plane and the potential location for the crow on the map. Interestingly, I had evidence that Melvin could also drew on his quantitative meanings when locating the crow on the map. However, he abandoned this meaning because there was no unique location for the crow on the map when using the quantitative information (i.e., the crow's DfA and DfC) that he could reveal from the given point on the plane. This suggested that he held two competing meanings (i.e., NMO and QMO) when figuring out where the crow would be on the map given a point on the plane. By abandoning the meaning he recently developed (i.e., QMO), Melvin chose to draw on his initial meanings (i.e., iconic translation; NMO) because he was able to easily locate the crow on the map since its location was unique. From his perspective, he was able to get to an answer by using iconic rather than quantitative meanings.

For another illustration of competing meanings, I present an instance from Naya's activity in Swimming Pool Task (SPT). Recall that Naya represented QMO in a NCP (even before the measurement activity) in DABT. She represented the quantities' magnitudes horizontally and vertically on the plane by using the axes as her reference rays. Also recall that Naya was able to represent QMO in a CCP after she went through the intervention including MGT. For example, in SPT, she was able to represent the quantities' magnitudes (i.e., AoW and DoW) on the axes of the plane and create a single point on the plane where the projections of the quantities' magnitudes intersected when the quantities were static. When asked to create a representation on the plane when both quantities varied in tandem, she first imagined varying bars on the axes of the plane, however, she could not imagine how the corresponding point could move on the plane. Naya claimed that the point could be anywhere on the plane as she imagined the bars moved on the axes. This idea later didn't seem reasonable to her as she said "I changed my mind. I was gonna say the point could be anywhere, but it can't I guess." Since she could not imagine where the point would be on the plane as she imagined varying bars on the axes, she could not draw a graph. By drawing on her canonical meanings, Naya could have imagined a moving point at the intersection of the projections of the bars and be able to trace that point on the plane as the magnitudes varied on the axes. She couldn't enact these actions because she was perturbed by the dynamics of the setting. Naya then abandoned her canonical meanings and draw on her original meaning of non-canonical plane because her original meaning helped her to reach her goal (i.e., graphing the relationship between AoW and DoW). She attempted to draw on her canonical meanings at first by imagining the varying bars on the axes, but she could not keep up with this dynamic image because she couldn't imagine (and trace) the moving point on the plane (i.e., "the point can be anywhere"). Then, she tried the other meaning and it worked.

Note that there was no way for me to perturb Naya when she was representing QMO in NCP. When she already had a stable non-canonical meaning, I wanted her to develop a different meaning (i.e., canonical) without perturbing the other meaning. Consequently, she drew on the original meaning when the new meaning did not afford her to solve the given task.

These two examples from my study showed that there are different meanings students can draw on in solving problems. Then, our job, as a researcher, is to identify in what ways these meanings are potentially compatible and incompatible and how we can create opportunities for those students to consciously deal with those compatibilities and incompatibilities. The different meanings Naya demonstrated were compatible (i.e., QMO in CCP vs. QMO in NCP) for the reasons I discussed in the previous section (i.e., operationally they are identical except a nuance difference in labelling convention). I did not want to create a perturbation that could imply she was wrong because she was not wrong. Her mathematical operations were sophisticated. They were just not canonical. Thus, I did not have much of a choice other than showing the canonical way of graphing, although I hesitated to do that because there was no intellectual need for that. What I would have liked to do is to have Naya engage in both meanings and have her reflect on her meanings. I would have hold both canonical and non-canonical graphs next to each other and engage Naya to reflect on the similarities and differences between the two and identify that it is a matter of conventional choice. However, at the time of data collection, I did not necessarily see that. And even now, reflecting retrospectively, I am not sure whether or not that would have been an appropriate move because middle school students might be too young for that kind of meta-reflective thought.

Implications for Research, Teaching, and Curriculum

Framework for Representing Multiplicative Objects

In my study, I investigated the types of reasoning that might engender productive meanings for representing quantitative relationships in one- or two-dimensional spaces. I identified various ways students organize two-dimensional space to construct or make sense points within those spaces. Drawing on these findings, I constructed an empirically grounded framework that characterizes students' meanings of points in terms of multiplicative objects. In addition to articulating important ways of thinking, this framework will enable researchers and teachers to be more attentive to those meanings students hold for their representational activity in both their research and instruction.

Representing NMO

Researchers (i.e., David et al., 2018; Frank, 2016, 2017; Lobato et al., 2012; Thompson & Carlson, 2017; Thompson et al., 2017) pointed out that students might be able to successfully plot points on the plane by carrying out a certain procedure (e.g., over and up). Echoing these aforementioned researchers, although students could successfully plot points, I suggest researchers and teachers should seek to understand students' thinking as their meanings of points might include solely an ordered pair of numbers and/or a location on the plane that do not symbolize or unite two quantities' magnitudes and measures.

For example, in their classroom-based study, Lobato et al., (2012) compared two classrooms in which students in each class attended to different features of points on a graph, lines, and slopes. When dealing with points on graphs, students in Class 2 noticed *physical locations*. For example, their graphing activity included "plotting the point (2, 5) by going over 2 'boxes' on the coordinate grid system and up 5 boxes (regardless of the scale of the axes)" (p. 448). On the other hand, students in Class 1 noticed *pairs of quantities*. For example, their graphing activity included "plotting a point by extending lines to the axes and speaking of the quantities represented by the axes, for example, explaining that (2, 5) means 5 m in 2 s" (p. 448). Lobato et al. also provided evidence of how those different meanings of a point on a graph influenced students' noticing in later classes when they work on lines and slopes. When working with lines, students in Class 2 noticed *physical objects* whereas students in Class 1 noticed "a collection of paired quantities and quantitative relationships between the values represented by points on the line." They also reported that, when dealing with slopes, Class 2 students noticed the *steepness of physical objects* (not surprisingly), whereas students in Class 1 noticed a *measure of a relationship between quantities*. Their result supports the idea that students might be able to plot or locate a point on the plane given a coordinate pair by carrying out a certain procedure (e.g., over and up), but the meaning of these points might not be connected to the quantities at all. Being able to represent information about two quantities as a single object requires more than a procedure. I believe their result also suggests that we should have students develop productive meanings for a single point in a coordinate system prior to developing meanings for graphs (e.g., lines).

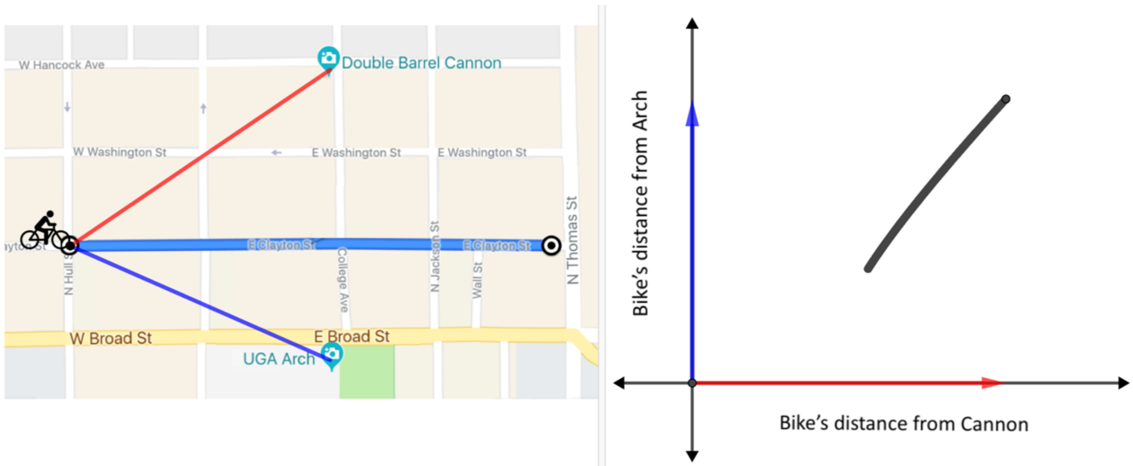
Researchers (e.g., Janvier, 1998; Hadjidemetriou & Williams, 2002; Leinhardt et al., 1990; McDermott et al., 1987) have been considering iconic translation as a difficulty or challenge in students' graphing activities. However, I consider those instances in which students make (transformed) iconic translation as student's sense making process. Students solved the given task (e.g., DAT) by assimilating what I perceived to be a graph on a Cartesian plane as a picture of a situation (or a transformed version of it) and envisioning the points as a location/object. Furthermore, by distinguishing transformed iconic translation from iconic translation, I was able to create a more sophisticated model of students' activity when they are asked to create or interpret graphs. Creating such a construct allowed me to name students' brilliance and report out how their activity was viable for them to reach their goal although they were not able to reason with quantities at the moment. It was my job, as a teacher-researcher, to create environments that could perturb them and develop quantitative meanings for points on the plane.

Moreover, students who represent a NMO assimilated the segments drawn in the situation and on the plane as a visual connection (i.e., bars as a shape/wire) between two locations, as opposed to being representative of distances, and assimilated points as objects through (transformed) iconic translation. The meanings of the segments and the meanings of the points were compatible so that they formed a system that is internally viable for students. When working with students, especially when having a situation where the quantities' magnitudes are visually available, we need to make sure how students could assimilate those visuals. This result is especially important as there is a growing body of researchers that emphasize and suggest having students working with quantities magnitudes.

Representing SQMO

By introducing SQMO in my study, I develop a model of my students' graphing activity in which they envisioned points on the plane as a location/object by focusing on the object's quantitative properties and engaging in quantitative reasoning. As an implication of this construct, we can now extend the idea of multiplicative object from the coordinate plane to the real-life situations. For example, with the idea of SQMO, we can consider a moving object's path as a graph as it also represents a quantitative relationship. I elaborate what I mean as follows.

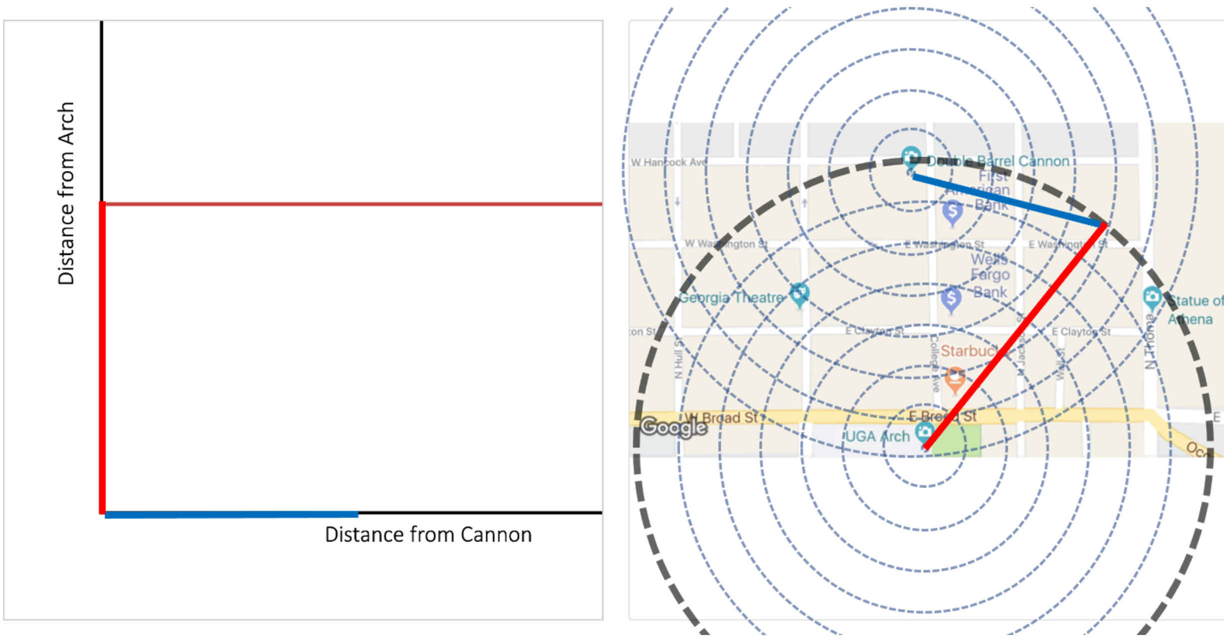
Assume that we are given DABT (see Figure 0.6a) and asked to graph the relationship between the bike's DfA and DfC. We can think about the situation as composed of two magnitudes. We can conceive these magnitudes in a certain relationship dictated by the bike's path on the map. Then, we can choose to represent these quantities in a Cartesian plane (Figure 0.6b). Thus, we can put those magnitudes on Cartesian axes and imagine them varying and make sure we preserve the same relationship. Then, we could produce a point to simultaneously represent the two magnitudes on the plane, and create a trace (i.e., a graph; see https://youtu.be/bedz94TjoPY). This activity can be considered as representing QMO in a CCP (or in a quantitative coordinate system; Lee et al., 2020).



(a) (b)

Figure 0.6. (a) DABT and (b) A graph that represents the bike's DfA and DfC on a Cartesian coordinate system.

Assume that, now, we are given a graph on a CCP that represents the relationship between the bike's DfA and DfC (see Figure 0.7a) and asked to draw an appropriate path on the map that the bike could take according to this graph. To do that, we can think about the graph as composed of two magnitudes. We can conceive these magnitudes in a certain relationship dictated by the graph on the Cartesian plane. Then, we can represent these quantities in a bi-radial coordinate system overlaid onto the map and imagine them varying in a way that maintains their relationship. Then, we can produce a point, which will be the bike in this case, to simultaneously represent the two bi-radial magnitude on the map and create a trace (i.e., the path; see Figure 0.7b), which is an indication of representing a SQMO. Therefore, an object on a map can also be a representation of a multiplicative object as long as the person reason with the object's measurable attributes. Thus, creating such a construct, SQMO, not only allowed me to develop a model of my students' thinking, but also provided an opportunity for us to consider creating the road/path on the map as being equivalent to a graphing activity and be able to see the path as a graph that represents certain relationship between two quantities.



(a) (b)

Figure 0.7. (a) A graph that represents the bike's DfA and DfC on Cartesian coordinate system and (b) A two-center bipolar coordinate system overlaid onto the map.

Representing QMO: Type 1 and Type 2

Given the importance of reasoning emergently in developing productive meanings for graphs (Frank, 2017; Moore & Thompson, 2015), by distinguishing two types of representing QMO (i.e., Type 1 and Type 1), I note that conceiving a point as a multiplicative object is necessary, but not sufficient in envisioning "graphs as composed of points, each of which record the simultaneous state of two quantities that covary continuously" (Saldanha & Thompson, 1998, p. 298). As I illustrated in the results, students whom I classify engaging in representing Type 1 QMO can conceive a point in terms of multiplicative object, but they do not imagine a trace being produced as representative of that multiplicative object. Given there are numerous students (i.e., about 89% of secondary students [*N*=1798], as reported in Kerslake, 1981) not conceiving of infinitely many points on a line and believing there is no point on a line until they are plotted (Mansfield, 1985), it becomes important for us to be able to determine which type of multiplicative object the students are forming and representing. In doing so, we can inform our instruction to foster and support students in developing productive meanings for graphs.

Although I report on findings from middle school students, the implication of the results of my study regarding Type 1 QMO is important for both the secondary and the undergraduate mathematics education community. I am drawing on the construct, emergent shape thinking, that was initially developed from undergraduate students. The results of this study regarding middle school students' meanings afford us a better understanding of students' graphing activity, including their construction of emergent shape thinking. Prior researchers (e.g., Frank, 2017) have reported on students' difficulty with constructing an emergent meaning for graphs, and the case of Zane suggests that one source of such difficulty can be students' meanings for lines and points in general. If students conceived of a line as showing *one* point's movement, it would be difficult for them to assimilate a line graph as an emergent trace that includes *infinitely many* points, despite being able to conceive each point as a representation of two quantities' magnitudes (and/or values) at a particular moment. Consequently, they would need to physically plot points to show quantities' magnitudes (and/or values) at any specified moments (i.e., a pointwise meaning for a graph). Thus, I conjecture that meanings for lines and points consistent with Zane's might help explain students' difficulties with constructing and representing smooth and continuous images of covariation in graphical contexts (Castillo-Garsow, Johnson, & Moore, 2013).

My analysis also supports the affordance of Castillo-Garsow's (2012) distinction between "a problem situation, the method used to solve it, and the reasoning that derives or selects that method" (p. 56) when characterizing students' covariational reasoning. Zane conceptualized two quantities' covariation continuously regarding the pool situation; however, he used a discrete method (i.e., plotting points) when constructing his graph due to a constrain implied by his meanings of lines. I argue that it is important for researchers to be aware of such a discrepancy and be aware that a pointwise graph activity (i.e., plotting the points first, and then connecting them) does not necessarily imply that their images of quantities' covariation is discrete, and vice versa. When characterizing students' covariational reasoning, it is important for us to take into account students' activities and reasoning in various context (e.g., situations, graphs in different coordinate systems, and number lines) before we make claims about their covariational reasoning. I believe identifying this inconsistency and its causes can allow us to better advance students' understandings of graphs.

Students' Reasoning in Dynamic Situations

In this study, I illustrated various ways in which students reasoned in dynamic situations (i.e., quantitative covariational reasoning, spatial proximity reasoning, and matching the perceptual or spatial features of motion in two difference spaces). These different ways of conceiving dynamic situations suggest that it is really important for us, as researchers and teachers, to understand how students conceptualize the objects varying attributes in the situation. We should not assume that the variation that students might conceive is the same as *given* in the dynamic situation. Thompson and Carlson (2017) pointed out that researchers might conceive the varying quantities in a situation; however, it might not be the case for students. In their extensive literature review, Leinhardt et al. (1990) reported that researchers often thought that variation has something to do with the situation only as opposed to acknowledging the importance of conceptualizing the variation in students' mind. Thus, merely offering dynamic situations to students does not imply that students will engage in quantitative covariational reasoning. As my study showed there are many ways students could reason, we need to understand how they conceive dynamic situations and how they imagine the quantities' magnitudes vary.

I believe the distinction I made in this study regarding students' quantitative covariational reasoning and spatial proximity reasoning could provide a lens for researchers to investigate students' graphing activity especially in contexts in which a *motion* occurs. Since these embodied activities often involve graphing with a motion sensor technology, students could either coordinate an object's distance from the motion detector or coordinate the object's proximity to the motion detector when creating or interpreting a graph. Researchers (e.g., Nemirovsky et al., 1998) reported that students could produce a normative graph (or interpret a graph normatively) as a representation of the motion although they don't attend to the quantities in the situation, such as distance between the object and the motion detector. For example, when engaging in creating graphs generated by a motion detector representing how a handheld button's distance from the detector varied over time, Nemirovsky et al. (1998) reported that students determined "the line on the screen becomes higher when one moves the button away from the tower and lower when one moves the button" (p. 122). Nemirovsky et al. interpreted that, for the students in their study, distance between the button and the tower (i.e., motion detector) was not related to meters, instead it "concerned the possibility and impossibility of creating certain graphical responses" (p. 164), which is somewhat related to spatial proximity reasoning. They noted that distance "is about the impossibility of certain actions that would be possible if the objects were nearby, not a matter of meters" (p. 164). Therefore, I believe the construct, spatial proximity reasoning, can be used as an additional lens when investigating students' representation of motion as a graph.

Developing Meanings for Graphs from Magnitude Lines (i.e., Empty Number Lines)

A relationship between two quantities is often represented in a Cartesian plane formed by two perpendicular number lines creating a two-dimensional space. The idea of representing quantities' values or magnitudes on number lines is often taken for granted by researchers when discussing students' construction of graphs (Lee, Hardison, & Paoletti, 2018). This is not surprising given that the focus of instruction is given to bar graphs and pointwise readings in elementary and secondary education. The researchers (Earnest, 2015; Schliemann et al. 2013) who have studied the relationship between the conception of number lines and coordinate systems have focused on individual points on the number lines and the intervals between them. Their focus did not give attention to representing quantities' magnitudes and continuously changing quantities.

In light of the essential role of quantitative and covariational reasoning in mathematical development, I design my study so that I can understand opportunities for students to participate in engaging with quantities' magnitudes in dynamic real-world situations. Relatedly, in this study, I provided opportunities for my students to engage in representing quantities' magnitudes by varying length of directed bars placed on parallel empty number lines, what I call *magnitude lines*, in one-dimensional space (i.e., a dynamic tool designed in GeoGebra). Thus, I extend the aforementioned work (e.g., Earnest, 2015; Schliemann et al., 2013) by investigating the nature and extend of students' ability to represent varying quantities' magnitudes on number lines, and how those abilities influence their meanings of graphs.

I reported developmental shifts of four middle school students' graphing activities as they engaged in dynamic situations. Students' initial meanings of graphs included iconic translation (i.e., picture of the situation) and representing the literal motion of the object that moved in a situation. As I implemented the instructional sequence that incorporated the magnitude lines tool, their meanings shifted to include quantities and their relationships, including how a graph is a record of the simultaneous variation of two quantities' magnitudes. I infer my students' actions to suggest that the experience of making parallel lines orthogonal was significant moment for them because, after this moment, they did not engage in iconic translation and/or coordinating spatial distances from the axes of the plane. Furthermore, representing quantities on magnitude lines played a significant role in their development as they subsequently and frequently referred to the variation of quantities' magnitudes represented on the axes when explaining how their graphs showed certain covariational relationship on the plane. This suggested that students should be provided opportunities to construct productive meanings for coordinate systems building on their meanings of number lines prior to graphing relationships on a *given* two-dimensional system.

Developing meanings for coordinate systems prior to doing so for graphs is helpful for students to develop "of meanings of each and the construction of contextual interrelationships among them" (Thompson, 2008, p. 47), which yields a coherence in a logical progression of topics. Merely having coordinate systems before graphs as an independent topic might not result in coherence. Instead, the ideas, meanings, and ways of reasoning should be emphasized in order to connect the goals of learning for each topic as pointed out by Schmidt et al. (2005). They asserted that school mathematics curriculum can be coherent if the practices of school mathematics "are articulated over time as a sequence of topics of topics and performances consistent with the logical and if appropriate, hierarchical nature of the disciplinary content from which the subject-matter derives" (p. 528). Therefore, the intention of being coherent in an instructional design about graphs makes more sense if we think about coherence as the product of the meanings driving the instruction on coordinate systems creating the essential understanding of graphs. Relatedly, we must apply the same understanding of coherence into the connection between the number line and Cartesian planes as the construction of a Cartesian plane requires conceiving of two number lines—one for each quantity whose magnitude or value varies on it—and using them to create a two-dimensional space (Lee et al., 2018).

Directions for Future Research

Developing epistemic students

In this study, I reported on six middle school students' reasoning as they constructed and represented relationships between covarying quantities through graphs. Building on my work with middle school students' mathematical learning, I will test the viability of identified ways of thinking by working with students at various grade levels and by using different tasks with different contextual features. This work will contribute to the integral part of viability of my models of student thinking and developing epistemic subjects. Moving forward, building on my work with students' mathematical learning in small-group settings, I would like to shift to mid-scale, design-based exploratory studies in whole-classroom settings. I would like to identify what support structures, in different environments, improve student learning and examine ways in which teachers/instructors can be supported to foster productive meanings for determining and representing quantitative relationships.

Extending the use of magnitude lines in connecting multiple representations

Expanding my focus beyond students' graphical meanings, I would like to use this work to inform the research and practice regarding multiple representations of functions with the intention of developing a more coherent system of representations of quantitative relationships. For example, I would like to investigate how engaging with the *magnitude line tool* could support students to construct quantitative structures, graphs and form equations, in turn, their learning of graphing and solving equations, inequalities, and functions

In-depth analysis of student's meanings of lines

Type 1 and Type 2 conceptualizations of lines and points could be related to Lakoff and Núñez's (2000) distinction between two very different conceptualizations of space: *naturally continuous space* and *the-set-of-points conception of space.* They claim that naturally continuous space is "our normal conceptualization" because we function in the everyday world by using our body and brain unconsciously. They considered the-set-of-points conception of space as a reconceptualization of the first, via conceptual metaphor. This reconceptualization process is called "the program of discretization." The program of discretization is a strategy to replace naturally continuous space with an infinite set of points for the sake of arithmetization, symbolization, and formalization to move from intuitive to rigorous mathematics. Since there is a rich history of different conceptions of lines and points, we should expect to see important and potentially related distinctions with students' thinking. Identification of Type 1 and Type 2 in my work suggests as such and further research is necessary to have in depth investigation of students' meanings of points and lines in the light of this historical trail of the different conceptions of the line. Moreover, further research is necessary to investigate how a student whose meaning aligned with Type 1 could develop a conception of a line as being created by infinitely many points and/or conception of points comprising a line.

Potential implications of Type 1 on learning other topics

In this study, I found a student, Zane, who interpreted his activity of connecting points on his graph as representing a direction of movement of one point from one location to another; hence he did not imagine a segment or a line as including infinitely many points and representing every moment of the quantities' covariation. In turn, he did not perceive an emergent trace generated by a continuously moving point as a line. Thus, his meanings for lines and points (i.e., Type 1 QMO) constrained him from developing a meaning of graphs consistent with emergent shape thinking. Future research is needed to understand if the inability to see a line consisting of infinitely many points create any implications or difficulties in students' learning of other topics, such as linear equations in the context of graphing or line of best fit in the context of a scatterplot. For example, maybe Type 1 and Type 2 has different implications on the nature of students' approaches in solving systems of linear equations. Maybe, students who don't imagine a line as including infinitely many points tend to use algebraic approaches more than graphical approaches when solving linear equations.

Furthermore, there is a good chance that Zane's Type 1 meaning for a line might be related to his meaning constructed outside a graphing context prior to the study. Paul Klee—a famous painter and a scholar in German art schools (e.g., Bauhaus and Kunstakademie Düsseldorf) whose ideas inspired many school curriculum—defined that "a line is a dot that takes a walk" (Ringe, 2021). Moreover, in the National Core Arts Standards, a line is defined as "the one-dimensional path of a dot through space used by artists to control the viewer's eye movement; a thin mark made by a pencil, pen, or brush" (Washington Arts K–12 Learning Standards, 2017; p. 136). These definitions are compatible with my model of Zane's meaning of a line (i.e., "a line shows where the dots go"). Given the results provided by this study and other researchers (e.g., Manfield, 1985; Kerslake, 1981) regarding students' understanding of lines and points, it is important that we take into account students' meanings for lines and points that they might learn outside of a math class and investigate how those meanings may influence learning math concepts that includes lines and points.

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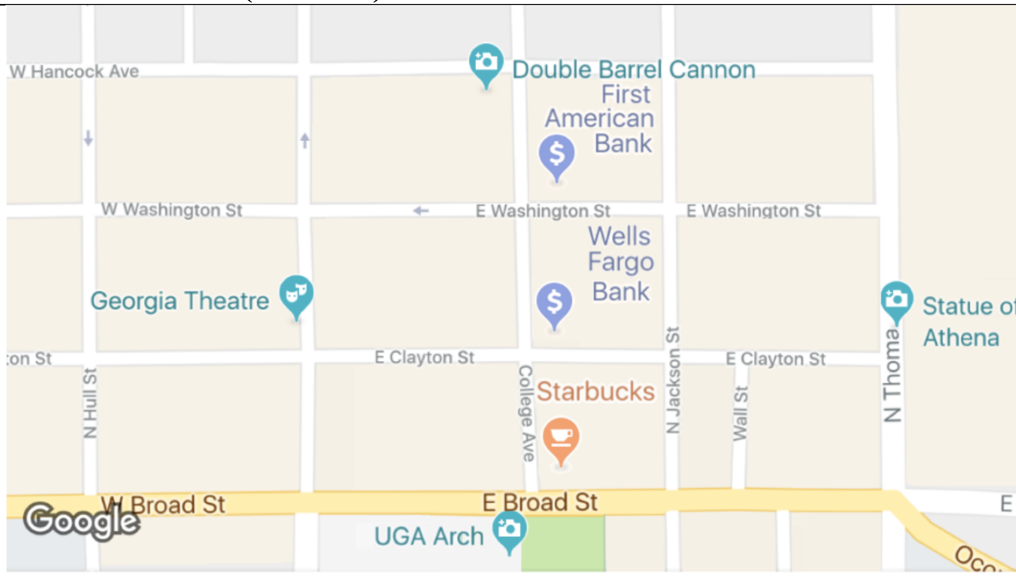
APPENDIX A

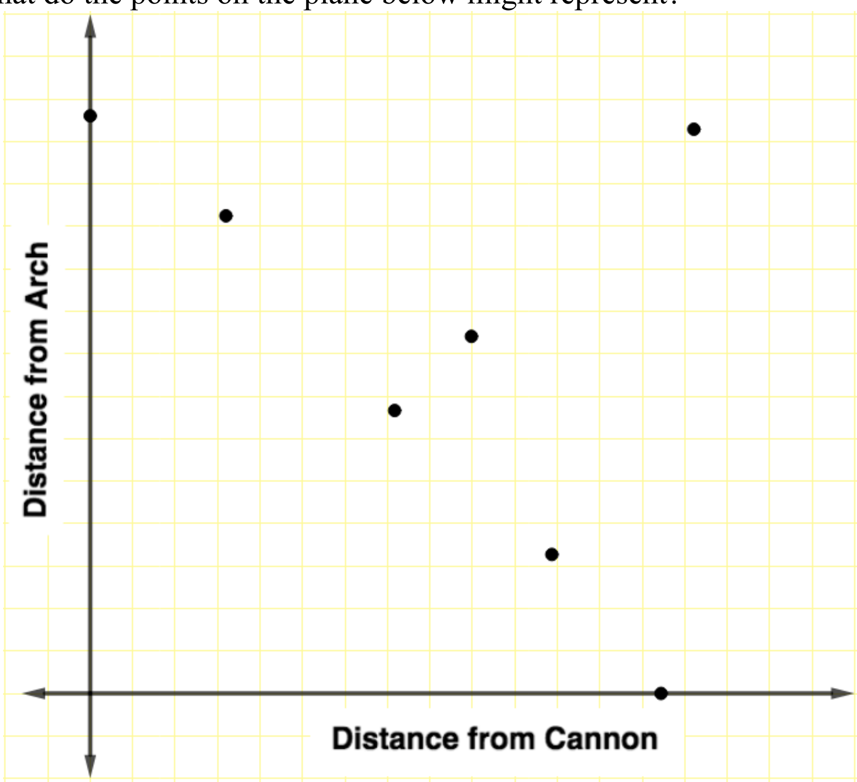
TASK PROTOCOLS AND HANDOUTS

Downtown Athens Task (Protocol)
Open Downtown Athens Task (https://www.geogebra.org/m/qpvf5ytb). *Show the coordinate system only. Hide the screen 1.* 1. What do each of these points tell you? 2. What does these lines might tell us? Label? 3. Does this image remind you anything that you learn in the school? Show the map 4. Any observations? Ask to point to the Arch and the Cannon. 5. How do you think what the points here (on the Cartesian coordinate plane) might represent? 6. What do you mean this point is [Starbucks]? What does that mean? How do you know? 7. Pick a point and ask the following question: What does this point represent? What information does this point give us? 8. How do you see distance from Arch and Cannon on the map? For example, how could you make sense (measure) Starbucks' distance from Arch? *If they don't come up with "direct distance," show them the following demonstration.* a. Open Google Map. Go to Athens Downtown. b. Right-click on your starting point (UGA Arch). c. Choose Measure distance. d. Click anywhere (Starbucks) on the map to create a path to measure. e. Optional: Drag a point or path to move it, or click a point to remove it. f. At the bottom, you'll see the total distance in miles (mi) and kilometers (km). **Extension:** Open Crow Task (https://www.geogebra.org/m/kzddhqs8). Students see a crow that they can control freely by dragging it and see how the corresponding point in the coordinate system changes. Move the crow by dragging it. 1. What is going on? Brainstorm observations and questions. 2. Why do you think the point on the coordinate system gets closer to the axis "Distance from Arch" as the crow flies toward to Cannon? 3. Why do you think the point on the coordinate system gets closer to the axis "Distance from Cannon" as the crow flies toward to Arch? 4. Why do you think that the point on the coordinate system moves the way it moves as the crow files in equidistance from each location (Arc and Cannon)?

Downtown Athens Task (Handout)



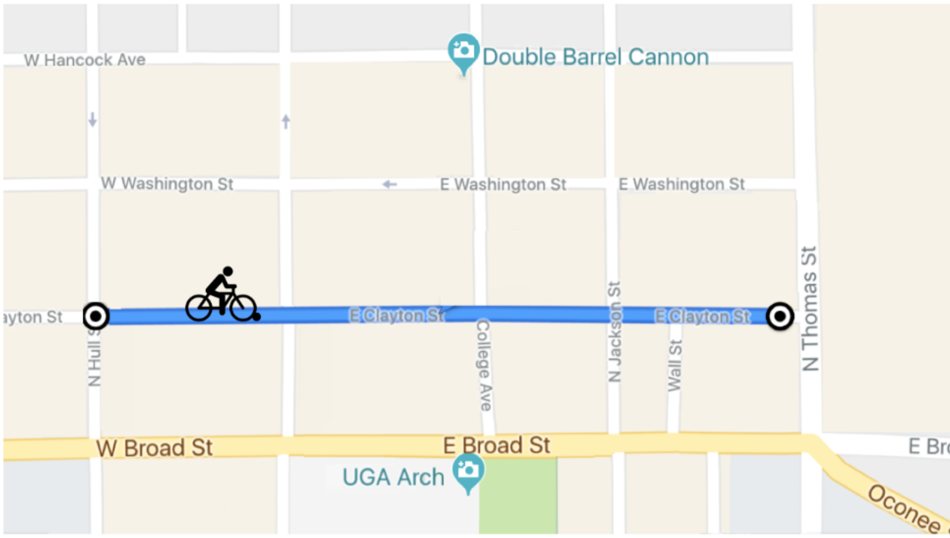
The map above shows Downtown Athens. There are seven locations labeled, UGA Arch, Cannon, Georgia Theater, First American Bank, Wells Cargo Bank, Statue of Athena, and Starbucks. What do the points on the plane below might represent?



Downtown Athens Bike Task (Protocol)

Open DABT (https://www.geogebra.org/m/pzxambfg). The bike rides on Clayton St in Athens Downtown at a constant speed starting from the West side of the street. The animation represents the bike's ride back and forth.
 	
1. What's going on here?
• (supporting questions) What are the things you could consider varying and possible to measure?
  o Tell each other, and then me, some quantities in the video that were changing and some that were unchanging as the bike rides on Clayton St. back and forth?
  o Say more.
• What's meant by distance from Arch? Distance from Cannon? [*see how they are interpreting "distance" (e.g., distance traveled, the crow flies distance)*].
a. Describe how the bike's distance from Arch changes as the bike moves along the road. How do see it (decrease/increase) in the situation?
b. Describe how the bike's distance from Cannon changes as the bike moves along the road.
c. Describe how the bike's distance from Cannon changes in relation to the bike's distance from the Arch.
 	
2. Focus on the bike's distance from Arch (or Cannon) as it travels on Clayton St back and forth.
a. Place your index fingers on the table. Your left index fingers will be fixed on the table. As the bike rides along the road, move your right index fingers left to right to indicate an increase or decrease in the quantity's magnitude in a way that the distance between the index fingers represent the bike's distance from Arch (or Cannon).
b. [*If they mimic the bike on the map*] Now, I want you to place your fingers on the table side by side horizontally. You will still move your right index finger and the distance between your index fingers will still represent the bike's distance form Arch and Cannon, but you will move your right index finger horizontally.
 	
3. Focus on the bike's distance from Arch and Cannon as the bike rides on Clayton St back and forth. Work in pairs.
a. [*Students will be assigned to each quantity*] Using the pointer fingers, move your hands so that the distance between your pointer fingers represents the bike's distance from Arch and Cannon accurately as the bike rides along the street.
b. What if the bike was travelling on Washington St. back and forth? How would you change your answer?

Downtown Athens Bike Task (Handout)



The bike rides on Clayton St in Athens Downtown at a constant speed starting from the West side of the street. The animation represents the bike's ride back and forth.

1. What's going on here?
    a. What are the things you could consider varying and possible to measure?

2. Focus on the bike's distance from Arch as it travels on Clayton St back and forth.
    a. Place your index fingers on the table.
    b. Your left index fingers will be fixed on the table.
    c. Move your right index fingers left to right so that the distance between your index fingers represents the bike's distance from Arch as the bike rides along the road.

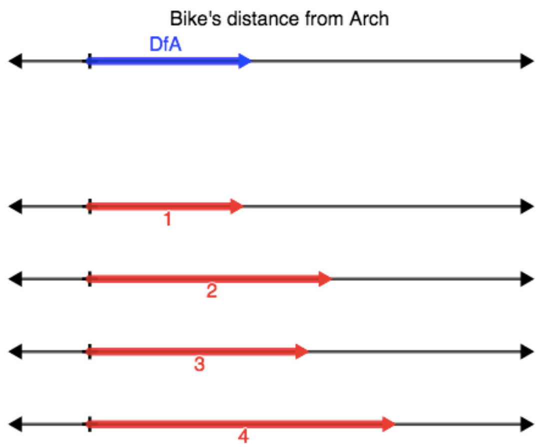
Measurement Activity (Protocol)

Open Measurement Activity (https://www.geogebra.org/m/rme8w8yk). The animation represents bike's ride on Clayton St. Now, the bike stops when it arrives to N. Thomas St. We want to represent the bike's distance from Arch with a bar.
1. Look at the map and focus your attention to the bike's distance from Arch. Then, go to next page (do not see the map), and drag the point and locate the bar's endpoint on the line in a way that it has approximately the same length of the distance from Arch by estimating what you see on the map.
    a. Compare your estimation with your friend. Explain, if any, the strategy that you have to place the bar.
2. Look at the map again. Use a piece of string or a straight edge (that is longer than the magnitude of the bikes' distance from Arch) and drag and locate the bar's endpoint on the line in a way that it exactly the same length of the distance from Arch. Work in pairs.
3. Go back to the map. Change the location of the bike. Use the tools that you pick [*e.g., the orange (or green) segments of the same length that is less than the bike's distance from Arch, but each group of students will pick a different fixed-length line segments as a unit of measure that has the different length*] and drag the point and locate the bar's endpoint on the line in a way that it is exactly the same length of the distance from Arch. Work in pairs.
  a. How did you come up with your result?
  b. Compare the length of your bar with your friends.
    i. (If they are not the same, ask them to work on together. They should be the same)
    ii. (If they are the same) What is the same and what is the difference?
    iii. What would be the value of the bike's DfA if you use a unit that is one-third of the green segment? [*ask them to write it down on paper together with other values*]
    iv. Explain what is changing and what is not changing?
4. What do the endpoint of the bar represent? What do the direction of the bar represent?
5. Change the length of the bar and find where the bike exactly should be on the map. [*Put a length of bar in a way that it is shorter that the bike's minimum and maximum distance from Arch along the road, and then find the location where the bike should be on the map*].
6. Hit the play. Look at the varying bar together with the animation that shows the bike that is moving accordingly.
  a. Describe how the bar changes as the bike changes its location.
  b. Does your description match with the description that you made in the previous task (when engaging in the situation)?
  c. What do the end point of the varying bar represent? How do you know it represents that on the line?
  d. How does this point correspond to the situation? Can you show me on the situation?
  e. When coming back, the bike moves to the left. But the bar moves to the right at some point. How can that happen?

Which One Task (Protocol)
Open the task (https://www.geogebra.org/m/fnmjxa6d). Hide the screen with the bars. Students only see the bike animation. The animation represents the bike's ride on Clayton St in Downtown Athens. 1. Describe in words how the bike's distance from Cannon changes in relation to the bike's distance from the Arch. Show the screen with bars. The blue bar represents the bike's distance from Arch as it travels along the road. The red bars changes depending on the blue bar. 2. Identify which of the four red bars, if any, accurately represent the bike's distance from Cannon as the bike's distance from Arch varies. You can change the length of the blue bar by moving the bike in the situation or by clicking the "play" button. You are not allowed to measure. After you decide which one, label it accordingly. 3. Hide the bike. Do question #1. In this version, I am adding more challenge. Basically, the task is the same, but students will not be able to see the bike. Make sure the students conceptualize both quantities changing at the same time simultaneously. Emphasize the tight coupling idea. A change in DfA means necessarily a change in DfC. Pass the handout with grid. • Take a couple of minutes and draw a rough sketch of the relationship between the bike's distance from Canon and the bike's distance from the Arch on the paper provided. Label your graph as "a first draft." [*keep the animation running*] Extra questions: • [*move the bike in a place on the map and ask*] Does your graph show this moment? How do you know? • Why do your graphs consist of points? • How confident are you about your idea? • Does each part of the graph 'fit' with your idea? • Can you convince someone else that your explanation of the graph is sound?

Which One Task (Handout)

Which one?

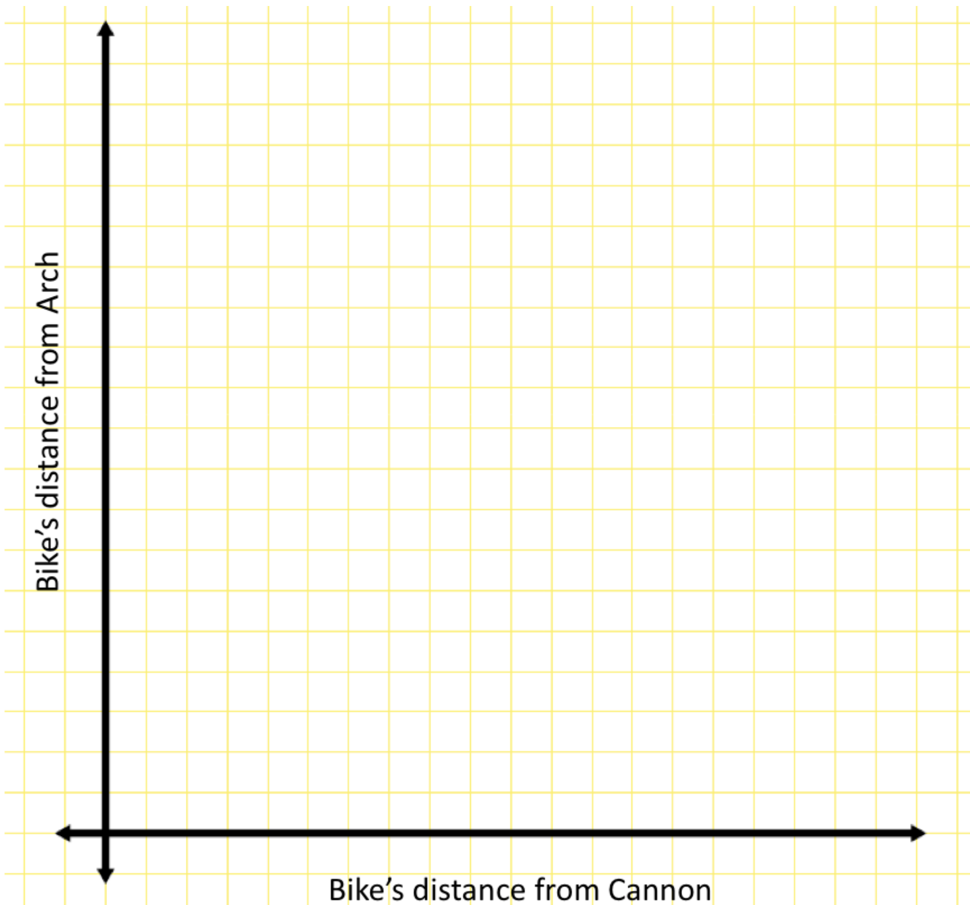


The blue bar represents the bike's distance from Arch as it travels along the road. The red bars changes depending on the blue bar.

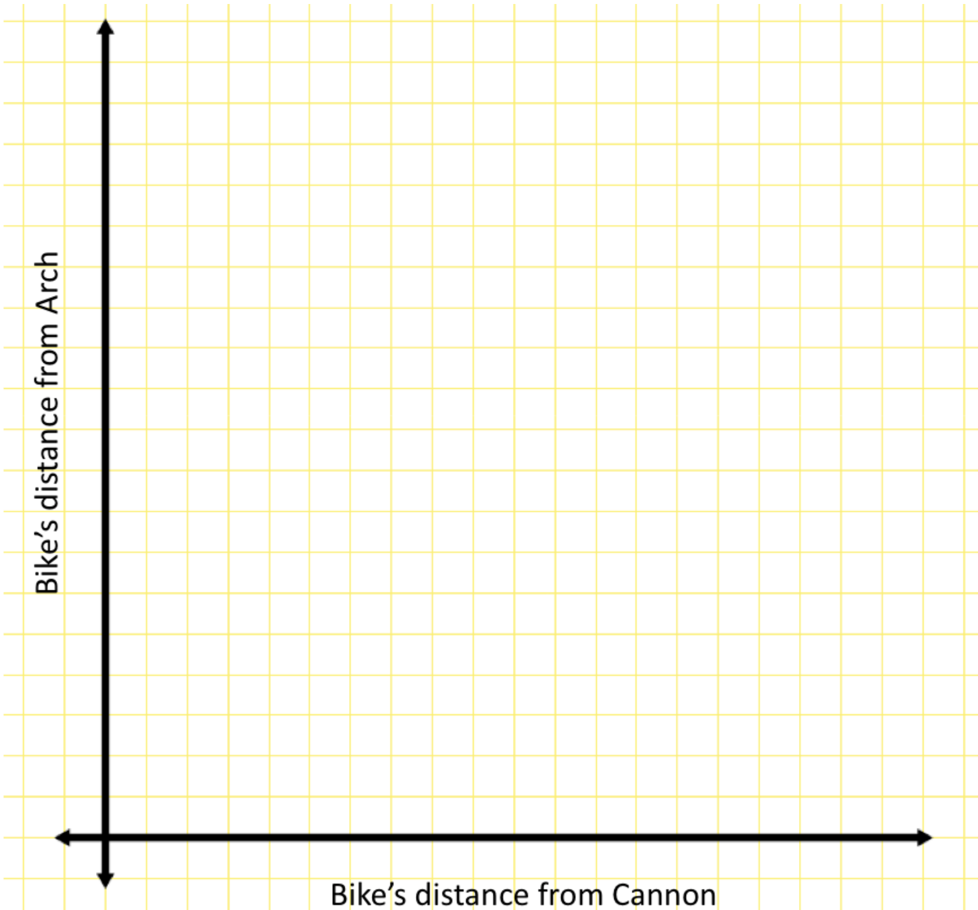
1. Identify which of the four red bars, if any, accurately represent the bike's distance from Cannon as the bike's distance from Arch varies. You can change the length of the blue bar by moving the bike in the situation or by clicking the "play" button. You are not allowed to measure. After you decide which one, label it accordingly.

2. Hide the bike. Do question #1.

3. Take a couple of minutes and draw a rough sketch of the relationship between the bike's distance from Canon and the bike's distance from the Arch on the paper provided as you watch the animation. Label your graph as "a first draft."



4. Take a couple of minutes and draw a rough sketch of the relationship between the bike's distance from Canon and the bike's distance from the Arch on the paper provided as you watch the animation. Label your graph as "a second draft."



Swimming Pool Task (Protocol)

Open the task (https://www.geogebra.org/m/fyfbvfcd). Pass the handout #1.

A side view of the pool is shown above and on the tablet screen.

1. What different measurements are changing in the pool as you increase the depth of water?
2. Describe how the amount of the water varies from the moment that empty pool begins to fill.
3. You also want to know whether there is a relationship between the amount of water and the depth of water. What do you think what kind of relationship?
   a. [*they may be focusing on correspondence, ask the following question to see if they focus on how they change*] Is there a relationship between the depth of water and the amount of water as you begin to fill the empty pool (as they change)? Explain?

Ask the second question in the handout #1.

   b. [*if they still focus on correspondence, ask the following question*] Could you describe how the amount of water changes as you increase the depth of water from the moment that empty pool begins to fill?

Pass the handout #2.

4. Sketch the relationship. Label your sketch as "a first draft."

If they have hard time or after they are done, pass the handout #3.

5. Sketch the relationship.

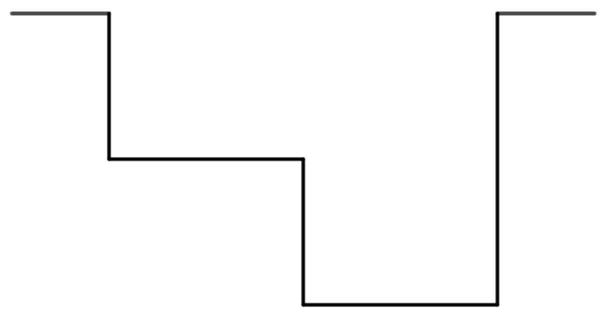
You described the relationship as "the amount of water increases as the depth of water increase."

1. Could you tell me more about this relationship? We know that the amount of water increases as we increase the depth of water in the pool. Can we have more information about how the amount of water is increasing in relation to the depth of water?
2. [*if they have no idea about what I meant, partition the depth equally by checking the box and ask the following question*]. For each unit increase in the depth of water, could you describe how amount of water increases?
   a. [*supporting questions*] As the depth of the water increases by this much, say one unit, could you **show** me how much the amount of water increases by?
   a. Does it increase by the same amounts for each unit increase in the depth of water? Or increase by greater and greater amounts? Or increase by smaller and smaller amounts?
3. As you increase the depth of water, do you see a difference in how AoW increases for the different part of the pool?
   b. What happens if I change the size of my unit? Could you describe how amount of water increases for each (new) unit increase in the depth of water?

Pass the handout #4.

4. Sketch the relationship. Label your sketch as "a second draft."

Swimming Pool Task (Handout #1)

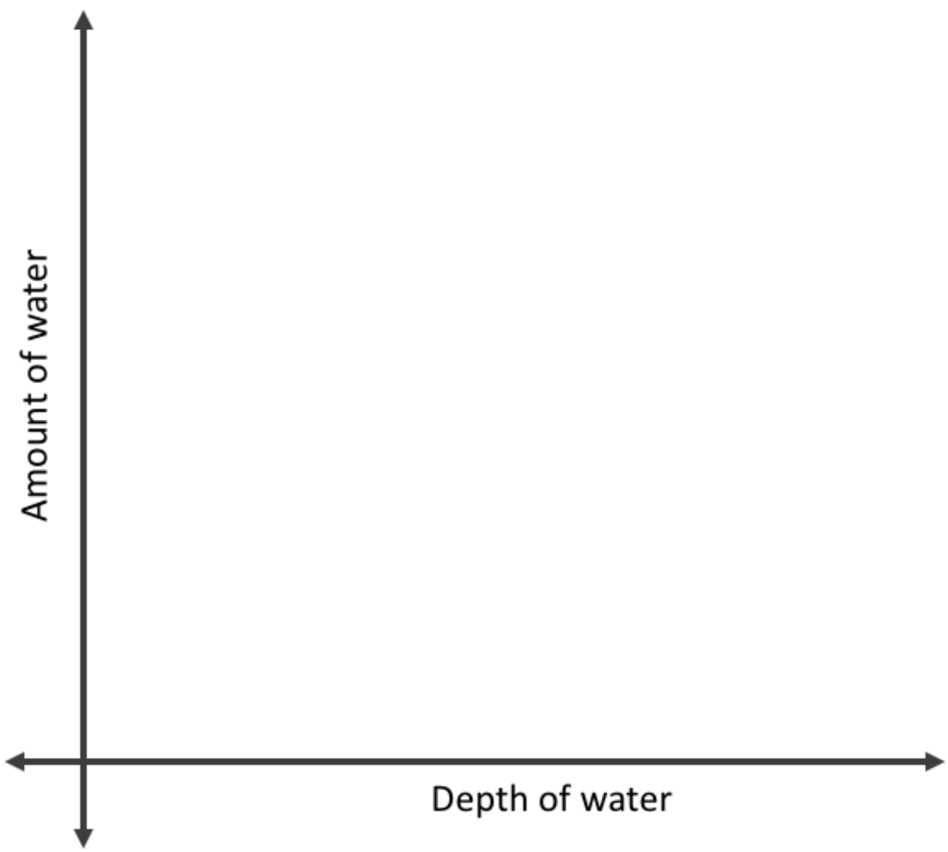


1. Describe how the amount of the water varies from the moment that empty pool begins to fill.

2. Could you describe how the amount of water changes as you increase the depth of water from the moment that empty pool begins to fill?

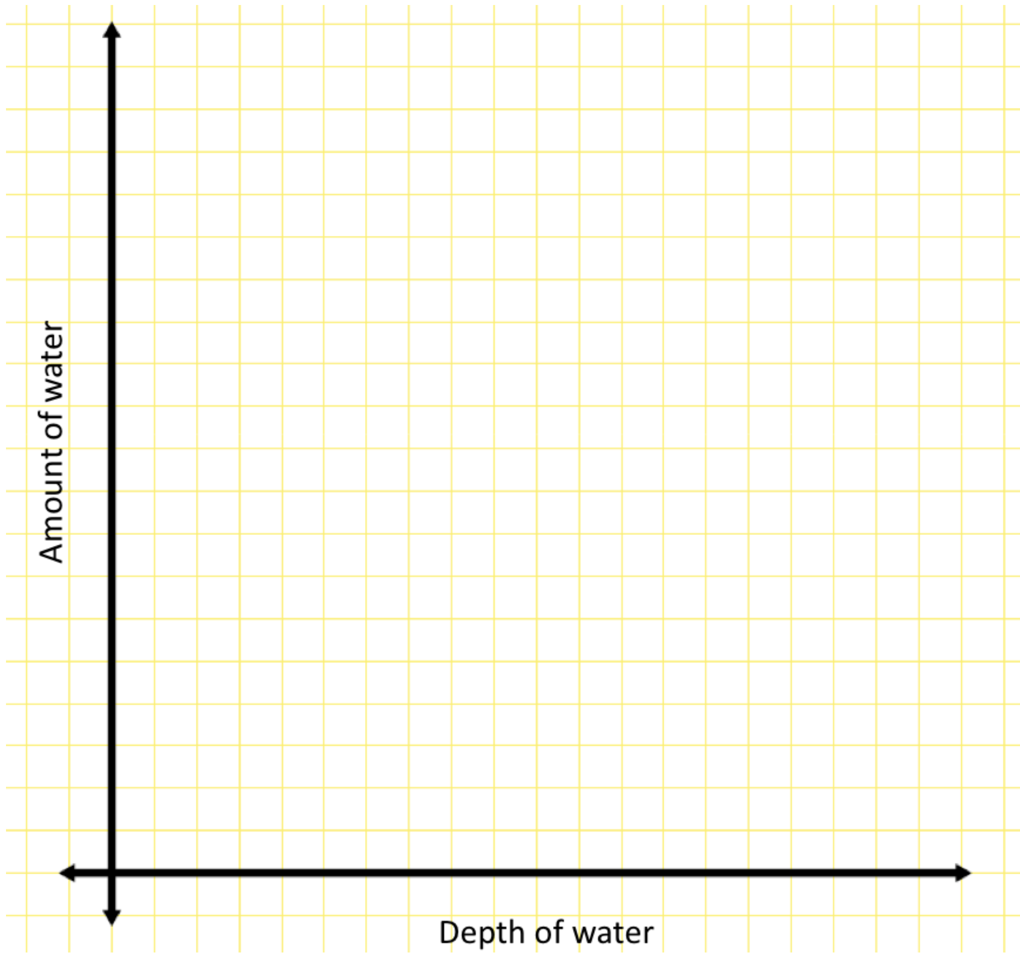
Swimming Pool Task (Handout #2)

1. Take a couple of minutes and draw a rough sketch of the relationship between the amount of water and the depth of water on the paper provided as you watch the animation. Label your graph as “a first draft.”



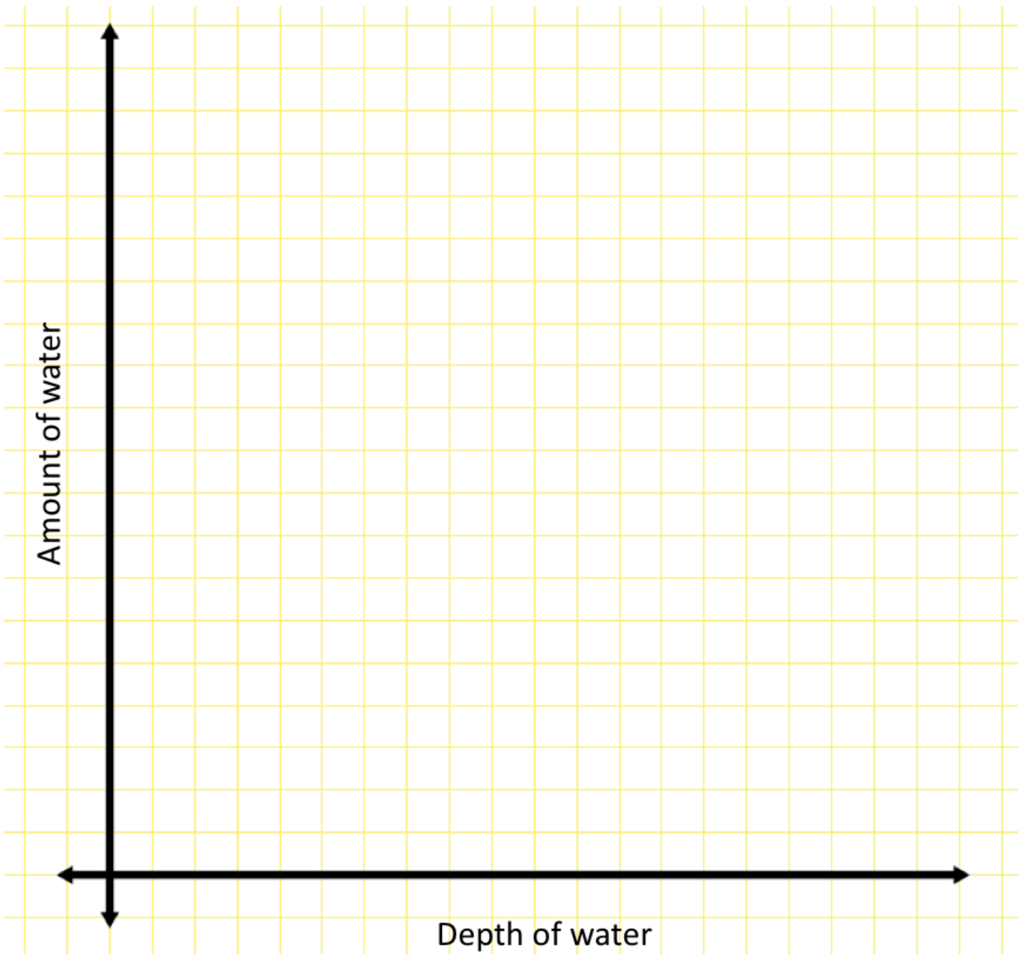
Swimming Pool Task (Handout #3)

1. Take a couple of minutes and draw a rough sketch of the relationship between the amount of water and the depth of water on the paper provided as you watch the animation. Label your graph as “a first draft.”



Swimming Pool Task (Handout #4)

1. Draw a rough sketch of the relationship between the amount of water and the depth of water on the paper provided as you watch the animation. Label your graph as "a second draft."



Which One Task in SPT (Protocol)

Open the task (https://www.geogebra.org/m/hefw2gnt)

Pass the handout #1 FOLDED

1. Identify which of the red bars, if any, accurately represent the amount of the water as the depth of the water varies. You can change the length of the blue bar by dragging its endpoint or by clicking the "play" button.
2. [*Only have #2 and #3 in red bars*]. Talk about how they change. Try to put tick marks to show how both quantities are changing.
    a. Any idea how much bigger the distance between tick marks in the second part than the ones in the first part?

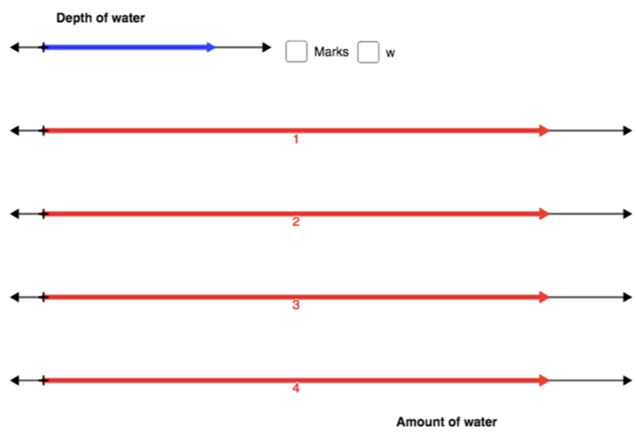
UNFOLD the handout #1. Then, bring #4 next to blue bar.

3. What would be the pool design for #4?

Pass the handout #2. Only have the blue one and #3 on the screen. Emphasize these are the ones for this pool. And mention about the relationship they determined right before asking them to graph the relationship.

4. Sketch a graph that shows this relationship.

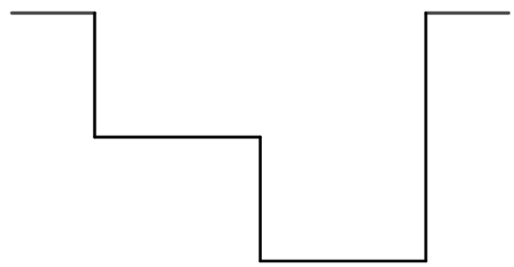
Which One Task in SPT (Handout #1)



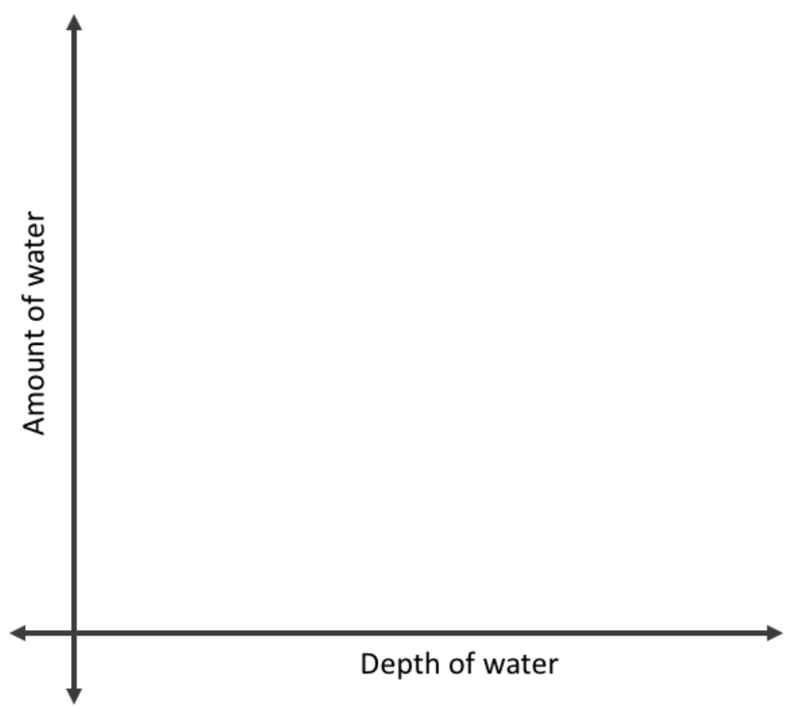
1. Identify which of the red bars, if any, accurately represent the amount of the water as the depth of the water varies. You can change the length of the blue bar by dragging its endpoint (the arrow) or by clicking the “play” button.

2. Design a pool in the box below so that the relationship between amount of water and the depth water as you fill the pool will be the same as the relationship between the blue bar and #4.

Which One Task in SPT (Handout #2)



Remember #3 represents how the amount of water changes depending on the blue bar (the depth of water) for this pool. Sketch the relationship between the amount of water and the depth of water on the paper provided as you watch the animation. Label your graph as “a third draft.”



Matching Game Task – Part 1 (Protocol)
Open the task (https://www.geogebra.org/m/wtg8de9g). In this task, the bike is riding along the College Avenue in Downtown Athens starting from North to South and backwards. *Do not show a, b, and c yet. Only the bike animation.* 1. What is going on? Could you describe how DfA and DfC are changing?   a. Break into three parts and talk about the covariational relationship. *Show a, b, and c, by taking turns.* 2. The magnitude of the blue bar represents the bike's distance from the Arch and the magnitude of the red bar represents the bike's distance from Cannon. Given the two bars in each case (i.e., (a), (b), and (c)), find where the bike would be (approximately) on the College Avenue in the map.

Matching Game Task – Part 2 (Protocol)

Open the task (https://www.geogebra.org/m/fn8p4tvv).
 	
Matching game with two players:
1. Play the animation. Each controls a bar by using the sliders.
    a. Change the length of the bars accordingly as the bike's location changes
    b. How are you moving? How do you know how to move? What are you considering when you move?
 	
Matching game with one player:
2. Control the blue bar with your left hand and the red bar with the right hand. Move your hands accordingly as the bike rides along the road back and forth.
    a. How are you moving? How do you know how to move? What are you considering when you move?
    b. Is there any place that you stop moving? Stop moving one of them? How do you know you need to move them at the same time?

Matching Game Task – Part 3 (Protocol)

Open the task (https://www.geogebra.org/m/nk3wranf).

Hide the bars and tick marks on the magnitude lines.

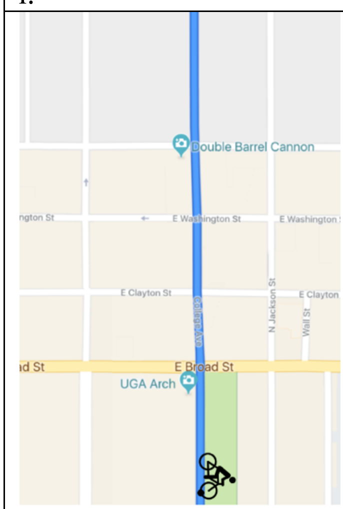
1. CREATE TWO POINTS TO REPRESENT TWO QUANTITIES IN 1D

a. Given (static) position of the bike on the map shown below (pick one), could you guess what would be the length of the bars on each number line? Insert tick marks on the number lines in a place where you think the heads of the bars are (i.e., the arrow).
   i. Explain what these tick marks on the number lines mean in terms of the situation?
   ii. [*show the tick marks and let students check their answer—and change the bike's position*] What is happening to these tick marks? How do you know?

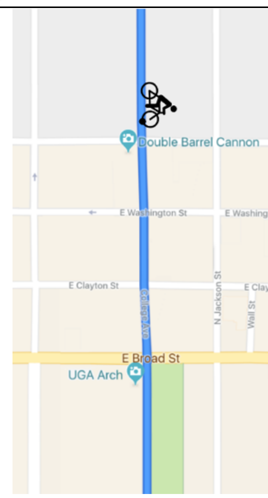
b. [*Hide the bike*] Would it be possible to know exactly where the bike is with one tick mark? [*Emphasize the importance of two quantities by pointing out that we need those two to talk about where the bike is*]

c. Change the bike position, and repeat (ii) and (iii).

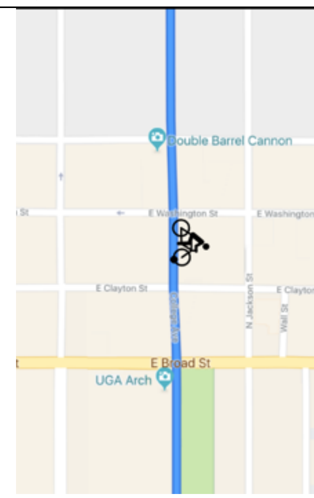
i.



ii.



iii.



2. CREATING A SINGLE POINT TO REPRESENT TWO QUANTITIES IN 2D

Show the bike and bars and make sure to have tick marks on the lines too.

a. Could you think of any way to show both the bike's DfA and the bike's DfC by using *a single point* so that anyone who see the point that you create should understand where the bike is on the map?

b. This point needs to record both information. If I have this point [*point to the tick mark on the magnitude line for DfA*], this one only records information about DfA. Same as other point. Now, is there any way for you to create a new point anywhere that record both information. Anyone who see the point that you create should understand where the bike is on the map.

c. You can freely move/rotate the number lines by dragging the small black points at the side of each number line or simply dragging/rotating the line itself if necessary. Also, feel free to draw and use GeoGebra tools.

If they create a point, hide the bars, only leave the point that they created.

d. Anticipate where the bike would be based on the visible point. How do you know?

If they do not create a point take the following steps.

e. Show the grid. Can you think of any way that we can do this? Doing two points is not really efficient. It would be nice if there is a way to have one point that shows both quantities, do you know any way?

If that does not help, in a different computer or tablet,
Open Crow Task (https://www.geogebra.org/m/kzddhqs8).

f. What does the black point in Cartesian coordinate system show in terms of the situation?
g. Move the crow and let them explore.
   i. Show the crow's DfA and DfC on the map [*ask them to draw on the map*]
   ii. Then, place a tick mark on each axis to show the crow's DfA and the crow's DfC. How do you know? [*Turn on the tick marks on the coordinate system*]
   iii. Show those segments (that you drew on the map) on the coordinate system. [*Turn on the bars on the coordinate system*]
   iv. [*Hide the black dot*] Move the crow. What is going on here on the coordinate system? What do you think where the black dot would be?
   v. [*Hide the bars and tick marks on coordinate system*] Move the crow to a different position. Now, where would the black dot be on the plane? How do you know? What does this point show us?
   vi. [*if they do not show it*] Place a tick mark on each axis to show both the crow's DfA and DfC. Show me how could you draw bars on each axis. Now, ask them where the black dot would be. [*make sure to emphasize the point has information about both quantities. One tick mark has the information about only one of them, this point has the information about two of them*]

Show them the other tablet screen with two parallel magnitude lines.

3. Are these two tasks related at all? Could you relate them? Now, could you think of any way to show both the bike's DfA and the bike's DfC by using a single point?

4. Go back to the Crow Task. [*Hide the black dot*]
   a. Could you plot a point on this coordinate plane in a way that that point should give us information about Wells Fargo Bank?
   b. Could you plot a point on this coordinate plane in a way that that point should give us information about Starbucks?

Show the dots in Crow Task hide the black point and the crow. This basically makes Crows Task DAT.

5. Ask them if they could label these dots appropriately.

Downtown Athens Bike Task v.2 (Protocol)

Open the task (https://www.geogebra.org/m/m8hhbhyh)
1. Review the previous task. Represent DfA and DfC by *one point*.
2. Construct a representation that represents how the bike's distance from Arch and Cannon change together during its trip in Downtown.
    a. Is there anything that was similar to how you were thinking about the two bars in matching games. How come?
 	
If students could not construct the graph, I plan to use the following steps.
 	
Show "Finger Tool" in the app.
The magnitude of the red bar on the horizontal axis represents the bike's distance from Cannon and the magnitude of the blue bar on the vertical axis represents the bike's distance from the arc. The bars and the bike in the situation are not synced now.
1. Given different positions of the bike on the map, adjust the length of the bars on each axis in the Cartesian coordinate system match with the biker's position on the College Avenue in the map.
    a. [*Place the bike in a certain position on the map and show the correspondence point on the plane for DfA and DfC*] What does this point represents?
Hide the point but ask students to pretend it is still there.
    b. [*Place the biker in a different position on the map and show the correspondence point on the plane for DfA and DfC*] What does this point represents?
Hide the point on the plane but ask students to pretend it is still there.
    c. [*Place the biker in a different position*] Point a finger at the correspondence point's position.
    d. [*Place the biker in a different position*] Point a finger at the correspondence point's new position.
 	
Hide the "Finger Tool" and show the linked bars on the axis that move as the bike move. Don't show the point on the plane. Tell students that I will play the animation in loop
 	
2. Keep a finger pointed at your imagined correspondence point as you watch the animation.
 	
3. Repeat the same task by using a pen at this time leaving a trace of the imagined correspondence point. What does this trace represent in terms of your finger movement and in terms of the quantities?