PRE-SERVICE TEACHERS' BELIEFS AND EXPECTATIONS TOWARD STUDENTS' MATHEMATICAL PROOF

by

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(Under the Direction of AnnaMarie Conner)

ABSTRACT

This study elaborates on pre-service teachers' beliefs and expectations toward students' mathematical proof. This is a qualitative case study using semi-structured interviews. The research participants were three 8-12 grade pre-service teachers in the Bachelor of Mathematics Education program at a university in Indonesia. The research instruments were an interview guide including four mathematics questions and several hypothetical arguments created following Stylianides' (2009) sophistication levels of reasoning. Participants defined proof as a tool to verify and explain the validity of a statement, as well as communicate and systematize mathematics. Their criteria for proof seemed relevant to a within-cluster definition of proof as described by Czocher and Weber (2020). Participants accepted demonstrations and generic arguments as proof, rationales and empirical arguments as non-proof, and others as incorrect proof. Finally, participants expected that proof is difficult for secondary students but is doable for some topics and students with some backgrounds.

INDEX WORDS: Proof, Teacher's Beliefs, Teacher's Expectations, Secondary School

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TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

Research Background

From the mathematics perspective, proof is generally defined as a convincing argument to show the truth of a certain mathematical statement using deductive reasoning, logic, and the accepted standards of the mathematical community (Sundtrom, 2016). Proof has essential functions in mathematics because it demonstrates and explains the truth value of some prepositions (Knuth, 2002a, 2002b; Solow, 2014; Stylianides, 2018). Accordingly, proof is considered central to the mathematics discipline (Knuth, 2002b).

Due to proof playing a significant role in mathematics education, it is implemented in standards documents worldwide (e.g., NCTM, 2000). The first reason for this is that proof can help students verify and construct their mathematics knowledge without merely accepting the information from textbooks or teachers (Knuth, 2002a, 2002b). Secondly, proof can help students show the meaning of a statement and connect it to a new idea (Hanna, 1995; Martin & Harel, 1989). Besides, proof can help students systematize students understanding (Knuth, 2002a) and can be an intellectual challenge that can sharpen students' thinking skills (Knuth, 2002b; Solow, 2014). Finally, proving can record and enhance students' understanding (Knuth, 2002a, 2002b; Varghese, 2009). Accordingly, secondary school students should develop their skills in proof (Guler, 2016; Ko & Knuth, 2009; NCTM, 2000).

Almost all researchers share a common expectation of what mathematical proof constitutes, but different perspectives regarding the criteria of proof occur among mathematicians, mathematics education researchers, and mathematics educators (Balacheff, 2008; Hanna, 1983). In line with that, even though proof may have some strict regulations in mathematics, the acceptance of evidence always depends on the mathematical community (Sundtrom, 2016). Sommerhoff and Ufer (2019) found that the acceptance criteria for validating mathematical proofs differed, with increasing standards from school students to university students, mathematician teachers, and mathematician researchers. Hanna (1995) proposed that proof in the classroom must be correct, explanatory, well-structured, and teachable, and its level of detail depends on students' experiences and classroom context. Typically, proof in secondary school is more intuitive and emphasizes less informal mathematical reasoning and argumentation (Rocha, 2019). Accordingly, teachers probably expect different standards of proof in the classroom compared to mathematicians or mathematics education researchers.

Even though numerous studies agree it is important to address mathematical proof at all levels of school, proof is not fully implemented in school because it requires an adaptation and understanding of proof methods and functions (Rocha, 2019). Moreover, many studies find it difficult for students to learn proof and teachers to teach proof, so systemic changes in classroom culture and instruction are needed (Nardi & Knuth, 2017).

Previous studies have found consistencies between teachers' beliefs and how they teach mathematics and problem-solving in their lessons, arguing that beliefs influence teachers' actions in the classroom (Stipek et al., 2001). Accordingly, the teaching and learning of proof can be addressed from a smaller scope by elaborating on teachers' beliefs and expectations toward students' proving skills. Thus, this study discusses pre-service teachers' beliefs and expectations toward students' mathematical proof by exploring the following research questions:

1. How do pre-service teachers define proof?
2. What are pre-service teachers' criteria for proof?
3. What are pre-service teachers' expectations toward students' proofs?

CHAPTER 2: LITERATURE REVIEW AND RESEARCH FRAMEWORK

The Roles of Proof

According to Knuth (2002a), mathematics plays at least five important roles in mathematics as follows:

1. Proof as Verification
2. Proof as Explanation
3. Proof of Communicating Mathematics
4. Proof of Creating Knowledge
5. Proof as Systematization

The roles above are also found in several studies. First of all, proof as verification is defined as merely demonstrating a statement's correctness (Knuth, 2002a) which is probably the most obvious role of proof (Ko, 2010). Among three senses of proof is verification, which concerns the truth of a proposition (Bell, 1976). This statement is supported by Hanna (1983), who said that proof is typically understood as a process to demonstrate the truth of a theorem in an axiomatic system. Furthermore, De Villers (1999) stated that proof is used mainly to remove either personal doubt or the doubt of skeptics. Besides, from the mathematics perspective, Solow (2014) stated that proof is a convincing argument stating the validity of a statement using mathematical language.

Secondly, as a tool of explanation, proof does not merely state that a statement is true, but it also explains the reasons behind its validity (Knuth, 2002a). Bell (1976) stated that the second sense of proof conveys an insight into why a statement is true. This role of proof is particularly important in teaching proof.

Thirdly, proof has a social aspect in communicating mathematics to others (Knuth, 2002a). Solow (2014) wrote that while mathematics is the language of mathematicians, proof is a method of communicating a mathematical truth to other mathematicians. This is supported by a widely agreed criterion that the acceptance of proof depends on the mathematical community (Sundtrom, 2016).

Fourthly, proof can create knowledge, meaning it is a means by which one can discover or invent new mathematical concepts (Knuth, 2002a). De Villiers (1999) noted that numerous new results in the history of mathematics were discovered or invented in a purely deductive manner.

Finally, proof can systematize results into a deductive system of definitions, axioms, and theorems (Knuth, 2002a). According to Lakatos (1979, as cited in Hanna, 1983), the purpose of proof in mathematical practice is not to approve a theorem that has already been proved. Instead, the proof is more closely related to a thought experiment used to examine a theorem in its original state to refine it. Besides, the proof results can be organized into a deductive system of axioms, major concepts, theorems, and minor results, which is probably the most characteristically mathematical sense of proof (Bell, 1976). Examples of these five roles of proof will be discussed in the section of a case study of proof.

A number of studies investigate the roles of proof according to secondary school pre-service or in-service teachers. Varghese (2009) asked 17 student teachers and found some equivalent roles to the Knuth’s (2002a) framework above, namely verification, justification (similar to explanation in Knuth's framework), derivation (systematization), and discovery (creating knowledge) in addition to other roles, namely logical argument, and explanation (means to enhance understandings in Varghese’s study). Moreover, using a literature review, Ko

(2010) concluded a similar framework as above but with an additional category, namely intellectual challenge, since individuals can obtain self-realization and fulfillment from constructing the proof.

A Definition of Proof within the Cluster Account

In classical categories, an object can be classified as a term if it satisfies a set of properties with no exception in the definition of the term (Lakoff, 1987, as cited in Czocher and Weber, 2020). On the other hand, Gaut (2000) said that in a cluster model, an object could be a member of a category if it satisfies most or all properties and cannot be a member if it satisfies none of the properties, while no individual property is mandatory.

Because of much disagreement regarding the definition of proof among mathematicians and mathematics educators, Czocher and Weber (2020) proposed one definition of proof that positions proof as a cluster concept. That is, to be proof, an argument should satisfy most of the following:

1. Convincing justification: the answer removes all doubts that a theorem is valid for a knowledgeable mathematician.
2. Perspicuous justification: the answer is comprehensible by a knowledgeable mathematician and provides the reader with an understanding of why a theorem is true.
3. Priori justification: the answer shows that a theorem is a logically necessary consequence (i.e., a deductive consequence) of axioms, assumptions, or previously established claims.
4. Transparent justification: any sufficiently knowledgeable mathematician can fill in every gap (or believes, in principle, that he or she can do so given sufficient time,

motivation, and content knowledge), perhaps to the level of being a formal derivation.

5. Sanctioned justification: the answer has been sanctioned by the mathematical community.

Among the strengths of this cluster model is that there may be justifications that cannot clearly be classified as proofs or non-proofs, namely justifications that meet many, but not all, of the criteria (Czocher & Weber, 2020). For example, picture proofs are problematic because they may satisfy all criteria except points 3 and 4, elaborated further in the visual proof section. On the other hand, computer-assisted proofs may contest properties 2, 3, and 4 (Czocher & Weber, 2020), which then cause a debate in the mathematics community. A case in point, one way to prove the four-color theorem is by checking possible map configurations, which are easier to be checked one by one by computer than manually. This proof can remove the doubts and can be rechecked by some mathematicians. However, not all mathematicians may find it convincing because using a computer is not necessarily surveyable by human mathematicians and does not explain why the theorem is true, for it does not rely on mathematical axioms, theorems, and previously established claims.

Levels of Sophistication of Reasoning

According to Stylianides' (2009) hierarchy of arguments, an argument can be classified as proof (demonstrations and generic arguments), non-proof (rationale and empirical arguments), and neither proof nor non-proof. In the framework, proof is considered to have a higher level of sophistication compared to non-proof arguments (Stylianides, 2009). Besides, the following arguments are arranged in descending order according to their levels of sophistication of reasoning: demonstrations, generic arguments, rationale, and empirical arguments (Stylianides, 2009).

The Stylianides' (2009) framework distinguishes between two types of proof, namely demonstrations, which have the highest sophistication level, and generic examples, which have a lower level of sophistication. Generic arguments use a particular case to represent the general case (Stylianides, 2009). Generic arguments are important for students as powerful and understandable arguments that enable students to establish mathematical claims even if they lack mathematical language or formal proof methods (Stylianides, 2009).

Using a higher level of mathematical language and cognitive constructions, the sophistication level of generic examples can be improved by decontextualization or eliminating examples (Balacheff, 1988). Proof that does not rely on the representativeness of a particular case, such as valid arguments by counterexample, contradiction, reductio ad absurdum, mathematical induction, contraposition, and exhaustion, is categorized as demonstrations. This is similar to Harel and Sowder's (1998) axiomatic proof (Stylianides, 2009). According to Harel and Sowder (1998), an axiomatic proof is a justification that starts from facts or statements accepted without proof, such as undefined terms and axioms.

In the Stylianides' (2009) framework, a non-proof argument is a valid argument for or against mathematical claims that do not qualify as proofs, namely empirical arguments with the lowest sophistication level and rationales with a higher level of sophistication. An argument counts as a rationale instead of proof if it does not make explicit reference to key accepted truths that it uses or if it uses statements that do not belong to the set of accepted truths of a particular community (Stylianides, 2009). On the other hand, if a justification only shows a proper subset of all the possible cases covered by the claim, it is categorized as an empirical argument similar to Balacheff's (1988) construct of naïve empiricism (Stylianides, 2009). Finally, responses that have circular reasoning, show minimal engagement, are irrelevant or are not evidently relevant, etc. do not count as either proof or non-proof arguments (Stylianides, 2009).

Visual Proof

Mathematics can be perceived as the science of designs, connections, generalized descriptions, and recognizable structure in space, numbers, and other patterns, relationships, and abstracted entities (Borwein & Jörgenson, 2001). Accordingly, in addition to written words, mathematics can be explained using mathematical representations, such as patterns, symbols, diagrams, and figures. According to Borwein & Jörgenson (2001), some criteria of acceptable visual proof are but are not limited to

1. reliability (the result is unvarying with each inspection)
2. consistency (the means and end are consistent with other known facts, beliefs, and proofs)
3. repeatability (others may confirm the proof).

Even though proof does not depend on sentential representation alone, visual, and sentential reasoning are not mutually exclusive (Barwise & Etchemendy, 1996). While the focus on logical structures and sentential reasoning may lead to the neglect of many other forms of mathematical thinking, such as diagrams, charts, nets, maps, and pictures, that do not fit the traditional inferential model, it is impossible to build logically sound and even rigorous arguments upon such visual representations (Barwise & Etchemendy, 1996).

A pluralistic perspective regarding the sophistication of visual proof occurs since, as stated in the previous section, visual proof may be convincing, perspicuous, and can be sanctioned, but not necessarily a priori and transparent (Czocher & Weber, 2020). Picture proofs are not considered a priori because they typically rely on the reader's intuition about two-dimensional space, which is not the actual underlying concept (Czocher & Weber, 2020). Accordingly, it may not be transparent, namely, formalized in a logical system (Czocher & Weber, 2020).

An Example of Several Modes of Proof

We will discuss an example of the roles of proof and visual proof in this section. Consider the following **statement 1**: "The sum of the first n natural numbers is $\frac{n(n+1)}{2}$." Statement 1 can be proven using multiple methods. Firstly, its truth can be shown instantly using a computer program that verifies the theorem but does not necessarily explain why the theorem is valid. To justify the validity of statement 1, it can be demonstrated formally in multiple ways, such as using mathematical induction to show the underlying mathematical concepts. However, not everyone may understand mathematical induction, thus proving that using this principle is not necessarily a good way to communicate mathematics for some communities. Middle-grade students, for example, probably find it more comprehensible if statement 1 is proven using a visual representation like **Error! Reference source not found.** as follows with some additional explanations.

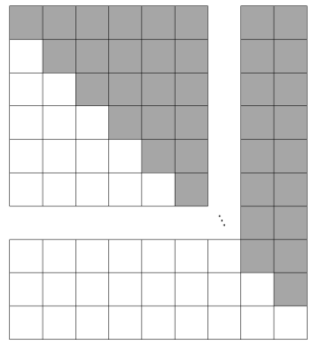


Figure 1. An illustration that the sum of the first $\boldsymbol{n}$ natural numbers is $\frac{\boldsymbol{n(n+1)}}{\boldsymbol{2}}$

Unfortunately, as explained in the previous section, this visual proof is probably not acceptable to all mathematicians since it is neither priori nor transparent. Instead of using properties of additions and integers, **Error! Reference source not found.** relies on the properties of a rectangle, which may not be helpful enough for students in building their understanding of the properties of integers. This method is different from formal ones, such as an algebraic approach that can help to create new knowledge. For instance, students can use their ability in proving statement 1 to elaborate on the following **statement 2**: “the sum of the first n odd natural numbers is n^2.” They can use statement 1 in their algebraic calculations or think how a similar algebraic manipulation can be used to prove statement 2. Moreover, given that this statement can be confirmed by others and is used widely, this statement is systematized as the formula of the sum of the first n natural numbers.

CHAPTER 3: RESEARCH METHODOLOGY

Research Approach

This research uses the qualitative approach with the case-study method. Qualitative research has four major criteria, namely, (1) the focus is on the process, understanding, and meaning; (2) the researcher is the primary instrument of data collection and analysis; (3) the process is inductive, and (4) the product is richly descriptive (Merriam & Tisdell, 2016). Besides, Creswell (2013, in Merriam and Tisdell, 2016) states that case study research is a qualitative exploration of one or multiple bounded systems over time through detailed, in-depth data collection involving multiple sources of information and reports a case description and case-based themes.

Research Instrument

The interviews were conducted in two sessions with different interview protocols (see Appendix 2). The first protocol included questions about the definition and roles of proof and their expectations of students' ability to prove. The second protocol focused on the participants' criteria of acceptable and convincing proof. The participants were given four mathematics questions (each asked for proof) with several students' answers (arguments) for each question. After reading each answer, the participants were asked whether the answer convinced them and how they would improve the answer. After reading a couple of answers, the participants were asked to compare and contrast the convincing levels of the answers. Each interview lasted between 90 and 120 minutes. The first interview included protocol 1 and the first two mathematics questions of protocol 2, while the second interview included the last two mathematics questions of protocol 2.

The questions prepared were on four different topics, namely (1) the formula of the area of a trapezoid, (2) the divisibility by three of the sum of three consecutive natural numbers, (3) the value of an algebraic inequality with an absolute value, and (4) the sum of opposite and non-adjacent interior angles in a triangle. Those topics were selected because they have many applications in secondary school and can be proven using multiple entry points. The researcher created the hypothetical student answers following Stylianides' (2019) levels of sophistication of an argument: demonstrations, generic arguments, rationales, empirical arguments, and neither proof nor non-proof. Those answers were created using multiple proving methods (e.g., direct proof, proof by contradiction, proof by cases, etc.), presenting techniques (e.g., traditional paragraphs, two-column proof, back-solving method, etc.), and working tools (mathematical symbols, pictures, diagrams, other symbols, technology, hands-on tools, etc.). The questions are below, and the hypothetical student answers are in Appendix 3.

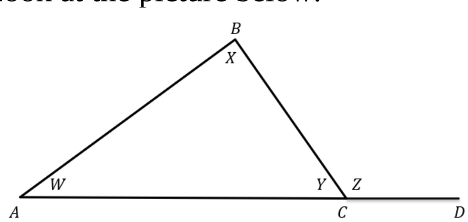
Question 1
Prove that the area of a trapezoid is

$$\frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Question 2
Prove that the sum of three consecutive natural numbers is divisible by 3.

Question 3
For all real numbers x, prove that $x + |x - 5| \geq 5$.

Question 4
Look at the picture below.



Prove that $Z = W + X$.

Figure 2. Mathematics Questions Used in Interviews

The thesis advisor validated the instrument based on the necessity and adequacy of the questions, the validity of sample solutions, and the classification based on the sophistication levels. After that, the algebraic question (Question 3) was tested in a class discussion to see the proper order of the solutions and the potential to address all the research questions in a limited time. Next, pilot interviews were done to try out questions. These pilot interviews are crucial in practicing interviews, learning which questions are confusing and need rewording, yield useless data, are suggested by participants, and are removed because of time concerns.

Data Collection

The research participants are three 8-12 grade pre-service teachers in the Bachelor of Mathematics Education program at one university in Indonesia. All participants have completed the Fundamentals of Mathematics course containing logic and proving in mathematics as one fundamental knowledge before assessing proof. Pre-service teachers were selected because their perceptions will influence their decisions in the classroom that will affect students' learning and understanding. Besides, since they were still in a preparation program, most theoretical and practical knowledge and skills they acquired were from the university instead of their direct experiences. Thus, this study can be one way to reflect on their understanding of the materials and practices they gained during their teacher preparation.

Data was collected using a semi-structured interview. DeMarrais (2004, in Merriam and Tisdell, 2016) defined a research interview as a process in which a researcher and participant engage in a conversation focused on questions related to a research study. Moreover, in the semi-structured interview, the interviewer asks only a few pre-determined questions, and the rest of the questions are not planned and can be following-up by participants' answers (Merriam & Tisdell, 2016). This study used the person-to-person encounter, in which the researcher asked each participant directly and individually through Zoom. The interviewees were recorded using Zoom's recording tool. The interviewees were conducted in Indonesian with English mathematical questions and answers based on the participants' preferences.

Data Analysis

Data gathered were examined using qualitative research methods. The interviews were transcribed in Indonesian and then translated into English. The Indonesian transcription and English translations were validated. After that, the translated interview transcriptions were color-coded based on some categories initially based on the research questions, with additional categories arising from the data. The data were interpreted using some frameworks elaborated in the literature review, and if the information inferred did not fit into one of the items in the frameworks, other codes were added. Participants' definitions, roles, and criteria of proof were interpreted with help from the frameworks of Weber and Czocher (2019), Knuth (2002a), and Stylianides (2009), respectively. Table 1 below is the codebook.

Table 1

Codebook

Category	Codes
The Definitions of Proof (Knuth, 2002a)	To verify that a statement is true
	To explain why a statement is true
	To communicate mathematical knowledge
	To discover or create new mathematics
	To systematize statements into an axiomatic system
The Roles of Proof	To encourage reflections and communications

	To allow the use of tools
	To leave some important residues
Students' Ability in Proof	They are able
	They are able under some conditions
	They are not able
Factors that Influence Students' Abilities in Proving	Their ability to find ideas
	Their understanding of the prerequisite materials
	Their practices
	Their teachers' delivery of the material
	Their ability to connect ideas
Proof/Not proof	Proof
	Not Proof
Criteria of Acceptable Proof	**STEPS**
	The steps of the proof are well organized.
	Every step in the proof is clearly explained
	The main idea/concept of the proof is emphasized.
	The goal of the proof is clear.
	Every step of the proof is written based on mathematical concepts.
	The conclusion of the proof should be written
	PICTURES
	The pictures in the proof can explain the ideas intended.
	Some pictures are added to the proof when needed.

	SYMBOLS AND WORDS
	The symbols are well defined.
	Every word in the proof is not ambiguous
	EXAMPLES
	The proof should prove in general/not just take examples
	The examples in the proof are more systematic or have a pattern (it makes the answer is more convincing but does not necessarily make it a proof)
	The pattern in the proof should be clearly stated
	Examples can be added to ease understanding a complex explanation (not mandatory)
	OTHERS
	The proof is easy to understand.
	The proof is mathematically correct.
	The mathematical calculation in the proof is broken down (sometimes it is a must, sometimes it is just a complementary)
	The proof is written effectively

CHAPTER 4: RESULTS AND DISCUSSION

Results

This section will elaborate on the research findings regarding definitions of proof, criteria of proof, different perspectives on proof, and expectations about students' proving skills. Appendix 1 contains a table summarizing how each item of each research framework was found across participants and questions.

Definitions of Proof

The participants' perspectives on the definitions of proof were inferred in two ways. Firstly, in the interview based on protocol 1, the participants explained the definition of proof when asked about its definition and roles. Additionally, their opinion can be found through the rest of the discussion when they examined and reviewed their justification on mathematics questions and answers prepared. In both interviews, the participants added to their definitions of proof by mentioning the functions or roles as they analyzed student arguments.

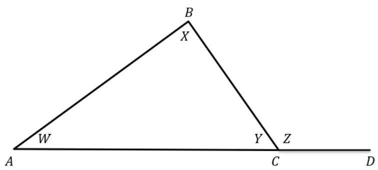
To verify the validity of a statement. Budi and Cita defined proof as verifying the validity of a statement. According to Budi, in the first session, the verification can be done by re-finding or reiterating the process to find a theorem using the agreed concepts.

00:03:00 Budi: For me, maybe it is because what is usually proven is a theorem. So, perhaps proof means to re-find the process so that this theorem can exist. As proof, it is usually the proof is based on…, so it is reiterated whether this theorem can be proven or not.[1]

[1] For readability and clarity without changing the meaning of all transcripts in this manuscript, some fillers (uhm, hmmm, like, sis) were removed and some bracketed words were inserted during the transcribing and translation. Some examples of added words are the mathematical model of the participants' mathematical explanations and the reference for the pronouns the participants mentioned based on their previous sentences or pictures that are being looked at.

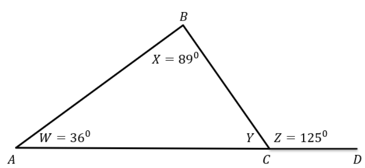
Budi appeared to use the converse of his statement, namely when an answer was considered inadequate or inaccurate to verify a statement, it was not proof. A case in point: MQ4.SA[2] and MQ4.SB (in **Error! Reference source not found.**) were not considered proof because they used a measurement that may have some errors.

Question 4
Look at the picture below.



Prove that $Z = W + X$.

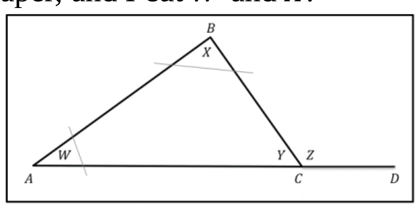
Student A
I measured the angles using a protractor. I found:



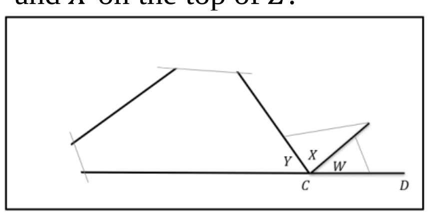
$$125^0 = 36^0 + 89^0$$

So, $Z = W + X$.

Student B
I drew the triangle on paper, and I cut W and X:



I arranged and stuck W and X on the top of Z:



We see that W and X covers Z.

Figure 3. Student A's Answer to Mathematical Question 4

[2] For simplicity, in this thesis, MQX.SY means (hypothetical) Student Y's answer to Mathematics Question X.

Moreover, verifying a proper subset of elements or cases in the domain was not considered proof because it did not say anything about the rest of the elements, according to Budi. Below is Budi's response after reading MQ3.SD (in **Error! Reference source not found.**).

> 00:16:10 Budi: On the question, it is for real x, right. It means all must apply in general. Then, the proof presented by Student D only takes one example, that is, when the x is 8. However, there is no guarantee that other xs will apply like that. It means the mathematics is not correct for others. So, for me, it is not categorized.

Cita also defined proof to justify whether a statement is right or wrong. She used her definition mentioned in session 1 to justify whether some students' answers in session 2 were proof. She even recommended multiple times to write the conclusion to clarify the validity of a statement. For example, Cita expected the conclusion of MQ3.SA (in **Error! Reference source not found.**) below to be written so that its truth value is explicit.

Question 3
For all real numbers x, prove that $x + |x - 5| \geq 5$.

Student A

	$x + \|x - 5\| \geq 5$		
$\Leftrightarrow$	$\|x - 5\| \geq 5 - x$		
$\Leftrightarrow$	$x - 5 \geq 5 - x$	or	$5 - x \geq 5 - x$
$\Leftrightarrow$	$2x \geq 10$	or	$5 - x \geq 5 - x$
$\Leftrightarrow$	$x \geq 5$	or	$5 - x \geq 5 - x$
	True because $\|x - 5\| = x - 5$ means $x \geq 5$.		True because $5 - x = 5 - x$

Figure 4. Student A’s Answer to Mathematical Question 3

> 00:04:51 Int: Is there anything that can be improved from student A's answer?
>
> 00:04:59 Cita: Maybe for me, clarify it turns out the statement was true or false. What I understand is they already got the value of x. Then they did not explain whether the previous statement was true or false.

00:05:25 Int:	Okay, how to clarify whether it was true or false?
00:05:33 Cita:	Giving a conclusion at the end.
00:05:36 Int:	Is writing a conclusion mandatory or just complementary?
00:05:49 Cita:	Actually, for some people, it is just a complement. No, in my opinion, it is a must.
00:06:01 Int:	Does it mean that if, let us say, a grade is given, can this student lose some points because they do not write a conclusion?
00:06:11 Cita:	Yes.

To Explain the Validity of a Statement. Not only does proof simply verify the truth of a statement, but it also should explain the reasons behind the verification. In the first session, Atika said that proof should be broken down into some steps, and each step clearly describes the underlying concepts. Her statement was aligned with Cita, stating that an answer should not have missing information, and the reason must rely on mathematical concepts, such as theorems and definitions.

Even though Budi did not mention explanation in the first session, all participants emphasized explanation as one criterion of proof throughout the second session. For instance, Cita considered that MQ3.SB was proof because it "explained from the beginning why they stated that this statement was true, then emphasized that the statement was true" (00:02:00). How explanation is used to establish proof will be explained further in the section on the criteria of proof.

To Communicate Mathematical Knowledge. Proof is expected to communicate mathematical knowledge from the author to the readers. One way to do so is by ensuring the understandability of proof. Even though none of the participants explicitly stated the importance of the ease of explanation in the first session, the idea came up while assessing students' answers. For instance, after assessing question 1 and a couple of solutions for question 2, Atika explicitly added a criterion that it should be easy to understand by teachers and students.

01:18:17 Atika: First, maybe I want to give an addition to the criteria of good proof. One point, it is easy to understand.
01:18:33 Int[3]: Okay, okay, okay. Okay, interesting, easy to understand. To whom do you think is it easy to understand?
01:18:46 Atika: For all. For those who read the proof. So, it could be for the teacher, or it could be for classmates.

The proof audience was also mentioned by Cita when commenting on MQ1.SA. She stated, "Some perspectives that proof is whether it is understandable, easy to understand by the readers from what field do you call it? Experts in that field as well." Similar to Cita's statement, according to Budi, after reading MQ2.SG, if someone does not understand a well-presented proof, they probably have not acquired the prerequisite materials.

00:16:07 Budi: In my opinion, from the beginning, the conditions must be systematic. It is clear where each step comes from. If you have followed the rules, it means the proof will be very clear. So, the readers most likely will also know where the steps came from. Even if later some do not understand, maybe it is because the readers did not study the previous material or because of something else.

To Systematize Statements into an Axiomatic System. Finally, all participants agreed that proof is to systematize statements into an axiomatic system. They all believed that proving is associated with knowing the origin of an idea. Reflecting on her personal experience, Cita stated that proof might help students satisfy their curiosity about a concept and its content, foundation, and application to other questions. Additionally, both Atika and Budi contrasted proving from memorizing. Atika said that proving requires an understanding of the concept so that the memory of the concept understood can last even after the exam, which is different from memorization. In line with Atika's statement, Budi believed that realizing the foundation of a mathematical

[3] Interviewer

statement or formula would attach the ideas longer than memorizing them. Below is the interview transcript with Budi.

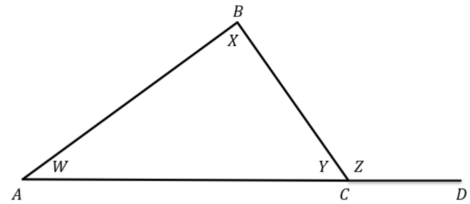
> 00:06:30 Budi: From what I understand, the origins of something are more attached. I mean, for example, when we know the formula for the area of a triangle that originated from the square formula, we know how to prove it, right? We know that the formula will be attached, meaning it is easier to remember because we know the process because it is different if we just know the formula or mathematical sentence but do not [know] how the sentences, how the formula can exist. With the origin, the understanding will be different because mathematics is not possible by just memorizing it.

Criteria of Proof

In the first session, participants only mentioned a few criteria of acceptable proof. The number then significantly increased after participants looked at several sample arguments. The ones mentioned in the first session are probably the most prominent criteria that the participants believed. After that, the sample answers inspired or reminded them of other criteria. The previous section states that proof is to explain the reason behind the truth of a statement and that it should be easy to understand. Besides, all participants considered the explanation the major criterion to determine whether an answer is proof. Some criteria to support clarity and explanation are discussed in the following.

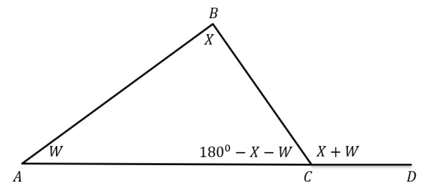
Mathematical Concepts. All participants put the clarity of explanation as one criterion of proof in many arguments. Although none explicitly mentioned transparency in the first session, they all stated that every step of proof should be written based on mathematical concepts. While reading the prepared answers, participants said that proof should be broken down into steps and that every step should be clearly explained. To illustrate, look at MQ4.SC, MQ4.**SD,** MQ4.SF and MQ4.SH (in Figure 5)**Error! Reference source not found.** below.

Question 4
Look at the picture below.



Prove that $Z = W + X$.

Student C



Student D

$$Z = 180^0 - Y = 180^0 - (180^0 - W - X) = 180^0 - 180^0 + W + X$$
$$= W + X.$$

Student F

$Y + Z = 180^0$ ... (**i**) $W + X + Y = 180^0$ ... (**ii**) $180^0 = 180^0$ $Y + Z = W + X + Y$ $Z = W + X$	Y and Z are supplementary. W, X, and Y form a triangle. Obvious (i) and (ii) Subtract both sides by Y.

Student H

$$W + X = Z$$
$$W + X + Y = Z + Y$$
$$180^0 = 180^0$$

This proves that $Z = W + X$.

Figure 5. Student C's, Student D's, student F's, and Student H's Answers to Mathematical Question 4

All participants agreed that MQ4.SF **(**in Figure 5**)** was proof because the explanation of each step was clearly written. However, this indicator was not used all the time. Some answers were justified regardless of the clarity of their explanation. For instance, unlike MQ4.SF, MQ4.SH **(**in Figure 5**)** only presents mathematical calculations without stating the mathematical concepts. Atika said that MQ4.SH was not proof because the answer started from the goal. She did not complain that no concept was written and instead appreciated students' application of such concepts. In other words, as illustrated in the following transcript, Atika did not seem concerned about the clarity of the concepts.

01:35:08 Int:	Is this proof or not?
01:35:08 Atika:	Not
01:35:10 Int:	Why?
01:35:10 Atika:	Because they use information that we are supposed to prove, but at least they can use a concept here. They know the concept of the sum of the angles in a triangle. Besides, they also know the concept of supplementary angles.

Furthermore, Budi and Cita suggested that MQ4.SH **(**in Figure 5**)** lacked explanation. However, they accepted the answer as proof that it just needed an improvement by stating mathematical concepts underlying each step. Below is the interview transcript with Budi.

01:42:04 Budi:	Proof maybe.
01:42:07 Int:	Why?
01:42:08 Budi:	This starts from such and such, yes, from what you want to prove. Then add the two together with the angle Y, and then they will be 180 degrees. However, I think it needs to be added for the explanation. It is 180. Then it is 180 because the picture does not show either. So, maybe the explanation can be added from that point to make it clearer.

All participants also discussed the necessity of breaking down mathematical calculations to clarify an answer. All participants agreed that MQ4.SD (in Figure 5**Error! Reference source not found.**) was proof because of its complete algebraic calculation. On the other hand, even though MQ4.SC (in Figure 5**Error! Reference source not found.**) was considered proof by all participants, Atika and Budi recommended writing down the mathematical process. From the interview script below, it seemed that Budi suggested improving the clarity and convincing level of the argument.

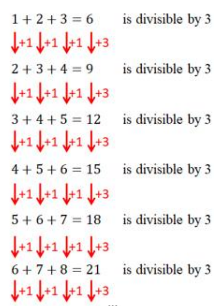
01:32:14 Budi:	In my opinion, it is better to write it down [referring to MQ4.SD] because it [referring to MQ4.SC] is just an illustration because it is not that convincing. It is not really clear to those who do not understand. So, it should be there.

Logical Explanation. The opinion that proof must be understood easily underlies some criteria of acceptable proof. Atika stated that proof should be written concisely and efficiently to ease the understanding. She said that "The plot is clear and not long-winded. Students may give an explanation using a lot of sentences but with very little content." Besides, Atika and Cita agreed that words used in the proof should not be ambiguous.

There was an expectation to provide well-structured proof. Atika mentioned this notion a few times. For instance, even though Atika said that neither MQ2.SC nor MQ2.SD in Figure 6 was proof, she found MQ2.SC to be more convincing than MQ2.SD because of MQ2. SC's systematic explanation and coherent flow.

Question 2
Prove that the sum of three consecutive natural numbers is divisible by 3.

Student C



6 is divisible by 3. The sum of next group is added by 3. So, it is also divisible by 3.

Student D
If there are 3 consecutive numbers, the median is the mean. So, the mean is a natural number. Thus, their sum is divisible by 3.

Figure 6. Student C's and D's Answers to Mathematical Question 2

In the first session, Budi explained that systematic, well-structured, and well-organized proof was convincing.

01:04:37 Budi: It has to be systematic as well. The proof looks like that. So, the steps cannot be random because the steps have to be arranged. The steps of proof must be organized. That is the rule.

Even though Cita did not mention structure in the first session, Budi and Cita used this criterion to assess several arguments. One reason multiple answers on questions 2 and 3 were considered proof was their structure. On the other hand, one suggestion for some answers to questions 2 and 3 that were not considered proof was to lead the explanation to the conclusion better. To further understand how participants perceived good structure, they were asked their opinion on the back-solving method. Figure 7 below contains MQ3.SA and MQ3.SE.

Question 3

For all real numbers x, prove that $x + |x - 5| \geq 5$.

Student A

	$x + \lvert x - 5\rvert \geq 5$		
$\Leftrightarrow$	$\lvert x - 5\rvert \geq 5 - x$		
$\Leftrightarrow$	$x - 5 \geq 5 - x$	or	$5 - x \geq 5 - x$
$\Leftrightarrow$	$2x \geq 10$	or	$5 - x \geq 5 - x$
$\Leftrightarrow$	$x \geq 5$	or	$5 - x \geq 5 - x$
	True because $\lvert x - 5\rvert = x - 5$ means $x \geq 5$.		True because $5 - x = 5 - x$

Student E

Take any real number x. We will prove that $x + |x - 5| \geq 5$.

Case I. For $x \geq 5$, since $|x - 5| \geq 0$,

$$x + |x - 5| = x + x - 5 = 2x - 5 \geq 2(5) - 5 = 10 - 5 = 5.$$

Hence, by the transitive property, $x + |x - 5| \geq 5$.

Case II. For $x < 5$, we have $|x - 5| = 5 - x$. Thus,

$$x + |x - 5| = x + 5 - x = 5 \geq 5.$$

Hence, by the transitive property, $x + |x - 5| \geq 5$.

Therefore, in all cases, $x + |x - 5| \geq 5$.

Figure 7. Student A's Answer and E's Answer to Mathematical Question 3.

All participants agreed that MQ3.SA (in Figure 7) was proof, and none of them problematized the answer started from the conclusion. However, this does not necessarily mean that doing mathematics in any order preserves its convincing level. According to the script below, Cita said that both MQ3.SA and MQ3.SE (in Figure 7) were correct, but MQ3.SE, with its forward-solving method, was more convincing than MQ3.SA with its backward-solving one.

01:03:26 Int: Okay, for example, student E. Student E, this part is on the bottom, part *x is greater than or equal to five* [$x \geq 5$], above. Compared to this, it is the other way around. The five on the bottom, this one on the top. So, what do you think, which one is more convincing?

01:04:20 Cita: The one that's more convincing, the greater or equal to five, is on top.

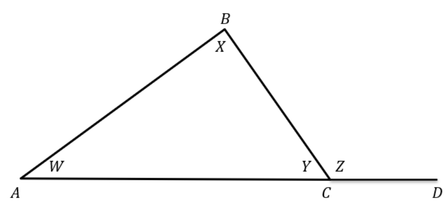
01:04:30 Int: Yes. Okay. Well, but even though it is more convincing, is it still true?

01:04:40 Cita: Yes.

Cita preferred to start from the given information because doing the mathematical operation the other way around did not indicate a good structure. Look at MQ4.SH (in Figure 8) below, along with its interview transcript. For Cita, to have a good structure, a proof should start from the attributes given and conclude with the matched objective.

Question 4

Look at the picture below.



Prove that $Z = W + X$.

Student H

$$W + X = Z$$
$$W + X + Y = Z + Y$$
$$180^0 = 180^0$$

This proves that $Z = W + X$.

Figure 8. Student H’s Answer to Mathematical Question 7

01:04:42 Int: So, it is the same as your explanation earlier. One of your answers earlier is to be systematic. So, is it still considered systematic even though the order is reversed?

01:05:09 Cita: Still systematic even though the plot is reversed. Yes.

01:05:16 Int: So, what do you mean by systematic here?

01:05:25 Cita: From the experience of the proof, later at the end, we will find that, uh, the statement is exactly the same as the final result. The definition of systematic that I understand means that the reverse order is actually not. It is not correct. Like the statement earlier, the previous ones are more convincing. How to explain this.

01:06:6 Int:	For them, the conclusion is at the end. What about those whose conclusion is placed at the beginning, is it considered systematic?
01:06:33 Cita:	No.
01:06:34 Int:	Okay or another question, what is meant by systematic here?
01:06:44 Cita:	In order. They are sorted and it is clear at the beginning where they took all the things they used in the proof. Then at the end, they will find it matches the statement. Becomes a conclusion.

The back-solving methods should be discussed more frequently with students to make sure their realization that not all statements preserved their truth value when done in the reverse order. For example, Cita said that MQ2.SF (in Figure 9) below was proof, while others said it was not proof because of the unclear symbols or lack of explanation. None of them realized that MQ2.SF proved the converse of the theorem instead. However, this may be because the participants read the answer in a small amount of time, so they focused more on the explanation and did not anticipate any tricky responses.

Question 2 Prove that the sum of three consecutive natural numbers is divisible by 3.
Student F If I have a multiple of 3, say $3k$, I can share it into three numbers: $3k-1, 3k, 3k+1$ They are consecutive numbers, and their total is divisible by 3.

Figure 9. Student F's Answer to Mathematical Question 2

Furthermore, there was an expectation to emphasize the main idea of proof. Atika brought up this idea multiple times in the second session. In some cases, she even expected the student to write down their approach, like her statement below, while assessing MQ1.SE.

00:48:00 Atika:	They should explain it. For example, they used this approach. Then, they could first mention it. For example, based on the formula for the area of a rectangle, this is like this. Then the width is the height, and the length is plus the top. So, it would be clearer if the concept used appeared here.

Generality and Example Taking. In the second session, all participants stated that an explanation should show all elements or possible cases to be considered proof. One way to represent all elements in the domain is using symbols. For example, Budi thought that MQ2.SG (in Figure 10) was proof because its symbols indicated that the statement applied in general.

Question 2 Prove that the sum of three consecutive natural numbers is divisible by 3.
Student G Suppose that the three numbers are $a-1$, a, and $a+1$. Because $a-1+a+a+1=3a$, the sum is divisible by 3.

Figure 10. Student G's Answer to Mathematical Question 2

00:08:12 Budi: Maybe the explanation for the numbers, is for less than 5, the number is between 0 and 5. So a number that is less than 5 will still satisfy it. It means we can use a symbol to describe that *x* is less than 5. Automatically, when *x* is less than 5, it will always be positive. It means that the absolute value will be subtracted by 5. So, let us say 2. That would be min 3 plus, uh 2, uh 3 plus 2 [|-3|+2=3+2]. So, the result will still be 5 because the one inside the absolute value is subtracted by 5. I do not really know how to explain this in detail.

Moreover, Budi suggested simply changing numbers to symbols on MQ3.SB (in Figure 11) made the statement proven in general.

Question 3 For all real number x, prove that $x + \|x - 5\| \geq 5$.
Student B

x	$x + \|x - 5\|$	Conclusion
-2	5	≥ 5
0	5	≥ 5
3	5	≥ 5
4.5	5	≥ 5
5	5	≥ 5
10	15	≥ 5
70	135	≥ 5

From the table above, for all values of x, negative, zero, positive, decimal, less than five, five, and greater than five, $x + |x - 5| \geq 5$. So, it is true that $x + |x - 5| \geq 5$ for all $x \in R$.

Figure 11. Student B's Answer to Mathematical Question 3

Interestingly, all participants shared one common similarity in understanding generality. When discussing the area of a trapezoid in question 1, all participants agreed that students might use any trapezoid and did not need to show all possible shapes because the formula applied in general. Participants said choosing a specific trapezoid (for instance, an isosceles trapezoid) was a valid choice within the proof. Below is one interview script with Budi.

00:51:44 Budi: Uhm, for me, it is more up to the students because we know that the formula of the area of a trapezoid applies in general, meaning that it can be proven using any trapezoid. But maybe to make it easier, you can use an isosceles one or whatever because it is easier to illustrate than other shapes because it depends on the student. The important point is how it is directed to the proof because sometimes some people use the same trapezoids but with different proof. It depends on the students' thoughts and previously obtained information. As we see, one student used a triangle, another student used a rectangle. Maybe so. It can be any trapezoid probably.

As explained earlier, proof should show all elements or possible cases. Accordingly, using empirical evidence, that is showing a proper subset of all elements in the domain, was not recognized as proof. For instance, all participants agreed that MQ2.SA (in Figure 12) below was not proof because it only included a couple of examples that did not represent all numbers in the domain.

Question 2 Prove that the sum of three consecutive natural numbers is divisible by 3.
Student A $1 + 2 + 3 = 6$ is divisible by 3. $5 + 6 + 7 = 18$ is divisible by 3. $23 + 24 + 25 = 72$ is divisible by 3. So, the sum of three consecutive natural numbers is divisible by 3.

Figure 12. Student A's Answer to Mathematical Question 2

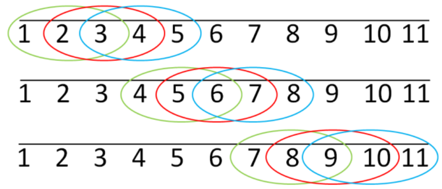
Even though the previous paragraphs emphasized the need for generality, it does not mean that providing selected examples is always wrong. Look at MQ2.SB and MQ2.SC (in Figure 14). All participants agreed that the examples provided in MQ2.SC (in Figure 13) showed a pattern and guaranteed the validity of all numbers. Thus, Budi and Cita said it was proof, although Atika did not think so and instead said it needed more symbols as she explained below.

> 01:10:03 Atika: Actually, it is systematic and clear enough regarding where this is going. Well, it is just a lack of… I think they need to say something like this, the sequence of the results forms a $3n$ pattern, so for any n, it must be divisible by 3.

Nevertheless, example-taking was sometimes problematic, and participants did not have the same opinion in every case. For example, they had different degrees of acceptance on MQ2.SB (in Figure 13). For Cita, MQ2.SB was proof because the examples were structured and guaranteed larger numbers. However, others opined that it did not adequately illustrate what happened to the larger numbers even though it was structured.

Question 2
Prove that the sum of three consecutive natural numbers is divisible by 3.

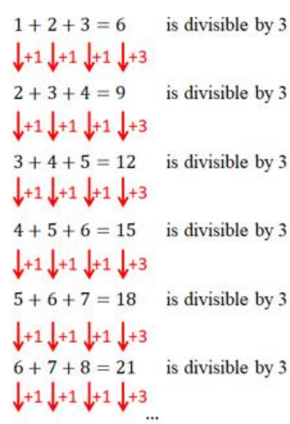
Student B
Look at the following number line:



The first three groups of three consecutive numbers always have 3 in them. The second three groups of three consecutive numbers always have 6 in them. The third three groups of three consecutive numbers always have 9 in them. So, the following groups of three consecutive numbers always have a number divisible by 3.

1st group : All left of 3 : $1 + 2 = 3$ is divisible by 3.
2nd group : Left and right of 3 : $2 + 4 = 6$ is divisible by 3.
3rd group : All right of 3: $4 + 5 = 9$ is divisible by 3.
The sum of other numbers in every group is also divisible by 3.

Student C



6 is divisible by 3. The sum of next group is added by 3. So, it is also divisible by 3.

Figure 13. Students B's Answer and C's Answers for Mathematical Question 2

There was also a recommendation to improve students' explanations using examples. When discussing question 4, Cita needed examples after valid proof at multiple points. However, she clarified that using examples was more complimentary than compulsory. To give an instance, look at MQ3.SG (in Figure 14) below and the discussion following it. Cita meant that giving examples could ease students' understanding of a complex explanation since it could be one way to interpret what an algebraic relationship means.

Question 3
For all real numbers x, prove that $x + |x - 5| \geq 5$.

Student F
Using GeoGebra, below is the graph $y = x + |x - 5|$. We can see that the graph is coincident or above the line $y = 5$. So, $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

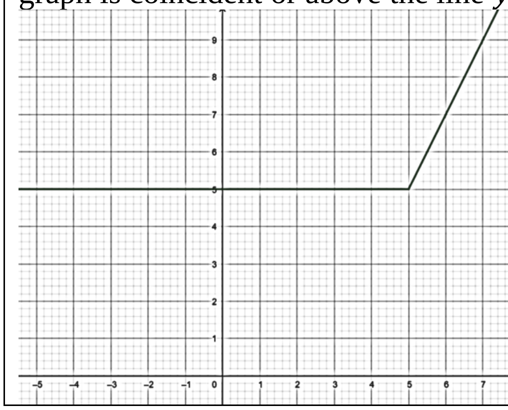


Figure 14. Student F's Answer to Mathematical Question 3

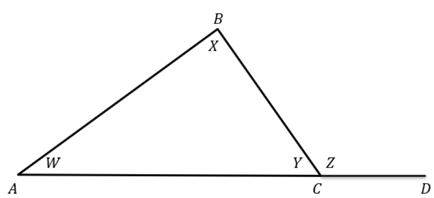
00:21:41 Int:	Before going to student G, is there anything that can improve student F?
00:22:17 Cita:	Maybe, like inputting a value. They draw, describe the picture at several points.
00:22:29 Int:	What do you mean?
00:22:30 Cita:	For example, for x, based on the graph, if x is one, then y is this. If x is seven, the y is this. Maybe for some cases.
00:23:02 Int:	So, is giving examples mandatory or just complementary?
00:23:04 Cita:	Complementary.
00:23:13 Int:	So even though examples are not given, is the answer still considered proof?
00:23:14 Cita:	Yes.
00:23:29 Int:	Okay. Why do you think it is considered as proof? Why would it be better if they added some examples like your suggestion?
00:23:29 Cita:	More convincing.
00:23:32 Int:	Okay, why is it considered more convincing?
00:23:37 Cita:	Because some people will assume, "oh, this is a symbol, then this is a symbol, what does it actually look like in a real example?" So, if you give an example, it is easier to imagine easier to understand what the writer meant for me.

Explanatory Tools. To improve the explanation in proof, sometimes tools are needed. All participants expected explanations, but these did not have to always be in words or long sentences. Instead, participants were satisfied with symbols, pictures, or diagrams in addition to written sentences. Participants said that a couple of answers proved the associated question even though the answers did not articulate the explanation.

The participants believed that some arguments could have benefitted from diagrams. For example, MQ4.SE (in Figure 15) below was considered proof by Budi and Cita because the diagram was easy to understand.

Question 4

Look at the picture below.



Prove that $Z = W + X$.

Student E

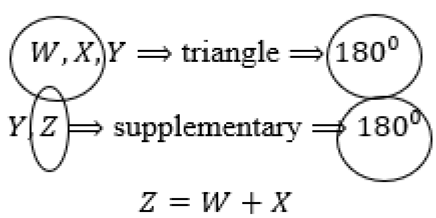


Figure 15. Student E’s Answer to Mathematical Question 4

Participants believed that pictures in the proof should be able to explain the ideas intended. For example, all participants agreed that the picture at MQ2.SE (in Figure 16) below was hard to understand. Budi and Cita said that because of insufficient information, MQ2.SE was not proof. On the other hand, Atika first said that MQ2.SE is not proof. However, after understanding the answer, she said it was proof but needed more explanation.

Question 2
Prove that the sum of three consecutive natural numbers is divisible by 3.

Student E

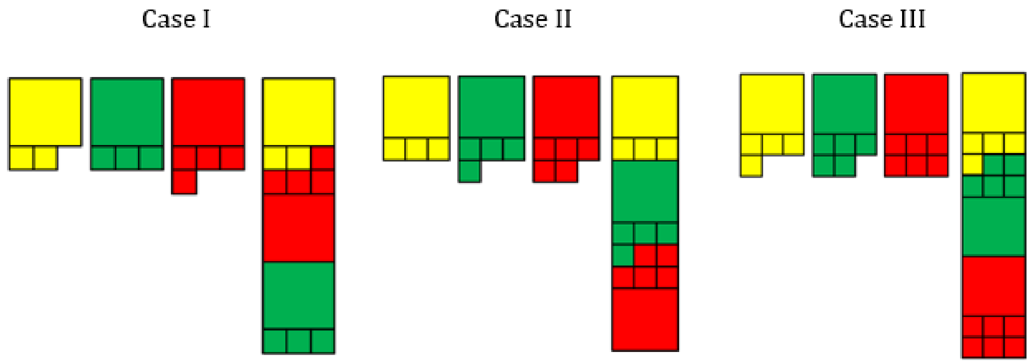


Figure 16. Student E's Answer to Mathematical Question 2

All participants said that explanation should be clarified with some pictures when needed. For example, all participants noted that MQ1.SB (in Figure 17) needed an improvement by adding a picture. Nevertheless, only Cita said it was not proof but a description instead. Atika needed to sketch an illustration before understanding the answer. She stated that most geometry questions required pictures along with some explanation.

Question 1
Prove that the area of a trapezoid is

$$\frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Student B
I can draw a diagonal in a trapezoid. I will have two triangles whose base is each parallel side of the trapezoid. The formula of the area of the triangle is $\frac{1}{2} \times \text{height} \times \text{base}$. So, if I add both triangles, its area is $\frac{1}{2} \times \text{the height} \times$ the sum of the parallel lines.

Figure 17. Student B's Answer to Mathematical Question 1

In addition to pictures or diagrams, participants also discussed symbols as a tool. All participants suggested that symbols must be well defined. For instance, one reason Atika suggested that MQ2.SF (Figure 19) was not proof because the starting point of the value of *k* was unclear.

Question 2
Prove that the sum of three consecutive natural numbers is divisible by 3.

Student F
If I have a multiple of 3, say $3k$, I can share it into three numbers:

$$3k-1, 3k, 3k+1$$

They are consecutive numbers, and their total is divisible by 3.

Figure 18. Student F's Answer to Mathematical Question 2

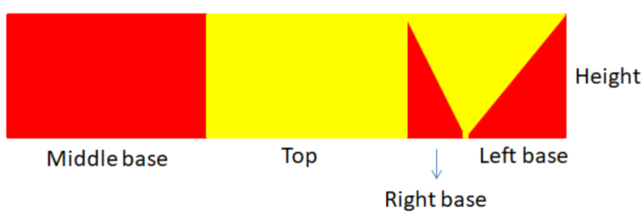
Sometimes the combination of some tools was needed. For instance, according to Budi, MQ1.SE (Figure 20) should be clarified with some symbols indicating the areas and other components of the pictures. Besides, Cita said that adding some right-angle symbols for the partitions might be helpful to clarify the representative.

Question 1
Prove that the area of a trapezoid is

$$\frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Student E
I make two congruent trapezoids on a piece of paper and cut them.

I arrange them like this.



So, the area is

$$\frac{1}{2} \times (\text{Top} + \text{Base}) \times \text{Height} = \frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Figure 19. Student E's Answer to Mathematical Question 1

Incorrect Proof. Look at MQ3.SH (in Figure 21) below.

Question 3
For all real numbers x, prove that $x + |x-5| \geq 5$.

Student H
$x + |x-5| = x + x - 5 = 2x - 5 = 2(10) - 5 = 15$. This is greater than 5.

Figure 20. Student H's Answer to Mathematical Question 3

When examining MQ3.SH (in Figure 21), Cita explained that an answer could be justified as proof from the depth of the explanation. It could be valued as correct or incorrect proof depending on the mathematical concepts' accuracy. In other words, for her, an answer did not have to be accurate to be considered proof as explained below.

00:26:57 Cita:	[Reading Student H's answer] proof.
00:27:06 Int:	Okay, proof. Why?
00:27:11 Cita:	Describe how they determine whether this statement is true or false. However, the process of proving it is wrong.
00:27:30 Int:	Okay, it is considered proof, but the process is wrong. Which part is wrong?
00:27:39 Cita:	Maybe the concept of the use of absolute values, which is not clearly described in this writing, is this writing of the proof.
00:28:01 Int:	Okay, so for you, the concept does not need to be true to be considered proof?
00:28:11 Cita:	The concept must be correct. Oh, do you mean, if, for example, in proving it turns out that the concept used is wrong, is that a proof procedure or not? Like that?
00:28:33 Int:	Yes.
00:28:35 Cita:	Still proof.
00:28:37 Int:	Okay. So why is it still proof even though the concept is wrong?
00:28:46 Cita:	From my point of view, what does it actually look like? Pouring the way of thinking of students. However, it turns out that from there, we can understand that students still do not understand this material.
00:29:07 Int:	Okay, that means from their process, it is categorized as proof. So, is it then considered as correct proof or incorrect proof?
00:29:23 Cita:	Incorrect proof.

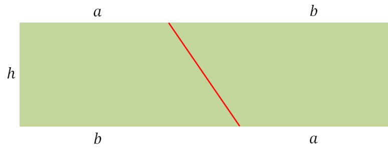
Different Perspectives in proof

Participants had various satisfaction levels of mathematical foundation and judgment when they considered the hypothetical students' solutions. Participants' responses to MQ1.SA, MQ1.SB, MQ1.SC, MQ1.SD, and MQ1.SE (in Figure 21) provide an example.

Question 1

Prove that the area of a trapezoid is $\frac{1}{2} \times \text{height} \times \text{sum of the parallel lines}$

Student A

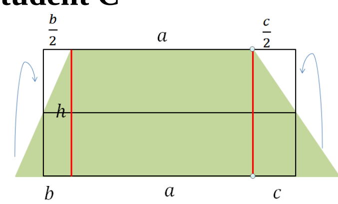


Area: $\frac{1}{2} \times \text{the height} \times$ the sum of the parallel lines

Student B

I can draw a diagonal in a trapezoid. I will have two triangles whose base is each parallel side of the trapezoid. The formula of the area of the triangle is $\frac{1}{2} \times \text{height} \times \text{base}$. So, if I add both triangles, its area is $\frac{1}{2} \times \text{the height} \times$ the sum of the parallel lines.

Student C



Area:

$$h \times \left(a + \frac{b}{2} + \frac{c}{2}\right)$$

$$= h \times \left(\frac{a}{2} + \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2}\right)\right)$$

$$= \frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

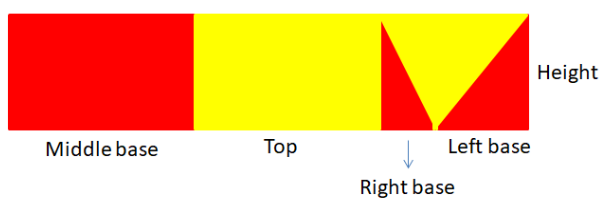
Student D

A trapezoid is similar to a parallelogram, but its parallel sides are not at the same length. So, to make it balanced, I take the average of the parallel sides.

Student E

I make two congruent trapezoids on a piece of paper and cut them.

I arrange them like this.



So, the area is $\frac{1}{2} \times (\text{Top} + \text{Base}) \times \text{Height} = \frac{1}{2} \times \text{the height} \times$ the sum of the parallel

Figure 21. Student A’s, Student B’s, Student C’s, Student D’s, and Student E’s Answers to Mathematical Question 3

All participants believed that MQ1.SE (in Figure 21) was proof even though they suggested an improvement on the explanation. Besides, all of them said that MQ1.SA and

MQ1.SD (in Figure 21) had a lack of mathematical foundation. Nevertheless, Cita still considered both answers proof, while others said neither was proof.

On the other hand, Atika needed more explanation on MQ1.SC (in Figure 21) while others felt satisfied. Cita said that MQ1.SB (in Figure 21) is not proof because it does not explain enough, while others agreed that its adequate explanation makes it proof. The potential reasons for these conflicting assessments of the convincing level of proof are discussed in the following sections.

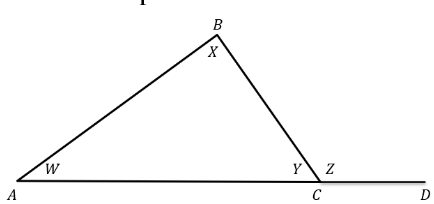
The Familiarity of the Question. Two answers may be written in similar ways but may be perceived differently. Examples of responses using a one-line mathematical calculation without explanation are MQ3.SC (in Figure 22) and MQ4.SD (in Figure 23) below.

Question 3
For all real numbers x, prove that $x + |x - 5| \geq 5$.

Student C
For $x \geq 5$, $x + |x - 5| \geq 5 + 0 = 5$ and for $x < 5$, $x + |x - 5| = x + 5 - x = 5$.

Figure 22. Student C's Answer to Mathematical Question 3

Question 4
Look at the picture below.



Prove that $Z = W + X$.

Student D

$$Z = 180^0 - Y = 180^0 - (180^0 - W - X) = 180^0 - 180^0 + W + X = W + X.$$

Figure 23. Student D's Answer to Mathematical Question 4

Even though all participants suggested adding more explanations to both answers, they agreed that MQ4.SD (in Figure 23) was proof while MQ3.SC (in Figure 22) was not. This opinion occurred possibly because MQ3.SC used more complicated algebraic properties and less familiar concepts. The hypothesis that participants' familiarity with a question may influence their assessment was strengthened by the finding that sometimes participants did not rely entirely on information written explicitly in an answer. In some parts, participants used their understanding of the problem to conclude that the answer was explained well. To illustrate, look at MQ3.SD (in Figure 24) and the transcript following it. Cita said that even though MQ3.SD only showed $x = 8$, it proved for $x \geq 8$ because a higher value substituted for x will result in a higher value.

Question 3 For all real numbers x, prove that $x + \lvert x - 5 \rvert \geq 5$.
Student D Test for $x = 8$, $x + \lvert x - 5 \rvert \geq 8 + \lvert 8 - 5 \rvert = 8 + \lvert 3 \rvert = 8 + 3 = 11 \geq 5.$ So, it is true that $x + \lvert x - 5 \rvert \geq 5$ for all $x \in \mathbb{R}$.

Figure 24. Student D’s Answer to Mathematical Question 3

00:13:32 Cita:	They took one of the cases and assumed that the x equals eight. However, this has not represented all the possibilities. So, their limitation is that it can be concluded for, let us say, numbers that are more than eight or equal to eight. Oh, it means this statement is correct. However, the case for less than eight has not yet been described, perhaps.
00:14:11 Int:	Okay, for you, why does this satisfy for more than eight or equal to eight but does not satisfy for less than eight?
00:14:24 Cita:	Because right here, they took equal to eight, meaning automatically the influence value is eight plus three. Actually, it is beyond what they give that I am thinking of. So, it means oh from eight plus three. It turns out it could be nine plus three. That means it will be bigger than five.

An argument that is hard to comprehend may be less convincing than the more understandable one. For instance, Atika found MQ1.SA was more convincing than MQ1.SB because the former one was easier to understand for its explanation and illustration. Cita also said that MQ1.SA was more convincing than MQ1.SC because student A's answer was more comprehensible and applied a more familiar technique.

00:17:35 Int:	Whose answer is more convincing?
00:17:45 Cita:	Student A.
00:17:47 Int:	What are the advantages?
00:17:50 Cita:	Student A's answer is compared to [student C's]. Student A's is easier to understand and is used more often than student C's, in my opinion.
00:18:03 Int:	Okay, so if it is easier to understand, is it more convincing?
00:18:17 Cita:	Yes.

The Number of Examples. There was probably a relationship between the number of examples taken and the convincing level of an answer. Look at MQ3.SB and MQ3.SD (in Figure 25).

Question 3

For all real numbers x, prove that $x + |x - 5| \geq 5$.

Student B

x	$x + \|x - 5\|$	Conclusion
-2	5	≥ 5
0	5	≥ 5
3	5	≥ 5
4.5	5	≥ 5
5	5	≥ 5
10	15	≥ 5
70	135	≥ 5

From the table above, for all values of x, negative, zero, positive, decimal, less than five, five, and greater than five, $x + |x - 5| \geq 5$. So, it is true that $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

Student D

Test for $x = 8$,

$$x + |x - 5| \geq 8 + |8 - 5| = 8 + |3| = 8 + 3 = 11 \geq 5.$$

So, it is true that $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

Figure 25. Student B’s Answer to Mathematical Question 3

Atika and Cita were more convinced by MQ3.SB (in Figure 25) than MQ3.SD (in Figure 25) because MQ3.SB provided more examples. According to Cita's interview transcript below, MQ3.SB was more convincing due to more examples showing more possibilities.

00:14:56 Int:	So, I want to compare this (Student D) to this one (Student A). Student A also took some numbers. Student B also took some numbers. Student D only took one number. For you, which answer is more convincing, Student B or Student D?
00:15:15 Cita:	Student B shows more numbers because it shows several possibilities.
00:15:27 Int:	Okay, does it mean that for you, the more examples are taken, the more convincing the proof will be?
00:15:43 Cita:	Yes.

However, sometimes participants mentioned not only the quantity of examples but also the method to choose the examples. From Atika's explanation below, MQ3.SB (in Figure 25) was more convincing because it took five examples that were well distributed on each interval.

01:19:06 Int:	Okay. This (student D) uses examples. Some other students also use examples. Which is more convincing than this student, student A or student B?
01:19:20 Atika:	Student B.
01:19:23 Int:	Why?
01:19:42 Atika:	The reason is for the number of x values they substituted. Student B gave us five examples. Student D only gave one example. Then the student, Student B, gave examples of negative, zero, and positive x values. Meanwhile, Student D just gave one example: a positive x-value, yes, eight. Then Student B also explained negative, zero, and positive. They said [it was] always greater than or equal to five. While student D... I think student B made more effort.

On the other hand, Budi said that it was not necessarily true that the convincing level of proof was determined by the number of examples shown. According to his statement below, neither a few nor many examples can guarantee the statement's validity for all domains.

01:27:29 Budi:	For me, it is influential but not really because example taking is like in the greyish area. Because maybe the example taken is correct by chance, by chance. Because let us say the statement is for x is this but turns out incorrect if we use an example. Besides, it means that the examples taken are many, but later, there is a counterexample. It means the number of examples is not influential, maybe.

Expectations about Students' Proving Skills

All participants agreed that proving requires the comprehension of prerequisite materials and the capacity to generate ideas. Nevertheless, they had different expectations about students' ability to prove. Atika said that middle grade and high school students do not seem ready to prove because of the lack of concepts and skills. She expected teachers to deliver prerequisite concepts and ensure students' understanding before giving a proving task.

Even though Cita also admitted that middle and high school students might experience difficulties proving, she suggested that some high school topics look doable. A student may be able to prove trigonometric identities but finds it hard to prove factorials. She also reflected on her experience in high school that some tasks were hard to prove because they went beyond the curriculum or standards taught, like the properties of real numbers. Cita suggested that, given her experience, a practice may help improve students' skills.

Budi had a higher expectation of students' proving ability. He believed that high school students might be ready to prove depending on several factors, such as their previous education, basic knowledge, critical thinking, ability to select and connect helpful concepts, and the frequency of practice. Additionally, the teaching style plays an important role. In addition to delivering materials, teachers need to present and exemplify proof in a way that they expect students to do. If teachers expect students to present well-structured proof, they should not trivialize their presentation structure in the classroom as stated on the following transcript.

00:08:05 Budi: In high school, it might be a little bit doable. In middle school, it could be. Because it also depends on where they come from, previous education because some students have the good basic knowledge. It means that they understand basic things and something. If it is taught well, the students are helped to understand well about materials related to the proof. Then I think they can do that proof. Proof cannot be done without practice because math requires much practice. Then from the teachers' delivery, it must also be in accordance with what is expected to be done by students. For example, we said earlier that proof must be systematic, meaning the teacher must also present proof like that because some teachers might think that that is not really important. So, they just present the main points without showing where those things are derived from. So maybe it also depends on the teachers.

Budi's opinion on how mathematics is taught was also reflected in his assessment of one answer provided. Budi accepted MQ3.SB (in Figure 26) as proof even though only one example from each interval is taken because it indicates that the samples were taken systematically. Besides, this is related to how the prerequisite material, inequality, is answered in writing. Students only test one point on each interval to justify whether all values in that interval are positive or negative.

Question 3
For all real numbers x, prove that $x + |x - 5| \geq 5$.

Student B

x	$x + \|x - 5\|$	Conclusion
-2	5	≥ 5
0	5	≥ 5
3	5	≥ 5
4.5	5	≥ 5
5	5	≥ 5
10	15	≥ 5
70	135	≥ 5

From the table above, for all values of x, negative, zero, positive, decimal, less than five, five, and greater than five, $x + |x - 5| \geq 5$. So, it is true that $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

Figure 26. Student B's Answer to Mathematical Question 3

00:05:36 Budi: For this, this is, but actually, this is quite tidy. Because for student B, the ones taken are examples. Like yesterday, they just took samples. So, there is no, there is no

> guarantee that it applies to all. However, from the way it takes this part, it is explained for the negative, then zero, less than five, then five, and more than five, all of them apply. So, probably the sampling is structured. It means there is a consideration. It is not random. So, it can be categorized as proof. Because when we do a question about inequality, students usually test one point only to determine whether it is positive or negative for that interval.

All participants agreed that the convincing level of an argument depended on the content, not the method (e.g., direct, contradictory, two-column proof, etc.), forms (words, pictures, diagram, a combination of them), and approach (e.g., rectangular, the composition of rectangles and triangles, etc. to prove the area of a trapezoid). However, the participants had different views on using the computer to prove. All participants agreed that MQ3.SF (in Figure 27) was proof. However, while Atika said that technology and manual calculation were equally convincing, others stated that it was better to use manual calculation since it could demonstrate students' understanding better.

Question 3
For all real numbers x, prove that $x + |x - 5| \geq 5$.

Student F
Using GeoGebra, below is the graph $y = x + |x - 5|$. We can see that the graph is coincident or above the line $y = 5$. So, $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

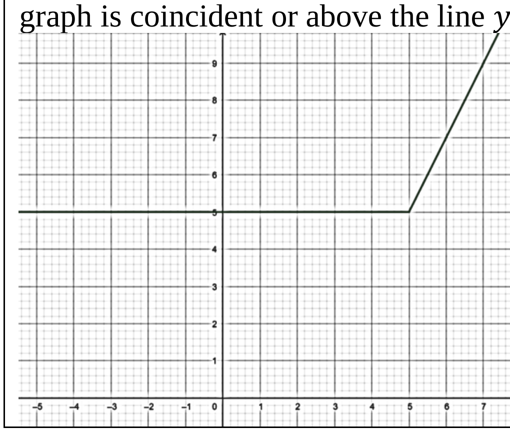
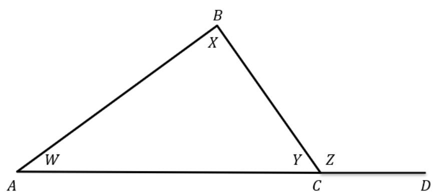


Figure 27. Student F’s Answer to Mathematical Question 3

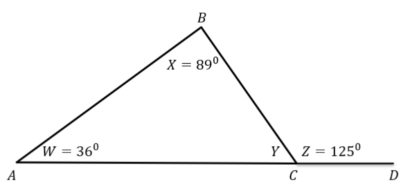
Furthermore, all of them agree that proving by measuring angles like on MQ4.SA (in Figure 28) below should be avoided. Cita stated that students should not always rely on tools because that would be hard when no tools were available. Others were concerned that measuring the triangles did not guarantee validity because it did not apply to general triangles, and some measurement errors may occur.

Question 4
Look at the picture below.



Prove that $Z = W + X$.

Student A
I measured the angles using a protractor. I found:



$$125^0 = 36^0 + 89^0$$

So, $Z = W + X$.

Figure 28. Student A’s Answer to Mathematical Question 4

Discussion

Various and Cluster Definitions of Proof

According to (Sundtrom, 2016), defining a term or phrase means agreeing that they stand for something we have spoken about often, like an object, property, or concept. In other words, the definition, roles, and criteria of proof cannot be discussed separately because some define a term using its criteria (stipulative definition) or roles.

In the first session, participants only mentioned a few criteria of acceptable proof and were inspired and recalled some others after looking at several sample arguments. There were some similarities and differences among participants in defining proof and mentioning its criteria. Besides, satisfactory levels of explanation varied among questions and participants. This disagreement is commonly experienced in mathematics education because of the absence of formal definition and proof properties. Mathematicians agree that every proof is a method with a set of rules for deducing statements, but there has never been a consensus on what constitutes an acceptable proof (Hanna, 1983). In line with this, even though almost all researchers shared a common expectation on what mathematical proof constitutes, some differences may occur among the mathematics education community, especially teachers (Balacheff, 2008).

The participants defined proof as a tool to verify and explain the validity of a statement, as well as communicate and systematize mathematics. Involving sixteen in-service secondary school mathematics teachers, Knuth (2002a) also found this variety of roles in addition to creating mathematics. Describing various roles suggested participants' diverse and potentially powerful understanding of the function of proof in mathematics (Knuth, 2002a).

One criterion of proof mentioned by all participants is a clear explanation at every step. However, drawing a line between acceptable and unacceptable explanations is confusing. In many parts, participants said that an answer is a proof because of its complete explanation. However, the inverse of the statement is not immediately true. Some answers are not proof because they lack explanation, but some others are proof even though it needs some more explanation. Some contradictive points also occur. For example, an answer is considered proof even though it does not satisfy some criteria mentioned.

The participants probably use their criteria as a rough guideline rather than evaluative criteria (Czocher & Weber, 2020). Accordingly, they rated proof flexibly, meaning an answer only needs to satisfy some of the criteria mentioned to be proof. This finding is related to the explanation of Gaut (2000) that in a cluster model, an object can be a member of a category if it satisfies most or all properties and cannot be a member if it satisfies none of the properties while no individual property is mandatory. Czocher and Weber (2020) offered and evaluated a proof cluster definition, namely, a proof must satisfy most of the following properties: convincing, perspicuous, priori, transparent, and has been sanctioned by the mathematical community. Accordingly, a proof may not be transparent, meaning that it does not fill every step with some formal derivation (Czocher & Weber, 2020).

This flexible classification probably also relates to whether an answer is easy or difficult to comprehend. It is found that participants tended to have a higher expectation of an answer that requires a more advanced or less familiar topic. The hypothesis is strengthened by the finding that sometimes, participants did not rely entirely on what is written explicitly in an answer. In some parts, participants used their understanding of the problem to conclude that the answer was explained well. However, this assumption is different from the finding that students' acceptance of either an inductive or a deductive argument was not dependent on their familiarity with the context.

All participants expected explanations, but these did not have to always be in words or long sentences. Instead, participants were satisfied with symbols, pictures, or diagrams in addition to written sentences. Participants said that a couple of answers proved the associated question even though the answers did not articulate the explanation. Participants expected a diagram showing a clear connection among the items, pictures demonstrating properties, and symbols that are well defined and show generalization. Sometimes a combination of tools is needed, like symbols can be used to clarify the properties of pictures. These findings are related to a suggestion on some criteria of an acceptable visual proof by Borwein and Jörgenson (2001). Some pictures in this study led to a disagreement about whether they were proof of their lack of explanation or difficulty comprehending. Nevertheless, picture arguments can satisfy the

Czocher and Weber (2020) definition of proof within the cluster account. Even though picture arguments are difficult to show logically necessary consequences and formalize in a formal logical system, they satisfy most criteria, namely can remove all doubt, is comprehensible, and can be sanctioned by the mathematical community.

Sophistication Levels of an Argument

According to Stylianides (2009), depending on the content and explanatory level, an argument can be classified into five categories: demonstrations, generic arguments, rationales, empirical arguments, and none of the four categories. Each section below examines the criteria of arguments satisfying each of those categories.

Participants' Assessments on Rationales and Demonstrations. All participants considered the explanation the major criterion to determine whether an argument is proof. They suggested a proof should include the underlying mathematics concept, emphasize the main ideas, and break down the mathematics calculations. On the other hand, if an answer does not satisfy most criteria mentioned, it is not considered proof. This opinion is supported by Stylianides (2009) that an argument counts as a rationale (one class of non-proof) instead of a demonstration (one class of proof) if it does not make explicit reference to fundamentally accepted truths or omits necessary steps.

Participants' Assessments on Empirical Arguments and Generic Arguments. All participants stated that an explanation should show all elements or possible cases to be considered proof. Accordingly, showing a few examples randomly, measuring selected figures, or presenting a proper subset of a graph can certainly be determined as not proof. Nevertheless, it does not mean that providing selected examples was never accepted. The participants opined that taking examples can guarantee generality if a pattern is clearly shown or explains why it applies to all members. In one case, a participant suggested that simply changing the specific number in a question to a variable can directly show the statement in general. This finding relates to the hierarchy of (Stylianides, 2009) that an empirical argument (one class of non-proof) validates a claim in a proper subset of all the possible cases covered by the claim, while a generic example (one class of proof) uses a particular case seen as representative of the general case.

When discussing the area of a trapezoid, all participants agreed that students could use any trapezoid and do not need to show all possible shapes. Participants said that since the formula generally applies, choosing a specific trapezoid (for instance, an isosceles trapezoid) was a valid choice within the proof. One hypothesis to explain this unexpected finding is that the participants used what might be considered circular reasoning. Since their task is to prove the generality of the area, they should not assume that the formula applies in general at the beginning of the proof. Assuming the conclusion to prove the conclusion is one common undergraduate misconception in proof (Idrus, 2017; Stavrou, 2014). One reason behind this is that students are often allowed to use the back solving method, namely starting with the conclusion, working, and reaching the given information in some mathematics tasks, but some students do not realize that the method does not apply in all cases (Idrus, 2017). Another hypothesis to explain this finding is that all participants shared one common similarity in understanding generality. They believed that students should use an arbitrary trapezoid to prove the area of any trapezoid. However, they did not consider that the arbitrary trapezoid, in this case, is independent of some special attributes, like the right angles and congruent sides.

A disagreement on the relationship between the amount of empirical evidence and the convincing level of an argument occurred. One participant believed that proof should apply to all, so the number of examples shown did not influence her belief in an argument. On the other hand, two participants felt more convinced when more examples were given in an answer. One of them also preferred an answer with well-distributed examples. This finding is connected to two basic ways of proof, namely empirical and logical (Fischbein, 1980). Showing all elements is the one accepted in mathematics since the only way to validate a mathematical statement is by using formal logical inferences (Fischbein, 1980). Nevertheless, believing that the convincing level depends on the number of examples is not surprising since the more confirmatory facts and the variety accumulated them, the more convincing empirical evidence is (Fischbein, 1980). However, instead of mathematics, this way of proof is more used by scientists who work in generalizations and predictions (Fischbein, 1980).

One participant recommended giving some examples to illustrate a complex idea after giving complete proof. This recommendation was also found by Martin and Harel (1989) that some individuals need to be convinced using a combination of a deductive argument and empirical evidence. Moreover, Fischbein and Kedem (1982, as cited in Martin & Harel, 1989) more specifically suggested that even after being convinced by deductive proof, many students wanted further empirical evidence. Supplementing valid proofs with empirical evidence was also found by Stavrou (2014). Even though students commonly generate examples to convince themselves further or understand the statement, the examples should not be included with formal proof (Stavrou, 2014).

Participants' Assessments on Arguments Satisfying None of the Four Categories. Some hypothetical student answers were mathematically wrong, showed minimal engagement, or presented irrelevant responses. Unfortunately, almost all of them were not noticed by the participants. This finding may result from one of the following limitations: the time for any particular question in an interview was limited, so participants did not have very long to examine any particular hypothetical student solution; additionally, participants may have assumed that all answers given were already correct because they were presented in a typed and finished form. However, this finding may also be related to Martin and Harel’s (1989) finding that even though students correctly accepted a general proof verification, many might not reject false-proof verification because they focused more on the appearance and ritualistic aspect than the correctness of the argument.

Only one participant noticed one answer with a wrong mathematics concept. She said that the answer is incorrect proof, which is proof in the process but incorrect in the concept. She suggested that simply changing the concept to the correct one can result in valid proof. This is different from Stylianides’s (2009) hierarchy that circular or non-genuine responses (like showing minimal engagement, irrelevant responses, or potentially relevant responses, but their relevance is not evident by the solver) do not qualify as non-proof or proof.

Expectations about Students' Proving Skills

Participants had different expectations about students' ability to prove themselves. They all believed that middle-grade students do not seem ready to prove by themselves. While one thought that high school students do not seem ready either, others believed that some students might have adequate knowledge and skills to prove, and some topics look possible to be proven by them. He believed that high school students might be ready to prove depending on several factors, such as their previous education, basic knowledge, critical thinking, ability to select and connect helpful concepts, and frequency of practice. In addition to understanding the prerequisite materials and skills, students needed the capacity to generate ideas.

All participants agreed that the convincing level of an argument depends on the content, not the method (e.g., direct, contradictory, two-column proof, etc.), forms (words, pictures, diagram, a combination of them), and approach (e.g., rectangular, the composition of rectangles and triangles, etc. to prove the area of a trapezoid). However, the participants had different views on using the computer to prove. One participant said that technology and manual calculation are equally convincing because they stressed different points, namely analytical interpretation for the former and concept application for the latter. On the other hand, others preferred manual calculation because it can demonstrate students' thought processes and logical justification. Previous researchers found different opinions regarding technology among mathematicians. Of 94 mathematicians, 37 agreed that the computer-assisted proof was proof, but the remaining 57 disagreed (Weber & Czocher, 2019). The disagreement occurs because the computer does not adequately show the necessary consequences of mathematical concepts at every step and cannot be sanctioned by the mathematical community (Weber & Czocher, 2019).

The participants suggested that teachers deliver prerequisite concepts and ensure students' understanding before giving a proving task to meet the expectations. They also need to avoid tasks beyond the curriculum or standards taught, like the properties of real numbers. The practice may help improve students' skills.

Additionally, how mathematics is taught is important. Teachers need to present and exemplify proof in a way that they expect students to do. If teachers expect students to present well-structured proof, they should not trivialize their presentation structure in the classroom. Besides, one participant accepted an inequality argument as proof even though it only tests one value from each interval. This is because in the class, determining whether the value of an interval in an inequality problem is positive or negative is more of a practical method. The students do not get used to writing or even understanding what happens on a number line in each interval.

CHAPTER 5: CONCLUSIONS AND IMPLICATIONS

Conclusions

Definitions of Proof

To accumulate, the participants defined proof as a tool to verify and explain the validity of a statement, as well as communicate and systematize mathematics. Describing various roles suggested participants' diverse and potentially powerful understandings of the function of proof in mathematics.

There were some similarities and differences in proof definitions and criteria. Besides, the satisfactory levels of explanation varied among questions and participants. This disagreement is commonly experienced in mathematics education because of the absence of formal definition and proof properties. The participants probably use their criteria as a rough guideline rather than evaluative criteria and a definition of proof within the cluster account by Czocher and Weber (2020). This flexible classification probably also relates to whether an answer is easy or difficult to comprehend because sometimes participants used their understanding of the problem to conclude that the answer was explained well.

Levels of Sophistication of an Argument

All participants suggested that proof should include the underlying mathematics concept, emphasize the main ideas, and break down the mathematics calculations; it can be in the form of symbols, pictures, or diagrams in addition to written sentences. This opinion is supported by Stylianides (2009) that an argument counts as a rationale (one class of non-proof) instead of a demonstration (one class of proof) if it does not make explicit reference to fundamentally accepted truths or omits necessary steps.

All participants stated that an explanation should show all elements or possible cases to be considered proof. Nevertheless, the participants opined that taking examples can guarantee generality if a pattern is clearly shown or explains why it applies to all members. This finding relates to the hierarchy of (Stylianides, 2009) that an empirical argument (one class of non-proof) validates a claim in a proper subset of all the possible cases covered by the claim, while a generic example (one class of proof) uses a particular case seen as representative of the general case. When discussing the area of a trapezoid, all participants agreed that students could use any trapezoid and do not need to show all possible shapes. The participants probably used circular reasoning or did not consider that the arbitrary trapezoid, in this case, is independent of some special attributes, like the right angles and congruent sides.

There was a disagreement on the relationship between the amount of empirical evidence and the convincing level of an argument. This finding is connected to two basic ways of proof, namely empirical and logical (Fischbein, 1980). One participant needed to be convinced using a combination of a deductive argument and empirical evidence.

Only one participant noticed one answer with a wrong mathematics concept and said that it was an incorrect proof, meaning it was proof in the process but incorrect in the concept. This is different from Stylianides's (2009) hierarchy in which irrelevant answers do not qualify as non-proof or proof.

Expectations about Students' Proving Skills

All participants had different expectations about students' ability to prove. They all believed that middle-grade students do not seem ready to prove. While one thought that high school students do not seem ready either, others believed that some students might have adequate knowledge and skills to prove, and some topics look possible to be proven by them.

The participants suggested that teachers deliver prerequisite concepts and ensure students' understanding before giving a proving task to meet the expectations. In addition to understanding the prerequisite materials and skills, students needed the capacity to generate ideas.

All participants agreed that the convincing level of an argument depends on the content, not the method (e.g., direct, contradictory, two-column proof, etc.), forms (words, pictures, diagram, a combination of them), and approach (e.g., rectangular, the composition of rectangles and triangles, etc. to prove the area of a trapezoid). However, the participants had different views on using the computer to prove, which is also a common debate among mathematicians.

Limitations

There were some limitations of this study. Firstly, the number of participants was few, while having more participants from different educational backgrounds can give broader results. Secondly, one question involved inequality and the absolute value at the same time, but it turned out that having two less familiar topics simultaneously made participants focus more on the mathematics content than the proving technique. Hence, it would be better either use common topics or make sure that the participants are familiar with the prerequisite materials in the mathematics problems. Thirdly, the participants did not notice almost all the answers that were mathematically wrong, showed minimal engagement, or presented irrelevant responses because those answers' flaws were less clear and straightforward. Using more familiar questions and handwriting instead of a typing form probably lessens participants’ tendency to assume that all answers are already correct.

Implications

Understanding teachers' beliefs of proof can recommend how mathematical reasoning and what sophistication level should be taught in the classroom. Since preservice teachers' understanding of generality in proof seemed different from the proof perspective, more experiences related to generality are needed. The research shows that the participants only mentioned a couple of criteria for proof in the first session, and they recalled or were inspired by many more criteria while assessing hypothetical answers. Thus, in considering how proof might be taught in pre-service education, asking pre-service teachers to analyze different types of arguments in order to identify criteria for proof is a promising approach.

A more rigorous investigation is needed to find a specific set of criteria for acceptable proof at the secondary level. This study elaborated expectations for 8-12 students in general, while there are probably different beliefs and expectations on proof for middle grade and high school students since the acceptance criteria for validating mathematical proofs differed, with increasing standards from school students to university students, mathematician teachers, and mathematician researchers (Sommerhoff & Ufer, 2019). There might also be different expectations on students' verbal, written, and combination of verbal and written proof.

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APPENDICES

A. SUMMARY OF THE CRITERIA OF PROOF

Note

S1 : Session 1

MQ X : Mathematical Question Number X

A-J : The associated criterion is mentioned by the associated interviewee while assessing student A-J's answer to the associated mathematical question

Letter Colours

Red : Explicitly stated that the criterion is complementary but more preferred

Blue : Explicitly stated that the criterion is mandatory

Green : Explicitly stated that the criterion is acceptable but less preferred

Orange: Explicitly stated that the criterion is a wrong method

Purple : Explicitly stated that the criterion is not really put into consideration

Black : Others

CRITERION		**Atika**					**Budi**					**Cita**				
		Session 1	**MQ 1**	**MQ 2**	**MQ 3**	**MQ 4**	**Session 1**	**MQ 1**	**MQ 2**	**MQ 3**	**MQ 4**	**Session 1**	**MQ 1**	**MQ 2**	**MQ 3**	**MQ 4**
TECHNIQUE																
a	The proof should prove in general/not just take examples or some cases			A, B, E, G	A, B, C, D, H, I, K	A			A, B, G	D, F, K	A			A, B, D, E, G	D, E, I	C
a	The examples in the proof are more systematic or have pattern		C	B					C	B, I				C, E		
a	How convincing a proof is not determined by	E			E, F, J			F		J				F		

		the method/approach chosen															
	a	The more examples/cases are given, the more convincing the proof will be.				D					F					D, I	
	a	The proof starts from the given information to the conclusion.				E	H				E						H
		EXPLANATION															
	a	The steps of the proof are well organized/structured.		C	D			S1	C, F	B	A, J, K, L	H			C	C, E, G, H	E
	a	Every step in the proof is clearly explained		C, F	G	F, J, K	B, D, E, F, H		D	B, E, F	A, B, C, E, F, H, J, K, L	C, D, E, F, G, H		B, C, E, F	C, D, E	A, B, C, E, G, H, I, J, K	A, B, D, E, H
	a a a	Every step of the proof is written based on the mathematical concepts.	S1	A, B, C, D, E	D	C, G	C, E	S1	A, B, C, D, E, F	D, A	A	C, H	S1	A, B, C, D, E		E	C
	a	The mathematical calculation in the proof is broken down.			F, G	F	C, D		C	F	C, L	C, D			G	L	D
	a	Examples are added to ease understanding a complex explanation													D, F, G	F, G, L	
		TOOLS															
	a	The pictures in the proof are able to explain the ideas intended.		A, B, E	E				A, C, E, F	E				E	E	G, K	
	a	Some pictures are added to the proof when needed.		B		A	G		B			D		A, B, D			

	a	The pattern in the proof should be clearly stated			C										B		
	a	The symbols are well defined.		E	F, G				F					E, G			
	a	Every word in the proof is not ambiguous		E										F			
PRESENTATION																	
	a	The main idea/concept of the proof is emphasized.		A, E	C, E, G				E								
	a	The goal of the proof is clear.		C, D					B	F			S1				D
	a	The conclusion of the proof is written								C, E	C, L					A, B, C, E, G, I, J, K, L	E, F
	a	The proof is written efficiently	S1		D, G	I											
OTHERS																	
	a	The proof is easy to understand.		B, C, E	C, D, E, G	E, K	E		B, D, E		L			C	F	C, F	
	a	The proof is mathematically correct.		C		B, E, I			B		H,	A, B				H	
	a	Using tools/technology to prove				E					F					F	A, B

Summary of the Meaning of Proof

Y: The associated interviewee mentioned the associated role during the interviews

	Role	Interviewee		
		Atika	Budi	Cita
A	to verify that a statement is true		Y	Y
A	to explain why a statement is true	Y		Y
A	to communicate mathematical knowledge	Y		
A	to discover or create new mathematics			
A	to systematize statements into an axiomatic system	Y		

B. INTERVIEW PROTOCOLS

Session 1 (Participants' beliefs and expectations toward proof)

Preliminary Questions

1. Do you have experience in teaching mathematics either in the formal or non-formal setting?
2. What did you learn about proof when you were in school?
3. What did you learn about proof in college?

Main Questions

1. What is proof?
2. What does a good proof include?
3. Do you think students should prove? If yes, what topic?
4. What can be good ways to develop students' ability to prove?
5. What are the criteria of acceptable students' proof? Do you believe that students can give such acceptable proof? What might be students' difficulties in providing such satisfactory proof?
6. Any other things about proof you would like to share?

Session 2 (How participants assess proof)

Questions and a set of solutions are given.

For each solution: What do you think about this argument? Does it convince you? What do you like about it? Is it a proof? Why do/don't you think it is a proof? What would you add/take away? If a student gives you this answer, how would you evaluate it?

After some solutions: Which of them you like the best? Which is most convincing?

C. MATHEMATICAL QUESTIONS AND SOLUTIONS

Question 1

Prove that the area of a trapezoid is

$$\frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Student A

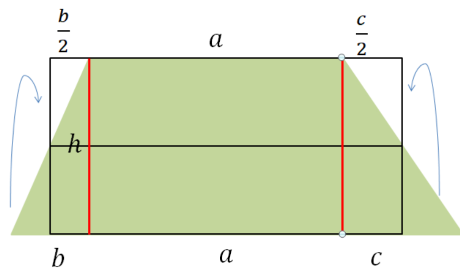
Area:

$$\frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Student B

I can draw a diagonal in a trapezoid. I will have two triangles whose base is each parallel side of the trapezoid. The formula of the area of the triangle is $\frac{1}{2} \times \text{height} \times \text{base}$. So, if I add both triangles, its area is $\frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$.

Student C



Area:

$$h \times \left(a + \frac{b}{2} + \frac{c}{2}\right)$$

$$= h \times \left(\frac{a}{2} + \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2}\right)\right)$$

$$= \frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Student D

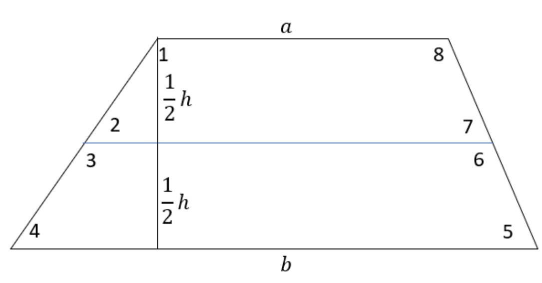
A trapezoid is similar to a parallelogram, but its parallel sides are not at the same length. So, to make it balance, I take the average of the parallel sides.

Student E

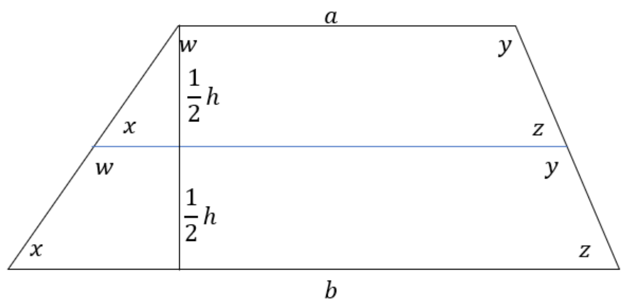
This is a proof and the picture is helpful in representing the relationships between figures. However, they need to explain each step and clarify each word. For them, students can use any kind of trapezoids to prove the formula.

Student F

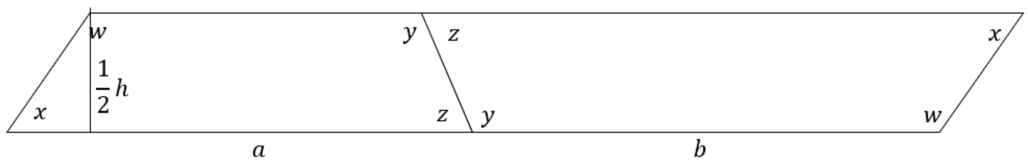
Look at the following trapezoid.



The blue line bisects the height of the trapezoid and is parallel to the parallel sides of the trapezoid. According to a rule of two parallel lines cut with a transversal line, because ∠1 and ∠3 are corresponding, $m\angle 1 = m\angle 3 = w$. Because $\angle 2$ and $\angle 4$ are corresponding, $m\angle 2 = m\angle 4 = x$. Because $\angle 5$ and $\angle 7$ are corresponding, $m\angle 5 = m\angle 7 = y$. Because $\angle 6$ and $\angle 8$ are corresponding, $m\angle 6 = m\angle 8 = z$. Look at the following;



Moreover, because $\angle 2$ and $\angle 3$ are supplementary, $m\angle 2 + m\angle 3 = w + x = 180^0$. Because $\angle 6$ and $\angle 7$ are supplementary, $m\angle 6 + m\angle 7 = y + z = 180^0$. This, the trapezoid can be rearranged perfectly into the following:



Accordingly, the area of the trapezoid is

$$\frac{1}{2} \times \text{the height} \times \text{the sum of the parallel lines}$$

Question 2

Prove that the sum of three consecutive natural numbers is divisible by 3.

Student A

$1 + 2 + 3 = 6$ is divisible by 3.

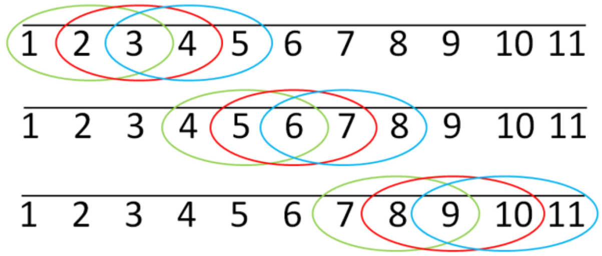
$5 + 6 = 7 = 18$ is divisible by 3.

$23 + 24 + 25 = 72$ is divisible by 3.

So, the sum of three consecutive natural numbers is divisible by 3.

Student B

Look at the following number line:



The first three groups of three consecutive numbers, always has 3 in it. The second three groups of three consecutive numbers, always has 6 in it. The third three groups of three consecutive numbers, always has 9 in it. So, the next groups of three consecutive numbers always have a number divisible by 3.

1st group	: All left of 3	: $1 + 2 = 3$ is divisible by 3.
2nd group	: Left and right of 3	: $2 + 4 = 6$ is divisible by 3.
3rd group	: All right of 3:	$4 + 5 = 9$ is divisible by 3.

The sum of other numbers in every group is also divisible by 3.

Student C

$1 + 2 + 3 = 6$ is divisible by 3

↓+1 ↓+1 ↓+1 ↓+3

$2 + 3 + 4 = 9$ is divisible by 3

↓+1 ↓+1 ↓+1 ↓+3

$3 + 4 + 5 = 12$ is divisible by 3

↓+1 ↓+1 ↓+1 ↓+3

$4 + 5 + 6 = 15$ is divisible by 3

↓+1 ↓+1 ↓+1 ↓+3

$5 + 6 + 7 = 18$ is divisible by 3

↓+1 ↓+1 ↓+1 ↓+3

$6 + 7 + 8 = 21$ is divisible by 3

↓+1 ↓+1 ↓+1 ↓+3

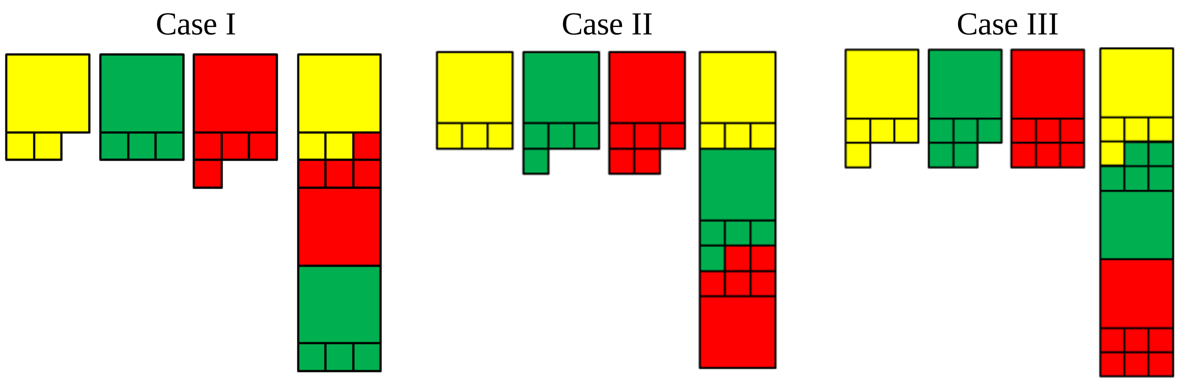
...

6 is divisible by 3. The sum of next group is added by 3. So, it is also divisible by 3.

Student D

If there are 3 consecutive number, the median is the mean. So, the mean is a natural number. Thus, their sum is divisible by 3.

Student E



Student F

If I have a multiply of 3, say $3k$, I can share it into three numbers:

$$3k-1,3k,3k+1$$

They are consecutive numbers, and their total is divisible by 3.

Student G

Suppose that the three numbers are $a-1$, a, and $a+1$. Because $a-1+a+a+1=3a$, the sum is divisible by 3.

Question 3

For all real number x, prove that $x + |x - 5| \geq 5$.

Student A

	$x + \|x - 5\| \geq 5$		
$\Leftrightarrow$	$\|x - 5\| \geq 5 - x$		
$\Leftrightarrow$	$x - 5 \geq 5 - x$	or	$5 - x \geq 5 - x$
$\Leftrightarrow$	$2x \geq 10$	or	$5 - x \geq 5 - x$
$\Leftrightarrow$	$x \geq 5$	or	$5 - x \geq 5 - x$
	True because $\|x - 5\| = x - 5$ means $x \geq 5$.		True because $5 - x = 5 - x$

Student B

x	$x + \|x - 5\|$	Conclusion
-2	5	≥ 5
0	5	≥ 5
3	5	≥ 5
4.5	5	≥ 5
5	5	≥ 5
10	15	≥ 5
70	135	≥ 5

From the table above, for all values of x, negative, zero, positive, decimal, less than five, five, and greater than five, $x + |x - 5| \geq 5$. So, it is true that $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

Student C

For $x \geq 5$, $x + |x - 5| \geq 5 + 0 = 5$ and for $x < 5$, $x + |x - 5| = x + 5 - x = 5$.

Student D

Test for $x = 8$,

$$x + |x - 5| \geq 8 + |8 - 5| = 8 + |3| = 8 + 3 = 11 \geq 5.$$

So, it is true that $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

Student E

Take any real number x. We will prove that $x + |x - 5| \geq 5$.

Case I. For $x \geq 5$, since $|x - 5| \geq 0$,

$$x + |x - 5| = x + x - 5 = 2x - 5 \geq 2(5) - 5 = 10 - 5 = 5.$$

Hence, by the transitive property, $x + |x - 5| \geq 5$.

Case II. For $x < 5$, we have $|x - 5| = 5 - x$. Thus,

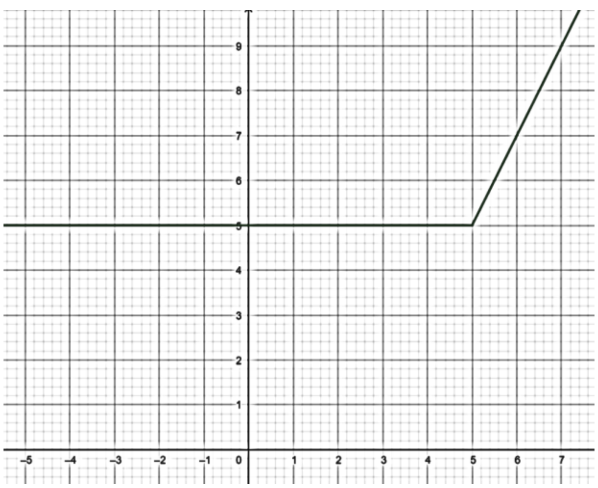
$$x + |x - 5| = x + 5 - x = 5 \geq 5.$$

Hence, by the transitive property, $x + |x - 5| \geq 5$.

Therefore, in all cases, $x + |x - 5| \geq 5$.

Student F

Using Geogebra, below is the graph $y = x + |x - 5|$. We can see that the graph is coincident or above the line $y = 5$. So, $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.



Student G

We will create the graph $y = x + |x - 5|$.

For x-intercept, $y = 0 = x + |x - 5|$

For $x > 5$,

$x + x - 5 = 0$

$2x - 5 = 0$

$x = \frac{5}{2}$

Contradicts

For $x < 5$,

$x - x + 5 = 0$

$5 = 0$

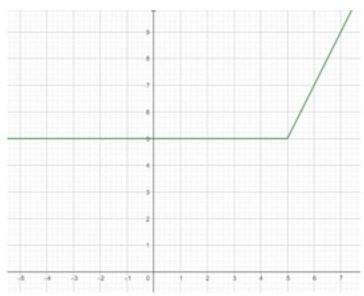
Contradicts

So, the graph does not have x-intercepts.

For y-intercept, $y = x + |x - 5| = 0 + |0 - 5| = 5$. So the graph intercepts the y-axis at (5,0).

For $x \geq 5$, $x + |x - 5| = x + x - 5 = 5$ and for $x < 5$, $x + |x - 5| = x + 5 - x = 5$.

Therefore, below is the graph.



We can see that the graph is coincident or above the line $y = 5$. So, $x + |x - 5| \geq 5$ for all $x \in \mathbb{R}$.

Student H

$x + |x - 5| = x + x - 5 = 2x - 5 = 2(10) - 5 = 15$. This is greater than 5.

Student I

If x is more than 5, $x + |8 - 5|$ means adding a positive value to x that is already greater than 5. So $x + |8 - 5|$ is still greater than 5.

If x is less than 5, say 3. The notation $|3 - 5|$ is reversed to $5 - 3$, but we will also add it with 3. So, both 3s cancel out. The remaining is 5. I can change number 3 with any number less than 5.

Therefore, $x + |x - 5| \geq 5$.

Student J

Suppose to the contrary that there exists $x \in \mathbb{R}$ such that $x + |x - 5| < 5$ ... (i). Now, we see

$x + \|x - 5\| < 5$ $\|x - 5\| < 5 - x$ $0 \leq \|x - 5\| < 5 - x$ $0 < 5 - x$ $x < 5$	(i) Subtract both sides by x A property of absolute values Transitive Property Add both sides by x

Since $x < 5$, by the definition of absolute values, $|x - 5| = 5 - x$…(ii). Thus,

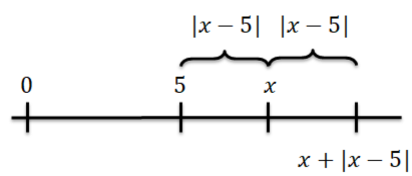
$5 > x + \|x - 5\|$ $5 > x + 5 - x$ $5 > 5$	(i) (ii)

The above shows a contradiction. Accordingly, the inverse of our assumption is true, that is $x + |x - 5| \geq 5$.

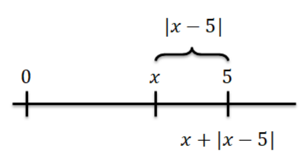
Student K

Without losing generalization, suppose that $x \geq 0$. We will prove that $x + |x - 5| \geq 5$.

Case I. For $x \geq 5$,



For $x < 5$,



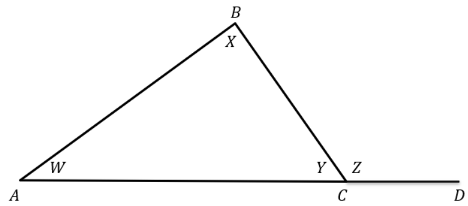
Hence, by all cases, $x + |x - 5| \geq 5$.

Student L

The value of $x - 5$ can be positive or negative. If it is positive, $x + |x - 5| \geq 5$, and if it is negative, $x + |x - 5| = 5$.

Question 4

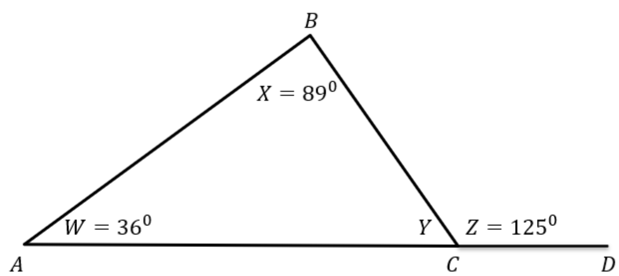
Look at the picture below.



Prove that $Z = W + X$.

Student A

I measured the angles using a protractor. I found:

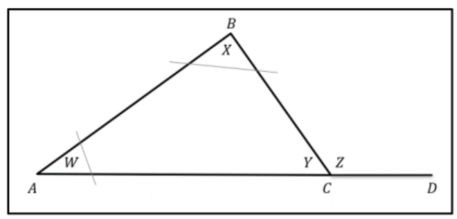


$$125^0 = 36^0 + 89^0$$

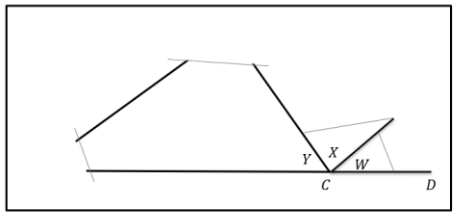
So, $Z = W + X$.

Student B

I drew the triangle on paper, and I cut W and X:

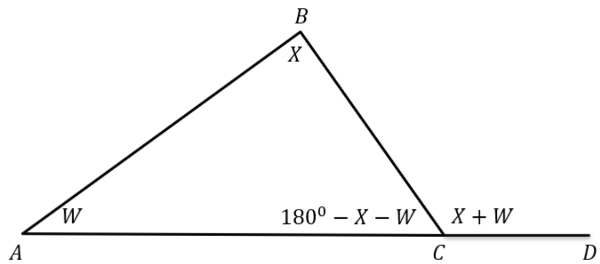


I arranged and sticked W and X on the top of Z:



We see that W and X covers Z.

Student C



Student D

$$Z = 180^0 - Y = 180^0 - (180^0 - W - X) = 180^0 - 180^0 + W + X = W + X.$$

Student E

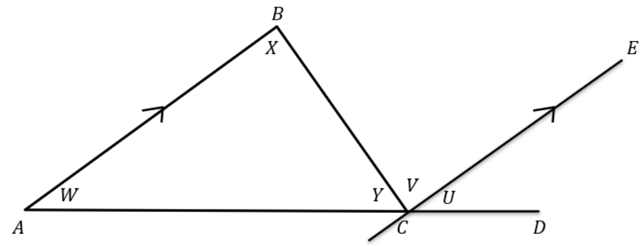
$$W, X, Y \Longrightarrow \text{triangle} \Longrightarrow 180^0$$
$$Y, Z \Longrightarrow \text{supplementary} \Longrightarrow 180^0$$
$$Z = W + X$$

Student F

$Y + Z = 180^0$ ... (**i**)	Y and Z are supplementary.
$W + X + Y = 180^0$ ... (**ii**)	W, X, and Y form a triangle.
$180^0 = 180^0$	Obvious
$Y + Z = W + X + Y$	**(i)** and **(ii)**
$Z = W + X$	Subtract both sides by Y.

Student G

Draw a segment CE which is parallel to the segment AB.



Observe AD as a transversal line cutting the parallel lines AB and CE. Since W and U are corresponding angles, $W = U$.

Observe BC as a transversal line cutting the parallel lines AB and CE. Since X and V are alternate interior angles, $X = V$.

Thus, $W + X = V + U = Z$.

This proves that $Z = W + X$.

Student H

$$W + X = Z$$

$$W + X + Y = Z + Y$$

$$180^0 = 180^0$$

This proves that $Z = W + X$.