Cohomology and Representation Theory for Lie Superalgebras

by

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(Under the Direction of Daniel Nakano)

Abstract

This dissertation consists of two parts. In the first, we describe the cohomology groups for the subalgebra $\mathfrak{n}^+$ relative to the BBW parabolic subalgebras constructed by D. Grantcharov, N. Grantcharov, Nakano and Wu, essentially with these calculations essentially providing the first steps towards an analogue of Kostant's theorem for Lie superalgebras. We express these groups in terms of their weight space decompositions relative to the torus, with the weights corresponding to each superalgebra included in the appendix. In the second part, based on joint work with Nakano, we analyze the sheaf cohomology groups $\mathrm{R}^i \operatorname{ind}_B^G L_{\mathfrak{f}}(\lambda)$, where G is a supergroup scheme, B a BBW parabolic subgroup scheme, and $L_{\mathfrak{f}}(\lambda)$ is an irreducible representation for the detecting subalgebra $\mathfrak{f}$. We provide a parametrization of simple G-modules and give analogues for the BBW theorem and Kempf's vanishing theorem for sufficiently large λ, as well as a criterion for when $\operatorname{soc}_G \mathrm{H}^1(\lambda)$ is simple.

Index words: [Cohomology, Lie Superalgebra, Representation Theory, Homological Algebra, Detecting Subalgebra]

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Contents

Chapter 1

Introduction

1.1 Background

For $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$, J a subset of simple roots and $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J$ the corresponding parabolic subalgebra, a famous theorem of Kostant demonstrates that

$$\mathrm{H}^k(\mathfrak{u}_J, L(\mu)) = \bigoplus_{w \in W^J,\, l(w)=k} L_J(w \cdot \mu),$$

where $L(\mu)$ is an irreducible module of highest weight μ for $\mathfrak{g}$ and $L_J(w \cdot \mu)$ is an irreducible finite-dimensional module corresponding to the Levi factor $\mathfrak{l}_J$ for J [BBC+09]. Kostant's theorem is piece of a larger picture where in the (parabolic) Category $\mathcal{O}_J$ one has the isomorphism:

$$\mathrm{Ext}^n_{\mathcal{O}_J}(Z_J(\lambda), L(\mu)) \cong \mathrm{Hom}_{\mathfrak{l}_J}(L_J(\lambda), \mathrm{H}^n(\mathfrak{u}_J, L(\mu))), \tag{1.1.1}$$

where $Z_J(\lambda)$ is a (parabolic) Verma module arising from inducing a finite-dimensional $\mathfrak{l}_J$-module $L_J(\lambda)$ and $L(\mu)$ is an irreducible representation in $\mathcal{O}_J$. It is a deep theorem that these extension groups in (1.1.1) can be computed via Kazhdan-Lusztig polynomials [Kum02].

In the case when $\mathfrak{g}$ is a classical simple Lie superalgebra one would like to have a Kazhdan-Lusztig theory and a Kostant-type theorem in the context of a Category $\mathcal{O}$ theory. This was accomplished in specific cases by various authors [Bru03, CZ04, CK08, CL10, CKW10, CLW11].

Recently, D. Grantcharov, N. Grantcharov, Nakano, and Wu [GGNW21] introduced a family of parabolic subgroups for G called *BBW parabolic* subgroups. These parabolic subgroups arise naturally when considering the detecting subalgebras as defined by Boe, Kujawa and Nakano in the mid 2000's [BKN09]. In [GGNW21], it was shown that if B is a BBW parabolic, the polynomial $p_{G,B}(t) = \sum_{i=0}^{\infty} \dim R^i\text{ind}_B^G\mathbb{C}\, t^i$ is equal to a Poincaré polynomial for a finite reflection group $W_{\bar{1}}$ specialized at a power of t. The existence of the BBW parabolics was also used in [GGNW21] to resolve a 15 year old conjecture posed in [BKN09] on the realization of the cohomological support varieties for the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ as a rank variety over a detecting subalgebra. Additionally, there exists a natural triangular decomposition of $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{f} \oplus \mathfrak{n}^+$ where $\mathfrak{b} = \mathfrak{f} \oplus \mathfrak{n}^+$ and the Lie superalgebras $\mathfrak{n}^\pm$ are nilpotent subalgebras.

Moreover, Lai, Nakano and Wilbert have recently constructed a Category $\mathcal{O}_{\mathfrak{f}}$ via this triangular decomposition and have proved an analog to (1.1.1). Other efforts have been made in understanding a Category $\mathcal{O}$ for Lie superalgebras on a case-by-case basis; however, up to this point there has not been a unified treatment. A fundamental question is to compute $\mathrm{H}^n(\mathfrak{n}^+, L(\lambda))$, where $L(\lambda)$ is a finite-dimensional $\mathfrak{g}$-module, and to determine if there is a Kostant-type theorem in $\mathcal{O}_{\mathfrak{f}}$.

Now let G be a reductive algebraic group over an algebraically closed field k. If B is a Borel subgroup of G then the sheaf cohomology groups

$$\mathrm{H}^\bullet(\lambda) := \mathcal{H}^\bullet(G/B, \mathcal{L}(\lambda)) \cong R^\bullet\text{ind}_B^G\,\lambda$$

play a central role in the representation theory for G. It is well-known that the irreducible (finite-dimensional) G-modules are indexed by dominant integral weights, X_+, and can be realized as the socles of $\mathrm{H}^0(\lambda)$. More precisely, for any $\lambda \in X_+$, one has $L(\lambda) = \text{soc}_G\,\mathrm{H}^0(\lambda)$. Another result that holds over arbitrary k is Kempf's vanishing theorem which states that for $\lambda \in X_+$, $\mathrm{H}^n(\lambda) = 0$ for $n > 0$. When the field k is of characteristic zero, the rational representations of G are completely reducible, and a description of $\mathrm{H}^\bullet(\lambda)$ is given via the classical Bott-Borel-Weil (BBW) theorem. For fields of characteristic $p > 0$, the general vanishing behavior for $\mathrm{H}^\bullet(\lambda)$ is not known, and it is not clear how to formulate an appropriate generalization of the BBW theorem. The only known case of additional information in regards to vanishing behavior is due to Andersen for $n = 1$ [And79] where he described $\mathrm{soc}_G \mathrm{H}^1(\lambda)$ for all weights λ.

Now consider the more general case when G is a supergroup scheme with Lie $G = \mathfrak{g}$ where $\mathfrak{g}$ is a classical "simple" Lie superalgebra over $\mathbb{C}$, and P a parabolic sub-(super)group scheme of G. A central problem in the super-representation theory is to understand the behavior of the sheaf cohomology groups $R^i\mathrm{ind}_P^G(-)$. Zubkov has published a general discussion on this topic [Zub06]. Specific calculations of sheaf cohomology have been made for specific supergroups such as $GL(m|n)$, $OSP(m|2n)$, and $Q(n)$ with certain parabolic/Borel subgroups [Zub06, GS10, GS13, Pen88, PS97, Ser96, Ser14]. From these computations, it was not clear whether there was a general theory that could be applied for all classical simple Lie superalgebras like the one for reductive algebraic groups where computations of sheaf cohomology could be related to the combinatorics of finite reflection groups.

1.2 Outline

The goals of this dissertation are thus twofold. First, we aim to take the first steps towards the computation of the Lie superalgebra cohomology groups $\mathrm{H}^n(\mathfrak{n}, \mathbb{C})$, in particular, the groups corresponding to degrees one and two. Second, we wish to study the behavior of the cohomology groups $R^\bullet\mathrm{ind}_B^G(L_\mathfrak{f}(\lambda))$ where $L_\mathfrak{f}(\lambda)$ is an irreducible representation for the detecting subalgebra $\mathfrak{f}$.

To that end, this dissertation is outlined as follows. Chapter 1 gives an introduction to the paper as a whole, and to the basic notions of Lie superalgebras, Lie superalgebra cohomology, and detecting subalgebras. Chapter 2 discusses the fundamentals of sheaf cohomology and parabolic subalgebras. In Chapter 3, a Hochschild-Serre type spectral sequence is constructed for each of the infinite families of classical simple Lie superalgebras, which in each case is shown to collapse. Chapters 4 and 5 discuss the first and second degree cohomology groups $\mathrm{H}^n(\mathfrak{n}, \mathbb{C})$ for each of these simple Lie superalgebras, respectively. Chapter 6 goes through the construction of several key spectral sequences which will be used to compute sheaf cohomology for various supergroups. In Chapter 7, we parametrize the simple G-modules $L(\lambda)$ via a correspondence with the socles of the induced modules $H^0(\lambda)$. In Chapter 8, we analyze the behavior of BBW parabolics. In particular, we obtain analogs of Kempf's vanishing theorem and the BBW theorem for supergroups. Finally, Chapter 9 consists of tables characterizing the cohomology groups $H^n(\mathfrak{n}, \mathbb{C})$ described in Chapters 4 and 5 in terms of their weight-space decomposition.

1.3 Preliminaries

1.3.1 Notation

Throughout this dissertation, all vector spaces, unless otherwise noted, will be over $\mathbb{C}$.

Definition 1.3.1. A *superspace* is a vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a $\mathbb{Z}_2$-grading. An element $v \in V_{\bar{0}}$ is referred to as *even*, and an element in $V_{\bar{1}}$ as *odd*. Such an element in either $V_{\bar{0}}$ or $V_{\bar{1}}$ is referred to as *homogeneous*. If v is homogeneous, we define the degree $|v|$ of v as the element $i \in \mathbb{Z}_2$ such that $v \in V_i$.

Definition 1.3.2. A *Lie superalgebra* is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with a bilinear multiplication $[\cdot, \cdot]$ satisfying the following properties:

1. $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$
2. $[a, b] = -(-1)^{|a| \cdot |b|}[b, a]$
3. $[a, [b, c]] = [[a, b], c] + (-1)^{|a| \cdot |b|}[b, [a, c]]$,

where properties 2 and 3 hold for homogeneous elements, and the multiplication is extended to all of $\mathfrak{g}$ linearly [CW12, Definition 1.3].

Definition 1.3.3. If $\mathfrak{g}$ is a Lie superalgebra, a $\mathfrak{g}$-*module* M is a superspace equipped with an action by $\mathfrak{g}$ that is compatible with the $\mathbb{Z}_2$-grading and with the multiplication on $\mathfrak{g}$.

The notion of a universal enveloping algebra generalizes to the superalgebra case as well. Given a Lie superalgebra $\mathfrak{g}$ let $T(\mathfrak{g})$ denote the tensor algebra on $\mathfrak{g}$. Let I denote the ideal generated by elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$$

Let $U(\mathfrak{g}) = T(\mathfrak{g})/I$ and let i be the canonical embedding of $\mathfrak{g}$ into $U(\mathfrak{g})$. Then $U(\mathfrak{g})$ satisfies the universal property that if $j : \mathfrak{g} \to M$ is any linear map satisfying

$$j([x, y]) = j(x)j(y) - (-1)^{|x||y|} j(y)j(x),$$

where M is an associative algebra, then there is a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to M$ such that $\phi \circ i = j$. We let $I\mathfrak{g}$ denote the augmentation ideal of $U(\mathfrak{g})$, i.e. the kernel of the canonical map from $U(\mathfrak{g})$ to $\mathbb{C}$.

1.3.2 Lie superalgebra cohomology

We define the Lie superalgebra cohomology of $\mathfrak{g}$ with coefficients in a module M as follows. Consider the Koszul complex whose cochain groups are given as

$$C^n(\mathfrak{g}, M) = \mathrm{Hom}(\Lambda_s^n(\mathfrak{g}), M),$$

where $\Lambda_s^n(\mathfrak{g})$ denotes the superexterior algebra

$$\Lambda_s^n(\mathfrak{g}) := \bigoplus_{i+j=n} \Lambda^i(\mathfrak{g}_{\bar{0}}) \otimes S^j(\mathfrak{g}_{\bar{1}}).$$

The differential maps $d^n : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$, for homogeneous f, are given by the formula

$$\begin{aligned} df(\omega_0 \wedge \cdots \wedge \omega_n) = &\sum_{i=0}^{n} (-1)^{\tau_i} \omega_i \cdot f(\omega_0 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_n) \\ &+ \sum_{i<j} (-1)^{\sigma_{i,j}} f([\omega_i, \omega_j] \wedge \omega_0 \wedge \cdots \widehat{\omega_i} \cdots \widehat{\omega_j} \cdots \wedge \omega_n), \end{aligned} \tag{1.3.1}$$

where

$$\tau_i = i + |\omega_i|(|\omega_0| + \cdots + |\omega_{i-1}| + |f|),$$

and

$$\sigma_{i,j} = i + j + |\omega_i||\omega_j| + |\omega_i|(|\omega_0| + \cdots + |\omega_{i-1}|) + |\omega_j|(|\omega_0| + \cdots + |\omega_{j-1}|),$$

and which is then extended linearly to all of $C^n(\mathfrak{g}, M)$. It follows that $d^n \circ d^{n-1} = 0$, and so we define the nth cohomology group as

$$\mathrm{H}^n(\mathfrak{g}, M) = \ker d^n / \operatorname{im} d^{n-1}.$$

Letting $\mathbb{C}$ denote the $\mathfrak{g}$-module of dimension 1 concentrated in the even component on which $\mathfrak{g}$ acts trivially, we define the cohomology of $\mathfrak{g}$ as $\mathrm{H}^n(\mathfrak{g}, \mathbb{C})$.

1.3.3 Detecting and nilpotent subalgebras

We now define the notion of a detecting subalgebra, essentially an analog of the Cartan subalgebra in the classical case, following D. Grantcharov, N. Grantcharov, Nakano, and Wu [GGNW21].

Definition 1.3.4. A Lie superalgebra $\mathfrak{g}$ is *classical* if there is a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$ and if the action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ differentiates to the adjoint action. Furthermore, a Lie superalgebra $\mathfrak{g}$ is *basic classical* if it is classical with a nondegenerate invariant supersymmetric even bilinear form.

Definition 1.3.5. A point $x \in \mathfrak{g}_{\bar{1}}$ is *semisimple* if the orbit $G_{\bar{0}} \cdot x$ is closed in $\mathfrak{g}_{\bar{1}}$.

Definition 1.3.6. The action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is *stable* if there exists an open dense subset of $\mathfrak{g}_{\bar{1}}$ consisting of semisimple points.

If $\mathfrak{g}$ is a classical Lie superalgebra, $\mathfrak{g}_{\bar{1}}$ admits a stable action by $G_{\bar{0}}$. Following the construction in [BKN09, Section 8.9], fix a generic element $x_0 \in \mathfrak{g}_{\bar{1}}$ and set $H = \mathrm{Stab}_{G_{\bar{0}}} x_0$. We define $\mathfrak{f}_{\bar{1}} = \mathfrak{g}_{\bar{1}}^H$ and $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$ and let $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$ be the *detecting subalgebra*.

Moreover, as per [BKN09, Section 8], we can make the odd roots corresponding to $\mathfrak{f}$ explicit. Let Ω denote the set of odd positive roots of $\mathfrak{f}$. Then

$$\mathfrak{f}_{\bar{1}} = \{\sum_{\alpha \in \Omega} (u_\alpha x_\alpha + v_\alpha x_{-\alpha}) \mid u_\alpha, v_\alpha \in \mathbb{C}\},$$

where x_α is a nonzero element of the root space corresponding to α. $\mathfrak{f}_{\bar{0}}$ can then be obtained by taking brackets.

Let ϵ_i and δ_j be linear functionals on diagonal matrices

$$a = \mathrm{diag}(a_1, \cdots, a_{n+m})$$

which satisfy

$$\epsilon_i(a) = a_i$$

and

$$\delta_j(a) = a_{m+j}.$$

By convention, let r denote the minimum of m and n in the table below. Then for each of the classical simple Lie superalgebras, we have the following values for Ω, following the notation found in [GGNW21, Section 3.2] for the exceptional Lie superalgebras:

$\mathfrak{g}$	Ω
$\mathfrak{gl}(m\|n)$	$\{\epsilon_i - \delta_i \mid 1 \leq i \leq r\}$
$\mathfrak{sl}(m\|n)$	$\{\epsilon_i - \delta_i \mid 1 \leq i \leq r\}$
$\mathfrak{psl}(n\|n)$	$\{\epsilon_i - \delta_i \mid 1 \leq i \leq n\}$
$\mathfrak{osp}(2m+1\|2n)$	$\{\epsilon_i - \delta_i \mid 1 \leq i \leq r\}$
$\mathfrak{osp}(2m\|2n)$	$\{\epsilon_i - \delta_i \mid 1 \leq i \leq r\}$
$D(2,1;\,\alpha)$	$\{(\epsilon, -\epsilon, \epsilon)\}$
$G(3)$	$\{(\omega_1, -\epsilon)\}$
$F(4)$	$\{(\omega_3, -\epsilon)\}$

In the case of $\mathfrak{q}(n)$ we let $\mathfrak{f}_{\bar{1}}$ be the collection of all matrices whose odd part is diagonal.

Looking at the adjoint action of the maximal torus in $\mathfrak{f}_{\bar{0}}$ on $\mathfrak{g}$ produces a root-space decomposition of $\mathfrak{g}$, and letting $\mathfrak{n}$ denote the space of positive roots and $\mathfrak{n}^-$ the space of negative ones, we obtain a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{f} \oplus \mathfrak{n}$. In table 1.3.3 below, we list the collection of root spaces corresponding to $\mathfrak{n}^-$ for each of the the classical Lie superalgebras, where $\Phi_{\bar{1}}$ refers to the set of odd roots for $\mathfrak{n}^-$.

$\mathfrak{g}$	$\Phi_{\bar{1}}^-$
$\mathfrak{gl}(m\|n)$	$\{\epsilon_i - \delta_j, -\delta_i + \epsilon_j \mid i < j\}$
$\mathfrak{sl}(m\|n)$	$\{\epsilon_i - \delta_j, -\delta_i + \epsilon_j \mid i < j\}$
$\mathfrak{osp}(2m+1\|2n)$	$\{-\epsilon_i + \delta_j,\ -\delta_i + \epsilon_j,\ -\epsilon_k - \delta_l,\ -\delta_t \mid i < j\}$
$\mathfrak{osp}(2m\|2n)$	$\{\epsilon_i - \delta_j, -\delta_i + \epsilon_j, -\epsilon_k - \delta_l \mid i < j\}$
$\mathfrak{q}(n)$	$\{\epsilon_i + \epsilon_j \mid i < j\}$
$D(2,1;\alpha)$	$\{(-\epsilon, -\epsilon, -\epsilon), (-\epsilon, -\epsilon, \epsilon), (\epsilon, -\epsilon, -\epsilon)\}$
G(3)	$\{(-\omega_1 + \omega_2, -\epsilon)$,$(2\omega_1 - \omega_2, -\epsilon)$,$(0, -\epsilon)$,$(\omega_1 - \omega_2, -\epsilon)$,$(-2\omega_1 + \omega_2, -\epsilon)$,$(-\omega_1, -\epsilon)\}$
F(4)	$\{(\omega_2 - \omega_3, -\epsilon)$, $(\omega_1 - \omega_2 + \omega_3, -\epsilon)$, $(\omega_1 - \omega_3, -\epsilon)$, $(-\omega_2 + \omega_3, -\epsilon)$ $(-\omega_1 + \omega_2 - \omega_3, -\epsilon)$, $(-\omega_1 + \omega_3, -\epsilon)$, $(-\omega_3, -\epsilon)\}$

Chapter 2

Sheaf Cohomology

2.1 Notation

We will use and summarize the conventions developed in [BKN09, BKN10, BKN11, LNZ11, GGNW21]. For more details we refer the reader to [BKN09, Section 2].

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over the complex numbers $\mathbb{C}$ with supercommutator $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. At times, we will impose more conditions on $\mathfrak{g}$, such as requiring it to be classical or basic classical.

The super-analogs for reductive groups will be supergroup schemes that arise from classical "simple" Lie superalgebras. As far as the author knows, there has not been a formal theory developed for reductive supergroup schemes. The classical "simple" Lie superalgebras are not always simple, simple in the sense of [GGNW21]. These Lie superalgebras have appeared frequently in the literature and are of general interest. When we refer to a simple superalgebra, it will be one of "simple" superalgebras listed below:

- $\mathfrak{gl}(m|n), \mathfrak{sl}(m|n), \mathfrak{psl}(n|n)$ [*general and special linear Lie superalgebras*],

- $\mathfrak{osp}(m,n)$ [*ortho-symplectic Lie superalgebras*],

- $D(2,1,\alpha), F(4), G(3)$ [*exceptional Lie superalgebras*],

- $\mathfrak{q}(n), \mathfrak{psq}(n)$ [*queer Lie superalgebras*],

- $\mathfrak{p}(n), \widetilde{\mathfrak{p}}(n)$ [*periplectic Lie superalgebras*].

For the queer Lie superalgebras, $\mathfrak{q}(n)$ will be the Lie superalgebra with even and odd parts $\mathfrak{gl}_n$, while $\mathfrak{psq}(n)$ is the corresponding simple subquotient of $\mathfrak{q}(n)$ (cf. [PS97]). The periplectic Lie superalgebras include $\mathfrak{p}(n)$ with even component $\mathfrak{sl}_n$ and its enlargement $\widetilde{\mathfrak{p}}(n)$ having even component $\mathfrak{gl}_n$.

Let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. Modules over Lie superalgebras can be viewed as unital modules for $U(\mathfrak{g})$. If M and N are $\mathfrak{g}$-modules one can employ the properties of $U(\mathfrak{g})$ as a super Hopf algebra to define a $\mathfrak{g}$-module structure on the dual M^* and the tensor product $M \otimes N$.

The cohomology theory of $\mathfrak{g}$-modules has a natural interpretation when one uses relative cohomology. The projective objects are relatively projective $U(\mathfrak{g}_{\bar{0}})$-modules, and every $U(\mathfrak{g})$-module admits a relatively projective $U(\mathfrak{g}_{\bar{0}})$-resolution. By using these facts, given $\mathfrak{g}$-modules, M, N, one can define the relative extension groups $\operatorname{Ext}^n_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, N)$ by taking a relatively projective $U(\mathfrak{g}_{\bar{0}})$-resolution for M. Furthermore,

$$\operatorname{Ext}^n_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, N) \cong \mathrm{H}^n(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M^* \otimes N)$$

where $\mathrm{H}^n(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M^* \otimes N)$ denotes relative Lie superalgebra cohomology which can be computed using an explicit complex (cf. [Kum02, 3.1.8 Corollary, 3.1.15 Remark], [BKN09, Section 2.3]).

2.2 Rational modules

Let G be an affine supergroup scheme over $\mathbb{C}$ and $\mathrm{Mod}(G)$ be the category of rational modules for G. Let H be a closed subgroup scheme of G and $R^j\mathrm{ind}_H^G(-)$ be the higher right derived functors of the induction functor $\mathrm{ind}_H^G(-)$. For a general overview about supergroup schemes and induction, the reader is referred to work of Brundan and Kleshchev. See [BK03, Sections 2,4,5] [Bru06, Section 2].

In the case when $\mathfrak{g}$ is a classical Lie superalgebra and $\mathfrak{g} = \mathrm{Lie}\, G$, the category $\mathrm{Mod}(G)$ is equivalent to locally finite integral modules for $\mathrm{Dist}(G) = U(\mathfrak{g})$ (cf. [BK03, Corollary 5.7]). In particular, if $\mathfrak{g}$ is a classical Lie superalgebra, then $\mathrm{Mod}(G)$ is equivalent to $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ (i.e., the category of $\mathfrak{g}$-supermodules that are completely reducible over $\mathfrak{g}_{\bar{0}}$). The projectives in the category $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ are relatively projective

$U(\mathfrak{g}_{\bar{0}})$-modules. Therefore, if M and N are rational G-modules then

$$\mathrm{Ext}^n_G(M, N) \cong \mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, N)$$

for all $n \geq 0$.

2.3 Parabolic subalgebras

Let $\mathfrak{g}$ be a classical simple Lie superalgebra, and $\mathfrak{t}$ be a fixed maximal torus in $\mathfrak{g}_{\bar{0}}$. One can use the action of $\mathfrak{t}$ on $\mathfrak{g}$ to obtain a set of roots Φ. We can now invoke the ideas presented by Grantcharov and Yakimov in [GY13] to define parabolic subsets S that correspond to parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$. For precise details, see [GY13], [GGNW21, 3.1, 3.2]. Given a parabolic subalgebra $\mathfrak{p}$, one has a decomposition of $S = S_0 \sqcup S^-$ with $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where (i) $\mathfrak{l}$ is the *Levi subalgebra* with roots in S_0, and (ii) $\mathfrak{u}$ is the *nilradical* of $\mathfrak{p}$ with roots in S^-.

A parabolic subalgebra $\mathfrak{b}$ arises from taking a principal parabolic subset given by $S = S(\mathcal{H}) = S^0 \sqcup S^-$, where $\mathcal{H}$ is listed in [GGNW21, Table 7.1.2]. In this case, $\mathfrak{b} \cong \mathfrak{f} \oplus \mathfrak{u}$ where the Levi subalgebra is the detecting subalgebra $\mathfrak{f}$ that was first introduced in [BKN09]. The Lie subalgebras $\mathfrak{f}$ (resp. $\mathfrak{u}$) are given in [GGNW21, Table 7.1.1] (resp. [GGNW21, Table 7.1.3]).

The subalgebra $\mathfrak{b}$ is a parabolic subalgebra and technically is not a Borel subalgebra. In this paper, we will view $\mathfrak{b}$ as analogous to a Borel subalgebra for a simple Lie algebra arising from an algebraic group. The detecting subalgebra $\mathfrak{f}$ will be analogous to a maximal torus. There exists a natural triangular decomposition of $\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{f} \oplus \mathfrak{u}$ where the roots in $\mathfrak{u}^+_{\bar{1}}$ (resp. $\mathfrak{u}_{\bar{1}}$) coincide with $-(S^-)$ (resp. S^-). Note the BBW parabolic subalgebra identifies with $\mathfrak{b} = \mathfrak{f} \oplus \mathfrak{u}$, and the BBW parabolic subalgebras are defined for classical simple Lie superalgebra that are not of type P.

Example 2.3.1. Consider the case when $G = GL(n|n)$ with $\mathfrak{g} = \mathfrak{gl}(n|n)$ or $G = Q(n)$ with $\mathfrak{g} = \mathfrak{q}(n)$. The BBW parabolic $\mathfrak{b}$ can be realized in $\mathfrak{g}$ as

$$\mathfrak{b} = \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathfrak{g} : A,\ B,\ C,\ D \in L_n(\mathbb{C}) \right\}$$

where $L_n(\mathbb{C})$ is the set of $n \times n$ lower triangular matrices. There exists a supergroup scheme B with Lie $B = \mathfrak{b}$ that corresponds to $\text{Dist}(B) = U(\mathfrak{b})$.

In [GGNW21, Theorem 4.10.1], the sheaf cohomology $R^\bullet\text{ind}_B^G\mathbb{C}$ was completely described as a G-module. Its Poincaré series was shown to be equal to the Poincaré series in a variable s of a finite reflection group, specialized either at $s = t$ for $Q(n)$ or $s = t^2$ for $GL(n|n)$.

Chapter 3

Hochschild-Serre Spectral Sequence

As in the case of classical Lie algebra cohomology, letting $\mathfrak{h}$ denote an ideal of $\mathfrak{g}$, we construct an analogue of the Hochschild-Serre spectral sequence for Lie superalgebras.

Consider a short exact sequence of Lie superalgebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

and functors:

$$\mathcal{F} : \mathfrak{g}/\mathfrak{h}\text{-mod} \to \mathbb{C}\text{-mod}$$

$$\mathcal{G} : \mathfrak{g}\text{-mod} \to \mathfrak{g}/\mathfrak{h}\text{-mod},$$

which are given by $\mathcal{F}(-) = \mathrm{H}^0(\mathfrak{g}/\mathfrak{h}, -)$ and $\mathcal{G}(-) = \mathrm{H}^0(\mathfrak{h}, -)$. Both $\mathcal{F}$ and $\mathcal{G}$ satisfy the conditions given in [Jan03, Proposition 4.1], and so we obtain a Grothendieck spectral sequence:

$$E_2^{p,q} = R^p\mathcal{F}(R^q(\mathcal{G}(-))),$$

which converges to $R^{p+q}(\mathcal{F}\mathcal{G})(-)$. As $\mathcal{F} \circ \mathcal{G} = \mathrm{H}^0(\mathfrak{g}, -)$, this simplifies to

$$E_2^{p,q} = \mathrm{H}^p(\mathfrak{g}/\mathfrak{h}, \mathrm{H}^q(\mathfrak{h}, -)) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}, -).$$

3.1 Infinite families

In this section, we provide a basis for $\mathfrak{n}$ for each of the infinite families of classical simple Lie superalgebras, and define an ideal $\mathfrak{I}$ of $\mathfrak{n}$. As a consequence, for each family we will obtain a short exact sequence

$$0 \to \mathfrak{I} \to \mathfrak{n} \to \mathfrak{n}/\mathfrak{I} \to 0,$$

which will give rise to a Hochschild-Serre spectral sequence

$$E_2^{ij} = \mathrm{H}^i(\mathfrak{n}/\mathfrak{I}, \mathrm{H}^j(\mathfrak{I}, \mathbb{C})) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{n}, \mathbb{C}).$$

We then show in the following section that each of these spectral sequences collapses.

3.2 $\mathfrak{gl}(m|n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ where $m \geq n$ and let $\mathfrak{n}^- \oplus \mathfrak{f} \oplus \mathfrak{n}$ be its triangular decomposition. Following [CW12, Section 1.1.2] we label the rows and columns of elements of $\mathfrak{gl}(m|n)$ by elements of the set $\{\bar{1}, \cdots \bar{m}, 1, \cdots n\}$. We let E_{ij} denote the elementary matrix for row i and column j. Then $\mathfrak{n}$ is spanned by

$$\begin{cases} E_{\bar{i},\bar{j}} & (\epsilon_i - \epsilon_j) \quad 1 \leq i < j \leq m \\ E_{i,j} & (\delta_i - \delta_j) \quad 1 \leq i < j \leq n \\ E_{\bar{i},j} & (\epsilon_i - \delta_j) \quad 1 \leq i \leq m, 1 \leq j \leq n, i < j \\ E_{i,\bar{j}} & (\delta_i - \epsilon_j) \quad 1 \leq i \leq n, 1 \leq j \leq m, i < j, \end{cases}$$

where the quantity in parentheses denotes the corresponding weight under the action of the maximal torus.

We let $\mathfrak{I} \subseteq \mathfrak{n}$ be the subalgebra spanned by elements $E_{\bar{i},\bar{m}}$, $E_{\bar{i},n}$, $E_{i,\bar{m}}$, and $E_{i,n}$ in the case where $m = n$, and by just $E_{\bar{i},\bar{m}}$ and $E_{i,\bar{m}}$ when $m > n$, with the appropriate bounds on i. Using the supercommutator identity:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-1)^{|E_{ij}| \cdot |E_{kl}|} \delta_{li} E_{kj},$$

it is a simple computation to show that $\mathfrak{I}$ is an ideal of $\mathfrak{n}$.

3.3 $\mathfrak{osp}(2m + 1|2n)$

Let $m \geq n$. We may view $\mathfrak{osp}(2m + 1|2n)$ as being a subalgebra of $\mathfrak{gl}(2m + 1|2n)$, and so we may describe its spanning set by means of the same elementary matrices. In particular, $\mathfrak{osp}(2m + 1|2n)$ will be the span of the root vectors and maximal torus as described in [CW12, Section 1.2.4]. Restricting our view to the weight spaces listed in Table 1.3.3, let $\mathfrak{n}$ be the subalgebra whose odd component is spanned by the elements:

$$\begin{cases} E_{k+n,\overline{i+m}} + E_{\bar{i},k} \quad (-\epsilon_i + \delta_j) \\ -E_{\overline{i+m},k+n} + E_{k,\bar{i}} \quad (-\delta_i + \epsilon_j) \\ E_{k+n,\bar{l}} - E_{\overline{l+m},k} \quad (-\epsilon_k - \delta_l) \\ E_{2n+1,\bar{t}} + E_{\overline{t+m},2n+1} \quad (\delta_t), \end{cases}$$

where $1 \leq i \leq m$ and $1 \leq k \leq n$, and whose even component is the direct sum of the nilpotent radicals of $\mathfrak{so}(2m + 1)$ and $\mathfrak{sp}(2n)$.

We let $\mathfrak{I}$ be the subalgebra of $\mathfrak{n}$ spanned by all root vectors with weights containing an ϵ_m or a δ_n term. Again, it may be shown that this constitutes an ideal of $\mathfrak{n}$.

3.4 $\mathfrak{osp}(2m|2n)$

The $\mathfrak{n}$ arising from $\mathfrak{osp}(2m|2n)$ has a similar basis as in the $\mathfrak{osp}(2m+1|2n)$ case, with an odd part given by:

$$\begin{cases} E_{k+n,\overline{i+m}} + E_{\bar{i},k} \quad (-\epsilon_i + \delta_j) \\ -E_{\overline{i+m},k+n} + E_{k,\bar{i}} \quad (-\delta_i + \epsilon_j) \\ E_{k+n,\bar{l}} - E_{\overline{l+m},k} \quad (-\epsilon_k - \delta_l) \end{cases}$$

where $1 \leq i \leq m$ and $1 \leq k \leq n$ and an even part given by the direct sum of the nilpotent radicals of $\mathfrak{so}(2m)$ and $\mathfrak{sp}(2n)$.

We may define an ideal just as we did for $\mathfrak{osp}(2m+1|2n)$, letting $\mathfrak{I}$ be the collection of all root vectors corresponding to weights of $\mathfrak{n}$ containing an ϵ_m term.

3.5 $\mathfrak{q}(n)$

We may view $\mathfrak{q}(n)$ as the subalgebra of $\mathfrak{gl}(n|n)$ spanned by the elements:

$$\widetilde{E}_{ij} := E_{\bar{i}\bar{j}} + E_{ij} \quad (\epsilon_i - \epsilon_j), \quad \overline{E}_{ij} := E_{i\bar{j}} + E_{\bar{i},j} \quad (\epsilon_i' - \epsilon_j'), \quad 1 \leq i, j \leq n.$$

Then $\mathfrak{n}$ is the subalgebra spanned by all $\widetilde{E}_{ij}$ and $\overline{E}_{ij}$ where $i < j$. Let $\mathfrak{I}$ be the subalgebra of $\mathfrak{n}$ generated by all $\widetilde{E}_{in}$ and $\overline{E}_{in}$. Again, it is not too difficult to show that $\mathfrak{I}$ is an ideal of $\mathfrak{n}$.

3.6 Collapsing

As a result of how the ideals were chosen in each case, we obtain the following result.

Theorem 3.6.1. *For any of the infinite families of classical Lie superalgebras $\mathfrak{g}$, the corresponding spectral sequence E_r^{ij} collapses on the $r = 2$ page.*

Proof. Recall that the differentials d_r on the rth page of a spectral sequence have bidegree $(r, 1-r)$, sending E_r^{ij} to $E_r^{i+r,j-r+1}$. Our goal is to show that for each page $r \geq 2$, the differentials must all be 0. First, note that we may decompose all E_r^{ij} into a direct sum of weight spaces under the action of the maximal torus of $\mathfrak{f}$. The differentials respect this action, and so to show that d_r is identically 0, it is sufficient to show that no weight in E_r^{ij} appears in $E_r^{i+r,j-r+1}$. To demonstrate this, we split the proof up into different cases for each classical superalgebra.

1. $\mathfrak{gl}(m|n)$ Consider an arbitrary differential from the E_2 page: $d_2 : E_2^{ij} \to E_2^{i+2,j-1}$.

   The term E_2^{ij} is a subquotient of $\Lambda_s^i(\mathfrak{n}/\mathfrak{I})^* \otimes \Lambda_s^j(\mathfrak{I})^*$, and so any weight of E^{ij} must also be a weight of $\Lambda_s^i(\mathfrak{n}/\mathfrak{I})^* \otimes \Lambda_s^j(\mathfrak{I})^*$. As the weights of $\mathfrak{n}/\mathfrak{I}$ are of the form $\epsilon_k - \delta_l$, $\delta_k - \epsilon_l$, $\epsilon_k - \epsilon_l$ and $\delta_k - \delta_l$ for $1 < k < l < n$ and the weights of $\mathfrak{I}$ are of the form $\epsilon_i - \delta_n$, $\delta_i - \delta_n$, $\epsilon_i - \epsilon_n$ and $\delta_i - \epsilon_n$ for $1 < i < n$, the weights of E^{ij} all have j summands containing either ϵ_n or δ_n. As the weights of $E^{i+2,j-1}$ have only $j-1$ such summands, d_2 must be the zero map.

   We therefore have that $E_3^{ij} = E_2^{ij}$ for all i and j. However, we can apply the same argument to the differentials on the E_r page for any arbitrary $r > 1$. Namely, if the weights in the domain of d_r have j copies of ϵ_m or δ_n, then those in the image have only $j - r$ such copies. Thus, d_r must again be the 0 map. Thus for all $r > 2$, $E_r^{ij} = E_2^{ij}$, and so the spectral sequence collapses.

2. $\mathfrak{sl}(m|n)$ The collection of weights corresponding to the $\mathfrak{n}$ in $\mathfrak{sl}(m|n)$ are identical to those for $\mathfrak{gl}(m|n)$. Hence, we may take the same ideal of $\mathfrak{n} \subseteq \mathfrak{sl}(m|n)$ and the same spectral sequence will collapse.

3. $\mathfrak{osp}(2m+1|2n)$ The ideal $\mathfrak{I}$ is spanned by all weight spaces of a root containing ϵ_m. Thus an arbitrary weight of E_r^{pq} must have a total of q copies of or ϵ_m, whereas those in $E_r^{p+r,q+(1-r)}$ have only $q+1-r$ copies. Thus any differential d_r must be 0, $r > 1$, and so the spectral sequence collapses on the E_2 page.

4. $\mathfrak{osp}(2m|2n)$ We defined the ideal for $\mathfrak{osp}(2m|2n)$ similarly to how it was defined for $\mathfrak{osp}(2m+1|2n)$, and so the above argument follows in the same way.

5. $\mathfrak{q}(n)$ As E_2^{ij} is a subquotient of $\Lambda_s^i(\mathfrak{n}/\mathfrak{I}^*) \otimes \Lambda_s^j \mathfrak{I}^*$, all of its weights must contain j total summands containing either copy of ϵ_n, whereas $E_2^{i+r,j+1-r}$ only contains $j+1-r$ such copies, and thus an arbitrary differential $d_r : E_2^{ij} \to E_2^{i+2,j-1}$ must be 0 for $r > 1$, so the spectral sequence again collapses.

As this covers all cases, this completes the proof. □

Chapter 4

$\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ Cohomology

4.1 Superderivations

It is well known that in the case of ordinary Lie algebras, $\mathrm{H}^1(\mathfrak{g}, M)$ corresponds to derivations from $\mathfrak{g}$ to M modulo inner derivations [HS97]. This situation generalizes to the Lie superalgebra case.

We define a *superderivation* from a Lie superalgebra $\mathfrak{g}$ to a $\mathfrak{g}$-module M to be a linear map ϕ satisfying

$$\phi([x, y]) = x \cdot \phi(y) - (-1)^{|x||y|} y \cdot \phi(x).$$

An *inner superderivation* is a derivation of the form $\phi_a(x) = x \cdot a$ for some $a \in M$. We define $\mathrm{SupDer}(\mathfrak{g}, M)$ to the set of all superderivations from $\mathfrak{g}$ to M, and then $\mathrm{InnSupDer}(\mathfrak{g}, M)$ to be the set of all inner superderivations. Based on these definitions, we obtain the following two results.

Proposition 4.1.1. $\mathrm{SupDer}(\mathfrak{g}, M) \cong \mathrm{Hom}(I\mathfrak{g}, M)$.

Proof. Let $d : \mathfrak{g} \to M$ be a superderivation. Consider the map $f_d' : T(\mathfrak{g}) \to M$ given by $f_d(x_1 \otimes \cdots \otimes x_n) = x_1 \circ \cdots \circ d(x_n)$ and which sends $T^0(\mathfrak{g})$ to 0. It follows immediately that f_d' vanishes on I and thus defines a morphism on $U(\mathfrak{g})$ which restricts to a homomorphism $f_d : I\mathfrak{g} \to M$.

Conversely, given a homomorphism $f : I\mathfrak{g} \to M$, we can extend it to a map on all of $U(\mathfrak{g})$ by setting $f(T^0(\mathfrak{g})) = 0$ and letting $d_f = f \circ i$. It is straightforward to show that $f_{d_f} = f$ and $d_{f_d} = d$, and so the map sending f to d_f is an isomorphism between $\mathrm{SupDer}(\mathfrak{g}, M)$ and $\mathrm{Hom}(I\mathfrak{g}, M)$. $\square$

Proposition 4.1.2. $\mathrm{H}^1(\mathfrak{g}, M) \cong \mathrm{SupDer}(\mathfrak{g}, M)/\,\mathrm{InnSupDer}(\mathfrak{g}, M)$.

Proof. From the augmentation map, we obtain the following short exact sequence:

$$0 \to I\mathfrak{g} \to U(\mathfrak{g}) \to \mathbb{C} \to 0.$$

From the corresponding long exact sequence in cohomology, we obtain that

$$\mathrm{H}^1(\mathfrak{g}, M) \cong \mathrm{coker}(\mathrm{Hom}(U(\mathfrak{g}), M) \to \mathrm{Hom}(I\mathfrak{g}, M)) \cong \mathrm{SupDer}(\mathfrak{g}, M)/\,\mathrm{im}(\mathrm{Hom}(U(\mathfrak{g}), M)).$$

However, if $f \in \mathrm{Hom}(U(\mathfrak{g}), M)$, and $f(1) = a$, then the corresponding derivation is $d_f(x) = x \cdot a$, and thus $\mathrm{H}^1(\mathfrak{g}, M) \cong \mathrm{SupDer}(\mathfrak{g}, M)/\,\mathrm{InnSupDer}(\mathfrak{g}, M)$. $\square$

In particular, when using trivial coefficients, we have the following result:

Theorem 4.1.3. $\mathrm{H}^1(\mathfrak{g}, \mathbb{C}) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.

4.2 Explicit calculations

By the above theorem, to compute the first $\mathfrak{n}$-cohomology, it is sufficient to describe both $\mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}]$. As we have already provided bases for $\mathfrak{n}$ in Section 3, below we do the same for $[\mathfrak{n}, \mathfrak{n}]$ and give formulas for the dimensions of $\mathfrak{n}$, $[\mathfrak{n}, \mathfrak{n}]$, and $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$. A table of corresponding weights is given in Section 6.

4.2.1 $\mathfrak{gl}(m|n)$

We have that the elementary matrices E_{ij} that span $\mathfrak{n}$ will be in $[\mathfrak{n}, \mathfrak{n}]$ precisely when $j - i \geq 2$, and so $[\mathfrak{n}, \mathfrak{n}]$ will have a basis given by

$$\begin{cases} E_{\bar{i},\bar{j}} & 1 \leq i, j \leq m, j - i \geq 2 \\ E_{i,j} & 1 \leq i, j \leq n, j - i \geq 2 \\ E_{\bar{i},j} & 1 \leq i \leq m, 1 \leq j \leq n, j - i \geq 2 \\ E_{i,\bar{j}} & 1 \leq i \leq n, 1 \leq j \leq m, j - i \geq 2. \end{cases}$$

The Lie superalgebra $\mathfrak{n}$ has dimension $\binom{m}{2} + n \cdot (m - n) + 3 \cdot \binom{n}{2}$ and $[\mathfrak{n}, \mathfrak{n}]$ has dimension

$$\binom{m-1}{2} + 2 \cdot \binom{n-1}{2} + n \cdot (m - n - 1) + \binom{n}{2},$$

and so $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ has dimension $m - 1 + n - 1 + n - 1 + n = m + 3n - 3$. The weights of $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ can be found by using the information listed in the previous section and are included in the tables in the appendix.

4.2.2 $\mathfrak{sl}(n|n)$

The weight space decomposition for $\mathfrak{n}$ is identical to that in the $\mathfrak{gl}(n|n)$ case, and thus the above dimension formula and weight space decomposition hold.

4.2.3 $\mathfrak{osp}(2m|2n)$

The derived subalgebra $[\mathfrak{n}, \mathfrak{n}]$ is spanned by the elements

$$\begin{cases} E_{j,\bar{i}} - E_{\overline{i+n},j+m} \\ E_{j+m,\overline{i+n}} - E_{\bar{i},j} \\ E_{l,\overline{k+n}} - E_{\overline{k+n},l+m} \\ E_{\bar{i},\overline{i+n}} \\ E_{\bar{i},\overline{k+n}} + E_{\bar{k},\overline{i+n}} \\ E_{\bar{i},\bar{k}} - E_{\overline{k+n},\overline{i+n}} \\ E_{j,l+m} - E_{l,j+m} \\ E_{jl} - E_{l+m,j+m}, \end{cases}$$

where $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$ and $j - i \geq 2$. The quotient by this subalgebra consists of root vectors solely with the corresponding weights $\epsilon_j - \delta_{j+1}$, $\delta_j - \epsilon_{j+1}$, $\epsilon_m + \delta_n$, $2\delta_n$, $\delta_i - \delta_{i+1}$, and $\epsilon_i - \epsilon_{i+1}$. As a result, $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ has dimension

$$2(m-1) + 2(n-1) + 2 = 2m + 2n - 2.$$

4.2.4 $\mathfrak{osp}(2m+1|2n)$

The only difference in terms of dimension between this and the preceding case is the existence of a root in $\mathfrak{n}$ not found in $[\mathfrak{n}, \mathfrak{n}]$. Thus, the dimension calculation may proceed in essentially the same way, yielding a dimension formula for $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ of $2m + 2n - 1$.

4.2.5 $\mathfrak{q}(n)$

Much like in the case of $\mathfrak{gl}(m|n)$, if $\mathfrak{g} = \mathfrak{q}(n)$, then $[\mathfrak{n}, \mathfrak{n}]$ is spanned by the matrices:

$$\begin{cases} \overline{E}_{i,j} & 1 \leq i, j \leq n,\ j - i \geq 2 \\ \widetilde{E}_{i,j} & 1 \leq i, j \leq n,\ j - i \geq 2 \end{cases}$$

Hence, the dimension of $[\mathfrak{n}, \mathfrak{n}]$ is $2 \cdot \binom{n-1}{2} = (n-1)(n-2)$. As the dimension of $\mathfrak{n}$ is $2 \cdot \binom{n}{2} = n(n-1)$, this implies that the dimension of $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ is

$$n(n-1) - (n-2)(n-1) = 2(n-1).$$

4.2.6 $D(2,1,\alpha)$, $G(3)$, and $F(4)$

For each of the exceptional superalgebras, we may look at the weights given in the Table 1.3.3. As no two weights add up to a third, it follows that the bracket is 0 on $\mathfrak{n}_{\bar{1}}$, and so $\mathfrak{n}_{\bar{1}}$ is abelian and thus $\mathfrak{n} \cong \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$.

Chapter 5

$\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ Cohomology

5.1 Central Extensions

As in the case of $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$, the classical Lie algebra interpretation of equivalence classes of extensions extends to the superalgebra case. On the cochain complex $C^n(\mathfrak{g}, M)$ we set the following $\mathbb{Z}_2$ grading:

$$C^n(\mathfrak{g}, M)_\alpha = \{f \in \mathrm{Hom}(\Lambda_s^n(\mathfrak{g}, M)) | f(\Lambda_s^n(\mathfrak{g}))_\beta \subseteq M_{\alpha+\beta}\},$$

where α and β are elements of $\mathbb{Z}_2$. As the differential map preserves this grading, this gives rise to a $\mathbb{Z}_2$ grading on $\mathrm{H}^n(\mathfrak{g}, M)$ as well.

If M is a $\mathfrak{g}$-module, regarding M as an abelian superalgebra, we say that $\mathfrak{h}$ is an extension of $\mathfrak{g}$ by M if there is an exact sequence of $\mathfrak{g}$-modules:

$$0 \to M \to \mathfrak{h} \to \mathfrak{g} \to 0,$$

where $\mathfrak{h}$ is a Lie superalgebra. Two such extensions are said to be equivalent if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\varphi} & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & M & \xrightarrow{\varphi'} & \mathfrak{h}' & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}.$$

Given an even 2-cocycle h, we define the extension E_h via the short exact sequence

$$0 \to M \to \mathfrak{g} \oplus M \to \mathfrak{g} \to 0,$$

where the product in $\mathfrak{g} \oplus M$ is given by

$$[(x, m), (y, n)] = ([x, y], xn - (-1)^{|m||y|}ym + h(x, y)).$$

Every extension will be equivalent to E_h for some even 2-cocycle h [Mus12, Section 16.4]. Moreover, one can show that two extensions E_h and $E_{h'}$ are equivalent if and only if there is some even linear map $f : \mathfrak{g} \to M$ such that $df = h - h'$, and thus the equivalence classes of extensions are in one-to-one correspondence with $\mathrm{H}^2(\mathfrak{g}, M)_{\bar{0}}$ [Mus12, Section 16.4].

5.2 Computing H^2

Computing the $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ cohomology involves a term mixing together both odd and even elements, and thus requires much more care than the H^1 case. The main idea will be to compute the dimension of these groups recursively. For simplicity's sake, let us restrict our attention to $\mathfrak{g} = \mathfrak{gl}(n|n)$, and let $\mathfrak{n}(n)$ denote the corresponding nilpotent radical. From the collapsing of the Hochschild-Serre spectral sequence, we have that:

$$\mathrm{H}^2(\mathfrak{n}(n), \mathbb{C}) \cong \mathrm{H}^0(\mathfrak{n}(n)/\mathfrak{I}, \mathrm{H}^2(\mathfrak{I}, \mathbb{C})) \oplus \mathrm{H}^1(\mathfrak{n}(n)/\mathfrak{I}, \mathrm{H}^1(\mathfrak{I}, \mathbb{C})) \oplus \mathrm{H}^2(\mathfrak{n}(n)/\mathfrak{I}, \mathrm{H}^0(\mathfrak{I}, \mathbb{C})), \quad (5.2.1)$$

where $\mathfrak{I}$ is the ideal described in Section 3. As $\mathfrak{I}$ is abelian, the cohomology groups $\mathrm{H}^n(\mathfrak{I}, \mathbb{C})$ can be easily computed. Additionally, there is a natural isomorphism between $\mathfrak{n}(n)/\mathfrak{I}$ and $\mathfrak{n}(n-1)$. Thus, in the above decomposition, the first term can be computed directly, viewing it as the set of fixed points of $\mathrm{H}^2(\mathfrak{I}, \mathbb{C})$ under the action of $\mathfrak{n}(n-1)$, and the third can be computed recursively. Thus, the main issue is the computation of $\mathrm{H}^1(\mathfrak{n}(n)/\mathfrak{I}, \mathrm{H}^1(\mathfrak{I}, \mathbb{C}))$, which is isomorphic to $\mathrm{H}^1(\mathfrak{n}(n-1), \mathfrak{I}^*)$.

5.3 Low-Dimension Examples

As an example where all of the computations are relatively straightforward, let us first consider the case of $\mathfrak{gl}(2|2)$ where we wish to compute $\mathrm{H}^2(\mathfrak{n}(2), \mathbb{C})$. As $\mathfrak{n}(2)$ is abelian, all of the differentials in the cochain complex

$$C^0 \to C^1 \to C^2 \to \cdots$$

are 0, where $C^k \cong \Lambda_s^k(\mathfrak{n}(2)^*)$. As such, for any k, $\mathrm{H}^k(\mathfrak{n}(2), \mathbb{C}) \cong \Lambda_s^k(\mathfrak{n}(2)^*)$. In particular,

$$\mathrm{H}^2(\mathfrak{n}, \mathbb{C}) \cong \Lambda_s^2(\mathfrak{n}^*) \cong \bigoplus_{i+j=2} \Lambda^i(\mathfrak{n}_{\bar{0}}) \otimes \mathrm{S}^j(\mathfrak{n}_{\bar{1}}).$$

Using the formulas for the dimensions of exterior and symmetric algebras on a vector space of dimension n, namely

$$\dim \Lambda^i(V) = \binom{n}{i}$$

and

$$\dim \mathrm{S}^j(V) = \binom{n+j-1}{j},$$

we obtain

$$\dim \mathrm{H}^2(\mathfrak{n}, \mathbb{C}) = \dim \Lambda_s^2(\mathfrak{n}) = 1 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = 8.$$

Now consider the case where $\mathfrak{g} = \mathfrak{gl}(3|3)$, and we wish to compute $\mathrm{H}^2(\mathfrak{n}(3), \mathbb{C})$. Letting $\mathfrak{n}$ denote $\mathfrak{n}(2)$, note that as $\mathfrak{n}$ is abelian, $\mathfrak{n}_{\bar{0}}$ is an ideal of $\mathfrak{n}$, and so we obtain a short exact sequence

$$0 \to \mathfrak{n}_{\bar{0}} \to \mathfrak{n} \to \mathfrak{n}_{\bar{1}} \to 0.$$

This gives rise to a second Hochschild-Serre spectral sequence:

$$E_2^{i,j} = \mathrm{H}^i(\mathfrak{n}_{\bar{1}}, \mathrm{H}^j(\mathfrak{n}_{\bar{0}}, \mathfrak{J}_{\bar{0}}^* \otimes \mathfrak{J}_{\bar{1}}^*)) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{n}, \mathfrak{J}_{\bar{0}}^* \otimes \mathfrak{J}_{\bar{1}}^*).$$

Again appealing to an argument with weights, the differential d^2 sends $E_2^{0,1}$ to 0. As the spectral sequence is in the first quadrant, all subsequent differentials must do the same. Thus, we have that

$$\mathrm{H}^1(\mathfrak{n}, \mathfrak{J}_{\bar{0}}^* \otimes \mathfrak{J}_{\bar{1}}^*) \cong E_2^{0,1} \oplus E_2^{1,0}.$$

As $E_2^{1,0} = \mathrm{H}^1(\mathfrak{n}_{\bar{1}}, \mathrm{H}^0(\mathfrak{n}_{\bar{0}}, \mathfrak{J}_{\bar{0}}^* \oplus \mathfrak{J}_{\bar{1}}^*)) = \mathrm{H}^1(\mathfrak{n}_{\bar{1}}, \mathbb{C}^{\oplus 4})$, we can simplify this as $\mathrm{H}^1(\mathfrak{n}_{\bar{1}})^{\oplus 4}$. As $\mathfrak{n}_{\bar{1}}$ is abelian of dimension 2, $E_2^{1,0}$ must have dimension 8. On the other hand, $E_2^{0,1} \cong \mathrm{H}^0(\mathfrak{n}_{\bar{1}}, \mathrm{H}^1(\mathfrak{n}_{\bar{0}}, \mathfrak{J}^*))$. However, as $\mathfrak{n}_{\bar{0}}$ is a classical Lie algebra, by Kostant's theorem,

$$\mathrm{H}^1(\mathfrak{n}_{\bar{0}}, \mathfrak{J}^*) \cong \bigoplus_{l(w)=1,\, j \in J} w \cdot \lambda_j,$$

where w is an element of the Weyl group of $\mathfrak{n}_{\bar{0}}$ and $\mathfrak{J}^* = \bigoplus_{j \in J} \mathrm{L}(\lambda_j)$ as a direct sum of $\mathfrak{n}_{\bar{0}}$ modules. (Viewing $\mathfrak{J}^*$ as an $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$-module shows it is isomorphic to $\mathrm{L}((1,0)) \oplus \mathrm{L}((1,0)) \oplus \mathrm{L}((0,1)) \oplus \mathrm{L}((0,1))$.) As the Weyl group of $\mathfrak{n}_{\bar{0}}$ is isomorphic to $\Sigma_2 \times \Sigma_2$, there are 2 elements of length 1, and so $E_2^{0,1} = \mathrm{H}^0(\mathfrak{n}_{\bar{1}}, s_\alpha \cdot \mathfrak{J}^*)$, which has dimension 4. Thus, altogether $\mathrm{H}^1(\mathfrak{n}_{\bar{0}}, \mathfrak{J}^*)$ has dimension 8, from which an easy computation shows that the dimension of the set of fixed points under the action of $\mathfrak{n}_{\bar{1}}$ is 4, which implies $\mathrm{H}^1(\mathfrak{n}, \mathfrak{J}^*)$ to have a total dimension of 12. Using the argument below, we can see that $\mathrm{H}^0(\mathfrak{n}, \Lambda_s^2(\mathfrak{J}^*))$ has dimension 8 and we already know $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ has dimension 8, so altogether, this implies that $\mathrm{H}^2(\mathfrak{n}(3), \mathbb{C})$ has dimension 8+12+8= 28. However, the argument for computing the dimension of $\mathrm{H}^1(\mathfrak{n}, \mathfrak{J}^*)$ was only valid because $\mathfrak{n}$ was abelian. For general $\mathfrak{gl}(n|n)$ this isn't the case, so $\mathfrak{n}_{\bar{0}}$ is not necessarily an ideal of $\mathfrak{n}$.

5.4 Explicit Calculations

5.4.1 $\mathfrak{gl}(n|n)$

Before beginning with the more general case of $\mathfrak{gl}(m|n)$, we start with the more special case of $\mathfrak{gl}(n|n)$. As in the general case above, we may compute $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ by means of the direct sum decomposition from the spectral sequence, i.e.,

$$\mathrm{H}^2(\mathfrak{n}, \mathbb{C}) \cong \mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(\mathfrak{I}^*)) \oplus \mathrm{H}^1(\mathfrak{n}/\mathfrak{I}, \mathfrak{I}^*) \oplus \mathrm{H}^2(\mathfrak{n}/\mathfrak{I}, \mathbb{C}).$$

The first term can be identified with the set of fixed points of $\Lambda_s^2(\mathfrak{I}^*)$ under the action of $\mathfrak{n}/\mathfrak{I}$, i.e., all $x \in \Lambda_s^2(\mathfrak{I}^*)$ such that $(\mathfrak{n}/\mathfrak{I}) \cdot x = 0$. This set is not particularly difficult to calculate, and we get the following result.

Proposition 5.4.1. *For all n, $\mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(\mathfrak{I}^*))$ has dimension 8.*

Proof. Note that if $a \in \mathfrak{n}/\mathfrak{I}$ and $x \in \Lambda_s^2(\mathfrak{I}^*)$ are weight vectors of weights λ and μ, then $a \cdot x$ has weight $\lambda + \mu$, and so a sends distinct weight spaces to distinct weight spaces. In particular, if $x_1 + \cdots + x_n$ is a sum of weight vectors of distinct weights in $\Lambda_s^2(\mathfrak{I}^*)$ and $a \cdot (x_1 + \cdots + x_n) = 0$, then $a \cdot x_i$ must equal 0 for all i. Since the standard basis for $\Lambda_s^2(\mathfrak{I}^*)$ consists of root vectors all of distinct weights, it suffices to look at which basis elements are sent to 0 by $\mathfrak{n}/\mathfrak{I}$.

Recall that $\mathfrak{I}^*$ has a basis given by $E^*_{i,n}$, $E^*_{\bar{i},n}$, $E^*_{i,\bar{n}}$, and $E^*_{\bar{i},\bar{n}}$, for $1 \leq i \leq n-1$. Based on the supercommutator identity, if $E_{i,j}$ is in $\mathfrak{n}/\mathfrak{I}$ and $E^*_{k,n}$ or $E^*_{k,\bar{n}}$ is in $\mathfrak{I}^*$, $E_{i,j} \cdot E^*_{k,n}$ doesn't vanish precisely when $i = k$. In particular, as there are no elements $E_{i,j}$ in $\mathfrak{n}/\mathfrak{I}$ where $i = n-1$ or $\overline{n-1}$, it is precisely the basis elements $E^*_{n-1,n}$, $E^*_{\overline{n-1},n}$, $E^*_{n-1,\bar{n}}$, and $E^*_{\overline{n-1},\bar{n}}$ that are sent to 0 for all $a \in \mathfrak{n}/\mathfrak{I}$. Any element of $\Lambda_s^2(\mathfrak{I}^*)$ that is sent to 0 is the superexterior product of two such basis elements of $\mathfrak{I}^*$, and as there are two even and two odd such basis elements, viewing $\Lambda_s^2(\mathfrak{I}^*)$ as $\Lambda^2(\mathfrak{I}^*_{\bar{0}}) \oplus (\mathfrak{I}^*_{\bar{0}} \otimes \mathfrak{I}^*_{\bar{1}}) \oplus \mathrm{S}^2(\mathfrak{I}^*_{\bar{1}})$, the total dimension of $\mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(\mathfrak{I}^*))$ is $1 + 2 \cdot 2 + 3 = 8$. □

Moreover, the third term may be computed recursively, using the fact that $\mathfrak{n}/\mathfrak{I}$ is isomorphic to $\mathfrak{n}_{n-1}$. Thus, it remains to compute the middle term.

Let us consider the cochain complex

$$C^0 \to C^1 \to C^2 \to \cdots ,$$

where $C^i \cong \Lambda_s^i(\mathfrak{n}/\mathfrak{I}^*) \otimes \mathfrak{I}^*$ and where the differentials are as in the introduction. Then the middle term $\mathrm{H}^1(\mathfrak{n}/\mathfrak{I}, \mathfrak{I}^*)$ is given by the cohomology of the complex at C^1. Since the differentials preserve the action of the torus, it follows that we may break up C^i into its weight spaces. The weights of $(\mathfrak{n}/\mathfrak{I})^*$ are of the form $\alpha_j - \beta_k$, where α and β correspond to either ϵ or δ, and $k < j < n$. The weights of $\mathfrak{I}^*$ are of the form $\alpha'_n - \beta'_i$, where $i < n$ and α' and β' again correspond to either ϵ or δ. All weights of C^i will be sums of weights of these forms. Actually, using the fact that the cohomology will be a subquotient of $\mathfrak{n}/\mathfrak{I}/[\mathfrak{n}/\mathfrak{I}, \mathfrak{n}/\mathfrak{I}]^* \otimes \mathfrak{I}^*$, we need only consider those weights of $(\mathfrak{n}/\mathfrak{I})^*$ of the form $\alpha_{j+1} - \beta_j$. As a shorthand, given a weight $\alpha_i - \beta_j$, we let $F_{i,j}$ and $G_{i,j}$ denote the basis vector of $(\mathfrak{n}/\mathfrak{I})^*$ of weight $\alpha_i - \beta_j$, or more explicitly:

$$F_{i,j},\ G_{i,j} = \begin{cases} E^*_{\bar{j},\bar{i}} & \alpha = \epsilon,\ \beta = \epsilon \\ E^*_{j,i} & \alpha = \delta,\ \beta = \delta \\ E^*_{j,\bar{i}} & \alpha = \epsilon,\ \beta = \delta \\ E^*_{\bar{j},i} & \alpha = \delta,\ \beta = \epsilon. \end{cases}$$

Proposition 5.4.2. *The dimension for a weight space of C^1 is at most 2.*

Proof. Suppose two basis vectors for C^1, $F_{j+1,j} \otimes G'_{n,k}$ with weight $(\alpha_{j+1} - \beta_j) + (\alpha'_n - \beta'_k)$ and $F_{l+1,l} \otimes G'_{n,m}$ with weight $(\alpha_{l+1} - \beta_l) + (\alpha'_n - \beta'_m)$ actually had the same weight. As the ϵ_i, δ_j are linearly independent, any weight has a unique representation as a sum of ϵ_i's and δ_j's. This leads to two cases:

1. If α_{j+1}, β_j, α'_n, and β'_k are all distinct, these must be, in some order, the same weights as ζ_{l+1}, η_l, ζ'_n, and η'_m. Since $l+1 < n$, it follows that $\zeta'_n = \alpha'_n$ and $\alpha_{j+1} = \zeta_{l+1}$, so $j = l$. Thus, either η_l equals either β_j or β'_k, which leads to two possible basis vectors of the same weight, giving a total dimension of at most two.

2. If $\alpha_{j+1} = \beta'_k$, then $\zeta'_n = \alpha'_n$, $\eta_l = \beta_l$ and $\zeta_{l+1} = \eta'_m$. Since this forces $l+1$ to equal $j+1$ and m to equal $l+1$, ζ_{l+1} can equal only ϵ_{j+1} or δ_{j+1}, which yields at most two basis vectors.

□

With this in mind, we aim to determine the dimension of the image of d^0 and kernel of d^1. To do this, we will determine which weights appear in both C^0 and C^1 and which appear in C^1 but not C^2. For the former calculation, to calculate the dimension of the image of d^0, first note that since its image is in C^1, the differential defined in Equation 1.3.1 simplifies to

$$d^0 f(\omega_0) = (-1)^{\tau_i} \omega_0 \cdot f(1),$$

where a function $f : \mathbb{C} \to \mathfrak{I}^*$ is identified with an element of $\mathfrak{I}^*$ via the map sending f to $f(1)$. What this means is that so long as there exists an element x of $\mathfrak{n}\mathfrak{I}$ such that $x \cdot f(1) \neq 0$, then d^0 does not map f to 0. If $f(1) \in \mathfrak{I}^*$ and $x \in \mathfrak{n}/\mathfrak{I}$ are nonzero weight vectors, this condition holds if the sum of the weights of $f(1)$ and x is again a weight of $\mathfrak{I}^*$.A weight $\alpha'_n - \beta'_k$ of a basis vector $G_{n,k}$ of $\mathfrak{I}^*$ may be written as a weight in C^1 precisely when $k < n-1$. In particular, $G_{n,k}$ will map to an element in the linear span of the root vectors $F'_{k+1,k} \otimes G'_{n,k+1}$ and $F''_{k+1,k} \otimes G''_{n,k+1}$ corresponding to $(\epsilon_{k+1} - \beta'_k) + (\alpha'_n - \epsilon_{k+1})$ and $(\delta_{k+1} - \beta'_k) + (\alpha'_n - \delta_{k+1})$, respectively. Since the differential preserves weights, and $\mathfrak{I}^*$ has $4(n-1) - 4 = 4(n-2)$ weights of the above form, the dimension of the image of d^0 is $4(n-2)$.

To compute the dimension of the kernel, we rely heavily on the differential defined in Equation 1.3.1 and note that a generic weight will be of the form $\alpha_{j+1} - \beta_j + \alpha'_n - \beta'_i$, where $j < n-1$. So long as $i < n-1$, this weight may be written as $(\alpha'_n - \alpha'_{i+1}) + (\alpha'_{i+1} - \beta'_i) + (\alpha_{j+1} - \beta_j)$, and so the differential will send the weight vector corresponding to $(\alpha_{j+1} - \beta_j) + (\alpha'_n - \beta'_i)$ to a nonzero element of C^2. Thus the only weight vectors in the kernel have $\mathfrak{I}^*$ component with $i = n-1$. There are four basis elements of $\mathfrak{I}^*$ with $i = n-1$ and there are $4(n-2)$ basis elements of $(\mathfrak{n}/\mathfrak{I}/[\mathfrak{n}/\mathfrak{I}, \mathfrak{n}/\mathfrak{I}])^*$, so the one-dimensional weight spaces in the kernel contribute total dimension $4(n-2) \cdot 4 = 16(n-2)$. Note, however, that none of these elements are in the image of d^0. Besides those corresponding to weights $\alpha'_n - \beta'_{n-1}$, which are already included in the span of the root vectors listed above, each of these adds 1 more dimension to the kernel. As there are $4(n-3)$ such elements, this gives the kernel a total dimension of at least $20(n-2) - 4$.

To show that no other elements are in the kernel, let $F_{j+1,j} \otimes G_{n,j}$ of weight $(\alpha_{j+1} - \epsilon_j) + (\alpha'_n - \delta_j)$ and $F'_{j+1,j} \otimes G'_{n,j}$ of weight $(\alpha_{j+1} - \delta_j) + (\alpha'_n - \epsilon_j)$ be two basis vectors of the same weight, where $j < n-1$. Identify these basis elements with functions f and g from $\mathfrak{n}/\mathfrak{I}$ to $\mathfrak{I}$. Using the action of the differential, we see that df will send some element $F_{j+1,j} \wedge H_{j+1,j}$ of weight $\alpha_{j+1} - \epsilon_j + \beta_{j+1} - \delta_j$ to a root vector of weight $\alpha'_n - \beta_{j+1}$, where β is either ϵ or δ, depending on what α is not. However, dg will send the same element to 0. Similarly, dg will send $F'_{j+1,j} \wedge \bar{H}_{j+1,j}$ of weight $\alpha_{j+1} - \delta_j + \beta_{j+1} - \epsilon_j$ to $\alpha'_n - \beta_{j+1}$ while df sends the same element to 0. As df and dg are nonzero on different subsets of the basis elements, it follows that they must be linearly independent, and hence there is no nontrivial linear combination of df and dg equal to 0. Since f and g span their weight space, any other nonzero element of that weight space gets mapped to a linear combination of df and dg, and so cannot be mapped to 0 and is thus not in the kernel. Therefore, any weight of the form $(\alpha_{j+1} - \epsilon_j) + (\alpha'_n - \delta_j)$ does not appear in the kernel and thus the kernel must have dimension exactly $20(n-2) - 4$, and so the dimension of the first cohomology is

$$\dim \ker d^1 - \dim \operatorname{im} d^0 = 20(n-2) - 4 - 4(n-2) = 16(n-2) - 4.$$

Combining this with the fact that the first term in the direct sum decomposition above has dimension 8, we have that when $n > 2$, the dimension of $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ equals

$$8 + \sum_{i=3}^{n} (16(i-2) - 4 + 8),$$

which simplifies to

$$8 + \sum_{i=3}^{n} 16(i - 28) = 8 + 8(n^2 + n) - 48 - 28(n - 2) = 8n^2 - 20n + 16.$$

5.4.2 $\mathfrak{gl}(m|n)$

We now proceed to the general case of $\mathfrak{gl}(m|n)$, where we assume that $m > n \geq 2$. Note that in this case the ideal $\mathfrak{I}$ is defined slightly differently from how it is in the case where $m = n$, leading to $\mathfrak{n}/\mathfrak{I}$ being isomorphic to the $\mathfrak{n}$ from $\mathfrak{gl}(m-1|n)$. Thus, using the spectral sequence decomposition

$$\mathrm{H}^2(\mathfrak{n}, \mathbb{C}) \cong \mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(\mathfrak{I}^*)) \oplus \mathrm{H}^1(\mathfrak{n}/\mathfrak{I}, \mathfrak{I}^*) \oplus \mathrm{H}^2(\mathfrak{n}/\mathfrak{I}, \mathbb{C})$$

we can compute $\mathrm{H}(\mathfrak{n}, \mathbb{C})$ recursively, working our way up from the $\mathfrak{n}$ corresponding to $\mathfrak{gl}(n|n)$.

From here, the principles behind the computation are largely the same as in the $\mathfrak{gl}(n|n)$ case, where $\mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(\mathfrak{I}^*))$ is computed by looking at the fixed points of $\Lambda_s^2(\mathfrak{I}^*)$ and $\mathrm{H}^1(\mathfrak{n}/\mathfrak{I}, \mathfrak{I}^*)$ is computed by observing how the differentials act on weights. Putting this all together, we obtain the following formulas for the dimension of $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ corresponding to $\mathfrak{gl}(n+p|n)$:

$$\dim \mathrm{H}^2(\mathfrak{n}, \mathbb{C}) = \begin{cases} 8n^2 - 12n + 8, & p = 1 \\ 8n^2 - 8n + 8, & p = 2 \\ 8n^2 - 8n + 8 + 4n(p-2) + \frac{(p-3)^2 + (p-3)}{2}, & p > 2. \end{cases}$$

5.4.3 $\mathfrak{q}(n)$

The calculation of the dimension of the second cohomology for $\mathfrak{q}(n)$ is similar to that for $\mathfrak{gl}(n|n)$. Note first that when $n = 2$, $\mathfrak{n}$ is a 2-dimensional, abelian Lie superalgebra, and so the ith cohomology will be isomorphic to $\Lambda_s^i(\mathfrak{n}^*)$. Since both $\mathfrak{n}_{\bar{0}}^*$ and $\mathfrak{n}_{\bar{1}}^*$ have dimension 1, $\Lambda^i(\mathfrak{n}_{\bar{0}}^*) = 0$ for all $i > 0$ and $\mathrm{S}^j(\mathfrak{n}_{\bar{1}}^*)$ has dimension 1 for all j, so $\Lambda_s^i(\mathfrak{n}^*)$ is 2-dimensional for all i. For general n, we may use the same direct sum decomposition derived from the spectral sequence as in Equation 5.2.1.

To compute $\mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(I^*))$, which corresponds to fixed points of $\Lambda_s^2(I^*)$ under the action of $\mathfrak{n}/\mathfrak{I}$, note that again the only weight vectors of $\Lambda_s^2(I^*)$ that will vanish under the action of all elements $\mathfrak{n}/\mathfrak{I}$ will be superexterior products involving maximal even root and maximal odd root vectors, in particular, $\widetilde{E}_{n-1,n}^*$ and $E_{n-1,n}^*$. Unlike in the $\mathfrak{gl}(n|n)$ case however, here there is only one such even root vector and one such odd root vector, so $\mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(I^*))$ is spanned by $\widetilde{E}_{n-1,n}^* \otimes E_{n-1,n}^*$ and $E_{n-1,n}^* \otimes E_{n-1,n}^*$. Thus, in the case where $n > 2$, the dimension of $\mathrm{H}^0(\mathfrak{n}/\mathfrak{I}, \Lambda_s^2(I^*))$ is equal to 2.

In computing the middle term $\mathrm{H}^1(\mathfrak{n}/\mathfrak{I}, \mathfrak{I}^*)$, we may again use the fact that we can decompose the terms of the cochain complex into their weight spaces, and the differentials will still preserve the action of the torus. Again, we may look solely at weights from $(\mathfrak{n}/[\mathfrak{n}/\mathfrak{I}, \mathfrak{n}/\mathfrak{I}])^* \otimes I^*$ and argue as we did in the $\mathfrak{gl}(n|n)$ case. Here, the kernel of d^1 will have dimension $4(n-2) + 2(n-3)$ and the image of d^0 will have dimension $2(n-2)$, giving $\mathrm{H}^1(\mathfrak{n}/\mathfrak{I}, \mathfrak{I}^*)$ a dimension of $4n - 10$.

Combining these terms together, we have the dimension of $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ equals

$$2 + \sum_{i=3}^{n} (4i - 8),$$

which simplifies to

$$2n^2 - 6n + 6.$$

5.4.4 $\mathfrak{osp}(2m|2n)$

The same principles apply in computing the second cohomology for the $\mathfrak{osp}(2m|2n)$ superalgebras. As before, we may decompose the cohomology into its direct sum decomposition as in Equation 5.2.1. Note that the last term is again computed recursively, starting with the base case $\mathfrak{osp}(2|2n)$. In this case, $\mathfrak{n}_{\bar{0}}$ is abelian, and so we obtain the direct sum decomposition:

$$H^2(\mathfrak{n}, \mathbb{C}) \cong H^2(\mathfrak{n}_{\bar{0}}, \mathbb{C}) \oplus H^1(\mathfrak{n}_{\bar{0}}, \mathfrak{n}_{\bar{1}}^*) \oplus H^2(\mathfrak{n}_{\bar{0}}, \mathrm{S}^2(\mathfrak{n}_{\bar{1}}^*)).$$

Since $\mathfrak{n}_{\bar{0}}$ is the nilpotent radical of an ordinary Lie algebra, these cohomologies may be computed via Kostant's theorem, which can be shown to sum up to have dimension $\frac{3n^2+n+4}{2}$. Using the fact for $\mathfrak{osp}(2m|2n)$, the $\mathfrak{n}/\mathfrak{I}$ is isomorphic to the $\mathfrak{n}$ from $\mathfrak{osp}(2(m-1)|n)$, the dimensions and weight space expressions for $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ for $m > 1$ may then be computed recursively as in the case for $\mathfrak{gl}(m|n)$ and $\mathfrak{q}(n)$. These are listed in the tables in Section 6.

5.4.5 $\mathfrak{osp}(2m+1|2n)$

We begin again with the direct sum decomposition from Equation 5.2.1. Much of the calculation is similar to that in the case of $\mathfrak{osp}(2m|2n)$. We begin with the base case of $\mathfrak{osp}(3|2n)$ and use the recurrence from the direct sum formula to determine the weight space decomposition for any higher $\mathfrak{osp}(2m+1|2n)$.

5.4.6 $D(2,1,\alpha)$, $G(3)$, and $F(4)$

Just as in the case of $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$, the second cohomology for $D(2,1,\alpha)$, $G(3)$, and $F(4)$ can be easily computed using the fact that the corresponding subalgebras $\mathfrak{n}$ are abelian. In particular, in each case $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ is isomorphic to $C^2(\mathfrak{n}, \mathbb{C})$ in the corresponding cochain complex. A description in terms of its weight space decomposition is given in the tables below.

Chapter 6

Spectral Sequence Constructions

6.1 Spectral Sequence I

Let G be a supergroup scheme and H be a closed subgroup scheme in G. Given an H-module, M, a natural question to ask is whether one can express $R^\bullet \mathrm{ind}_H^G M$ when considered as a $G_{\bar{0}}$-module in terms of $R^\bullet \mathrm{ind}_{H_{\bar{0}}}^{G_{\bar{0}}}(-)$. In [Bru06, Corollary 2.8], Brundan showed that this can be accomplished in the Grothendieck group of $G_{\bar{0}}$-modules by looking at alternating sums via Euler characters. This presents some difficulties when one wants to analyze $R^n \mathrm{ind}_H^G M$ for a fixed n. The following theorem relates $R^n \mathrm{ind}_H^G M$ for a fixed n as a $G_{\bar{0}}$-module to certain cohomologies for $R^\bullet \mathrm{ind}_{H_{\bar{0}}}^{G_{\bar{0}}}(-)$ via a spectral sequence. Our construction was inspired by the result stated for the structure sheaf by Sam and Snowden (cf. [SS21, Proposition 2.1]), and employs the work in [Bru06, Section 2].

Theorem 6.1.1. *Let G be a supergroup scheme and H be a closed subgroup scheme of G, with $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{h} = \mathrm{Lie}\, H$. If M is an H-module then there exists a spectral sequence*

$$E_1^{i,j} = R^{i+j} \operatorname{ind}_{H_{\bar{0}}}^{G_{\bar{0}}} [M \otimes \Lambda^j (\mathfrak{g}_{\bar{1}}/\mathfrak{h}_{\bar{1}})^*] \Rightarrow [R^{i+j} \operatorname{ind}_H^G M]|_{G_{\bar{0}}},$$

where $|_{G_{\bar{0}}}$ denotes restriction down to $G_{\bar{0}}$.

Proof. We will apply the spectral sequence construction given in [Kum02, E9 Theorem, Appendix E]. In order to do so we need to construct a convergent cochain filtration, F, bounded above on the cochain complex, C, whose cohomology is $[R^\bullet \operatorname{ind}_H^G M]|_{G_{\bar{0}}}$. This will yield a convergent spectral sequence where $E_1^{i,j} = \mathrm{H}^{i+j}(F^iC/F^{i+1}C)$.

Recall that $R^\bullet \mathrm{ind}_H^G M = \mathrm{H}^\bullet(H, M \otimes k[G]) = \mathrm{Ext}_H^\bullet(k, M \otimes k[G])$ (cf. [FP86, Section 1]). Let

$$0 \to M \to I_0 \to I_1 \to \dots .$$

be an injective H-resolution of M. By tensoring by $k[G]$ one has an injective H-resolution for $M \otimes k[G]$:

$$0 \to M \otimes k[G] \to I_0 \otimes k[G] \to I_1 \otimes k[G] \to \dots .$$

Now one filters $k[G]$ by powers of $\mathcal{I} = k[G]k[G]_{\bar{1}}$. Note that $\mathcal{I}$ is an H-$G_{\bar{0}}$-bimodule.

This induces a filtration on $C^n = \mathrm{H}^0(H, I_n \otimes k[G])$:

$$C^n \supseteq \mathrm{H}^0(H, I_n \otimes \mathcal{I}) \supseteq \mathrm{H}^0(H, I_n \otimes \mathcal{I}^2) \supseteq \dots . \tag{6.1.1}$$

Since I_n is injective, $\mathrm{H}^1(H, I_n \otimes \mathcal{I}^k) = 0$, thus

$$\mathrm{H}^0(H, I_n \otimes \mathcal{I}^k/\mathcal{I}^{k+1}) \cong \mathrm{H}^0(H, I_n \otimes \mathcal{I}^k/I_n \otimes \mathcal{I}^{k+1}) \cong \mathrm{H}^0(H, I_n \otimes \mathcal{I}^k)/\mathrm{H}^0(H, I_n \otimes \mathcal{I}^{k+1}).$$

By applying the construction described in the first paragraph, there exists a spectral sequence

$$E_1^{i,j} = \mathrm{H}^{i+j}(H, M \otimes \mathcal{I}^i/\mathcal{I}^{i+1}) \Rightarrow \mathrm{H}^{i+j}(H, M \otimes k[G]).$$

The result now follows by applying the isomorphisms given in [Bru06, Theorem 2.7]

$$\mathrm{H}^s(H, M \otimes \mathcal{I}^t/\mathcal{I}^{t+1}) \cong \mathrm{H}^s(H_{\bar{0}}, M \otimes \Lambda^t((\mathfrak{g}_{\bar{1}}/\mathfrak{h}_{\bar{1}})^*) \otimes k[G_{\bar{0}}]).$$

□

One of the immediate consequences of this spectral sequence is the following fact. Let G be a supergroup scheme arising from a classical Lie superalgebra. In this case $G_{\bar{0}}$ is reductive. Let P be a parabolic subgroup scheme which implies that $G_{\bar{0}}/P_{\bar{0}}$ is a projective variety. Then $R^n\text{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}$ takes finite-dimensional rational $P_{\bar{0}}$-modules to finite-dimensional rational $G_{\bar{0}}$-modules. It now follows from Theorem 6.1.1 that if M is a finite-dimensional rational P-module then $R^n\text{ind}_P^G M$ is a finite-dimensional rational G-module for all $n \geq 0$.

6.2 Applications

In this section we demonstrate how several key results in [GGNW21, Propositions 4.1.1 and 4.1.2] can be steamlined with shorter and more efficient proofs by using the the spectral sequence in Theorem 6.1.1[1].

Corollary 6.2.1. *Let $\mathfrak{g} = \text{Lie}\, G$ be a classical simple Lie superalgebra and P be a parabolic subgroup with M a P-module.*

(a) Assume that $R^n \,\text{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^j((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)] = 0$ when $n \neq j$. Then

$$(R^n \,\text{ind}_P^G M)|_{G_{\bar{0}}} \cong R^n \,\text{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)]$$

for $n \geq 0$.

(b) Assume that $M \cong \mathbb{C}$ and $R^n \,\text{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[\Lambda^j((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)] = 0$ for $n \neq j$. Then

$$(R^n \,\text{ind}_P^G \mathbb{C})|_{G_{\bar{0}}} \cong R^n \,\text{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)]$$

for $n \geq 0$.

[1]In the original statement of [GGNW21, Proposition 4.1.1], i is used instead of j. In Corollary 6.2.1, we use j to facilitate a smoother transition from the notation used in the spectral sequence given in Theorem 6.1.1.

Proof. Observe that part (b) which is [GGNW21, Proposition 4.1.1(b)] follows immediately from part (a). Also, note that part (a) is a stronger version of [GGNW21, Proposition 4.1.1(a)].

For part (a), set $H = P_{\bar{0}}$ and apply the spectral sequence given in Theorem 6.1.1. Under the assumption, one has $E_1^{i,j} = 0$ when $i + j \neq j$ or equivalently $E_1^{i,j} = 0$ unless $i = 0$. The spectral sequence lives on the vertical axis (i.e., $E_1^{0,j}$ for $j \geq 0$). Using the fact that the bidgrees of d_r are $(r, 1-r)$ (cf. [Kum02, E.9 Theorem, proof]), it follows that the spectral sequence collapses and yields the isomorphism. □

Corollary 6.2.2. *Let* $\mathfrak{g} = \operatorname{Lie} G$ *be a classical simple Lie superalgebra and* P *be a parabolic subgroup with* M *a* P*-module. Assume that* $R^n \operatorname{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)] = 0$ *for* $n > 0$*. Then*

$$(R^n \operatorname{ind}_P^G M)|_{G_{\bar{0}}} \cong R^n \operatorname{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)]$$

for $n \geq 0$*.*

Proof. Set $H = P_{\bar{0}}$ and apply the spectral sequence given in Theorem 6.1.1. In this case, one has $E_1^{i,j} = 0$ unless $i + j = 0$ or $j = -i$. The spectral sequence collapses because the bidgrees of d_r are $(r, 1-r)$ for $r \geq 1$, and the result follows. □

6.3 Spectral Sequence II

One can use the theorem in [Jan03, I. 4.1 Proposition] to construct a spectral sequence that relates the composition of two induction functors.

Theorem 6.3.1. *Let* G *be a supergroup scheme with* $H \leq K \leq G$ *an inclusion of closed subgroup schemes contained in* G*. If* N *is an* H*-module then there exists a first quadrant spectral sequence*

$$E_2^{i,j} = R^i \operatorname{ind}_K^G R^j \operatorname{ind}_H^K N \Rightarrow R^{i+j} \operatorname{ind}_H^G N.$$

6.4 Spectral Sequence III

The third spectral sequence below was constucted in [GGNW21, Proposition 6.2.1] and relates the relative Lie superalgebra cohomology with sheaf cohomology. The standard construction involves a composition of left exact functors. This spectral sequence is a first quadrant spectral sequence and the differentials also have bidegree $(r, 1-r)$. This spectral sequence can be viewed analogously to the one relating cohomology for algebraic groups and sheaf cohomology presented in [Jan03, I.4.5 Proposition].

Theorem 6.4.1. *Let G be a supergroup scheme where $\mathfrak{g} = \operatorname{Lie} G$ is a classical simple Lie superalgebra, and H be a closed subgroup scheme of G with $\mathfrak{h} = \operatorname{Lie} H$. If M_1 is a G-module and M_2 is an H-module then there exists a first quadrant spectral sequence:*

$$E_2^{i,j} = \operatorname{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, R^j \operatorname{ind}_H^G M_2) \Rightarrow \operatorname{Ext}^{i+j}_{(\mathfrak{h},\mathfrak{h}_{\bar{0}})}(M_1, M_2).$$

Chapter 7

Irreducible Representations via $\mathrm{H}^0(\lambda)$

7.1 Preliminaries

Throughout this chapter, we will assume that G is a classical simple algebraic supergroup scheme and B is a BBW parabolic for G. In particular, we will be tacitly assuming that G is not of type P. Recall that one has a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{f} \oplus \mathfrak{u}^+$ with corresponding supergroup schemes G, U, F and U^+. The supergroup schemes U and U^+ are unipotent, that is, the only finite-dimensional simple module for these subgroup schemes is $\mathbb{C}$.

There exists a maximal torus $T_{\bar{0}}$ contained in the even part of F. Set $X = X(T_{\bar{0}})$. Then there exists a subset of weights $X_F \subseteq X$ that indexes the set of finite-dimensional irreducible representations for F. For $\lambda \in X_F$, let $L_{\mathfrak{f}}(\lambda)$ be the corresponding simple F-module. One can then inflate this module to $B = F \ltimes U$, and consider $H^n(\lambda) = R^n \mathrm{ind}_B^G L_{\mathfrak{f}}(\lambda)$.

The goal of this chapter is to show how to classify finite-dimensional simple G-modules via G-socles of $H^0(\lambda)$. The proofs follow are the same lines as those in [Jan03, Section II, Chapter 2] and generalize the statement of the theorem for $Q(n)$ in [Bru06, Theorem 4.4].

7.2 Simple F-modules

For the algebraic supergroup scheme F where Lie $F = \mathfrak{f}$ is a detecting subalgebra, one can determine the set X_F.

Example 7.2.1. Let $G = Q(n)$. In this case $F \cong Q(1) \times Q(1) \times \cdots \times Q(1)$, and $X_F = X(T_{\bar{0}})$. The irreducible representations are given by Clifford modules $\pi(\lambda)$ [Bru06].

Example 7.2.2. Let $G = GL(n|n)$. The subgroup $F \cong GL(1|1) \times GL(1|1) \times \cdots \times GL(1|1)$ and $X_F = X(T_{\bar{0}})$. The irreducible representations are formed by taking outer tensor products of simple $GL(1|1)$-representations which are either one-dimensional or two-dimensional.

7.3 Induced Modules

Let L be a finite-dimensional simple G-module. Then for some $\lambda \in X_F$, $\mathrm{Hom}_B(L, L_{\mathfrak{f}}(\lambda)) \neq 0$. Therefore, by Frobenius reciprocity $0 \neq \mathrm{Hom}_G(L, H^0(\lambda))$, and $L \hookrightarrow H^0(\lambda)$ for some $\lambda \in X_F$. Let $X_{F,+} = \{\lambda \in X_F : H^0(\lambda) \neq 0\}$.

Proposition 7.3.1. *Let $\lambda \in X_{F,+}$. Then $H^0(\lambda)^{U^+} \cong L_{\mathfrak{f}}(\lambda)$.*

Proof. We first consider a more general idea about induction. Let M be a rational B-module and let $\epsilon_M : \mathrm{ind}_B^G M \to M$ be the evaluation homomorphism. Using the same proof in [Jan03], one can show that $[\mathrm{ind}_B^G M]^{U^+} \hookrightarrow M$ under ϵ_M. This is a monomorphism of F-modules.

Now apply this to the case when $M = L_{\mathfrak{f}}(\lambda)$. The statement of the proposition now follows since $L_{\mathfrak{f}}(\lambda)$ is simple as an F-module and the U^+-fixed points of $H^0(\lambda)$ cannot be zero for $\lambda \in X_{F,+}$. □

7.4 Parametrization

We can now give a parametrization of simple G-modules.

Theorem 7.4.1. *Let G be a classical simple algebraic group scheme. Then there is a 1-1 correspondence between simple G-modules and $X_{F,+}$ given by $L(\lambda) = \mathrm{soc}_G\, H^0(\lambda)$.*

Proof. First we need to show that if $\lambda \in X_+$ then $\mathrm{soc}_G H^0(\lambda)$ is simple. This can easily be seen because if L_1 and L_2 are simple G-modules with $L_1 \oplus L_2 \hookrightarrow H^0(\lambda)$, then one can take U^+-fixed points to get a monomorphism of F-modules: $L_1^{U^+} \oplus L_2^{U^+} \hookrightarrow L_{\mathfrak{f}}(\lambda)$. Since U_+-fixed points on L_j, $j = 1, 2$ are non-trivial and $L_{\mathfrak{f}}(\lambda)$ is a simple F-module, one obtains a contradiction.

Let $L = \mathrm{soc}_G H^0(\lambda)$. Then $L^{U^+} \cong L_{\mathfrak{f}}(\lambda)$. This shows that the socles of $H^0(\lambda)$ and $H^0(\mu)$ where $\lambda, \mu \in X_{F,+}$ are not isomorphic unless $\lambda = \mu$. Therefore, for $\lambda \in X_{F,+}$, one can set $L(\lambda) = \mathrm{soc}_G H^0(\lambda)$ to obtain the desired bijective correspondence. □

7.5 Example: $Q(n)$

We will now indicate how one can parametrize the simple modules using this setup for $G = Q(n)$. Let M be a G-module and $M = \oplus_{\mu \in X} M_\mu$ be its weight space decomposition. We have $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$ with $\mathfrak{f}_{\bar{0}} \cong \mathfrak{t}$, and $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$. This implies that the weight space M_μ is an F-module.

Now let M be a simple $Q(n)$-module. Then for some $\lambda \in X_F$,

$$0 \neq \mathrm{Hom}_G(M, \mathrm{ind}_B^G L_{\mathfrak{f}}(\lambda)) = \mathrm{Hom}_B(M, L_{\mathfrak{f}}(\lambda)).$$

It follows that $L_{\mathfrak{f}}(\lambda)$ has to appear in the head of M as B-module and λ must be the highest weight of M. The ordering is given by the roots $\Delta_{\bar{0}} = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n\}$. Furthermore,

$$0 \neq \mathrm{Hom}_B(M, L_{\mathfrak{f}}(\lambda)) \subseteq \mathrm{Hom}_{B_{\bar{0}}}(M, L_{\mathfrak{f}}(\lambda)) = \mathrm{Hom}_{G_{\bar{0}}}(M, \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} L_{\mathfrak{f}}(\lambda)).$$

Now $L_{\mathfrak{f}}(\lambda) = \oplus \lambda$ as a $B_{\bar{0}}$-module, so that it follows that λ must be in $(X_{\bar{0}})_+$, i.e. it is a dominant integral weight. The upshot of this analysis is that $L(\lambda) = \mathrm{soc}_G H^0(\lambda)$ where $\lambda \in (X_{\bar{0}})_+$ and λ is the highest weight of $L(\lambda)$ (cf. [Bru06, Theorem 4.18]).

For G not of type Q the weight spaces no longer yield F-modules, so this analysis would not work. An interesting problem would be to provide explicit parametrization of simple modules involving weights for the other classical simple Lie superalgebras. Moreover, once one has an explicit parametrization, an interesting problem would be to develop a theory of decomposition numbers (e.g., $[H^0(\lambda) : L(\mu)]$ for $\lambda, \mu \in X_{F,+}$).

Chapter 8

Generic Behavior for BBW Parabolics

8.1 Redux: GGNW Computations

Assume throughout this section that $\mathfrak{g}$ is a classical simple Lie superalgebra not of type P. Furthermore, let G be a supergroup scheme with $\mathfrak{g} = \operatorname{Lie} G$, and B a BBW parabolic subgroup of G. Set

$$p_{G,B}(t) = \sum_{i=0}^{\infty} \dim R^i \operatorname{ind}_B^G \mathbb{C}\, t^i.$$

For the detecting subalgebra $\mathfrak{f}$ associated to $\mathfrak{b}$, there is an isomorphism of rings given by the restriction map:

$$S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}} \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N.$$

where N is a reductive algebraic group. If N_0 is the connected component of the identity in N then $W_{\bar{1}} = N/N_0$ is a finite reflection group. Let $p_{W_{\bar{1}}}(s) = \sum_{w \in W_{\bar{1}}} s^{l(w)}$ be the Poincaré polynomial for $W_{\bar{1}}$.

A fundamental result in [GGNW21, Sections 4.2-4.9] was the calculation of $R^\bullet \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$. It was shown that

$$R^n \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^j((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0 \;\text{ for } n \neq j. \tag{8.1.1}$$

Furthermore, in the case when $n = j$, $R^n\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^n((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ is a direct sum of trivial modules whose number is prescribed by the coefficients of $p_{W_{\bar{1}}}(s)$. These results in conjunction with Corollary 6.2.1 yield the calculation of $R^\bullet\mathrm{ind}_B^G\mathbb{C}$ which is summarized below (cf. [GGNW21, Theorem 4.10.1]).

Theorem 8.1.1. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra with $\mathfrak{g} = \mathrm{Lie}\, G$. Assume that $\mathfrak{g}$ is not isomorphic to $P(n)$. Let B be the parabolic subgroup such that $\mathfrak{b} = \mathrm{Lie}\, B$ where $\mathfrak{b}$ is a BBW parabolic subalgebra. Then*

(a) $R^\bullet\, \mathrm{ind}_B^G\, \mathbb{C}$ is a direct sum of trivial modules.

(b) The number of trivial modules in $R^n\, \mathrm{ind}_B^G\, \mathbb{C}$ is given by the coefficient in front of t^n in

$$p_{G,B}(t) = p_{W_{\bar{1}}}(s)$$

where $s = t$ when G is of type Q, and $s = t^2$ otherwise.

8.2 An Analog of Kempf's Vanishing Theorem

Let $T_{\bar{0}}$ the a maximal torus in $G_{\bar{0}}$, $X = X(T_{\bar{0}})$ and $(X_{\bar{0}})_+$ the dominant integral weights. The Weyl group of $G_{\bar{0}}$ is denoted by $W_{\bar{0}}$ with identity element $1 \in W_{\bar{0}}$.

Moreover, let V be a $T_{\bar{0}}$-module and $V = \oplus_{\gamma\in X} V_\gamma$ be its weight space decomposition. Set $\mathrm{wt}(V) = \{\gamma \in X : V_\gamma \neq 0\}$ (i.e., the set of weights of V). We start off this section by stating a key definition.

Definition 8.2.1. Let $\lambda \in X_F$ and $w \in W_{\bar{0}}$.

(a) The weight λ is *very dominant* if $\mu + \sigma \in X_+$ for all $\mu \in \mathrm{wt}(L_{\mathfrak{f}}(\lambda))$ and $\sigma \in \mathrm{wt}(\Lambda^\bullet((\mathfrak{g}/\mathfrak{b})^*))$.

(b) The set of very dominant weight will be denoted by X_{++}.

(c) Set $\Gamma(\lambda, w) = \mathrm{wt}(L_{\mathfrak{f}}(\lambda) \otimes w^{-1}\Lambda^\bullet((\mathfrak{g}/\mathfrak{b})^*))$. For $\gamma \in \Gamma(\lambda, w)$, let $m^\lambda_{\gamma,w}$ be the multiplicity of the weight γ in $L_{\mathfrak{f}}(\lambda) \otimes w^{-1}\Lambda^\bullet((\mathfrak{g}/\mathfrak{b})^*)$.

As a consequence of Theorem 6.1.1, we can provide a criterion for the vanishing of the higher sheaf cohomology groups for weights that are very dominant.

Theorem 8.2.2. *Let $\lambda \in X_{++}$, and 1 be the identity element in $W_{\bar{0}}$. Then*

(a) $R^n \operatorname{ind}_B^G L_{\mathfrak{f}}(\lambda) = 0$ *for* $n > 0$.

(b) $\operatorname{ind}_B^G L_{\mathfrak{f}}(\lambda)|_{G_{\bar{0}}} \cong \oplus_{\gamma \in \Gamma(\lambda,1)} [\operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \gamma]^{\oplus m^{\lambda}_{\gamma,1}}$ *as a* $G_{\bar{0}}$*-module.*

Proof. One can apply the spectral sequence in Theorem 6.1.1 with $H = B$ and $M = L_{\mathfrak{f}}(\lambda)$. Under the condition that $\lambda \in X_{++}$, one has $E_1^{i,j} = 0$ for $i + j > 0$. Therefore, the spectral sequence degenerates and yields part (a). Part (b) follows because under the assumption that $\lambda \in X_{++}$, one has $R^1\operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\gamma = 0$ for all $\gamma \in \Gamma(\lambda, 1)$. □

We now illustrate how this theorem works for $\mathfrak{q}(n)$.

Example 8.2.3. Let $\mathfrak{g} = \mathfrak{q}(n)$, $G = Q(n)$ and B be a BBW parabolic subgroup. For $\lambda \in X_F$, $L_{\mathfrak{f}}(\lambda) \cong \lambda^{\oplus \dim L_{\mathfrak{f}}(\lambda)}$ (direct sum of copies of $\mathbb{C}_\lambda$) as a $B_{\bar{0}}$-module.

First observe that $\lambda = 0$ is not very dominant because $0 \neq R^1\operatorname{ind}_B^G\mathbb{C} = R^1\operatorname{ind}_B^G\lambda$ by Theorem 8.1.1. Let $\lambda \in X_{++}$. In this case, $\lambda \in X_{++}$ if and only if $\lambda + \sigma \in (X_0)_+$ for all $\sigma \in \operatorname{wt}(\Lambda^\bullet((\mathfrak{g}/\mathfrak{b})^*))$. Since σ can be zero, one has $X_{++} \subseteq (X_{\bar{0}})_+$.

The weight σ is a sum of distinct roots from the set $-\Phi_{\bar{1}}^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}$. Now the simple roots for $G_{\bar{0}}$ are given by $\Delta_{\bar{0}} = \{\alpha_1, \dots, \alpha_n\}$ where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ where $i = 1, 2, \dots, n-1$. The condition that $\lambda + \sigma \in (X_0)_+$ is equivalent to $\langle \lambda + \sigma, \alpha^\vee \rangle \geq 0$ for $\alpha \in \Delta_{\bar{0}}$.

A direct calculation shows that $-\langle \sigma, \alpha^\vee \rangle \geq n + 1$, and it follows that

$$\{\lambda \in X : n + 1 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Delta_{\bar{0}}\} \subseteq X_{++} \subseteq (X_{\bar{0}})_+.$$

8.3 An Analog of the Bott-Borel-Weil Theorem

For $w \in W_{\bar{0}}$, recall that the dot action on X is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half-sum of the positive roots of $\mathfrak{g}_{\bar{0}}$. Let $\overline{C}_{\mathbb{Z}}$ for $G_{\bar{0}}$ be defined as in [Jan03, II. 5.5].

For a given $w \in W_{\bar{0}}$, set

$$\Omega(w) = \{\lambda \in X_F : \mu + w^{-1}\sigma \in \overline{C}_{\mathbb{Z}} \text{ for all } \mu \in \operatorname{wt}(L_{\mathfrak{f}}(\lambda)) \text{ and } \sigma \in \operatorname{wt}(\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*))\}.$$

Observe that $\Omega(w) \subseteq \overline{C}_{\mathbb{Z}}$ for all $w \in W_{\bar{0}}$ since 0 is a weight of $\Lambda^{\bullet}((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^{*})$. We say that a weight μ is a *generic weight* if and only if $\mu \in \cup_{w \in W_{\bar{0}}} w \cdot \Omega(w) =: \Omega$. The set Ω will be called the set of generic weights.

We can now prove a version of the Bott-Borel-Weil Theorem for generic weights. Note that this theorem encompasses Theorem 8.2.2 which coincides with how the ordinary BBW Theorem encompasses the classical Kempf's vanishing theorem (see [Jan03, II. Chapters 4 and 5]).

Theorem 8.3.1. *Let $w \in W_{\bar{0}}$ and $w \cdot \lambda$ is a generic weight where $\lambda \in \Omega(w)$. Then*

$$(R^n \operatorname{ind}_B^G L_{\mathfrak{f}}(w \cdot \lambda))|_{G_{\bar{0}}} \cong \begin{cases} \bigoplus_{\gamma \in \Gamma(\lambda, w)} [\operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \gamma]^{\oplus m_{\gamma,w}^{\lambda}} & n = l(w) \\ 0 & n \neq l(w). \end{cases}$$

Proof. Let $\mu + w^{-1}\sigma \in \overline{C}_{\mathbb{Z}}$ where μ is a weight of $L_{\mathfrak{f}}(w \cdot \lambda)$ and σ a weight of $\Lambda^{\bullet}((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^{*})$. According to the ordinary BBW Theorem [Jan03, II 5.5 Corollary], one has

$$R^n \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} w \cdot (\mu + w^{-1}\sigma) \cong \begin{cases} 0 & \text{if } n \neq l(w) \\ \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \mu + w^{-1}\sigma & \text{if } n = l(w). \end{cases} \tag{8.3.1}$$

Now apply the spectral sequence in Theorem 6.1.1 with $H = B$ and $M = L_{\mathfrak{f}}(w \cdot \lambda)$. From (8.3.1), it follows that $E_1^{i,j} = 0$ for $i + j \neq l(w)$. One can now apply the same reasoning as in the proof of Theorem 8.2.2. The spectral sequence degenerates and yields the desired result. □

8.4 Generic Weights

Let $G = Q(n)$. Since $\mathrm{wt}(L_{\mathfrak{f}}(\lambda)) = \{\lambda\}$, one has for a given $w \in W_{\bar{0}}$,

$$\Omega(w) = \{\lambda \in X : \lambda + w^{-1}\sigma \in \overline{C}_{\mathbb{Z}} \text{ for all } \sigma \in \mathrm{wt}(\Lambda^{\bullet}((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^{*}))\}.$$

We now show that Ω can be obtained by translating $\Omega(1)$ by the ordinary action of the Weyl group $W_{\bar{0}}$.

Lemma 8.4.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let σ be a sum of distinct negative roots of $\mathfrak{g}$. Then for all w in the Weyl group W of $\mathfrak{g}$, $w \cdot \sigma$ is also a sum of distinct negative roots.*

Proof. Let A_w be the set of all roots α in Φ^+ such that $w(\alpha) \in \Phi^-$, and let B_w be the set of all roots β in Φ^+ such that $w(\beta) \in \Phi^+$. If σ is sum of distinct negative roots, then it may be written as

$$\sigma = -\alpha_1 - \cdots - \alpha_n - \beta_1 - \cdots - \beta_m$$

where $\alpha_i \in A_w$ for all i and $\beta_j \in B_w$ for all j. Then we have that

$$w \cdot \sigma = w(\sigma) + w \cdot 0 = -w(\alpha_1) - \cdots - w(\alpha_n) + w(-\beta_1) + \cdots + w(-\beta_m) + w \cdot 0.$$

Notice that

$$w \cdot 0 = \sum_{\alpha \in A_w} w(\alpha),$$

and so

$$-w(\alpha_1) - \cdots - w(\alpha_n) + w \cdot 0$$

is a sum of distinct negative roots

$$-w(\alpha_1) - \cdots - w(\alpha_n) + w \cdot 0 = w(\gamma_1) + \cdots + w(\gamma_l),$$

where the γ_k are all in A_w. Moreover, since β_j is in B_w, each $w(-\beta_j)$ is a negative root, so this implies that $w \cdot \sigma$ is a sum of negative roots. Finally, since the γ_k and $-\beta_j$ are all distinct roots, so too are the $w(\gamma_k)$ and $w(-\beta_j)$, so σ is a sum of distinct negative roots. □

Proposition 8.4.2. *Let $G = Q(n)$. The generic regions $w \cdot \Omega(w)$ are conjugate under the regular action of the Weyl group $W_{\bar{0}}$. Consequently, $\Omega = \cup_{w \in W_{\bar{0}}} w(\Omega(1))$.*

Proof. It is enough to show that $w \cdot \Omega(w)$ is equal to $w(\Omega(1))$. Let $\lambda \in \Omega(1)$. Then for all positive roots α and all sums of distinct negative roots σ,

$$\langle \lambda + \sigma + \rho, \alpha^\vee \rangle \geq 0.$$

By the above lemma, $w^{-1} \cdot \sigma$ is a sum of distinct negative roots, and so

$$\langle \lambda + w^{-1} \cdot \sigma + \rho, \alpha^\vee \rangle \geq 0.$$

Now

$$\lambda + \rho + w^{-1} \cdot \sigma = [\lambda + w^{-1}\rho - \rho] + w^{-1}\sigma + \rho.$$

However, the condition for μ to be in $\Omega(w)$ is that

$$\langle \mu + w^{-1}\sigma + \rho, \alpha^\vee \rangle \geq 0,$$

so $\lambda + w^{-1}\rho - \rho$ is an element of $\Omega(w)$. This is equal to $w^{-1} \cdot (w\lambda)$, and so $w\lambda \in w \cdot \Omega(w)$. Thus, $w(\Omega(1)) \subseteq w \cdot \Omega(w)$. The other direction follows similarly. $\square$

8.5 Example: $G = Q(2)$

In this case $G_{\bar{0}} \cong GL_2(\mathbb{C})$ and $W_{\bar{0}} \cong \Sigma_2 = \{1, s_\alpha\}$. Moreover, $\mathrm{wt}(\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)) = \{0, -\alpha\}$. Using the definition of $\Omega(w)$, one can directly show that

$$\begin{aligned} \Omega(1) &= \{(\lambda_1, \lambda_2) : \lambda_1 - \lambda_2 \geq 1\} \\ \Omega(s_\alpha) &= \{(\lambda_1, \lambda_2) : \lambda_1 - \lambda_2 \geq -1\} \end{aligned}$$

Therefore,

$$\Omega = \bigcup_{w \in W_{\bar{0}}} w \cdot \Omega(w) = \{\mu = (\mu_1, \mu_2) : \mu_1 - \mu_2 \neq 0\}.$$

It follows that for $\mu \in \Omega$, $H^n(\mu)|_{G_{\bar{0}}}$ can be computed for all n by Theorem 8.3.1. This agrees with the calculation for $G = Q(2)$ given in [Bru06, Lemma 4.4].

8.6 Example: $G = Q(3)$

One has $G_{\bar{0}} \cong GL_3(\mathbb{C})$ and

$$W_{\bar{0}} \cong \Sigma_3 = \{1, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}, s_{\alpha_2}s_{\alpha_1}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}\}.$$

Moreover, the generic region $\Omega = \bigcup_{w \in W_{\bar{0}}} w \cdot \Omega(w)$ where

$$\begin{aligned}
\Omega(1) &= \{\lambda \in X : \langle \lambda, \alpha_1^\vee \rangle \geq 2, \langle \lambda, \alpha_2^\vee \rangle \geq 2\} \\
\Omega(s_{\alpha_1}) &= \{\lambda \in X : \langle \lambda, \alpha_1^\vee \rangle \geq 3, \langle \lambda, \alpha_2^\vee \rangle \geq 0\} \\
\Omega(s_{\alpha_2}) &= \{\lambda \in X : \langle \lambda, \alpha_1^\vee \rangle \geq 0, \langle \lambda, \alpha_2^\vee \rangle \geq 3\} \\
\Omega(s_{\alpha_1}s_{\alpha_2}) &= \{\lambda \in X : \langle \lambda, \alpha_1^\vee \rangle \geq 3, \langle \lambda, \alpha_2^\vee \rangle \geq -1\} \\
\Omega(s_{\alpha_2}s_{\alpha_1}) &= \{\lambda \in X : \langle \lambda, \alpha_1^\vee \rangle \geq -1, \langle \lambda, \alpha_2^\vee \rangle \geq 3\} \\
\Omega(s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) &= \{\lambda \in X : \langle \lambda, \alpha_1^\vee \rangle \geq 0, \langle \lambda, \alpha_2^\vee \rangle \geq 0\}.
\end{aligned}$$

Therefore, it can be shown that

$$\Omega = \bigcup_{w \in W} w\{\lambda \in X : \langle \lambda, \alpha_1^\vee \rangle \geq 2, \langle \lambda, \alpha_2^\vee \rangle \geq 2\},$$

which are the W-conjugates of $\Omega(1)$ under the regular action.

8.7 Comparison of Cohomology for $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ and $(\mathfrak{b}, \mathfrak{b}_{\bar{0}})$

For reductive algebraic groups, one can use the induction functor to compare cohomology for G to P where P is any parabolic subgroup [Jan03, I. 4.5 Proposition]. Using Theorem 6.4.1 and 8.3.1, one can obtain a comparison theorem for extensions between modules for $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ and $(\mathfrak{b}, \mathfrak{b}_{\bar{0}})$.

Theorem 8.7.1. *Let G be a supergroup scheme where $\mathfrak{g} = \operatorname{Lie} G$ is a classical simple Lie superalgebra, and B be a BBW parabolic subgroup of G. Moreover, let $w \in W_{\bar{0}}$, $w \cdot \lambda$ be a generic weight where $\lambda \in \Omega(w)$ and M be a G-module. Then for $i \geq 0$,*

$$\operatorname{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, R^{l(w)} \operatorname{ind}_B^G L_{\mathfrak{f}}(w \cdot \lambda)) \cong \operatorname{Ext}^{i+l(w)}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, L_{\mathfrak{f}}(w \cdot \lambda)).$$

Proof. Consider the spectral sequence in Theorem 6.4.1 with $H = B$. Under the condition that $w \cdot \lambda$ is a generic weight, $R^j \operatorname{ind}_B^G L_{\mathfrak{f}}(w \cdot \lambda) \neq 0$ when $j \neq l(w)$. Therefore, the spectral sequence collapses, and $E_2^{i,l(w)} \cong \operatorname{Ext}^{i+l(w)}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, L_{\mathfrak{f}}(w \cdot \lambda))$ for all $i \geq 0$. □

Chapter 9

Results on $\mathrm{H}^1(\lambda)$

For the moment, assume that G is a reductive algebraic group and B is a Borel subgroup (arising from the negative roots) [Jan03]. If λ is a weight then Andersen [And79] proved that $H^1(\lambda) = \mathrm{ind}_B^G \lambda$ is either zero or has a simple G-socle. The socle of $H^1(\lambda)$, $\mathrm{soc}_G H^1(\lambda)$, can be explicitly described [Jan03, II 5.15 Proposition]. For $H^n(\lambda)$, $n \geq 2$, the vanishing behavior remains an open question over fields of characteristic $p > 0$.

Let us now return to the situation where G is a supergroup scheme with Lie $G = \mathfrak{g}$ where $\mathfrak{g}$ is a simple classical Lie superalgebra and B is a BBW parabolic subgroup in G. In Section 7, we proved that $H^0(\lambda)$ is either zero or has simple socle. In dramatic contrast to the situation for reductive groups, Theorem 8.1.1 demonstrates that $H^1(\lambda)$ need not have simple socle. For example, if $G = Q(3)$ then $H^1((0,0,0)) = R^1\mathrm{ind}_B^G \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

9.1 Socles for $H^1(\lambda)$

Let $P = L_P \ltimes U_P$ be a parabolic subgroup such that $B \subseteq P \subseteq G$. For any $\sigma \in X_{F,+}$, let $\bar{\sigma}$ be the weight in X with $L_P(\bar{\sigma}) = L(\sigma)^{U_P}$ where $L_P(\bar{\sigma})$ is the inflation of a simple L_P-module.

Theorem 9.1.1. *Let G be a supergroup arising from a simple Lie superalgebra $\mathfrak{g}$, B be a BBW parabolic subgroup and $\lambda \in X$. Suppose there exists a parabolic subgroup scheme P in G with $B \subseteq P \subseteq G$ with*

$R^0 \operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda) = 0$. *Then for all* $\sigma \in X_{F,+}$

$$[\operatorname{soc}_G H^1(\lambda) : L(\sigma)] = [\operatorname{soc}_L R^1 \operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda) : L_P(\bar{\sigma})]$$

Proof. Suppose that $R^0 \operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda) = 0$. Consider the spectral sequence given in Theorem 6.3.1 with $K = P$, $H = B$ and $N = L_{\mathfrak{f}}(\lambda)$. One has a five term exact sequence of the form

$$0 \to E_2^{1,0} \to E_1 \to E_2^{0,1} \to E_2^{2,0} \to E_2.$$

The assumption implies that $E_2^{i,0} = 0$ for $i \geq 0$. Therefore,

$$H^1(\lambda) \cong \operatorname{ind}_P^G[R^1\operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda)].$$

In order to compute the socle we need to consider homomorphisms of $L(\sigma)$ into $H^1(\lambda)$:

$$\begin{aligned}
\operatorname{Hom}_G(L(\sigma), H^1(\lambda)) &\cong \operatorname{Hom}_G(L(\sigma), \operatorname{ind}_P^G[R^1\operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda)]) \\
&\cong \operatorname{Hom}_P(L(\sigma), R^1\operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda)) \\
&\cong \operatorname{Hom}_{L_P}(k, \operatorname{Hom}_{U_P}(k, L(\sigma)^* \otimes R^1\operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda))) \\
&\cong \operatorname{Hom}_{L_P}(k, \operatorname{Hom}_{U_P}(k, L(\sigma)^*) \otimes R^1\operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda)) \\
&\cong \operatorname{Hom}_{L_P}(k, L_P(\bar{\sigma})^* \otimes R^1\operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda)) \\
&\cong \operatorname{Hom}_{L_P}(L_P(\bar{\sigma})), R^1\operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda)).
\end{aligned}$$

The statement of the theorem follows from this chain of isomorphisms. $\square$

Additionally, the following result uses Theorem 9.1.1 and provides a criterion for the irreducibility of $\operatorname{soc}_G H^1(\lambda)$.

Corollary 9.1.2. *Let G be a supergroup arising from a simple classical Lie superalgebra $\mathfrak{g}$, B be a BBW parabolic subgroup and $\lambda \in X$. Suppose there exists a parabolic subgroup scheme P in G with $B \subseteq P \subseteq G$ satisfying:*

(a) $R^0 \operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda) = 0$,

(b) $R^1 \operatorname{ind}_B^P L_{\mathfrak{f}}(\lambda)$ *has simple L-socle.*

Then $\operatorname{soc}_G H^1(\lambda)$ *is simple.*

9.2 Applications

Let $G = Q(2)$ and $\sigma = (\sigma_1, \sigma_2)$ be a weight in $(X_{\bar{0}})_+$. In [Pen86, Section 7], Penkov computed the characters of all the irreducible $Q(2)$-modules $L(\sigma)$ of highest weight σ. In particular, suppose σ is not a nonzero integer multiple of ρ. Then

$$\operatorname{char} L(\sigma) = \begin{cases} e^0 & \sigma = 0 \\ 2 \operatorname{char} L_{\bar{0}}(\sigma) + 2 \operatorname{char} L_{\bar{0}}(\sigma - \alpha) & \sigma_1 - \sigma_2 \neq 1 \\ 2 \operatorname{char} L_{\bar{0}}(\sigma) & \sigma_1 - \sigma_2 = 1 \end{cases},$$

where $L_{\bar{0}}(\sigma)$ is the irreducible $G_{\bar{0}}$-module of highest weight σ, with

$$\operatorname{char} L_{\bar{0}}(\sigma) = e^{\sigma} + e^{\sigma - \alpha} + \cdots + e^{w\sigma + \alpha} + e^{w\sigma}.$$

If σ is a nonzero integer multiple of ρ, then

$$\operatorname{char} L(\sigma) = 2 \operatorname{char} L_{\bar{0}}(\sigma).$$

On the other hand, from [Bru06, Lemma 4.4] the character of the induced module $\mathrm{ind}_B^G L_{\mathfrak{f}}(\sigma)$ for nonzero dominant σ is

$$\mathrm{char}\,\mathrm{ind}_B^G L_{\mathfrak{f}}(\sigma) = 2(e^{\sigma} + 2e^{\sigma-\alpha} + \cdots + 2e^{w\sigma+\alpha} + e^{w\sigma}).$$

Therefore, when σ is not an integer multiple of ρ with $\sigma_1 \neq \sigma_2 + 1$, $\mathrm{ind}_B^G L_{\mathfrak{f}}(\sigma)$ is an irreducible module isomorphic to $L(\sigma)$. Otherwise, $\mathrm{ind}_B^G L_{\mathfrak{f}}(\sigma)$ has length 2, with composition factors $L(\sigma)$ and $L(\sigma - \alpha)$. Moreover, by Serre duality [Bru06, Theorem 5.1], $H^1(\sigma) \cong H^0(-\sigma)^*$. If $w\sigma \in (X_{\bar{0}})_+$, then $H^1(\sigma)$ is either irreducible or has socle isomorphic to $L(w\sigma - \alpha)$. In summary, if $H^1(\sigma) \neq 0$ then $H^1(\sigma)$ will have simple G-socle.

Let $G = Q(n)$, and fix a root α in $\Delta_{\bar{0}} = \{\epsilon_i - \epsilon_{i+1} : i = 1, 2, \ldots, n\}$. Let P_α be a minimal parabolic subgroup containing B such that Lie P_α is the Lie superalgebra generated by $\mathfrak{b} = \mathrm{Lie}\, B$ along with the root space $\mathfrak{g}_\alpha$. Then $P_\alpha = L_\alpha \ltimes U_\alpha$ where L_α is a supergroup scheme of type $Q(2)$.

Now assume that $\lambda \in X$ with $\langle \lambda, \alpha^\vee \rangle < 0$. Then by Theorem 6.1.1,

$$R^0\mathrm{ind}_B^{P_\alpha} L_{\mathfrak{f}}(\lambda)|_{L_\alpha} \cong R^0\mathrm{ind}_{B\cap L_\alpha}^{L_\alpha} L_{\mathfrak{f}}(\lambda)|_{L_\alpha} = 0.$$

One can now invoke Theorem 9.1.1

$$[\mathrm{soc}_G\, H^1(\lambda) : L(\sigma)] = [\mathrm{soc}_{L_\alpha}\, R^1\, \mathrm{ind}_B^{P_\alpha}\, L_{\mathfrak{f}}(\lambda) : L_{P_\alpha}(\bar{\sigma})]. \tag{9.2.1}$$

The analysis for $Q(2)$ shows that $R^1\, \mathrm{ind}_B^{P_\alpha}\, L_{\mathfrak{f}}(\lambda)$ has a simple L_α-socle. We can now state the following theorem.

Theorem 9.2.1. *Let $G = Q(n)$ and $\lambda \in X$ where $\langle \lambda, \alpha^\vee \rangle < 0$ for some $\alpha \in \Delta_{\bar{0}}$. Then $H^1(\lambda)$ has a simple G-socle.*

9.3 Open Problems

When $\mathfrak{g}$ is a simple classical Lie superalgebra of type other than Q, a similar type of analysis can be done for minimal parabolic subgroups $P_\alpha = L_\alpha \ltimes U_\alpha$ where L_α is of type $GL(2|2)$. This motivates the following open problem.

Problem 9.3.1. Determine when $\text{soc}_G H^1(\lambda)$ is simple for $G = GL(2|2)$.

The solution to the aforementioned problem in conjunction with Corollary 9.1.2 would provide necessary insights into solving the more general problem.

Problem 9.3.2. Compute $\text{soc}_G H^1(\lambda)$ for all $\lambda \in X$.

The sheaf cohomology groups $H^n(\lambda)$ for $n \geq 0$ are central objects for the cohomology and representation theory of G. As demonstrated in this dissertation, these sheaf cohomology groups unify the theory of Lie superalgebra representations. Their vanishing behavior is tied in with the combinatorics of the Weyl group for $G_{\bar{0}}$ acting on odd roots. Furthermore, concrete calculations are, for general λ, highly dependent on the use of the detecting subalgebra $\mathfrak{f}$ along with the finite reflection group $W_{\bar{1}}$. This produces a unique mixture of the odd and even theories. Further investigations along these lines should yield solutions to the open questions about these G-modules and provide new insights into the representation theory for classical simple Lie superalgebras.

Chapter 10

Appendix: Tables of Weights and Dimensions

In the tables below, we compile a list of all of the weights appearing in the first and second cohomologies for the Lie superalgebras used above, as well as their dimensions. As a shorthand, we use the following notation. For $\mathfrak{gl}(m|n)$, we let α_i be the weight $\epsilon_{i+1} - \epsilon_i$, α_i' be the weight $\delta_{i+1} - \delta_i$, β_i be the weight $\delta_{i+1} - \epsilon_i$, and β_i' be the weight $\epsilon_{i+1} - \delta_i$. In the case of $\mathfrak{gl}(m|n)$, we assume that $m > n$. In the case of $\mathfrak{osp}$, we let $\mu_1, \cdots, \mu_m$ denote the simple weights of B_m or D_m, and let $\nu_1, \cdots, \nu_n$ be the simple weights of C_n. Again, for the exceptional Lie superalgebras, we follow the notation in [GGNW21, Section 3.2].

10.1 $\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ Cohomology

$\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ Cohomology (Classical Cases)		
Lie Superalgebra	**Corresponding Even Weights**	**Corresponding Odd Weights**
$\mathfrak{gl}(n\|n)$	$\alpha_i, 1 \leq i \leq n-1,$ $\alpha'_j, 1 \leq j \leq n-1$	$\beta_i, 1 \leq i \leq n-1,$ $\beta'_j, 1 \leq j \leq n-1$
$\mathfrak{gl}(m\|n)$	$\alpha_i, 1 \leq i \leq m-1,$ $\alpha'_j, 1 \leq j \leq n-1$	$\beta_i, 1 \leq i \leq n-1,$ $\beta'_j, 1 \leq j \leq n$
$\mathfrak{osp}(2m\|2n)$	$-\epsilon_i - \epsilon_{i+1}, 1 \leq i \leq m-1,$ $-\delta_{i+1} - \delta_i, 1 \leq i \leq n-1,$ $-2\delta_n$	$\epsilon_{i+1} - \delta_i, 1 \leq i \leq m-1,$ $\delta_{i+1} - \epsilon_i, 1 \leq i \leq n-1,$ $\epsilon_m + \delta_n$
$\mathfrak{osp}(2m+1\|2n)$	$-\epsilon_i - \epsilon_{i+1}, 1 \leq i \leq m-1,$ $-\delta_{i+1} - \delta_i, 1 \leq i \leq n-1,$ $-\delta_n$	$\epsilon_{i+1} - \delta_i, 1 \leq i \leq m-1,$ $\delta_{i+1} - \epsilon_i, 1 \leq i \leq n-1,$ $\epsilon_m + \delta_n$
$\mathfrak{q}(n)$	$\epsilon_{i+1} - \epsilon_i,$ $1 \leq i \leq n-1$	$\delta_{i+1} - \delta_i,$ $1 \leq i \leq n-1$

$\mathrm{H}^1(\mathfrak{n}, \mathbb{C})$ Cohomology (Exceptional Cases)		
Lie Superalgebra	**Corresponding Even Weights**	**Corresponding Odd Weights**
$D(2,1,\alpha)$	$-\mu_1, -\mu_2, -\mu_3$	$(-\epsilon, -\epsilon, -\epsilon)$, $(-\epsilon, -\epsilon, \epsilon)$, $(\epsilon, -\epsilon, -\epsilon)$
$G(3)$	$-\mu_1, -\alpha, -\beta$	$(-\omega_1 + \omega_2, -\epsilon)$, $(2\omega_1 - \omega_2, -\epsilon)$, $(0, -\epsilon)$, $(\omega_1 - \omega_2, -\epsilon)$, $(-2\omega_1 + \omega_2, -\epsilon)$, $(-\omega_1, -\epsilon)$
$F(4)$	$-\mu_1, \quad -\nu_1, \quad -\nu_2, -\nu_3$	$(\omega_2 - \omega_3, -\epsilon)$, $(\omega_1 - \omega_2 + \omega_3, -\epsilon)$, $(\omega_1 - \omega_3, -\epsilon)$, $(-\omega_2 + \omega_3, -\epsilon)$, $(-\omega_1 + \omega_2 - \omega_3, -\epsilon)$, $(-\omega_1 + \omega_3, -\epsilon)$, $(-\omega_3, -\epsilon)$

$H^1(\mathfrak{n}, \mathbb{C})$ Cohomology Dimensions (Classical Cases)			
Lie Superalgebra	**Even**	**Odd**	**Total**
$\mathfrak{gl}(n\|n)$	$2(n-1)$	$2(n-1)$	$4(n-1)$
$\mathfrak{gl}(m\|n)$	$m-1+n-1$	$2n-1$	$m+3n-3$
$\mathfrak{osp}(2m\|2n)$	$m+n-1$	$m+n-1$	$2m+2n-2$
$\mathfrak{osp}(2m+1\|2n)$	$m+n-1$	$m+n-1$	$2m+2n-2$
$\mathfrak{q}(n)$	$n-1$	$n-1$	$2n-2$

$H^1(\mathfrak{n}, \mathbb{C})$ Cohomology Dimensions (Exceptional Cases)			
Lie Superalgebra	**Even**	**Odd**	**Total**
$D(2,1,\alpha)$	3	3	6
$G(3)$	3	6	9
$F(4)$	4	7	11

10.2 $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ Cohomology

Every weight in $\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ corresponds to the sum of two roots of the Lie superalgebra. Those weights which are the sum of two odd or two even roots are listed as even, and those which are the sum of an odd and even root are listed as odd. Note that some weights may appear twice in the list. For example, in the case of $\mathfrak{gl}(3|3)$, the weight $\epsilon_2 + \delta_2 - \epsilon_1 - \delta_1$ may be written as both $(\epsilon_2 - \epsilon_1) + (\delta_2 - \delta_1)$ and as $(\epsilon_2 - \delta_1) + (\delta_2 - \epsilon_1)$, corresponding to $\alpha_1 + \alpha_1'$ and $\beta_1 + \beta_1'$ in the table below. In this case, this corresponds to the weight space having dimension 2.

For the sake of brevity, the cases $\mathfrak{gl}(m|m+1)$ and $\mathfrak{gl}(m|m+2)$ are omitted.

$\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ Cohomology (Classical Cases)		
Lie Superalgebra	**Even Weights**	**Odd Weights**
$\mathfrak{gl}(n\|n)$	$\alpha_i + \alpha_j$, $1 \leq i < j \leq n-1$, $\alpha'_i + \alpha'_j$, $1 \leq i < j \leq n-1$, $\alpha_i + \alpha'_j$, $1 \leq i,\, j \leq n-1$, $\beta_i + \beta_j$, $1 \leq i \leq j \leq n-1$, $\beta'_i + \beta'_j$, $1 \leq i \leq j \leq n-1$, $\beta'_i + \beta_j$, $1 \leq i,\, j \leq n-1$, $\|i-j\| \neq 1$	$\alpha_i + \beta_j$, $1 \leq i,\, j \leq n-1$ $\alpha_i + \beta'_j$, $1 \leq i,\, j \leq n-1$ $\alpha'_i + \beta_j$, $1 \leq i,\, j \leq n-1$, $i-j \neq 1$, $\alpha'_i + \beta'_j$, $1 \leq i,\, j \leq n-1$, $j-1 \neq 1$
$\mathfrak{gl}(m\|n)$, $m-n>2$	$\alpha_i + \alpha_j$, $1 \leq i < j \leq m-1$, $\alpha'_i + \alpha'_j$, $1 \leq i < j \leq n-1$, $\alpha_i + \alpha'_j$, $1 \leq i \leq m-1$, $1 \leq j \leq n-1$, $\beta_i + \beta_j$, $1 \leq i \leq j \leq n-1$, $\beta'_i + \beta'_j$, $1 \leq i \leq j \leq n$, $\beta'_i + \beta_j$, $1 \leq i \leq n$, $1 \leq j \leq n-1$, $\|i-j\| \neq 1$, $2\alpha_i + \alpha_{i+1}$, $n < i < m$, $\alpha_i + 2\alpha_{i+1}$, $n < i < m$	$\alpha_i + \beta_j$, $1 \leq i, \leq m-1$ $1 \leq j \leq n-1$ $\alpha_i + \beta'_j$, $1 \leq i \leq m-1$, $1 \leq j \leq n$ $\alpha'_i + \beta_j$, $1 \leq i \leq n-1$, $1 \leq j \leq n-1$ $i-j \neq 1$, $\alpha'_i + \beta'_j$, $1 \leq i \leq n-1$, $1 \leq j \leq n$, $j-1 \neq 1$ $\alpha_n + (\epsilon_{n+1} - \delta_n)$

$H^2(\mathfrak{n}, \mathbb{C})$ Cohomology (Classical Cases)		
Lie Superalgebra	**Even Weights**	**Odd Weights**
$\mathfrak{osp}(2m\|2n)$	$\mu_i + \mu_j$, $1 \leq i < i+1 < j \leq m-1$, $\nu'_i + \nu'_j$, $1 \leq i < i+1 < j \leq n-1$	$\mu_i + (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$ $\mu_i - (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$ $\nu_i + (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$ $\nu_i - (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$
$\mathfrak{osp}(2m+1\|2n)$	$\mu_i + \mu_j$, $1 \leq i < i+1 < j \leq m-1$, $\nu'_i + \nu'_j$, $1 \leq i < i+1 < j \leq n-1$	$\mu_i + (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$ $\mu_i - (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$ $\nu_i + (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$ $\nu_i - (\delta_{j+1} - \epsilon_j)$, $1 \leq i < i+1 < j \leq r-1$
$\mathfrak{q}(n)$	$\alpha_i + \alpha_j$, $1 \leq i < j \leq n-1$, $\alpha'_i + \alpha'_j$, $1 \leq i \leq j \leq n-1$, $j - i \neq 1$	$\alpha_i + \alpha'_j$, $1 \leq i, j \leq n-1$, $j - i \neq 1$

$\mathrm{H}^2(\mathfrak{n}, \mathbb{C})$ Cohomology (Exceptional Cases)			
Lie Superalgebra	**Even+Even Weights**	**Odd+Odd Weights**	**Odd+Even Weights**
$D(2, 1, \alpha)$	Sums of any distinct two of the following weights: $-\mu_1, -\mu_2, -\mu_3$	Sums of one weight from left column with one from right column	Sums of any two of the following weights: $(-\epsilon, -\epsilon, -\epsilon)$, $(-\epsilon, -\epsilon, \epsilon)$, $(\epsilon, -\epsilon, -\epsilon)$
$G(3)$	Sums of any distinct two of the following weights: $-\mu_1, -\alpha, -\beta$	Sums of one weight from left column with one from right column	Sums of any two of the following weights: $(-\omega_1 + \omega_2, -\epsilon)$, $(2\omega_1 - \omega_2, -\epsilon)$, $(0, -\epsilon)$, $(\omega_1 - \omega_2, -\epsilon)$, $(-2\omega_1 + \omega_2, -\epsilon)$, $(-\omega_1, -\epsilon)$
$F(4)$	Sums of any distinct two of the following weights: $-\mu_1, -\nu_1, -\nu_2, -\nu_3$	Sums of one weight from left column with one from right column	Sums of any two of the following weights: $(\omega_2 - \omega_3, -\epsilon)$, $(\omega_1 - \omega_2 + \omega_3, -\epsilon)$, $(\omega_1 - \omega_3, -\epsilon)$, $(-\omega_2 + \omega_3, -\epsilon)$, $(-\omega_1 + \omega_2 - \omega_3, -\epsilon)$, $(-\omega_1 + \omega_3, -\epsilon)$, $(-\omega_3, -\epsilon)$

$H^2(\mathfrak{n}, \mathbb{C})$ Cohomology Dimensions (Classical Cases)			
Lie Superalgebra	Even Dimension	Odd Dimension	Total Dimension
$\mathfrak{gl}(n\|n)$	$4n^2 - 10n + 8$	$4n^2 - 10n + 8$	$8n^2 - 20n + 16$
$\mathfrak{gl}(m\|n)$	$\binom{m+n-2}{2} + \binom{2n}{2} + m - 2n$	$(m+n-2)(2n-1)+1$	$8n^2 - 8n + 8 + 4n(m-n-2) + \frac{(m-n-3)^2+(m-n-3)}{2}$
$\mathfrak{osp}(2m\|2n)$	$\frac{1}{2}((n-1)^2 + (n-1) + (m-1)^2 + 2((m-n)^2 + (m-n))$	$(n+m-2)(2(m-n))$	$\frac{1}{2}((n-1)^2 + (n-1) + (m-1)^2 + (m-1) + (n+m-2)(2(m-n)) + 2((m-n)^2 + (m-n))$
$\mathfrak{osp}(2m+1\|2n)$	$\frac{1}{2}((n-1)^2 + (n-1) + (m-1)^2 + 2((m-n)^2 + (m-n))$	$(n+m-2)(2(m-n))$	$\frac{1}{2}((n-1)^2 + (n-1) + (m-1)^2 + (m-1) + (n+m-2)(2(m-n)) + 2((m-n)^2 + (m-n))$
$\mathfrak{q}(n)$	$n^2 - 3n + 3$	$n^2 - 3n + 3$	$2n^2 - 6n + 6$

$H^2(\mathfrak{n}, \mathbb{C})$ Cohomology Dimensions (Exceptional Cases)			
Lie Superalgebra	**Even**	**Odd**	**Total**
$D(2,1,\alpha)$	12	6	18
$G(3)$	21	21	42
$F(4)$	34	28	62

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