Geometry of tropical compactifications of moduli spaces

by

Nolan Schock

(Under the Direction of Valery Alexeev)

Abstract

We develop techniques for studying tropical compactifications of closed subvarieties of tori by introducing a broad class of such tropical compactifications, called *quasilinear* tropical compactifications, which satisfy a number of remarkable properties generalizing compactifications of complements of hyperplane arrangements. We apply these techniques to study the birational geometry and intersection theory of certain compactifications of moduli spaces, namely, the moduli spaces $M(r,n)$ of hyperplane arrangements and $Y(3,n)$ of marked del Pezzo surfaces. In particular, we prove a conjecture of Keel and Tevelev that the stable pair compactification of $M(r,n)$ for $r=2$ or $r=3$ and $n \leq 8$ is the log canonical compactification, and we describe the intersection theory and cohomology of tropical compactifications of $M(r,n)$ for $r=2$ or $r=3$, $n \leq 8$ and tropical compactifications of $Y(3,n)$ for $n \leq 7$.

Index words: Moduli space, tropical compactification, log canonical, tropical fan, Chow ring, hyperplane arrangement, del Pezzo surface, stable pair.

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Chapter 1

Introduction

1.1 Tropical compactifications

Let Y be a d-dimensional closed subvariety of an n-dimensional algebraic torus T. It is natural to attempt to compactify Y by compactifying T, i.e. by taking the closure $\overline{Y}$ of Y in a (possibly non-complete) toric variety $X(\Sigma)$ with torus T. One is then led to ask when the compactification $\overline{Y}$ is well-behaved, and how close the geometry of $\overline{Y}$ is to the geometry of the ambient toric variety $X(\Sigma)$.

In general one should of course not expect much in this direction; for instance, any quasiprojective variety can be viewed in this way. However, it was observed by Tevelev that there is a family of nice compactifications $\overline{Y} \subset X(\Sigma)$ are described by fans Σ supported on a combinatorial object called the *tropicalization* $\mathrm{trop}(Y)$ of Y [Tev07]. This is the support of a d-dimensional fan in $\mathbb{R}^n$ which encodes the behavior of all well-behaved compactifications of Y in toric varieties. More precisely, Tevelev defined a notion of a *tropical compactification* of

$Y \subset T$ as a compactification $\overline{Y} \subset X(\Sigma)$ such that $\overline{Y}$ is proper and the induced multiplication map $m : \overline{Y} \times T \to X(\Sigma)$ is flat and surjective; then the fan Σ is necessarily supported on $\text{trop}(Y)$ [Tev07]. The main point of this definition is that for tropical compactifications, the stratification of $X(\Sigma)$ by torus orbits pulls back to a stratification of $\overline{Y}$, which one can use to study the geometry of $\overline{Y}$. Tropical compactifications of closed subvarieties of tori are the main object of study of this dissertation.

Even for a tropical compactification $\overline{Y} \subset X(\Sigma)$, the geometry of $\overline{Y}$ can be very far from the geometry of the ambient toric variety $X(\Sigma)$. For instance, any smooth projective variety in $\mathbb{P}^n$ can be viewed as a tropical compactification of its intersection with the open dense torus. However, the prototypical example of a tropical compactification is when Y is the complement of a hyperplane arrangement, in which case $\overline{Y} \subset X(\Sigma)$ satisfies a number of remarkable properties:

1. All strata of $\overline{Y}$ are also complements of hyperplane arrangements [KP11].

2. There is an isomorphism of Chow rings $A^*(\overline{Y}) \cong A^*(X(\Sigma))$ [Gro15]. If $X(\Sigma)$ is smooth, then so is $\overline{Y}$, and one further has

$$H^*(\overline{Y}) \cong A^*(\overline{Y}) \cong A^*(X(\Sigma)) \cong H^*(X(\Sigma)).$$

The first main result of this dissertation is a generalization of these properties to a much broader class of closed subvarieties of tori, which we call *quasilinear varieties*. We show the following.

Theorem 1.1.1 (Theorem 4.3.20, Theorem 4.3.22). *Let $\overline{Y} \subset X(\Sigma)$ be a tropical compactification of a quasilinear variety $Y \subset T$. Then*

1. *All strata of $\overline{Y}$ are also quasilinear.*

2. *There is an isomorphism $A^*(X(\Sigma)) \xrightarrow{\sim} A^*(\overline{Y})$. If $X(\Sigma)$ is smooth, then so is $\overline{Y}$, and*

$$H^*(\overline{Y}) \cong A^*(\overline{Y}) \cong A^*(X(\Sigma)) \cong H^*(X(\Sigma)).$$

Our interest in quasilinear varieties comes from a desire to understand the geometry of certain compactifications of moduli spaces, as we explain now.

1.2 Compactifications of moduli spaces

The first examples of moduli spaces in algebraic geometry are the moduli space $M_{0,n}$ of n points on $\mathbb{P}^1$, and its compactification $\overline{M}_{0,n}$, the moduli space of stable n-pointed rational curves [Knu83]. It is known that $M_{0,n} \subset \overline{M}_{0,n}$ satisfies a number of remarkable properties not typically satisfied by other compactifications of moduli spaces:

1. $\overline{M}_{0,n}$ is a smooth projective variety of dimension $n-3$ [Knu83].

2. $\overline{M}_{0,n}$ is the *log canonical compactification* of $M_{0,n}$; roughly speaking, this is the smallest compactification with reasonable boundary singularities [KT06].

3. The intersection theory of $\overline{M}_{0,n}$ has an explicit presentation, and looks like the intersection theory of a toric variety [Kee92].

The primary goal of this dissertation is to study the analogues of these properties for compactifications of certain moduli spaces of higher-dimensional varieties. As we will later explain, the fundamental reason why these properties hold is that $M_{0,n}$ is a complement of a hyperplane arrangement, and $\overline{M}_{0,n}$ is a tropical compactification given by the coarsest fan structure on $\operatorname{trop}(M_{0,n})$. The higher-dimensional analogues of these properties will be obtained by realizing the particular moduli spaces of higher-dimensional varieties not as complements of hyperplane arrangements, but as *quasilinear varieties* as discussed above.

1.2.1 Moduli of hyperplane arrangements

From the point of view of modular compactifications of moduli spaces and the minimal model program, the natural higher-dimensional generalization of $M_{0,n} \subset \overline{M}_{0,n}$ is the moduli space $\overline{M}(r,n)$ of *stable hyperplane arrangements*, compactifying the space $M(r,n)$ of arrangements of n hyperplanes in $\mathbb{P}^{r-1}$ in general position [HKT06; Ale15]. (Indeed, $\overline{M}(2,n) = \overline{M}_{0,n}$.) Unfortunately, the geometry of $\overline{M}(r,n)$ is typically much more complicated than that of $\overline{M}_{0,n}$: in general, $\overline{M}(r,n)$ has multiple irreducible components, strata of its main irreducible component $\overline{M}^m(r,n)$ can have arbitrary singularities, and $\overline{M}^m(r,n)$ is usually not the log canonical compactification [HKT06; KT06]. Keel and Tevelev have conjectured that the compactification $\overline{M}^m(r,n)$ is well-behaved only in a few small cases.

Conjecture 1.2.1 ([KT06, Conjecture 1.6]). *The compactification $M(r,n) \subset \overline{M}^m(r,n)$ is the log canonical compactification $\iff$ $r = 2$ or $r = 3$ and $n \leq 8$ (as well as the cases occuring by duality $\overline{M}^m(r,n) \cong \overline{M}^m(n-r,n)$).*

The forward direction of this conjecture was shown by Keel and Tevelev [KT06]: they showed that $M(r,n) \subset \overline{M}^m(r,n)$ is not log canonical for $r=3$, $n \geq 9$ and $r \geq 4$, $n \geq 8$. In the backward direction, the case $r=2$ is well-known as mentioned above, cf. [KT06], and the cases $r=3$, $n \leq 7$ have also been shown via work of Luxton [Lux08] and Corey [Cor21]. As an application of Theorem 1.1.1, we settle the remaining case and unify the proofs of the previous cases.

Theorem 1.2.2 (Theorem 7.2.3)**.** *If $r=2$ or $r=3$ and $n \leq 8$, then $M(r,n)$ is a quasilinear variety.*

In the cases of the theorem, $\overline{M}^m(r,n)$ is a tropical compactification of $M(r,n)$ given by the coarsest fan structure on $\operatorname{trop}(M(r,n))$ (Theorem 7.2.4). This implies the solution to Keel and Tevelev's conjecture.

Corollary 1.2.3 (Theorem 7.2.2)**.** *If $r=2$ or $r=3$ and $n \leq 8$, then $\overline{M}^m(r,n)$ is normal, has toroidal singularities, and is the log canonical compactification of $M(r,n)$.*

The intersection theory of $\overline{M}(r,n)$ is of great interest, as it gives the first higher-dimensional version of the intersection theory of the moduli space of curves. As another consequence of Theorems 1.1.1 and 1.2.2, we are able to describe the intersection theory in the above cases.

Corollary 1.2.4 (Theorem 7.3.1)**.** *Assume $r=2$ or $r=3$ and $n \leq 8$. Let $\overline{M}^\Sigma(r,n) \subset X(\Sigma)$ be any tropical compactification of $M(r,n)$. Then $A^*(\overline{M}^\Sigma(r,n)) \cong A^*(X(\Sigma))$. If $X(\Sigma)$ is smooth, then $\overline{M}^\Sigma(r,n)$ is a resolution of singularities of $\overline{M}^m(r,n)$, and*

$$H^*(\overline{M}^\Sigma(r,n)) \cong A^*(\overline{M}^\Sigma(r,n)) \cong A^*(X(\Sigma)) \cong H^*(X(\Sigma)).$$

The Chow ring of a (nonsingular) toric variety has a well-known explicit presentation [Bri96; KP08]. Thus the above corollary in principle describes the intersection theory of $\overline{M}^m(r,n)$ and its resolutions, but in practice the fans Σ supported on $\mathrm{trop}(M(r,n))$ are too complicated to be useful. Nevertheless in Section 7.4 we completely describe the situation for the first nontrivial higher-dimensional case, namely, $\overline{M}(3,6)$.

1.2.2 Moduli of marked del Pezzo surfaces

From a combinatorial point of view, a better-behaved higher-dimensional generalization of $M_{0,n}$ is the moduli space $Y(3,n)$ of marked del Pezzo surfaces of degree $9-n$ ($n \leq 8$). In [HKT09], Hacking, Keel, and Tevelev constructed, for $n \leq 7$, the log canonical compactification $\overline{Y}(3,n)$ as a tropical compactification using combinatorics of the root system E_n. As another application of Theorem 1.1.1, we describe the intersection theory of $\overline{Y}(3,n)$.

Theorem 1.2.5 (Theorem 8.1.12)**.** *If $n \leq 7$, then $Y(3,n)$ is a quasilinear variety.*

Corollary 1.2.6 (Theorem 8.2.1)**.** *Assume $n \leq 7$. Let $\overline{Y}^{\Sigma}(3,n) \subset X(\Sigma)$ be any tropical compactification of $Y(3,n)$. Then $A^*(\overline{Y}^{\Sigma}(3,n)) \cong A^*(X(\Sigma))$. If $X(\Sigma)$ is smooth, then so is $\overline{Y}^{\Sigma}(3,n)$, and*

$$H^*(\overline{Y}^{\Sigma}(r,n)) \cong A^*(\overline{Y}^{\Sigma}(r,n)) \cong A^*(X(\Sigma)) \cong H^*(X(\Sigma)).$$

In particular, we have an explicit presentation of the Chow ring of the log canonical compactification $\overline{Y}(3,n)$.

Remark 1.2.7. We expect the above results also hold for $n = 8$, but this is currently only a conjecture.

Remark 1.2.8. We also expect that stable pair compactifications of $Y(3, n)$ (or more precisely, of the open subvariety $Y^\circ(3, n)$ parameterizing marked del Pezzo surfaces where the union of the (-1)-curves is a normal crossings divisor) can be described as tropical compactifications. This is known for $n = 5, 6$ [HKT09; GKS21]. Thus the above corollary would also describe the intersection theory of moduli of stable marked del Pezzo surfaces.

1.3 Outline

This dissertation is organized as follows. There are two main parts.

1. In Part I we study tropical geometry and geometry of tropical intersection theory, leading to the notion of quasilinear tropical compactifications mentioned above.
   (a) Chapter 2 is a review of background material in tropical geometry. We introduce the basic objects in tropical geometry, tropical fan cycles, and their natural tropical Chow homology and cohomology groups. We also discuss the notion of a tropical modification, which was first introduced by Mikhalkin in [Mik07b], and will be our main technical tool in Part I of this dissertation.
   (b) Chapter 3 is a review of background material on tropicalizations and tropical compactifications of closed subvarieties of tori.

(c) Chapter 4 discusses tropical intersection theory and its connections to the intersection theory of tropical compactifications. This chapter contains the first original results of this dissertation. The chapter is split into three sections.

   i. Section 4.1 is a review of some basic notions in intersection theory, especially intersection theory of toric varieties.

   ii. In Section 4.2 we introduce the general setup of tropical intersection theory of tropical fan cycles, and review previous work by Fulton and Sturmfels [FS97], Allermann and Rau [AR10], and Shaw [Sha13], on classes of tropical fan cycles which admit a well-behaved tropical intersection theory. Then in Section 4.2.3, we introduce a new, broader class of such tropical fan cycles, called *quasilinear tropical fan cycles*.

   iii. In Section 4.3 we prove a general theorem giving criteria for the intersection theory of a tropical compactification to be the same as the tropical intersection theory of the corresponding tropical fan (Theorem 4.3.3). We review the previous setting for which such a result was known to hold, namely, compactifications of complements of hyperplane arrangements, and then in Section 4.3.2, we define and prove the main properties of our primary objects of interest: *quasilinear tropical compactifications*.

2. In Part II we turn our attention to compactifications of moduli spaces.

   (a) In Chapter 5, we review the basics of log canonical compactifications, and stable pair compactifications of moduli spaces.

(b) In Chapter 6, we review the well-understood case of the compactification $M_{0,n} \subset \overline{M}_{0,n}$.

(c) In Chapter 7, we study the moduli space of stable hyperplane arrangements. This starts with a brief review of the definitions and construction in Section 7.1, and then in the remaining sections we prove our main results about $\overline{M}(r, n)$ as mentioned above.

(d) In Chapter 8, we study tropical compactifications of the moduli space of marked del Pezzo surfaces. We begin with a summary of the construction of the log canonical compactification by Hacking, Keel, and Tevelev [HKT09], and then we prove our main results on the intersection theory of tropical compactifications of the moduli space of marked del Pezzo surfaces.

The original work of this dissertation appears in the preprint [Sch21], and the article [Sch22].

1.4 Setup and notation

1.4.1 Algebraic geometry

We assume familiarity with algebraic geometry and unless otherwise stated work over the field $\mathbb{C}$. (This is largely done for simplicity of exposition; many of the results of this dissertation can be adapted to other base fields without great difficulty.) Unless otherwise stated we assume varieties are irreducible, i.e. that a variety is an integral finite-type scheme over $\mathbb{C}$.

1.4.2 Toric geometry

We will assume some basic familiarity with fans and toric varieties as described in [Ful93; CLS11]. We mostly follow the notations of [Ful93; AP21].

In particular throughout this dissertation we fix a torus $T \cong (\mathbb{C}^*)^n$, and denote by $N = \mathrm{Hom}(\mathbb{C}^*, T)$ and $M = \mathrm{Hom}(T, \mathbb{C}^*)$ the cocharacter and character lattices respectively; additionally we write $N_\mathbb{R} = N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$. By a *fan* Σ in $N_\mathbb{R}$ we mean a collection of finitely many strongly convex rational polyhedral cones, such that

1. each face of a cone in Σ is also a cone in Σ, and

2. the intersection of two cones in Σ is a face of each.

We denote by $X(\Sigma)$ the corresponding toric variety. For a cone $\sigma \in \Sigma$, we write $O(\sigma) \subset X(\Sigma)$ for the corresponding torus orbit and $V(\sigma)$ for its closure.

A cone σ is *simplicial* if it is generated by linearly independent vectors. If its generators additionally form part of a basis of N, then σ is called *unimodular* [AP21] or *strictly simplicial* [HKT09]. The fan Σ is *simplicial* (resp. *unimodular*) if each cone of Σ is simplicial (resp. unimodular). Thus simplicial (resp. unimodular) fans correspond to simplicial (resp. nonsingular) toric varieties.

For a cone σ, we write $N_\sigma = \langle \sigma \rangle \cap N$ for the sublattice of N spanned by σ. The *dimension* of the cone σ is the dimension of N_σ. We write Σ_k for the set of k-dimensional cones of a fan Σ, and $|\Sigma|$ for the *support* of Σ, i.e. the union of all cones of Σ.

A cone of Σ is *maximal* if it is not contained in any other cone of Σ. The *dimension* of Σ is the largest dimension of a maximal cone of Σ. The fan Σ is *pure-dimensional* if all maximal cones have the same dimension.

If σ, τ are cones of Σ with τ a face of σ, then we write $\tau \prec \sigma$, and we write $n_{\sigma,\tau}$ for any lattice point in the relative interior of σ whose image generates the one-dimensional quotient lattice N_σ/N_τ. Additionally we write Σ^σ for the *star fan* of Σ at σ, i.e. the fan in $N_{\sigma,\mathbb{R}}$ whose cones are the images of the cones of Σ which have σ as a face.

A *subfan* Δ of a fan Σ is a fan such that $|\Delta| \subset |\Sigma|$ and every cone of Δ is contained in a cone of Σ. A *refinement* of Σ is a subfan Δ such that $|\Delta| = |\Sigma|$.

Let ρ be a ray generated by a vector in the relative interior of a cone σ of a fan Σ. The *stellar refinement* of Σ with respect to ρ is the refinement $\Sigma_{(\rho)}$ obtained by replacing the cone σ with the collection of cones $\tau + \rho$ for τ a face of σ. The stellar coarsening is the reverse of the stellar refinement. Stellar refinements and coarsenings correspond to toric blowups and blowdowns in toric geometry. A *stellar refinement* or *coarsening* of Σ at σ is any stellar refinement or coarsening of Σ with respect to a ray ρ generated by a vector in the relative interior of σ. Further details of these constructions are not important for our purposes; we refer to e.g. [Ful93; CLS11; AP21] for more details.

Remark 1.4.1. It is occasionally useful in tropical geometry to consider more generally fans where the cones are not necessarily strongly convex [GKM09, Example 2.5(i)]. Since we are only interested in fans which correspond to toric varieties, we restrict to fans as defined above.

It is also occasionally useful to consider fans with infinitely many cones, but again we will not be concerned with such fans.

Part I

Geometry of tropical compactifications

Chapter 2

Tropical geometry

In this expository chapter we study the tropical geometry of fans as a subject in its own right, following mainly [Mik07b; GKM09; AR10; AP21]; see the book in progress [MR18] for a more detailed and general introduction to this perspective on tropical geometry.

2.1 Tropical fans and fan cycles

Minkowski weights

Definition 2.1.1 ([FS97])**.** A *Minkowski weight* of dimension k on a fan Σ is a function $\omega : \Sigma_k \to \mathbb{Z}$ such that

$$\sum_{\substack{\sigma\in\Sigma_k \\ \sigma\succ\tau}} \omega(\sigma) n_{\sigma,\tau} = 0 \mod N_\tau \tag{2.1}$$

for all $\tau \in \Sigma_{k-1}$. The relation (2.1) is referred as the *balancing condition*. Write $M_k(\Sigma)$ for the group of Minkowski weights of dimension k on Σ, and $M_*(\Sigma) = \bigoplus_k M_k(\Sigma)$.

Minkowski weights were introduced by Fulton and Sturmfels to give a combinatorial description of the intersection theory of toric varieties [FS97]. One motivation for this is given by the following proposition, which will be explained in Section 4.1.2.

Proposition 2.1.2 ([FS97])**.** *Let* Σ *be any fan in* $N_{\mathbb{R}}$*. Then*

$$M_k(\Sigma) \cong \mathrm{Hom}(A_{n-k}(X(\Sigma)), \mathbb{Z}).$$

From the perspective of tropical geometry, the groups $M_k(\Sigma)$ are the more natural Chow homology groups compared to the classical Chow groups $A_k(X(\Sigma))$.

Remark 2.1.3. Since $M_*(\Sigma)$ is dual to $A_*(X(\Sigma))$, from the toric perspective $M_*(\Sigma)$ is perhaps better interpreted as the Chow *co*homology, cf. Section 4.1. But the toric side is also dual to the tropical side, so from the tropical perspective $M_*(\Sigma)$ is best thought of as the Chow homology.

2.1.1 Tropical fans

Definition 2.1.4 ([GKM09; AR10])**.** A *tropical fan* is a pure d-dimensional fan $\Sigma \subset N_{\mathbb{R}}$, together with a fixed nonzero Minkowski weight $\omega : \Sigma_d \to \mathbb{Z}$ on the top-dimensional cones, called the *fundamental weight*.

Reduced tropical fans

Definition 2.1.5. A tropical fan (Σ, ω) is *reduced* if $\omega(\sigma) = 1$ for all top-dimensional cones σ.

Remark 2.1.6. The terminology differs slightly in our main references [AR10; AP21]. In [AP21] all tropical fans are assumed to be reduced. In [AR10], fans are allowed to have weight zero on some cones, and a fan is said to be reduced if all weights are nonzero. Our terminology seem more appropriate for our purposes.

Reduced tropical fans will be our primary fans of interest.

Example 2.1.7. Any complete fan Σ is a reduced tropical fan; the fundamental weight defines the degree map on the Chow group $A_0(X(\Sigma))$ of the corresponding toric variety. Thus whenever we refer to a complete fan, we view it as a reduced tropical fan.

Example 2.1.8. The 1-dimensional fan Σ pictured in Figure 2.1a cannot be made tropical with any weight, since

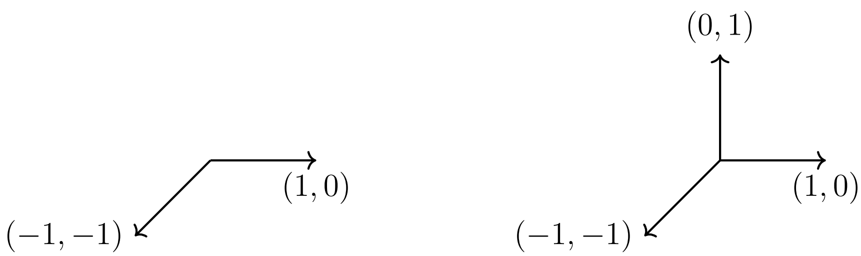
$$a(1,0) + b(-1,-1) = (a-b,-b)$$

is nonzero unless $a = b = 0$. However, adding the ray through $(0,1)$ as pictured in Figure 2.1b now makes this a tropical fan with the constant weight 1, since

$$(1,0) + (-1,-1) + (0,1) = (0,0).$$

This fan is called the *standard tropical line* (in $\mathbb{R}^2$).

Example 2.1.9. The line $2y = 3x$ has a unique fan structure Σ with two rays, as pictured in Figure 2.2. This is a tropical fan with weight 1 on each ray.



(a) A fan which is not tropical. **(b)** The standard tropical line.

Figure 2.1. Tropical and not-tropical fans.

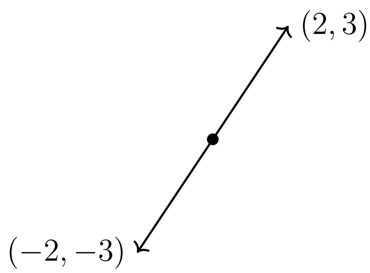


Figure 2.2. The unique reduced tropical fan structure on the line $2y = 3x$.

Example 2.1.10. The fan Σ pictured in Figure 2.3 cannot be made into a reduced tropical fan, since

$$(1,0) + (0,1) + (-1,-2) = (0,-1)$$

is nonzero. However, it is a tropical fan with the weights shown in the picture, since now

$$(1,0) + 2(0,1) + (-1,-2) = (0,0).$$

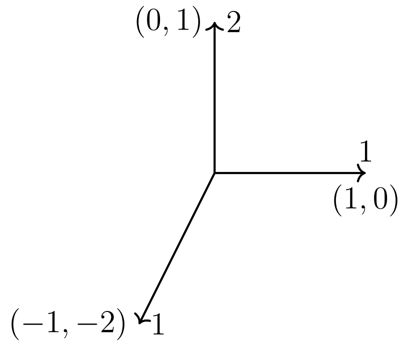


Figure 2.3. A non-reduced tropical fan.

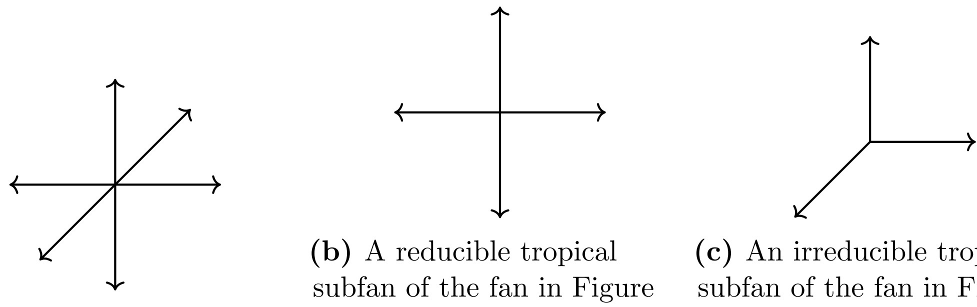
Irreducible tropical fans

Definition 2.1.11 ([GKM09])**.** A reduced tropical fan Σ is *irreducible* if there is no reduced tropical fan Σ' of the same dimension with $|\Sigma'| \subsetneq |\Sigma|$.

Example 2.1.12. All of the reduced tropical fans given in the previous examples are irreducible.

Example 2.1.13. The fan Σ pictured in Figure 2.4a is not irreducible; there are several tropical subfans of the same dimension. Some are shown in Figures 2.4b, 2.4c. Note this example also indicates that there is not an obvious well-behaved notion of the irreducible components of a tropical fan, cf. [GKM09, Remark 2.19], [AP21, Definition 3.10, Remark 3.12]

Remark 2.1.14. Alternative definitions of irreducibility are given in [RSS16; AP21]. It is easily verified that these definitions are equivalent to the one given above. (We point this out as we will later use results from [RSS16; AP21].)



(a) A reducible tropical fan.

(b) A reducible tropical subfan of the fan in Figure 2.4a.

(c) An irreducible tropical subfan of the fan in Figure 2.4a.

Figure 2.4. Some reducible and irreducible tropical fans.

One can also define irreducibility in the same way for possibly nonreduced tropical fans, but we will only need the notion in the reduced case, where it is somewhat better behaved.

Local properties

Note that if (Σ, ω) is a tropical fan, then all of its star fans are also tropical, with the fundamental weight inherited from ω.

Definition 2.1.15 ([AP21, Section 5.4])**.** A property $\mathcal{P}$ of tropical fans is *local* or *stellar-stable* if whenever a tropical fan Σ satisfies $\mathcal{P}$, all of its star fans Σ^σ also satisfy $\mathcal{P}$.

In particular the property of being reduced is automatically local. On the other hand, irreducibility is not a local property, see [AP21, Example 11.3].

Definition 2.1.16 ([GKM09; RSS16; AP21])**.** A reduced tropical fan Σ is *locally irreducible* if all of its star fans are irreducible.

Example 2.1.17. All of the irreducible tropical fans given in the previous examples are also locally irreducible.

Products

Example 2.1.18. If (Σ, ω) and (Σ', ω') are two tropical fans, then their product $\widetilde{\Sigma} = \Sigma \times \Sigma'$ is also tropical, with weight $\widetilde{\omega}$ given by $\widetilde{\omega}(\sigma \times \sigma') = \omega(\sigma)\omega(\sigma')$ [GKM09, Example 2.9(iii)]. In particular the product of reduced tropical fans is reduced. The product of (locally) irreducible tropical fans is also (locally) irreducible [GKM09, Proposition 2.20].

2.1.2 Tropical fan cycles

Tropical fan cycles on $N_{\mathbb{R}}$

Recall that a subfan Δ of a fan Σ is a fan such that $|\Delta| \subset |\Sigma|$ and each cone of Δ is contained in a cone of Σ. For a cone $\tau \in \Delta$, we denote by $\tau_{\Delta,\Sigma}$ the unique inclusion-minimal cone of Σ containing τ, cf. [GKM09, Definition 2.6].

Definition 2.1.19 ([GKM09, Definition 2.10])**.** Let (Σ, ω) and $(\widetilde{\Sigma}, \widetilde{\omega})$ be two tropical fans. We say $(\widetilde{\Sigma}, \widetilde{\omega})$ is a *tropical refinement* of (Σ, ω) if $\widetilde{\Sigma}$ is a refinement of Σ (i.e. a subfan of Σ with the same support), and

$$\widetilde{\omega}(\widetilde{\sigma}) = \omega(\widetilde{\sigma}_{\Delta,\Sigma}) \text{ for all maximal cones } \widetilde{\sigma} \in \widetilde{\Sigma}.$$

Remark 2.1.20. Suppose in the above definition that the weights ω and $\widetilde{\omega}$ are not necessarily balanced (so (Σ, ω) and $(\widetilde{\Sigma}, \widetilde{\omega})$ are only weighted fans, not necessarily tropical). Then (Σ, ω) is a tropical fan $\iff$ $(\widetilde{\Sigma}, \widetilde{\omega})$ is a tropical fan [GKM09, Example 2.11(iv)]. In particular if

(Σ, ω) is a tropical fan and $\widetilde{\Sigma}$ is any refinement of Σ, then $\widetilde{\Sigma}$ is made into a tropical fan with fundamental weight $\widetilde{\omega}$ as given in Definition 2.1.19.

Say two tropical fans are *equivalent* if they have a common tropical refinement. This is indeed an equivalence relation, cf. [GKM09, Example 2.11], [AR10, Lemma 2.11].

Definition 2.1.21 ([GKM09, Definition 2.10], [AR10, Definition 2.12]). A *tropical (fan) cycle* (on $N_{\mathbb{R}}$) is an equivalence class of tropical fans (on $N_{\mathbb{R}}$) up to common tropical refinement.

A *fan structure* or *fan supported* on a tropical fan cycle is a choice of representative of the tropical fan cycle. The *support* of a tropical fan cycle is the support of any fan structure on the tropical fan cycle.

Thus we view tropical fan cycles as pairs $(\mathcal{F}, \omega)$ consisting of the support $\mathcal{F}$ of a (pure-dimensional) fan in $N_{\mathbb{R}}$, together with a nonzero function $\omega : \mathcal{F} \to \mathbb{Z}$ which makes some fan supported on $\mathcal{F}$ into a tropical fan. In particular, whenever we refer to the support of a tropical fan, we view it as a tropical fan cycle with the inherited fundamental weight.

Remark 2.1.22. In general there is no distinguished choice of representative of a tropical fan cycle, i.e. there is no coarsest fan structure on a tropical fan cycle. For instance the function $\omega : \mathbb{R}^n \to \mathbb{Z}$ sending every point to 1 makes any complete fan into a tropical fan (Example 2.1.7). See [MS15, Example 3.5.4] for a more nontrivial example. Tropical fan cycles which *do* have a coarsest fan structure will play an important role in the second part of this dissertation, cf. Section 5.1.3.

Remark 2.1.23. Likewise, not every fan whose support is a tropical fan cycle $\mathcal{F}$ necessarily defines a tropical fan structure on $\mathcal{F}$, cf. Example 2.2.3.

Remark 2.1.24. One can make the same definitions not just for fans but for rational polyhedral complexes in general [AR10, Section 5]—this is what is meant by a tropical cycle rather than a tropical fan cycle. In [AHR16] it is shown that every tropical cycle can be decomposed into a sum of tropical fan cycles and their translates, cf. Section 2.2.1, Remark 2.2.5.

Properties of tropical fan cycles

Definition 2.1.25. A property $\mathcal{P}$ of tropical fans is *intrinsic to the support* if whenever Σ and Σ' are two tropical fans supported on the same tropical fan cycle, then Σ satisfies $\mathcal{P}$ $\iff$ Σ' satisfies $\mathcal{P}$.

Remark 2.1.26. We emphasize that in the above definition, since Σ and Σ' are supported on the same tropical fan cycle, they by assumption also have the same fundamental weight.

In order for a property $\mathcal{P}$ of tropical fans to give a well-behaved property of tropical fan cycles, it is clearly necessary to ask that $\mathcal{P}$ is intrinsic to the support.

Definition 2.1.27. Let $\mathcal{P}$ be a property of tropical fan which is intrinsic to the support. Then a tropical fan cycle $\mathcal{F}$ satisfies $\mathcal{P}$ if some (hence any) fan structure on $\mathcal{F}$ satisfies $\mathcal{P}$.

Local properties

Definition 2.1.28 ([Gub12, A.6])**.** Let $\mathcal{F} \subset N_{\mathbb{R}}$ be a tropical fan cycle and $w \in \mathcal{F}$. The *local fan cycle* of $\mathcal{F}$ at w is the tropical fan cycle

$$\mathcal{F}^w = \{v \in N_{\mathbb{R}} \mid v + \varepsilon w \in \mathcal{F} \text{ for all sufficiently small } \varepsilon\}.$$

In the above definition, if Σ is a fan structure on $\mathcal{F}$ such that w is in the relative interior of a cone σ of Σ, then

$$\mathcal{F}^w = |\Sigma^\sigma| \times \mathbb{R}^{\dim \sigma},$$

hence $\mathcal{F}^w$ is indeed a tropical fan cycle with the fundamental weight inherited from $\mathcal{F}$ (cf. Example 2.1.18).

Definition 2.1.29. A property $\mathcal{P}$ of tropical fan cycles is *local* if whenever a tropical fan cycle $\mathcal{F}$ satisfies $\mathcal{P}$, all of its local fan cycles $\mathcal{F}^w$ also satisfy $\mathcal{P}$.

Let $\mathcal{P}$ be a property of tropical fans which is intrinsic to the support and local. One would like $\mathcal{P}$ to define a local property of tropical fan cycles. However, this is not necessarily well-behaved: if Σ and Σ' are two fan structures on $\mathcal{F}$ such that w is in the relative interiors of $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$, then

$$\mathcal{F}^w = |\Sigma^\sigma| \times \mathbb{R}^{\dim \sigma} = |(\Sigma')^{\sigma'}| \times \mathbb{R}^{\dim \sigma'},$$

and *a priori* one of Σ^σ or $(\Sigma')^{\sigma'}$ could satisfy $\mathcal{P}$ while the other does not. Thus $\mathcal{P}$ defines a well-behaved local property of tropical fan cycles $\iff$ $\mathcal{P}$ is *stably invariant* in the following sense.

Definition 2.1.30 ([AP21, Section 3.2.3])**.** Let $\mathcal{P}$ be a property of tropical fans which is intrinsic to the support. We say $\mathcal{P}$ is *stably invariant* if for any tropical fan Σ and positive integer k, tropical fans supported on $|\Sigma|$ satisfy $\mathcal{P}$ $\iff$ tropical fans supported on $|\Sigma| \times \mathbb{R}^k$ satisfy $\mathcal{P}$.

We summarize the above discussion in the following proposition.

Proposition 2.1.31. *A property $\mathcal{P}$ of tropical fans defines a well-behaved local property of tropical fan cycles $\iff$ $\mathcal{P}$ is intrinsic to the support, local, and stably invariant.*

Proposition 2.1.32. *The property of being a reduced (resp. locally irreducible) tropical fan is intrinsic to the support, local, and stably invariant.*

Proof. Immediate, cf. Example 2.1.18. □

Thus we can take about reduced and locally irreducible tropical fan cycles. (Irreducibility is also intrinsic to the support, so we can talk about irreducible tropical fan cycles as well, although this is not a local notion.)

When a property of tropical fans is local, there is a direct criterion to show whether it is also intrinsic to the support. Recall the definitions of stellar refinements and coarsenings from Section 1.4.2.

Theorem 2.1.33 ([AP21, Theorem 5.7])**.** *Let $\mathcal{P}$ be a local property of tropical fans. Then $\mathcal{P}$ is intrinsic to the support $\iff$ $\mathcal{P}$ is preserved by stellar refinements and coarsenings along cones whose star fans also satisfy $\mathcal{P}$.*

Proof sketch. The proof follows from the Weak Factorization Theorem for fans, which says one can move between any two fans with the same support by a sequence of stellar refinements and coarsenings. We refer to [AP21, Theorem 5.7] for more details. □

The above theorem will play an important role later in this dissertation.

2.2 Tropical fan cycles and cocycles

2.2.1 The group of tropical fan cycles

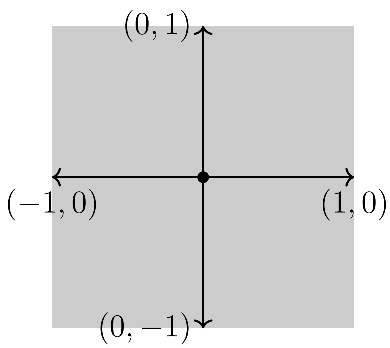
Definition 2.2.1 ([AR10, Definition 2.15])**.** Let $(\mathcal{F}, \omega)$ be a tropical fan cycle. A *k-dimensional tropical fan (sub)cycle on $\mathcal{F}$* is a k-dimensional tropical fan cycle $(\mathcal{F}', \omega')$ such that $\mathcal{F}' \subset \mathcal{F}$. Denote by $M_k(\mathcal{F})$ the group of all k-dimensional tropical fan subcycles on $\mathcal{F}$, and $M_*(\mathcal{F}) = \bigoplus_k M_k(\mathcal{F})$.

It is not obvious in the above definition that $M_k(\mathcal{F})$ is actually a group—one needs to know how to add tropical fan cycles. An explicit construction of the sum of tropical fan cycles is given in [AR10, Construction 2.13]. The basic idea is to take an appropriately fine fan structure so that the union of the two representatives is a fan, and define the fundamental weight on the union by the sum of the fundamental weights. Here we give a less explicit construction which will be more relevant to our perspective, cf. [GS21]. First we define tropical fan cycles on a tropical fan.

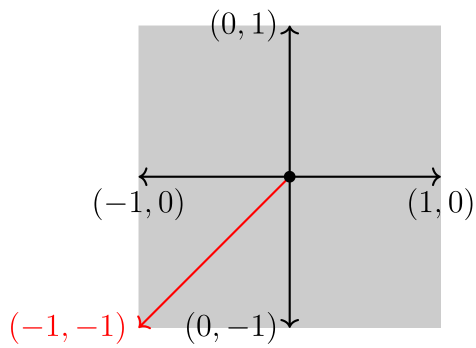
Definition 2.2.2. Let (Σ, ω) be a tropical fan. The *group of k-dimensional tropical fan cycles* on Σ is the group $M_k(\Sigma)$ of k-dimensional Minkowski weights on Σ.

Indeed, a k-dimensional Minkowski weight $\omega' \in M_k(\Sigma)$ defines a k-dimensional tropical fan cycle with support contained in $|\Sigma|$, as the equivalence class of the tropical fan (Σ', ω'), where Σ' is the union of the k-dimensional cones of Σ on which ω' is nonzero. But a tropical fan cycle whose support is contained in $|\Sigma|$ does not necessarily define a Minkowski weight on Σ, as some refinements may be necessary (Example 2.2.3). So $M_k(\Sigma)$ only captures those tropical fan cycles with a representative Σ' such that every cone of Σ' is a cone of Σ. For this reason we prefer tropical fan cycles on $N_{\mathbb{R}}$ to tropical fans in $N_{\mathbb{R}}$.

Example 2.2.3. The tropical fan cycle $\mathcal{F}$ on $\mathbb{R}^2$ defined by the standard tropical line of Figure 2.1b is not a tropical fan cycle on the complete fan Σ pictured in Figure 2.5a, since not every cone of $\mathcal{F}$ is a cone of Σ—one must refine Σ to the fan $\widetilde{\Sigma}$ of Figure 2.5b.



(a) A complete tropical fan Σ.



(b) A refinement $\widetilde{\Sigma}$ of Σ such that $\mathcal{F}$ is a tropical fan cycle on $\widetilde{\Sigma}$.

Figure 2.5. A refinement necessary to describe a tropical fan cycle.

Proposition 2.2.4 ([GS21])**.** *Let $\mathcal{F}$ be a tropical fan cycle. Then*

$$M_k(\mathcal{F}) = \varinjlim_{|\Sigma|=\mathcal{F}} M_k(\Sigma) = \varinjlim_{\substack{|\Sigma|=\mathcal{F} \\ \Sigma \text{ unimodular}}} M_k(\Sigma).$$

Proof. Recall that if $\widetilde{\Sigma}$ is a refinement of Σ, then there is a natural map $M_k(\Sigma) \to M_k(\widetilde{\Sigma})$ given by Definition 2.1.19, cf. Remark 2.1.20. Thus the limits are well-defined. An element of the first limit is a function $\omega : \mathcal{F} \to \mathbb{Z}$ inducing a k-dimensional Minkowski weight on some fan structure Σ on $\mathcal{F}$. An element of the second limit is the same, except now the fan structure must be unimodular. In particular an element of the second limit is also an element of the first, and an element of the first defines an element of the second by taking a unimodular tropical refinement. Thus both limits are the same, and they are equal to $M_k(\mathcal{F})$ by definition of a tropical fan cycle. □

In particular, the proposition implies that $M_k(\mathcal{F})$ is indeed a group, with the sum of two tropical fan subcycles given by taking a sufficiently fine fan structure so that one can add the corresponding Minkowski weights.

Remark 2.2.5. The Chow homology groups $A_k(X)$ in algebraic geometry consist of algebraic cycles modulo *rational equivalence*. No such equivalence relation is necessary for tropical fan cycles. In a sense this is because $M_k(\mathcal{F})$ already takes rational equivalence of tropical cycles into account—at least for $\mathcal{F} = N_{\mathbb{R}} \cong \mathbb{R}^n$, it can be viewed as the group of tropical cycles (in the sense of the previous remark), modulo *bounded rational equivalence* [AHR16].

2.2.2 The ring of tropical fan cocycles

Definition 2.2.6. The *ring of tropical fan cocycles* $A^*(\Sigma)$ on a tropical fan Σ is the (operational [Ful98, Chapter 17]) Chow ring $A^*(X(\Sigma))$ of the corresponding toric variety.

The *ring of tropical fan cocycles* $A^*(\mathcal{F})$ on a tropical fan cycle $\mathcal{F}$ is

$$A^*(\mathcal{F}) = \varinjlim_{|\Sigma| = \mathcal{F}} A^*(\Sigma).$$

Remark 2.2.7. If $\widetilde{\Sigma}$ is a refinement of Σ, then there is an induced proper morphism $X(\widetilde{\Sigma}) \to X(\Sigma)$ of toric varieties, hence a pullback morphism $A^*(\Sigma) \to A^*(\widetilde{\Sigma})$. Thus the limit in the definition of $A^*(\mathcal{F})$ is defined.

Remark 2.2.8. If Σ is a unimodular fan, then $A^*(\Sigma)$ is isomorphic to the ring $A^*_{pw}(\Sigma) = PP^*(\Sigma)/LPP^*(\Sigma)$ of piecewise polynomials on Σ modulo piecewise linear polynomials and there is an explicit presentation of $A^*(\Sigma)$ (Theorem 4.1.15), see [Bri96; KP08]. This is the prevalent definition of $A^*(\Sigma)$ in tropical geometry [Fra12]. As in Proposition 2.2.4 the two definitions agree for $A^*(\mathcal{F})$, since we can write

$$A^*(\mathcal{F}) = \varinjlim_{\substack{|\Sigma|=\mathcal{F} \\ \Sigma \text{ unimodular}}} A^*(\Sigma).$$

However, $A^*_{pw}(\Sigma)$ is not the correct definition for fans which are not unimodular (or at least simplicial), cf. Remark 4.2.5.

2.2.3 Cap products

Recall from Proposition 2.1.2 that $M_j(\Sigma) \cong \mathrm{Hom}(A_{n-j}(X(\Sigma)), \mathbb{Z})$.

Definition 2.2.9. Let Σ be a tropical fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$. Define the *cap product*

$$A^k(\Sigma) \times M_j(\Sigma) \to M_{j-k}(\Sigma)$$

by

$$A^k(X(\Sigma)) \times \mathrm{Hom}(A_{n-j}(X(\Sigma)), \mathbb{Z}) \to \mathrm{Hom}(A_{n-j+k}(X(\Sigma)), \mathbb{Z}),$$
$$(\alpha, \omega) \mapsto (\beta \mapsto \omega(\alpha \cap \beta)),$$

where $\alpha \cap \beta$ is the usual cap product between $\alpha \in A^k(X(\Sigma))$ and $\beta \in A_{n-j+k}(X(\Sigma))$.

Remark 2.2.10. In [Fra12], Francois gives a combinatorial definition of a cap product using piecewise polynomials, see also [AR10; KP08]. When Σ is unimodular the two cap products agree (cf. Remark 2.2.8).

Definition 2.2.11. Let $\mathcal{F}$ be a tropical fan cycle. The *cap product* $\alpha \cap \beta \in M_{j-k}(\mathcal{F})$ of $\alpha \in A^k(\mathcal{F})$ and $\beta \in M_j(\mathcal{F})$ is defined by choosing a fan structure Σ on $\mathcal{F}$ sufficiently fine so that $\alpha \in A^k(\Sigma)$ and $\beta \in M_j(\Sigma)$, and taking (the equivalence class of) $\alpha \cap \beta \in M_{j-k}(\Sigma)$.

2.3 Morphisms of tropical fan cycles

2.3.1 Morphisms

For a fan $\Sigma \subset N_{\mathbb{R}}$, let $N_{\Sigma,\mathbb{R}}$ be the subspace of $N_{\mathbb{R}}$ spanned by the first lattice points of the rays of Σ. If $\widetilde{\Sigma}$ is a refinement of Σ, then any new ray of $\widetilde{\Sigma}$ lies in the span of a collection of rays of Σ, thus it also makes sense to talk about the subspace $N_{\mathcal{F},\mathbb{R}}$ spanned by a tropical fan cycle $\mathcal{F}$. We let $N_\Sigma = N \cap N_{\Sigma,\mathbb{R}}$ (resp. $N_{\mathcal{F}} = N \cap N_{\mathcal{F},\mathbb{R}}$).

Remark 2.3.1. By definition, N_Σ (resp. $N_{\mathcal{F}}$) is spanned by a subset of a basis of N, thus we do not need to worry about finite index sublattices (cf. [CLS11, Proof of Proposition 3.3.9]). This is the reason for defining N_Σ (resp. $N_{\mathcal{F}}$) as above rather than as the sublattice spanned by the rays.

We can view Σ either as a fan in N_Σ or as a fan in N; write $X(\Sigma, N_\Sigma)$ and $X(\Sigma, N)$ for the corresponding toric varieties. Extending a basis of N_Σ to a basis of N implies that

$$X(\Sigma, N) \cong X(\Sigma, N_\Sigma) \times (\mathbb{C}^*)^{\dim N - \dim N_\Sigma},$$

see [CLS11, Proposition 3.3.9]. In particular,

$$A_*(X(\Sigma, N)) \cong A_*(X(\Sigma, N_\Sigma) \times (\mathbb{C}^*)^n) \cong A_*(X(\Sigma, N_\Sigma))$$

(cf. Section 4.1), so $M_*(\Sigma)$ is independent of the choice of N or N_Σ as the ambient lattice. For this reason we can replace N with N_Σ (resp. $N_\mathcal{F}$) without affecting the tropical geometry. Thus by abuse of notation we may often write N in place of N_Σ (resp. $N_\mathcal{F}$) when notationally convenient.

Example 2.3.2. Consider the unique fan Σ on the line $2y = 3x$ as considered in Example 2.1.9. Then a basis for N_Σ is given by the vector $(2, 3)$; this can be extended to e.g. the basis $(2, 3)$, $(1, 1)$ of $N = \mathbb{Z}^2$. We have

$$X(\Sigma, N) \cong X(\Sigma, N_\Sigma) \times \mathbb{C}^* \cong \mathbb{P}^1 \times \mathbb{C}^*.$$

Definition 2.3.3. Let $\Sigma \subset N_\mathbb{R}$ and $\Sigma' \subset N'_\mathbb{R}$ be two fans. A *morphism* $f : \Sigma \to \Sigma'$ is a map $f : |\Sigma| \to |\Sigma'|$ induced by a linear map $f : N_\Sigma \to N_{\Sigma'}$, such that $f(\sigma)$ is contained in a cone of Σ' for all $\sigma \in \Sigma$ [CLS11, Section 3.3].

Let $\mathcal{F} \subset N_\mathbb{R}$ and $\mathcal{F}' \subset N'_\mathbb{R}$ be two tropical fan cycles. A *morphism* $f : \mathcal{F} \to \mathcal{F}'$ is a map $f : \mathcal{F} \to \mathcal{F}'$ induced by a morphism of fans $\Sigma \to \Sigma'$ for some fan structures Σ on $\mathcal{F}$ and Σ' on $\mathcal{F}'$.

Remark 2.3.4. There is no condition on the fundamental weights for a morphism of tropical fans or tropical fan cycles, cf. [AR10, Definition 4.1].

2.3.2 Pushforwards

Definition 2.3.5 ([GKM09; AR10])**.** Let $f : \mathcal{F} \to \mathcal{F}'$ be a morphism of tropical fan cycles. Let $\omega \in M_k(\mathcal{F})$ be a tropical fan cycle on $\mathcal{F}$. The *pushforward* $f_*\omega \in M_k(\mathcal{F}')$ is the tropical fan cycle on $\mathcal{F}'$ defined as follows. Choose fan structures Σ and Σ' on $\mathcal{F}$ and $\mathcal{F}'$ so that $f(\sigma)$ is a cone of Σ' for all $\sigma \in \Sigma$. Then for a k-dimensional cone $\sigma' \in \Sigma'$, we define

$$f_*\omega(\sigma') = \sum_{\substack{\sigma \in \Sigma \\ f(\sigma)=\sigma'}} \omega(\sigma) \cdot [N_{\sigma'} : f(N_\sigma)].$$

This is a group morphism $f_* : M_k(\mathcal{F}) \to M_k(\mathcal{F}')$.

Remark 2.3.6. In particular if $\dim \mathcal{F}' < k$ then the pushforward of a k-dimensional fan cycle on $\mathcal{F}$ is 0.

Example 2.3.7. The projection from the standard tropical line Σ in $\mathbb{R}^2$ to $\mathbb{R}^1$ given by $f(x, y) = x + y$ sends Σ to the unique fan on $\mathbb{R}^1$. There are two rays of Σ mapping to each ray of $\mathbb{R}^1$, so the weights on the pushforward are 2. On the other hand, the pushforward along the function $f(x, y) = x$ has weights 1.

Example 2.3.8. Consider the fan $\Sigma \subset \mathbb{R}^2$ from Example 2.1.10 and the projection $f : \mathbb{R}^2 \to \mathbb{R}^1$, $f(x, y) = y$. For the ray σ_3 through $(-1, -2)$, we have $N_{\sigma_3} = \{(x, 2x) \mid x \in \mathbb{Z}\}$, projecting to $f(N_{\sigma_3}) = 2\mathbb{Z} \subset \mathbb{Z}$. Thus $[N_{f(\sigma_3)} : f(N_{\sigma_3})] = 2$. Recall from Example 2.1.10 that the weights on the other two cones σ_1 and σ_2 are 1 and 2, respectively. It follows that $f_*\Sigma$ is the unique fan on $\mathbb{R}^1$, but with weight 2 rather than weight 1.

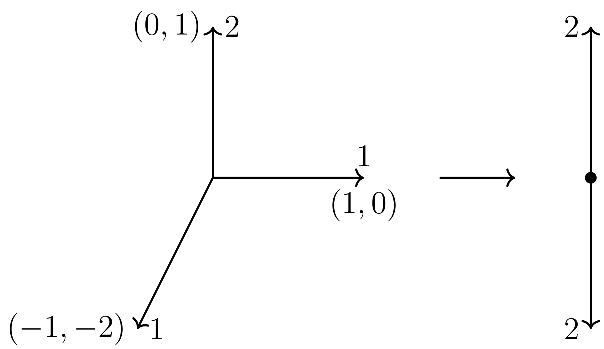


Figure 2.6. The projection $f(x, y) = y$

2.3.3 Pullback of tropical fan cocycles

Definition 2.3.9. Let $f : \mathcal{F} \to \mathcal{F}'$ be a morphism of tropical fan cycles. The *pullback* $f^* : A^k(\mathcal{F}') \to A^k(\mathcal{F})$ is obtained by choosing fan structures Σ and Σ' on $\mathcal{F}$ and $\mathcal{F}'$ so that f induces a morphism of fans, and then taking the pullback morphism $f^* : A^*(\Sigma') = A^*(X(\Sigma')) \to A^*(X(\Sigma)) = A^*(\Sigma)$.

2.3.4 Projection formula

Proposition 2.3.10 (cf. [AR10, Proposition 4.8])**.** *Let $f : \mathcal{F} \to \mathcal{F}'$ be a morphism of tropical fan cycles. Then*

$$f_*(f^*\alpha \cap \beta) = \alpha \cap f_*\beta$$

for $\alpha \in A^k(\mathcal{F}')$, $\beta \in M_j(\mathcal{F})$.

Proof. This follows from the projection formula in the classical setting for toric varieties. □

2.3.5 Isomorphisms

Definition 2.3.11. Let $(\Sigma, \omega) \subset N_{\mathbb{R}}$ and $(\Sigma', \omega') \subset N'_{\mathbb{R}}$ be two tropical fans. An *isomorphism* $f : (\Sigma, \omega) \xrightarrow{\sim} (\Sigma', \omega')$ is a morphism induced by an isomorphism $f : N_\Sigma \xrightarrow{\sim} N_{\Sigma'}$, such that $f(\sigma)$ is a cone of Σ' for all $\sigma \in \Sigma$, $f^{-1}(\sigma')$ is a cone of Σ for all $\sigma' \in \Sigma'$, $f_*\omega = \omega'$, and $(f^{-1})_*\omega' = \omega$ [AP21].

Let $(\mathcal{F}, \omega) \subset N_{\mathbb{R}}$ and $(\mathcal{F}', \omega') \subset N'_{\mathbb{R}}$ be two tropical fan cycles. An *isomorphism* $f : (\mathcal{F}, \omega) \xrightarrow{\sim} (\mathcal{F}', \omega')$ is a morphism $f : \mathcal{F} \to \mathcal{F}'$ induced by an isomorphism of tropical fans for some fan structures Σ on $\mathcal{F}$ and Σ' on $\mathcal{F}'$.

Remark 2.3.12. In our definition, an isomorphism of fans $\Sigma \subset N_{\mathbb{R}}$ and $\Sigma' \subset N'_{\mathbb{R}}$ does not necessarily induce an isomorphism of toric varietes $X(\Sigma, N) \cong X(\Sigma, N')$, but instead an isomorphism of toric varieties $X(\Sigma, N_\Sigma) \cong X(\Sigma', N_{\Sigma'})$, cf. Remark 2.3.1.

Example 2.3.13. Let (Σ, ω) be the unique reduced tropical fan on the line $2y = 3x$ as in Examples 2.1.9, 2.3.2. The map $N_\Sigma \to \mathbb{Z}$, $(x, y) \mapsto y - x$ induces an isomorphism of Σ with (the unique fan structure on) $\mathbb{R}^1$. On the other hand, the map $f : N_\Sigma \to \mathbb{Z}$, $(x, y) \mapsto x$ does not induce an isomorphism of Σ with $\mathbb{R}^1$, as in this case we have $f(N_\sigma) = 2\mathbb{Z} \subset \mathbb{Z}$ for each ray σ of Σ, so $f_*\omega(\sigma) = 2$ instead of 1. Likewise for the map $N_\Sigma \to \mathbb{Z}$, $(x, y) \mapsto y$, the pushforward of the fundamental weight is 3 instead of 1.

Proposition 2.3.14. *Suppose Σ and Σ' (resp. $\mathcal{F}$ and $\mathcal{F}'$) are isomorphic tropical fans (resp. tropical fan cycles). Then $M_*(\Sigma) \cong M_*(\Sigma')$ and $A^*(\Sigma) \cong A^*(\Sigma')$ (resp. $M_*(\mathcal{F}) \cong M_*(\mathcal{F}')$ and $A^*(\mathcal{F}) \cong A^*(\mathcal{F}')$).*

Proof. This is essentially immediate from the definitions, cf. Proposition 2.1.2 and Remarks 2.3.1, 2.3.12. ☐

2.4 Divisors and tropical modifications

2.4.1 Rational functions and tropical Cartier divisors

Definition 2.4.1 ([AR10])**.** A *rational function* on a tropical fan Σ is a piecewise integral linear function on Σ, i.e. a continuous function $\varphi : |\Sigma| \to \mathbb{R}$ such that for each cone $\sigma \in \Sigma$, $\varphi|_\sigma$ is identified with the restriction of an integral linear function $\varphi_\sigma \in M_\sigma = \mathrm{Hom}(N_\sigma, \mathbb{Z})$. The *group of rational functions* on Σ is denoted $PP^1(\Sigma)$.

A *rational function* on a tropical fan cycle $\mathcal{F}$ is a continuous function $\varphi : \mathcal{F} \to \mathbb{R}$ which is a rational function on some tropical fan structure on $\mathcal{F}$. The *group of rational functions* on $\mathcal{F}$ is denoted $PP^1(\mathcal{F})$. Thus as in Proposition 2.2.4, Remark 2.2.8,

$$PP^1(\mathcal{F}) = \varinjlim_{|\Sigma|=\mathcal{F}} PP^1(\Sigma) = \varinjlim_{\substack{|\Sigma|=\mathcal{F} \\ \Sigma \text{ unimodular}}} PP^1(\Sigma).$$

Remark 2.4.2. From the tropical perspective the rational functions as defined above are all *nonzero*; the zero function should be thought of as the constant function ∞ [AR10, Remark 3.2].

Remark 2.4.3. In [AR10], rational functions are defined not to be piecewise integral linear, but more generally to be piecewise integral *affine*, i.e. they can take the form $\varphi + c$ where φ is piecewise integral linear and $c \in \mathbb{R}$. The situation is somewhat simplified by only considering piecewise integral linear functions, and this will be the only case of interest for us.

In particular, the main construction [AR10, Construction 3.3] of the tropical Weil divisor associated to a rational function (see Definition 2.4.13 below) already only uses the linear part.

Definition 2.4.4 ([AR10])**.** A *linear function* on a tropical fan Σ (resp. a tropical fan cycle $\mathcal{F}$) is a continuous function $\varphi : |\Sigma| \to \mathbb{R}$ (resp. $\varphi : \mathcal{F} \to \mathbb{R}$) which is the restriction of a linear function $\varphi \in M = \operatorname{Hom}(N, \mathbb{Z})$. Thus the group of linear functions on Σ (resp. $\mathcal{F}$) is just M.

Proposition 2.4.5 ([KP08, Theorem 4.5])**.** *Let Σ be a tropical fan and $\mathcal{F}$ a tropical fan cycle. There are isomorphisms*

$$A^1(\Sigma) \cong PP^1(\Sigma)/M \quad \textit{and} \quad A^1(\mathcal{F}) \cong PP^1(\mathcal{F})/M.$$

Remark 2.4.6. $PP^1(\Sigma)/M$ is also identified with the Picard group of the corresponding toric variety $X(\Sigma)$, thus the above proposition implies $\operatorname{Pic} X(\Sigma) \cong A^1(X(\Sigma))$ for *any* toric variety $X(\Sigma)$ [KP08, Corollary 4.6]

Definition 2.4.7 ([AR10])**.** A *(tropical) Cartier divisor* on a tropical fan Σ or a tropical fan cycle $\mathcal{F}$ is an element of $A^1(\Sigma)$ or $A^1(\mathcal{F})$.

Remark 2.4.8. On the toric side $A^1(X(\Sigma))$ is the group of Cartier divisors modulo principal Cartier divisors. But on the tropical side $A^1(\Sigma)$ actually describes tropical Cartier divisors, not just Cartier divisor classes. This is because the linear functions on a tropical fan Σ

actually define *trivial* tropical divisors, in contrast to the nontrivial principal Cartier divisors appearing on an algebraic variety. In fact for many tropical fans of interest, all tropical divisors will be principal, cf. Definition 2.4.13. See [AR10, Section 6] for more details.

2.4.2 Tropical Weil divisors

Tropical Weil divisors

Definition 2.4.9. A *(tropical) Weil divisor* on a d-dimensional tropical fan Σ or tropical fan cycle $\mathcal{F}$ is an element of $M_{d-1}(\Sigma)$ or $M_{d-1}(\mathcal{F})$, i.e. a codimension one tropical fan cycle on Σ or $\mathcal{F}$.

Principal tropical Weil divisors

Definition 2.4.10 ([AR10])**.** Let φ be a rational function on a d-dimensional tropical fan (Σ, ω). The *order of vanishing* of φ along a cone $\tau \in \Sigma_{d-1}$ is

$$\operatorname{ord}_\tau(\varphi) = \varphi_\tau \left(\sum_{\substack{\sigma \succ \tau \\ \dim \sigma = \dim \tau + 1}} \omega(\sigma) n_{\sigma,\tau} \right) - \sum_{\substack{\sigma \succ \tau \\ \dim \sigma = \dim \tau + 1}} \varphi_\sigma(\omega(\sigma) n_{\sigma,\tau})$$

Remark 2.4.11. The cones τ where $\operatorname{ord}_\tau(\varphi)$ is positive (resp. negative) should be interpreted as the zeros (resp. poles) of the rational function φ.

Lemma 2.4.12 ([AR10, Proposition 3.7], [AP21, Propositions 4.1, 4.2])**.** *The function $\Sigma_{d-1} \to \mathbb{Z}$, $\tau \mapsto \operatorname{ord}_\tau(\varphi)$ is a well-defined (independent of the choice of $n_{\sigma,\tau}$) $(d-1)$-dimensional Minkowski weight on Σ.*

Definition 2.4.13 ([AR10, Definition 3.4])**.** Let φ be a rational function on a d-dimensional tropical fan (Σ, ω). The *principal tropical Weil divisor* associated to φ is the tropical fan cycle $\operatorname{div}(\varphi) \in M_{d-1}(\Sigma)$ defined by $\operatorname{div}(\varphi)(\tau) = \operatorname{ord}_\tau(\varphi)$.

If φ is a rational function on a d-dimensional tropical fan cycle $(\mathcal{F}, \omega)$, then the *principal tropical Weil divisor* assocated to φ is the tropical fan cycle $\operatorname{div}(\varphi) \in M_{d-1}(\mathcal{F})$ obtained by choosing a tropical fan structure Σ on $\mathcal{F}$ such that $\varphi \in A^1(\Sigma)$, and taking $\operatorname{div}(\varphi) \in M_{d-1}(\Sigma)$.

When we wish to view $\operatorname{div}(\varphi)$ as a tropical fan rather than a tropical fan cycle, we write it as $\operatorname{div}(\varphi) = (\Delta, \delta)$, where Δ is the support of $\operatorname{div}(\varphi)$, i.e the $(d-1)$-dimensional fan consisting of the cones $\tau \in \Sigma_{d-1}$ for which $\operatorname{ord}_\tau(\varphi) \neq 0$, and $\delta(\tau) = \operatorname{ord}_\tau(\varphi)$. Thus by abuse of notation $\operatorname{div}(\varphi)$ could refer to either the tropical fan cycle $\operatorname{div}(\varphi) \in M_{d-1}(\Sigma)$ or the tropical fan (Δ, δ).

Example 2.4.14. Consider the function $\varphi = \min\{2y, 3x\}$ on the complete fan $\Sigma \subset \mathbb{R}^2$ pictured in Figure 2.7. Then $\operatorname{div}(\varphi) \in M_1(\Sigma)$ has weight 4 on the rays ρ_2 and ρ_4 and weight 0 on the rays ρ_1 and ρ_3.

In particular the weights on $\operatorname{div}(\varphi)$ are not necessarily the minimal weights, as the support of $\operatorname{div}(\varphi)$ in this case can also be viewed as a reduced tropical fan, cf. Example 2.1.9.

Example 2.4.15. Consider the function $\varphi = \min\{2x, 0\}$ on the complete fan $\Sigma \subset \mathbb{R}^2$ pictured in Figure 2.8. Then $\operatorname{div}(\varphi) \in M_1(\Sigma)$ has weight 2 on the rays ρ_2 and ρ_4 and weight 0 on the rays ρ_1 and ρ_3.

Now consider function $\varphi' = \min\{x, 0\}$ on the same fan Σ. Then $\operatorname{div}(\varphi')$ has the same support as $\operatorname{div}(\varphi)$, but the weights are 1 for φ' as opposed to 2 for φ.

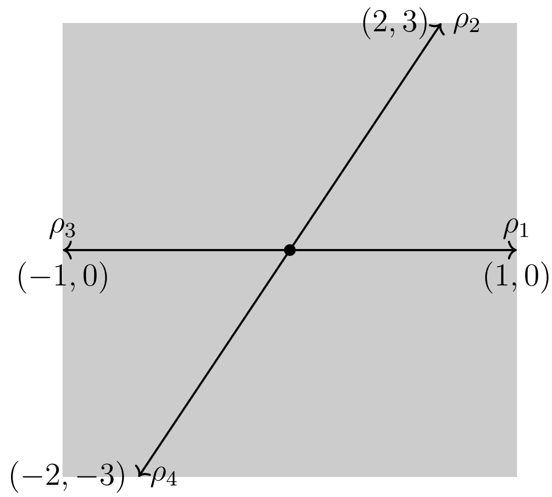


Figure 2.7. The divisor of $\varphi = \min\{2y, 3x\}$ is supported on the rays ρ_2 and ρ_4, with weight 4

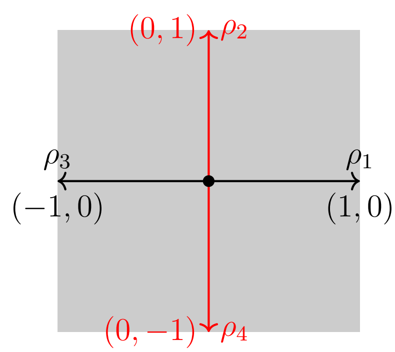


Figure 2.8. The divisor of $\varphi = \min\{2x, 0\}$ is supported on the rays ρ_2 and ρ_4, with weight 2. The divisor of $\varphi' = \min\{x, 0\}$ has the same support but weight 1.

Example 2.4.16. Not every tropical Weil divisor on a tropical fan need be principal. Consider the tropical fan $\Sigma \subset \mathbb{R}^3$ pictured in Figure 2.9. Then the reduced tropical divisor $\{y = z = 0\}$ pictured in red in Figure 2.9 is not principal [AP21, Example 11.5].

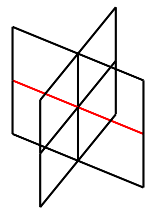


Figure 2.9. A non-principal divisor on a tropical fan.

Definition 2.4.13 defines a map $\text{div} : A^1(\Sigma) \to M_{d-1}(\Sigma)$. Another such map is given by the cap product discussed in Section 2.2.3. These two maps are related as follows.

Proposition 2.4.17 ([AP21, Proposition 4.8])**.** *Let* (Σ, ω) *be a d-dimensional tropical fan and let* $\varphi \in A^1(\Sigma)$*. Then*

$$\varphi \cap \omega = -\operatorname{div}(\varphi)$$

The same holds if instead Σ *is a tropical fan cycle* $\mathcal{F}$*.*

Example 2.4.18. Consider the function $\varphi = \min\{x, y, 0\}$ on the complete fan $\Sigma \subset \mathbb{R}^2$ pictured in Figure 2.10. Then $\text{ord}_{\rho_i}(\varphi) = 1$ for $i = 1, 2, 3$, so $\text{div}(\varphi) \in M_1(\Sigma)$ is the Minkowski weight given by $\text{div}(\varphi)(\rho_i) = 1$ for $i = 1, 2, 3$. On the other hand, one can compute $\varphi \cap \omega$ using the formula given in [KP08, Theorem 1.4]. In this case one finds that $\varphi \cap \omega \in M_1(\Sigma)$ is the Minkowski weight given by $(\varphi \cap \omega)(\rho_i) = -1$ for $i = 1, 2, 3$. Thus $\varphi \cap \omega = -\operatorname{div}(\varphi)$.

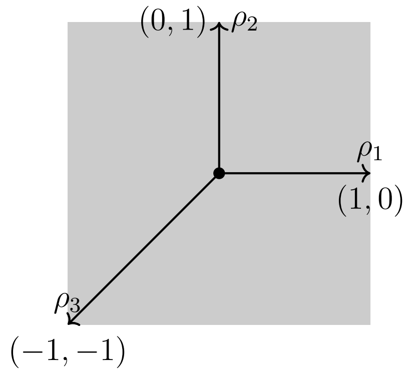


Figure 2.10. The divisor of $\varphi = \min\{2x, 0\}$ is supported on the rays ρ_2 and ρ_4, with weight 2. The divisor of $\varphi' = \min\{x, 0\}$ has the same support but weight 1.

Remark 2.4.19. Our definition of $\mathrm{ord}_\tau(\varphi)$ and $\mathrm{div}(\varphi)$ agrees with the definition given in [AP21] (in the case of reduced fans) but is the negative of the one given in [AR10]. This is because our perspective on tropical geometry is dual to the perspective taken in [AR10]—we work with the "min-plus" convention, while [AR10] works with the "max-plus" convention. The above proposition gives one reason why one might prefer the "max-plus" convention; however, we use the "min-plus" convention so that our pictures of tropical fans match up with the usual pictures of fans associated to toric varieties.

2.4.3 Tropical modifications

The following definition originally due to Mikhalkin [Mik07b] will play a fundamental role in this dissertation.

Definition 2.4.20 ([Mik07b; AR10; AP21])**.** The *tropical modification* of a tropical fan (Σ, ω) in $N_\mathbb{R}$ with respect to a rational function φ is the tropical fan $\mathcal{TM}_\varphi(\Sigma) = (\widetilde{\Sigma}, \widetilde{\omega})$ in $\widetilde{N}_\mathbb{R} = N_\mathbb{R} \times \mathbb{R}$ with cones

1. $\widetilde{\sigma} = \{(x, \varphi(x)) \mid x \in \sigma\}$ for $\sigma \in \Sigma$, and

2. $\tau_\geq = \widetilde{\tau} + \{0\} \times \mathbb{R}_{\geq 0}$ for $\tau \in \mathrm{div}(\varphi)$,

and weights $\widetilde{\omega}(\widetilde{\sigma}) = \omega(\sigma)$ and $\widetilde{\omega}(\tau_\geq) = \mathrm{ord}_\tau(\varphi)$.

If φ is a rational function on a tropical fan cycle $\mathcal{F}$ in $N_\mathbb{R}$, then the tropical modification $\mathcal{TM}_\varphi(\mathcal{F})$ is the tropical fan cycle in $\widetilde{N}_\mathbb{R}$ obtained by first choosing a fan structure Σ on $\mathcal{F}$ such that $\varphi \in A^1(\Sigma)$, and then taking the equivalence class up to common tropical refinement of the tropical modification $\mathcal{TM}_\varphi(\Sigma)$.

One needs to check that the tropical modification is actually a tropical fan (resp. tropical fan cycle).

Lemma 2.4.21 ([Mik07b; AR10; AP21])**.** *Let Σ (resp. $\mathcal{F}$) be a tropical fan (resp. tropical fan cycle), and φ a rational function on Σ (resp. $\mathcal{F}$). Then $\mathcal{TM}_\varphi(\Sigma)$ (resp. $\mathcal{TM}_\varphi(\mathcal{F})$) is a tropical fan (resp. tropical fan cycle), and the natural projection $p : \widetilde{N}_\mathbb{R} \to N_\mathbb{R}$ induces a morphism of tropical fans (resp. tropical fan cycles) $p : \mathcal{TM}_\varphi(\Sigma) \to \Sigma$ (resp. $p : \mathcal{TM}_\varphi(\mathcal{F}) \to \mathcal{F}$).*

Tropical modifications are the tropical versions of graphs of rational functions on an algebraic variety. If φ is a rational function on a fan Σ, then the graph of φ, though still a fan, may no longer tropical—the induced weight can to satisfy the balancing condition. The cones where $\mathrm{ord}_\tau(\varphi)$ is nonzero are precisely the cones where the graph fails the balancing condition, and adding the cones $\tau_{\geq}$ for $\tau \in \mathrm{div}(\varphi)$ restores the balancing condition [AR10, Construction 3.3]. This is the motivation for Definitions 2.4.10 and 2.4.20, and the proof of Lemma 2.4.21 essentially follows by working out the details of this construction. See [AR10, Construction 3.3], [AP21, Section 5.1] for more details.

Example 2.4.22. Consider the function $\varphi(x) = \min\{x, 0\}$ on $\mathbb{R}^1$. Note $\mathrm{ord}_0(\varphi) = 1$. The graph of φ is the fan pictured in Figure 2.1a, which is not tropical. The fan is made tropical by adding the ray through $(0, 1)$ as in Figure 2.1b. This is the tropical modification of $\mathbb{R}^1$ with respect to the function $\varphi(x)$.

Example 2.4.23. The tropical modification of $\mathbb{R}^1$ with respect to the function $\varphi(x) = \min\{2x, 0\}$ is the fan of Example 2.1.10. In particular note $\mathrm{div}(\varphi)$ has weight 2 rather than weight 1, reflecting that the tropical modification is not reduced.

Example 2.4.24. The tropical modification of the standard tropical line with respect to the function $\varphi(x, y) = \min\{x, y, 0\}$ is shown in Figure 2.11. This could be called the standard tropical line *in* $\mathbb{R}^3$.

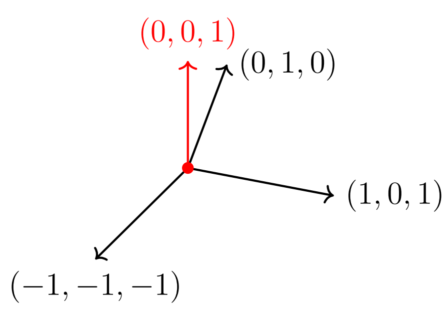


Figure 2.11. The tropical modification of the standard tropical line with respect to $\varphi(x, y) = \min\{x, y, 0\}$.

Example 2.4.25. The tropical modification of $\mathbb{R}^2$ with respect to the function $\varphi(x, y) = \min\{2x, 0\}$ is pictured in Figure 2.12a. Since the weights on $\text{div}(\varphi)$ are 2 (Example 2.4.15), the weights on the two added maximal cones of the tropical modification are also 2.

On the other hand, the tropical modification of $\mathbb{R}^2$ with respect to the function $\varphi'(x, y) = \min\{x, 0\}$ is shown in Figure 2.12b. It looks similar to the tropical modification with respect to φ, since $\text{div}(\varphi)$ and $\text{div}(\varphi')$ have the same support. However, the weights on $\text{div}(\varphi')$ are 1, so the tropical modification with respect to φ' is reduced.

(a) The tropical modification of $\mathbb{R}^2$ with respect to $\varphi(x, y) = \min\{2x, 0\}$.

(b) The tropical modification of $\mathbb{R}^2$ with respect to $\varphi'(x, y) = \min\{x, 0\}$.

Figure 2.12. Two distinct tropical modifications along two divisor with the same unweighted support.

Example 2.4.26. The tropical modification of $\mathbb{R}^2$ with respect to the function $\varphi(x, y) = \min\{x, y, 0\}$ is the two-dimensional tropical fan cycle in $\mathbb{R}^3$ as pictured in Figure 2.13. Since $\text{div}(\varphi)$ is reduced by Example 2.4.18, it follows that this tropical modification is also reduced.

Example 2.4.27. Suppose $\mathcal{F}$ is a tropical fan cycle obtained as a tropical modification. Not every fan structure on $\mathcal{F}$ need be obtained as a tropical modification as well. For instance, let Σ be the tropical modification of the previous example, and let $\widetilde{\Sigma}$ be the refinement obtained

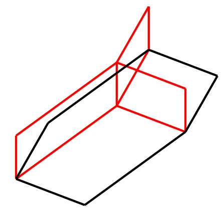


Figure 2.13. The tropical modification of $\mathbb{R}^2$ with respect to $\varphi(x,y) = \min\{x,y,0\}$.

by adding an extra ray as shown in blue in Figure 2.14. Then there is no way to write $\widetilde{\Sigma}$ as a tropical modification.

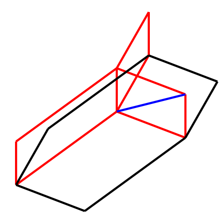


Figure 2.14. A fan structure which cannot be written as a tropical modification.

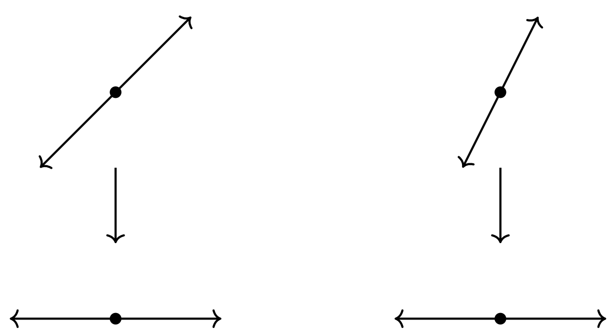
Remark 2.4.28. The tropical modification crucially depends on the rational function φ and not just the underlying tropical Weil divisor $\operatorname{div}(\varphi)$, as Example 2.4.25 shows, see also [BM12, Example 4.2]

On the other hand, if $\operatorname{div}(\varphi)$ is reduced and $\operatorname{div} : A^1(\Sigma) \to M_{d-1}(\Sigma)$ is an isomorphism then this does not happen. Indeed, in this case if φ and φ' are two rational functions with $\operatorname{div}(\varphi) = \operatorname{div}(\varphi')$ a reduced principal tropical Weil divisor, then φ and φ' differ by a linear function, which induces an isomorphism between the two tropical modifications [AP21, Section 5.1.5]. In this situation, we write $\mathcal{TM}_\Delta(\Sigma)$ (resp. $\mathcal{TM}_\mathcal{D}(\mathcal{F})$) for any tropical modification of Σ (resp. $\mathcal{F}$) along a rational function whose corresponding principal tropical Weil divisor is the *reduced* divisor Δ (resp. $\mathcal{D}$). This will be the primary situation of interest.

Remark 2.4.29. A particular special case is when the divisor $\text{div}(\varphi)$ is trivial, i.e. has empty support, in which case the tropical modification is said to be *degenerate* [AP21, Section 5.1.4]. In general a degenerate tropical modification can be different from the original tropical fan cycle, and somehow seems to be a "desingularization," with the appropriate notion of nonsingularity [AP21, Example 11.4]. On the other hand, the main situation where degenerate tropical modifications occur is when φ is linear, in which case $\mathcal{TM}_\varphi(\mathcal{F}) \cong \mathcal{F}$ [AP21, Section 5.1.4]. This will again be our primary situation of interest.

Example 2.4.30. The tropical modification of $\mathbb{R}^1$ with respect to the function $\varphi(x) = x$ is a degenerate tropical modification pictured in Figure 2.15a. It is the line $y = x$.

The tropical modification of $\mathbb{R}^1$ with respect to the function $\varphi(x) = 2x$ is also a degenerate tropical modification, pictured in Figure 2.15b. It is the line $y = 2x$. Note this modification is still a reduced tropical fan, isomorphic to $\mathbb{R}^1$ via the projection $\mathbb{R}^2 \to \mathbb{R}^1$, $(x, y) \mapsto x$.



(a) The degenerate tropical modification of $\mathbb{R}^1$ with respect to $\varphi(x) = x$

(b) The degenerate tropical modification of $\mathbb{R}^1$ with respect to $\varphi(x) = 2x$.

Figure 2.15. Two isomorphic degenerate tropical modifications.

Example 2.4.31. The tropical modification of the standard tropical line with respect to the function $\varphi(x, y) = x$ is a degenerate tropical modification pictured in Figure 2.16. We remark that this is also isomorphic to the (nondegenerate) tropical modification of the line $y = x$ at the origin.

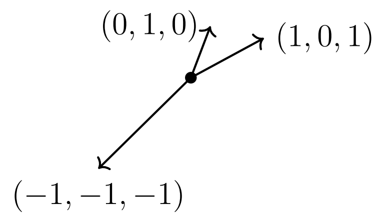


Figure 2.16. The degenerate tropical modification of the standard tropical line with respect to $\varphi(x, y) = x$.

Star fans of tropical modifications

Proposition 2.4.32 ([AP21, Proposition 5.2])**.** *Let φ be a rational function on a tropical fan Σ, and let $\widetilde{\Sigma} = \mathcal{TM}_\varphi(\Sigma)$. Let $\Delta = \operatorname{div}(\varphi)$ (possibly trivial). The star fans of $\widetilde{\Sigma}$ are described as follows.*

1. *If $\sigma \in \Sigma$, then $\widetilde{\Sigma}^{\widetilde{\sigma}} \cong \mathcal{TM}_{\varphi_\sigma}(\Sigma^\sigma)$. In particular, if $\sigma \notin \Delta$, then $\widetilde{\Sigma}^{\widetilde{\sigma}}$ is a degenerate tropical modification of Σ^σ.*

2. *If $\tau \in \Delta$, then $\widetilde{\Sigma}^{\tau_\geq} \cong \Delta^\tau$.*

Properties preserved by tropical modifications

Definition 2.4.33. A property $\mathcal{P}$ of tropical fans (resp. tropical fan cycles) is *preserved by tropical modifications* if whenever Σ (resp. $\mathcal{F}$) is a tropical fan (resp. tropical fan cycle) satisfying $\mathcal{P}$ and φ is a rational function on Σ (resp. $\mathcal{F}$) such that $\mathrm{div}(\varphi)$ is either trivial or also satisfies $\mathcal{P}$, then $\mathcal{TM}_\varphi(\Sigma)$ (resp. $\mathcal{TM}_\varphi(\mathcal{F})$) satisfies $\mathcal{P}$.

We emphasize that for a property to be preserved by tropical modifications, it is in particular preserved by *degenerate* tropical modifications.

Theorem 2.4.34 ([AP21, Theorems 5.6, 5.9])**.** *The properties of being a reduced or locally irreducible tropical fan (resp. tropical fan cycle) are preserved by tropical modifications.*

The proof of the theorem is essentially by direct verification, see. [AP21] for details.

Remark 2.4.35. Recall that for a property $\mathcal{P}$ of tropical fans to define a well-behaved local property of tropical fan cycles, $\mathcal{P}$ must be intrinsic to the support, local, and stably invariant (Proposition 2.1.31). Now we add another requirement: "nice" properties should also be preserved by tropical modifications.

It follows from Proposition 2.1.32 and Theorem 2.4.34 that the properties of being reduced or locally irreducible fit into this nice class of properties.

In [AP21], Amini and Piquerez study in depth a number of additional nice properties, leading to the notion of *shellability* of properties of tropical fans. We will see more important examples of such properties in Section 4.2.

Chapter 3

Tropicalizations and tropical compactifications

In this expository chapter we turn our attention to connections between algebraic and tropical geometry, discussing tropicalizations and tropical compactifications of closed subvarieties of tori. The main references for this chapter are [Tev07; HKT09; LQ11; Cue12]; other useful references include [MS15; Gub13].

Definition 3.0.1 ([Tev07])**.** A variety Y is *very affine* if it admits a closed embedding $Y \subset T$ into a torus $T \cong (\mathbb{C}^*)^n$.

It is natural to attempt to compactify a very affine variety $Y \subset T$ by compactifying T, i.e. by taking the closure of Y in a toric variety with torus T. It is a remarkable fact that such compactifications are often well-behaved and describe degenerations of the variety Y. Such well-behaved compactifications are known as *tropical compactifications*, and are described combinatorially by a tropical fan cycle in $N_{\mathbb{R}} = \mathrm{Hom}(\mathbb{C}^*, T)$ known as the *tropicalization* of Y.

3.1 Tropicalizations

There are multiple equivalent ways to define the tropicalization. Historically it was first defined as a *logarithmic limit set* [Ber71; GKZ94; Bru+; Ite+19]. We will not use this perspective and instead refer the interested reader to the aforementioned references as well as [MS15, Section 1.4]. For our purposes it is more appropriate to describe the tropicalization of $Y \subset T$ in terms of initial degenerations and geometric tropicalization, as developed and explained in [MS15; Gub13; HKT09; Cue12].

3.1.1 Initial degenerations

Throughout we fix a torus $T \cong (\mathbb{C}^*)^n$ and denote by $M = \mathrm{Hom}(T, \mathbb{C}^*)$ and $N = \mathrm{Hom}(\mathbb{C}^*, T)$ the character and cocharacter lattices of T.

Definition 3.1.1. For $f = \sum a_{\mathbf{v}} t^{\mathbf{v}} \in \mathbb{C}[T] \cong \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and $\mathbf{w} \in N_{\mathbb{R}}$, the *initial form* of f is

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\substack{\mathbf{v} \in M_{\mathbb{R}} \\ \langle \mathbf{v}, \mathbf{w} \rangle \text{ is minimal}}} a_{\mathbf{v}} t^{\mathbf{v}}.$$

Example 3.1.2. 1. Consider $f = x + 2y + 3z$ on $T = (\mathbb{C}^*)^3$. If $\mathbf{w} = (0,0,0)$, then $\mathrm{in}_{\mathbf{w}}(f) = x + 2y + 3z = f$. If $\mathbf{w} = (2,1,1)$, then $\mathrm{in}_{\mathbf{w}}(f) = 2y + 3z$.

2. Now consider $f = x + y^2 + z^3$. If $\mathbf{w} = (2,1,1)$, then $\text{in}_{\mathbf{w}}(f) = x + y^2$. If $\mathbf{w} = (3,2,1)$ then $\text{in}_{\mathbf{w}}(f) = x + z^3$.

Definition 3.1.3. For an ideal $I \subset \mathbb{C}[T]$ and for $\mathbf{w} \in N_{\mathbb{R}}$, the *initial ideal* of I is the ideal

$$\text{in}_{\mathbf{w}}(I) = \langle \text{in}_{\mathbf{w}} f \mid f \in I \rangle.$$

For a closed subscheme $Y \subset T$, the *initial degeneration* of Y with respect to $\mathbf{w} \in N_{\mathbb{R}}$ is the variety

$$\text{in}_{\mathbf{w}}(Y) = \text{Spec}(\mathbb{C}[T]/\,\text{in}_{\mathbf{w}}(I)),$$

where $Y = \text{Spec}\,\mathbb{C}[T]/I$.

Definition 3.1.4. The *tropicalization* of a closed subscheme $Y \subset T$ is the set

$$\text{trop}(Y) = \{\mathbf{w} \in N_{\mathbb{R}} \mid \text{in}_{\mathbf{w}}(Y) \neq \emptyset\}.$$

In order for the tropicalization to deserve its name, it must be a tropical fan cycle, i.e. the support of a tropical fan. This is achieved as follows.

Theorem 3.1.5 ([MS15, Theorem 3.3.5])**.** *If $Y \subset T$ is a very affine variety, then there is a pure $(\dim Y)$-dimensional fan Σ supported on $\text{trop}(Y)$ such that, for all cones $\sigma \in \Sigma$, $\text{in}_{\mathbf{w}}(Y)$ is constant on the relative interior of σ. For a top-dimensional cone $\sigma \in \Sigma$, let $\omega(\sigma)$ be the sum of the multiplicities of the irreducible components of $\text{in}_{\mathbf{w}}(Y)$, for any $\mathbf{w}$ in the relative interior of σ. Then (Σ, ω) is a tropical fan, hence $\text{trop}(Y)$ is a tropical fan cycle.*

Example 3.1.6. Consider $Y = \{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$. One computes that trop$(Y)$ is the standard tropical line of Example 2.1.8, with corresponding initial degenerations as pictured in Figure 3.1. This explains why this tropical fan is called a "line".

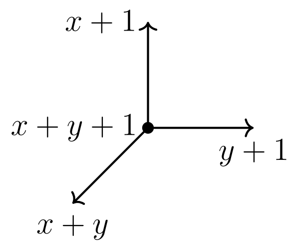


Figure 3.1. Tropicalization and initial degenerations of $Y = \{x + y + 1 = 0\}$

Example 3.1.7. Consider $Y = \{y = x^2\} \subset (\mathbb{C}^*)^2$. Note the only nontrivial initial degeneration of Y is Y itself (since x, y are nonvanishing). The tropicalization of Y is therefore the line $y = 2x$, with weight 1.

Example 3.1.8. By the same logic as the previous example, if $Y = \{y^2 = x^3\} \subset (\mathbb{C}^*)^2$, then trop$(Y)$ is the line $2y = 3x$, with weight 1, cf. Example 2.1.9.

Remark 3.1.9. There is a natural notion of the tropicalization of a rational function f on a very affine variety Y, by replacing multiplication with addition and addition with taking the minimum [MS15]. For instance, following the previous example, if $f = y^2 - x^3$, then $\mathrm{trop}(f) = \min\{2y, 3x\}$. If f is a regular function on a torus T, then trop(div(f)) is identified with the support of div(trop(f)) [MS15, Theorem 3.1.3], but in general the weights are different—continuing with the example $f = y^2 - x^3$, we computed in Example 2.4.14 that the weights on div(trop(f)) are 4, as opposed to the weight 1 on trop(div(f)).

Remark 3.1.10. The fan Σ in Theorem 3.1.5 can be taken to be the *Gröbner fan*, a fan defined exactly to have the desired property that $\mathrm{in}_{\mathbf{w}}(Y)$ is constant on the relative interior of each cone of Σ [MS15, Section 2.5]. There is not a unique tropical fan structure on $\mathrm{trop}(Y)$—clearly any refinement of the Gröbner fan is also tropical; furthermore, as we will see, the Gröbner fan need not be the coarsest tropical fan structure on $\mathrm{trop}(Y)$. We will get a better intuition for the different fan structures on $\mathrm{trop}(Y)$ and the weights below. For more on computing the tropicalization, we refer to [MS15; Gub13].

Proposition 3.1.11 ([OP13, Definition 2.5.2, Proposition 2.5.7])**.** *Let $Y \subset T$ be any pure-dimensional closed subscheme and $Y = Y_1 \cup \cdots \cup Y_k$ its decomposition into irreducible components, with multiplicities $a_i > 0$. Then*

$$\mathrm{trop}(Y) = \mathrm{trop}(Y_1) \cup \cdots \cup \mathrm{trop}(Y_k)$$

with weights

$$\omega(\sigma) = a_1\omega_1(\sigma) + \cdots + a_k\omega_k(\sigma),$$

where ω_i is the induced weight on $\mathrm{trop}(Y_i)$, extended to all of $\mathrm{trop}(Y)$ by $\omega_i(\sigma) = 0$ if σ is not a cone of $\mathrm{trop}(Y_i)$.

The following corollary will be useful in Chapter 4.

Corollary 3.1.12. *Let $Y \subset T$ be a pure-dimensional closed subscheme such that $\mathrm{trop}(Y)$ is reduced and irreducible. Then Y is irreducible and generically reduced.*

Proof. Let $Y = Y_1 \cup \cdots \cup Y_k$ be the decomposition of Y into irreducible components, with multiplicities $a_i > 0$. We need to show $k = 1$ and $a_i = 1$.

By Proposition 3.1.11,

$$\mathrm{trop}(Y) = \mathrm{trop}(Y_1) \cup \cdots \cup \mathrm{trop}(Y_k),$$

with

$$\omega(\sigma) = a_1\omega_1(\sigma) + \cdots + a_k\omega_k(\sigma),$$

where ω is the induced weight on $\mathrm{trop}(Y)$, ω_i is the induced weight on $\mathrm{trop}(Y_i)$, σ is a top-dimensional cone of a tropical fan structure on $\mathrm{trop}(Y)$ and $\omega_i(\sigma) = 0$ if $\mathrm{trop}(Y_i)$ does not contain σ.

Since $\mathrm{trop}(Y)$ is reduced and irreducible and $\mathrm{trop}(Y_i)$ is a tropical fan subcycle of $\mathrm{trop}(Y)$ of the same dimension, $\mathrm{trop}(Y_i) = \mathrm{trop}(Y)$ as sets. Furthermore, reducedness means that $\omega(\sigma) = 1$ for all top-dimensional cones σ, which implies by irreducibility that for each i there is some constant $\lambda_i > 0$ such that $\omega_i(\sigma) = \lambda_i$ for all top-dimensional cones σ [GS12, Lemma 2.20]. Thus

$$1 = a_1\lambda_1 + \cdots + a_k\lambda_k,$$

with all $a_i, \lambda_i > 0$. This implies that $k = 1$ and $a_i = \lambda_i = 1$. □

Example 3.1.13. In $(\mathbb{C}^*)^2$ consider $Y_1 = \{x = y\}$, $Y_2 = \{x = -y\}$, and $Y = Y_1 \cup Y_2 = \{x^2 = y^2\}$. Then $\mathrm{trop}(Y_1) = \mathrm{trop}(Y_2)$ is the line $y = x$ with weight 1. One directly computes that

$\mathrm{trop}(Y) = \mathrm{trop}(Y_1) = \mathrm{trop}(Y_2)$ as sets, but the weight function on $\mathrm{trop}(Y)$ is 2, corresponding to the sum of the weight functions on $\mathrm{trop}(Y_1)$ and $\mathrm{trop}(Y_2)$.

Remark 3.1.14. Implicit in this section is that we are taking the tropicalization over the field $\mathbb{C}$ with the *trivial valuation*, cf. Section 3.1.2 below. This ensures that the tropicalization is indeed a tropical fan cycle. More generally one can consider tropicalizations over fields with nontrivial valuations, e.g. the field $\mathbb{C}\{\{t\}\}$ of Puiseux series, in which case the tropicalization will only be (the support of) a balanced polyhedral complex, i.e. a more general tropical cycle—see [MS15; Gub13] for more details. Additionally, we use the *min* convention when defining the initial degeneration, which says that we are working with the min convention in tropical geometry, rather than the max convention.

3.1.2 Geometric tropicalization

Geometric tropicalization was first developed by Hacking, Keel, and Tevelev in [HKT09] as a method to compute the tropicalization of a very affine variety $Y \subset T$ from any simple normal crossings compactification of Y. The method was further developed by Cueto to describe the weights on the geometric tropicalization [Cue12]. In this section we mainly follow the exposition of [Cue12]; the other main references are [HKT09, Section 2] and [MS15, Section 6.5].

Normal crossings compactifications

By a *compactification* $Y \subset \overline{Y}$ of a variety Y, we always mean a complete variety containing Y as an open dense subset. We say a compactification $Y \subset \overline{Y}$ has *divisorial boundary* if $\overline{Y} \setminus Y$ is a union of codimension one subvarieties of $\overline{Y}$, called *boundary divisors*.

Definition 3.1.15 ([Tev07])**.** Let Y be any variety. A compactification $Y \subset \overline{Y}$ of Y has *combinatorial normal crossings* if it has divisorial boundary and for all k, the intersection of k boundary divisors of $\overline{Y}$ has codimension k. The compactification has *simple normal crossings* if in addition each such intersection of boundary divisors is transversal.

Definition 3.1.16. Let $Y \subset \overline{Y}$ be a combinatorial normal crossings compactification. The *boundary complex* Δ of $Y \subset \overline{Y}$ is the abstract simplicial complex with a vertex v_i for each boundary divisor of $Y \subset \overline{Y}$, with a collection of k vertices forming a k-dimensional simplex if the corresponding boundary divisors intersect.

Divisorial valuations

Now fix a very affine variety $Y \subset T$ and a combinatorial normal crossings compactification $Y \subset \overline{Y}$. Assume furthermore that $\overline{Y}$ is normal and $\mathbb{Q}$-factorial. Then to each boundary divisor D there is an associated discrete valuation $\mathrm{val}_D : \mathbb{C}(Y) \to \mathbb{Z}$, called the *divisorial valuation* at D, given by the order of zeros and poles along D. More precisely, if U is an affine open of $\overline{Y}$ intersecting D, and $\mathfrak{p}$ is the prime ideal of $\mathbb{C}[U]$ defining $D \cap U$, then since $\overline{Y}$ is normal and $D \subset \overline{Y}$ has codimension one, it follows that $\mathbb{C}[U]_{\mathfrak{p}}$ is a discrete valuation ring, so val_D is just the associated discrete valuation on the fraction field $\mathbb{C}(Y)$ of $\mathbb{C}[U]_{\mathfrak{p}}$. (See [MS15, Section 6.5] for more details.)

Example 3.1.17 ([MS15, Example 6.5.1])**.** Let $\overline{Y} = \{x_0 + x_1 + x_2 + x_3 = 0\} \subset \mathbb{P}^3$ and $Y = \overline{Y} \cap (\mathbb{C}^*)^3$. The boundary of $Y \subset \overline{Y}$ consists of 4 lines, the intersections of $\overline{Y}$ with the coordinate hyperplanes in $\mathbb{P}^3$. Let $D = \overline{Y} \cap \{x_1 = 0\}$. To compute val_D we work in the affine open $U = \{x_0 \neq 0\}$, so that

$$\mathbb{C}[U] = \frac{\mathbb{C}[y_1, y_2, y_3]}{(1 + y_1 + y_2 + y_3)}$$

where $y_i = x_i/x_0$. Then $D \cap U = \{y_1 = 0\} \subset U$, and for $f \in \mathbb{C}(Y)$, $\mathrm{val}_D(f) = m$, where we write $f = y_1^m f'$ with $m \in \mathbb{Z}$ and neither the numerator nor the denominator of f' is divisible by y_1.

Geometric tropicalization

Recall that $M = \mathrm{Hom}(T, \mathbb{C}^*)$ and $N = \mathrm{Hom}(\mathbb{C}^*, T) = \mathrm{Hom}(M, \mathbb{Z})$ denote the character and cocharacter lattice of the torus T. For a boundary divisor D of $Y \subset \overline{Y}$, we define an element $[D] \in N_{\mathbb{R}}$ by

$$[D](m) = \mathrm{val}_D(m|_Y)$$

for $m \in M$. (Here $m|_Y$ is a morphism $Y \to \mathbb{C}^*$, viewed as an element $\mathbb{C}(Y)$.)

Recall that Δ denotes the boundary complex of $Y \subset \overline{Y}$. For a simplex $\sigma \in \Delta$ corresponding to an intersection of boundary divisors D_i, let $[\sigma] \in N_{\mathbb{R}}$ be the cone spanned by the $[D_i]$.

Theorem 3.1.18 ([HKT09, Section 2], [Cue12, Section 2])**.** *Let $Y \subset T$ be a smooth very affine variety, $Y \subset \overline{Y}$ a combinatorial normal crossings compactification, and Δ the boundary complex. Then*

$$\operatorname{trop}(Y) = \bigcup_{\sigma \in \Delta} \mathbb{R}_{\geq 0}[\sigma]$$

If $\mathbb{R}_{\geq 0}[\sigma]$ is a maximal cone of $\operatorname{trop}(Y)$, so σ corresponds to the intersection of boundary divisors $D_1, \ldots, D_m$, then the weight on $\mathbb{R}_{\geq 0}[\sigma]$ given by

$$\omega\left(\mathbb{R}_{\geq 0}[\sigma]\right) = \omega(\sigma) = (D_1 \cdots D_m)[\mathbb{R}[\sigma] \cap N : \mathbb{Z}[\sigma]]$$

agrees with the weight defined in Theorem 3.1.5, and in particular makes $\operatorname{trop}(Y)$ into a tropical fan.

Remark 3.1.19. The above theorem is what is meant by "geometric tropicalization." Roughly speaking, it says that the tropicalization of a smooth very affine variety is the (cone over the) boundary complex of any combinatorial normal crossings compactification, embedded into $N_{\mathbb{R}}$ via the divisorial valuations. The weight on a maximal cone is just given by the intersection product of the corresponding boundary divisors, times the index factor $[\mathbb{R}[\sigma] \cap N : \mathbb{Z}[\sigma]]$ accounting for the specific embedding into the lattice.

Remark 3.1.20. Without the smoothness assumption on Y, the cone over the boundary complex may strictly contain $\operatorname{trop}(Y)$—for instance if too many boundary divisors intersect at a point, then the dimension will be larger than $\dim Y$, cf. [ST08, Remark 2.7]. Likewise if $\overline{Y}$ is only a partial compactification (i.e. $\overline{Y}$ is not complete), then $\operatorname{trop}(Y)$ can be bigger than the cone over the boundary complex.

Remark 3.1.21. Geometric tropicalization requires resolution of singularities in order to work with a combinatorial normal crossings compactification—otherwise, one needs to use a simple normal crossings compactification [Cue12, Section 2].

Example 3.1.22. The line $\overline{Y} = \{x + y + z = 0\} \subset \mathbb{P}^2$ is a tropical compactification of $Y = \overline{Y} \cap (\mathbb{C}^*)^2$. It has 3 boundary divisors at $0, 1, \infty \in \overline{Y} \cong \mathbb{P}^1$, corresponding to the three intersection points of $\overline{Y}$ with the coordinate lines in $\mathbb{P}^2$ (see Figure 3.2). For each boundary divisor D, computing $\mathrm{val}_D(m|_Y)$ for the basis vectors of M, one finds the corresponding rays to be the rays of the standard tropical line. Thus using geometric tropicalization, we recover that the tropicalization of a line in $(\mathbb{C}^*)^2$ is the standard tropical line in $\mathbb{R}^2$, as in Example 3.1.6.

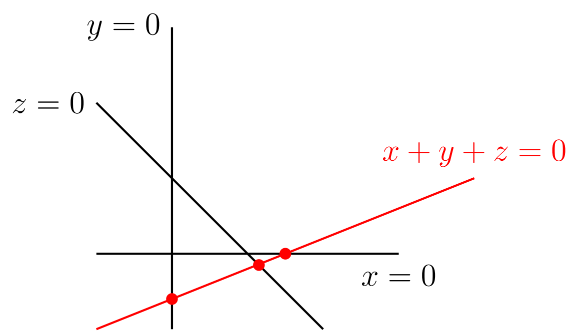


Figure 3.2. The line $\overline{Y} = \{x + y + z = 0\} \subset \mathbb{P}^1$ gives a combinatorial normal crossings compactification of $Y = \{x + y + 1 = 0\} \subset \mathbb{C}^*$

3.1.3 Maps between tropicalizations

A morphism of tori $\pi : T \to T'$ is the same as a corresponding linear map

$$p : N = \mathrm{Hom}(\mathbb{C}^*, T) \to \mathrm{Hom}(\mathbb{C}^*, T') = N',$$

which can be described explicitly as follows. Suppose $T \cong (\mathbb{C}^*)^n$, $T' \cong (\mathbb{C}^*)^m$, and π is given by $w_i = \mathbf{z}^{\mathbf{a_i}}$, where $\mathbf{a_i} \in \mathbb{Z}^n$. Then the $m \times n$ matrix A whose rows are given by the vectors $\mathbf{a_i}$ defines the linear map $p : N \cong \mathbb{Z}^n \to \mathbb{Z}^m \cong N'$. We also denote by p the induced map $N_\mathbb{R} \to N'_\mathbb{R}$.

Lemma 3.1.23 ([Tev07, Proposition 3.1], [ST08, Proposition 2.8])**.** *Suppose $Y \subset T$ and $Y' \subset T'$ are closed subvarieties, and $\pi : T \to T'$ induces a dominant morphism $Y \to Y'$. Then under the induced map $p : N_\mathbb{R} \to N'_\mathbb{R}$, one has $p(\mathrm{trop}(Y)) = \mathrm{trop}(Y')$ (as sets).*

Recall from Definition 2.3.5 that if $p : (\mathcal{F}, \omega) \to (\mathcal{F}', \omega')$ is a morphism of tropical fan cycles of the same dimension, then the pushforward $p_*\mathcal{F}$ of $\mathcal{F}$ is given by choosing fan structures Σ, Σ' on $\mathcal{F}$ and $\mathcal{F}'$ such that p maps each cone of Σ to a cone of Σ', and then defining, for a top-dimensional cone $\sigma' \in \mathcal{F}'$

$$p_*\omega(\sigma') = \sum_{\substack{\sigma \in \Sigma \\ p(\sigma) = \sigma'}} \omega(\sigma)[N_{\sigma'} : p(N_\sigma)].$$

Theorem 3.1.24 ([ST08, Theorem 3.12])**.** *Suppose $Y \subset T$ and $Y \subset T'$ are closed subvarieties, and $\pi : T \to T'$ induces a dominant, generically finite morphism $Y \to Y'$ of degree δ. Then*

$$\mathrm{trop}(Y') = \frac{1}{\delta} p_* \,\mathrm{trop}(Y).$$

Example 3.1.25. Let $Y = \{y = x^2\} \subset (\mathbb{C}^*)^2$. Then $\mathrm{trop}(Y)$ is the line $y = 2x$ as described in Example 3.1.7. The projection $\pi : (\mathbb{C}^*)^2 \to \mathbb{C}^*$, $\pi(x, y) = x$ sends Y isomorphically onto

$\mathbb{C}^*$, and $p_* \operatorname{trop}(Y) = \operatorname{trop}(Y')$. On the other hand, the projection $\pi : T \to \mathbb{C}^*$, $\pi(x, y) = y$ realizes Y as a $2 : 1$ cover of $\mathbb{C}^*$, and $p_* \operatorname{trop}(Y) = 2 \operatorname{trop}(Y')$.

Example 3.1.26. Let $Y = \{y^2 = x^3\} \subset (\mathbb{C}^*)^2$. Then $\operatorname{trop}(Y)$ is the line $2y = 3x$ as described in Example 3.1.8. The projection $\pi : (\mathbb{C}^*)^2 \to \mathbb{C}^*$, $\pi(x, y) = y/x$ sends Y isomorphically onto $\mathbb{C}^*$ and $\operatorname{trop}(Y)$ isomorphically onto $\mathbb{R}^1$, cf. Example 2.3.13.

Remark 3.1.27. Theorem 3.1.24 goes under the name *tropical elimination theory*. Elimination theory is concerned with computing the image of a variety $Y \subset T$ under a morphism $T \to T'$. Tropical elimination theory gives a combinatorial approach to this by first computing the tropicalization, and using this to recover the image, see [ST08].

3.1.4 Intrinsic tori

When considering tropicalizations above, we assumed a fixed closed embedding of a very affine variety Y in a torus T. But for a given very affine variety Y, there are potentially many possible closed embeddings of Y in tori, which can give different tropicalizations.

Example 3.1.28. $Y = \mathbb{C}^*$ could be viewed as a closed subvariety of itself, or embedded into the torus $T = (\mathbb{C}^*)^2$ in many different ways, e.g. as $Y = \{y = x^2\}$, $Y = \{y^2 = x^3\}$, or $Y = \{x + y + 1 = 0\}$. As discussed previously, the first three embeddings give distinct but isomorphic tropicalizations (all isomorphic to $\mathbb{R}^1$), while the last embedding gives a different tropicalization, namely the standard tropical line.

The most important ambient torus for a very affine variety Y is the *intrinsic torus* T_Y, described as follows [Tev07; HKT09]. For *any* variety Y, let $M = \mathcal{O}^*(Y)/\mathbb{C}^*$. Then a splitting of the exact sequence

$$0 \to \mathbb{C}^* \to \mathcal{O}^*(Y) \to M \to 0$$

induces an evaluation map $Y \to T_Y := \mathrm{Hom}(M, \mathbb{C}^*)$. The torus T_Y is called the intrinsic torus, and Y is very affine $\iff$ the evaluation map $Y \to T_Y$ is a closed embedding. Note there are many possible choices of evaluation map, differing by translation. Any other closed embedding $Y \subset T$ factors through $Y \subset T_Y$ and a morphism of tori $\pi : T_Y \to T$. In particular, if $p : N_{Y,\mathbb{R}} \to N_{\mathbb{R}}$ is the linear map corresponding to π, then $\mathrm{trop}(Y \subset T) = p(\mathrm{trop}(Y \subset T_Y))$ by Lemma 3.1.23.

Example 3.1.29. A complement Y of an arrangement of $n+1$ hyperplanes in $\mathbb{P}^d$, not all meeting at a point, is a very affine variety. Taking equations for the hyperplanes gives an embedding $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$ realizing the ith hyperplane in $\mathbb{P}^d$ as the complement of the ith coordinate hyperplane in $\mathbb{P}^n$, hence realizing Y as a closed subvariety of the open dense torus $(\mathbb{C}^*)^n \subset \mathbb{P}^n$. This is the embedding of Y in its intrinsic torus, cf. [Tev07].

More generally, if Y is the complement in a torus T of k hypersurfaces $H_1, \dots, H_k$, then the intrinsic torus of Y is $T_Y = T \times (\mathbb{C}^*)^k$, and the embedding of Y in T_Y is obtained by taking the graph of the equations of the H_i, see [HKT09, Lemma 6.1].

This example will play a fundamental role in the remainder of this dissertation.

Example 3.1.30 ([MS15, Example 6.4.3(4)]). Consider $Y = \{x^3 + y^3 - 2x^2y - 2x + 1 = 0\} \subset (\mathbb{C}^*)^2$. The tropicalization of this embedding is the standard tropical line in $\mathbb{R}^2$, but with weights 3 instead of 1. This is not the embedding of Y in its intrinsic torus: $1 - x + y$

is also a unit of Y, and the embedding of Y in its intrinsic torus is given by taking the graph of $1 - x + y$ in $(\mathbb{C}^*)^3$:

$$Y = \{x^3 + y^3 - 2x^2y - 2x + 1 = 0, z = 1 - x + y\} \subset (\mathbb{C}^*)^3.$$

Remark 3.1.31. It is an interesting problem to compare different tropicalizations of a very affine variety. In some sense the intrinsic tropicalization is universal, as it maps to all other tropicalizations. In another sense the correct universal object to consider is really the *analytification*, a more complicated object encoding *all* tropicalizations [Pay09].

More concretely, if $Y \subset T$ and $Y \subset T'$ are two closed embeddings of Y, then both are mapped to from the embedding in the intrinsic torus $Y \subset T_Y$. So there are maps

$$\begin{array}{ccc} & \mathrm{trop}(Y \subset T_Y) & \\ \swarrow & & \searrow \\ \mathrm{trop}(Y \subset T) & & \mathrm{trop}(Y \subset T'). \end{array}$$

One could ask whether it is possible to understand the relationship between $\mathrm{trop}(Y \subset T)$ and $\mathrm{trop}(Y \subset T')$ via these maps. For instance, can the above diagram be factored into a sequence of tropical modifications and their inverses? We are not aware of any serious results in this direction.

3.2 Tropical compactifications

We have already seen that, by geometric tropicalization, the tropicalization of a smooth very affine variety $Y \subset T$ is determined by a combinatorial normal crossings compactification of Y. We now go in the other direction, and show how the tropicalization of Y describes reasonable ("tropical") compactifications of Y.

Tropical compactifications were introduced by Tevelev in [Tev07] and have been studied in more depth in [HKT09; LQ11].

3.2.1 Tropical compactifications

Proposition 3.2.1 ([MS15, Proposition 6.4.7])**.** *Let $Y \subset T$ be a very affine variety and $\overline{Y}$ the closure of Y in a toric variety $X(\Sigma)$ with torus T.*

1. *$\overline{Y}$ is proper $\iff$ $|\Sigma| \supset \mathrm{trop}(Y)$.*

2. *Assume the equivalent conditions of part (1) hold. Then $\mathrm{trop}(Y) = |\Sigma| \iff$ for all $\sigma \in \Sigma$, $\overline{Y} \cap O(\sigma)$ is nonempty of pure dimension $\dim Y - \dim \sigma$.*

In other words, a compactification $\overline{Y}$ of Y in a toric variety $X(\Sigma)$ is proper and intersects toric strata properly $\iff$ Σ is supported on $\mathrm{trop}(Y)$. Thus the stratification of $X(\Sigma)$ by torus orbits pulls back to a stratification of $\overline{Y}$. Note if $X(\Sigma)$ is smooth this also implies that $\overline{Y}$ is a combinatorial normal crossings compactification. However, one can run into unexpected singularities on $\overline{Y}$: even a compactification in a smooth toric variety can have "bad" singularities.

Example 3.2.2 ([MS15, Example 6.4.16])**.** Let $\overline{Y} \subset \mathbb{P}^n$ be any projective variety which is not Cohen-Macaulay at a point p. Then one can arrange coordinates on $\mathbb{P}^n$ so that $\overline{Y}$ is a compactification of $Y = \overline{Y} \cap (\mathbb{C}^*)^n$ in a smooth toric variety with fan supported on $\operatorname{trop}(Y)$, but $p = \overline{Y} \cap O(\sigma)$ is a non-Cohen-Macaulay point of $\overline{Y}$.

Tropical compactifications eliminate situations like the above example.

Definition 3.2.3 ([Tev07, Definition 1.2])**.** Let $Y \subset T$ be a very affine variety. Let $X(\Sigma)$ be a toric variety with torus T, and let $\overline{Y}$ be the closure of Y in $X(\Sigma)$. Then $\overline{Y}$ is a *tropical compactification* of Y if $\overline{Y}$ is proper and the induced multiplication map $\overline{Y} \times T \to X(\Sigma)$ is flat and surjective.

Theorem 3.2.4 ([Tev07, Theorem 1.2, Proposition 2.5])**.** *If $\overline{Y} \subset X(\Sigma)$ is a tropical compactification of a very affine variety, then Σ is supported on* $\operatorname{trop}(Y)$*. If Σ' is any refinement of Σ, then the closure $\overline{Y}'$ of Y in $X(\Sigma')$ is also a tropical compactification, and is obtained as the pullback of $\overline{Y}$ by the proper toric map $X(\Sigma') \to X(\Sigma)$.*

Additionally if $X(\Sigma)$ is smooth, then $\overline{Y}$ has at worst Cohen-Macaulay singularities on the boundary $\overline{Y} \setminus Y$.

Remark 3.2.5. In particular, if $\overline{Y} \subset X(\Sigma)$ is a tropical compactification, then $\overline{Y}$ intersects each torus orbit in $X(\Sigma)$ properly, cf. Proposition 3.2.1.

Example 3.2.6. Consider the line $Y = \{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$. The tropicalization of Y is the standard tropical line in $\mathbb{R}^2$ as shown in Examples 3.1.6, 3.1.22. There is a unique fan structure Σ on $\operatorname{trop}(Y)$, and the corresponding toric variety $X(\Sigma)$ is isomorphic to $\mathbb{P}^2$ minus the coordinate points. The corresponding tropical compactification $\overline{Y}$ of Y is isomorphic to

$\mathbb{P}^1 = \{x + y + z = 0\} \subset \mathbb{P}^2$, with the three boundary points given by the intersections of $\overline{Y}$ with the three coordinate lines in $\mathbb{P}^2$, as in Example 3.1.22, Figure 3.2.

We refer to [MS15] for some more examples, see e.g. [MS15, Examples 6.4.10, 6.4.11]. We will see many more examples of tropical compactifications throughout the rest of this dissertation.

It is a nontrivial theorem that tropical compactifications indeed exist.

Theorem 3.2.7 ([Tev07, Theorem 1.2])**.** *Any very affine variety has a tropical compactification in a smooth toric variety.*

Remark 3.2.8. This theorem is closely related to Theorem 3.1.5, that there is a fine enough fan structure on $\text{trop}(Y)$ so that weights can be defined on maximal cones—indeed, the so-called "Gröbner fan" mentioned in Remark 3.1.10 works to find a tropical compactification; see [Tev07] and [MS15, Proposition 6.4.17] for more details. The idea of the proof is related to Kapranov's "visible contours" construction introduced in [Kap93] and further considered in [HKT06]. We will see this construction come up again in the second part of this thesis.

The connection between geometric tropicalization and tropical compactifications is now hopefully obvious—a (suitably nonsingular) combinatorial normal crossings compactification $\overline{Y}$ of $Y \subset T$ determines a particular fan structure Σ on the tropicalization $\text{trop}(Y)$ realizing $\overline{Y}$ as the tropical compactification defined by the fan Σ, and other tropical compactifications of Y are determined by other fan structures on $\text{trop}(Y)$. The connection between tropical compactifications and the tropicalization is made even stronger by the following proposition.

Proposition 3.2.9. *Let $\overline{Y} \subset X(\Sigma)$ be a tropical compactification of a very affine variety Y, let σ be a cone of Σ, and let $\mathbf{w}$ be a point in the relative interior of σ. Then*

1. (a) $\operatorname{trop}(\overline{Y} \cap O(\sigma) \subset O(\sigma)) = |\Sigma^\sigma|$.

   (b) $\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(Y)) = \operatorname{trop}(Y)^{\mathbf{w}}$, *the local cycle of* $\operatorname{trop}(Y)$ *at* $\mathbf{w}$ *(cf. Definition 2.1.28).*

2. *If $\mathbf{w}$ is any point in the relative interior of the cone σ, then*

$$\operatorname{in}_{\mathbf{w}}(Y) = (\overline{Y} \cap O(\sigma)) \times T_\sigma,$$

   and in particular

$$\overline{Y} \cap O(\sigma) = \operatorname{in}_{\mathbf{w}}(Y)/T_\sigma,$$

   where T_σ is the subtorus $N_\sigma \otimes \mathbb{C}^* \subset T$ (so $O(\sigma) = T/T_\sigma$).

3. *If σ is a top-dimensional cone of Σ, then the weight $\omega(\sigma)$ on σ as given in Theorem 3.1.5 is the length of the 0-dimensional scheme $\overline{Y} \cap O(\sigma) = \overline{Y} \cap V(\sigma)$.*

Proof.

1. (a) See [Gub13, Proposition 14.3].

   (b) See [Spe05, Proposition 2.2.3].

2. Recall from Section 2.1.2 that the local cycle of $\operatorname{trop}(Y)$ at $\mathbf{w}$ is equal to $|\Sigma^\sigma| \times \mathbb{R}^{\dim \sigma}$. Thus this follows from the first part. See also [MS15, page 308].

3. Immediate from the second part.

□

Remark 3.2.10. The weight $\omega(\sigma)$ can also be viewed as the degree of the intersection product $[\overline{Y}] \cdot [V(\sigma)]$ in the toric variety $X(\Sigma)$ [ST08, Proposition 3.2]. (Although in general $X(\Sigma)$ need not be proper, one can still consider the degree of $[\overline{Y}] \cdot [V(\sigma)]$ by e.g. viewing it as a refined intersection product in the proper scheme $\overline{Y} \cap V(\sigma)$. See also [Kat08, Section 9].) From this perspective the connection between the definitions of the weight in Theorems 3.1.18 and 3.1.5 is obvious.

The above results justify our intuition that the tropicalization is "a combinatorial object encoding the information of all tropical compactifications of $Y \subset T$."

3.2.2 Schön compactifications

We turn our attention now to a particularly well-behaved class of tropical compactifications called *schön* compactifications. These will be our primary tropical compactifications of interest in the remainder of this dissertation.

Definition 3.2.11 ([Tev07])**.** A tropical compactification $\overline{Y} \subset X(\Sigma)$ of a very affine variety $Y \subset T$ is *schön* if all strata $\overline{Y} \cap O(\sigma)$ are smooth.

Theorem 3.2.12.

1. *A tropical compactification $\overline{Y} \subset X(\Sigma)$ is schön $\iff$ the multiplication map $\overline{Y} \times T \to X(\Sigma)$ is smooth and surjective.*

2. *If $\overline{Y} \subset X(\Sigma)$ is a schön tropical compactification, then it is regularly embedded, normal, and has toroidal singularities, and if $X(\Sigma)$ is nonsingular then so is $\overline{Y}$. The log canonical bundle $\omega_{\overline{Y}}(B)$, where $B = \overline{Y} \setminus Y$ is the boundary, is globally generated and equal to the determinant of the normal bundle of the regular embedding $\overline{Y} \subset X(\Sigma)$. If in addition Σ'*

is any refinement of Σ *and* $\overline{Y}' \subset X(\Sigma')$ *is the corresponding tropical compactification, then* $\overline{Y}' \to \overline{Y}$ *is log crepant, meaning the pullback of the log canonical bundle is the log canonical bundle.*

Proof. 1. See [Hac08, Lemma 2.7].

2. See [Tev07, Theorem 1.4].

□

Theorem 3.2.13 ([Tev07, Theorem 1.4], [LQ11, Theorem 1.5])**.** *If a very affine variety* $Y \subset T$ *admits a schön compactification, then* any *fan supported on* $\mathrm{trop}(Y)$ *gives a schön compactification.*

In particular, it makes sense to call a very affine variety schön. Furthermore with schön compactifications we do not need to worry about situations like Example 3.2.2—now any fan on $\mathrm{trop}(Y)$ gives a (schön) tropical compactification.

Remark 3.2.14. The above theorem should not be surprising, since strata of any tropical compactification are described by initial degenerations, so being schön is equivalent to all initial degenerations being smooth—this does not depend on the fan structure on $\mathrm{trop}(Y)$ (see [Cor21] for more on this perspective).

Example 3.2.15. The tropical compactification of the line $Y = \{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$ as discussed in Example 3.2.6 is schön.

Example 3.2.16. If $\overline{Y} \subset \mathbb{P}^2$ is a (sufficiently general) nodal cubic, then $\overline{Y}$ can be viewed as a tropical compactification of $Y = \overline{Y} \cap (\mathbb{C}^*)^2$ in $\mathbb{P}^2$ minus the coordinate points, see [MS15, Exercise 6.8.11]. Since $\overline{Y}$ is singular but the ambient toric variety is nonsingular, this cannot be a schön compactification.

Theorem 3.2.17 ([LQ11, Theorem 1.4])**.** *Any variety contains an open very affine schön subvariety.*

Chapter 4

Intersection theory

In this chapter we discuss the intersection theory of tropical fans and tropical compactifications, and connections to the intersection theory of toric varieties. This chapter contains the first original results of this dissertation, in Theorem 4.3.3 and Sections 4.2.3 and 4.3.2.

4.1 Preliminaries

4.1.1 Algebraic preliminaries

Recall our conventions from the introduction: we work over $\mathbb{C}$ unless otherwise stated.

Kronecker duality

Suppose $\overline{Y}$ is any complete variety. Then there is a well-defined degree morphism $\int$: $A_0(\overline{Y}) \to \mathbb{Z}$, inducing a *Kronecker duality morphism*

$$A^k(\overline{Y}) \xrightarrow{\mathcal{D}_{\overline{Y}}} \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}), \quad \alpha \mapsto \left(\beta \mapsto \int \alpha \cap \beta\right).$$

The analogous map between singular cohomology and homology is always an isomorphism (up to torsion), but generally $\mathcal{D}_{\overline{Y}}$ is very far from being an isomorphism, even if $\overline{Y}$ is nonsingular [Ful+95]. However, Fulton-MacPherson-Sturmfels-Sottile, as well as Totaro, have studied a large class of varieties, including all complete toric varieties, for which $\mathcal{D}_{\overline{Y}}$ is an isomorphism [Ful+95; Tot14].

Definition 4.1.1. A complete variety $\overline{Y}$ *satisfies Kronecker duality* if the Kronecker duality morphism is an isomorphism for all k.

Definition 4.1.2. A variety X satisfies *Chow-Künneth* if for all finite-type schemes Z, the natural map $A_*(X) \otimes A_*(Z) \to A_*(X \times Z)$ is an isomorphism.

Proposition 4.1.3 ([Ful+95, Corollary to Theorem 2])**.** *If a complete variety $\overline{Y}$ is nonsingular and satisfies Chow-Künneth, then the cycle class map $A_*(\overline{Y}) \to H_*(\overline{Y})$ is an isomorphism.*

In particular, the proposition implies that any nonsingular complete variety $\overline{Y}$ which satisfies Chow-Künneth also satisfies Kronecker duality. If $\overline{Y}$ is singular, then $A_*(\overline{Y}) \to H_*(\overline{Y})$ can fail to be an isomorphism, but remarkably $\overline{Y}$ will still satisfy Kronecker duality.

Proposition 4.1.4 ([Ful+95, Section 4, Proposition]). *If a complete variety $\overline{Y}$ satisfies Chow-Künneth, then the Kronecker duality morphism*

$$\mathcal{D}_{\overline{Y}} : A^k(\overline{Y}) \to \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z})$$

is an isomorphism for all k.

Weakly linear varieties

Definition 4.1.5 ([Tot14]). A variety Y is *weakly linear* if it is isomorphic to affine space, it is the complement of a weakly linear variety in a weakly linear variety, or it contains a weakly linear variety Z such that the complement $Y \setminus Z$ is also weakly linear.

It follows from the definition that a variety stratified by weakly linear varieties is itself weakly linear.

Remark 4.1.6. Weakly linear varieties are called "linear" in [Tot14; Jan06]. We rename them "weakly linear" to avoid confusion with our notion of linear varieties to be discussed below.

Proposition 4.1.7 ([Tot14, Proposition 1]). *Weakly linear varieties satisfy Chow-Künneth.*

Corollary 4.1.8. *Let $\overline{Y}$ be a complete, weakly linear variety. Then $\overline{Y}$ satisfies Chow-Künneth and Kronecker duality, and if $\overline{Y}$ is nonsingular then the cycle class map $A_*(\overline{Y}) \to H_*(\overline{Y})$ is an isomorphism.*

Proof. Immediate from the propositions. □

Poincaré duality

Proposition 4.1.9 ([Ful98, Corollary 17.4]). *Let $\overline{Y}$ be a nonsingular complete algebraic variety of dimension d. Then $\overline{Y}$ is* Poincaré*: the cap product with the fundamental class $[\overline{Y}]$ induces an isomorphism*

$$- \cap [\overline{Y}] : A^k(\overline{Y}) \xrightarrow{\sim} A_{d-k}(\overline{Y})$$

for all k.

This can be used to *define* the intersection product on $A_*(X)$ via the ring structure on $A^*(X)$; by [Ful98, Corollary 17.4], the intersection product defined in this fashion agrees with the expected one constructed in [Ful98, Chapter 6]. For this reason, Poincaré duality will play an important role in our discussion of tropical intersection theory below.

Remark 4.1.10. It is common in algebraic geometry to abuse the isomorphism $A_k(\overline{Y}) \cong A^{d-k}(\overline{Y})$ for complete nonsingular varieties $\overline{Y}$ to view the cocycles as codimension $d-k$ cycles and write the cup product on $A^*(\overline{Y})$ as the intersection product. In this setting, the Kronecker duality morphism $\mathcal{D}_{\overline{Y}} : A^k(\overline{Y}) \to \operatorname{Hom}(A_k(\overline{Y}), \mathbb{Z})$ is identified with the morphism

$$A^k(\overline{Y}) \to \operatorname{Hom}(A^{d-k}(\overline{Y}), \mathbb{Z}), \quad \alpha \mapsto \left(\beta \mapsto \int \alpha \cdot \beta \cap [\overline{Y}]\right).$$

Thus in this case the statement that $\overline{Y}$ satisfies Kronecker duality is equivalent to the statement that the intersection pairing

$$A^k(\overline{Y}) \times A^{d-k}(\overline{Y}) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \int \alpha \cdot \beta \cap [\overline{Y}]$$

is a perfect pairing. This is what is often referred to in algebraic geometry as Poincaré duality, although for our purposes it is perhaps better referred to as *Kronecker-Poincaré duality*.

Chow-free varieties

Definition 4.1.11 ([PV15])**.** A variety Y is *Chow-free* if $A_{\dim Y}(Y) \cong \mathbb{Z}$ and $A_k(Y) = 0$ for $k \neq \dim Y$.

Lemma 4.1.12. *An open subvariety of a Chow-free variety is Chow-free.*

Proof. Let Y be Chow-free, $U \subset Y$ open, and $Z = Y \setminus U$. Then $\dim Z < \dim Y$, so the result is immediate from the exact sequence

$$A_k(Z) \to A_k(Y) \to A_k(U) \to 0.$$

□

The main examples of Chow-free varieties are affine space and open subvarieties of affine space (cf. the definition of weakly linear varieties). Our interest in Chow-free varieties comes from the following proposition, whose proof is essentially given in [Ful93, Section 5.2].

Proposition 4.1.13. *Let Z be a variety with a stratification by locally closed strata which are all Chow-free. Then $A_k(Z)$ is generated by the classes of the closures k-dimensional strata.*

Proof. We induct on $n = \dim Z$. When $n = 0$ the claim is obvious. Suppose the result holds for all $i < n$.

Let Z_i be the union of the closed strata of dimension $\leq i$. This gives a filtration

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z.$$

Each $Z_i \setminus Z_{i-1}$ is the disjoint union of the strata of dimension i.

Consider the exact sequence

$$A_k(Z_{i-1}) \to A_k(Z_i) \to A_k(Z_i \setminus Z_{i-1}) \to 0.$$

If $k = i$, then for dimension reasons $A_i(Z_{i-1}) = 0$, so $A_i(Z_i) \cong A_i(Z_i \setminus Z_{i-1})$, hence by assumption is generated by the classes of the closures of the i-dimensional strata.

If $k < i$, then by assumption $A_i(Z_i \setminus Z_{i-1}) = 0$, so $A_k(Z_i)$ is generated by $A_k(Z_{i-1})$.

By induction, $A_k(Z)$ is generated by $A_k(Z_k)$, which is generated by the classes of the closures of the k-dimensional strata. □

4.1.2 Intersection theory of toric varieties

Chow homology

Recall a toric variety $X(\Sigma)$ is stratified in dimension $n-k$ by the torus orbits $O(\sigma) \cong (\mathbb{C}^*)^{n-k}$ for $\sigma \in \Sigma_k$. In particular, all strata are Chow-free and weakly linear. This implies the following.

Proposition 4.1.14 ([FS97, Proposition 2.1]). *Let Σ be any fan in $N_{\mathbb{R}}$. Then $A_{n-k}(X(\Sigma))$ is generated by the classes $[V(\sigma)]$ for $\sigma \in \Sigma_k$, with relations*

$$\sum_{\substack{\sigma \succ \tau \\ \dim \sigma = \dim \tau + 1}} \langle u, n_{\sigma,\tau} \rangle [V(\sigma)] = \operatorname{div}(\chi^u)$$

for all $\tau \in \Sigma_{k-1}$ and all $u \in M(\tau) = \tau^{\perp} \cap M$, where χ^u denotes the corresponding character of $V(\tau)$.

This proposition motivates the balancing condition of Definition 2.1.1, and implies Proposition 2.1.2, that

$$M_k(\Sigma) \cong \operatorname{Hom}(A_{n-k}(X(\Sigma)), \mathbb{Z}).$$

Chow cohomology

Theorem 4.1.15 ([Bri96; KP08]). *Let Σ be a unimodular fan in $N_{\mathbb{R}}$. Then*

$$A^*(X(\Sigma)) \cong H^*(X(\Sigma)) \cong PP^*(\Sigma)/LPP^*(\Sigma) \cong \frac{\mathbb{Z}[D_\sigma \mid \sigma \in \Sigma_1]}{\textit{the following relations}}$$

1. *(Linear relations) $\sum_{\sigma \in \Sigma_1} \langle u, n_\sigma \rangle D_\sigma = 0$ for all $u \in M$, where n_σ is a primitive generator of the ray σ.*

2. *(Multiplicative relations) $D_{\sigma_1} \cdots D_{\sigma_k} = 0$ unless $\sigma_1, \ldots, \sigma_k$ form a cone of Σ.*

If Σ is simplicial, the same result holds with rational coefficients.

Cap product and Kronecker-Poincaré duality

Proposition 4.1.16 ([FS97]). *Let Σ be any fan in $N_{\mathbb{R}}$. Then $X(\Sigma)$ satisfies Chow-Künneth. Now assume Σ is complete. Then $X(\Sigma)$ also satisfies Kronecker duality, and if Σ is unimodular then $X(\Sigma)$ satisfies Kronecker-Poincaré duality and $A_*(X(\Sigma)) \cong H_*(X(\Sigma))$.*

Proof. Recall $X(\Sigma)$ is stratified by weakly linear varieties $O(\sigma) \cong (\mathbb{C}^*)^{n-k}$, so $X(\Sigma)$ is itself weakly linear. Then the result follows from the discussions of Kronecker and Poincaré duality above. □

4.2 Tropical intersection theory

Let Σ (resp. $\mathcal{F}$) be a tropical fan (resp. tropical fan cycle). Recall that in Chapter 2 we defined a ring $A^*(\Sigma)$ (resp. $A^*(\mathcal{F})$) and a group $M_*(\Sigma)$ (resp. $M_*(\mathcal{F})$), which respectively play the roles of the Chow cohomology ring and Chow homology group in tropical geometry. Our above discussion of Poincaré duality for complete nonsingular algebraic varieties motivates the following definition.

Definition 4.2.1. A reduced tropical fan (Σ, ω) of dimension d is *Poincaré* if the cap product with the fundamental weight induces an isomorphism

$$- \cap \omega : A^k(\Sigma) \xrightarrow{\sim} M_{d-k}(\Sigma)$$

for all k. The tropical fan is *star-Poincaré* if all of its star fans are Poincaré.

Remark 4.2.2. Recall tropical fans usually are not complete. The definition above could be thought of as saying that the intersection theory of a Poincaré tropical fan behaves like the intersection theory of a complete nonsingular variety, even in cases where the fan is neither complete nor unimodular.

Recall from Proposition 2.1.31 and Remark 2.4.35 that for a property $\mathcal{P}$ of tropical fans to define a nice local property of tropical fan cycles, $\mathcal{P}$ should be intrinsic to the support, local, stably invariant, and preserved by tropical modifications.

Theorem 4.2.3. *The property of being a star-Poincaré tropical fan is intrinsic to the support, local, stably invariant, and preserved by tropical modifications.*

Proof sketch. The property of being star-Poincaré is local by definition. To show it is also intrinsic to the support, one uses Theorem 2.1.33—the key is to show that being star-Poincaré is preserved by stellar refinements and coarsenings. This is the most nontrivial part of the proof and this property has played a fundamental role in the development of combinatorial Hodge theory, see [AHK18; ADH20; AP20; AP21]. In particular we refer to [AP21, Section 6.7.3] for more details on the proof.

The proofs of stable invariance and preservation by tropical modifications are obtained by direct verification. For stable invariance, this follows by the Chow-Künneth property for toric varieties as discussed in Section 4.1, cf. [AP21, Section 6.7.1]. For preservation by tropical modifications, this follows by a direct computation of the tropical Chow cohomology ring and Chow homology group of a tropical modification, cf. [AP21, Theorem 6.8, Section 6.7.2]. □

Remark 4.2.4. Let Σ be a star-Poincaré tropical fan. Since by definition Σ is reduced and $A^1(\Sigma^\sigma) \cong M_{d-1}(\Sigma^\sigma)$ for all $\sigma \in \Sigma$, we avoid the subtle technical issues mentioned in Remarks 2.4.28, 2.4.29—any tropical Weil divisor on Σ is principal, any two tropical modifications of Σ along the same reduced tropical Weil divisor are isomorphic, and degenerate tropical modifications are isomorphic to Σ.

Remark 4.2.5. As is evidenced by the proof sketch, the above theorem is essentially shown in [AP21, Theorem 6.11]. The only difference is that in our version we work more generally with possibly non-unimodular tropical fans. This difference is immaterial when discussing tropical fan cycles as below, cf. Proposition 2.2.4 and Remark 2.2.8. However, if one works with $A^*_{pw}(\Sigma)$ instead of $A^*(\Sigma)$, as has previously been the convention in tropical geometry (cf. Remark 2.2.8), then one runs into problems with non-unimodular fans. Indeed, as we will see, all complete tropical fans are star-Poincaré, but by [KP08], there are non-unimodular complete fans Σ such that $A^*(\Sigma) \ncong A^*_{pw}(\Sigma)$—this implies that there are non-unimodular complete fans which are not even Poincaré if one uses $A^*_{pw}(\Sigma)$ instead of $A^*(\Sigma)$ (see also [AP21, Example 11.9]). This is why we prefer the definition $A^*(\Sigma) = A^*(X(\Sigma))$, rather than the previously suggested definition in terms of piecewise polynomials.

Definition 4.2.6. A reduced tropical fan cycle $\mathcal{F}$ of dimension d is *star-Poincaré* if some (hence any) fan structure on $\mathcal{F}$ is star-Poincaré.

The isomorphism of tropical fan cycles and cocycles on a star-Poincaré tropical fan/fan cycle allows us to define an intersection product on tropical fan cycles.

Definition 4.2.7. The *tropical intersection product* on a star-Poincaré tropical fan Σ or tropical fan cycle $\mathcal{F}$ is the product on $M_*(\Sigma)$ or $M_*(\mathcal{F})$ induced by the Poincaré duality isomorphism with the ring $A^*(\Sigma)$ or $A^*(\mathcal{F})$.

Typically one is interested in giving an alternative definition of the tropical intersection product, from which one can extract more meaningful geometric information. In the remainder of this section, we discuss three classes of star-Poincaré tropical fan cycles for which this can be done. The first two classes are *complete tropical fan cycles* (Section 4.2.1) and *linear tropical fan cycles* (Section 4.2.2). These are the examples which were previously understood in the literature. In Section 4.2.3 we introduce a new class of star-Poincaré tropical fan cycles which we call *quasilinear tropical fan cycles*.

Remark 4.2.8. From our perspective, star-Poincaré tropical fan cycles form the natural class of *smooth* tropical fan cycles. The definition of smoothness in tropical geometry has been a nontrivial topic of study. Originally, smooth tropical fan cycles were taken to be those which are *linear* as discussed below, in part because they are star-Poincaré and admit a well-behaved tropical intersection product [Mik07b; Sha13; FR13; GS21]. We observed, following *a priori* unrelated computations in [Sch22], that there are non-linear tropical fans which are still star-Poincaré, suggesting a broader possible definition of smooth tropical fan cycles. Around the same time, Amini and Piquerez made the similar observation in the realm of tropical homology and cohomology, and proposed a definition of smooth tropical fans as those which satisfy a local Poincaré duality between tropical homology and cohomology [AP21]. The definition of Amini and Piquerez, which differs subtly from ours, is perhaps the best definition of a smooth tropical fan cycle; however, for our purposes it will be best to stick to star-Poincaré tropical fan cycles. We refer to [Ite+19; AP21] for a more in depth discussion of tropical homology and smoothness in tropical geometry.

4.2.1 Complete tropical fan cycles

The following theorem is immediate from Proposition 4.1.16.

Theorem 4.2.9 ([FS97])**.** *Complete tropical fan cycles are star-Poincaré.*

A tropical intersection product on complete tropical fan cycles was first proposed by Mikhalkin in [Mik07b], and the details were first worked out by Allermann and Rau in [AR10], where they proposed an alternative definition of the tropical intersection product. A third definition, well-suited to explicit computation, was given by Jensen and Yu in [JY16]. It has been shown that all three of these definitions coincide and are in fact given by the fan displacement rule introduced by Fulton and Sturmfels in [FS97] to describe the intersection theory of toric varieties [Kat12; Rau16]. From our perspective the definition in terms of the fan displacement rule makes it obvious (following [FS97]) that the tropical intersection product on $M_*(\mathbb{R}^n)$ coincides with the intersection product induced by $A^*(\mathbb{R}^n)$. Thus we give a brief description of the fan displacement rule below, although we note that it is not necessary for our later results. We refer to the aforementioned references for the alternative definitions of the tropical intersection product on $\mathbb{R}^n$.

Theorem 4.2.10 ([Kat12, Theorem 4.5], [Rau16, Theorem 1.9]). *Let Σ be a complete fan, $\omega_1 \in M_{k_1}(\Sigma)$, and $\omega_2 \in M_{k_2}(\Sigma)$. Then*

$$\omega_1 \cdot \omega_2 = \sum_{\substack{\sigma_1 \in \Sigma_{k_1}, \sigma_2 \in \Sigma_{k_2} \\ \tau \prec \sigma_1, \sigma_2}} m^{\tau}_{\sigma_1,\sigma_2} \omega_1(\sigma_1)\omega_2(\sigma_2),$$

where

$$m^{\tau}_{\sigma_1,\sigma_2} = \begin{cases} [N : N_{\sigma_1} + N_{\sigma_2}], & \text{if } (\sigma_1 + v) \cap \sigma_2 \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for any fixed choice of generic vector $v \in N$ in the sense of [FS97].

It is a nontrivial statement that the intersection product constructed in the theorem is well-defined, i.e. does not depend on the choice of generic vector v—see [FS97]. Since the fan displacement rule will not come up again, the details of this construction are not be necessary for our purposes, and so we omit further explanation. See [FS97; Kat12] for more details and examples, especially [Kat12, Example 2.2], [FS97, Example 4.3].

Remark 4.2.11. Note the property of being a complete tropical fan is obviously intrinsic to the support, local, and stably invariant; however, it is not preserved by tropical modifications.

4.2.2 Linear tropical fan cycles

Linear tropical fan cycles are tropical fan cycles defined by supports of Bergman fans of matroids (to be defined momentarily). That linear tropical fan cycles are star-Poincaré is by now a well-established fact which has played a fundamental role in tropical intersection theory and combinatorial and tropical Hodge theory [Fra12; GS21; AHK18; ADH20; Adi19; BES20; AP21]. We will sketch a proof below following [Sha13; AP21]; essentially the same proof will be repeated to show our broader class of quasilinear tropical fan cycles defined in Section 4.2.3 is also star-Poincaré.

Definitions of a tropical intersection product on linear tropical fan cycles were given by Shaw in [Sha13] and separately by Francois and Rau in [FR13]. The details of these definitions are not important for our purposes, so we omit them, although we remark on Shaw's definition below (Remark 4.2.30).

Matroids

Matroids are combinatorial objects abstracting the notion of linear independence. A detailed introduction can be found in [Oxl92]. We will be brief in our exposition and mainly follow [Sha13], see also [Ale15, Chapter 3].

Definition 4.2.12. A *matroid* M is a pair (E, r) consisting of a finite set $E = \{0, 1, \ldots, n\}$ and a nonnegative function $r : 2^E \to \mathbb{Z}_{\geq 0}$ satisfying the following properties.

1. For any $I \subset E$, $r(I) \leq |I|$.

2. If $I \subset J$ then $r(I) \leq r(J)$.

3. $r(I \cup J) + r(I \cap J) \leq r(I) + r(J)$.

Definition 4.2.13.

- A *base* of a matroid $M = (E, r)$ is a subset $B \subset E$ such that $|B| = r(B) = r(E)$.

- A *loop* of a matroid $M = (E, r)$ is an element $i \in E$ such that i is not contained in any base of M.
- A *flat* of a matroid $M = (E, r)$ is a subset $I \subset E$ such that $r(I \cup j) > r(I)$ for all $j \notin I$.
- The *rank* of a matroid is the maximal length of a nested chain of flats.

Definition 4.2.14. Let $M = (E, r)$ be a matroid and $i \in E$.

1. The *deletion* with respect to i is the matroid $M \setminus i = (E \setminus i, r|_{E\setminus i})$.
2. The *restriction* with respect to i is the matroid $M|_i = (E \setminus i, r')$, where $r'(I) = r(I \cup i) - r(i)$.

Example 4.2.15. To an arrangement $\mathcal{A}$ of $n+1$ hyperplanes $H_0, H_1, \dots, H_n$ in $\mathbb{P}^d$ one can associate a loopless matroid $M_{\mathcal{A}}$ on $E = \{0, \dots, n\}$ via the rank function

$$r(I) = \operatorname{codim}\left(\bigcap_{i \in I} H_i\right).$$

The hyperplane arrangement is *essential* if $\bigcap_{i=0}^{n} H_i = \emptyset$. This is equivalent to the rank of the matroid $M_{\mathcal{A}}$ being $d+1$.

The flats of $M_{\mathcal{A}}$ correspond to the intersections of the hyperplanes. The bases correspond to rank $d+1$ subsets $I \subset E$ such that $\bigcap_{i \in I} H_i = \emptyset$. (Loops correspond to degenerate hyperplanes defined by equations $f_i = 0$; we assume this does not occur.)

The deletion matroid $M_{\mathcal{A}} \setminus i$ is the matroid associated to the hyperplane arrangement obtained by removing the hyperplane H_i. The restriction matroid $M_{\mathcal{A}}/i$ is the matroid associated to the hyperplane arrangement obtained by restricting to the hyperplane H_i.

A matroid associated to a hyperplane arrangement is called *realizable*. These will be the primary matroids of interest for us.

Example 4.2.16. The *uniform matroid* $U_{d,n}$ is the matroid whose bases are given by all rank d subsets of $[n] = \{1, \ldots, n\}$. It is the realizable[1] matroid associated to the arrangement of $n+1$ hyperplanes in general position in $\mathbb{P}^{d-1}$. An important special case is when $d = n-1$, in which case the corresponding hyperplane arrangement can be taken to be the coordinate hyperplanes.

Example 4.2.17. The *Fano plane* is the projective plane over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. It has 7 points and 7 lines as pictured in Figure 4.1. This arrangement defines a matroid which is realizable over $\mathbb{F}_2$ but not realizable over any field of characteristic different from 2.

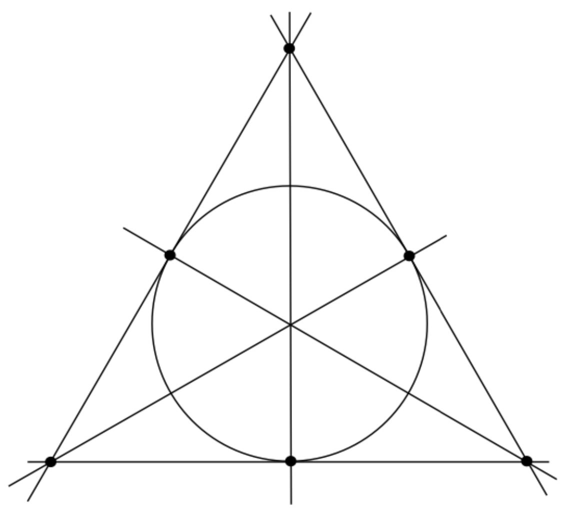


Figure 4.1. The Fano matroid (picture taken from [Ale15, Figure 3.1]).

[1] Assuming the base field is sufficiently large, but we assume the base field is $\mathbb{C}$ unless otherwise mentioned.

Linear tropical fan cycles

Definition 4.2.18 ([AK06; ARW05; Sha13])**.** Let $M = (E, r)$ be a loopless matroid on $E = \{0, 1, \ldots, n\}$. The *Bergman fan* of M is the fan $\Sigma_M \subset \mathbb{R}^n$ defined as follows. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$ and let $e_0 = -e_1 - \cdots - e_n$. For a flat F of M, let $e_F = \sum_{i \in F} e_i$. Then the k-dimensional cones of Σ_M are the cones $\sigma_{F_1,\ldots,F_k}$ spanned by $e_{F_1}, \ldots, e_{F_k}$, for $\emptyset \neq F_1 \subset \cdots \subset F_k \neq E$ a chain of flats of M.

Remark 4.2.19. The loops of a matroid correspond to cones "at infinity" and the correct notion of the Bergman fan for a matroid with loops is not a fan in $N_{\mathbb{R}}$, but rather a fan in the tropical projective space $\mathbb{T}N_{\mathbb{R}}$ [Sha13]. We will not be concerned with this situation, and therefore will assume unless otherwise stated that all of our matroids are loopless.

Lemma 4.2.20 ([Sha13])**.** *The Bergman fan of a loopless matroid is a reduced and locally irreducible tropical fan.*

Definition 4.2.21. A *linear (tropical) fan* is a reduced tropical fan supported on the Bergman fan of a (loopless) matroid.

Example 4.2.22. The Bergman fan of the uniform matroid $U_{n-1,n}$ is the normal fan to the *permutohedron*. It is a complete fan, and the corresponding toric variety is the blowup of $\mathbb{P}^{n-2}$ at the intersections of the coordinate hyperplanes, in increasing order of dimension, cf. [Kap93].

Theorem 4.2.23. *The property of being a linear tropical fan is intrinsic to the support, local, and stably invariant.*

Proof. The property of being a linear tropical fan is intrinsic to the support by definition, and it is local and stably invariant by [Sha13] and [AP21, Theorem 5.11, Proposition 5.14]. □

Definition 4.2.24. A tropical fan cycle $\mathcal{F}$ is *linear* if some (hence any) fan structure on $\mathcal{F}$ is linear. In other words, $\mathcal{F}$ is the support of the Bergman fan of a matroid.

Proposition 4.2.25 ([Sha13])**.** *Let M be a loopless matroid. Then Σ_M is a tropical modification of $\Sigma_{M\setminus i}$ along $\Sigma_{M/i}$.*

Example 4.2.26. When M is realizable by a hyperplane arrangement $H_0, \ldots, H_n$ in $\mathbb{P}^d$, $M \setminus 0$ is realizable by the hyperplane arrangement $H_1, \ldots, H_n$. The tropical modification $\Sigma_M \to \Sigma_{M\setminus 0}$ corresponds to adding the hyperplane H_0 to the arrangment of hyperplanes $H_1, \ldots, H_n$, cf. Proposition 4.3.9.

Example 4.2.27. Let Y be the complement in $\mathbb{P}^2$ of the coordinate hyperplanes as well as the hyperplanes $x_i = x_j$. We will see in Chapter 6 that $Y = M_{0,5}$ is the moduli space of 5 points on $\mathbb{P}^1$. If M denotes the corresponding matroid, then Σ_M is a 2-dimensional fan in $\mathbb{R}^5$ which can be identified with the cone over the Petersen graph. It is obtained from a complete fan in $\mathbb{R}^2$ by a sequence of three tropical modifications; however, determining the fan structure from this sequence is very subtle, see [Sha13; RSS16].

Example 4.2.28. Let M be the Fano matroid as in Example 4.2.17. This is a rank 3 matroid on $[7] = \{1, \ldots, 7\}$, which is not realizable over $\mathbb{C}$. Let the conic in Figure 4.1 be labeled by 7. Then $M \setminus 7$ is the hyperplane arrangement considered in the previous example; in particular it is realizable. The restriction matroid $M/7$ is also realizable. Thus $\Sigma_M = \mathcal{TM}_{\Sigma_{M/7}}\Sigma_{M\setminus 7}$ is the tropical modification of a realizable Bergman fan along a realizable Bergman fan, but Σ_M is not realizable. The problem is essentially that the two realizations are not compatible—they would have to fit together to form the Fano arrangement, which is not possible over $\mathbb{C}$.

Theorem 4.2.29 ([GS21; AP21])**.** *Linear tropical fan cycles are star-Poincaré.*

Proof. By Proposition 4.2.25, the Bergman fan of any matroid can be obtained from the (complete) Bergman fan of the uniform matroid $U_{n-1,n}$ by a sequence of tropical modifications along Bergman fans. Since complete fans are star-Poincaré, the result follows by induction and Theorem 4.2.3. □

Remark 4.2.30. In [Sha13], Shaw uses Proposition 4.2.25 to inductively define a tropical intersection product on linear tropical fan cycles by starting with the intersection product on $\mathbb{R}^n$.

Remark 4.2.31. We note that Proposition 4.2.25 does not imply that the property of being a linear tropical fan is preserved by tropical modifications. Indeed, this is false, as observed by Amini and Piquerez [AP21, Example 11.11]. Thus neither the properties of being complete nor being linear are preserved by tropical modifications (Remark 4.2.11). Since the property of being star-Poincaré *is* preserved by tropical modifications, this indicates that the classes of complete or linear tropical fans are too restrictive. This issue is fixed by the next class of tropical fans below.

4.2.3 Quasilinear tropical fan cycles

Theorem 4.2.3 and Remark 4.2.31 motivate the following definition.

Definition 4.2.32. A reduced tropical fan cycle $\mathcal{F}$ is *quasilinear* if it is isomorphic to a complete tropical fan cycle or a tropical modification of a quasilinear tropical fan cycle $\mathcal{F}'$ along a trivial or quasilinear tropical divisor $\mathcal{D}$. A *quasilinear tropical fan* is any (reduced) tropical fan supported on a quasilinear tropical fan cycle.

Remark 4.2.33. The tropical Weil divisor $\mathcal{D}$ on $\mathcal{F}'$ can be viewed either as a tropical fan cycle on $\mathcal{F}'$ or on the ambient space. In the definition we take the latter viewpoint.

Remark 4.2.34. If $\mathcal{F}$ is isomorphic to a degenerate tropical modification of $\mathcal{F}'$, then in fact $\mathcal{F} \cong \mathcal{F}'$, see Remark 4.2.41 below. If $\mathcal{F}$ is isomorphic to a nondegenerate tropical modification of $\mathcal{F}'$ along $\mathcal{D}$, then $\dim N_{\mathcal{D}} \leq \dim N_{\mathcal{F}'} = \dim N_{\mathcal{F}} - 1$. Thus the definition of quasilinearity is inductive on dimension of the ambient space.

Example 4.2.35. It follows from Proposition 4.2.25 and induction that linear tropical fan cycles are also quasilinear.

Remark 4.2.36. We comment on some differences between linear and quasilinear tropical fan cycles.

1. Allowing degenerate modifications leads to a much larger class of tropical modifications than just the matroidal tropical modifications considered in Proposition 4.2.25. The same effect is achieved by also working up to isomorphism, see Remark 4.2.41.

2. The image of a linear tropical fan cycle under any coordinate projection is also linear [Sha13]. This need not be true for quasilinear tropical fan cycles. Indeed, for quasilinearity, all one needs is that, after composing with an isomorphism, some projection is quasilinear. This leads to a much broader class of tropical fan cycles.

See below for more examples of quasilinear tropical fan cycles. We now discuss the main properties.

Theorem 4.2.37. *The property of being a quasilinear tropical fan is intrinsic to the support, local, stably invariant, and preserved by tropical modifications.*

Proof. Quasilinearity is intrinsic to the support and preserved by tropical modifications by definition. It follows from Propositions 4.2.38 and 4.2.39 that it is also local and stably invariant. □

Proposition 4.2.38. *If $\mathcal{F}$ and $\mathcal{G}$ are two reduced tropical fan cycles, then $\mathcal{F}\times\mathcal{G}$ is quasilinear $\iff$ $\mathcal{F}$ and $\mathcal{G}$ are both quasilinear.*

Proof. Note that $\mathcal{F}\times\mathcal{G}$ is complete $\iff$ $\mathcal{F}$ and $\mathcal{G}$ are both complete. Thus we can assume without loss of generality that $\mathcal{F}\times\mathcal{G}$ and $\mathcal{G}$ are not complete. We will prove the result by induction on the dimension of the ambient vector space of $\mathcal{F}\times\mathcal{G}$. Note the base case $n = 2$ is trivial.

($\Longrightarrow$) Suppose $\mathcal{F}$ and $\mathcal{G}$ are both quasilinear. By assumption $\mathcal{G}$ is a non-complete quasilinear tropical fan cycle, so we can write $\mathcal{G} \cong \mathcal{TM}_{\mathcal{D}}\mathcal{G}'$ for some quasilinear tropical fan cycle $\mathcal{G}'$ and trivial or quasilinear divisor $\mathcal{D}$. Then

$$\mathcal{F}\times\mathcal{G} \cong \mathcal{F}\times\mathcal{TM}_{\mathcal{D}}\mathcal{G}' \cong \mathcal{TM}_{\mathcal{F}\times\mathcal{D}}(\mathcal{F}\times\mathcal{G}'),$$

so by induction $\mathcal{F}\times\mathcal{G}$ is quasilinear.

($\Longleftarrow$) Suppose $\mathcal{F}\times\mathcal{G}$ is quasilinear. By assumption both $\mathcal{F}\times\mathcal{G}$ and $\mathcal{G}$ are not complete, so (after composing with an isomorphism if necessary), we can assume the projection to the last coordinate realizes $\mathcal{F} \times \mathcal{G}$ as a tropical modification of a quasilinear tropical fan $\mathcal{F} \times \mathcal{G}'$ along a trivial or quasilinear divisor $\mathcal{D}$. By induction $\mathcal{G}'$ is quasilinear. If $\mathcal{D}$ is trivial then $\mathcal{G}$ is a degenerate tropical modification of $\mathcal{G}'$, so we are done. Suppose $\mathcal{D}$ is nontrivial. Then $\mathcal{D} = \mathcal{F} \times \mathcal{D}' + \mathcal{D}'' \times \mathcal{G}$ for some tropical divisors $\mathcal{D}'$ on $\mathcal{G}'$ and $\mathcal{D}''$ on $\mathcal{F}$ [AP21, Proposition 6.3]. But $\dim \mathcal{D} = \dim \mathcal{F} \times \mathcal{D}' = \dim \mathcal{D}'' \times \mathcal{G}$, and since $\mathcal{D}$ is quasilinear it is reduced and irreducible, thus either $\mathcal{D} = \mathcal{F} \times \mathcal{D}'$ or $\mathcal{D} = \mathcal{D}'' \times \mathcal{G}$. Since $\mathcal{F} \times \mathcal{G} \cong \mathcal{TM}_{\mathcal{D}}(\mathcal{F} \times \mathcal{G}')$, it follows that $\mathcal{D} = \mathcal{F} \times \mathcal{D}'$. We conclude that

$$\mathcal{F} \times \mathcal{G} \cong \mathcal{TM}_{\mathcal{F} \times \mathcal{D}'}(\mathcal{F} \times \mathcal{G}') \cong \mathcal{F} \times \mathcal{TM}_{\mathcal{D}'}\mathcal{G}',$$

so $\mathcal{G} = \mathcal{TM}_{\mathcal{D}'}\mathcal{G}'$. Since by induction $\mathcal{D}'$ and $\mathcal{G}'$ are quasilinear, it follows that $\mathcal{G}$ is quasilinear.

□

Proposition 4.2.39. *Star fans of quasilinear tropical fans are quasilinear.*

Proof. Recall (Proposition 2.4.32) that the star fans of a tropical modification $\mathcal{TM}_{\Delta}(\Sigma)$ are all isomorphic to either star fans of Δ or (possibly degenerate) tropical modifications of star fans of Σ along star fans of Δ. Thus if the star fans of Σ and Δ are all quasilinear, then the star fans of $\mathcal{TM}_{\Delta}(\Sigma)$ are quasilinear. Since star fans of complete fans are complete (hence quasilinear), the result follows by induction and the fact that quasilinearity is stably invariant by the previous proposition. □

Theorem 4.2.40. *Quasilinear tropical fan cycles are locally irreducible and star-Poincaré.*

Proof. By definition a quasilinear tropical fan cycle is obtained from a complete tropical fan cycle by a sequence of tropical modifications along trivial or quasilinear tropical divisors. Since complete tropical fan cycles are locally irreducible and star-Poincaré, and these properties are preserved by tropical modifications along divisors which are trivial or satisfy the same properties (Theorems 2.4.34, 4.2.3), the result follows by induction. □

Remark 4.2.41. The theorem implies that any two tropical modifications of a quasilinear tropical fan cycle along the same reduced tropical divisor are isomorphic, and a degenerate tropical modification of a quasilinear tropical fan cycle is isomorphic to the original fan cycle, cf. Remark 4.2.4.

In particular, in the definition of quasilinearity, we could have only considered tropical modifications along quasilinear tropical divisors, rather than trivial or quasilinear tropical divisors. We include trivial divisors in the definition to better emphasize the difference between quasilinearity and linearity. (Degenerate tropical modifications will also be useful for showing a given tropical fan cycle is quasilinear, see Theorem 4.3.23 and Sections 7.2.1, 8.1.1.)

Remark 4.2.42. The properties of quasilinear fans discussed above should not come as a surprise—in a sense, quasilinearity is defined in order to force these properties to be true. Quasilinear tropical fans also satisfy a number of additional desirable properties discussed in [AP21]; for instance, they are tropically smooth in the sense of *op. cit.* The class of quasilinear tropical fans should also be compared with the (*a priori* larger) class of *shellable* tropical fans introduced in [AP21].

Remark 4.2.43. In principle one could follow the arguments of Shaw [Sha13] to inductively define a tropical intersection product on quasilinear tropical fan cycles, cf. Remark 4.2.30.

However, the technicalities of this construction are likely to be quite subtle, as *a priori* they depend on the choice of tropical modification. We do not pursue this construction any further.

Examples

Example 4.2.44. The standard tropical line of Example 2.1.8 is quasilinear, since it is the tropical modification of $\mathbb{R}^1$ with respect the function $\varphi(x) = \min\{x, 0\}$, as shown in Example 2.4.22. Note however the tropical modification of $\mathbb{R}^1$ with respect to $\varphi(x) = \min\{2x, 0\}$ is not reduced, hence not quasilinear, cf. Example 2.4.23.

Example 4.2.45. Let Σ be the line $ay = bx$, $a, b \in \mathbb{Z} \setminus 0$, with unique reduced tropical fan structure given by the rays through the points (a, b) and $(-a, -b)$. After scaling down the equation if necessary, we can assume that a and b are coprime. Then there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$. The map

$$N_\Sigma \to \mathbb{Z}, \quad (a, b) \mapsto ax + by = 1$$

therefore induces an isomorphism of Σ with $\mathbb{R}^1$, so Σ is quasilinear (cf. Example 2.3.13). Note however that Σ cannot be written as a degenerate tropical modification of $\mathbb{R}^1$ unless either a/b or b/a is an integer.

Example 4.2.46. The degenerate tropical modification of the standard tropical line, as shown in Example 2.4.31, is quasilinear.

Example 4.2.47. The tropical modification of the standard tropical line with respect to the function $\varphi(x,y) = \min\{x,y,0\}$, as shown in Example 2.4.24, is quasilinear.

Example 4.2.48. The tropical modification of $\mathbb{R}^2$ with respect to the functions $\varphi(x,y) = \min\{x,0\}$ is quasilinear, but the modification with respect to $\varphi(x,y) = \min\{2x,0\}$ is not quasilinear, cf. Example 2.4.25.

Example 4.2.49. The tropical modification of $\mathbb{R}^2$ with respect to the function $\varphi(x,y) = \min\{x,y,0\}$ is quasilinear, cf. Example 2.4.26.

Example 4.2.50. In Chapters 7 and 8, we will show that the tropicalizations of the moduli spaces $M(3,n)$ of $n \leq 8$ lines in $\mathbb{P}^2$ and $Y(3,n)$ of marked del Pezzo surfaces of degree $9-n$ for $n \leq 7$ are quasilinear. These moduli spaces at first glance are very far from linear, and this result is one of the main results of this dissertation (and one of the motivations for the definition of quasilinearity).

4.3 Intersection theory of tropical compactifications

Let $i : \overline{Y} \hookrightarrow X(\Sigma)$ be a tropical compactification of a d-dimensional very affine variety $Y \subset T$. The goal of this section is to give criteria under which the intersection theory of the algebraic variety $\overline{Y}$ is the same as the intersection theory of the tropical fan Σ. In general one might not expect much in this direction, since the intersection theory of Σ is essentially the same as the intersection theory of the toric variety $X(\Sigma)$, and $\overline{Y}$ can be very far from toric: any smooth projective variety $\overline{Y} \subset \mathbb{P}^n$ can be thought of as a tropical compactification of $Y = \overline{Y} \cap (\mathbb{C}^*)^n$ [MS15, Exercise 6.8.11]. On the other hand, the prototypical examples of tropical compactifications are *linear* tropical compactifications, where it was observed by A. Gross that the intersection theories of $\overline{Y}$ and Σ agree [Gro15]. We present a generalization of this result to broader classes of tropical compactifications.

Before stating our main theorem, we make a few preliminary comments.

1. Since $A^*(\Sigma) = A^*(X(\Sigma))$, there is a natural pullback morphism $i^* : A^*(\Sigma) \to A^*(\overline{Y})$ offering a comparison of the Chow cohomologies.

2. In general there is no analogous obvious comparison map $i_* : A_*(\overline{Y}) \to M_*(\Sigma)$. There is a pushforward of cycles from $A_*(\overline{Y})$ to $A_*(X(\Sigma))$, but since the dimension of $\overline{Y}$ is typically much smaller than the dimension of $X(\Sigma)$, this pushforward is typically trivial. The tropical Chow homology group $M_*(\Sigma)$ is preferable in this setting because $\dim \overline{Y} = \dim \Sigma$; the isomorphism $M_k(\Sigma) \cong \mathrm{Hom}(A_{n-k}(\Sigma), \mathbb{Z})$ shifts the grading so that the pushforward can occur in the appropriate dimension.

   To define the pushforward, we need to assume $i : \overline{Y} \hookrightarrow X(\Sigma)$ is a regular embedding. (This occurs for instance if $\overline{Y}$ is a schön tropical compactification (Theorem 3.2.12), or if both $\overline{Y}$ and $X(\Sigma)$ are nonsingular.) Then by [Ful98, Section 6.2], there exists a Gysin pullback morphism

   $$i^* : A_{n-d+k}(X(\Sigma)) \to A_k(\overline{Y}).$$

   Dualizing, we get a morphism

   $$\mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to \mathrm{Hom}(A_{n-d+k}(X(\Sigma)), \mathbb{Z}) \cong M_{d-k}(\Sigma).$$

We view this as the correct pushforward of cycles, and denote it by $i_* : \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to M_{d-k}(\Sigma)$.

Proposition 4.3.1. *Assume $i : \overline{Y} \hookrightarrow X(\Sigma)$ is a regular embedding. Then the following diagram is commutative.*

$$\begin{array}{ccc} A^k(\Sigma) & \xrightarrow{-\cap\omega} & M_{d-k}(\Sigma) \\ \big\downarrow{\scriptstyle i^*} & & {\scriptstyle i_*}\big\uparrow \\ A^k(\overline{Y}) & \xrightarrow{\mathcal{D}_{\overline{Y}}} & \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}) \end{array} \tag{4.1}$$

Proof. Unwinding definitions, we are asking that for $\alpha \in A^k(X(\Sigma))$, $\beta \in A_{n-d+k}(X(\Sigma))$,

$$\omega(\alpha \cap \beta) = \int_{\overline{Y}} i^*\alpha \cap i^*\beta,$$

where $i^*\alpha$ is the pullback on Chow cohomology and $i^*\beta$ is the Gysin pullback on Chow homology. But $i^*\alpha \cap i^*\beta = i^*(\alpha \cap \beta)$, and the result follows simply by definition of ω, cf. Proposition 3.2.9, Remark 3.2.10. □

Remark 4.3.2. The composition

$$A^k(\overline{Y}) \xrightarrow{\mathcal{D}_{\overline{Y}}} \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}) \xrightarrow{i_*} M_{d-k}(\Sigma)$$

can reasonably be interpreted as the tropicalization of cocycles on $\overline{Y}$, cf. [Gro18]. In particular it sends $[\overline{Y}] \in A^0(\overline{Y}) \cong \mathbb{Z} \cdot [\overline{Y}]$ to the fundamental weight on Σ, i.e. to $\mathrm{trop}(Y)$. As a generalization of this idea one can define the *extended tropicalization* of any quasiprojective variety, not necessarily very affine, see [Pay09].

3. In order for Σ to have a well-defined tropical intersection theory, we require that Σ is star-Poincaré. This already imposes strong conditions on Σ, and since Σ is supported on $\operatorname{trop}(Y)$ and the property of being star-Poincaré is intrinsic to the support (Theorem 4.2.3), it also imposes strong conditions on Y.

4. To say that the intersection theories of $\overline{Y}$ and Σ are "the same," we ask that both $i^* : A^*(\Sigma) \to A^*(\overline{Y})$ and $i_* : \operatorname{Hom}(A_*(\overline{Y}), \mathbb{Z}) \to M_*(\Sigma)$ are isomorphisms. Thus we are asking that all arrows in the commutative diagram (4.1), except for $\mathcal{D}_{\overline{Y}} : A^k(\overline{Y}) \to \operatorname{Hom}(A_k(\overline{Y}), \mathbb{Z})$, are isomorphisms. This of course implies $\mathcal{D}_{\overline{Y}}$ is also an isomorphism, i.e. $\overline{Y}$ satisfies Kronecker duality. So this also imposes very strong conditions on $\overline{Y}$.

Theorem 4.3.3. *Assume $i : \overline{Y} \hookrightarrow X(\Sigma)$ is a regular embedding and Σ is star-Poincaré.*

1. *If all strata $\overline{Y} \cap O(\sigma)$ of $\overline{Y}$ are irreducible and Chow-free, then $i_* : \operatorname{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to M_{d-k}(\Sigma)$ is an isomorphism.*

2. *If in addition to condition 1 one of the following conditions holds, then $i^* : A^k(\Sigma) \to A^k(\overline{Y})$ is also an isomorphism, and $\overline{Y}$ satisfies Kronecker duality.*

   (a) All strata $\overline{Y} \cap O(\sigma)$ are weakly linear.

   (b) $\overline{Y}$ and $X(\Sigma)$ are both nonsingular.

3. *If* all *of the above conditions hold, then we also have isomorphisms*

$$H^*(\overline{Y}) \cong A^*(\overline{Y}) \cong A^*(\Sigma) \cong H^*(X(\Sigma)).$$

Proof. The condition that Σ is star-Poincaré implies that $- \cap \omega : A^k(\Sigma) \to M_{d-k}(\Sigma)$ is an isomorphism, which by the commutative diagram 4.1 implies that $i^* : A^k(\Sigma) \to A^k(\overline{Y})$ is injective and $i_* : \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to M_{d-k}(\Sigma)$ is surjective.

1. Assume all strata $\overline{Y} \cap O(\sigma)$ are irreducible and Chow-free. We need to show $i_* : \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to M_{d-k}(\Sigma)$ is injective. It is enough to show the Gysin pullback $i^* : A_{n-d+k}(X(\Sigma)) \to A_k(\overline{Y})$ is surjective. But $A_{n-d+k}(X(\Sigma))$ is generated by $[V(\sigma)]$ for $\sigma \in \Sigma_{d-k}$ (Propositions 4.1.13, 4.1.14) and $A_k(\overline{Y})$ is generated by $[\overline{Y} \cap V(\sigma)]$ for $\sigma \in \Sigma_{d-k}$ (Proposition 4.1.13). Furthermore, since $\overline{Y}$ is a tropical compactification, it intersects torus orbits properly, so $i^*[V(\sigma)] = [\overline{Y} \cap V(\sigma)]$ The result follows.

2. (a) If all strata $\overline{Y} \cap O(\sigma)$ are weakly linear, then $\overline{Y}$ satisfies Kronecker duality (Corollary 4.1.8), so all arrows in the commutative diagram 4.1 are isomorphisms except for $i^* : A^k(\Sigma) \to A^k(\overline{Y})$. This implies i^* is also an isomorphism.

   (b) If $\overline{Y}$ and $X(\Sigma)$ are both nonsingular, then we can identify $A_{n-d+k}(X(\Sigma)) \cong A^{d-k}(\Sigma)$ and $A_k(\Sigma) \cong A^{d-k}(\overline{Y})$. Then $i_* : \mathrm{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to \mathrm{Hom}(A_{n-d+k}(X(\Sigma)), \mathbb{Z})$ is just the dual of

   $$i^* : A^{d-k}(\Sigma) \to A^{d-k}(\overline{Y}).$$

   So in this case i^* is an isomorphism for all $k \iff i_*$ is an isomorphism for all k, and the result follows. (Alternatively, we can now say that $A^k(\Sigma)$ is generated by $[V(\sigma)]$, $\sigma \in \Sigma_k$, and $A^k(\overline{Y})$ is generated by $[\overline{Y} \cap V(\sigma)]$, $\sigma \in \Sigma_k$, and $i^*[V(\sigma)] = [\overline{Y} \cap V(\sigma)]$ so i^* is surjective.)

3. All we need to show now is that $H^*(\overline{Y}) \cong A^*(\overline{Y})$ and $A^*(\Sigma) = A^*(X(\Sigma)) \cong H^*(X(\Sigma))$. But this follows because $\overline{Y}$ and $X(\Sigma)$ are both nonsingular and stratified by weakly linear varieties (Corollary 4.1.8).

□

Remark 4.3.4. The theorem should be compared with the work of A. Gross on intersection theory of tropicalizations of toroidal embeddings [Gro18].

In the remainder of this chapter we describe two classes of tropical compactifications for which all of the results of Theorem 4.3.3 hold. These are the compactifications of varieties whose tropicalizations are linear or quasilinear tropical fan cycles as discussed in the previous section.

Remark 4.3.5. Of course, the results also trivially hold for varieties $Y \subset T$ whose tropicalizations are complete tropical fan cycles: then $Y = T$ and $\overline{Y} = X(\Sigma)$ is a complete toric variety.

4.3.1 Linear tropical compactifications

Definition 4.3.6. A very affine variety $Y \subset T$ is *linear* if $\mathrm{trop}(Y)$ is a linear tropical fan cycle.

Recall from Example 3.1.29 that if Y is the complement of $n+1$ hyperplanes in $\mathbb{P}^d$, then Y is very affine and taking equations for the hyperplanes gives an embedding of Y in its intrinsic torus $(\mathbb{C}^*)^n$.

Proposition 4.3.7 ([KP11])**.** *A very affine variety $Y \subset T$ is linear $\iff$ Y is a complement of a hyperplane arrangement.*

Theorem 4.3.8 ([Gro15])**.** *Let $\overline{Y} \subset X(\Sigma)$ be a tropical compactification of a linear variety (a* linear tropical compactification*). Then*

1. *Σ is star-Poincaré.*
2. *all strata of $\overline{Y}$ are also linear, and in particular smooth, irreducible, Chow-free, and weakly linear.*

Thus Theorem 4.3.3 applies, i.e.

$$H^*(\overline{Y}) \cong A^*(\overline{Y}) \cong A^*(\Sigma) \cong H^*(X(\Sigma)).$$

Proofs of the above propositions will follow from our discussion of quasilinear tropical compactifications below. The proofs we will give are essentially generalizations of the same ideas used in [KP11; Gro15]. The key idea for our generalization is that the results can be obtained inductively: recall if Σ_M is the Bergman fan of a matroid, then Σ_M is a tropical modification of $\Sigma_{M\setminus i}$ along $\Sigma_{M/i}$ (Proposition 4.2.25). The following algebraic analog, whose proof is straightforward, offers intuition for this statement, cf. Example 4.2.26.

Proposition 4.3.9. *Let $Y \subset (\mathbb{C}^*)^n$ be a linear variety. Then either $Y = (\mathbb{C}^*)^n$ or $Y = \Gamma_f \cap (\mathbb{C}^*)^n$, where $\Gamma_f \subset Y' \times \mathbb{A}^1 \subset (\mathbb{C}^*)^{n-1} \times \mathbb{A}^1$ is the graph of a linear function f on a linear variety $Y \subset (\mathbb{C}^*)^{n-1}$.*

4.3.2 Quasilinear tropical compactifications

Quasilinear varieties

Definition 4.3.10. A very affine variety $Y \subset T$ is *quasilinear* if $\mathrm{trop}(Y)$ is a quasilinear tropical fan cycle.

As the class of quasilinear tropical fan cycles is much larger than the class of linear tropical fan cycles, so too is the class of quasilinear very affine varieties much larger than the class of linear very affine varieties. See below for some basic examples, as well as Chapters 7 and 8 in the second part of this dissertation for some more involved examples.

The theory of quasilinear tropical cycles and compactifications is essentially analogous to that of linear tropical cycles and compactifications, except for some subtle technical details. First of all, we need to work up to tropical isomorphism, cf. Definitions 4.2.32, 2.3.11.

Definition 4.3.11. Two very affine varieties $Y \subset T$ and $Y' \subset T'$ are *tropically isomorphic* if there is an isomorphism of tropicalizations $\mathrm{trop}(Y) \cong \mathrm{trop}(Y')$ inducing an isomorphism of algebraic varieties $Y \cong Y'$.

Remark 4.3.12. Let $Y \subset T$ be a very affine variety, and suppose $\mathrm{trop}(Y)$ spans a proper subspace V of $N_{\mathbb{R}}$. Let T' be the subtorus of T corresponding to V. Then a translate Y' of Y is contained in T', and $\mathrm{trop}(Y) \cong \mathrm{trop}(Y')$ [Jel20, Proposition 5.3]. In particular, Y is tropically isomorphic to Y'.

Definition 4.3.13. Let f be a regular function on a very affine variety $Y \subset (\mathbb{C}^*)^{n-1}$. The *very affine graph* of f on Y is $\widetilde{Y} = \Gamma_f \cap (\mathbb{C}^*)^n \subset (\mathbb{C}^*)^n$, where $\Gamma_f \subset Y \times \mathbb{A}^1 \subset (\mathbb{C}^*)^{n-1} \times \mathbb{A}^1$ is the usual graph of f.

Remark 4.3.14. Note the very affine graph of f is identified via the natural projection with the complement $Y \setminus \operatorname{div}(f)$.

Theorem 4.3.15. *Let $\widetilde{Y}$ be a quasilinear variety. Then either $\widetilde{Y}$ is a torus, or $\widetilde{Y}$ is tropically isomorphic to the very affine graph of a regular function f on a quasilinear variety Y, such that either f is nonvanishing or $D = \operatorname{div}(f)$ is also quasilinear.*

Proof. Following Remark 4.3.12 we can reduce to the case where $\widetilde{Y} \subset (\mathbb{C}^*)^n$ and $\operatorname{trop}(\widetilde{Y}) \subset \mathbb{R}^n$ is the tropical modification of a quasilinear tropical cycle $\mathcal{F} \subset \mathbb{R}^{n-1}$ along a trivial or quasilinear divisor $\mathcal{D}$. Let $\pi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}$ be the projection of tori corresponding to the projection $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$, and let $Y = \overline{\pi(\widetilde{Y})} \subset (\mathbb{C}^*)^{n-1}$. Then $\pi : \widetilde{Y} \to Y$ is dominant, so by Lemma 3.1.23, $\operatorname{trop}(Y) = p(\operatorname{trop}(\widetilde{Y})) = \mathcal{F}$. In particular, $\dim Y = \dim \operatorname{trop}(Y) = \dim \operatorname{trop}(\widetilde{Y}) = \dim \widetilde{Y}$, so $\pi : \widetilde{Y} \to Y$ is a dominant morphism of varieties of the same dimension, hence it is generically finite, say of degree δ. Then by Theorem 3.1.24, $\operatorname{trop}(Y) = \frac{1}{\delta} p_* \operatorname{trop}(\widetilde{Y})$. But since $\operatorname{trop}(\widetilde{Y}) \to \operatorname{trop}(Y)$ is a tropical modification of reduced tropical fan cycles, $p_* \operatorname{trop}(\widetilde{Y}) = \operatorname{trop}(Y)$ and $\delta = 1$. Thus $\pi : \widetilde{Y} \to Y$ is dominant and generically finite of degree 1, i.e. π is birational.

Since $\operatorname{trop}(Y) = \mathcal{F}$ is quasilinear, Y is quasilinear. If $Y = (\mathbb{C}^*)^{n-1}$, then Y is smooth and rational. In particular, Y is normal, so the birational morphism $\pi : \widetilde{Y} \to Y$ is an isomorphism with an open dense subset of Y. Thus $\widetilde{Y}$ is also smooth and rational. By induction (on

$n - \dim \widetilde{Y}$), we conclude that even if $Y \subsetneq (\mathbb{C}^*)^{n-1}$, both Y and $\widetilde{Y}$ are smooth and rational, and $\pi : \widetilde{Y} \to Y$ is an isomorphism $\pi : \widetilde{Y} \xrightarrow{\sim} U$ with an open dense subset U of Y.

If $U = Y$, then $\widetilde{Y}$ is the graph of the nonvanishing regular function on Y given by the composition $Y \xrightarrow{\pi^{-1}} \widetilde{Y} \to \mathbb{C}^*$, where $\widetilde{Y} \to \mathbb{C}^*$ is induced by the projection complementary to $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}$. So we are done in this case.

Suppose $U \subsetneq Y$. Note U is actually an affine open subset of Y, since the inclusion $U \hookrightarrow Y$ is identified with the morphism of (very) affine varieties $\pi : \widetilde{Y} \to Y$. Since in addition Y is nonsingular, it follows that $Y \setminus U$ is an effective Cartier divisor D [SPA21, Tag 0BCW]. Since Y is rational, all of its divisors are principal, so $D = \operatorname{div}(f)$ for some regular function f on Y. Then by construction $\operatorname{trop}(D) = \mathcal{D}$, so D is quasilinear, and $\widetilde{Y}$ is tropically isomorphic to the very affine graph of f on Y. □

We extract from the above theorem and its proof some fundamental properties of quasilinear varieties below.

Theorem 4.3.16. *Quasilinear varieties are smooth, rational, Chow-free, and weakly linear.*

Proof. We have shown in the proof of the previous theorem that quasilinear varieties are smooth and rational. To show they are Chow-free and weakly linear, we use induction on the dimension of the ambient torus. If $\widetilde{Y} \subset (\mathbb{C}^*)^n$ is quasilinear, then by the previous theorem, either $\widetilde{Y} = (\mathbb{C}^*)^n$, or $\widetilde{Y} \cong Y \setminus D$ for some quasilinear variety $Y \subset (\mathbb{C}^*)^{n-1}$ and trivial or quasilinear divisor D on Y. By induction both Y and D are Chow-free and weakly linear, hence $\widetilde{Y}$ is Chow-free and weakly linear. □

Remark 4.3.17. Of course since quasilinear varieties are quasiaffine and rational, Chow-free is already immediate. For most of our applications it is enough to know quasilinear varieties are smooth and rational.

Remark 4.3.18. The above results should be compared with, and are inspired by, [KP11, Section 4], [Jel20, Section 5], where the analogous results for *linear* varieties are shown.

Quasilinear compactifications

Definition 4.3.19. A tropical compactification $\overline{Y} \subset X(\Sigma)$ of a very affine variety $Y \subset T$ is *quasilinear* if $Y \subset T$ is quasilinear.

Theorem 4.3.20. *All strata of quasilinear compactifications are quasilinear varieties. In particular, they are smooth, irreducible, rational, Chow-free, and weakly linear.*

Proof. Let $\overline{Y} \subset X(\Sigma)$ be a quasilinear compactification, and denote the strata of $\overline{Y}$ by $Y_\sigma = \overline{Y} \cap O(\sigma)$. By Proposition 3.2.9 $\operatorname{trop}(Y_\sigma) = |\Sigma^\sigma|$ and the weights on the tropical fan Σ^σ are induced by Y_σ. Since Σ is quasilinear, so is Σ^σ by Proposition 4.2.39. In particular, Σ^σ is reduced and irreducible by Theorem 4.2.40, so Y_σ is irreducible and generically reduced by Corollary 3.1.12. Let Y_σ^{red} be the reduced induced scheme structure on Y_σ. Then $\operatorname{trop}(Y_\sigma^{\text{red}}) = \operatorname{trop}(Y_\sigma) = \Sigma^\sigma$ (with the same weights, since the weights on $\operatorname{trop}(Y_\sigma)$ are 1), so since Σ^σ is quasilinear, Y_σ^{red} is a quasilinear variety. In particular Y_σ^{red} is smooth and irreducible by Theorem 4.3.16. Therefore, by [Car12, Theorem 11], Y_σ is also reduced, i.e. $Y_\sigma = Y_\sigma^{\text{red}}$, hence Y_σ is a quasilinear variety. We have shown that strata of quasilinear compactifications are quasilinear varieties. The last statement of the theorem follows from Theorem 4.3.16. $\square$

The following corollary and theorem will play an important role in Part II of this dissertation.

Corollary 4.3.21. *Quasilinear varieties are schön.*

Proof. The above theorem shows that all strata of any quasilinear compactification are smooth. The result follows by definition. □

Theorem 4.3.22. *Let $\overline{Y} \subset X(\Sigma)$ be a quasilinear compactification. Then $i^* : A^*(\Sigma) \to A^*(\overline{Y})$ is an isomorphism and $\overline{Y}$ satisfies Kronecker duality. If in addition $X(\Sigma)$ is smooth, then $\overline{Y}$ is also smooth and*

$$H^*(\overline{Y}) \cong A^*(\overline{Y}) \cong A^*(X(\Sigma)) \cong H^*(X(\Sigma)).$$

Proof. Since quasilinear tropical fans are star-Poincaré (Theorem 4.2.40) and strata of quasilinear tropical compactifications are irreducible, Chow-free, and weakly linear (Theorem 4.3.20), the first statement follows immediately from Theorem 4.3.3. The last statement also follows from Theorem 4.3.3 and the above Corollary 4.3.21 that quasilinear varieties are schön. □

Criteria for quasilinearity

In this section we give criteria for showing a very affine variety is quasilinear. The main advantage of these criteria is that they do not require directly computing the tropicalization, which in general can be computationally challenging.

The following partial converse to Theorem 4.3.15 is our main criterion.

Theorem 4.3.23. *Let $Y \subset T$ be a quasilinear variety, and f a regular function on Y such that either f is nonvanishing, or $D = \mathrm{div}(f) \subset Y \subset T$ is also quasilinear. Then the very affine graph $\widetilde{Y}$ of f on Y is quasilinear.*

Proof. Since $\mathrm{trop}(Y)$ is quasilinear, it is reduced, locally irreducible, and star-Poincaré by Theorem 4.2.40. It is shown in [RSS16, Proposition 4.1], that the properties that f is regular on Y and $\mathrm{trop}(Y)$ is locally irreducible together imply that the fibers of the natural projection $\mathrm{trop}(\widetilde{Y}) \to \mathrm{trop}(Y)$ are all either a single point, or a half-line in the upwards direction; furthermore, the locus of points with infinite fibers is precisely $\mathrm{trop}(D)$. Since $\mathrm{trop}(Y)$ is principal, $\mathrm{trop}(D) = \mathrm{div}(g)$ for some rational g on $\mathrm{trop}(Y)$, and $\mathrm{trop}(\widetilde{Y})$ is isomorphic to the tropical modification of $\mathrm{trop}(Y)$ along $\mathrm{trop}(D)$, with respect to g. □

Remark 4.3.24. The rational function g in the proof above is not necessarily the tropicalization of the regular function f, cf. Example 2.4.14, Remarks 2.4.28, 3.1.9, [RSS16, Section 4]. It is also not necessarily a tropically regular function, cf. [Sha13].

This theorem can be used to show a very affine variety $\widetilde{Y} \subset T^n$ is quasilinear by the following inductive procedure: find a projection $\pi : T^n \to T^{n-1}$ realizing $\widetilde{Y}$ as the graph of a regular function f on a variety $Y \subset T^{n-1}$, then show that $Y \subset T^{n-1}$ and $D = \mathrm{div}(f) \subset Y \subset T^{n-1}$ are quasilinear by the same procedure. In practice this procedure can be computationally difficult, as one has to check an increasing number of cases. The steps of this procedure can be simplified by the following criteria.

Theorem 4.3.25. *Let $Y \subset T$ be a quasilinear variety and $D_1, \dots, D_k \subset Y$ quasilinear hypersurfaces on Y such that all nonempty intersections of the D_i are also quasilinear in T. Then $Y \setminus (D_1 \cup \cdots \cup D_k) \subset T \times (\mathbb{C}^*)^k$ is also quasilinear.*

Proof. The proof is by induction on k. The base case $k = 1$ is immediate from Theorem 4.3.23. Assume the result for $k - 1$. Then $Y' = Y \setminus (D_1 \cup \cdots \cup D_{k-1})$ is quasilinear by induction, and $Y \setminus (D_1 \cup \cdots \cup D_k) = Y' \setminus D_k'$, wher $D_k' = D_k \setminus ((D_k \cap D_1) \cup \cdots \cup (D_k \cap D_{k-1}))$. Since D_k and all intersections of the D_i's are quasilinear, D_k' is also quasilinear by induction. Thus $Y' \setminus D_k'$ is quasilinear. □

Theorem 4.3.26. *Let $Y_1 \subset T_1$ and $Y_2 \subset T_2$ be two very affine varieties. Then $Y_1 \times Y_2 \subset T_1 \times T_2$ is quasilinear $\iff$ both $Y_1 \subset T_1$ and $Y_2 \subset T_2$ are quasilinear.*

Proof. We have $\operatorname{trop}(Y_1 \times Y_2) = \operatorname{trop}(Y_1) \times \operatorname{trop}(Y_2)$ [Cue10, Theorem 3.3.4], so the result is immediate from Proposition 4.2.38. □

Theorem 4.3.27. *Linear varieties are quasilinear.*

Proof. Let Y be a linear variety, given as the complement of $n + 1$ hyperplanes in $\mathbb{P}^d$. Let M be the matroid corresponding to the hyperplane arrangement. Then $\operatorname{trop}(Y) = |\Sigma_M|$. By [Sha13], Σ_M is a tropical modification of $\Sigma_{M\setminus i}$ along $\Sigma_{M/i}$. The result follows by induction on n, where the base case is $n = d$, in which case Σ_M is complete. □

Remark 4.3.28. If one wishes, one could avoid the tropical side in the above proof, by invoking Theorem 4.3.23: a linear variety Y is the complement of a hyperplane in a linear variety Y', and the result follows by induction.

Since quasilinearity is defined in terms of the tropicalization, it depends not just on the very affine variety Y, but on the specific embedding $Y \subset T$, as different embeddings of Y into tori can have different tropicalizations. In some cases, one can reduce to a smaller torus.

Proposition 4.3.29. *Suppose $Y \subset T' \subset T$ are closed subvarieties. Then $Y \subset T'$ is quasilinear $\iff$ $Y \subset T$ is quasilinear.*

Proof. Since $Y \subset T'$, $\operatorname{trop}(Y \subset T')$ is contained in $\operatorname{trop}(T' \subset T)$. Thus by [Jel20, Proposition 5.3], $\operatorname{trop}(Y \subset T') \cong \operatorname{trop}(Y \subset T)$. In particular, $Y \subset T'$ is quasilinear $\iff$ $Y \subset T$ is quasilinear. □

Examples

Compare this section with the examples of quasilinear tropical fan cycles in Section 4.2.3.

Example 4.3.30. As mentioned above, linear varieties are quasilinear, cf. Example 4.2.35.

Example 4.3.31. In $(\mathbb{C}^*)^2$, the varieties $Y = \{ax + by + c = 0\}$ and $Y = \{y^a = cx^b\}$, $a, b \in \mathbb{Z} \setminus 0$, $c \in \mathbb{C}^*$ are quasilinear. Indeed, the first is linear, and the second is tropically isomorphic to $\mathbb{C}^*$. The corresponding tropicalizations are the standard tropical line and the line $ay = bx$, cf. Examples 4.2.44, 4.2.45.

Example 4.3.32. Consider $Y = \{y^2 = x^3\}$, which is quasilinear by the previous example. Indeed, $\operatorname{trop}(Y)$ is the line $2y = 3x$ (Example 3.1.8), which is quasilinear by Example 4.2.45. Since there is a unique fan structure Σ on $\operatorname{trop}(Y)$, there is a unique tropical compactification $\overline{Y} \subset X(\Sigma)$ of Y. Since Y is quasilinear, it is schön, so since $X(\Sigma)$ is smooth, $\overline{Y}$ must be smooth. Indeed, $\overline{Y} \cong \mathbb{P}^1$, as follows for instance from the isomorphism of $\operatorname{trop}(Y)$ with $\mathbb{R}^1$ and the isomorphism $X(\Sigma) \cong \mathbb{P}^1 \times \mathbb{C}^*$ (Example 2.3.2).

Remark 4.3.33. Note in the previous example that the more naive compactification of Y by the plane cuspidal cubic $\{y^2 z = x^3\} \subset \mathbb{P}^2$ is not a tropical compactification; rather, the tropical compactification in this case is the normalization.

Example 4.3.34. In $(\mathbb{C}^*)^3$, varieties defined by two monomial equations, a monomial equation and a linear equation, or two linear equations, are quasilinear. They correspond respectively to a classical line in $\mathbb{R}^3$, a degenerate tropical modification of the standard tropical line in $\mathbb{R}^2$, and a nondegenerate tropical modification of the standard tropical line in $\mathbb{R}^2$, cf. Example 4.2.46, 4.2.47.

Example 4.3.35. In $(\mathbb{C}^*)^3$, the varieties

$$Y = \{z^\ell = ax^m y^n\}, \;\; Y = \{z = y - ax^m\}, \;\; Y = \{z = ax + by + c\}$$

are all quasilinear. They correspond respectively to a classical plane in $\mathbb{R}^3$, and to the quasilinear fan cycles in Examples 4.2.48 and 4.2.49.

Example 4.3.36. In Chapters 7 and 8 we will show that the moduli spaces $M(3, n)$, $n \leq 8$ and $Y(3, n)$, $n \leq 7$ are quasilinear, cf. Example 4.2.50.

Part II

Geometry of compactifications of moduli spaces

Chapter 5

Log canonical compactifications and compactifications of moduli spaces

5.1 Tropical and log canonical compactifications

In this section we review some results on tropical and log canonical compactifications. These compactifications are important from the point of view of the minimal model program, as the log canonical compactification of a variety is essentially the smallest compactification with reasonable boundary singularities.

5.1.1 Log canonical pairs and compactifications

Let $(X, B = \sum b_i B_i)$ be a pair of a normal variety X and a $\mathbb{Q}$- or $\mathbb{R}$-divisor B with all $b_i \geq 0$. The *log canonical divisor* of (X, B) is the divisor $K_X + B$. Let $f : Y \to X$ be a resolution of singularities of X. If the log canonical divisor $K_X + B$ is $\mathbb{Q}$- or $\mathbb{R}$-Cartier, then one can pull it back, and one obtains

$$K_Y = f^*(K_X + B) + \sum a_j D_j$$

for some divisors D_j on Y and coefficients a_j, and where the equality denotes rational equivalence [KM98, Section 2.3].

Definition 5.1.1 ([KM98, Definition 2.34])**.** The pair (X, B) has *log canonical* singularities if all $a_j \geq -1$.

Example 5.1.2. If (X, B) is a pair of a (normal) toric variety X and its boundary $B = X \setminus T$, then the pair (X, B) has log canonical singularities [Kol96, Proposition 3.7]. More generally a pair (X, B) has *toroidal singularities* if it is locally isomorphic to a pair of a toric variety and its boundary. Thus toroidal singularities are log canonical. This will be the only example which appears for us.

Definition 5.1.3 ([KM98, Section 3.8])**.** A pair (X, B) is a *log canonical model* if it has log canonical singularities and the $\mathbb{R}$-Cartier divisor $K_X + B$ is ample.

The motivation for this definition is as follows. Suppose (X, B) is a pair with log canonical singularities and such that X is proper. The *abundance conjecture* asserts that if $K_X + B$ is nef, then for some $m > 0$, the linear system $|m(K_X + B)|$ is basepoint-free [KM98, Conjecture 3.12]. This implies that the *log canonical ring*

$$\bigoplus_{m=0}^{\infty} H^0(X, m(K_X + B))$$

is finitely generated. Then the pair of

$$X_{lc} = \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty} H^0(X, m(K_X + B))\right)$$

and the image B_{lc} of B under the natural rational map $X \dashrightarrow X_{lc}$ is a log canonical pair, called the *log canonical model* of X. The intuition behind this is that $X \dashrightarrow X_{lc}$ contracts all of the curves C on which $K_X + B$ is 0, hence making $K_X + B$ ample. Thus the pair (X_{lc}, B_{lc}) should be thought of as the smallest model of (X, B) which still has log canonical singularities.

5.1.2 Log canonical compactifications

Definition 5.1.4 ([HKT09])**.** Let Y be a smooth variety. Then Y is *log minimal* if there is a normal crossings compactification $(\overline{Y}, B)$ of Y such that for some $m > 0$, the natural rational map

$$Y \dashrightarrow \mathbb{P}\left(H^0(\overline{Y}, m(K_{\overline{Y}} + B))^\vee\right)$$

is an embedding.

If Y is log minimal, then the natural rational map $Y \dashrightarrow \overline{Y}_{lc}$ is regular and an embedding, thus $Y \subset \overline{Y}_{lc}$ is a natural compactification of Y, called the *log canonical compactification*. Phrased another way, we have the following definition.

Definition 5.1.5. Let Y be a smooth, log minimal variety. A compactification $Y \subset \overline{Y}_{lc}$ is the *log canonical compactification* of Y if the pair $(\overline{Y}_{lc}, B = \overline{Y}_{lc} \setminus Y)$ is a log canonical model, i.e. it has log canonical singularities and $K_{\overline{Y}_{lc}} + B$ is ample.

Again our intuition for the log canonical compactification is that it is the smallest compactification of Y with log canonical singularities.

5.1.3 Hübsch tropical compactifications

Recall from Chapter 3 that if $Y \subset T$ is a closed subvariety of a torus, then there is a natural class of well-behaved compactifications of Y called tropical compactifications, obtained by taking the closures of Y in appropriate toric varieties associated to fans supported on the tropicalization of Y. The nicest situation is when Y is schön, meaning all strata of any tropical compactification of Y are smooth—this implies that *any* fan supported on $\mathrm{trop}(Y)$ gives a schön tropical compactification.

Definition 5.1.6 ([HKT09, Definition 1.9])**.** A schön very affine variety $Y \subset T$ is *hübsch* if Y is log minimal and the log canonical compactification $\overline{Y}_{lc}$ is a tropical compactification of Y. The corresponding fan structure Σ_{lc} on $\mathrm{trop}(Y)$ is called the *log canonical fan*.

In general there is no canonical coarsest fan structure on the tropicalization of a very affine variety, cf. Remark 2.1.22. However, if Y is hübsch, then the log canonical fan gives the coarsest fan structure.

Theorem 5.1.7 ([HKT09, Theorem 1.10])**.** *Suppose $Y \subset T$ is a hübsch very affine variety. Then any fan supported on* $\mathrm{trop}(Y)$ *is a refinement of the log canonical fan.*

Thus in this setting the log canonical compactification should be thought of as the smallest tropical compactification. This gives a strategy for finding the log canonical compactification of a very affine variety $Y \subset T$ using tropical geometry:

1. Show $Y \subset T$ is schön.

2. Show there is a unique coarsest fan structure on $\mathrm{trop}(Y)$, and that this gives the log canonical compactification.

In practice the first step (showing that Y is schön) is often the most difficult. In Chapter 4 we gave a method for showing Y is schön by showing it is quasilinear, which essentially reduces the problem to direct (yet tedious) computations with equations.

For the second step, we use the following strategy developed by Hacking, Keel, and Tevelev [HKT09], see also [LQ11; Cor21].

Definition 5.1.8 ([HKT09, Definition 1.15])**.** A fan Σ is *convexly disjoint* if any convex subset of $|\Sigma|$ is contained in a cone of Σ.

Clearly if Σ is convexly disjoint then it is the coarsest fan on its support.

Theorem 5.1.9 ([LQ11, Theorem 4.9])**.** *Let Y be a schön very affine variety admitting a tropical compactification all of whose strata are irreducible. If there is a convexly disjoint fan Σ supported on Y, then Y is hübsch and $\overline{Y} \subset X(\Sigma)$ is the log canonical compactification.*

Proof sketch. Hacking, Keel, and Tevelev have shown that if $Y \subset T$ is schön, then either Y is log minimal or Y is preserved by translation by a nontrivial subtorus of T [HKT09, Theorem 3.1]. The second condition is equivalent to $\mathrm{trop}(Y)$ being preserved by a proper linear subspace of $N_{\mathbb{R}}$. The condition that Σ is convexly disjoint immediately implies that this does not occur for any star fans of Σ, hence all strata of $\overline{Y} \subset X(\Sigma)$ (including Y itself) are log minimal. $\square$

5.2 Stable pair compactifications

In this section we briefly review the theory of stable pair (also known as KSBA) compactifications of moduli spaces, following the treatment of [Ale15, Chapter 1]. The basic idea is that a moduli space M of smooth varieties or pairs is not complete, since it fails to see singular limits. The canonical singular limits from the point of view of the minimal model program are the so-called stable pairs, as defined below. The prototypical example is when M is a moduli space of curves, in which case the stable pair compactification is the usual Deligne-Mumford compactification [DM69; Knu83].

5.2.1 Moduli of stable pairs

Definition 5.2.1. A pair $(X, B = \sum b_i B_i)$ of a variety X and a $\mathbb{Q}$- or $\mathbb{R}$-divisor B on X is *semi-log canonical (slc)* if the following hold.

1. X satisfies Serre's condition S_2.
2. X has only double normal crossings (locally analytically isomorphic to $xy = 0$) in codimension one, and the double locus has no components in common with the B_i's.
3. $K_X + B$ is $\mathbb{R}$-Cartier.

4. If $\nu : X^\nu \to X$ denotes the normalization, then the pair $(X^\nu, \sum b_i \nu^{-1}(B_i) + D^\nu)$ has log canonical singularities, where D^ν is the preimage of the double locus of X.

Remark 5.2.2 ([Ale15, Remark 1.3.2])**.** Recall that a variety is normal if it satisfies Serre's condition S_2 and is regular in codimension one. Thus the first two conditions in the definition of slc above should be thought of as a generalization of normality.

Example 5.2.3 ([Ale15, Example 1.3.5])**.** If X is a curve, then the pair $(X, B = \sum b_i B_i)$ is slc $\iff$ X has at worst nodal singularities, the divisors B_i do not contain the nodes, and for every non-nodal point $x \in X$, one has $\operatorname{mult}_x B = b_i \sum \operatorname{mult}_x B_i \leq 1$.

Definition 5.2.4. A pair $(X, B = \sum b_i B_i)$ of a projective variety X and an $\mathbb{R}$-divisor B on X is a *stable pair* if the following two conditions hold.

1. (Singularities) (X, B) is slc.

2. (Numerical) $K_X + B$ is ample.

Example 5.2.5. If X is a curve, then the pair $(X, B = \sum b_i B_i)$ is stable if it satisfies the description of the singularities in Example 5.2.3 and $K_X + B$ is ample. The latter condition is equivalent to saying that for every irreducible component $E \subset X$,

$$\deg(K_X + B)|_E = 2p_a(E) - 2 + E \cdot (X - E) + \sum_{B_i \in E} b_i > 0,$$

see [Ale15, Section 1.1]. For instance if the arithmetic genus of X is 0 and all $b_i = 1$, then X is a tree of $\mathbb{P}^1$'s attached at nodes, such that each $\mathbb{P}^1$ has at least 3 nodes or marked points on it.

The reason for this definition is the following theorem.

Theorem 5.2.6. *Let* $\pi^0 : (X^0, B^0) \to S^0$ *be a family of irreducible stable pairs over a punctured curve* $S^0 = S \setminus 0$. *Then* π^0 *can be uniquely completed to a complete family* $\pi : (X, B) \to S$ *with central fiber a (possibly reducible) stable pair.*

This theorem allows one to construct complete moduli of stable pairs. The details of this construction are more involved, as the definition of a family of stable pairs is nontrivial, and then the steps to construct the moduli space are somewhat involved and require the minimal model program in dimension one higher than the dimension of the stable pairs, see [Ale15, Section 1.4] for an outline and [Kol21] for a detailed treatment. For our purposes the details are not important, as we will only consider moduli spaces which can be explicitly constructed through more direct means.

Chapter 6

Moduli of n points on $\mathbb{P}^1$

Let $M_{0,n}$ denote the moduli space of n distinct points on $\mathbb{P}^1$. Fixing the last point at ∞ and scaling coordinates for the remaining points, one obtains an open embedding $M_{0,n} \subset \mathbb{P}^{n-3}$ as the complement of the 2×2 minors of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ x_1 & x_2 & \cdots & x_{n-1} & 1 \end{bmatrix}. \tag{6.1}$$

Here the coordinates on $\mathbb{P}^{n-3}$ are $(x_1, \ldots, x_{n-1})$ with $\sum x_i = 0$. Thus $M_{0,n}$ is the complement in $\mathbb{P}^{n-3}$ of the hyperplanes $x_i = x_j$.

6.1 Stable pair compactification

The stable pair compactification $\overline{M}_{0,n}$ of $M_{0,n}$ is the first example of a geometrically meaningful compactification of a moduli space [Knu83]. As mentioned in Example 5.2.5, stable n-pointed curves of genus 0 are trees of projective lines attached by nodes, with at least 3 nodes or marked points on each irreducible component.

Example 6.1.1. The boundary divisors of $\overline{M}_{0,n}$ parameterize stable curves consisting of two $\mathbb{P}^1$'s meeting in a point, with at least 2 points on each component. We label these divisors by $D_{I,J}$, $|I| \geq 2$, $|J| \geq 2$, $I \amalg J = [n]$, where I and J denote the set of points on the two components. For instance, $D_{12,345} \subset \overline{M}_{0,5}$ has points $1, 2$ on one component, and $3, 4, 5$ on the other component.

Note two boundary divisors intersect $\iff$ there is a common degeneration of the corresponding stable curves. It follows that $D_{I,J}$ and $D_{I',J'}$ intersect $\iff$ $I \subset I'$, $I \subset J'$, $J \subset I$, or $J \subset J'$ [Kee92].

A construction of $\overline{M}_{0,n}$ as a tropical compactification was given by Kapranov in [Kap93]. It is the closure of $M_{0,n}$ in the toric variety associated to a unimodular fan known as the Dressian $Dr(2,n)$, which is also equal to the tropical Grassmannian $TG(2,n)$ [SS04]. This is the coarsest fan structure on $\text{trop}(M_{0,n})$. We will discuss this construction in more depth in Chapter 7.

Remark 6.1.2. Kapranov's construction of $\overline{M}_{0,n}$ in [Kap93] appeared long before the notion of a tropical compactification was introduced in [Tev07]. In a sense Kapranov's construction is the first example of a tropical compactification, and the motivation for studying tropical comapctifications in the first place.

6.2 Log canonical compactification

As described above, $M_{0,n}$ is linear and $\overline{M}_{0,n}$ is the closure of $M_{0,n}$ in the toric variety associated to the coarsest fan structure $Dr(2,n) = TG(2,n)$ on $\text{trop}(M_{0,n})$. Furthermore, this fan structure is convexly disjoint (a proof of this will be given in Chapter 7). This implies the following.

Theorem 6.2.1 ([KT06; Tev07; HKT09]). *$M_{0,n}$ is hübsch and $\overline{M}_{0,n}$ is the log canonical compactification.*

6.3 Intersection theory

The intersection theory of $\overline{M}_{0,n}$ was explicitly described by Keel in [Kee92].

Theorem 6.3.1 ([Kee92]).

$$H^*(\overline{M}_{0,n}) \cong A^*(\overline{M}_{0,n}) \cong \frac{\mathbb{Z}[D_{I,J} \mid |I|, |J| \geq 2, I \amalg J = [n]]}{\textit{the following relations}}$$

1. *(Linear relations)* $\sum_{\substack{i,j\in I\\ k,l\in J}} D_{I,J} = \sum_{\substack{i,k\in I\\ j,l\in J}} D_{I,J} = \sum_{\substack{i,l\in I\\ j,k\in J}} D_{I,J}$.

2. *(Multiplicative relations)* $D_{I,J} \cdot D_{I',J'} = 0$ *unless* $I \subset I'$, $I \subset J'$, $J \subset I'$, *or* $J \subset J'$.

Remark 6.3.2. The linear relations are just the pullbacks of the relations $0 = 1 = \infty$ on $\overline{M}_{0,4} \cong \mathbb{P}^1$ via the natural forgetful maps $f : \overline{M}_{0,n} \to \overline{M}_{0,4}$.

The multiplicative relations are just the obvious ones, cf. Example 6.1.1.

Remark 6.3.3. Since $M_{0,n}$ is linear and $\overline{M}_{0,n}$ is a tropical compactification of $M_{0,n}$ in a nonsingular toric variety, the above theorem follows from Theorems 4.3.22 and 4.1.15. Keel's original proof instead uses an explicit construction of $\overline{M}_{0,n}$ as a sequence of blowups of $\overline{M}_{0,n-1} \times \overline{M}_{0,4}$ [Kee92]. In Chapter 7 we will see higher-dimensional generalizations of both proofs, cf. Theorem 7.3.1, Section 7.4.

6.3.1 Tautological classes and intersection theory of $\overline{\mathcal{M}}_{g,n}$

Although the intersection theory of $\overline{M}_{0,n}$ has a nice explicit description by Theorem 6.3.1, in general the intersection theory of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable genus g curves with n marked points is far more complicated for $g > 0$. Indeed, there is typically no hope in finding an explicit presentation of $A^*(\overline{\mathcal{M}}_{g,n})$ analogous to Theorem 6.3.1. For this reason, when studying the intersection theory of $\overline{\mathcal{M}}_{g,n}$, most attention is placed on studying a smaller subring of $A^*(\overline{\mathcal{M}}_{g,n})$ called the *tautological ring*. The basic idea is that, since $\overline{\mathcal{M}}_{g,n}$ is a moduli space, it has a number of natural classes in its Chow ring coming from the geometry of the moduli problem. These classes are called the *tautological classes*, and the tautological ring is the subring generated by them. This idea was first developed by Mumford in [Mum83]. By [Fab99], all top intersections of tautological classes on a given $\overline{\mathcal{M}}_{g,n}$ are governed by the intersection numbers of certain tautological classes called *ψ-classes* ψ_i on *all* $\overline{\mathcal{M}}_{g,n}$. In turn, these intersection numbers are determined by the famous Witten's conjecture/Kontsevich's theorem [Wit91; Kon92].

On $\overline{M}_{0,n}$, the intersection numbers of the ψ-classes have a particularly nice description. To explain this, we first introduce some notation. Let $f_i : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ denote the forgetful map, forgetting the ith marked point (then restabilizing if necessary). It turns out that $f_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ is the universal family over $\overline{M}_{0,n}$ [Knu83], and it has n sections $\sigma_1, \dots, \sigma_n$ corresponding to the n marked points.

Definition 6.3.4 ([Wit91])**.** Define $\psi_i = c_1(\sigma_i^*(\omega_{f_{n+1}}))$, where $\omega_{f_{n+1}}$ is the relative dualizing sheaf of the universal family $f_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$.

Thus the ith ψ-class ψ_i is the first Chern class of the line bundle on $\overline{M}_{0,n}$ whose fiber at a stable curve $(C, p_1, \dots, p_n)$ is the cotangent line to C at p_i.

Proposition 6.3.5 ([Wit91])**.** *Let $f_k : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ be a forgetful map, and suppose $i \neq k$. Then*

$$\psi_i = f_k^*(\psi_i) + D_{ik},$$

where D_{ik} is the divisor in $\overline{M}_{0,n}$ parameterizing stable curves with two irreducible components, marked points i and k on one component, and all other marked points on the other component.

Corollary 6.3.6 ([Get98, Section 4])**.** *On $\overline{M}_{0,n}$, one has*

$$\psi_i = \sum_{i \in I, j,k \in J} D_{I,J}.$$

Corollary 6.3.7 (String equation [Wit91])**.**

$$\int \psi_1^{k_1} \cdots \psi_n^{k_n} \cap [\overline{M}_{0,n+1}] = \sum_{i=1}^{n} \int \psi_1^{k_1} \cdots \psi_i^{k_i - 1} \cdots \psi_n^{k_n} \cap [\overline{M}_{0,n}]$$

Corollary 6.3.8 ([Wit91])**.**

$$\int \psi_1^{k_1} \cdots \psi_n^{k_n} \cap [\overline{M}_{0,n}] = \binom{n-3}{k_1, \ldots, k_n}.$$

We will investigate higher-dimensional versions of the ψ-classes in Section 7.3.2.

Remark 6.3.9. The linear system associated to ψ_n defines a birational map $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ described on the interior by the matrix (6.1). In [Kap93], Kapranov factors this birational map as a sequence of blowups of the intersections of the hyperplanes $x_i = x_j$, in increasing order of dimension.

Chapter 7

Moduli of hyperplane arrangements

Let $M(r,n)$ denote the moduli space of arrangements of n hyperplanes in general position in $\mathbb{P}^{r-1}$. Note $M(2,n) = M_{0,n}$, so $M(r,n)$ is a natural higher-dimensional generalization of $M_{0,n}$. As we saw for $M_{0,n}$, fixing the last r hyperplanes and scaling equations for the remaining hyperplanes, we obtain an open embedding $M(r,n) \subset (\mathbb{P}^{n-r-1})^{r-1}$ as the complement of the $r \times r$ minors of the matrix

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ x_1^1 & \cdots & x_{n-r}^1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{r-1} & \cdots & x_{n-r}^{r-1} & 0 & \cdots & 1 \end{bmatrix} \tag{7.1}$$

In particular, $M(r,n)$ is a very affine variety with intrinsic torus $(\mathbb{C}^*)^{\binom{n}{r}-n}$.

7.1 Moduli of stable hyperplane arrangements

The stable pair compactification $\overline{M}(r,n)$ of $M(r,n)$ was constructed by Hacking, Keel, and Tevelev in [HKT06]. It is called the moduli space of *stable hyperplane arrangements*, and is the first example of a stable pair compactification of a moduli space of higher-dimensional varieties/pairs.

7.1.1 Stable hyperplane arrangements

Stable hyperplane arrangements have a concrete interpretation in terms of so-called *matroid subdivisions of the hypersimplex* [HKT06; Ale15]. This works as follows.

Definition 7.1.1. Let M be a rank r matroid on $[n]$. The *matroid (base) polytope* of M is the convex hull in $\mathbb{R}^n$ of the vectors $e_I = \sum_{i\in I} e_i$ for $I \subset [n]$ a base of the matroid M.

Example 7.1.2. The *hypersimplex* $\Delta(r,n)$ is the matroid polytope of the uniform matroid $U_{r,n}$, i.e. the convex hull in $\mathbb{R}^n$ of the vectors e_I for all $I \subset [n]$, $|I| = r$. It can also be described as

$$\Delta(r,n) = \left\{(x_1,\dots,x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1, \sum x_i = r\right\}.$$

It follows from the above example that the matroid polytope of any rank r matroid on $[n]$ is a subpolytope of the hypersimplex $\Delta(r,n)$. A *matroid subdivision* of $\Delta(r,n)$ is a tiling of $\Delta(r,n)$ by these matroid polytopes.

The connection to hyperplane arrangements is the following. Recall (Example 4.2.15) that an arrangement of n hyperplanes $H_1,\dots,H_n$ in $\mathbb{P}^{r-1}$ is the same as a rank r matroid on $[n]$

whose bases consist of the rank r subsets $I \subset [r]$ such that $H_I = \bigcap_{i \in I} H_i = \emptyset$. The uniform matroid $U_{r,n}$ corresponds to an arrangement of n hyperplanes in $\mathbb{P}^{r-1}$ *in general position*. Thus the hypersimplex $\Delta(r,n)$ corresponds to the moduli space $M(r,n)$ of hyperplanes in general position. Each matroid in a (realizable) matroid subdivision of $\Delta(r,n)$ corresponds to a degenerate hyperplane arrangement, so a matroid subdivision of $\Delta(r,n)$ corresponds to a union of degenerate hyperplane arrangements glued in a particular way.

The correspondence described above is best understood through examples.

Example 7.1.3. Figure 7.1, shows a matroid subdivision of the hypersimplex $\Delta(2,4)$ and the corresponding stable curve. There are three matroid polytopes in this subdivision. For example, the matroid polytope shown on the left corresponds to the curve where points 3 and 4 coincide, because $34 \subset [4]$ is not a base of the corresponding matroid.

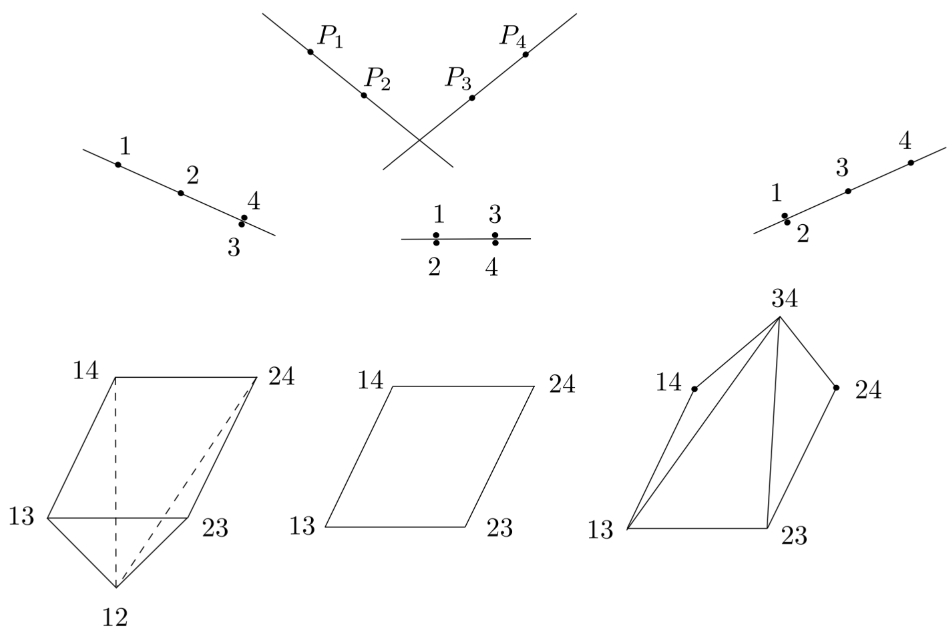


Figure 7.1. A matroid subdivision of $\Delta(2,4)$ and the corresponding stable curve (picture taken from [Ale15, Figure 4.7]).

Following the description of [Ale15], each matroid polytope in a matroid subdivision of $\Delta(r, n)$ can be described by a collection of inequalities of the form

$$\sum_{i=1}^{n} x_i = r, \quad x_i \geq 0 \text{ for } i \in [n],$$

$$x_I = \sum_{i \in I} x_i \leq r(I), \text{ for all nondegenerate flats } I \neq \emptyset, [n].$$

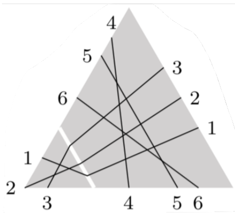
The first two types of inequalities are always present, so we only write inequalities of the third type $x_I \leq r(I)$, called the *essential inequalities* [Ale15, Definition 4.2.3].

Example 7.1.4.

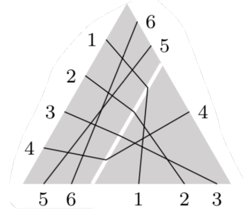
1. When $r = 2$, the essential inequalities are those of the form $x_I \leq 1$, parameterizing when the points on $\mathbb{P}^1$ indexed by I coincide.

2. When $r = 3$, the essential inequalites are those of the forms $x_I \leq 1$, parameterizing when the lines on $\mathbb{P}^2$ indexed by I coincide, and $x_I \leq 2$, parameterizing when the lines on $\mathbb{P}^2$ indexed by I meet in a point.

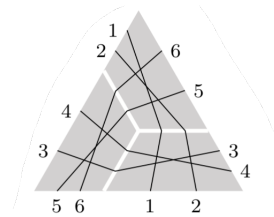
Example 7.1.5. Three examples of stable hyperplane arrangements for $\overline{M}(3, 6)$ are shown in Figure 7.2. The inequalities for the corresponding matroid subdivisions are labeled in the figure. These three stable hyperplane arrangements are representative of the boundary divisors of $\overline{M}(3, 6)$, see Section 7.4. All stable hyperplane arrangements for $\overline{M}(3, 6)$ are listed in [Ale15, Figures 5.12, 5.13].



(a) $\{x_{456} \leq 1\}, \{x_{123} \leq 2\}$



(b) $\{x_{56} \leq 1\}, \{x_{1234} \leq 2\}$



(c) $\{x_{12} \leq 1, x_{1234} \leq 2\}$, $\{x_{34} \leq 1, x_{3456} \leq 2\}$, $\{x_{56} \leq 1, x_{1256} \leq 2\}$

Figure 7.2. Examples of stable hyperplane arrangements for $\overline{M}(3,6)$ (pictures taken from [Ale15, Figure 5.12]).

7.1.2 Construction of the main irreducible component

The moduli space $\overline{M}(r,n)$ is constructed in [HKT06] using the *multigraded Hilbert scheme*. This is a generalization of Kapranov's Chow quotient construction of $\overline{M}_{0,n}$ [Kap93]. In general $\overline{M}(r,n)$ has many irreducible components. The *main irreducible component* $\overline{M}^m(r,n)$, i.e. the closure of $M(r,n)$ in $\overline{M}(r,n)$, will be the only component we consider in this chapter. The construction of $\overline{M}^m(r,n)$ is somewhat simpler than the construction of the whole moduli space $\overline{M}(r,n)$, and is more relevant for our purposes, so we briefly review it now.

The basic idea of the construction is to formalize the correspondence between stable hyperplane arrangements and matroid subdivisions of the hypersimplex as described in the previous section. A fact which essentially dates back to Kapranov in this setting [Kap93; GKZ94] is that there is a fan $\mathrm{Sec}(\Delta(r,n))$ whose cones parameterize all regular[1] subdivisions of the hypersimplex $\Delta(r,n)$. A subfan $Dr(r,n)$ of $\mathrm{Sec}(\Delta(r,n))$, known as the *Dressian*,

[1] We have not defined what we mean by a regular subdivision of a polytope, as it will not be necessary for our purposes. See e.g. [GKZ94] for more details.

parameterizes all regular matroid subdivisions of $\Delta(r,n)$ [Her+09]. The fan $Dr(r,n)$ should be interpreted as a sort of tropical moduli space. The corresponding toric variety $X(Dr(r,n))$ also has a modular interpretation, as a moduli space of *stable toric varieties*—to each polytope in a matroid subdivision of $\Delta(r,n)$, one can associate a projective toric variety, and the data of how the polytopes meet can be used to glue together these projective toric varieties to form a stable pair, see [Ale15, Chapter 2] for details. In particular $X(Dr(r,n))$ has a universal family. Now the key observation is that $M(r,n)$ is contained in the dense open torus in $X(Dr(r,n))$, and that the pullback of the universal family over $X(Dr(r,n))$ to the closure $\overline{M}^m(r,n)$ of $M(r,n)$ in $X(Dr(r,n))$ is the desired universal family of stable hyperplane arrangements [Kap93; HKT06; Ale15]. This is known as Kapranov's *visible contours* construction, and we refer to *op. cit.* for more details.

Remark 7.1.6. The tropicalization $\mathrm{trop}(M(r,n))$ is in fact the *tropical Grassmannian* $TG(r,n)$ of [SS04], which parameterizes matroid subdivisions of $\Delta(r,n)$ by *realizable* matroid polytopes. It is well-known that $TG(r,n) \subset Dr(r,n)$, but in general $Dr(r,n)$ is much larger [Her+09; HJS14]. This is reflected in our setting by the fact that $M(r,n) \subset \overline{M}^m(r,n)$ is typically not a tropical compactification [KT06]—$\overline{M}^m(r,n)$ can intersect nontrivially torus orbits of $X(Dr(r,n))$ corresponding to cones not contained in $TG(r,n)$.

Example 7.1.7. $TG(2,4) = Dr(2,4)$ is the standard tropical line of Example 2.1.8. The three rays correspond to the three different matroid subdivisions of $\Delta(2,4)$.

7.2 Log canonical compactifications

As we have mentioned above, although the compactification $M(r,n) \subset \overline{M}^m(r,n)$ is obtained by taking the closure of $M(r,n)$ in the toric variety $X(Dr(r,n))$, this compactification is usually neither tropical nor log canonical [KT06]. However, Keel and Tevelev have made the following conjecture.

Conjecture 7.2.1 ([KT06, Conjecture 1.6])**.** *If $r = 2$ or $r = 3$ and $n \leq 8$, then (the normalization of) $\overline{M}^m(r,n)$ is the log canonical compactification of $M(r,n)$.*

Furthermore, Keel and Tevelev showed that (up to the natural duality $\overline{M}^m(r,n) \cong \overline{M}^m(n-r,n)$), the cases of the conjecture are the only cases for which $\overline{M}^m(r,n)$ is possibly the log canonical compactification [KT06].

Of course, for the case $r = 2$, $\overline{M}^m(2,n) = \overline{M}_{0,n}$, and it is well-known that this is the log canonical compactification, cf. Chapter 6. Furthermore, $\overline{M}^m(3,4)$ is a point and $\overline{M}^m(3,5) \cong \overline{M}^m(2,5)$. Thus the only nontrivial cases of the conjecture are when $r = 3$ and $n = 6,7,8$. The cases $r = 3$, $n = 6,7$ were shown respectively by Luxton [Lux08] and Corey [Cor21], both using tropical techniques. In this section we use the theory developed in Chapter 4 to simplify the proofs of these cases and settle the remaning case $n = 8$.

Theorem 7.2.2. *Let $r = 2$ or $r = 3$ and $n \leq 8$. Then $\overline{M}^m(r,n)$ is normal, has toroidal singularities, and is the log canonical compactification of $M(r,n)$.*

To prove the theorem, we use the strategy outlined in Section 5.1.3. There are two steps.

Theorem 7.2.3. *Let $r = 2$ or $r = 3$ and $n \leq 8$. Then $M(r,n)$ is quasilinear. In particular, $M(r,n)$ is schön and all strata of any tropical compactification of $M(r,n)$ are smooth, irreducible, and rational.*

Theorem 7.2.4. *Let $r = 2$ or $r = 3$ and $n \leq 8$. Then the Dressian $Dr(r,n)$ induces a convexly disjoint fan structure Σ_{lc} on $\operatorname{trop}(M(r,n))$, and $\overline{M}^m(r,n)$ is the closure of $M(r,n)$ in $X(\Sigma_{lc})$.*

Assuming these two theorems, let us prove Theorem 7.2.2.

Proof of Theorem 7.2.2. Since $M(r,n)$ is schön by Theorem 7.2.3, any fan on $\operatorname{trop}(M(r,n))$ induces a schön tropical compactification of $M(r,n)$ by Theorem 3.2.13. Thus by Theorem 7.2.4, $\overline{M}^m(r,n)$ is the schön tropical compactification defined by the fan structure Σ_{lc} on $\operatorname{trop}(M(r,n))$. It follows by Theorem 3.2.12 that $\overline{M}^m(r,n)$ is normal and has toroidal singularities. Furthermore, since all strata of $\overline{M}^m(r,n)$ are irreducible by Theorem 7.2.3 and Σ_{lc} is convexly disjoint by Theorem 7.2.4, it follows by Theorem 5.1.9 that $\overline{M}^m(r,n)$ is the log canonical compactification. □

7.2.1 Quasilinearity of $M(r,n)$

In this section we prove Theorem 7.2.2. We refer back to Chapter 4, and in particular Section 4.3.2 for our discussion of quasilinear varieties and how to show a given very affine variety is quasilinear (and therefore schön). The basic idea is to write $M(r,n)$ as a sequence of very affine graphs of quasilinear regular functions, starting with the torus $(\mathbb{C}^*)^{(r-1)(n-r-1)}$.

This is done by removing the $r \times r$ minors of the matrix (7.1) one at a time, in an appropriate order.

Quasilinearity of $M(2, n)$

We showed in Chapter 6 that $M(2, n) = M_{0,n}$ is linear, and in particular quasilinear.

Remark 7.2.5. An inductive construction of $\mathrm{trop}(M_{0,n})$ as a sequence of tropical modifications is also given in [Her07]. The construction given there is a tropical version of Keel's construction of $\overline{M}_{0,n}$ [Kee92]. Our inductive construction of $\mathrm{trop}(M_{0,n})$ is a tropical version of Kapranov's construction of $\overline{M}_{0,n}$ [Kap93].

Setup for $M(3, n)$

For $r = 3$, the matrix (7.1) can be written as

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ x_1 & \cdots & x_{n-4} & x_{n-3} & x_{n-2} & 1 & 0 \\ y_1 & \cdots & y_{n-4} & y_{n-3} & y_{n-2} & 0 & 1 \end{bmatrix}. \tag{7.2}$$

This yields an open embedding $M(3, n) \subset \mathbb{P}^{n-4} \times \mathbb{P}^{n-4}$ realizing $M(3, n)$ as the complement of the 3×3 minors of the matrix. Here the coordinates on $\mathbb{P}^{n-4} \times \mathbb{P}^{n-4}$ are $(x_1, \ldots, x_{n-2}) \times (y_1, \ldots, y_{n-2})$ with $\sum x_i = \sum y_i = 0$. There are two types of minors:

- Hyperplanes $X_{ij} = \{x_i = x_j\}$ and $Y_{ij} = \{y_i = y_j\}$.
- Hypersurfaces $Q_{ijk} = \{(x_i - x_k)(y_j - y_k) = (x_j - x_k)(y_i - y_k)\}$.

Remark 7.2.6. One can obtain more familiar bihomogeneous coordinates on $\mathbb{P}^{n-4} \times \mathbb{P}^{n-4}$ by setting $x_{n-2} = y_{n-2} = 0$; further setting $x_{n-3} = y_{n-3} = 1$ gives affine coordinates for an open embedding $M(3,n) \subset \mathbb{A}^{2(n-4)}$. We will use these coordinates in the appendix and in Section 8.1.1.

Denote by $M_1 = M_1(3,n)$ the complement of all of the hyperplanes. Then M_1 embeds into its intrinsic torus $T_1 = T^{\binom{n-2}{2}-1} \times T^{\binom{n-2}{2}-1}$, with bihomogeneous coordinates x_{ij}, y_{ij}, via the equations

$$x_{ij} = x_{i,n-2} - x_{j,n-2}, \quad y_{ij} = y_{i,n-2} - y_{j,n-2}.$$

Thus M_1 is linear. (In fact, $M_1 = M_{0,n-1} \times M_{0,n-1}$.) In M_1 we can write equations for the remaining hypersurfaces as

$$Q_{ijk} = \{x_{ik}y_{jk} = x_{jk}y_{ik}\} = \{x_{ij}y_{ik} = x_{ik}y_{ij}\} = \{x_{ij}y_{jk} = x_{jk}y_{ij}\}.$$

Lemma 7.2.7. *All Q_{ijk}'s are quasilinear in M_1.*

Proof. Dehomogenizing the coordinates on M_1 by setting $x_{ij} = y_{ij} = 1$ gives a linear equation $Q_{ijk} = \{x_{ik} = y_{ik}\}$ for Q_{ijk} in the linear variety M_1. □

Remark 7.2.8. The dehomogenization in the above proof amounts to setting columns i and j in the above matrix to $(1,1,1)^T$ and $(1,0,0)^T$, cf. Remark 7.2.6.

For $n \geq 6$, it is not possible to dehomogenize or choose coordinates so that all Q_{ijk} are simultaneously linear, in which case M itself would be linear.

To show $M(3,n)$ is quasilinear, it is now enough to show that all intersections of the Q_{ijk}'s in M_1 are quasilinear. We will do this on a case-by-case basis; in order to improve readability, some of the details are postponed to the appendix. However, as we will see below, the statement is false for $M(3,8)$, where another step is necessary.

We will make frequent use of the following observation.

Observation 7.2.9. Let $i,j,k,l \in [n-2]$ be distinct. Then the intersection of any two of $Q_{ijk}, Q_{ijl}, Q_{ikl}, Q_{jkl}$ in M_1 is the same as the intersection of all four, and is quasilinear.

Proof. The intersection of any two of these hypersurfaces is naturally identified with the moduli space of n lines in the plane such that no two lines coincide, any number of the first $n-2$ lines are allowed to meet in a point, there can be no triple intersections involving the last two lines, and lines i,j,k,l meet in a point. This is clearly also the intersection of all four hypersurfaces.

Quasilinearity can be checked directly with equations, as in the previous lemma: dehomogenizing by setting $x_{ij} = y_{ij} = 1$ reveals that $Q_{ijk} \cap Q_{ijl} = \{x_{ik} = y_{ik}, x_{il} = y_{il}\}$ is defined by linear equations in the linear variety M_1. □

Quasilinearity of $M(3,6)$

There are four Q_{ijk}'s in $M_1(3,6)$, for $ijk \subset [4]$. By Observation 7.2.9, the intersection of any two is the same as the intersection of all four, and is quasilinear. Therefore, $M(3,6)$ is quasilinear.

Quasilinearity of $M(3,7)$

There are ten Q_{ijk}'s in $M_1(3,7)$, for $ijk \subset [5]$. Up to symmetry, there are three types of intersections: $Q_{145} \cap Q_{245} \cap Q_{345}$, $Q_{245} \cap Q_{345}$, and $Q_{125} \cap Q_{345}$.

1. Dehomogenizing by setting $x_{45} = y_{45} = 1$ reveals that

$$Q_{145} \cap Q_{245} \cap Q_{345} = \{x_{15} = y_{15}, x_{25} = y_{25}, x_{35} = y_{35}\} \subset M_1,$$

$$Q_{245} \cap Q_{345} = \{x_{25} = y_{25}, x_{35} = y_{35}\} \subset M_1,$$

   so both of these intersections are linear.

2. To show $Z = Q_{125} \cap Q_{345}$ we work directly with equations. This is slightly more difficult than the previous cases, as one cannot write linear equations for Z. We show Z is quasilinear by performing a sequence of projections reducing to the $M(3,6)$ case, see Appendix A.1.1 for details.

We have shown that all intersections of the Q_{ijk}'s in M_1 are quasilinear. It follows that $M(3,7)$ is quasilinear.

Quasilinearity of $M(3,8)$

There are 20 Q_{ijk}'s in $M_1(3,8)$, for $ijk \subset [6]$. Not all intersections of the Q_{ijk}'s are quasilinear: the intersection

$$Q_{126} \cap Q_{346} \cap Q_{135} \cap Q_{245}$$

has two irreducible components. Therefore, another step is necessary to show $M(3,8)$ is quasilinear.

First let $M_2 = M_2(3,8) \subset T_2 = T^{38}$ be the complement in M_1 of the 10 hypersurfaces Q_{ijk} for $ijk \subset [5]$. An identical argument to the $M(3,7)$ case shows that M_2 is quasilinear. (In fact, the forgetful map dropping the line 6 on $M_2(3,8)$ identifies $M_2(3,8)$ with a complement of hyperplanes in $M(3,7) \times T^2$.)

The only remaining hypersurfaces in M_2 are Q_{ij6} for $ij \subset [5]$. Up to symmetry, there are only two types of intersections of these hypersurfaces in M_2: $Q_{126} \cap Q_{456}$, and Q_{456}.

1. $Z = Q_{126} \cap Q_{456} \subset M_2$ can be interpreted as the moduli space of 8 lines in $\mathbb{P}^2$, with two triple intersection points 126 and 346, but no further degenerate arrangements allowed. Then dropping line 6 gives an arrangement of 7 lines in general position in $\mathbb{P}^2$, and conversely, adding to such an arrangement the line through points 12 and 45, one obtains Z. Therefore $Z \cong M(3,7)$. Furthermore, under this isomorphism, the embedding $Z \subset T^{38}$ factors through the embedding $M(3,7) \subset T^{28}$ of $M(3,7)$ in its intrinsic torus. We conclude by Proposition 4.3.29 that $Z \subset T^{38}$ is quasilinear; see Appendix A.1.2 for more details.

2. To show $Q_{456} \subset M_2$ is quasilinear, we show that $Q_{456} \subset M_1$ intersects any collection of the Q_{ijk}, $ijk \subset [5]$, quasilinearly. There are 7 cases, which we split up into two types.

(a) (Linear cases) $Q_{145} \cap Q_{245} \cap Q_{345} \cap Q_{456}$, $Q_{145} \cap Q_{245} \cap Q_{456}$, $Q_{145} \cap Q_{456}$. Dehomogenizing by setting $x_{45} = y_{45} = 1$, we get linear equations

$$Q_{145} \cap Q_{245} \cap Q_{345} \cap Q_{456} = \{x_{15} = y_{15}, x_{25} = y_{25}, x_{35} = y_{35}, x_{46} = y_{46}\} \subset M_1,$$

$$Q_{145} \cap Q_{245} \cap Q_{456} = \{x_{15} = y_{15}, x_{25} = y_{25}, x_{46} = y_{46}\} \subset M_1,$$

$$Q_{145} \cap Q_{456} = \{x_{15} = y_{15}, x_{46} = y_{46}\} \subset M_1.$$

(b) (Nonlinear cases) $Q_{124} \cap Q_{135} \cap Q_{456}$, $Q_{123} \cap Q_{124} \cap Q_{456}$, $Q_{124} \cap Q_{456}$, $Q_{123} \cap Q_{456}$. We show each of these are quasilinear directly from the equations, by performing a sequence of projections to reduce to the $M(3, 7)$ case; see Appendix A.1.2 for details.

Completion of proof of Theorem 7.2.3

Proof of Theorem 7.2.3. We have shown in the above subsections that $M(r, n)$ is quasilinear when $r = 2$ or when $r = 3$ and $n \leq 8$. Quasilinear varieties are schön and all strata of their tropical compactifications are smooth, irreducible, and rational, by Theorem 4.3.20 and Corollary 4.3.21. □

7.2.2 Convexly disjoint fan on $\text{trop}(M(r, n))$

Lemma 7.2.10. *Let Σ be a convexly disjoint fan, and Δ a fan such that every cone of Δ is a cone of Σ. Then Δ is convexly disjoint.*

Proof. Let C be a convex subset of $|\Delta|$. Then since $|\Delta| \subset |\Sigma|$, C is a convex subset of Σ. Since Σ is convexly disjoint, C is contained in a cone σ of Σ. If C is not contained in a cone of Δ, then it intersects the relative interiors of two distinct cones of Δ. Since these are also cones of Σ by construction, we obtain a contradiction—C cannot be contained in a cone σ of Σ and also meet the relative interiors of two distinct cones in Σ. □

Proposition 7.2.11. *Suppose* $\mathrm{trop}(M(r,n))$ *admits a coarsest fan structure* Δ. *Then* Δ *is convexly disjoint.*

Proof. The intersection of any $r-2$ hyperplanes in general position in $\mathbb{P}^{r-1}$ gives a line ℓ, and the intersection of ℓ with four more hyperplanes gives a configuration of four points in general position on $\mathbb{P}^1$. This yields, for any $M(r,n)$, a collection of *cross-ratio maps* $M(r,n) \to M_{0,4}$. By [Lux08, Proposition 3.1.3], the product of all such cross-ratio maps induces a closed embedding

$$M(r,n) \hookrightarrow \prod M_{0,4},$$

and an injection

$$N \hookrightarrow \prod N_4$$

from the cocharacter lattice N of the intrinsic torus of $M(r,n)$, to the product over all cross-ratios of the cocharacter lattice N_4 of the intrinsic torus of $M_{0,4}$. Let Σ_4 be the unique fan structure on $\mathrm{trop}(M_{0,4})$. (This is the standard tropical line as in Example 2.1.8.) Viewing N as a sublattice of $\prod N_4$, define the *intersection fan*

$$\Sigma = N_{\mathbb{R}} \cap \prod \Sigma_4 \subset N_{\mathbb{R}}.$$

By [Lux08, Proposition 3.1.3], $X(\Sigma)$ can be identified with its image in $X\left(\prod \Sigma_4\right)$, cf. [HKT09, Proof of Theorem 9.14]. Since the closure of $\prod M_{0,4}$ in $X\left(\prod \Sigma_4\right)$ is $\prod \overline{M}_{0,4}$, it follows that the closure $\overline{M}^{\Sigma}(r,n)$ of $M(r,n)$ in $X(\Sigma)$ is the same as the closure of $M(r,n)$ in $\prod \overline{M}_{0,4}$; in particular, $\overline{M}^{\Sigma}(r,n)$ is proper. Thus by [Tev07, Proposition 2.3], $\operatorname{trop}(M(r,n)) \subset |\Sigma|$. The fan Σ is convexly disjoint (cf. [HKT09, Proof of Theorem 9.14]), so it is the coarsest fan structure on its support. But since Δ is the coarsest fan structure on $\operatorname{trop}(M(r,n))$, it follows that each cone of Δ is a cone of Σ. Thus by the previous lemma, Δ is convexly disjoint. □

Remark 7.2.12. In fact, by [ST21, Corollary 3.5], $\overline{M}^{\Sigma}(r,n)$ is (up to normalization) the same as $\overline{M}^{m}(r,n)$.

Proof of Theorem 7.2.4. It is well-known that for $r=2$ or $r=3$ and $n=6$, the Dressian $Dr(r,n)$ is actually supported on the tropical Grassmannian $TG(r,n)=\operatorname{trop}(M(r,n))$, see [SS04; Lux08]. For $r=3$ and $n=7,8$, the only cones of $Dr(r,n)$ whose relative interiors do not meet $TG(r,n)$ are cones which parameterize non-realizable matroid subdivisions of the hypersimplex, see [Her+09] for $n=7$ and [HJS19, Proposition 5.5], [Ben+20, Remark 4.6] for $n=8$. In particular, it follows by the discussion of Section 7.1.2 that $\overline{M}^{m}(r,n)$ does not meet the torus orbits of $X(Dr(r,n))$ corresponding to cones not contained in the $TG(r,n)$. Removing these cones defines the coarsest fan structure Σ_{lc} on $TG(r,n)$ by [Her+09; Ben+20], and by the previous sentence $\overline{M}^{m}(r,n)$ is the closure of $M(r,n)$ in $X(\Sigma_{lc})$. Finally, Σ_{lc} is convexly disjoint by the proposition. □

Note that with this proof we have also concluded the proof of Theorem 7.2.2.

Remark 7.2.13. The description of the coarsest fan structure on $\mathrm{trop}(M(r,n))$ for $r = 3$, $n = 7, 8$ in [Her+09; Ben+20] is very complicated and involves massive computer computations, especially for the $(3, 8)$ case. It would be interesting to understand the coarsest fan structure by more theoretical means.

7.2.3 The log canonical compactification in general

As mentioned previously, $\overline{M}^m(r,n)$ is not the log canonical compactification in general. However, since $M(r,n)$ is still log minimal [KT06, Proposition 2.18], it (conjecturally) still has a log canonical compactification $\overline{M}^{lc}(r,n)$. Determining the log canonical compactification in general would be a significant result, as it would give a reasonable birational model of a space $\overline{M}^m(r,n)$ with arbitrary boundary singularities, cf. [KT06, Question 1.7]. We cannot offer anything in this direction other than speculation, so we content ourselves with a few comments.

1. If $\overline{M}^\Sigma(r,n)$ is a tropical compactification of $M(r,n)$, then there is a natural map $\overline{M}^\Sigma(r,n) \to \overline{M}^m(r,n)$ [Tev07, Theorem 5.4]. Thus the geometry of $\overline{M}^m(r,n)$ could still be studied by studying tropical compactifications of $M(r,n)$.

2. Corey has constructed an initial degeneration of $M(3,9)$ with two connected components [Cor21, Theorem 1.4]. This implies $M(3,9)$ is not quasilinear. However, the corresponding stratum of a tropical compactification of $M(3,9)$ is 0-dimensional (i.e. consists of two distinct points). This does not rule out the possibility (however un-

likely) that $M(r,n)$ is schön or that the log canonical compactification is a tropical compactification.

3. Our proof of quasilinearity of $M(r,n)$ (for $r = 2$ or $r = 3$ and $n \leq 8$) describes $\operatorname{trop}(M(r,n))$ via a sequence of tropical modifications of a complete fan; each tropical modification corresponds to removing a single hypersurface. In general the proof fails because at some point one has to remove a hypersurface which is not quasilinear. If one stops before this point, then one obtains a quasilinear variety M' containing $M(r,n)$ as an open dense subset. Thus there is a surjection of tropicalizations $\operatorname{trop}(M(r,n)) \to \operatorname{trop}(M')$ (Lemma 3.1.23), and $\operatorname{trop}(M')$ is well-understood since it is quasilinear. In theory this could be used to understand $\operatorname{trop}(M(r,n))$, but in practice it may not be useful outside of some small cases, since M' could still be much larger than $M(r,n)$.

7.3 Intersection theory

Theorem 7.3.1. *Let $r = 2$ or $r = 3$ and $n \leq 8$, and let $\overline{M}^{\Sigma}(r,n) \subset X(\Sigma)$ be any tropical compactification of $M(r,n)$. Then $i^* : A^*(\Sigma) \to A^*(\overline{M}^{\Sigma}(r,n))$ is an isomorphism and $\overline{M}^{\Sigma}(r,n)$ satisfies Kronecker duality. Furthermore if $X(\Sigma)$ is smooth then $\overline{M}^{\Sigma}(r,n)$ is a resolution of singularities of $\overline{M}^{m}(r,n)$ and*

$$H^*(\overline{M}^{\Sigma}(r,n)) \cong A^*(\overline{M}^{\Sigma}(r,n)) \cong A^*(X(\Sigma)) \cong H^*(X(\Sigma)).$$

Proof. Immediate from Theorems 7.2.3, 7.2.2 and 4.3.22, see also Theorems 3.2.12 and 5.1.7. □

In principle Theorem 7.3.1 gives an explicit presentation of the Chow ring of e.g. small resolutions of $\overline{M}^m(r,n)$ ($r=2$ or $r=3$ and $n \leq 8$) via Theorem 4.1.15. However, in practice fan structures on $\operatorname{trop}(M(r,n))$ are quite complicated and we are not aware of an explicit description of any unimodular fan structure on $\operatorname{trop}(M(r,n))$ for $r=3$, $n=7,8$. From this perspective the closely related moduli space of marked del Pezzo surfaces, which we will study in the next chapter, is more well-behaved. Additionally, the details are all fully understood for the case $r=3$, $n=6$ as we will discuss below.

More generally, since $M(r,n) \subset \overline{M}^m(r,n)$ is not even a tropical compactification outside of the aforementioned cases, one cannot expect a result like Theorem 7.3.1 to hold in general. The problem of understanding the intersection theory of $\overline{M}(r,n)$ is of compelling interest as it gives one of the first higher-dimensional generalizations of the intersection theory of $\overline{\mathcal{M}}_{g,n}$, and can be thought of as the first step towards a generalization of Gromov-Witten theory to moduli spaces of stable maps from surfaces. We outline below two possible approaches to understanding the intersection theory of $\overline{M}(r,n)$ in general.

7.3.1 Tropical intersection theory

One approach to understanding the intersection theory of $\overline{M}(r,n)$ is to instead study the tropical intersection theory of $\operatorname{trop}(M(r,n))$. Since for a tropical compactification $\overline{M}^\Sigma(r,n)$ there is a map $\overline{M}^\Sigma(r,n) \to \overline{M}^m(r,n)$ induced by a morphism of toric varieties [Tev07, Theorem 5.4], and since $\operatorname{trop}(M(r,n))$ encodes the information of all tropical compactifications of $M(r,n)$, the intersection theory of $\text{trop}(M(r,n))$ could be viewed as a sort of virtual intersection theory for $M(r,n)$. This would also likely be related to the intersection theory of the log canonical compactification, cf. Section 7.2.3. Additionally, $\text{trop}(M(r,n))$ is the tropical Grassmannian [SS04], so its intersection theory is also of great interest from the tropical point of view.

However, there are a couple of problems with this approach.

1. Corey's example [Cor21, Theorem 1.4] of an initial degeneration of $M(3,9)$ with two connected components implies that $\text{trop}(M(3,9))$ is not star-Poincaré. Thus $\text{trop}(M(r,n))$ doesn't even have a well-define tropical intersection theory in general. It is possible that this problem could be fixed by working with rational rather than integral coefficients, but currently this is only speculation.

2. Since fan structures Σ on $\text{trop}(M(r,n))$ are already very difficult to understand, $A^*(\text{trop}(M(r,n)))$ is likely too large to be useful in general. Indeed, we do not even have an explicit description of the Chow rings of unimodular fans on $\text{trop}(M(r,n))$ in the known cases $r=3$, $n=7,8$.

The second problem would be solved by the following, more classical, approach.

7.3.2 Tautological classes

Recall the discussion of tautological classes, and in particular ψ-classes, from Section 6.3.1. This discussion suggests an approach to studying the intersection theory of $\overline{M}(r,n)$ by introducing higher-dimensional versions of ψ-classes and studying their intersection numbers.

Since $\overline{M}(r,n)$ is a moduli space, it has a universal family $\pi : (\mathbb{S}, \mathbb{B} = \sum_{i=1}^{n} \mathbb{B}_i) \to \overline{M}(r,n)$, whose fibers are stable hyperplane arrangements. By [HKT06, Proposition 5.1], there are $\binom{n}{r-1}$ sections $\sigma_I : \overline{M}(r,n) \to \mathbb{S}$ of π, for $I \subset [n]$ with $|I| = r-1$, with images $\mathbb{B}_I = \bigcap_{i \in I} \mathbb{B}_i$. Furthermore, at a fiber $(S, B = \sum B_i)$ of π, S is smooth and B has normal crossings at the point $B_I = \bigcap_{i \in I} B_i$.

Definition 7.3.2. With notation as above, define $\mathbb{L}_I = \sigma_I^*(\omega_\pi)$ and $\phi_I = c_1(\mathbb{L}_I)$, where ω_π denotes the relative dualizing sheaf of the universal family $\pi : \mathbb{S} \to \overline{M}(r,n)$.

Observe that $\mathbb{L}_I$ is a vector bundle whose fiber at a stable hyperplane arrangement (S, B) is the cotangent space to S at B_I.

Definition 7.3.3. For $i \in I$, define $\mathbb{L}_{I,i} = \sigma_I^*(\omega_\pi|_{\mathbb{B}_i})$ and $\psi_{I,i} = c_1(\mathbb{L}_{I,i})$.

By adjunction, the curve $C_{I\setminus i} = \bigcap_{j \in I \setminus i} B_j$ on a stable hyperplane arrangment (S, B) is a stable $(n-r+2)$-pointed curve of genus zero, where the marked points are $P_k = B_k \cap C_{I\setminus i}$ for $k \notin I \setminus i$. This implies that there is a natural restriction morphism

$$r_{I\setminus i} : \overline{M}(r,n) \to \overline{M}_{0,n-r+2},$$

sending (S, B) to $C_{I\setminus i}$.

Observe that $\mathbb{L}_{I,i}$ is a line bundle whose fiber at (S, B) is the cotangent line to $C_{I\setminus i}$ at B_I. There is a decomposition

$$\mathbb{L}_I = \bigoplus_{i \in I} \mathbb{L}_{I,i}, \quad \phi_I = \sum_{i \in I} \psi_{I,i}.$$

To understand the vector bundle $\mathbb{L}_I$ and its first Chern class ϕ_I, it is therefore enough to understand the individual line bundles $\mathbb{L}_{I,i}$ and their first Chern classes $\psi_{I,i}$.

Example 7.3.4. When $r = 2$, $I = \{i\}$, and we write $\psi_{I,i} = \psi_i$. Then $\phi_I = \phi_i = \psi_i$ is just the usual ψ-class on $\overline{M}_{0,n}$.

Example 7.3.5. When $r = 3$, $I = \{i, j\}$, and we write $\psi_{I,j} = \psi_{ij}$. Note that $\psi_{ij} = r_i^*(\psi_j)$. The class $\phi_{ij} = \psi_{ij} + \psi_{ji}$ is a symmetric version of ψ_{ij}.

Example 7.3.6. Generalizing the previous examples, on any $\overline{M}(r, n)$ one can write $\psi_{I,i} = r_{I\setminus i}^*(\psi_i)$, where $r_{I\setminus i} : \overline{M}(r, n) \to \overline{M}_{0,n-r+2}$ is the restriction to the curve $C_{I\setminus i}$.

The ϕ_I and $\psi_{I,i}$ are thus the natural higher-dimensional versions of the ψ-classes on $\overline{M}_{0,n}$. This makes the following question of compelling interest, as it would offer a higher-dimensional analogue of Witten's conjecture/Kontsevich's theorem [Wit91; Kon92].

Question 7.3.7. Is there a nice combinatorial formula for the intersection numbers of the ϕ_I or $\psi_{I,i}$ on $\overline{M}(r, n)$, analogous to Corollary 6.3.8?

As a partial result, we have a pullback formula analogous to Proposition 6.3.5.

Proposition 7.3.8 ([Sch22, Lemma 6.6])**.** *On $\overline{M}(r, n)$ $(r \geq 3)$, one has*

$$\psi_{I,i} = f_k^*(\psi_{I,i}) + r_{I\setminus i}^*(D_{ik})$$

Proof. The result is trivial for $n = r + 1$, because $\overline{M}(r, r + 1)$ is a point.

For $n \geq r+2$, the following diagram commutes.

$$\begin{array}{ccc} \overline{M}(r,n) & \xrightarrow{r_{I\setminus i}} & \overline{M}_{0,n-r+2} \\ \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_k} \\ \overline{M}(r,n-1) & \xrightarrow{r_{I\setminus i}} & \overline{M}_{0,n-r+1} \end{array}$$

The pullback formula $\psi_i = f_k^*(\psi_i) + D_{ik}$ for $\overline{M}_{0,n}$ together with the formula $\psi_{I,i} = r_{I\setminus i}^*(\psi_i)$ (Example 7.3.6) gives

$$\psi_{I,i} = r_{I\setminus i}^*(f_k^*(\psi_i) + D_{ik}) = r_{I\setminus i}^* f_k^* \psi_i + r_{I\setminus i}^* D_{ik} \text{ on } \overline{M}(r,n).$$

Commutativity implies that

$$r_{I\setminus i}^* f_k^* \psi_i = f_k^*(r_{I\setminus i}^* \psi_i) = f_k^*(\psi_{I,i}),$$

so the result follows. □

Unfortunately, the intersections of the $r_{I\setminus i}^*(D_{ik})$ are typically nonzero, which makes recursive computations of intersections of ψ-classes on $\overline{M}(r,n)$ more complicated than in the rank 2 case, and we have not been able to obtain an analogue of e.g. the string equation (Corollary 6.3.7).

In principle, we could use Theorem 7.3.1 to compute intersection numbers of ψ-classes on $\overline{M}(3,n)$ for $n \leq 8$. We carry this out for $\overline{M}(3,6)$ in Theorem 7.4.10 below. However, we have found no pattern to these intersection numbers, and even the cases $n = 7, 8$ are out of reach for purely computational reasons.

Remark 7.3.9. One approach to computing intersection numbers of ψ-classes on $\overline{M}(r,n)$ is to define analogous classes on the tropicalization $\mathrm{trop}(M(r,n))$ and compute the intersections there using tropical geometry. This is done for $\overline{M}_{0,n}$, where it is known the tropical and classical intersection numbers coincide, in [Kat12; GKM09; KM09; Mik07a]. This approach is of great interest, both because it seems like the most feasible approach, and because it would describe a sort of "tautological intersection theory," for the tropical Grassmannian as well, cf. Section 7.3.1.

7.4 Moduli of six lines on the plane

Although in general the geometry of $\overline{M}(r,n)$ is very complicated, even for the nice cases $\overline{M}(3,7)$ and $\overline{M}(3,8)$ as described above, everything is completely understood in the first higher-dimensional case $\overline{M}(3,6)$ [Lux08; Sch22].

7.4.1 Boundary and singularities

Boundary

The tropicalization $\mathrm{trop}(M(3,6))$ is the tropical Grassmannian $TG(3,6)$ described explicitly in [SS04]. The coarsest fan structure on $TG(3,6)$ is described as follows. Define an abstract simplicial complex Δ as the flag complex on the graph whose vertices are labeled as e_{ijk}, $ijk \subset [6]$, f_{ij}, $ij \subset [6]$, and $g_{ij,kl,mn}$, $ij \amalg kl \amalg mn = [6]$, and whose edges are given as follows.

1. 90 edges like $\{e_{123}, e_{145}\}$ and 10 edges like $\{e_{123}, e_{456}\}$.

2. 45 edges like $\{f_{12}, f_{34}\}$.

3. 15 edges like $\{g_{12,34,56}, g_{12,56,34}\}$.

4. 60 edges like $\{e_{123}, f_{45}\}$ and 60 edges like $\{e_{123}, f_{12}\}$.

5. 180 edges like $\{e_{123}, g_{12,34,56}\}$.

6. 90 edges like $\{f_{12}, g_{12,34,56}\}$.

The maximal simplices of this complex are all 3-dimensional, except for 15 4-dimensional simplices of the form

$$\{f_{12}, f_{34}, f_{56}, g_{12,34,56}, g_{12,56,34}\}.$$

Following [SS04; Lux08], the cone over this simplicial complex induces the coarsest fan Σ_{lc} structure on $TG(3,6)$. This is a purely 4-dimensional fan in $\mathbb{R}^{14}$ which is unimodular except at 15 top-dimensional cones which are not even simplicial, corresponding the 15 simplices above. Each such cone looks like the cone over the triangular bipyramid of Figure 7.3a.

From Theorems 7.2.2, 7.2.4, the closure of $M(3,6)$ in $X(\Sigma_{lc})$ is $\overline{M}(3,6)$ and is also the log canonical compactification. Thus Δ describes the boundary complex of $\overline{M}(3,6)$. There are 65 boundary divisors, labeled as $D_{ijk,lmn}$ (corresponding to e_{ijk}), $D_{ij,klmn}$ (corresponding to f_{ij}), and $D_{ij,kl,mn}$ (corresponding to $g_{ij,kl,mn}$).

Example 7.4.1. The stable hyperplane arrangements parameterized by $D_{456,123}$, $D_{56,1234}$, and $D_{12,34,56}$, are shown in Figure 7.2.

Singularities and resolutions

Since $\overline{M}(3,6) \subset X(\Sigma_{lc})$ is a schön tropical compactification (Theorem 7.2.3) and Σ_{lc} is unimodular except at the 15 non-simplicial top-dimensional cones mentioned above, it follows that $\overline{M}(3,6)$ is smooth with normal crossings boundary except for the 15 boundary points of the form

$$P_{12,34,56} = D_{12,3456} \cap D_{34,1256} \cap D_{56,1234} = D_{12,34,56} \cap D_{12,56,34}.$$

The singularities of the corresponding points of $\overline{M}(3,6)$ look like $0 \in C(\mathbb{P}^1 \times \mathbb{P}^2)$, where $C(\mathbb{P}^1 \times \mathbb{P}^2)$ denotes the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ [Lux08], [Sch22, Proposition 2.1].

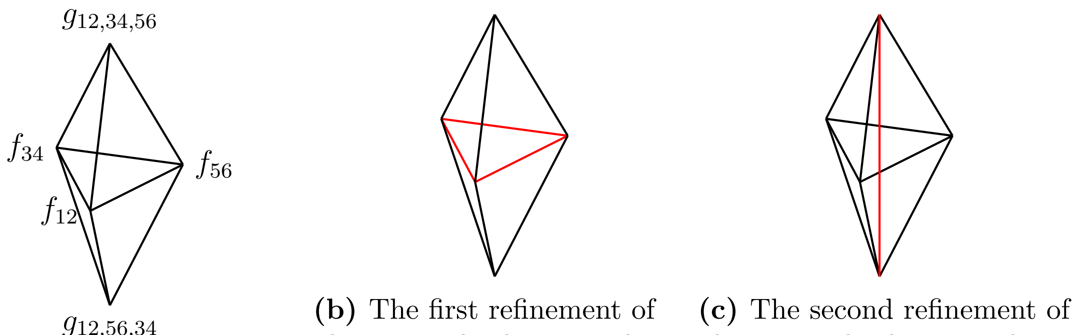
There are two minimal ways to refine the triangular bipyramid to turn Σ_{lc} into a unimodular fan.

1. Split along the center triangle into two smaller polytopes, as shown in Figure 7.3b. This corresponds to the small resolution replacing the singular point $P_{ij,kl,mn}$ with a line $\mathbb{P}^1$. This resolution makes the strict transforms of $D_{ij,kl,mn}$ and $D_{ij,mn,lk}$ disjoint.

2. Split down the middle into three smaller polytopes, as shown in Figure 7.3c. This corresponds to the small resolution replacing the singular point $P_{ij,kl,mn}$ with a plane $\mathbb{P}^2$. This resolution makes the strict transforms of $D_{12,3456}$, $D_{34,1256}$, and $D_{56,1234}$ disjoint.

We can make either choice of refinement for each of the 15 non-simplicial top-dimensional cones [Sch22, Section 2.2]. If S_1 and S_2 form a partition of the 15 non-simplicial top-dimensional cones, we write $\widetilde{\Sigma}_{S_1,S_2}$ for the refinement which splits the cones in S_1 into two polytopes and the cones in S_2 into three polytopes, and we write $\widetilde{M}_{S_1,S_2}(3,6)$ for the closure of $M(3,6)$ in the corresponding nonsingular toric variety $X(\widetilde{\Sigma}_{S_1,S_2})$. Thus $\widetilde{M}_{S_1,S_2}(3,6)$ is a small resolution of $\overline{M}(3,6)$ with fiber $\mathbb{P}^1$ over the singular points indexed by S_1 and fiber $\mathbb{P}^2$ over the singular points indexed by S_2. The $\widetilde{M}_{S_1,S_2}(3,6)$ describe all the small resolutions of $\overline{M}(3,6)$; there are 2^{15} in total.

There are two special small resolutions, where S_1 consists of all non-simplicial cones and S_2 is empty, and vice-versa. We denote the fans in these cases by $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$, and denote the corresponding small resolutions of $\overline{M}(3,6)$ by $\widetilde{M}_1(3,6)$ and $\widetilde{M}_2(3,6)$.



(a) The triangular bipyramid.

(b) The first refinement of the triangular bipyramid: split along the center triangle.

(c) The second refinement of the triangular bipyramid: split along the line through the two endpoints.

Figure 7.3. The two minimal unimodular refinements of the triangular bipyramid.

We refer to [Sch22] for a more detailed discussion of the singularities and resolutions of $\overline{M}(3,6)$.

7.4.2 Blowup construction

Recall that Kapranov has factored the map $|\psi_i| : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ into a sequence of blowups [Kap93], cf. Remark 6.3.9. In [Sch22] we discovered an analogous result for the map $|\phi_{56}| : \overline{M}(3,6) \to \mathbb{P}^2 \times \mathbb{P}^2$. To state the result, let coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$ be given by $(x_1, \ldots, x_4) \times (y_1, \ldots, y_4)$ as in (7.2), and for any $\mathbb{P}^2$ with coordinates $(z_1, \ldots, z_4)$, write

$$P_{ijk} = \{z_i = z_j = z_k\}, \quad L_{ij} = \{z_i = z_j\}.$$

Additionally, we will need the notion of the dominant transform of a subvariety. If $\pi : \widetilde{X} \to X$ is a blowup of a variety X along a subvariety Z, then the dominant transform of a subvariety $Y \subset X$ is either the inverse image of Y, if $Y \subset Z$, or the strict transform of Y, if $Y \not\subset Z$.

Theorem 7.4.2 ([Sch22, Theorem 3.4])**.** *The small resolution $\widetilde{M_1}(3,6)$ is obtained from $\mathbb{P}^2 \times \mathbb{P}^2$ via the following sequence of blowups.*

1. *Blowup the eight surfaces $P_{ijk} \times \mathbb{P}^2$, $\mathbb{P}^2 \times P_{ijk}$.*

2. *Blowup the dominant transforms of the four points $P_{ijk} \times P_{ijk}$.*

3. *Blowup the dominant transforms of the six surfaces $L_{ij} \times L_{ij}$.*

4. *Blowup the dominant transform of the surface $\Delta(\mathbb{P}^2)$.*

5. *Blowup the dominant transforms of the 30 lines $P_{ijk} \times L_{ij}$, $L_{ij} \times P_{ijk}$, $\Delta(L_{ij})$.*

Remark 7.4.3. It turns out that each blowup in the above construction is along a union of codimension two subvarieties intersecting transversally.

Remark 7.4.4. This blowup construction could be thought of as an algebraic version of the sequence of tropical modifications used to obtain $\mathrm{trop}(M(3,6))$ in the proof that $M(3,6)$ is quasilinear.

We refer to [Sch22, Section 3] for more details on this blowup construction and a proof of Theorem 7.4.2.

7.4.3 Intersection theory

Since we have explicit descriptions of the fans Σ_{lc} and $\widetilde{\Sigma}_{S_1,S_2}$, Theorems 7.3.1, 4.1.15 imply explicit descriptions of the Chow rings of $\overline{M}(3,6)$ and all of its small resolutions.

Theorem 7.4.5 ([Sch22, Theorem 4.1])**.** *Let $\widetilde{M}_{S_1,S_2}(3,6)$ be any small resolution of $\overline{M}(3,6)$.*

1.

$$A^*(\widetilde{M}_{S_1,S_2}(3,6)) = \frac{\mathbb{Z}[D_{ijk,lmn}, D_{ij,klmn}, D_{ij,kl,mn}]}{\textit{the following relations}}$$

(a) (Linear relations)

i. $D_{ij,kl,mn} = D_{mn,ij,kl} = D_{kl,mn,ij}$.

ii. $f^*(0) = f^*(1) = f^*(\infty)$*, where* f *is any composition of restriction and forgetful maps* $\widetilde{M}_{S_1,S_2}(3,6) \to \overline{M}(3,6) \xrightarrow{r_i} \overline{M}_{0,5} \xrightarrow{f_j} \overline{M}_{0,4} = \mathbb{P}^1$.

(b) (Multiplicative relations) $\prod D_i = 0$ *if* $\bigcap D_i = \emptyset$ *in* $\widetilde{M}_{S_1,S_2}(3,6)$ *(see Remark 7.4.7 below).*

2. The nontrivial (i.e. $\neq 0, 1$) ranks of the Chow groups are

$$\operatorname{rk} A^1(\widetilde{M}_{S_1,S_2}(3,6)) = 51,$$

$$\operatorname{rk} A^2(\widetilde{M}_{S_1,S_2}(3,6)) = 127 + |S_2|,$$

$$\operatorname{rk} A^3(\widetilde{M}_{S_1,S_2}(3,6)) = 51.$$

3. (a) $\operatorname{Pic} \widetilde{M}_{S_1,S_2}(3,6)$ *is generated by the boundary divisors, modulo the linear relations.*

(b) A basis for $\operatorname{Pic} \widetilde{M}_{S_1,S_2}(3,6)$ *is given by*

i. $D_{156,234}, D_{256,134}, D_{345,126}, D_{346,125}, D_{356,124}, D_{456,123}$*,*

ii. all 15 $D_{ij,klmn}$*,*

iii. all 30 $D_{ij,kl,mn}$*.*

4. (Over $\mathbb{C}$) The map $cl : A_*(\widetilde{M}_{S_1,S_2}(3,6)) \to H_*(\widetilde{M}_{S_1,S_2}(3,6))$ *is an isomorphism.*

Remark 7.4.6. The relations $D_{ij,kl,mn} = D_{mn,ij,kl} = D_{kl,mn,ij}$ reflect that these divisors are all the same. We will assume these relations implicitly.

Remark 7.4.7. The explicit descriptions of the fans $\widetilde{\Sigma}_{S_1,S_2}$ imply that multiplicative relations on $A^*(\widetilde{M}_{S_1,S_2}(3,6))$ are as follows.

1. (Relations from $\overline{M}(3,6)$)

   (a) $D_{ijk,lmn} D_{abc,def} = 0$ if $|ijk \cap abc| = 2$.

   (b) $D_{ij,klmn} D_{ab,cdef} = 0$ if $|ij \cap ab| = 1$.

   (c) $D_{ij,kl,mn} D_{ab,cd,ef} = 0$ unless $\{ij, kl, mn\} = \{ab, cd, ef\}$.

(d) $D_{ijk,lmn}D_{ab,cdef} = 0$ if $|ijk \cap ab| = 1$.

(e) $D_{ijk,lmn}D_{ab,cd,ef} = 0$ unless $ijk = abc$ or $ijk = abd$ (after sufficient cyclic permutation).

(f) $D_{ij,klmn}D_{ab,cd,ef} = 0$ unless $ij = ab$ or cd or ef.

2. (Relations from S_1) $D_{ij,kl,mn}D_{ij,mn,kl} = 0$ for $P_{ij,kl,mn} \in S_1$.

3. (Relations from S_2) $D_{ij,klmn}D_{kl,ijmn}D_{mn,ijkl} = 0$ for $P_{ij,kl,mn} \in S_2$.

By Theorem 7.3.1, we also have an isomorphism $A^*(X(\Sigma_{lc})) \cong A^*(\overline{M}(3,6))$, but since Σ_{lc} is not even simplicial, it is more difficult to obtain an explicit presentation in this case. However, one can use the proper birational map $X(\widetilde{\Sigma}_1) \to X(\Sigma_{lc})$ or the corresponding small resolution $\widetilde{M}_1(3,6) \to \overline{M}(3,6)$ to obtain a description of $A^*(\overline{M}(3,6))$ as a subring of $A^*(\widetilde{M}_1(3,6))$.

Define

$$\delta_{ijk,lmn} = D_{ijk,lmn},$$

$$\delta_{ij,k,lmn} = D_{ij,klmn} + D_{kl,ij,mn} + D_{km,ij,ln} + D_{kn,ij,lm} \text{ for } k < l,m,n,$$

$$\delta_{ij,kl,mn} = D_{ij,kl,mn} - D_{kl,ij,mn} \text{ for } k < l,m,n.$$

The conditions $k < l,m,n$ are so we do not have to worry about permuting the indices. Note that there are 20 $\delta_{ijk,lmn}$, 15 $\delta_{ij,k,lmn}$, and 15 $\delta_{ij,kl,mn}$. One can show that these are Cartier divisors on $\overline{M}(3,6)$ [Sch22, Proposition 2.2].

Theorem 7.4.8. *1. $A^*(\overline{M}(3,6))$ is the subring of $A^*(\widetilde{M}_1(3,6))$ described by $A^k(\overline{M}(3,6)) = A^k(\widetilde{M}_1(3,6))$ for $k \neq 1$, and*

$$A^1(\overline{M}(3,6)) = \{\alpha \in A^1(\widetilde{M}_1(3,6)) \mid \alpha|_{L_{ij,kl,mn}} = 0 \text{ for all } P_{ij,kl,mn}\}.$$

2. The nontrivial (i.e. $\neq 0,1$) ranks of the Chow groups are

$$\operatorname{rk} A^1(\overline{M}(3,6)) = 36,$$

$$\operatorname{rk} A^2(\overline{M}(3,6)) = 127,$$

$$\operatorname{rk} A^3(\overline{M}(3,6)) = 51.$$

3. (a) $\operatorname{Pic} \overline{M}(3,6) = A^1(\overline{M}(3,6))$ and is generated by the $\delta_{ijk,lmn}, \delta_{ij,k,lmn}, \delta_{ij,kl,mn}$, modulo the linear relations $f^*(0) = f^*(1) = f^*(\infty)$ for any composition $f : \overline{M}(3,6) \xrightarrow{r_i} \overline{M}_{0,5} \xrightarrow{f_j} \overline{M}_{0,4}$.

(b) A basis for $\operatorname{Pic} \overline{M}(3,6)$ is

i. $\delta_{156,234}, \delta_{256,134}, \delta_{345,126}, \delta_{346,125}, \delta_{356,124}, \delta_{456,123}$,

ii. all 15 $\delta_{ij,k,lmn}$,

iii. all 15 $\delta_{ij,kl,mn}$.

4. $A^*(\overline{M}(3,6))$ is generated by $A^1(\overline{M}(3,6))$.

We refer to [Sch22, Section 5] for a proof of Theorem 7.4.8.

Remark 7.4.9. Both Theorems 7.4.5 and 7.4.8 can be proven classically (i.e. without any tropical geometry), by using the blowup construction of $\widetilde{M}_1(3,6)$ in Theorem 7.4.2. This is how we first proved these theorems in [Sch22], and it was the observation that the presentations look like the presentation of the Chow ring of a toric variety which eventually led to our defining quasilinearity and discovering the main results of this dissertation.

In fact, in [Sch22], we proved all stated facts about $\overline{M}(3,6)$ using pure algebraic geometry, with no tropical geometry involved. In particular we also gave a proof, just using the blowup construction of Theorem 7.4.2, that $\overline{M}(3,6)$ is the log canonical compactification.

Tautological classes

The explicit presentations of the Chow rings of $\overline{M}(3,6)$ and its small resolutions above also allow us to completely describe all possible intersections of the ψ-classes ψ_{ij} on $\overline{M}(3,6)$, cf. Sections 6.3.1, 7.3.2.

Theorem 7.4.10 ([Sch22, Theorem 6.9])**.** *On $\overline{M}(3,6)$, $\psi_{i_1j_1}\cdots\psi_{i_4j_4}=0 \iff$ at least 3 of the i_k's coincide.*

The nonzero intersection numbers of the ψ_{ij} are listed, up to S_6-symmetry, in Table 7.1.

Table 7.1. Intersections of ψ-classes on $\overline{M}(3,6)$

n	Product
1	$\psi_{12}\psi_{12}\psi_{21}\psi_{21}$, $\psi_{12}\psi_{12}\psi_{21}\psi_{31}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{23}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{32}$, $\psi_{12}\psi_{12}\psi_{31}\psi_{41}$, $\psi_{12}\psi_{12}\psi_{32}\psi_{32}$, $\psi_{12}\psi_{12}\psi_{34}\psi_{34}$, $\psi_{12}\psi_{12}\psi_{34}\psi_{43}$
2	$\psi_{12}\psi_{12}\psi_{21}\psi_{23}$, $\psi_{12}\psi_{12}\psi_{21}\psi_{32}$, $\psi_{12}\psi_{12}\psi_{21}\psi_{34}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{24}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{31}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{34}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{41}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{42}$, $\psi_{12}\psi_{12}\psi_{23}\psi_{43}$, $\psi_{12}\psi_{12}\psi_{31}\psi_{32}$, $\psi_{12}\psi_{12}\psi_{31}\psi_{34}$, $\psi_{12}\psi_{12}\psi_{31}\psi_{42}$, $\psi_{12}\psi_{12}\psi_{31}\psi_{43}$, $\psi_{12}\psi_{12}\psi_{31}\psi_{45}$, $\psi_{12}\psi_{12}\psi_{32}\psi_{34}$, $\psi_{12}\psi_{12}\psi_{32}\psi_{42}$, $\psi_{12}\psi_{12}\psi_{32}\psi_{43}$, $\psi_{12}\psi_{12}\psi_{34}\psi_{35}$, $\psi_{12}\psi_{12}\psi_{34}\psi_{45}$, $\psi_{12}\psi_{12}\psi_{34}\psi_{54}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{31}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{41}$, $\psi_{12}\psi_{13}\psi_{23}\psi_{32}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{42}$, $\psi_{12}\psi_{13}\psi_{41}\psi_{51}$, $\psi_{12}\psi_{13}\psi_{45}\psi_{54}$
3	$\psi_{12}\psi_{12}\psi_{23}\psi_{45}$, $\psi_{12}\psi_{12}\psi_{32}\psi_{45}$, $\psi_{12}\psi_{12}\psi_{34}\psi_{56}$, $\psi_{12}\psi_{21}\psi_{31}\psi_{41}$, $\psi_{12}\psi_{21}\psi_{34}\psi_{43}$, $\psi_{12}\psi_{23}\psi_{42}\psi_{52}$, $\psi_{12}\psi_{32}\psi_{42}\psi_{52}$
4	$\psi_{12}\psi_{13}\psi_{21}\psi_{23}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{24}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{32}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{34}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{42}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{43}$, $\psi_{12}\psi_{13}\psi_{21}\psi_{45}$, $\psi_{12}\psi_{13}\psi_{23}\psi_{24}$, $\psi_{12}\psi_{13}\psi_{23}\psi_{34}$, $\psi_{12}\psi_{13}\psi_{23}\psi_{41}$, $\psi_{12}\psi_{13}\psi_{23}\psi_{42}$, $\psi_{12}\psi_{13}\psi_{23}\psi_{43}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{25}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{34}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{41}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{43}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{45}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{51}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{52}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{54}$, $\psi_{12}\psi_{13}\psi_{41}\psi_{52}$, $\psi_{12}\psi_{13}\psi_{41}\psi_{54}$, $\psi_{12}\psi_{13}\psi_{41}\psi_{56}$, $\psi_{12}\psi_{13}\psi_{42}\psi_{43}$, $\psi_{12}\psi_{13}\psi_{42}\psi_{45}$, $\psi_{12}\psi_{13}\psi_{42}\psi_{52}$, $\psi_{12}\psi_{13}\psi_{42}\psi_{54}$, $\psi_{12}\psi_{13}\psi_{45}\psi_{46}$, $\psi_{12}\psi_{13}\psi_{45}\psi_{56}$, $\psi_{12}\psi_{13}\psi_{45}\psi_{65}$, $\psi_{12}\psi_{21}\psi_{31}\psi_{42}$, $\psi_{12}\psi_{21}\psi_{31}\psi_{43}$
5	$\psi_{12}\psi_{21}\psi_{34}\psi_{45}$, $\psi_{12}\psi_{21}\psi_{31}\psi_{45}$, $\psi_{12}\psi_{23}\psi_{34}\psi_{53}$, $\psi_{12}\psi_{23}\psi_{42}\psi_{53}$, $\psi_{12}\psi_{23}\psi_{43}\psi_{53}$
6	$\psi_{12}\psi_{13}\psi_{23}\psi_{45}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{35}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{53}$, $\psi_{12}\psi_{13}\psi_{24}\psi_{56}$, $\psi_{12}\psi_{13}\psi_{42}\psi_{53}$, $\psi_{12}\psi_{13}\psi_{42}\psi_{56}$, $\psi_{12}\psi_{21}\psi_{34}\psi_{54}$, $\psi_{12}\psi_{21}\psi_{34}\psi_{56}$, $\psi_{12}\psi_{23}\psi_{31}\psi_{41}$, $\psi_{12}\psi_{23}\psi_{34}\psi_{41}$, $\psi_{12}\psi_{23}\psi_{34}\psi_{52}$, $\psi_{12}\psi_{23}\psi_{42}\psi_{56}$, $\psi_{12}\psi_{23}\psi_{43}\psi_{54}$, $\psi_{12}\psi_{32}\psi_{42}\psi_{56}$
7	$\psi_{12}\psi_{23}\psi_{34}\psi_{45}$, $\psi_{12}\psi_{23}\psi_{34}\psi_{54}$, $\psi_{12}\psi_{23}\psi_{43}\psi_{56}$, $\psi_{12}\psi_{23}\psi_{45}\psi_{56}$
8	$\psi_{12}\psi_{23}\psi_{34}\psi_{56}$, $\psi_{12}\psi_{23}\psi_{45}\psi_{65}$, $\psi_{12}\psi_{32}\psi_{45}\psi_{65}$
9	$\psi_{12}\psi_{23}\psi_{31}\psi_{45}$

Chapter 8

Moduli of marked del Pezzo surfaces

Let $Y(2,n) = M(2,n) = M_{0,n}$, and for $n \leq 8$ let $Y(3,n)$ denote the moduli space of n *points* in general position in $\mathbb{P}^2$, meaning

1. no two points coincide,
2. no three points lie on a line,
3. no six points lie on a conic, and
4. no eight points lie on a cubic singular at one of the points.

By projective duality, $Y(3,n) \subset M(3,n)$ is the complement of the locus where six points lie on a conic or eight points lie on a cubic singular at one of the points.

Recall a del Pezzo surface S of degree d is a smooth projective surface S with $-K_S$ ample and $K_S^2 = d$. It is a classical fact that $1 \leq d \leq 9$, and with the exception of the degree 8 del Pezzo surface $\mathbb{P}^1 \times \mathbb{P}^1$, every del Pezzo surface of degree d is the blowup of $\mathbb{P}^2$ at $n = 9 - d$ points in general position [DP80]. Furthermore, if $S \neq \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$, then S has finitely many (-1)-curves, which can be labeled according to a blowup construction $S = Bl_n\mathbb{P}^2$. We call such a labeling a *marking*, and we say a pair (S, B) of a del Pezzo surface S and the union B of its (-1)-curves, marked as described, a *marked del Pezzo surface.* It follows that $Y(3, n)$ can be identified with the moduli space of marked del Pezzo surfaces of degree $9 - n$ [HKT09].

Throughout this section we fix $r = 2$ or $r = 3$ and $n \leq 8$ unless otherwise mentioned. In [HKT09], Hacking, Keel, and Tevelev studied interesting tropical compactifications of $Y(r, n)$ using combinatorics of the root systems $E_{2,n} = D_n$, $E_{3,n} = E_n$. These moduli spaces form another example of higher-dimensional generalizations of $\overline{M}_{0,n}$, which from the combinatorial point of view are better behaved than the moduli spaces $\overline{M}(r, n)$ studied in the previous chapter. In this Chapter we use the technical results developed in Chapter 4 to study the intersection theory of tropical compactifications of $Y(r, n)$.

8.1 Log canonical compactifications

Recall we write $E_{2,n} = D_n$ and $E_{3,n} = E_n$ for the root systems D_n and E_n.

Definition 8.1.1. For $r = 2$ or $r = 3$, $n \leq 8$, define a simplicial complex $\Delta_{r,n}$ as follows. The vertices of $\Delta_{r,n}$ are proper root subsystems $\Theta \times \Theta^\perp \subset E_{r,n}$ whose Dynkin diagram looks like the Dynkin diagram obtained by deleting a single vertex from the extended Dynkin diagram of $E_{r,n}$. A collection $\{\Theta_1 \times \Theta_1^\perp, \ldots, \Theta_k \times \Theta_k^\perp\}$ forms a simplex of $\Delta_{r,n} \iff$ for all $i \neq j$, $\Theta_i \subset \Theta_j$, $\Theta_i \subset \Theta_j^\perp$, $\Theta_i^\perp \subset \Theta_j$, or $\Theta_i^\perp \subset \Theta_j^\perp$.

Additionally, for $r = 3$, $n = 7$, remove the simplices formed by 7-tuples of $A_1 \times A_1^{\perp}$'s with the A_1's pairwise orthogonal, and for $r = 3$, $n = 8$, remove the simplices formed by 7- or 8-tuples of $A_1 \times A_1^{\perp}$'s with the A_1's pairwise orthogonal, and simplices involving $D_4 \times A_1^{\times 4}$'s. (These correspond to root systems which do not give du Val singularities on del Pezzo surfaces.)

Example 8.1.2. 1. The vertices of $\Delta_{2,n}$ correspond to root subsystems $D_k \times D_{n-k} \subset D_n$ for $k \geq 2$. Note these correspond in an obvious way to the boundary divisors of $\overline{M}_{0,n}$, and the simplicial complex $\Delta_{2,n}$ is naturally identified with the boundary complex of $\overline{M}_{0,n}$ [HKT09, Lemma 7.2].

2. There are 76 vertices of $\Delta_{3,6}$, in 2 $W(E_6)$-classes: 36 of the form $A_1 \times A_5$, and 40 of the form $A_2 \times A_2 \times A_2$ [Nar82; HKT09].

3. There are 1065 vertices of $\Delta_{3,7}$, in 4 $W(E_7)$-classes:

   (a) 63 $A_1 \times E_6$'s.

   (b) 336 $A_2 \times A_5$'s.

   (c) 630 $A_3 \times A_1 \times A_3$'s.

   (d) 36 A_7's.

4. There are 227911 vertices of $\Delta_{3,8}$, in 8 $W(E_8)$-classes.

   (a) 120 $A_1 \times E_7$'s.

   (b) 1120 $A_2 \times E_6$'s.

(c) 7560 $A_3 \times D_5$'s.

(d) 12096 $A_4 \times A_4$'s.

(e) 201600 $A_5 \times A_1 \times A_2$'s.

(f) 4320 $A_7 \times A_1$'s.

(g) 960 A_8's.

(h) 135 D_8's.

Remark 8.1.3. A slightly different definition of the complex $\Delta_{r,n}$ is given for $r = 2$ and $r = 3$, $n \leq 7$ in [HKT09], but it is easily seen to give the same complex as in the above definition.

Definition 8.1.4. Let $\mathbb{Z}^{\mathcal{A}_1(E_{r,n})}$ denote the free $\mathbb{Z}$-module with basis $[\alpha] = [\pm\alpha]$ for α a positive root of $E_{r,n}$ ($\{\pm\alpha\}$ is an A_1 root subsystem of $E_{r,n}$).

For a root subsystem $\Theta \times \Theta^\perp \subset E_{r,n}$, define $\rho_{\Theta,\Theta^\perp}$ in $\mathbb{R}^{\mathcal{A}_1(E_{r,n})}$ as the ray passing through the point

$$\sum_{\substack{\alpha \in \Theta \times \Theta^\perp \\ \alpha \text{ a positive root}}} [\alpha].$$

Define a collection of cones $\widetilde{\Sigma}_{r,n}$ in $\mathbb{R}^{\mathcal{A}_1(E_{r,n})}$ by saying $\rho_{\Theta_1,\Theta_1^\perp}, \ldots, \rho_{\Theta_k,\Theta_k^\perp}$ span a cone $\iff$ $\{\Theta_1 \times \Theta_1^\perp, \ldots, \Theta_k \times \Theta_k^\perp\}$ form a simplex of $\Delta_{r,n}$.

Let $X(E_{r,n})$ denote the complement of the Coxeter hyperplane arrangement

$$\{(x_\alpha = 0) \mid \alpha \text{ a positive root of } E_{r,n}\} \subset \mathbb{A}^n.$$

Since $X(E_{r,n})$ is the complement of a hyperplane arrangement, it is very affine and linear; its intrinsic torus is $(\mathbb{C}^*)^{\mathcal{A}_1(E_{r,n})}$. In [HKT09, Section 6], Hacking, Keel, and Tevelev define a morphism $\Psi : X(E_{r,n}) \to Y(r,n)$ as follows.

Definition 8.1.5 ([HKT09]).

- Define $\Psi : X(E_{2,n}) \to Y(2,n)$ by sending $(x_1, \dots, x_n) \in X(E_{2,n})$ to the n points $x_1^2, \dots, x_n^2 \in \mathbb{P}^1$. Note points i, j coincide $\iff x_i = \pm x_j$, but these are exactly the root hyperplanes.
- Define $\Psi : X(E_{3,n}) \to Y(3,n)$ by sending $(x_1, \dots, x_n) \in X(E_{3,n})$ to the n points $(x_i : x_i^3 : 1) \in \mathbb{P}^2$. These n points lie on a cuspidal plane cubic with a cusp at $(0:0:1)$, and the conditions for the points to be in general position are described exactly by the root hyperplanes.

The morphism $\Psi : X(E_{r,n}) \to Y(r,n)$ induces a morphism of tropicalizations $\psi : \mathrm{trop}(X(E_{r,n})) \to \mathrm{trop}(Y(r,n))$. For $r = 2$ or $r = 3$, $n \leq 7$, Ψ is dominant, so ψ is surjective by Lemma 3.1.23. However, for $r = 3$, $n = 8$, Ψ is not dominant, so we cannot say anything about ψ.

Definition 8.1.6. Define a collection of cones $\Sigma_{r,n}$ in $N(Y(r,n))_{\mathbb{R}}$ as the image under ψ of the cones of $\widetilde{\Sigma}_{r,n}$.

Theorem 8.1.7 ([HKT09]). *If $r = 2$ or $r = 3$ and $n \leq 7$, then $\Sigma_{r,n}$ is a unimodular convexly disjoint fan supported on* $\mathrm{trop}(Y(r,n))$.

Theorem 8.1.8 ([HKT09]). *If $r = 2$ or $r = 3$ and $n \leq 7$, then $Y(r,n)$ is schön.*

Corollary 8.1.9 ([HKT09]). *If $r = 2$ or $r = 3$ and $n \leq 7$, then $Y(r,n)$ is hübsch, the log canonical fan is $\Sigma_{r,n}$, and the log canonical compactification is nonsingular.*

Hacking, Keel, and Tevelev prove these results by using geometric tropicalization as discussed in Section 3.1.2: they consider nice compactifications of $Y(r,n)$ and use them to show $Y(r,n)$ is schön and describe the tropicalizaton [HKT09]. In the next section we will give another proof that $Y(r,n)$ is schön.

Remark 8.1.10. The collection of cones $\widetilde{\Sigma}_{r,n}$ is not supported on $\text{trop}(X(E_{r,n}))$. It is not *a priori* obvious that $\widetilde{\Sigma}_{r,n}$ is even a fan; furthermore, one can check even in small cases that $\widetilde{\Sigma}_{r,n}$ is not tropical (i.e. does not satisfy the balancing condition).

Remark 8.1.11. Note the above results are not stated for $r = 3$, $n = 8$; this case is still open. The reason for this is that the map $\Psi : X(E_{3,8}) \to Y(3,8)$ is not dominant, as mentioned above, so the techniques of Hacking, Keel, and Tevelev do not work. We expect however that the above stated results also hold for $Y(3,8)$; in particular note that $\Sigma_{3,8}$ is also still defined.

8.1.1 Quasilinearity of $Y(r,n)$

Theorem 8.1.12. *If $r = 2$ or $r = 3$ and $n \leq 7$, then $Y(r,n)$ is quasilinear.*

Note this theorem implies Theorem 8.1.8, that $Y(r,n)$ is schön. The proof of the theorem is by the same strategy as the proof for $M(r,n)$ in Section 7.2.1. Note $Y(2,n) = M_{0,n}$ and $Y(3,5) = M(3,5)$, so the only new cases are $Y(3,6)$ and $Y(3,7)$. We describe these as complements of hypersurfaces in $M(3,n)$.

Quasilinearity of $Y(3,6)$

$Y(3,6)$ is the complement in $M(3,6)$ of the hypersurface Q parameterizing the locus where 6 points lie on a line. Recall from Section 7.2.1 and Remark 7.2.6 that $M(3,6)$ is the complement of a collection of 10 hypersurfaces in T^4, where coordinates on T^4 are x_1, x_2, y_1, y_2. In these coordinates,

$$Q = \left\{ \begin{vmatrix} 1 & 1 & 1 \\ x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \end{vmatrix} = 0 \right\}.$$

The automorphism

$$\varphi : T^4 \to T^4$$
$$(x_1, x_2, y_1, y_2) \mapsto \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{y_1}, \frac{1}{y_2} \right)$$

sends $Q \subset T^4$ to

$$Q_{123} = \{(x_1 - 1)(y_2 - 1) = (x_2 - 1)(y_1 - 1)\}$$

and vice-versa. It sends each of the remaining 9 hypersurfaces to itself. Let $M' = M'(3,6) \subset T^{13}$ be the complement of these 9 remaining hypersurfaces. Then M' is quasilinear by the arguments of Section 7.4, and φ induces an automorphism of T^{13}, restricting to an automorphism of M' which swaps the divisors Q and Q_{123}. The composition of this automorphism with the projection showing Q_{123} is quasilinear in M' (cf. Section 7.4) therefore shows that Q is quasilinear in M'. One directly verifies that Q and Q_{123} are disjoint in M'. Thus we conclude that $Q \subset M(3,6)$ is quasilinear, hence $Y(3,6) = M(3,6) \setminus Q$ is quasilinear.

Remark 8.1.13. The automorphism φ is induced by the Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $(x : y : z) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$ centered at the coordinate points. This Cremona transformation sends the conic through the 6 points to the line through the images of the three non-coordinate points.

Remark 8.1.14. Note $Q \cong M_{0,6}$. However, the embedding $Q \subset M(3,6) \subset T^{14}$ is different from the embedding of $M_{0,6}$ in its intrinsic torus T^9; $Q \subset T^{14}$ is not linear, while $M_{0,6} \subset T^9$ is. On the other hand, they have isomorphic tropicalizations, and under the isomorphism $Q \cong M_{0,6}$, the embedding $Q \subset T^{14}$ factors through the embedding $M_{0,6} \subset T^9$.

Quasilinearity of $Y(3,7)$

$Y(3,7)$ is the complement in $M(3,7)$ of the seven hypersurfaces $Q_1, \ldots, Q_7$, where Q_i parameterizes the locus where the six points $p_j \neq p_i$ lie on a conic.

By symmetry, the Q_i's in $M(3,7)$ are all isomorphic to each other. Furthermore, the intersection of any two of the Q_i's is the same as the intersection of all seven; it is the locus where all seven points lie on a conic. Therefore, to show the Q_i's and their intersections are quasilinear, we will restrict our attention to $i = 1, 2, 3, 4$, cf. Remark 8.1.15.

Recall from Section 7.2.1 and Remark 7.2.6 that $M(3,7)$ is the complement of an arrangement $\mathcal{A}$ of 22 hypersurfaces in T^6, where coordinates on T^6 are $x_1, x_2, x_3, y_1, y_2, y_3$. The automorphism

$$\varphi : T^6 \to T^6$$
$$(x_1, x_2, x_3, y_1, y_2, y_3) \mapsto \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{y_1}, \frac{1}{y_2}, \frac{1}{y_3}\right)$$

gives a correspondence

$$Q_1 \leftrightarrow Q_{234}, \quad Q_2 \leftrightarrow Q_{134}, \quad Q_3 \leftrightarrow Q_{124}, \quad Q_4 \leftrightarrow Q_{123},$$

and sending each remaining hypersurface $H \in \mathcal{A}' = \mathcal{A} \setminus \{Q_{123}, Q_{124}, Q_{134}, Q_{234}\}$ to itself. Let $M' = M'(3,7) \subset T^{24}$ be the complement of the hypersurfaces in $\mathcal{A}'$. Then M' is quasilinear by the arguments of Section 7.2.1, and φ induces an automorphism of T^{24} which restricts to an automorphism of M' swapping Q_i, $i \in [4]$ and Q_{jkl}, $jkl \subset [4]$ as above. The compositions of this automorphism with the projections showing the Q_{ijk}, $ijk \in [4]$, and their intersections, are quasilinear (cf. Section 7.2.1), therefore shows that Q_i, $i \in [4]$, and their intersections, are quasilinear in M'. Thus in order to show that the Q_i and their intersections are quasilinear in $M = M(3,7)$, all that remains is to show that the intersections of the Q_i's and Q_{jkl}'s in M' are quasilinear.

It is an immediate verification that Q_i and Q_{jkl} are disjoint in M', where i, j, k, l are distinct. Since the intersection of any two of the Q_i's is the same as the intersection of all four, and likewise for the Q_{jkl}'s, this implies that the only nontrivial case, up to symmetry, is $Q_4 \cap Q_{234}$. But the forgetful map dropping point 3 induces a projection $T^{24} \to T^{13}$ identifying $Q_4 \cap Q_{234} \subset T^{24}$ with $M'(3,6) \subset T^{13}$, where $M'(3,6)$ is as in the proof of the $Y(3,6)$ case above. This identification induces inclusions

$$Q_4 \cap Q_{234} \cong M'(3,6) \subset T^{13} \subset T^{24},$$

so since $M'(3,6) \subset T^{13}$ is quasilinear, we conclude by Proposition 4.3.29 that $Q_4 \cap Q_{234} \subset T^{24}$ is also quasilinear. See Appendix A.2 for details.

Since we have shown all intersections of the Q_i's, $i \in [4]$, and Q_{jkl}'s, $jkl \subset [4]$, in $M'(3,7)$ are quasilinear, we conclude that Q_1, Q_2, Q_3, Q_4 and their intersections are quasilinear in $M(3,7)$; by symmetry, the same holds for all Q_i's. Thus the complement $Y(3,7)$ of the Q_i's in $M(3,7)$ is quasilinear.

Remark 8.1.15. Under the automorphism $\varphi : T^6 \to T^6$ above, the hypersurfaces Q_5, Q_6, and Q_7 do not correspond to any of the hypersurfaces in $\mathcal{A}$. We know by symmetry that Q_5, Q_6, and Q_7 are quasilinear in $M(3,7)$. To show this directly with an argument as above, one would have to change coordinates or work with a different automorphism, e.g. by swapping points 1,2,3 with points 5,6,7.

Completion of proof

Proof of Theorem 8.1.12. We have shown above that $Y(r,n)$ is quasilinear for $r = 2$ and $r = 3$, $n \leq 7$. $\square$

8.2 Intersection theory

Theorem 8.2.1. *Assume $r = 2$ or $r = 3$ and $n \leq 7$. If $\overline{Y}^{\Sigma}(r,n) \subset X(\Sigma)$ is any tropical compactification of $Y(r,n)$, then $A^*(\overline{Y}^{\Sigma}(r,n)) \cong A^*(\Sigma)$ and $\overline{Y}^{\Sigma}(r,n)$ satisfies Kronecker duality. If in addition $X(\Sigma)$ is smooth, then so is $\overline{Y}^{\Sigma}(r,n)$, and*

$$H^*(\overline{Y}^{\Sigma}(r,n)) \cong A^*(\overline{Y}^{\Sigma}(r,n)) \cong A^*(X(\Sigma)) \cong H^*(X(\Sigma)).$$

Proof. Immediate from Theorem 8.1.12 and Theorem 4.3.22. □

As a particular case, we get an explicit description of the Chow ring of the log canonical compactification $\overline{Y}(r,n)$.

Theorem 8.2.2. *Let $r = 2$ or $r = 3$ and $n \leq 7$. Then*

$$A^*(\overline{Y}(r,n)) \cong H^*(\overline{Y}(r,n)) \cong \frac{\mathbb{Z}[D_{\Theta,\Theta^\perp}]}{\textit{the following relations}}$$

- *(Linear relations) $f^*(0) = f^*(1) = f^*(\infty)$ for any cross-ratio map $f : \overline{Y}(r,n) \to \overline{M}_{0,4}$.*
- *(Multiplicative relations) $D_{\Theta_1,\Theta_1^\perp} \cdot D_{\Theta_2,\Theta_2^\perp}$ unless $\Theta_1 \subset \Theta_2$, $\Theta_1 \subset \Theta_2^\perp$, $\Theta_1^\perp \subset \Theta_2$, or $\Theta_1^\perp \subset \Theta_2^\perp$.*

Proof. The fan $\Sigma_{r,n}$ can be viewed as the intersection fan for the product of all cross-ratio maps $\overline{Y}(r,n) \to \overline{M}_{0,4}$ (obtained by restricting to a D_4 root subsystem) [HKT09], cf. Proof of Proposition 7.2.11. Then the given presentation describes the Chow ring of $\Sigma_{r,n}$ by Theorem 4.1.15. It is the same as the Chow ring of $\overline{Y}(r,n)$ by Theorem 8.2.1. □

Remark 8.2.3. Partial results on the Chow ring of $\overline{Y}(3,6)$ were previously obtained by Colombo and van Geemen in [CG04]. Additionally, Luxton has shown that $\overline{Y}(3,6)$ is obtained from the small resolution $\widetilde{M}_1(3,6)$ of $\overline{M}(3,6)$ by blowing up the ten surfaces $D_{ijk,lmn} \cap D_{lmn,ijk}$ (see Section 7.4). This implies the results on the Chow ring of $\overline{Y}(3,6)$ given the results on the Chow ring of $\widetilde{M}_1(3,6)$ from Theorem 7.4.5.

8.3 Stable pair compactifications

In this section we summarize partial results due to Hacking, Keel, and Tevelev [HKT09], and Gallardo, Kerr, and Schaffler [GKS21] on understanding modular compactifications of moduli of marked del Pezzo surfaces and their connections to tropical compactifications. There are no results in this section which are original to this dissertation, but see Remark 8.3.8.

In a general a marked del Pezzo surface (S, B) is not a stable pair, even if S is smooth, because B can fail to be a normal crossings divisor.

Example 8.3.1. An *Eckardt point* of a cubic surface S is a point where three lines intersect. The locus of cubic surfaces with an Eckardt point has codimension one in the moduli space of smooth cubic surfaces; it is the union of 45 irreducible hypersurfaces [Dol, Section 9.4.5], [Nar82]. An example of a cubic surface with Eckardt points is the *Clebsch cubic surface* in $\mathbb{P}^4$, defined by the equations

$$x_0 + x_1 + x_2 + x_3 + x_4 + x + 5 = 0, \quad x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

This surface has 10 Eckardt points [Dol, Section 9.5.4].

Lemma 8.3.2. *Fix $5 \leq n \leq 8$. Let (S, B) be a smooth marked del Pezzo surface of degree $9 - n$ such that B is a normal crossings divisor. Then (S, cB) is a stable pair for all rational c in the ranges shown in Table 8.1.*

Table 8.1. Ranges for stable marked del Pezzo surfaces.

n	c
5	$(1/4, 1]$
6	$(1/9, 1]$
7	$(1/28, 1]$
8	$(1/240, 1]$

Proof. The pair (S, cB) has log canonical singularities for all $c \in \mathbb{Q} \cap (0, 1]$, since S is smooth and B is a normal crossings divisor. Thus we just need to determine when $K_S + cB$ is ample. By definition $-K_S$ is ample. One directly computes (e.g. using a blowup construction of S) that $K_S + cB$ is as in Table 8.2. The result follows. □

Table 8.2. Values of $K_S + cB$.

n	$K_S + cB$
5	$(4c - 1)(-K_S)$
6	$(9c - 1)(-K_S)$
7	$(28c - 1)(-K_S)$
8	$(240c - 1)(-K_S)$.

Definition 8.3.3. Fix $5 \leq n \leq 8$. A *smooth weighted stable marked del Pezzo surface* of degree $9 - n$ and weight c is a pair (S, cB) such that (S, B) is a marked del Pezzo surface of degree $9 - n$ with B a normal crossings divisor, and c is a rational number in the range given in Table 8.1.

Let $Y_c(3, n)$ denote the moduli space of smooth weighted stable marked del Pezzo surfaces of degree $9 - n$ and weight c.

Remark 8.3.4. We do not consider other values of n because they give no moduli.

Note that regardless of the weight c, the moduli space $Y_c(3,n)$ is identified with the open subvariety $Y^\circ(3,n) \subset Y(3,n)$ parameterizing smooth marked del Pezzo surfaces (S,B) with B a normal crossings divisor.

We would like to construct and understand the stable pair compactification $\overline{Y}_c(3,n)$ of $Y_c(3,n)$. The general expectation is that the weight interval from Table 8.1 splits into finitely many subintervals, called *chambers*, such that $\overline{Y}_c(3,n)$ is the same for all c in the interior of a given chamber. The boundaries between the chambers are called *walls*, and one expects that as the weight decreases $c > c'$ crossing a wall, there is a birational morphism $\overline{Y}_c(3,n) \to \overline{Y}_{c'}(3,n)$. In particular, the compactification $\overline{Y}_c(3,n)$ for c in the lowest chamber should be the smallest of these compactifications. This motivates the following conjecture.

Conjecture 8.3.5 ([HKT09, Remark 1.3(4)]). *The log canonical compactification $\overline{Y}(3,n)$ of $Y(3,n)$ is isomorphic to (the normalization of) the stable pair compactification $\overline{Y}_c(3,n)$ for the* minimal weight *c. That is, $\overline{Y}(3,n) = \overline{Y}_c(3,n)$ for c as in Table 8.3.*

Table 8.3. Minimal weights for marked del Pezzo surfaces.

n	Minimal c
5	$\frac{1}{4}+\varepsilon$
6	$\frac{1}{9}+\varepsilon$
7	$\frac{1}{28}+\varepsilon$
8	$\frac{1}{240}+\varepsilon$

Progress on the conjecture has been made by Hacking, Keel, and Tevelev [HKT09], and Gallardo, Kerr, and Schaffler [GKS21].

Theorem 8.3.6 ([HKT09; GKS21]). *We have $\overline{Y}(3,5) = \overline{Y}_{\frac{1}{4}+\varepsilon}(3,5) = \overline{Y}_1(3,5)$ and $\overline{Y}(3,6) = \overline{Y}_{\frac{1}{9}+\varepsilon}(3,6)$.*

Note the theorem implies that there are no walls for $n = 5$. However, there are walls when $n = 6$.

Theorem 8.3.7 ([HKT09]). *There is a unimodular refinement $\Sigma^1_{3,6}$ of the log canonical fan $\Sigma_{3,6}$ such that $\overline{Y}_1(3,6)$ is the closure of $Y_1(3,6)$ in $X(\Sigma^1_{3,6})$.*

The refinement $\Sigma^1_{3,6}$ is explicitly described in [HKT09, Definition 10.4].

Remark 8.3.8. In particular, Theorems 8.2.1 and 8.2.2 give explicit presentations of the Chow rings of the stable pair compactifications $\overline{Y}_{\frac{1}{4}+\varepsilon}(3,5) = \overline{Y}_1(3,5)$, $\overline{Y}_{\frac{1}{9}+\varepsilon}(3,6)$, and $\overline{Y}_1(3,6)$.

Remark 8.3.9. The construction of $\overline{Y}_1(3,6)$ by Hacking, Keel, and Tevelev [HKT09] is reminiscent of their construction of $\overline{M}^m(r,n)$ described in Section 7.1.2, [HKT06]. Indeed, $\overline{Y}_1(3,6)$ is obtained as the closure of $Y_1(3,6)$ in a toric variety $X(\Sigma^1_{3,6})$; this toric variety has an interpretation as a moduli space of certain stable toric varieties, and the universal family over $\overline{Y}_1(3,6)$ is the pullback of the universal family over $X(\Sigma^1_{3,6})$.

An alternative construction of $\overline{M}(r,n)$ was given by Alexeev in [Ale08; Ale15]. We speculate that an adaptation of these ideas to marked del Pezzo surfaces could be used to construct all weighted stable pair compactifications $\overline{Y}_c(3,n)$ as tropical compactifications whose corresponding fans are refinements of the log canonical fan $\Sigma_{3,n}$.

Recall that the forgetful map $f_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ realizes $\overline{M}_{0,n+1}$ as the universal family over $\overline{M}_{0,n}$. There is a similar result for weighted stable marked del Pezzo surfaces with weight 1. First note that contracting a (-1)-curve on a del Pezzo surface gives a natural forgetful map $f : Y(3,n+1) \to Y(3,n)$.

Theorem 8.3.10 ([HKT09, Theorem 1.2])**.** *Assume $n = 5$ or 6. Then forgetful map $f : Y(3, n+1) \to Y(3, n)$ induces a commutative diagram*

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \overline{Y}(3, n+1) \\ \downarrow & & \downarrow \\ \overline{Y}_1(3, n) & \longrightarrow & \overline{Y}(3, n), \end{array}$$

where the horizontal maps are isomorphisms for $n = 5$ and birational log crepant morphisms for $n = 6$, and $\mathcal{S}$ is the universal family over $\overline{Y}_1(3, n)$. In particular $\overline{Y}_1(3, 6) \to \overline{Y}_1(3, 5)$ is the universal family.

Note the theorem is also suggestive of our expectation that $\overline{Y}_1(3, 7)$ is obtained as a tropical compactification by a refinement of the fan $\Sigma^1_{3,7}$. But currently we have no further results on the weighted stable pair compactifications of $Y_c(3, n)$; in particular nothing is known for $n = 7$ or 8.

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Appendix A

Quasilinearity computations

A.1 Computations for $M(3,7)$, $M(3,8)$

In this appendix we perform in detail the computations showing $M(3,7)$ and $M(3,8)$ are quasilinear. The basic idea is to perform a sequence of projections to reduce from the $M(3,n)$ case to the $M(3,n-1)$ case. We refer to Section 7.2.1 for the notation and setup.

A.1.1 $M(3,7)$

Recall from Section 7.2.1 that we only need to show that $Z = Q_{125} \cap Q_{345} \subset M_1(3,7)$ is quasilinear. For ease of notation, we relabel $x_i = x_{i5}$, $y_i = y_{i5}$, and dehomogenize by setting $x_4 = y_4 = 1$. This amounts to using the following matrix to get coordinates on $M(3,7)$ as an open subset of $\mathbb{A}^6$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ x_1 & x_2 & x_3 & 1 & 0 & 1 & 0 \\ y_1 & y_2 & y_3 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{A.1}$$

Then equations for $Z \subset T_1 = T^{18}$ are as follows.

$$\begin{aligned} x_{12} = x_1 - x_2, \quad x_{13} = x_1 - x_3, \quad x_{23} = x_2 - x_3, \\ x_{14} = x_1 - 1, \quad x_{24} = x_2 - 1, \quad x_{34} = x_3 - 1, \\ y_{12} = y_1 - y_2, \quad y_{13} = y_1 - y_3, \quad y_{23} = y_2 - y_3, \\ y_{14} = y_1 - 1, \quad y_{24} = y_2 - 1, \quad y_{34} = y_3 - 1, \\ x_1 y_2 = x_2 y_1, \quad x_3 = y_3. \end{aligned}$$

The projection $T^{18} \to T^{16}$ dropping y_3 and y_{34} realizes Z as the graph of the nonvanishing regular functions x_3 and x_{34} on the subvariety $Z' \subset T^{16}$ defined by the equations

$$\begin{aligned} x_{12} = x_1 - x_2, \quad x_{13} = x_1 - x_3, \quad x_{23} = x_2 - x_3, \\ x_{14} = x_1 - 1, \quad x_{24} = x_2 - 1, \quad x_{34} = x_3 - 1, \\ y_{12} = y_1 - y_2, \quad y_{13} = y_1 - y_3, \quad y_{23} = y_2 - y_3, \\ y_{14} = y_1 - 1, \quad y_{24} = y_2 - 1, \\ x_1 y_2 = x_2 y_1. \end{aligned}$$

Thus it suffices to show $Z' \subset T^{16}$ is quasilinear. For this consider the projection $T^{16} \to T^{11}$ dropping coordinates x_{13}, x_{23}, x_{34}, y_{13}, y_{23}. Then Z' is the complement of the hyperplanes

$$\{x_1 = x_3\}, \quad \{x_2 = x_3\}, \quad \{x_3 = 1\}, \quad \{y_1 = x_3\}, \quad \{y_2 = x_3\} \tag{A.2}$$

on the variety $Z'' \subset T^{11}$ defined by the equations

$$\begin{aligned} x_{12} = x_1 - x_2, \quad x_{14} = x_1 - 1, \quad x_{24} = x_2 - 1, \\ y_{12} = y_1 - y_2, \quad y_{14} = y_1 - 1, \quad y_{24} = y_2 - 1, \\ x_1 y_2 = x_2 y_1. \end{aligned}$$

Observe that $Z'' \cong Q_{124} \times T^1_{x_3} \subset M_1(3,6) \times T^1_{x_3}$. In particular, Z'' is quasilinear. Now all that remains is to show Z'' intersects any collection of the hyperplanes in (A.2) quasilinearly, but this is a direct verification which we leave to the reader.

A.1.2 $M(3,8)$

Recall from Section 7.2.1 that $M_2(3,8)$ is the complement in $M_1(3,8)$ of the hypersurfaces Q_{ijk}, $ijk \subset [5]$, and to show $M(3,8)$ is quasilinear, it suffices to show that $Q_{456} \subset M_2(3,8)$ and $Q_{126} \cap Q_{456} \subset M_2(3,8)$ are quasilinear.

$Q_{456} \subset M_2(3,8)$

To show $Q_{456} \subset M_2(3,8)$ is quasilinear, we show $Q_{456} \subset M_1(3,8)$ intersects any collection of the $Q_{ijk} \subset M_1(3,8)$, $ijk \subset [5]$, quasilinearly. As discussed in Section 7.2.1, it suffices show

$Q_{123} \cap Q_{124} \cap Q_{456}$, $Q_{124} \cap Q_{135} \cap Q_{456}$, $Q_{124} \cap Q_{456}$, and $Q_{123} \cap Q_{456}$ are all quasilinear in $M_1(3,8) \subset T_1 = T^{28}$. For ease of notation, we can by symmetry replace $Q_{123} \cap Q_{124} \cap Q_{456}$, $Q_{124} \cap Q_{135} \cap Q_{456}$, and $Q_{124} \cap Q_{456}$ with $Q_{356} \cap Q_{456} \cap Q_{126}$, $Q_{125} \cap Q_{136} \cap Q_{456}$, and $Q_{126} \cap Q_{456}$ respectively.

Now, simplify the coordinates on M_1 by setting $x_i = x_{i6}$, $y_i = y_{i6}$, and dehomogenizing by $x_5 = y_5 = 1$. This amounts to using the following matrix to get coordinates on $M(3,8)$ as an open subset of $\mathbb{A}^8$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 & 1 & 0 & 1 & 0 \\ y_1 & y_2 & y_3 & y_4 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{A.3}$$

Then $M_1 \subset T^{28}$ is defined by the following equations.

$$x_{12} = x_1 - x_2, \quad x_{13} = x_1 - x_3, \quad x_{14} = x_1 - x_4, \quad x_{23} = x_2 - x_3, \quad x_{24} = x_2 - x_4,$$

$$x_{34} = x_3 - x_4, \quad x_{15} = x_1 - 1, \quad x_{25} = x_2 - 1, \quad x_{35} = x_3 - 1, \quad x_{45} = x_4 - 1,$$

$$y_{12} = y_1 - y_2, \quad y_{13} = y_1 - y_3, \quad y_{14} = y_1 - y_4, \quad y_{23} = y_2 - y_3, \quad y_{24} = y_2 - y_4,$$

$$y_{34} = y_3 - y_4, \quad y_{15} = y_1 - 1, \quad y_{25} = y_2 - 1, \quad y_{35} = y_3 - 1, \quad y_{45} = y_4 - 1.$$

The four subvarieties of M_1 are given by the following equations.

$$Q_{456} \cap Q_{356} \cap Q_{126} = \{x_4 = y_4, x_3 = y_3, x_1 y_2 = x_2 y_1\} \subset M_1,$$

$$Q_{125} \cap Q_{136} \cap Q_{456} = \{x_4 = y_4, x_1 y_3 = x_3 y_1, x_{15} y_{25} = x_{25} y_{15}\} \subset M_1,$$

$$Q_{126} \cap Q_{456} = \{x_4 = y_4, x_1 y_2 = x_2 y_1\} \subset M_1,$$

$$Q_{123} \cap Q_{456} = \{x_4 = y_4, x_{13} y_{23} = x_{23} y_{13}\} \subset M_1.$$

The projection $T^{28} \to T^{26}$ dropping coordinates y_4, y_{45} identifies $Q_{456} \subset M_1$ with the graph of the nonvanishing regular functions x_4, x_{45} on the linear variety $Q \subset T^{26}$ defined by the following equations.

$$x_{12} = x_1 - x_2, \quad x_{13} = x_1 - x_3, \quad x_{14} = x_1 - x_4, \quad x_{23} = x_2 - x_3, \quad x_{24} = x_2 - x_4,$$

$$x_{34} = x_3 - x_4, \quad x_{15} = x_1 - 1, \quad x_{25} = x_2 - 1, \quad x_{35} = x_3 - 1, \quad x_{45} = x_4 - 1,$$

$$y_{12} = y_1 - y_2, \quad y_{13} = y_1 - y_3, \quad y_{14} = y_1 - x_4, \quad y_{23} = y_2 - y_3, \quad y_{24} = y_2 - x_4,$$

$$y_{34} = y_3 - x_4, \quad y_{15} = y_1 - 1, \quad y_{25} = y_2 - 1, \quad y_{35} = y_3 - 1.$$

Now the projection $T^{26} \to T^{19}$ dropping coordinates x_{14}, x_{24}, x_{34}, x_{45}, y_{14}, y_{24}, y_{34} identifies $Q \subset T^{26}$ with the complement of the hyperplanes

$$\{x_1 = x_4\}, \quad \{x_2 = x_4\}, \quad \{x_3 = x_4\}, \quad \{x_4 = 1\}, \quad \{y_1 = x_4\}, \quad \{y_2 = x_4\}, \quad \{y_3 = x_4\} \tag{A.4}$$

in the linear variety $Q' \subset T^{19}$ defined by the following equations.

$$x_{12} = x_1 - x_2, \quad x_{13} = x_1 - x_3, \quad x_{23} = x_2 - x_3,$$

$$x_{15} = x_1 - 1, \quad x_{25} = x_2 - 1, \quad x_{35} = x_3 - 1,$$

$$y_{12} = y_1 - y_2, \quad y_{13} = y_1 - y_3, \quad y_{23} = y_2 - y_3,$$

$$y_{15} = y_1 - 1, \quad y_{25} = y_2 - 1, \quad y_{35} = y_3 - 1.$$

Observe that $Q' \cong M_1(3,7) \times T^1_{x_4}$. The images of the four subvarieties of interest under these projections are as follows.

$$\begin{aligned}
Q_{456} \cap Q_{356} \cap Q_{126} &: \{x_3 = y_3, x_1 y_2 = x_2 y_1\} \subset Q' \\
&\cong (Q_{345} \cap Q_{125}) \times T^1 \subset M_1(3,7) \times T^1, \\
Q_{125} \cap Q_{136} \cap Q_{456} &: \{x_1 y_3 = x_3 y_1, x_{15} y_{25} = x_{25} y_{15}\} \subset Q' \\
&\cong (Q_{135} \cap Q_{124}) \times T^1 \subset M_1(3,7) \times T^1, \\
Q_{126} \cap Q_{456} &: \{x_1 y_2 = x_2 y_1\} \subset Q' \\
&\cong Q_{125} \times T^1 \subset M_1(3,7) \times T^1, \\
Q_{123} \cap Q_{456} &: \{x_{13} y_{23} = x_{23} y_{13}\} \subset Q' \\
&\cong Q_{123} \times T^1 \subset M_1(3,7) \times T^1.
\end{aligned}$$

In particular, these projections are all quasilinear. All that remains is to show that each projection intersects any collection of the hyperplanes in (A.4) quasilinearly, but this is an immediate verification we leave to the reader.

$Q_{126} \cap Q_{456} \subset M_2$

Recall from Section 7.2.1 that $Z = Q_{126} \cap Q_{456} \subset M_2(3,8)$ is naturally identified with $M(3,7)$ via the forgetful map dropping line 6. We describe this isomorphism explicitly with coordinates.

We take our starting coordinates on T^8 to be given by the matrix (A.3). The forgetful map dropping line 6 is induced by the rational map

$$f_6 : T^8 \dashrightarrow T^6,$$
$$(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \mapsto \left(\frac{x_1 - 1}{x_4 - 1}, \frac{x_2 - 1}{x_4 - 1}, \frac{x_3 - 1}{x_4 - 1}, \frac{y_1 - 1}{y_4 - 1}, \frac{y_2 - 1}{y_4 - 1}, \frac{y_3 - 1}{y_4 - 1} \right).$$

This is obtained by dropping column 6 in the matrix (A.3), then performing matrix transformations to write the resulting matrix in the form of matrix (A.1). If coordinates on T^6 are labeled by $z_1, z_2, z_3, w_1, w_2, w_3$, then f_6 induces the following correspondence between hypersurfaces on T^8 and hypersurfaces on T^6.

$$\{x_i = x_j\} \leftrightarrow \{z_i = z_j\}, \quad \{y_i = y_j\} \leftrightarrow \{w_i = w_j\},$$
$$\{x_i = x_4\} \leftrightarrow \{z_i = 1\}, \quad \{y_i = y_4\} \leftrightarrow \{w_i = 1\},$$
$$Q_{ijk} \subset T^8 \leftrightarrow Q_{ijk} \subset T^6, \quad ijk \subset [5].$$

It sends the remaining hyperplanes $\{x_i = 1\}$ and $\{y_i = 1\}$ to all of T^6. We conclude that $f_6 : T^8 \to T^6$ induces a projection $f_6 : T^{38} \to T^{28}$ mapping $M_2(3,8) \subset T^{38}$ onto

$M(3,7) \subset T^{28}$. Furthermore, f_6 restricts to an isomorphism of

$$Z = Q_{126} \cap Q_{456} = \{x_1 y_2 = x_2 y_1, x_4 = y_4\} \subset M_2(3,8)$$

with $M(3,7)$; using the computer algebra system Magma, we computed that a regular inverse to $f_6 : Z \to M(3,7)$ is induced by the following rational map $g : T^6 \dashrightarrow T^8$.

$$\begin{aligned}
x_1 &= \frac{(z_1 - z_2)(w_1 - z_1)}{z_1 w_2 - z_2 w_1}, \\
x_2 &= \frac{(z_1 - z_2)(w_2 - z_2)}{z_1 w_2 - z_2 w_1}, \\
x_3 &= \frac{w_2(z_1 - z_3) + w_1(z_3 - z_2) + z_3(z_2 - z_1)}{z_1 w_2 - z_2 w_1}, \\
x_4 &= \frac{(z_1 - 1)(w_2 - 1) - (z_2 - 1)(w_1 - 1)}{z_1 w_2 - z_2 w_1}, \\
y_1 &= \frac{(w_1 - w_2)(w_1 - z_1)}{z_1 w_2 - z_2 w_1}, \\
y_2 &= \frac{(w_1 - w_2)(w_2 - z_2)}{z_1 w_2 - z_2 w_1}, \\
y_3 &= \frac{z_2(w_1 - w_3) + z_1(w_3 - w_2) + z_3(w_2 - w_1)}{z_1 w_2 - z_2 w_1}, \\
y_4 &= \frac{(z_1 - 1)(w_2 - 1) - (z_2 - 1)(w_1 - 1)}{z_1 w_2 - z_2 w_1}.
\end{aligned}$$

This is regular on $M(3,7)$ because the locus $Q_{125} = \{z_1 w_2 = z_2 w_1\}$ where the denominator vanishes is removed in $M(3,6)$. In particular, g induces inclusions

$$Z \cong M(3,7) \subset T^{28} \subset T^{38}.$$

Since $Z \cong M(3,7)$ is quasilinear in T^{28}, we conclude that it is quasilinear in T^{38} by Proposition 4.3.29.

A.2 Computations for $Y(3,7)$

In this appendix we give the details showing that $Q_4 \cap Q_{234} \subset M'(3,7) \subset T^{24}$ is quasilinear, cf. Section 8.1.1. The strategy is essentially the same as the strategy for $Q_{126} \cap Q_{456} \subset M_2(3,8)$ given in the previous section.

The projection

$$f_3 : T^6 \to T^4,$$

$$(x_1, x_2, x_3, y_1, y_2, y_3) \mapsto (x_1, x_2, y_1, y_2)$$

induces the following correspondence between hypersurfaces in T^6 and hypersurfaces in T^4.

$$x_i = 1 \leftrightarrow x_i = 1, \quad y_i = 1 \leftrightarrow y_i = 1, \quad i = 1, 2$$

$$x_1 = x_2 \leftrightarrow x_1 = x_2, \quad y_1 = y_2 \leftrightarrow y_1 = y_2,$$

$$Q_{125} \leftrightarrow Q_{124}, \quad Q_{145} \leftrightarrow Q_{134}, \quad Q_{245} \leftrightarrow Q_{234}.$$

It sends the remaining hypersurfaces for $M'(3,7)$ to all of T^4. We conclude that $f_3 : T^6 \to T^4$ induces a projection $f_3 : T^{24} \to T^{13}$ sending $M'(3,7)$ onto $M'(3,6)$, where $M'(3,6) \subset T^{13}$ is the complement of all hypersurfaces for $M(3,6)$ except for Q_{123}, cf. Section 8.1.1. Furthermore, f_3 restricts to an isomorphism of $Q_4 \cap Q_{234} \subset M'(3,7)$ with $M'(3,6)$; using the computer algebra system Magma, we compute that an inverse to $f_3 : Q_4 \cap Q_{234} \to M'(3,6)$ is induced by the rational map $g : T^4 \dashrightarrow T^6$ given by

$$\begin{aligned} x_1 = x_1, \quad x_2 = x_2, \\ x_3 = \frac{x_1x_2y_1 - x_1x_2y_2 - x_1y_1y_2 + x_1y_2^2}{(x_1y_2 - x_2y_1)(y_2 - 1)}, \\ y_1 = y_1, \quad y_2 = y_2, \\ x_3 = \frac{x_1x_2y_1 - x_1y_1y_2 - x_2^2y_1 + x_2y_1y_2}{(x_1y_2 - x_2y_1)(y_2 - 1)}. \end{aligned}$$

This is regular on $M'(3,6)$, because the locus $\{x_1y_2 = x_2y_1\} \cup \{y_2 = 1\}$ where the denominators vanish is removed to obtain $M'(3,6)$. In particular, g induces inclusions

$$Q_4 \cap Q_{234} \cong M'(3,6) \subset T^{13} \subset T^{24},$$

and since $M'(3,6)$ is quasilinear in T^{13}, we conclude that $Q_4 \cap Q_{234}$ is quasilinear in T^{24} by Proposition 4.3.29.