Optimal Trading Rules under Mean Reversion and GBM with Regime Switching

by

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(Under the Direction of Qing Zhang)

Abstract

This dissertation contains two parts: the first chapter is concerned with identifying the highs and lows of prices for stock trading. The underlying asset prices fluctuate in a mean reversion fashion. The purpose is to maximize the overall profit in the long run. Ideally, we want to sell high and buy low. However, it is extremely difficult to identify when is low and when is high in practice. Under the mean reversion model, we follow a dynamic programming approach and determine these key thresholds to optimize our profit. In the second chapter, we discuss an optimal pairs trading rule. A pairs position consists of a long position in one stock and a short position in the other. The problem is to find stopping times to open and then close the pairs position to maximize expected reward functions. We consider the optimal pairs trading rule with one round trip. The underlying stock prices follow a general geometric Brownian motion with regime switching. The optimal policy is characterized by threshold curves obtained by solving the associated HJB equations (quasi-variational inequalities). Moreover, numerical examples are provided to illustrate optimal policies.

Index words: [Mean Reverting, Pairs Trading, Dynamical Programming, HJB equation, Regime Switching]

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世有伯乐，然后有千里马.

—— Han Yu, Tang Dynasty

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Chapter 1

MEAN-REVERTING MODEL WITH CUT LOSS

1.1 Introduction

This chapter is concerned with identifying the highs and lows with a cut loss line when trading an asset that is subject to random fluctuation in its price. Trading is concerned with buy and sell. The purpose is to maximize the profit in the long run. Ideally, we want to sell high and buy low. However, it is extremely difficult to identify when is low and when is high in practice. In this chapter, we consider the case in a mean reversion model, follow a dynamic programming approach to determine these key thresholds to optimize our profit.

Economists would call a market as a "place" where buyers and sellers meet to exchange products. A financial market is where "money" is traded. The spot price is affected by the supply and demand relationship in a free-market economy. The geometric Brownian motion model is often used to capture the price fluctuation. Mean-reversion models are often used in financial markets to capture price movements that have the tendency to move toward an “equilibrium” level.In addition to stock markets, mean-reversion models are used for stochastic interest rates also in energy markets.

Mathematical trading rules have been studied for many years. For example, [Q. Zhang, 2001] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. In [Q. Zhang, 2001], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. [Guo and Zhang, 2005] studied the optimal selling rule under a model with switching Geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type.[Dai, 2010] developed a trend following rule based on a conditional probability indicator. They showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated Hamilton–Jacobi–Bellman (HJB) equations.

In [Q. Zhang, 2001] paper, they studied the problem using the dynamic programming approach and establish the associated HJB equations (quasi-variational inequalities) for the value functions. The smooth-fit technique is applied to derive algebraic equations for the threshold levels in the corresponding optimal stopping times problem. They also provided sufficient conditions that guarantee the optimality of the corresponding optimal stopping times in the form of a verification theorem. In this chapter we will continue their study with a cut loss line. A percentage transaction is imposed on each transaction. We show that the corresponding optimal stopping times can be determined by three threshold levels x_0, x_1, and x_2. These key levels can be obtained by solving a set of algebraic like equations similar in [Q. Zhang, 2001]. We show that the optimal pairs trading rule can be given in terms of two intervals: $I_1 = [x_0, x_1]$ and $I_2 = (M, x_2)$. Here M is the given stop-loss level and $M < x_0 < x_1 < x_2$. The idea to initiate a trade whenever the state process X_t enters I_1 and hold the position till X_t exits I_2.

This chapter is organized as follows. In §1.2, the problem is formulated. In §1.3, we study properties of the value functions. In §1.4 and §1.5, the associated HJB equations are established and their solutions are obtained. In §1.6, we provide a set of sufficient conditions that guarantee the optimality of our trading rule. §1.7, we state some numerical examples.

1.2 Problem Formulation

Let $X_t \in R, t \geq 0$, denote a mean-reversion process governed by

$$dX_t = a\,(b - X_t)\,dt + \sigma dW_t, X_0 = x$$

where $a > 0$ is the rate of reversion, b is the equilibrium level, $\sigma > 0$ is the volatility, and W_t is a standard Brownian motion. The asset price at time t is given by $S_t = \exp{(X_t)}$.

Let $i = 0, 1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell the stock before acquiring any shares. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should first buy a stock before selling any shares. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$.

To detect the threshold of cut loss, we define τ_M:

$$\begin{aligned} \tau_M &= \inf\{t : x_t \notin (M, \infty)\} \\ 0 &\leq \tau_1^b \leq \tau_1^s \leq \tau_2^b \leq \tau_2^s \leq \cdots \leq \tau_M. \end{aligned}$$

Then our reward function is :

$$J_i(x,\Lambda_i) = \begin{cases} E\Big\{ \sum_{n=1}^{\infty} \Big[e^{-\rho\tau_n^s} S_{\tau_n^s}(1-K) - e^{-\rho\tau_n^b} S_{\tau_n^b}(1+K) \Big] \Big\} I_{\tau_n^b \le \tau_M}, & \text{if} \quad i=0, \\ E\Big\{ e^{-\rho\tau_1^s} S_{\tau_1^s}(1-K) & \\ \quad + \sum_{n=2}^{\infty} \Big[e^{-\rho\tau_n^s} S_{\tau_n^s}(1-K) - e^{-\rho\tau_n^b} S_{\tau_n^b}(1+K) \Big] \Big\} I_{\tau_n^b \le \tau_M}, & \text{if} \quad i=1. \end{cases}$$

For simplicity, the term $E\sum_{n=1}^{\infty}\xi_n$ for random variables ξ_n is interpreted as

$$\limsup_{N\to\infty} E\sum_{n=1}^{N}\xi_n.$$

Let $V_i(x)$ denote the value functions with the initial net positions $i=0,1$ and initial state $X_0=x$. That is

$$V_i(x) = \sup_{\Lambda_i} J_i\left(x,\Lambda_i\right).$$

1.3 Properties of Value Functions

First, note that the sequence $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_1^s, \ldots)$ can be regarded as a combination of buying in at τ_1^b then followed by the sequence of stopping times $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$. By definition of value functions:

$$\begin{aligned} V_0(x) \ge & J_0\left(x,\Lambda_0\right) \\ = & E\Bigg\{ e^{-\rho\tau_1^s} S_{\tau_1^s}(1-K) + \sum_{n=2}^{\infty} \Big[e^{-\rho\tau_n^s} S_{\tau_n^s}(1-K) - e^{-\rho\tau_n^b} S_{\tau_n^b}(1+K) \Big] \Bigg\} I_{\tau_n^b\le\tau_M} \\ & - Ee^{-\rho\tau_1^b} S_{\tau_1^b}\left(1+K\right) I_{\tau_n^b\le\tau_M} \\ = & J_1\left(X_{\tau_1^b},\Lambda_1\right) - Ee^{-\rho\tau_1^b} S_{\tau_1^b}\left(1+K\right) I_{\tau_n^b\le\tau_M}. \end{aligned}$$

Setting $\tau_1^b = 0$ (recall that $S_t = \exp\left(X_t\right)$), and taking supremum over all Λ_1, we get

$$V_0(x) \ge V_1(x) - e^x(1+K).$$

Similarly,

$$\begin{aligned} V_1(x) &\ge J_1\left(x,\Lambda_1\right) \\ &= J_0\left(X_{\tau_1^s},\Lambda_0\right) + Ee^{-\rho\tau_1^s} S_{\tau_1^s}\left(1-K\right) I_{\tau_n^b\le\tau_M}. \end{aligned}$$

By setting $\tau_1^s = 0$, and taking supremum over all Λ_0, we get

$$V_1(x) \geq V_0(x) + e^x(1-K).$$

There exist constants K_0, K_1, K_2, and K_3 such that

$$0 \leq V_0(x) \leq K_0, \quad \text{and} \qquad 0 \leq V_1(x) \leq K_1 e^x + K_2.$$

Proof: It is clear that they are nonnegative. It remains to establish their upper bounds.
In view of the definition of $J_i(x)$:

$$J_i(x) = E\left\{\sum_{n=1}^{\infty}\left[e^{-\rho\tau_n^s}S_{\tau_n^s}(1-K) - e^{-\rho\tau_n^b}S_{\tau_n^b}(1+K)\right]\right\} I_{\tau_n^b \leq \tau_M}.$$

Using Ito's formula, we have

$$d\left(e^{-\rho t}S_t\right) = e^{-\rho t}S_t\left(-\rho + a\left(b - X_t\right) + \frac{\sigma^2}{2}\right)dt + e^{-\rho t}S_t\sigma dW_t.$$

.

Integrate both sides from τ_n^s to τ_n^b, focusing on $e^{-\rho t}S_t\sigma dW_t$, we have:

$$E\left[\int_{\tau_n^b}^{\tau_n^s}\sigma e^{-\rho t}S_t dW_t I_{\tau_n^b < \tau_M}\right] = \quad E\int_{\tau_n^b}^{\tau_n^s}\sigma e^{-\rho_t}S_t dW_t - E\left[\int_{\tau_n^b}^{\tau_n^s}\sigma e^{-\rho t}S_t dW_t I_{\tau_n^s = \tau_n^b = \tau_M}\right] = 0.$$

It follows from Dynkin's formula, that:

$$Ee^{-\rho\tau_n^s}S_{\tau_n^s} - Ee^{-\rho\tau_n^b}S_{\tau_n^b} = E\int_{\tau_n^b}^{\tau_n^s} e^{-\rho t}e^{X_t}\left(A - aX_t\right)dt.$$

Note that the function $e^x(A - ax)$ is bounded above on R. Let C be an upper bound. It follows that

$$E\left[\left(e^{-\rho\tau_n^s}S_{\tau_n^s} - \mathrm{e}^{-p\tau_n^b}S_{\tau_M}\right) I_{\tau_n^b<\tau_M}\right]$$
$$= E\left[\int_{\tau_n^b}^{\tau_n^s} e^{\,s_t(A-ax_t)dt)I_{\tau_n^b<\tau_M}}\right]$$
$$\leq CE\left[\int_{\tau_n^b}^{\tau_n^s} e^{-\rho t}dt_{\tau_n^b<\tau_M}\right]$$
$$\leq CE\int_{\tau_n^b}^{\tau_n^s} e^{-\rho t}dt.$$

Therefore,

$$V_0(x) = \sup_{\Lambda_0} J_0\left(x, \Lambda_0\right) \leq \sum_{n=1}^{\infty} CE\int_{\tau_n^b}^{\tau_n^s} e^{-\rho t}dt \leq C\int_0^{\infty} e^{-\rho t}dt = \frac{N}{\rho} := K_0.$$

This implies that $V_0(x) \leq K_0$. Similarly, we have that

$$J_1\left(x, \Lambda_1\right) \leq K_0 + Ee^{-\rho\tau_1^s}S_{\tau_1^s}(1-K).$$

we can show that by taking $\tau_0 = 0$ that

$$Ee^{-\rho\tau_1^s}e^{X_{\tau_1^s}} - e^x \leq \frac{C}{\rho}.$$

This implies that

$$V_1(x) \leq K_0 + (1-K)\left(e^x + \frac{C}{\rho}\right) := K_1e^x + K_2.$$

Therefore, $0 \leq V_1(x) \leq K_1e^x + K_2$. This completes the proof.

1.4 HJB equation

The generator $\mathcal{A}$ of X_t is given by

$$\mathcal{A} = a(b-x)\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}.$$

Formally, the associated HJB equations should have the form

$$\begin{cases} \min\left\{\rho v_0(x) - \mathcal{A}v_0(x), v_0(x) - v_1(x) + e^x(1+K)\right\} = 0, \\ \min\left\{\rho v_1(x) - \mathcal{A}v_1(x), v_1(x) - v_0(x) - e^x(1-K)\right\} = 0. \end{cases} \tag{1.1}$$

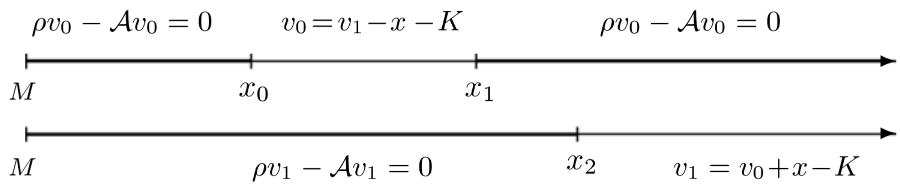

Figure 1.1: Continuous Region

for $x \in (M, \infty)$, with the boundary conditions $v_0(M) = 0$ and $v_1(M) = e^M \beta_s$. If $i = 0$, then one should only buy when the price is low, then the continuation region should include $(M, x_0) \cup (x_1, \infty)$ for when x enters $[x_0, x_1]$ one should buy where we have $V_0(x) = V_1(x) - e^x(1 + K)$. In addition, one should not establish any new position if X_t is close to the stop-loss level M.

If $i = 1$, then one should only sell when the price is high (greater than or equal to $x_2 > x_1$), which implies $v_1(x) = v_0(x) + x(1 - K)$ and the continuation region (given by $\rho v_1(x) - \mathcal{A}v_1(x) = 0$) should be (M, x_2).

Furthermore, on the boundary, we have $v_0(M) = 0, \quad v_1(M) = e^M \beta_s$ where $\beta_b = 1 + K$ and $\beta_s = 1 - K$. On the other hand, $v_i(x)$ has to satisfy the following conditions to qualify for being solutions to the HJB equations

$$
\begin{cases}
v_0(x) \geq v_1(x) - e^x(1 + K) & \text{on } (M, x_0) \cup (x_1, \infty), \\
v_1(x) \geq v_0(x) + e^x(1 - K) & \text{on } (M, x_2), \\
(\rho - \mathcal{A})v_0(x) \geq 0 & \text{on } (x_0, x_1), \\
(\rho - \mathcal{A})v_1(x) \geq 0 & \text{on } (x_2, \infty).
\end{cases}
$$

1.5 Solving HJB

In this section, we will obtain the threshold levels (x_0, x_1, x_2) by solving the HJB equation. We first solve the equations $\rho v_i(x) - \mathcal{A}v_i(x) = 0$ with $i = 0, 1$. Let

$$
\begin{cases}
\phi_1(x) = \int_0^\infty \eta(t) e^{-\kappa(b-x)t} dt, \\
\phi_2(x) = \int_0^\infty \eta(t) e^{\kappa(b-x)t} dt,
\end{cases}
$$

where $\kappa = \sqrt{2a}/\sigma$, $\lambda = \rho/a$, and $\eta(t) = t^{\lambda-1} \exp(-t^2/2)$. Then the general solution of $\rho v_i(x) - \mathcal{A}v_i(x) = 0$ is given by a linear combination of these functions.

First, consider the interval (x_1, ∞) and suppose the solution is given by $A_1\phi_1(x) + A_2\phi_2(x)$, for some A_1 and A_2. Recall the upper bound for $V_0(x)$, $v_0(\infty)$ should be bounded above. This implies that $A_1 = 0$ and $v_0(x) = A_2\phi_2(x)$ on (x_1, ∞).

On the interval (M, x_0), suppose $v_0(x) = B_1\phi_1(x)+B_2\phi_2(x)$, for some B_1 and B_2. On the interval (M, x_2), suppose $v_1(x) = C_1\phi_1(x) + C_2\phi_2(x)$, for some C_1 and C_2.

It is easy to see that these functions are twice continuously differentiable on their continuation regions. We follow the smooth-fit method which requires the solutions to be continuously differentiable. In particular, it requires v_0 to be continuously differentiable at x_0. Therefore, at x_0:

$$\begin{cases} B_1\varphi_1(x_0) + B_2\varphi_2(x_0) = C_1\varphi_1(x_0) + C_2\varphi_2(x_0) - e^{x_0}\beta_b, \\ B_1\varphi_i'(x_0) + B_2\varphi_2'(x_0) = C_1\varphi_1'(x_0) + C_2\varphi_2'(x_0) - e^{x_0}\beta_b. \end{cases} \tag{1.2}$$

at x_1:

$$\begin{cases} C_1\varphi_1(x_2) + C_2\varphi_2(x_2) = A_2\varphi_2(x_2) + e^{x_2}\beta_s, \\ C_1\varphi_1'(x_2) + C_2\varphi_2'(x_2) = A_2\varphi_2'(x_2) + e^{x_2}\beta_s. \end{cases} \tag{1.3}$$

at x_2:

$$\begin{cases} C_1\varphi_1(x_2) + C_2\varphi_2(x_2) = A_2\varphi_2(x_2) + e^{x_2}\beta_s, \\ C_i\varphi_1'(x_2) + C_2\varphi_2'(x_2) = A_2\varphi_2'(x_2) + e^{x_2}\beta_s. \end{cases} \tag{1.4}$$

at M:

$$\begin{cases} B_1\varphi_1(M) + B_2\varphi_2(M) = 0, \\ C_1\varphi_1(M) + C_2\varphi_2(M) = e^M\beta_s. \end{cases} \tag{1.5}$$

Let

$$\Phi(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix}.$$

Writing (1.2)-(1.5) in term of matrix notations:

$$\Phi(x_0)\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi(x_0)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - e^{x_0}\beta_b\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\Phi(x_1)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2\begin{pmatrix} \varphi_2(x_1) \\ \varphi_2'(x_1) \end{pmatrix} + e^{x_1}\beta_b\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\rho(x_2)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2\begin{pmatrix} \varphi_2(x_2) \\ \varphi_2'(x_2) \end{pmatrix} + e^{x_2\beta}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$(\varphi_1(M), \varphi_2(M)) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0,$$

$$(\varphi_1(M), \varphi_2(M)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = e^M \beta_s.$$

Note that : $\frac{\sigma^2}{2} v_i''(x) + a(b-x)v_i'(x) - \rho v_i(x) = 0$. Recall that solution of $\rho v_i(x) - \mathcal{A} v_i(x) = 0$ is given by a linear combination of $\phi_1(x)$ and $\phi_2(x)$, form their Wronskian:

$$W(\phi_1, \phi_2) = det \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix} = Ce^{-\frac{2ax}{\sigma^2} + \frac{a}{\sigma^2} x^2}.$$

which is non-zero for all x.

Therefore, $\Phi(x)$ is invertible for all x. Also, let

$$R(x) = \Phi^{-1}(x) \begin{pmatrix} \phi_2(x) \\ \phi_2'(x) \end{pmatrix}, \quad P(x) = \Phi^{-1}(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

After rearrangement we have :

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - e^{x_0} \beta_b P(x_0), \tag{1.6}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2 R(x_1) + e^{x_1} \beta_b P(x_1), \tag{1.7}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2 R(x_2) + e^{x_2} \beta_s P(x_2), \tag{1.8}$$

$$(\varphi_1(M), \varphi_2(M)) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0, \tag{1.9}$$

$$(\varphi_1(M), \varphi_2(M)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = e^M \beta_s. \tag{1.10}$$

It follows from (1.9) and (1.10), and the definition of $P(x)$, we have

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - e^M \beta_s P(x_0). \tag{1.11}$$

Therefore,multiplying both sides with $(\varphi_1(M), \varphi_2(M))$:

$$(\varphi_1(M), \varphi_2(M)) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = (\varphi_1(M), \varphi_2(M)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - e^{x_0} \beta_b (\varphi_1(M), \varphi_2(M)) P(x_0).$$

we have

$$e^{x_0} \beta_b (\varphi_1(M), \varphi_2(M)) P(x_0) = e^M \beta_s.$$

It follows that

$$(\varphi_1(M), \varphi_2(M)) P(x_0) = \frac{e^M - x_0}{\beta}. \tag{1.12}$$

This equation could be used to solve x_0.
To solve for x_1 and x_2, also note that:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2 R(x_1) + e^{x_1} \beta_b P(x_1), \tag{1.13}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2 R(x_2) + e^{x_2} \beta_b P(x_2). \tag{1.14}$$

Combining (1.12) and (1.13):

$$A_2 R(x_1) + e^{x_1} \beta_b P(x_1) \quad = A_2 R(x_2) + e^{x_2} \beta_s P(x_2).$$

Also notice that, multiplying both sides with $(\varphi_1(M), \varphi_2(M))$:

$$(\varphi_1(M), \varphi_2(M)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2 (\varphi_1(M), \varphi_2(M)) R(x_2) + e^{x_2} \beta_s (\varphi_1(M), \varphi_2(M)) P(x_2).$$

This leads to

$$A_2 = \frac{e^M \beta_s - e^{x_2} \beta_s (\varphi_1(M), \varphi_2(M)) P(x_2)}{(\varphi_1(M), \varphi_2(M)) R(x_2)}.$$

Combine the above to obtain:

$$[R(x_1) - R(x_2)] \left[\frac{e^M \beta_s - e^{x_2} \beta_s \varphi_1(M), \varphi_2(M)] \, P(x_2)}{(\varphi_1(M), \varphi_2(M)) \, R(x_2)} \right] = e^{x_2} \beta_s P(x_2) - e^{x_1} \beta_b P(x_1). \tag{1.15}$$

Solving this will give us the thresholds x_1 and x_2.

We need additional conditions for x_1 and x_2. Note that $v_i(x)$ has to satisfy the following inequalities for being solutions to the HJB equations :

$$\begin{cases} \rho v_0(x) - \mathcal{A} v_0(x) \geq 0, \\ \rho v_1(x) - \mathcal{A} v_1(x) \geq 0, \\ v_0(x) \geq v_1(x) - e^x \beta_b, \\ v_1(x) \geq v_0(x) + e^x \beta_s, \end{cases} \tag{1.16}$$

for all $x \geq M$. Next, we examine each of these inequalities on intervals $(M, x_0), (x_0, x_1), (x_1, x_2)$, and (x_2, ∞).

First, on (M, x_0), the top two inequalities in (1.16) the conditions $v_1(x) - \mathcal{A} v_1(x) = 0$ become equalities. We only need the last two inequalities to hold. Therefore, we have

$$e^x(1-K) \leq v_1(x) - v_0(x) \leq e^x(1+K) \quad \text{on } (M, x_0).$$

Then,

$$|v_1 - v_0 - e^x| \leq e^x K.$$

By the definition of v_0:

$$|\, ((C_1 - B_1) \varphi_1(x) + (C_2 - B_2) \varphi_2(x) - e^x \,| \leq e^x K.$$

On (x_0, x_1), note that $v_0(x) = v_1(x) - e^x(1+K)$ implies $v_1(x) \geq v_0(x) + e^x(1-K)$. We only need $\rho v_0(x) - \mathcal{A} v_0(x) \geq 0$. Again, using $v_0(x) = v_1(x) - e^x - K$ and $\rho v_1(x) - \mathcal{A} v_1(x) = 0$ on this interval, we have

$$\begin{aligned} \rho v_0 - A v_0 &= \rho (v_1 - e^x \beta_b) - A (v_1 - e^x \beta_b) \\ &= (\rho v_1 - A v_1) - \beta_b (\rho e^x - A e^x) \\ &= -\beta_b (\rho e^x - A e^x) \\ &= -\beta_b e^x \left[\rho - \left(a(b - x) + \frac{\sigma^2}{2} \right) \right] \geq 0. \end{aligned}$$

In view of this, $\rho v_0(x) - \mathcal{A}v_0(x) \geq 0$ on (x_0, x_1) is equivalent to

$$x_1 \leq \frac{1}{a}\left(ab + \frac{\sigma^2}{2} - \rho\right).$$

On (x_1, x_2), we need to satisfy the first two inequalities in (1.16), $|v_1(x) - v_0(x) - e^x| \leq e^x K$, we will have :

$$|C_1\phi_1(x) + (C_2 - A_2)\,\phi_2(x) - e^x| \leq e^x K.$$

On (x_2, ∞), we need to satisfy the first and last inequalities in (1.16), the first inequality is satisfied automatically because $v_1(x) = v_0(x) + e^x(1-K)$. Using $(\rho - \mathcal{A})v_0(x) = 0$, the last inequality becomes:

$$x_2 \geq \frac{1}{a}\left(ab + \frac{\sigma^2}{2} - \rho\right).$$

1.6 Verification Theorem

In this section, we give a verification theorem to show that the solution $v_i(x)$, $i = 0, 1$, of (1.1) is equal to the value functions $V_i(x)$, $i = 0, 1$, respectively, and sequences of optimal stopping times can be constructed from the triple (x_0, x_1, x_2).

Let (x_0, x_1, x_2) be a solution to (1.15) and (1.12) satisfy

$$x_1 \leq \frac{1}{a}\left(ab + \sigma^2/2 - \rho\right) \quad \text{and} \quad x_2 \geq \frac{1}{a}\left(ab + \sigma^2/2 - \rho\right).$$

Let A_2, B_1, B_2, C_1, and C_2 be constants given in the eariler part satisfying the inequalities. Let

$$\begin{cases} v_0(x) = \begin{cases} B_1\phi_1(x) + B_2\phi_2(x) & \text{if } x \in [M, x_0)\,, \\ C_1\phi_1(x) + C_2\phi_2(x) - e^x(1+K) & \text{if } x \in [x_0, x_1)\,, \\ A_2\phi_2(x) & \text{if } x \in [x_1, \infty)\,, \end{cases} \\ v_1(x) = \begin{cases} C_1\phi_1(x) + C_2\phi_2(x) & \text{if } x \in [M, x_2)\,, \\ A_2\phi_2(x) + e^x(1-K) & \text{if } x \in [x_2, \infty)\,. \end{cases} \end{cases}$$

Let

$$v_i(x) = V_i(x), i = 0, 1$$

If initially $i = 0$,let $\Lambda_0^* = (\tau_1^*, \sigma_1^*, \tau_2^*, \sigma_2^*, \ldots)$, where the stopping times $\tau_1^* = \inf\{t \geq 0 : X_t \in [x_0, x_1]\} \wedge \tau_M$,
$\sigma_n^* = \inf\{t \geq \tau_n^* : X_t = x_2\} \wedge \tau_M$, and $\tau_{n+1}^* = \inf\{t > \sigma_n^* : X_t \in [x_0, x_1]\} \wedge \tau_M$ for $n \geq 1$. Similarly, if initially $i = 1$, let $\Lambda_1^* = (\sigma_1^*, \tau_2^*, \sigma_2^*, \tau_3^*, \ldots)$, where the stopping times $\sigma_1^* = \inf\{t \geq 0 : X_t \geq x_2\} \wedge$

τ_M, $\tau_n^* = \inf\left\{t > \sigma_{n-1}^* : X_t \in [x_0, x_1]\right\} \wedge \tau_M$, and $\sigma_n^* = \inf\{t \geq \tau_n^* : X_t = x_2\} \wedge \tau_M$ for $n \geq 2$. Then Λ_0^* and Λ_1^* are optimal.

Proof: We divide the proof into two steps. In the first step, we show that $v_i(x) \geq J_i(x, \Lambda_i)$ for all Λ_i. Then in the second step, we prove that $v_i(x) = J_i(x, \Lambda_i^*)$, which implies $v_i(x) = V_i(x)$ and Λ_i^* is optimal.

Using Dynkin's formula, and Fatou's lemma, we have, for any stopping times $0 \leq \gamma_1 \leq \gamma_2 \leq \tau_M$,

$$E\left(e^{-\rho\gamma_1} v_i\left(X_{\gamma_1}\right)\right) \geq E\left(e^{-\rho\gamma_2} v_i\left(X_{\gamma_2}\right)\right),$$

$$E\left(e^{-\rho\gamma_1} v_i\left(X_{\gamma_1}\right) I_{\{\gamma_1 < \tau_M\}}\right) \geq E\left(e^{-\rho\gamma_2} v_i\left(X_{\gamma_2}\right) I_{\{\gamma_1 < \tau_M\}}\right).$$

It follows, for the position $i = 0$, that

$$\begin{aligned}
Ee^{-\rho\tau_1^b} v_0\left(X_{\tau_1^b}\right) &\geq Ee^{-\rho\tau_1^b}\left(v_1\left(X_{\tau_1^b}\right) - S_{\tau_1^b}(1+K)\right) I_{\tau_n^b \leq \tau_M} \\
&\geq Ee^{-\rho\tau_1^s} v_1\left(X_{\tau_1^s}\right) I_{\tau_n^b \leq \tau_M} - Ee^{-\rho\tau_1^b}\left(S_{\tau_1^b}(1+K)\right) I_{\tau_n^b \leq \tau_M} \\
&\geq Ee^{-\rho\tau_1^s}\left(v_0\left(X_{\tau_1^s}\right) + S_{\tau_1^s}(1-K)\right) I_{\tau_n^b \leq \tau_M - Ee^{-\rho\tau_1^b}\left(S_{\tau_1^b}(1+K)\right) I_{\tau_n^b \leq \tau_M}} \\
&\geq Ee^{-\rho\tau_2^b} v_0\left(X_{\tau_2^b}\right) I_{\tau_n^b \leq \tau_M + E\left[e^{-\rho\tau_1^s}\left(S_{\tau_1^s}(1-K)\right) - e^{-\rho\tau_1^b}\left(S_{\tau_1^b}(1+K)\right)\right] I_{\tau_n^b \leq \tau_M}}.
\end{aligned}$$

Again with $E\left(e^{-\rho\gamma_1} v_i\left(X_{\gamma_1}\right) I_{\{\gamma_1 < \tau_M\}}\right) \geq E\left(e^{-\rho\gamma_2} v_i\left(X_{\gamma_2}\right) I_{\{\gamma_1 < \tau_M\}}\right)$, we have:

$$Ee^{-\rho\tau_2^b} v_0\left(X_{\tau_2^b}\right) \geq Ee^{-\rho\tau_2^s} v_0\left(X_{\tau_2^s}\right) + E\left[e^{-\rho\tau_2^s}\left(S_{\tau_2^s}(1-K)\right) - e^{-\rho\tau_2^b}\left(S_{\tau_2^b}(1+K)\right)\right].$$

After iterations, we have shown:

$$v_0(x) \geq E\left(\sum_{n=1}^{N}\left[e^{-\rho\tau_n^s}\left(S_{\tau_n^s}(1-K)\right) - e^{-\rho\tau_n^b}\left(S_{\tau_n^b}(1+K)\right)\right] I_{\left\{\tau_n^b < \tau_M\right\}}\right).$$

Sending $N \to \infty$ to obtain $v_0(x) \geq J_0(x, \Lambda_0)$ for all Λ_0. Therefore, $v_0(x) \geq V_0(x)$.

Similarly, when $i = 1$, we can show :

$$v_1(x) \geq Ee^{-\rho\tau_1^s}\left(S_{\tau_1^s}(1-K)\right) + E\left(\sum_{n=2}^{N}\left[e^{-\rho\tau_n^s}\left(S_{\tau_n^s}(1-K)\right) - e^{-\rho\tau_n^b}\left(S_{\tau_n^b}(1+K)\right)\right] I_{\left\{\tau_n^b < \tau_M\right\}}\right).$$

Then we have finished proving $v_i(x) \geq J_i(x, \Lambda_i)$.

Now we proceed to equalities. It again is equivalent to

$$v_0(x) = E\left(e^{-\rho\tau_N^{s*}} v_0\left(X_{\tau_N^{s*}}\right)\right) + E\left(\sum_{n=1}^{N}\left[e^{-\rho\tau_n^{s*}}\left(S_{\tau_n^{s*}}(1-K)\right) - e^{-\rho\tau_n^{b*}}\left(S_{\tau_n^{b*}}(1+K)\right)\right] I_{\left\{\tau_n^{b*} < \tau_M\right\}}\right).$$

Define $\tau_1^{b*} = \inf\{t \geq 0 : X_t \in [x_0, x_1]\} \wedge \tau_M$. We observe that :

$$v_0(x) = v_1(x) - e^x(1+K) \quad \text{on } [x_0, x_1],$$

and when $i = 0$ then $X_t \in (M, x_0] \cup [x_1, \infty)$ for all $t \in [0, \tau_1^*]$, which implies $(\rho - \mathcal{A})v_0(X_t) = 0$ for all $t \in [0, \tau_1^*]$. Therefore,

$$v_0\left(X_{\tau_1^{b*}}\right) I_{\left\{\tau_1^{b*} < \tau_M\right\}} = \left(v_1\left(X_{\tau_1^{b*}}\right) - \left(S_{\tau_1^{b*}}(1+K)\right)\right) I_{\left\{\tau_1^{b*} < \tau_M\right\}}.$$

Then,

$$\begin{aligned}
v_0(x) =& E e^{-\rho\tau_1^{b*}} v_0\left(X_{\tau_1^{b*}}\right) \\
=& E\left(e^{-\rho\tau_1^{b*}} v_0\left(X_{\tau_1^{b*}}\right) I_{\left\{\tau_1^{b*} < \tau_M\right\}}\right) \\
=& E\left(e^{-\rho\tau_1^{b*}}\left(v_1\left(X_{\tau_1^{b*}}\right) - \left(S_{\tau_1^{b*}}(1+K)\right)\right) I_{\left\{\tau_1^{b*} < \tau_M\right\}}\right) \\
=& E\left(e^{-\rho\tau_1^{b*}} v_1\left(X_{\tau_1^{b*}}\right) I_{\left\{\tau_1^{b*} < \tau_M\right\}}\right) - E\left(e^{-\rho\tau_1^{b*}}\left(S_{\tau_1^{b*}}(1+K)\right) I_{\left\{\tau_1^{b*} < \tau_M\right\}}\right).
\end{aligned}$$

Again, since $(\rho - \mathcal{A})v_0(X_t) = 0$ for all $t \in [\sigma_1^*, \tau_2^*]$, we have that $E e^{-\rho\sigma_1^*} v_0\left(X_{\sigma_1^*}\right) = E e^{-\rho\tau_2^*} v_0\left(X_{\tau_2^*}\right)$. If we keep iterating this process, we have :

$$\begin{aligned}
v_0(x) =& E\left(e^{-\rho\tau_N^{s*}} v_0\left(X_{\tau_N^{s*}}\right)\right) + E\left(\sum_{n=1}^{N}\left[e^{-\rho\tau_n^{\tau_n^*}}\left(S_{\tau_n^{s*}}(1-K)\right)\right.\right. \\
&\left.\left. - e^{-\rho\tau_n^{b*}}\left(S_{\tau_n^{b*}}(1+K)\right)\right] I_{\left\{\tau_n^{b*} < \tau_M\right\}}\right).
\end{aligned}$$

Recall that $P(\tau_M < \infty) = 1$. This implies $\lim_{N\to\infty} \tau_N^{S*} = \tau_M$, Recall also that $v_0(M) = 0$. It follows that $E\left(e^{-\rho\tau_n^{s*}} v_0\left(X_{\tau_n^{s*}}\right)\right) \to 0$. This completes the proof.

1.7 Numerical Example

In this section, we consider a numerical example with the following specifications:

$$a = 1, b = 0, \sigma = 0.5, \rho = 0.5, K = 0.001, M = -0.8.$$

The threshold (x_0, x_1, x_2) is $(-0.779661, -0.427661, -0.345661)$. Next, we vary one of the parameters at a time and examine the dependence of the triple (x_0, x_1, x_2) on these parameters.

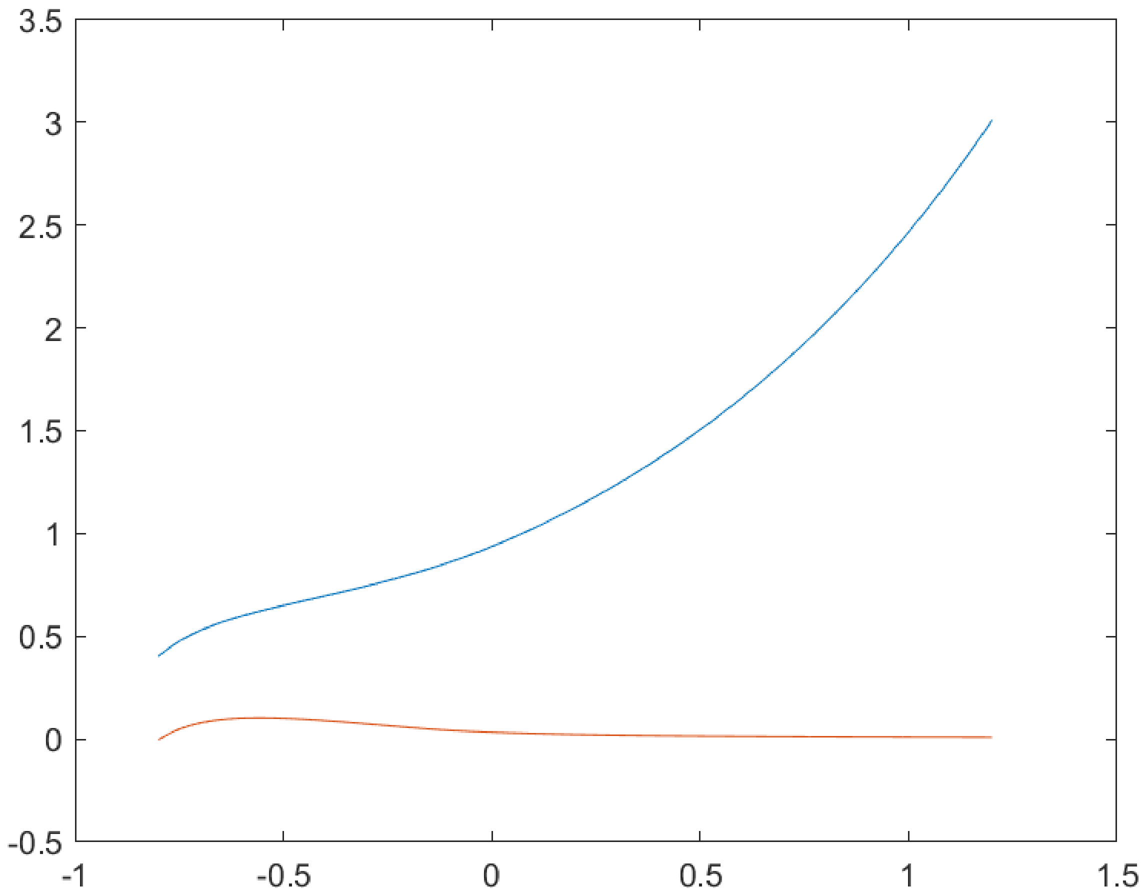

Figure 1.2: Value functions

a	0.8	0.9	1	1.1	1.2
x_0	-0.778498	-0.778888	-0.779661	-0.780367	-0.781021
x_1	-0.536498	-0.496887	-0.427661	-0.388367	-0.359021
x_2	-0.424498	-0.354887	-0.345661	-0.316367	-0.287021

A larger a implies larger pulling rate back to the equilibrium level $b = 2$. It can be seen that the lower buying level x_0 decreases and the higher buying level x_1 increases as a increases. This leads to a larger buying interval $[x_0, x_1]$ resulting greater buying opportunities. The selling level x_2 increases but the the interval $[x_1, x_2]$ decreases which suggests one should take profit sooner as a gets bigger.

ρ	0.3	0.4	0.5	0.6	0.7
x_0	-0.778270	-0.779197	-0.779661	-0.780588	-0.781052
x_1	-0.246270	-0.3471963	-0.427661	-0.538588	-0.649052
x_2	-0.114270	-0.215196	-0.345661	-0.436588	-0.527052

Next, we vary the discount rate ρ. Larger ρ means quicker profits. It shows that larger ρ leads to a smaller x_0, smaller x_1, and x_2.

σ	0.3	0.4	0.5	0.6	0.7
x_0	-0.788177	-0.784169	-0.779661	-0.775513	-0.771276
x_1	-0.526177	-0.472169	-0.427661	-0.403512	-0.349276
x_2	-0.404177	-0.390169	-0.345661	-0.251512	-0.177276

Next, we vary the volatility σ. The volatility is the source forcing the price to go away from its equilibrium. The larger the σ, the further the price fluctuates. As a result, x_1 increases and the pair (x_0, x_2) decreases in σ resulting in a smaller buying interval $[x_0, x_1]$ and a higher profit target x_2.

K	0.3	0.4	0.5	0.6	0.7
x_0	-0.788177	-0.784169	-0.779661	-0.775513	-0.771276
x_1	-0.526177	-0.472169	-0.427661	-0.403512	-0.349276
x_2	-0.404177	-0.390169	-0.345661	-0.251512	-0.177276

Larger K transaction cast somehow led threshold level to increase.

b	-0.2	-0.1	0	0.1	0.2
x_0	-0.777903	-0.778975	-0.779661	-0.780535	-0.781446
x_1	-0.645903	-0.526975	-0.427661	-0.348534	-0.259446
x_2	-0.52390	-0.444975	-0.345661	-0.216534	-0.107446

In the end we compute the threshold levels (x_0, x_1, x_2) associated with varying b. Bigger equilibrium level would bigger threshold levels. It can be seen that the pair (x_1, x_2) is monotonically increasing in b.

Chapter 2

PAIRS TRADING UNDER GEOMETRIC BROWNIAN MOTIONS WITH REGIME SWITCHING

2.1 Introduction

This chapter is concerned with pairs trading of stocks. The idea behind pairs trading is to track the price movements of a pairs of stocks over time and compare their relative price strengths. A pairs position consists of a short position in the stronger stock and a long position in the weaker one. A pairs trade is about buying and then selling such pairs positions. The strategy bets on the reversal of their price strength. What makes the strategy attractive is its 'market neutral' nature in the sense that it can be profitable under any market conditions. Pairs trading was initially introduced by Bamberger and followed by Tartaglia's quantitative group at Morgan Stanley in the 1980s; see Gatev et al. [E. Gatev and Rouwenhorst., 2006] for related history and background details. There are many in-depth discussions in connection with the cause of the divergence and subsequent convergence; see the book by Vidyamurthy [Vidyamurthy, 2018] and references therein.

Mathematical trading rules have been studied for many years. For example, Zhang [Q. Zhang, 2001] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. In [Q. Zhang, 2001], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [Guo and Zhang, 2005] studied the optimal selling rule under a model with switching geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type. Dai et al. [Dai, 2010] developed a trend following rule based on a conditional probability indicator. They showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated Hamilton-Jacobi-Bellman (HJB) equations. A similar idea was developed following a confidence interval approach by Iwarere and Barmish [Iwarere and Barmish, 2010]. Besides, Merhi and Zervos [Merhi and Zervos, 2007] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a geometric Brownian motion market model. In connection with mean reversion trading, Zhang and Zhang [H. Zhang and Zhang, 2008] obtained a buy-low and sell-high policy by characterizing the 'low' and 'high' levels in terms of the mean reversion parameters. Song and Zhang [Song and Zhang, 2013] studied pairs trading under a mean reversion model. It is shown that the optimal trading rule can be determined by threshold levels that can be obtained by solving a set of algebraic equations. A set of sufficient conditions are also provided to establish the desired optimality. Deshpande and Barmish [Deshpande and Barmish, 2016] introduced a control-theoretic approach. In particular, they were able to relax the requirement for spread functions and showed that their trading algorithm produces positive expected returns. Other related pairs technologies can be found in Elliott et al. [R.J. Elliott and Malcolm, 2005] and Whistler [Whistler, 2004]. Recently, Tie et al.[Tie and Zhang, 2020] studied an optimal pairs trading rule under geometric Brownian motions.. The objective is to initiate and close the positions of the pair sequentially to maximize a discounted payoff function. Using a dynamic programming approach, they studied the problem under a geometric Brownian motion model and proved that the buying and selling can be determined by two threshold curves in closed form. They also demonstrate the optimality of their trading strategy.

Market models with regime switching are important in market analysis. In this chapter, we consider a geometric Brownian motion with regime switching. The market mode is represented by a two-state Markov chain. In a recent paper, Tie and Zhang [Tie and Zhang, 2018] treated the selling part of pairs trading that generalizes the results of Hu and Oksendal [Hu and ksendal, 1998] by incorporating models with regime switching. They showed that the optimal selling rule can be determined by two threshold curves and established a set of sufficient conditions that guarantee the optimality of the policy. To complete the circle of pairs trading, one has to come up with the buying part of the trading rule to determine how much divergence is needed that triggers the entry of the position. It is the focus of this chapter. In particular, we study pairs trading under geometric Brownian motions with regime switching. The objective is to buy and then sell a pairs position to maximize the expected return. Using a smooth-fit method, we characterize the trading policies in terms of threshold curves which can be determined by a set of algebraic equations, We also provide a set of sufficient conditions for the optimality of the trading policy. Finally, we present numerical examples to illustrate the results.

This chapter is organized as follows. In §2.2 and §2.3, we formulate the pairs trading problem under consideration and the property of value function. In §2.4, we study the associated HJB equations and

their solutions, and key steps for pairs selling rules are given. In §2.5, we provide a set of sufficient conditions that guarantee the optimality of our trading rule. Numerical examples are given in §2.6. Some concluding remarks are given in §2.7.

2.2 Problem Formulation

Our pairs trading strategy involves two stocks $\mathbf{S}^1$ and $\mathbf{S}^2$. Let $\{X_t^1, t \geq 0\}$ denote the prices of stock $\mathbf{S}^1$ and $\{X_t^2, t \geq 0\}$ that of stock $\mathbf{S}^2$. They satisfy the following stochastic differential equation:

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix}\left[\begin{pmatrix} \mu_1(\alpha_t) \\ \mu_2(\alpha_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(\alpha_t) & \sigma_{12}(\alpha_t) \\ \sigma_{21}(\alpha_t) & \sigma_{22}(\alpha_t) \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}\right], \tag{2.1}$$

where $\alpha_t \in \mathcal{M} = \{1, 2\}$ is a two-state Markov chain and (W_t^1, W_t^2) a two-dimensional standard Brownian motion. Here, for $\alpha = 1, 2$, $\mu_i(\alpha)$, $i = 1, 2$, are the return rates and $\sigma_{ij}(\alpha)$, $i, j = 1, 2$, the volatility constants.

Let Q be the generator of α_t given by $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, with $\lambda_1 > 0$ and $\lambda_2 > 0$. We assume α_t and (W_t^1, W_t^2) are independent.

In this chapter, we assume, for simplicity, a pairs position consists of one-share long position in stock $\mathbf{S}^1$ and one-share short position in stock $\mathbf{S}^2$. Let $\mathbf{Z}$ denote the corresponding pairs position. One share in $\mathbf{Z}$ represents the combination of one share long position in $\mathbf{S}^1$ and one share short position in $\mathbf{S}^2$.

We consider one round trip pairs trading. The net position at any time can be either long (with one share of $\mathbf{Z}$) or flat (no stock position of either $\mathbf{S}^1$ or $\mathbf{S}^2$). Let $i = 0$, 1 denote the initial net position and let τ_0, τ_1, τ_2 denote stopping times with $\tau_1 \leq \tau_2$.. If initially the net position is flat $(i = 0)$, then one should start to buy a share of $\mathbf{Z}$. That is, to first buy at τ_1 and then sell at τ_2. The decision is denoted by $\Lambda_0 = \{\tau_1, \tau_2\}$. If initially the net position is long $(i = 1)$, then one should sell $\mathbf{Z}$. The corresponding decision is denoted by $\Lambda_1 = \{\tau_0\}$. Let K denote the fixed percentage of transaction costs associated with buying or selling of stocks $\mathbf{S}^i$, $i = 1, 2$. For example, the cost to establish the pairs position $\mathbf{Z}$ at $t = t_1$ is $(1+K)X_{t_1}^1 - (1-K)X_{t_2}^2$ and the proceeds to close it at a later time $t = t_2$ is $(1-K)X_{t_2}^1 - (1+K)X_{t_2}^2$. For ease of notation, let $\beta_{\text{b}} = 1 + K$ and $\beta_{\text{s}} = 1 - K$.

Given the initial state (x_1, x_2, α), the initial net position $i = 0, 1$, and the decision variables Λ_0 and Λ_1, the corresponding reward functions

$$\begin{aligned} J_0(x_1, x_2, \alpha, \Lambda_0) &= E\left\{\left[e^{-\rho\tau_2}\left(\beta_{\text{s}} X_{\tau_2}^1 - \beta_{\text{b}} X_{\tau_2}^2\right) I_{\{\tau_2 < \infty\}} - e^{-\rho\tau_1}\left(\beta_{\text{b}} X_{\tau_1}^1 - \beta_{\text{s}} X_{\tau_1}^2\right) I_{\{\tau_1 < \infty\}}\right]\right\} \\ J_1(x_1, x_2, \alpha, \Lambda_1) &= E\left\{e^{-\rho\tau_0}\left(\beta_{\text{s}} X_{\tau_0}^1 - \beta_{\text{b}} X_{\tau_0}^2\right) I_{\{\tau_0 < \infty\}}\right\}, \end{aligned} \tag{2.2}$$

where $\rho > 0$ is a given discount factor and I_A is the indicator function of an event A. Let $\mathcal{F}_t = \sigma\{(X_r^1, X_r^2, \alpha_r) : r \leq t\}$. The problem is to find $\{\mathcal{F}_t\}$ stopping times τ_0, τ_1, and τ_2, to maximize J_i. For $i = 0, 1$, let $V_i(x_1, x_2, \alpha)$ denote the value functions with the initial state $(X_0^1, X_0^2, \alpha_0) = (x_1, x_2, \alpha)$ and initial net positions $i = 0, 1$. That is, $V_i(x_1, x_2, \alpha) = \sup_{\Lambda_i} J_i(x_1, x_2, \alpha, \Lambda_i)$, $i = 0, 1$

Remark 1.. We would like to point out that our 'one-share' pair position is not as restrictive as it appears. For example, one can consider any pairs with n_1 shares of long position in $\mathbf{S}^1$ and n_2 shares of short position in $\mathbf{S}^2$. To treat this case, one only has to make change of the state variables $(X_t^1, X_t^2) \to (n_1 X_t^1, n_2 X_t^2)$. Due to the nature of GBMs, the corresponding system equation in (2.1) will remain the same. The modification only affects the reward function in (2.2) implicitly.

We make the following assumptions in this chapter: **(A1)** $\rho > \mu_j(\alpha)$, for $\alpha = 1, 2$ and $j = 1, 2$. Under these conditions, we can establish the lower and upper bounds for the value functions as follows.

2.3 Properties of Value Functions

Lemma 1. For some constant C, the inequalities hold

$$0 \le V_0\left(x_1, x_2, \alpha\right) \le C x_2. \tag{2.3}$$

In addition, we have

$$\beta_{\mathrm{s}} x_1 - \beta_{\mathrm{b}} x_2 \le V_1\left(x_1, x_2, \alpha\right) \le \beta_{\mathrm{s}} x_1. \tag{2.4}$$

Proof. We first consider the inequalities in (2.3). Clearly, $V_0 \ge 0$. To see $V_0 \le C x_2$, note that

$$\begin{aligned} &J_0\left(x_1, x_2, \alpha, \Lambda_0\right) \le E\left\{\left[e^{-\rho\tau_2}\left(X_{\tau_2}^1 - X_{\tau_2}^2\right) I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(X_{\tau_1}^1 - X_{\tau_1}^2\right) I_{\{\tau_1<\infty\}}\right]\right\} \\ &= E\left[e^{-\rho\tau_2} X_{\tau_2}^1 I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1} X_{\tau_1}^1 I_{\{\tau_1<\infty\}}\right] - E\left[e^{-\rho\tau_2} X_{\tau_2}^2 I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1} X_{\tau_1}^2 I_{\{\tau_1<\infty\}}\right]. \end{aligned}$$

Following from the proof of Lemma 3.1 of Tie et al. [14], we can show the first term above is less than or equal to 0. To find an upper bound for the second term, it suffices to show

$$E \int_{\tau_1}^{\tau_2} e^{-\rho t} X_t^2 \left(\rho - \mu_2\left(\alpha_t\right)\right) dt \le C x_2.$$

To this end, let $\mu_{\min} = \min\{\mu_2(1), \mu_2(2)\}$ and $\mu_{\max} = \max\{\mu_2(1), \mu_2(2)\}$. Then, we have

$$E \int_{\tau_1}^{\tau_2} e^{-\rho t} X_t^2 \left(\rho - \mu_2\left(\alpha_t\right)\right) dt \le \left(\rho - \mu_{\min}\right) \int_0^{\infty} e^{-\rho t} E X_t^2 dt.$$

Note that

$$E X_t^2 = x_2 + E \int_0^t X_s^2 \mu_2\left(\alpha_s\right) ds \le x_2 + \mu_{\max} \int_0^t E X_s^2 ds.$$

Use Gronwall's inequality to obtain $EX_t^2 \leq x_2 e^{\mu_{\max} t}$. It follows that

$$\int_0^\infty e^{-\rho t} E X_t^2 dt = \frac{x_2}{\rho - \mu_{\max}}.$$

Therefore, we have

$$E \int_{\tau_1}^{\tau_2} e^{-\rho t} X_t^2 \left(\rho - \mu_2\left(\alpha_t\right)\right) dt \leq \frac{\left(\rho - \mu_{\min}\right) x_2}{\rho - \mu_{\max}} =: C x_2.$$

Similarly, the inequalities in (2.4) can be obtained.

2.4 HJB equations

In this chapter, we follow the dynamic programming approach and focus on the associated HJB equations. For $i = 1, 2$, let

$$\mathcal{A}_i = \frac{1}{2}\left[a_{11}(i) x_1^2 \frac{\partial^2}{\partial x_1^2} + 2 a_{12}(i) x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + a_{22}(i) x_2^2 \frac{\partial^2}{\partial x_2^2}\right] + \mu_1(i) x_1 \frac{\partial}{\partial x_1} + \mu_2(i) x_2 \frac{\partial}{\partial x_2} \quad (2.5)$$

where $a_{11}(i) = \sigma_{11}^2(i) + \sigma_{12}^2(i)$, $a_{12}(i) = \sigma_{11}(i)\sigma_{21}(i) + \sigma_{12}(i)\sigma_{22}(i)$, and $a_{22}(i) = \sigma_{21}^2(i) + \sigma_{22}^2(i)$.

Formally, the associated H.JB equations have the form:

$$\begin{cases} \min\left\{\left(\rho - \mathcal{A}_1\right) v_0\left(x_1, x_2, 1\right) - \lambda_1\left(v_0\left(x_1, x_2, 2\right) - v_0\left(x_1, x_2, 1\right)\right)\right. \\ \qquad \left. v_0\left(x_1, x_2, 1\right) - v_1\left(x_1, x_2, 1\right) + \beta_b x_1 - \beta_s x_2\right\} = 0, \\ \min\left\{\left(\rho - \mathcal{A}_2\right) v_0\left(x_1, x_2, 2\right) - \lambda_2\left(v_0\left(x_1, x_2, 1\right) - v_0\left(x_1, x_2, 2\right)\right)\right. \\ \qquad \left. v_0\left(x_1, x_2, 2\right) - v_1\left(x_1, x_2, 2\right) + \beta_b x_1 - \beta_s x_2\right\} = 0, \end{cases} \quad (2.6)$$

$$\begin{cases} \min\left\{\left(\rho - \mathcal{A}_1\right) v_1\left(x_1, x_2, 1\right) - \lambda_1\left(v_1\left(x_1, x_2, 2\right) - v_1\left(x_1, x_2, 1\right)\right), v_1\left(x_1, x_2, 1\right) - \beta_{\mathrm{s}} x_1 + \beta_{\mathrm{b}} x_2\right\} = 0, \\ \min\left\{\left(\rho - \mathcal{A}_2\right) v_1\left(x_1, x_2, 2\right) - \lambda_2\left(v_1\left(x_1, x_2, 1\right) - v_1\left(x_1, x_2, 2\right)\right), v_1\left(x_1, x_2, 2\right) - \beta_{\mathrm{s}} x_1 + \beta_{\mathrm{b}} x_2\right\} = 0. \end{cases}$$

For ease of notation, let $u_1 = v_0\left(x_1, x_2, 1\right), u_2 = v_0\left(x_1, x_2, 2\right), u_3 = v_1\left(x_1, x_2, 1\right)$, and $u_4 = v_1\left(x_1, x_2, 2\right)$.

To solve the above HJB equations, we first convert them into single variable equations. Let $y = x_2/x_1$ and $u_i\left(x_1, x_2\right) = x_1 w_i\left(x_2/x_1\right)$, for some function $w_i(y)$ and $i = 1, 2, 3, 4$. Then we have by direct calculation that

$$\frac{\partial u_i}{\partial x_1} = w_i(y) - yw_i'(y), \frac{\partial u_i}{\partial x_2} = w_i'(y),$$
$$\frac{\partial^2 u_i}{\partial x_1^2} = \frac{y^2 w_i''(y)}{x_1}, \frac{\partial^2 u_i}{\partial x_2^2} = \frac{w_i''(y)}{x_1}, \text{ and } \frac{\partial^2 u_1}{\partial x_1 \partial x_2} = -\frac{yw_i''(y)}{x_1}.$$

Write $\mathcal{A}_j u_i$ in terms of w_i to obtain

$$\mathcal{A}_j u_i = x_1 \left\{ \sigma_j y^2 w_i''(y) + [\mu_2(j) - \mu_1(j)]\, yw_i'(y) + \mu_1(j) w_i(y) \right\},$$

where $\sigma_j = (a_{11}(j) - 2a_{12}(j) + a_{22}(j))/2$.

Then, the HJB equations can be given in terms of y and w_i as follows:

$$\begin{aligned}
&\min\{(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y), w_1(y) - w_3(y) + \beta_{\rm b} - \beta_{\rm s} y\} = 0,\\
&\min\{(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y), w_2(y) - w_4(y) + \beta_{\rm b} - \beta_{\rm s} y\} = 0,\\
&\min\{(\rho + \lambda_1 - \mathcal{L}_1)\, w_3(y) - \lambda_1 w_4(y), w_3(y) + \beta_{\rm b} y - \beta_{\rm s}\} = 0,\\
&\min\{(\rho + \lambda_2 - \mathcal{L}_2)\, w_4(y) - \lambda_2 w_3(y), w_4(y) + \beta_{\rm b} y - \beta_{\rm s}\} = 0,
\end{aligned} \tag{2.7}$$

where

$$\mathcal{L}_j [w_i(y)] = \sigma_j y^2 w_i''(y) + [\mu_2(j) - \mu_1(j)]\, yw_i'(y) + \mu_1(j) w_i(y). \tag{2.8}$$

In this chapter, we only consider the when $\sigma_j \neq 0, j = 1, 2$. If either $\sigma_1 = 0$ and/or $\sigma_2 = 0$, the problem reduces to a (partial) first order case and can be treated in a similar and simpler way. Next, we consider the joint equations $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = \lambda_1 w_2$ and $(\rho + \lambda_2 - \mathcal{L}_2)\, w_2 = \lambda_2 w_1$. Combine them to obtain

$(\rho + \lambda_1 - \mathcal{L}_1)(\rho + \lambda_2 - \mathcal{L}_2)\, w_2 = \lambda_1 \lambda_2 w_2$ and $(\rho + \lambda_2 - \mathcal{L}_2)(\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = \lambda_1 \lambda_2 w_1$ Both w_1 and w_2 must satisfy

$$[(\rho + \lambda_1 - \mathcal{L}_1)(\rho + \lambda_2 - \mathcal{L}_2) - \lambda_1 \lambda_2]\, w = 0.$$

Note that the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are the Euler type and the solutions to the above equation are of the form $w_i = y^\delta$. Thus, δ must satisfy the equation

$$P(\delta) := [\rho + \lambda_1 - A_1(\delta)]\,[\rho + \lambda_2 - A_2(\delta)] - \lambda_1 \lambda_2 = 0, \tag{2.9}$$

,

where

$$A_j(\delta) = \sigma_j \delta(\delta - 1) + \left[(\mu_2(j) - \mu_1(j)\right] \delta + \mu_1(j) = \sigma_j \delta^2 - \left[\sigma_j + \mu_1(j) - \mu_2(j)\right] \delta + \mu_1(j) \quad (2.10)$$

.

Note that $\rho + \lambda_1 - A_1(\zeta) = 0$ and $\rho + \lambda_2 - A_2(\nu) = 0$ have roots, respectively,

$$\begin{aligned} \zeta_1 &= \frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1} + \sqrt{\left(\frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \frac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}}, \\ \zeta_2 &= \frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1} - \sqrt{\left(\frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \frac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}}, \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \nu_1 &= \frac{1}{2} + \frac{\mu_1(2) - \mu_2(2)}{2\sigma_2} + \sqrt{\left(\frac{1}{2} + \frac{\mu_1(2) - \mu_2(2)}{2\sigma_2}\right)^2 + \frac{\rho + \lambda_2 - \mu_1(1)}{\sigma_2}}, \\ \nu_2 &= \frac{1}{2} + \frac{\mu_1(2) - \mu_2(2)}{2\sigma_2} - \sqrt{\left(\frac{1}{2} + \frac{\mu_1(2) - \mu_2(2)}{2\sigma_2}\right)^2 + \frac{\rho + \lambda_2 - \mu_1(1)}{\sigma_2}}. \end{aligned} \quad (2.12)$$

Note also that $\zeta_1 > 1$ and $\nu_1 > 1$, and $\zeta_2 < 0$ and $\nu_2 < 0$. It is elementary to show that the equation $P(\delta) = 0$ has four distinct roots $\delta_j, 1 \le j \le 4$ with $\delta_4 < \delta_3 < 0 < 1 < \delta_2 < \delta_1$. The δ_j, ζ_j and ν_j should have relation

$$\delta_4 < \min\{\zeta_2, \nu_2\}, \quad 0 > \delta_3 > \max\{\zeta_2, \nu_2\}, 0 < \delta_2 < \min\{\zeta_1, \nu_1\}, \quad \text{and} \quad \delta_1 > \max\{\zeta_1, \nu_1\}.$$

The general solutions of the equations

$$(\rho + \lambda_1 - \mathcal{L}_1) w_1 = \lambda_1 w_2 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2) w_2 = \lambda_2 w_1$$

can be given as

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j}, \quad \text{and} \quad w_2 = \sum_{j=1}^{4} c_{2j} y^{\delta_j},$$

for constants c_{ij}. Substituting them into the original equations leads to

$$\sum_{j=1}^{4} c_{1j} (\rho + \lambda_1 - A_1(\delta_j)) y^{\delta_j} = \lambda_1 \sum_{j=1}^{4} c_{2j} y^{\delta_j} \quad \text{and} \sum_{j=1}^{4} c_{2j} (\rho + \lambda_2 - A_2(\delta_j)) y^{\delta_j} = \lambda_2 \sum_{j=1}^{4} c_{1j} y^{\delta_j}.$$

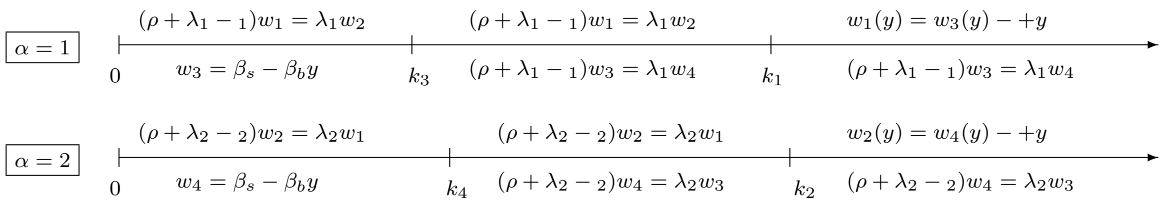

Figure 2.1: Equalities of HJB equations

Hence, we have

$$c_{1,j}\left(\rho + \lambda_1 - A_1\left(\delta_j\right)\right) = \lambda_1 c_{2j} \quad \text{and} \quad c_{2j}\left(\rho + \lambda_2 - A_2\left(\delta_j\right)\right) = \lambda_2 c_{1j}.$$

Let $\eta_j = \left(\rho + \lambda_1 - A_1\left(\delta_j\right)\right)/\lambda_1$. Then, we have

$$\eta_j = \frac{\rho + \lambda_1 - A_1\left(\delta_j\right)}{\lambda_1} = \frac{\lambda_2}{\rho + \lambda_2 - A_2\left(\delta_j\right)},$$

Necessarily, $c_{2j} = \eta_j c_{1j}$. Hence,

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j} \quad \text{and} \quad w_2 = \sum_{j=1}^{4} \eta_j c_{1j} y^{\delta_j}.$$

Similarly we can show the general solutions of $(\rho + \lambda_1 - \mathcal{L}_1)\, w_3 = \lambda_1 w_4$ and $(\rho + \lambda_2 - \mathcal{L}_2)\, w_4 = \lambda_2 w_3$ are given by

$$w_3 = \sum_{j=1}^{4} d_{1j} y^{\delta_j} \quad \text{and} \quad w_4 = \sum_{j=1}^{4} \eta_j d_{1j} y^{\delta_j},$$

for constants d_{ij}. By direct computation, we can show

$$\{y > 0 : w_1 - w_3 + \beta_b - \beta_s y = 0\} \cap \{y > 0 : w_3 + \beta_b y - \beta_s = 0\} = \emptyset,$$
$$\{y > 0 : w_2 - w_4 + \beta_b - \beta_s y = 0\} \cap \{y > 0 : w_4 + \beta_b y - \beta_s = 0\} = \emptyset.$$

Intuitively, if X_t^1 is small and X_t^2 is large, then one should buy $\mathbf{S}^1$ and sell (short) $\mathbf{S}^2$, i.e., to open a pairs position $\mathbf{Z}$. On the other hand, if X_t^1 is large and X_t^2 is small, then one should close the position $\mathbf{Z}$ by selling $\mathbf{S}^1$ and buying back $\mathbf{S}^2$.

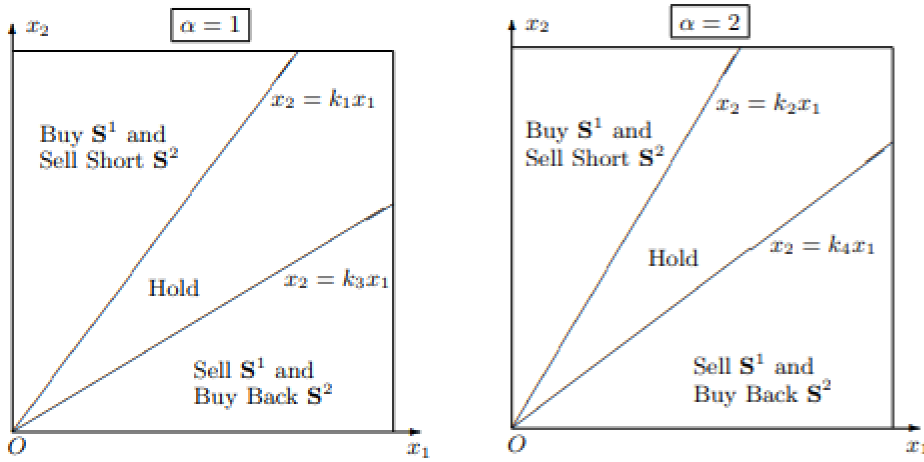

Figure 2.2: Switching Regions $\alpha = 1$ (left) and $\alpha = 2$ (right)

In view of this, if $\alpha = 1$, we divide the first quadrant into three regions $\{(x_1, x_2) > 0 : x_2 \leq k_3 x_1\}$ (close position region), $\{(x_1, x_2) > 0 : k_3 x_1 < x_2 < k_1 x_1\}$, (hold region) and $\{(x_1, x_2) > 0 : x_2 \geq k_1 x_1\}$ (open position region), for some positive constants k_1 and k_3. If $\alpha = 2$, we can do so similarly with regions $\{(x_1, x_2) > 0 : x_2 \leq k_4 x_1\}$ (close position region), $\{(x_1, x_2) > 0 : k_4 x_1 < x_2 < k_2 x_1\}$ (hold region), and $\{(x_1, x_2) > 0 : x_2 \geq k_2 x_1\}$ (open position region), for some positive k_2 and k_4. Note here $k_3 < k_1$ and $k_4 < k_2$. As a result, recall the change of variables ($y = x_2/x_1$), the equations in (2.7) can be specified as follows:

$$\begin{cases} w_3 = \beta_s - \beta_b y \text{ and } (\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = \lambda_1 w_2 & \text{when } y < k_3, \\ (\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = \lambda_1 w_2 \text{ and } (\rho + \lambda_1 - \mathcal{L}_1)\, w_3 = \lambda_1 w_4 & \text{when } k_3 < y < k_1, \\ w_1 = w_3 + \beta_s y - \beta_b \text{ and } (\rho + \lambda_1 - \mathcal{L}_1)\, w_3 = \lambda_1 w_4 & \text{when } y > k_1, \end{cases}$$

$$\begin{cases} w_4 = \beta_s - \beta_b y \text{ and } (\rho + \lambda_2 - \mathcal{L}_2)\, w_2 = \lambda_2 w_1 & \text{when } y < k_4, \\ (\rho + \lambda_2 - \mathcal{L}_2)\, w_2 = \lambda_2 w_1 \text{ and } (\rho + \lambda_2 - \mathcal{L}_2)\, w_4 = \lambda_2 w_3 & \text{when } k_4 < y < k_2, \\ w_2 = w_4 + \beta_s y - \beta_b \text{ and } (\rho + \lambda_2 - \mathcal{L}_2)\, w_4 = \lambda_2 w_3 & \text{when } y > k_2. \end{cases}$$

Each of these intervals and the corresponding equalities are given in Figure 1. We have four threshold parameters k_1, k_2, k_3 and k_4 to be determined. There are a number of ways to order them. Recall that $k_3 < k_1$ and $k_4 < k_2$. The largest is either k_1 or k_2 and the smallest is either k_3 or k_4. If k_3 is the smallest, then we can place k_1 at three different places. So this will lead to the following three cases.

$$k_3 \leq k_1 \leq k_4 \leq k_2, \quad k_3 \leq k_4 \leq k_1 \leq k_2, \quad k_3 \leq k_4 \leq k_2 \leq k_1.$$

Similarly if k_4 is the smallest, then we can place k_2 at three different places. Hence the next three possibilities:

$$k_4 \leq k_2 \leq k_3 \leq k_1, \quad k_4 \leq k_3 \leq k_2 \leq k_1, \quad k_4 \leq k_3 \leq k_1 \leq k_2.$$

On the region $(0, k_1 \wedge k_2]$ with $k_1 \wedge k_2 = \min\{k_1, k_2\}$, we have

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = \lambda_1 w_2 \text{ and } (\rho + \lambda_2 - \mathcal{L}_2)\, w_2 = \lambda_2 w_1,$$

this implies

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j} \quad \text{and} \quad w_2 = \sum_{j=1}^{4} \eta_j c_{1j} y^{\delta_j}.$$

in this region. Recall Lemma 1 and $\delta_3 < 0, \delta_4 < 0$. It follows that the coefficients for y^{δ_3} and y^{δ_4} have to be zero. Thus, we have

$$w_1 = C_1 y^{\delta_1} + C_2 y^{\delta_2} \text{ and } w_2 = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}.$$

Similarly, in the region $[k_3 \vee k_4, \infty)$ with $k_3 \vee k_4 = \max\{k_3, k_4\}$,

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_3 = \lambda_1 w_4 \quad \text{and } (\rho + \lambda_2 - \mathcal{L}_2)\, w_4 = \lambda_2 w_3,$$

the linear growth conditions (recall $\delta_1, \delta_2 > 1$) yield

$$w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4} \text{ and } w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4}.$$

To solve the HJB equations, we first note that w_3 and w_4 are not coupled with w_1 and w_2 and can be found separately. This is treated as a pure selling problem in Tie and Zhang [Tie and Zhang, 2018]. In this chapter, we first consider the case $(k_3 < k_4)$ and provide key steps for this case in Appendix for the sake of completeness then $(k_3 > k_4)$.

Solving for w_1 and w_2. In this section, we solve for w_1 and w_2 using the solution w_3 and w_4. Recall that w_1 and w_2 satisfy the HJB equations

$$\begin{aligned} &\min\{(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y), w_1(y) - w_3(y) + \beta_\mathrm{b} - \beta_\mathrm{s} y\} = 0, \\ &\min\{(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y), w_2(y) - w_4(y) + \beta_\mathrm{b} - \beta_\mathrm{s} y\} = 0. \end{aligned}$$

To find threshold type solutions, we are to determine k_1 and k_2 so that on $(0, k_1) : (\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) = 0$ and $w_1(y) - w_3(y) + \beta_\mathrm{b} - \beta_\mathrm{s} y \geq 0$; on $[k_1, \infty) : (\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) \geq 0$ and $w_1(y) - w_3(y) + \beta_\mathrm{b} - \beta_\mathrm{s} y = 0$; on $(0, k_2) : (\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y) = 0$ and $w_2(y) - w_4(y) + \beta_\mathrm{b} - \beta_\mathrm{s} y \geq 0$; and on $[k_2, \infty) : (\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y) \geq 0$ and $w_2(y) - w_4(y) + \beta_\mathrm{b} - \beta_\mathrm{s} y = 0$.

Recall that $k_4 < k_2$ and $k_3 < k_1$. Recall also the condition $k_4 < k_3$, $k_4 < k_2$ and $k_3 < k_1$, we need to further consider the three cases:

$$k_4 < k_2 < k_3 < k_1, \quad k_4 < k_3 < k_2 < k_1, \quad \text{and} \quad k_4 < k_3 < k_1 < k_2$$

2.4.1 CASE I: $k_3 < k_1 < k_4 < k_2$

First, we consider the case when $k_3 < k_1 < k_4 < k_2$. For $0 < y < k_1$, we have $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) = 0$ and $(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y) = 0$. Their general solutions have the form:

$$w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_2(y) = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}.$$

For $k_1 \leq y \leq k_2$, we have $w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y$ and $(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y) = 0$. For $k_2 \leq y < \infty$, we have $w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y$ and $w_2(y) = w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y$. Recall that the solution $w_3(y)$ and $w_4(y)$ in 2.4.7. This leads to, on $[k_1, k_4]$, $w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\mathrm{b}} + (\beta_{\mathrm{s}} - a_2)\, y$ and $w_2(y)$ satisfies

$$(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) = \lambda_2 w_1(y) = \lambda_2 \left[E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\mathrm{b}} + (\beta_{\mathrm{s}} - a_2)\, y \right].$$

Then the solution $w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p_1}(y)$, where $B_1 y^{\nu_1} + B_2 y^{\nu_2}$ is the general solution of the homogeneous differential equation $(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) = 0$ with ν_1 and ν_2 given in (2.12). A particular solution of

$$(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) = \lambda_2 w_1(y) = \lambda_2 \left[E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\mathrm{b}} + (\beta_{\mathrm{s}} - a_2)\, y \right],$$

can be given by

$$w_{2,p_1}(y) = \frac{\lambda_2 E_1}{\rho + \lambda_2 - A_2(\zeta_1)} y^{\zeta_1} + \frac{\lambda_2 E_2}{\rho + \lambda_2 - A_2(\zeta_2)} y^{\zeta_2} + \frac{\lambda_2 (a_1 - \beta_{\mathrm{b}})}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 (\beta_{\mathrm{s}} - a_2)}{\rho + \lambda_2 - \mu_2(2)} y.$$

Next, on the interval $[k_4, k_2]$, $w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y$ and $w_2(y)$ satisfies the inhomogenous equation $(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) = \lambda_2 w_1(y) = \lambda_2 \big(C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y \big)$. Similarly, a general solution $w_2(y) = D_1 y^{\nu_1} + D_2 y^{\nu_2} + w_{2,p_2}(y)$, where $w_{2,p_2}(y)$ is the particular solution given by

$$w_{2,p_2}(y) = \frac{\lambda_2 C_3}{\rho + \lambda_2 - A_2(\delta_3)} y^{\delta_3} + \frac{\lambda_2 C_4}{\rho + \lambda_2 - A_2(\delta_4)} y^{\delta_4} - \frac{\lambda_2 \beta_{\mathrm{b}}}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 \beta_{\mathrm{s}} y}{\rho + \lambda_2 - \mu_2(2)}.$$

Recall that $\eta_3 = \lambda_2 / (\rho + \lambda_2 - A_2(\delta_3))$ and $\eta_4 = \lambda_2 / (\rho + \lambda_2 - A_2(\delta_4))$. It follows that

$$w_{2,p_2}(y) = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \frac{\lambda_2 \beta_{\mathrm{b}}}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 \beta_{\mathrm{s}} y}{\rho + \lambda_2 - \mu_2(2)}.$$

Finally, on the interval $[k_2, \infty)$, $w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y$ and $w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y$. These computations can be summarized as follows:

$$\begin{aligned}
\text{On } (0, k_1): \quad & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2}, \\
& w_2(y) = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}, \\
\text{On } [k_1, k_4): \quad & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_{\rm b} + (\beta_{\rm s} - a_2) y, \\
& w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p_1}(y), \\
\text{On } [k_4, k_2]: \quad & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y, \\
& w_2(y) = D_1 y^{\nu_1} + D_2 y^{\nu_2} + w_{2,p_2}(y), \\
\text{On } (k_2, \infty): \quad & w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y, \\
& w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \beta_{\rm b} + \beta_{\rm s} y,
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
w_{2,p_1}(y) &= \frac{\lambda_2 E_1}{\rho + \lambda_2 - A_2(\zeta_1)} y^{\zeta_1} + \frac{\lambda_2 E_2}{\rho + \lambda_2 - A_2(\zeta_2)} y^{\zeta_2} + \frac{\lambda_2 (a_1 - \beta_{\rm b})}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 (\beta_{\rm s} - a_2)}{\rho + \lambda_2 - \mu_2(2)} y, \\
w_{2,p_2}(y) &= C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \frac{\lambda_2 \beta_{\rm b}}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 \beta_{\rm s} y}{\rho + \lambda_2 - \mu_2(2)}.
\end{aligned}$$

We follow the smooth-fit method to determine parameters $C_1, C_2, B_1, B_2, D_1, D_2, k_1$ and k_2. The continuity of $w_1(y), w_2(y), w_1'(y)$ and $w_2'(y)$ at k_1 yields

$$\begin{aligned}
C_1 k_1^{\delta_1} + C_2 k_1^{\delta_2} &= w_3(k_1) + \beta_{\rm s} k_1 - \beta_{\rm b}, \\
C_1 \delta_1 k_1^{\delta_1 - 1} + C_2 \delta_2 k_1^{\delta_2 - 1} &= w_3'(k_1) + \beta_{\rm s}, \\
C_1 \eta_1 k_1^{\delta_1} + C_2 \eta_2 k_1^{\delta_2} &= B_1 k_1^{\nu_1} + B_2 k_1^{\nu_2} + w_{2,p_1}(k_1), \\
C_1 \eta_1 \delta_1 k_1^{\delta_1 - 1} + C_2 \eta_2 \delta_2 k_1^{\delta_2 - 1} &= B_1 \nu_1 k_1^{\nu_1 - 1} + B_2 \nu_2 k_1^{\nu_2 - 1} + w_{2,p_1}'(k_1).
\end{aligned}$$

The continuity of $w_2(y)$ and $w_2'(y)$ at k_4 yields

$$\begin{aligned}
B_1 k_4^{\nu_1} + B_2 k_4^{\nu_2} + w_{2,p_1}(k_4) &= D_1 k_4^{\nu_1} + D_2 k_4^{\nu_2} + w_{2,p_2}(k_4), \\
B_1 \nu_1 k_4^{\nu_1 - 1} + B_2 \nu_2 k_4^{\nu_2 - 1} + w_{2,p_1}'(k_4) &= D_1 \nu_1 k_4^{\nu_1 - 1} + D_2 \nu_2 k_4^{\nu_2 - 1} + w_{2,p_2}'(k_4).
\end{aligned}$$

The continuity of $w_2(y)$ and $w_2'(y)$ at k_2 yields

$$\begin{aligned}
D_1 k_2^{\nu_1} + D_2 k_2^{\nu_2} + w_{2,p_2}(k_2) &= w_4(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2, \\
D_1 \nu_1 k_2^{\nu_1 - 1} + D_2 \nu_2 k_2^{\nu_2 - 1} + w_{2,p_2}'(k_2) &= w_4'(k_2) + \beta_{\rm s}.
\end{aligned}$$

Let

$$\Lambda = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \text{ and } \Phi(t, s_1, s_2) = \begin{pmatrix} t^{s_1} & t^{s_2} \\ s_1 t^{s_1} & s_2 t^{s_2} \end{pmatrix}$$

Then, we have

$$\Phi^{-1}(t,s_1,s_2)=\frac{1}{s_2-s_1}\left(\begin{array}{cc} s_2t^{-s_1} & -t^{-s_1}. \\ -s_1t^{-s_2} & t^{-s_2}\end{array}\right)$$

Using these matrices, we can write the first four equations at k_1 as

$$\Phi(k_1,\delta_1,\delta_2)\left(\begin{array}{c} C_1 \\ C_2\end{array}\right)=\left(\begin{array}{c} w_3(k_1)+\beta_\mathrm{s}k_1-\beta_\mathrm{b} \\ k_1\left[w_3'(k_1)+\beta_\mathrm{s}\right]\end{array}\right),$$
$$\Phi(k_1,\delta_1,\delta_2)\Lambda\left(\begin{array}{c} C_1 \\ C_2\end{array}\right)=\Phi(k_1,\nu_1,\nu_2)\left(\begin{array}{c} B_1 \\ B_2\end{array}\right)+\left(\begin{array}{c} w_{2,p_1}(k_1) \\ k_1w_{2,p_1}'(k_1)\end{array}\right).$$

It follows, by solving for C_1, C_2, B_1 and B_2, that

$$\left(\begin{array}{c} C_1 \\ C_2\end{array}\right)=\Phi^{-1}(k_1,\delta_1,\delta_2)\left(\begin{array}{c} w_3(k_1)+\beta_\mathrm{s}k_1-\beta_\mathrm{b} \\ k_1\left[w_3'(k_1)+\beta_\mathrm{s}\right]\end{array}\right),$$
$$\left(\begin{array}{c} B_1 \\ B_2\end{array}\right)=\Phi^{-1}(k_1,\nu_1,\nu_2)\Bigg[\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2)\left(\begin{array}{c} w_3(k_1)+\beta_\mathrm{s}k_1-\beta_\mathrm{b} \\ k_1\left[w_3'(k_1)+\beta_\mathrm{s}\right]\end{array}\right)$$
$$-\left(\begin{array}{c} w_{2,p_1}(k_1) \\ k_1w_{2,p_1}'(k_1)\end{array}\right)\Bigg].$$

In addition, simple calculation yields

$$\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2)=\frac{1}{\delta_2-\delta_1}\left(\begin{array}{cc} \eta_1\delta_2-\eta_2\delta_1 & \eta_2-\eta_1 \\ \delta_1\delta_2(\eta_1-\eta_2) & \eta_2\delta_2-\eta_1\delta_1\end{array}\right).$$

Note that this matrix is independent of k_1. Moreover, we can write (from the continuity of w_2 and w_2' at k_4)

$$\Phi(k_4,\nu_1,\nu_2)\left(\begin{array}{c} B_1-D_1 \\ B_2-D_2\end{array}\right)=\left(\begin{array}{c} w_{2,p_2}(k_4)-w_{2,p_1}(k_4) \\ k_4\left[w_{2,p_2}'(k_4)-w_{2,p_1}'(k_4)\right]\end{array}\right).$$

This yields

$$\left(\begin{array}{c} B_1-D_1 \\ B_2-D_2\end{array}\right)=\Phi^{-1}(k_4,\nu_1,\nu_2)\left(\begin{array}{c} w_{2,p_2}(k_4)-w_{2,p_1}(k_4) \\ k_4\left[w_{2,p_2}'(k_4)-w_{2,p_1}'(k_4)\right]\end{array}\right).$$

Finally, follow from the continuity of w_2 and w_2' at k_2, we write

$$\Phi(k_2,\nu_1,\nu_2)\left(\begin{array}{c} D_1 \\ D_2\end{array}\right)=\left(\begin{array}{c} w_4(k_2)-w_{2,p_2}(k_2)-\beta_\mathrm{b}+\beta_\mathrm{s}k_2 \\ k_2\left[w_4'(k_2)-w_{2,p_2}'(k_2)+\beta_\mathrm{s}\right]\end{array}\right).$$

This gives

$$\left(\begin{array}{c} D_1 \\ D_2\end{array}\right)=\Phi^{-1}(k_2,\nu_1,\nu_2)\left(\begin{array}{c} w_4(k_2)-w_{2,p_2}(k_2)-\beta_\mathrm{b}+\beta_\mathrm{s}k_2 \\ k_2\left[w_4'(k_2)-w_{2,p_2}'(k_2)+\beta_\mathrm{s}\right]\end{array}\right).$$

Eliminate $(B_1, B_@)$ to obtain the following equations for k_1 and k_2 :

$$\Phi^{-1}(k_1,\nu_1,\nu_2)\left[\Phi(k_1,\delta_1,\delta_2)\Lambda\Phi^{-1}(k_1,\delta_1,\delta_2)\begin{pmatrix} w_3(k_1)+\beta_s k_1-\beta_b \\ k_1 w_3'(k_1)+\beta_s k_1\end{pmatrix}-\begin{pmatrix} w_{2,p_1}(k_1) \\ k_1 w_{2,p_1}'(k_1)\end{pmatrix}\right]$$
$$=\Phi^{-1}(k_4,\nu_1,\nu_2)\begin{pmatrix} w_{2,p_2}(k_4)-w_{2,p_1}(k_4) \\ k_4\left[w_{2,p_2}'(k_4)-w_{2,p_1}'(k_4)\right]\end{pmatrix}$$
$$+\Phi^{-1}(k_2,\nu_1,\nu_2)\begin{pmatrix} w_4(k_2)-w_{2,p_2}(k_2)-\beta_b+\beta_s k_2 \\ k_2\left[w_4'(k_2)-w_{2,p_2}'(k_2)+\beta_s\right]\end{pmatrix}. \tag{2.14}$$

This yields two equations of k_1 and k_2. The existence of k_1 and k_2 can proved.. Once we find k_1 and k_2 and note that the constants B_1, B_2, C_1, C_2, D_1, and D_2 can be written as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. In view of this, k_1 and k_2 have to be determined so that the following variational inequalities are satisfied:

$$\begin{array}{ll} \text{On } (0,k_1): & w_1(y)-w_3(y)+\beta_b-\beta_s y\geq 0, \\ & w_2(y)-w_4(y)+\beta_b-\beta_s y\geq 0, \\ \text{On } [k_1,k_2]: & (\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)\geq 0, \\ & w_2(y)-w_4(y)+\beta_b-\beta_s y\geq 0, \\ \text{On } (k_2,\infty): & (\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)\geq 0, \\ & (\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0. \end{array} \tag{2.15}$$

To facilitate numerical computations, we provide equivalent inequalities for those involving the differential operators $\mathcal{L}_j$. First, we consider the two inequalities on the interval $[k_2,\infty)$:

$$(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)\geq 0 \quad \text{and} \quad (\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0.$$

Recall that $w_1(y)=w_3(y)-\beta_b+\beta_s y$ and $w_2(y)=w_4(y)-\beta_b+\beta_s y$, and we apply $\mathcal{L}_1$ to $w_1(y)$ and $\mathcal{L}_2$ to $w_2(y)$ to get

$$\begin{aligned}(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)&=\lambda_1 w_4(y)+(\rho+\lambda_1-\mu_2(1))\,\beta_s y-(\rho+\lambda_1-\mu_1(1))\,\beta_b,\\ (\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)&=\lambda_2 w_3(y)+(\rho+\lambda_2-\mu_2(2))\,\beta_s y-(\rho+\lambda_2-\mu_1(2))\,\beta_b.\end{aligned}$$

Then (23) is equivalent to

$$\begin{aligned}(\rho+\lambda_1-\mu_2(1))\,\beta_s y-(\rho+\lambda_1-\mu_1(1))\,\beta_b&\geq\lambda_1(\beta_s y-\beta_b),\\ (\rho+\lambda_2-\mu_2(2))\,\beta_s y-(\rho+\lambda_2-\mu_1(2))\,\beta_b&\geq\lambda_2(\beta_s y-\beta_b).\end{aligned}$$

Simplify to obtain

$$(\rho-\mu_2(1))\,\beta_{\rm s}y-(\rho-\mu_1(1))\,\beta_{\rm b}\geq 0 \quad \text{and} \quad (\rho-\mu_2(2))\,\beta_{\rm s}y-(\rho-\mu_1(2))\,\beta_{\rm b}\geq 0.$$

These inequalities hold as long as

$$k_2 \geq \frac{(\rho-\mu_1(j))\,\beta_{\rm b}}{(\rho-\mu_2(j))\,\beta_{\rm s}} \quad \text{for } j=1,2.$$

Next, we consider the inequality involving $\mathcal{L}_1$, i.e, $(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)\geq 0$ on $[k_1,k_2]$. Recall that $w_1=w_3-\beta_{\rm b}+\beta_{\rm s}y$ and w_2 satisfies $(\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)=\lambda_2 w_1(y)$ on this interval. Applying $(\rho+\lambda_1-\mathcal{L}_1)$ to w_1 yield

$$(\rho+\lambda_1-\mathcal{L}_1)\,w_1=(\rho+\lambda_1-\mathcal{L}_1)\,w_3+(\rho+\lambda_1-\mu_2(1))\,\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\,\beta_{\rm b}.$$

Recall that $k_3<k_1<k_4<k_2$ and $(\rho+\lambda_1-\mathcal{L}_1)\,w_3=\lambda_1 w_4$. It follows that

$$(\rho+\lambda_1-\mathcal{L}_1)\,w_1=\lambda_1 w_4+(\rho+\lambda_1-\mu_2(1))\,\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\,\beta_{\rm b}.$$

Recall also that $w_4=\beta_{\rm s}-\beta_{\rm b}y$ on the interval $[0,k_4]$. Hence on interval $[k_1,k_4]\subset[0,k_4]$, $(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)\geq 0$ is equivalent to

$$\lambda_1\,(\beta_{\rm s}-\beta_{\rm b}y)+(\rho+\lambda_1-\mu_2(1))\,\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\,\beta_{\rm b}\geq\lambda_1 w_2.$$

Since $w_2(y)=B_1y^{\nu_1}+B_2y^{\nu_2}+w_{2,p_1}(y)$ on the interval $[k_1,k_4]$, the above inequality is equivalent to

$$B_1y^{\nu_1}+B_2y^{\nu_2}+w_{2,p_1}(y)\leq\left[\frac{\rho-\mu_2(1)}{\lambda_1}\beta_{\rm s}+\beta_{\rm s}-\beta_{\rm b}\right]y-\left[\frac{\rho-\mu_1(1)}{\lambda_1}\beta_{\rm b}+\beta_{\rm b}-\beta_{\rm s}\right].$$

Similarly on the interval $[k_4,k_2]$, $w_2(y)=D_1y^{\nu_1}+D_2y^{\nu_2}+w_{2,p_2}(y)$, and the inequality is equivalent to

$$D_1y^{\nu_1}+D_2y^{\nu_2}+w_{2,p_2}(y)\leq\left[\frac{\rho-\mu_2(1)}{\lambda_1}\beta_{\rm s}+\beta_{\rm s}-\beta_{\rm b}\right]y-\left[\frac{\rho-\mu_1(1)}{\lambda_1}\beta_{\rm b}+\beta_{\rm b}-\beta_{\rm s}\right].$$

2.4.2 CASE II: $k_3<k_4<k_1<k_2$.

Next, we treat the case $(k_3<k_4<k_1<k_2)$. Note that, for $0<y<k_1$, we have $(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)=0$ and $(\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)=0$. Their general solutions are of the forms

$$w_1(y)=C_1y^{\delta_1}+C_2y^{\delta_2} \quad \text{and} \quad w_2(y)=C_1\eta_1y^{\delta_1}+C_2\eta_2y^{\delta_2}.$$

For $k_1 \leq y \leq k_2$, we have $w_1(y) = w_3(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y \quad$ and $\quad (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) = 0$. For $k_2 < y < \infty$, we have $w_1(y) = w_3(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y \quad$ and $\quad w_2(y) = w_4(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y$. Recall also the solutions $w_3(y)$ and $w_4(y)$ in (39) (Appendix): It follows that, on the interval $[k_1, k_2]$, $w_1(y) = w_3(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_\mathrm{b} + (\beta_\mathrm{s} - a_2) y$; and $w_2(y)$ satisfies the equation $(\rho + \lambda_2 - \mathcal{L}_2) w_2(y) = \lambda_2 w_1(y) = \lambda_2 \left[E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_\mathrm{b} + (\beta_\mathrm{s} - a_2) y\right]$. Then the general solution $w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p}(y)$ where the particular solution

$$w_{2,p}(y) = \frac{\lambda_2 E_1}{\rho + \lambda_2 - A_2(\zeta_1)} y^{\zeta_1} + \frac{\lambda_2 E_2}{\rho + \lambda_2 - A_2(\zeta_2)} y^{\zeta_2} + \frac{\lambda_2 (a_1 - \beta_\mathrm{b})}{\rho + \lambda_2 - \mu_1(2)} + \frac{\lambda_2 (\beta_\mathrm{s} - a_2)}{\rho + \lambda_2 - \mu_2(2)} y.$$

In this chapter, the use of parameters A_i, B_i, C_i, etc is limited to the particular section. They may be different across sections if no confusion arises.

Finally, on the interval (k_2, ∞), we have

$$\begin{aligned}
& w_1(y) = w_3(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_\mathrm{b} + \beta_\mathrm{s} y, \\
& w_2(y) = w_4(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \beta_\mathrm{b} + \beta_\mathrm{s} y. \\
\text{On } (0, k_1): & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2}, \\
& w_2(y) = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}, \\
\text{On } [k_1, k_2]: & w_1(y) = w_3(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - \beta_\mathrm{b} + (\beta_\mathrm{s} - a_2) y, \\
& w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p}(y), \\
\text{On } (k_2, \infty): & w_1(y) = w_3(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_\mathrm{b} + \beta_\mathrm{s} y, \\
& w_2(y) = w_4(y) - \beta_\mathrm{b} + \beta_\mathrm{s} y = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \beta_\mathrm{b} + \beta_\mathrm{s} y.
\end{aligned} \tag{2.16}$$

Next, we use the smooth-fit conditions to determine the parameters C_1, C_2, B_1, B_2, k_1 and k_2. First, the continuity of $w_1(y), w_2(y), w_1'(y)$ and $w_2'(y)$ at k_1 yields

$$\begin{aligned}
C_1 k_1^{\delta_1} + C_2 k_1^{\delta_2} &= w_3(k_1) + \beta_\mathrm{s} k_1 - \beta_\mathrm{b}, \\
C_1 \delta_1 k_1^{\delta_1 - 1} + C_2 \delta_2 k_1^{\delta_2 - 1} &= w_3'(k_1) + \beta_\mathrm{s}, \\
C_1 \eta_1 k_1^{\delta_1} + C_2 \eta_2 k_1^{\delta_2} &= B_1 k_1^{\nu_1} + B_2 k_1^{\nu_2} + w_{2,p}(k_1), \\
C_1 \eta_1 \delta_1 k_1^{\delta_1 - 1} + C_2 \eta_2 \delta_2 k_1^{\delta_2 - 1} &= B_1 \nu_1 k_1^{\nu_1 - 1} + B_2 \nu_2 k_1^{\nu_2 - 1} + w_{2,p}'(k_1).
\end{aligned}$$

Similarly, the continuity of $w_2(y)$ and $w_2'(y)$ at k_2 yields

$$\begin{aligned}
B_1 k_2^{\nu_1} + B_2 k_2^{\nu_2} + w_{2,p}(k_2) &= w_4(k_2) - \beta_\mathrm{b} + \beta_\mathrm{s} k_2, \\
B_1 \nu_1 k_2^{\nu_1 - 1} + B_2 \nu_2 k_2^{\nu_2 - 1} + w_{2,p}'(k_2) &= w_4'(k_2) + \beta_\mathrm{s}.
\end{aligned}$$

We can write them in matrix form:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Phi^{-1}(k_1, \delta_1, \delta_2) \begin{pmatrix} w_3(k_1) + \beta_s k_1 - \beta_b \\ k_1 [w_3'(k_1) + \beta_s] \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_1, \nu_1, \nu_2) \Bigg[\Phi(k_1, \delta_1, \delta_2) \Lambda \Phi^{-1}(k_1, \delta_1, \delta_2) \begin{pmatrix} w_3(k_1) + \beta_s k_1 - \beta_b \\ k_1 [w_3'(k_1) + \beta_s] \end{pmatrix} - \begin{pmatrix} w_{2,p}(k_1) \\ k_1 w_{2,p}'(k_1) \end{pmatrix} \Bigg].$$

The continuity of w_2 and w_2' at k_2 leads to the equations

$$\Phi(k_2, \nu_1, \nu_2) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} w_4(k_2) - w_{2,p}(k_2) - \beta_b + \beta_s k_2 \\ k_2 \left[w_4'(k_2) - w_{2,p}'(k_2) + \beta_s \right] \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_2, \nu_1, \nu_2) \begin{pmatrix} w_4(k_2) - w_{2,p}(k_2) - \beta_b + \beta_s k_2 \\ k_2 \left[w_4'(k_2) - w_{2,p}'(k_2) + \beta_s \right] \end{pmatrix}.$$

Eliminate B_1 and B_2 to obtain the equations for k_1 and k_2 :

$$\Phi^{-1}(k_1, \nu_1, \nu_2) \Bigg[\Phi(k_1, \delta_1, \delta_2) \Lambda \Phi^{-1}(k_1, \delta_1, \delta_2) \begin{pmatrix} w_3(k_1) + \beta_s k_1 - \beta_b \\ k_1 [w_3'(k_1) + \beta_s] \end{pmatrix} - \begin{pmatrix} w_{2,p}(k_1) \\ k_1 w_{2,p}'(k_1) \end{pmatrix} \Bigg] = \Phi^{-1}(k_2, \nu_1, \nu_2) \begin{pmatrix} w_4(k_2) - w_{2,p}(k_2) - \beta_b + \beta_s k_2 \\ k_2 \left[w_4'(k_2) - w_{2,p}'(k_2) + \beta_s \right] \end{pmatrix}. \tag{2.17}$$

Recall that the constants $B_1, B_2, C_1,$ and C_2 can be represented as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. Therefore, k_1 and k_2 need to be determined so that the following variational inequalities are satisfied:

$$\begin{aligned} \text{On } (0, k_1): \quad & w_1(y) - w_3(y) + \beta_b - \beta_s y \geq 0, \\ & w_2(y) - w_4(y) + \beta_b - \beta_s y \geq 0, \\ \text{On } [k_1, k_2]: \quad & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, \\ & w_2(y) - w_4(y) + \beta_b - \beta_s y \geq 0, \\ \text{On } (k_2, \infty): \quad & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, \\ & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0. \end{aligned} \tag{2.18}$$

Next, we consider equivalent inequalities for those involving the differential operators $\mathcal{L}_j$. First, on the interval $[k_2, \infty)$, the variational inequalities are equivalent to

$$(\rho + \lambda_1 - \mu_2(1))\, \beta_{\rm s} y - (\rho + \lambda_1 - \mu_1(1))\, \beta_{\rm b} \geq \lambda_1 (\beta_{\rm s} y - \beta_{\rm b}),$$
$$(\rho + \lambda_2 - \mu_2(2))\, \beta_{\rm s} y - (\rho + \lambda_2 - \mu_1(2))\, \beta_{\rm b} \geq \lambda_2 (\beta_{\rm s} y - \beta_{\rm b}).$$

as in Case I. The equivalent conditions for these inequalities to hold are

$$k_2 \geq \frac{(\rho - \mu_1(j))\, \beta_{\rm b}}{(\rho - \mu_2(j))\, \beta_{\rm s}} \quad \text{for } j = 1, 2$$

Move on to the interval $[k_1, k_2]$ and recall $w_1 = w_3 - \beta_{\rm b} + \beta_{\rm s} y$. Apply $(\rho + \lambda_1 - \mathcal{L}_1)$ to w_1 to obtain

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = (\rho + \lambda_1 - \mathcal{L}_1)\, w_3 + (\rho + \lambda_1 - \mu_2(1))\, \beta_{\rm s} y - (\rho + \lambda_1 - \mu_1(1))\, \beta_{\rm b}.$$

In addition, recall that $k_3 < k_4 < k_1 < k_2$ and $(\rho + \lambda_1 - \mathcal{L}_1)\, w_3 = \lambda_1 w_4$. It follows that

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = \lambda_1 w_4 + (\rho + \lambda_1 - \mu_2(1))\, \beta_{\rm s} y - (\rho + \lambda_1 - \mu_1(1))\, \beta_{\rm b}.$$

Recall also that $w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4}$ for $y \geq k_4$ and $w_2(y) = B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p}(y)$. Hence the inequality $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) \geq 0$ is equivalent to

$$B_1 y^{\nu_1} + B_2 y^{\nu_2} + w_{2,p}(y) \leq C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} + \left[\frac{\rho + \lambda_1 - \mu_2(1)}{\lambda_1}\right] \beta_{\rm s} y - \left[\frac{\rho + \lambda_1 - \mu_1(1)}{\lambda_1}\right] \beta_{\rm b}.$$

2.4.3 CASE III: $k_3 < k_4 < k_2 < k_1$

Next, we consider the last case ($k_3 < k_4 < k_2 < k_1$). For $0 < y < k_2$, we have the equations

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) = 0 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y) = 0.$$

Their general solutions can be given by

$$w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_2(y) = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}.$$

For $k_1 \leq y \leq k_2$, we have

$$w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y) = 0.$$

For $k_2 < y < \infty$, we have

$$w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s} y \quad \text{and} \quad w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s} y.$$

Recall the solutions w_3 and w_4 given in (39) (Appendix). It follows that, on the interval $[k_2, k_1]$

$$w_2(y) = w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y.$$

and $w_1(y)$ satisfies

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) = \lambda_1 w_2(y) = \lambda_1 \left[C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y\right].$$

Then the general solution $w_1(y) = B_1 y^{\zeta_1} + B_2 y^{\zeta_2} + w_{1,p}(y)$ where the particular solution

$$w_{1,p}(y) = \frac{\lambda_1 C_3 \eta_3}{\rho + \lambda_1 - A_1(\delta_3)} y^{\delta_3} + \frac{\lambda_1 \eta_4 C_4}{\rho + \lambda_1 - A_1(\delta_4)} y^{\delta_4} - \frac{\lambda_1 \beta_\text{b}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1 \beta_\text{s}}{\rho + \lambda_1 - \mu_2(1)} y.$$

Note that $\lambda_1 / (\rho + \lambda_1 - A_1(\delta_3)) = 1/\eta_3$ and $\lambda_1 / (\rho + \lambda_1 - A_1(\delta_4)) = 1/\eta_4$. These imply

$$w_{1,p}(y) = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \frac{\lambda_1 \beta_\text{b}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1 \beta_\text{s} y}{\rho + \lambda_1 - \mu_2(1)} = w_3(y) - \frac{\lambda_1 \beta_\text{b}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1 \beta_\text{s} y}{\rho + \lambda_1 - \mu_2(1)}.$$

Finally, on the interval $[k_1, \infty)$, we have

$$\begin{aligned} w_1(y) &= w_3(y) - \beta_\text{b} + \beta_\text{s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y, \\ w_2(y) &= w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y. \end{aligned}$$

$$\begin{aligned} \text{On } (0, k_2) : & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2}, \\ & w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2}, \\ \text{On } [k_2, k_1] : & w_1(y) = B_1 y^{\zeta_1} + B_2 y^{\zeta_2} + w_{1,p}(y), \\ & w_2(y) = w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y, \\ \text{On } (k_1, \infty) : & w_1(y) = w_3(y) - \beta_\text{b} + \beta_\text{s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y, \\ & w_2(y) = w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y. \end{aligned} \tag{2.20}$$

Next, we apply the smooth-fit method to determine the parameters C_1, C_2, B_1, B_2, k_1 and k_2. First, the continuity of $w_1(y), w_2(y), w_1'(y)$ and $w_2'(y)$ at k_2 yields

$$\begin{aligned} C_1 k_2^{\delta_1} + C_2 k_2^{\delta_2} &= B_1 k_2^{\zeta_1} + B_2 k_2^{\zeta_2} + w_{1,p}(k_2), \\ C_1 \delta_1 k_2^{\delta_1 - 1} + C_2 \delta_2 k_2^{\delta_2 - 1} &= B_1 \zeta_1 k_2^{\zeta_1 - 1} + B_2 \zeta_2 k_2^{\zeta_2 - 1} + w_{1,p}'(k_2), \\ C_1 \eta_1 k_2^{\delta_1} + C_2 \eta_2 k_2^{\delta_2} &= w_4(k_2) + \beta_\text{s} k_2 - \beta_\text{b}, \\ C_1 \eta_1 \delta_1 k_2^{\delta_1 - 1} + C_2 \eta_2 \delta_2 k_2^{\delta_2 - 1} &= w_4'(k_2) + \beta_\text{s}. \end{aligned}$$

The continuity of $w_1(y)$ and $w_1'(y)$ at k_1 yields

$$B_1 k_1^{\zeta_1} + B_2 k_1^{\zeta_2} + w_{1,p}(k_1) = w_3(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1,$$
$$B_1 \zeta_1 k_1^{\zeta_1 - 1} + B_2 \zeta_2 k_1^{\zeta_2 - 1} + w_{1,p}'(k_1) = w_3'(k_1) + \beta_{\mathrm{s}}.$$

Solve for C_1, C_2, B_1 and B_2 to obtain

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Lambda^{-1}\Phi^{-1}(k_2, \delta_1, \delta_2) \begin{pmatrix} w_4(k_2) + \beta_{\mathrm{s}} k_2 - \beta_{\mathrm{b}} \\ k_2 \left[w_4'(k_2) + \beta_{\mathrm{s}}\right] \end{pmatrix},$$
$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_2, \zeta_1, \zeta_2) \Bigg[\Phi(k_2, \delta_1, \delta_2)\Lambda^{-1}\Phi^{-1}(k_2, \delta_1, \delta_2) \begin{pmatrix} w_4(k_2) + \beta_{\mathrm{s}} k_2 - \beta_{\mathrm{b}} \\ k_2 \left[w_4'(k_2) + \beta_{\mathrm{s}}\right] \end{pmatrix}$$
$$- \begin{pmatrix} w_{1,p}(k_2) \\ k_2 w_{1,p}'(k_2) \end{pmatrix}\Bigg].$$

The continuity of w_1 and w_1' at k_1 yields the system

$$\Phi(k_1, \zeta_1, \zeta_2) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} w_3(k_1) - w_{1,p}(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1 \\ k_1 \left[w_3'(k_1) - w_{1,p}'(k_1) + \beta_{\mathrm{s}}\right] \end{pmatrix}.$$

This gives

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Phi^{-1}(k_1, \zeta_1, \zeta_2) \begin{pmatrix} w_3(k_1) - w_{1,p}(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1 \\ k_1 \left[w_3'(k_1) - w_{1,p}'(k_1) + \beta_{\mathrm{s}}\right] \end{pmatrix}.$$

Eliminate B_1 and B_2 to obtain the following equations for k_1 and k_2 :

$$\Phi^{-1}(k_2, \zeta_1, \zeta_2) \left[\Phi(k_2, \delta_1, \delta_2)\Lambda^{-1}\Phi^{-1}(k_2, \delta_1, \delta_2) \begin{pmatrix} w_4(k_2) + \beta_{\mathrm{s}} k_2 - \beta_{\mathrm{b}} \\ k_2 \left[w_4'(k_2) + \beta_{\mathrm{s}}\right] \end{pmatrix} - \begin{pmatrix} w_{1,p}(k_2) \\ k_2 w_{1,p}'(k_2) \end{pmatrix}\right]$$
$$= \Phi^{-1}(k_1, \zeta_1, \zeta_2) \begin{pmatrix} w_3(k_1) - w_{1,p}(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1 \\ k_1 \left[w_3'(k_1) - w_{1,p}'(k_1) + \beta_{\mathrm{s}}\right] \end{pmatrix}. \tag{2.21}$$

Again, note that the constants B_1, B_2, C_1, and C_2 can be given as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. Therefore, k_1 and k_2 need to be determined so that the following variational inequalities are satisfied:

$$\begin{aligned} \text{On } (0,k_2): & \, w_1(y)-w_3(y)+\beta_{\mathrm{b}}-\beta_{\mathrm{s}}y \geq 0, \\ & \, w_2(y)-w_4(y)+\beta_{\mathrm{b}}-\beta_{\mathrm{s}}y \geq 0, \\ \text{On } [k_2,k_1]: & \quad w_1(y)-w_3(y)+\beta_{\mathrm{b}}-\beta_{\mathrm{s}}y \geq 0, \\ & \quad (\rho+\lambda_2-\mathcal{L}_2)\, w_2(y)-\lambda_2 w_1(y) \geq 0, \\ \text{On } (k_1,\infty): & \, (\rho+\lambda_1-\mathcal{L}_1)\, w_1(y)-\lambda_1 w_2(y) \geq 0, \\ & \, (\rho+\lambda_2-\mathcal{L}_2)\, w_2(y)-\lambda_2 w_1(y) \geq 0. \end{aligned} \tag{2.22}$$

Finally, to see equivalent conditions for the above inequalities involving $\mathcal{L}_j$, we first note that, on the interval (k_1,∞), the variational inequalities are equivalent to (as in Case II by switching the roles of k_1 and k_2, (and w_1 and w_2),

$$k_1 \geq \frac{(\rho-\mu_1(j))\,\beta_{\mathrm{b}}}{(\rho-\mu_2(j))\,\beta_{\mathrm{s}}} \quad \text{for } j=1,2.$$

Next, on the interval $[k_2,k_1]$, to relate $(\rho+\lambda_2-\mathcal{L}_2)\, w_2(y)-\lambda_2 w_1(y) \geq 0$, recall that $w_1(y) = B_1 y^{\zeta_1}+B_2 y^{\zeta_2}+w_{1,p}(y)$ and $w_2(y)=w_4(y)-\beta_{\mathrm{b}}+\beta_{\mathrm{s}}y$ on $[k_2,k_1]$. Apply $(\rho+\lambda_2-\mathcal{L}_2)$ to w_2 to obtain

$$(\rho+\lambda_2-\mathcal{L}_2)\, w_2(y)=\lambda_2 w_3-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}+(\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y.$$

Hence, $(\rho+\lambda_2-\mathcal{L}_2)\, w_2(y)-\lambda_2 w_1(y) \geq 0$ is equivalent to

$$B_1 y^{\zeta_1}+B_2 y^{\zeta_2}+w_{1,p}(y) \leq C_3 y^{\delta_3}+C_4 y^{\delta_4}-\left[\frac{\rho+\lambda_2-\mu_1(2)}{\lambda_2}\right]\beta_{\mathrm{b}}+\left[\frac{\rho+\lambda_2-\mu_2(2)}{\lambda_2}\right]\beta_{\mathrm{s}}y.$$

Finally, to see equivalent conditions for the above inequalities involving $\mathcal{L}_j$, we first note that, on the interval (k_1,∞), the variational inequalities are equivalent to (as in Case II by switching the roles of k_1 and k_2, (and w_1 and w_2),

$$k_1 \geq \frac{(\rho-\mu_1(j))\,\beta_{\mathrm{b}}}{(\rho-\mu_2(j))\,\beta_{\mathrm{s}}} \quad \text{for } j=1,2.$$

Next, on the interval $[k_2,k_1]$, to relate $(\rho+\lambda_2-\mathcal{L}_2)\, w_2(y)-\lambda_2 w_1(y) \geq 0$, recall that $w_1(y) = B_1 y^{\zeta_1}+B_2 y^{\zeta_2}+w_{1,p}(y)$ and $w_2(y)=w_4(y)-\beta_{\mathrm{b}}+\beta_{\mathrm{s}}y$ on $[k_2,k_1]$. Apply $(\rho+\lambda_2-\mathcal{L}_2)$ to w_2 to obtain

$$(\rho+\lambda_2-\mathcal{L}_2)\, w_2(y)=\lambda_2 w_3-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}+(\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y.$$

Hence, $(\rho + \lambda_2 - \mathcal{L}_2)\, w_2(y) - \lambda_2 w_1(y) \geq 0$ is equivalent to

$$B_1 y^{\zeta_1} + B_2 y^{\zeta_2} + w_{1,p}(y) \leq C_3 y^{\delta_3} + C_4 y^{\delta_4} - \left[\frac{\rho + \lambda_2 - \mu_1(2)}{\lambda_2}\right] \beta_{\mathrm{b}} + \left[\frac{\rho + \lambda_2 - \mu_2(2)}{\lambda_2}\right] \beta_{\mathrm{s}} y.$$

2.4.4 CASE IV: $k_4 < k_2 < k_3 < k_1$

In the subsectoin §2.4.7, we calculated w_3 and w_4:

$$\begin{array}{ll} [0,k_3]: & w_3 = \beta_s - \beta_b y, \\ [k_3,\infty): & w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4}, \\ [0,k_4]: & w_4 = \beta_s - \beta_b y, \\ [k_4,k_3]: & w_4 = D_1 y^{\nu_1} + D_2 y^{\nu_2} + b_1 - b_2 y, \\ [k_3,\infty): & w_4 = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4}. \end{array} \tag{2.23}$$

Similarly to the previous computation, we have on the interval $0 \le y \le k_2$:

$$w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_1(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2}.$$

On the interval $k_2 \le y \le k_1$, we have

$$w_2(y) = w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y \quad \text{and} \quad (\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) = 0.$$

On the interval $k_1 \le y < \infty$, we have

$$w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y \quad \text{and} \quad w_2(y) = w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y.$$

Then combine with w_3 and w_4, we can find w_2 first:

$$\begin{array}{ll} [k_2,k_3]: & w_2 = w_4 - \beta_b + \beta_s y = D_1 y^{\nu_1} + D_2 y^{\nu_2} + (b_1 - \beta_b) - (b_2 - \beta_s)\, y, \\ [k_3,\infty): & w_2 = w_4 - \beta_b + \beta_s y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_b + \beta_s y. \end{array}$$

Next we shall find w_1. On the interval $[k_2, k_3]$:

$$w_2(y) = D_1 y^{\nu_1} + D_2 y^{\nu_2} + (b_1 - \beta_b) - (b_2 - \beta_s)\, y \quad \text{and} \quad (\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) = 0.$$

implies the solution $w_1(y) = E_1 y^{\tau_1} + E_2 y^{\tau_2} + w_{1,p_1}(y)$ where $E_1 y^{\tau_1} + E_2 y^{\tau_2}$ is the general solution of the homogeneous differential equation $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) = 0$ and the particular solution

$$w_{1,p_1}(y) = \frac{\lambda_1 D_1}{\rho + \lambda_1 - A_1(\nu_1)} y^{\nu_1} + \frac{\lambda_1 D_2}{\rho + \lambda_1 - A_1(\nu_2)} y^{\nu_2} + \frac{\lambda_1 (b_1 - \beta_{\mathrm{b}})}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1 (\beta_{\mathrm{s}} - b_2)}{\rho + \lambda_1 - \mu_2(1)} y.$$

On the interval $[k_3, k_1]$

$$w_2(y) = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_b + \beta_s y \quad \text{and} \quad (\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) = 0.$$

implies the solution $w_1(y) = F_1 y^1 + F_2 y^2 + w_{1,p_2}(y)$ where $F_1 y^1 + F_2 y^2$ is the general solution of the homogeneous differential equation $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) = 0$ and the particular solution

$$\begin{aligned} w_{1,p_2}(y) &= \frac{\lambda_1 \eta_3 C_3}{\rho + \lambda_1 - A_1(\delta_3)} y^{\delta_3} + \frac{\lambda_1 \eta_4 C_4}{\rho + \lambda_1 - A_1(\delta_3)} y^{\delta_4} - \frac{\lambda_1 \beta_{\mathrm{b}}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1 \beta_{\mathrm{s}}}{\rho + \lambda_1 - \mu_2(1)} y \\ &= C_3 y^{\delta_3} + C_4 y^{\delta_4} - \frac{\lambda_1 \beta_{\mathrm{b}}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1}{\rho + \lambda_1 - \mu_2(1)} y. \end{aligned}$$

Finally, on the interval $[k_1, \infty)$, we have

$$w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y.$$

We summarize the computation about $w_1(y)$ and $w_2(y)$:

$$\begin{aligned} [0, k_2]: \quad & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2}, \\ & w_2(y) = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}, \\ [k_2, k_3]: \quad & w_1(y) = E_1 y^{\tau_1} + E_2 y^{\tau_2} + w_{1,p_1}(y), \\ & w_2(y) = w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = D_1 y^{\nu_1} + D_2 y^{\nu_2} + (b_1 - \beta_b) - (b_2 - \beta_s)\, y, \\ [k_3, k_1]: \quad & w_1(y) = F_1 y^{\tau_1} + F_2 y^{\tau_2} + w_{1,p_2}(y), \\ & w_2(y) = w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \beta_b + \beta_s y, \\ [k_1, \infty): \quad & w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y, \\ & w_2(y) = w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y. \end{aligned} \tag{2.24}$$

Here

$$\begin{aligned} w_{1,p_1}(y) &= \frac{\lambda_1 D_1}{\rho + \lambda_1 - A_1(\nu_1)} y^{\nu_1} + \frac{\lambda_1 D_2}{\rho + \lambda_1 - A_1(\nu_2)} y^{\nu_2} + \frac{\lambda_1 (b_1 - \beta_{\mathrm{b}})}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1 (\beta_{\mathrm{s}} - b_2)}{\rho + \lambda_1 - \mu_2(1)} y, \\ w_{1,p_2}(y) &= C_3 y^{\delta_3} + C_4 y^{\delta_4} - \frac{\lambda_1 \beta_{\mathrm{b}}}{\rho + \lambda_1 - \mu_1(1)} + \frac{\lambda_1 \beta_{\mathrm{s}}}{\rho + \lambda_1 - \mu_2(1)} y. \end{aligned}$$

We next use the continuity of w_1, w_2 and their derivatives at k_2, k_3 and k_1 to get

$$\begin{aligned} C_1 k_2^{\delta_1} + C_2 k_2^{\delta_2} &= E_1 k_2^{\tau_1} + E_2 k_2^{\tau_2} + w_{1,p_1}(k_2), \\ C_1 \delta_1 k_2^{\delta_1} + C_2 \delta_2 k_2^{\delta_2} &= E_1 k_2^{\tau_1} + E_2 k_2^{\tau_2} + w_{1,p_1}'(k_2)\, k_2, \\ C_1 \eta_1 k_2^{\delta_1} + C_2 \eta_2 k_2^{\delta_2} &= w_4(k_2) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2, \\ C_1 \eta_1 \delta_1 k_2^{\delta_1} + C_2 \eta_2 \delta_2 k_2^{\delta_2} &= w_4'(k_2) + \beta_{\mathrm{s}} k_2. \end{aligned}$$

The continuity of w_1 and its derivative at k_3 yields

$$\begin{aligned} E_1 k_3^{\tau_1} + E_2 k_3^{\tau_2} + w_{1,p_1}(k_3) &= F_1 k_3^{\tau_1} + F_2 k_3^{\tau_2} + w_{1,p_2}(k_3), \\ E_1 k_3^{\tau_1} + E_2 k_3^{\tau_2} + k_3 w_{1,p_1}'(k_3) &= F_1 k_3^{\tau_1} + F_2 k_3^{\tau_2} + k_3 w_{1,p_2}'(k_3). \end{aligned}$$

The continuity of w_1 and its derivative at k_1 yields

$$F_1 k_1^{\tau_1} + F_2 k_1^{\tau_2} + w_{1,p_2}(k_1) = w_3(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1,$$
$$F_1 k_1^{\tau_1} + F_2 k_1^{\tau_2} + w'_{1,p_2}(k_1) = k_1 w'_3(k_1) + \beta_{\mathrm{s}} k_1.$$

We write the matrix form of the above eight equations as follows

$$\begin{aligned}
\Phi(k_2;\delta_1,\delta_2)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \Phi(k_2;1,2)\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} + \begin{pmatrix} w_{1,p_1}(k_2) \\ w'_{1,p_1}(k_2)\,k_2 \end{pmatrix}, \\
\Phi(k_2;\delta_1,\delta_2)\begin{pmatrix} \eta_1 C_1 \\ \eta_2 C_2 \end{pmatrix} &= \begin{pmatrix} w_4(k_2) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2 \\ w'_4(k_2) + \beta_{\mathrm{s}} k_2 \end{pmatrix}, \\
\Phi(k_3;\tau_1,\tau_2)\begin{pmatrix} E_1 - F_1 \\ E_2 - F_2 \end{pmatrix} &= \begin{pmatrix} w_{1,p_2}(k_3) - w_{1,p_1}(k_3) \\ \left[w'_{1,p_2}(k_3) - w'_{1,p_1}(k_3)\right] k_3 \end{pmatrix}, \\
\Phi(k_1;\tau_1,\tau_2)\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} &= \begin{pmatrix} w_3(k_1) - w_{1,p_2}(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1 \\ \left[w'_3(k_1) - w'_{1,p_2}(k_1) + \beta_{\mathrm{s}}\right] k_1 \end{pmatrix}.
\end{aligned}$$

We can solve the above system backward and get

$$\begin{aligned}
\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} &= \Phi^{-1}(k_1;\tau_1,\tau_2)\begin{pmatrix} w_3(k_1) - w_{1,p_2}(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1 \\ \left[w'_3(k_1) - w'_{1,p_2}(k_1) + \beta_{\mathrm{s}}\right] k_1 \end{pmatrix}, \\
\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} &= \Phi^{-1}(k_3;\tau_1,\tau_2)\begin{pmatrix} w_{1,p_2}(k_3) - w_{1,p_1}(k_3) \\ \left[w'_{1,p_2}(k_3) - w'_{1,p_1}(k_3)\right] k_3 \end{pmatrix}, \\
&\quad + \Phi^{-1}(k_1;\tau_1,\tau_2)\begin{pmatrix} w_3(k_1) - w_{1,p_2}(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1 \\ \left[w'_3(k_1) - w'_{1,p_2}(k_1) + \beta_{\mathrm{s}}\right] k_1 \end{pmatrix}, \\
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\eta_1} & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix}\Phi^{-1}(k_2;\delta_1,\delta_2)\begin{pmatrix} w_4(k_2) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2 \\ k_2\left[w'_4(k_2) + \beta_{\mathrm{s}}\right] \end{pmatrix}.
\end{aligned}$$

We can use the formula for C_1, C_2, E_1 and E_2 and the first system to get

$$\begin{aligned}
&\Phi(k_2;\delta_1,\delta_2)\begin{pmatrix} \frac{1}{\eta_1} & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix}\Phi^{-1}(k_2;\delta_1,\delta_2)\begin{pmatrix} w_4(k_2) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2 \\ k_2\left[w'_4(k_2) + \beta_{\mathrm{s}}\right] \end{pmatrix} - \begin{pmatrix} w_{1,p_1}(k_2) \\ w'_{1,p_1}(k_2)\,k_2 \end{pmatrix} \\
&= \Phi(k_2;\tau_1,\tau_2)\Bigg[\Phi^{-1}(k_3;\tau_1,\tau_2)\begin{pmatrix} w_{1,p_2}(k_3) - w_{1,p_1}(k_3) \\ \left[w'_{1,p_2}(k_3) - w'_{1,p_1}(k_3)\right] k_3 \end{pmatrix} \\
&+ \Phi^{-1}(k_1;\tau_1,\tau_2)\begin{pmatrix} w_3(k_1) - w_{1,p_2}(k_1) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1 \\ \left[w'_3(k_1) - w'_{1,p_2}(k_1) + \beta_{\mathrm{s}}\right] k_1 \end{pmatrix}\Bigg].
\end{aligned} \tag{2.25}$$

This yields two equations of k_1 and k_2. The existence of k_1 and k_2 can proved by following the method in Lemma 4.2 of [12]. Once we find k_1 and k_2 and note that the constants B_1, B_2, C_1, C_2, D_1, and D_2 can be written as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. In view of this, k_1 and k_2 have to be determined so that the following variational inequalities are satisfied:

$$\begin{aligned}
\text{On } (0,k_2):\quad & w_1(y)-w_3(y)+\beta_{\rm b}-\beta_{\rm s}y\geq 0,\\
& w_2(y)-w_4(y)+\beta_{\rm b}-\beta_{\rm s}y\geq 0,\\
\text{On } [k_2,k_1]:\quad & (\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0,\\
& w_2(y)-w_4(y)+\beta_{\rm b}-\beta_{\rm s}y\geq 0,\\
\text{On } (k_1,\infty):\quad & (\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)\geq 0,\\
& (\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0.
\end{aligned} \tag{2.26}$$

On (k_2, k_1), because w_1 and w_2 are defined differently on (k_2, k_3) and (k_3, k_1), the differential operators $\mathcal{L}_j$. First, we consider the two inequalities on the interval $[k_2, \infty)$:

$$(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y)-\lambda_1 w_2(y)\geq 0 \quad \text{and} \quad (\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0.$$

Recall that $w_1(y) = w_3(y) - \beta_{\rm b} + \beta_{\rm s}y$ and $w_2(y) = w_4(y) - \beta_{\rm b} + \beta_{\rm s}y$, and we apply $\mathcal{L}_1$ to $w_1(y)$ and $\mathcal{L}_2$ to $w_2(y)$ to get

$$\begin{aligned}
(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y) &= \lambda_1 w_4(y)+(\rho+\lambda_1-\mu_2(1))\,\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\,\beta_{\rm b},\\
(\rho+\lambda_2-\mathcal{L}_2)\,w_2(y) &= \lambda_2 w_3(y)+(\rho+\lambda_2-\mu_2(2))\,\beta_{\rm s}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\rm b}.
\end{aligned}$$

Then (23) is equivalent to

$$\begin{aligned}
(\rho+\lambda_1-\mu_2(1))\,\beta_{\rm s}y-(\rho+\lambda_1-\mu_1(1))\,\beta_{\rm b} &\geq \lambda_1\left(\beta_{\rm s}y-\beta_{\rm b}\right),\\
(\rho+\lambda_2-\mu_2(2))\,\beta_{\rm s}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\rm b} &\geq \lambda_2\left(\beta_{\rm s}y-\beta_{\rm b}\right).
\end{aligned}$$

Simplify to obtain

$$(\rho-\mu_2(1))\,\beta_{\rm s}y-(\rho-\mu_1(1))\,\beta_{\rm b}\geq 0 \quad \text{and} \quad (\rho-\mu_2(2))\,\beta_{\rm s}y-(\rho-\mu_1(2))\,\beta_{\rm b}\geq 0.$$

These inequalities hold as long as

$$k_2 \geq \frac{(\rho-\mu_1(j))\,\beta_{\rm b}}{(\rho-\mu_2(j))\,\beta_{\rm s}} \quad \text{for } j=1,2$$

Next, we consider the inequality involving $\mathcal{L}_2$, i.e, $(\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0$ on $[k_2, k_1]$. Recall that $w_2 = w_4 - \beta_{\rm b} + \beta_{\rm s}y$ and w_2 satisfies $(\rho+\lambda_1-\mathcal{L}_1)\,w_1(y) = \lambda_1 w_2(y)$ on this interval.

Applying $(\rho+\lambda_2-\mathcal{L}_2)$ to w_2 yield

$$(\rho+\lambda_2-\mathcal{L}_2)\,w_2=(\rho+\lambda_2-\mathcal{L}_2)\,w_4+(\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}.$$

Recall that $k_4<k_2<k_3<k_1$ and$(\rho+\lambda_2-\mathcal{L}_2)\,w_4=\lambda_2 w_3$. It follows that

$$(\rho+\lambda_2-\mathcal{L}_2)\,w_2=\lambda_2 w_3+(\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}.$$

Recall also that $w_3=\beta_{\mathrm{s}}-\beta_{\mathrm{b}}y$ on the interval $[0,k_3]$.
Hence on interval $[k_2,k_3]\subset[0,k_3]$,

$$(\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0$$

is equivalent to

$$\lambda_2\left(\beta_{\mathrm{s}}-\beta_{\mathrm{b}}y\right)+(\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}\geq\lambda_2(E_1y^{\tau_1}+E_2y^{\tau_2}+\frac{\lambda_1 D_1}{\rho+\lambda_1-A_1\left(\nu_1\right)}y^{\nu_1}$$
$$+\frac{\lambda_1 D_2}{\rho+\lambda_1-A_1\left(\nu_2\right)}y^{\nu_2}+\frac{\lambda_1\left(b_1-\beta_{\mathrm{b}}\right)}{\rho+\lambda_1-\mu_1(1)}+\frac{\lambda_1\left(\beta_{\mathrm{s}}-b_2\right)}{\rho+\lambda_1-\mu_2(1)}y). \tag{2.27}$$

On (k_3,k_1) w_3 is defined differently as $w_3=C_3y^{\delta_3}+C_4y^{\delta_4}$, then $(\rho+\lambda_2-\mathcal{L}_2)\,w_2(y)-\lambda_2 w_1(y)\geq 0$ is equivalent to

$$\lambda_2\left(C_3y^{\delta_3}+C_4y^{\delta_4}\right)+(\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}\geq\lambda_2 w_1. \tag{2.28}$$

2.4.5 CASE V: $k_4 < k_3 < k_2 < k_1$

In the subsectoin §1.4.7, we calculated w_3 and w_4:

$$\begin{aligned}
[0,k_3] :& w_3 = \beta_s - \beta_b y,\\
[k_3,\infty) :& w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4},\\
[0,k_4] :& w_4 = \beta_s - \beta_b y,\\
[k_4,k_3] :& w_4 = D_1 y^{\nu_1} + D_2 y^{\nu_2} + b_1 - b_2 y,\\
[k_3,\infty) :& w_4 = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4}.
\end{aligned}$$

This case is relatively simpler than the previous case. Similarly on the interval $[0, k_2]$, we have

$$w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2}.$$

On the interval $[k_2, k_1]$, $w_2(y) = w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_b + \beta_s y$ and $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) = 0$ imply that $w_1(y) = F_1 y^{\tau_1} + F_2 y^{\tau_2} + w_{1,p}(y)$ where $F_1 y^{\tau_1} + F_2 y^{\tau_2}$ is the general solution of the homogeneous differential equation $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) = 0$ and the particular solution

$$\begin{aligned}
w_{1,p}(y) &= \frac{\lambda_1\eta_3 C_3}{\rho+\lambda_1 - A_1(\delta_3)} y^{\delta_3} + \frac{\lambda_1\eta_4 C_4}{\rho+\lambda_1 - A_1(\delta_3)} y^{\delta_4} - \frac{\lambda_1\beta_\text{b}}{\rho+\lambda_1-\mu_1(1)} + \frac{\lambda_1\beta_\text{s}}{\rho+\lambda_1-\mu_2(1)} y\\
&= C_3 y^{\delta_3} + C_4 y^{\delta_4} - \frac{\lambda_1\beta_\text{b}}{\rho+\lambda_1-\mu_1(1)} + \frac{\lambda_1\beta_\text{s}}{\rho+\lambda_1-\mu_2(1)} y.
\end{aligned}$$

Finally, on the interval $[k_1, \infty)$, we have

$$\begin{aligned}
w_1(y) &= w_3(y) - \beta_\text{b} + \beta_\text{s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y,\\
w_2(y) &= w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_b + \beta_s y.
\end{aligned}$$

Let's summarize :

$$\begin{aligned}
[0,k_2]: \quad & w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2},\\
& w_2(y) = C_1\eta_1 y^{\delta_1} + C_2\eta_2 y^{\delta_2},\\
[k_2,k_1]: \quad & w_1(y) = F_1 y^{\tau_1} + F_2 y^{\tau_2} + w_{1,p}(y),\\
& w_2(y) = w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_b + \beta_s y,\\
[k_1,\infty): \quad & w_1(y) = w_3(y) - \beta_\text{b} + \beta_\text{s} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y,\\
& w_2(y) = w_4(y) - \beta_\text{b} + \beta_\text{s} y = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4} - \beta_\text{b} + \beta_\text{s} y.
\end{aligned} \tag{2.29}$$

The continuity of w_1 and w_2, and their derivatives at k_2 yield

$$\begin{aligned} C_1 k_2^{\delta_1} + C_2 k_2^{\delta_2} &= F_1 k_2^{\tau_1} + F_2 k_2^{\tau_2} + w_{1,p}(k_2), \\ C_1 \delta_1 k_2^{\delta_1} + C_2 \delta_2 k_2^{\delta_2} &= F_1 k_2^{\tau_1} + F_2 k_2^{\tau_2} + k_2 w_{1,p}'(k_2), \\ C_1 \eta_1 k_2^{\delta_1} + C_2 \eta_2 k_2^{\delta_2} &= w_4(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2, \\ C_1 \eta_1 \delta_1 k_2^{\delta_1} + C_2 \eta_2 \delta_2 k_2^{\delta_2} &= \left[w_4'(k_2) + \beta_{\rm s}\right] k_2. \end{aligned}$$

The continuity of w_1 and its derivative at k_1 imply

$$\begin{aligned} F_1 k_1^{\tau_1} + F_2 k_1^{\tau_2} + w_{1,p}(k_1) &= w_3(k_1) - \beta_{\rm b} + \beta_{\rm s} k_1, \\ F_1 k_1^{\tau_1} + F_2 k_1^{\tau_1} + k_1 w_{1,p}'(k_1) &= k_1\left[w_3'(k_1) + \beta_{\rm s}\right]. \end{aligned}$$

We can find that

$$\begin{aligned} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} &= \Phi^{-1}(k_1; \tau_1, \tau_2) \begin{pmatrix} w_3(k_1) - \beta_{\rm b} + \beta_{\rm s} k_1 - w_{1,p}(k_1) \\ k_1\left[w_3'(k_1) - w_{1,p}'(k_1) + \beta_{\rm s}\right] \end{pmatrix}, \\ \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\eta_1} & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix} \Phi^{-1}(k_2; \delta_1, \delta_2) \begin{pmatrix} w_4(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2 \\ k_2\left[w_4'(k_2) + \beta_{\rm s}\right] \end{pmatrix}. \end{aligned}$$

Then we can get two equations of k_1 and k_2 in matrix form:

$$\begin{aligned} &\Phi(k_2; \delta_1, \delta_2) \begin{pmatrix} \frac{1}{\eta_1} & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix} \Phi^{-1}(k_2; \delta_1, \delta_2) \begin{pmatrix} w_4(k_2) - \beta_{\rm b} + \beta_{\rm s} k_2 \\ k_2\left[w_4'(k_2) + \beta_{\rm s}\right] \end{pmatrix} \\ =&\Phi(k_2; \tau_1, \tau_2)\Phi^{-1}(k_1; \tau_1, \tau_2) \begin{pmatrix} w_3(k_1) - \beta_{\rm b} + \beta_{\rm s} k_1 - w_{1,p}(k_1) \\ k_1\left[w_3'(k_1) - w_{1,p}'(k_1) + \beta_{\rm s}\right] \end{pmatrix} + \begin{pmatrix} w_{1,p}(k_2) \\ k_2 w_{1,p}'(k_2) \end{pmatrix}. \end{aligned} \tag{2.30}$$

Recall that the constants B_1, B_2, C_1, and C_2 can be represented as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. Therefore, k_1 and k_2 need to be determined so that the following variational inequalities are satisfied:

$$\begin{aligned} \text{On } (0, k_2): \quad & w_1(y) - w_3(y) + \beta_{\rm b} - \beta_{\rm s} y \geq 0, \\ & w_2(y) - w_4(y) + \beta_{\rm b} - \beta_{\rm s} y \geq 0, \\ \text{On } [k_2, k_1]: \quad & (\rho + \lambda_1 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0, \\ & w_1(y) - w_3(y) + \beta_{\rm b} - \beta_{\rm s} y \geq 0, \\ \text{On } (k_1, \infty): \quad & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, \\ & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0. \end{aligned}$$

Next, we consider equivalent inequalities for those involving the differential operators $\mathcal{L}_j$. First, on the interval $[k_2, \infty)$, the variational inequalities are equivalent to

$$\begin{aligned}(\rho+\lambda_1-\mu_2(1))\,\beta_{\mathrm{s}}y-(\rho+\lambda_1-\mu_1(1))\,\beta_{\mathrm{b}} &\geq \lambda_1\left(\beta_{\mathrm{s}}y-\beta_{\mathrm{b}}\right),\\ (\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}} &\geq \lambda_2\left(\beta_{\mathrm{s}}y-\beta_{\mathrm{b}}\right).\end{aligned}$$

as in Case IV. The equivalent conditions for these inequalities to hold are

$$k_2 \geq \frac{(\rho-\mu_1(j))\,\beta_{\mathrm{b}}}{(\rho-\mu_2(j))\,\beta_{\mathrm{s}}} \quad \text{for } j=1,2.$$

Move on to the interval $[k_2, k_1]$ and recall $w_2 = w_4 - \beta_{\mathrm{b}} + \beta_{\mathrm{s}}y$. Apply $(\rho+\lambda_2-\mathcal{L}_2)$ to w_2 to obtain

$$(\rho+\lambda_2-\mathcal{L}_2)\,w_2 = (\rho+\lambda_2-\mathcal{L}_2)\,w_4 + (\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}.$$

In addition, recall that $k_4 < k_3 < k_2 < k_1$ and $(\rho+\lambda_2-\mathcal{L}_2)\,w_4 = \lambda_2 w_3$. It follows that

$$(\rho+\lambda_2-\mathcal{L}_2)\,w_2 = \lambda_2 w_3 + (\rho+\lambda_2-\mu_2(2))\,\beta_{\mathrm{s}}y-(\rho+\lambda_2-\mu_1(2))\,\beta_{\mathrm{b}}.$$

Recall also that $w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4}$ for $y \geq k_3$. Hence the inequality $(\rho+\lambda_2-\mathcal{L}_2)\,w_2(y) - \lambda_2 w_1(y) \geq 0$ is equivalent to

$$w_1 \leq C_3 y^{\delta_3} + C_4 y^{\delta_4} + \left[\frac{\rho+\lambda_1-\mu_2(1)}{\lambda_2}\right]\beta_{\mathrm{s}}y - \left[\frac{\rho+\lambda_1-\mu_1(1)}{\lambda_2}\right]\beta_{\mathrm{b}}.$$

2.4.6 CASE VI: $k_4 < k_3 < k_1 < k_2$

On the interval $[0, k_1]$,

$$w_1(y) = C_1 y^{\delta_1} + C_2 y^{\delta_2} \quad \text{and} \quad w_1(y) = C_1 \eta_1 y^{\delta_1} + C_2 \eta_2 y^{\delta_2}.$$

On the interval $[k_1, k_2]$,

$$w_1(y) = w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y \text{ and } \left(\rho + \lambda_2 - {}_2\right) w_2(y) = \lambda_2 w_1(y).$$

This will yield

$$w_2(y) = D_1 y^{\nu_1} + D_2 y^{\nu_2} + w_{2,p}(y).$$

with

$$w_{2,p}(y) = w_4(y) - \frac{\beta_{\mathrm{b}} \lambda_2}{\rho + \lambda_2 - \mu_1(2)} + \frac{\beta_{\mathrm{s}} \lambda_2}{\rho + \lambda_2 - \mu_2(2)} y.$$

Finally, on the interval $[k_1, \infty)$, we have

$$\begin{aligned} w_1(y) &= w_3(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_3 y^{\delta_3} + C_4 y^{\delta_4} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y, \\ w_2(y) &= w_4(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} t = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} - \beta_b + \beta_s y. \end{aligned}$$

The continuity of w_1 and w_2, and their derivatives at k_1 yield

$$\begin{aligned} C_1 k_1^{\delta_1} + C_2 k_1^{\delta_2} &= w_3\left(k_1\right) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_1, \\ C_1 \delta_1 k_1^{\delta_1} + C_2 \delta_2 k_1^{\delta_2} &= \left[w_3'\left(k_1\right) + \beta_{\mathrm{s}}\right] k_1, \\ C_1 \eta_1 k_1^{\delta_1} + C_2 \eta_2 k_1^{\delta_2} &= D_1 k_1^{\nu_1} + D_2 k_1^{\nu_2} + w_{2,p}\left(k_1\right), \\ C_1 \eta_1 \delta_1 k_1^{\delta_1} + C_2 \eta_2 \delta_2 k_1^{\delta_2} &= D_1 \nu_1 k_1^{\nu_1} + D_2 \nu_2 k_1^{\nu_2} + k_1 w_{2,p}'\left(k_1\right). \end{aligned}$$

The continuity of w_2 and its derivative at k_2 imply

$$\begin{aligned} D_1 k_2^{\nu_1} + D_2 k_2^{\nu_2} + w_{2,p}\left(k_2\right) &= w_4\left(k_2\right) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2, \\ D_1 \nu_1 k_2^{\nu_1} + D_{22} k_2^{\nu_2} + k_2 w_{2,p}'\left(k_2\right) &= k_2\left[w_4'\left(k_2\right) + \beta_{\mathrm{s}}\right]. \end{aligned}$$

We can find that

$$\begin{aligned} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} &= \Phi^{-1}\left(k_2; \nu_1, \nu_2\right) \begin{pmatrix} w_4\left(k_2\right) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2 - w_{2,p}\left(k_2\right) \\ k_2\left[w_4'\left(k_2\right) - w_{2,p}'\left(k_2\right) + \beta_{\mathrm{s}}\right] \end{pmatrix}, \\ \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \Phi^{-1}\left(k_1; \delta_1, \delta_2\right) \begin{pmatrix} w_3\left(k_1\right) + \beta_{\mathrm{s}} k_1 - \beta_{\mathrm{b}} \\ k_1\left[w_3'\left(k_1\right) + \beta_{\mathrm{s}}\right] \end{pmatrix}. \end{aligned}$$

Then we can get two equations of k_1 and k_2 in matrix form:

$$\begin{aligned}&\Phi\left(k_1;\delta_1,\delta_2\right)\left(\begin{array}{cc}\eta_1 & 0\\ 0 & \eta_2\end{array}\right)\Phi^{-1}\left(k_1;\delta_1,\delta_2\right)\left(\begin{array}{c}w_3\left(k_1\right)-\beta_{\mathrm{b}}+\beta_{\mathrm{s}}k_1\\ k_1\left[w_3'\left(k_1\right)+\beta_{\mathrm{s}}\right]\end{array}\right)\\ =&\Phi\left(k_1;\nu_1,\nu_2\right)\Phi^{-1}\left(k_2;\tau_1,\tau_2\right)\left(\begin{array}{c}w_4\left(k_2\right)-\beta_{\mathrm{b}}+\beta_{\mathrm{s}}k_2-w_{2,p}\left(k_2\right)\\ k_2\left[w_4'\left(k_1\right)-w_{2,p}'\left(k_2\right)+\beta_{\mathrm{s}}\right]\end{array}\right)+\left(\begin{array}{c}w_{2,p}\left(k_1\right)\\ k_1 w_{2,p}'\left(k_1\right)\end{array}\right).\end{aligned} \tag{2.31}$$

Recall that the constants B_1, B_2, C_1, and C_2 can be represented as functions of k_1 and k_2. So are functions $w_1(y)$ and $w_2(y)$. Therefore, k_1 and k_2 need to be determined so that the following variational inequalities are satisfied:

$$\begin{aligned}\text{On }(0,k_1):&\quad w_1(y)-w_3(y)+\beta_{\mathrm{b}}-\beta_{\mathrm{s}}y\geq 0,\\ &\quad w_2(y)-w_4(y)+\beta_{\mathrm{b}}-\beta_{\mathrm{s}}y\geq 0,\\ \text{On }[k_1,k_2]:&\quad\left(\rho+\lambda_1-\mathcal{L}_1\right)w_1(y)-\lambda_1 w_2(y)\geq 0,\\ &\quad w_1(y)-w_3(y)+\beta_{\mathrm{b}}-\beta_{\mathrm{s}}y\geq 0,\\ \text{On }(k_2,\infty):&\quad\left(\rho+\lambda_1-\mathcal{L}_1\right)w_1(y)-\lambda_1 w_2(y)\geq 0,\\ &\quad\left(\rho+\lambda_2-\mathcal{L}_2\right)w_2(y)-\lambda_2 w_1(y)\geq 0.\end{aligned} \tag{2.32}$$

Finally, to see equivalent conditions for the above inequalities involving $\mathcal{L}_j$, we first note that, on the interval (k_2, ∞), the variational inequalities are equivalent to (as in Case II by switching the roles of k_1 and k_2, (and w_1 and w_2),

$$k_1 \geq \frac{(\rho - \mu_1(j))\, \beta_{\mathrm{b}}}{(\rho - \mu_2(j))\, \beta_{\mathrm{s}}} \quad \text{for } j = 1, 2.$$

Move on to the interval $[k_1, k_2]$ and recall $w_1 = w_3 - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y$. Apply $(\rho + \lambda_1 - \mathcal{L}_1)$ to w_1 to obtain

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = (\rho + \lambda_1 - \mathcal{L}_1)\, w_3 + (\rho + \lambda_1 - \mu_2(1))\, \beta_{\mathrm{s}} y - (\rho + \lambda_1 - \mu_1(1))\, \beta_{\mathrm{b}}.$$

In addition, recall that $k_4 < k_3 < k_1 < k_2$ and $(\rho + \lambda_1 - \mathcal{L}_1)\, w_3 = \lambda_1 w_4$. It follows that

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_1 = \lambda_1 w_4 + (\rho + \lambda_1 - \mu_2(1))\, \beta_{\mathrm{s}} y - (\rho + \lambda_1 - \mu_1(1))\, \beta_{\mathrm{b}}.$$

Recall also that $w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4}$ for $y \geq k_4$. Hence the inequality $(\rho + \lambda_1 - \mathcal{L}_1)\, w_1(y) - \lambda_1 w_2(y) \geq 0$ is equivalent to

$$w_2(y) \leq C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} + \left[\frac{\rho + \lambda_1 - \mu_2(1)}{\lambda_1}\right] \beta_{\mathrm{s}} y - \left[\frac{\rho + \lambda_1 - \mu_1(1)}{\lambda_1}\right] \beta_{\mathrm{b}}.$$

2.4.7 the solution w_3 and w_4

$k_3 < k_4$ **(for CASES I, II, III)**:
We sketch the key steps in derivation of solutions w_3 and w_4. Recall the corresponding HJB equations:

$$\min \{(\rho + \lambda_1 - \mathcal{L}_1)\, w_3(y) - \lambda_1 w_4(y), w_3(y) + \beta_{\mathrm{b}} y - \beta_{\mathrm{s}}\} = 0,$$
$$\min \{(\rho + \lambda_2 - \mathcal{L}_2)\, w_4(y) - \lambda_2 w_3(y), w_4(y) + \beta_{\mathrm{b}} y - \beta_{\mathrm{s}}\} = 0.$$

First, we divide the interval $(0, \infty)$ into three subintervals:

$$\Gamma_1 = (0, k_3)\,, \quad \Gamma_2 = (k_3, k_4)\,, \quad \text{and} \quad \Gamma_3 = [k_4, \infty)\,.$$

Note that $w_3 = w_4 = \beta_s - \beta_b y$ on Γ_1

$$w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4} \quad \text{and} \quad w_4 = \eta_3 C_3 y^{\delta_3} + \eta_4 C_4 y^{\delta_4} \quad \text{on} \quad \Gamma_3,$$

and $w_4 = \beta_s - \beta_b y$ and $(\rho + \lambda_1 - \mathcal{L}_1)\, w_3(y) = \lambda_1 w_4(y)$ on Γ_2. To solve the non-homogeneous linear equation of Euler type:

$$(\rho + \lambda_1 - \mathcal{L}_1)\, w_3(y) = \lambda_1 w_4(y) = \lambda_1 (\beta_s - \beta_b y)\,,$$

let

$$a_1 = \frac{\lambda_1 \beta_s}{\rho + \lambda_1 - \mu_1(1)} \quad \text{and} \quad a_2 = \frac{\lambda_1 \beta_b}{\rho + \lambda_1 - \mu_2(1)}.$$

Then a particular solution can be given as $w_{3,p}(y) = a_1 - a_2 y$. The general solution is given by

$$w_3 = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - a_2 y,$$

where ζ_1 and ζ_2 are given by (12). Next we apply smooth-fit conditions to find the parameters C_1, C_2, E_1, E_2, k_3 and k_4. First the continuity of w_4 and its derivative at k_4 yield

$$\begin{aligned} \beta_s - \beta_b k_4 &= \eta_3 C_3 k_4^{\delta_3} + \eta_4 C_4 k_4^{\delta_4}, \\ -\beta_b &= \eta_3 \delta_3 C_3 k_4^{\delta_3 - 1} + \eta_4 \delta_4 C_4 k_4^{\delta_4 - 1}. \end{aligned}$$

The continuity of w_3 and its derivative at k_3 and k_4 yield

$$\begin{aligned} \beta_s - \beta_b k_3 &= E_1 k_3^{\zeta_1} + E_2 k_3^{\zeta_2} + a_1 - a_2 k_3, \\ -\beta_b &= E_1 \zeta_1 k_3^{\zeta_1 - 1} + E_2 \zeta_2 k_3^{\zeta_2 - 1} - a_2, \\ E_1 k_4^{\zeta_1} + E_2 k_4^{\zeta_2} + a_1 - a_2 k_4 &= C_3 k_4^{\delta_3} + C_4 k_4^{\delta_4}, \\ E_1 \zeta_1 k_4^{\zeta_1 - 1} + E_2 \zeta_2 k_4^{\zeta_2 - 1} - a_2 &= \delta_3 C_3 k_4^{\delta_3 - 1} + \delta_4 C_4 k_4^{\delta_4 - 1}. \end{aligned}$$

Let

$$\Phi(t, s_1, s_2) = \begin{pmatrix} t^{s_1} & t^{s_2} \\ s_1 t^{s_1} & s_2 t^{s_2} \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}.$$

Then the above system can be rewritten as

$$\Phi(k_4, \delta_3, \delta_4) \Lambda \begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} \beta_{\mathrm{s}} - \beta_{\mathrm{b}} k_4 \\ -\beta_{\mathrm{b}} k_4 \end{pmatrix}, \Phi(k_3, \zeta_1, \zeta_2) \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} (\beta_{\mathrm{s}} - a_1) - (\beta_{\mathrm{b}} - a_2) k_3 \\ -(\beta_{\mathrm{b}} - a_2) k_3 \end{pmatrix},$$
$$\Phi(k_4, \zeta_1, \zeta_2) \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} + \begin{pmatrix} a_1 - a_2 k_4 \\ -a_2 k_4 \end{pmatrix} = \Phi(k_4; \delta_3, \delta_4) \begin{pmatrix} C_3 \\ C_4 \end{pmatrix}.$$

Eliminate the parameters C_1, C_2, E_1, and E_2 to obtain the equations for k_3 and k_4 :

$$\begin{aligned} &\Phi(k_4, \zeta_1, \zeta_2) \Phi^{-1}(k_3, \zeta_1, \zeta_2) \begin{pmatrix} (\beta_{\mathrm{s}} - a_1) - (\beta_{\mathrm{b}} - a_2) k_3 \\ -(\beta_{\mathrm{b}} - a_2) k_3 \end{pmatrix} + \begin{pmatrix} a_1 - a_2 k_4 \\ -a_2 k_4 \end{pmatrix} \\ &= \Phi(k_4, \delta_3, \delta_4) \Lambda^{-1} \Phi^{-1}(k_4, \delta_3, \delta_4) \begin{pmatrix} \beta_{\mathrm{s}} - \beta_{\mathrm{b}} k_4 \\ -\beta_{\mathrm{b}} k_4 \end{pmatrix}. \end{aligned} \tag{2.33}$$

Let $r = k_4/k_3$. Some simple calculations yield

$$\Phi\left(k_4,\zeta_1,\zeta_2\right)\Phi^{-1}\left(k_3,\zeta_1,\zeta_2\right) = \frac{1}{\zeta_1-\zeta_2}\begin{pmatrix} \zeta_1 r^{\zeta_2}-\zeta_2 r^{\zeta_1} & r^{\zeta_1}-r^{\zeta_2} \\ \zeta_1\zeta_2\left(r^{\zeta_2}-r^{\zeta_1}\right) & \zeta_1 r^{\zeta_1}-\zeta_2 r^{\zeta_2} \end{pmatrix},$$

$$\Phi\left(k_4,\delta_3,\delta_4\right)\Lambda^{-1}\Phi^{-1}\left(k_4,\delta_3,\delta_4\right) = \frac{1}{\eta_1\eta_2\left(\delta_3-\delta_4\right)}\begin{pmatrix} \eta_1\delta_3-\eta_2\delta_4 & \eta_2-\eta_1 \\ \delta_3\delta_4\left(\eta_1-\eta_2\right) & \eta_2\delta_3-\eta_1\delta_4 \end{pmatrix}.$$

We can rewrite these (2.33) as follows

$$\frac{1}{\zeta_1-\zeta_2}\begin{pmatrix} \left(\zeta_2-1\right)\left(\beta_{\mathrm{b}}-a_2\right)k_3-\zeta_2\left(\beta_{\mathrm{s}}-a_1\right) & \zeta_1\left(\beta_{\mathrm{s}}-a_1\right)+\left(1-\zeta_1\right)\left(\beta_{\mathrm{b}}-a_2\right)k_3 \\ \zeta_1\left[\left(\zeta_2-1\right)\left(\beta_{\mathrm{b}}-a_2\right)k_3-\zeta_2\left(\beta_{\mathrm{s}}-a_1\right)\right] & \zeta_2\left[\zeta_1\left(\beta_{\mathrm{s}}-a_1\right)+\left(1-\zeta_1\right)\left(\beta_{\mathrm{b}}-a_2\right)k_3\right] \end{pmatrix}\begin{pmatrix} r^{\zeta_1} \\ r^{\zeta_2} \end{pmatrix}$$
$$=\frac{1}{\eta_1\eta_2\left(\delta_3-\delta_4\right)}\begin{pmatrix} \eta_1\delta_3-\eta_2\delta_4 & \eta_2-\eta_1 \\ \delta_3\delta_4\left(\eta_1-\eta_2\right) & \eta_2\delta_3-\eta_1\delta_4 \end{pmatrix}\begin{pmatrix} \beta_{\mathrm{s}}-\beta_{\mathrm{b}}k_4 \\ -\beta_{\mathrm{b}}k_4 \end{pmatrix}-\begin{pmatrix} a_1-a_2k_4 \\ -a_2k_4 \end{pmatrix}.$$

Let

$$\alpha_1 = \left(\zeta_2-1\right)\left(\beta_{\mathrm{b}}-a_2\right)k_3-\zeta_2\left(\beta_{\mathrm{s}}-a_1\right) \quad \text{and} \quad \alpha_2 = \left(\zeta_2-1\right)\left(\beta_{\mathrm{b}}-a_2\right)k_3-\zeta_2\left(\beta_{\mathrm{s}}-a_1\right).$$

The matrix on the lefthand side is

$$\frac{1}{\zeta_1-\zeta_2}\begin{pmatrix} \alpha_1 & \alpha_2 \\ \zeta_1\alpha_1 & \zeta_2\alpha_2 \end{pmatrix} \text{ with inverse } \begin{pmatrix} -\frac{\zeta_2}{\alpha_1} & \frac{1}{\alpha_1} \\ \frac{\zeta_1}{\alpha_2} & -\frac{1}{\alpha_2} \end{pmatrix}.$$

This yields

$$\begin{pmatrix} r^{\zeta_1} \\ r^{\zeta_2} \end{pmatrix} = \begin{pmatrix} -\frac{\zeta_2}{\alpha_1} & \frac{1}{\alpha_1} \\ \frac{\zeta_1}{\alpha_2} & -\frac{1}{\alpha_2} \end{pmatrix}\left[\frac{1}{\eta_1\eta_2(\delta_3-\delta_4)}\begin{pmatrix} \eta_1\delta_3-\eta_2\delta_4 & \eta_2-\eta_1 \\ \delta_3\delta_4(\eta_1-\eta_2) & \eta_2\delta_3-\eta_1\delta_4 \end{pmatrix}\begin{pmatrix} \beta_s-\beta_b k_4 \\ -\beta_b k_4 \end{pmatrix}-\begin{pmatrix} a_1-a_2k_4 \\ -a_2k_4 \end{pmatrix}\right].$$

Simplify them to obtain

$$\begin{aligned} &[\zeta_2(\beta_s-a_1)+(1-\zeta_2)(\beta_b-a_2)k_3]r^{\zeta_1}+\zeta_2a_1+(1-\zeta_2)a_2k_4 \\ &\quad= \tfrac{-\delta_4\beta_s+(\delta_4-1)\beta_b k_4}{\eta_3(\delta_3-\delta_4)}(\zeta_2-\delta_3)+\tfrac{\delta_3\beta_s+(1-\delta_3)\beta_b k_4}{\eta_4(\delta_3-\delta_4)}(\zeta_2-\delta_4), \\ &[-\zeta_1(\beta_s-a_1)+(\zeta_1-1)(\beta_b-a_2)k_3]r^{\zeta_2}+(\zeta_1-1)a_2k_4-\zeta_1a_1 \\ &\quad= \tfrac{-\delta_4\beta_s+(\delta_4-1)\beta_b k_4}{\eta_3(\delta_3-\delta_4)}(\delta_3-\zeta_1)+\tfrac{\delta_3\beta_s+(1-\delta_3)\beta_b k_4}{\eta_4(\delta_3-\delta_4)}(\delta_4-\zeta_1). \end{aligned}$$

Let

$$\begin{aligned} A_1 &= \tfrac{-\delta_4\beta_s(\zeta_2-\delta_3)}{\eta_3(\delta_3-\delta_4)}+\tfrac{\delta_3\beta_s(\zeta_2-\delta_4)}{\eta_4(\delta_3-\delta_4)}-\zeta_2a_1, \quad B_1 = \tfrac{(\delta_4-1)(\zeta_2-\delta_3)\beta_b}{\eta_3(\delta_3-\delta_4)}+\tfrac{(1-\delta_3)\beta_b(\zeta_2-\delta_4)}{\eta_4(\delta_3-\delta_4)}-(1-\zeta_2)a_2; \\ A_2 &= \tfrac{-\delta_4\beta_s(\delta_3-\zeta_1)}{\eta_3(\delta_3-\delta_4)}+\tfrac{\delta_3\beta_s(\delta_4-\zeta_1)}{\eta_4(\delta_3-\delta_4)}+\zeta_1a_1, \quad B_2 = \tfrac{(\delta_4-1)(\delta_3-\zeta_1)\beta_b}{\eta_3(\delta_3-\delta_4)}+\tfrac{(1-\delta_3)\beta_b(\delta_4-\zeta_1)}{\eta_4(\delta_3-\delta_4)}-(\zeta_1-1)a_2. \end{aligned}$$

Then we can rewrite the above system as

$$\begin{aligned}
[\zeta_2(\beta_s - a_1) + (1-\zeta_2)(\beta_b - a_2)k_3]r^{\zeta_1} &= A_1 + B_1 k_4, \\
[-\zeta_1(\beta_s - a_1) + (\zeta_1 - 1)(\beta_b - a_2)k_3]r^{\zeta_2} &= A_2 + B_2 k_4.
\end{aligned}$$

Since $k_4 = rk_3$, we can obtain

$$k_3 = \frac{A_1 - \zeta_2(\beta_s - a_1)r^{\zeta_1}}{(1-\zeta_2)(\beta_b - a_2)r^{\zeta_1} - B_1 r} = \frac{A_2 + \zeta_1(\beta_s - a_1)r^{\zeta_2}}{(\zeta_1 - 1)(\beta_b - a_2)r^{\zeta_2} - B_2 r}, \tag{2.34}$$

and

$$k_4 = rk_3 = \frac{A_1 r - \zeta_2(\beta_s - a_1)r^{\zeta_1+1}}{(1-\zeta_2)(\beta_b - a_2)r^{\zeta_1} - B_1 r} = \frac{A_2 r + \zeta_1(\beta_s - a_1)r^{\zeta_2+1}}{(\zeta_1 - 1)(\beta_b - a_2)r^{\zeta_2} - B_2 r}. \tag{2.35}$$

To solve for r:

$$\frac{A_1 - \zeta_2(\beta_s - a_1)r^{\zeta_1}}{(1-\zeta_2)(\beta_b - a_2)r^{\zeta_1} - B_1 r} = \frac{A_2 + \zeta_1(\beta_s - a_1)r^{\zeta_2}}{(\zeta_1 - 1)(\beta_b - a_2)r^{\zeta_2} - B_2 r}. \tag{2.36}$$

Since we assume that $k_3 < k_4$, we need to show that the above equation has a unique solution $r > 1$. Once we find r, we can find k_3 and k_4 from k3 and k4. Then C_1, C_2, E_1 and E_2 can be given as follows:

$$\begin{aligned}
C_3 &= \tfrac{-\delta_4\beta_s + (\delta_4 - 1)\beta_b k_4}{\eta_3(\delta_3 - \delta_4)k_4^{\delta_3}}, & C_4 &= \tfrac{\delta_3\beta_s + (1-\delta_3)\beta_b k_4}{\eta_4(\delta_3 - \delta_4)k_4^{\delta_4}}, \\
E_1 &= \tfrac{-\zeta_2(\beta_s - a_1) - (1-\zeta_2)(\beta_b - a_2)k_3}{(\zeta_1 - \zeta_2)k_3^{\zeta_1}}, & E_2 &= \tfrac{\zeta_1(\beta_s - a_1) - (\zeta_1 - 1)(\beta_b - a_2)k_3}{(\zeta_1 - \zeta_2)k_3^{\zeta_2}}.
\end{aligned}$$

We summarize the solutions w_3 and w_4 as follows:

$$\begin{aligned}
(0, k_3): &\quad w_3 = \beta_s - \beta_b y, \\
[k_3, k_4]: &\quad w_3 = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - a_2 y, \\
(k_4, \infty): &\quad w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4}, \\
[0, k_4]: &\quad w_4 = \beta_s - \beta_b y, \\
(k_4, \infty): &\quad w_4 = C_3\eta_3 y^{\delta_3} + C_4\eta_4 y^{\delta_4},
\end{aligned}$$

where a_1 and a_2 are given by a1a2, C_1, C_2, E_1 and E_2 are given by CE; and η_1 and η_2 are given by etas. In addition, we assume the inequalities to hold:

$$\begin{aligned}
(0, k_3): &\quad (\rho + \lambda_1 - \mathcal{L}_1) w_3(y) - \lambda_1 w_4(y) \geq 0, \\
[k_3, k_4]: &\quad w_3 = E_1 y^{\zeta_1} + E_2 y^{\zeta_2} + a_1 - a_2 y \geq \beta_s - \beta_b y, \\
[k_4, \infty): &\quad w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4} \geq \beta_s - \beta_b y, \\
(0, k_4): &\quad (\rho + \lambda_2 - \mathcal{L}_2) w_4(y) - \lambda_2 w_3(y) \geq 0, \\
[k_4, \infty): &\quad w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} \geq \beta_s - \beta_b y.
\end{aligned} \tag{2.37}$$

$k_3 > k_4$ **(for CASES IV, V, VI)**:

In this case, we divide the region $(0, \infty)$ into three parts:

$$\Gamma_1 = [0, k_4], \quad \Gamma_2 = [k_4, k_3], \quad \text{and} \quad \Gamma_3 = [k_3, \infty).$$

In the region $\Gamma_1 = [0, k_4]$, $w_3 = w_4 = \beta_s - \beta_b y$; and in the region $\Gamma_3 = [k_3, \infty)$,

$$w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4} \quad \text{and} \quad w_4 = \eta_3 C_3 y^{\delta_3} + \eta_4 C_4 y^{\delta_4}.$$

In the region $\Gamma_2 = [k_4, k_3]$, $w_3 = \beta_s - \beta_b y$ and $(\rho + \lambda_2 - {}_2) w_4(y) = \lambda_2 w_3(y)$. We need to solve the non-homogeneous equation

$$(\rho + \lambda_2 - {}_2) w_4(y) = \lambda_2 w_3(y) = \lambda_2(\beta_s - \beta_b y).$$

We shall find the particular solution first. By the method of undetermined coefficients, we let one particular solution of w_4 is of the form

$$w_4 = b_1 - b_2 y.$$

Then

$$(\rho + \lambda_2 - {}_2) w_4 = (\rho + \lambda_2 - \mu_1(2)) b_1 - (\rho + \lambda_2 - \mu_2(2)) b_2 y.$$

This implies

$$b_1(\rho + \lambda_2 - \mu_1(2)) = \lambda_2 \beta_s \quad \text{and} \quad b_2(\rho + \lambda_2 - \mu_2(2)) = \lambda_2 \beta_b.$$

This yields

$$b_1 = \frac{\lambda_2 \beta_s}{\rho + \lambda_2 - \mu_1(2)} \quad \text{and} \quad b_2 = \frac{\lambda_2 \beta_b}{\rho + \lambda_2 - \mu_2(2)}.$$

Summarize the computation, we have one particular solution

$$w_4 = \frac{\lambda_2 \beta_s}{\rho + \lambda_2 - \mu_1(2)} - \frac{\lambda_2 \beta_b}{\rho + \lambda_2 - \mu_2(2)} y.$$

Then the general solution of the non-homogeneous equation is the above particular solution plus the solution of the homogeneous equation $(\rho + \lambda_2 - {}_2) w_4 = 0$. The general form of w_4 is w4inR2 w4 $= D_1 y^{\nu_1} + D_2 y^{\nu_2} + \frac{\lambda_2 \beta_s}{\rho+\lambda_2-\mu_1(2)} - \frac{\lambda_2 \beta_b}{\rho+\lambda_2-\mu_2(2)} y Wewillfindtheparameters$C$_3$, C_4, D_1, D_2, k_3 and k_4 by the method of smooth-fitting.

The continuity of w_3 and its derivative at k_3 yields

$$\begin{aligned} \beta_s - \beta_b k_3 &= C_3 k_3^{\delta_3} + C_4 k_3^{\delta_4}, \\ -\beta_b &= \delta_3 C_3 k_3^{\delta_3 - 1} + \delta_4 C_4 k_3^{\delta_4 - 1}. \end{aligned}$$

The continuity of w_4 and its derivative at k_3 and k_4 yield

$$\begin{aligned}
\beta_s - \beta_b k_4 &= D_1 k_4^{\nu_1} + D_2 k_4^{\nu_2} + b_1 - b_2 k_4, \\
-\beta_b &= D_1 \nu_1 k_4^{\nu_1 - 1} + D_2 \nu_2 k_4^{\nu_2 - 1} - b_2, \\
D_1 k_3^{\nu_1} + D_2 k_3^{\nu_2} + b_1 - b_2 k_3 &= \eta_3 C_3 k_3^{\delta_3} + \eta_4 C_4 k_3^{\delta_4}, \\
D_1 \nu_1 k_3^{\nu_1 - 1} + D_2 \nu_2 k_3^{\nu_2 - 1} - b_2 &= \eta_3 \delta_3 C_3 k_3^{\delta_3 - 1} + \eta_4 \delta_4 C_4 k_3^{\delta_4 - 1}.
\end{aligned}$$

We will solve C_3 and C_4 in term k_3, D_1 and D_2 in term of k_4 from the first four equations to get

$$\begin{aligned}
C_3 &= \tfrac{-\delta_4 \beta_s + (\delta_4 - 1)\beta_b k_3}{(\delta_3 - \delta_4) k_3^{\delta 3}}, & C_4 &= \tfrac{\delta_3 \beta_s + (1 - \delta_3)\beta_b k_3}{(\delta_3 - \delta_4) k_3^{\delta 4}}, \\
D_1 &= \tfrac{\nu_2(\beta_s - b_1) + (1 - \nu_2)(\beta_b - b_2) k_4}{(2-1) k_4^{\nu 1}}, & D_2 &= \tfrac{-\nu_1(\beta_s - b_1) + (\nu_1 - 1)(\beta_b - b_2) k_4}{(2-1) k_4^{\nu 2}}.
\end{aligned}$$

We can get a system of equations of k_3 and k_4 by inserting C_3, C_4, D_1 and D_2 into the last two equations:

$$\begin{aligned}
& \tfrac{\nu_2(\beta_s - b_1) + (1 - \nu_2)(\beta_b - b_2) k_4}{(\nu_2 - \nu_1)} \left(\tfrac{k_3}{k_4}\right)^{\nu_1} + \tfrac{-\nu_1(\beta_s - b_1) + (\nu_1 - 1)(\beta_b - b_2) k_4}{(2-1)} \left(\tfrac{k_3}{k_4}\right)^{\nu_2} + b_1 - b_2 k_3 \\
= \; & \tfrac{-\delta_4 \beta_s + (\delta_4 - 1)\beta_b k_3}{(\delta_3 - \delta_4)} \eta_3 + \tfrac{\delta_3 \beta_s + (1 - \delta_3)\beta_b k_3}{(\delta_3 - \delta_4)} \eta_4, \\
& \tfrac{\nu_2(\beta_s - b_1) + (1 - \nu_2)(\beta_b - b_2) k_4}{(\nu_2 - \nu_1)} \nu_1 \left(\tfrac{k_3}{k_4}\right)^{\nu_1} + \tfrac{-\nu_1(\beta_s - b_1) + (\nu_1 - 1)(\beta_b - b_2) k_4}{(\nu_2 - \nu_1)} \nu_2 \left(\tfrac{k_3}{k_4}\right)^{\nu_2} - b_2 k_3 \\
= \; & \tfrac{-\delta_4 \beta_s + (\delta_4 - 1)\beta_b k_3}{(\delta_3 - \delta_4)} \eta_3 \delta_3 + \tfrac{\delta_3 \beta_s + (1 - \delta_3)\beta_b k_3}{(\delta_3 - \delta_4)} \eta_4 \delta_4.
\end{aligned}$$

We first simplify the system to

$$\begin{aligned}
& [\nu_2(\beta_s - b_1) + (1 - \nu_2)(\beta_b - b_2) k_4] \left(\frac{k_3}{k_4}\right)^{\nu_1} + \nu_2 b_1 - (\nu_2 - 1) b_2 k_3 \\
= & \frac{-\delta_4 \beta_s + (\delta_4 - 1)\beta_b k_3}{(\delta_3 - \delta_4)} \eta_3 (\nu_2 - \delta_3) + \frac{\delta_3 \beta_s + (1 - \delta_3)\beta_b k_3}{(\delta_3 - \delta_4)} \eta_4 (\nu_2 - \delta_4), \\
& [-\nu_1(\beta_s - b_1) + (\nu_1 - 1)(\beta_b - b_2) k_4] \left(\frac{k_3}{k_4}\right)^{\nu_2} + (\nu_1 - 1) b_2 k_3 - \nu_1 b_1 \\
= & \frac{-\delta_4 \beta_s + (\delta_4 - 1)\beta_b k_4}{(\delta_3 - \delta_4)} \eta_3 (\delta_3 - \nu_1) + \frac{\delta_3 \beta_s + (1 - \delta_3)\beta_b k_4}{(\delta_3 - \delta_4)} \eta_4 (\delta_4 - \nu_1).
\end{aligned}$$

Next one can introduce a new parameter $r = k_4/k_3$. This will reduce the above system to a linear system of k_3 and k_4. We can solve k_3 and k_4 in term of r and finally $r = k_4/k_3$ will yields an equation of

r. If we can show the existence of r, we can find k_3 and k_4. We shall first simplify the system as

$$[\nu_2(\beta_s - b_1) + (1-\nu_2)(\beta_b - b_2)k_4]r^{-\nu_1} = \left[\frac{-\eta_3\delta_4\beta_s(\nu_2-\delta_3)}{(\delta_3-\delta_4)} + \frac{\eta_4\delta_3\beta_s(\nu_2-\delta_4)}{(\delta_3-\delta_4)} - \nu_2 b_1\right]$$
$$+ \left[\frac{\eta_3(\delta_4-1)(\nu_2-\delta_3)\beta_b}{(\delta_3-\delta_4)} + \frac{\eta_4(1-\delta_3)\beta_b(\nu_2-\delta_4)}{(\delta_3-\delta_4)} - (1-\nu_2)b_2\right] k_3$$
$$[-\nu_1(\beta_s - b_1) + (\nu_1-1)(\beta_b - b_2)k_4]r^{-\nu_2} = \left[\frac{-\eta_3\delta_4\beta_s(\delta_3-\nu_1)}{(\delta_3-\delta_4)} + \frac{\eta_4\delta_3\beta_s(\delta_4-\nu_1)}{(\delta_3-\delta_4)} + \nu_1 b_1\right]$$
$$+ \left[\frac{\eta_3(\delta_4-1)(\delta_3-\nu_1)\beta_b}{(\delta_3-\delta_4)} + \frac{\eta_4(1-\delta_3)\beta_b(\delta_4-\nu_1)}{(\delta_3-\delta_4)} - (\nu_1-1)b_2\right] k_3.$$

We introduce some new parameters to simplify the notations:

$$\begin{aligned}
A_1' &= \tfrac{-\eta_3\delta_4\beta_s(\nu_2-\delta_3)}{(\delta_3-\delta_4)} + \tfrac{\eta_4\delta_3\beta_s(\nu_2-\delta_4)}{(\delta_3-\delta_4)} - \nu_2 b_1,\\
B_1' &= \tfrac{\eta_3(\delta_4-1)(\nu_2-\delta_3)\beta_b}{(\delta_3-\delta_4)} + \tfrac{\eta_4(1-\delta_3)\beta_b(\nu_2-\delta_4)}{(\delta_3-\delta_4)} - (1-\nu_2)b_2;\\
A_2' &= \tfrac{-\eta_3\delta_4\beta_s(\delta_3-\nu_1)}{(\delta_3-\delta_4)} + \tfrac{\eta_4\delta_3\beta_s(\delta_4-\nu_1)}{(\delta_3-\delta_4)} + \nu_1 b_1,\\
B_2' &= \tfrac{\eta_3(\delta_4-1)(\delta_3-\nu_1)\beta_b}{(\delta_3-\delta_4)} + \tfrac{\eta_4(1-\delta_3)\beta_b(\delta_4-\nu_1)}{(\delta_3-\delta_4)} - (\nu_1-1)b_2.
\end{aligned}$$

Then we can rewrite the above system as

$$\begin{aligned}
[\nu_2(\beta_s - b_1) + (1-\nu_2)(\beta_b - b_2)k_4]r^{-\nu_1} &= A_1' + B_1'k_3,\\
[-\nu_1(\beta_s - b_1) + (\nu_1-1)(\beta_b - b_2)k_4]r^{-\nu_2} &= A_2' + B_2'k_3.
\end{aligned}$$

Since $k_4 = rk_3$, we can obtain

$$k_3 = \frac{A_1' - \nu_2(\beta_s - b_1)r^{-\nu_1}}{(1-\nu_2)(\beta_b + b_2)r^{1-\nu_1} - B_1'} = \frac{A_2' + \nu_1(\beta_s - b_1)r^{-\nu_2}}{(\nu_1-1)(\beta_b + b_2)r^{1-\nu_2} - B_2'}, \tag{2.38}$$

and

$$k_4 = rk_3 = \frac{A_1'r - \nu_2(\beta_s - b_1)r^{1-\nu_1}}{(1-\nu_2)(\beta_b + b_2)r^{1-\nu_1} - B_1'} = \frac{A_2'r + \nu_1(\beta_s - b_1)r^{1-\nu_2}}{(\nu_1-1)(\beta_b + b_2)r^{1-\nu_2} - B_2'}. \tag{2.39}$$

The second equality will yield an equation about r:

$$\frac{A_1' - \nu_2(\beta_s - b_1)r^{-\nu_1}}{(1-\nu_2)(\beta_b - b_2)r^{1-\nu_1} - B_1'} = \frac{A_2' + \nu_1(\beta_s - b_1)r^{-\nu_2}}{(\nu_1-1)(\beta_b - b_2)r^{1-\nu_2} - B_2'}. \tag{2.40}$$

Once we find r, we can obtain k_3 and k_4, and other parameters.

We will need w_3 and w_4 to derive w_1 and w_2, we summerize what we have find for w_3 and w_4 here:

$$\begin{array}{ll} [0, k_3]: & w_3 = \beta_s - \beta_b y, \\ [k_3, \infty): & w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4}, \\ [0, k_4]: & w_4 = \beta_s - \beta_b y, \\ [k_4, k_3]: & w_4 = D_1 y^{\nu_1} + D_2 y^{\nu_2} + b_1 - b_2 y, \\ [k_3, \infty): & w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4}. \end{array} \tag{2.41}$$

We assume the following inequalities to hold :

$$\begin{array}{ll} [0, k_3]: & (\rho + \lambda_1 - \mathcal{L}_1) w_3(y) - \lambda_1 w_4(y) \geq 0, \\ [k_3, \infty): & w_3 = C_3 y^{\delta_3} + C_4 y^{\delta_4} \geq \beta_s - \beta_b y, \\ [0, k_4]: & (\rho + \lambda_2 - \mathcal{L}_2) w_4(y) - \lambda_1 w_3(y) \geq 0, \\ [k_4, k_3]: & w_4 = D_1 y^{\nu_1} + D_2 y^{\nu_2} + b_1 - b_2 \geq \beta_s - \beta_b y, \\ [k_3, \infty): & w_4 = C_3 \eta_3 y^{\delta_3} + C_4 \eta_4 y^{\delta_4} \geq \beta_s - \beta_b y. \end{array} \tag{2.42}$$

2.5 Verification Theorems

Theorem 1. (Selling Rule $k_3 < k_4$). Assume (A1). Let k_3 and k_4 be given in (2.34) and (2.35), resp. Let $w_3(y)$ and $w_4(y)$ be given as in (2.36) such that the variational inequalities in (2.37) are satisfied. Then, $v_1(x_1, x_2, 1) = x_1 w_3(x_2/x_1) = V_1(x_1, x_2, 1)$ and $v_1(x_1, x_2, 2) = x_1 w_4(x_2/x_1) = V_1(x_1, x_2, 2)$. Let $D_S = \{(x_1, x_2, 1) : x_2 > k_3 x_1\} \cup \{(x_1, x_2, 2) : x_2 > k_4 x_1\}$. Let $\tau_0^* = \inf\{t : (X_t^1, X_t^2, \alpha_t) \notin D_S\}$. Then τ_0^* is optimal.

Theorem 2. (Buying Rule $k_3 < k_4$). Assume (A1). Let k_1 and k_2 be given by (2.14) in Case I (by (2.17) in Case II and (2.20) in Case III, resp.). Let also $w_1(y)$ and $w_2(y)$ be given by (2.13) in Case I (by (2.16) in Case II and (2.19) in Case III, resp.) Suppose the variational inequalities in (2.15) hold (Case I) (in (2.18) (Case II) and (2.21) (Case III), resp.). Then, $v_0(x_1, x_2, 1) = x_1 w_1(x_2/x_1) = V_0(x_1, x_2, 1)$ and $v_0(x_1, x_2, 2) = x_1 w_2(x_2/x_1) = V_0(x_1, x_2, 2)$. Let $D_B = \{(x_1, x_2, 1) : x_2 < k_1 x_1\} \cup \{(x_1, x_2, 2) : x_2 < k_2 x_1\}$. Define $\tau_1^* = \inf\{t : (X_t^1, X_t^2, \alpha_t) \notin D_B\}$ and $\tau_2^* = \inf\{t \geq \tau_1^* : (X_t^1, X_t^2, \alpha_t) \notin D_S\}$. Then $\Lambda_0 = (\tau_1^*, \tau_2^*)$ is optimal.

Theorem 3. (Selling Rule $k_4 < k_3$). Assume (A1). Let k_3 and k_4 be given in (2.38) and (2.39), resp. Let $w_3(y)$ and $w_4(y)$ be given as in (2.41) such that the variational inequalities in (2.42) are satisfied. Then, $v_1(x_1, x_2, 1) = x_1 w_3(x_2/x_1) = V_1(x_1, x_2, 1)$ and $v_1(x_1, x_2, 2) = x_1 w_4(x_2/x_1) = V_1(x_1, x_2, 2)$. Let $D_S = \{(x_1, x_2, 1) : x_2 > k_3 x_1\} \cup \{(x_1, x_2, 2) : x_2 > k_4 x_1\}$. Let $\tau_0^* = \inf\{t : (X_t^1, X_t^2, \alpha_t) \notin D_S\}$. Then τ_0^* is optimal.

Theorem 4. (Buying Rule $k_4 < k_3$). Assume (A1). Let k_1 and k_2 be given by (2.25) in Case 4 (by (2.30) in Case 5 and (2.31) in Case 6, resp.). Let also $w_1(y)$ and $w_2(y)$ be given by (2.24) in Case 4 (by (2.29) in Case 5, resp.) Suppose the variational inequalities in (2.26) hold (Case 4) (in (2.26) (Case

5), since case 4 and 5 share the same inequalities, and (2.32) (Case 6), resp.). Then, $v_0(x_1,x_2,1) = x_1 w_1(x_2/x_1) = V_0(x_1,x_2,1)$ and $v_0(x_1,x_2,2) = x_1 w_2(x_2/x_1) = V_0(x_1,x_2,2)$. Let $D_{\mathrm{B}} = \{(x_1,x_2,1) : x_2 < k_1 x_1\} \cup \{(x_1,x_2,2) : x_2 < k_2 x_1\}$. Define $\tau_1^* = \inf\{t : (X_t^1, X_t^2, \alpha_t) \notin D_{\mathrm{B}}\}$ and $\tau_2^* = \inf\{t \geq \tau_1^* : (X_t^1, X_t^2, \alpha_t) \notin D_{\mathrm{S}}\}$. Then $\Lambda_0 = (\tau_1^*, \tau_2^*)$ is optimal.

Proof. We sketch key steps for the sake of completeness. First, we show $v_i(x_1,x_2,\alpha) \geq J_i(x_1,x_2,\alpha,\Lambda_i)$. To this end, note that, in view of the variational inequalities in the HJB equations, for any stopping times $0 \leq \theta_1 \leq \theta_2$, a.s.,

$$E\left(e^{-\rho\theta_1} v_i\left(X_{\theta_1}^1, X_{\theta_1}^2, \alpha_{\theta_1}\right) I_{\{\theta_1<\infty\}}\right) \geq E\left(e^{-\rho\theta_2} v_i\left(X_{\theta_2}^1, X_{\theta_2}^2, \alpha_{\theta_2}\right) I_{\{\theta_2<\infty\}}\right), \text{ for } i = 0, 1.$$

Given $\Lambda_0 = (\tau_1, \tau_2)$, it follows that

$$\begin{aligned}
v_0(x_1,x_2,\alpha) &\geq E\left(e^{-\rho\tau_1} v_0\left(X_{\tau_1}^1, X_{\tau_1}^2, \alpha_{\tau_1}\right) I_{\{\tau_1<\infty\}}\right) \\
&\geq E\left(e^{-\rho\tau_1}\left(v_1\left(X_{\tau_1}^1, X_{\tau_1}^2, \alpha_{\tau_1}\right) - \beta_{\mathrm{b}} X_{\tau_1}^1 + \beta_{\mathrm{s}} X_{\tau_1}^2\right) I_{\{\tau_1<\infty\}}\right) \\
&= E\left(e^{-\rho\tau_1} v_1\left(X_{\tau_1}^1, X_{\tau_1}^2, \alpha_{\tau_1}\right) I_{\{\tau_1<\infty\}} - e^{-\rho\tau_1}\left(\beta_{\mathrm{b}} X_{\tau_1}^1 - \beta_{\mathrm{s}} X_{\tau_1}^2\right) I_{\{\tau_1<\infty\}}\right) \\
&\geq E\left(e^{-\rho\tau_2} v_1\left(X_{\tau_2}^1, X_{\tau_2}^2, \alpha_{\tau_2}\right) I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_{\mathrm{b}} X_{\tau_1}^1 - \beta_{\mathrm{s}} X_{\tau_1}^2\right) I_{\{\tau_1<\infty\}}\right) \\
&\geq E\left(e^{-\rho\tau_2}\left(\beta_{\mathrm{s}} X_{\tau_2}^1 - \beta_{\mathrm{b}} X_{\tau_2}^2\right) I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_{\mathrm{b}} X_{\tau_1}^1 - \beta_{\mathrm{s}} X_{\tau_1}^2\right) I_{\{\tau_1<\infty\}}\right) \\
&= J_0(x_1,x_2,\alpha,\Lambda_0).
\end{aligned}$$

Next, we establish the equality $v_i(x_1,x_2) = J_i(x_1,x_2,\Lambda_i^*)$. Recall that $\tau_1^* = \inf\{t \geq 0 \colon (X_t^1, X_t^2, \alpha_t) \in D_{\mathrm{B}}\}$ and $\tau_2^* = \inf\{t \geq \tau_1^* : (X_t^1, X_t^2, \alpha_t) \in D_{\mathrm{S}}\}$. Using Dynkin's formula over the intervals $(0, \tau_1^*)$ and (τ_1^*, τ_2^*) to obtain

$$\begin{aligned}
v_0(x_1,x_2,\alpha) &= E\left[e^{-\rho\tau_1^*} v_0\left(X_{\tau_1^*}^1, X_{\tau_1^*}^1, \alpha_{\tau_1^*}\right) I_{\{\tau_1^*<\infty\}}\right] \\
&= E\left[e^{-\rho\tau_1^*}\left(v_1\left(X_{\tau_1^*}^1, X_{\tau_1^*}^1, \alpha_{\tau_1^*}\right) - \beta_{\mathrm{b}} X_{\tau_1^*}^1 + \beta_{\mathrm{s}} X_{\tau_2^*}^2\right) I_{\{\tau_1^*<\infty\}}\right].
\end{aligned}$$

We have also

$$\begin{aligned}
E\left(e^{-\rho\tau_1^*} v_1\left(X_{\tau_1^*}^1, X_{\tau_1^*}^2, \alpha_{\tau_1^*}\right) I_{\{\tau_1^*<\infty\}}\right) &= E\left(e^{-\rho\tau_2^*} v_1\left(X_{\tau_2^*}^1, X_{\tau_2^*}^2, \alpha_{\tau_2^*}\right) I_{\{\tau_2^*<\infty\}}\right) \\
&= E\left(e^{-\rho\tau_2^*}\left(\beta_{\mathrm{s}} X_{\tau_2^*}^1 - \beta_{\mathrm{b}} X_{\tau_2^*}^2\right) I_{\{\tau_2^*<\infty\}}\right).
\end{aligned}$$

Combine these two equalities to obtain $v_0(x_1,x_2,\alpha) = J_0(x_1,x_2,\alpha,\Lambda_0^*)$.

2.6 Numerical examples

CASE1: $(k_3 < k_1 < k_4 < k_2)$:

Let's look at one numerical example:

$$\mu_1(1) = 0.30, \quad \mu_2(1) = 0.27, \quad \mu_1(2) = -0.43, \quad \mu_2(2) = -0.66,$$
$$\sigma_{11}(1) = 0.44, \quad \sigma_{12}(1) = 0.27, \quad \sigma_{21}(1) = 0.31, \quad \sigma_{22}(1) = 0.60,$$
$$\sigma_{11}(2) = 0.19, \quad \sigma_{12}(2) = 0.65, \quad \sigma_{21}(2) = 0.28, \quad \sigma_{22}(2) = 0.15,$$
$$\lambda_1 = 6.0, \quad \lambda_2 = 10.0, \quad K = 0.001, \quad \rho = 0.50.$$

First, we solve k_3 and k_4 , then k_1 and k_2. We obtain $k_1 = 0.597020, k_2 = 0.690976, k_3 = 0.578407$, and $k_4 = 0.601707$. Using these to calculate the rest parameters to get $B_1 = -1082.994378$, $B_2 = 0.002139$, $C_1 = 6.721641$, $C_2 = -0.043221$, $C_3 = 0.189389$, $C_4 = -0.000004$, $D_1 = -0.078520$, $D_2 = -0.0007050$, $E_1 = 1.377957$, and $E_2 = 4.440166$.

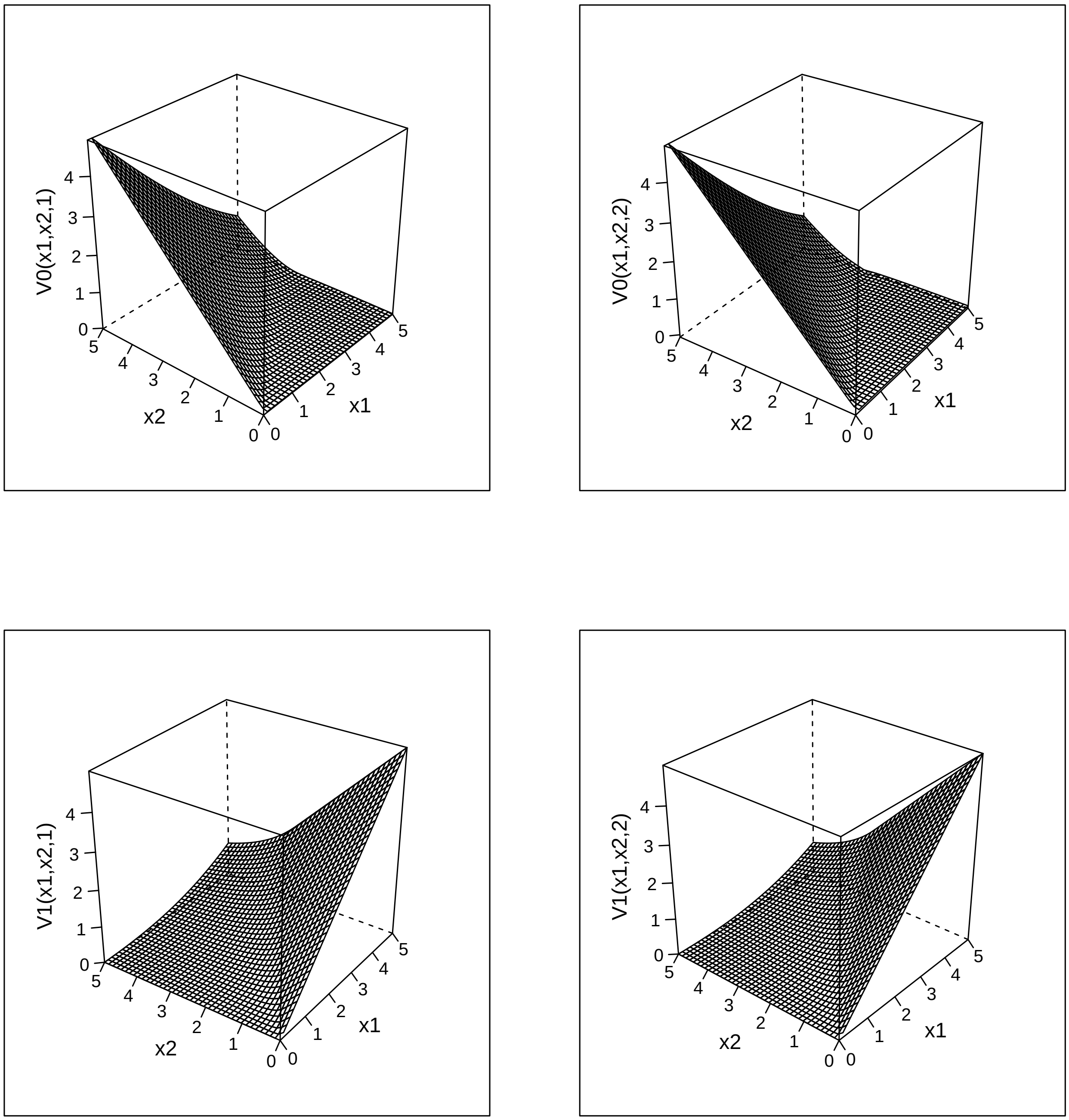

Figure 2.3: Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

CASE2: $(k_3 < k_4 < k_1 < k_2)$: For this case:

$$\begin{array}{llll} \mu_1(1) = -0.26, & \mu_2(1) = -0.56, & \mu_1(2) = -0.4, & \mu_2(2) = 0.22 \\ \sigma_{11}(1) = 0.37, & \sigma_{12}(1) = 0.46, & \sigma_{21}(1) = 0.59, & \sigma_{22}(1) = 0.59 \\ \sigma_{11}(2) = 0.47, & \sigma_{12}(2) = 0.31, & \sigma_{21}(2) = 0.28, & \sigma_{22}(2) = 0.68 \\ \lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50. \end{array}$$

Similarly as in Example 1, we solve obtain $k_1 = 0.929500$, $k_2 = 0.962000, k_3 = 0.678861$, and $k_4 = 0.810852$. Then, we calculate and get $B_1 = 0.295000, B_2 = 0.021266, C_1 = 0.078164, C_2 = 0.048996, C_3 = 0.097388, C_4 = -0.000156$, $E_1 = 0.225207$, and $E_2 = 0.000199$. Plugging these numbers to obtain the corresponding value functions. We verify that all the variational inequalities are satisfied. Finally, the graphs of these value functions are given :

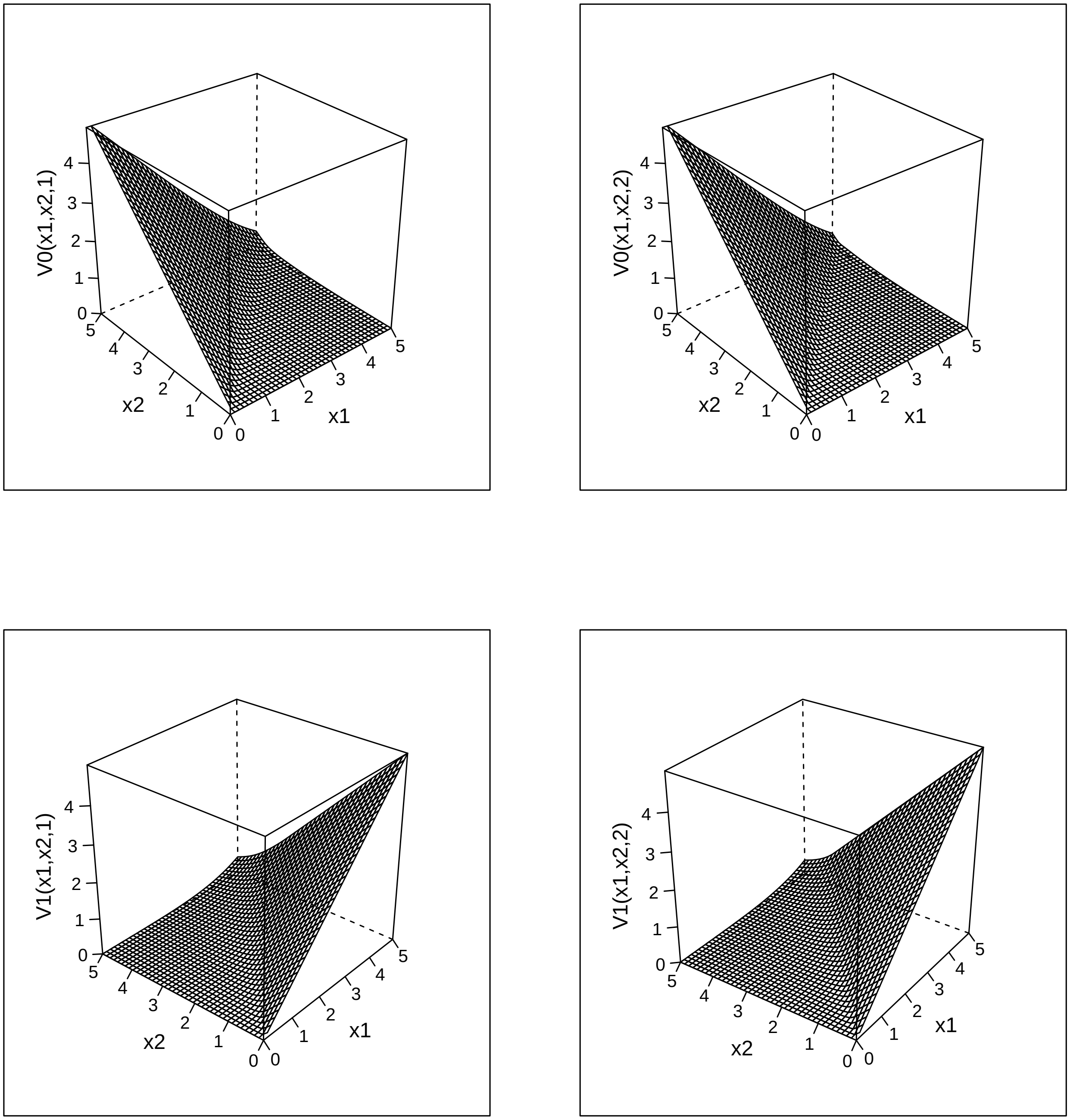

Figure 2.4: Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

CASE3: $k_3 < k_4 < k_2 < k_1$:

$$\begin{array}{llll}
\mu_1(1) = 0.20, & \mu_2(1) = 0.25, & \mu_1(2) = -0.30, & \mu_2(2) = -0.35, \\
\sigma_{11}(1) = 0.30, & \sigma_{12}(1) = 0.10, & \sigma_{21}(1) = 0.10, & \sigma_{22}(1) = 0.35 \\
\sigma_{11}(2) = 0.40, & \sigma_{12}(2) = 0.20, & \sigma_{21}(2) = 0.20, & \sigma_{22}(2) = 0.45 \\
\lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50.
\end{array}$$

Similarly as in previous examples, we solve to obtain $k_1 = 1.379000$, $k_2 = 1.212000$, $k_3 = 0.723277$, and $k_4 = 0.737941$. Plugging these numbers to obtain the corresponding value functions. We verify that all the variational inequalities are satisfied. Finally, the graphs of these value functions are given :

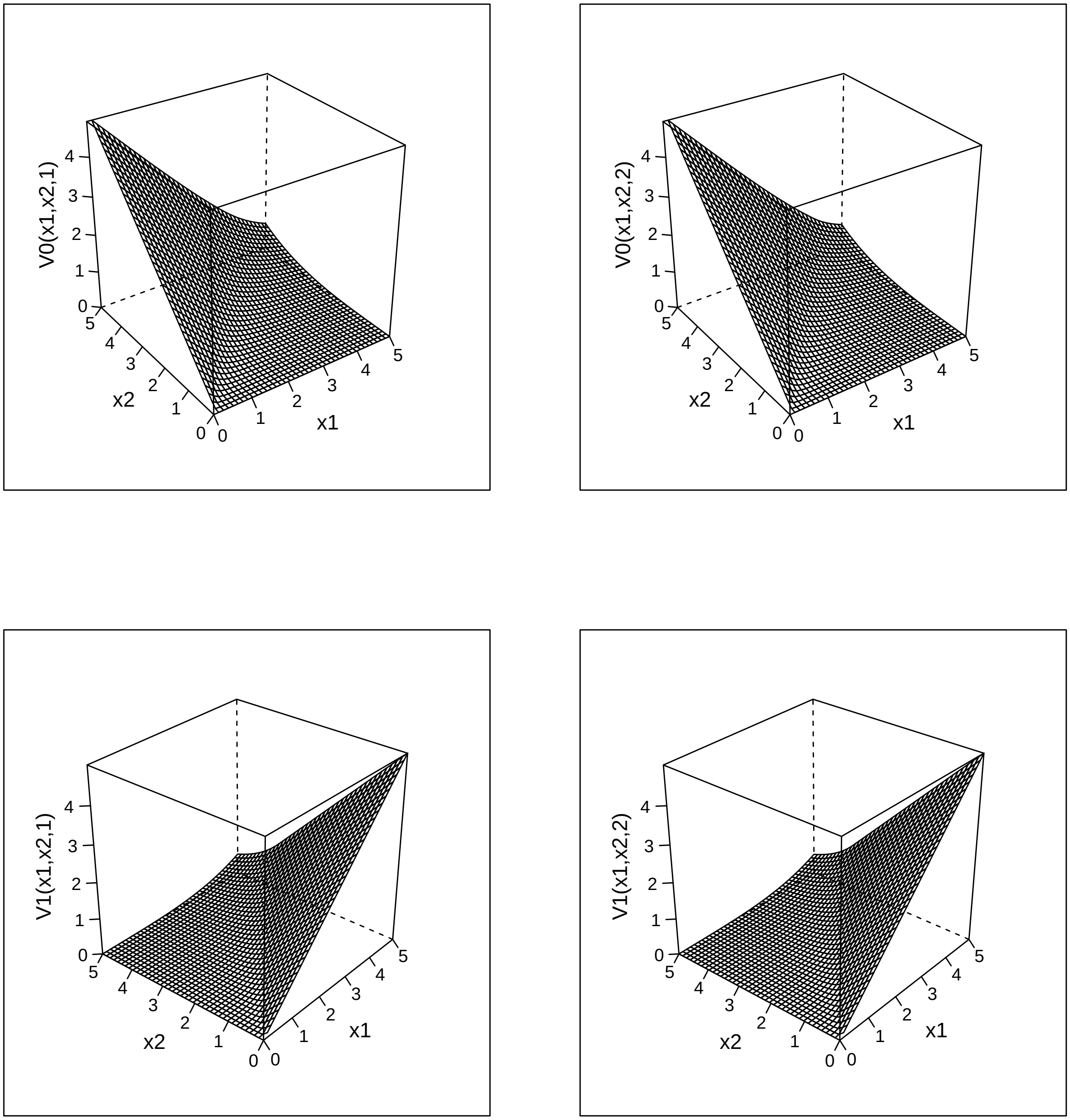

Figure 2.5: Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

CASE4: $k_4 < k_2 < k_3 < k_1$:
Let's look at a numerical example:

$$
\begin{array}{llll}
\mu_1(1) = 0.839, & \mu_2(1) = 0.61, & \mu_1(2) = -0.7, & \mu_2(2) = -0.5, \\
\sigma_{11}(1) = 0.31, & \sigma_{12}(1) = 0.27, & \sigma_{21}(1) = 0.31, & \sigma_{22}(1) = 0.60, \\
\sigma_{11}(2) = 0.19, & \sigma_{12}(2) = 0.65, & \sigma_{21}(2) = 0.28, & \sigma_{22}(2) = 0.25, \\
\lambda_1 = 3.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50.
\end{array}
$$

We obtain $k_1 = 0.813146556050517$,$k_2 = 0.607323219314284$, $k_3 = 0.187445927589516$,$k_4 = 0.0515053606625219$ Using these to calculate the rest parameters to get $B_1 = -1082.994378$, $B_2 = 0.002139$, $C_1 = 6.721641$, $C_2 = -0.043221$, $C_3 = 0.189389$, $C_4 = -0.000004$, $D_1 = -0.078520$, $D_2 = -0.0007050$, $E_1 = 1.377957$, and $E_2 = 4.440166$. Plugging these numbers to obtain the corresponding value functions. We verify that all the variational inequalities are satisfied. Finally, the graphs of these value functions are given :

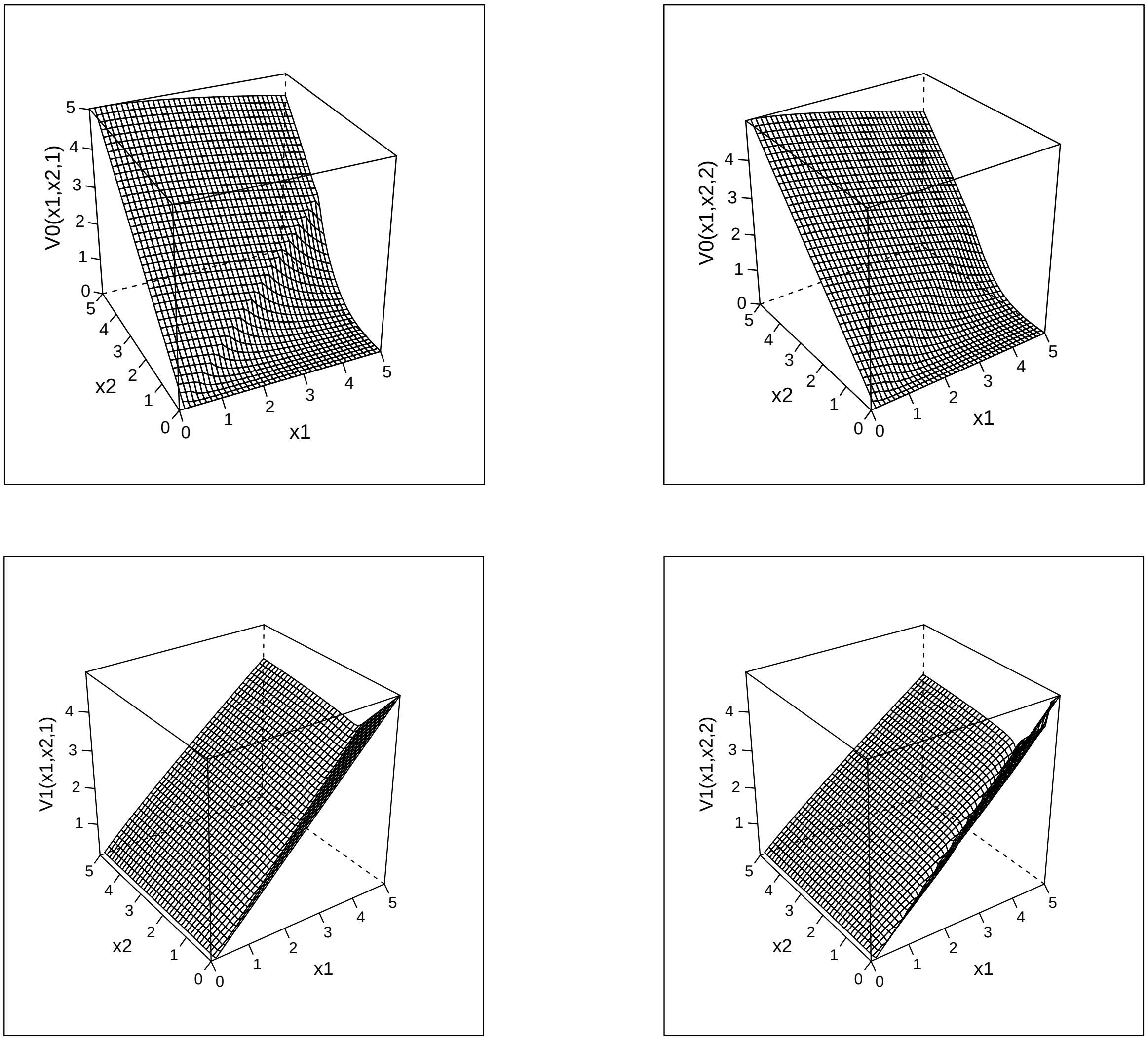

Figure 2.6: Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

CASE5: $k_4 < k_3 < k_2 < k_1$:
Let's look at a numerical example:

$$\begin{array}{llll}
\mu_1(1) = 0.55, & \mu_2(1) = 0.44, & \mu_1(2) = -0.35, & \mu_2(2) = -0.65, \\
\sigma_{11}(1) = 0.40, & \sigma_{12}(1) = 0.22, & \sigma_{21}(1) = 0.30, & \sigma_{22}(1) = 0.90, \\
\sigma_{11}(2) = 0.65, & \sigma_{12}(2) = 0.05, & \sigma_{21}(2) = 0.28, & \sigma_{22}(2) = 0.5, \\
\lambda_1 = 3.0, & \lambda_2 = 9.0, & K = 0.001, & \rho = 0.50.
\end{array}$$

We obtain $k_1 = 0.810634931470992$, $k_2 = 0.492463166582152$, $k_3 = 0.0440799199487903$, $k_4 = 0.0387559973146809$ Plugging these numbers to obtain the corresponding value functions. We verify that all the variational inequalities are satisfied. Finally, the graphs of these value functions are given :

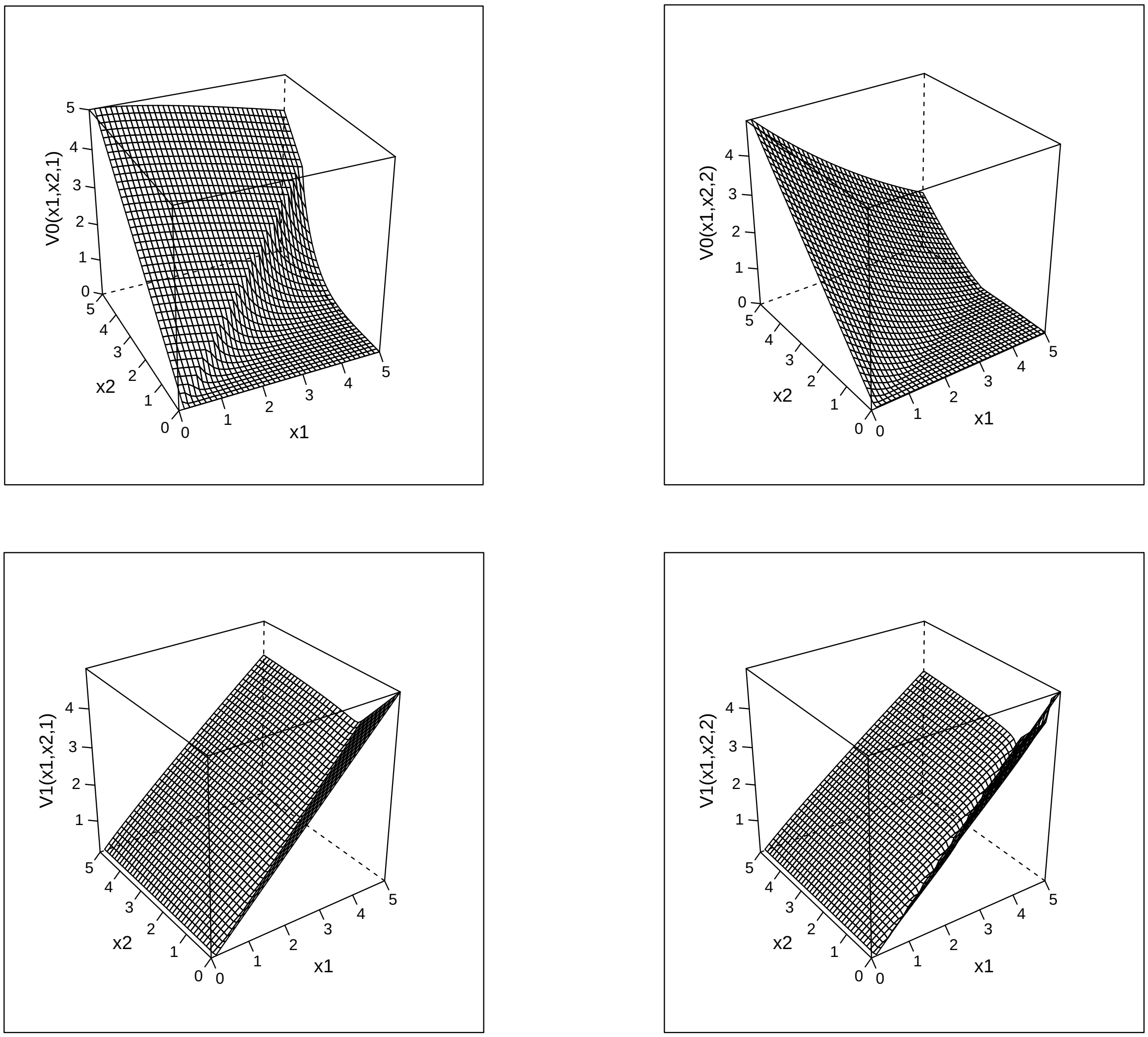

Figure 2.7: Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

CASE6: $k_4 < k_3 < k_1 < k_2$:

$$
\begin{array}{llll}
\mu_1(1) = 0. - 0.26, & \mu_2(1) = -0.56, & \mu_1(2) = -0.4, & \mu_2(2) = 0.22, \\
\sigma_{11}(1) = 0.37, & \sigma_{12}(1) = 0.46, & \sigma_{21}(1) = 0.59, & \sigma_{22}(1) = 0.59, \\
\sigma_{11}(2) = 0.47, & \sigma_{12}(2) = 0.31, & \sigma_{21}(2) = 0.28, & \sigma_{22}(2) = 0.68, \\
\lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50.
\end{array}
$$

We obtain $k_1 = 0.105684239991422$, $k_2 = 0.258372296578252$, $k_3 = 0.0888776438049554$, $k_4 = 0.0516269777215502$. Plugging these numbers to obtain the corresponding value functions. We verify that all the variational inequalities are satisfied. Finally, the graphs of these value functions are given :

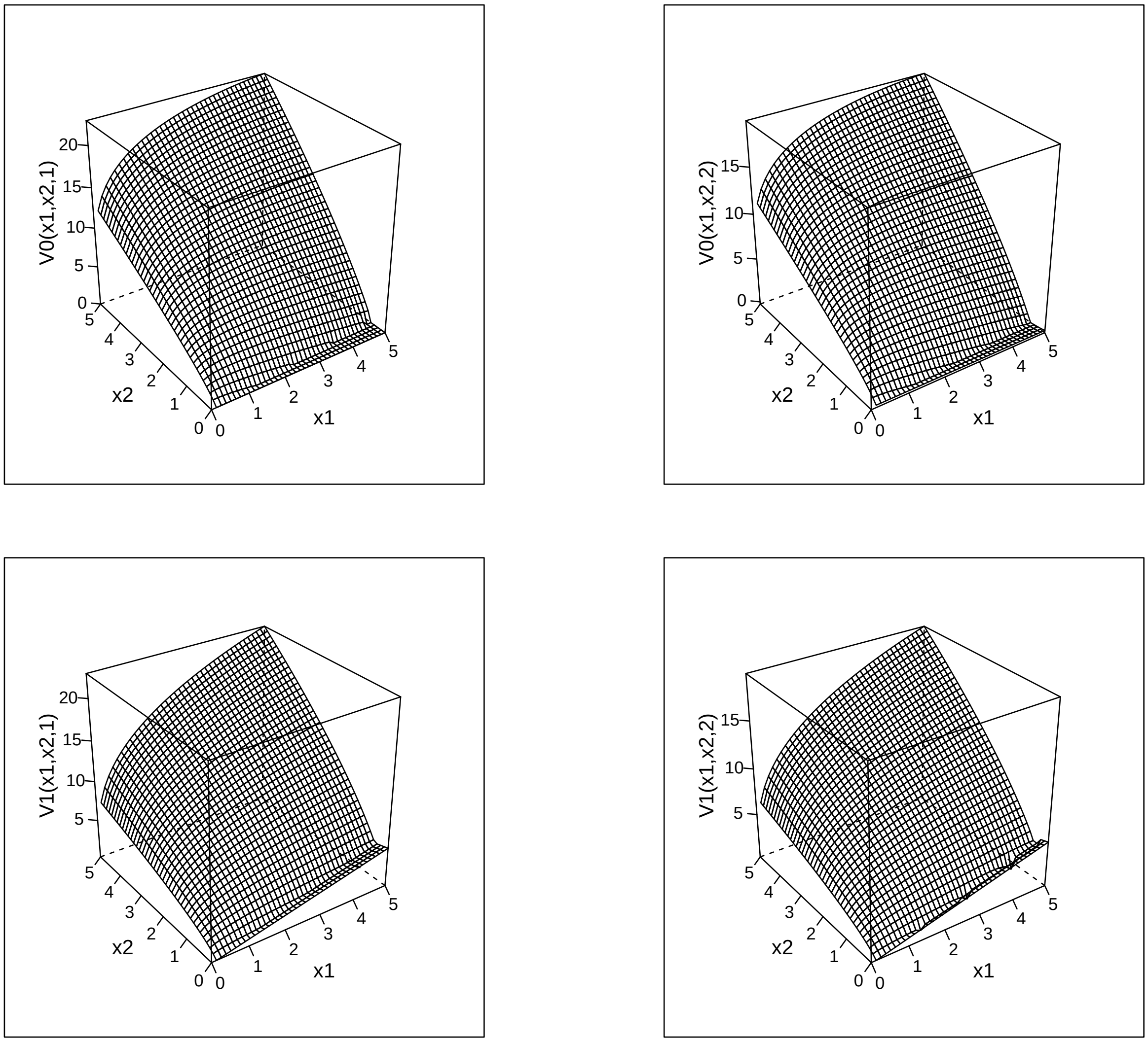

Figure 2.8: Value Functions $V_0(x_1, x_2, 1)$, $V_0(x_1, x_2, 2)$, $V_1(x_1, x_2, 1)$, and $V_1(x_1, x_2, 2)$

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