MEANING AND COLLECTIVE ARGUMENTATION IN MATHEMATICS:

INVESTIGATING INFERENTIALISM'S POTENTIAL CONTRIBUTIONS

by

JAMES DRIMALLA

(Under the Direction of AnnaMarie Conner)

ABSTRACT

This theoretical, methodological, and empirical networking study investigated the potential of inferentialism to contribute to the study of meaning and collective argumentation in mathematics. I carefully attended to my worldview and explicated my philosophical process for identifying inferentialism as my theory of choice. I then drew on Prediger et al.'s (2008b) networking strategies to theoretically, methodologically, and empirically compare and contrast inferentialism with radical constructivism and the sociocultural perspective. Inferentialism and radical constructivism were compared with respect to the meaning of mathematical concepts; inferentialism and the documenting collective activity (DCA) methodology (Rasmussen & Stephan, 2008), which is based on sociocultural theories, were compared with respect to collective argumentation. The extant data I used to empirically network the theories were from two related sources. The first source was video data from the first of three content courses, each paired with a pedagogy course, for prospective secondary mathematics teachers (PSTs) prior to student teaching. Amidst the content course, clinical interviews were performed with nine of the PSTs as part of an overarching teaching experiment (Steffe & Thompson, 2000b). The PSTs were interviewed twice—near the beginning of the semester and near the end of the semester.

These clinical interviews were my second source of extant data. As a result of my study, I clarified the identity of inferentialism, explicated an inferentialist analytic methodology, and furthered inferentialist research on collective argumentation and students' mastery of multiple mathematical concepts. I also identified the affordances and limitations of inferentialism in comparison to the other theories. Radical constructivism has a more established tradition of research to draw on, but inferentialism enabled me to analyze the social and contextual factors at play in students' mathematical reasoning. Furthermore, my analysis underlined the power of DCA's ability to move across grain sizes to make general claims about the collective learning of a classroom. Inferentialism, however, allowed me to foreground individuals' learning within collective argumentation and simultaneously attend to how ideas received normative status in the classroom. Implications for future inferentialist research in mathematics education are discussed.

INDEX WORDS: Inferentialism, Networking Theories, Collective Argumentation, Mathematical Meaning, Epistemology

MEANING AND COLLECTIVE ARGUMENTATION IN MATHEMATICS:

INVESTIGATING INFERENTIALISM'S POTENTIAL CONTRIBUTIONS

by

JAMES DRIMALLA

B.A., Trinity International University, 2015

M.S., Loyola University of Chicago, 2018

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2023

© 2023

James Drimalla

All Rights Reserved

MEANING AND COLLECTIVE ARGUMENTATION IN MATHEMATICS:

INVESTIGATING INFERENTIALISM'S POTENTIAL CONTRIBUTIONS

by

JAMES DRIMALLA

Major Professor:	AnnaMarie Conner
Committee:	Carlos Nicolas Gómez Marchant Kevin C. Moore Denise A. Spangler

Electronic Version Approved:

Ron Walcott
Vice Provost for Graduate Education and Dean of the Graduate School
The University of Georgia
August 2023

ACKNOWLEDGEMENTS

A dissertation cannot be completed without the help and support of many people and communities throughout my life. I am largely a product of the people and communities that have formed me, and I offer the next few paragraphs as gratitude to them.

I am thankful for the holistic education I received at Trinity International University through professors like Dr. Bialek, Dr. Matthias, and Dr. Worfel and friends like Matt, Jeff, Kevin, and Zach. In my time at Trinity, I not only developed as a mathematics educator but as a person. My life has much more depth because of the people I studied with there.

I am thankful for my friends and colleagues at Christian Academy of Louisville. I truly learned how to be a reflective practitioner at lunchtime with Kyle, Carson, Brian D., Brian N., Troy, and Curt and grew professionally and spiritually because of the thoughtful and open space they cultivated.

I am thankful for my professors at Loyola University of Chicago, especially Dr. Linda Brazdil. Linda forever altered my career path when she gave me the opportunity to work as a graduate assistant in Loyola's Center for Science and Math Education. The FYRE program that we coordinated made me a better educator, scholar, and person. I am also thankful for my Chicago friends, especially Jeff, Dan, Carrie, Sam, and Adam. They showed me what a life committed to social justice might look like. I am also thankful for my colleagues at Math Circles of Chicago and Latin School of Chicago. My teaching of mathematics improved exponentially in both of those spaces.

I am thankful for my peers and professors at the University of Georgia. Dr. White, Dr. Spangler, Dr. Kevin Moore, Dr. Kelly Edenfield, Dr. Amy Ellis, Dr. Cameron Byerley, Dr. Megan Wongkamalasai, and Dr. AnnaMarie Conner taught me how to think critically about mathematics education. I especially want to thank AnnaMarie Conner for her relentless support of me and doctoral students in general. There are dozens of sentences that begin with "Without Anna Conner, I would not have been able to…" I also want to thank my makeshift doctoral cohort: Jenna, Anne, Claire, Brady, and Mike. You all accepted me like a younger brother and showed me the ropes.

I am thankful for my family and close friends. So many of the opportunities I have had are a direct result of the love of my mom, my dad, my grandparents, Jean, and Matt. You all made innumerable sacrifices—with your time, your empathy, your resources, and your homes—just so I could have a chance to pursue my goals. I am also thankful for (a) friendships with people like Christian, Eric Russ, Jake, Hayden, and Ben that do not fit neatly into any of the previous categories; (b) my ongoing friendships with Ricky (Rick) Betori, Calvin Klint, Jeff Hawarny, and Matt Heber; and (c) my more recently added family members Joanie, Dennie, Bryce, Lisi, Brigette, and Anton. Loving relationships with these friends and family are central to my flourishing and enabled me to pursue the arduous goal of a Ph.D.

I am thankful for my wife Shara. She is my best friend, and my life would be like chaotic waters without her. Regardless of our circumstances, life with her is like a quiet garden. She has shown and taught me graciousness and loyal love, and her heart and intellect are unmatched. My dissertation would not have been possible without her.

Finally, as a follower of Jesus, I am grateful to the one who creatively loves, defends the dignity of all human beings, and sustains my own experience of human flourishing.

TABLE OF CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

Page

CHAPTER 1

RATIONALE AND GOALS

Bikner-Ahsbahs and colleagues, in the opening chapter of their book *Networking of Theories as a Research Practice in Math Education* (2014), described a dynamic view of theorizing. This view

> regards a theory as a tool in use rooted in some kind of philosophical background which constantly has to be developed in a suitable way in order to answer a specific question about an object. In this sense the notion of theory is embedded in the practical work of researchers. (p. 6)

Because theory is dynamic, active, and contextual, Bikner-Ahsbahs et al. (2014) suggested that the phrase theoretical approach is preferred to the word theory; it only makes sense in light of its practical use. Bikner-Ahsbahs et al. also suggested theory is “rooted in some kind of philosophical background” (p. 6) or, put another way, a *worldview* (Creswell, 2007). Because theory is necessary for any research (Spangler & Williams, 2019), all research is performed with a worldview that undergirds the theoretical and empirical work. Unfortunately, worldview issues are often avoided by mathematics education researchers (Stinson, 2020). Often, mathematics education research is like the scene at the front of a gourmet restaurant’s kitchen: dishes emerge as finished, tidy products with an aesthetically pleasing presentation. Ernest (1995) used this restaurant metaphor to describe mathematics research. Ernest described how, contrary to the façade at the front of the kitchen, “in the back, mathematicians cook up new knowledge amid mess, chaos and all the inescapably associated features of human striving” (p. 8). Ernest suggested mathematicians should accept the fallibility of mathematics and that the messy processes are just as essential to the discipline as the tidy products.

Like mathematics research, the work in the back of the mathematics education research community's kitchen is anything but clean. Yet, the field of mathematics education often avoids the dynamic nature of theory and the omnipresence of researchers' worldviews. Researchers often do not make their theoretical and philosophical processes products—the methods they use in conference rooms and personal conversations are rarely made explicit (Siy, 2019). This is a particularly harrowing problem given researchers' worldviews inform every aspect of their research and all research uses a theory that has a philosophical background. In response to this problem, Cobb (2007) called for mathematics education researchers to put philosophy to work; Siy (2019) suggested researchers explicitly document how they decide between different theories and coordinate them with their own subjectivities; and Stinson (2020) suggested "considerations of ontology, epistemology, ethics, values, subjective and ideological grounds, and so on—that is, the researcher's worldview—should precede not follow theoretical and methodological considerations" (p. 13).

In this dissertation, I carefully attend to my worldview and a new theory in mathematics education research—inferentialism—along with its philosophical background. I explicate my philosophical process for identifying inferentialism as my theory of choice and then, through theoretical and empirical analyses, I network inferentialism with other theories to investigate inferentialism's potential contributions to research on meaning and collective argumentation in mathematics education. My networking methodology and inferentialist empirical analyses led to methodological insights for robust inferentialist analysis that has hitherto been undocumented in the literature.

Inferentialism: Emergence and Areas for Research

Inferentialism's Emergence in Mathematics Education

In the last ten years, inferentialism—a theory of meaning that addresses linguistic and epistemic issues—has been adopted by several mathematics education researchers to study a variety of topics, often instead of the more dominant constructivist and sociocultural learning theories in the field. In short, inferentialism describes how the content of a concept (e.g., a mathematical concept like function or inverse) exists in the social game of giving-and-asking-for-reasons with an emphasis on the inferential relation between concepts.

Although inferentialism is new by name in the field of mathematics education, it also has ties to the social turn in mathematics education research. Inferentialism, developed predominantly by philosopher Robert Brandom, is indebted to many of Ludwig Wittgenstein's ideas; and Lerman (2001) also cited Wittgenstein's work as crucial to understanding language and meaning from a sociocultural perspective. Lerman argued Wittgenstein's work in *Philosophical Investigations* prompted the "move to a cultural, discursive psychology" (Lerman, 2001, p. 90), has been crucial in understanding the difference between cognitive constructivism and social theories of learning (Lerman, 2000), and has been key to understanding the "strong social turn" in mathematics education research (Lerman, 2006).

Bakker and Hußmann (2017), however, were clear: inferentialism is not a learning theory. It was developed for philosophical purposes. This is why Bakker and Derry (2011) noted that inferentialism may appear superfluous to a field full of theoretical options: "At first sight, inferentialism might seem to offer little more than other criticisms of representationalism found in contemporary literature (e.g., Cobb et al., 1992; Sfard, 2008)" (p. 24). Bakker and Derry (2011), however, also argued that inferentialism goes beyond prior critiques and has far-reaching implications. Inferentialism, as a theory of meaning, offers "an interesting resource with which to approach thorny old issues in mathematics education research with fresh eyes, in particular where it comes to epistemological topics such as concepts, knowledge, or reason" (Bakker & Hußmann, 2017; p. 396).

Fertile Areas for Inferentialist Research

Despite ties to previously established social theories in mathematics education research, some philosophical issues have resurfaced with the emergence of inferentialism (Radford, 2017). More specifically, inferentialist researchers have questioned constructivists' (both social and cognitive) descriptions of the intertwined social and individual nature of humanity (Noorloos et al., 2017), constructivists' unwillingness to address ontological issues regarding descriptions of reality (Noorloos et al., 2017), and the relationship between mind and world (Derry, 2017). I attempt to address each of these inferentialist-related philosophical questions in Chapters 3 and 4.

Inferentialist researchers have noted numerous other fertile areas for further research in mathematics education. First, Seidouvy and Schindler (2020), in their paper on an inferentialist account of student collaboration, argued that inferentialism provides a more productive account of how the social and individual nature of collaboration are intertwined and highlighted the possible deficiencies of other accounts of collaboration. They also acknowledged that they did not attempt to compare their inferentialist framework with theoretical and analytic frameworks from other philosophical backgrounds. They said such a comparison (between analytic frameworks) would be important because "there is reason to believe that studies that use two separate frameworks to capture the social and the individual may overlook the intertwined nature of the social and individual in collaboration" (Seidouvy & Schindler, 2020, p. 17). As previously noted, initial theoretical comparisons between inferentialism and other perspectives have been performed (e.g., Noorloos et al., 2017; Taylor et al., 2017; Radford, 2017), but further comparative work can be done at both the theoretical and empirical level.

Second, Noorloos et al. (2014) highlighted the need for further research in argumentation and reasoning and proof from an inferentialist perspective. Meyer (2018) has begun such research by attempting to integrate elements of Wittgenstein's language games and Brandom's inferentialism with Toulmin's work on the structure of arguments. More specifically, Meyer took Wittgenstein's concept of language-game and Brandom's inferentialist insights and described how "rules" (per Wittgenstein's language-games) are established via Toulmin diagrams. In order to make such connections, Meyer substituted Wittgenstein's concept of "rules" for Toulmin's "warrants," treating the two as one and the same. Thus, Meyer adopted Toulmin's focus on the structure of collective argumentation, attempted to integrate Wittgensteinian insights, and subsequently made claims about students' understanding of concepts by adopting inferentialist ideas. These results are meaningful but restricted because Meyer did not discuss his entire process for integration. A fuller account could go beyond the typical focus of the structure of arguments in collective argumentation research and more carefully network Wittgenstein, Brandom, and Toulmin-styled collective argumentation research.

A third topic in significant need of further inferentialist research is methodology and methods. Taylor et al. (2017) explicitly called for the development of inferentialist empirical methods and Radford (2017), in his capstone piece to the special inferentialist issue of the Mathematics Education Research Journal, named methodology as one of the shortcomings of the inferentialist research presented in the special issue. Although the articles in the special edition explicitly conceptualized learning, Radford believed there needed to be detailed inferentialist examples of learning "in order for us to have a clear idea of the way in which learning is specifically investigated both theoretically and empirically" (p. 502). Thus, further work on inferentialist methodology and the determination of practical methods is needed.

Fourth and finally, although numerous papers in the inferentialist literature address philosophical topics directly (Derry, 2017; Bakker & Hußmann, 2017; Noorloos et al., 2017; Taylor et al., 2017; etc.), Radford (2017) still expressed two major philosophical concerns. Radford was concerned about inferentialism's overly rationalist depiction of humans and inferentialist researchers' inability to articulate their overarching ontology. He recommended an alternative philosophical anthropology and said, "it will be up to our mathematics education inferentialist theoreticians to be explicit in their future work about the kind of ontology that they are ready to hold" (Radford, 2017, p. 504).

Need for Networking

Seidouvy and Schindler's (2020) call for comparative studies implies there needs to be explicit methods for how different theories can be compared and contrasted. The importance of an intentional methodology for comparing and contrasting can, at times, be forgotten. As noted, Meyer (2018) attempted to integrate some ideas from Wittgenstein and inferentialism with Toulmin-styled collective argumentation research, but he did not discuss his entire process for integration.

These sorts of concerns are addressed in the networking literature of mathematics education research. In their opening editorial to a special edition of ZDM entitled *Comparing, Combining, Coordinating – Networking Strategies for Connecting Theoretical Approaches*, Prediger et al. (2008a) said that the diversity of theoretical options poses significant challenges to researchers, most notably: how should researchers deal with this diversity? All the editors of the special issue, however, expressed commitment to the idea that the diversity of theoretical options is a valuable resource. The special issue's purpose, then, was to present various networking strategies for connecting theoretical approaches developed in European contexts. The editors explicitly did not aim to achieve one totalizing and unified theory but believed "there [was] a *wide spectrum of strategies for connecting theories* which respect the pluralism of autonomous theories without isolationism" (Prediger, et al., 2008a, p. 163).

The most notable American attempt at something akin to the networking attempts in the *ZDM* special edition was Paul Cobb's *Putting Philosophy to Work* (2007). Cobb discussed how "competing, incommensurable theories" can be "appropriated and adapted to our purposes as mathematics educators" (p. 4). Cobb (2007) used the metaphor of "co-existence and conflict" (p. 31) to describe how the field should continue in light of the diversity of theories and suggested two criteria for comparing and contrasting theoretical perspectives: 1) how each perspective considers the nature of the individual and 2) how each theoretical perspective contributes to Cobb's idea of design science, where he measured the "potential contributions of each perspective to the collective enterprise of formulating, testing, and revising designs for supporting learning" (p. 31).

Within the *ZDM* special edition and other networking literature, various related topics are also addressed. Bikner-Ahsbahs & Prediger (2010) described how connecting and networking a theory can further a theory's development by 1) requiring the theory's philosophical base to be made as explicit as possible, 2) increasing the stability of the theory through meaningful empirical work, and 3) establishing interconnectivity with other theories. Prediger et al. (2008b) described a methodological framework for networking and said that other works within the *ZDM* special issue provide examples of finer methods such as "cross-experimentation, dialectical consideration, three-by-two comparison, [and] creating research designs" (Prediger et al., 2008b, p. 175). Kidron et al. (2008) described one of the finer methods for comparing and contrasting three theoretical approaches by performing a theoretical comparison and then attempting to apply all three frameworks to a common data set.

Radford's (2008) concluding comments in his *Connecting theories in mathematics education: Challenges and possibilities*, provide a helpful summary of networking in mathematics education. Radford suggested that the topic and issues surrounding the practice of networking theories was interesting for the mathematics education research community for two main reasons. First, by characterizing the *identity* of different theories through networking, the field will attain a better understanding of their similarities and differences. Furthermore, Radford (2008) suggested that further research on the identity of theories through networking "would require the elaboration of new tools and concepts and the corresponding meta-language to describe them" (p. 325). He also suggested that these new tools, concepts, and language "may emerge from the study of a same set of classroom data as seen from the point of view of various theories. But they can also emerge from the investigation of theoretical problems" (Radford, 2008, pp. 325–326).

Networking methods are thus ideal for the advancement of inferentialist research in mathematics education. Per Bikner-Ahsbahs & Prediger (2010) and Radford (2008), networking research practices can (a) require inferentialism's philosophical bases to be made more explicit, (b) increase the stability of inferentialism through meaningful empirical work, (c) establish interconnectivity between inferentialism and other theories, (d) characterize and clarify inferentialism's identity in the field of mathematics education, and (e) prompt the development and elaboration of an inferentialist analytic methodology for empirical work.

Research Questions

Given the various strands discussed in this introduction (theory, worldview, the emergence of inferentialism, gaps in inferentialist research, the need for networking), three relevant research questions emerged that guided my study. Each guiding question contributes to a better understanding of inferentialist research in mathematics education. The research questions that guided the study were:

1. Using networking research practices, what is inferentialism's identity in relation to
    a. radical constructivism?
    b. Toulmin-styled collective argumentation research from the sociocultural perspective?
2. How does an inferentialist researcher robustly analyze
    a. students' mastery of meanings in clinical interviews?
    b. the game of giving and asking for reasons (GoGAR) and conceptual content within collective argumentation?
3. Given the inferentialist methods developed in response to research question 2 and the empirical analyses from research question 1, what can an inferentialist perspective add to the field's understanding of the interplay of individual and collective activity?

The first question, building on the research practice of networking theories, seeks to establish inferentialism's position amongst the plethora of available theoretical options in the field. Specific networking strategies and methods are explicated in the methodology section, but it is important to note that my attempt to network included empirical comparisons. Thus, there were four research sub-questions beneath my first research question—a pair for the comparison between inferentialism and radical constructivism and a pair for the comparison between inferentialism and sociocultural, Toulmin-styled collective argumentation research. They were:

1. Given radical constructivism's conception of meaning, how have individual students constructed mathematical meanings related to function and inverse within the clinical interviews?
2. Given inferentialism's conception of meaning, how have individual students mastered mathematical meanings related to function and inverse within the clinical interviews?
3. Given the sociocultural perspective's conception of function as-if-shared mathematical ideas and collective activity, what collective mathematical growth has occurred within episodes of collective argumentation?
4. Given inferentialism's conception of concepts and claims, what content have students mastered within episodes of collective argumentation?

My investigation of the first research question provided methodological insights for the second research question. My response to the second question addresses expressed concerns regarding inferentialist methodologies by describing the inferentialist methodology I developed so that it can be applied in future research. Finally, the methodological results explicated in response to my second research question and the inferentialist empirical results from my first research question informed the results related to the third question. My responses to these research questions are preceded by an acknowledgement of the autobiographical nature of qualitative research and an exploration of my worldview. I discuss how my worldview is compatible with inferentialism and respond to concerns in the literature about inferentialism's philosophical anthropology and inferentialist researchers' lack of specificity regarding ontology.

CHAPTER 2

METHODOLOGY AND METHODS

The methodology and methods I used to answer my research questions has two parts. First, my methodology and specific methods are predominantly informed by Prediger and colleagues' (2008b) framework for connecting and networking theories. However, I also see a limitation of their work—namely the lack of attention given to researchers' worldviews. Radford (2008) highlighted this limitation when he, in his capstone to the *ZDM* special issue, discussed the role of worldviews in research questions. He said researchers cannot determine whether or not two theories' research questions are incompatible without looking into the underlying principles and argued that "research questions are not stated in a conceptual vacuum: research questions are stated within a world-view and this world-view is defined by the explicit and implicit principles of any given theory" (Radford, 2008, p. 325)

Consequently, my methodology and methods are also informed by Stinson's (2020) call for researchers to both interrogate and follow their worldview in their research. I have titled this part "confessional" because it is an honest acknowledgement and statement of my worldview and social imaginary. In the subsequent sections, I explicate the details of these two parts of my methodology and methods: my confessional and Prediger et al.'s (2008b) framework for connecting and networking theories.

My Confessional: Worldview Interrogation and Inclusion

As researchers, interrogating and confessing the stories that shape us (social imaginaries) and our subsequent philosophical frames (worldviews) is an act of honesty done in love for our conversation partners. This act of confession is like the reflexivity called for in qualitative research (Creswell, 2007), but more philosophical. It includes awareness of the typical categories of positionality (cultural, social, gender, class, etc.) but goes beyond the generic worldview options described in qualitative research (Creswell, 2007). Cornel West (1999) echoed the importance of such philosophical confession when he wrote,

> We must acknowledge our finitude, fallenness and sinfulness as human beings. This acknowledgment entails that when we say we "know" that a particular scientific or religious description, version or theory of the self, world and God is true, we are actually identifying ourselves with a particular group of people, community of believers or tradition of social practices. There indeed may be good reasons why we identify ourselves with particular groups, communities or traditions. But there are, ultimately, no reasons with the force of logical necessity or universal obligation that could rationally compel others to join us. In this sense, there is no true description, version or theory of the self, world and God that all must and should acknowledge as inescapably true. (p. 418)

Because, per West's description, there are no inescapably true and logically certain reasons that people must admit, researchers ought to confess the community and tradition of belief and social practices with which they identify. Without such confession, misunderstanding will abound.

My confessional is not a checklist item—something completed to meet some standard of legitimate qualitative research. It is an integral part of the dissertation, for me and the reader. It exposes unfamiliar aspects of research and causes discomfort that illustrates the complexity of this type of qualitative research (Pillow, 2003).

Worldview

Stinson (2020) wrote, "In the end, considerations of ontology, epistemology, ethics, values, subjective and ideological grounds, and so on—that is, the researcher's worldview—should precede not follow theoretical and methodological considerations" (p. 13). Notice here that one's worldview encompasses a whole host of philosophical questions and concerns—it is how a researcher (or collection of researchers) views the world. I will use two specific philosophical categories, ontology and epistemology, throughout my dissertation and consequently need to define them. The task is difficult because, even in my descriptions of the categories, I will signal my positionality and worldview.

Ontology

Ontology is a philosophical category mathematics education researchers do not give sustained attention (Stinson, 2020). Broadly speaking, ontology is the discussion of "the nature of reality and its characteristics" (Creswell, 2007, p. 16). This basic philosophical category is mostly ignored in mathematics education research because of the complications surrounding the "realness" of mathematical objects and the nature of humanity. It is further complicated by different philosophical traditions' boundaries for the concept of reality and consequently also entails postmodern and poststructuralist rejections of the real (Stinson, 2020). (Oftentimes, postmodernists and poststructuralists reject previous conceptions of what is real or how it is knowable.)

Epistemology

Epistemology is the discussion of knowledge and how knowledge is known (Creswell, 2007). It is slightly downstream from ontology because different traditions' definitions of knowledge are usually dependent on their definitions of what is real. Despite its complexities, epistemology is a topic historically more attended to in mathematics education research (Stinson, 2020). Citing Foucault, Stinson (2020) argued this strong disposition to engage with epistemology seems to be a leftover remnant from the Enlightenment—the Age of Reason.

Social Imaginary

I believe an articulated worldview is shaped by social imaginaries (Taylor, 2004). Taylor (2004) said a social imaginary is "much broader and deeper than the intellectual schemes people may entertain when they think about social reality in a disengaged mode" (p. 23). A social imaginary comprises the pre-theoretical commitments humans make and the practices they participate in; every community narrates their existence and participates in practices that accord with their social imaginary. Taylor said that, while the relationship between pre-theoretical understandings and practices is codependent, "it is the practice that largely carries the understanding" (Taylor, 2004, p. 25).

Taylor used the word imaginary because he adopted some of Benedict Anderson's (2006) language from Imagined Communities, but also because of its pre-theoretical nature. Taylor (2004) wrote,

> It [the social imaginary] is in fact that largely unstructured and inarticulate understanding of our whole situation, within which particular features of our world show up for us in the sense they have. It can never be adequately expressed in the form of explicit doctrines because of its unlimited and indefinite nature. That is another reason for speaking here of an imaginary and not a theory. (p. 25)

Consequently, even the worldviews and philosophical considerations directing the production of knowledge are already embedded in an imagination; the theories researchers produce are bounded by a community's imagination. A social imaginary, then, is "not expressed in theoretical terms, but is carried in images, stories, and legends" (Taylor, 2004, p. 23).

For example, the word psychology has a normative meaning in academia which aligns with the *second* entry for the word in the Oxford English Dictionary. In short, psychology, as normatively defined in the dictionary and in contemporary academic settings, is the scientific study of the human mind and human behavior (Oxford University Press, n.d.; American Psychological Association, n.d.). This definition, however, is overdetermined by late modernist pre-theoretical commitments. The language and categories that shape contemporary understandings of psychology are bounded by an unnecessary story of the world.

Taylor's (2004) concept of social imaginary is helpful here. Late modernity can be thought of as a social imaginary: it was (and is) the (pre)disposition to pursue Western notions of progress predominantly through scientific methods. Much of mathematics education research—specifically research that draws on Piaget, Vygotsky, or Dewey—is still bound within this social imaginary. Prior to being written about explicitly, it was passed on through stories, practices, and art detailing human progress through scientific technology. Thus, contemporary psychology is described as scientific and predominantly studies the relations between the mind, behavior, and the environment through neutral means of observation, experimentation, and testing.

Contrast that with the *first* entry for psychology in the Oxford English Dictionary: "the study or consideration of the soul or spirit" (Oxford University Press, n.d.). This first entry is dependent on the Greek and Latin root word psyche, which now carries mixed meanings because there is no normative consensus on the meaning of soul and spirit. This definitional difference is important: even the explicit articulation of philosophical ideas, theories, and definitions are subject to pre-theoretical commitments like stories and social practices.

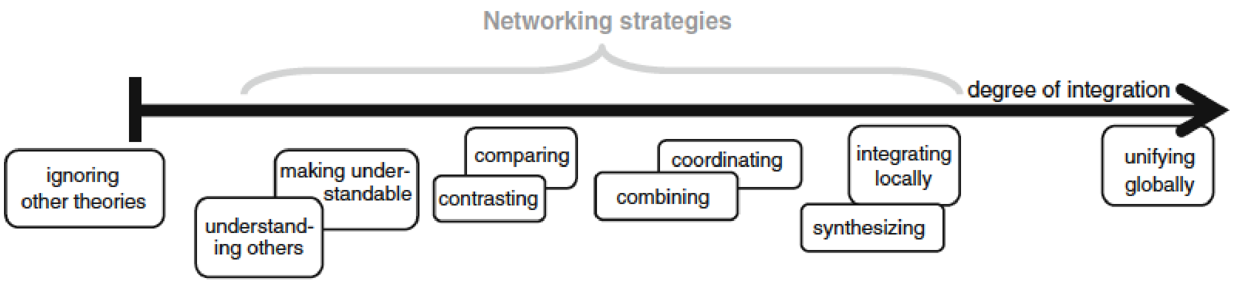
Networking

Following the confession of my worldview and social imaginary, inferentialism and the variety of theories available in mathematics education research require networking. In Prediger et al.'s (2008b) discussion of networking strategies, the authors offered a compelling framework for *connecting* theories. By connecting, the authors meant to suggest "the overall notion for all strategies that put theories into relation (including the non-relation of ignoring other theories)" (Prediger et al., 2008b, p. 170). They described how three questions guided their considerations and laid the groundwork for all 11 papers in the special *ZDM* issue on networking theoretical approaches. The three questions were: 1) "What exactly are theories, and for what do we need them?", 2) "Why is diversity of theoretical approaches a challenge and a resource?", and 3) "Why is connecting theoretical approaches an important aim?" (Prediger et al., 2008b, p. 165).

Prediger and colleagues (2008b) then carefully addressed each question by discussing 1) a dynamic view of theory, 2) the challenge of dealing with so many theories as well as the diversity of theories being "a resource for coping with the complexity of the research field" (p. 176), and 3) identifying "that there is a need for connecting theories, and we propose networking between theories as a more systematic way of interacting with theories" (p. 176). Through the networking of theories, Prediger et al. (2008b) argued the field of mathematics education can better establish an open-minded conversation regarding theories, theory development, and the quality of theories for research.

Figure 1

Landscape for Connecting Theoretical Approaches



Note. Reprinted from "Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework," by Prediger et al., 2008, *ZDM – The International Journal on Mathematics Education*, *40*(2), p.170. Copyright 2008 by Springer Nature. Reprinted with permission.

Most important to Prediger et al.'s framework for connecting theories was their spectrum of strategies for connecting (see Figure 1). The authors described how, in their framework, networking strategies are a subset of connecting strategies. So, while "ignoring other theories" and "unifying [theories] globally" are on the spectrum of connecting theories, networking strategies involve the interaction and subsequent friction between different theories. Table 1 provides a summary of each networking strategy.

Table 1

Prediger et al.'s (2008b) Networking Strategies and Descriptions

Networking Strategy	Description (Quotes From Prediger et al., 2008b)
Understanding Others and Making Understandable	"All inter-theoretical communication and especially all attempts to connect theories must start with the hard work of understanding others and reciprocally, with making the own theory understandable.… Understanding a theory means to understand their

	articulation in research practices which are full of implicit aspects" (p. 171).
Comparing and Contrasting	"Whereas comparing refers to similarities and differences in a more neutral way of perceiving theoretical components, contrasting is more focused on stressing differences" (p. 171). Can include "the role of well chosen implicit or explicit aspects in the theoretical structures (more general level)," "the articulation of the mutual theory in the practices of empirical research (more concrete level)," or "a priori defined criteria for quality of theories" (p. 172).
Combining and Coordinating	"Coordinating and combining are mostly used for a networked understanding of an empirical phenomenon or a piece of data" (p. 172). "Combining theoretical approaches does not necessitate the complementarity or even the complete coherence of the theoretical approaches in view. Even theories with conflicting basic assumptions can be combined in order to get a multi-faceted insight into the empirical phenomenon in view" (p. 173).
Synthesizing and Integrating Locally	"Whereas the strategies of combining and coordinating aim at a deeper insight into an empirical phenomenon, the strategies of synthesizing and integrating locally are focused on the development of theories by putting together a small number of theoretical approaches into a new framework" (p. 173).

Prediger et al.'s (2008b) networking strategies provide the broad contours of part of my dissertation as I network three theories (inferentialism, radical constructivism, and the sociocultural perspective). My overarching purpose for networking, however, is slightly different than Prediger et al. Prediger et al. implicitly suggested that the integration of theories is the purpose of networking. In his comments on Prediger et al.'s (2008b) approach, Radford (2008) used the word "plots" to describe the themes and purposes of networking theories. He identified Prediger et al.'s (2008b) plot as integration and suggested that the *identity* of a theory can also be a plot of networking. When identity is the main plot, the overarching goal is not the integration of theories; instead, networking is used to clarify a theory's identity. Radford (2008) wrote, "By reacting to our own theory and our claims about it, the other theories make visible some elements that may have remained in the background of our theory" (p. 319).

Although Prediger et al.'s (2008b) main plot for networking was integration, they were also explicit that total unification was not their ultimate purpose. "Unifying [theories] globally" was on their spectrum of integration, but they said any attempt to globally unify theories would likely "usurp the richness of theories by one dominant approach" (p. 170). Furthermore, Prediger et al. (2008b) did not pursue any global unification because they believed it was "doubtful whether theoretical approaches with contradictory fundamental assumptions in their core (concerning for example their general assumptions on learning) could be globally unified without abandoning the core of one theory" (p. 170). Thus, I adopt Prediger et al.'s (2008b) networking strategies as an approach but adopt Radford's emphasis on the plot of identity. Consequently, I do not attempt to synthesize and locally integrate the inferentialism with the other theories but, instead, explicate the identity of inferentialism in relation to the other theories.

By centering the identity plot in my networking of inferentialism, I create opportunities for "extraction" (Radford, 2008, p. 319). Radford explained that, when theories are put into dialogue with one another, researchers can extract or "pull out things from the brackets of common sense (the brackets of things that we take for granted in our theory to the extent that we no longer even notice them)" (p. 319). Through this dialogue and extraction process, the ideas that were previously implicit can be newly scrutinized (Radford, 2008). Thus, Radford's work helps connect the dots between my two research questions. By putting inferentialism in dialogue with two other dominant theories, the implicit "brackets of common sense" aspects of inferentialism will be extracted. In the process of responding to my first research question, the stage will be set to respond to my second research question. In a subsequent section, I further detail why methodological insights, as compared to other insights, can be learned through the networking process. In the sections that follow, I describe how I network the three theories with Prediger et al.'s (2008b) networking strategies.

Understanding Others and Making Understandable

The specific theories I network are inferentialism, radical constructivism, and the sociocultural perspective. In Chapter 4, titled "Understanding Others and Making Understandable," I present each theory in a holistic way. I discuss the theories' philosophical backgrounds, their theoretical development in mathematics education, their methodologies for empirical research, and their results in relevant empirical studies. Although I provide an in-depth analysis of each of the perspectives in a subsequent section, an initial justification of my decision to include these three theories is necessary. As previously noted, inferentialism is an emerging theory in mathematics education research that can be "an interesting resource with which to approach thorny old issues in mathematics education research with fresh eyes, in particular where it comes to epistemological topics such as concepts, knowledge, or reason" (Bakker & Hußmann, 2017; p. 396). I also previously noted that numerous topics are ripe for inferentialist research.

Given inferentialism's focus on epistemology and issues of concepts, knowledge, and reasoning (Bakker & Hußmann, 2017), I believe it is best to situate inferentialism with respect to two of the dominant theories in the field that address such topics: radical constructivism and the sociocultural perspective. Noorloos et al. (2017) performed an initial theoretical comparison between inferentialism and social constructivism and Taylor et al. (2017) offered an inferentialist metaphor for learning in contrast to the participation and acquisition metaphors used in constructivism and the sociocultural perspective. A more in-depth attempt, however, to network inferentialism, radical constructivism, and the sociocultural perspective is needed. My in-depth attempt follows Prediger et al.'s (2008b) networking spectrum by 1) initially making all three theories and their philosophical underpinnings understandable, 2) comparing and contrasting the theories theoretically and methodologically, and 3) attempting to coordinate the theories empirically given a piece of extant data.

Comparing and Contrasting

After the holistic overview of each theory in Chapter 4, I compare and contrast inferentialism with radical constructivism in Chapter 5 and with the sociocultural perspective in Chapter 6. Both Chapter 5 and Chapter 6 begin by comparing and contrasting theoretical components as instructed by Prediger et al.'s (2008b) "comparing and contrasting" networking strategy. I first highlight the similarities between inferentialism and the other perspectives given the material discussed in the "Understanding Others and Making Understandable" section and then contrast inferentialism with each perspective in order to identify the main differences.

As I theoretically compare and contrast the theories, I focus on the theories' philosophical anthropology (i.e., how they respond to the question "what does it mean to be human?"), epistemology, and the research produced. My focus on philosophical anthropology relates to one of Cobb's stated criteria for investigating different theories in mathematics education research. He said that "the notion of the individual is conceptually relative" (Cobb, 2007, p. 12) and that different perspectives characterize humans in different ways. How a theory conceives of humans is important because it will largely determine epistemology, the topics researched, and the research questions asked. (Cobb's choice of the words "the individual" may already indicate specific ideas about what it means to be human.)

Combining and Coordinating

After making the theories understandable and comparing them, I combine and coordinate the theories empirically. In Chapter 5, I combine and coordinate inferentialism with radical constructivism and in Chapter 6 I combine and coordinate inferentialism with the sociocultural perspective. To empirically combine and coordinate different perspectives, more specific methods (in addition to Prediger et al.'s (2008b) networking strategies) are necessary. Prediger et al. (2008b) wrote, "Similarly, the more general networking strategies require specific methods to be developed for their concrete application. This section illustrates different methods that the authors of this issue have elaborated for developing or applying a specific networking strategy" (p. 173). Prediger et al. described how several different methods within these networking strategies differ due to varying foci of attention, centering different concepts, using different techniques, and having different aims.

Before I describe my specific methods to empirically combine and coordinate the different perspectives, I want to acknowledge an inherent limitation in my networking methodology. I—an individual with a worldview and specific theoretical inclinations—describe, compare, contrast, combine, and coordinate theories that I do not fully ascribe to. Some readers may fear that my interpretation and use of the theories are thus disingenuous. Although I did not attempt to evade my limitations—there is no objective perspective of a theory—I took substantial lengths to ensure the accuracy of my portrayal of each theory. In my process, I (a) described each theory, (b) thoroughly analyzed data from each perspective, and (c) discussed my theoretical descriptions and analyses with a more experienced researcher from that perspective.

Empirical Methods

In addition to adopting Prediger et al.'s (2008b) approach to networking, I adopt Kidron et al.'s (2008) more precise methods to empirically combine and coordinate. Kidron et al. (2008) aimed to network three theoretical approaches through both theoretical means and empirical means. The three theoretical approaches the authors attempted to network were "the theory of didactic situations (TDS), the nested epistemic actions model for abstraction in context (RBC+C), and the theoretical approach of interest-dense situations (IDS)" (p. 247). To compare the theoretical approaches empirically, the authors used data (more specifically, a video recorded lesson that was part of a long-term project on Italian high school students) from another team's research project. None of the authors were part of the team that generated the data originally (i.e., they used extant data), yet all the authors had been in conversation with each other and other colleagues about comparing, combining, and contrasting the different theoretical approaches. So, while the authors did not participate in the data's generation, they all had knowledge of the other theoretical approaches (Kidron et al., 2008).

When comparing the theoretical approaches via empirical means, Kidron et al. (2008) suggested that subsequent attempts to network theories should "find out what ideas each pair of theoretical approaches share" (p. 262), identify "a common, but not precisely defined aspect" (p. 262), and subsequently compare each pair according to the common aspect. Table 2 details how I follow Kidron et al.'s (2008) suggestions.

Table 2

Pairs of Theories to Network and Common Aspects for Comparison

Pair of Theories	Common Aspect for Comparison
Inferentialism and Radical Constructivism	Meaning of Mathematical Concepts
Inferentialism and Sociocultural Perspective	Collective Argumentation

Kidron et al. (2008) also noted how their methods were helpful for gaining methodological insights. They described how, after each of them attempted to make sense of the data from their perspective, a common response was that the video data was insufficient to appropriately analyze from their theory and attending frameworks. The most interesting thing, from Kidron et al.'s (2008) perspective, was that the different theoretical lenses were missing different data. Researchers from different perspectives needed different data. Thus, Kidron et al. (2008) suggested that by "investigating for each framework the information that the researchers claim that they miss in the data offered by the video, we learn about the existing positions of the frameworks on the subject of social interactions in learning processes" (p. 256).

Kidron et al.'s method revealed methodological insights regarding necessary data and appropriate analyses. The tensions produced by analyzing extant data clarified what analysis looked like for the different theories. I gained similar methodological insights via Kidron et al.'s methods by using inferentialism to analyze data that were collected for a different purpose and by networking inferentialist empirical analyses with analyses from other theories. In fact, Bikner-Ahsbahs and Prediger (2014) suggested "the methodological reflection of possibilities, benefits, and limits constantly accompanies the dialogue between theoretical approaches. In this sense, networking practices also aim at increased methodological awareness" (p. 244).

Extant Data

The use of extant data in networking is part of Kidron et al.'s (2008) approach and is commended by Radford (2008) when he wrote, "They [themes within the networking of theories] may emerge from the study of a same set of classroom data as seen from the point of view of various theories" (p. 326). Bikner-Ahsbahs and Prediger (2014) described data collected from a different theoretical perspective as "alien data" (p. 236) and, like Kidron et al. (2008), suggested that the use of alien data has the potential to generate significant methodological insights. My use of extant data is also an attempt to follow St. Pierre and Jackson's (2014) advice to "do more with less data" by "focusing on the difficult work of [theoretical] analysis rather than on conducting more and more interviews" (p. 715).

The extant data I use are from two highly related sources. The first source is video data from the first of three content courses, each paired with a pedagogy course, for prospective secondary mathematics teachers (PSTs) prior to student teaching at a large southeastern university in the United States. The course met twice weekly for a 15-week semester and focused on quantitative and covariational reasoning through the study of functions and inverses. The data, which were part of a larger study, are video recordings of small group and whole class discussions for all class sessions of this first content course. The overarching research question that informed the project in which this data were collected was: "How can beginning teachers learn to support productive collective argumentation in secondary mathematics classrooms?" A research sub-question that prominently informed collection of this video data was: "What opportunities do the participants have to learn about collective argumentation during their mathematics education coursework?" These research questions and the study's purpose meant that the video recordings focused on the PSTs' learning experiences in the classroom and how the instructor supported, modeled, or explicitly discussed collective argumentation in the classroom. Thus, there was a camera recording the instructor throughout each class session, a camera recording whole group discussions, and multiple cameras recording most of the small group discussions. The instructor wore a clip-on microphone and each small group of students being recorded had a microphone recording their conversations.

Amidst the content course, the mathematics teacher educator and a research assistant performed clinical interviews with nine of the PSTs as part of an overarching teaching experiment (for a description of teaching experiments, see Steffe & Thompson, 2000b). Each PST was interviewed twice—near the beginning of the semester and near the end of the semester—and each interview lasted between 60 and 120 minutes. These clinical interviews are my second source of extant data. The clinical interviews were conducted from a radical constructivist perspective and focused on each PST's ability to reason about covarying quantities as well as their meanings for function, inverse, and graphs. More specifically, the goal of the interviews was to characterize the nine PSTs' quantitative reasoning and their meanings for inverse and function so that the researchers could better promote fundamental shifts in all the PSTs' thinking within the content course classroom.

Summary of Methodology and Methods

To summarize, in this section, I described my confessional and networking methodology and methods. In the subsequent chapters, I (a) provide a full account of my worldview and social imaginary in Chapter 3 and (b) holistically characterize inferentialism, radical constructivism, and the sociocultural perspective in Chapter 4. I then further network inferentialism with radical constructivism and the sociocultural perspective to better characterize inferentialism's identity. I compare and contrast inferentialism and radical constructivism and combine and coordinate the perspectives empirically in Chapter 5. I then follow the same process with inferentialism and the sociocultural perspective in Chapter 6. By empirically comparing the perspectives, I develop inferentialist analytical methods that have not been made explicit in the inferentialist literature by decontextualizing the methods I used to analyze the data. Given the newly developed inferentialist methods and empirical analyses of the extant data, I provide unique characterizations of the interplay of individual and collective activity from an inferentialist perspective in Chapter 7. Table 3 provides a summary of the research questions, data, and methods of my dissertation.

Table 3

Summary of Research Questions, Data Sources, and Analytic Methods

Research Question	Data Source(s)	Analytic Method(s)
1.a. Using networking research practices, what is inferentialism's identity in relation to radical constructivism?	Theoretical, methodological, and empirical literature on inferentialism and radical constructivism Beyond the broader radical constructivist literature, I focus on Moore and colleagues' work on modeling students' meanings (e.g., Moore et al., 2016; 2019a; 2019b)	Prediger et al.'s (2008b) networking strategies
1.a.i. Given radical constructivism's conception of meaning, how have individual students constructed mathematical meanings related to function and inverse within the clinical interviews?	Extant clinical interview data	Building second order models of students' understandings and meanings (Steffe & Thompson, 2000b; Thompson, 2008; Thompson et al., 2014)

1.a.ii. Given inferentialism's conception of meaning, how have individual students demonstrated mastery of mathematical meanings related to function and inverse within the clinical interviews?	Extant clinical interview data	I draw on Bakker et al. (2017) and Schindler et al.'s (2017) methodological emphasis on explicit reasons and inferences. I also draw on inferentialism's concept of the GoGAR, inference, and social objectivity.
1.b. Using networking research practices, what is inferentialism's identity in relation to Toulmin-styled collective argumentation research from the sociocultural perspective?	Theoretical, methodological, and empirical literature on inferentialism and the sociocultural perspective Beyond the broader sociocultural literature, I focus on Rasmussen and colleagues' work on documenting collective activity (e.g., Rasmussen & Stephan, 2008; Cole et al., 2012)	Prediger et al. (2008b) networking strategies
1.b.i. Given the sociocultural perspective's conception of functioning as-if-shared mathematical ideas and collective activity, what collective mathematical growth has occurred within episodes of collective argumentation?	Extant classroom data	Documenting collective activity with Toulmin's model of argumentation and Rasmussen's analytic methods (Rasmussen & Stephan, 2008; Cole et al., 2012)
1.b.ii. Given inferentialism's conception of concepts and claims, what content have students mastered within episodes of collective argumentation?	Extant classroom data	I draw on Seidouvy et al. (2019) and Seidouvy and Schindler's (2020) attempts to make sense of collaboration in the classroom. I also draw on inferentialism's concept of the GoGAR, inference, and social objectivity.

2.a. How does an inferentialist researcher robustly analyze students' mastery of meanings in clinical interviews?	My process from 1.a.ii	I draw on Bakker et al. (2017) and Schindler et al.'s (2017) methodological emphasis on explicit reasons and inferences. I also follow Kidron et al. (2008), Bikner-Ahsbahs and Prediger (2014), and Radford's (2008) suggestions about how to draw methodological conclusions via networking.
2.b. How does an inferentialist researcher robustly analyze the game of giving and asking for reasons (GoGAR) and conceptual content within collective argumentation?	My process from 1.b.ii	I draw on Seidouvy et al. (2019) and Seidouvy and Schindler's (2020) attempts to make sense of collaboration in the classroom. I also follow Kidron et al. (2008), Bikner-Ahsbahs and Prediger (2014), and Radford's (2008) suggestions about how to draw methodological conclusions via networking.
3. Given the inferentialist methods developed in response to research question 2 and the empirical analyses from research question 1, what can an inferentialist perspective add to the field's understanding of the interplay of individual and collective activity?	Extant classroom and clinical interview data; results from research question 1	Methods developed in response to question 2

CHAPTER 3

MY POSITIONALITY, SOCIAL IMAGINARY, AND WORLDVIEW

I begin this chapter with some comments on my positionality because my social imaginary and worldview need to be contextualized by an initial statement about myself. I then describe my social imaginary and subsequently describe the relevant aspects of my worldview. The description of my social imaginary is brief because it is rooted in the broad Christian tradition and my dissertation is not a theological attempt to delineate the particulars of different Christian traditions and their doctrines. Additionally, a social imaginary, given its pre-theoretical nature, is difficult to make explicit without long-form stories and artistic images.

Positionality

For the last ten years, I have wanted to formally study Christian theology or philosophy. I never had a specific career in mind as an end-goal (e.g., theologian, monk, etc.), but I longed for time to ruminate on theological and philosophical texts. My curriculum vitae, however, only shows degrees, achievements, and jobs in mathematics education because, at every major turn, I made decisions to formally pursue mathematics and mathematics education instead of theology or philosophy. So, alongside the enjoyment I have experienced working as a mathematics teacher, mathematics teacher educator, and educational researcher, I also have enjoyed reading theology and philosophy and coordinating my learning between disciplines.

I am incredibly grateful for my committee members and various faculty members at the University of Georgia (UGA) because my desire to wrestle with philosophical questions resonated with multiple scholars in the mathematics education program. Within my first two years at UGA, I received provocative pushback from radical constructivist professors—both current and retired—and had the opportunity to take two independent, philosophical seminars with my advisor and a member of the philosophy department. For me, the whole process was a natural extension of my identity as a follower of Jesus and my pursuit of beauty, truth, and justice.

Before further exploring the implications of my identity as a follower of Jesus, it is important to discuss my identity as a White man from a family with economic security in the United States and the privileges that follow in American society and academia. Scholars of Color, international scholars, scholars from less economically secure backgrounds, and scholars of different gender are generally not afforded the same built-in systemic advantages. I have not faced the exclusion experienced by many scholars (Gomez Marchant & Jones, 2021), I have been freer to take career risks because of familial economic safety nets, and I have been mostly affirmed in academia. I have not had to assimilate and have experienced limited friction when I pushed back against some of the norms of academia.

It is important to discuss my identity as a follower of Jesus within the context of these other identities because some may presume that my identity as a follower of Jesus is part and parcel of the dominant, white supremacist culture in Western society. Yet I believe the Christianity predominantly on display in Western society (and in the United States in particular) is not fueled by the life and person of the first century Jewish man who lived under an oppressive empire in an occupied land. The American misappropriation of Jesus is due to a White supremacist (Douglass, n.d.), capital-loving (West, 1999), Enlightenment-fueled civil religion (Gorman, 2011) and social imaginary (Smith, 2013). Gorman (2011) described how this civil religion and social imaginary treats national flags as sacred, blends Christian and American artistic imagery, preserves national holidays like sacred holy days, uses sacred language to describe war, uses biblical language to describe America (e.g., "city on a hill," "light of the world" [p. 52]), sings about God's direct relationship with America (e.g., "God Bless America"), reveres American texts as sacred (e.g., Declaration of Independence, the Constitution, the Bill of Rights, "biblical texts that seem to underwrite national values such as freedom and redemptive violence" [p. 53]), and conflates the liturgy of the church with civil rituals (e.g., pledge of allegiance in church, patriotic music in worship, sermons on patriotic themes).

Frederick Douglass (n.d.) described the disconnect between Jesus and the religion in America that appropriated his name when he wrote:

> What I have said respecting and against religion, I mean strictly to apply to the slaveholding religion of this land, and with no possible reference to Christianity proper; for, between the Christianity of this land, and the Christianity of Christ, I recognize the widest possible difference—so wide, that to receive the one as good, pure, and holy, is of necessity to reject the other as bad, corrupt, and wicked. To be the friend of the one, is of necessity to be the enemy of the other. I love the pure, peaceable, and impartial Christianity of Christ: I therefore hate the corrupt, slaveholding, women-whipping, cradle-plundering, partial and hypocritical Christianity of this land. Indeed, I can see no reason, but the most deceitful one, for calling the religion of this land Christianity. I look upon it as the climax of all misnomers, the boldest of all frauds, and the grossest of all libels. (pp. 184–185)

Moving ahead in history, although slavery has been abolished in America, the remnant of the slaveholders' religion has not. Thus, I do not dismiss the damage done by some who confess

Christianity but believe it is important to distinguish between “the Christianity of this land” and “the Christianity of Christ.”

My Social Imaginary

My social imaginary begins with Jesus of Nazareth. I believe, quoting the late Pope Benedict XVI, that “Christianity is not an intellectual system, a collection of dogmas, or moralism. Christianity is instead an encounter, a love story” (Benedict XVI, 2004, p. 685). Thus, the stories and images in which I intentionally immerse myself are those of Jesus and his Jewish social imaginary. These stories, central to the Christian tradition, are about the divine’s love of humanity and what it means to be a human reflecting the divine. They are primarily found in the Christian Bible, which is akin to a library—filled with poetry, songs, speeches, letters, and narratives—that emerged out of the history of ancient Israel. Collectively, the various writings speak of the hope for a Messiah: a person that would renew all things, rescue humanity from its own chaos and violence, and liberate humanity from all oppression. Christians believe Jesus is that Messiah and God incarnate.

Cornel West (1999) summarized the centrality of Jesus and the pre-theoretical nature of the Christian confession:

> At this point, it is appropriate for me to cast off my dispassionate philosophical disposition and openly acknowledge my own membership in the Christian community…Jesus Christ is the Truth or Reality and can only be existentially appropriated by fallen human beings caught in their finite descriptions. (p. 419)

Within the Christian social imaginary, the purpose of humanity is, in West’s words, to existentially appropriate Jesus. The purpose of humanity is not demonstratable nor does it carry reasons "with the force of logical necessity or universal obligation that could rationally compel others to join us" (West, 1999, p. 418). It is confessional.

My Worldview

From this social imaginary, my responses to different philosophical questions comprise my worldview. In this section, I discuss my anthropology, mathematical ontology, and epistemology.

Philosophical Anthropology

I begin with a discussion of my philosophical anthropology, responding to the question: What does it mean to be human? My account is dependent upon the belief that all humans reflect, or image, the divine. Although I do not go into great theological detail, the two subsequent subsections on humans are grounded in Christian doctrine on God and humanity. It is important to note that these doctrines are not arbitrary religious beliefs. In the words of Henri Nouwen (1975), Christian "doctrines are not alien formulations which [Christians] must adhere to but the documentation of the most profound human experiences which, transcending time and place, are handed over from generation to generation as a light in our darkness" (p. 89).

Humans Reflect the Divine: Communal and Individual

Several aspects of my philosophical anthropology could be noted (e.g., the necessarily embodied nature of humans, humans' predisposition to create culture, the political nature of humans, etc.), but my commitment to humans as reflections of a triune God—and thus communal and individual beings created for love—is highly relevant to my dissertation. A brief theological foray on the Christian doctrine of the Trinity is thus necessary.

Christians believe God is a being that exists as three persons: Father, Jesus, and Holy Spirit. Fleming Rutledge (2015) wrote,

> "The whole point of the original doctrine of the Trinity was that God (God's *ousia*) simply does not exist except as three persons. Vice versa, the divine persons are not other than the divine *ousia*, they *are* the *ousia*"…God's inner being or essence (*ousia*) is interrelational, intradynamic, interpersonal. In other words, "God is love" (I John 4:16). (pp. 12–13, italics in original)

Quoting Catherine LaCugna, Rutledge described how God is one being yet exists in three persons (Father, Jesus, and Holy Spirit). Furthermore, there is and always has been a loving relationship between these three persons. To boil down the technical language to a single phrase: God is love. As Michael Reeves (2012) said "God is love because God is a Trinity" (p. 9). God is a relational and lifegiving being, a community of love that humanity reflects.

The idea that humans reflect the triune God of love has implications: "made in the image of this God, [humans] are created to delight in harmonious relationship, to love God, to love each other" (Reeves, 2012, pp. 64-65). Humans are necessarily communal and individual; they always exist within community, in relationships, and as unique persons. This necessarily blurs categories and echoes the idea that the Trinity is mysterious. Humanity's reflection of the interrelational, intradynamic, and interpersonal Trinity leads into my subsequent: humans' purpose, or *telos*, is rightly ordered love.

Humans as Lovers

I believe humans at their core, because they reflect a God who is love, are lovers. The single English word love, however, is somewhat insufficient to capture the various themes of love in Christianity (e.g., love as an intimate relationship between God and humans, love as an intimate relationship between humans, love as the virtue of self-sacrifice and kindness, love as desire, etc.). I will focus on the themes of love as desire and love as virtue by following

Augustine—a 4th century North African bishop—and his 20th century interpreter, Hannah Arendt. Augustine described how, “love is indeed nothing else than to crave something for its own sake” and “a kind of motion…toward something” (as cited in Arendt et al., p. 9). In his definition, Augustine highlighted two important aspects of love: it is both a desire (a craving) and movement toward an object of love. One aspect—desire—is passive while the other—movement—is active.

Augustine (2009) described how love operated at the communal level as well as the individual level. Adjusting a definition from Cicero, he described how a community can be characterized by what it loves:

> …for example, if one should say, ‘A people is the association of a multitude of rational beings united by a common agreement on the objects of their love,’ then it follows that to observe the character of a particular people we must examine the objects of its love. (§19.24)

From the Augustinian perspective, communities are first and foremost a collection of lovers defined by the objects of their love. The same principle applies to individuals: humans can be generally characterized by the objects of their love. Arendt and Augustine also highlighted how the connection between a lover and the object of their love transforms the lover. Thus, Augustine wrote, “Such is each as is his love.”

This transformative effect of humans’ loves can be either positive or negative because loves are either *caritas* (rightly ordered) or *cupiditas* (disordered) (Arendt, 1996). Disordered loves lead to destruction and dehumanization whereas rightly ordered loves give a “foretaste of the pinnacle of our existence” (Benedict XVI, 2006, para. 8). In the Christian tradition, rightly ordered love is summed up by Jesus’ teaching to love God and to love one’s neighbor.

Francis Su (2021) provided a helpful account of rightly ordered loves and disordered loves in the context of mathematics. Su recalled how he had "been groomed by society to see math as a way of drawing a circle and putting myself in it, believing that math was a showcase for flaunting talent rather than a playground for building virtue" (p. 204). Su had been trained to do mathematics out of a desire for prestige, personal significance, and to showcase himself as better than others and, as a result, "tasted only [mathematics'] bitter remains...I had lost my joy" (p. 203). In contrast to doing mathematics out of a love for external goods, Su described how mathematics should be done out of a love for beauty, justice, truth, community, play, exploration, and freedom. In short, humans should do mathematics out of a desire for human flourishing, which will provide a foretaste of the pinnacle of human existence.

Mathematical Ontology

The work of Satyan Devadoss—a mathematician and artist at the University of San Diego—and Francis Su is paradigmatic of my mathematical ontology as inspired by a Christian social imaginary and worldview. Devadoss has described how the Enlightenment's push for reasoning and critical thought caused an unnecessary divergence between ideas and anything physical (Harvey Mudd College, 2016). Bertrand Russell demonstrated the epitome of this dichotomy in mathematics when he said,

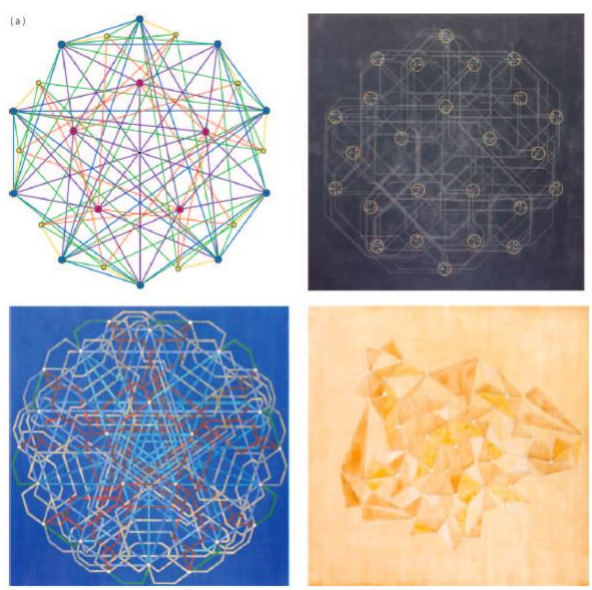
> Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the actual world. (Russell, 1919, para. 5)

Russell delighted in mathematics' ability to escape any sort of embodied or emotional state. He praised the ordered cosmos of pure thought.

In contrast to Russell, John Littlewood said, "A heavy warning used to be given that pictures are not rigorous. This has never had its bluff called and has permanently frightened its victims" (Littlewood, 1986, p. 54). Devadoss has used Littlewood's promotion of mathematical pictures as a launching pad to artistically explore mathematics through the physical. Devadoss has created artistic mathematical works and proofs (e.g., Devadoss & Hoffoss, 2019; Devadoss & Schuh, 2019, see Figure 2) and has often discussed various artists' use of mathematics in their paintings (e.g., Piero Della Fancesca, Marcel Duchamp, Salvador Dali, Julie Mehretu). Devadoss has also described how physical and artistic manifestations of mathematics occur in nature. He described how several natural phenomena (e.g., bees' construction of honeycomb, genome sequences, the unfolding of leaves, protein folding) *are* mathematical and are also ripe for humans' mathematical and artistic attention (Harvey Mudd College, 2016).[1]

Figure 2

Three Illustrator Renderings of T(5), the Space of Tree for Five Species



[1] Although Gutiérrez's (2017) work approaches the mathematics of other-than-human persons from a different worldview, there are touchpoints between her mathematx and Devadoss's vision.

Note. Reprinted from "Cartography of Tree Spaces," by Devadoss & Schuh, 2019, *Leonardo 52*(3), p.282. Copyright 2019 by The MIT Press. Reprinted with permission.

Devadoss has also drawn on Pope John Paul II's *Theology of the Body* (1997) to promote an embodied mathematics (Fuller Theological Seminary, 2018). In *Theology of the Body*, Pope John Paul II described how, in the Christian tradition, there is no line between the physical and the spiritual. The physical presents the spiritual in the physical domain and the physical is not considered less than the spiritual. Contrary to classical Greek conceptions of the soul, the Hebrew word often translated as soul (nephesh) denotes the entire human and the totality of their being: body, desires, emotions, mind, and spirit. From the Jewish and Christian perspective, a human does not merely have a body; a human is an embodied being.

These sorts of claims, about the soul and the relation between the spiritual and the physical, have been at the center of philosophical conversations about mathematics for thousands of years. Radford (2004) described how a perspective's mathematical ontology often needs to be considered in terms of the perspective's philosophical anthropology. For example, Radford connects Aristotle's beliefs about the soul and its immaterial nature with Aristotle's emphasis on abstraction and the immaterial within Aristotle's mathematical ontology. Thus, a renewed emphasis on the physical aspects of mathematics neatly connects to other Christian anthropological convictions.

Complementing Devadoss's work, Francis Su (2021) has neatly summarized humans' mathematics as "the science of patterns and the art of engaging the meaning of those patterns" (p. 44). As a science, mathematics is precise and investigative. As an art, mathematics is reflective and fueled by aesthetics. A similar (albeit implicit) mathematical ontology is evident in Lockhart's (2009) *A Mathematician's Lament*. Lockhart described mathematics as both creative expression of the self and delightful encounters of something other than the self. Lockhart (2009) communicated his conception of mathematics through words and phrases like: “invention,” “discovery,” “creative expression,” “aesthetic choices,” “desire,” “conjecture,” and “to be awed.” Along these lines, I believe mathematics exists in linguistic, physical, symbolic, and mental forms in humans’ creative acts and in things outside of human construction.

Finally, perhaps most important to my mathematical ontology is its telos, or purpose. As previously mentioned, Su has argued mathematics is for human flourishing—a wholeness of life in mind, body, spirit, and character. Su (2021) suggested “the proper practice of mathematics cultivates virtues that help people flourish” (p. 10) and is tied to basic human desires. Human beings desire meaning, play, beauty, permanence, truth, struggle, power, justice, freedom, community, and love; Su has described how specific virtues are developed when the practice of mathematics is grounded in these basic human desires. This vision of mathematics neatly coincides with my claim that humans are lovers who are defined by their longings. When humans practice mathematics with rightly ordered loves, they pursue things like justice, beauty, freedom, and love and develop virtues along the way.

In summary, my view does not align with Platonists who believe that mathematical objects exist independently of the physical world in an ideal dimension (Ernest et al., 2016) nor does it align with constructivists (cognitive or social) who believe that mathematics is solely a product of human constructions. I acknowledge that common human mathematical practices like counting, locating, measuring, designing, playing, and explaining occur across cultures and throughout history (Bishop 1991) but also acknowledge, with Gutiérrez (2017), the importance of non-human mathematics. I embrace Devadoss’s promotion of the embodied nature of mathematics, his account of non-human mathematics, and the connections he makes between a

Christian philosophical anthropology and a mathematical ontology. I also embrace Su's description of the purpose of mathematics (human flourishing) as well as his succinct summation of mathematics.

Epistemology of Love

My epistemology begins with love because humans are lovers. Esther Meek (2011) said that "love is prior to knowing" (p. 429) and, like Augustine, described both a passive and active aspect of love in relation to knowledge. The passive aspect of an epistemology of love is delight. In the Christian tradition, reality was created out of divine delight and subsequently any knowledge must start with delight. Quoting Parker Palmer's *To Know as We Are Known*, Meek described the active aspect of an epistemology of love: "…the act of knowing *is* an act of love, the act of entering and embracing the reality of the other, or allowing the other to enter and embrace our own" (as cited in Meek, 2011, p. 428). Thus, an epistemology of love acknowledges affective delight—being enamored with something—and active exploration.

NT Wright, a British theologian, helpfully described further characteristics of an epistemology of love. First, Wright (2019)—following Francis Bacon, Nietzsche, and Foucault—argued many claims to knowledge can be "unmasked as claims to power" (p. 209). My Christian epistemology of love acknowledges that knowledge and claims to power cannot be disentangled. Any claim to absolute knowledge, passivity, or some sort of detached observation (e.g., "simply following the data") is either naïveté or a power play. In contrast, knowledge informed by rightly ordered love is knowledge that seeks to serve and not to be served.

Second, related to humanity's communal nature, Wright (2019) argued that all knowing takes place in community. No human "thinks for themselves" as the modern proverb suggests; all knowing involves participation in a community of knowers and is relational. This is directly related to the belief that humans reflect the divine. If humans, in light of the Trinity, are interrelational, intradynamic, and interpersonal, then any epistemological account must describe knowledge in interrelational, intradynamic, and interpersonal terms. Knowledge must be seen as dynamic cooperation within the context of a community and within the context of human-to-human relationships.

Finally, Wright argued knowing involves all aspects of being human; it is a "whole-person activity" (Wright, 2019, p. 206). This is related to my mathematical ontology and is echoed by James K.A. Smith (2004), who described how knowledge, in the Christian tradition, is never reduced to cognitive activity. Contrary to the more narrow, dominant, Western epistemologies that privilege cognitive ways of knowing and denigrate the physical, my Christian epistemology of love embraces numerous modes of knowing such as the "affective, tactile, [or] sensible" (Smith, 2004, p. 224). Smith (2004) related these dominant epistemologies back to Descartes. Descartes is most known for his pursuit of certainty and his statement "*Cogito, ergo sum.*" (Translated: "I think, therefore I am.") Descartes' anthropological statement privileges the mind; it is precisely Descartes' ability to think that grounds his existence. Smith (2004) pointed out how Descartes' work set a trajectory for subsequent philosophy that ultimately reduced knowledge to cognitive activity.

Mathematics Education and Religion

Given my confessional claims related to Christianity, I would like to briefly address the concept of religion. The word religion can carry various connotations and is rarely addressed in mathematics education literature. One exception is Leatham, Ulrich, and Steffe's comments in Steffe's (2015) article entitled *Can a Radical Constructivist Be Religious? - Yes!* In the article, Steffe (2015) responded to Andreas Quale's (2014) assertion that a radical constructivist cannot be religious. Steffe quoted from personal conversations with Keith Leatham, a radical constructivist and adherent to the Mormon faith, and Katy Ulrich, another religious radical constructivist. Leatham, discussing the relation between religious belief and radical constructivism, provided an explanation that seems to separate spiritual knowledge from that which "has to do with our mortal existence" (p. 134). Additionally, Ulrich seemed to put radical constructivism prior to her religious commitments given her statement, "I can construct a first-order model intersubjective with the model of a religion and then join the religion" (p. 134).

In short, I believe Leatham, Ulrich, and Steffe's comments in the article suggest a modernist view of religion epitomized by Steffe's final comment: "Isn't the basic motivation that drives anyone, including a radical constructivist, to be religious is for the self-conscious "I" to survive beyond the survival of the body that the "I" inhabits?" (Steffe, 2015, p. 134). While there is nothing wrong with adopting such a notion of religion, my confessional approach is different. Smith (2004) described how, in the Christian tradition, life is religion: "worship or doxology is not confined to a religious compartment of human existence but rather spills over into every sphere of human activity" (p. 170). Consequently, while Christianity is often viewed as a religion through a modernist lens, my confessional claims about Jesus are all-encompassing. My Christian worldview informs every aspect of my life, not just my beliefs about potential existence after death. My theoretical choices in mathematics education "begin from a Christian worldview [in order] to reflect the fullness of Christian reflection" (Smith, 2004, p. 177).

Inferentialism as my "Fecund Discursive Space"

The ideas I discussed in the previous sections lead to my entertainment of inferentialism as a helpful philosophical backdrop and theoretical tool for mathematics education research. Inferentialism (as a specific branch of philosophical pragmatism) offers me a space to operate within as a mathematics education researcher. Cornel West has used a similar approach in his philosophical work. Introduced to pragmatism in his undergraduate and graduate studies, West (1999) described how he developed his own "prophetic pragmatism"

> when I dipped this tradition [pragmatism] into the furnace of black suffering and resistance in America. Yet prophetic pragmatism is not my philosophy or particular vision of the world. Rather, it is a fecund discursive space in which I can put forward many voices and viewpoints. It is the philosophical space occupied by my Chekhovian Christian perspective. (p. 141)

Thus, West's self-developed theory of prophetic pragmatism is not his worldview. He has operated within prophetic pragmatism as a philosophical space to interact with other scholars.

As a branch of pragmatism, I believe inferentialism affords similar possibilities for me. Although I discuss inferentialism and its core claims in more depth in a subsequent section, I believe some preliminary comments regarding its philosophical claims are necessary. To begin, my philosophical anthropology is different than inferentialism's. I agree with Radford's (2017) suggestion that "the most controversial assumption of inferentialism might rest on its answer to the questions of what makes us really human" (p. 506). Radford (2017) rightfully noted that, in inferentialism, humans are characterized in an overly rationalistic and logical way. Yet I do not believe it is necessary to adopt Brandom's rationalist anthropology in order to appropriate some inferentialist insights. Brandom's descriptions of concepts and the GoGAR are not entirely dependent on his assertion that the giving and asking of reasons is at the core of what it means to be human (Smith, 2014).

Regarding ontological concerns, Radford (2017), in his capstone article of the *Mathematics Education Research Journal*'s special inferentialist issue, said it is difficult to articulate inferentialism's ontological commitments. Based on the papers in the special issue, Radford (2017) believed inferentialists hold to a realist mathematical ontology. Radford also bemoaned the general lack of precision in the discussion of ontological commitments made by mathematics education researchers and believed that the specifics of inferentialism's ontology, following Brandom's work, need to be more clearly explained. Radford said the work of spelling out ontological commitments is not "an irrelevant and pedantic academic inquiry" and that "it will be up to our mathematics education inferentialist theoreticians to be explicit in their future work about the kind of ontology that they are ready to hold" (Radford, 2017, pp. 504–505). Following Radford's suggestions, I have clearly described my mathematical ontology and believe it is compatible with inferentialism. Inferentialism advocates for a type of realist ontology[2] but leaves space for more specific ontological commitments; it does not prescribe precise details of an inferentialist ontological account.

The epistemology of inferentialism concurs with my philosophical commitments as well. First, following Wittgenstein, inferentialists believe claims, like all language, are caught up in an embodied form of life. This idea matches my emphasis on the physical in my mathematical ontology. Second, Taylor and colleagues' (2017) inferentialist learning metaphor—mastery—describes how an individual masters the social norms of a concept or practice. The metaphor is an attempt to overcome the individual-social dichotomy because it describes "the learning activity of the student simultaneously and essentially in both cognitive and social terms" (Noorloos et al., 2017, p. 441). This idea—a learning metaphor that simultaneously describes both the individual and social aspects of learning—matches my commitments about humans

---

[2] The realist ontology that inferentialism espouses is not the naïve realism that accompanied the various forms of positivism. Inferentialism does not believe that the universe posits itself in a way that can be objectively read by any human or that humans' truth claims transcend context.

reflecting the Trinity. Third and finally, Hußmann et al. (2019) used inferentialism to describe how knowledge, social norms, and power relations are intertwined in the learning of mathematics. This idea matches my commitments about the importance of love.

Ultimately, I believe inferentialism gives me space to confess my Christian anthropological, ontological, and epistemological commitments. Following the lead of West (1999) and his prophetic pragmatism, inferentialism is my "fecund discursive space." It is a philosophical and theoretical space with sufficiently similar boundaries and commitments as my own worldview that allows me to converse with other mathematics education scholars.

CHAPTER 4

UNDERSTANDING OTHERS, MAKING UNDERSTANDABLE, AND LITERATURE REVIEW

This chapter is dedicated to making inferentialism understandable and understanding radical constructivism and the sociocultural perspective. As noted in Table 1, Prediger et al. (2008b) claimed "all inter-theoretical communication and especially all attempts to connect theories must start with the hard work of understanding others and reciprocally, with making their own theory understandable" (p. 171). Considering inferentialism's recent emergence in the field of mathematics education, radical constructivism's debated history, and the plethora of social theories in mathematics education, this chapter is not trivial. I interweave each theory's philosophical backdrop with its key concepts, methodology, methods, and a survey of empirical results because "understanding a theory means to understand their articulation in research practices" (Prediger et al., 2008b, p. 171).

Inferentialism

To begin, I provide a summary of the most important aspects of inferentialism relevant to this project: (a) a philosophical discussion of inferentialism and its key concepts, (b) a review of methodologies and research methods used by inferentialist researchers in mathematics education, (c) a review of inferentialist research results in mathematics education, and (d) an overview of my inferentialist methodology.

Inferentialist Philosophy and Key Concepts

At the heart of inferentialism is Robert Brandom's philosophical work, which builds on the work of pragmatist philosophers Ludwig Wittgenstein, Wilfrid Sellars, and Richard Rorty (Sellars et al., 1997). Brandom (2000) has used various insights from each of these figures to develop the theory of inferentialism and has said the theory is "about the use and content of concepts" (p. 1) and that it emphasizes the role of collective reasoning in understanding the meaning of language. More specifically, Brandom privileges "inference over reference in the order of semantic explanation" (p. 1). That is, Brandom has claimed the meaning of concepts exist in their inferential relation to each other and he has rejected referential views of language. A few key features of Brandom's project are worth discussing further, particularly his description of 1) the relationship between mind and language, 2) conceptual activity as expression, 3) the role of inferences in understanding the conceptual, 4) semantic holism, 5) the game of giving and asking for reasons, 6) his theory of social objectivity, and 7) the mastery learning metaphor.

Mind and Language

In different philosophical traditions, either the mind or language is considered the locus of content. When the mind is the locus, language is reduced to a convenient tool to communicate an individual's fully formed thoughts. When language is the locus, "judgment, rather, is the interiorization of the external act of assertion" (Dummett, 1973, as cited in Brandom, 2000, p. 5). For Brandom, however, neither language nor the mind has conceptual priority. Brandom quoted Donald Davidson, who wrote, "neither language nor thinking can be fully explained in terms of the other, and neither has conceptual priority. The two are, indeed, linked in the sense that each requires the other in order to be understood" (as cited in Brandom, 2000, pp. 5–6). Brandom

(2000) has claimed neither verbal claims nor mental states can be made sense of without the other; knowledge is necessarily tied to both thought and language because the two cannot be separated from one another. As Derry (2017) said, inferentialism “is a theory which forms part of a move in thought which sees mind as inseparable from world and language” (p. 404).

Conceptual Activity as Expression

Brandom (2000) has also claimed verbal expression can be thought of as making explicit what is implicit. Brandom wrote, “This can be understood in a pragmatist sense of turning something we can initially only do into something we can say: codifying some sort of knowing how in the form of a knowing that” (p. 8). Here Brandom signaled how he views knowledge more broadly than many modern epistemologies. The inferentialist category of knowledge includes embodied know-how, not merely verbal know-that. All verbal expressions, then, are attempts to make explicit (express a know-that) something that was implicit in a know-how.

Inferences and Concepts

For Brandom (2000), expressing something and making it explicit means it can “both serve as and stand in need of reasons” (p. 11). Thus, grasping a concept is equivalent to knowing its inferential implications. Mastering a concept’s inferential use means to know “what else one would be committing oneself to by applying the concept, what would entitle one to do so, and what would preclude such entitlement” (Brandom, 2000, p. 11). This type of inferential reasoning is what Brandom has claimed distinguishes humans from other creatures. He has claimed argumentative practices and the ability to articulate inferences are the defining attributes of being human.

Semantic Holism

Related to his emphasis on inference, Brandom has argued for semantic holism in contrast to semantic atomism. Semantic atomists suggest content can be assigned to concepts one by one—like a mapping of one element directly to another. But in semantic holism, "one cannot have any concepts unless one has many concepts. For the content of each concept is articulated by its inferential relations to other concepts" (Brandom, 2000, pp. 15–16). Thus, specific concepts only have meaning from their use in a web of other concepts.

The Game of Giving and Asking for Reasons

Perhaps the most applicable inferentialist topic for mathematics education research is what Brandom has called the game of giving and asking for reasons. The term is adopted from Wilfrid Sellars (Sellars et al., 1997; 2007) and is abbreviated as GoGAR. Seidouvy and Schindler (2020) succinctly described the GoGAR: "The GoGAR is a discursive practice in which peers exchange claims via interpretations, challenges, and endorsements" (p. 7). The meaning of these claims, because of Brandom's emphasis on inference, is dependent on their inferential relation to other claims.

The claims made in the GoGAR are specifically tied to social norms. Brandom (2000) wrote, "For this reason we can understand making a claim as taking up a particular sort of normative stance toward an inferentially articulated content. It is endorsing it, taking responsibility for it, committing oneself to it" (p. 192). Thus, understanding the content of a claim is equivalent to knowing the consequences of a claim or what else one is committing to by making the claim (Brandom, 2000). Participants in the GoGAR, through a process Brandom has called deontic scorekeeping, calibrate meaning by keeping track of commitments, entitlements, and endorsements (Derry, 2017). The phrase deontic scorekeeping indicates the obligation or necessity for people to track commitments, entitlements, and endorsements in order to make sense of concepts.

Commitments. Commitments, entitlement, and endorsement are important terms within inferentialism and for my analysis. Brandom (2000) has described claims as similar to other practices—they are a doing, an action—but the act of making a claim is also unique. Making a claim is equivalent to committing to something and, when a commitment is articulated, the person who made the claim is responsible for justifying it. Seidouvy and Schindler (2020) have said a commitment is "an individual claim with propositional content that one utters in discursive practices" (Seidouvy & Schindler, 2020, p. 12), which the individual takes responsibility for. However, I believe a commitment, according to Brandom, can also include something other than verbally articulated claims. Brandom (2000) wrote,

> It must be possible for some performances to have the practical significance of undertaking commitments. For asserting something is committing oneself to it, and the beliefs those assertions express involve a kind of commitment. It is such commitments that, in the first instance, stand in practical inferential relations—such as that by committing oneself overtly (assertionally) to Leo's being a lion, one thereby implicitly commits oneself (whether one realizes it or not) to Leo's being a mammal. And it is the contents of those commitments that stand in semantic inferential relations that can be made explicit by the use of conditionals. (p. 43)

At the beginning of this passage, Brandom used the term "performances" to describe what can have the practical significance of a commitment; he then suggested commitments can also be implicit and unarticulated. Although I believe Seidouvy and Schindler are correct to suggest an uttered claim is a commitment, I believe commitments are a larger category that also include actions or performances related to mathematical conceptual content like gestures, drawings, or writings.

Verbal, explicit claims are a focus of the GoGAR, but Brandom has also said "the sort of algebraic understanding characteristic of mature mathematized sciences—the sort for which analytic philosophers long—is pragmatically dependent…on everyday hermeneutic understanding, which accordingly cannot be replaced by, or reduced to, the more technical kind" (Brandom, 2008, p. 212–213). What is often considered rigorous and formal knowledge is pragmatically dependent on other modes of knowing and on interpretation.

Entitlements. Brandom's concept of entitlement is tied to his concept of commitment; once an individual commits to a claim, other people within the GoGAR have to assess the commitment—they have to determine whether the individual is entitled to the commitment. The practice of giving reasons for commitments only makes sense if people may or may not be entitled to their commitments. Brandom (2000) wrote, "Giving reasons for a claim is producing other assertions that license or entitle one to it, that justify it" (p. 193). Brandom (2000) has described how a commitment must be fleshed out both downstream and upstream. Downstream, an individual must discuss the "inferential consequences" of their claim, and upstream, an individual must be able to articulate "inferential antecedents" that "serve as premises from which entitlement to the original content can be inherited" (p. 194). People are not entitled to their commitments when incompatible inferential relations exist. Incompatible inferential relations exist because language-games have norms. Brandom (2000) wrote, "We can say that two assertible contents are incompatible in case commitment to one precludes entitlement to the other" (p. 194). These socially identified incompatibilities are at the heart of Brandom's account of objectivity, which I explain further in a subsequent section.

Endorsement. The final term relevant to the GoGAR is endorsement. To endorse a commitment is to take a commitment as true and undertake the same commitment oneself. Endorsements occur when an individual explicitly affirms a peer's commitment or when someone uses someone else's "commitments as premises in one's own inferences" (Brandom, 2000, p. 120).

Nilsson (2020) outlined a mathematical example to illustrate how the content of a claim emerges in the GoGAR. If a student claims the sum of the interior angles of a polygon is 180°, a teacher or peer may press them to describe the antecedent to their claim. If the student can articulate the inferential antecedent (e.g., the polygon is a triangle), they may subsequently be pressed to describe the inferential consequences of their claim (e.g., the algebraic determination of the measure of individual angles) or describe what is incompatible with the claim (e.g., the polygon is not a quadrilateral). Ultimately, the content of the original claim is dependent upon this web of inferentially related concepts that is only illuminated in the GoGAR.

A Social Objectivity

From his description of the GoGAR, Brandom has claimed that social objectivity exists. Because there are social norms that govern the giving and asking of reasons and commitments and entitlements are social statuses conferred by a community, humans within the same community (or language-game) can declare two claims incompatible. Brandom (2000) described how "commitment to the content expressed by the sentence 'The swatch is red' rules out entitlement to the commitment…'The swatch is green.'" (p. 194). The two statements ("the swatch is red" and "the swatch is green") are not incompatible because they both refer to an objective reality. Rather, within a social community with specific norms, the statements infer different things and thus cannot be held together. (Thus, the theory is called inferentialism, not referentialism.)

Mastery

Brandom (2000) has offered an inferentialist metaphor to describe what it means to grasp a concept in the GoGAR. He said navigation and mastery of the web of reasons is equivalent to *mastering* a concept's inferential use:

> knowing (in the practical sense of being able to distinguish, a kind of knowing how) what else one would be committing oneself to by applying the concept, what would entitle one to do so, and what would preclude such entitlement. (p. 11)

Mastery is thus a measurement of conceptual understanding that depends on one's knowledge of the: (a) downstream inferential consequences of a concept, (b) upstream justifications that entitle the use of a concept, and (c) potential incompatible inferences related to a concept.

Previous Inferentialist Research

The inferentialist literature in mathematics education research can broadly be categorized as theoretical and empirical. Theoretical inferentialist researchers have done several things, including

- described Brandom's inferentialism (Derry, 2017; Bakker & Hußmann, 2017),
- compared inferentialism to theoretical alternatives in mathematics education (Noorloos et al., 2017; Taylor et al., 2017; Radford, 2017),
- explained how inferentialism is relevant for research in mathematics education (Bakker & Hußmann, 2017; Noorloos et al., 2017),
- proposed mastering as an inferentialist metaphor for learning (Taylor et al., 2017),
- and proposed an inferentialist ontology of reasons (Uegatani & Otani, 2019).

Inferentialism-based empirical researchers have studied

- authority (Seidouvy et al., 2019),
- student collaboration (Seidouvy & Schindler, 2020),
- the coordination of action and reasons in making statistical inferences (Bakker et al., 2017),
- how students reason inferentially based on previous experiences both in and outside the classroom (Schindler et al., 2017),
- the redesigning of statistics classes via an inferentialist approach (Bakker & Derry, 2011),
- students' conceptual development in mathematics with an original inferentialist analytic framework (Hußmann et al., 2019),
- the quality of procedural and conceptual knowledge according to an inferentialist framework (Nilsson, 2020),
- and how inferentialism "can illuminate epistemological dimensions in current multilingual mathematics education contexts" (Ryan & Parra, 2019, p. 153).

In the subsequent sub-sections, I further describe the conclusions of the theoretical studies, the methods used within the empirical studies, and the results of the empirical studies.

Conclusions of Inferentialist Theoretical Studies

Derry (2017) described practical implications of inferentialism for mathematics education classrooms and mathematics teachers. She said that "when we are concerned with formal education and the development of judgment and with it appropriate responsiveness, the adoption of a stance in an inferentially articulated space becomes crucial" (p. 410). As a theory of meaning, Derry argued inferentialism has significant implications for teaching because teachers must understand what their students are saying and what they understand. She claimed an inferentialist approach encourages teachers to pay attention to students' knowledge by attending to their reasoning and "showing the learners what they have committed themselves to and what is entailed by their commitments" (p. 409). Derry ultimately concluded this requires deontic scorekeeping: teachers and students keeping track of everyone's endorsements and commitments within a GoGAR.

Bakker and Hußmann (2017) described how inferentialism is well suited to address how people learn concepts, a notable challenge in mathematics education research. They argued inferentialism and the way it explains concepts ("in terms of reasoning arising within a social game of giving and asking for reasons" [Bakker & Hußmann, 2017, p. 398]) can offer important insights into the discussion of sociomathematical norms (Yackel & Cobb, 1996). They also offered practical implications of inferentialism for teaching: inferentialism "argue[s] against atomistic teaching approaches in which students learn concepts and graphical representations one by one. Rather, inferentialism suggests students need to become familiar with systems of judgments, in which the meanings of several concepts become inferentially articulated" (p. 398). The authors also argued inferentialism can be used to design learning environments through the design of tasks and specific learning processes. They claimed inferentialism is uniquely able to meet this challenge because it can make sense of mathematical content and individual's knowledge within social norms and practices. Inferentialism is valuable to the field of mathematics education and its research, they claimed, because "reason and inference are the bread and butter of mathematics and hence also of learning and teaching the subject. A richer account of reason will hopefully help to develop the field" (Bakker & Hußmann, 2017, p. 399).

Noorloos et al. (2017) performed a theoretical comparison of inferentialism with social constructivism and claimed inferentialism overcomes three prominent weaknesses of social constructivist research in mathematics education. First, they claimed social constructivist research in mathematics education perpetuates a false dichotomy between the social and individual aspects of learning. In contrast, they argued inferentialism successfully accounts for the dynamic between the social and the individual because the two are interrelated in inferentialism: “Every reason depends on a social context if it is to be uttered or understood. There are no reasons in the absence of others with whom to discuss, share, or establish them in the first place” (p. 449). In this way, inferentialism can be used to describe the learning activity of a student “simultaneously and essentially in both cognitive and social terms” (p. 441). Second, Noorloos et al. claimed social constructivism is prone to relativism and does not successfully account for the relation between the human mind and the world. In defense of inferentialism, the authors cited Brandom’s theory of social objectivity and claimed it successfully describes how knowledge claims are constrained by something outside of individuals’ interpretations but also always exist within a social context. Finally, the authors claimed social constructivism’s construction metaphor for learning is too vague. They briefly introduced the inferentialist metaphor for learning—mastery—in its stead.

Taylor et al. (2017), building on Brandom’s description of mastery, suggested mastery as an inferentialist metaphor to overcome the deadlock in mathematics education research between the acquisition metaphor and participation metaphor (Sfard, 1998). They outlined how the inferentialist mastery metaphor for learning simultaneously accounts for the social and individual aspects of learning. An individual is a member of the GoGAR, and the social world is the “socio-cognitively structured setting or context of human reason” (p. 780) in which the individual operates. An individual’s learning is developmental, entails mastery of socio-cognitive capacities, and is ratified in linguistic practice—which is inherently social. Finally, knowledge is the socially evaluated status of one's mastery of a concept or practice as indicated by the ability to reason with it.

Finally, Uegatani and Otani (2019) proposed a new inferentialist "ontology of reasons." In contrast to other inferentialist mathematics education researchers, they argued "there is no real reason for why someone does something unless she herself indicates her reason explicitly" (p. 191). That is, unless a reason is made explicit, researchers doing retrospective analysis risk "mistakenly characterizing students' behaviors" (p. 188).

Methods Used in Inferentialist Empirical Studies

Within inferentialist empirical studies, researchers' methods have varied. Bakker and Derry (2011) used data from teaching experiments in sixth-grade mathematics classrooms in the United States. The teaching experiments were used to revise a mathematics curriculum that used TinkerPlots, which is a data visualization and modeling tool, and to provide feedback to the designers of TinkerPlots. The teaching experiments were performed in two mathematics classes, one for eighteen 42-minute periods and one for twenty 42-minute periods, by videotaping all the class periods and audio recording interviews with students and the teacher. The purpose of the teaching experiments was to simultaneously revise instructional materials with inferentialist ideas in mind while also engaging students in meaningful statistical reasoning. The authors gave empirical examples of how students learned to reason inferentially about several statistical concepts, but they did not describe their analytic methods.

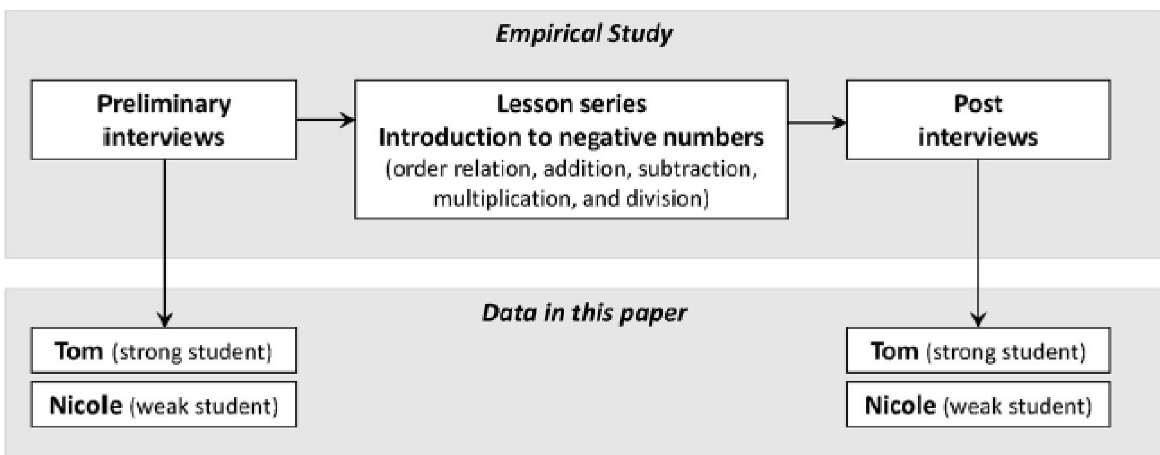
Bakker et al. (2017) performed a case study with Sam, a Dutch intern in a hospital laboratory that was part of a "senior secondary vocational laboratory school" (p. 459). Their data consisted of video recordings of (a) six meetings between Bakker, three students, and their teacher, (b) Sam's final presentation for a project and post-interview with the researcher, and (c) audio recordings of interviews between Sam and his workplace supervisor. The authors' goal was to identify the reasons and actions that Sam used to make a valid statistical inference within his research project. In their interviews and meetings, the authors said they were intent on asking for reasons for Sam's actions. Their data analysis consisted of four steps. First, they familiarized themselves with the relevant disciplinary knowledge of chemistry and laboratory work. Second, they identified any action or reason of Sam's from transcripts of the meetings, presentation, or interviews. Third, the research team discussed their conjectures and conclusions about Sam's actions and reasons and, when uncertain, checked their analysis with Sam or his teacher. Fourth and finally, they visually summarized Sam's statistical reasons and actions with web diagrams.

Ryan and Parra (2019) investigated epistemological aspects of multiple multilingual mathematics education contexts from an inferentialist perspective. They described moments from two different episodes: "a workshop held during a revitalisation project in Colombian indigenous communities and a Grade 5 mathematics classroom in Sweden" (p. 153). Due to the vastly different contexts in which the authors explored various linguistic and cultural aspects of students' mathematics, there are significant intricacies of their research methods unnecessary to explore here. The first episode was an "indigenous initiative, structured by Nasa culture, worldview and ways of validating and registering knowledge" (p. 159) and the second episode involved a student who spoke Persian and Swedish at home yet resided in a classroom that was socially and linguistically diverse in a variety of ways. Their analysis of the first episode was a vignette based on Parra's field notes from the workshop in Colombia. Their analysis of the second episode featured a transcription analysis of an audio recording of an exchange between Ryan and the student from the Swedish classroom that spoke Persian and Swedish. The authors did not describe their analytic methods.

Schindler et al. (2017) researched how students reasoned inferentially based on previous experiences both in and outside the classroom. They performed clinical interviews with two students out of a larger data set from a previous empirical study carried out by Schindler. The authors wanted to analyze how the students reasoned about symbolic representations of negative numbers via the interviews. The previous study was with sixth-grade students—six boys and two girls—in a German secondary school aged 11, 12, and 13. The lesson series from that study was developed by the researchers as part of an attempt to teach mathematics through the use of real contexts. Schindler et al.'s design and methods are best illustrated by Figure 3 below.

Figure 3

Design of Schindler et al.'s (2017) Previous Empirical Study and Data Used



Note. Reprinted from "Sixth-grade students' reasoning on the order relation of integers as influenced by prior experience: an inferentialist analysis," by Schindler et al., 2017, *Mathematics Education Research Journal*, *29*(4), p. 479. Copyright 2017 Schindler et al. Reprinted with permission.

Schindler et al. (2017) made pointed suggestions on how inferentialism informed their interview methodology. Because inferentialism is focused on inferential reasoning, they described how "an inferentialist design requires situations in which students are explicitly encouraged to make explicit their reasons and inferences, the origins of their inferences, and the prior experiences that they draw on" (Schindler et al., 2017, pp. 478–479). Consequently, they conducted semi-structured, task-oriented clinical interviews and asked questions like: "Why's that?" "How did you think about it?" "How did you get the idea to…" and "Can you explain to me how you think about this?" (p. 479). The interviews were videotaped and transcribed, and the students' written notes were scanned. The authors did not outline their analytic methods but said their data analysis focused on three epistemological considerations they discussed in their theoretical framework. The three epistemological considerations were (a) the idea that students' reasoning is situated within or derived from the GoGAR and thus always social, (b) students' reasoning in the GoGAR was implicitly normative, and (c) students' concepts were not isolated from one another.

Seidouvy et al. (2019), in their investigation of authority via inferentialism, performed something like a series of focus groups. The video recorded data featured in their article came from group work sessions of fifth graders in a primary school in Sweden, and the students' written work was collected as well. Seidouvy et al. (2019) described how, "for the experiment, two groups were allocated in the classroom and the other three groups were each assigned to a study room (smaller than a classroom)" (p. 32). Each group was given a mathematical task designed by the authors on the topic of data generation. The task required students to generate data about the length that different sized paper frog models could jump. The authors analyzed the data in three steps. First, they identified authoritative student claims that were acknowledged or undertaken by other students. They also compiled the related speech acts and actions that led to the authoritative claims. Second, they categorized each authoritative instance as content-based, person-based, or observational authority. Third and finally, they applied an inductive analysis that further characterized the authority of the claims in relation to statistics. The authors also claimed to provide a methodological contribution to inferentialist research by showing how inferentialism could be used to identify and classify authority in student collaboration.

Several inferentialist empirical studies (Hußmann et al., 2019; Meyer, 2018; Nilsson, 2020; Seidouvy & Schindler, 2020; Uegatani & Otani, 2019) discuss a theoretical or analytic framework and then illustrate its use with examples from whole class interactions between students, small group interactions between students, or a clinical interview. Seidouvy & Schindler (2020) used inferentialism to describe student collaboration in mathematics classrooms as a process of commitment and deontic scorekeeping in the GoGAR. They also used empirical examples to illustrate their inferentialist account of student collaboration; the data was from video-recorded small group work in fifth and seventh grade classrooms in Sweden. In the small groups, the fifth-grade students worked on two different tasks focused on data generation. The authors first used empirical examples to demonstrate how inferentialism was relevant to the learning of mathematics in a classroom. They then used empirical examples to illustrate how their inferentialist framework could be operationalized for research purposes. The authors did not describe their analytic methods and did not state empirical results. Rather, they used their empirical analyses to argue that their inferentialist-provided account of the social and individual nature of collaboration was more productive than other accounts of collaboration.

Hußmann et al. (2019) proposed an inferentialist analytic framework to analyze students' conceptual development in mathematics. The framework was an initial step in the translation of inferentialism's epistemological ideas to an inferentialist theory of learning mathematical concepts and focused on coding commitments and entitlements in short-term episodes of the game of giving and asking for reasons. The authors used short empirical episodes from a clinical interview with two teenage German students to illustrate how their framework captures both the individual and social facets of student learning. The clinical interview focused on the students' understanding of decimal numbers and was part of a sequence of clinical interviews within a design research study.

Like Hußmann et al., Nilsson (2020) proposed an inferentialist framework to analyze procedural and conceptual knowledge in light of the GoGAR and then included empirical examples to illustrate the framework's use. To analyze the teaching and learning of fractions in classroom data from a sixth-grade mathematics class in Sweden, Nilsson performed three steps. First, he ordered the data into teaching episodes in which the teacher and students engaged in a task together. Second, he coded the episode as procedural or conceptual depending on the type of knowledge that was featured in the episode. Finally, he coded the episodes as rich or limited GoGARs based on his theoretical descriptions of limited and rich GoGARs.

Results of Inferentialist Empirical Studies

The empirical studies' results can largely be characterized by their preliminary nature. Most of them gave new descriptions of learning through an inferentialist lens or demonstrated the use of a new inferentialist framework. Bakker and Derry (2011) listed three major challenges in statistics education: (a) to avoid rote knowledge, (b) to foster coherent student understanding in light of atomistic approaches in textbooks, and (c) to sequence topics appropriately. They then provided three empirical illustrations of sixth-grade students' informal statistical reasoning as evidence that an inferentialist approach to teaching statistics overcame these challenges. Their study illustrated what statistics education can be when semantic holism is privileged over semantic atomism and when inference is privileged over reference. They found students were able to draw improved and sensible inferences due to their specific inferentialist-designed instruction.

Bakker et al. (2017), in their case study of the hospital intern Sam, used inferentialism to analyze Sam's statistical decision-making and inferences in a real-life context. The authors summarized Sam's actions in coordination with his emotions and various types of reasoning (statistical, mathematical, practical, medical) and claimed inferentialism helped them coordinate these things while avoiding dichotomies between them. They argued similar fine-grained inferentialist analyses could have implications for designing learning environments that attempt to holistically develop students' knowledge, skills, and attitudes in general education.

Meyer (2018) integrated Toulmin's argumentation diagrams with insights from Wittgenstein and Brandom and thus made novel contributions to Toulmin studies in mathematics education, inferentialist research, and Wittgensteinian research. He described episodes from a German fourth grade class and a German tenth grade class and claimed they showed how mathematical concepts are established by developing rules (or, in Toulmin's words, warrants) for the inferential use of concepts within a language-game. Thus, Meyer took ideas from Toulmin (the structure of arguments), Brandom (the emphasis on inference), and Wittgenstein (language-games) and claimed "understanding (students') conceptual learning processes means to understand the rules they use in order to establish meaning" (p. 306).

Hußmann et al. (2019) described the conceptual development of students' understanding of decimals after introducing their analytic framework. The author's empirical results primarily served to validate the usefulness of the analytic framework. Alongside their written empirical analyses, Hußmann et al. (2019) provided companion diagrams to demonstrate how their framework can analyze a dialogue chronologically or as a re-structured inferential web of reasons. In their analysis, they also described how the norms of the GoGAR include social norms distinct to the students' particular community and power relations. Throughout the article, the framework was only used to analyze short episodes for fine-grained analyses, but the authors said the framework could be used for a long-term analysis by compiling all short-term analyses over a certain period of time.

Nilsson (2020) presented an inferentialist framework to characterize the quality of procedural and conceptual GoGARs; the author's empirical results primarily served to validate the usefulness of the framework. To classify the quality of both procedural and conceptual knowledge, Nilsson distinguished between rich and limited GoGARs. Rich mathematical GoGARs give space for students to make explicit connections between concepts in a holistic way. Limited mathematical GoGARs are primarily characterized by a teacher asking factual questions and funneling students towards an answer. In limited mathematical GoGARs, students' reasoning is often implicit and their use of concepts is atomistic. Nilsson effectively used the framework for empirical analysis and assessed the quality of several GoGARs according to the type of inferential connections made between concepts and procedures. He sought to determine the quality of knowledge by the quality of the GoGAR it emerged within and insisted procedural knowledge is not less valuable than conceptual knowledge.

Ryan and Parra (2019), in two episodes from multilingual contexts, showed how inferentialism can help researchers make sense of multilingual mathematical situations. Through a vignette based on field notes of interactions with researchers in an Indigenous Nasa settlement in the Cauca state of Columbia, the authors described how the English mathematical concept *unit of measure* had no directly translatable word in Nasa. In an episode from a Swedish Grade 5 mathematics class, the authors described a similar phenomenon: Aldrin, a student who spoke

Persian and Swedish at home, troubled the meaning of the word *long* in the concept *half as long*. Ultimately, Aldrin's use of the mathematical concept was "not merely a matter of translation or code switching" (p. 163). The authors' analysis showed how epistemology was necessarily intertwined with language in multilingual mathematics education and how inferentialism has the tools to make sense of both the epistemological and linguistic aspects.

The purpose of Schindler et al.'s (2017) study was to show how inferentialism can be used to make sense of sixth grade students' mathematical reasoning from both out-of-school and in-school experiences. Based on the analysis of their interviews with two students, they claimed inferentialism offered a framework adept at analyzing "students' manifold and sophisticated ways to reason on the order relation of negative numbers" (p. 490). The authors also claimed they were better equipped to understand why students provided specific answers and reasons for their decisions because they used inferentialism to design the study and their analysis instead of other theoretical perspectives.

Finally, Seidouvy et al. (2019), in their investigation of authority via inferentialism, established several results. First, they established that inferentialism can be used to categorize authority in student collaboration. To identify authority, they observed when and how students stopped questioning one another's claims. They categorized authority types into one of three previously established inferentialist categories: content-based authority, observational authority, and person-based authority. Second, they established five new types of authority in their attempt to further characterize authority in the context of statistics through inductive analysis of their data. The five types were authority of (a) socially negotiated content (acceptable data), (b) logical inferences, (c) variability, (d) motion and location in space, and (e) procedural expertise. Third and finally, the authors concluded that their collaborative data generation activity provided students the opportunity to appeal to content-based authority as well as observational authority. Students did not merely make observations in the activity; their observations were regulated by their conceptual knowledge of measurement, units, and natural numbers.

My Inferentialist Methodology

To empirically analyze the extant data and respond to my set of methodological research questions (How does an inferentialist researcher robustly analyze students' mastery of meanings in clinical interviews? How does an inferentialist researcher robustly analyze the game of giving and asking for reasons and conceptual content within collective argumentation?), I developed an inferentialist analytic methodology. My analytic framework and methodology share some similarities with Hußmann et al.'s (2019) inferentialist analytic framework (which was developed to trace students' conceptual development in mathematics), but also diverges from it. Hußmann et al.'s framework has two analytical units, commitments and entitlements, both of which are foundational concepts that Brandom created. However, Brandom also used precise distinctions between the types of inferences people can make (upstream, downstream, and incompatible) and described endorsements as another action within the GoGAR. Hußmann et al. did not attend to Brandom's distinctions between different types of inferences and instead focused on different ways to structure episodes of dialogues with different diagrams. I believe the distinctions between types of inferences is crucial to a detailed account of a student's mastery of a concept and contributes to inferentialism's explanatory power. However, I adopt several of Hußmann et al.'s insights regarding social norms and the dynamic between the individual and the social.

I introduce the general outline of my analytic methodology (a result of my dissertation) in this chapter to prepare the reader for my inferentialist analyses in the subsequent chapters where

I use this methodology to analyze both the extant clinical interview data as well as the extant classroom data. In my final chapter, I describe the methodological conclusions drawn from networking inferentialism with radical constructivism and the sociocultural perspective.

My inferentialist methodology begins with transcribing the entire clinical interview or classroom session. Within the transcription, I am careful to describe any bodily movements or actions related to the mathematics or any mathematical explanation. I then use the complete transcription as data to code for several inferentialist concepts. My analytic decisions for coding these inferentialist concepts are based on a close reading of Brandom's work and, more specifically, his idea of deontic scorekeeping. As previously noted, participants in the GoGAR, through the process of deontic scorekeeping, calibrate meaning by keeping track of commitments, inferences, entitlements, and endorsements (Derry, 2017). The word deontic is an adjective which suggests something is duty-bound or obligatory ("deontic", n.d.). Thus, deontic scorekeeping indicates the obligation or necessity for people to track commitments, inferences, entitlements, and endorsements to make sense of concepts. Throughout the subsequent descriptions of my inferentialist codes, I provide specific justifications from Brandom (2000).

My first inferentialist analytic code is *commitments*: students' verbal claims or actions (committal performances) related to mathematical conceptual content. I code certain actions as commitments in addition to verbally articulated claims because Brandom suggested commitments can also be implicit and unarticulated (Brandom, 2000). By tracking commitments instead of claims, I also better capture the meaning of explicit claims and the content of concepts. Within the code, neither verbal claims nor actions are given priority over the other. Examples of actions that indicate commitment are the production of a graph, written work, hand gestures, or rotations of a paper. In addition to students' commitments, I also code implicit commitments within the prompts or problems given by the classroom instructor or interviewer. Finally, although I often have a specific mathematical concept in mind for analysis, other mathematical concepts related to the concept under analysis are also tracked. For example, for the mathematical concept inverse, commitments related to the concepts function, equation, set, input, output, mapping, graph, *x*, *y*, *x*-axis, *y*-axis, variables, relationship, quantity, reflection, and coordinate pairs must be tracked. This analytic decision reflects inferentialism's emphasis on semantic holism: specific concepts only have meaning from their use in a web of other concepts.

The next inferentialist analytic code is *inferences*; I document any inference made from a person's commitment as one of three sub-categories. A *downstream inference* occurs when a person commits to the inferential consequences of another commitment. An *upstream inference* occurs when a person commits to the inferential antecedents of another commitment (justifications) or when a student describes the qualifying circumstances necessary for their commitment. As I track downstream and upstream inferences, I also record which original commitment the inference is downstream or upstream from. Finally, *incompatible inferences* occur when someone is committed to two things but cannot be entitled to both. Incompatible inferences are socially determined, and I do not code incompatible inferences based on my understanding of correct mathematics. I identify (a) what commitments are identified as incompatible by the people in the GoGAR, (b) what linguistic or epistemological norms cause the incompatibility, and (c) how the incompatibility is resolved.

The next inferentialist analytic codes are *questions* and *questions of entitlement*. I track general questions and prompts but questions of entitlement, a sub-category of questions, have priority. Through questions of entitlement, people question whether other people are entitled to their commitments and inferences. Brandom (2000) wrote,

> Asking for reasons for a claim is asking for its warrant, what entitles one to that commitment...Indeed, I take it that liability to demands for justification—that is, demonstration of entitlement—is another major dimension of the responsibility one undertakes, the commitment one makes, in asserting something. (p. 193)

Consistent with Brandom's understanding of entitlement, I code an utterance as a question of entitlement when a question aimed to prompt or prompts an upstream inference. When I code questions of entitlement, I also identify what commitment or inference is at stake and how the situation is resolved. Following a question of entitlement, either the question is ignored, subsequent inferences are provided and endorsed, or an incompatibility arises.

The final inferentialist codes are *entitlement* and *endorsement*. I code entitlements whenever a person's commitment or inference receives implicit approval from other people in the GoGAR. An entitlement is a normative status conferred when someone's commitment or inferences are implicitly approved and not explicitly rejected; the collective indicates that the person is entitled to their idea. I code endorsements whenever one person endorses another person's commitment or inference. An endorsement may be an explicit acknowledgement and positive assessment of a commitment, or it can occur implicitly when someone is willing to use someone else's "commitments as premises in one's own inferences" (Brandom, 2000, p. 120). Entitlements require other people to think a person is entitled to their idea(s); endorsements require someone to endorse and adopt someone else's ideas as their own. Table 4 provides a summary of my inferentialist codes.

Table 4

Codes in Inferentialist Analytic Methodology

Inferentialist Code	Description of Code
Commitment	Verbal claim or action (committal performance) related to mathematical conceptual content
Downstream Inference	Commitment to the inferential consequences of another commitment
Upstream Inference	Commitment to the inferential antecedents of another commitment (a justification) or when someone describes the qualifying circumstances necessary for their commitment
Incompatible Inference	Socially identified situation when someone is committed to two things but cannot be entitled to both
Question of Entitlement	Question aimed to prompt or prompts an upstream inference; questions whether an individual is entitled to their commitments and inferences
Entitlement	Commitment or inference is implicitly approved and not explicitly rejected; the collective indicates that someone is entitled to their idea
Endorsement	Explicit acknowledgement and positive assessment of a commitment or when someone uses someone else's "commitments as premises in one's own inferences" (Brandom, 2000, p. 120)

Radical Constructivism

I now give an overview of radical constructivism. Radical constructivism is a specific, influential theoretical tradition within cognitive constructivism that was birthed out of Ernst von Glasersfeld's reading of Kant and Piaget and his collaboration with members (faculty and doctoral students) of the University of Georgia's mathematics education department like Les Steffe and Pat Thompson. Cognitive constructivism is the more appropriate term to understand the wider theoretical tradition and international movement and I occasionally include relevant comments and results from cognitive constructivist researchers. However, I have chosen to use the term radical constructivism because I mainly focus on von Glasersfeld, Steffe, and Thompson's interpretation of Piaget.

Philosophical Background

Piaget

Paul Cobb, in *Putting Philosophy to Work: Coping with Multiple Theoretical Perspectives*, gave an overview of four different theoretical traditions including cognitive constructivism, or what he called "cognitive psychology" (Cobb, 2007, p. 19). Cobb helpfully traced the history of cognitive psychology from an 18th century Italian philosopher named Giambattista Vico to Immanuel Kant to the perspective's most significant contributor: Jean Piaget.

Piaget's work was part of the structuralism movement—a philosophical and psychological movement focused on studying permanent, underlying, abstract structures—that he grounded in mathematical, physical, and biological structures. Piaget (1968) wrote, "A critical account of structuralism must begin with a consideration of mathematical structures, not only for logical but even for historical reasons" (p. 17). Piaget described how group theory and formal logical structures are prime examples of structures. However, he also claimed physical structures were more foundational than mathematical and logical structures. Piaget said, "In sum, there are physical structures which, though independent of us, correspond to our operational structures, especially in sharing the quasi-intellectual trait of covering the possible and locating the real within a system of virtuals" (Piaget, 1968, p. 43).

Physical structures were the penultimate bedrock for Piaget. Organic, or biological, structures were at the heart of his account of structure. Piaget (1968) argued, "In short, biological wholes and self-regulating systems, though 'material' and of physico-chemical content, enable us to understand the connection between 'structures' and 'the subject,' because it is the organism which is the latter's source" (p. 51). Piaget saw the biological organism as paradigmatic; he believed a more thorough account of biological organisms would provide a general theory of structure for psychology because organisms were the originators of behavior.

Piaget's emphasis on biology and biological organisms was a continuation of his early training as a biologist. As Cobb (2007) wrote, "Piaget drew on his early training as a biologist to characterize intellectual development as an adaptive process in the course of which children reorganize their sensory-motor and conceptual activity (Piaget, 1980)" (p. 20). Consequently, much of Piaget's work used biological concepts like ontogenesis (the development of an individual organism from its beginning) and phylogenesis (the development of a group of organisms from their beginning) to make sense of human learning (Piaget, 1968).

The tradition's "constructivism" label comes from Piaget's divergence from Kant. Kant believed fundamental structures were *a priori,* but Piaget gave an account of the fundamental categories of thought through the constructs of assimilation, accommodation, and equilibration (Cobb, 2007). So, while Piaget followed several Kantian threads, Piaget believed the structures and categories that Kant argued were present prior to experience (*a priori*) were in fact constructed. His structuralism was founded on the principle: "there is no structure apart from construction" (Piaget, 1970, p. 140).

von Glasersfeld

Piaget's constructivism is prominent in mathematics education largely because of Ernst von Glasersfeld. von Glasersfeld saw genius in Piaget's constructivism; he wrote,

> Piaget was not the first to suggest that we construct our concepts and picture of the world we live in, but no thinker before him had taken a developmental approach. To someone who is driven to ask about the source and validity of knowledge by circumstances of experience (in my case, the plurality of languages), it seems obvious that the best and perhaps only way to find out how knowledge is built up, would be to investigate how children do it. For traditional philosophers, of course, this would be committing an unforgivable sin, because to justify knowledge through its development rather than by a timeless logic, is what they call a 'genetic fallacy'. But Piaget was not a traditional philosopher. (von Glasersfeld, 1995, p. 13)

Although this quote is long, it is an important quote from von Glasersfeld that shows both his draw to Piaget and Piaget's distinctive philosophical approach to epistemology. von Glasersfeld highlighted how Piaget rejected Enlightenment notions of idealized reason and logic—the epistemic currency within the economy of more rationalistic philosophy—and instead looked to the ontogenesis of children. This piqued von Glasersfeld's interest because of his personal experiences as a multi-lingual child and his draw towards epistemology and the philosophy of language.

von Glasersfeld (1995) explained how, for Piaget, knowledge was not about mirroring reality, but about adaptation. von Glasersfeld believed the translated Piagetian term *adaptation* was best understood with the biological concept *viable*. "Actions, concepts, and conceptual operations are viable if they fit the purposive or descriptive contexts in which we use them" (von

Glasersfeld, 1995, p. 14). von Glasersfeld (1995) quoted a passage from Piaget's *Genetic Epistemology* (1970) in which Piaget wrote:

> So, in sum, genetic epistemology deals with both the formation and the meaning of knowledge. We can formulate our problem in the following terms: by what means does the human mind go from a state of less sufficient knowledge to a state of higher knowledge? The decision of what is lower or less adequate knowledge, and what is higher knowledge, has of course formal and normative aspects. It is not up to psychologists to determine whether or not a certain state of knowledge is superior to another state. That decision is one for logicians or specialists within a given realm of science. (pp. 12-13)

von Glasersfeld's notion of viability is helpful for interpreting Piaget here. When Piaget suggested there were more superior states of knowledge, he meant some states of knowledge were more viable than others. Some actions, concepts, and conceptual operations are more adaptative to their environment than others. Piaget also said logicians and specialists within given sciences can assess how some forms of knowledge are more viable than others. von Glasersfeld said that Piaget, in his "logicians or specialists within a given realm of science," comment, was "simply explaining that it must be tested for logical consistency (non-contradictions) and for experiential validity (e.g., in experiments)" (von Glasersfeld, 1995, p. 14). Thus, radical constructivist's account of truth is coherentist and maintains remnants of a scientism or empiricism. As von Glasersfeld (1995) remarked: "the concept of viability in the domain of experience, takes the place of the traditional philosopher's concept of Truth, that was to indicate a 'correct' representation of reality" (p. 14).

This makes it difficult to determine what radical constructivists mean by reality (a metaphysical or ontological concern) and how humans construct knowledge of it. Consider von Glasersfeld's (1995) comments on how many believers in the representational account of truth respond to views like radical constructivism:

> They immediately assume that giving up the representational view is tantamount to denying reality, which would indeed be a foolish thing to do. The world of our experience, after all, is hardly ever quite as we would like it to be. But this does not preclude that we ourselves have constructed our knowledge of it. (p. 14-15)

von Glasersfeld believed an "anything goes" relativism would be foolish: humans do not dictate to the world and everything in it how they ought to behave. Yet von Glasersfeld (1995), referencing Piaget, also suggested

> The elements the thinking subject coordinates are by definition present *in* the subject's system because they are 'experiential'. The system has no access to items which, from an observer's point of view, are seen as external, 'environmental' causes of the system's experiences. Coordination, thus, is a strictly internal affair and, therefore, it is always subjective to the coordinator…Like all cognizing organisms, [philosophers] draw conclusions from their own sensorimotor and conceptual experience, and any explanation of their conclusions, i.e., their 'knowledge', must be in terms of *internal* events and cannot draw on elements posited elsewhere. (p. 72, italics in original)

von Glasersfeld emphasized humans do not have access to anything that causes their experiences and *all knowledge is entirely internal*. von Glasersfeld (1991) acknowledged his debt to skeptics like Hume and Kant for adopting this radical view.

Critics of radical constructivism, concerned about classifying knowledge as entirely internal and concerned with granting epistemological status to all individuals' conceptions, brought their concerns to von Glasersfeld regarding these topics. He briefly responded in a footnote:

> Recently is has been suggested that radical constructivism is contradictory because it attacks realism and at the same time assumes a realist position by admitting that an ontological reality may constrain human action (e.g., Matthews, 1992, p. 186). In the usual language of philosophers, 'realists' are those who believe that they can obtain knowledge of a world as it is in itself. This I deny, and admitting 'ontic' constraints does not contradict it, because while they may determine what is impossible, they do not determine the ways of acting and thinking that can be constructed within them. (von Glasersfeld, 1995, p. 52)

von Glasersfeld attempted to address confusions surrounding radical constructivism, but he did not specify how reality ("ontic constraints") limits and impacts humans' acting and thinking. Commenting further, von Glasersfeld (1995) believed his critics "refuse to consider that this theory of knowing is intended as a tool that should be tested for its usefulness rather than taken as a metaphysical proposal" (p. 51).

Philosophical Anthropology

Much of radical constructivism's philosophical background can be helpfully summarized with a few comments on its philosophical anthropology. von Glasersfeld (1995) wrote radical constructivism "places the responsibility for actions and thoughts where it belongs: on the individual thinker" (p. 19). Furthermore, a person's cognition in the radical constructivist account is "an instrument of adaptation, as a tool for fitting ourselves into the world of our experience" (von Glasersfeld, 1995, p. 14). In radical constructivism, humans are autonomous individuals who construct viable models to biologically adapt to the world of their experiences for the sake of survival. The word autonomous is from the Greek words *auto,* meaning self, and *nomos*, meaning law or principles governing human conduct ("autonomous," n.d.). If individuals are fundamentally creatures that create principles to govern the self, then individuals' thoughts and their verbal claims cannot be dismissed as mere psychological content—as some type of mental activity that carries no weight. Individuals' thoughts and verbal claims are knowledge and have epistemological content; they are valid ways of adapting in the world.

Methodology and Conceptual Frameworks: Models and Schemes

Radical constructivist theorists appropriated "Piaget's general constructivist orientation to the process of development" and "have necessarily had to adapt his theoretical constructs because their concern is to gain insight into the process of students' mathematical learning rather than to address problems of genetic epistemology" (Cobb, 2007, p. 20). These comments by Cobb implicitly suggest how the tradition's philosophical engagement has waned. Radical constructivism's rejection of traditional philosophical categories and pursuits (i.e., metaphysics and ontology) contributed to an end of the Kant-Piaget-von Glasersfeld philosophical lineage. Noddings (1990) identified this trend and said, "I intend no criticism in these remarks about cognitive science advances, but I want to emphasize the psychological and pedagogical aspects of these advances; they are not epistemological" (p. 10.) Rather than engage in contemporary philosophical dialogues, radical constructivists accept "a [Piagetian] psychological foundation for epistemology" and subsequently build "models of students' mathematics" (Norton & Wilkins, 2012, p. 557).

Models

von Glasersfeld and Steffe (1991) suggested mathematics teachers and mathematics education researchers adopt the methods of scientists. Hence radical constructivists, to make sense of student thinking, build second-order models of student thinking. von Glasersfeld and Steffe (1991) chose the language of modeling precisely because of its scientific meaning; they described how a model of a student's thinking "simulates reality." The model is treated as if it were an accurate picture of the student's thinking "but has the actual function of making experimental results and other experiential elements compatible with the general assumptions that are inherent in the research program's core" (p. 95). The models play a specific role in a scientific world where experiments, simulations, and predictions are key (von Glasersfeld, 1995). Because they believe it is futile to pursue truth as a correct representation of reality, the models are constructed and treated as if they were an accurate picture of the real world.

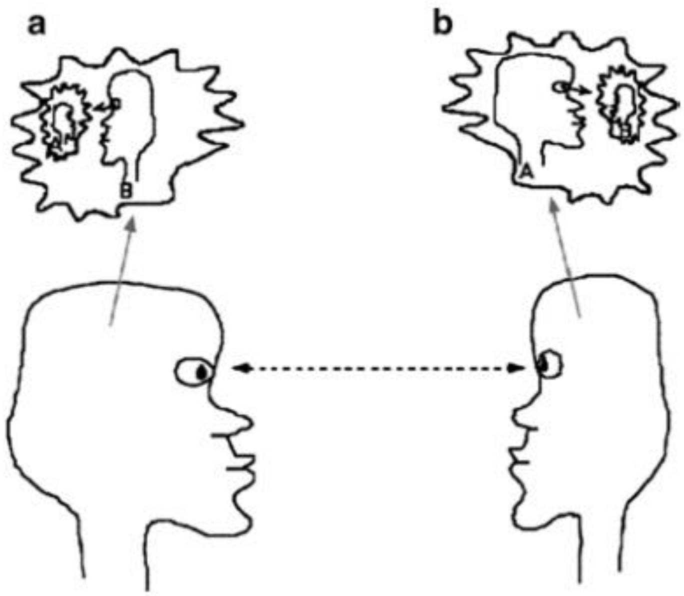
Radical constructivists differentiate between psychological and epistemological content but believe all individuals' first-order models (i.e., their thoughts and subsequently any verbalized contributions in a classroom) have epistemological content (Confrey, 1991). By suggesting students' thinking has epistemological content and not just psychological content, constructivism always gives validity to students' methods and names it as genuine knowledge (Confrey, 1991). Furthermore, radical constructivists believe conceptions belong to students and that it is inappropriate to label a student's thinking as a misconception. Confrey (1991) explained in detail the internal debate among radical constructivists on how to label or discuss a student's model (i.e., how to construct a second-order model). Radical constructivists decided to never describe a student's thinking as a misconception because the label does not take into "consideration the perspective of the student, for whom the belief may explain all instances under consideration and fail only in cases to which s/he is not privy" (Confrey, 1991, p. 121). They also considered labeling some students' models "alternative conceptions" but recognized the language of alternative conceptions would implicitly give them a lower status in comparison to "normal conceptions." Instead, Confrey described how radical constructivists decided to simply describe students' models as conceptions: "For the constructivist, this difficulty can be resolved by using the term 'conception' while always declaring a frame of reference (observer, expert, or participant) and indicating whether it seems adequate from that person's perspective" (Confrey, 1991, p. 121). For example, Moore et al. (2019), in their discussion of pre-service teachers' (PSTs) potentially contradicting actions and claims related to the meanings of graphs, are clear to name the perspective from which the PSTs' actions and claims appear contradictory. They describe how a PST's claims may appear contradictory to the researchers but are not contradictory from the PST's perspective. Like Confrey, Moore et al. (2019) deconstructed notions of contradiction by discussing how PSTs' conceptions are viable to the PSTs.

Scheme, Assimilation, and Accommodation

Radical constructivists build second-order models of students' meanings based on Piaget's concepts of scheme, assimilation, and accommodation (Moore et al., 2016). I follow Thompson et al.'s (2014) definition of scheme; they surveyed Piaget's six different uses of the concept and suggested a scheme is "an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity" (pp. 9–10). The definition is dense and worth unpacking. First, schemes are a theoretical construct used to model a subject's knowing based on observations of the subject's behavior (Thompson, 2008; Thompson et al., 2014). It is important to note they are *mental* structures, or organizations of *mental* activity. The focus on inner mental activity is, as previously noted, the defining feature of the radical constructivist tradition and is artistically illustrated by Thompson's (2013) drawing that summarized his radical constructivist perspective on the intersubjective operations involved in the communication of meaning (Figure 4).

Figure 4

Summary of the Intersubjective Operations Involved in the Communication of Meaning



Note. Reprinted from "In the absence of meaning…," by Thompson, 2013, in *Vital directions for mathematics education research*, p. 64. Copyright 2013 by Springer. Reprinted with permission.

Second, the components organized within a scheme—actions, operations, and images—need to be unpacked as well. Actions are not just observable behaviors. Instead, Thompson et al. (2014), quoting Piaget, explained an action is "all movement, all thought, or all emotions that respond to a need (Piaget, 1968, p. 6)" (p. 9). Operations are mental activities or doings; some specific mental activities—such as partitioning, splitting, and iterating—have been written about extensively in mathematics education research (Norton & Wilkins, 2012). Images are related to memory; they are developed “recollections of ‘momentary states’ in having reasoned” (Thompson et al., 2014, p. 10).

In the final part of their definition of scheme, Thompson et al. (2014) described how schemes are related to expectations; they are “anticipations of outcomes.” For example, a student may partition a unit into three parts (a mental operation) and then partition each of those parts into five parts (another mental operation). They may then *anticipate* a total of fifteen one-fifteenth parts (Norton & Wilkins, 2012). This anticipation aspect of a scheme is highly related to radical constructivists’ epistemology; von Glasersfeld wrote: “If [another person] does what we predict, we may say that the piece of knowledge was found to be viable not only in our own sphere of actions but also in that of the other” (as cited in Steffe & Thompson, 2000a, p. 120). In radical constructivism, knowledge is viable because what was experienced cohered with what was anticipated (Thompson, 2008).

Assimilation and accommodation are two other prominent concepts in radical constructivism’s theory of meaning and epistemology; both are types of cognitive equilibration made in the cognitive realm by an individual (von Glasersfeld, 1974). Assimilation is *giving meaning* and is focused on the inferences made when something is mentally attended to. Piaget wrote, “Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complete system of inferences, even when it is simply a question of verifying a fact” (as cited in Thompson et al., 2014). Thus, assimilating involves making new inferences to conceptual structures that a person already has (von Glasersfeld, 1974).

Accommodation, in contrast, involves the construction of novel conceptual structures, or fuller accommodations to a scheme; it occurs when an individual recognizes the idiosyncratic nature of their use of a word (von Glasersfeld, 1974; Thompson et al., 2014; Schmidt, 2002). von Glasersfeld (1974), summarizing the distinction between the two concepts, wrote, “In other words, assimilation and accommodation are operative on every level of cognitive activity; what differentiates the two is the relative novelty of the constructs to which they give rise” (p. 9).

In order to build second order models of students’ mathematical schemes, radical constructivist researchers perform clinical interviews and teaching experiments (Steffe & Thompson, 2000b). Clinical interviews are used to understand a student’s current mathematical understanding; whereas teaching experiments are used to progress student’s understanding over multiple teaching episodes. More specifically, teaching experiments are a sequence of teaching episodes where a teacher (or researcher) scientifically investigates one or more students’ mathematics. As researchers meet with a student, they develop and refine hypotheses (i.e., second-order models) about the student’s mathematical understandings and create tasks that may perturb the student’s relevant schemes. The perturbations may subsequently result in the student giving new meanings (assimilating to their scheme) or constructing new conceptual structures (accommodating their scheme). Ultimately, both clinical interviews and teaching experiments provide opportunities for researchers to perform retrospective conceptual analysis of a student’s mathematical meanings.

Relevant Empirical Findings

I now present a review of cognitive and radical constructivist empirical findings related to prospective and in-service teachers’ understanding of functions, inverses, and covarying quantities. The review provides a backdrop for the empirical investigations in the subsequent chapters. Although multiple mathematical topics are featured in the extant data, I have chosen to focus on three: functions, inverses, and covarying quantities. I chose these three concepts because of their mathematical importance and role in the secondary mathematics curriculum. I primarily focus on functions and inverses but also discuss covariational reasoning because of its close connection to the topic of function.

Pat Thompson and Marilyn Carlson are prominent radical constructivist researchers and have explained why quantities, variation, and covariation are foundational to the concept of function (Thompson & Carlson, 2017). People construct quantities through a process called quantification, which Thompson (2011) described as “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute’s measure entails a proportional relationship (linear, bi-linear, multi-linear) with its unit” (p. 37). Quantitative reasoning, then, is how people conceive of quantities and the relationships between quantities. Covariational reasoning is “the cognitive activities involved in coordinating two varying quantities while attending to the way they change in relation to each other” (Carlson et al., 2002, p. 354). Carlson et al. (2002) created a covariational reasoning framework that Thompson and Carlson (2017) later updated and refined. The updated framework listed levels of covariational reasoning that ranged from “no coordination” in which someone has “no image of variables varying together” to “smooth continuous covariation” in which someone “envisions increases or decreases (hereafter, changes) in one quantity’s or variable’s value (hereafter, variable) as happening simultaneously with changes in another variable’s value” (Thompson & Carlson, 2017, p. 441). Thompson and Carlson (2017) explained that continuous covariational reasoning (the highest level in their framework) was crucial to mathematician’s invention of various concepts that eventually led to the modern understanding of the concept of function. Furthermore, they argued a robust understanding of continuous covariation is necessary for a robust understanding of functions because functions are a specific kind of relationship between two sets (or quantities).

The concept of function, according to Freudenthal (as cited in Even, 1990), has two essential features: arbitrariness and univalence. Freudenthal claimed the arbitrariness of a function refers to two things. First, functions—as a relationship between two sets—do not have to be described by a specific expression or a graph with a particular shape. Second, the sets on which the function is defined are arbitrary: they do not have to be sets of numbers. The other essential feature of a function, which is often more explicitly featured in textbook definitions, is the univalence requirement. The univalence requirement states that for each element in the domain, there exists only one image in the range (Even, 1990).

Even (1990) claimed mathematical concepts like function are successful and powerful when they are unique and open new opportunities. Citing Freudenthal, Even explained how the ability to compose and invert functions created new mathematical possibilities and, subsequently, made the function a powerful mathematical concept. Composing and inverting functions helped create new functions and contributed to the study of differentials and integrals. The mathematical concept of inverse function thus carries similar significance because the power of the concept of function partially rests in the ability to invert a function. The formal definition of the inverse function f^{-1} of a function f is the function for which $f(f^{-1}(x)) = x$ for any x.

There is an extensive history of cognitive and radical constructivist researchers studying PSTs' understanding of functions and inverses starting in the early 1990s. Vidakovic (1993) presented a preliminary genetic decomposition for inverse function and then, after refining it through the analysis of her data, described a function schema and an inverse function schema. Her function schema consisted of a web of related concepts—including domain, range, rule of transformation, and graphs—that needed to be constructed to enhance the meaning of the concept of function. Her inverse function schema consisted of three parts. First, the inverse function is the reverse of the function process. Second, the inverse function is the coordination of two function processes that result in the identity. And third, the inverse function is the action of switching *x* and *y* and solving for *y*.

Even (1990, 1992) explained the importance of the concepts of function and inverse for PSTs and studied PSTs' understanding of the concepts. She specifically noted (a) the PSTs in her study displayed a lack of connected meanings for function across contexts, (b) the PSTs' informal meaning for inverse functions as "undoing" was insufficient, (c) the PSTs needed better connections between procedural and conceptual knowledge of inverse functions, and (d) that "understanding of the concept of function must, therefore, include an understanding of the composition of functions and the inverse function" (1990, p. 535).

Since Even and Vidakovic's foundational work, there have been numerous studies on functions and inverses. These studies often stress the disconnected meanings students or PSTs have for the two concepts across different contexts. Thompson (2013) described how most secondary students in the United States have not internalized the interconnected system of meaning within conventional function notation. He gave recommendations to teachers on how to meaningfully engage students with the different components within function notation. Lucus (2005), in his study on PSTs and in-service teachers' understanding of functions, the composition of functions, and inverses, described how the PSTs and in-service teachers in his study often had compartmentalized knowledge of inverses and their understanding was sometimes limited to a switching procedure for finding an inverse. Similarly, Carlson et al. (2015) described how many pre-calculus and calculus students' meanings for inverse were insufficient to solve novel problems. They argued students who conceptualize functions as a process mapping input values from the domain to output values in the range are better equipped to understand and use function compositions and function inverses. Finally, Paoletti et al. (2018) performed and analyzed the clinical interviews that I used as extant data. Their goal was to address the research gap related to PSTs' meanings for inverse function and to better understand how the PSTs' techniques to solve tasks "indicate they maintain connected systems of meanings for 'function inverse' across several tasks" (p. 95). They said most of their interviewees exhibited inconsistent techniques across tasks, indicating disconnected meanings for inverse function across contexts. They also described the complexities of the meaning of inverse function PSTs should understand and suggested new notation for inverse functions.

Sociocultural Perspective

To begin my discussion of the sociocultural perspective, I must briefly describe the reasoning behind my choice of words. The sociocultural perspective and the situated perspective are similar, but not the same. In their forward to Lave and Wenger's *Situated Learning: Legitimate Peripheral Participation,* Roy Pea and John Seely Brown wrote, "The situated nature of learning, remembering, and understanding is a central fact" (1991, p. 11). They further explained how the dominant cognitive theories prominent in education at the time did not pay sufficient attention to how "human minds develop in social situations, and that they use the tools and representational media that culture provides to support, extend, and reorganize mental functioning" (p. 11). Lave and Wenger's work (Lave & Wenger, 1991; Lave, 1988; Wenger, 1999) was influential for the social turn in mathematics education research and, per the forward of their book quoted above, they saw learning as *situated* within *social* and *cultural* situations.

From a mile-high perspective, it seems the language of “situated perspective” and “sociocultural perspective” are interchangeable. However, different uses of the terms emerge within different spheres of mathematics education research. For my purposes, I will broadly use the phrase “the sociocultural perspective” because it is the language used most frequently with the perspective I plan to compare with inferentialism (Documenting Collective Activity, or DCA). Research that uses the language of *situated* or *situative* is included in a discussion of the social turn of mathematics education research but will not be centered.

Philosophical Background

Vygotsky and Leont’ev

Cobb (2007) described how much of sociocultural theorizing draws on the work of Vygotsky and Leont’ev; both had significant insights that sociocultural researchers have used to guide their work. First, Vygotsky followed Marx in suggesting “it is the making and use of tools that serves to differentiate humans from other animal species” (Cobb, 2007, p. 22). Thus, Vygotsky believed “human history is the history of artifacts such as language, counting systems, and writing that are not invented anew by each generation but are instead passed on and constitute the intellectual bequest of one generation to the next” (Cobb, 2007, p. 22).

Cobb claimed Leont’ev saw intellectual development as “synonymous with the process by which the child becomes a full participant in particular cultural practices” (2007, p. 23). This description of learning fluidly connects to the well-known work of Lave and Wenger (1991) on situated learning. While they did not explicitly cite Leont’ev, their suggestion “that learning is an integral and inseparable aspect of social practice” (Lave & Wenger, 1991, p. 31) paralleled Leont’ev’s work. Lave and Wenger’s work also emphasized that anthropology is at the heart of the situated perspective on learning. Lave is a social anthropologist and has described her work as a correction to previous psychologies of education (Lave, 1988).

Wittgenstein

In addition to Vygotsky and Leont'ev, Lerman (2001) claimed Wittgenstein's work in *Philosophical Investigations* (1953) was crucial in prompting (a) the "move to a cultural, discursive psychology" (Lerman, 2001, p. 90) and (b) the strong social turn in mathematics education research (Lerman, 2006). Additionally, Lerman saw Wittgenstein's work as key to understanding the difference between cognitive constructivism and social theories of learning (Lerman, 2000). Lerman, borrowing Wittgenstein's term language-games, wrote: "A discursive, cultural psychology locates its interpretation of the individual at the intersection of overlapping language games in which the person has developed and thus is necessarily rooted in the study of cultures and histories" (Lerman, 1998, p. 67). By discursive, Lerman referred to varied modes of discourse, which is why he stressed the "intersection of overlapping language games."

For background, Wittgenstein proposed the concept of language-games in contrast to referential theories of language. The concept was meant to emphasize that language is not independent from context nor is it referencing some outer reality. Wittgenstein used the metaphor of games and how games can take a variety of forms and have all different sorts of rules to illustrate how language is similar to games. However, Wittgenstein did not want games to just be a metaphor for language; he referred to any linguistic activity as a language-game. By referring to language as language-games, Wittgenstein "meant to bring into prominence the fact that the speaking of language is part of an activity, or of a form of life" (1953/1958, §23). As a result of being a part of a form of life, language-games have linguistic norms (or rules) that are often implicit, flexible, and varied (Hallett, 1977). The norms are "not a product of ratiocination" nor are they “read off from reality” (Hallett, 1977, p. 69). Instead, they are learned through participation in a community of practice that maintains a certain form of life. The meaning of words, from this perspective, is thus largely found in their use in humans’ embodied activities and social practices. Hence Lerman (2001) argued: “Searching for evidence of, or ways to bring about, mathematical ‘understanding’ as a decontextualised mental process might best be abandoned” (p. 98).

Philosophical Anthropology

A renewed sense of the communal, contextual, historical, and cultural nature of humanity helped birth the sociocultural perspective. Surveying the field around him, Vygotsky (1962) argued psychology needed to be posed in different terms: language must take center stage alongside thought and psychology should be recast as social psychology. Furthermore, following Marx, Vygotsky argued the social and individual natures of humans were inseparable (Planas & Valero, 2016). Learning subsequently began to be put in terms of communities of practice, legitimate peripheral participation, and apprenticeship because the social and cultural nature of humanity was elevated. Lave (1988) in *Cognition in Practice: Mind, Mathematics and Culture in Everyday Life*, summarized the importance of anthropology for the movement when she wrote, “The project is a ‘social anthropology of cognition’ rather than a ‘psychology’ because there is reason to suspect that what we call cognition is in fact a complex social phenomenon” (p. 1).

Sociocultural Areas of Research, Conceptual Framework, and Methodology

Planas and Valero (2016) have identified PME 29 in 2005 as a turning point for research in mathematics education. At the General Assembly of the conference, “a proposal was approved to remove from the PME constitution the preference to consult psychology as the fundamental field of scholarship for the PME community” (Planas & Valero, 2016, p. 452). Since then, the sociocultural perspective has been used for a wide variety of purposes. One prominent use of the sociocultural perspective in mathematics education research is the study of mathematics teacher education. Peressini et al. (2004) described how the *situative* perspective (a term they use to describe a set of theories with Lave and Wenger's work at the forefront) helped them "better understand the ways in which the various contexts of teacher education and early career teaching made a difference in the professional development of the mathematics teachers in our project" (p. 90). Furthermore, the situative perspective's emphasis on norms and contexts allowed Peressini et al. (2004) to make sense of the varied knowledges and practices that novice teachers learn; they wrote, "the situative perspective helped to bring the numerous and varied contexts of teacher education into focus and supported our characterization of teacher learning as a single trajectory through these multiple contexts" (p. 90).

The sociocultural perspective is also regularly used to study mathematical practices in the classroom. Although there is some research that solely uses the sociocultural perspective to analyze classroom mathematical practices, the topic of mathematical practices has been extensively explored via the emergent perspective (Cobb & Yackel, 1996). The emergent perspective attempts to coordinate radical constructivism and the sociocultural perspective, but adherents to the emergent perspective often view classroom practices with a sociocultural framework. As Cobb (2007) wrote, "sociocultural theorists usually frame people's reasoning as acts of participation in relatively broad systems of cultural practices" (p. 25). Thus, researchers like Rasmussen (Rasmussen et al., 2015) have used the sociocultural perspective to answer questions like, "What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?" (p. 262) as they explored how a classroom community participated in the practices of the larger mathematical community.

Collective Argumentation

Collective argumentation has been an object of sociocultural mathematics education research for approximately 30 years (Duval, 1989) and is the topic within sociocultural research I explore in this dissertation. Much of the field's work on the concept of collective argumentation has been influenced by Krummheuer's landmark piece *The Ethnography of Argumentation*. Krummheuer (1995) surveyed a wide array of literature on the concepts of argument and argumentation and stressed argumentation's importance in mathematics education. He suggested argumentation was "the method or technique to (re)establish the challenged claim of an assertion" (p. 239) and that collective argumentation was argumentation accomplished by several persons. In Krummheuer's (2015) later work, he continued to stress the importance of collective argumentation for learning and even argued collective mathematical argumentation was a pre-condition of learning, not merely an outcome.

Toulmin's (1958/2003) work on argumentation is also frequently used in collective argumentation research. Toulmin, in *The Uses of Argument*, was not concerned with logical theory, but with logical practice. He was interested in the structure of everyday people's arguments rather than the work of logicians. Logic for Toulmin (1958/2003) was "the soundness of the claims we make—with the solidity of the grounds we produce to support them, the firmness of the backing we provide for them" (p. 7). Already evident in this definition of logic is Toulmin's method of structuring arguments. For Toulmin, arguments are comprised of several components including claims, data, warrants, and backings. Claims are statements whose validity is being established in the argument, data are evidence provided for the claims, warrants are the reasoning that connect data with claims, and backings are further reasoning given to justify a warrant.

From these theoretical roots, collective argumentation research has gone in numerous directions. Conner and Singletary (2021), describing their expanded Toulmin diagrams, described how “expanded Toulmin diagrams offer an analytical mechanism for researchers to investigate the ways the actors in the mathematics classroom (teachers and students) participate in the construction of the arguments” (p. 216). Both Wood (1999) and Staples and Newton (2016) argued collective argumentation is crucial for concept development. Rasmussen (Rasmussen et al., 2015; Stephan & Rasmussen, 2002; Yackel & Rasmussen, 2002) has heavily leaned on the emergent perspective and its accompanying frameworks in his research on collective argumentation. Rasmussen et al.’s (2015) interpretive framework uses classroom practices, norms, and Toulmin’s argumentation scheme to make sense of collective argumentation.

Documenting Collective Activity: A Methodology

Rasmussen and Stephan (2008) introduced a methodological approach, using Toulmin’s model of argumentation as an analytic tool, to document the collective activity of a classroom community. Their approach was the methodology used to compare a sociocultural investigation of collective argumentation with an inferentialist investigation of collective argumentation. Rasmussen and Stephan’s approach involves three phases to document collective activity and three criteria to determine if an idea functions as-if-shared. They define collective activity as the normative ways of reasoning in the classroom community and use the phrase as-if-shared to emphasize how ideas function in the classroom. The ideas “function as if everyone shares this way of reasoning. However, there is individual variation within collective activity” (2008, p. 196). The first phase requires researchers to transcribe every whole-class discussion of the class periods under consideration and then create a Toulmin argumentation scheme for each claim made. The collection of Toulmin argumentation schemes is then used to create an argumentation log.

In the second phase, researchers look across the argumentation log and determine what mathematical ideas became part of the classroom community's normative way of reasoning. To do that, Rasmussen and Stephan created two criteria to determine when an idea functions as-if-shared and have more recently added a third. The criteria are: 1) "When the backings and/or warrants for particular claim initially are present but then drop off," 2) "When any of the four parts of an argument (the data, warrant, claim, or backing) shifts position within subsequent arguments," or 3) "When a particular idea is repeatedly used as either data or warrant for different claims across multiple days" (Cole et al., 2012, pp. 199–200). Researchers then create a chart detailing the confirmed mathematical ideas from the classroom and include researchers' other notes and ideas.

Finally, in the third phase of analysis, researchers take the confirmed mathematical ideas and relate them to common mathematical practices that the students were engaged in when the ideas emerged. These common mathematical practices are called *classroom mathematical practices* and represent the collective mathematical growth of the classroom community.

Relevant Empirical Findings

The DCA methodology has been used by researchers in mathematics and science education. Stephan and Rasmussen (2002) used DCA to analyze the classroom mathematical practices in a differential equations course. They performed a design experiment over the first 22 meetings of the course and identified six mathematical practices that emerged. Each classroom mathematical practice involved multiple mathematical ideas that functioned as-if-shared and the authors described how each idea began to function as-if-shared. The authors also identified two theoretical contributions of their study. Through their analysis, they learned that classroom mathematical practices could be non-sequential in relation to time and structure. Classroom mathematical practices can emerge in a non-linear and overlapping way and ideas that function as-if-shared can contribute to the development of multiple mathematical practices.

Rasmussen et al. (2004) coordinated an analysis of classroom mathematical practices with an analysis of student and teacher gesturing. The authors re-analyzed the same data from Stephan and Rasmussen's (2002) study with the intention to identify instances when gesturing no longer appeared as ideas began to function as-if-shared and to describe how particular gesturing shifted function in argumentation. The authors' theoretical orientation was unique because they did not focus on specific gestures but rather the activity of gesturing as an indivisible part of human activity and argumentation. Furthermore, they expanded on Cobb and Yackel's (1996) taken-as-shared construct to include gesturing activity. In their empirical analyses, they identified one instance of gesturing shifting function in argumentation: gesturing related to the shifting of the slope was initially constituted as data and later used as a warrant. The authors did not identify any instances in which gesturing no longer appeared as ideas began to function as-if-shared.

Stephan et al. (2003) used DCA as part of their attempt to coordinate the individual and social aspects of learning in a classroom teaching experiment on linear measurement. The authors described five mathematical practices that emerged in the classroom community: (a) measuring by covering distance, (b) partitioning distance with a collection of units, (c) measuring by accumulating distance, (d) measuring with a strip of 100, and (e) reasoning with a strip of 100. Within their description of the emergence of the five mathematical practices, the authors also analyzed the learning of two individual students. They described how individual students contributed to the emergence of the mathematical practices and how the same students' learning occurred as they participated in the practices. After describing their empirical results, the authors contrasted their study with purely cognitive analyses of individuals' learning in classrooms that are individualistic.

Rasmussen and Stephan (2008), in their full description of the methodology, used empirical examples from two different classroom teaching experiments to illustrate the two criteria within DCA to determine if an idea functioned as-if-shared in the respective classroom communities. Their first example was from a first-grade classroom and illustrated their first criterion. In the class, two students each made a claim about the length of an item based on two different methods of measuring and both students provided a warrant connecting their claim to their data. After a teacher-led discussion in which backings were provided in support of each argument, the class came to a shared conclusion. Subsequently, one of the warrants previously used to justify a method for measuring a distance became implicit: the warrant was no longer explicitly needed and, according to the DCA methodology, the mathematical idea about measuring functioned as-if-shared.

Rasmussen and Stephan's (2008) second example was from the previously described 15 week differential equations course and illustrated their second criterion. In the class, a student claimed slopes were invariant across time and connected his claim to relevant data with a warrant. Another student used the first student's claim as data to make a new claim and the classroom conversation shifted to assess the second student's new claim. Thus, the first student's claim shifted positions in a subsequent argument (from claim to data) and functioned as-if-shared.

Cole et al. (2012) provided empirical examples from an undergraduate chemistry course to illustrate DCA's third criterion to determine if an idea functioned as-if-shared. The authors constructed four Toulmin argumentation schemes in which students repeatedly used an idea (that the motion of particles in solids, liquids, and gases are relative) as either data or as a warrant to make claims about chemical properties. The idea was used repeatedly for different claims across multiple days and thus functioned as-if-shared.

More recently, Conner et al. (2022) used both DCA and extended Toulmin diagrams (ETDs) to analyze engagement and argumentation in a mathematics content course on fractals and chaos for mathematics education graduate students. Their empirical analyses focused on two parts of a whole class discussion that occurred after students worked in small groups on an activity related to Sierpiński's Triangle. They used DCA to identify five mathematical ideas that functioned as-if-shared in the two parts of the whole class discussion and used ETDs to highlight the instructor's role in supporting the argumentation. The authors also characterized how members of the classroom community (including the instructor and students) engaged with others' ideas and how their characterization of engagement related to argumentation. They created diagrams in which their codes for engagement were coordinated with their two methodological approaches (DCA and ETDs).

Conclusion

In this chapter, I provided an overview of the background philosophy, key theoretical concepts, methodology, methods, and a survey of relevant empirical results for inferentialism, radical constructivism, and the sociocultural perspective. The overviews are the first step in putting inferentialism in conversation with radical constructivism and the sociocultural perspective. In the next chapter, I network inferentialism and radical constructivism in a more focused way by comparing and contrasting them theoretically and empirically.

CHAPTER 5

INFERENTIALISM AND RADICAL CONSTRUCTIVISM

In this chapter, I (a) compare and contrast inferentialism with radical constructivism and (b) combine and coordinate inferentialism with radical constructivism. In the compare and contrast section, I describe the theoretical similarities and differences between inferentialism and radical constructivism. In the combine and coordinate section, I use each theory to analyze empirical phenomena and explicate the similarities and differences between the analyses.

Comparing and Contrasting: Inferentialism and Radical Constructivism

In this section, I compare and contrast the theoretical components of inferentialism and radical constructivism described in Chapter 4. My exploration of similarities and differences is part of my employment of Prediger et al.'s (2008b) Comparing and Contrasting strategy and responds to my first research question: *Using networking research practices, what is inferentialism's identity in relation to radical constructivism?* In the comparison section, I describe the three prominent similarities between the two perspectives: the attention to meaning, the rejection of referential accounts of language, and the ability to focus on mathematics content in research. In the contrasting section, I describe how the perspectives differ in four ways: their respective histories of research in mathematics education, their philosophical approach to ontology, their accounts of the individual and the social, and their epistemologies. By putting these theories in dialogue with one another and scrutinizing their often-implicit ideas, I clarify their identity (i.e., their philosophical and theoretical backgrounds and the way they have been developed to answer mathematics education research questions).

Comparing

Attention to Meaning

Radical constructivism and inferentialism have several similarities. To begin, both radical constructivist and inferentialist researchers recognize the importance of meaning and language. More specifically, both have rigorous frameworks for understanding concepts. The Piagetian concept of scheme has been powerfully adopted by radical constructivists like von Glasersfeld, Steffe, and Thompson for conceptual analysis and has been used to highlight the relationship between knowledge and meaning. In fact, von Glasersfeld's entire life and his philosophical and psychological work was heavily shaped by his multi-lingual childhood and his theoretical attempts to make sense of it. Inferentialist research has also focused on meaning and the relationship between thought and language. Brandom's (2000) most basic description of inferentialism focused on the meaning of linguistic expressions, the content of concepts, and the relation between meaning and reasoning.

Rejection of Referential Accounts of Language

Second, both radical constructivist and inferentialist researchers reject referential accounts of language and instead emphasize the importance of inference in their accounts of meaning. This similarity is partially due to the perspectives' shared pragmatist philosophical heritage. The identity of radical constructivism is almost entirely dependent on the rejection of language as referential and knowledge as an attempt to represent reality. For radical constructivists, reality is not mirrored; language does not directly refer to reality. Instead, models are constructed that simulate reality and learning is framed by an individual inferring new meaning to their environment (assimilating to and accommodating schemes). Similarly, inferentialists, following Wittgenstein and Richard Rorty (1979), reject referential accounts of language and the idea that language and ideas reflect reality. Instead, understanding a concept is equivalent to mastering its inferences within a social and semantic web of downstream and upstream inferences. This similarity has implications for their research methodologies: proponents of both perspectives believe mathematical tasks, within the context of clinical interviews and teaching experiments, should prompt inferences (Schindler et al., 2017; Moore et al., 2019).

Ability to Focus on Mathematical Content

Finally, both perspectives can be used to focus on mathematical content in research. Radical constructivist researchers, since the theory's inception, have had the ability to perform fine-grained analyses of students' learning of mathematical content. In contrast, a common feature of sociocultural traditions of research in mathematics education is the focus on practices and activity that are done for a general objective (Bakker et al., 2017). Some have suggested this focus on social and interactive issues poses problems for researchers who want to focus on mathematical content (Cobb, 2006; Lerman, 2006). Inferentialism, through its focus on the content of concepts, "meets the need to take seriously the social nature of learning in reform mathematics, without losing sight of the mathematical content" (p. 596, Nilsson, 2020). Both perspectives can provide fine-grained analyses on the teaching and learning of mathematical content.

Contrasting

In the following subsections, I detail several differences between radical constructivism and inferentialism. Some of the divergences between inferentialism and radical constructivism were anticipated by Lerman's (1996) challenge to radical constructivism and the subsequent back-and-forth between Lerman, Steffe, and Thompson about intersubjectivity (Lerman, 2000;

Steffe & Thompson, 2000a). As I argued in Chapter 1, Lerman's Wittgensteinian perspective was a forerunner to inferentialist research in mathematics education and, consequently, I use some of his theoretical analysis and precisely drawn distinctions between his perspective and radical constructivism to inform my contrast of inferentialism and radical constructivism.

History of Research in Mathematics Education

An obvious difference between radical constructivism and inferentialism is that radical constructivism has a much longer history of research in mathematics education. Inferentialist research in mathematics and statistics education only began in the last decade. Although Lerman's work—a forerunner to inferentialism—has been around for over 30 years, inferentialism, with its unique ability to analyze the teaching and learning of mathematical content while attending to social dynamics, is still in the developmental stages. Radical constructivism, on the other hand, has been an established theory in mathematics education for over 30 years. The sheer depth and breadth of radical constructivist research makes it a powerful theory to address a wide range of topics.

For example, cognitive and radical constructivist researchers have historically had the unique ability to focus on people's understanding of the concepts *function* and *inverse*. Nearly all researchers on the topics have performed a conceptual analysis that focused on the meaning of the concepts from a cognitive or radical constructivist perspective. The ubiquity of cognitive and radical constructivism in mathematics education research also hides the fact that these conceptual analyses of students, PSTs, and in-service teachers' understandings of function and inverse have an underlying philosophical and theoretical perspective on the meaning of concepts. In contrast, my attempt to explore PSTs' understanding of these concepts from an inferentialist perspective is novel.

Thus, if a researcher wants immediate insight into a student's understanding of specific mathematical content (e.g., quantities, covariational reasoning, etc.), then radical constructivism may be the best theoretical option. Radical constructivism already has tools available for use and ready-made responses to complex questions regarding individuals' understanding of mathematical concepts. The continued development of inferentialist research is, of course, still valuable. As Hußmann et al. (2019) said about developing inferentialism:

> It takes a long time for philosophy to become productive as a learning theory (see, e.g., the history of constructivism). Yet we believe such theory development needs to be done and can be productive, not just to better understand knowing and learning but also to design learning arrangements in a productive manner for the purposes of mathematics education. (p. 134)

Space for Ontology

Another difference between radical constructivism and inferentialism is the amount of space they leave for reflection on ontology. Radical constructivists have rejected ontology; von Glasersfeld did not offer a description of the "ontic constraints" he said existed. When pressed on the matter, von Glasersfeld said any attempt at ontology was either unnecessary or unproductive. He said his critics did not understand that radical constructivism was a theory of knowledge "intended as a tool that should be tested for its usefulness" and was not an ontological project (von Glasersfeld, 1995, p. 51). Although von Glasersfeld did not explicitly articulate an ontology, his phrase "testing for usefulness" displayed remnants of a scientific ontological account that was central in Piaget's work and attempted to describe everything that exists in scientific terms.

In contrast, Derry (2017) and Noorloos et al. (2017) emphasized the importance of considering the world and descriptions of reality from an inferentialist perspective. While Radford (2017) said inferentialist researchers in mathematics education need to clarify their ontological position, the importance of explicitly discussing ontology remains central to the inferentialist project. This major divergence on an important issue further supports Lerman's (1996) claim that radical constructivism and Wittgensteinian social perspectives are two different worldviews and irreconcilable. Radical constructivist researchers are focused on psychology and settled in the philosophical foundations laid by Piaget and von Glasersfeld. Inferentialist researchers are committed to further philosophical and psychological research.

The Social, the Individual, and the Location of Meaning

Next, radical constructivist researchers emphasize the mental activity of autonomous individuals while inferentialist researchers emphasize the lived, embodied give and take of social interactions. Radical constructivists are clear: models constructed within a subject's internal mental system are knowledge; they have epistemological content. Inferentialism is also clear: knowledge is the social status conferred upon an individual's mastery of a concept's use within the GoGAR. Similarly, in radical constructivism, meaning—the substance of language, practices, and thoughts—is solely constructed by the individual; it exists within their mental schemes. In inferentialism, the meaning of concepts exists within social contexts, practices and, more specifically, in the GoGAR where the concept's upstream and downstream inferences are socially calibrated.

The emphases of the two perspectives are different, but both perspectives take the individual and the social into consideration. Lerman (1996), contrasting his perspective with radical constructivism, summed it up well:

> Thus both views place the individual and the social life at the center of their theories, and it is unhelpful to claim that either view ignores or downplays the individual or the social life; the terms carry different significations, however. I have argued above and elsewhere (Lerman, 1992b, 1994b) that the difference is encapsulated in their identification of the source of meaning, the one identifying the cognizing individual and the other cultural and discursive practices. (p. 147)

Neither radical constructivists nor inferentialists ignore the individual or the social, but each perspective defines those terms differently. Radical constructivists identify the individual as the autonomous meaning-maker and the social world as part of the environment that individuals interact with and adapt to (Steffe & Thompson, 2000a). Inferentialists locate meaning within sociocultural and discursive practices—the socially structured context of human intersubjectivity and reason (Lerman, 1996, 2000; Taylor et al., 2017). So, the individual is not a meaning-maker, but a member of the GoGAR.

Radical constructivism, for its part, sees only two alternatives: their perspective that focuses on the autonomous cognizing individual *or* an absolutism of the platonic or positivist variety (Lerman, 1996). Thompson (2013) signaled this dichotomy in his description of any account of meaning that resides outside the individual:

> As we think about teaching and the conveyance of mathematical meaning, it will be productive to look for useful ways to imagine how "conveyance" happens. Is meaning on a printed page? Written on a whiteboard? Does it appear on a computer screen? Is meaning conveyed to students by directing their attention to "real world" referents? Each of these stances puts meaning in the world, so that there are "correct" meanings to be had and any meanings that depart from them are incorrect…I maintain that any stance that

> puts meaning outside of individuals is less helpful for purposes of instructional and curricular design, teacher preparation, and professional development than a stance that puts meaning within individuals. (pp. 61-62)

Thompson was adamant: he claimed any stance that puts meaning outside of the individual is less helpful for instruction, curriculum design, teacher preparation, and professional development. Lerman (1996) described how, to radical constructivists, “the notion of the mind as constituted in social and cultural experience is only seen as another form of absolutism where there is no mechanism for internalization” (p. 138).

In Thompson’s rejection of any stance that puts meaning outside the individual, his examples of meaning on a printed page, on a whiteboard, on a computer screen, and in real world referents seem like caricatures. They display deficient accounts of meaning that are entirely referential and static. Inferentialism and Lerman’s account of meaning, however, is not referential or static; it is inferential and dynamic. Thompson accepted the false dichotomy of (a) radical constructivism or (b) some form of absolutism or positivism. His examples indicate he believed any account of meaning outside the individual must be another attempt at absolutism or positivism.

Epistemology, Ethics, and Power Relations

Although Hackenberg (2005) has described the ethical motivation for teachers to decenter and reduce students’ feelings of depletion by theoretically combining radical constructivism with Noddings’ notion of caring relations, radical constructivism does not address how all knowledge rests on trust and is necessarily ethical (Ernest, 2012). In inferentialism, the meaning of a concept in the GoGAR is often dependent on reasons that are related to motives, emotions, or power relations (Hußmann et al., 2019; Bakker et al., 2017). For example, Siy (2019) recounted an exchange from a mathematics education conference:

> I watched a senior scholar press a graduate student to justify their use of the word *x*. The senior scholar resisted the graduate student's answer. They explained *x* means *y*. I read the scholars' work on *x* and the graduate student could not save face unless they said '*x* meant *y*.' (p. 19)

The meaning of *x* in the exchange was dependent on the senior scholar's insistence they were right. Siy, reflecting on the exchange, claimed researchers in the field needed to practice academic humility and resist arrogance. Inferentialism can account for the intertwined epistemological and ethical aspects of exchanges like this. Inferentialists specify that knowledge is socially conferred and that reasons given in the GoGAR are not necessarily discipline specific. The same concern for ethics prompted Lerman (1996) to use two empirical episodes that highlight the relation between the ethical and epistemological in his challenge to radical constructivism. It is also why he wrote, "Power relations and the particular ways in which people are positioned are carried in, and regulated through, discursive practices" (p. 143).

Combining and Coordinating: Inferentialism and Radical Constructivism

The differences between inferentialism and radical constructivism regarding the meaning of concepts are further illustrated in my empirical analyses of PSTs' understanding of the mathematical concepts of function and inverse. In this section, I detail the precise empirical differences between a radical constructivist analysis and an inferentialist analysis of two PSTs' understandings of function and inverse. I employ Prediger et al.'s (2008b) Combining and Coordinating strategy and report empirical analyses of two episodes from the clinical interview extant data described in my methods section in Chapter 2. These episodes were selected because the events that take place within them further clarify inferentialism's identity in relation to radical constructivism. My analyses of these episodes help answer my first overarching research question and two of my research sub-questions:

1. Using networking research practices, what is inferentialism's identity in relation to radical constructivism?
    a. Given radical constructivism's conception of meaning, how have individual students constructed mathematical meanings related to function and inverse within the clinical interviews?
    b. Given inferentialism's conception of meaning, how have individual students mastered mathematical meanings related to function and inverse within the clinical interviews?

My analyses of these two episodes illustrate how each theory practically makes sense of individuals' understanding of mathematical concepts. The inferentialist analyses specifically illustrate how inferentialist researchers (a) track mathematical meanings within the GoGAR, (b) might operationalize the mastery metaphor for learning, (c) locate meaning in the social space of giving and asking for reasons, (d) attend to incompatibilities, and (e) attend to social and power dynamics in the GoGAR. The radical constructivist analyses specifically illustrate how radical constructivist researchers (a) build second order models of students' mathematical meanings, (b) have a rich history of research in conceptual analysis that can be built on, (c) locate meaning within the mind of the individual, (d) attend to apparent contradictions, and (e) report on social and power dynamics given meaning and knowledge reside in the individual. The conclusions of my analyses further clarify the similarities and differences between inferentialism and radical constructivism that were described in the Comparing and Contrasting section.

As a reminder, clinical interviews were performed with nine of the PSTs from the content course described in my methodology section. The first clinical interview was performed at the beginning of the content course and the second clinical interview was performed near the end of the content course. The first episode I analyzed is an excerpt of the first clinical interview with a PST named Susan (all names are pseudonyms) and the second episode is an excerpt of the second clinical interview with a PST named Elliot.

I already described the radical constructivist methodology for performing and analyzing clinical interviews and briefly detailed my inferentialist methodology for performing and analyzing clinical interviews. Prior to describing the episodes and my analysis, however, I would like to describe the specific analytic methods I used to empirically analyze extant data from two different perspectives. After watching the clinical interviews, I chose three PSTs who were both interviewed and actively participated in the content course. I performed an initial analysis of the first and second clinical interview for each of the three PSTs—noting anything that stood out to me as interesting—and then transcribed the entirety of the six clinical interviews. After transcribing, I completed an inferential analysis of all six interviews, which involved multiple iterations. While analyzing these data, I regularly returned to important texts like Brandom's philosophical work (2000), various inferentialist mathematics education research articles, and Lerman's (1996, 2000) scholarly exchanges with Steffe and Thompson (2000a) to ensure my interpretation of the data and the texts were consistent.

Then, after completing the inferentialist analysis, I completed a radical constructivist analysis of all six interviews. During the radical constructivist analysis, I read and reread important and relevant texts from Steffe and Thompson (Thompson, 2008, 2013; Thompson et al., 2014; Steffe & Thompson, 2000a, 2000b). Multiple radical constructivist researchers from the original research group (Moore et al., 2019; Paoletti et al., 2018) have analyzed this extant data. I waited to read their analyses until after I completed my radical constructivist analysis. After completing my radical constructivist analysis, I discussed my analysis with an experienced radical constructivist researcher who was also a member of the original research group.

Episode 1: Susan and the Concept of Inverse

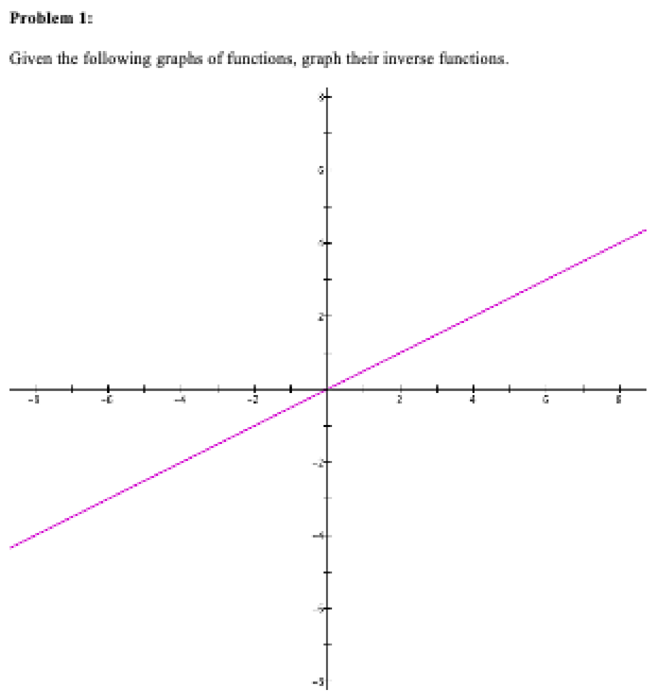
Context of Episode

The initial question the interviewer asked Susan was: "When you hear the word inverse, what's that make you think of? Everyday connotation or mathematics." Susan responded in two ways: first she described an algebraic procedure and, second, she related inverses to the mathematical concept of reciprocals. Susan's algebraic procedure was as follows: when given an equation in y-intercept form, switch the *x* and *y* variables and then solve for *y* again. She then gave an example of what she called a reciprocal ($\frac{1}{2}$ and -2) and said that sometimes she got confused between reciprocals and the procedure to find an inverse.

After this initial conversation, the interviewer provided Susan with the first part of the first problem (Figure 5); the prompt of the problem said, "Given the following graphs of functions, graph their inverse functions." The first problem consisted of four parts, each with the same prompt. In each part, Susan was shown a different graph of a function in order to graph the inverse function. Via the prompt, the interviewer implicitly suggested that an inverse is (or can be) a function and that an inverse function can be graphed on the same coordinate grid as the original function. (I have identified this second implicit claim—that an inverse function can be graphed on the same coordinate grid as the original function—because the formatting of the task infers it. The provided prompt and graph of a function extends beyond two-thirds of the paper's length with approximately one-inch margins around the edges. The remaining space does not include a blank Cartesian coordinate grid nor sufficient space to draw a reasonably sized coordinate grid to graph an inverse function.) Neither of these implicit suggestions fazed Susan and she seemingly accepted them without hesitation.

Figure 5

Part 1 of Problem 1

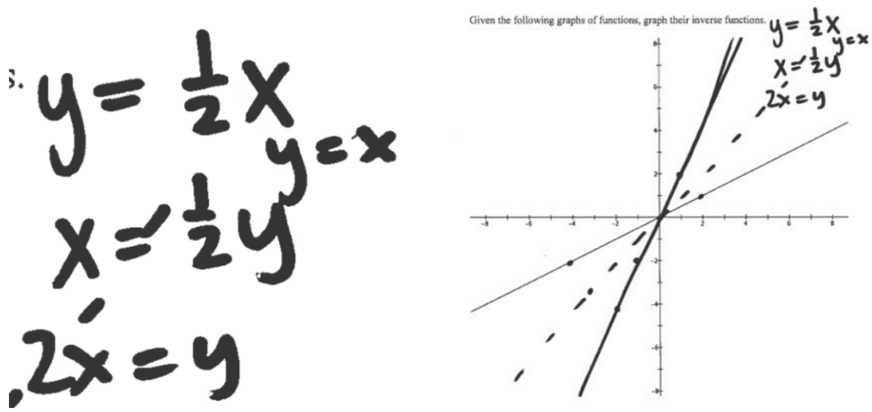


To complete the first two parts of Problem 1, Susan derived an equation of the provided graphs, followed her previously described algebraic procedure, and graphed her inverse functions. The problem did not provide an equation for the lines or curves; it only provided a graph of a function on horizontal and vertical axes. Susan's procedural work and her inverse graph for the first part of the problem is shown in Figure 6. In addition to following her procedure and producing graphs for the first two parts of problem 1, Susan noticed that both the original graph and the graph of her inverse function were equidistant from the line $y = x$. Based on the evidence, Susan claimed that an inverse was also a reflection of a graph over the line $y =$

x. Susan also made preliminary inferences about the isometric relationship between the graph of a function and the graph of its inverse (Susan said the graph of the inverse "is gonna, like this [the graph of the function] basically but it's gonna be, well, this way (Susan rotated graph 90 degrees counterclockwise).")

Figure 6

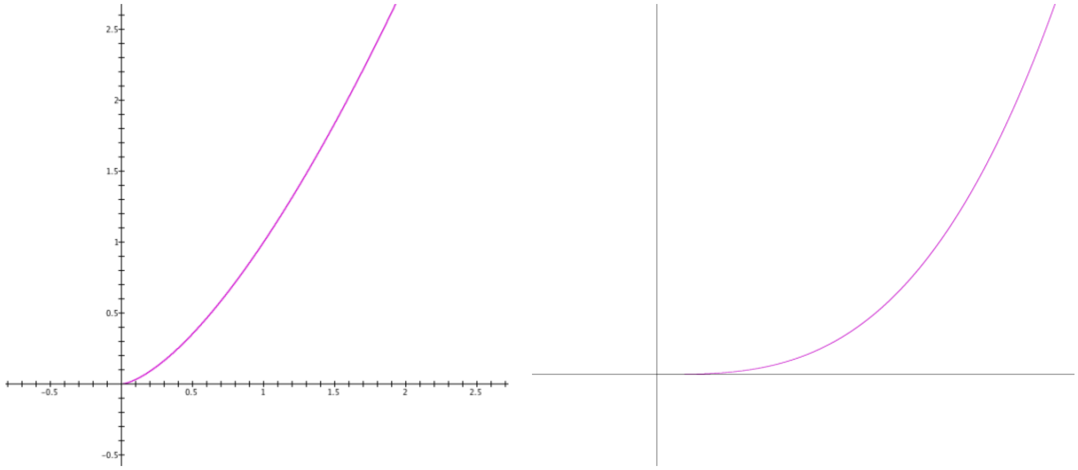
Susan's Procedural Work and Graph of the Inverse Function, Part 1 of Problem 1



To complete the third part of the problem, Susan attempted to follow the same steps she used to complete the first two parts of the problem. The third part of the problem, shown on the right in Figure 7, consisted of a vertical axis and a horizontal axis with a curve that began at the origin and slowly ascended toward the upper right corner of quadrant I.

Figure 7

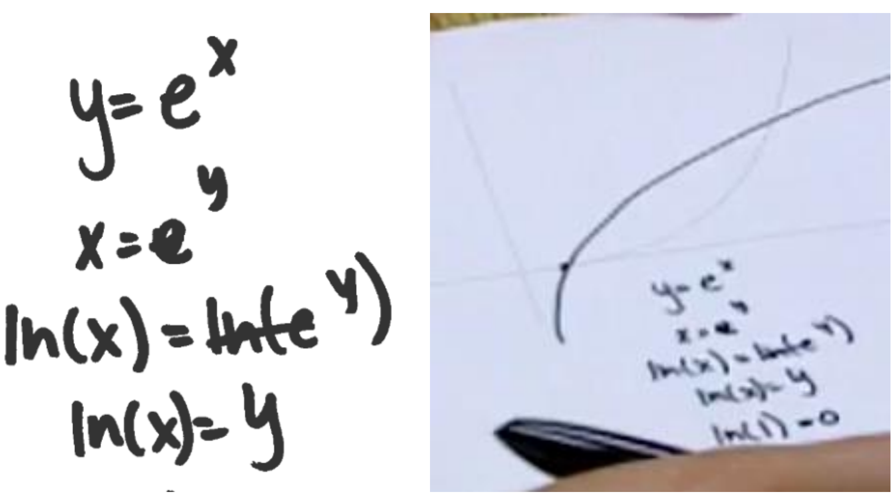
Parts 2 and 3 of Problem 1 (left and right, respectively)



Susan attempted to derive an equation based on the provided graph, followed her previously described algebraic procedure, and attempted to graph her inverse function. However, Susan admitted that she was unable to graph her inverse function, ln(x) = y, without a calculator. She said the graph she drew was based on her memory of a natural log function. Figure 8 shows Susan's procedural work and the graph of the inverse function she drew.

Figure 8

Susan's Procedural Work and Graph of the Inverse Function, Part 3 of Problem 1



The interviewer then asked a question about Susan's graph and her previous claim about reflecting the original graph over the line $y = x$. The transcript of the ensuing dialogue is below followed by my inferentialist and radical constructivist analyses of the transcribed episode.

11:47 Interviewer: Now relative to this one, you mentioned, you know there might be a reflection over y = x line. In this case, do we have something like that or?

12:05 Susan: Uhh, well if you draw the y = x line, it's going to be something like this. [Draws y = x line as dashed] So in this case, the answer is no. But that could be because I drew it wrong. It might be more of like, more of, more like that, [draws a new graph of the natural log of x] but I'm not really entirely sure. I don't think that's correct [tapping on the newly drawn graph].

12:07 Interviewer: Okay.

12:20 Susan: So I don't know, maybe it doesn't apply to exponential functions 'cause those were both, well, this is kind of just me grabbing things out of thin air here so.

12:22 Interviewer: No perfect, that's what I'm looking for.

12:23 Susan: It might not apply to exponential functions then, because they're kind of, they follow a different set of rules, whereas like x and x squared [Interviewer: Mhm.] and x cubed, probably they follow the same rules. 'Cause they're all, you're doing the same operation, not the same operation, but you're just increasing the exponent, whereas this is a whole other type of, or another whole type of graph. [Interviewer: Okay.] Because you're involving logs and L–N's of things so. [Interviewer: Okay, okay.] There's a lot.

12:56 Interviewer: So you said like there's probably different rules for it,

12:57 Susan: Yeah, there might be.

13:00 Interviewer: And then for, like the quadratic example and things like that.

13:05 Susan: Right, maybe it's reflected over a different line. Or maybe you find the inverse differently and I totally did it wrong. I don't know exactly.

13:08 Interviewer: Okay, cool, that makes sense.

Inferentialist Analysis

Description of GoGAR. Prior to addressing the transcript, it is important to recognize that the sheet of paper, Part 3 of Problem 1, was part of the interviewer's participation in the GoGAR. It was the third in a series of sub-tasks that prompted Susan to graph the inverse function of the provided graph of a function. Also, from the beginning of the interview, the interviewer asked Susan to "think out loud" and stated the more she thought out loud, the less questions he would ask her. When the interviewer gave the first sheet of paper to Susan, it was an initial move in the GoGAR: it claimed that the line drawn (Figure 5) was a function and asked Susan to graph the inverse function and explain her thinking.

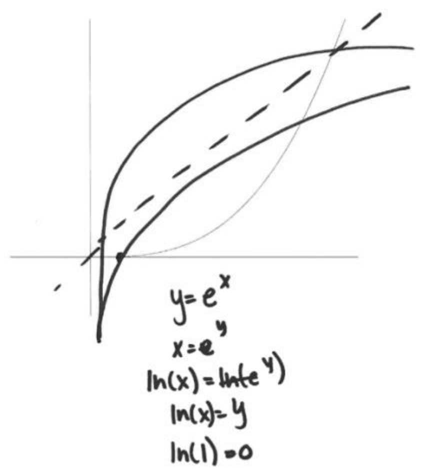
Part 3 of Problem 1 was also different from the previous parts in Problem 1. Parts 1 and 2 of Problem 1 (Figure 5 and 7) had scaled axes: units marked with tick marks and numbers that denoted specific values. Part 3 of Problem 1(Figure 7) did not have scaled axes; there were no units marked with tick marks or numbers. This was another implicit move in the GoGAR: it asked Susan to graph the inverse function in an ambiguous context. She did not know what quantities the axes represented and, subsequently, what the curve represented. The interviewer, through the task, did not explicitly ask Susan to solve the task using a specific method. However, the lack of information provided in the sub-task proved problematic for Susan's commitment to the process of (1) deriving an equation of the provided graphs, (2) following her algebraic procedure to find an equation of the inverse function, and (3) graphing her inverse function. In spite of the difficulties she faced graphing her inverse function based on her equation, Susan did not initially question her procedure as the ideal strategy to answer the problem.

After Susan drew a curve (a commitment about the inverse of the graphed function), the interviewer, in his question at 11:47 ("Now relative to this one you mentioned, you know there might be a reflection over $y = x$ line. In this case, do we have something like that or?"), asked Susan if she was still entitled to her previous commitment about inverses as reflections of graphs over the line $y = x$ (a question of entitlement). After drawing $y = x$ as a dashed line, Susan realized that her previous commitment was incompatible with the graph she drew (Figure 8). The graph of the inverse function she drew was not a reflection of the original graph. Her original graph for Part 3 of Problem 1 represented an incompatible inference.

Susan attempted to resolve the incompatibility and entertained the idea that she incorrectly drew the graph of the inverse. She drew a new graph of the graph of the natural log of *x* by reflecting it over the line *y* = *x* (a new commitment), but said, “I don’t think that’s correct.” Susan did not give any reasons why she thought her new graph (see Figure 9 below) was incorrect, but she already said that her first attempt to draw the inverse was based on her memory of the graph of the natural log function. We can potentially infer that she did not think this new graph was correct because it did not align with her memory of the graph of the natural log function (an upstream inference that served as justification for her originally drawn graph).

Figure 9

Susan’s Attempts to Graph the Inverse Function



Susan then attempted to resolve the incompatibility by everything downstream from the two incompatible inferences. First, she adjusted her previous commitment that an inverse was a reflection of a graph over the line *y* = *x*. She said this understanding of inverse may not apply to exponential functions (a commitment). She justified her new commitment by describing how exponential functions are different than linear, quadratic, and cubic functions (an upstream inference). Rather than reflecting over the line *y* = *x*, Susan said the inverse of an exponential function may be a reflection over a different line (a commitment). She also suggested that her procedure for finding inverses that she used from the beginning of the interview could be wrong (a potential adjustment to her original commitment). At the end of this clip, Susan acknowledged that she was not certain about any of her newly proposed commitments or her original commitment about finding an inverse.

Susan's Mastery of Conceptual Content. In the interview, Susan explored the concept of inverse and, in this specific episode, she navigated a pair of incompatible inferences. She previously claimed the graph of an inverse was a reflection of the original graph over the line $y = x$ and had justified that commitment. But the graph she produced of the inverse function for Part 3 of Problem 1 conflicted with her prior commitment. After the interviewer asked her about her previous commitment, Susan recognized the incompatibility and sought to address it.

Susan's recognition of the incompatibility and her ability to navigate the web of inferences indicate a high level of mastery. After recognizing the incompatibility between the two commitments, Susan attempted to adjust the first graph she drew in response to the prompt (on the right in Figure 8). After drawing a new graph that was reflected over the line $y = x$ (Figure 9) and considering its correctness, Susan said she did not think it was correct. Rather than giving up, Susan attempted to address the incompatibility from the other side by re-evaluating her other commitment. If her graph was correct, then perhaps her previous commitment about inverses being a reflection over the line $y = x$ needed to be adjusted. She suggested multiple ways in which that commitment could be adjusted that would still entitle her to both it and her graph. At the end of the clip, the unresolved incompatibility along with her commitment to her graph caused uncertainty about her original procedure to find an inverse. She was so committed to her original graph of *ln(x)* that she said: “maybe you find the inverse differently and I totally did it wrong.”

Although Susan did not arrive at a solid conclusion about the concept of inverse by the end of this clip, she was able to (a) recognize the incompatibility between her two commitments after the interviewer asked her a question of entitlement, (b) recognize a previous commitment that the incompatibility put in jeopardy, and (c) logically address the incompatibility with new commitments and justifications. She was able to trace, in multiple inferential directions, the effects of the incompatibility and how it called into question various commitments and inferences. She also fluidly made inferences across contexts. So, despite not coming to a clean conclusion by the end of the task, she recognized how all the mathematical ideas—her various commitments and inferences about the graph of an inverse function, inverses as reflections, her algebraic procedure to find an inverse, and different types of functions—were connected, and she demonstrated that she could navigate the game of giving and asking for reasons about inverses and their graphs. At the same time, the lack of finality in her commitments indicated she did not believe she was fully entitled to them.

It is difficult to further assess Susan’s mastery for two reasons. First, the problem began with the prompt, “Given the following graphs of functions, graph their inverse functions.” The graph in the third part of the problem, however, did not have defined units on either axis and consequently did not have defined quantities. The initial claim that the graph was a function was meaningless without defined quantities—it merely was a picture of a curve on perpendicular lines. In light of this, Susan could have questioned whether the interviewer was entitled to the claim that the graph represented a function (an incompatibility). Perhaps because of social norms (Susan may have expected the task to be solvable), the power imbalance between the interviewer and Susan (Susan may have felt unqualified to question a professional mathematics education researcher), or she simply did not recognize the incompatibility, Susan did not question the initial commitment and proceeded with the problem. The second difficulty in assessing Susan’s mastery of the concept of inverse function is due to a conflict between the underlying assumptions of the inferentialist mastery metaphor and the purpose of the clinical interview. I discuss this methodological difficulty in Chapter 7 when I reflect on the limitations of my inferentialist methodology.

Radical Constructivist Analysis

Leading up to the episode, Susan described her predominant meaning for inverse as switching variables and solving for the original "output." This was evidence for the core of Susan’s scheme for inverse: the procedure she described (an action) and some image (a developed recollection of reasoning in a certain way) of switching and solving in response to being given an equation and asked to find the inverse. Based on the first 12 minutes of the interview, it is uncertain what she anticipated from this activity.

Susan's work on each of the four graphs in Task 1 indicated something significant. Like Thompson's (2008) description of elementary students’ use of guess-and-check to assimilate to their current scheme, Susan likely assimilated her environment according to her current scheme for inverse. She was given graphs that were not accompanied with equations and, every time, she began by attempting to create an equation for each of the graphs so that she could complete her procedure. She continued to assimilate to her switch-and-solve activity despite constructing new meaning for inverse: she had already conjectured and confirmed that an inverse was, in a graphical context, the reflection of a graph over the line $y = x$ yet she did not use that meaning to solve subsequent parts of Problem 1. The reflection was likely an operation newly constructed in her scheme: she mentally envisioned how the graph reflected over the line and anticipated that inverses should always do so.

Susan's in-the-moment (Thompson et al., 2014) meaning for inverse in the context of a graphical representation indicates Susan's scheme for inverse was not solely the switch-and-solve procedure. She fluidly made new meaning for inverse after seeing how both lines in Part 1 of Problem 1 were equidistant from the line $y = x$. There were also indications that Susan was developing a new operation in her scheme for inverse: she saw how the rotation of the graph of a function was related to the graph of the inverse. Much later in the interview she indicated that a reflection was needed after the rotation of the graph of the inverse in order to match the graph of the original function. Again, this suggests that Susan's scheme for inverse, leading up to the transcribed episode, was not merely the switch-and-solve procedure.

In her attempt to make sense of Part 3 of Task 1 (Figure 7) Susan relied on the switch-and-solve procedure to find the equation of the inverse but she struggled to graph her equation of the inverse. Rather than attend to two quantities (which were not actually represented on the graph), she relied on her memory of a figurative meaning for the graph of the natural log of x that was dependent on its direction and shape (Moore et al., 2019b). After producing her graph of the inverse function, the interviewer asked Susan the question at the beginning of the transcript: "Now relative to this one you mentioned, you know there might be a reflection over y = x line. In this case, do we have something like that or?" Prior to this question from the interviewer, there was no evidence that Susan anticipated an issue between her graph as currently drawn and her previously espoused meaning for inverse within a graphical context (a reflection over the line $y = x$). After the interviewer asked the question, Susan investigated and anticipated that the reflection over the line $y = x$ would hold true. (Her anticipation was further evidence that a reflection over the line $y = x$ in a graphical context had become part of Susan's scheme for inverse.)

In Susan's investigation of the potential contradiction, she held to her initially drawn graph of the inverse. She claimed her original graph was the graph of the natural log because she remembered the shape and direction of the graph of the natural log. Susan's confidence in her figurative meaning for a different mathematical concept prompted her to accommodate her scheme for inverse. (For a more detailed account of graphical thinking that focuses on shape and direction—figurative thought—as compared to a focus on quantities, see Moore et al., 2019b.) She abandoned the idea that an inverse, within a graphical context, was always a reflection over the line $y = x$ and retained an adaptable connection between inverse and reflection. Susan's accommodation relates to Even's (1990) claim that understanding other mathematical concepts is an integral part of understanding the concept of function and, by extension, inverse. A more productive meaning for graphs and more specifically the graph of the natural log of x could have preserved Susan's developing and more productive meanings for inverse.

Susan had not yet explicated a strong, underlying connection between her switch-and-solve procedure for inverse in the algebraic context and her meaning for inverse in the graphical context (a reflection over the line $y = x$). However, her adjustment of her reflection-related meaning for inverse in the graphical context made any subsequent attempt to connect across contexts more difficult. At the end of the episode, her meaning for inverse was primarily *deconstructed*: Susan did not have clear expectations about her conjectured meanings for inverse in the graphical context and she began to question her meanings for inverse in an algebraic context as well. Her final comment is a succinct summary of her deconstruction: "Right, maybe it's reflected over a different line. Or maybe you find the inverse differently and I totally did it wrong. I don't know exactly."

Explicating Similarities and Differences Between the Analyses of Episode 1

In this subsection, I coordinate the empirical analyses of Susan's clinical interview episodes with the theoretical similarities and differences described in the Comparing and Contrasting section.

Similarities

To begin, both analyses focused on the meaning of Susan's thoughts, language, and actions. While each analysis conceptualized meaning differently and tracks Susan's inferences in different ways, both analyses focused on her conceptual understanding of the mathematical concepts of inverse and function. Neither perspective reduced Susan's conceptual understanding of inverse or function to a memorized definition or a rote procedure.

The inferentialist analysis of Susan's interview focused on her understanding of the concept of inverse in interaction with the interviewer. It detailed how Susan—in all her commitments and inferences related to the inverse function—justified her claims, drawn graphs, and written equations with the interviewer, and how she inferred logical consequences of her claims, graphs, and equations. The radical constructivist analysis of Susan's interview similarly focused on Susan's understanding of the concept of inverse within the clinical interview. In it, I began to create a model of Susan's techniques for solving the problem and how she constructed meaning for the concept of inverse across contexts (graphical, verbal, equation) through her reasoning.

Differences

In addition to the similarities, there are obvious differences between the two perspectives' analyses. First, radical constructivism has a longer history of research in mathematics education compared to inferentialism and both analyses of the episode illustrated this fact. In the radical constructivist analysis of the episode, there were several previous radical constructivist studies that shed light on Susan's meanings for inverse. Thompson (2008) offered a representative example of students assimilating a new environment to their current scheme that enriched my analysis of Susan's meanings for inverse within graphical tasks. Thompson et al. (2014) provided meaningful language (e.g., scheme, in-the-moment meaning, etc.) for my description of Susan's meaning for inverse within a particular context. Moore et al.'s (2019b) work on figurative and operative meanings for graphs helped me model Susan's understanding of graphs. And Even's (1990) claim that students' understanding of other mathematical concepts was important for their understanding of the concepts of function and inverse supported my analytic conclusions about Susan's work on Part 3 of Task 1. In contrast, the entirety of my inferentialist analysis of Susan's work was solely informed by theoretical inferentialist research and obliquely related empirical research. In fact, my inferentialist analyses are the first attempts to operationalize the mastery metaphor for learning that Taylor et al. (2017) described.

Second, the location of meaning in both analyses showed how meaning, from a radical constructivist perspective, exists within the mental schemes of the autonomous individual and, from an inferentialist perspective, exists within the GoGAR. The radical constructivist analysis of the episode focused on Susan's developing scheme for inverse whereas the inferentialist analysis focused on her conceptual mastery within the social give and take with the interviewer and the prompts provided by the interviewer. This difference in the locus of meaning is also related to the perspectives' different mathematical ontologies and epistemologies. From the radical constructivist perspective, Susan's meanings for inverse were legitimate because they were hers; any perceived contradiction within Susan's meanings must solely be a perception from an outside observer. Inferentialists, in contrast, identify the GoGAR as the location of meaning and, in my inferentialist analysis, I acknowledged that Susan made incompatible mathematical commitments: Susan could not be entitled to both her graph and her previous claim about inverses as a reflection over $y = x$. Susan's attempt to address the incompatibility between these two commitments showed she knew the incompatibility existed and that it needed to be addressed before proceeding with confidence.

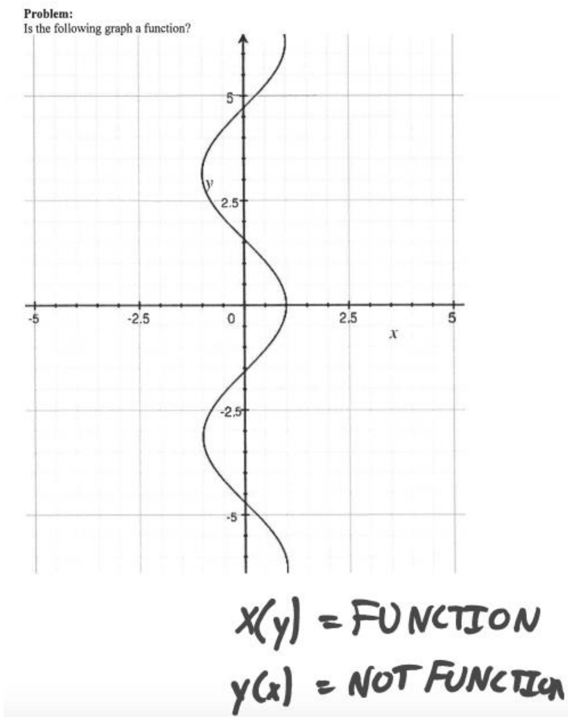
Episode 2: Elliot and the Concept of Function

Context of Episode

The episode with Elliot took place approximately 25 minutes into his second clinical interview after he had completed several tasks related to inverses, functions, and covarying quantities. Within the first 25 minutes of the interview, Elliot said that a function must have a unique output for each input and claimed that graphs may or may not be functions depending on how they are interpreted. For example, Figure 10 shows Elliot's interpretation of the functional status of a graph. Because a function must have a unique output for each input, Elliot said the graph in Figure 9 may or may not be a function. He claimed the graph would be a function if y is the input and x is the output, but the graph would not be a function if x is the input and y is the output. (Elliot previously described how the notation $x(y)$ means that y is the input and x is the output of the function.)

Figure 10

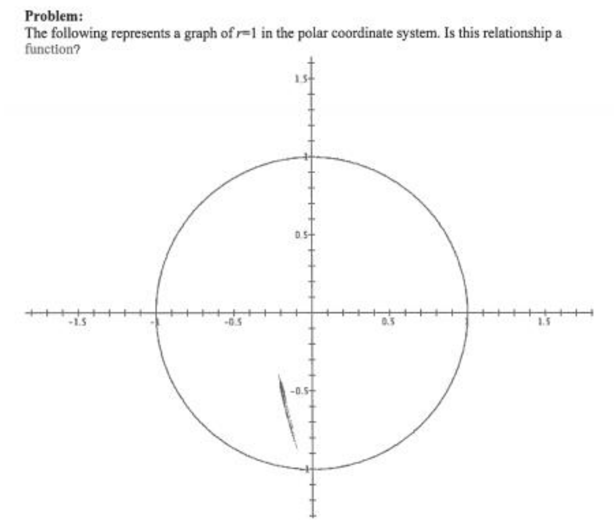
Elliot's Solution to Previous Problem



In the problems leading up to the transcribed excerpt below, Elliot had also encountered multiple non-normative mathematical representations including coordinate planes with (a) axes of different scales, (b) the vertical axis at x = -3, and (c) a horizontal axis that represented the positive real numbers to the right of the origin *and* to the left of the origin. At the end of the interview, Elliot referred to one of the non-normative problems as a "trick" that "tripped [him] up." In the transcribed episode below, Elliot encountered another atypical problem. Given a graph and a prompt (see Figure 11), he reasoned about the relationship between quantities and whether the relationship was a function. This problem occurred two problems after the problem featured in Figure 10.

Figure 11

The Problem and Elliot’s Work



24:49 Interviewer: What do you think about it?

24:57 Elliot: [inaudible, reading prompt]

25:17 Elliot: I would say it's not a function, it's, either way, you don't have a unique input, er or unique output for each input. And you know 'cause it fails the vertical line test that way [motions along the vertical axis]. And it fails it that way [motions along the horizontal axis].

25:31 Interviewer: OK.

25:32 Elliot: I would say I mean, a circle is not a function. I don't, maybe it changes when it's in the polar coordinate system. But I don't think so.

25:44 Interviewer: So it's not a function?

25:46 Elliot: I mean, it probably is [laughs], but I don't think it is.

25:47 Interviewer: So what I'm saying is, why do you think it probably is?

25:50 Elliot: Because y’all are giving me the sheet.

25:52 Interviewer: Just the fact that I'm giving you the sheet makes you think it might be a function?

25:56 Elliot: Right. The question is so easy for it, if it's just vertical line test just like that. That isn’t hard so...I feel like it is a function, but I don't know why.

26:06 Interviewer: OK. So you feel like it might be a function, but you're not sure why.

26:09 Elliot: Right. The only reason I would say that is because y'all gave me the sheet though.

26:13 Interviewer: Just because, so that's the only reason why you think, not even, the sheet you're saying it might be a function.

26:17 Elliot: Yeah.

Inferentialist Analysis

Description of GoGAR. To begin, the problem the interviewer gave to Elliot is a move within the GoGAR. The interviewer, via the task, made a commitment in the form of a written claim: the image (in Figure 11) represents a graph of $r = 1$ in the polar coordinate system. He then asked a question: is this relationship a function? Elliot responded to the question by claiming "it's not a function" (a commitment). He then justified his commitment by claiming there would not be a unique output for each input regardless of which axis was interpreted as the input and the output (an upstream inference). Elliot's upstream inference was supported by another upstream inference related to the vertical line test. Elliot, in a later task, described what his hand motions along the vertical and horizontal axes meant and how those actions connected to the vertical line test. He was performing a vertical line test and a horizontal line test by visually checking to see if any vertical line or any horizontal line crossed two points of the circle. He was checking to see if the graph could represent a function with either the vertical axis or the horizontal axis representing the input or the output (an upstream inference to support his previous upstream inference). Elliot provided another separate justification for his initial commitment by claiming that a circle was not a function (an upstream inference).

In contrast to his initial line of reasoning, Elliot also suggested the circle may be a function (a potential commitment) because "it's in the polar coordinate system" (an upstream inference to support the potential commitment). He then quickly dismissed that reasoning. He did not, however, dismiss the potential commitment; he said, "it probably is [a function], but I don't think it is." The interviewer followed up by asking a question of entitlement: "Why do you think it probably is [a function]?" Elliot responded: "because y'all are giving me the sheet" (an upstream inference to support the potential commitment). The interviewer's presentation of the problem within a series of problems with non-normative graphical representations within a series of interviews on the concept of inverses, functions, and covariational reasoning, gave Elliot sufficient reason to believe the relationship was a function. The interviewer questioned Elliot on this point (another question of entitlement) and Elliot provided further justification: the problem was too easy if his initial idea was correct and the graph was not a function (an upstream inference to support his upstream inference that supported his potential commitment). He "[felt] like it is a function" but he did not have any other explicit reasons to offer. At the end of the exchange, Elliot made very clear his reasoning: "The only reason I would say that [it might be a function] is because y'all gave me the sheet though."

Elliot's Mastery of Conceptual Content. In this episode, Elliot's mastery of the conceptual content of function is difficult to discern. He provided multiple reasons why a circle was not a function with an operable definition of function and interpretation of the circle in the graph. However, to more fully understand how Elliot used and reasoned about the concept of function in the GoGAR, it would have been necessary to hear Elliot further investigate his potential upstream inference about the polar coordinate system. The interviewer, however, did not press Elliot for further reasoning about the upstream inference and Elliot did not further explore how the polar coordinate system might have justified the relationship's status as a function. Because Elliot brought up the polar coordinate system as a potential justification for why the relationship was a function, he may not have been solely committed to the idea that functions, within graphical contexts, are determined by horizontal and vertical line tests.

Elliot did illustrate mastery of an implicit aspect of the conceptual content of function within this particular setting. Elliot's experiences with the researchers—in his first interview, in his time in the content course, and in the first 25 minutes of the second interview—cultivated another aspect of the meaning of function. Within these settings, the concept of function was associated with mathematical problems and activities that had unexpected solutions. Elliot unveiled the researchers' methodology and motives: the problems were in fact created to perturb interviewees' thinking. Elliot then used the interviewer's motives as reasoning for his commitment: the relationship defined by the equation $r = 1$ in the polar coordinate system was a function because, from Elliot's perspective, (a) "the question is so easy for it if it's just vertical line test just like that" and (b) "y'all gave me the sheet." The social context was part of how the concept of function was used in these settings and Elliot gave multiple legitimate reasons for why the relationship in the problem represented a function.

As Hußmann et al. (2019) said:

> To identify the discursive webs of reasons, one has to regard the norms of the game of giving and asking for reasons. Here, among others, the composition of the particular community, power relations, and linguistic regularities have an influence (see Schindler and Seidouvy, to appear 2018). Nevertheless, one has to be aware that this is only an analytic differentiation. In particular situations, individual and social practices are inseparably combined. (p. 142)

In this instance with Elliot, the setting, power relations, and the interviewer's motives were part of the norms of the GoGAR. Elliot considered a new commitment because there was a power imbalance: Elliot knew the interviewer had a more in-depth understanding of the concept of function and he knew that the researcher often gave tasks that revealed more intricate notions of mathematical concepts. He originated a mathematical idea because of the norms of the GoGAR.

Radical Constructivist Analysis

Elliot's scheme for function, as evidenced by the episode under analysis and his work in the interview leading up to the episode, revolved around a formal definition of function, the mental and physical action of performing a line test, his schemes for input and output, and an image of the functional status of a circle. To begin, much of Elliot's activity was tied to his definition of function: a function must have a unique output for each input. He smoothly made the connection from this formal and theoretical context to the graphical context. Figure 10 illustrates how Elliot used that definition within a graphical context: the graph's status as a function depended on which variable was the input and which was the output. Elliot claimed if x—represented on the horizontal axis—was the input and y was the output, then the graph would not represent a function. He also said if y—represented on the vertical axis—was the input and x was the output, then the graph would represent a function.

The episode under analysis sheds light on Elliot's mental activity. Elliot said, "I would say it's not a function, it's, either way, you don't have a unique input, er or unique output for each input." Elliot's immediate, in-the-moment reasoning seemed to be a mental visualization of a vertical and horizontal line test. He claimed the relationship did not have a unique output for each input because he visualized a series of vertical lines crossing through the circle and a series of horizontal lines crossing through the circle. This mental visualization was signaled by his comment "either way." Prior to articulating that it did not have a unique output for each input and prior to his physical gestures along the vertical and horizontal axes, Elliot had visualized a vertical and horizontal line test. His mental activity was followed by a physical presentation: Elliot motioned along the vertical and horizontal axes demonstrating a vertical and horizontal line test.

This seems to be evidence of Elliot’s scheme for input and output, which was part of his scheme for function. Elliot read the prompt and the rule that described the function in the polar coordinate system: $r = 1$. However, Elliot’s scheme for input and output within a joint algebraic and graphical context seemed to be dependent on the graph’s axes. He operationalized his definition of function (“unique output for each input”) by visualizing and enacting vertical and horizontal line tests. He did not assign meaning for inputs and outputs from the polar coordinate system equation but instead interpreted the vertical and horizontal axes as graphical representations of the input and output. He had previously demonstrated that he could interpret either axis as the input or the output but did not envision how the input or the output may exist without reference to the graph’s axes.

Elliot’s scheme for function also included an image: a recollection of instances related to a circle’s status as a function. Elliot indicated that the shape of the graph was indicative of its status as a function; he said, “I mean, a circle is not a function.” Elliot’s thinking, like Susan’s graph of the natural log, was indicative of figurative thought. He was focused on the shape of the graph rather than the quantities, and he had previously reasoned about graphs shaped like a circle.

Finally, Elliot indicated a willingness to accommodate his scheme for function. He explicitly stated that his understanding of function might need to be adjusted because “maybe it changes when it's in the polar coordinate system.” However, Elliot then said, “but I don't think so,” and the interviewer did not probe his thinking any further on the topic. Although there was a willingness to reconsider, there was no deconstruction of his image related to circle, his scheme for input and output, or his actions related to line tests.

A radical constructivist analysis of the episode is focused on Elliot's mathematical scheme for function; his suggestion that the relationship was a function because of social factors ("because y'all are giving me the sheet," "the question is so easy for it," etc.) is not within the bounds of radical constructivist mathematics education research as it currently stands. Mathematics education researchers in the radical constructivist tradition do not attend to affective and social domains with radical constructivism; they focus on individuals' mental constructs—their mathematical schemes—as Piaget did. Instead, radical constructivist researchers often lean on constructs from other theories for interpretation of affective and social factors (M. Tallman, personal communication, September 8, 2022; K. Moore, personal communication, September 8, 2022). For example, Yoon et al. (2021) did not describe the participants' interactions with COVID-19 quantitative data representations (QDRs) with a schema of action or any other type of schema related to social interactions. Instead, for participants who justified their interpretations of the QDRs with social and political reasoning, they described how their mathematical schemes were superseded by their political beliefs.

Episode 2: Explicating Similarities and Differences Between the Analyses

In this subsection, I coordinate the empirical analyses of Elliot's clinical interview episodes with the theoretical similarities and differences described in the Comparing and Contrasting section.

Similarities

Both analyses focused on the meaning of Elliot's thoughts, language, and actions. Although each analysis conceptualized meaning differently and tracked Elliot's inferences in different ways, both analyses focused on Elliot's conceptual understanding of the mathematical concepts of inverse and function. Neither perspective reduced Elliot's conceptual understanding of inverse or function to a memorized definition or a rote procedure.

The inferentialist analysis of Elliot focused on his understanding of function within the context of the interview and interactions with the interviewer. It detailed the reasons Elliot gave to the interviewer explaining why he thought the relationship described and represented in the problem may or may not have been a function. The radical constructivist analysis of Elliot also focused on his understanding of the concept of function. In it, I began to accumulate evidence to model his scheme for the concept of function and the concepts of input and output based on his utterances and actions.

Differences

In addition to the similarities, there was one prominent difference between the two analyses. Despite the long history of conceptual analysis in radical constructivist research, the theory was unable to shed light on Elliot's social and contextual reasoning about the concept of function. Radical constructivist researchers must lean on other (potentially incompatible) theories to make sense of phenomenon like Elliot's understanding of the concept of function; they only have the tools to focus on students' mathematical schemes. In contrast, with inferentialism, Elliot's social and contextual reasons can be categorized alongside all of Elliot's other commitments and reasons. As Hußmann et al. (2019) suggested, the composition of a particular community, power relations, linguistic regularities, routines, and other norms influence the game of giving and asking for reasons. This difference points to inferentialism's ability to make sense of the intertwined nature of power, knowledge, and social context.

Conclusion

In this chapter, I described the theoretical similarities and differences between inferentialism and radical constructivism and subsequently used each theory to analyze empirical phenomena. Both perspectives (a) emphasize the relationship between knowledge and meaning, (b) reject referential accounts of language and representational accounts of knowledge, and (c) have the unique ability to perform fine-grained analyses of the learning of mathematical content. These similarities were evidenced by my theoretical and empirical analyses: each perspective was able to produce a distinct, in-depth semantic and epistemic analysis of Susan’s understanding of inverse and Elliot’s understanding of function.

The two perspectives also have clear differences. First, radical constructivism has a longer and more established tradition of research as compared to inferentialism. Unlike my inferentialist analyses, I was able to draw on previous radical constructivist results to support my empirical analyses. Second, in contrast to radical constructivism, inferentialism explicitly addresses ontological issues. Inferentialists conceive of mathematics as necessarily social: there are social, epistemic constraints on individuals’ use of mathematical concepts. Relatedly, inferentialists claim the GoGAR is the locus of meaning whereas radical constructivists claim autonomous individuals are meaning-makers. These points of tension were spotlighted by the two perspectives’ handling of Susan’s commitments about inverse. From the inferentialist perspective, Susan’s commitments were incompatible; from the radical constructivist perspective, her meanings for inverse were legitimate and any contradiction was only perceived. The points of tension were also spotlighted by the two perspectives’ handling of Elliot’s contextual reasoning about function. Inferentialism had the internal tools to assess Elliot’s reasoning; radical constructivism needed another theory. In light of these differences, I will discuss further implications about the relationship between ethics, power relations, and epistemology in the final chapter.

In the next chapter, I repeat this chapter’s process with inferentialism and the sociocultural perspective. Then, in the final chapter, I draw implications and conclusions from comparing, contrasting, combining, and coordinating the three theories.

CHAPTER 6

INFERENTIALISM AND THE SOCIOCULTURAL PERSPECTIVE

In this chapter, I (a) compare and contrast inferentialism with the sociocultural perspective and (b) combine and coordinate inferentialism with the sociocultural perspective. In the compare and contrast section, I describe the theoretical similarities and differences between inferentialism and the sociocultural perspective. In the combine and coordinate section, I use each theory to analyze an empirical phenomenon and explicate the similarities and differences between the empirical analyses.

Comparing and Contrasting: Inferentialism and the Sociocultural Perspective

In this section, I compare and contrast the theoretical components of inferentialism and the sociocultural perspective that I described in Chapter 4. This exploration of similarities and differences is part of my employment of Prediger et al.'s (2008b) Comparing and Contrasting strategy and responds to my first research question: *Using networking research practices, what is inferentialism's identity in relation to Toulmin-styled collective argumentation research from the sociocultural perspective?* In the comparison section, I describe several similarities between the two perspectives by drawing connections between (a) inferentialism and Lerman's Wittgensteinian research, (b) inferentialism and Vygotsky, and (c) inferentialism and the documenting collective activity (DCA) methodology. In the contrasting section, I describe how inferentialism and DCA differ in three ways. I detail (a) the philosophical differences between Brandom and Toulmin, (b) the differences in the way the perspectives address the individual and social aspects of learning, and (c) how the perspectives operate on different grain sizes. By putting these theories in dialogue with one another and scrutinizing their often-implicit ideas, I clarify their identity (i.e., their philosophical backgrounds and the way they have been developed to answer mathematics education research questions).

Comparing

Inferentialism, Lerman, and Wittgenstein

As I indicated in Chapter 1, inferentialism has philosophical ties with Wittgenstein and, consequently, inferentialist research has several similarities with Lerman's research. Lerman (2001; 2006) said Wittgenstein's work played a crucial role in the strong social turn in mathematics education and Brandom has adopted Wittgenstein's resistance to referential accounts of language (Brandom, 2000, p. 34) and Wittgenstein's idea of language games. (Brandom's GoGAR—the *game* of giving and asking for reasons—was an homage to Wittgenstein.) Combined with his adoption of Wittgenstein's ideas on language, Brandom's belief that the mind and language are inseparably linked also indicates a strong connection to Lerman' sociocultural tradition of research. Lerman, in his challenge to radical constructivism, said that "it is in discourses, subjectivities, significations, and positionings that psychological phenomena actually exist (Evans & Tsatsaroni, 1994)" (Lerman, 1996, p. 136). Like Brandom's theory of meaning, where the meaning of a concept exists within the discursive space of the GoGAR, Lerman said psychologists needed to focus on social and cultural practices because they were the location of meaning and purposes. Foreshadowing the inferentialist GoGAR, Lerman said understanding the meaning of a concept "can be interpreted as that of an individual coming to share in that meaning through negotiation and discussion" (Lerman, 1996, p. 146).

Brandom's account of social objectivity also parallels Lerman's reflections on objectivity and meaning in mathematics education. Lerman (1996) argued the mathematical meanings that teachers want students to learn are specific: children should be active in offering ideas and the "teacher and the rest of the class [should] take these on board and examine them, but there must come a stage when those ideas are extended and compared with other interpretations and meanings from other discourses" (p. 146). Thus, Lerman, like Brandom, believed there was a type of objectivity that was contextually and socially dependent. Socially "objective" knowledge, in this sense, is a social status conferred by communities because there are social norms (i.e., specific interpretations and meanings) that mathematical ideas in the classroom are subject to.

The inferentialist mastery metaphor is another touchpoint between inferentialism and Lerman's Wittgensteinian perspective. Lerman, quoting Davydov, said children master the speech of adults and that learning takes place "in interaction with more knowledgeable others" (Lerman, 1996, p. 138). Similarly, from an inferentialist perspective, an individual grows in their mastery of a concept through interactions with others that more fully understand its inferential web (i.e., the justifications and implications of a concept).

Inferentialism and Vygotsky

In addition to inferentialism's connections with Wittgenstein, inferentialism can be coordinated with Vygotsky's social psychology. Derry (2013), in her argument for inferentialism's potential to contribute to social epistemology, made multiple connections between Brandom's and Vygotsky's work. She connected Vygotsky's ideas about systems of relationships between concepts with the inferentialist concepts of commitments and entitlements and wrote, "For Vygotsky concepts depend for their meaning on the system of judgements (the infrastructure of commitments and entitlements) within which they are disclosed" (Derry, 2013, p. 230). Based on Vygotsky and Brandom's shared perspective regarding the meaning of concepts, Derry provided pedagogical advice: "teaching involves providing opportunities for learners to use concepts in the space of reasons within which they fall and within which their meaning is constituted" (Derry, 2013, p. 230).

Inferentialism and DCA

Finally, there are multiple similarities between inferentialism and the theoretical orientation of DCA. First, like inferentialism, DCA was built on Wittgenstein's ideas about discourse and meaning (Rasmussen & Stephan, 2008). More specifically, both inferentialists and the creators of DCA maintain that meaning exists in embodied social interactions between people within cultural contexts. Second, DCA is a methodology intended to document the normative ways of reasoning in a classroom and Brandom has emphasized the importance of social norms in his inferentialist account of knowledge and objectivity. Although their accounts of normative ways of reasoning differ—Rasmussen and Stephan (2008) used Toulmin argumentation schemes to describe the normative ways of reasoning whereas Brandom developed his own inferentialist system with the concepts of commitment, upstream inference, downstream inference, endorsement, and entitlement—both perspectives ground learning in discourse and students' participation in normative ways of reasoning.

Philosophically, DCA was built on Toulmin's (1958/2003) philosophical work (more specifically, his work on the pattern and general layout of arguments) which has similarities with inferentialism. Toulmin's work on the topic of argumentation was grounded in his larger philosophical framework, which he explicated in a later work (Toulmin, 1999). In his broader philosophical account, Toulmin gave an account of Western epistemology and described how Vygotsky and his successors could shed light on epistemology and activity theory. In his account of Western epistemology, Toulmin said that a convergence between the later work of Wittgenstein and the work of Vygotsky would provide a paradigm for human activity. Toulmin believed Wittgenstein made an important shift: a philosophical shift away from the focus on the minds of single individuals and a shift toward a focus on language and practical activities within collective forms of life. Toulmin praised Wittgenstein's account of knowledge and meaning because he agreed that "all meanings are created in the public domain in the context of collective situations and activities" (Toulmin, 1999, p. 58).

These Wittgensteinian ideas that influenced Toulmin are the same ideas that influenced Brandom's inferentialist account of meaning and concepts. Brandom (2000; 2008) has described how Wittgenstein's later ideas about (a) practical knowledge, (b) "everyday hermeneutic understandings" (p. 213), and (c) the meaning of language within forms of life have been foundational for his inferentialist project. Thus, Toulmin's account of arguments had the same philosophical influences as Brandom's inferentialist account of concepts.

Contrasting

Brandom and Toulmin

Because DCA was built on Toulmin's philosophical work, it is important to identify the contrasts between Toulmin and Brandom's work. To begin, Toulmin's work—used as theoretical support for DCA—was focused on practical logic and argumentation (Toulmin, 1958/2003). He wanted to explore the validity of arguments by examining general patterns in the layout of arguments and drew on theories of arguments from jurisprudence and formal logic. In contrast, Brandom gave an account of the content, or meaning, of concepts that was dependent on reasoning. Although there is a touchpoint between Toulmin's work on arguments and Brandom's work on the meaning of concepts (reasoning is featured prominently in both), the two performed philosophical investigations of different phenomena. Thus, their goals and the terminology they used do not overlap. Toulmin used the language of data, claims, warrants, backings, rebuttals, and qualifiers to describe general patterns of logic in everyday arguments. And Brandom used the language of commitments, claims, entitlement, endorsements, upstream inferences, downstream inferences, and incompatible inferences to describe the inferential relations that comprise a concept's content in a semantic holist account. Consequently, any application of these theories to mathematics education research will have different emphases. Both perspectives are suitable for research on reasoning, but research that draws on Toulmin will focus on reasoning in relation to the structure of arguments whereas research that draws on Brandom will focus on reasoning in relation to conceptual understanding.

Social and Individual

Rasmussen and Stephan (2008) developed DCA as a methodology designed to document the intellectual activity of a classroom community—not the learning of individuals. They emphasized that the mathematical activity of the classroom community documented with DCA may differ from the intellectual achievement of individual students and that cognitive constructivist perspectives should be used to study the learning of individual students. In contrast, Taylor et al. (2017) described how the inferentialist mastery metaphor for learning simultaneously accounts for the social and individual aspects of learning. In their words, an individual is a member of the social GoGAR and the social world is the "socio-cognitively structured setting or context of human reason" (p. 780) in which the individual operates. Put another way, the *game* of *giving* and *asking* for reasons cannot be played alone. Thus, "any description of individual reasoning is predicated on an understanding of the social norms that reasoning is subject to" and "any description of social interaction must be based on an appreciation of the rational relations between moves in the social game" (pp. 779, 781). In this way, inferentialism can describe the learning activity of a student "simultaneously and essentially in both cognitive and social terms" (Noorloos et al., 2017, p. 441).

Grain Size

Finally, DCA and inferentialism are meant to be used to research phenomena of different grain size. In the first phase of DCA, researchers create an argumentation log by using Toulmin's argumentation model to create argumentation schemes for every claim that was made in the classroom over multiple days. In the second phase, researchers use the argumentation log as data itself to identify ideas that function as-if-shared according to the three DCA criteria Rasmussen and colleagues developed. Finally, in the third phase, researchers view all the ideas that function as-if-shared across all the class periods and organize them around common mathematical activities. Throughout these different phases, the DCA methodology enables researchers to analyze the data at the grain size of (a) arguments, (b) mathematical ideas that recur across arguments over multiple days, and (c) common mathematical activities related to the recurring mathematical ideas.

In contrast, because of inferentialism's focus on individual students' activity within social contexts, it does not lend itself to be used beyond specific interactions within the GoGAR. Inferentialism is used for a close analysis of how mathematical concepts are reasoned about within moment-by-moment interactions between people in the GoGAR. The accumulated analyses of moment-by-moment interactions between people in the GoGAR can be compiled, but—unlike DCA—there is no mechanism to move from one grain size to another.

Combining and Coordinating: Inferentialism and the Sociocultural Perspective

In this section, I employ Prediger et al.'s (2008b) Combining and Coordinating strategy and report empirical analyses of three pairs of episodes from the content class extant data described in my methodology section. I report on *three* pairs of episodes because DCA's three different criterion to determine if a mathematical idea functions as-if-shared operate on the same analytic grainsize as my inferentialist methodology. DCA's three criterion are the touchpoint with inferentialism. I report on three *pairs* of episodes because DCA's three criterion require something to happen across multiple episodes of argumentation. The specific pairs of episodes were selected because the events that take place within them further clarify inferentialism's identity in relation to DCA. My analyses for each of the pairs of episodes are presented in the order of (1) the context of the pairs of episodes, (2) my DCA analysis across the pairs of episodes, (3) my inferentialist analysis of what happens in the pairs of episodes, and (4) an explication of the similarities and differences between the DCA and inferentialist analyses. After I analyze each pair of episodes, I give a summary of DCA's subsequent analytic phase and an example of a long-form inferentialist analysis. Ultimately, my analyses help answer an overarching research question and two of my research sub-questions:

4. Using networking research practices, what is inferentialism's identity in relation to Toulmin-styled collective argumentation research from the sociocultural perspective?
    a. Given the sociocultural perspective's conception of function as-if-shared mathematical ideas and collective activity, what collective mathematical growth has occurred within episodes of collective argumentation?
    b. Given inferentialism's conception of concepts and claims, what content have students mastered within episodes of collective argumentation?

My analyses illustrate how each theory practically makes sense of student reasoning within collective argumentation. The inferentialist analyses specifically illustrate how inferentialist researchers foreground individual students' mastery of different concepts within the social background of the GoGAR. The DCA analyses illustrate how DCA researchers identify ideas that function as-if-shared in the classroom and summarize the collective mathematical growth of the students in the classroom. The conclusions of my analyses further clarify the similarities and differences between inferentialism and the sociocultural perspective that were described in the Comparing and Contrasting section.

Prior to describing the episodes and my analysis I would like to describe the specific analytic methods I used to empirically analyze extant data from two different perspectives. After watching all the meetings of the content course, I transcribed two specific classes that prominently featured collective argumentation between students on the topics of function, inverse, and covarying quantities—a feature necessary for DCA analysis. After transcribing, I completed an inferential analysis of both class meetings. Then, after completing the inferentialist analysis, I used DCA to analyze the data. During my DCA analysis, I read and reread relevant texts from Rasmussen (Rasmussen & Stephan, 2008; Cole et al., 2012) and Toulmin (1958/2003; 1999).

Broader Context of Episodes

Before describing the more precise context of the pairs of episodes and providing my analyses from both perspectives, I need to provide broader context of the extant data under analysis. As a reminder, the classroom data are from the first of three content courses for prospective secondary mathematics teachers (PSTs) prior to student teaching at a large southeastern university in the United States. The course met twice weekly for a 15-week semester and focused on quantitative and covariational reasoning through the study of functions and inverses. The data, which were part of a larger study, consisted of video recordings of small group and whole class discussions for every class meeting.

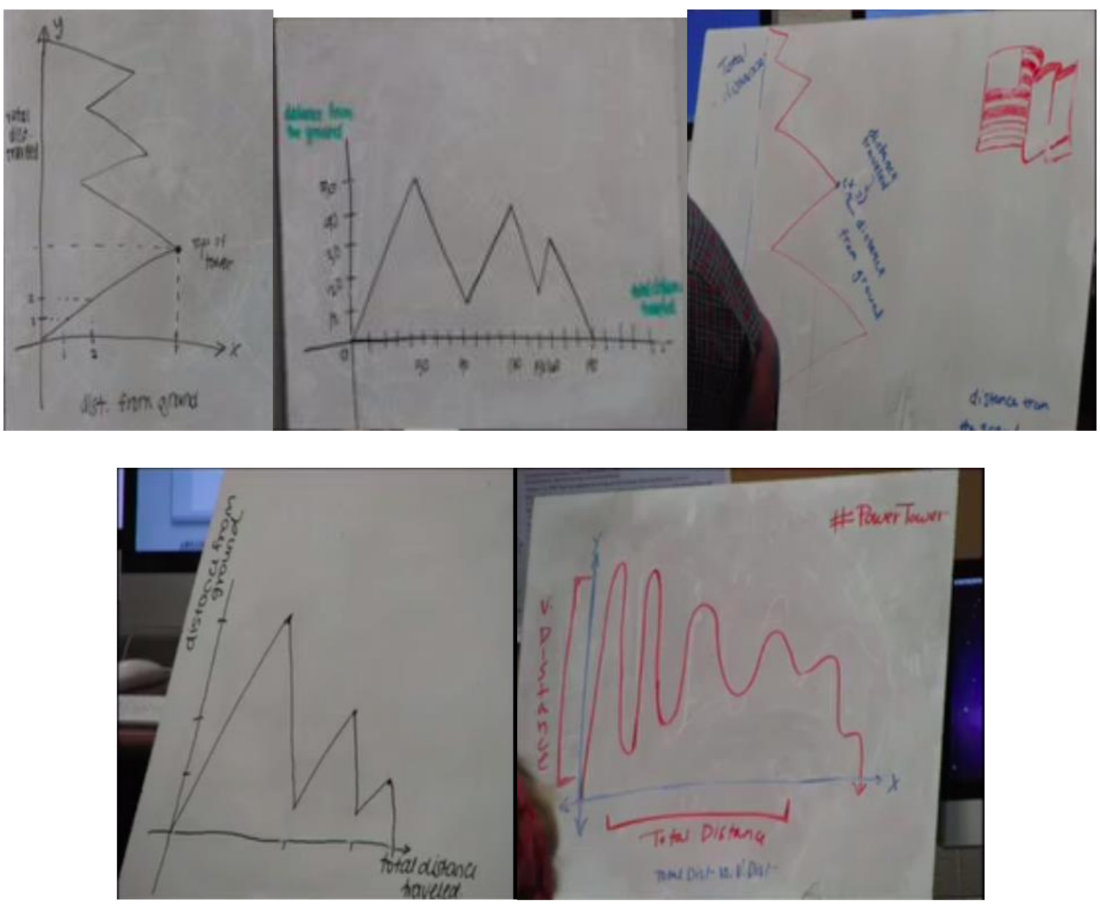
The three pairs of episodes I selected all occurred in one of two consecutive meetings of the content course. The first class meeting included collective argumentation in student small groups and in a whole class discussion. The second class meeting included collective argumentation in a whole class discussion. In the first of these class meetings, the instructor began the class by showing a video of the "Power Tower," a ride at an amusement park that launched riders up along vertical towers and subsequently dropped them. The up and down process repeated several times. The PSTs were arranged in groups of three or four at tables and, after watching the video, each small group of PSTs was given one of two handouts. The PSTs were not aware that different handouts were distributed to different groups. One handout asked the PSTs to "*Graph the distance from the ground of an individual (vertical axis) vs. the individual's total distance traveled (horizontal axis) (assume their feet were touching the ground at the beginning of the video).*" The other handout asked the PSTs to "*Graph the individual's total distance traveled (vertical axis) vs. distance from the ground of an individual (horizontal axis) (assume their feet were touching the ground at the beginning of the video).*" Both handouts then asked the students to give a justification for their graphs.

The PSTs then had approximately 25 minutes to work in their small groups to create their graph on a large whiteboard and to begin to respond to subsequent extension questions listed on their handout (see Appendix A). Out of the six small groups, three small groups were videorecorded for the entire 25 minutes. The first small group's conversations about the task focused on (a) whether the graph should be comprised of straight lines or curves, (b) the graph's average rate of change, and (c) conjectures about the extension questions. The second small group's conversations about the task focused on (a) what their graph's axes should represent and (b) whether the graph should be comprised of straight lines or curves. The third small group's conversations about the task focused on (a) whether their graph represented a function and (b) whether their graph should represent the relationship between the two variables or if it should represent the motion of the ride.

After 25 minutes in small groups, the instructor asked each group to display their graph around the room to form a gallery walk. Five of the groups' graphs are pictured in Figure 12. The PSTs then had five minutes to walk around the room, observe other groups' graphs, and make notes from their observations. After the PSTs returned to their seats, the instructor asked the students to discuss their observations with each other and to ask each other questions about their graphs. The instructor said he would participate in the discussion as little as possible.

Figure 12

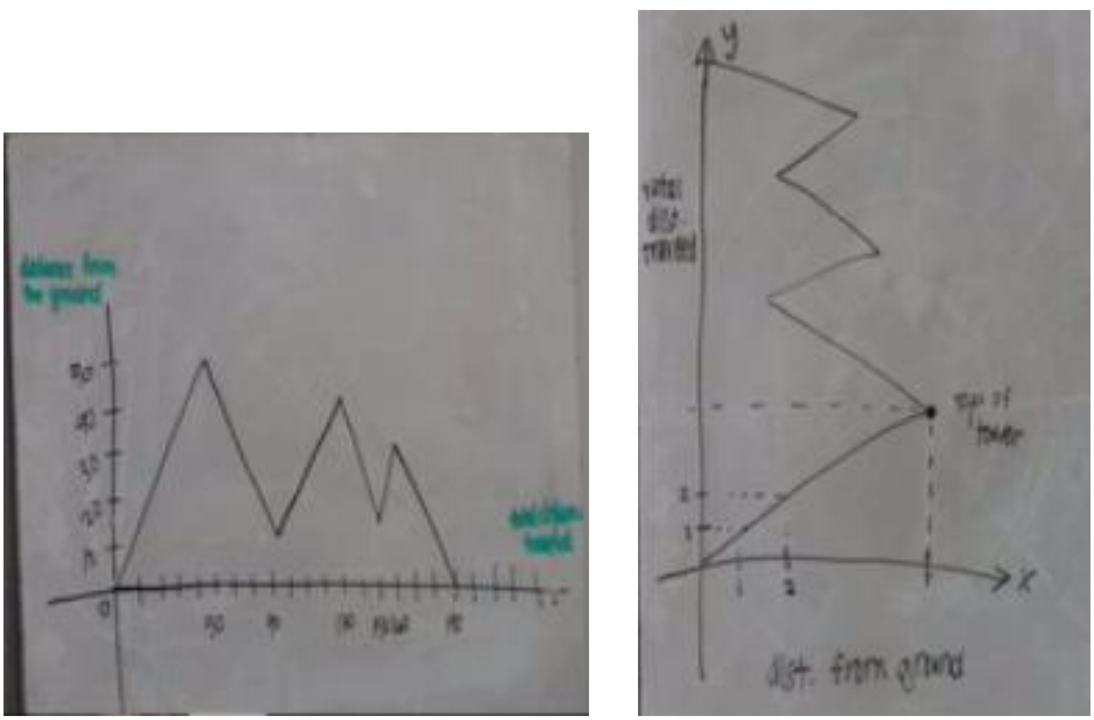
Graphs Displayed by Small Groups for Gallery Walk



For the next 20 minutes, the PSTs discussed (a) whether the graph should have been comprised of straight lines or curved, (b) whether the speed of the ride should influence the graph, (c) why the width between the graphs' peaks was not constant, and (d) the slope of the graph. After these discussions, the instructor pointed out that two of the created graphs (shown in Figure 13) looked different. The PSTs discussed how the graphs' appearances differed and Rachel (all names are pseudonyms) revealed that different small groups were given different prompts. A PST named Reilly then brought up the idea of function while comparing the two graphs and the instructor facilitated a 10-minute discussion about the meaning of function with respect to graphs. The instructor asked the students questions about functions and graphs with related PowerPoint slides and then concluded the class.

Figure 13

Graphs Emphasized by Instructor



In the first 20 minutes of the subsequent meeting of the class, the instructor recapped what happened in the previous class and facilitated another discussion about functions and graphs. He also facilitated a discussion about the PSTs' responses to the final questions from their handouts. The final questions asked the PSTs to make a conjecture about what would happen to their graph if (a) the height of the Power Tower ride was doubled and (b) the speed of the Power Tower ride was doubled. The remainder of the time was unrelated to the Power Tower task.

Structure of Subsequent Sections

As a reminder, DCA and inferentialist research operate on two grain-sizes, depicted in Figure 14 and Figure 15. In the DCA methodology, episodes of argumentation from the content course need to be analyzed in pairs so the collective mathematical growth of the class is clearly displayed from one episode to the next. DCA researchers do not have anything to say about any one specific episode of argumentation from the content course. The DCA methodology, however, does have a subsequent analytic phase that advances to a larger grain size and looks across all the episodes of argumentation. In that subsequent analytic phase, every idea that was identified as functioning as-if-shared across multiple episodes of argumentation is recorded in a table, and the ideas are subsequently organized around common mathematical practices.

Figure 14

DCA Units of Analysis

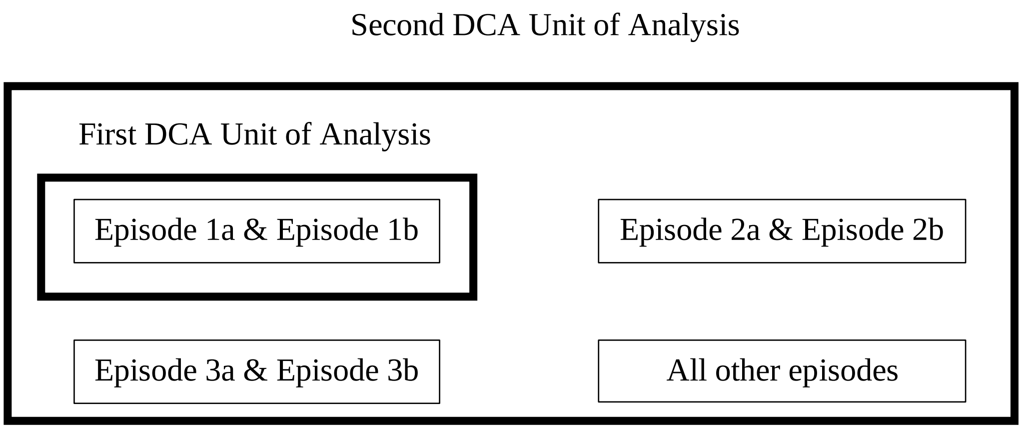
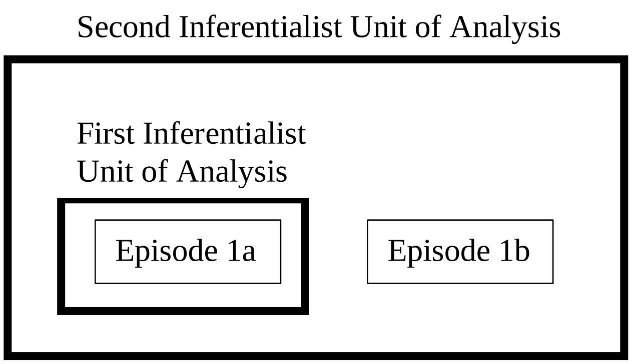


Figure 15

Inferentialist Units of Analysis

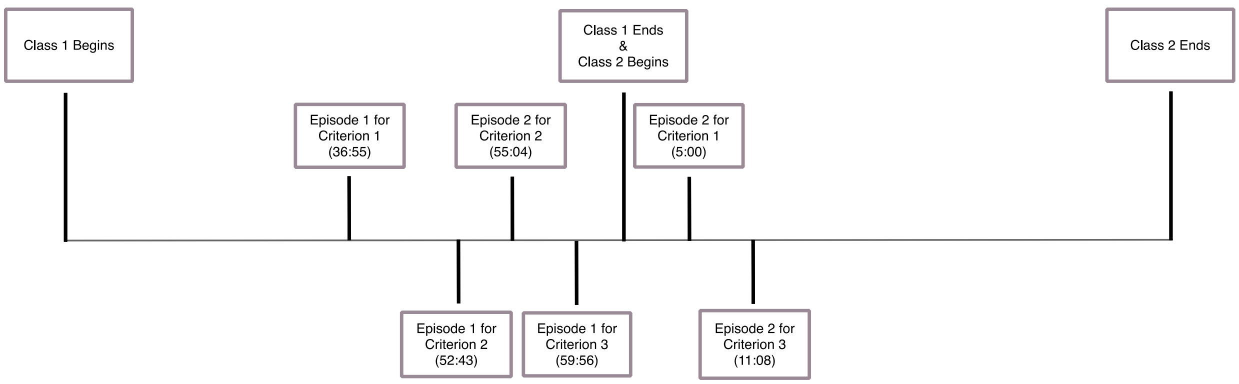


In contrast, I used my inferentialist methodology to describe each episode of argumentation and then compiled those descriptions about students' learning across a pair of episodes of argumentation. Inferentialism is capable of long-form analysis so, after my summary of DCA's subsequent analytic step, I gave an example of an inferentialist long-form analysis. The purpose of inferentialist-based mathematics education research, however, is not to advance to a larger grain size and offer an overarching analysis of collective growth like DCA.

Thus, the subsequent sections of this chapter are structured in such a way so that the analyses of the two perspectives are comparable. First, there is a general description along with transcripts of a pair of episodes of argumentation from the content class extant data. (Figure 16 shows a timeline of the episodes included in my analysis.) I then analyze the pair of episodes using DCA followed by an inferentialist analysis of the same pair of episodes. After the analyses, I explicate the similarities and differences between the DCA and inferentialist analyses and give an overall summary of my DCA and inferentialist analyses.

Figure 16

Timeline of Episodes Included in Analysis



First Pair of Episodes

The first episode within the first pair of episodes occurred during the student-led discussion about the slope of the graphs with straight lines. William made a comment about the slope of the graphs while several students were still trying to understand why the graph was comprised of straight lines. After the class agreed that the graphs should be comprised of straight lines, Reilly expanded on William's comment about slope at the instructor's request. Another student, Elliot, then questioned Reilly and William's reasoning and Susan provided further justification.

36:55 William: But the slope would still be the same on each because it's 1 and 1. Or 1 and negative (one).

…

38:21 Jason: Umm. Because we're only concerned with distance traveled. Not speed, in any way. And we just wanted our graph to represent visually what was happening, but that wasn't actually what the graph is asking for. Well, what the question was asking for.

38:47 Instructor: And William, what did you say a minute ago?

38:50 William: (Unclear)

38:51 Instructor: Reilly, did you hear him?

38:52 Reilly: Yeah.

38:53 Instructor: What did William say, Reilly?

38:54 Reilly: It was about the slope and, every, every time like you move, say, 1 in total distance, you only move 1 in the vertical distance. And even when you start going up, then you start going down, you're still moving over one. So it's 1 to 1. So it's gonna be constant no matter what. Cause you have 1 over 1. And the slope will never change.

…

41:21 Elliot: How do we know the slope is one though? I don't understand that.

41:25 William: Well, it'd be one for when it's going up and then it'd be negative one for when it's going down. Cause you're kinda taking the absolute value of the distance from the ground and adding it to the total distance traveled. Does that make sense?

41:37 Susan: Because like for every foot you travel, like up this thing, you're going one more [motioning up with hand] in your vertical distance. And then when you go down, you're just going another foot in your total distance but your vertical distance is decreasing. But you're just, you're adding another foot to your total distance. So like you go down a couple feet [motioning with hand] so you add that feet to your total distance. Does that make sense?

The second episode within this pair of episodes took place in the subsequent meeting of the class when the instructor led a discussion about the rate of change of the two quantities from the initial Power Tower problem. Jason, the instructor, and another student co-constructed the idea that the two quantities involved were distances and unrelated to time. The instructor then asked about the rate of change and Susan said it would be 1 or -1. After the instructor expanded on Susan’s claim, Rachel explained why the rate of change was 1 or -1.

5:00 Instructor: But in this case, so you're saying in this case there's still a rate of change, it's just a comparison of?

5:11 Jason: Not involving time.

5:12 Instructor: Not involving time. Specifically, though, involving?

5:15 Unidentified Student: Distances.

5:17 Instructor: Distance and distance right. And the rate of change here was what? Numerically. It was constant. Kinda constant.

5:25 Susan: It's 1 and negative 1.

5:25 Instructor: 1 and negative 1, right? So kinda constant. Over certain intervals it's constant, right? And over those certain intervals it's always 1 or negative 1. And the reason for that was what?

5:37 Rachel: For every like change in total distance, there was the same change in distance from the ground.

In the subsequent subsections, I analyze this pair of episodes from a DCA perspective and an inferentialist perspective. I offer analytic descriptions as well as conclusions from both perspectives and then explicate the similarities and differences between the analyses.

DCA Analysis: A Backing Becomes Implied

In their first criterion for determining if a mathematical idea functions as-if-shared in the classroom, Rasmussen and Stephan (2008) wrote:

> When the backings and/or warrants for an argumentation no longer appear in students' explanations (i.e., they become implied rather than being stated explicitly or called for), no member of the community challenges the argumentation, and/or if the argumentation is contested and the student's challenge is rejected, we consider that the mathematical idea expressed in the core of the argument stands as self-evident. (p. 205)

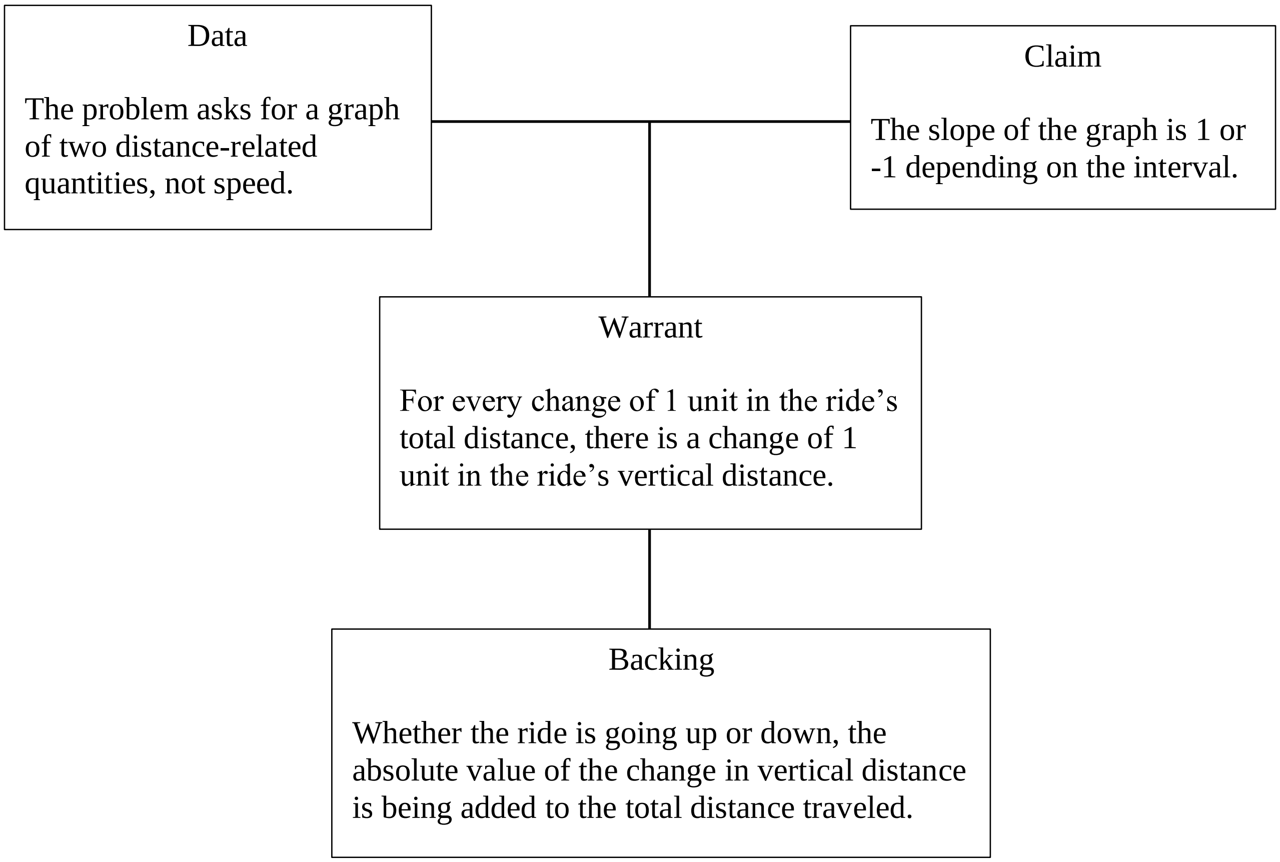
As demonstrated in the following diagrams, this criterion was fulfilled in this pair of episodes because a backing provided in the argument in the first episode became implied in the argument in the second episode in the subsequent class.

To describe the first episode in Toulmin terms, William initially claimed the slope of the graphs would be 1 or -1 depending on the interval. Reilly and Susan articulated a warrant for William's claim: for every change of 1 unit in the ride's total distance, there is a change of 1 unit in the ride's vertical distance. Susan and William then co-constructed a backing for the warrant:

Whether the ride is going up or down, the absolute value of the change in vertical distance is being added to the total distance traveled. Figure 17 shows a summary of the first argument.

Figure 17

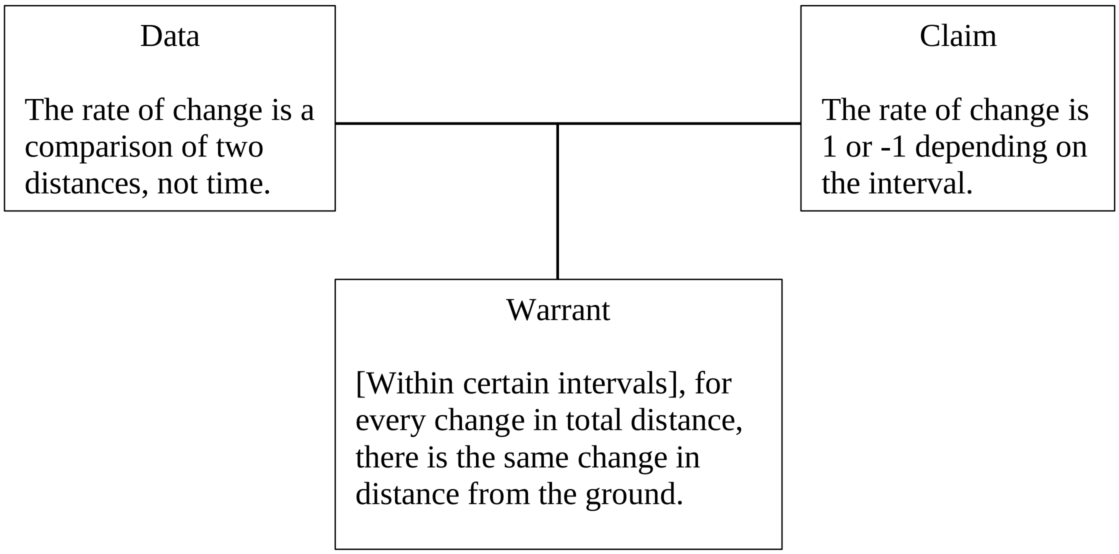
Summary of Argument from First Episode, First DCA Criterion



To summarize the second episode in Toulmin terms, Susan and the instructor co-constructed a claim: the rate of change was 1 or -1 depending on the interval. Rachel then connected their claim to the idea that the rate of change was a comparison of two distances and was not related to time. The warrant Rachel used to connect the claim with the data was: for every change in total distance, there was the same change in distance from the ground. Figure 18 shows a summary of the argument.

Figure 18

Summary of Argument from Second Episode, First DCA Criterion



By the time of the second argument, the backing from the first argument used to justify the warrant became implicit and was not called for. Thus, the core of the argument—the use of the warrant as a step from the data to the claim—stood as self-evident and functioned as-if-shared in the classroom.

Inferentialist Analysis

Description of GoGAR. At the beginning of the first episode, William said the slope of the graph would always be either 1 or -1 (a commitment). A minute and a half later, Reilly endorsed William's commitment about the slope of the graph and then justified part of the commitment with an upstream inference: the rate of change would be 1 to 1 because "even when you start going up, then you start going down, you're still moving over one." In response to William's commitment and Reilly's upstream inference, Elliot expressed uncertainty and asked a question of entitlement at 41:21: "How do we know the slope is one though?" In response to Elliot's question, William more fully justified his initial commitment with his own upstream inference: whether the ride is going up or down, the absolute value of the change in vertical distance is being added to the total distance traveled. Susan subsequently endorsed Reilly and William's upstream inferences and, by connection, William's original commitment about the slope of the graph.

Near the beginning of the second episode, Jason claimed time was not one of the two quantities being compared in the rate of change (a commitment). Another student then claimed the quantities in the rate of change were two distances (another commitment) and the instructor endorsed that student's commitment. Susan then claimed the rate of change was 1 and -1 (a commitment) and the instructor endorsed her commitment as well. The instructor, however, followed up and asked why Susan was entitled to her commitment: "And the reason for that was what?" Rachel justified Susan's commitment and said the rate of change was 1 and -1 because "for every like change in total distance, there was the same change in distance from the ground" (an upstream inference and an endorsement of Susan's commitment).

Student Mastery of Conceptual Content. Altogether, in this pair of episodes several students engaged in the GoGAR and together evidenced mastery of slope and covarying quantities within the context of the problems. More specifically, (a) William made a commitment about slope and related it to the covarying quantities; (b) William, Reilly, and Susan each made inferences from William's original commitment about the slope of the graph; (c) Reilly and Susan endorsed William's contributions; (d) Elliot asked a question of entitlement to determine if William and Reilly were entitled to their claims about slope; and (e) Rachel endorsed Susan's commitment about the rate of change and made an upstream inference from it. So, William, Reilly, and Susan—in response to Elliot's question of entitlement—demonstrated entitlement to their commitments, endorsements, and inferences about slope and how the two quantities (distance from the ground and total distance traveled) co-varied. All three helped explain why the slope was 1 or -1 because the absolute value of any change in the ride's distance from the ground was added to the ride's total distance traveled. In the subsequent class, Susan restated that the rate of change was 1 or -1 and Rachel re-articulated that, for every change in the ride's total distance, there was the same change in the ride's distance from the ground.

These students' commitments, inferences, explicit endorsements, and questions of entitlement illustrate growth in their ability to reason with the concept of slope and covarying quantities because they knew what a commitment about a concept entailed, what upstream inferences would entitle somebody to use the concept, and what inferences would preclude such entitlement. Because, from an inferentialist perspective, knowledge is the socially evaluated status of one's mastery of a concept as indicated by the ability to reason with it, then the lack of explicit objection to the students' reasoning from the instructor and their peers in the class implied entitlement. The students who spoke showed evidence of mastery of slope and the covarying quantities within the context of the problem. They explained—with detail sufficient to receive their peers' entitlement—how the two quantities simultaneously changed. The rest of the class—by not rejecting these students' reasoning—grant entitlement to the creation of norms of reasoning about slope and the covarying quantities.

Explicating Similarities and Differences Between the Analyses of the First Pair of Episodes

Similarities. The most notable similarity between the DCA and inferentialist analyses of the pair of episodes was the shared assumption used to make a claim about the class's collective growth and mastery. In the DCA analysis, the backing from the first argument used to justify the warrant was not used or called for in the second argument. The implicit class-wide support for the validity of the core of the argument was the reason why the idea was categorized as functioning as-if-shared. The phrase functioning as-if-shared further signifies the inability to make a strong claim about the learning of any individual in the collective: within the classroom, the core idea of the argument (the data: the rate of change is a comparison of two distances; the claim: the rate of change is 1 or -1; and the warrant: for every change in total distance, there is the same change in distance from the ground) passed as if everyone shared in an understanding of the idea because no one objected to the argument when the backing no longer appeared.

In the same way, the attribution of entitlement and lack of explicit objection to the reasoning of William, Reilly, Susan, Elliot, and Rachel from other people in the class was the sole evidence used to support the learning of students who did not say anything in the GoGAR. The students that remained silent may have been actively engaged in the GoGAR by keeping track of all the commitments, inferences, entitlements, and explicit endorsements related to the concepts of slope, rate of change, and covarying quantities. But there is nothing in the extant data—such as students' written notes—that might provide additional evidence of other students' mastery. Thus, both DCA and inferentialism encounter the same problem: there are significant limitations to any analytic claim about the growth in knowledge of a collective group of people when not all the people fully participated in the collective argumentation.

Another similarity between the two analyses is their focus on students learning a collection of ideas. In the DCA analysis, the core of the argument shown in Figures 17 and 18—the *claim* that the rate of change was 1 or -1 depending on the interval; the *data* that the rate of change was a comparison of two distances and was not related to time; and the *warrant* that, for every change in total distance, there was the same change in distance from the ground—functioned as-if-shared in the classroom. The content of the students' learning is summarized by the connection between the data and the claim with the warrant. Similarly, in the inferentialist analysis, the students' ability to reason about slope and the covarying quantities is measured by their ability to connect ideas. For example, Reilly and William showed their mastery of the conceptual content of rate of change by drawing an upstream inference from William's idea that the slope of the graph would always be either 1 or -1; they both gave reasons for why William was entitled to his claim. Thus, in both analyses, the content of students' learning is characterized by connections between ideas and cannot be reduced to a single piece of information.

Differences. The most significant difference between the DCA analysis and the inferentialist analysis was the analytic treatment of individuals and the collective in the classroom. My DCA analysis focused on the data, claim, warrant, and backing provided in the first episode and the data, claim, and warrant provided in the second episode. I did not need to attend to the individuals that contributed the components because DCA is solely concerned with the conclusion: the core of the argument—the use of the warrant as a step from the data to the claim—stood as self-evident and functioned as-if-shared in the classroom. The DCA analysis can be visually summarized with Figures 17 and 18, which do not include the names of any students because the focus of the methodology is on collective mathematical growth.

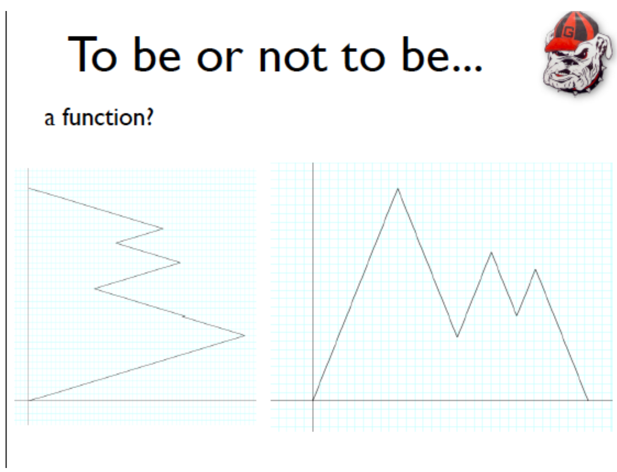
In contrast, the inferentialist analysis foregrounded specific individuals' participation in the GoGAR. It was comprised of detailed descriptions of specific students' commitments, inferences, endorsements, and questions of entitlement. However, because the students participated in the GoGAR, the analysis did not neglect the social component of the exchanges. The GoGAR was the social background where the individuals reasoned about slope, rate of change, and covarying quantities. In the GoGAR, students had their ideas challenged and received entitlement for their ideas from their peers. Thus, my inferentialist methodology did not solely focus on collective mathematical growth but instead focused on (a) how individual students reasoned about the mathematical ideas and (b) how their ideas received normative status in the classroom.

Second Pair of Episodes

The first episode within the second pair of episodes occurred at the beginning of the instructor-led discussion on the meaning of the concept of function in the context of graphs, and the second episode occurred two minutes later. In the first episode, the instructor displayed images of two graphs that were simplified versions of the PST-produced graphs in Figure 13 (see Figure 19); the instructor had anticipated what the PSTs' graphs would look like. He then asked the class if the graphs were functions. Cathy and Jill initially responded to the instructor's question and, 40 seconds later, Reilly elaborated on Cathy and Jill's initial responses.

Figure 19

Slide Displayed by Instructor



52:43 Instructor: Okay, so we, we hit on something here. So I heard Jill say, well, she brought up the idea of function right? So there's kind of two simplified graphs, right? [Puts a new PowerPoint slide up: Figure 19.] Functions? Not functions?

52:46 Cathy: It just depends on how the axes are labeled.

52:49 Jill: Depends on what you call your input and what you call your output.

52:49 Cathy: Yeah.

52:49 Rachel: Yeah.

…

53:28 Reilly: Depends which axis is your input and which axis is your output.

53:30 Instructor: So say a little more Reilly.

53:46 Reilly: Uhh, okay. If your, umm, I guess functions, the graph on the left, if your inputs were actually your vertical axis, your outputs are your horizontal axis then it would be a function.

A little over a minute later in the subsequent episode, the instructor claimed the question "Is that graph a function?" was an ambiguous question. The instructor then asked: Why is the question "Is that graph a function?" an ambiguous question? Rachel responded to the question and, upon the instructor's request, provided further reasoning.

55:04 Instructor: Okay. So, the point here is, right so it depends, answering that question depends on how we conceive the graph, right? How we, how we actually conceptualize what we are representing. Because in each case, right, we have this, we're representing the same covariational relationship, right? On both graphs. Same relationship, right? They're describing the situation, they're about the same quantities. And their numerical pairs are the same except [gestures a swap motion with his hands] just, one case, if we keep them in the proper order of horizontal then vertical pair, right? Then the second case is just gonna be the reverse pair of the other case [gestures indicating swap with hands]. But the pairs should be the same. That one quantity goes with the other quantity. And so we mentioned that, well, really, so really the question we asked when we look at a graph and say, "Is that graph a function?", that's kind of an ambiguous question. Is it not? If we say, is that graph a function?

55:52 Susan: Uh-huh.

55:56 Instructor: Why is that an ambiguous question if I say, "Is the graph a function?"

56:00 Rachel: Cause it's open to interpretation. Like, it's just how you see it.

56:05 Instructor: So say a little bit more…about how you see it. Why that's open to your interpretation?

56:25 Rachel: It's how, well like, it depends on how you conceive it. You want your x-axis, if you want your input to be the vertical axis and your output to be your horizontal axis, then it might be a function or it might not be. Depending on where you put your input and output.

In the subsequent subsections, I analyze this pair of episodes from a DCA perspective and an inferentialist perspective. I offer analytic descriptions as well as conclusions from both perspectives and then explicate the similarities and differences between the analyses.

DCA Analysis: A Shift from Claim to Warrant

In their second criterion for determining if a mathematical idea functions as-if-shared in the classroom, Rasmussen and Stephan (2008) wrote:

> When any of the four parts of an argument (the data, the warrant, claim, or backing) shift position (i.e., function) in subsequent arguments and are unchallenged (or, if contested, the challenges are rejected), the mathematical idea functions as if it were shared by the classroom community. For example, when students use a previously justified claim as an unchallenged justification (the data, warrant, or backing) for future arguments, we conclude that the mathematical idea expressed in the claim becomes a part of the group's normative ways of reasoning. (p. 209)

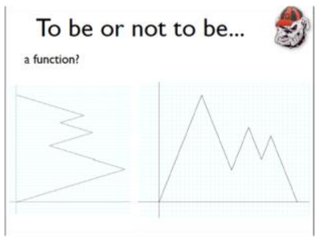
As demonstrated in the following diagrams, this criterion was met in this pair of episodes because the claim from the earlier argument shifted position and was then used as a warrant in the later argument.

To put the first episode in Toulmin terms, Cathy and Jill together co-constructed a claim: a graph's status as a function depends on which axes are called input and output. Reilly then gave a warrant that connected their claim to the data (the picture of the two graphs); he said the graph on the left would be a function if the input was represented on the vertical axis and the output was represented on the horizontal axis. Figure 20 shows a summary of the first argument.

Figure 20

Summary of Argument from First Episode, Second DCA Criterion

Data



Claim

A graph's status as a function depends on which axes are called input and output.

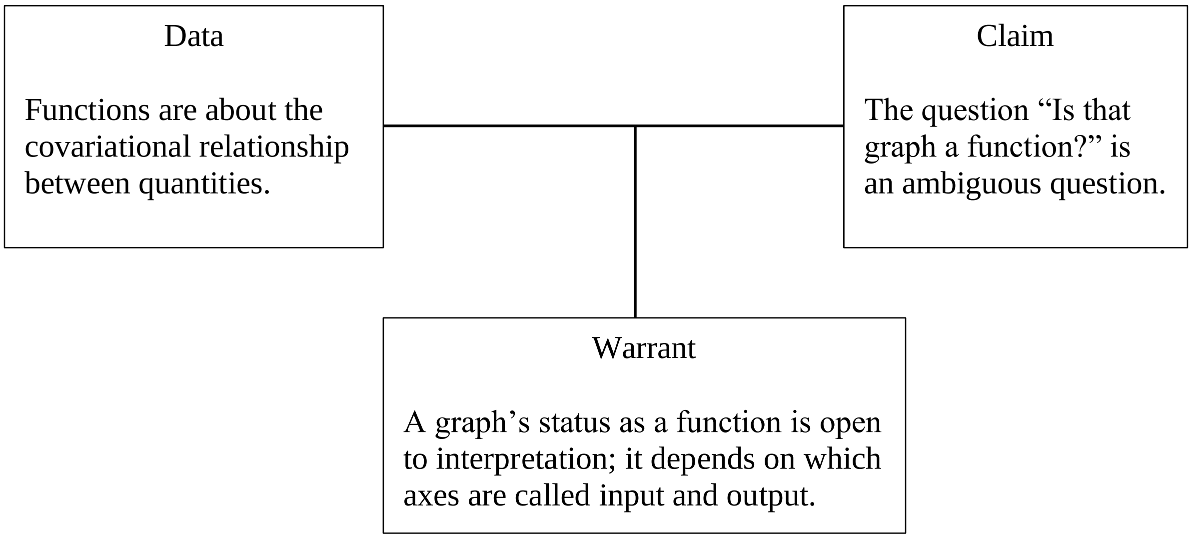
Warrant

The graph on the left will be a function if the input is represented on the vertical axis and the output is represented on the horizontal axis.

To summarize the second episode in Toulmin terms, Rachel used the instructor's idea that functions are really about the relationship between quantities as data and connected that data to the instructor's claim with a warrant: the question is ambiguous because a graph's status as a function is open to interpretation; it depends on which axis represents the input and which axis represents the output. Figure 21 shows a summary of the second argument.

Figure 21

Summary of Argument from Second Episode, Second DCA Criterion



Thus, the claim from the first argument (a graph's status as a function depends on which axes are called input and output) co-constructed by Cathy and Jill was used by Rachel as a warrant only minutes later. The idea shifted position without challenge and, consequently, functioned as-if-shared by the classroom community.

Inferentialist Analysis

Description of GoGAR. The first episode began with a question: the instructor asked if the two graphs were functions. In response, Cathy said the graphs' status as functions depended on how the axes were labeled (a commitment). Jill followed up Cathy's commitment with a commitment of her own: the graphs' status as functions depends on what is called the input and the output. Then Cathy and Rachel both endorsed Jill's commitment and Reilly endorsed both Cathy and Jill's commitments by merging them together. The instructor then asked Reilly to say more about his explicit endorsement of Cathy and Jill's commitments; this request to "say a little more" functioned as a question of entitlement because it prompted Reilly to give reasons for why he, Jill, and Cathy were entitled to those commitments. Reilly explained his endorsement by providing examples as downstream inferences. He said the status of the graphs as functions depended on which axis was the input and which axis was the output because, for example, the graph on the left would be a function if the inputs were represented on the vertical axis and the outputs were represented on the horizontal axis.

In the second episode, the instructor said the question "Is that graph a function?" was an ambiguous question. The instructor's claim was a downstream inference from the students' claim that the graphs' status as functions depended on what was called the input and the output. He then asked the class to explain why the question was ambiguous (a question of entitlement). In response, Rachel said the question was ambiguous because it was open to interpretation (an upstream inference). The instructor then asked Rachel to explain why she was entitled to her upstream inference with another question of entitlement: why is it open to interpretation? In response, Rachel described how, depending on which axis represented the input and the output, the graph may or may not be a function (an upstream inference from her upstream inference).

Students' Mastery of Conceptual Content. In this pair of episodes several students engaged in the GoGAR and together evidenced mastery of the concept of function and a function's relationship with a graph within the context of the problems. More specifically, (a) Cathy and Jill committed to the idea that a graph's status as a function depended on which axes represented the input and output; (b) Reilly endorsed Cathy's and Jill's commitments and provided downstream inferences in which he described which axes needed to represent the inputs and outputs in order for the graph to represent a function; and (c) Rachel responded to back-to-back questions of entitlement with upstream inferences that explained why the question "Is that graph a function?" was open to interpretation. These students' commitments, downstream inferences, upstream inferences, and endorsements illustrated their ability to reason with the concept of function because they knew what would entitle somebody to make commitments about functions and the graphs. From an inferentialist perspective, Cathy and Jill were entitled to their commitments about functions because Reilly's endorsement and subsequent downstream inferences went unchallenged by the instructor and the rest of the class. Furthermore, Rachel was entitled to her two upstream inferences that justified the ambiguity of the question "Is that graph a function?" because the instructor and the rest of the class did not challenge her any further. The rest of the class—by not rejecting these students' reasoning—grant entitlement to the creation of norms of reasoning about functions, inputs, and outputs.

Explicating Similarities and Differences Between the Analyses of the Second Pair of Episodes

Similarities. The primary similarity between the analyses of the second pair of episodes is their focus on the interconnectivity of ideas. In the DCA analysis, the claim from the first argument (a graph's status as a function depends on which axes are called input and output), co-constructed by Cathy and Jill, was later used by Rachel as a warrant. The idea shifted position without challenge and subsequently functioned as-if-shared by the classroom community. In the DCA methodology, an idea can be identified as functioning as-if-shared if students first justify the idea as a claim and then use the idea to justify another claim. Thus, students must connect one idea with multiple other ideas to meet the second criterion.

In the inferentialist analysis, students' mastery of the concept of function was measured by a similar sort of interconnected reasoning. For example, in the second episode, Rachel used Cathy's and Jill's commitment along with Reilly's upstream inference to justify a different commitment. Rachel's ability to draw inferences from other people's commitments and inferences showed her interconnected understanding of functions, graphs, and inputs and outputs. Her learning was measured by her ability to connect her ideas with others' ideas.

Differences. Interestingly, the primary difference between the analyses of the second pair of episodes seems to contradict the similarity. Although the second DCA criterion for determining if an idea functioned as-if-shared values the interconnectivity of ideas, the DCA analysis concludes with only one idea functioning as-if-shared in the classroom. With DCA's second criterion for determining if a mathematical idea functions as-if-shared in the classroom, a single idea—that a graph's status as a function depends on which axes are called input and output—is identified as functioning as-if-shared.

The inferentialist analysis of the second pair of episodes maintained a focus on the interconnectivity of a collection of ideas. Any description of a student's learning from an inferentialist perspective could not be reduced to a piece of information because of the idea of semantic holism. In semantic holism, "one cannot have any concepts unless one has many concepts. For the content of each concept is articulated by its inferential relations to other concepts" (Brandom, 2000, pp. 15–16). So, from an inferentialist perspective, Rachel did not just show evidence of learning that a graph's status as a function depends on which axes are called input and output. Because of the inferences she made, Rachel also showed evidence of learning how to interpret a graph and that the question "Is that graph a function?" is an ambiguous question.

Third Pair of Episodes

The first episode in the third pair of episodes occurred several minutes into the whole class discussion on the concept of function in graphical contexts. The second episode occurred in the subsequent class meeting while the class recapped the same ideas. In the first episode, the instructor asked if the total distance traveled in the Power Tower ride would be a function of the ride's vertical distance. Several PSTs said total distance traveled would not be a function of vertical distance and that it would not matter which graph was associated with the question. Rachel and William provided the justification: if vertical distance was the input and total distance traveled was the output, there would be multiple outputs for a single input.

0:59:56 Instructor: What about this question? [Displays the question: "A function? Distance Traveled a function of Vertical Distance?"]

1:00:02 Multiple students: Mmm, no.

1:00:03 Instructor: Does it matter what graph I look at?

1:00:05 Reilly: No.

1:00:07 Instructor: No, cause we defined what to be the input in that situation?

1:00:08 Reilly: Vertical distance.

1:00:14 Instructor: Vertical distance in, we choose a vertical distance, why doesn't it work?

1:00:15 Rachel: Cause you have different…

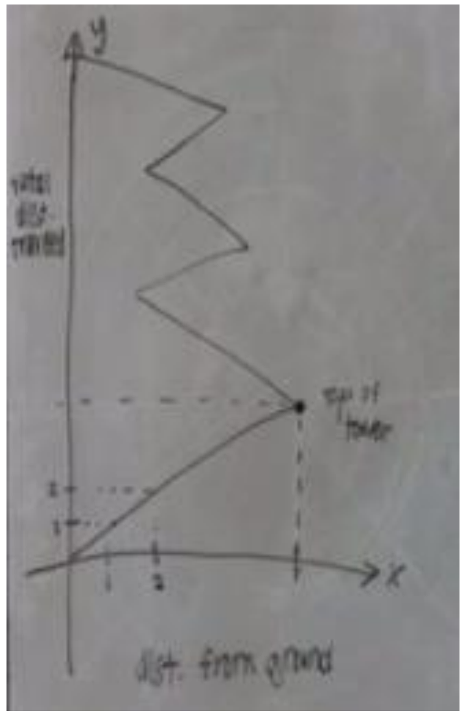
1:00:15 William: Multiple outputs.

1:00:21 Instructor: Not a unique total distance, right? Okay.

In the next class meeting, the instructor facilitated another discussion about functions and graphs. The instructor drew the PSTs' attention to one of the graphs produced in the previous class meeting (Figure 22) and asked the PSTs what claims they could make about it. Susan responded and explained that the graph would represent a function if the *y*-axis was the input. At the instructor's prompting, Susan elaborated: if the *y*-axis represented the input, then for each distance traveled, there would be only one distance from the ground.

Figure 22

Graph Under Discussion (Vertical Axis: Total Distance Traveled, Horizontal Axis: Distance from the Ground)



11:08 Instructor: Now what if I did something, well here, let's, so here's another graph right? And we could say, we could do, people presented a graph like this, right? So, in that case, what did we kind of talk about and conclude?

11:27 Susan: That if you make your y-axis your input then you will have a function.

11:36 Kevin: So, make the y-axis, go ahead Susan.

11:36 Susan: Yeah, just, every time, because like, for each total distance traveled we only have one distance from the ground. For each point, we only have one distance from the ground.

In the subsequent subsections, I analyze this pair of episodes from a DCA perspective and an inferentialist perspective. I offer analytic descriptions as well as conclusions from both perspectives and then explicate the similarities and differences between the analyses.

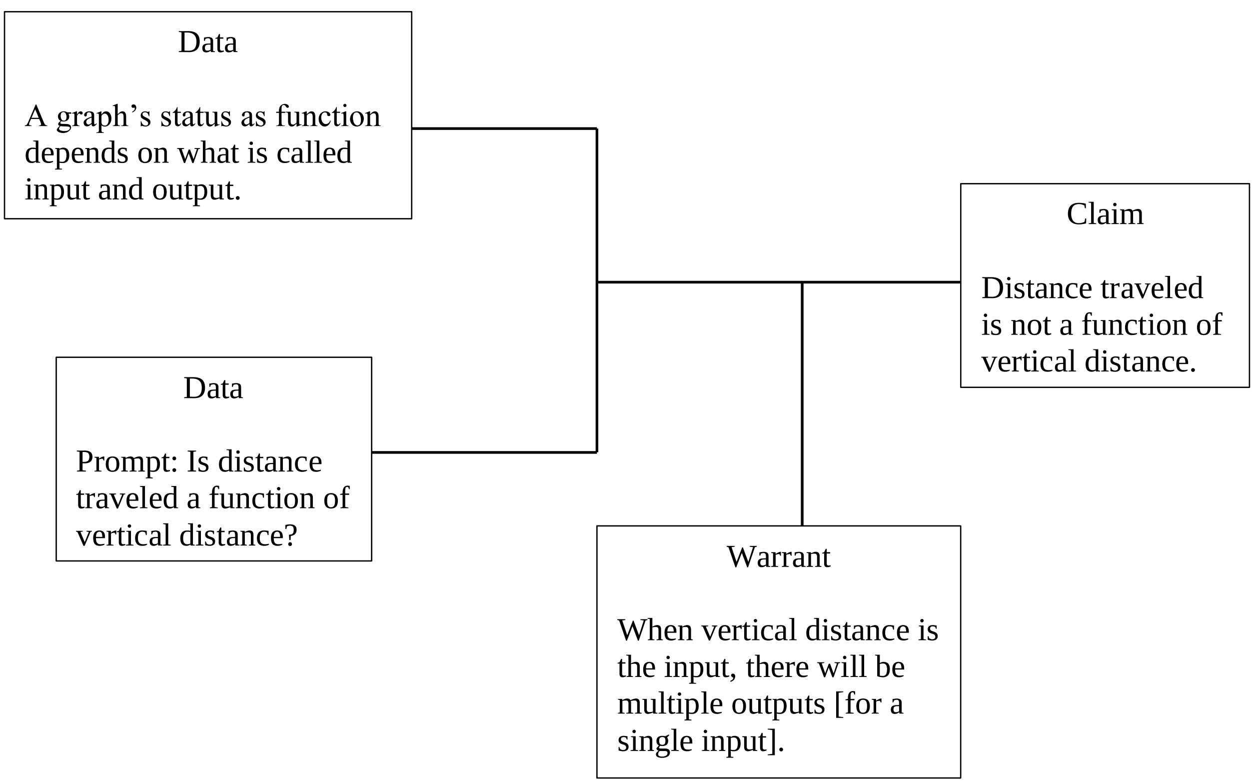
DCA: Reuse of Warrant

In a later-developed third criterion for determining if an idea functions as-if-shared in the classroom via the DCA methodology, Cole et al. (2012) wrote that an idea functioned as-if-shared if “a particular idea is repeatedly used as either data or warrant for different claims across multiple days” (p. 200). This criterion is met in this pair of episodes because part of the definition of function was used as a warrant across multiple days. The part of the definition of function used was: For a mapping to be considered a function, each element from the set of inputs must be mapped to exactly one element from the set of outputs.

To summarize the first episode in Toulmin terms, Rachel, William, and the instructor together used a part of the definition of function as warrant for Reilly’s claim that distance traveled is not a function of vertical distance. The data that Rachel, William, and the instructor drew on was (a) the prompt from the instructor’s slide and (b) the already established idea that a graph’s status as function depended on what was called input and output. Figure 23 shows a summary of the first argument.

Figure 23

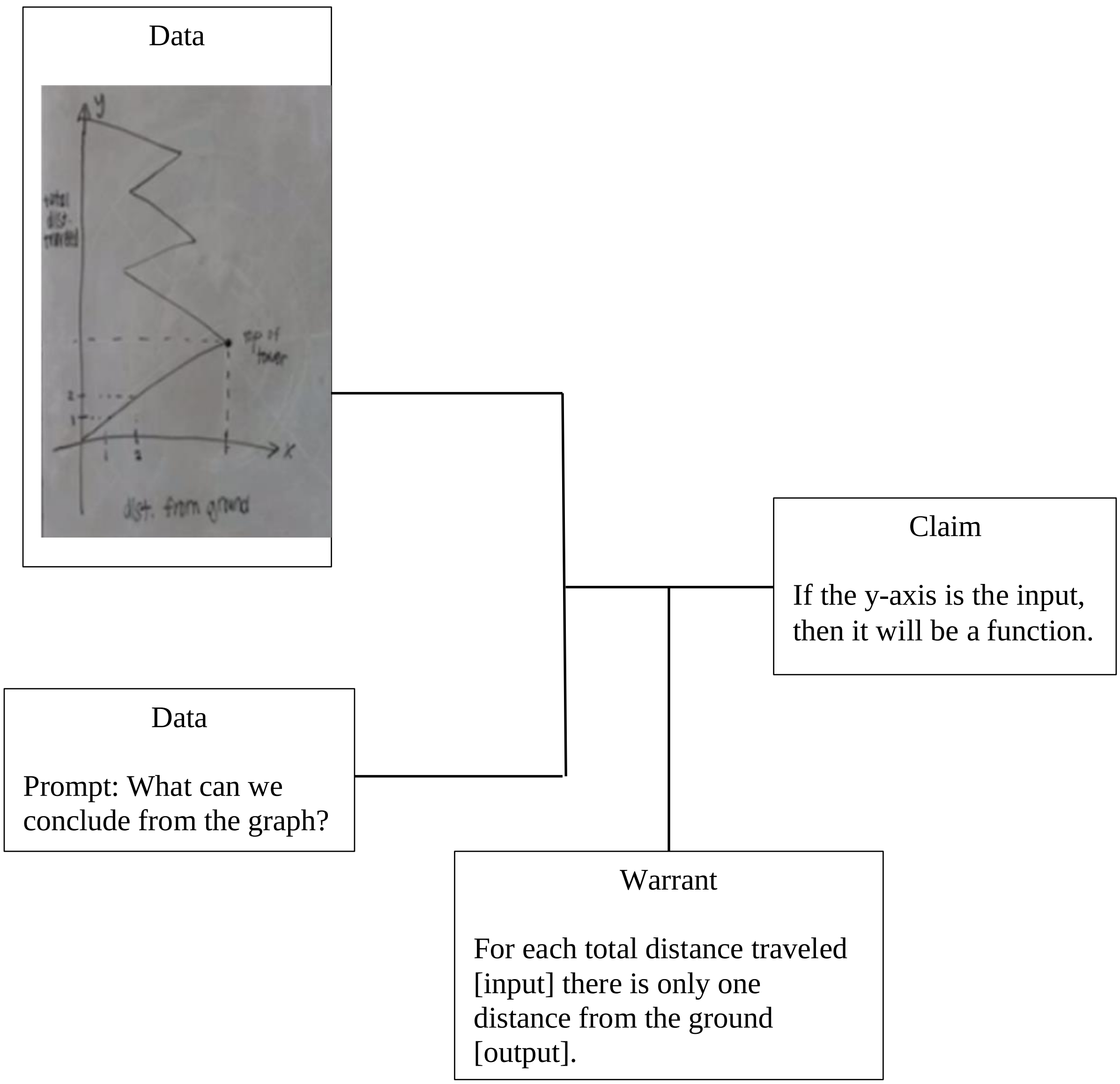
Summary of Argument from First Episode, Third DCA Criterion



In the second episode, Susan claimed the graph would represent a function if the y-axis was the input and justified her claim by referring to the same part of the definition of function as a warrant. She said that, if the y-axis was the input, then "for each total distance traveled [input] we only have one distance from the ground [output]" and the warrant was not challenged by anyone in the class. Figure 24 shows a summary of the second argument.

Figure 24

Summary of Argument from Second Episode, Third DCA Criterion



In the two episodes, the idea that a relationship between inputs and outputs is a function if each input is mapped to exactly one output was used as a warrant to justify two different—albeit similar—claims. Thus, the idea functioned as-if-shared in the classroom.

Inferentialist Analysis

Description of GoGAR. At the beginning of the first episode, the instructor asked whether distance traveled was a function of vertical distance. Numerous students responded that it was not a function (a commitment). The instructor then asked the students a different question: does the students' conclusion depend on which graph is viewed? Reilly responded no (a commitment) and Reilly, Rachel, William, and the instructor together justified Reilly's commitment by explaining that, because vertical distance was the input, there would be multiple outputs for a single input (an upstream inference).

In the second episode, the instructor asked what claims they could make about the graph in Figure 22. Susan said the quantitative relationship depicted in the graph would be a function (a commitment) if the y-axis was the input (an upstream inference). She then further justified her commitment and upstream inference and said that for each total distance traveled there would only be one distance from the ground (an upstream inference).

Students' Mastery of Conceptual Content. Altogether, in this pair of episodes multiple students engaged in the GoGAR and together evidenced mastery of aspects of the concepts of function, input, and output. More specifically, (a) Reilly, Susan, and several other students committed to the idea that distance traveled was not a function of vertical distance, (b) Rachel and William (along with the instructor) used part of the definition of function as an upstream inference to justify why the other students' prior commitment was true, (c) Susan showed that she knew the necessary circumstances for a relationship to be a function which, in Brandom's terms, is another form of an upstream inference, and (d) Susan justified her own upstream inference with an additional upstream inference drawing on the definition of a function—that each input must be mapped to exactly one output. Because each of the commitments and upstream inferences went unchallenged by anyone in the class, the students are entitled to them, and the commitments and inferences are evidence of mastery of aspects of the concepts of function, input, and output within the context of the problems. The rest of the class—by not rejecting these students' reasoning—grant entitlement to the creation of norms of reasoning about function, inputs, and outputs.

Explicating Similarities and Differences Between the Analyses of the Third Pair of Episodes

Similarities. The main similarity between the two analyses is their focus on using one idea to justify different claims. In DCA, an idea functions as-if-shared if the idea is repeatedly used as either data or warrant *for different claims* across multiple days. For example, if Susan merely repeated the warrant that Rachel and William provided in the same context to support the same claim, then the idea would not be coded as functioning as-if-shared. In DCA, an idea does not function as-if-shared if it is repeatedly regurgitated as a warrant for a repeated claim.

In a similar way, from an inferentialist perspective, repeatedly using the same upstream inference to establish a repeated commitment demonstrates minimal understanding. Further understanding, from an inferentialist perspective, is demonstrated by a student using the same idea to justify multiple commitments which increases the interconnected nature of the student's understanding of the inferential web of a concept like function.

Differences. One of the differences between the DCA and inferentialist analyses of the second pair of episodes reappeared in the analyses of the third pair of episodes. DCA's third criterion is used to identify a single idea—that a relationship between inputs and outputs is a function if each input is mapped to exactly one output—that functioned as-if-shared. In contrast, the inferentialist analysis of the third pair of episodes maintained a focus on a collection of ideas. For example, Susan's use of the definition of function in the second episode as an upstream inference to justify her commitment "that if you make your y-axis your input then you will have a function" was evidence of her learning multiple things. She demonstrated understanding of that aspect of the definition of function, the meaning of the term input, an ability to interpret the graph, and an ability to reason about the covarying quantities.

Another difference between the analyses is DCA's emphasis on a particular idea being repeatedly used as either data or warrant *across multiple days*. For DCA's third criterion, an idea does not function as-if-shared if it is repeatedly used as data or warrant within the same class period. Susan's use of the idea, after Rachel and William used it in the previous class, triggered the fulfillment of the third criterion. In contrast, from an inferentialist perspective, Rachel and William exhibited understanding of the idea when they together used it as an upstream inference to justify why distance traveled was not a function of vertical distance. Inferentialism did not require the idea to be used across multiple class meetings for the idea to be categorized as learned.

DCA's Subsequent Analytic Phase

After identifying ideas that function as-if-shared, DCA has a subsequent analytic phase that advances to a larger grain size. Every idea that is identified as functioning as-if-shared in a classroom is recorded in a table and the ideas are subsequently organized around common mathematical practices. DCA researchers then use the common mathematical practices to describe the collective mathematical growth of a class. Table 5 lists the ideas that functioned as-if-shared across the two classes along with the related common mathematical practices. (In the previous sections of this chapter, I showed how three different ideas function as-if-shared in the classroom. The class's collective mathematical growth in relation to the three ideas can be described as growth in reasoning about functions and reasoning about covarying quantities.)

Table 5

Ideas that Function As-If-Shared and the Related Common Mathematical Practices

Ideas that function as-if-shared	Common mathematical practice
The graph should only represent distances. For every change of 1 unit in the ride's total distance, there is a change of 1 unit in the ride's vertical distance. The rate of change is constant. The problem asks for two distance-related quantities, not speed. Whether the ride is going up or down, the absolute value of the change in vertical distance is being added to the total distance traveled.	Covariational Reasoning
Both graphs show the same relationship [between quantities]. The graph should not be influenced by speed. The slope of the graph is 1 when the ride is going up and negative 1 when the ride is going down.	Covariational Reasoning / Graphical Reasoning
Different parts of the graph are wider and other parts are skinnier. Some parts of the graph are wider because it is a bigger distance traveled. Some parts of the graph are shorter because the ride did not go up as high.	Graphical Reasoning / Covariational Reasoning
The graphs' status as a function depends on what is called input and what is called output. The graph on the left will be a function if the input is on the vertical axis and the output is on the horizontal axis. For a function, there must be a unique output for every input.	Reasoning About Functions

Inferentialist Long-Form Analysis

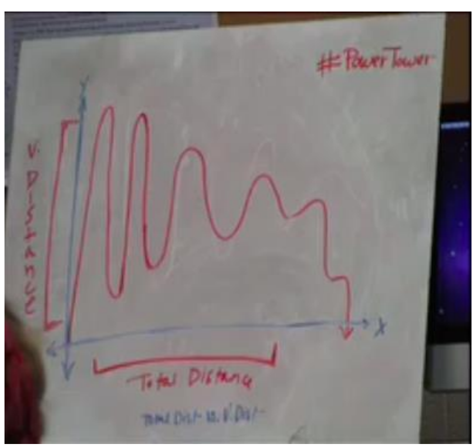
Inferentialism does not have a similar subsequent analytic step that advances the analysis to a larger grain size. Inferentialism can, however, be used to perform a long-form analysis on a specific student relative to a specific concept. Consider a brief example with Elliot and the concept of function. Elliot worked on the Power Tower problem in his small group with Melissa and Jason; their prompt asked them to "*Graph the distance from the ground of an individual (vertical axis) vs. the individual's total distance traveled (horizontal axis) (assume their feet were touching the ground at the beginning of the video)*." Elliot, near the beginning of their conversation, asked about the shape of the graph. Elliot, Melissa, and Jason were not certain what the shape of the graph would look like but Melissa asked Elliot and Jason: "This might sound dumb, but is it a function?" Elliot responded and asked Melissa an interesting question: "Why wouldn't it be a function?" Elliot's question acted as both a commitment and a question of entitlement; he committed to the idea that their graph would be a function and asked Melissa to justify why it might not be a function.

Elliot and Melissa then had a back-and-forth about whether their graph would be a function. In her argument for why it would not be a function, Melissa seemed to mistakenly swap the quantities assigned to the vertical and horizontal axes from their prompt. She said it might not be a function because the "distance" was going to increase and then decrease which meant the "x-value is gonna move back towards your y [axis]." Elliot responded: "Well, like the x-value is the total distance traveled, so if you go up [motions up with finger] that's distance, if you come down [motions down with finger] that's distance [motions zig zag up and down with finger]." Melissa then endorsed Elliot's reasoning and said, "So it's adding distance. Okay, never mind. Sorry."

There are several aspects of this exchange between Melissa and Elliot that I could analyze from an inferentialist perspective. But I believe it is important to note that, beneath Melissa's and Elliot's reasoning about the graph's status as function, there seemed to be an assumption about how to assess a graph's status as a function. They do not explicitly appeal to the vertical line test, but their conclusion seemed to be dependent on the assumption that a graph's status as a function can be visually determined by checking if the "x-value moved back toward the y-axis." Melissa agreed that the graph would be a function after Elliot explained why the x-value would not "move back toward the y-axis" and he gestured up and down in a way that resembled their final graph (Figure 25 is a picture of their final graph). Because that reasoning was sufficient to answer Melissa's question about the graph's status as a function, some type of visual assessment like the vertical line test—with its accompanying assumption that the input was represented on the horizontal axis and the output was represented on the vertical axis—was likely the unsaid upstream inference that justified their conclusion that the graph would represent a function.

Figure 25

Elliot's Gesture Resembled the Group's Final Graph



Later in the class, during the whole class discussion on functions and their relationship with graphs, Elliot remained mostly quiet. He only spoke up one time to provide an upstream inference in support of the commitment that, within the Power Tower problem, vertical distance was a function of time. The instructor asked the class: Is vertical distance a function of time? In response, some students said "yes," and some students said "no." After several students expressed uncertainty, Elliot said vertical distance would be a function of time because "there aren't going to be two vertical distances at the same time." Elliot's upstream inference was then endorsed by the instructor, Jill, and Reilly. The instructor responded to Elliot: "So if we chose a time, we should get…" Then Jill responded to the instructor: "One vertical distance." And Reilly said, "we get one answer."

Elliot's contributions in those exchanges above were his only contributions to the discussions of function in either of the two class meetings I analyzed. Yet they may indicate growth in his mastery of the concept of function. In the small group exchange about the graph's status as a function, both Elliot and Melissa seemed to implicitly assume something like the vertical line test—visually checking if a vertical line would intersect their curve more than once. In their justification for why the graph was a function, they did not discuss the relationship between the quantities. They only discussed how the quantities would visually appear on their graph. However, later in the whole class exchange about vertical distance as a function of time, Elliot did not rely on a visual assessment of a graphical representation to determine if vertical distance was a function of distance. Instead, Elliot said the relationship between the two quantities would be a function because there would not be multiple outputs ("two vertical distances") for a single input ("at a single time"). This may be evidence that Elliot grew in his mastery of the concept of function because he attended to the relationship between inputs and outputs and did not rely on a visual assessment of a graph with built-in assumptions about which axes represented the input and the output.

Conclusion

In this chapter, I described the theoretical similarities and differences between inferentialism and DCA—a particular sociocultural methodology—and subsequently used each theory to analyze empirical phenomena. I then explicated the similarities and differences between the perspectives' empirical analyses. Philosophically, inferentialism and DCA share roots in Wittgenstein and have theoretically compatible social psychologies. Empirically, both perspectives (a) are limited in their ability to make analytic claims about the growth in knowledge of a collective group of people and (b) highlight the interconnectivity of ideas.

The two perspectives also have clear differences. My theoretical and empirical analyses both showed how DCA is used to study *collective* mathematical growth whereas my inferentialist methodology foregrounded individual students' reasoning within the background of the GoGAR. In other words, DCA can only be used to make conclusions about a social group but the social component of my inferentialist methodology focused on how students' ideas received normative status in the classroom.

Second, DCA and my inferentialist methodology operate on different grain sizes. DCA works on multiple grain sizes and has built-in mechanisms to make conclusions about the collective mathematical growth of a class. In contrast, my inferentialist methodology can only be used to analyze individuals' use of concepts in the GoGAR, how individuals developed mastery of concepts, and the established normative ways of reasoning. All of these can be compiled to form a long-form analysis of a student's mastery of a concept.

Third and finally, my inferentialist empirical analyses consistently reflected the idea of semantic holism: “one cannot have any concepts unless one has many concepts. For the content of each concept is articulated by its inferential relations to other concepts” (Brandom, 2000, pp. 15–16). In contrast, some of the DCA analyses showed how *an* idea functioned as-if-shared in the classroom. The conclusions of those analyses described a single idea that was functioning as-if-shared without relation to any other ideas.

In the subsequent chapter—the final chapter—I state conclusions from my study and draw implications from the comparing, contrasting, combining, and coordinating of the three theories.

CHAPTER 7

DISCUSSION AND IMPLICATIONS

In this final chapter, I discuss my inferentialist methodology in light of networking inferentialism with radical constructivism and the sociocultural perspective. I also describe the implications of my inferentialist analyses for the field's understanding of the interplay of individual and collective activity. The inferentialist methodological discussion further responds to my methodological research questions: (a) How does an inferentialist researcher robustly analyze students' mastery of meanings in clinical interviews? and (b) How does an inferentialist researcher robustly analyze the game of giving and asking for reasons and conceptual content within collective argumentation? The implications from my inferentialist analyses address my final research question: Given the inferentialist methods developed in response to research question 2 and the empirical analyses from research question 1, what can an inferentialist perspective add to the field's understanding of the interplay of individual and collective activity?

My Inferentialist Methodology

Several researchers have called for the development of an inferentialist methodology and the determination of practical analytic methods (Taylor et al., 2017; Radford 2017). In response, I determined that networking inferentialism with radical constructivism and DCA was the ideal process to develop and elaborate an inferentialist methodology for empirical research. In chapter 4, I described (a) my inferentialist methodology, (b) the methodology's basis in Brandom's work (2000) and the work of inferentialist theoreticians (e.g., Derry 2017), (c) the similarities and differences between my methodology and Hußmann et al.'s (2019) analytic framework, and (d) my analytic codes (commitments, upstream inferences, downstream inferences, questions of entitlement, entitlements and endorsements). My subsequent networking of inferentialism with radical constructivism and the sociocultural perspective created an opportunity for “extraction” (Radford, 2008, p. 319). Radford explained that, when theories are put into dialogue with one another, researchers can extract or “pull out things from the brackets of common sense (the brackets of things that we take for granted in our theory to the extent that we no longer even notice them)” (p. 319). Thus, by putting inferentialism in dialogue with two other dominant theories, the implicit aspects of my inferentialist methodology can be extracted and explicated.

My inferentialist methodology was able to be networked with two drastically different theories because in inferentialism the individual and social components of knowledge and learning are intertwined. It was applicable in the analysis of clinical interviews and group discussions in a content course because individuals’ mastery of mathematical content was measured by their engagement with interlocuters in the social GoGAR and how others endorsed or entitled their commitments. Even in clinical interviews with a single interviewee, Susan and Elliot interacted with the interviewer, responded to prompts provided by the interviewer, and interacted with the socialized setting. Ultimately, it proved useful for analyzing how individual students mastered mathematical meanings within clinical interviews as well as how mathematical content was socially established and mastered within the content course.

In the subsequent subsections, I describe the implicit aspects of my inferentialist methodology that became apparent after networking inferentialism with radical constructivism and DCA respectively.

Methodological Takeaways from Networking Inferentialism with Radical Constructivism

The primary takeaway for my inferentialist analytic methodology from networking inferentialism with radical constructivism is the limitation on assessing a student's mastery of a concept within a clinical interview. It was difficult to assess Susan's and Elliot's mastery of the concepts of function and inverse function because of a conflict between the underlying assumptions of the inferentialist mastery metaphor and the original purpose of the clinical interviews. Inferentialists hold that knowledge is socially conferred, yet the clinical interviews with Susan and Elliot were conducted for the purpose of gaining insights into the meanings they individually constructed for function and inverse function. The research team intentionally asked open-ended questions to probe Susan's and Elliot's thinking but did not aim to promote shifts in their thinking (Paoletti et al., 2018). Consequently, the inferentialist conception of knowledge as socially conferred was somewhat undermined. For example, the interviewer may have disagreed with some of Susan's commitments and inferences, and, within a different setting, the interviewer may have questioned whether she was entitled to her commitments more frequently, pressed her for further reasoning, or suggested alternative commitments. The interviewer did ask Susan questions of entitlement (e.g., "Now relative to this one you mentioned, you know there might be a reflection over y = x line. In this case, do we have something like that or?") and there were tasks presented to Susan within the interview that pushed back on her commitments, but the purpose of the interview was not to determine Susan's mastery of the concept of inverse. If the purpose of the interview was to determine Susan's mastery of the concept of inverse, the interviewer would have been obliged to participate more fully in the GoGAR by explicating their commitments, making inferences, explaining why they did not think Susan was entitled to a commitment, etc. But, for the interviewer to more fully engage in the game of giving and asking for reasons, Susan would also need the freedom to question why the interviewer was entitled to their commitments.

The same limitation was on display in the clinical interview with Elliot. Elliot claimed a circle was not a function but also said “I don't, maybe it changes when it's in the polar coordinate system.” The interviewer, however, did not follow up on this idea from Elliot and instead asked Elliot about his other line of reasoning: “Just the fact that I'm giving you the sheet makes you think it might be a function?” Elliot’s unexplored upstream inference about the polar coordinate system points to the different underlying methodological assumptions between inferentialism and the purpose of the clinical interview. Ideally, from an inferentialist perspective, both of Elliot’s reasons would be followed up with further questions. While no interviewer can follow up on every claim from an interviewee, the interviewer may have chosen to avoid asking Elliot any further questions about his polar coordinate system upstream inference out of a desire to not promote shifts in Elliot’s thinking. Thus, Elliot’s mastery of the concept of function was difficult to fully assess because there was no question prompting Elliot to further explore the meaning of function in a graphical context and in light of the polar coordinate system.

The above examples pose a methodological difficulty for inferentialist clinical interview research and point to a fertile area for further research. If knowledge is socially determined—as inferentialists suggest it is—but the interviewer in a clinical interview is not participating fully in the GoGAR (e.g., the interviewer may not press an interviewee for further reasons even though the interviewer does not believe the interviewee is entitled to their commitment) then the GoGAR is contorted. The GoGAR is dependent on all participants actively engaging in making commitments, drawing inferences, asking questions of entitlement, and making endorsements. If one of the participants in the GoGAR is keeping some of their judgements silent, then the integrity of the GoGAR is lost because the knowledge established was not actually socially determined.

On the other hand, if interviewers continually press interviewees for further reasoning until the interviewer is satisfied with the interviewee's reasoning, the clinical interview may be less about what an interviewee knows and more like teaching. Or, if an interviewer constantly presses an interviewee for further reasoning, the interviewer may further exacerbate the power dynamics between themself and the interviewee that already exist in traditional clinical interviews (Kvale, 2006). Consider a hypothetical example based on the clinical interview with Susan. Susan reconsidered all her commitments and reasoning about inverses because she was confident in her graph of the inverse of the exponential function, which was based on her memory of the graph of the natural log function. Her confidence in the shape of the graph of the natural log function even caused her to reconsider her most basic understanding of inverse. Hypothetically, the interviewer may have disagreed with Susan's graph of the natural log function and could have openly questioned whether she was entitled to her graph. If Susan did not give an adequate justification for her graph, the interviewer could have continued to press her for reasons. However, persistent questioning from the interviewer—someone socially positioned as more knowledgeable about mathematics—would likely cause Susan to feel less confident in her graph and she may consider drawing an alternative graph solely because of the power dynamics.

This methodological difficulty due to power relations echoes qualitative studies that suggest power relations and domination are typically embedded in clinical interviews. Within clinical interviews the researcher rules over the interview, the interview is instrumental, the interview may be manipulative, and the researcher often has a monopoly on the interpretation

(Kvale, 2006; Stinson & Bullock, 2015). Although some mathematics education researchers have attempted to reposition researchers and participants in a more egalitarian methodological structure with co-constructed narratives and interview protocols (Valero & Zevenbergen, 2004; Stinson & Bullock, 2015), conceptual analysis studies maintain a pronounced contrast between the researcher and the participant. In clinical interviews in conceptual analysis studies on mathematical content, the researcher provides the tasks, the researcher poses the questions and follow-up questions, the researcher decides which of the interviewee's comments to follow-up on, and the researcher keeps their mathematical judgements to themselves. None of these methodological decisions, however, accord with the characteristics of the GoGAR.

Ultimately, these difficulties with clinical interviews show a lack of methodological development in inferentialist qualitative research and points to the need for further study. How might an inferentialist researcher ethically conduct an interview with a student to assess their mastery of a concept without warping the GoGAR? Because inferentialist researchers believe knowledge is socially conferred and considering the methodological difficulties outlined above, inferentialist researchers may need to perform clinical interviews with multiple interviewees simultaneously or study focus groups with several individuals if they want to study students' mastery of a mathematical concept. Additionally, inferentialist researchers may need to become fuller participants of the GoGAR by allowing interviewees to ask questions of entitlement and pose problems to the interviewer so that the interviewee can better understand the interviewer's commitments and inferences.

Methodological Takeaways from Networking Inferentialism with DCA

The primary takeaway for my inferentialist analytic methodology from networking inferentialism with DCA—a methodology from the sociocultural perspective—is the limitation of my inferentialist methodology to provide larger-grain analyses. To empirically network inferentialism with DCA, I had to identify a shared analytic grain-size so the two perspectives could converse with one another. DCA, however, operates across multiple grain sizes and, put simply, can do much more than inferentialism. DCA is comprised of multiple theories that are used for multiple stages of analysis and is founded on a variety of social theories (including activity theory, sociocultural theory, and socioconstructivist theory) and the desire to describe collective mathematical activity (Rasmussen & Stephan, 2008). Rasmussen & Stephan developed a three-phase approach for DCA with some of the components from Toulmin's structure of arguments (i.e., data, claims, warrants, and backing), their own empirically tested criteria for identifying ideas that function as-if-shared (e.g., backing disappeared, component shifted, etc.), and their own organizational strategy for categorizing mathematical activity. This multi-theory and multi-phase approach allows DCA researchers to move efficiently from fine-grained analyses to a large-grain analysis of the collective mathematical growth of a classroom of students.

In contrast, inferentialism solely operates on one, fine-grained level of analysis. Hußmann et al. (2019) suggested inferentialism could be used to perform long-term analyses by compiling all the fine-grained, detailed analyses over a longer time frame. But their suggested method lacks efficiency and effective communication of results because, unlike DCA, there is no analytic mechanism used to move from one grain size to the next. DCA researchers (a) track basic Toulmin components of an argument, (b) look across the episodes of argumentation to see if any ideas began to function as-if-shared, and (c) categorize the ideas that functioned as-if-shared according to mathematical practices. Thus, they have three analytic mechanisms to move from a smaller grain size to a larger grain size: (a) they do not create full Toulmin diagrams for each episode of argumentation over every class session they are analyzing but only identify data, claims, warrants, and backing, (b) they identify larger ideas that function as-if-shared according to the three criteria, and (c) they organize those ideas into larger categories of practices. To make inferentialist research viable at larger grain sizes, researchers must develop similar analytic mechanisms. A compilation of fine-grained inferentialist analyses over a long period of time is too cumbersome to communicate relevant results effectively and efficiently.

There was an additional, smaller takeaway from networking inferentialism with DCA. An inferentialist analysis of students' participation in a class needs as much video data as possible. In the extant data from the content course, I often needed more video footage of the students' written work during their time in small groups for my inferentialist analysis. A researcher can analyze collective activity and collective argumentation with DCA with minimal video of students' written work; they primarily need students' verbal utterances. A researcher analyzing students' commitments and inferences within collective argumentation with inferentialism, however, needs more video to track students' performances and writing. My inferentialist methodology will be more powerful if all students' committal performances are codable.

Implications of Inferentialist Research for the Interplay of Individual and Social Activity

My inferentialist analyses have implications for my final research question: Given the inferentialist methodology developed in response to research question 2 and the empirical analyses from research question 1, what can an inferentialist perspective add to the field's understanding of the interplay of individual and collective activity? In the subsequent subsections, I detail how inferentialism can contribute to the field's understanding of specific topics by emphasizing the intertwined nature of individual and collective activity. More specifically, compared to a radical constructivist approach to conceptual analysis, an inferentialist approach to conceptual analysis (a) more fully depicts how the learning of mathematical concepts is embedded within a social and ethical context and (b) offers an alternative framework to discuss contradictions (or incompatibilities) within students' conceptual understanding and mathematical reasoning. Furthermore, compared to sociocultural approaches that analyze collective argumentation, inferentialism offers a new—albeit limited—approach to analyzing students' participation in collective argumentation. An inferentialist approach enables researchers to foreground individual students' learning while simultaneously attending to the social nature of collective argumentation, rather than focusing on the structure of the argument (i.e., Toulmin) or the collective growth of the students in the classroom (i.e., DCA).

Implications Drawn from Networking Inferentialism with Radical Constructivism

Compared to a radical constructivist approach to conceptual analysis, an inferentialist approach to conceptual analysis (a) more fully illustrates how the learning of mathematical concepts is embedded within a social and ethical context and (b) offers an alternative framework to discuss meaning and contradictions in relation to students' conceptual understanding and mathematical reasoning.

The Meaning of Mathematical Concepts within Social and Ethical Contexts

The focus on meaning and its relationship with knowledge has made radical constructivism very useful in mathematics education research (Thompson, 2008). As I described in Chapters 4 and 5, radical constructivists follow Piaget and locate meaning within the mental schemes of autonomous, meaning-making individuals. Inferentialist researchers also focus on meaning but see the locus of meaning within social contexts, practices and, more specifically, in the GoGAR where the concept's upstream, downstream, and incompatible inferences are socially calibrated. This divergence is particularly meaningful and has practical implications for mathematics education research. I believe radical constructivism is a strong learning theory with a thoroughly developed epistemological background based on Piaget's and von Glasersfeld's work. But I also believe its emphasis on the cognizing individual who constructs reality limits its use because it does not attend to sophisticated accounts of meaning that reside outside the individual and does not attend to important social aspects of learning and knowledge.

The episode from Elliot's clinical interview in Chapter 5 illustrated how the meaning of a mathematical concept is embedded within a social and ethical context. The problem, the setting with the interviewer, Elliot's experiences in the previous clinical interview and in the content course, and the power relations between Elliot and the interviewer shaped Elliot's reasoning about the graph's status as a function. Clinical interviews—given the more-obviously perceived hierarchy between interviewer and interviewee—may seem more susceptible to exacerbating power relations that influence the content of a mathematical concept. But Elliot's contextual reasoning and his contextual understanding of the mathematical concept of function was not an isolated incident. Consider an example from the Hulu series *The Dropout* that illustrates how the meaning of the concept of outlier requires an understanding of the social context and the power relations between interlocuters. *The Dropout* is a streaming series about the medical technology company Theranos, its founder Elizabeth Holmes (who has been convicted of wire fraud conspiracy), and Erika Cheung and Tyler Shultz (the key whistle-blowers who worked at Theranos). Theranos claimed to have cutting-edge blood testing technology that could efficiently run a variety of tests with a few drops of blood and at a low cost (Griffith, 2021). Not only did the technology not work, but the *New York Times* has reported that, "in some cases, outlier results of the blood tests were deleted to ensure that Theranos's technology passed quality control tests" (Griffith, 2021, paragraph 6).

In the made-for-streaming-television representation of events in Theranos's labs (Hulu, 2022), the conceptual content of outlier existed in the social context of their lab and the power-laden exchanges between colleagues. Erika, a young lab assistant at Theranos, initiated a give-and-take with her workplace superior and her superior's superior about the meaning of outlier while the president and chief operating officer of Theranos stood in the background, visible through the glass walls of his office. Erika first approached her supervisor because she believed she was being asked to "cherry-pick" data, which she said she could not do. Her supervisor, confused by Erika's unwillingness to follow instructions, asked why she was unwilling to do that. Tyler, a fellow lab assistant, responded that, by cherry-picking data, they would not be able to accurately determine if the company's blood tests worked. Their supervisor responded by saying the "cherry-picked" data should be deleted because they were outliers.

The supervisor of Erika's and Tyler's supervisor then entered the conversation and Erika asked him, "What exactly is an outlier? Because it seems to me, like, it's just a data point that isn't doing what you want it to do and—." Their supervisor's supervisor cut Erika off and said, "We consider that an outlier." Erika explained that, if that was the meaning of outlier, there would be outliers in every data run generated. Their supervisor's supervisor confirmed Erika's inference but reiterated that those data points should be deleted. Erika became rather flustered and said she did not want to cause trouble. Their supervisor's supervisor refuted Erika, said she was causing trouble, and ended the conversation by saying: "We delete outliers. That's what we do. [Four second pause.] All good?" The camera then panned to show the CEO of Theranos watching their exchange from afar through the glass walls of his office. Erika and Tyler nodded and, as they walked away from the conversation, Erika said, "I'm going to get fired."

The conversation ended with Erika and Tyler seemingly endorsing their superiors' commitments and inferences related to the meaning of outlier. It was not until later (in the series and in real-life), after Erika brought her concerns to a federal agency and the agency investigated Theranos's labs, that the federal government declared that Theranos was not entitled to their commitments and inferences about outliers.

Erika, Tyler, their supervisor, and their supervisor's supervisor were engaged in the GoGAR—giving and asking for reasons about the meaning of the concept outlier—which was embedded within a social (and thus ethical) context that heavily shaped the meaning of the concept. Outliers were ultimately determined to be data points that did not accord with the higher-ups' desired outcome of an experiment and, consequently, were deleted. Tyler and Erika drew multiple downstream inferences: they said that this understanding of outlier would (a) not provide accurate information to determine if the blood tests work and (b) require them to delete data in every data run. Those reasons, however, were overruled by the commitment of a more powerful superior: "We delete outliers. That's what we do." Their supervisor's position in the company was the unsaid upstream inference that justified his commitment. Erika then endorsed this meaning for outlier to preserve her job.

In summary, the social setting, capitalist motives, and power relations between Erika, Tyler, and their superiors shaped the meaning of outlier within the context of the Theranos labs. Radical constructivism, as mentioned in Chapter 5, does not have the internal tools to make sense of this type of epistemic phenomena without other (potentially incompatible) theories. The epistemological complexities of the social aspect of mathematical concepts (such as contextual, relational, and power-related dynamics) are insufficiently accounted for within radical constructivism but can be captured with inferentialism.

Meaning and Contradictions

Inferentialism also offers an alternative framework to discuss meaning and contradictions within students' conceptual understanding and mathematical reasoning. I believe the radical constructivist account of meaning and contradictions—with the locus of meaning and knowledge within the individual—requires further refinement. For example, radical constructivists' rejection of the word *misconception* for the language of *current conceptions* helps them avoid a deficit perspective of students' learning: students' contributions are valued and descriptions of students' knowledge are not framed by how it may fall short from ideal standards. The aim of this radical constructivist perspective is noble but, in practice, I believe the perspective itself has an inherent contradiction. To illustrate this contradiction, I will briefly analyze a passage from Gutiérrez (2018) that is typical of a cognitive constructivist perspective on conceptions and misconceptions. Gutiérrez described how many teachers are trained to anticipate students' misconceptions in their lesson planning. Gutiérrez argued this practice is dehumanizing because

> Students don't have misconceptions. They have conceptions. And those conceptions make sense for them, until they encounter something that no longer works. They are only 'misconceptions' when we begin with the expectation that others need to come to *our* way of thinking or viewing the world. (p. 2)

Gutiérrez's use of the words *misconception*, *conceptions*, and *no longer works* are indicative of the cognitive constructivist perspective. She said students do not have misconceptions but instead have conceptions that are valid conceptions until the individual abandons them for other conceptions that allow them to better adapt in the world.

Although I believe Gutiérrez's intentions are good and I agree that mathematics education ought to be rehumanized, the cognitive constructivist perspective she espouses betrays itself. Consider a teacher or teacher educator who regularly uses the word *misconception* in their practice and then reads Gutiérrez's article. When Gutiérrez tells this reader their students do not have misconceptions, Gutiérrez implicitly says the reader has a misconception about the word misconception. The contradiction is more pronounced when enacted in an in-person setting. Imagine a conversation between a mathematics teacher educator and a mathematics teacher. The teacher uses the word misconception to describe some of her students' struggles in class and the mathematics teacher educator immediately corrects her by saying something like Gutiérrez: "Students do not have misconceptions. They have conceptions." In these cases, the perspective of the person being corrected (the reader or the mathematics teacher) does not have epistemological content. Their perspective is not considered valid and is not treated as something that makes sense to them.

Gutiérrez and the mathematics teacher educator are attempting to cultivate epistemic humility within a cognitive or radical constructivist framework, but their claims have the opposite effect. They can be exertions of power that invalidate another person's use of the word misconception. Cognitive and radical constructivists insist on this conception of conceptions because understanding and meaning are, from their perspective, located within the autonomous individual. As I described in Chapter 4, they reject traditional notions of objectivity and instead promote the idea of autonomous individuals whose thoughts and understandings always have epistemological content and are valid ways of adapting in the world. Like radical constructivists, I reject traditional notions of objectivity; but I also believe the radical constructivist theory of autonomous individuals constructing viable models of the world has issues that need to be addressed.

The inferentialist idea of social objectivity is a helpful alternative. In inferentialist terms, Gutiérrez's written claim and the mathematics teacher educator's verbal claim are commitments that they are bound to along with the immediate inferences. When they correct someone's use of the term *misconception*, they imply that the person had an incorrect understanding of the situation. But they cannot denounce the term *misconception* and be entitled to correct someone. The two commitments infer different things and cannot be held together. As Brandom (2000) said, "We can say that two assertible contents are incompatible in case commitment to one precludes entitlement to the other" (p. 194).

Inferentialism allows researchers to highlight incompatibilities—not all commitments and inferences have equivalent epistemological content—but still value students' ideas. Consider the episode from Susan's clinical interview analyzed in Chapter 5. Although Susan ended the episode with incomplete ideas about inverses, her ability to identify the incompatibility in her thinking and navigate her web of inferences illustrated a high level of mastery. She was able to trace, in multiple inferential directions, the effects of the incompatibility and how it called into question various commitments and inferences. This sort of adaptive reasoning exemplifies the ability to think logically about the relationships between concepts and representations (Kilpatrick et al., 2001) even though she did not come to a complete conclusion. Thus, inferentialism jointly accounts for the individual and social components of meaning and offers a nuanced account of contradictions and objectivity.

Implications Drawn from Networking Inferentialism with the Sociocultural Perspective

In this section, I describe three implications I have drawn from comparing inferentialism with the sociocultural perspective and, more specifically, DCA. First, DCA is more efficient and powerful in its ability to document the collective activity of a classroom community. Second, unlike inferentialism, DCA requires researchers to adopt irreconcilable perspectives in order to simultaneously make sense of the social and individual aspects of learning mathematics. Third and finally, my inferentialist methodology has the unique ability to account for social and individual aspects of learning across settings (i.e., clinical interviews and classrooms).

The Efficiency and Power of DCA

I want to state plainly: with its three phases that coherently move from (1) Toulmin diagrams to (2) the application of criteria to identify ideas that function as-if-shared to (3) the documentation of mathematical practices that relate to the ideas that function as-if-shared, DCA is a more efficient methodology to document collective mathematical growth in comparison to inferentialism. I was able to use my inferentialist methodology to assess students' mastery of different concepts in the content class, identify some normative ways of reasoning in the class, and perform a long-form analysis of Elliot's understanding of functions across two class meetings. However, I was unable to make general claims about the collective progress of the classroom community with my inferentialist methodology. To make such claims with my inferentialist methodology, I would need to develop analytic mechanisms that could move from fine-grained analyses of students' mastery of concepts within an instantiation of the GoGAR to a more general description of the content mastered by multiple students within every instance of the GoGAR across multiple class periods. DCA, however, already has elaborate and rigorously tested built-in mechanisms that enable researchers to efficiently document the collective mathematical growth of a classroom community.

Incompatible Theoretical Synthesis

Rasmussen and Stephan (2008) stressed that the DCA methodology describes the mathematical activity of a classroom community and does not describe the intellectual achievement of individual students in the classroom. They also said DCA could be used to better understand a classroom learning environment in conjunction with a cognitive constructivist account of individual students actively and mentally constructing meaning for mathematical concepts. As an example, Rasmussen, Wawro, and Zandieh (Rasmussen et al., 2015) used DCA in conjunction with radical constructivism to expand Cobb and Yackel's (1996) interpretive framework. The interpretive framework is rooted in the emergent perspective which sees mathematical development as "a process of active individual construction and a process of mathematical enculturation" (Rasmussen et al., 2015, p. 260). Rasmussen et al.'s explicitly stated goal was to expand the interpretive framework in order to coordinate empirical analyses of the individual and social mathematical progress of students in an undergraduate linear algebra course. The authors cited Prediger et al.'s (2008b) "combining and coordinating" networking strategy which, as described in my second chapter, "does not necessitate the complementarity or even the complete coherence of the theoretical approaches in view. Even theories with conflicting basic assumptions can be combined in order to get a multi-faceted insight into the empirical phenomenon in view" (p. 173).

However, because Rasmussen et al.'s coordination of their empirical analyses was part of an attempt to synthesize different perspectives to further develop a theory, their study more fittingly falls into Prediger et al.'s (2008b) "synthesizing and integrating locally" networking strategy. Prediger et al. (2008b) said "the strategies of synthesizing and integrating locally are focused on the development of theories by putting together a small number of theoretical approaches into a new framework" (p. 173). When Rasmussen et al.'s work is understood as a theoretical synthesis, it preserves a theoretical tradition that has failed to respond to Lerman's

(1996) claim that radical constructivism and social perspectives are from irreconcilable worldviews. Lerman said:

> There is a substantial body of literature that argues that there can be no resolution of different world views…Neither the mentalistic, individualistic psychology nor the cultural, discursive psychology can be discounted or disproved by the other. I want to argue, however, that a merger of these two views is incoherent and can only be attempted by not engaging fully with their distinct interpretations of the individual in her or his actions in the world. (p. 138)

To translate Lerman's words into Prediger's categories: it is incoherent to attempt to synthesize two perspectives before doing "the hard work of understanding others and reciprocally, with making [one's] own theory understandable" (Prediger et al., 2008b, p. 171).

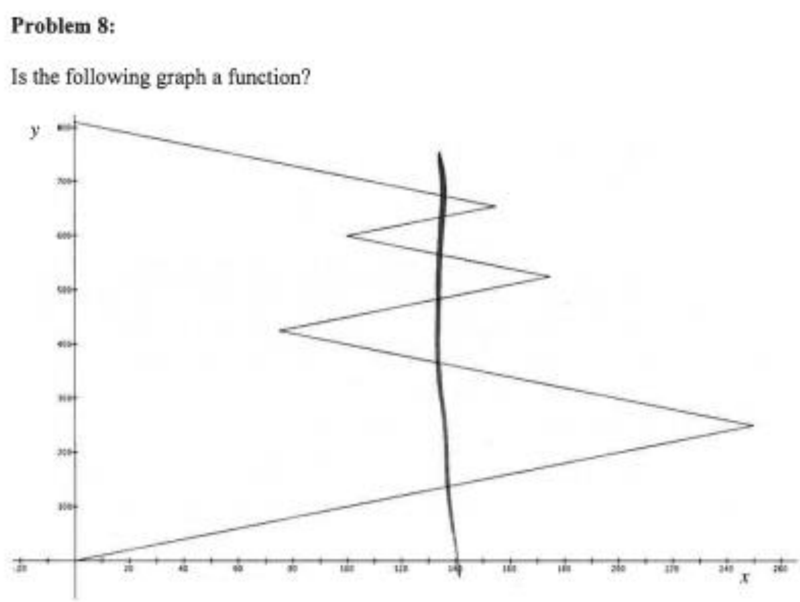
Inferentialism does not face this issue because it simultaneously accounts for individual and social aspects of learning mathematical concepts. In inferentialism, knowledge is the socially evaluated status of an individual's mastery of a concept or practice as indicated by the person's ability to reason with it. So, although my inferentialist analysis foregrounds individuals' commitments, inferences, questions of entitlement, entitlements, and endorsements in the GoGAR, a student's mastery of a mathematical concept is assessed according to socially determined normative ways of reasoning. A student has not demonstrated mastery until their peers have endorsed their ideas or granted them entitlement. Thus, inferentialist accounts of an individual's learning requires an account of their peers' engagement with their ideas.

Inferentialism Across Settings

Inferentialism's ability to simultaneously capture individual and social components of learning enable it to be used across different contexts. I used my inferentialist methodology to analyze Elliot's and Susan's mastery of the concepts of function and inverse within the context of their clinical interviews and used it to analyze students' mastery of content within episodes of collective argumentation within the context of the content course. The ability to use the same theory and methodology across contexts allows researchers to look across data and make more general inferences. For example, in his first clinical interview (which took place near the beginning of the semester of the content course), Elliot was given a problem that asked if the following graph (Figure 26) was a function. Elliot immediately said the graph was not a function and provided a single upstream inference: "Vertical line test."

Figure 26

A Problem and Elliot's Work from His First Clinical Interview

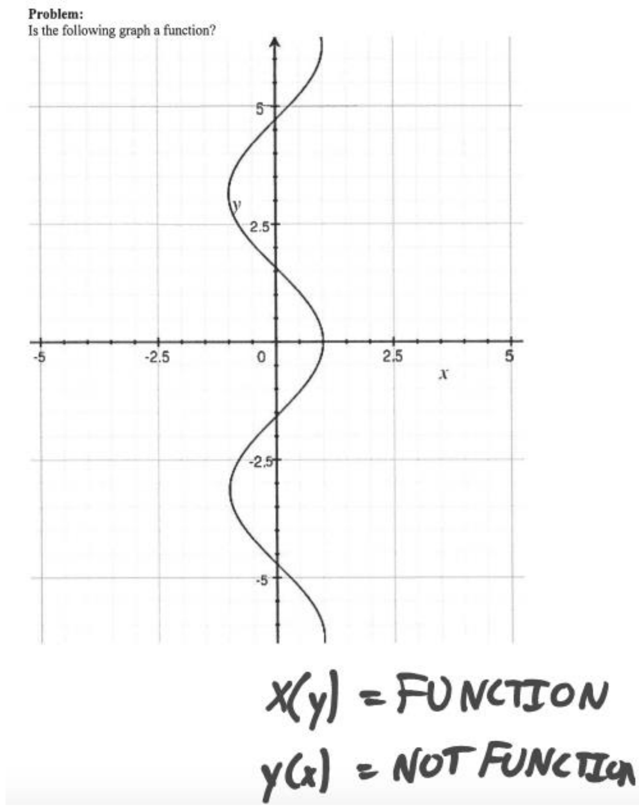


The classroom episodes from the content course featured in the previous chapter occurred between Elliot's first clinical interview and his second clinical interview. Near the end of the previous chapter, I described how Elliot's and Melissa's conversation in their small group may have been evidence that Elliot relied on the same upstream inference (a visual assessment like the vertical line test) to justify his conclusion that their graph would represent a function. However, later in the whole class exchange about vertical distance as a function of time, Elliot did not rely on a visual assessment of a graphical representation to determine if vertical distance was a function of distance. Instead, Elliot said the relationship between the two quantities would be a function because there would not be multiple outputs ("two vertical distances") for a single input ("at a single time"). I suggested this may be evidence that Elliot grew in his mastery of the concept of function because he attended to the relationship between inputs and outputs and did not rely on a visual assessment of a graph with built-in assumptions about which axes represented the input and the output.

Fast-forward to Elliot's second clinical interview, approximately a month and a half after the analyzed episodes from the content course. In the interview, Elliot is given a problem (Figure 27) that asked if the following graph was a function. Elliot, as seen in his writing at the bottom of the page, made a more nuanced commitment about the relation between functions and graphs and supported it with multiple upstream inferences. Elliot said the graph would not be a function if "you think of your x as your inputs and your y's as your outputs," but that it would be a function "if you have your y's as your inputs and your x's as your outputs."

Figure 27

A Problem and Elliot's Work from His Second Clinical Interview



These episodes provide snapshots of Elliot's growth in his mastery of the concept of function in graphical contexts. Through Elliot's engagement in the GoGAR in the content class, he grew in his understanding of the inferential implications of the concept of function. His commitment ("it's not [a function]") and sole upstream inference from the first clinical interview ("vertical line test") was like connecting two dots with a single line. In contrast, in the second clinical interview, his commitment—that the graph may or may not be a function—and two upstream inferences were like connecting several dots with multiple lines: his understanding of the concept of function was connected to the concepts of input and output. This sort of consistent analysis with a single theory and methodology across data from different contexts is a strength of inferentialism. It can provide an overarching explanatory narrative through the analysis of students' understanding of mathematical concepts in clinical interviews as well as students' understanding of mathematical concepts within collective argumentation in classroom settings.

Discussion

Personal Reflection

As I said in Chapter 1, philosophical and theoretical analysis is important for mathematics education research. The emergence of a new theory of meaning—inferentialism—inspired me to better establish the theory's identity and its merits more properly by networking it with radical constructivism and the sociocultural perspective. Prior to networking the theories, I discussed the autobiographical nature of my research and situated the entire project within a social imaginary and my worldview. More specifically, I described my Christian belief that humans are irreducibly communal and individual reflecting the triune nature of the divine. My desire to make sense of mathematics education in light of the Trinity stemmed from my perception of the deeply relational nature of humanity and mathematics. Humans should not be reduced to individuals siloed off from one another and the purpose of mathematics should be human flourishing. Human activity—the learning of mathematics included—is always relational and interpersonal, charged with social and ethical concerns. Inferentialist researchers claimed inferentialism could capture these varied aspects of learning mathematics and I chose to use inferentialism as my "fecund discursive space." Upon completion of my study, I have identified multiple limitations of using inferentialism for mathematics education research, but I still consider it a productive theory for my future work.

Contributions to the Field

My study contributes to the field by simultaneously clarifying the identity of inferentialism, explicating an inferentialist analytic methodology, furthering inferentialist research on collective argumentation and students' mastery of multiple mathematical concepts, pursuing human flourishing, and researching research. I networked inferentialism with radical constructivism and the sociocultural perspective and, as a result, responded to recent calls to compare and contrast inferentialism with more established theories in the field (International Commission on Mathematical Instruction, 2023). Radical constructivists see humans as autonomous individuals that construct meaning for themselves in order to adapt to their environment. Proponents of the sociocultural perspective focus on human participation in communities within social and cultural contexts, reframe psychology as a "social anthropology of cognition," and often adopt Wittgenstein's idea of language-games where the norms of reasoning and the meaning of words are embedded in a form of life. Proponents of inferentialism maintain similar ideas about language-games and humans' participation in communities within social and cultural contexts but focus more acutely on human participation in a more rational context called the game of giving and asking for reasons (GoGAR).

I also identified the affordances and limitations of inferentialism in comparison to the other theories. My theoretical and empirical analyses underlined the power of DCA's ability to move across grain sizes to make general claims about the collective learning of a classroom. But the GoGAR provides a new angle to study students' participation in collective argumentation in the mathematics classroom. Sociocultural researchers often focus on collective mathematical growth but, by focusing on individuals' participation in the GoGAR, inferentialist researcher can foreground individuals' learning within collective argumentation and simultaneously attend to how ideas receive normative status in the classroom.

As a semantic theory, inferentialism shares radical constructivism's emphasis on meaning. Proponents of both perspectives see the close connection between meaning and knowledge and can closely study the learning of mathematical concepts through inference. Although radical constructivism has a more established tradition of research to draw on, inferentialism enables researchers to analyze the social and contextual factors at play in students' mathematical reasoning. My theoretical and empirical inferentialist analyses showed how even the learning of tightly and rigorously defined mathematical concepts is social. Mathematical concepts are only used by humans in relationship with one another and the meaning of the concepts are caught up in the details of the social context and setting. Inferentialism also offers researchers new conceptions of contradictions and objectivity that contrast those of cognitive and radical constructivism.

Although my study focused on the importance of meaning and collective argumentation for the learning of mathematics, meaning and collective argumentation are primarily significant because of their role in human flourishing. Francis Su (2020) described how mathematics, when pursued out of a desire for meaning, develops the virtues of story building, abstract thinking, persistence, and contemplation. Su also described how mathematics, when pursued out of a desire for community, develops the virtues of hospitality, excellence in teaching, excellence in mentoring, disposition to affirm others, self-reflection, attention to people, and vulnerability. Thus, my investigation of the inferentialist account of meaning and collective argumentation in mathematics contributes to the pursuit of meaning and community. Different theories of meaning and collective argumentation emphasize different things, and researchers need to determine which theory best contributes to a larger vision of human flourishing.

Finally, my study contributes to larger conversations about researching research (Pais & Valero, 2012), putting philosophy to work (Cobb, 2007), and networking research practices (Prediger et al., 2008). I augmented Prediger et al.'s networking strategies by explicitly discussing my social imaginary and worldview before putting inferentialism, radical constructivism, and the sociocultural perspective in conversation with one another. My positionality, philosophical anthropology, mathematical ontology, epistemology, and religion helped explain why I chose to network the three theories and made my assessment of the theories more transparent. The criterion I used to assess the theories' affordances and constraints were partially dependent on my worldview, my assessment of the field, and my close theoretical, methodological, and empirical analyses of the theories.

Thus, my attempt to put philosophy to work continues the conversation initiated by Cobb (2007) about comparing and adopting theories. Cobb simultaneously emphasized that there was no neutral framework to compare incommensurable theoretical frameworks and that theoretical decisions should not be made according to "personal whim or taste" (p. 32). He attempted to transcend that dichotomy by suggesting the usefulness of theories be determined according to his description of mathematics education as a design science. In a slight divergence, I believe a researcher's explicit inclusion of their worldview is an important component in theorizing. Following Siy (2019) and Stinson (2020), I believe researchers should make their philosophical and theoretical processes products by explicitly documenting how their worldview and subjectivities informed their theoretical and methodological decisions.

Future Research

My dissertation and the discussion above pose interesting questions for future research about researching research and philosophical considerations of research. For example, to what extent are researchers responsible for investigating the philosophical underpinnings of a theory? And to what extent should researchers explicate their worldviews in their research? Furthermore, how can a field comprised of researchers with a plurality of worldviews create theoretical discursive spaces to collaborate? Over 15 years ago, Cobb initiated a conversation about how philosophy could be put to work to cope with multiple conflicting theoretical perspectives. My dissertation aims to reanimate that conversation among mathematics education researchers.

The inferentialist methodology I developed captures the individual and social aspects of learning mathematical concepts and can be a launchpad for future research. The methodology can be used (a) to investigate students' learning of other mathematical concepts, (b) in different contexts and with students of different ages than the ones featured in my analyses and (c) to simultaneously investigate students' conceptual understanding and mathematical reasoning. It could also be extended to include mechanisms to move from one grain size to another like DCA. Other inferentialist researchers should also propose different ways to operationalize the mastery learning metaphor.

Inferentialism's ability to capture social and power dynamics within mathematical reasoning should also be explored. For example, inferentialist researchers could research questions like: When are students' ideas granted entitlement because of social and power dynamics? How often is social status used as an upstream inference to justify a mathematical commitment? Like Elliot's reasoning that was dependent on the context of his clinical interview, are there characteristics of students' mathematical reasoning that are unique to a classroom context? Furthermore, how can inferentialism be used to address other issues related to equity and inclusion?

Finally, I networked inferentialism with two learning theories, but, because inferentialism is a theory of meaning, it could be networked with different theories. For example, inferentialism could be networked with theories that study the meaning of mathematics education, school mathematics, or mathematics education research. How might inferentialism assist scholars in exploring questions like: (a) What is the political significance of mathematics education in different countries and cultures? (b) Why is some mathematics deemed significant for study in formal school settings and other mathematics is not? (c) What counts as mathematics education research? These potential questions and directions make inferentialism a generative tool for future research in mathematics education.

REFERENCES

American Psychological Association. (n.d.) Psychology. In *APA Dictionary of Psychology*. Retrieved September 21, 2022, from https://dictionary.apa.org/psychology

Arendt, H. Scott, J.V., & Stark, J.C. (1996). *Love and Saint Augustine*. University of Chicago Press.

Augustine, S. (2009). *The city of God*. Hendrickson Publishers.

Autonomous. (n.d.). In *OED Online*. Retrieved January 11, 2023, from www.oed.com/view/Entry/13500.

Bakker, A. (2018). *What is design research in education?*. Routledge.

Bakker, A., & Derry, J. (2011). Lessons from inferentialism for statistics education. *Mathematical Thinking and Learning*, *13*(1-2), 5–26. https://doi.org/10.1080/10986065.2011.538293

Bakker, A., Ben-Zvi, D., & Makar, K. (2017). An inferentialist perspective on the coordination of actions and reasons involved in making a statistical inference. *Mathematics Education Research Journal*, *29*(4), 455–470. https://doi.org/10.1007/s13394-016-0187-x

Bakker, A., & Hußmann, S. (2017). Inferentialism in mathematics education: introduction to a special issue. *Mathematics Education Research Journal*, *29*(4), 395–401. https://doi.org/10.1007/s13394-017-0224-4

Benedict XVI. (2004). Funeral homily for Msgr. Luigi Giussani. *Communio: International Catholic Review 31*(4), 685–687. https://www.communio-icr.com/files/ratzinger31-4.pdf

Benedict XVI. (2006). *Deus Caritas Est*. https://www.vatican.va/content/benedict-xvi/en/encyclicals/documents/hf_ben-xvi_enc_20051225_deus-caritas-est.html

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—an approach for exploiting the diversity of theoretical approaches. In *Theories of mathematics education* (pp. 483–506). Springer.

Bikner-Ahsbahs, A., & Prediger, S. (2014). Networking as research practices: Methodological lessons learnt from the case studies. In Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 235–247). Springer.

Bikner-Ahsbahs, A., Prediger, S., Artigue, M., Arzarello, F., Bosch, M., Dreyfus, T. Gascón, Halverscheid, S., Haspekian, M., Kidron, I., Croblin-Lenfant, A., Meyer, A., Sabena, C., & Schäfer, I. (2014). Starting points for dealing with the diversity of theories. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 3–13). Springer.

Bishop, A. (1991). *Mathematical enculturation: A cultural perspective on mathematics education*. Springer Science & Business Media.

Brandom, R. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, Robert. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. *International Journal of Research in Undergraduate Mathematics Education, 1*(2), 209–233. https://doi.org/10.1007/s40753-015-0013-y

Cerulli, M., Trgalova, J., Maracci, M., Psycharis, G., & Georget, J. P. (2008). Comparing theoretical frameworks enacted in experimental research: TELMA experience. *ZDM*, *40*(2), 201–213. https://doi.org/10.1007/s11858-008-0075-2

Cobb, P. (2006). Discursive perspectives on mathematical learning: Commentary on Sfard's and Lerman's chapters. In J. Maasz & W. Schlöglmann (Eds.), *New mathematics education research and practice* (pp. 189–201). Sense. https://doi.org/10.1163/9789087903510_017

Cobb, P. (2007). Putting philosophy to work. In Lester, F. (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 3-38). Information Age Publishing.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31(*3-4), 175–190. https://doi.org/10.1080/00461520.1996.9653265

Cole, R., Becker, N., Towns, M., Sweeney, G., Wawro, M., & Rasmussen, C. (2012). Adapting a methodology from mathematics education research to chemistry education research: Documenting collective activity. *International Journal of Science and Mathematics Education*, *10*(1), 193–211. https://doi.org/10.1007/s10763-011-9284-1

Confrey, J. (1991). Learning to listen: A student's understanding of powers of ten. In *Radical constructivism in mathematics education* (pp. 111–138). Springer.

Conner, A., Singletary, L.M., Smith, R.C., Wagner, P.A., & Francisco, R.T. (2014) Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401–429. https://doi.org/10.1007/s10649-014-9532-8

Conner, A., Tabach, M. & Rasmussen, C. (2022). Collectively engaging with others' reasoning: Building intuition through argumentation in a paradoxical situation. *International Journal of Research in Undergraduate Mathematics Education*, 1–28. https://doi.org/10.1007/s40753-022-00168-x

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches* (2nd ed.). Sage Publications.

Deontic. (n.d.). In *OED Online*. Retrieved January 11, 2023, from https://www.oed.com/view/Entry/50152.

Derry, J. (2013). Can inferentialism contribute to social epistemology?. *Journal of Philosophy of Education*, *47*(2), 222–235. https://doi.org/10.1111/1467-9752.12032

Derry, J. (2017). An introduction to inferentialism in mathematics education. *Mathematics Education Research Journal*, *29*(4), 403–418. https://doi.org/10.1007/s13394-017-0193-7

Devadoss, S. L., & Hoffoss, D. (2019). Unfolding Humanity: Mathematics at Burning Man. *Notices of the American Mathematical Society*, *66*(4). https://dx.doi.org/10.1090/noti1857

Devadoss, S. & Schuh, O. (2019). Cartography of tree space. *Leonardo Journal*, *52*, 279–283. https://doi.org/10.1162/leon_a_01475

Douglass, F. (n.d.). *Narrative of the life of Frederick Douglass, an American slave*. Open Road Media.

Ernest, P. (1995). Values, gender and images of mathematics: A philosophical perspective. *International Journal of Mathematical Education in Science and Technology*, *26*(3), 449–462. https://doi.org/10.1080/0020739950260313

Ernest, P. (2012). What is our first philosophy in mathematics education? *For the Learning of Mathematics*, *32*(3), 8–14.

Ernest, P., Skovsmose, O., Van Bendegem, J. P., Bicudo, M., Miarka, R., Kvasz, L., & Moeller, R. (2016). *The philosophy of mathematics education*. Springer Nature.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*. *21*(6), 521–544. https://doi.org/10.1007/BF00315943

Even, R. (1992). The inverse function: Prospective teachers' use of 'undoing'. *International Journal of Mathematical Education in Science and Technology*. *23*(4), 557–562. https://doi.org/10.1080/0020739X.1992.10715689

Fuller Theological Seminary [FULLER studio]. (2018, May 8). *Blue collar mathematics | Satyan Devadoss* [Video]. YouTube. https://www.youtube.com/watch?v=yR6saOaGXNg

Gómez Marchant, C.N., & Jones, S.R. (2021). Let us be the healing of the wound/Seamos la Curación de la Herida. In Olanoff, D., Johnson, K., & Spitzer, S. M. (Eds.). *Proceedings of the forty-third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Gorman, M. J. (2011). *Reading Revelation responsibly: Uncivil worship and witness: Following the Lamb into the new creation*. Wipf and Stock Publishers.

Griffith, E. (2021, October 6). Theranos whistle-blower testifies she was alarmed by company's blood tests. *New York Times*. https://www.nytimes.com/2021/09/14/technology/elizabeth-holmes-trial-theranos.html

Gutiérrez, R. (2017). Living Mathematx: Towards a Vision for the Future. In Galindo, E., & Newton, J., (Eds.). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, IN.

Gutiérrez, R. (2018). Introduction: The Need to Rehumanize Mathematics. In I. Goffney, R. Gutiérrez, & M. Boston (Eds.), *Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (Annual Perspectives in Mathematics Education; Vol. 2018). National Council of Teachers of Mathematics.

Hackenberg, A. J. (2005). A model of mathematical learning and caring relations. *For the Learning of Mathematics, 25*(1), 45-51.

Hallett, G. L. (1977). *A companion to Wittgenstein's "Philosophical investigations"*. Cornell University Press.

Harvey Mudd College [HarveyMuddCollegeEDU]. (2016, November 23). *Satyan Devadoss: the shape of nature: bee, tree, origami* [Video]. YouTube. https://www.youtube.com/watch?v=hLhmCUJkqBM

Hulu. (2022, March 24). *The Dropout | Episode 6 Clip | Hulu* [Video]. YouTube. https://www.youtube.com/watch?v=pijhcZ-fhIM

Hußmann, S., Schacht, F., & Schindler, M. (2019). Tracing conceptual development in mathematics: epistemology of webs of reasons. *Mathematics Education Research Journal*, *31*, 133-149. https://doi.org/10.1007/s13394-018-0245-7

International Commission on Mathematical Instruction. (2023). *Topic Study Group 5.8: Philosophy of mathematics and mathematics education*. 15th International Congress on

Mathematical Education. https://icme15.org/icme-15-scientific-program/topic-study-groups/topic-study-group-5-8-philosophy-of-mathematics-and-mathematics-education/

John Paul II, Pope. (1997). *The theology of the body according to John Paul II: Human love in the divine plan*. Pauline Books and Media.

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *ZDM, 40*(2), 247–264. https://doi.org/10.1007/s11858-008-0079-y

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up. Helping children learn mathematics*. National Academy Press.

Kvale, S. (2006). Dominance through interviews and dialogues. *Qualitative inquiry*, *12*(3), 480-500.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm?. *Journal for Research in Mathematics Education*, *27*(2), 133–150. https://doi.org/10.2307/749597

Lerman, S. (1998). A moment in the zoom of a lens: Towards a discursive psychology of mathematics teaching and learning. In A. Olivier & K. Newstead (Eds.), *Proceedings of the twenty-second annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, 66-81). Stellenbosch, South Africa: Faculty of Education, The University of Steelenbosch.

Lerman, S. (2000). A case of interpretations of social: A response to Steffe and Thompson. *Journal for Research in Mathematics Education*, *31*(2), 210–227. https://doi.org/10.2307/749752

Lerman, S. (2001). Cultural, discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics*, *46*(1), 87–113. https://doi.org/10.1023/A:1014031004832

Lerman, S. (2006). Cultural psychology, anthropology and sociology: The developing ‘strong’ social turn. In J. Maasz & W. Schlöglmann (Eds.), *New Mathematics Education Research and Practice* (pp. 171–188). Sense. https://doi.org/10.1163/9789087903510_016

Littlewood, J. E. (1986). *Littlewood's miscellany*. Cambridge University Press.

Lockhart, P. (2009). *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form*. Bellevue Literary Press.

Lucus, C. A. (2005). *Composition of functions and inverse function of a function: Main ideas as perceived by teachers and preservice teachers* (Unpublished doctoral dissertation). Canada: Simon Fraser University, BC.

Meek, E. L. (2011). *Loving to know: Covenant epistemology*. Wipf and Stock Publishers.

Meyer, M. (2014). An inferential view on concept formation. In P. Liljedahl, S. Oesterle, C. Niole & D. Allen (Eds.), *Proceedings of PME 28 and PME-NA 36* (Vol. 4, pp. 217–224). Vancouver.

Meyer, M. (2018) Using rules for elaborating mathematical concepts. In: Ernest P. (eds) *The Philosophy of Mathematics Education Today*. ICME-13 Monographs. Springer. https://doi.org/10.1007/978-3-319-77760-3_18

Moore, K. C., LaForest, K. R., & Kim, H. J. (2016). Putting the unit in pre-service secondary teachers' unit circle. *Educational Studies in Mathematics*, *92*(2), 221–241. https://doi.org/10.1007/s10649-015-9671-6

Moore, K. C., Silverman, J., Paoletti, T., Liss, D., & Musgrave, S. (2019a). Conventions, habits, and US teachers' meanings for graphs. *The Journal of Mathematical Behavior*, *53*, 179–195. https://doi.org/10.1016/j.jmathb.2018.08.002

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L., & Liang, B. (2019b). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*, 100692. https://doi.org/10.1016/j.jmathb.2019.01.008

Nilsson, P. (2020). A Framework for Investigating Qualities of Procedural and Conceptual Knowledge in Mathematics—An Inferentialist Perspective. *Journal for Research in Mathematics Education*, *51*(5), 574–599. https://dx.doi.org/10.5951/jresematheduc-2020-0167

Noddings, N. (1990). Chapter 1: Constructivism in mathematics education. *Journal for Research in Mathematics Education. Monograph*, *4*, 7–18. https://doi.org/10.2307/749909

Noorloos, R., Taylor, S., Bakker, A., & Derry, J. (2014). An inferentialist alternative to constructivism in mathematics education. In Liljedahl, P., Nicol, C., Oesterle, S., & Allan, D. (Eds.). *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1). PME.

Noorloos, R., Taylor, S. D., Bakker, A., & Derry, J. (2017). Inferentialism as an alternative to socioconstructivism in mathematics education. *Mathematics Education Research Journal*, *29*(4), 437–453. https://doi.org/10.1007/s13394-017-0189-3

Norton, A., & Wilkins, J. L. (2012). The splitting group. *Journal for Research in Mathematics Education*, *43*(5), 557–583. https://doi.org/10.5951/jresematheduc.43.5.0557

Nouwen, H.J.M. (1975). *Reaching out: the three movements of the spiritual life*. Doubleday.

Pais, A., & Valero, P. (2012). Researching research: Mathematics education in the political. *Educational Studies in Mathematics*, *80*, 9-24. https://doi.org/10.1007/s10649-012-9399-5

Paoletti, T., Stevens, I.E., Hobson, N.L.F., Moore, K.C., & LaForest, K.R. (2018). Inverse function: Pre-service teachers' techniques and meanings. *Educational Studies in Mathematics*. *97*(1), 93–109. https://doi.org/10.1007/s10649-017-9787-y

Piaget, J. (1968). *Structuralism*. Press Universitaires de France.

Piaget, J. (1970). *Genetic epistemology*. Columbia University Press.

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (2008a) (Eds.). Comparing, Combining, Coordinating - Networking strategies for connecting theoretical approaches. *Thematic Issue of ZDM - The International Journal on Mathematics Education, 40* (2), 163–327. https://doi.org/10.1007/s11858-008-0093-0

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008b). Networking strategies and methods for connecting theoretical approaches – First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40* (2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Oxford University Press. (n.d.) Psychology. In *OED Online.* https://www.oed.com/view/Entry/153907?redirectedFrom=psychology#eid

Quale, A. (2014). Ethics: A Radical-constructivist Approach. *Constructivist Foundations*, *9*(2), 256–261.

Radford, L. (2004). Sensible things, essences, mathematical objects, and other ambiguities. *La Matematica e la Sua Didatica*, *1*, 4–23.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM - The International Journal on Mathematics Education*, *40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Radford, L. (2017). On inferentialism. *Mathematics Education Research Journal*, *29*(4), 493–508. https://doi.org/10.1007/s13394-017-0225-3

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *The Journal of Mathematical Behavior*, *23*(3), 301-323. https://doi.org/10.1016/j.jmathb.2004.06.003

Rasmussen, C., & Stephan, M. (2008). A Methodology for Documenting Collective Activity. In A. Kelly, R. Lesh, & J. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*, (pp. 195–215). Routledge.

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*(2), 259-281. https://doi.org/10.1007/s10649-014-9583-x

Reeves, M. (2012). *Delighting in the Trinity: An introduction to the Christian faith*. InterVarsity Press.

Rorty, R. (1979). *Philosophy and the Mirror of Nature*. Princeton University Press.

Russell, B. (1919). The study of mathematics. *Mysticism and logic: And other essays*, 60.

Rutledge, F. (2015). *The crucifixion: Understanding the death of Jesus Christ*. Wm. B. Eerdmans Publishing.

Ryan, U., & Parra, A. (2019). Epistemological aspects of multilingualism in mathematics education: an inferentialist approach. *Research in Mathematics Education*, *21*(2), 152–167. https://doi.org/10.1080/14794802.2019.1608290

Schindler, M., Hußmann, S., Nilsson, P., & Bakker, A. (2017). Sixth-grade students' reasoning on the order relation of integers as influenced by prior experience: an inferentialist analysis. *Mathematics Education Research Journal*, *29*(4), 471–492. https://doi.org/10.1007/s13394-017-0202-x

Schmidt, S. J. (2002). Ernst von Glasersfeld's philosophy of language: Roots, concepts, perspectives. In *Radical Constructivism in Action* (pp. 23-33). Routledge.

Seidouvy, A., Helenius, O., & Schindler, M. (2019). Authority in students' peer collaboration in statistics: an empirical study based on inferentialism. *Nordisk Matematikkdidaktikk, NOMAD: [Nordic Studies in Mathematics Education]*, *24*(2), 25–47.

Seidouvy, A., & Schindler, M. (2020). An inferentialist account of students' collaboration in mathematics education. *Mathematics Education Research Journal*, *32*(3), 411–431. https://doi.org/10.1007/s13394-019-00267-0

Sellars, W., Rorty, R., & Brandom, R. (1997). *Empiricism and the philosophy of mind*. Harvard University Press.

Sellars, W., Scharp, K., & Brandom, R. (2007). *In the space of reasons: selected essays of Wilfrid Sellars*. Harvard University Press.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational researcher*, *27*(2), 4-13.

Siy, E. (2019). Creating a healthy and rigorous culture of research by revealing our work. In Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (Eds.). *Proceedings of* *the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

Smith, J. K. (2004). *Introducing Radical Orthodoxy: mapping a post-secular theology*. Baker Academic.

Smith, J. K. (2013). *Imagining the Kingdom (Cultural liturgies): How worship works*. Baker Academic.

Smith, J. K. (2014). *Who's Afraid of Relativism? (The Church and Postmodern Culture): Community, Contingency, and Creaturehood*. Baker Academic.

Steffe, L. P. (2015). Can a Radical Constructivist Be Religious?—Yes!. *Constructivist Foundations, 11*(1), 131–134. http://constructivist.info/11/1/131

Steffe, L. P., & Thompson, P. W. (2000a). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*(2), 191–209. https://doi.org/10.2307/749751

Steffe, L. P., & Thompson, P. W. (2000b). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A.E. Kelly (Eds.), *Handbook of research design in mathematics and science education*, (pp. 267–306). Erlbaum.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*, *21*(4), 459-490. https://doi.org/10.1016/S0732-3123(02)00145-1

Stephan, M., Cobb, P., & Gravemeijer, K. (2003). Chapter 5: Coordinating social and individual analyses: Learning as participation in mathematical practices. *Journal for Research in Mathematics Education. Monograph*, *12*, 67-102.

Stinson, D. W., & Bullock, E. C. (2015). Critical postmodern methodology in mathematics education research: Promoting another way of thinking and looking. *Philosophy of Mathematics Education Journal [25th Anniversary Issue], 29,* 1–18.

Stinson, D. (2020). Scholars before researchers: Philosophical considerations in the preparation of mathematics education researchers. *Philosophy of Mathematics Education Journal*, 36.

St. Pierre, E. A., & Jackson, A. Y. (2014). Qualitative data analysis after coding. *Qualitative Inquiry, 20*(6), 715-719.

Su, F. (2020). *Mathematics for Human Flourishing*. Yale University Press.

Taylor, C. (2004). *Modern Social Imaginaries*. Duke University Press.

Taylor, S. D., Noorloos, R., & Bakker, A. (2017). Mastering as an inferentialist alternative to the acquisition and participation metaphors for learning. *Journal of Philosophy of Education, 51*(4), 769–784. https://doi.org/10.1111/1467-9752.12264

Temple, K. (2017). *Answering Dreyfus's challenge: toward a theory of concept without intellectualism* [Unpublished doctoral dissertation]. The New School.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlin, & L. L. Hatfield (Eds.), *WISDOM[e] monograph: Vol. 1. New perspectives and directions for collaborative research in mathematics education*: Papers from a planning conference for WISDOM[e] (pp. 33–57). University of Wyoming.

http://www.uwyo.edu/wisdome/_files/documents/qr_reasoningmathmodeling_thompson.pdf

Thompson, P. W. (2013). In the absence of meaning…. In Leatham, K. (Ed.), *Vital directions for mathematics education research* (pp. 57–93). Springer. https://doi.org/10.1007/978-1-4614-6977-3_4

Thompson, P., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra 1, 2, 3. In Steffe, L., Moore, K., & Hatfield, L. (Eds.) *Epistemic algebraic students: emerging models of students' algebraic knowing*, *4*, 1-24.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 421–456). National Council of Teachers of Mathematics.

Toulmin, S. (1958/2003). *The uses of argument*. Cambridge University Press.

Toulmin, S. (1999). Knowledge as shared procedures. In Y. Engeström, R. Miettinen, & R. Punamäki (Eds.), *Perspectives on Activity Theory* (pp. 53-64). Cambridge University Press. https://doi.org/10.1017/CBO9780511812774.005

Uegatani, Y., & Otani, H. (2019). A new ontology of reasons for inferentialism: redefining the notion of conceptualization and proposing an observer effect on assessment. *Mathematics Education Research Journal, 33*(1), 183–199. http://dx.doi.org/10.1007/s13394-019-00289-8

Valero, P., & Zevenbergen, R. (Eds.). (2004). *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology*. Springer.

Vidakovic, D. (1993). *Cooperative learning: Differences between group and individual processes of construction of the concept of inverse function* (Order No. 9403803). Available from ProQuest Dissertations & Theses A&I; ProQuest Dissertations & Theses Global. (304097741). https://www.proquest.com/dissertations-theses/cooperative-learning-differences-between-group/docview/304097741/se-2

von Glasersfeld, E. (1974). Piaget and the radical constructivist epistemology. In: Smock, C. D. & von Glasersfeld, E. (Eds.) *Epistemology and education* (pp. 1–24). Follow Through Publications.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning* (Vol. 6). Routledge.

von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *The Journal of Educational Thought (JET)/Revue de la Pensée Educative*, 91–103.

Vygotsky, L. S. (1962). *Thought and language*. MIT Press.

Wenger, E. (1999). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

West, C. (1999). *The Cornel West reader*. Civitas Books.

Wittgenstein, L. (1953). *Philosophical investigations*. John Wiley & Sons.

Wright, N.T. (2019). *History and eschatology: Jesus and the promise of natural theology*. Baylor University Press.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477. https://doi.org/10.2307/749877

Yoon, H., Byerley, C., Joshua, S., Moore, K., Park, M. S., Musgrave, S., Valaas, L. Drimalla, J. (2021). United States and South Korean Citizens Interpretation and Assessment of COVID-19 Quantitative Data. *The Journal of Mathematical Behavior, 62*, 100865. https://doi.org/10.1016/j.jmathb.2021.100865

APPENDIX A: CONTENT CLASS HANDOUTS

Prompts from Handout 1

Graph the distance from the ground of an individual (vertical axis) vs. the individual's total distance traveled (horizontal axis) (assume their feet were touching the ground at the beginning of the video).

Give a justification for your graph.

Choose a point on the graph and explain the meaning of that point.

Assuming that the ride was doubled in height, draw a second graph using the same axes above that reflects the relationship of an individual's distance from the ground vs. the individual's total distance traveled (assume their feet were touching the ground at the beginning of the video). Give a justification for your graph.

Assuming that the ride was doubled in speed, draw a second graph using the same axes above that reflects the relationship of an individual's distance from the ground vs. the individual's total distance traveled (assume their feet were touching the ground at the beginning of the video). Give a justification for your graph.

Prompts from Handout 2

Graph the individual's total distance traveled (vertical axis) vs. distance from the ground of an individual (horizontal axis) (assume their feet were touching the ground at the beginning of the video).

Give a justification for your graph.

Choose a point on the graph and explain the meaning of that point.

Assuming that the ride was doubled in height, draw a second graph using the same axes above that reflects the relationship of an individual's distance from the ground vs. the individual's total distance traveled (assume their feet were touching the ground at the beginning of the video). Give a justification for your graph.

Assuming that the ride was doubled in speed, draw a second graph using the same axes above that reflects the relationship of an individual's distance from the ground vs. the individual's total distance traveled (assume their feet were touching the ground at the beginning of the video). Give a justification for your graph.