INSIGHT INTO STUDENTS' COVARIATIONAL REASONING WITH ASYMPTOTES IN RATIONAL FUNCTIONS

by

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(Under the Direction of Kevin Moore)

ABSTRACT

Present research on the teaching and learning of asymptotes at the high school level is limited to applications to higher level mathematics courses or rational function guides for secondary teachers, leaving a gap in the research on how students perceive asymptotes and productive ways to construct student thinking around asymptotes. In this thesis, I engaged three high school students in semi-structured task-based interviews to give insight into how they conceive and construct understandings of asymptotes, specifically within rational functions, as well as how they develop definitions of approaching using variation in quantities. I discuss the models of each student's thinking that resulted from analysis of these interviews. Finally, I present suggestions for future research to help fill this gap in research.

INDEX WORDS: Quantitative Reasoning, Covariational Reasoning, Asymptotes, Rational Functions

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CHAPTER 1

INTRODUCTION

The goal of teaching mathematics well is having, at the end of the course, a group of students who can model and reason with the world around them. I know I have succeeded if my students can both *do* the mathematics (have procedural fluency) *and* reason with the mathematics (achieve conceptual understanding). Students deepen their mathematical understandings when they can connect ideas across different mathematics courses. Building on the ability of mathematics educators to connect concepts to create critical thinkers and problem solvers, I want to teach in a way that allows students to see how the concepts they learn in my class relate to and interact with systems they see in their regular, day-to-day, non-mathematics-class lives. Students should believe they are problem solvers and see mathematics as a tool to solve problems.

Problems in the real world do not limit themselves to whole numbers or situations that tie neatly into brown paper packages tied up with string. Students must be able to grapple with abstract notions in mathematics like infinity and undefinedness (Davis et al., 2011, Contrill et al., 1996). We do not want students shying away from abstract concepts because they are not equipped with the reasoning skills to make sense of them. The beauty of mathematics is that within abstract concepts lie deeper connections to how the world works. I want students to see how the mathematics they learn folds and connects to create deep, satisfying understandings. I also want students to use these tools and ways of thinking once they have left our classrooms so they can think more critically of the world around them. Students can use mathematics to model and make sense of the world around them—but this cannot happen if we cannot give them the tools to do so. My study focuses on just one of these tools: reasoning with asymptotes in rational functions.

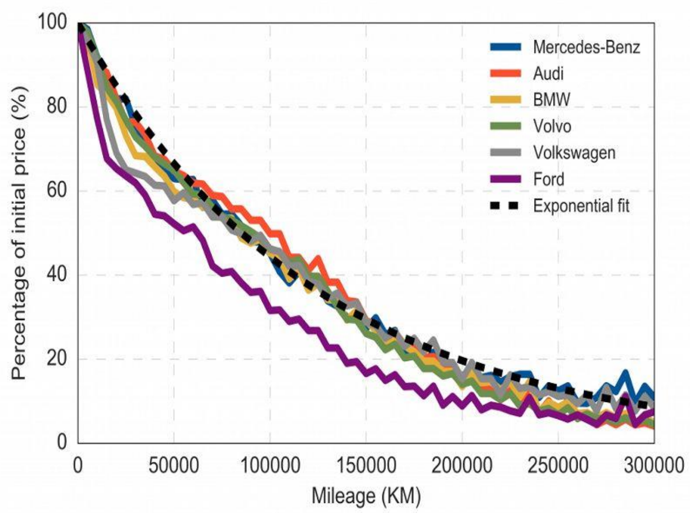
While the study of asymptotes is a chance for students to grapple with abstract mathematics at multiple levels, the concept of asymptote is often taught at a surface level (Hornsby & Cole, 1986). How students reason with the asymptotes of functions and build on those conceptions ripples into their understanding of higher levels of mathematics, including reasoning with limits in Calculus (Contrill et al., 1996). Studying asymptotes can be an avenue for students to consider function discontinuity, notions of infinity, and more. Studying asymptotes gives students a chance to grapple with higher levels of mathematics as they are challenged to think more abstractly about functions. Reasoning with asymptotes usually begins in early Algebra courses alongside the introduction of exponential functions. Teachers can push students to dig into more complex applications of asymptotes in later algebra courses when students begin learning increasingly complex function families, such as rational functions.

From exponential functions in Algebra 1, to early understandings of limits in calculus, to applications in the real world, asymptotic reasoning is vital in students' studies of mathematics as they begin to work with increasingly abstract concepts. Davis et al. (2011) observes, "The study of asymptotes constitutes one of the earliest and most significant encounters a high school student can have with infinity" (p. 1). In developing asymptotic reasoning, students are given the opportunity to explore abstract topics in the landscape of high school mathematics. In the classroom setting, asymptotic reasoning builds towards more advanced understandings of notions such as limits in calculus. Asymptotic reasoning deepens their understanding of why more complex functions, such as exponential functions, rational functions, trigonometric functions, behave the way that they do. If the goal of teaching mathematics well is to equip students with tools to model the world, developing students' asymptotic reasoning is a step towards that goal.

Students typically begin modeling with asymptotes when they learn exponential functions. Exponential functions are a foundational tool in modeling real-world phenomena that range from topics such as predicting population growth, understanding how a body absorbs medication, and financial planning through compound interest. These topics are then further explored when students expand from exponential functions to logarithmic functions (the inverse class of functions to that of exponential functions) in Algebra 2. In both exponential and logarithmic functions, students begin to explore the effects of varying kinds of covariation. For example, students who are studying the value of a car may observe that as mileage increases, not only does the car's value depreciate, but also decreasing is the amount by which the value decreases (see Figure 1). Although not always emphasized in common curricular approaches, students can use covariational reasoning (Carlson et al., 2002) to gain the ability to better understand the causes of characteristics unique to functions with asymptotes, such as the curved shape of exponential functions when graphed in the conventional Cartesian coordinate system.

Figure 1

Example of real-world situation with exponential decay.



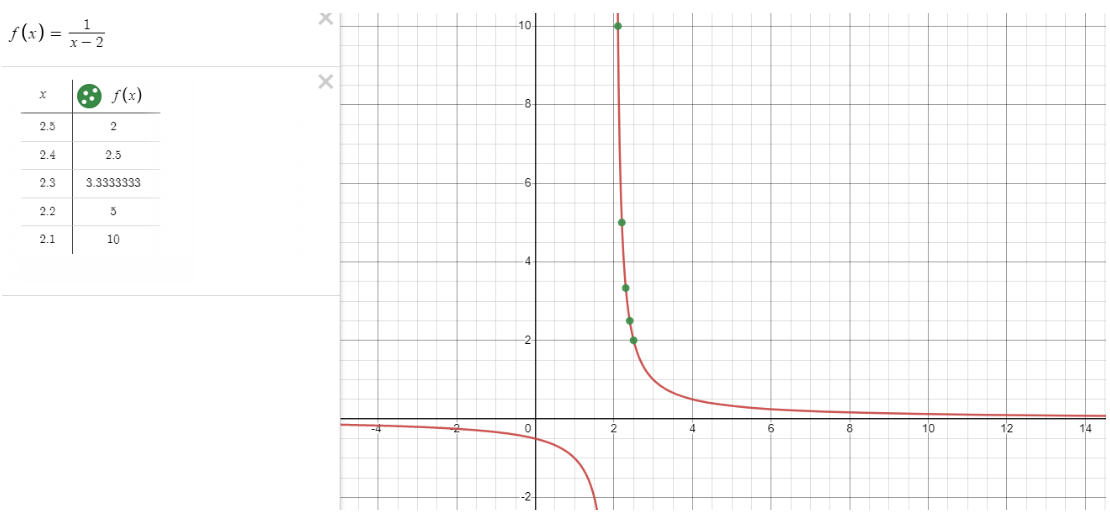
Note. Source *Graph of car values for various car manufacturers, n.d.*

Motivation and Study Goal

When I began teaching Algebra 2, I felt the same hesitation and confusion that students often feel when learning rational functions. I did not have a solid understanding of how these types of functions behaved, so I did not feel confident in reasoning through the complex shape of rational functions. Even though rational functions seemed daunting to teach, I loved seeing so many different parts of Algebra coming together to play key roles in this one function family. I wanted students to share that sense of awe in how the mathematical concepts they have been learning are connected in complex ways. I also wanted my students to feel like they could be successful in learning and explaining abstract mathematics—rational functions felt like the perfect way to do that. The breakthrough moment for my own understanding of rational functions came when I began to be more explicit about how the quantities of the function change in reference to the asymptote. For example, consider the function $f(x) = \frac{1}{x-2}$. If I consider input values that are greater than 2, but are decreasing and approaching 2, I see the function's output values are increasing. In fact, as I decrease toward 2 by equal amounts of change, the output values are increasing by increasing amounts. Said another way, the output is increasing by an increasing rate as the input values decrease towards 2. Examining the relationship between the input and output values motivates the shape of the function when graphed (See Figure 2 for a graph and table of values). Knowing how much clearer my own understanding of rational functions began using this kind of thinking, I wanted my students to share that same level of understanding.

Figure 2

Table and Graph (created in Desmos) of $f(x) = \frac{1}{x-2}$



Drawing on my experience, my goal is to explore, as few works have, students' understanding of asymptotic behavior in secondary-level mathematics courses, specifically regarding rational functions. While there is literature to help high school mathematics teachers effectively teach functions with asymptotes by providing lessons or ideas for applications in the classroom (Cook & Bossé, 2021; Davis et al., 2011; Kidron, 2020; Hornsby & Cole, 1986), these resources do not seem to be built from an understanding of how students reason with asymptotes organically. Since these studies do not explicitly engage with common student conceptions (or misconceptions), there exists a missing link between the valuable research within these studies and concrete ways we can help students in our classrooms today.

Studies like Cooke & Bossé (2019) and Giblin (1972) engage with mathematics which exist far beyond the understanding of secondary students and even some secondary teachers. These studies, which are supposedly meant for practical application, fall flat because there is not much focus on student understandings. These studies, and studies such as that by Hornsby Jr. and Cole (1986), define rational functions and give an overview of characteristics, but are not rooted in student understandings of vertical and horizontal asymptotes. The Hornsby Jr. and Cole (1986) study focuses on rational functions, specifically, because "[w]hile other functions such as exponential, logarithmic, and trigonometric reciprocal functions may also possess asymptotes, for the first time, the concept of asymptotes is formally introduced through rational functions" (Nair 2010).

Some studies also use rational functions as a vehicle for other study focuses, highlighting individual reasonings (Katalenić, Milin, & Čižmešija, 2020, 2023; Mpofu & Pournara, 2018). Many studies that focus student reasoning with asymptotes (Dahl, 2017; Mayes, 1993; Dick et al., 2022; Öçal, 2017) focus attention on student reasoning with graphs of rational functions, highlighting how students make connections between the algebraic and graphical representations of rational functions.

Research Questions

My interest in this research is to see how students reason with asymptotes in rational functions organically. Thus, my guiding research question is: In what ways do students conceptualize asymptotes in rational functions? More specifically, I want to better understand how students justify relationships between the input values and output values of rational functions, specifically when considering what happens when either quantity increases or decreases without bound. I discuss ways covariational reasoning between the input values and the output values can motivate more flexible definitions of vertical and horizontal asymptotes. Through student interviews and analyses of those interviews, I hope to gain insight into what kinds of reasonings students use to conceptualize asymptotic behavior in rational functions. The questions I hope to begin investigating through this research are (1) In what ways, if any, do students conceptualize asymptotes in rational functions using covariation of quantities? and (2) Does covariation play a role in students' definitions of boundlessness?

This thesis is written into the four following chapters. In Chapter 2, I discuss the existing research base on the teaching and learning of asymptotes and rational functions, as well as a conceptual analysis on covariational reasoning and asymptotes in rational functions. This conceptual analysis provides the basis of the tasks developed in my study. In Chapter 3, I outline my method of data collection, background of the students, and justification for the tasks. In Chapter 4, I outline three student interviews as they worked through a three-part task to gain insights into their covariational reasoning in rational functions. Finally, in Chapter 5, I discuss the implications of the results discussed in Chapter 4 as well as areas for further research.

CHAPTER 2

BACKGROUND

Literature Review

Compared to other topics that constitute secondary mathematics at the precalculus and function level, there is limited work published with a focus on student understanding of asymptotes (Mpofu & Pournara, 2018; Dahl, 2017; Katalenić, Milin, & Čižmešija, 2020). Published work for the teaching and learning of asymptotes falls into categories including connections to higher level mathematics, teaching guides, and implications of the use of technology when teaching rational functions. In this section, I synthesize available literature and illustrate why researchers, teachers, and students would benefit from additional information and research on this topic. Throughout this section, I discuss why my research, which focuses on student reasoning, fills a gap in the existing literature.

Concepts in mathematics build atop each another, and asymptotes are no exception. There is a subset of research that ties asymptotes to more complex concepts. Kidron (2010) studied one student's definition of horizontal asymptotes and tried to reconstruct that definition as it relates to the notion of limits. Cook & Bossé (2019) discussed the connection between asymptotes and topology, but endeavored to explain asymptotes using topology, rather than using asymptotes to explain topology. Bossé et al. (2008) studied curved asymptotes and non-linear asymptotes, using the method of dividing by the leading term to help motivate an "intuitive" definition of limits. Bossé et al. used the teaching and learning of asymptotes to highlight the steps of the mathematical process, thereby splitting the focus of the study in two. The problem that arises with connections to higher-level mathematics being the focus of research is that the findings are only valuable if the student already has a firm grasp of the advanced mathematical techniques being used. These studies connected asymptotes to higher level mathematics like limits and topology, but they evidence a lack of awareness in the gap of research in asymptotes. This research privileges the advanced mathematics or "on the other side of learning" perspective, and then tries to reverse engineer learning experiences based on that perspective. It is different to think about a topic from a developmental perspective that considered the mental operations available to a student, and then consider how those operations might be leveraged to have students construct some topic. I lay a foundation for exactly this work in my thesis.

Another category of research focuses on how practicing teachers can teach rational functions. Unfortunate gaps in literature remain here as well. Hornsby Jr. and Cole (1986) gave an overview of key topics in teaching rational functions but did not explore effective methods for how to teach rational functions to students. Rather, they provided an overview of mathematical concepts for teachers who may need a refresher on the mathematical concepts in order to feel confident in teaching rational functions. Davis et al. (2011) gave a practical lesson for teachers to explore rational functions with their students through a real-life application problem. Again, while research like this is useful for practicing teachers, there is still room for a look into how students understand these concepts using their mathematical backgrounds. Guidebook articles play an important and unique role in the published work on asymptotes. While these articles give teachers a way to better understand the concept of rational functions and asymptotes or concrete lessons to use in the classroom, they still leave unanswered the question of how students understand these concepts. Knowing how students learn or conceptualize the concept of asymptotes gives a foundation for creating mathematically appropriate tasks for teaching and learning, thus providing an appropriate, research-based foundation to articles which are focused on instructional strategies and tasks.

Within the literature base available on asymptotes, an important section highlights the use of technology, both in the teaching of asymptotes as well as how technology can be used to remedy so-called misconceptions in student understanding. These articles leaned on the dynamic nature of technology and how that dynamism lent itself to helping students create visual understandings of asymptotes. However, a reliance on technology alone could lead to weak concept definitions (Dahl, 2017). Mayes (1993) used the software Derive to help students construct definitions of asymptotes in rational functions. Students were shown different functions both algebraically and graphically and asked to form conjectures about the connections between the two forms of each function. Students then used the dynamic nature of the technology to test these conjectures and began to create definitions of asymptotes. Dick et al. (2022) focused on preservice teachers' understanding of how students were learning in a "technology mediated" lesson. While the focus of Dick et al. was not how students learned asymptotes, they did provide some insight into how the secondary students interacting with a Desmos task were conceptualizing asymptotes. The task used sliders to manipulate parameters for a given rational function. Using the sliders to change the parameters and see the changes in the graph of the function gave students the ability to create dynamic changes in the functions and see how these changes altered the graphs of the functions immediately. I add that while this use of technology can be useful in students seeing how the parameters affect the graph of the function, tasks like this could lead to static shape thinking, which I discuss later in this chapter. Similar to Mayes (1993), students could use this technology to create conceptualizations of vertical asymptotes.

Öçal (2017) highlighted how the use of technology to teach asymptotes leads to students creating more precise graphs. Öçal studied the precision of student sketches of graphs of three functions ($\frac{1}{x}, \ln(x), e^{x}$) when students were taught using either graphing technology or teacher sketches to visualize each graph. Students were less likely to have an asymptote misconception, or "have problems in approaching the arms of functions to the asymptotes during the sketching process," if they had been taught using the graphing software GeoGebra. Through student interviews, Öçal attributed these misconceptions to lack of precision on the part of the teacher or a lack of emphasis on precise graphing on exams due to a focus on solving. Collectively, the aforementioned studies illustrate the good and the bad of technology: its dynamism and precision can kickstart conceptions and remedy misconceptions, but its reliance on visual understanding can lead to weak concept definitions. Overall, these studies gave little attention to how students conceptualize asymptotic relationships in the covariation of the quantities. Although the aforementioned students centered on asymptotes, I note that they explored associations between changes in a graph and changes in a parameter value, or associations between features of a formula and features of a graph. As I describe in more detail later, this differs from a focus on students' meanings for asymptotes from a perspective that foregrounds the ways in which students conceive the underlying quantitative relationship that produces an asymptote.

In addition to the work mentioned above, several studies explored individuals' reasoning relevant to their meanings for asymptotes. Mpofu & Pournara (2018) described students as relying on “ritualized routines” to create graphs of functions. The students graphed the rational functions and created a table of values, but “their choices in *x*-values did not indicate the asymptotic behaviour of the function because they only chose integer values of *x*.” Mpofu & Pournara suggested this indicates the students memorized the shape of the graph of the rational function, but do not connect the coordination of values to the shape or the asymptotes. Katalenić, Čižmešija, & Šipuš (2023) saw a similar lack of asymptotic understanding in prospective mathematics teachers. Through a questionnaire on the notion of asymptotes, the researchers found the participants used different solution methods, including “reading-off” (the students use the graphs of the functions to make conclusions about the asymptotes) and “calculation” (the students used limits to calculate and justify the asymptotes). The same researchers (Katalenić, Šipuš, & Čižmešija, 2020) discussed how preservice teachers include asymptotes in their graphing of functions and curves, with the authors summarizing the methods they usually employ. Most preservice teachers used plotting points to create their graphs, but this method did not lead to all subjects acknowledging the existence of the asymptotes in the functions that were graphed.

Researchers, students, and teachers engage with asymptotes in different ways, as evidenced by the following studies that include asymptotes, not as their focus, but as the concept through which the researchers hope to learn more about their subjects. The engagement with asymptotes, both in rational functions as well as in other relationships, was used within research of other concepts. Jaafar & Lin (2017) discussed the utility of assessment of student learning in a calculus course, using assessments on asymptotes to better develop student understanding of various calculus concepts, such as limits or the chain rule. The pre-assessment used by Jaafar &

Lin revealed a lack of conceptual understanding among students, especially regarding horizontal and vertical asymptotes. After the learning period, some students could use the graphs of the functions and limits to define horizontal and vertical asymptotes. While the Sedaghatjou (2018) studied a blind student's ability to communicate mathematical ideas, having him create graphs of rational functions with raised paper and WikkiStix and then describe the creation of those graphs. While the student's asymptotic reasoning was not the focus of the study, there was some insight into how the student reasoned with the shape of the graph and the asymptote. The student would describe the graph of the function as "hugging" the vertical asymptote, using his arms to show this relationship. After looking at these studies, it is safe to conclude asymptotes are a popular and well-enough-known subject since they provide a suitable testing ground in which researchers will often do their actual research.

While the literature shows ways to connect asymptotic reasoning to advanced mathematics, gives suggestions on how to teach asymptotes in the high school classroom, or discusses the benefits of using technology in the teaching and learning of asymptotes, one question remains: how do students organically reason with asymptotes? As I have established, relatively little research questions how students make sense of asymptotes beyond students drawing connections between the rational functions and the graphs they produce. The research that does exist, instead, tends to cover advanced mathematics instead of early mathematics courses where students are first becoming aware of asymptotes. The research that concerns student learning in early math courses is valuable but focus attention on how to remedy student misconceptions of asymptotes using technology. Students may have a working knowledge of practice and procedure, but these studies seldom question how students make sense (or struggle to make sense) of asymptotes beyond a connection to the graphs of the functions.

Given most students are first exposed to asymptotes in Algebra 1 and then, up to two years later, revisit asymptotes in Algebra 2, the lack of applicable research on this topic at the high school level is a disservice to teachers and students. My study aims to begin a conversation about this gap in the literature and how we may begin to close it. I specifically accomplish two goals. First, I present a conceptual analysis of the role of covariational reasoning in understanding asymptotic relationships in rational functions. Second, I explore the kinds of reasonings students use to describe asymptotic behavior in rational functions and discuss the implications of the different kinds of reasonings students present, with a focus on covariational reasoning and shape thinking.

Conceptual Analysis

Quantitative and Covariational Reasoning

Two key skills for students to successfully reason with functions, and mathematics in general, are quantitative and covariational reasoning (Madison et al., 2015). Quantitative reasoning (QR) is how students conceptualize the measurable parts of a problem or situation (Thompson, 2011). Covariational reasoning (CR) is how students think about multiple changing quantities varying together (Carlson et al., 2002). In other words, QR broadly entails measurement comparisons of quantities, whether between static or dynamic states, whereas CR is a subset of QR specific to reasoning about how different quantities change in tandem. Both QR and CR open the door for constructing productive definitions of vertical and horizontal asymptotes through a lens of covariation. Using covariation to define vertical and horizontal asymptotes in rational functions creates a clear relationship between the input and output values of the function as well as highlights that the behavior of asymptotes, whether vertical or horizontal, can be thought of in a mathematically equivalent way; the difference between the two is which quantity is being considered independent. I return to this point in a subsequent section.

To illustrate how to use QR and CR to understand a real-world context, consider a savings account that results in a $2 increase each month, with the interest accrued being proportional within the month. Thus, the account gives a $2 increase of the original amount a month with a scaler increase within the month necessary to obtain that $2 across the month. Using QR, we can see the quantities are time (in months) and the amount of money saved. Using directional CR (Carlson et al., 2002), we see that as the number of months passed is increasing, the amount of money saved is increasing. Directional CR (Carlson et al., 2002) allows us to be specific in how number of months passed and money saved change *together*. We could simply say, "as time (in months) increases, the amount of money saved increases." Expanding to amounts of change CR (Carlson et al., 2002) deepens the understanding of *how* the amount of money is increasing. We know that for each month that passes, the student is saving an additional $2. Understanding this additive change of 2 underscores the role of QR with respect to CR, since it entails a comparison in quantities' states. Considering the constant additive increase of $2, we infer for unit changes in time, the amounts of change in money saved are equal.

What would happen if the student opened a savings account that accrued 2% interest on the amount of money in the account at the end of the month? The QR and CR can be approached in the same way. Now, there is a multiplicative change resulting in an increase of 0.02. With the latter account, the change is applied to the balance within the account through the month, increasing the total amount being accrued. The changing quantities are still time and amount saved, and both are still increasing. However, applying directional and amounts of change CR, we can see that how the quantities change together is different from the first savings account. Now, for equal length increases in time, the amounts of change of money saved increases (Carlson et al., 2002). Each time we accrue interest, we are adding more to the account than we did the last time. The difference in equal amounts of change versus increasing amounts of change is illustrated by the solid vertical in the graphs below (see Figure 4 and Figure 5).

In this chapter, I use the framework from Carlson et al. (2002) to explain covariational reasoning. Their framework aims to explain how specific behaviors are indicative of certain mental actions (see Figure 3). In the next few pages, I provide an explanation of the asymptotic nature of rational functions using covariational reasoning. I continue to touch on this framework in the analysis of my data. The following is an example of how to use this framework.

Figure 3

Carlson et al. (2002) Covariational Reasoning Table of Mental Actions

Table 1
Mental Actions of the Covariation Framework

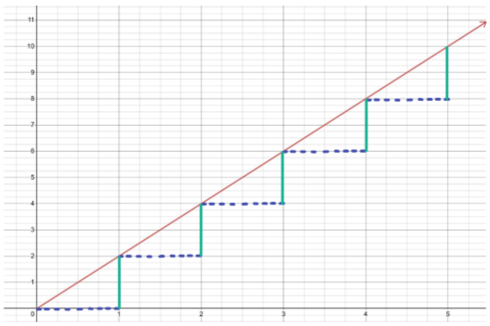
Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

To illustrate the application of the Covariation framework, let us go back to the first savings account example. Suppose there is a savings account in which one will save $2 each month. Mental Action 1 requires students to show an awareness of one quantity as the other quantity changes. This could be as time increases, amount saved increases. Mental Action 2 coordinates the directional change of the two quantities. As more time passes, the amount of money saved increases. Mental Action 3 considers the amounts of change happening for increases in time. For example, there is more money saved for the passing of two months than in the passing of one month. Alternatively, the student might coordinate amounts of change in one quantity when considering uniform changes in the other quantity. In this example, for each month that passes, the amount of money saved is equal ($2). Mental Action extends beyond MA3 by incorporating attention to rates of change and understanding that for any intervals of input, the average rate of change can be determined and compared. Finally, Mental Action 5 considers the instantaneous rate of change over the entire domain of the relationship. Here, the instantaneous rate of change would be a vertical line at 2.

Strength in both QR and CR gives students a sturdy foundation of reasoning when faced with more complex functions or relationships. Madison et al. (2015) stress, "Quantitative and covariational reasoning are two foundational ways of thinking that students engage in when *constructing, interpreting, and using* functions *meaningfully*" (pg. 58, emphasis mine), aligning themselves with similar cases made by several researchers, such as Weber et al. (2014), Moore (2014), Ellis (2007, 2009) and Paoletti & Moore (2018). Using quantitative reasoning as a mathematical orientation, rather than a method of solving, to conceptualize the varying quantities in functions bridges student understanding towards covariation in the quantities (Weber et al., 2014), rather than static shape reasoning (Paoletti & Moore, 2018).

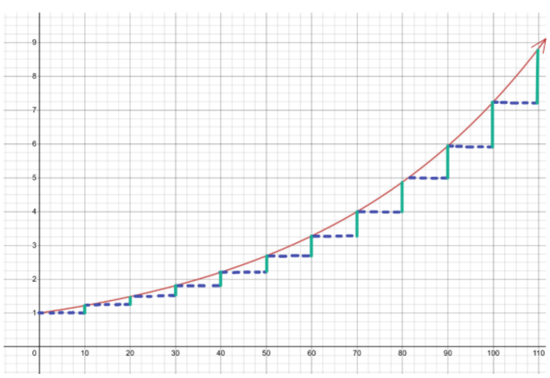
Figure 4

Graph (created in Desmos and annotated by me) of $y = 2x$



Note: x is the time (in months), and y is the amount of money saved. The dotted horizontal lines represent equal increases in time passed (1 month) and the solid vertical lines represent the increase in money saved after 1 month. Notice the solid verticals are equal in length, showing equals amounts of change.

Figure 5



Graph (created in Desmos and annotated by me) of $y = (1.02)^x$

Note: x is the time (in months) and y is the amount of money saved. The dotted horizontal lines represent equal increases in time passed (10 months) and the solid vertical lines represents the increase in money saved for after 10-month intervals. Notice the solid verticals are increasing in length, showing increasing amounts of change.

Several researchers, including Ellis (2009, 2007), Thompson and Carlson (2017), and Paoletti and Moore (2018) have illustrated that meanings for functions can be productively engendered through covariational and quantitative reasoning. In fact, Paoletti and Moore (2018) emphasized that developing students' understanding of functions through CR puts the "horse" in front of the "cart" of student reasoning, and they subsequently argued that educators may be overemphasizing function in US curriculum (Paoletti et al., in press). That is, the importance of students' covariational and quantitative reasoning need not be necessarily tied to the function concept, and instead should be valued in and of itself (Paoletti et al., in press).

The savings account example is a simple example of how two quantities change together, yet there is much depth to understanding the example. Consider how much more complex the situation would become if we began to consider quantities and how they change in reciprocal relationships, like those in rational functions.

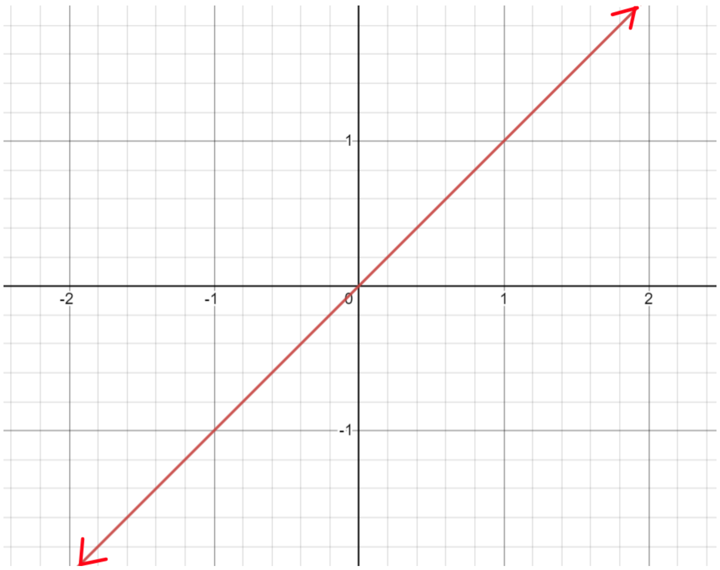
Shape Thinking

Another facet of student conceptualization of graphing is shape thinking (Moore & Thompson, 2015). Shape thinking categorizes how students make sense of graphs by sorting student conceptualizations of graphs into two ways of thinking: static and emergent shape thinking. Each of these two ways of thinking leads students to different kinds of meanings for what graphs represent. Students use static shape thinking when their primary meanings and actions with a graph are be based on "perceptual cues" and "the perceptual shape" of the graph (Moore, K. C., & Thompson, P. W., 2015). Static shape thinking is rooted in a conception of the graph as an object, and thus not directly consisting of relationship between quantities. Emergent shape thinking is when a student conceives a graph as both the graph itself as well as the covariation that leads to the graph; it involves understanding that a graph is a product of covariation, and thus the shape of the graph is a consequence of that covariation. Both static and emergent shape thinking have implications on how students conceive graphical representations of relationships (Moore et al., 2019).

Static shape thinking relies on students using the graph as a static image or a line with non-relational attributes. Students using static shape thinking conceive of relationship in the graph like that of physically manipulating a wire (Moore, 2021). An example of static shape thinking applied to Figure 6 could be describe the slope as positive because the slope is going up from left to right. Although this is technically correct for the graph given, if the same shape were presented in a non-conventional coordinate system (e.g., positive y-values oriented downward), an example of static shape thinking would be still conceiving the graph as having a positive slope because it is going up from left to right. The slope of the graph is attributed to the shape of the graph itself. Another example of static shape thinking within rational functions is a student describing the graph of the parent rational functions as being curved because each side is getting closer to each axis. The approach of the two axes is motivated by the shape of the graph, rather than the changing of the quantities, as I will describe later in this chapter.

Figure 6

Graph of y = x



Note: Created by me in Desmos.

Emergent shape thinking relies on the ability of students to conceive how quantities are changing. Emergent shape thinking lends itself to students using CR to reason about how those changes can be seen in the shape of the graph representing the relationship of the quantities. Students understand that the graph is the result of a tracing of the covariation between the quantities of the relationship. An example of emergent shape thinking regarding the slope of Figure 6 could be a student reasoning the positive slope of the graph is a result of each increase in the input value coordinating with an increase in the output value. A student reasoning emergently could then use the same reasoning to conclude that the same shape in a non-conventional coordinate system (e.g., positive *y*-values oriented downward) has a negative slope despite sloping upward left to right. Regarding rational functions, students can apply emergent shape thinking to motivate the relationships between the quantities that lead to the presence of asymptotes. For example, consider the parent rational function, $y = \frac{1}{x}$ (see Figure 7). Using emergent shape reasoning, we see that for increases in the input values, the output values decrease. Using amounts of change reasoning, for equal amounts of increase in the input values, the output values decrease by decreasing amounts of change. These decreasing amounts of change mean as the input value increases, the output values decrease towards 0. Graphically, we see this decrease towards 0 as the horizontal asymptote at $y = 0$.

Figure 7

Graph of $y = \frac{1}{x}$

Note: Created by me in Desmos.

Rational Functions

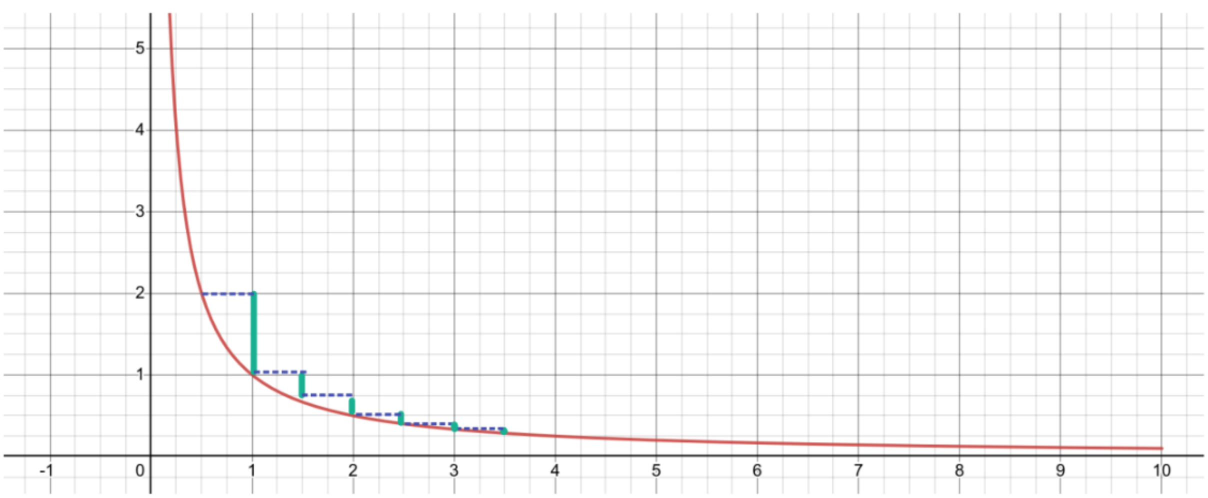
Formally, rational functions are some function $f(x) = \frac{g(x)}{q(x)}$, where $q(x)$ is a non-constant polynomial. In terms of CR, at least one of the quantities that is changing is in the denominator of the fraction forming the function's output value. The influence of variations of

$q(x)$ on variations in $f(x)$ is a reciprocal effect. An example of a reciprocal effect on changing quantities would be that if the quantity in the denominator increases, the output of the rational expression decreases (assuming a constant value in the numerator). The reciprocal nature of rational functions opens the door for all kinds of complex characteristics to explore. Not only does CR help to break down how the quantities change, considering the reciprocity of the changes, it can also help students explain characteristics of the functions, such as a value being undefined, the occurrence of vertical or horizontal asymptotes, and the end behavior. Students are often confused when asked to explain why asymptotes occur and what the difference is between vertical and horizontal asymptotes. Adding to their confusion, students are rarely given real-world scenarios that necessitate a rational function. This absence of application problems means that students are not given an opportunity to use QR or CR in meaningful ways.

Using an example to further illustrate CR's relevance to rational functions, assume 1 gallon of liquid flavoring is poured into a large vat. Some amount of water (in gallons) will be added to the vat and mixed with the flavoring to produce a beverage to be bottled and sold. How could we describe the ratio of flavoring to water in terms of how much water is poured into the vat? Using our QR from before, we can see the measurable quantities are the amount of water being poured into the vat (a varying quantity), the amount of flavoring in the vat (a fixed quantity), and the ratio of flavoring to water added (a varying quantity). We can write this relationship as $R = \frac{1}{x}$, where R is the ratio of flavoring to water and x is the amount of water added (measured in gallons). Since the amount of flavoring is constant, as the amount of water increases, the ratio of flavoring to water *decreases*. In other words, the more water added, the more diluted the flavoring will become. In this example, we can see how having the changing quantity in the denominator causes a reciprocal effect (as explained above) on how the ratio changes with relation to the amount of water increasing. The ratio itself should be conceptualized as a measurable quantity, showing the need for QR in rational functions as well. Using QR, we describe the ratio as a multiplicative comparison between flavoring and water. In other words, there is R times as much flavoring as there is water in the mixture[1]. Incorporating CR, not only is the ratio decreasing for equal amounts of increases of water, the ratio of flavoring to water is decreasing at a decreasing rate. Meaning, as more water is added in equal amounts, the flavoring is becoming less concentrated by decreasing amounts. For a visual representation of these changes, see Figure 8**.** This relationship will be discussed in more detail when motivating horizontal asymptotes.

Figure 8

Graph (created in Desmos and annotated by me) of $R = \frac{1}{x}$



[1] While I describe the ratio as the multiplicative relationship between flavoring and water, the ratio can represent multiple meanings in the relationship between flavoring and water. Another interpretation of the ratio could be: What percentage of a smaller amount would the flavoring be compared to the water if mixed well.

Note: x is the amount of water added to a mixture and R is the ratio of flavoring to water (assuming the maximum amount of water to be added is 10). The dotted horizontal lines represent equal increases in water added (.5 gallons of water) and the solid vertical lines represent the decrease in the ratio for .5 gallon increases of water. Notice the solid verticals are decreasing in length, showing decreasing amounts of change.

Using Covariation to Construct Asymptotes

Transitioning our attention to asymptotes, we can use CR to construct vertical asymptotes in rational functions. Still using the ratio of flavoring to water scenario, we see the ratio changes as the amount of water added increases. We can also consider how the ratio changes as the amount of water added decreases. Before doing so, let us consider what happens if we were to add 0 gallons of water to the flavoring. Here we come to an issue. If there is no water added, there is no ratio to be considered, resulting in an undefined situation. In other words, when x (the amount of water added in gallons) is 0, the ratio is undefined and thus the function is undefined.

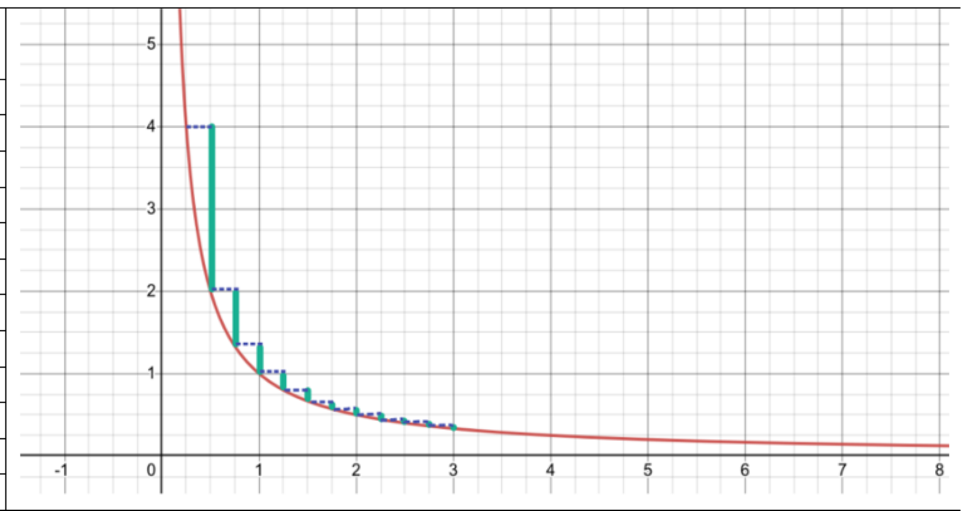
Using CR, let us now consider how the ratio varies as we approach to the value of the water added that causes the function to be undefined. In the context, we are considering what happens as we imagine adding less and less water. If less water is being added, the ratio of flavoring to water increases. Using the same reasoning as above, for equal amounts of less water being added, the ratio is increasing at an increasing rate (see Figure 9). As the amount of water being added decreases closer and closer to 0 (the value that would make the ratio undefined) by equal amounts of change, the ratio increases by more and more. In other words, as less water is added, the mixture gets more concentrated. However, there must always be some amount of water to be able to consider this a mixture of flavoring and water, so the amount of water added

(x) will not be 0. Furthermore, the ratio will continue to be a larger and larger value (or will increase without bound) as the amount of water added approaches 0. A defining feature of vertical and horizontal asymptotes is that as one quantity approaches a fixed value, the other quantity not only continues to increase or decrease, but it does so without bound[2]. The fact that the ratio is increasing without bound as x approaches 0 can be seen in the shape of the graph of the function (Figures 8 and 9). This reasoning highlights the mathematical properties that form the vertical asymptote of the function.

Figure 9

Table and Graph (created in Desmos and annotated by me) of $R = \frac{1}{x}$

x	R	Change in R
.25	4	--
.5	2	-2
.75	1.33	-0.67
1	1	-0.33
1.25	0.8	-0.2
1.5	0.67	-0.13
1.75	0.57	-0.10
2	0.5	-0.07
2.25	0.44	-0.06
2.5	0.4	-0.04
2.75	0.36	-0.04
3	0.33	-0.03



Note: x is the amount of water added to a mixture and R is the ratio of flavoring to water. The dotted horizontal lines represent equal changes in water added (.25 gallons of water) and the solid vertical lines represent the decrease in the ratio for .25 gallon increases of water. Notice the

[2] For the purposes of this thesis, I am restricting my focus on monotonic relationships as it relates to their asymptotes.

solid verticals are increasing in length as less water is added, or as the amount of water approaches 0, showing increasing amounts of change.

The key to understanding the relationship between the input values and output values that form a vertical asymptote is embedded in how the quantities vary together. As we saw above, as the input values of the function approached 0 from values larger than 0, the output values increase by increasing amounts. Emphasizing that the output values are increasing by increasingly large amounts of change implies that the output values increase without bound.

Ignoring context, we can think about the relationship between two variables x and y as defined by the function above. Doing so, we can also consider x as increasing and approaching 0 (i.e., becoming less negative). As the input values get closer to 0 (the value that would make the ratio undefined) by equal amounts of change, the output values decrease (i.e., become more negative). For equal amounts of change in the input values, the output values are decreasing by larger amounts of change. The output values will continue to be smaller and smaller (or will decrease without bound) as the input values get closer and closer to 0 (see Figure 10).

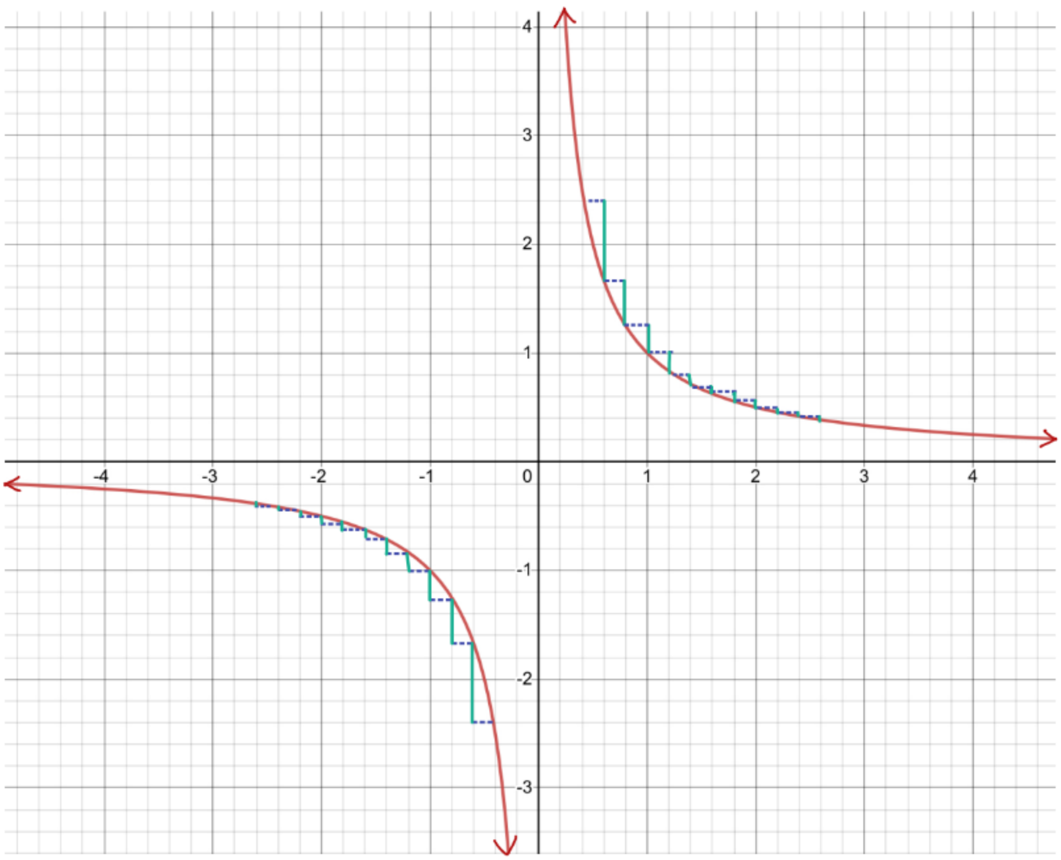
Generalizing beyond the beverage manufacturing example, we can use covariation to define a vertical asymptote. We can say a vertical asymptote occurs when, as the input of a function increases or decreases towards a specific value a, the function's output value increases or decreases without bound[3]. Furthermore, as the input values approach a by equal amounts of increase or decrease, the function's output values are increasing or decreasing by increasing amounts of change (in magnitude). Using covariation as the foundation for the definition of vertical asymptotes clarifies the difference between a vertical asymptote and a hole. While both

[3] This term implies a conventional Cartesian coordinate system, with the inputs and output values represented horizontally and vertically, respectively.

occur at input values where a function is undefined, the amounts of change in the output values of the function do not increase or decrease without bound as the input values increase or decrease towards the input value at which the hole occurs. The boundlessness of the growth in the output values is unique to vertical asymptotes.

Figure 10

Graph (created in Desmos and annotated by me) of $f(x) = \frac{1}{x}$



Note: the dotted horizontal lines represent equal changes in the input and the solid vertical lines represent the amounts of change in the output value. Notice the solid verticals are growing in length as the input values approach 0, showing amounts of change that are increasing without bound.

A key aspect of using QR and CR effectively is the ability to apply the reasoning in compatible ways, even if we change the notion of independent and dependent. Paoletti et al. (2018) highlight the work of Thompson and Carlson (2017) to show the need for flexibility in what one considers independent or dependent in the context of the problem; many situations do not include an inherent dependence between quantities. "What is independent and what is dependent will depend entirely on the person's conception of the situation and which way they envision dependence, if they envision dependence at all" (Thompson and Carlson, 2017, quoted in Paoletti et al., 2018, pg. 39). This begs the question: What if we considered equal amounts of change for the function's output, effectively considering a function's output value the independent variable?

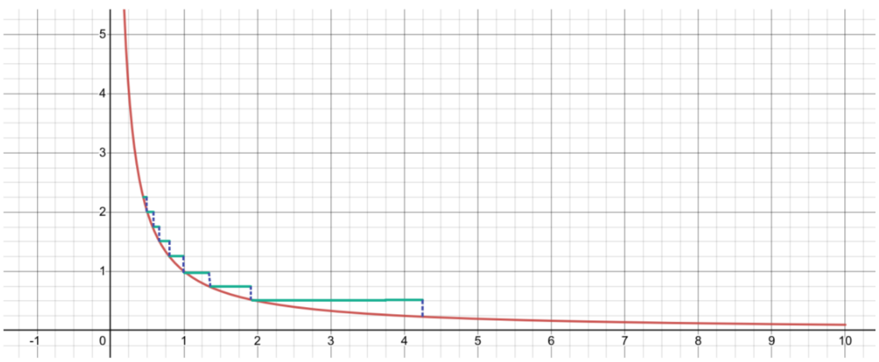
Looking back at the ratio of flavoring to water, we already saw that we cannot have an input of 0, because that would mean there is no water being added, so there is no ratio to consider. Changing our focus from the input quantity (water added) to the output quantity (the ratio of flavoring to water), we now consider what must happen for the ratio of flavoring and water to be 0 ($R(x) = 0$)[4]. The only way we would have a ratio that is equal to 0 is having no flavoring. Much like the value of 0 water added results in an undefined ratio, there is no amount of water added that can result in a concentration of 0; no matter how much water is added, some amount of flavoring is in the mixture. The relationship is thus undefined for $R(x) = 0.$ But, we can conclude that as the ratio of flavoring to water increases, the amount of water added must decrease. Also, as the ratio decreases in equal amounts, the amount of water (x) added is

[4] Such a context illustrates the importance of reasoning about a function's output as the independent variable. In practical terms, the beverage manufacturer might be interested in making a specific concentration, meaning we must figure out how much water to add to achieve that specific concentration.

increasing. More specifically, if we consider the ratio of flavoring to water to be decreasing by equal amounts of change, the amount of water added is increasing by increasing amounts of change (see Figure 11). As the ratio decreases to 0, the amount of water added to create that concentration increases by increasing amounts of change. Using the beverage manufacturing example of water added and the ratio of flavoring to water, we see similarities in the CR of vertical and horizontal asymptote.

Figure 11

Graph (created in Desmos and annotated by me) of $R = \frac{1}{x}$



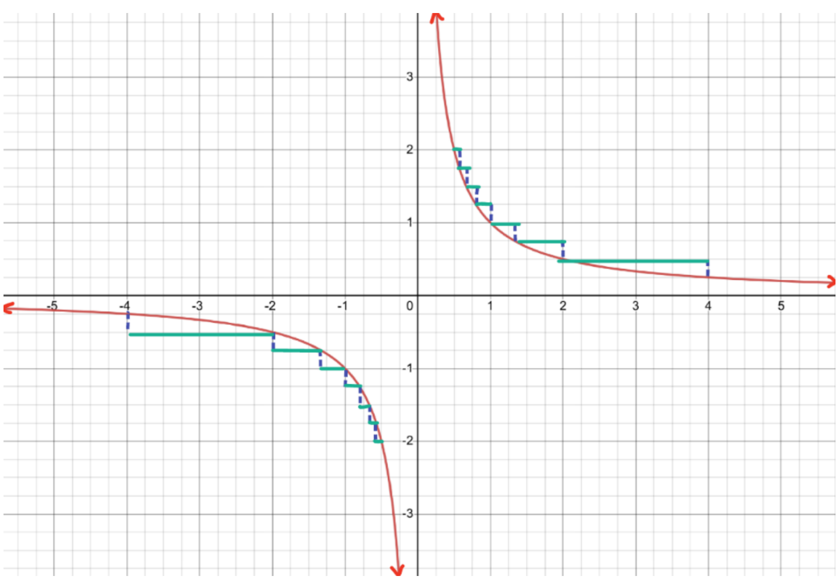
Note: x is the amount of water added to a mixture and R is the ratio of flavoring to water. The dotted vertical lines represent equal changes in the ratio (changes of 0.25) and the solid horizontal lines represent the increase in water added for the ratio to decrease by 0.25. Notice the solid horizontal lines are increasing in length, showing increasing amounts of change.

Using covariation, the reasoning underlying a horizontal asymptote is compatible with the reasoning underlying vertical asymptotes above. The only change is shifting our independent variable to be the output values of the function (i.e., the input values to the inverse function). With reference to the original function, as the output values of the function decrease towards 0 from values larger than 0, the input values increase at an increasing rate. In other words, for equal decreases in the output values as they approach 0, the input value is increasing by increasingly large amounts; the input values are increasing without bound.

Ignoring the context, the same reasoning can be applied to the output values as the function increases towards 0. As the output values increase to 0 by equal amounts of change, the input values decrease by increasing amounts of change. For equal amounts of change in the output values, the input values are decreasing by larger amounts of change. The input values will continue to decrease without bound as the output values increase to 0 (see Figure 12). Using covariation to define a horizontal asymptote, we can say a horizontal asymptote occurs when, as the output values of a function increase or decrease towards a specific value a by equal amounts of increase or decrease, the function's input values increase or decrease by growing (or boundless) amounts of change.

Figure 12

Graph (created in Desmos and annotated by me) of $f(x) = \frac{1}{x}$



Note: The dotted vertical lines represent equal changes in the output and the solid horizontal lines represent the amounts of change in the input values. Notice the solid horizontal lines are growing in length as the output values approach 0, showing amounts of change that are increasing without bound.

Two Methods of Determining Values at which Asymptotes Occur

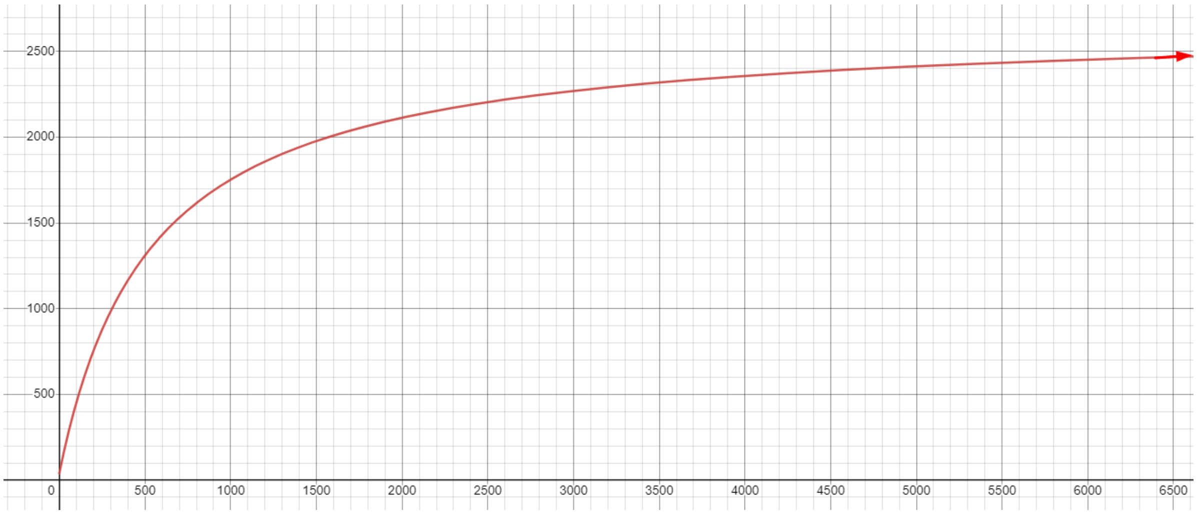
When finding the value of vertical asymptotes, we use the input values that cause the function to be undefined, which are typically input values that lead to a division by 0 in determining the corresponding output value. Applying compatible ways of thinking, horizontal asymptotes, then, occur at output values that cause the input values to be undefined, which are typically output values that lead to a division by 0 in determining the corresponding input values. Using values of one quantity that cause the other quantity to be undefined to find the asymptotes of a rational function is the method most often used for finding vertical asymptotes in current curricula (Hornsby Jr. & Cole, 1986). This method can lead to students having action conception of finding asymptotes (Cottrill et al., 1996), reducing finding vertical asymptotes to the step "make the denominator equal 0." This method also does not lend itself to easily considering the output value as the independent variable, a skill with which many students struggle. Thus, I present a second method that lends itself to easily switching between the input value and output value. This second method uses the long-term behavior of one quantity to look for increasing amounts of change in the other quantity, leading to the second quantity increasing (or decreasing) without bound.

In the following example, we can use long-term behavior and amounts of change to motivate an asymptote without the need for finding an undefined value. Assume the function

$P(t) = \frac{10(2t+16)}{0.0075t+4}$ models the growth of the population of deer on a research reservation, where t is time in years and $P(t)$ is the population after time as passed (Figure 13a). We are using QR here to define these measurable quantities. Directional CR highlights that as time passes, the population of deer increases (Carlson et al., 2002). Applying the definition above, a horizontal asymptote will occur at the output value such that, as the input values are increasing (or decreasing) by increasing amounts of change, the output values increase or decrease towards a specific value. In other words, for equal amounts of change in time, does the population of deer decrease towards a value? If time increases without bound, the population of deer increases, but as we start to consider boundless increase of time, we see the population values increases by decreasing amounts (see Figure 13b). In other words, the amounts of change in the population decreases in the *long-term* passing of time.

Figure 13a

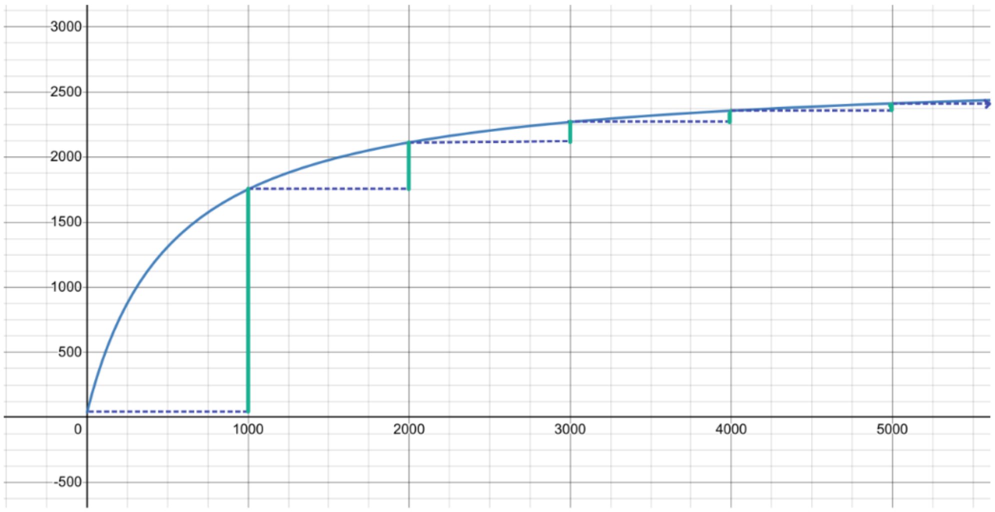
Graph (created in Desmos) of $P(t) = \frac{10(2t+16)}{0.0075t+4}$



Note: t is time in years and $P(t)$ is the population of deer on the reservation.

Figure 13b

Graph (created in Desmos and annotated by me) of $P(t) = \frac{10(2t+16)}{0.0075t+4}$



Note: t is time in years. The dotted horizontal lines represent large equal changes in time output and the solid vertical lines represent the amounts of change in the deer population values. Notice the solid horizontal lines are decreasing when we consider large values of time.

While undefinedness helps us determine where an asymptote could occur, we must consider the long-term behavior of the input and output values. The long-term behavior as the value approaches the potential asymptote must match the CR relationships, making CR a key part in determining whether a value truly defines an asymptote.

Long-term behavior is the boundless increase or decrease of either the input (or the output value) and how the other value behaves for that boundless increase or decrease. When we use CR in tandem with long-term behavior, the value of the asymptote is more clearly defined than when we only rely on method one (looking for values that cause undefinedness). Finding asymptotes is about identifying the undefined values (the input values that result in undefined output values, and the output values that result in undefined input values) and then exploring the long-term behavior or end-behavior to see if the CR is such it satisfies the definitions above.

We can use approximation of the function in the long-term increases or decreases in either value to consider how function behaves for boundless increase or decrease in the input and output values. To show how to use long-term behavior to find a horizontal asymptote, consider the function $m(x) = \frac{2x}{x^2-4}$ (Figure 14). First, only looking at the numerator, as x increases without bound, $2x$ will increase without bound. In the denominator, as x increases without bound, $x^2 - 4$ will also increase without bound, but it will increase by larger amounts than the numerator. Not only that, but for extremely large values of x, the -4 will have less of an impact on the overall value of $x^2 - 4$ when compared to the impact that x^2 will have on the output values of the function. If we are only considering the long-term behavior of x, the expression $\frac{2x}{x^2}$ can be used as a model for the behavior of $m(x)$. For boundless increases in the input value, the numerator will always be divided by a larger value, and the difference between the numerator and denominator will continue to increase as x increases. Connecting to directional CR, as x increases without bound, the ratio defined by $m(x)$ will decrease towards 0. If we consider amounts of change CR, for equal amounts of increase in the x-values, the ratio with decrease by increasing amounts of change.

Extending to Slant Asymptotes

What if we change the relationship between the numerator and the denominator? We can change the function above slightly to $g(x) = \frac{2x^2}{x^2-4}$ (Figure 15). There are two input values that result in a division of 0 in determining corresponding output value, causing the function to be undefined, $x = 2$ and $x = -2$. Looking to the long-term behavior of the function, as the input values decrease to -2, the output values increase by decreasing amounts of change. Also, as the input values increase to 2, the output values decrease by increasing amounts of change. Thus, for boundless decreases in the output values, the input values decrease to -2 and increase to 2. We can also consider boundless increases in the output values. As the input values increase to -2, the output values increase by increasing amounts of change. Also, as the input values decrease to 2, the output values increase by increasing amounts of change. Thus, when the output values increase without bound, the input values decrease to -2 and increase to 2.

Turning our attention to the long-term behavior of the input values, we can use similar reasoning to determine and justify the horizontal asymptote. The output value that causes the input values to be undefined (leading to a division by 0 in determining the corresponding input values) is 2. Consider the long-term behavior as the input values increase or decrease without bound, we want to see if the function's output values approach 2. First, let us focus on what happens as x increases without bound. Only looking at the numerator, as x increases without bound, $2x^2$ will increase without bound. In the denominator, as x increases without bound, $x^2 - 4$ will also increase without bound. Here we see a slightly different relationship between the numerator and denominator. For boundless increases in the input value, the numerator will always be divided by a value that is almost half its value and the output value approaches being two times as large as the input value[5]. Using long-term behavior, we can see that for equal amounts of change in the functions output, the function's input decreases by increasing amounts towards 2.

[5] This is QR with a dash of CR in that you are doing consecutive static comparisons of states.

Figure 14

Graph (created in Desmos) of $m(x) = \frac{2x}{x^2-4}$

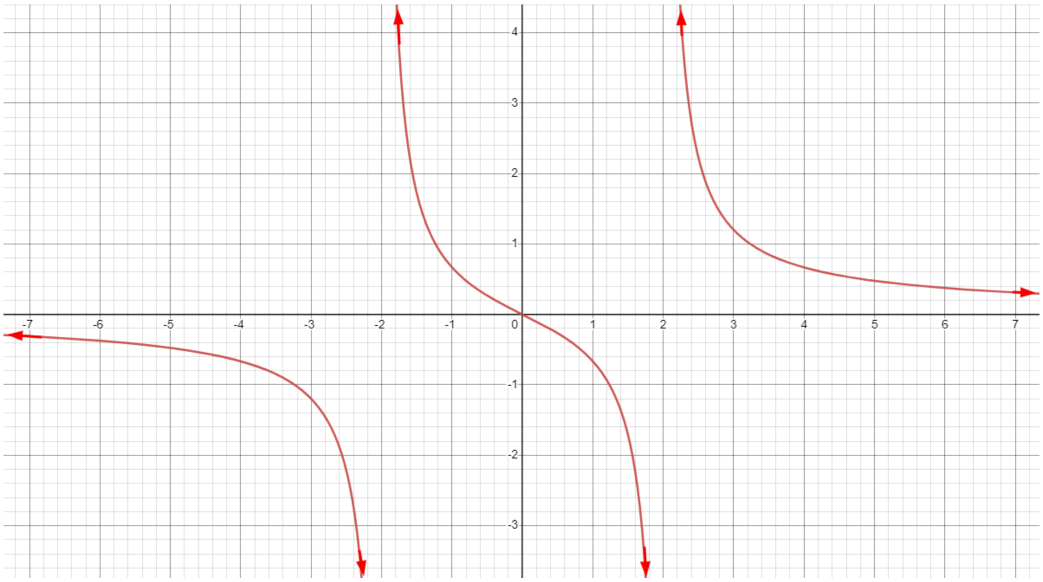
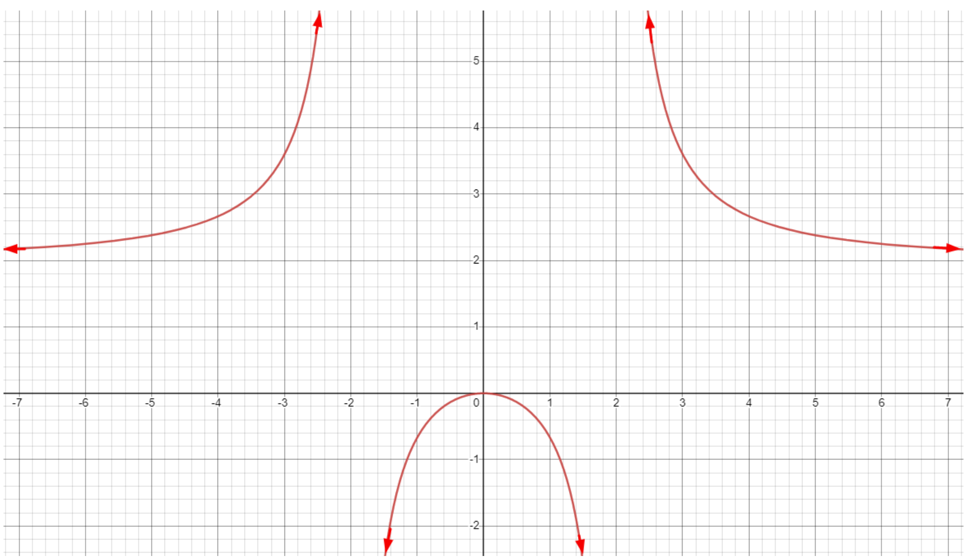


Figure 15

Graph (created in Desmos) of $g(x) = \frac{2x^2}{x^2-4}$



There is another relationship between the numerator and denominator to consider. What happens if, for boundless increase in the input values, the relationship is modeled by a function? In other words, let us consider what happens as the input values increase or decrease without bound, while the increase or decrease of output values is modeled by a function. We can explore these relationships using the examples $h(x) = \frac{2x^2-1}{4x}$ and $j(x) = \frac{x^3}{4x-1}$. First considering $h(x)$, as x increases without bound, the value of $2x^2 - 1$ is being divided by the increasing larger value of $4x$. More specifically, as the input values increase (or decrease) the output values begin to model the linear relationship $\frac{x}{2}$. This is a change from the previous relationship that led to a horizontal asymptote. For equal increases in the input values, the amounts of change of the output values are not increasing by increasing amounts. In fact, the amounts of change are not increasing at all. Considering our definition of a horizontal asymptote (for equal amounts of change in the output values, the input values changes by growing amounts) there is no horizontal asymptote present. Considering function $j(x)$, as the input values increase (or decrease) without bound, the output values are modeled by the quadratic relationship $\frac{1}{4}x^2 + \frac{1}{16}x + \frac{1}{64}$. For equal increases in the input values, the amounts of change of the output are increasing at a constant rate. Again, this is inconsistent with the definition of horizontal asymptotes above. While these two functions hold interesting relationships between the covariation of the input values and the output values, they are beyond the goal of this study.

The combination of long-term behavior and CR can also be applied to how students think about vertical asymptotes. Again, we can apply Thompson and Carlson’s (2017) flexibility in independence and dependence of variables. If we consider the long-term increase in a function’s output, the value of the vertical asymptote is where input values of the function decrease by growing amounts. Using CR to conceive asymptotes in rational functions creates a fluidity in understanding the function's input and output values.

Student Understandings

For students to be successful in conceptualizing asymptotes, both in the context of rational functions and in other mathematical applications, using CR is essential. Students need to think fluidly about how the input values and output values of the functions are changing, while also conceptualizing the notion of unbounded increases or decreases. Productive understandings for students would be: having emergent shape thinking (Moore & Thompson, 2015) when conceiving the relationships present in graphs of rational functions, using CR (Carlson et al., 2002) to motivate the asymptotic nature of rational functions, and developing fluidity in switching between the variables (Paoletti & Moore, 2018).

For both asymptotes, as one quantity increases or decreases by equal amounts, the other quantity increases or decreases towards a specific value by decreasing amounts (in magnitude). For both vertical and horizontal asymptotes, as one quantity increases or decreases without bound, the other quantity increases or decreases towards a value. Alternately, we can consider the reverse of the sentence: As one quantity increases or decreases towards a value, the other quantity increases or decreases without bound. Using these productive understandings helps students develop clear definitions of vertical and horizontal asymptotes that are mathematically equivalent. Using CR does away with the idea that vertical and horizontal asymptotes are distinct mathematical concepts.

Unproductive understandings for students would be using what Cottrill et al. (1996) call an “action conception.” Action conception is when students have a static view of mathematics and only deal with calculations (p. 172). Often, asymptotes in rational functions are presented as memorizing what part of the expression leads to a specific characteristic. Memorizing steps does not motivate clear definitions of each characteristic and can lead to confusion. It is also unproductive to encourage students to think of asymptotes as a line that a function “never touches.” Doing this sets up the expectation that asymptotes can never be crossed, which is incorrect (see Figure 14). This definition also does not speak to the deeper reasoning and behavior of the function. The focus should be on clear definitions of what quantities are changing and how they are changing. Both the use of action conception when working with rational functions and using equations and memorized steps as motivation for the asymptotic nature of rational functions likely promotes students engaging in static shape thinking when faced with the graphs of rational functions. Each foregrounds associations between shapes and memorized facts.

It is important to use contexts to motivate student understandings of asymptotes, but rational functions are difficult to contextualize at the high school level. Often, contexts only work in one direction. It is not always possible to have meaningful contexts that work in both the increasing and decreasing quantities. For example, in the beverage manufacturing problem, it is more difficult to conceptualize water added being a negative value, because in the context that would mean removing water that is not there.

CHAPTER 3

METHODS

The goal of this study is to explore what role covariational reasoning has in student understandings of asymptotes in rational functions. Using the conceptual analysis, a three-part task was designed and implemented in interviews with three students: Harper, Amy, and Steven (pseudonyms). The interviews were conducted with the researcher. While students worked through the tasks, I asked probing questions to gain insight to how students were reasoning during the task. I used prewritten questions to begin Part 1 of the task but asked specific or probing questions as necessary while students worked through the tasks. In this section, I discuss the setting of the study as well as the background of the participants. Then, I justify the tasks given to the students. Finally, I review the data collection and method of data analysis.

Settings and Participants

All three students are enrolled in a medium-sized public high school in a suburban area in the Southeast United States. Harper and Steven are both 9^{th} graders, while Amy is a 10^{th} grader. Each of the students had just completed Accelerated Geometry B/Algebra 2 with the researcher as their teacher with a class average of 90 or above. This course is the second high school level course on the accelerated track. The course structure focuses on Algebra 2 content for the first 3 months of the semester, and the last half of high school Geometry for the last 6 weeks of the semester.

All three students had a similar background in prior understandings of the concept of asymptotes in rational functions because these students learned to recognize asymptotes, create graphs, and explain characteristics, such as domain and range, of rational functions during this course. Each of the students is on track to take Precalculus next school year. These students were chosen from a pool of 10 volunteers and were not compensated for their participation. These participants were selected based on availability and ability to explain their thinking when working through a coursework as seen in interactions with them throughout the course.

Data Collection

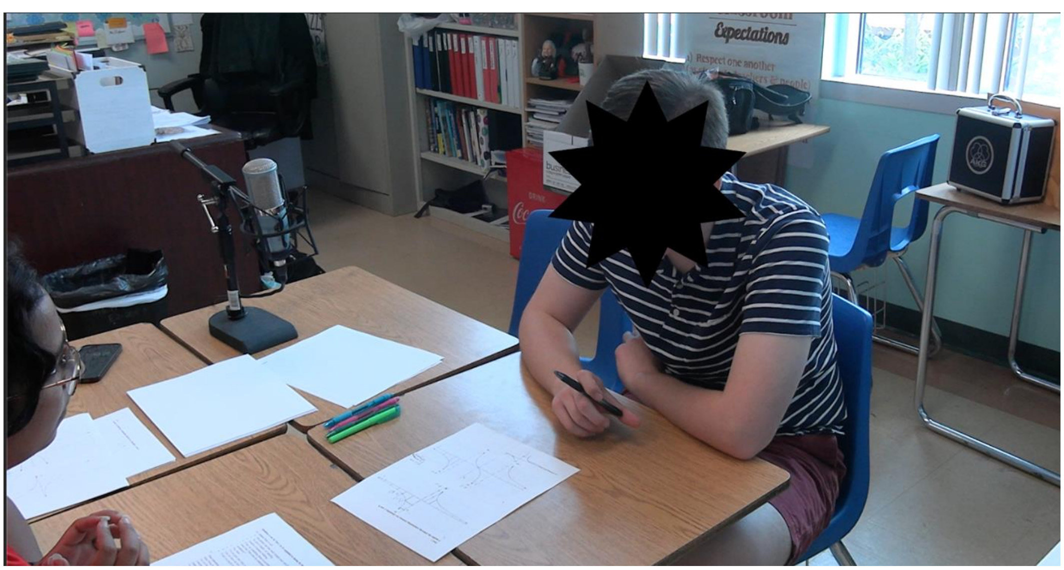
All three students participated in semi-structured task-based clinical interviews in which the students were interviewed by the researcher while they worked through tasks created beforehand (Clement, 2000; Goldin, 2000). I also created a list of questions to use as a rough script for Part 1 of the task, but intended to be flexible while questioning each student, given their thinking. Using the semi-structured task-based interviews, I probed the students' thinking as they worked through each task. The students had not seen this task before, and I encouraged them to fully explain their thinking as they worked through the tasks.

I chose semi-structured task-based interviews as my method because I wanted to give students a structured environment to think about the concepts at which this study focuses. Using the interviews, I listened to students, inferred aspects of their thinking about the tasks, and asked follow-up questions while they were working with the problem. The interviews provided me an immediate chance to gain insight into the students' thinking. The interviews did not push students to certain avenues of thinking or reasoning because a specific method of solving was not expected. Students had the freedom to work the tasks as they saw fit, as I asked for insight into their thinking. I did not guide them in how to complete the task.

The interviews were scheduled to occur after the conclusion of their Accelerated Geometry B/Algebra 2 course. Each interview was approximately 80 minutes. I video recorded each interview using one camera that captured the students' facial expressions, gestures, and verbal expression using an additional microphone. I kept record of their written work (see Figure 16). This provided data for analysis and maintained records of their written work progress. Amy's video and microphone cut out during the last half of Part 1 and all of Part 2 of the task, so there is a gap in the video collection of her interview. Immediately after her interview, I took detailed field notes to maintain some recollection of her actions during the uncaptured portion of the interview. All student written work was kept and scanned, then analyzed in tandem with the video footage.

Figure 16

Captured view of student interview



Tasks

The students worked on a three-part task, with each part consisting of sub parts. All three students worked through all of Part 1, the first two subparts of Part 2, and all of Part 3 (Appendix A). Students were told in advance that they did not need to feel pressure about getting a "correct" answer but should fully explain their thinking as they worked each part.

In Part 1, students were given the graphs of 5 different rational functions, each chosen in advance by me. I ordered each function to change either the vertical asymptote, horizontal asymptote, or both asymptotes than the graph prior. Thus, Graph A was the parent function, Graph B had two vertical asymptotes but shared the same horizontal asymptote as Graph A, Graph C had the same two vertical asymptotes as Graph B but did not have a horizontal asymptote at $y = 0$, Graph D had a vertical asymptote and a slant asymptote, and Graph E had a vertical asymptote and a non-linear asymptote. The goal of Part 1 was to gain insight into how students described the covariation between the Quantities A and B, specifically about what happens to one quantity as the other approaches a value or increases or decreases without bound. For each graph in Part 1, students were asked an open-ended question about what relationship they noticed between Quantities A and B. Part 1 was designed to see how students organically began working with the graphs, not to point them directly into a certain form of reasoning or focal point. Students worked on each graph individually, while, if necessary, I asked follow-up questions such as "Can you describe what happens as Quantity A decreases towards 0?" The focus of such questions was to push students to discuss what was happening to one of the quantities for a particular behavior in the other quantity if they did not spontaneously consider such relationships. I wanted to see if students could go into detail about how they interpreted what was happening to the function as the quantity approached a value. Another example of a question asked during the interview was, "Can you explain what happens as Quantity B increases without bound?" This question helped gain insight in how students interpreted a quantity increasing without bound. It also pushed students to consider what happened to Quantity A as Quantity B increased without bound to see if we could develop definitions of asymptotes consistent with above. After the student analyzed all 5 functions, the student was asked to compare the graphs.

Part 2 was in essence the inverse of Part 1. In Part 1, students made observations and gave statements regarding how one quantity behaved for specific behavior in the other quantity. In Part 2 of the task, students were given statements that described different relationships between Quantities A and B and then were asked to create a possible graph that showed these relationships. This task affords the students the opportunity to enact reasoning like that of Part 1, but without the presence of a graph. Students provided as many graphs as needed. Graph 1 only contained one horizontal asymptote and one vertical asymptote, while Graph 2 contained one horizontal asymptote and two vertical asymptotes. The statements describe one quantity increasing (or decreasing) without bound, while the other quantity approaches a value by either increasing or decreasing amounts of change. These statements are consistent with the definition of the asymptotes from the conceptual analysis and make students consider each quantity as the independent variable for at least two of the statements. None of the students worked on Graph 3 because of time.

Part 3 is a context task that leads to an asymptote. The context and questions of Part 3 were adapted from "The Canoe Trip, Variation 1" from Illustrative Mathematics. This task gave me a chance to see how students reason with asymptotes in a real-world context. Students are asked to read the prompt, which concerns how a canoe trip is affected by the speed of a current. The questions prompt the students to describe what the given equation given means in the context of Mike's canoe trip and then begin to consider what happens when the speed of the water equals the speed Mike can canoe. After the students discussed what happens to time if the speed of the current is equal to Mike's speed (150 ft/min), they are asked to consider what happens if the speed was less than 150 but approaches 150. I anticipated the students would use a table of values to see how the time changes as speed increased to 150, but none of them did. After the students worked through the questions, they were given the option to try to graph the situation if they felt it was possible. I chose to sequence this task last because I wanted to first gain insight into how students reasoned with the quantities within the rational functions using Part 1 of the task. Part 2 applied the reasoning students used in Part 1, thus was sequenced directly after Part 1.

In summary, all three of these tasks worked as a progression of gaining insight into the students' asymptotic reasoning including the extent to which their reasoning entailed covariational reasoning. Part 1 gave students an open-ended opportunity to explore how different quantities behaved and how one quantity affected another. Part 1 also gave students the chance to describe similar ways the quantities were covarying for different functions, showing comparable ways of understanding. Whereas Part 1 provided graphical representations, Part 2 provided statements of the covariation between Quantities A and B to gain insights into how the students might conceive those statements and produce graphs to capture such conceptions. In Part 1, students could use available figurative material to conceive the covariation between quantities A and B. In Part 2, students needed to have produced that figurative material from scratch to create the graphs of the relationships between A and B. Part 2 gives insight into what aspects are important to students when creating a graphical representation of the relationship for A and B. Students must also tend to multi-directional variations of the quantities. Part 3 asked students to apply these understandings to a real-world context and justify the relationships in context.

Analysis

After the completion of the data collection, I began an analysis of the data with multiple steps consistent with Thompson's notion of a conceptual analysis (Thompson, 2008). Through an initial rewatch of the interviews and analysis of their written work, I made notes of each student's general observations with special attention to how each student discussed covariation, approaching a value, or boundlessness. While watching the student interviews, I began to summarize each student's notable moments, gestures, evidence of reasoning with the quantities, and moments of struggle. I considered the actions of the students, my own understandings and the themes and concepts discussed in Chapter 2 of this paper. Using the same framework used by Carlson et al. (2002), I began to look for themes and summarize how the students used covariation within the tasks. I used The Table of Mental Actions of Covariation Framework (see Figure 17) as well as language of covariational reasoning set forth by Carlson et al. (2002) to describe the actions of the students through the lens of covariation. I also used The Table of Mental Actions to code my notes on the students' dialogue, gestures, and written work with the corresponding mental action that was being evidenced. The results present a breakdown of each student's interview, highlighting how students use covariation to describe the quantities changing within the relationship as well as how they explain concepts such as approaching or boundlessness. My analysis focuses on describing how students use covariational reasoning, rather than developing it. Using the language structures and coding of the students' mental actions, I synthesized each student's thinking and overall themes in Chapter 5. In cases the Carlson framework was not applicable for coding the students' thinking, I attempted to develop a model of a meaning that provided a viable explanation for the students' actions. Referring to the framework mentioned above, I drew themes from the analyses of the interviews.

Figure 17

Carlson et al. (2002) Covariational Reasoning Table of Mental Actions

Table 1
Mental Actions of the Covariation Framework

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Note: I used the description of Mental Actions as codes to categorize evidence of students thinking in Chapter 4.

CHAPTER 4

ANALYSIS

Using the framework presented by Carlson et al. (2002), I use work and drawings, excerpts from the interviews, and my own descriptions to make claims about the students' thinking. I use these reasonings to discuss productive and unproductive ways of thinking when the students were working with vertical and horizontal asymptotes. Recall that Parts 1 and 2 of the task centered on how each student used CR to reason with the Quantities A and B, either when a relationship is given graphically (Part 1) or through descriptive statements (Part 2). The students' work on Part 3 gives insight into how they reason with quantities in a real-world context that presents a scenario in which the input value produces an undefined output value. The results are presented participant by participant, ordering in the same sequence they were interviewed, with analysis of Amy's interview being first, followed by Harper, and then Steven.

Amy's Interview

Amy's interview had technical difficulties that deleted the video and audio recording of her working of the Graphs C – E in Part 1 as well as all of Part 2. For the analysis of these sections, I relied on notes taken immediately after the interview as well as her written work. As such, excerpts from the interview will only be from her work with Graphs A and B in Part 1 and Part 3 of the task.

Part 1

To begin Amy's interview, I asked her to share what relationship she noticed between Quantities A and B. Pointing to the labels on the graph, she explained they represented the x-axis and y-axis. When asked what she meant by that, she started to describe the graph's shape, pointing to the curved shape of the graph. She explained, "Each of them are going to each axis. Like it could either be approaching the y-axis or approaching the x-axis." I note that as Amy progressed through Parts 1 and 2 of the interview, she continued to use x and y when describing the quantities of the graph, as opposed to using A and B. To gain insight into how Amy understood a graph approaching something, I asked her to explain in more detail what she meant when she says "approaching", focusing on approaching the x-axis. Excerpt 1 shows her response.

Excerpt 1. Amy's initial understanding of approaching.

Researcher:	What are you thinking when you say it's approaching the x-axis?
Amy:	It's approaching 0 [*uses her finger to show the graph continuing off the page to the right*]. Like that line is not going to ever touch that, but it's approaching 0 [*uses her hands to create a gesture of her left hand move towards her right, but then being stopped*].
Researcher:	Ok, what do you feel like it means for it to approach 0? Can you say a little more about that?
Amy:	That it's getting -- the numbers in between the line and the -- or the x-axis [*moves hand left to right*] and the line [*points to the part of the graph in Quadrant 1*

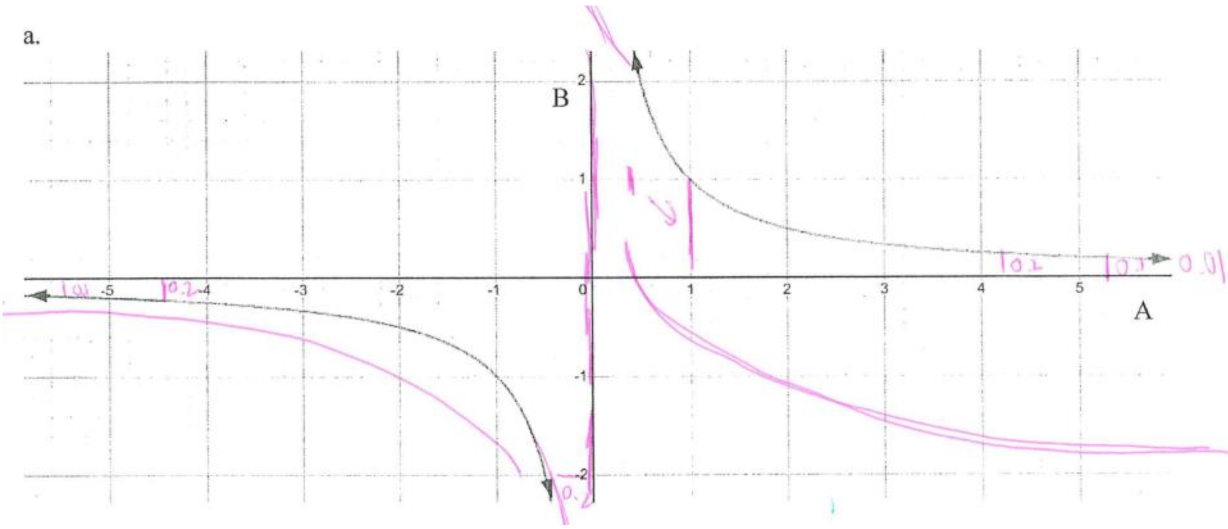
closest to the x-axis] are getting smaller, like their relationship [*makes a gesture of a gap closing with her hands*]

From her gestures and initial descriptions, I infer Amy was thinking about the line of the graph “approaching 0” as the line getting closer to the *x*-axis. She was persistent in saying that the line will never touch the 0, without specifying what quantity 0 is, and she did not reference *x* or *y* throughout her description beyond reference to an axis. However, Amy did reference a numerical value and she did allude to “numbers…getting smaller,” suggesting some attention to one quantity varying, though her conception of that quantity remained unclear.

To continue to gain insight into Amy’s thinking, I asked Amy if she could visually show how the graph’s line approached the axis. She labeled a few values on the line and eventually drew in vertical lines, which she called “spaces,” from the graph to the axis (see Figure 18). Amy drew in the spaces on both the right and left side of the graph (both places where the graph approached the *x*-axis). Amy described the spaces as getting smaller as the line continued off the page; however, she focused her description on the line of the graph itself changing. She explained that for the graph of the relationship to approach the x-axis, the space between the graph and the axis must keep getting smaller. She did not make references to quantities changing, so it remained unclear the extent to which she imagined quantities changing.

Figure 18

Amy's Graph A in Part 1



With Amy continuing to reference "spaces," I chose to pull her attention back to the Quantities A and B. Excerpt 2 provides this interaction.

Excerpt 2. Applying Quantities A and B to what Amy has drawn.

Researcher: What do you imagine is getting closer, like when you say this line is getting closer [*points to graph*]? In terms of A and B, are you imagining a specific thing?

Amy: [*Long pause*] Mm…wait, can you repeat that?

Researcher: Mhm, so, you're saying this line is getting closer [*points to graph*]. Can you talk about that in terms of A and B, or the quantities A and B [*points to labels on the axes*]?

Amy: Well, I mean, these lines are going to keep going [*points to graph*]. So, it's going to keep…the line is going to keep going like… [*uses her hands to mimic the shape of the graph, moving her right hand to the right of the page and her left hand up*

	the page] …I don't…Like it's going to keep getting closer and closer to that line and almost create this [*points to graph*] section of the coordinate plane.
Researcher:	Ok. So, what are you imagining A represents for this graph?
Amy:	Um… [*long pause*] …the *x*-axis? I don't know…

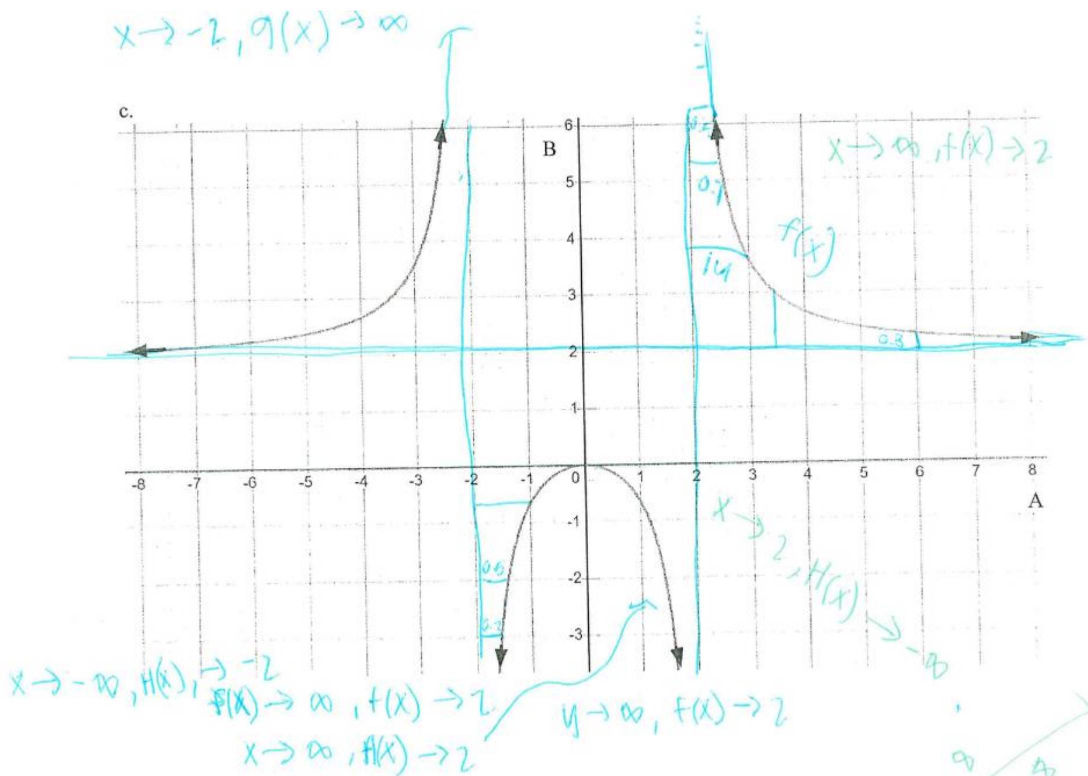
In this excerpt, Amy continued to describe the graph as "the line," and, when pressed, she indicated that she understood A (and B) as labels for the axes, but she expressed uncertainty as to what they might represent beyond being labels. In this interaction and her activity immediately surrounding it, she did not indicate that she conceived A or B as varying quantities. When asked about the relationship of the quantities, she described the motion or shape of the graph.

When Amy began working with Graph B, she drew in dashed lines where she perceived the graph of the relationship to "look like it was approaching" a certain value. When asked why she drew in these lines, she started to explain that the dashed lines are needed so that the graph's sections do not intersect, but she quickly abandoned this theory. Eventually, she stated, "I just remember seeing something like this in class, and it was like approaching a certain line." Amy seemed to be recalling her image of rational functions from her exposure in prior mathematics courses, but it was unclear how she conceptualized these lines outside of the concept image.

Although Amy's actions were consistent with viewing a graph as a static object when considering Graph A and B, her actions did suggest a quantitative and covariational focus during her engagement with Graph C. When discussing the relationship of the quantities in Graph C, she first identified the same spaces using vertical lines as she did in Graphs A and B. Initially, she still expressed confusion as to how to explain how the variables changed together. She began using an arrow notation to show a relationship between x and $f(x)$ (see Figure 19), thus providing written evidence of considering the directional changes of one quantity. For example, when she wrote $x \to 2, H(x) \to -\infty$, it seemed she showed an awareness of the output values decreasing and the input values approaching 2 *(MA2)*. Furthermore, as she described other examples, it was easier for her to describe how the output values were changing when the input values were increasing or decreasing without bound. Specifically referencing her arrow notation in the bottom of her work in Figure 19, Amy coordinated that as one quantity approached 2, the other approached infinity. Amy expressed difficulty in switching the inputs and outputs when considering the output value decreasing without bound in multiple places, leading to her use of *y* and *f(x)* within the same comparison of quantities.

Figure 19

Amy's Graph C in Part 1



Note: The arrow notation for each section of the graph shows 1) how she understood the input values and output values changing and 2) that she considered each section its own function.

In Graphs D and E, Amy focused on the end behavior of the functions. She described the end behavior of Graph D as increasing and decreasing without bound and wrote in ∞ and $-\infty$ to notate this observation. When asked to compare Graphs D and E, she compared the shapes of the graphs, using statements such as, “D has one side going up and one side going down for both” and “E has one side going opposite directions and one side going the same direction.” Amy used the term “exponentially” throughout her description of the graph, which seemed to be how she explained an increasing rate of change, however she did not explicitly equate exponential change to increasing amounts of change. For instance, Amy said Graph E was increasing exponentially in both directions while D was increasing without bound one way and decreasing without bound the other direction. Again, she used the arrow notation to denote how the quantities were changing overall (see Figures 20 and 21). Amy’s use of the arrow notation lends some evidence to her sustained awareness of some directional changes in a quantity *(MA2)*.

Figure 20

Amy’s Graph D in Part 1.

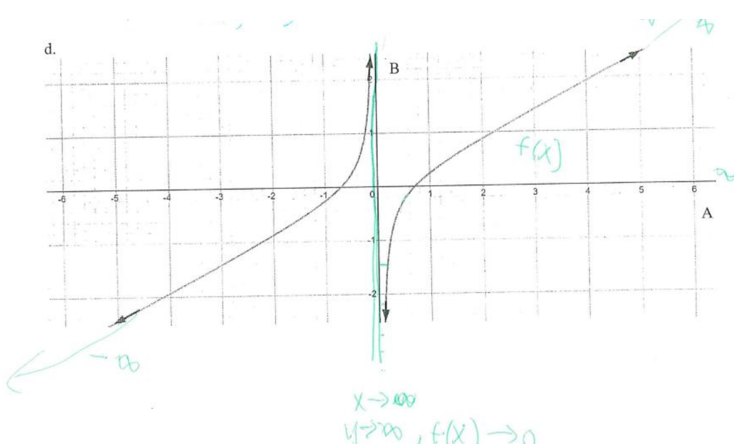
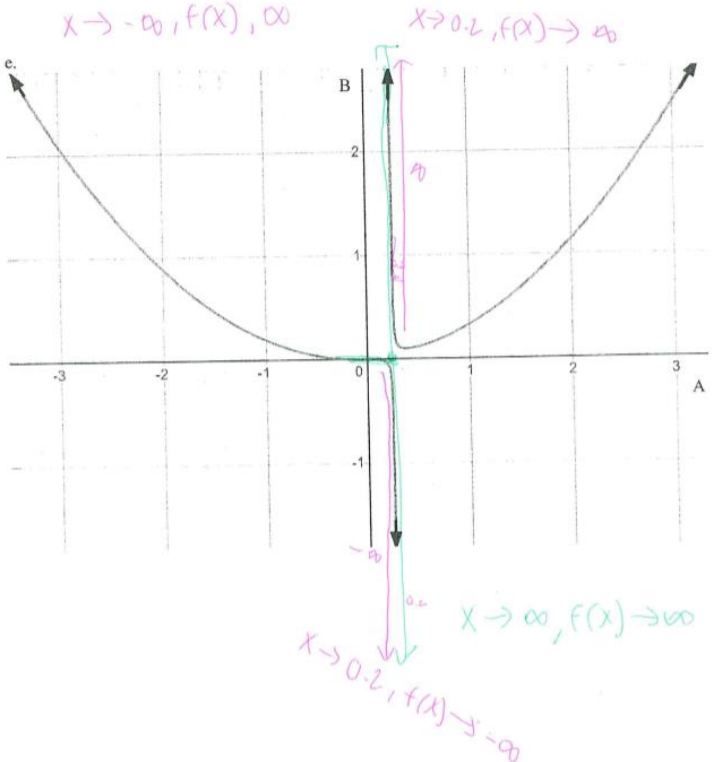


Figure 21

Amy's Graph E in Part 1.



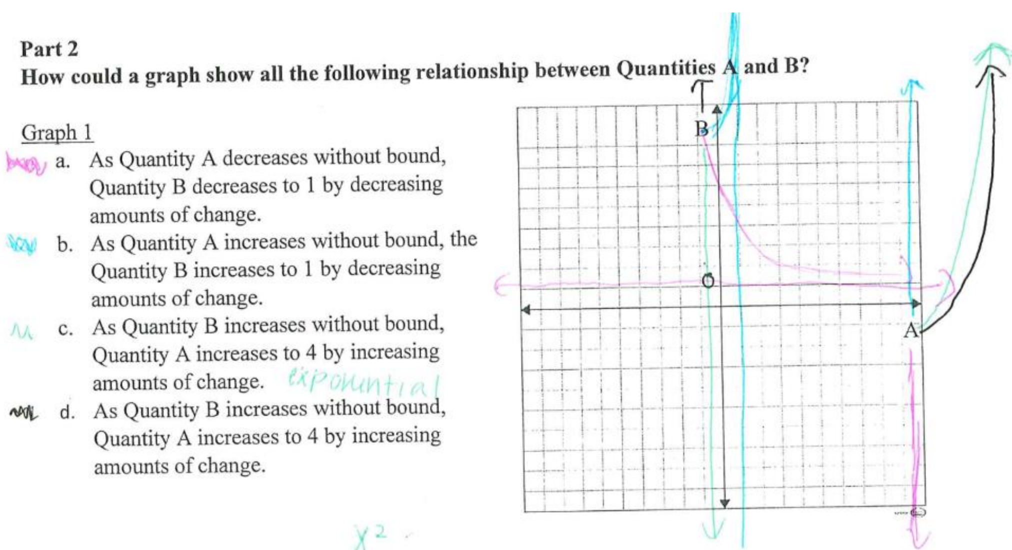
Part 2

During Part 2, Amy's actions provided evidence that her actions in Part 1 with respect to covariational reasoning were tenuous. As she worked through Graph 1 and Graph 2 of Part 2, she used the labels of the axis as the starting points for each "constraint," (see Figure 22) which she defined as the vertical or horizontal lines in the graph that lines approached. In each case, when considering how to draw her graph, she could not produce a graph she was satisfied with because "the constraints get in the way of each other." Furthermore, when she attempted to create the graphs, she did not exhibit actions consistent with covariational reasoning and the sustained focus how either quantity varied. Instead, she focused on the directional changes that would be true for each statement, while also considering each quantity in isolation. For example, in Figure 22, Amy drew in pink ink her relationship for statement a. Looking at the pink segments of her graph, she drew two separate lines. The first line begins at Label A and points down, which she used to show "As A decreases without bound." The second line is a curve beginning at the Label B and curves down to the line B = 1. There is not sufficient evidence to suggest why Amy drew in the horizontal line at B = 1.

Amy did show some evidence of awareness of the impact on the behavior of a relationship that has increasing amounts of change. Recall that in Part 1, Amy used the term "exponentially," seemingly to describe increasing amounts of change. She used the same term in Graph 1 of part one, writing it in next to statement c. She also drew a curve on her graph to align with statement c that begins at the Label A and curves up, seeming to show more evidence of Amy's awareness of the effect of increasing amounts of change on the graph's shape.

Figure 22

Amy's Graph 1 in Part 1.



Part 3

As Amy started working on Part 3, she first spent time explaining each part of the equation. She broke down each part of the equation given for T. For the numerator, she explained the maximum distance to travel was 30,000 feet. For the denominator, she knew Mike could paddle 150 feet per minute and the speed of the current (s) would slow Mike down, so “you have to subtract that [s] to determine what he can actually paddle depending on what the actual speed of the current is.” She also clarified the role of the input values and output values, stating “You’re trying to determine the time based on the speed of the current.”

When she began working part b, she first tried substituting 150 for *s* and solving the function algebraically; however, she was not sure how to interpret the algebraic solution. Excerpt 3 shows her making sense of the solution:

Excerpt 3. Amy’s substitutes 150 for speed of the current.

Amy:	So, the speed if 150…um…ok that would be 0 [*begins writing out work*]. So that would be 30,000 divided by 150 minus 150, which would be…0, I guess. Because 30,000… So, it won’t take him any shorter time because the current’s the same as…wait…[*pause*]…I don’t know that could be 0 though. Because it’s still going to take him time.
Researcher:	So, if it’s 0, what would that mean?
Amy:	Um, that it takes him 0 minutes, which makes no sense.
Researcher:	Ok. Why do you feel like it doesn’t make sense?
Amy:	Because 30,000 feet is kind of a lot to get there in 0 minutes. I mean I guess the

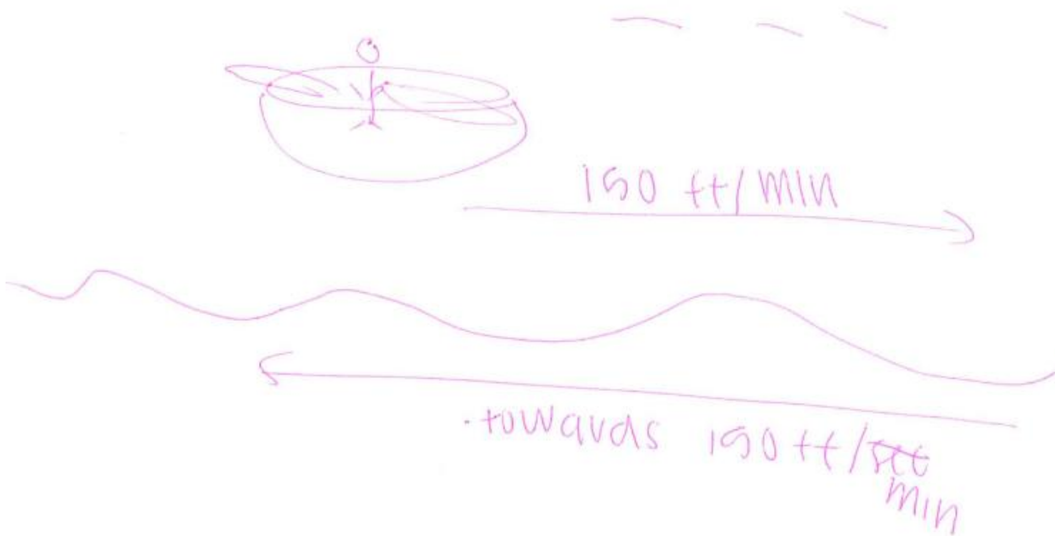
> speed of the current…well like, maybe he'll never get there. Because the feet he can paddle and the speed of the current is the exact same, so they are going against each other at the same speed, so it's like going to keep pulling him away and away from that 30,000 [*gestures with her hands to going against each other in opposite directions*].

Amy understood, using the context of the problem, that it would not make sense for Mike to take 0 minutes to travel 30,000 feet. The excerpt above is evidence of her having difficulty reconciling her algebraic solution and the meaning of that solution in context, saying "it takes him 0 minutes, which make no sense" and "30,000 feet is kind of a lot to get there in 0 minutes." After we discussed how the context impacted the solution, Amy decided that the distance Mike is travelling is 0, so he cannot reach his goal of 30,000 feet, because he is not moving. Talking through this interaction helped Amy solidify her thinking of the quantities in context and using the mathematics as a tool to better understand the context, not vice versa.

Amy also began working part c hoping to use an algebraic solution, eventually claiming she was not sure what values to use for s. After giving her some time to consider quantities, I asked her if she would like to try using specific values or drawing a picture. I provided this prompt to gain insight into how Amy may describe how the quantities were changing in the context. Recall from Part 1 that Amy used visual cues and specific numeric values to describe the relationships between the quantities. My hope was that through a picture or specific values, Amy would have more confidence in describing how she understood the quantities to be changing. Amy drew a picture to help her visualize what happened as the speed of the current increased (see Figure 23).

Figure 23

Amy's drawing of Mike and the current.



After drawing Figure 23, she began to make sense of what happened to Mike as the speed of the current varied, which can be seen in Excerpt 4:

Excerpt 4. Amy discusses what happens as the speed of the current increases to 150.

Amy:	I mean he'll get somewhere cause it's not the same current, like he can still move against the current in that way [*gestures in her drawing the boat going to the right*]. I don't really know what numbers to represent that with…[*pause*]
Researcher:	So, keep going with that idea, you said he'll get somewhere…
Amy:	He'll get somewhere because his speed is going fast than the current [*pointing to the boat*].
Researcher:	Ok
Amy:	So, it'll take him until this [*point at her drawing where she wrote the speed of the*

current] gets to 150, then he'll stay in one place. He won't really get anywhere.

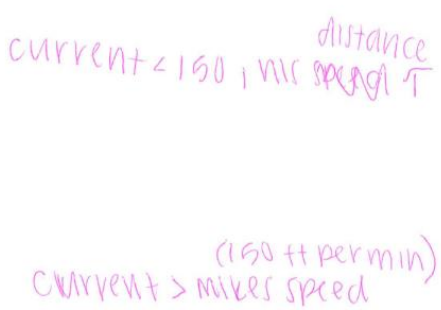
Researcher: Ok. So, you're saying if this was less [*points to the drawing where she wrote speed of the current*] the speed of the current is less than 150, then what happens?

Amy: Then, he'll keep going [*adds dashed line to drawing*]. Ok, so if this current [*starts writing under the original drawing (see Figure 23)*] is less than 150…um…his speed will increase. Or his speed will not increase but his distance will increase.

Amy went on to again explain that as the speed of the current increased to 150, the distance Mike could travel would decrease towards 0. Once the speed of the current reaches 150, Mike will stop gaining distance. When Amy began working on part d, she related Mike's distance as decreasing when the current speed is greater than 150, adding another inequality under her drawing (see Figure 24). These explanations and inequalities provide evidence that Amy considered directional changes of current speed and how those changes would influence Mike's distance *(MA2)*.

Figure 24

Amy's additions to her drawing after answer parts c and d.



At this point in the interview, Amy had not related the variation of the speed of the current to variations in time; she had considered how variations in the speed of the current would relate to variations in the distance Mike was able to paddle. To gain insight into how she might coordinate time into this context, I asked her to revisit parts b, c, and d and consider what was happening to time in each situation. She stated that when the speed of the current was 150, the time it takes Mike to reach his destination "is a long time because Mike is not moving." She went on to explain when the current speed is less than 150, the time will be less than when the current speed is equal to 150 because he is traveling faster or covering more distance. Finally, when the current speed is greater than 150, the time will be greater than when current speed is equal to 150. She used the inequality in Figure 25 to write out the comparison she made above. Here, she considered what is happening to time for specific situations, comparing each of the speeds to one another to consider how each speed affected Mike's time. She showed awareness that the quantity of time varies with the quantity of current speed.

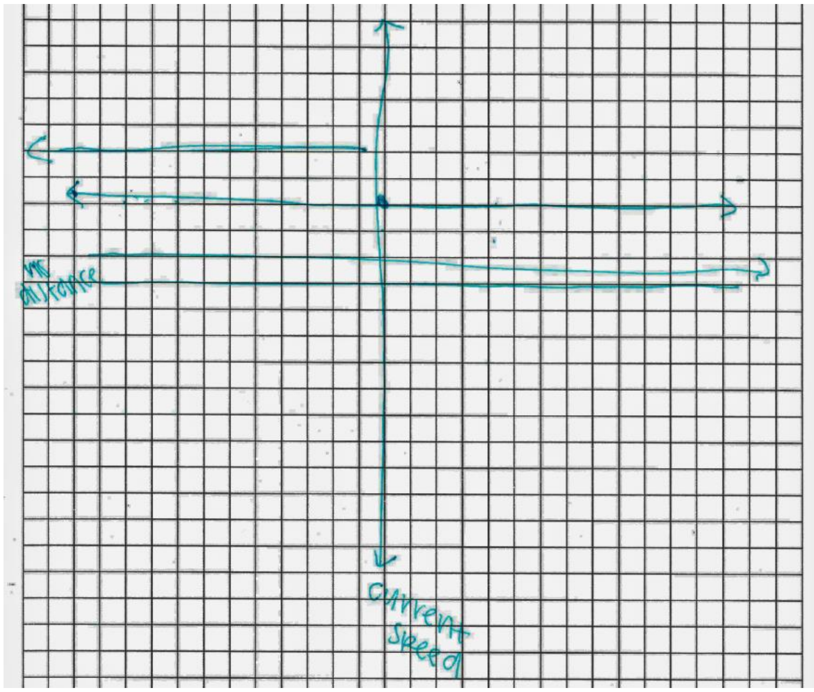
Figure 25

Amy's inequality for how Mike's time compares to each of the situations listed.

In the last stage of Amy's interview, I asked her to create a graph of the situation. She first referenced previous tasks and explained that the quantity of 150 is being approached, "so it will be like those other graphs [referring to the graphs in Parts 1 and 2]." She then graphed the relationship between the speed of the current and the amount of distance Mike can paddle (not Mike's distance towards or from the 30,000 feet goal), which is reflective of the original quantities she was coordinating when she began working this task. She drew a horizontal line at 150, seemingly because this was the quantity she explained earlier as being approached. From here, Amy experienced persistent difficulties constructing a graph with which she was satisfied. She knew there was a relationship between speed of the current and the distance Mike was able to paddle, but she was not sure how the quantities changed together. She stated, "I just think there is a correlation between his distance and the speed of the current, so like his distance is all dependent on that speed [*points to the line she drew at* $y = 150$] and whether it is above or below that speed." Eventually, she drew rays to show which direction each part of the graph would be going (see Figure 26). Under the line $y = 150$, she drew a ray pointing to the right because the distance Mike can paddle is increasing. Above the line, she drew a ray pointing to the left, because the distance Mike can paddle is decreasing (i.e., he can paddle less distance). She claimed she could not draw the specific shape because she did not know anything about the rate at which the quantities were changing, stating, "I don't know the rate for all that" before she started drawing her graph. Later, she stated, "I don't know if it's a straight line or anything, but like, his distance will keep increasing…," when she drew her ray for when the speed of the current was greater than 150. She did show that she considered the directional changes in the distance Mike was able to paddle for changes in the speed of the current *(MA2)*.

Figure 26

Amy's graph of speed of the current and distance Mike can paddle.



Note: The image has been edited for better clarity.

In Part 1 of the interview, Amy described changes in the graph with a focus on the shape of the graphs. She used vertical lines, which she called "spaces," to show how the graphs approached specific values. When explaining what "approached" means, her reasoning included the "spaces" getting smaller. She showed some evidence of directional changes in one quantity, using arrow notation to relate one quantity approaching a value when another quantity increased or decreased. In Part 2 of the interview, Amy showed a tenuous understanding of the relationship between Quantities A and B. Amy focused on directional changes for each quantity in isolation, rather than the covariation between both quantities together. When she produced a graph to show these isolated changes, she used the labels as starting points for her graphs. When working Part

3, she made sense of the quantities changing by using algebra to find a solution and then checking that solution against the context of the problem. When graphing Part 3, she related the quantities speed of the current and distance (rather than speed of the current and time). She reasoned that 150 was the value being approached and thus, harkening back to the graphs in Part 1, drew a dashed line at s=150. After this, however, she felt she needed more information to draw in the rest of the graph, specifically information about the rates of change on the quantities.

Harper's Interview

Part 1

To begin Harper's interview in Part 1, I asked her what relationships she noticed between A and B in Graph A. Like Amy, she first used x and y, instead of A and B, respectively, in reference to her graph. Her first observation involved considering *x* as increasing. She observed that the y-value was approaching 0 when [the x-value] "got bigger or smaller." When asked to clarify what she meant, she explained, "as the x-value is going away from 0 [*pointing to the origin*], the arrow's going closer and closer to one number, and it looks a lot like 0." In contrast to Amy, she continued to make explicit reference to *x* and *y* as values that vary as the interview moved forward. For instance, when I asked her to explain what was happening in the other direction, and she responded, "As the x-value is approaching 0, the y-value is approaching infinity or negative infinity." Her response showed evidence of Harper using directional covariation between the two quantities in the graph *(MA2)*. More notably, even though the y-value was varying in two different manners as the x-value approached 0, she considered both directions in her observations. She also expanded on this by using arrow notation, like Amy, (see Figure 27) to explain what she meant by "approaching."

Figure 27

Harper uses arrow notation to explain "approaching."

To gain insight into how Harper perceived the concept of approaching visually, I asked if she could draw the idea of approaching on the graph. Excerpt 5 illustrates Harper's conception of approaching as it relates to the graph.

Excerpt 5. How to show the graph is approaching 0.

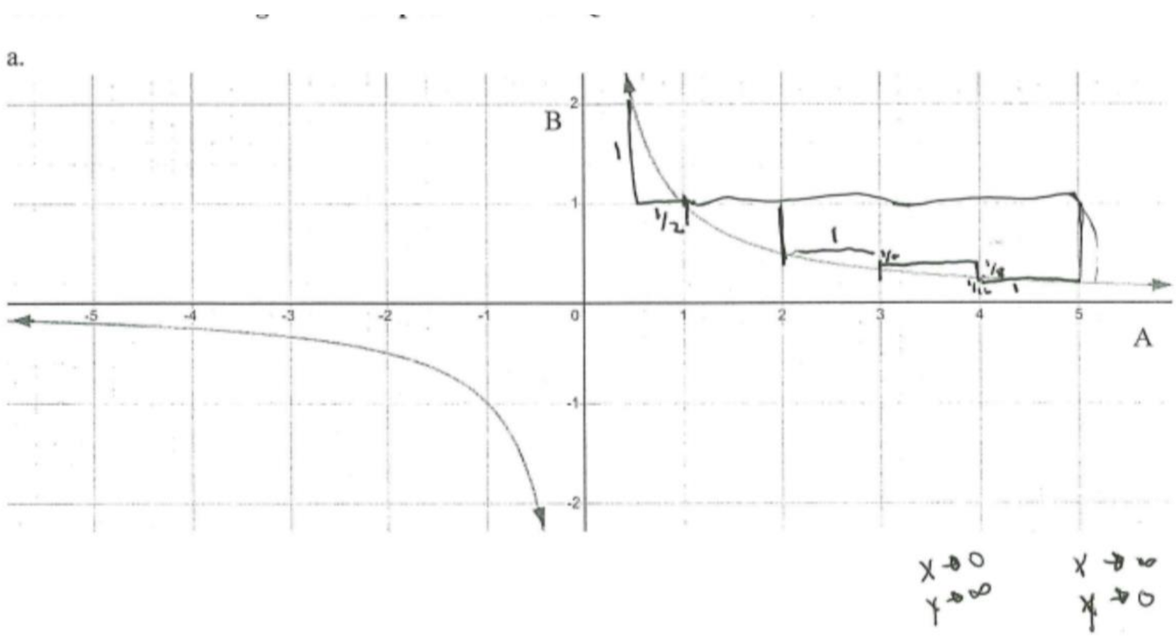
Harper:	[*long pause*] …I mean you could draw from like here [*gestures to a point on the graph*] to the last y to show how fast it's decreasing.
Researcher:	Show me what you mean by that.
Harper:	Like, kind of like we would do slope, except it's exponential probably, is what I could assume.
Researcher:	So, if you were to draw that in, what would that look like?
Harper:	Like we could do the lines like that [*draws in horizontal and vertical line connecting two points on the graph*] except it doesn't really work for exponential.
Researcher:	Why doesn't that work?
Harper:	Because it's…at each point it's going to be different, kind of.
Researcher:	Ok. Could you show that then? Splitting it up by points?
Harper:	Yeah, because you could use from, like, 4 to 5 it decreases this much [*points with her pen to the changes from $x = 4$ to $x = 5$*], and from 3 to 4…

Researcher: Ok. Could you show that?

The above excerpt suggests Harper considered the amounts of change of the y-values, but she showed hesitation in drawing them because the amounts of change varied. As the interview continued, she partitioned the right side of the graph into 1-unit changes in the x-value and constructed the corresponding change in the y-value (see Figure 28). She assigned values to these changes and used these values to justify that the amounts are decreasing, using 1-unit increases in the x-value to describe the changes in the y-value *(MA3)*. When asked to restate what change is happening in the y-values for equal changes in the x-values, she said, "it's decreasing…a lot."

Figure 28

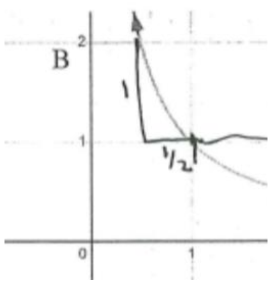
Harper's Graph A from Part 1.



To encourage her to continue using equal amounts of change in the x-value, I asked if she saw a similar relationship somewhere else in the graph. She observed, "I see that as you go closer to 0 on the x-value, the growth would be even bigger, and it would just keep on growing." She then drew in a change of ½ in the x-value and a change of 1 in the y-value. She explained she did not show more changes in this direction because there was not more of the graph visible (where B is increasing) to show the next change (see Figure 29).

Figure 29

Harper showing a ½-unit change in x corresponding to a 1-unit change in y in Graph A from Part 1.



After this, I began asking some of my prewritten questions. I changed the language of the questions to be the quantities x and y to be consistent with the language she used[6]. When asked what happens to the y-values as the x-values increase to 0, she conceived the y-values as decreasing and approaching negative infinity. I used this opportunity to gain insight into how she interpreted a function approaching negative infinity. Excerpt 6 illustrates Harper conceiving that increasing to infinity required increasing amounts of change in *y* as *x* varies incrementally *(MA3)*. I note that Harper used the word "jump" when describing the amounts of change. She

[6] I did not start using to "Quantities A and B" until Part 2.

goes on to explain the jumps were the amounts that the function's y −values are changing for "decimal" changes in x-values.

Excerpt 6. Harper's explanation of approaching infinity.

Researcher: So, when you're saying it's approaching negative infinity, or even here [*points to the arrow notation from the beginning*] you're saying it's approaching infinity, what does that mean?

Harper: It just means that like, to me like, I think of it as like, the growth just keeps on jumping. And it's like, increasingly just getting larger or smaller as it's negative, but um…it's like…[*pause*]…it's just going to keep on increasing, it's not going to stop, basically, at a number.

Researcher: Ok, so that increase…when you say it keeps jumping, are you imagining that value is going to keep getting bigger, like the y-values keep getting bigger or…?

Harper: [*starts nodding*]

Researcher: Do you imagine anything happening to the actual increase? Like the amount it's increasing by?

Harper: It keeps on just exponentially growing, by each, as each…as each like little decimal between -1 and 0, as that like keeps on getting closer to 0, it's just going to keep on having a bigger jump [*makes an up and down gesture with her hand*] between each like tenth and hundredth of a decimal.

Researcher: Ok. And when you say jump what do you mean by that?

Harper: I mean like the first one might be like at -1, it might be like 2, it might go up 2 on the… I mean down 2 on the y axis, and then it might…the next time it might be 4, and then it just keeps on like increasing.

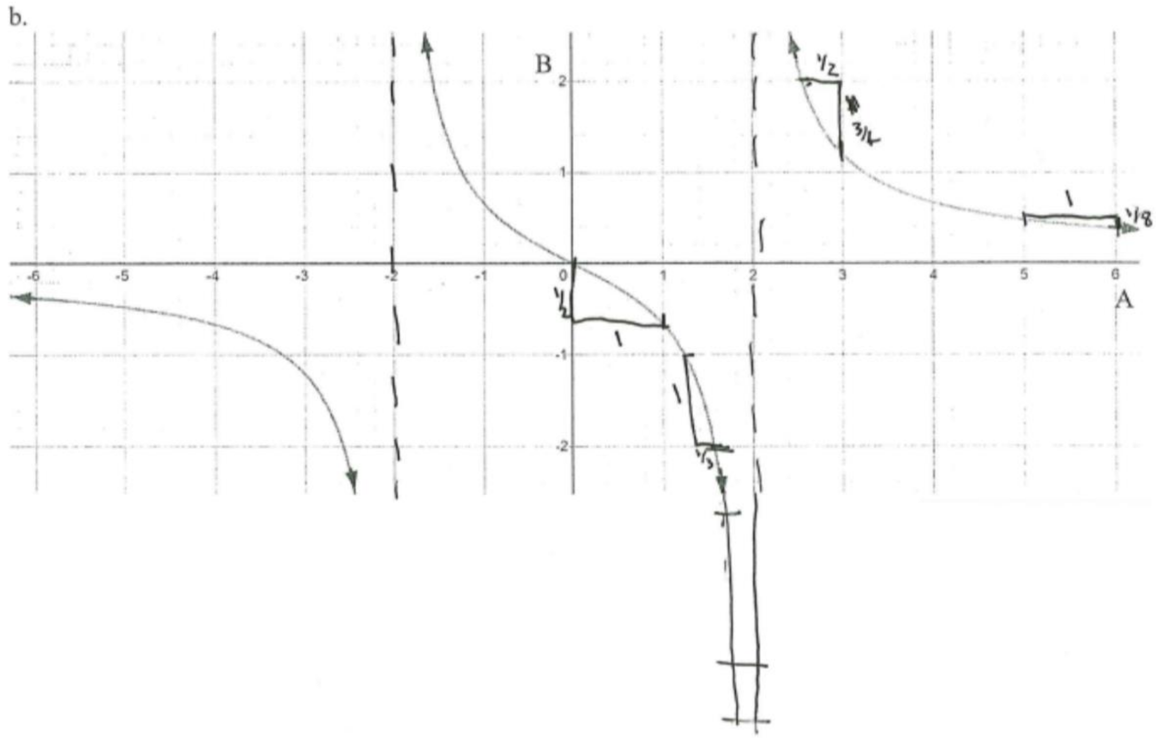
Researcher: Ok. So, by jumps you're thinking the amount that it's changing.

Harper: [*Nods*] Mhm, yes.

When Harper started Graph B, her initial observations surprised me based on the fluidity of her actions on Graph A. She began with a focus on the shape of the graph. She described the two sides were "going away" from the middle part, and the middle section "crossed the x-axis." She mentioned the right side of the graph was approaching the x-axis, so I asked if she could relate that to Graph A. She drew in segments to show change in the x-value and the y-value, but I highlight that she did not use 1-unit changes or consecutive intervals (see Figure 30). When comparing the two intervals, she explained the change from $x = 5$ to $x = 6$ was decreasing towards 0 "slower" than the interval from $x = 2.5$ to $x = 3$ *(MA3)*. To see how she would apply this reasoning to other parts of the graph, I asked her how she could apply the same ideas in the middle section. She began to draw the change in x-value and y-values from $x = 0$ to $x = 1$ and from $y = -1$ to $y = -2$ (see Figure 30). She explained that she used these to determine whether a specific section "will increase or decrease faster." She observed that "as [the function] approaches 0, it's slowing down." I note that although she used language of slopes and rate, her actions suggest that she conceives of amounts of change in the two quantities.

Figure 30

Harper's Graph B from Part 1.

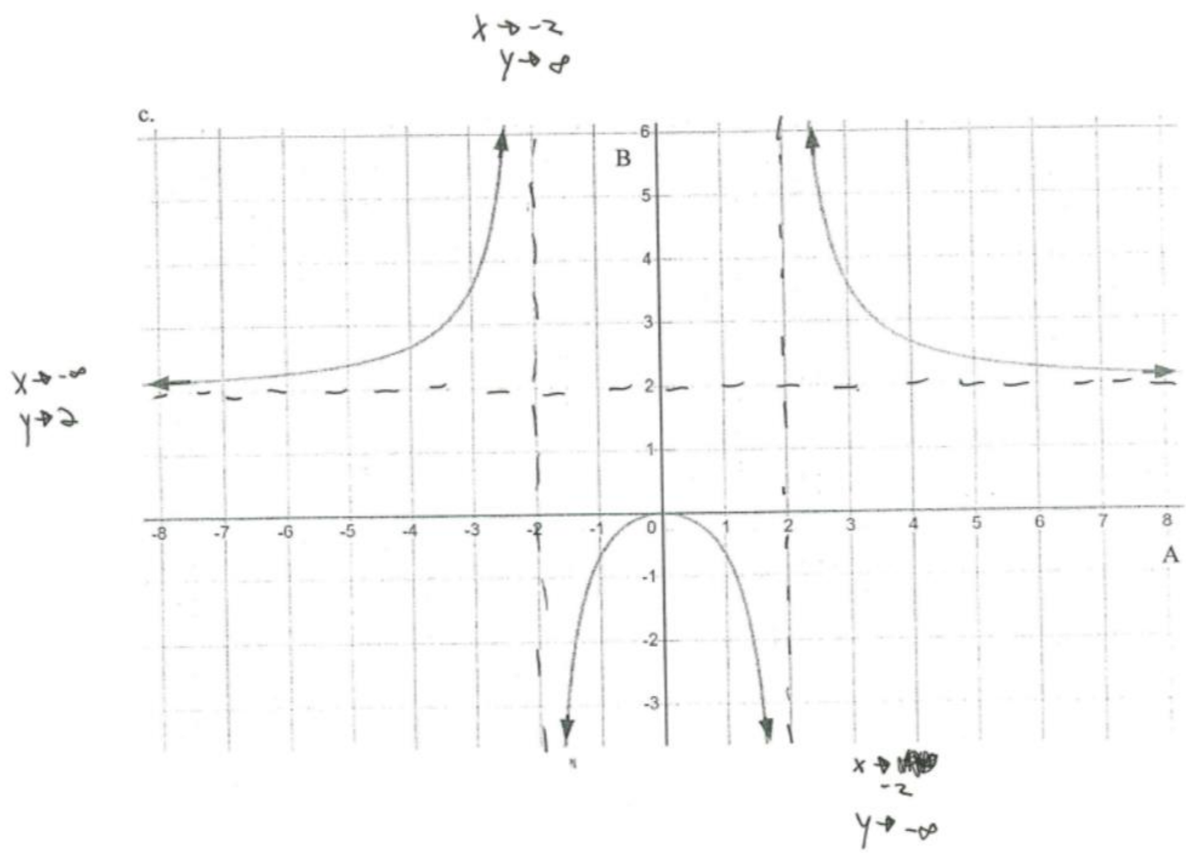


As we moved forward, and to gain additional insights into Harper's reasoning for the vertical asymptotes, since neither was at $x = 0$, I began by asking her to consider x-values increasing to 2. She referenced "jumps" to describe how the function was changing, explaining there were larger jumps in the y-values and illustrating the y-values were decreasing by larger amounts of change. Eventually, Harper drew the vertical lines at 2 and -2 , seemingly to help show the x-values were approaching those values as the y-values increase or decrease without bound. Harper began using her hands to visualize directional changes in specific quantities, and this method of visualizing directional change is present throughout the rest of her interview.

Consistent with that of Graph B, her initial observations of Graph C focused on the shape of the graph (i.e., the middle section looks like a quadratic). She then drew in the dashed lines for $y = 2$, $x = 2$, and $x = -2$. She explained these lines aided her in conceiving the graph by helping her know what the two quantities "are approaching", and she ultimately used the same arrow notation as previously to show the relationship between the x-values and the y-values for a few sections of the graph (see Figure 31).

Figure 31

Harper's Graph C from Part 1.



As Harper continued to describe Graph C, she did not illustrate changes on the graphs as she did with Graphs A and B. Rather, she used gestures to indicate changes, showing a focus on directional changes of the quantities *(MA2).* She understood that the quantities "were working together." In the following excerpt, Harper discussed what the arrow notation meant and gave some insight on what it meant to her that the quantities work together.

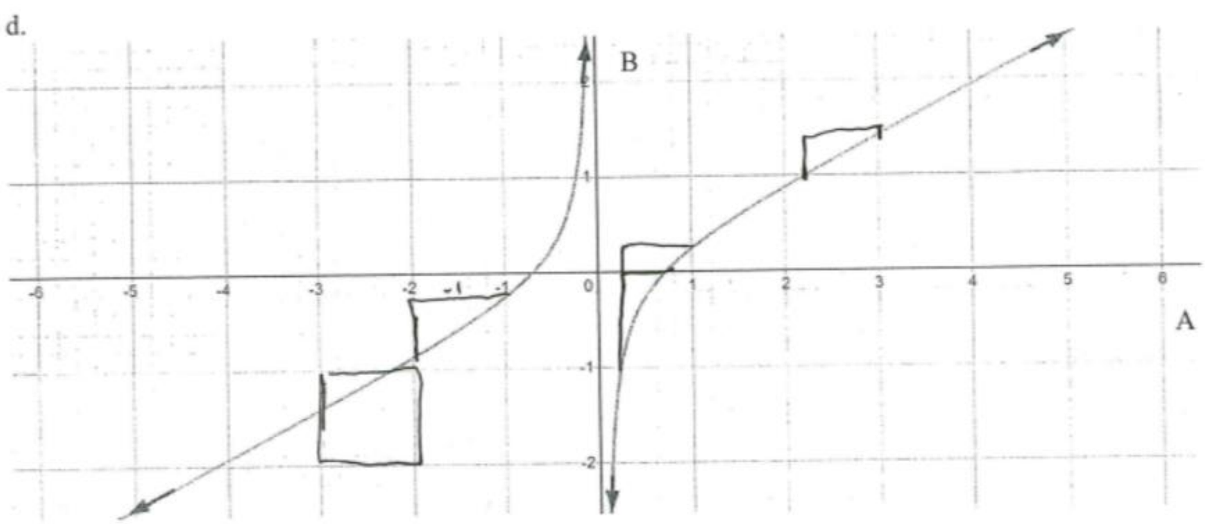
Excerpt 7. Harper described what the arrow notation means.

Researcher:	Can you just kind of briefly describe what you wrote or why you wrote those?
Harper:	Um…it's kind of to get like a gauge on like where everything is happening, I guess.
Researcher:	Ok.
Harper:	So, it's saying that like, it's going to keep on going towards -2 [*gestures to* $x \to -2, y \to \infty$ *and motions her hands getting closer together*] with the x right here, but as that happens, the y is going to be approaching infinity and it's now like a separate thing. They work together.
Researcher:	Ok. They work together. So, what do you mean by that when you say they work together?
Harper:	I mean as the y keeps on increasing, it's going to get it closer and closer to -2 for this function and so like they work together to [*gestures air quotes*] touch -2 but they don't actually touch it.
Researcher:	Ok. And when you think about y approaching infinity, what does that mean to you?
Harper:	I think of it as it just keeps on increasing without bound.

When starting Graph D, Harper first acknowledged that the ends of the graph did not seem to be approaching a number, saying that both ends "just increase or decrease without bound," but the middle section "looks really, really steep." To show the steepness, similar to how she worked her other graphs, she started to draw changes in x and the corresponding change in y (see Figure 32). Unlike her first two graphs, she did not assign number values to the changes. Harper used language relating to steepness and rate to compare different parts of the graph, but then justified the comparisons by considering how the quantities changed together. For example, when comparing what she drew on the right side of the graph, she stated, "As the x decreases, the y is changing less" *(MA3)*. Harper used similar language when comparing the changes in the quantities as x increased without bound versus as x approached 0. When comparing these "speeds" in x, Harper claimed, "…y is still increasing or decreasing a lot, but the x is slowly…like that [*gestures to the middle where x is approaching 0*] looks like it's going to get into a smaller and smaller decimal [*moves closer to 0 on the x-axis*]. And this [*points to where x is increasing*] just looks like it's just going to keep on going."

Figure 32

Harper's Graph D from Part 1.



When Harper began working on Graph E, she first observed the shape of the graph, noting the middle had a "hole" where the line and the parabola met. She did not show evidence beyond what she had shown in previous graphs, again relying on slope to compare different sections of the parabolic-shaped sections. She remarked that the y-values were increasing when the x-value both increased and decreased, showing some attention to directional changes in the quantities of Graph E *(MA2)*.

To conclude Part 1, I asked Harper to compare graph D and E. She first compared the end behaviors of Graphs D and E, where she observed that the two graphs have "slopes." When asked to say more, she explained, "It looks like a different, like, 'as the x changes the y changes.' It looks like one might be greater than the other…or one might be faster." She went on to explain that Graph E looked as though it had a faster rate of change but could not be confident because the scales of the graph were different, which underscores her attention to the quantities and how they covary. She explained she would need values or formulas to be more confident in which function had a greater rate of change. Comparing all the graphs, she noticed that each graph had values that were being approached. The following excerpt is Harper's explanation of what it means to her for a value to be approached based on all the graphs together.

Excerpt 8. Harper explains what happens when a value is being approached.

Researcher:	When you're saying the values are approaching it, what are you picturing there?
Harper:	I picture that the, like if the x is approaching 2, that the number is just going to keep on getting closer and closer, and it'll just keep on being like…the decimals will be longer, if that makes sense. But it's going to be like 1.99999999999 and like a 2 or something at the end, and then it'll just keep on getting like...and then

	there'll be a 3 at the end or a 4, and with every like value that happens, the y increases too.
Researcher:	Ok. So, you're saying as that decimal gets longer, so as you're getting closer to, you said, 2, the y is increasing? Just as an example?
Harper:	Increasing or decreasing.
Researcher:	Increasing or decreasing, ok.

I wanted to gain insight into how Harper used similar thinking as above to describe a quantity increasing without bound. I asked her to describe, based on what we discussed so far, what it meant for a quantity to increase without bound. Harper says she pictures the value "getting greater and greater." She used an example of a real-world graph to show increase in money and gestures with her hand the shape of a curve that had increasing amounts of change, saying that "the number just shoots up" and "keeps on going." From this interaction, it seemed Harper envisioned a single quantity increasing at an increasing rate when thinking of a boundless increase. Her gestures and description mirrored a relationship that had exponential growth. She considered a quantity increasing without bound as the quantity getting larger more quickly (again using rate or slope to describe how the quantity changes).

Part 2

Transition from Part 1 to Part 2, Harper continued to use *x* and *y* instead of A and B, starting this section by saying, "So to me, I think of A and B as, of course is *x* and *y*" As she worked through Part 2, she read the statements by substituting *x* and *y* for A and B, respectively.

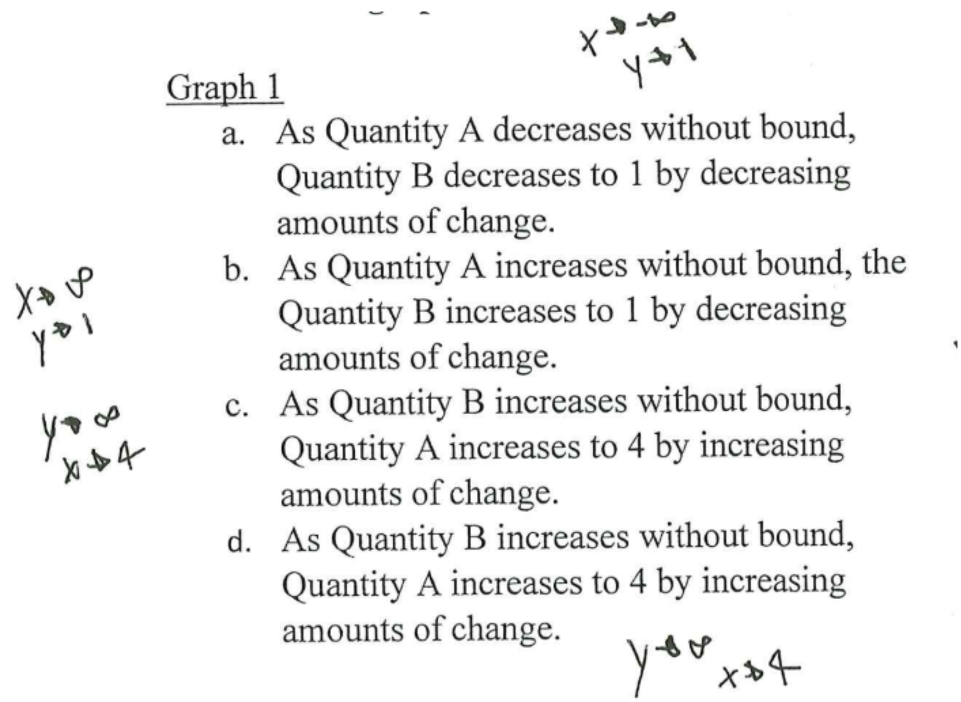
As she began Graph 1, she read the first statement then reworded the statement to help her make sense of the relationship of the quantities. The following statement is an example of how she reworded the statement:

> "As *x* decreases without bound, so I would think of an arrow…oh then *y* decreases to 1 by decreasing amounts of change [*pause*]. So, I would think that…as…*x* gets smaller, the *y* is going to…be decreasing…towards 1, …so it's going to be getting smaller towards 1. So, I'm just going to write that too [*rewrites the statement using the arrow notation*]" *(MA2).*

After she made sense of the first statement and continued to read and process the next three statements, Harper rewrote the statements with the symbolic arrow notation she used in Part 1 (see Figure 33). She used these rewritten statements, as well as hand gestures both here and later in the interview, to think about what the graph could look like, explaining, "First of all, like, the *y* looks a lot like…it's going to be going the same direction [*points up with hands*] for c and d, because they're both approaching infinity. Um, [*pause*] the *x*'s are going to go in different directions [*points one hand up and one hand down*], while the *y* approaches 1 in a and b." Harper used this notation to summarize how she understood the covariation of the quantities *x* and *y* as she read, using this notation to simplify the statements to what she considered key information regarding the quantities' covariation. She continued to refer to these statements as she began drawing her graph. Harper consistently used hand gestures throughout Part 2 (like she did in Part 1) both when thinking about and explained what direction she imagined for increases or decreases in each quantity. As she mentioned towards the end of Part 1, she knew that the quantities were varying together. The use of the arrow notation and her hand gestures gives insight into how she imagined the quantities changing together.

Figure 33

Harper's use of symbolic notation to make sense of the quantity statements.



Graph 1

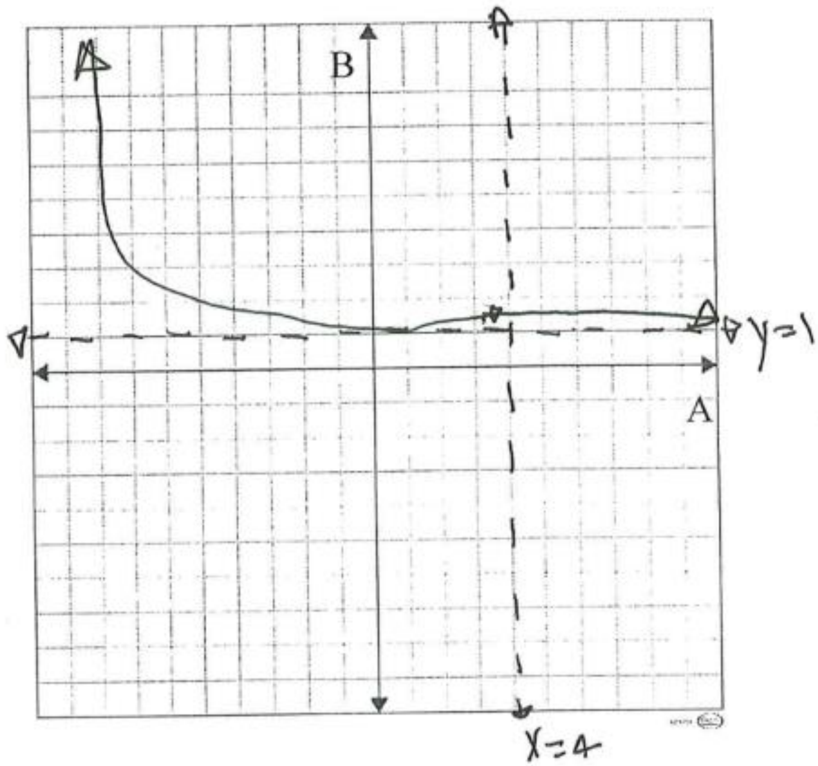
a. As Quantity A decreases without bound, Quantity B decreases to 1 by decreasing amounts of change.
b. As Quantity A increases without bound, the Quantity B increases to 1 by decreasing amounts of change.
c. As Quantity B increases without bound, Quantity A increases to 4 by increasing amounts of change.
d. As Quantity B increases without bound, Quantity A increases to 4 by increasing amounts of change.

For Graph 1, she began her graph by drawing in two dashed lines, which she labeled $y = 1$ and $x = 4$. To explain her lines, she explained, "it's saying that, like, when each of the quantities decrease or increase to a number, I'm thinking of, like, the dotted lines because it's not going to be, like, the solid lines because it's not going to touch it. So, it's just kind of like, a visual for me to know where I'm going with it." Harper gathered from the statements that the graph will approach $x = 4$ and $y = 1$.

In her first attempt in graphing, she drew a decreasing curve that approached $y = 1$ from values larger than 1. After she drew this curve, she hesitated, then decided to extend the line beyond $x = 4$. She reviewed the other statements to add on to her graph but decided to redraw her graph (see Figure 34). I asked her to explain why she changed her mind (Except 9).

Figure 34

Harper's first Graph 1.



Excerpt 9. Harper explains why she changed her mind when graphing Graph 1.

Harper:	Oh, I did that backwards, can I… [*grabbed blank graph paper*].
Researcher:	Yeah, of course. Why did you think you are doing it backwards?
Harper:	Because, um, I'm saying that as the *x* increases without bound that the *y*…actually, wait…wait a sec [*puts pen down*].
	[*whispers to herself as she uses hand gestures*] Ok. As the *x* decreases

without bound [*moves her hand from right to left*] …

Yea. Ok, because the *y* needs to also be decreasing towards 1 [*moves both hands from right to left*] so as *x* grows [*moves hands left to right*] *y* is going to grow [*moves right hand up*] away from 1.

Researcher: Ok.

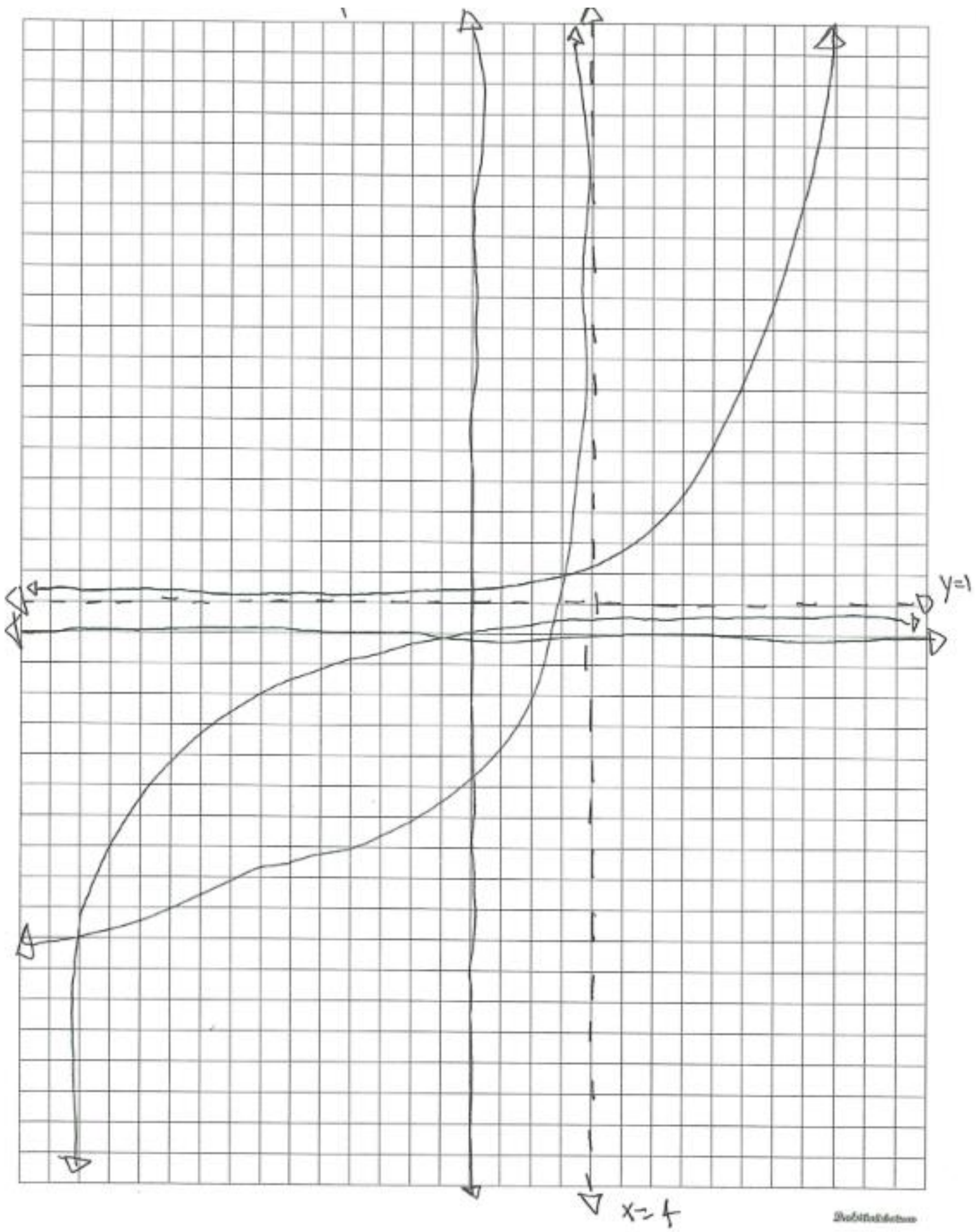
Harper: So… [*begins graphing on new graph paper*]

The excerpt above showed Harper anticipating the directional changes of the quantities through gestures *(MA2)*. Harper understood that the *y*-values were approaching 1 from values larger than 1, but she saw that she needed to also consider that *x* was decreasing without bound. Interestingly, she explained her change by reasoning what she thought the *y*-values should do as the *x*-values increased.

As Harper graphed her next iteration of Graph 1 (see Figure 35), she used her hands to consider the direction of each quantity as she graphed. As she considered how to graph the *x*-values decreasing without bound, she motioned her hands from right to left and drew an increasing, concave up curve that approached 1 from values greater than 1. Moving her hands from right to left showed evidence of Harper considering the directional changes in the *x*-values, where she equated the *x*-values decreasing as going to the right. Knowing this direction, she seemed to connect that directional change with the knowledge that as the *x*-values decrease, the *y*-values approach 1 from values larger than 1 (given statement a). This observation is consistent with her reasoning for how she drew the second curve (seen in Excerpt 10).

Figure 35

Harper's final Graph 1.



Excerpt 10. Harper explains her second curve.

Harper:	I chose to do the second one, um, this way because it says that as, I'm going to say *x* and *y* again, because as *x* is increasing [*traces her hand on the x axis from right to left*], so *x* is going towards infinity [*motions off the paper towards the right*], it's growing, um…*y* increases to 1 so that means it had to be negative before or 0 [*motions from the bottom of the graph up towards the line y = 1*]. So, it has to be going from a smaller number to 1 and getting closer to 1 [*motions to the right of the paper again*].

When she used gestures to visualize *x* increasing or decreasing without bound, she motioned her hand going towards the dashed line at $y = 1$. As mentioned above, Harper attended to the directional change of one quantity, in this case *x*, while knowing the other quantity, *y*, is approaching 1 and used gestures to help her visualize the relationship in her mind *(MA2)*.

As Harper read the last two statements and added onto her graph, she worked quietly and made small gestures as she read the statements, which were her statements from earlier thinking segments. These gestures could indicate her connecting certain changes in one quantity to certain directions. She went on to say that she could not draw a more specific graph because she lacked information about the graph. Harper's justification of the last curve she drew is in the following excerpt:

Excerpt 11. Harper's explanation of the last addition to her graph.

Harper:	I would probably draw something like that for the next one. Because again it's saying that *y* is growing, [*motions from the bottom of the graph to the top*] so it's going to be going further and further up, and *x* is approaching 4 [*points to line x =*

	4] and it's saying, um, increases to, so it means, again, it has to be going this way [*motions left to right*] to 4.
	And so, on the last one I would say… [*long pause*] …
	This is just saying the same thing. So, …[*pause*]…I would probably just keep this.
Researcher:	Ok.
Harper:	Or we could, with more information I would probably be able to determine if it was like quadratic, or, you know. But I don't think I have enough information to…to decipher that.
Researcher:	What kind of information do you feel like you would need to do that?
Harper:	Um…maybe what happens more this way [*points to the left*]. To find out if there is a change in direction. Or…if like this is just a segment of it…um…or a point on the graph to know.

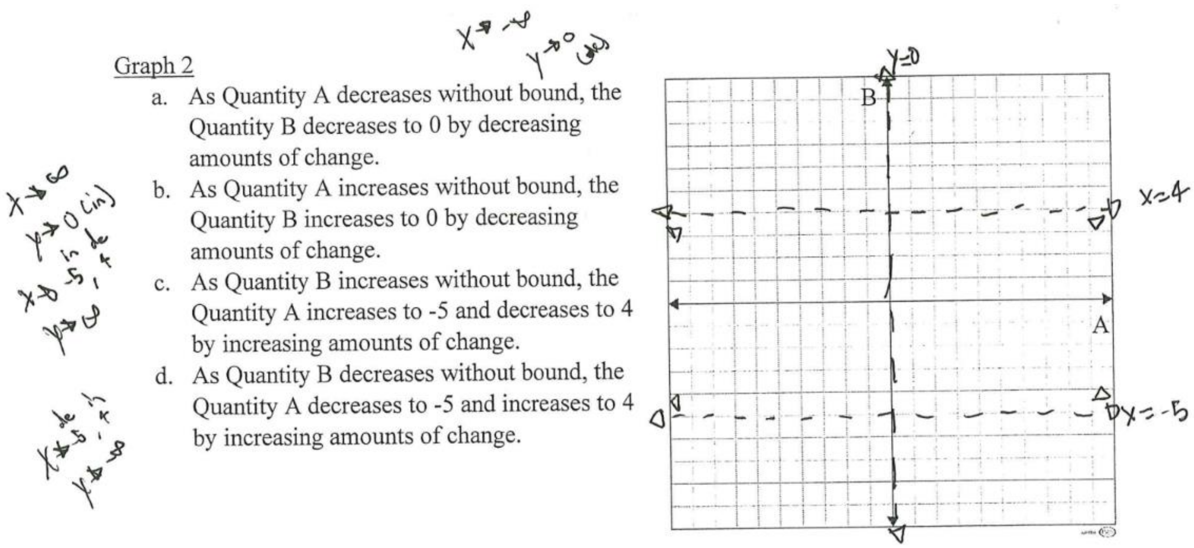
Harper's graph showed a different curve for each statement a, b, and c (see Figure 35). She did not draw a new curve for statement d because she decided that statement c and d would result in the same curve. To uncover how Harper might create a single curve with multiple statements, I asked if she could combine a and c to create a single curve. She responded, "I don't think so because *x* is…it can't decrease without bound if it's increasing to a number, if that makes sense. Because that's like…that's like hindering it from being without bound because that is a bound, saying that it must be going towards this number." It seems Harper was reasoning that each statement must be true on its own, and thus since statement a describes the *x*-values as decreasing without bound, approaching a number would contradict that statement.

We moved on to Graph 2. Harper started by rewriting the statements using *x* and *y* with the arrow notation from above. As she rewrote these statements, she continued to use her hands to anticipate and visualize the direction in which a specific quantity would increase or decrease. When she wrote these statements, she made a distinction for when the function was increasing or decreasing towards a value; this was the first time Harper used some indication of whether the quantity was increasing or decreasing within the arrow notation. She paused for a while after rewriting the statements, so I prompted her to see what she was thinking. She explained she knew part of the graph needed to be between two dashed lines, since the third and fourth statements have two values being approached. As she tried to explain what the graph would look like, she tried to move her hands in multiple ways but showed signs of being confused on how to proceed, eventually saying, "This is a confusing graph." She went on to say it was confusing to think about the *x*-values approaching two numbers. To help her find a starting point, I asked her what was happening as the *x*-value approached -5 and 4. She began to discuss the amounts of change, which seemed to be the first time she focused on the amounts of change given in the statements. To gain more insight on how she understood amounts of change, I asked her what the amounts of change element meant to her. She explained, "It means, basically, that like it's going to be -- I think of it as growing faster or slower. Yeah…like I think of it as increasing super high if it's got a high rate of change." Consistent with previously, Harper used speed when she would think of the amounts of change or rate of change. She seemed to understand that the amounts of change were the rate of change of the quantity and, to her, that rate of change moves quickly or slowly.

Harper began her graph by drawing in dashed lines (see Figure 36). She chose to start with statement c. She read the statement slowly, moving her hands towards the lines drawn. It seems like she did this to help her visualize the directional change of the quantities. Harper began silently rereading the statement and motioning with her hands and her pen, but not drawing anything new on the graph. After about 2 minutes of silence, she began to think about loud and explain where she was struggling, which is provided in the following excerpt.

Figure 36

Harper's rewriting the statements with arrow notation and first sketch of Graph 2.



Excerpt 12. Harper discusses why she is stuck on Graph 2.

Harper:	I'm trying to see if this is going to be like two lines that intersect [*makes an X with her hands*] or if it's just going to be one.
Researcher:	Ok.
Harper:	Because I know that the *y* is going to be going in different directions [*points her*

thumbs left and right] and the…as the *y* is going in different directions, as it increases [*motions her to the right with her thumb*] and decreases [*motions her to the left with her thumb*]…what *x* does switches, basically, because it, like, increases or decreases to these numbers, um, and I'm trying to think as like [*puts her pen down*] as the *y* goes up [*motions her left hand up*], the *x* is going to go to -5…this way [*moves her left from right to left*]. Oh, wait no, it's going to go…this way [*points to the right*] towards -5.

Researcher: And what do you mean by this way [*points to the right*]?

Harper: Um…to the right because it's saying that it's increasing.

Researcher: Ok. So, you said as *y* goes up…

Harper: Which also, I might have drawn these lines wrong. It might be this way [*holds her hands up vertical and parallel to each other*]. That is the biggest question to me right now. Because I feel like that might make sense.

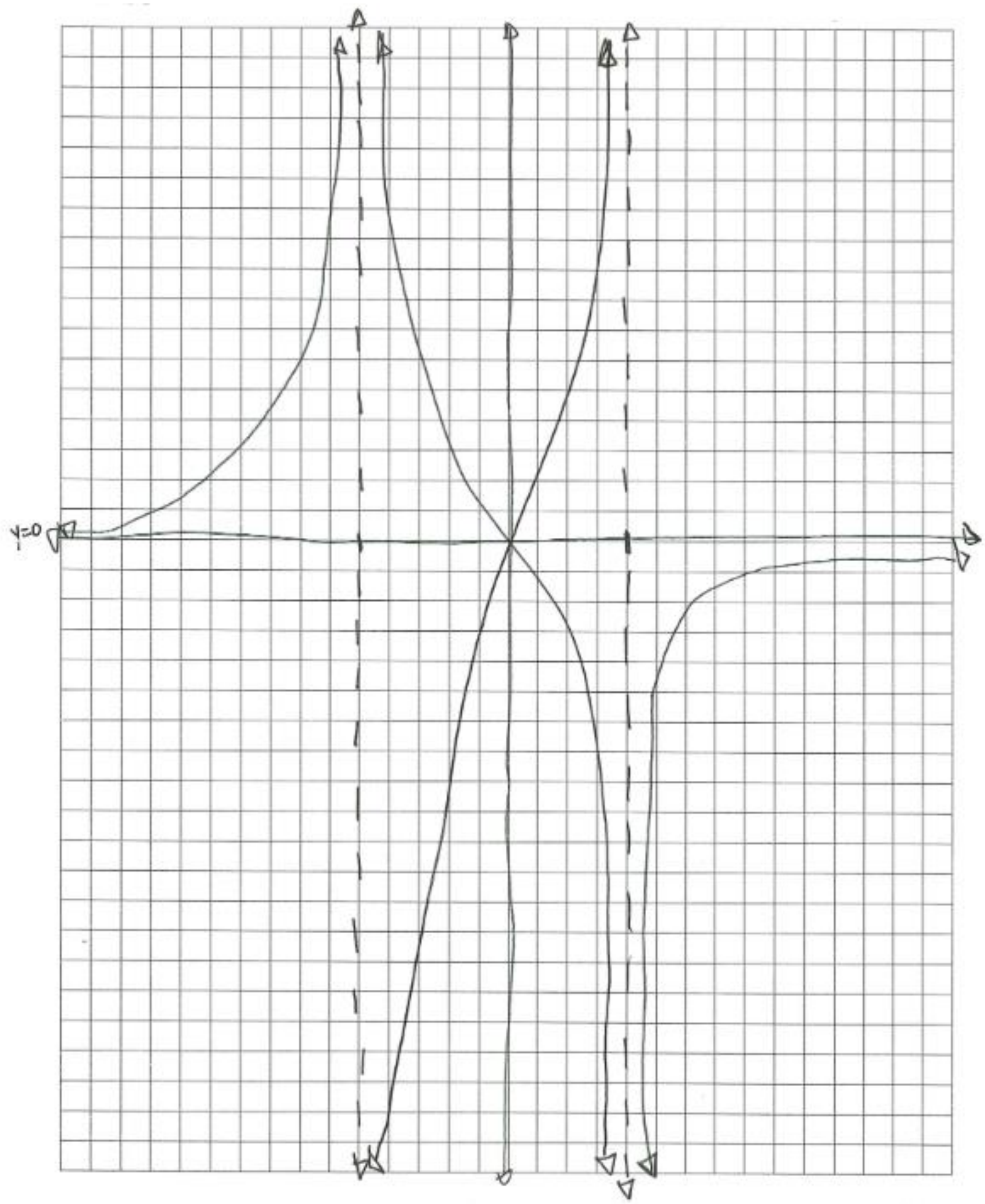
I referred to her Graph 1, in which she drew the lines in which *y* equaled a number horizontally and *x* equaled a number vertically. After looking back to Graph 1, she decided to redraw her Graph 2, this time changing the directions of her dashed lines. As she did this, she explained, "…because here [*point to the x-axis*] all the *y*s are 0. The *x* is going to change, but the *y* is going to be 0 every single value here."

With her next sketch, Harper returned to thinking about the *y*-values increasing, using her hands as she thought about the changes in the *y*-values. She drew a decreasing curve, bounded by $x = -5$ and $x = 4$, passing through the origin (see Figure 37). She explained that this curve showed values decreasing to $x = 4$ and increasing to $x = -5$, but she did not clarify which quantity decreased to $x = 4$ and increased to $x = -5$. She did point down for decreasing and up for increasing to show evidence of her awareness of the directional changes needed for decreasing or increasing values. She went on to draw an increasing curve, also bounded by $x = -5$ and $x = 4$ that passed through the origin. For her second curve, she reasoned that this curve showed values that decreased to $x = -5$ and increased to $x = 4$. Again, she did not clarify which quantity decreased to -5 and increased to 4. She seemed hesitant after she finished, explaining she did not feel sure about this section because she does not have enough information to know if there should be a change in direction.

She moved on to graphing statements a and statement b. As she drew each curve, she explained the increase or decrease of the *x*-values as "moving" to the right or to the left, respectively (see Figure 37). For example, as she drew in the curve from statement a, she explained, "…it would be *x* decreases, so going to towards the left [*motions her hand to the left*], then *y* would be decreasing towards 0." She also justified the *y*-values as increasing or decreasing to 0 as coming from below or above $y = 0$, respectively.

Figure 37

Harper's second sketch of Graph 2.



When Harper drew in the curves for statements a and b, she drew her curve so that it approached the vertical dashed lines (unlike on Graph 1). To understand why she made this change from Graph 1 to Graph 2, I asked her if she would explain the sections of the graph approaching the vertical lines, comparing it to her Graph 1. She explained that for this relationship to be a function, each *x*-value should only have one *y*-value. Thus, the lines cannot cross. She went on to say that she probably should have done the same in Graph 1. I then asked her what was happening in those parts of the graph to see how she would describe the quantities changing for that section of the graph. In the excerpt below, Harper described how the quantities are changing.

Excerpt 13. Harper describes how the quantities changed in the top left portion of her graph.

Researcher:	What's happening right here? [*points to the section of the graph such that the y-values are increasing, and the x-values are increasing and approaching 5*] Like, what did you draw right there?
Harper:	It's again, like, increasing a lot. Like the rate of change is going to grow.
Researcher:	So, which part's increasing?
Harper:	The *y*.
Researcher:	Ok. And, what's happening as the *y*-value increases?
Harper:	The *x* is going to be approaching -5.
Researcher:	Ok. So, this for section [*points to the same section of the graph*], as your *y*-values increase, the *x*-values approach -5?
Harper:	[*Long pause*] Yes.
Researcher:	Ok. And over here [*points to the other side of the dashed line defining x = -5*], what's happening?
Harper:	As the *x*-value approaches -5, um, the *y*-value increases.

Researcher: Ok. How are those different?

Harper: [*pause*] They're different because, first of all, they're like coming from different sides of the line. So, one's increasing and the other one's decreasing towards… [*motions her hands coming together*]. Um…

Researcher: Towards what?

Harper: Towards -5. Because this one [*points to the line on the right*] is… [*makes small motion with her hand moving left to right*] this one's increasing [*points to the curve on the left of the line*], this one's decreasing [*points to the curve on the right of the line*] towards -5. So, I would say that as *x* approaches -5 [*points to curve on the right of the line*] …the *y*…um…increases towards -5, the *y* goes towards infinity and then as the *x* decreases to -5 [*points to curve on the left of the line*], the *y* approaches negative infini--or infinity.

Harper recognized and described the boundless increase of the *y*-values in her graph. Harper evidenced an awareness of the amounts of change in the *y*-values, stating they are "increasing a lot" and that "the rate of change is going to grow" in as the *x*-values approach *(MA3)*. She was also able to explain the difference in the curves using directional descriptions in the *x*-values *(MA2)*.

Part 3

Harper began Part 3 by silently reading the question and beginning to describe the parts of the equation given. She explained the 30,000 feet represent the distance he needs to paddle, *T* represents the time it will take him to paddle to his destination, and the current speed is pushing him back, so the current speed will slow him down. Thus, the bottom of the equation represents how many feet per minute he can paddle, knowing the current speed will slow him down.

As Harper worked through this task, she used the context, often relating quantities current speed and Mike's change in distance. For example, when the current speed is 150, she explained that Mike will not move up the stream. She extended the effect of the current speed to Mike's change in distance, saying, "If he stopped [paddling] for one minute, he would go 150 feet backwards." I asked Harper what that speed would do to Mike's time to see how she might relate the quantity of time to the context. She responded, "there will be none, because there's no way to get upstream at that speed." When the current speed was increasing to 150, her initial observation was that Mike would gain distance as he begins his journey, but he would get slower as the current speed got closer to 150, possibly not even making it to his destination. Seeing how she would relate current speed to time for a current speed the is increasing to 150, I asked her how this current speed affect Mike's time. She explained that at the beginning of the trip, Mike would be going fast, so his time would be "small." As the speed of the current increased, Harper explains, "it would take more time to get the same amount of distance as it did in the beginning." When the speed of the current is greater than 150, Harper explained that he would move backwards, using her hands to show the opposing current speed and Mike's paddling speed. Relating this to time, Harper claimed that there would be no time to consider. She gave an example of plugging in a current speed of 151 would result in a negative number, and "you can't have a negative number for time." As Harper worked this portion of the task, she related her answers to the context of the problem. She reasoned with changes in current speed and their effect on the distance Mike can paddle.

When she began to graph this situation, she first attempted to relate the quantities time passed in minutes and what distance in feet Mike has paddled, but she decided to change the quantities because “we don’t really know time.” After some consideration, she returned to the given function and uses the quantities given, saying, “the input is going to be speed so *x* would have to be speed, I guess. Because the input to the function is speed, while the output is time” and later clarifying the input to be speed of the current. To start her graph, she explained that to graph the relationship, she would plug-in values for speed. I asked her if she could consider what the general relationship would be between speed of the current and time without calculating specific values. The following except shows how Harper reasoned with changes in speed of the current and time in a general way and justified her graph (see Figure 38).

Excerpt 14. Harper discusses the relationship between speed and time in a general way.

Harper:	I mean I would definitely say as speed is lower, so as it’s closer to 0, the time is going to be less. So, it’s going to grow. Both of them. As speed grows, time’s going to grow.
Researcher:	As speed grows, time grows.
Harper:	Yea.
Researcher:	Could you talk me through that?
Harper:	So, like, as the speed increases, the time it takes for him travel is going to take longer, because it’s going to be more resistance against him. So, it would be like…I mean, like, I could do a basic linear function. Just go straight [*gestures a line with her hands going up and to the right*].
Researcher:	So, as the speed increases, the time it takes to get there will…or as the speed of the

current increases, the time it will take time to get there will also increase.

Harper: Exactly.

Researcher: How could we use, like, these ideas of 150?

Harper: I would start on the *y*-axis at 150, probably, to show that like these is a starting…[*pause*]…wait…because it's not going to start at…let me think…yea because it would start at (0,0). Because it would be 0 time, 0 spee--…wait…well the time it would take, if the speed was 0, the time it would take would be 30,000 divided by 150, so we would have to start with that number.

Researcher: Ok. Whatever that number is.

Harper: Mhm. So, like, I would, if I just took a random guess, I would just put it, like, here [*plots a point on the y-axis*] or something. And then I would say, like, as the speed increases, the time it takes is going to increase, so I would just draw like that or something [*draws in ray from her point up and to the right*] and, I don't know, just put an arrow saying that it's going to keep on increasing.

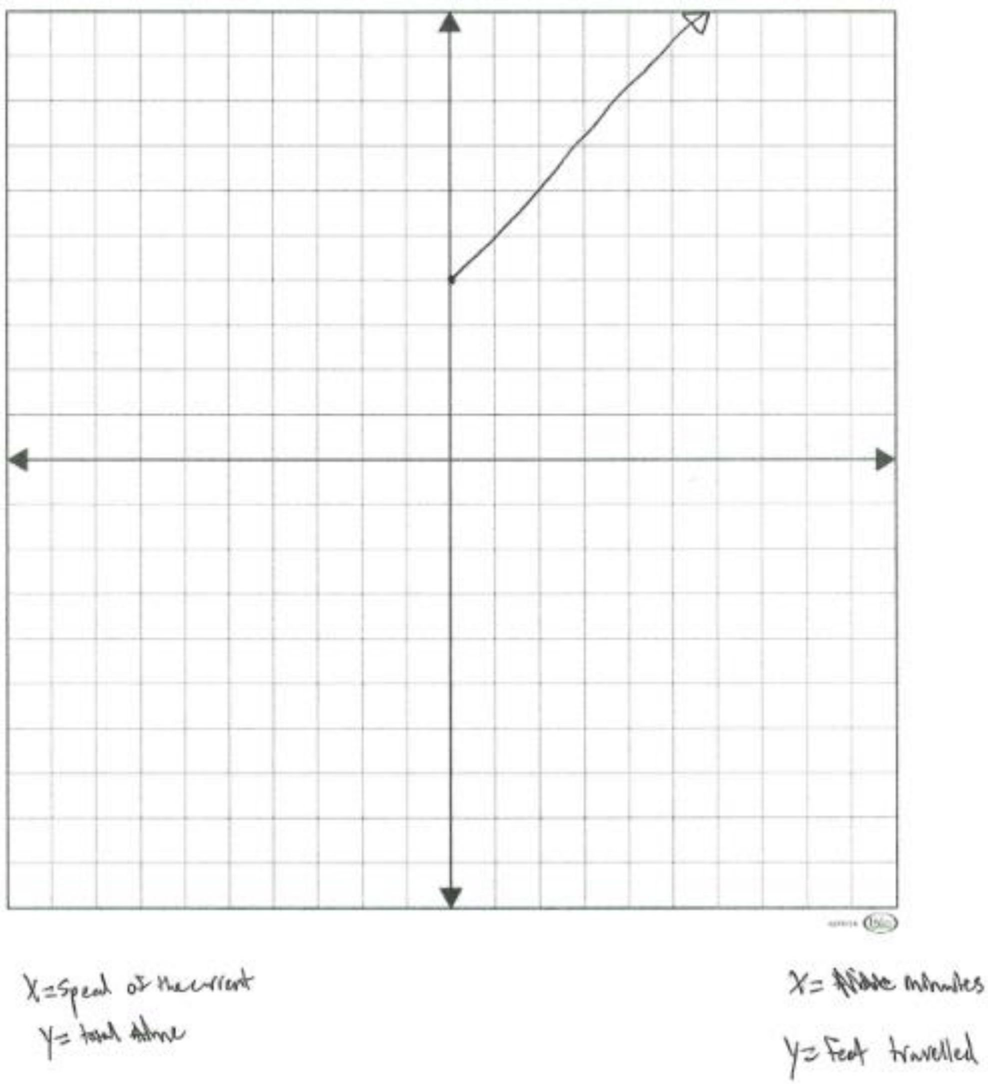
Researcher: Is it increasing towards anything?

Harper: …towards…[*pause*]…yea, towards 30,000. Because if it's, if he can only travel 1 foot per minute, it's going to take him 30,000 minutes. So, it's going to be going towards 30,000.

Harper used directional reasoning to describe how the quantities in her graph are changing, relating increases in speed of the current to increases in time. She also considered a reasonable starting point for her graph, describing the time it would take when the current speed is 0. Her graph reflects the directional relationship between time and speed of the current she described as well as the starting point of a current speed equal to 0.

Figure 38

Harper's Graph relating current speed to time it took Mike to reach his destination.



As Harper worked Part 1, she remained attuned to the variation of the quantities *x* and *y*, using arrow notation to describe how the quantities covaried. She expressed awareness of the quantities “working together” to create the graph. Harper also referred to “jumps” to reason with amounts of change. In Part 2, the graphs Harper produced were constrained. She seemed to use each statement in isolation, rather than considering how all the statements could describe different aspects of the same graph. As she worked Part 3, Harper used the context to verify how the quantities of speed of the current and the distance Mike is able to paddle affected one another. She explained that her graph related these two quantities was not complete because she did not have enough information about the “rate” of the relationship. Throughout her interview, Harper relied on directional reasoning between two quantities to describe the relationship of the quantities. She described how functions changed using language regarding slope and speed. She used her hands to illustrate directions to show how she visualized increases and decreases in different quantities.

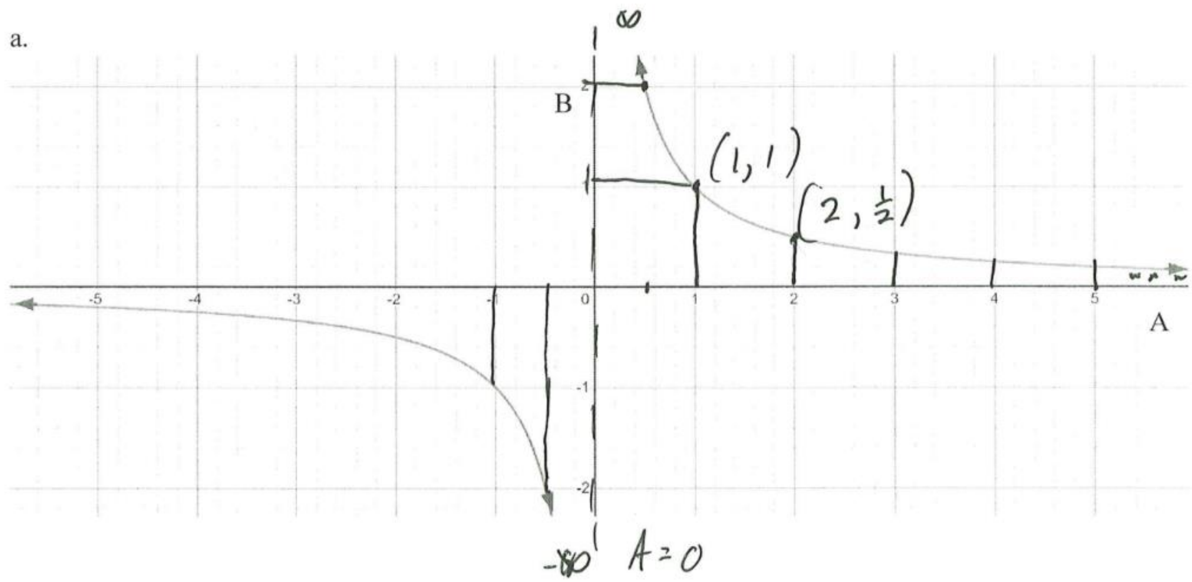
Steven’s Interview

Part 1

Consistent with the other interviews, I asked Steven what relationship he noticed between quantities A and B. Steven’s first observation was “both A and B are being approached on either side towards infinity.” Steven further explained that as Quantity A became larger and larger, or smaller and smaller, the line representing the relationship became “closer and closer” to the A-axis. He clarified that while it is getting closer, it “never reaches it.” Since he was relating the variations in Quantity A to variations in the graph, I asked him if he could illustrate his description (see Figure 39). In the following excerpt, Steven talks through how he could use the graph to explain how Quantity A is approaching the axis.

Figure 39

Steven's Graph A in Part 1.



Excerpt 15. Steven uses the graph to show quantity A approaching the axis.

Steven:	Ok, so, if we're thinking that the equation can…or like the quantities continue to go in this same way, so for example, this [*points to the section of graph in Quadrant I*] becomes…is going from a lot… a very large change to like, half as large a change, to a quarter as large a change, to an eighth as large a change for example [*as he talks through these, he is moving along the graph and pointing to different points with these changes; these changes get drawn in later*].
Researcher:	Ok.
Steven:	I don't know if that's exactly what's going on, but let's just say that's the example. So, like the distance from here to here [*plots two points on the graph*], this is the point (1,1) on the graph, for example [*labels the point (1,1)*]. And then, this is the

point (2, ½) [*labels the point (2, ½)*]. So, each time we go, like, one full space between any two points on, any two, like, full whole numbers on the Quantity A, the parts of Quantity B sort of decrease by a half. So, if we look at the third point on here, and this is the same in the other direction [*points the section of the graph in Quadrant III*] just instead of growing, it's shrinking, right? Or not necessarily, it's…they are technically both shrinking, but instead of like in this [*points to the right*] direction, it's going in this direction [*points to the left*], but it's actually looks like it's growing because we are reading from left to right [*moves finger from left to right*]. But…

Researcher: Sorry, real quick, when you say it's growing or it's shrinking, it's shrinking, what's shrinking?

Steven: You were saying, like, as it's reaching infinity, you're like telling me to explain that, so I'm saying, like, the distance between the two is shrinking, because it's approaching, right? So, as it's approaching infinity, like the distance is shrinking.

Researcher: Can you show the distance shrinking? Like is there a way to draw that in?

Steven: Uh, ok well. Between this point and this point the distance is 1 [*draws a vertical like from (1,1) to the A-axis*]. Between this point and this point, it's one half. [*draws vertical like from (2, ½) to the A-axis*] one-fourth, one-eighth, one-sixteenth [*continues to draw in vertical lines*]. And, like, it continues on that way.

Researcher: Ok.

Steven: It's like decreasing by one half each time. And that decrease becomes less and less noticeable as time goes on because, uhh, it's becoming smaller and smaller, so the difference is less and less, so like the further you go, it's still technically getting closer, but it doesn't look like it's getting closer.

As evidenced by the excerpt above, Steven reasoned that Quantity A was increasing, and that as Quantity A increased, the line that represents the function decreases. Steven did not specify the decreasing quantity as Quantity B until later, when I asked him to summarize what he noticed *(MA2)*. He summarized, "As value A is increasing, value B is becoming closer and closer to 0. And then, like, so as A approaches infinity, B approaches 0." Steven also showed evidence of Quantity B decreasing by decreasing amounts of change as Quantity A increased *(MA3),* stating amount that Quantity B is decreasing is becoming "smaller and smaller."

To see if he would be consistent with this thinking when considering a different section of the graph, I asked Steven if there is a similar relationship "somewhere else on the graph." I kept this question vague to let him choose what section of the graph to explore next. He then began considering changes in Quantity B (Excerpt 16). In this interaction, Steven shows evidence of coordinating the amounts of changes for each quantity *(MA3),* as well as flexibility in regarding each quantity as the independent variable due to him reasoning that he could show the same type of relationship of Quantity A with respect to Quantity B as Quantity B with respect to Quantity A *(MA3)*.

Excerpt 16. Steven highlights similarity in how Quantity A and Quantity B are changing.

Steven: It's very similar to A, so…in fact, if you turn the graph [*rotates the page 90°*], it looks very, very similar. It's kind of the same relationship. So, as B becomes greater and greater, it to, you see the distance here [*draws in horizontal distances from the graph to the B-axis*], that's a distance of 1, that's a distance of ½, so it's

---

like, it's like A, but in this direction.

Following this explanation, I used one of the prewritten questions to gain insight into how Steven perceived other aspects of the given graph. I first asked Steven to describe what occurs as Quantity A increases to 0. He answered that as Quantity A increased to 0, Quantity B is approaching negative infinity. When asked to clarify what it means for it to approach negative infinity, he referred to the changes from above, stating, "Kind of like the opposite from it decreases as much, it's increasing as much." To illustrate this relationship, he drew in vertical lines from the A-axis to the graph of the relationship in Quadrant III (see Figure 39). He concluded that as the Quantity A increased to 0, the changes in the Quantity B were getting greater and greater, so it is approaching negative infinity and the changes in the Quantity B were getting greater and greater (in magnitude) *(MA3)*.

After he described the changes in Quantity A and Quantity B, I asked him why he used 1-unit changes in each quantity. He explained using 1-unit changes made it easier to see how the other quantity changed. He went on to give ½ as alternative to the unit changes, saying the relationship would be the same. As he used the ½ example, he used his fingers to show consecutive changes of ½ in Quantity A. Collectively, Steven described the changes in Quantity A and B using both directional changes (e.g., as Quantity A increased, Quantity B decreased) as well as using amounts of change. When describing amounts of change, he used the distance of the graph of the relationship from an axis (represented by the horizontal and vertical lines) to describe how a quantity changed. He was strategic in where he drew the lines, using uniform changes *(MA3)*.

As Steven began Graph B, he described what one quantity approached as the other quantity increased or decreased without bound *(MA1)*. The following excerpt gives his observations.

Excerpt 17. Steven's initial observations for Graph B.

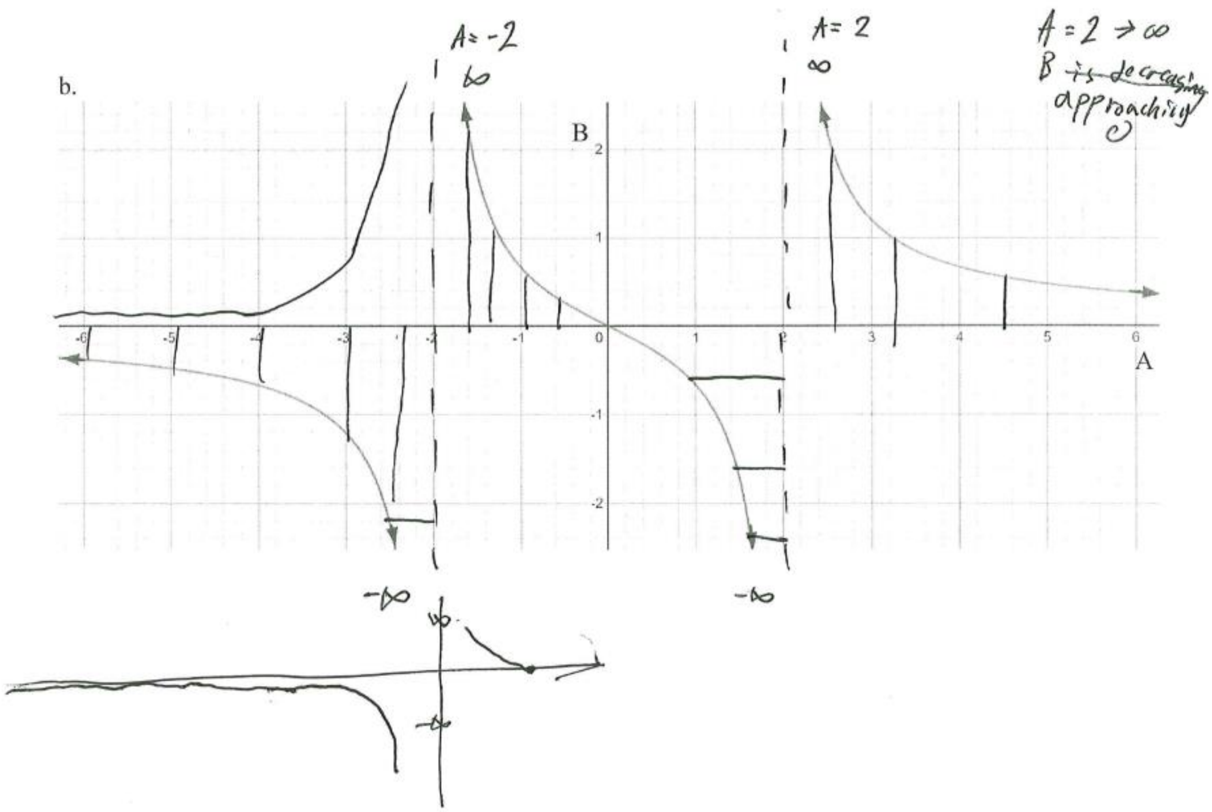
Steven:	…so, as A increases to infinity, B approaches 0. As A decreases, B approaches 0. As B increases, A approaches, uh, 2 and -2, I'm pretty sure. Yeah.
Researcher:	Will you say that one just one more time. So, you're saying as B what?
Steven:	So, I'm saying as B gets bigger and bigger, the values of A are approaching, uh, 2 and -2.
Researcher:	Ok.

I asked Steven if he could show the changes as he did in Graph A, stating where Quantity A approached A = 0, Quantity B approached negative infinity and infinity. Connecting to Graph B, he claimed that Quantity A approaches two values in the same way. He drew in dashed lines at A = 2 and A = -2 in Graph B, explaining as Quantity A approached -2 and 2, Quantity B approached infinity and negative infinity. Thus far, Steven's initial description the relationship of Quantity A and B was to state what was being approached (whether it was a value or positive/negative infinity).

As the interview continued, and consistent with his actions during Graph A, he used the amounts of change, or distances to justify his claim. For example, when justifying why Quantity B approaches 0 when Quantity A increases to infinity in Graph B, he drew in the vertical distances from the graph of the relationship to the A-axis (see Figure 40). Furthermore, as he was explaining how Quantity B is decreasing as A is increasing in Graph B, he made a distinction between Quantity B decreasing and Quantity B approaching 0, seen by his work in Figure 40 and highlighted in Excerpt 18.

Figure 40

Steven's Graph B in Part 1.



Excerpt 18. Steven makes a distinction between decreasing and approaching 0.

Steven: [*writing statement in top corner of Graph B*] From A equals 2 to infinity, B is decreasing. Or better to describe it [*crosses out decreasing*] approaching 0 [*writes in new conclusion*] because it doesn't actually reach 0.

Researcher: Ok. So, what is that distinction decreasing and approaching 0.

Steven: Because if it were just decreasing, it would get to point where crosses through 0, because it's decreasing by some certain amount. But when you say it's approaching 0, you're saying it's decreasing to a certain point where it's…dividing. I feel like there's a difference there. It's…when you're saying it's approaching 0, it's similar to saying it's decreasing, but it's sort of saying how it's decreasing.

Researcher: Can you say how it's decreasing?

Steven: Yea, it's decreasing by a certain, like, division stance. So, like, the distance between this one and this one [*points to vertical lines at B = 2 and B = 1*] it goes by a half. The distance between this one and this one [*points to vertical lines at B = 1 and B = 1/2*] goes by half. So, like, half each unit measurement. And so, each time it goes half, it becomes smaller and smaller, but it never reaches 0, so it's only approaching 0.

Through this distinction, I infer that as though Steven is alluding to decreasing amounts of change. Steven reasoned that for unit changes in Quantity A, the distances between the graph of the relationship and the A-axis were being divided by the same value. In other words, for each unit-change in A, the distance from the graph of the relationship to the A-axis was halved. Steven used a multiplicative relationship to quantify the change in distance by comparing successive changes in distance.

To gain insight into how Steven would reason with the relationship when neither quantity is increasing or decreasing without bound, I asked Steven to describe what happens as Quantity A increased towards 0. Illustrating directional change in one quantity, Steven sweeps his hand left to right and states, "Quantity A is increasing, so that's going this way" *(MA2)*. As he described the relationship, he recognized that while the function is decreasing over the interval (-∞, 0), there is a difference in how B is decreasing in the intervals (-∞, -2) and (-2,0). To illustrate that difference, he drew in the distances from the graph of the relationship to the A-axis in Quadrants II and III. The following excerpt is evidence of Steven's use of these distances as he described the difference in the two intervals.

Excerpt 19. Steven explains difference between decreasing intervals (-∞, -2) and (-2,0).

Steven:	… the distance [*draws in vertical lines from A = -6 to A = -2*] between these is increasing, right? And that's the same if it was negative or positive. But then, here [*draws in vertical lines from A = -2 to A = 0*] the distance is getting smaller as it goes along. That's the difference.
Researcher:	Ok. Could you restate that difference?
Steven:	So, as A is approaching 0, between negative infinity and -2, the distance between 0 [*points to A-axis*] and the value of B is getting larger, but then between -2 and 0, the distance between, uh, the value of B and 0 decreases until it reaches 0. So, when A is approaching between negative infinity to 2 it's getting away from 0, and then from -2 to 0, it's coming to 0.

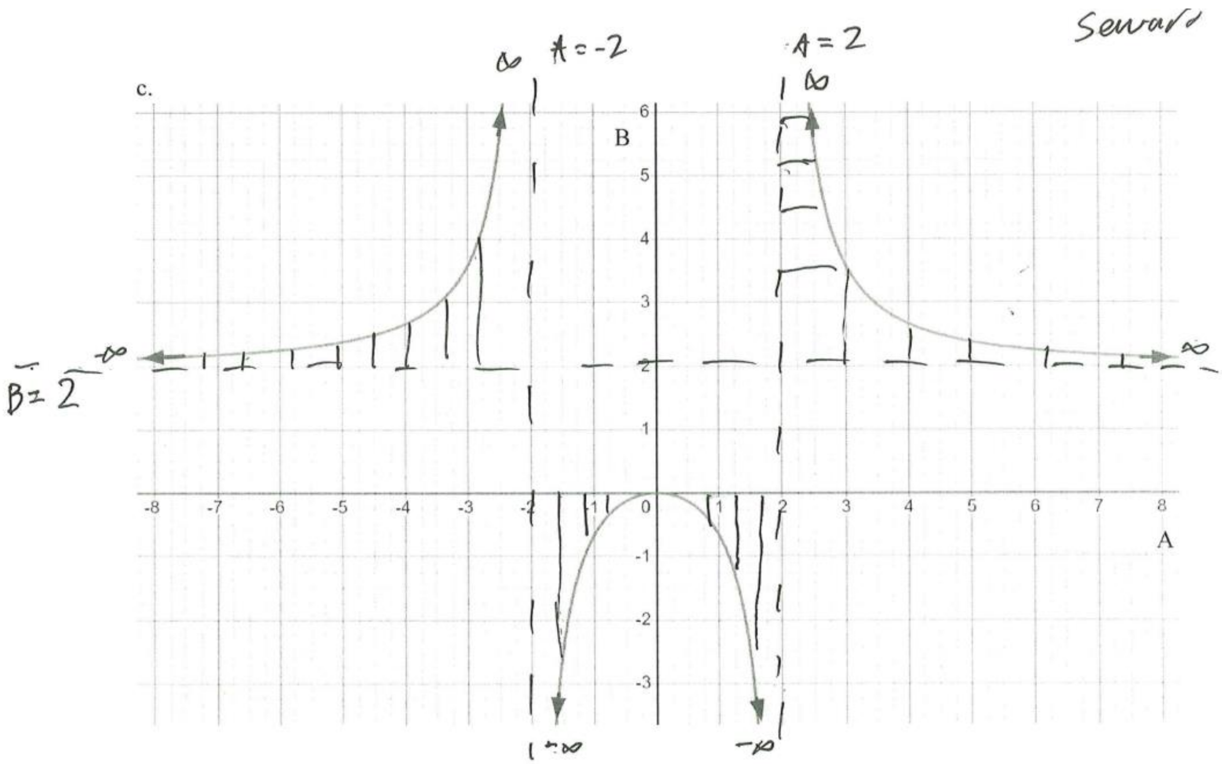
Steven distinguished between Quantity A decreasing without bound and Quantity A decreasing to a value by describing changes in the magnitude of Quantity B (e.g., "the distance is getting smaller"). He gave a visual for these amounts of change by drawing in the distances from the graph of the relationship to the A-axis *(MA2)*. I note that Steven also considered how Quantity A varied as Quantity B decreased without bound. He described that Quantity A approached 2 and -2 and drew in horizontal distances between the graph of the relationship and the dashed line he drew at A = 2, stating that these distances get smaller as Quantity B decreases (see Figure 40).

Moving onto Graph C (Figure 41), he began by describing what value a quantity is approaching when the other quantity is “approaching negative infinity or infinity.” As he described these relationships, he drew in dashed lines at the values A = 2, A = -2, and B = 2. When asked why he drew in the dashed lines, he explained they helped him visualize “what’s approaching what.” He went on to state, "I just put those in so it's easier to see when A is approaching infinity and when B is approaching infinity, what numbers they're approaching when the other is approaching infinity." As the interview continued, Steven continued with the same line of reasoning, referencing the distances to justify his claims, but not explicitly drawing them in the graph. For example, as Quantity A gets larger, Quantity B approaches 2 because “it’s becoming smaller and smaller towards 2” and then clarifying, “the distance between the value of B and the value of 2” as what is getting smaller. The reference to the distances also aided Steven’s flexibility in switching between Quantity A and B, because the distances were not reliant on starting from a specific place.

Steven generalized how to use these distances from the graph of the relationship to decide if a quantity is approaching a value or (positive or negative) infinity (see Excerpt 20). Steven’s generalization shows his awareness of the effect different magnitudes in the distances have on the behavior of the function *(MA2)*. This awareness aides him later in Part 2 when he is graphing the relationships.

Figure 41

Steven's Graph D.



Excerpt 20. Steven generalizes how amounts of change effect the relationship between Quantities A and B.

Researcher: What's happening to those distances that you're drawing [*points to the vertical lines in the middle section of the graph*]?

Steven: They [the distances] are becoming larger and larger, so that means they are approaching infinity. Uh, when the distances are becoming larger as they approach something, that means going to infinity, and when they [the distances] become smaller, that means they are approaching a number.

Researcher: Ok.

Steven: So, for example, these distances, in this quadrant, [*draws in vertical distances from the graph of the relationship and B = 2*], well it's not a quadrant, this section, these distances are becoming larger in this direction [from left to right], so that means it's approaching infinity, but if we look at it in going in the other direction [from right to left], it's approach--it--the distances are becoming smaller, so then it's approaching 2.

To see if Steven would apply this generalization to a graph with a slant asymptote, I moved on to Graph D. As he did with Graphs A, B, and C, he began by considering the how one quantity changed when the other quantity increases or decreases without bound. He observed, "as A approaches infinity, B isn't approaching the same number every single time. Same way when it's approaching negative infinity." Excerpt 21 shows how Steven compared Graphs A and D, and how he used that comparison to describe the variation in Quantity A and Quantity B.

Excerpt 21. Steven compares Graph A and Graph D.

Steven: It's similar to how it was in this example [*pointing to Graph A*], so like if we turn this example [Graph D] to be similarly distanced to this example [Graph A], we can see, uh, the lines are approaching it, like they are in this example [Graph A].

Researcher: Mhm.

Steven: So, I'm just going to put lines in like this [*begins drawing in lines to represent the distance from the graph of the relationship to the slanted dashed line he draws on the left side of the graph*] to be an example. And so, they're becoming bigger and bigger. And I'll go through there. So, as, uh…whatever…as A decreases, B is approaching uh, some number on this line [*points to slanted dashed line*]. So, whatever the slope of this line is…uh, I can't ok…we're just going to put a dot on the line here at (-2, -1) even though I draw it on this line. We are just going to say it's at (-2, -1) [*plots point and labels it on the dashed line*].

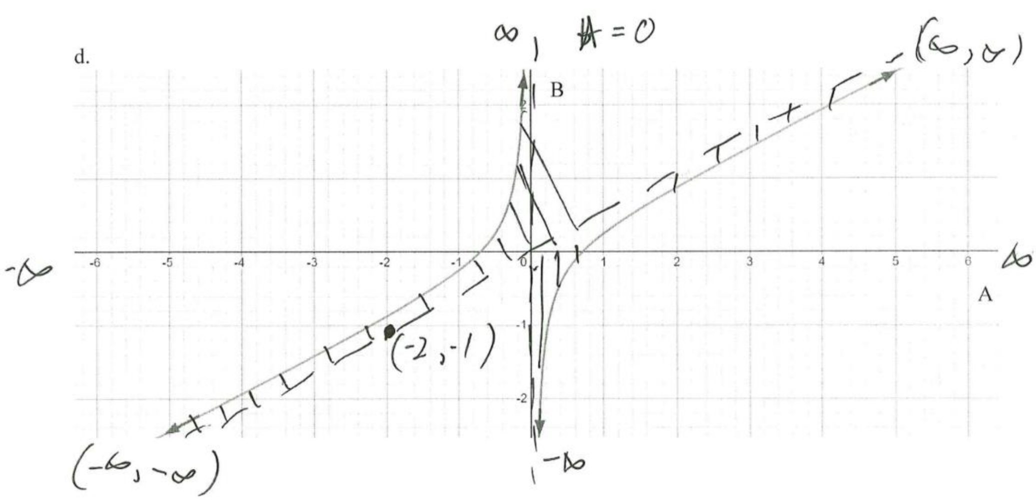
Researcher: Ok.

Steven: So, as A approaches (-2, -1), B is also approaching (-2, -1), because it's coming towards it. Because the distances are decreasing here. It's hard to show. But like, the length of this distance is shorter than the length of this distance [*points to two consecutive distances closer to the point (-2, -1)*], right? So, it's getting closer to this point on the line, but this line is also decreasing, so B is decreasing to a point where, as A decreases to negative infinity, B approaches some number on this sloped line [*points to slanted dashed line*]. It's similar to how it would have approached a single number on this line [*points to horizontal dashed line in Graph A*] or on this line [*points to horizontal dashed line in Graph B*] or on this like, imaginary line [*points to horizontal dashed line in Graph C*], but instead of a straight line, it's a sloped line. And then same this direction [*points to where the graph is increasing on the right*] except positive.

Since Steven had already given evidence of using the distances to inform how the graph approached specific A or B values, I did not spend time asking him to go into detail where Quantity A approached 0. To try to get Steven to go into more detail about how Quantity A and B were changing in general about the slanted line, I asked him to say more about the numbers Quantity B is approaching. He used the change in magnitude of the distances between the graph of the relationship and the slanted dashed line to confirm that the values of Quantity B were the line, making the claim that Quantity B was approaching a value on the dashed line. Recall that Steven's generalization from Graph C claimed that if the distances between the graph of the relationship and the dashed line were decreasing, the graph was approaching a value. He extended his generalization in that, while the function was approaching a value, the values being approached were approaching negative infinity *(MA2)*. He explained that the numbers are decreasing overall, eventually writing (-∞, -∞) and (∞, ∞) (see Figure 42) as "some point" the graph is approaching, since "that is the direction both sides are going towards." He added, "…just to show it's going somewhere, I'm saying that it's going to a point that's an infinitely large value of B and an infinitely large value of A…because this line [*the graph*] is increasing towards infinity in both sides." He also clarified how this is different from the other graphs, referring to Graph A and stating that the function is only increasing to infinity "in one way" whereas for Graph D it was increasing "in both ways."

Figure 42

Steven's Graph D.



Moving to Graph E, Steven first drew in dashed lines where it seemed to him the function was approaching a value or a specific relationship. He was also specific in noting all the places the one quantity increased. For example, when describing what happens when Quantity B approached positive infinity, he argued, “A is approaching this number slightly bigger than 1 *and* negative infinity *and* positive infinity.” As before, his observations consisted of what happened with one quantity when the other quantity approached infinity or negative infinity. Even though he did not draw in the distances, he referred to them in his descriptions of the variations on the quantities.

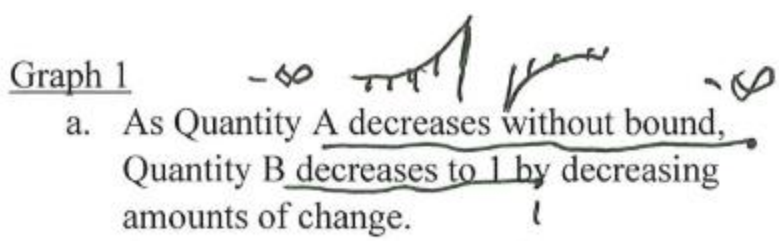
Since Steven had shown similar ways of thinking as in the other graphs in his initial observations, I did not spend much more time asking about Graph E specifically. Instead, I moved on to ask how he what similarities he saw when considering all the graphs of Part 1 together. He concluded “to see what the graphs are about, we want to look at what things they’re approaching,” whether that be positive infinity, negative infinity, or a specific value. He also restated his generalization of how to use the distances to decide if a value is being approached (when the distances decrease) or if the quantity is approaching infinity (when the distances increase). To conclude Part 1, I asked Steven if there was a difference in how he imagined a quantity approaching infinity versus approaching the non-vertical and non-horizontal dashed lines, such as in Graph D and Graph E. He explained that as Quantity A increased, Quantity B “approaching the same value as that parabola looking line, but that value is going towards infinity, so this value also goes towards infinity.”

Part 2

I suggest to Steven to focus on statements a – c in Graph 1 because of a typo in statement d I did not realize until Harper's interview. As Steven began working on Part 2, he continued his reasoning from Part 1 to inform his construction of the graphs. Working with Graph 1, as he read through the statements, he underlined key phrases, drew sketches to show how he was imagining certain relationships, and began drawing in dashed lines for what values were being approached before moving on to the next statement. For example, Figure 43 gives insight into what Steven emphasized as key information and shows his sketch of what it means for the Quantity B to have decreasing amounts of change (the sketch on the right). After reading statements a and b, he drew in a dashed line at B = 1. After reading statement c, he drew in a dashed line at A = 4.

Figure 43

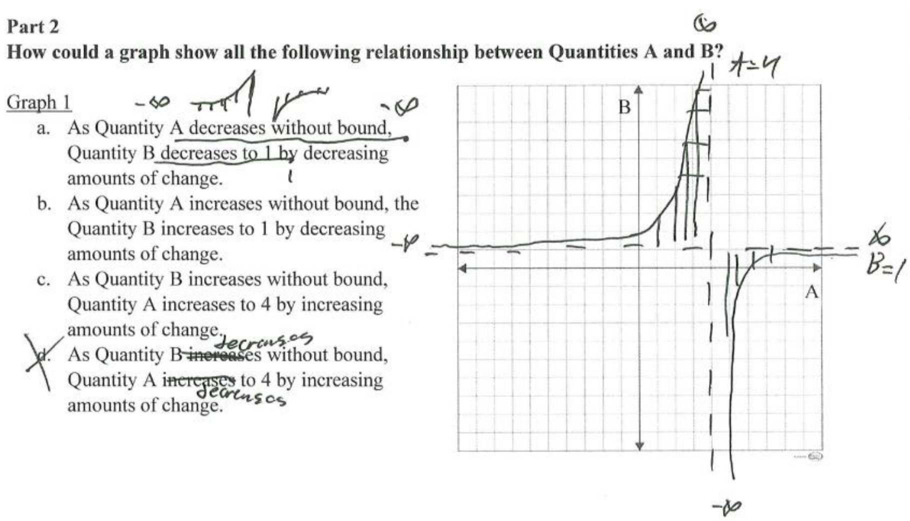
Steven underlines key phrases and creates sketches of what he imagines for increasing (left sketch) and decreasing (right sketch) amounts of change.



As Steven considered what the shape of the graph could be, he seemed to have some difficulties in how to incorporate statement c. He repeated the statement multiple times, at one point stating, "A is…increases to 4 by increasing amounts of change. That's weird." While making sense of statement c, like Harper, he gestures with his hands to help visualize the relationships between A and B. After some deliberation with statement c, he concludes, "Because the values of A are increasing, so as it goes towards 4, the values are increasing, so it's becoming bigger and bigger like that [*points to left sketch in Figure 43*]." After he felt he made sense of statement c, he drew in a sketch for Graph 1 (see Figure 44) and justified his sketch by checking back over the statements. Once he finished his initial sketch, I asked Steven to revisit the difficulties he seemed to be having with increasing amounts of change in statement c to gain insight into what caused that difficulty (see Excerpt 22).

Figure 44

Steve's Graph 1 in Part 2.



Excerpt 22. Steven explains why he had difficulty with statement c.

Steven:	So, I was thinking about it as if, like, the graph were flipped [*turns paper 90°*] and I'm like, "increasing? This isn't increasing, this are decreasing," right? These values [*draws in horizontal distances between graph of the relationship and A = 4*] are decreasing. But I'm then like, "wait a minute, we aren't looking at it like that" [*turns paper back to original orientation*]. The values are actually increasing,

> so I was just looking at it in the wrong direction. I was thinking about it flipped. Like the same way I drew B, like that, in my mind when I was explaining.

Steven continued to explain that he expected the amounts of change to be decreasing, since Quantity A was approaching the value 4 (applying his generalization from Part 1). He stated, “I just thought, it was, ‘oh it’s approaching 4, so the values must be getting smaller and smaller,’ because it’s approaching 4, but in all actuality, it’s approaching 4 but the values are getting bigger because it’s…B is approaching infinity as well.” Steven’s initial difficulties and then application of his generalization from Part 1 shows how he imagined the variation of both quantities A and B, even when having to be flexible in the application of his generalization to both.

As Steven concluded Graph 1, many of his actions were comparable to what he showed in Part 1, so I did not spend much time pursuing more insights from him. Instead, I chose to move on to Graph 2. As Steven began working with Graph 2, he drew in dashed lines at B = 0, A = -5, and A = 4 as he read each statement. When working with Graph 2, he used gestures to help visualize directional changes, pointing different directions on the graph as he read the statements. He had difficulty with each statement, mostly statements c and d, and used gestures to help parse the statements. After reading statement c, Steven claimed, “the increasing amounts of change, that’s what I was missing.” He began by drawing in the left-most curve on his graph. As he began to describe the variation in Quantity A and B to ensure it matched the statement, he connected back to Quantity B, stating, “…wait. No! Because it’s asking about Quantity B, so, ok, ok. So, as Quantity B increases without bound, so if we are thinking about it like B is increasing without bound [*points up*], …so as B increases without bound, Quantity A increases to -5 [*moves his finger towards A = -5 from the left*] and decreases to 4 [*points to A = 4*] …" Having this insight, Steven used this change in directional thinking to reconsider the other statements and began drawing his first sketch of Graph 2.

Steven began by graphing the Quantity A decreasing to -5 and increasing to 4. After he drew each curve, he would check it against the statements to ensure the curve he drew matched. Doing this made him second guess his first iteration of his graph. As he drew new iterations of Graph 2, he used different colors, starting with black, then green, and then pink (see Figure 45a). Because he used the same graph for his brainstorming, I have recreated each iteration in Desmos for visual clarity (I have made every effort to recreate his brainstorming as closely as possible) in Figures 45b-d. Each time Steven completed an iteration, he checked his graph against the statements and would be unsatisfied with the graph as it related to the statements.

Figure 45a

Steven's first three sketches of Graph 2 in Part 2.

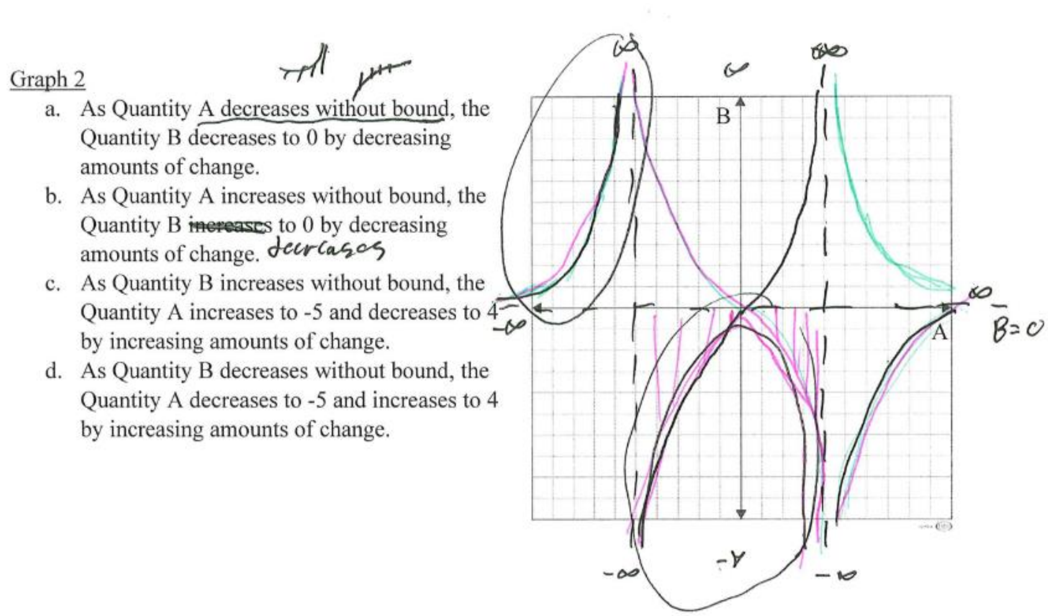
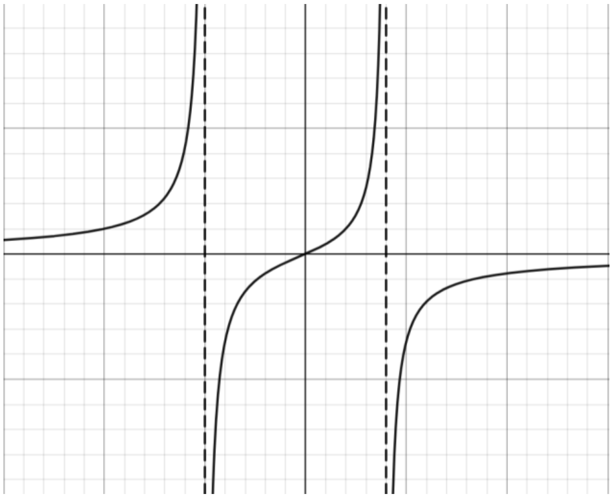


Figure 45b

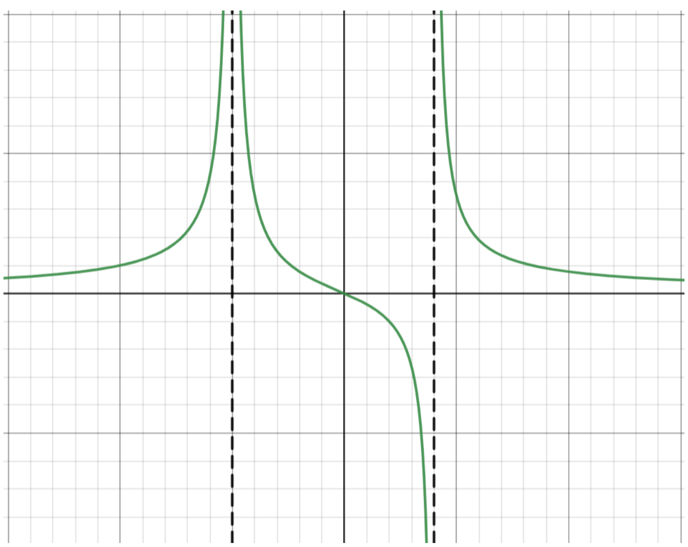
Steven's first iteration of Graph 2 in Part 2



Note: Recreated in Desmos by me for visual clarity.

Figure 45c

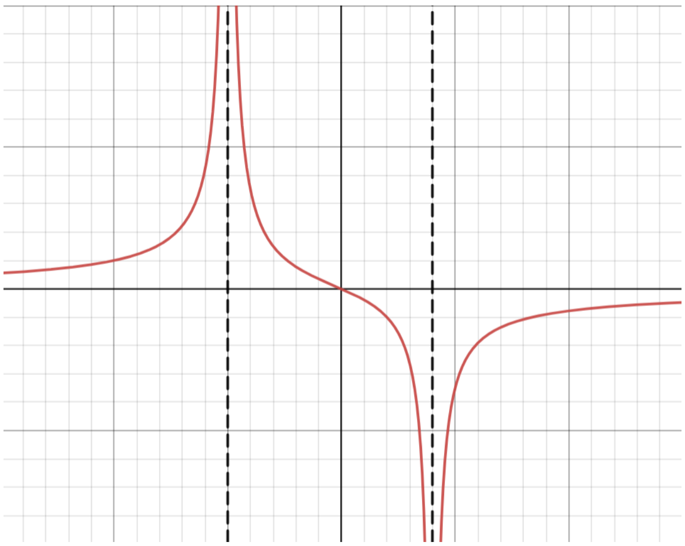
Steven's second iteration of Graph 2 in Part 2



Note: Recreated in Desmos by me for visual clarity.

Figure 45d

Steven's third iteration of Graph 2 in Part 2



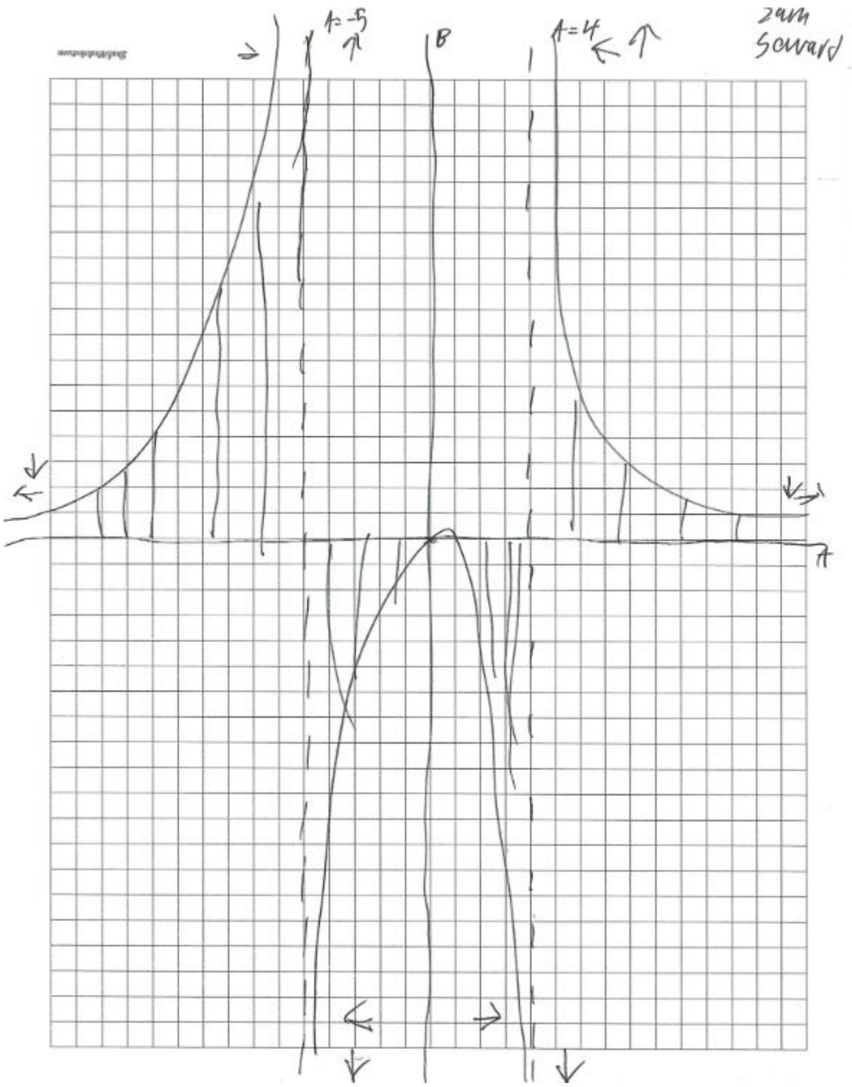
Note: Recreated in Desmos by me for visual clarity.

As Steven continued to second guess his graph, I suggest he focus on the middle. I prompted this because he had drawn two different curves (see Figures 45b and 45c), and I was interested how he could use the statements to create a curve he felt was satisfactory. As he compared his sketch to statement d, he drew in the distances (like in Part 1) to help visualize the increasing amounts of change piece. Once those distances were drawn in, Steven concluded that since Quantity B is decreasing without bound when Quantity A decreases to -5 and increases to 4, the graph must "point down" (see arrows in Figure 45 as more evidence of this conclusion). After drawing different sketches on top of one another, Steven drew a new sketch on a new piece of graph paper that he felt sufficiently captured the statements. He did change statement b because he did not feel that statement b and the latter part of statement c could be true at the same time (see Figure 45a).

We concluded Part 2 by discussing if Steven saw any benefit in being given information about the amounts of change for each statement. He explained that knowing the amounts of change helped him “prove” if his sketch was correct. He drew in the distance lines on his final sketch and discussed that these lines helped him recognize that his relationships reflected the “rules” he was given, stating, “It just helps you to make sure that it’s [the graph] working.”

Figure 46

Steve’s final sketch of Graph 2 in Part 2.



Part 3

As Steven began Part 3, he read the context aloud. As Steven considered the given equation, he first focused on redefining *s* as the speed of the current and *T* as the time it would take Mike to reach his destination. He then went on to describe what the values of 150 and 30,000 represent in context.

I prompted him to also explain what 150 - *s* represented to how he would contextualize that expression. Steven began by explaining, "He's paddling 150 feet per minute, but then the current is pulling him back however many feet the current speed is" and then used his explanation as a bridge to question b. Steven explained, "…if the speed of the current is 150, then his speed minus the current speed equals 0. So, he's not going anywhere." Steven elaborated to say that because he is not moving, Mike will never reach his destination, so there is no time to consider.

As Steven progressed through questions c and d of Part 3, he coordinated the speed of the current to the time it takes Mike to reach his destination. He related the change in the speed of the current speed to Mike's change in distance (see Excerpt 23) but then extended that to how Mike's change in distance affected the time the journey would take. Steven was the first student to coordinate the speed of the current with time. As he continued to part d, he first reasoned that Mike would be moving downstream when the current speed was greater than 150 and then used *s = 200* to give an example in context. Unlike Amy, Steven plugged in values into the equation as evidence of claims he made with the context, not vice versa. Harper did not use specific values at all, relying solely on the context as her justification.

Excerpt 23. Steven discusses changes in speed of the current on time for Mike's trip.

Steven:	As the speed of the current approaches 150 feet per minute, the time it takes for him to reach his destination increases as well. Because, so for example, when the speed of the current is 0 [*writes in the equation for time with 0 substituted in for s*], that's a lot faster than if his speed, or if the speed of the current was 149 [*writes in the equation for time with 149 substituted in for s*], right? Because there [*points to equation on the left*] he'd be moving 150 feet per minute, but here [*points to equation on the right*] he'd only be moving 1 foot per minute. So, as the speed of the current increases, the time it takes for him to go increases as well.

Steven used similar reasoning in this task as he did in Parts 1 and 2. He focused his thinking on the covariation between the speed of the current and the time the journey would take Mike, being explicit in his consideration of how changes in speed of the current affected the time of the journey. He used a dashed line on his graph to show value is being approached. Since he felt confident in the directional changes in the quantities (specifically, as speed of the current increases, the time of Mike's journey also increases), once Steven had a starting value, he was able to produce a graph. For his graph, Steven chose to relate time the journey takes (the vertical axis) to the speed of the current (the horizontal axis). He began by scaling his horizontal axis to reach at least 150, "because that's when the current surpasses Mike's ability to paddle." The following excerpt shows Steven reasoning as he draws his graph (see Figure 47).

Excerpt 24. Steven's explanation of his graph for Part 3.

Steven:	As the current's speed decreases, his time reach the destination also decreases, so how long does it take Mike to reach when the current is 0…so when the current is 0, that'd just be 30,000 divided by 150, that'd be…200. So, it would take 200

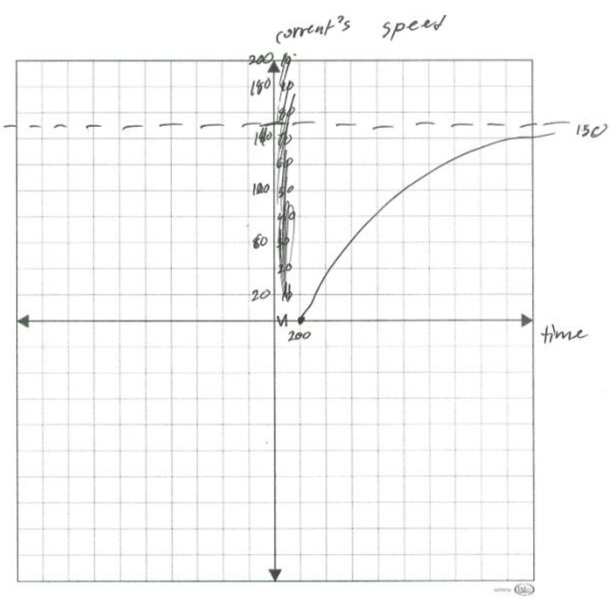
minutes when the current speed is 0. So current speed is 0, time is 200…I'll put a little thing there [*plots point (0, 200)*]

…

When the current speed is 150, the time it takes is infinite, right? Because he never makes it there. So, if we did like we did before with the [*draws in horizontal dashed line at 150*] …uh…imaginary line here, this is the current speed is 150, so… [*draws in increasing curve from plotted point to horizontal line*] something like that. Because, it's not exact, but since we know that when the current speed is 0, it would take 200 minutes and when the current speed is 150, it takes infinite amount of time, you can see that as, uh, as the current speed increases, the time it takes becomes, wait sorry no, as the current speed approaches 150, the time it takes becomes larger and larger and larger until it approaches infinity.

Figure 47

Steve's graph for Part 3.



Steven showed consistent evidence of *MA2* and *MA3* throughout his interview. In Part 1, Steven used focused his observations on what happens to one quantity as the other quantity increased or decreased without bound. He showed fluidity in changing between the two quantities as he reasoned the with the changes. Steven drew horizontal and vertical lines that represented the distances from the line that represented the relationship to an axis or dashed line. He then used comparisons of the lengths of these distances to reason that a quantity was increasing or decreasing. By the end of Part 1, Steven created a generalization that a graph is approaching a number if these distances were "getting smaller" and the graph is approaching infinity (or negative infinity) if these distances were "getting larger." In Part 2, Steven used the generalization above to produce graphs that coordinated Quantities A and B. He also considered all four statements in the production of his graph, revisited the statements throughout his sketches to verify if the relationships graphed were accurate. Working Part 3, Steven used the context to make claims about how the speed of the current affected the time of Mike's journey.

CHAPTER 5

DISCUSSION

In this chapter, I summarize the meanings each student presented during their interviews, giving evidence from the interviews. After summarizing each student's meanings through engagement with the task, I discuss broader takeaways from the interviews. Finally, I discuss areas of future research.

Amy

During Amy's engagement in the tasks, I inferred her to exhibit two primary meanings. First, she exhibited a view that considered the graph to be a line with non-relational parts, or a figure to use specific steps to describe the relationships, rather than the values changing together. Amy's conception of a graph in general as well as when the graph approached a value was consistent with Moore & Thompson's (2015) static shape thinking. For example, when asked questions about Quantities A and B, she referred to the axis lines as fixed, considering A and B to be labels of the axes. In the beginning of her interview, Amy referred to x and y as directions, rather than quantities. She did illustrate "spaces" tied to values being approached.

Second, she exhibited evidence of coordinating directional changes between two quantities. Amy used arrow notation to relate directional changes in quantities, which suggested repeated evidence of Mental Action 2 from the Carlson et al. (2002) framework. This arrow notation was most often used when one quantity was increasing or decreasing without bound or approaching a dashed line Amy drew in to represent a value being approached. This suggests

Amy's coordination of the quantities still relied on her static shape reasoning, using the dashed line as a reference line.

Due to Amy's shape-based reasoning being her primary source of reasoning with the quantities, her experience producing graphs in Part 2 was constrained, since there was not a graph given for her to reason about. Amy treated each section of each statement in isolation, used A and B as starting points. She showed some awareness of how increasing or decreasing amounts of change would affect the shape of the graph. For example, Amy correlated "exponential" with "increasing amounts of change," thus graphing a curve that had increasing amounts of change.

Harper

As Harper worked through the task, she exhibited a primary meaning consistently focused on the quantities. Like Amy, Harper relied on static shape reasoning to make initial observations about the graphs, but again, she would pull her focus to the quantities. Harper showed evidence of attention to directional changes in the quantities, using arrow notation like Amy's to describe the directional relation between two quantities (Mental Action 2 in the Carlson et al. (2002) framework). She also showed some considerations for amounts of change (Mental Action 3 in the Carlson et al. (2002) framework). For example, when describing areas the graph is approaching a value or comparing different parts of the graphs (or different graphs), Harper used slope or rate as ways to explain how the quantities were changing. Unlike Amy, Harper was consistent in her use of the arrow notation and understanding the changes in the quantities x and y "work together."

Moving from Part 1 to Part 2, Harper showed evidence of covarying two quantities, with hints of using amounts of change to justify certain kinds of changes. For example, she often related increasing amounts of change or increasing without bound to “jumps” in the graph getting larger. While she considered the covariation of the quantities within each statement in Part 2, she did not consider the implications of all the statements together. Despite the difficulties Harper had in creating the graphs, she showed a propensity to focus on quantities, through gestures and the use of the arrow notation. The difficulty was in representing the relationship. While Harper used covariational reasoning to motivate the existence of the asymptote in the graphs, static shape thinking constrained her graphs after the dashed lines represented the asymptotes were drawn in. Once the dashed lines were present, Harper conceptualized them as lines that could not be crossed, leading her to not consider how the multiple statements could combine to be drawn as a single curve.

Steven

Steven’s primary meaning focused on the use of distances to show if a function approached a value or increased (or decreased) without bound. Stephen used the distance of the graph of the relationship from an axis (represented by the horizontal and vertical lines) or asymptote, using these distances to create a generalization of the relation to the change in magnitude of the distances to the behavior of the graph. Steven generalized when a quantity is approaching a number, distances shrink in magnitude and when a quantity is approaching infinity, the distances grow in magnitude. This generalization played a key role in Steven’s success in Part 2. For example, when constructing the graphs in Part 2, he drew in small sketches using these distances to visualize information related the amounts of change of a quantity. Even when faced with a question that could not work in the context of the other statements in Part 2 (Graph 1, statement d), his meaning was stable enough that he knew it to be decreasing amounts of change.

Throughout the interview, he described the changes in Quantity A and B using both directional changes (e.g., as Quantity A increased, Quantity B decreased) as well as some evidence of amounts of change. Like Harper, Steven would use gestures with hands or pen to visualize directional changes. Unlike the other two interviews, he used unit-changes when first looking at amounts of changes in a quantity, being the only participant to show some evidence of Mental Action 4 in the Carlson et al. (2002) framework. Steven showed the fluidity in using each quantity as the independent variable, often restating statements such that each quantity was independent to consider variation in the other quantity.

Main Takeaways

The goal of this study was to better understand the ways students conceptualize asymptotic behavior in rational functions and the implications of those conceptualizations. Based on the data, students showed two primary methods of reasonings: static shape thinking (Moore & Thompson, 2015) and covariational reasoning (Carlson et al., 2002). Stephen showed some emergent shape thinking in his use of covariational reasoning. The task design of Part 1 gave insight into which of these types of reasonings students organically used. The implications of each of the ways students reasoned in Part 1, including constraints of each type of reasoning, was evidenced in the productions of the graphs for Part 2. Primary meanings using static shape reasoning led to more constraints in the conceptualizations of asymptotes in Part 2. Since the visual cues of the graphs were not present, there was no basis rooted in the changing quantities for the asymptotic relationship. Student conceptualizations of asymptotes were not a byproduct of the covariation of the quantities. Rather, the asymptotes were an inherent feature of the rational functions, and their existence implied specific relationships in the quantities. On the contrary, covariational reasoning led to stronger conceptualizations of the asymptotic relationship of rational functions. Covariational reasoning led to asymptotes being a product of a particular form of covariation between the quantities which could be reasoned with both graphically (Part 1) and verbally (Part 2).

Student evidence of covariational reasoning in Part 1 aided the conceptualization of the statements in Part 2 when producing a graph. The production of Amy's graphs showed benefits of using covariational reasoning, despite the constraints from using static shape thinking. For example, Amy drew an "exponential" curve to show the "increasing amounts of change" in Graph 1 statement c. Steven showed evidence of covariational reasoning as well as emergent shape thinking in his generalization of the role of the distances on the behavior of the graphs in Part 1. The use of distances and the generalizations by Steven from Part 1 aided his production of the graphs in Part 2. His ability to generalize the affect the distances had on the shape of the graph shows he attributes the shape to the variation of the quantities, not the other way around.

When students primary reasoning was static shape thinking, the asymptotes became an inherent feature of the graph, rather than the product of the covariation between the quantities. The presence of the asymptotes then led to specific statements about the relationship between the values. For example, both Amy and Harper showed directional covariation using arrow notation. Amy's directional covariation was motivated by the shape of the graphs in Part 1. Amy made claims such as "the line decreasing and approaching -2" when describing the relationships in Part

1, using the asymptotes as a reference line for her quantities. Harper's use of the arrow notation was based in an attention the quantities. Harper also used the asymptote as a line of reference, but her direction covariation did not rely on the image of the line being present. Applying these ways of static shape thinking in Part 2 presented constraints for Amy and Harper. The application of static shape thinking led to Amy using the labels of the axis as starting points of her graphs. Harper used directional covariation to produce the curves, but she was constrained by static shape thinking in connecting her curves to create a single graph. Harper saw the dashed lines as something that could not be crossed, once present on the graph. Static shape reasoning made the production of the graphs in Part 2 constrained because there was no visual cue for students to apply reasoning. Thus, with no graph present, inferences regarding quantities and their variations are not readily available. Also, static shape reasoning does not lend itself to be reversible since the properties or shape of the graph are not consequences of covariation between quantities.

There are two important notes to the nature of Part 2. First, Part 2 is more cognitively complex on the part of the reasoner. In Part 1, since the graph is given, you have the material to conceive of quantities and relationships. The reasoner has the ability to operate on the graph, like distances in Steven's case or "spaces" in Amy's case. When given statements (or equations or tables), the reasoner must produce any such material. They are placed in a situation in which they must construct the material that is constrained by how the quantities need to change together. With respect to a normative conception of a graph, they must construct and unite segments, produce a point, and then produce a trace for that point that captures the appropriate relationship (emergent shape thinking as defined by Moore & Thompson, 2015). Also, when charged with production of the graph given only directional and amounts of change information, the reader must consider an infinite number of options, an infinite amount correct graphs and an infinite not. Second, constraints in the production of the graphs in Part 2 do not necessarily imply we should disregard successful covariational reasoning in Part 1 (such as Harper's case). There is not enough information to know if the student had previously learned to associate covariational language with movement or shapes of the graph. There might be some attachment to anticipated quantities, but they might not have the graph conceived of two quantities the same way as emergent shape thinking is defined by Moore & Thompson (2015). Part 2 does give insight into the generative nature of the reasoner. The researcher can see to what extend the reasoner can apply covariational relationships.

Limitations

There are limitations to the study based on the population of students and the design of Part 2. All three students can from the same high school classroom with me as their instructor. While the students did not have any direct instruction on reasoning with amounts of change prior to participating in the interview, they may have been exposed to specific models of thinking through my teaching that could have impacted how they reasoned with this task. The selection of the students was based on pulling from a pool of volunteers from my classroom; thus, resulting in a specific population of students.

In the design of Task 2, some covariation statements were repetitions of a prior statement (e.g., Graph 1 statements c and d) or were not possible to graph as phrased in the task (e.g., Graph 1 statement c or Graph 2 statement b). These errors were not seen until the second interview, and I did not want to alter the tasks between interviews for the sake of consistency among the data. Even though Task 2 presented students with statements that were not possible to graph, I still gained insights into how students applied their CR in their understanding of the statements. With respect to Graph 1 statement c, if the student applied the "increasing amounts of change" to Quantity B's increase without bound, the relationship between Quantity A and Quantity B still results in a viable graph. If the student applied the "increasing amounts of change" to Quantity A as Quantity increased to 4 and the student notes this as an impossible relationship between Quantities A and B, I was given insight into their CR of the quantities and via their determining that covariational relationship is not viable. The student could use CR to justify why they must reject the problem statement, or they could use some alternative form of reasoning to consider the statement. Regarding Graph 2 statement b, while it is possible to create such a graph, the student must recognize the function crosses through the horizontal asymptote and then begins to increase to 0. Given the nature of the graphs in Part 1 and Graph 1 in Part 2, the students are not set up to perceive this as a viable relationship.

Future Research

This thesis presents a basic study, shedding light on how students may begin to use QR and CR in context of rational functions, but there is still work to be done. The nature of the gap in research on student thinking on asymptotes lends itself to the need for further research. Studies, such as this, provide students the opportunity to use covariation to explain boundlessness, and such an opportunity could work towards productive conceptions for limits. This study also illustrates the importance for students to be flexible in their thinking with quantities as far as dependent and independent variables. As illustrated above, asymptotes, whether vertical or horizontal, can be conceived as being mathematically equivalent. For this reason, rational functions may provide an important area of study promote such reasoning. More generally, the use of CR by the students in the task, even if not used in a sophisticated way, suggests that asymptotes could be a topic to help generate CR and emergent thinking in students. That is, rational functions and asymptotes could be a topical area by which to create a need for covariational reasoning, especially as a productive form of reasoning, and this should be explored in further studies.

Given the nature of the data, we are also presented the opportunity to consider the implications of shape reasoning versus covariation. I posit CR is a productive approach to the teaching and learning of rational functions and end behavior, based on the conceptual analysis and promise of QR and CR in other areas of mathematics (Carlson & Moore, 2015, Madison et al., 2015, Moore, 2014, Weber et al., 2014). However, this thesis alone is not enough evidence. Future work on supporting such an approach with students, both in the construction and the application, is needed to test the viability of this belief. Regarding constructing this approach, exploration into the learning on these concepts is needed. Future research can begin to consider what different meanings for asymptotes students construct. Considering what meanings students naturally consider gives insight into how students are reasoning with these concepts, and what prerequisite knowledge is informing those meanings. Future research can go on to consider how the meanings students construct influence the development of other related ideas. Referring to the current literature base, there is already a case for tying student reasoning of asymptotes to higher level mathematics (Cooke & Bossé, 2019; Giblin, 1972). Building from how students construct meanings builds a bridge to the current literature base.

If shown as a viable and productive mathematic orientation (Weber et al., 2014) in student conception of asymptotes, there is work to do to get it infused into school curriculum and instruction. I am aware the system at-large is complex, and change is slow and time consuming. To push for changes would be a huge and difficult undertaking, but I believe it is worth it. I deeply know the challenges and pressures students and teachers alike face in the current school climate. I know in my own classroom I have a difficult time justifying taking the time to recreate teaching approaches on top of all the other duties expected of me, even if I see the success in a lab setting. Even still, I restate my belief in the goal of teaching mathematics well. We want to create critical thinkers--students that not only *do* mathematics but *understand* mathematics. To do so, there is a need for change. For change, there is a need to better understand how we can better serve our students.

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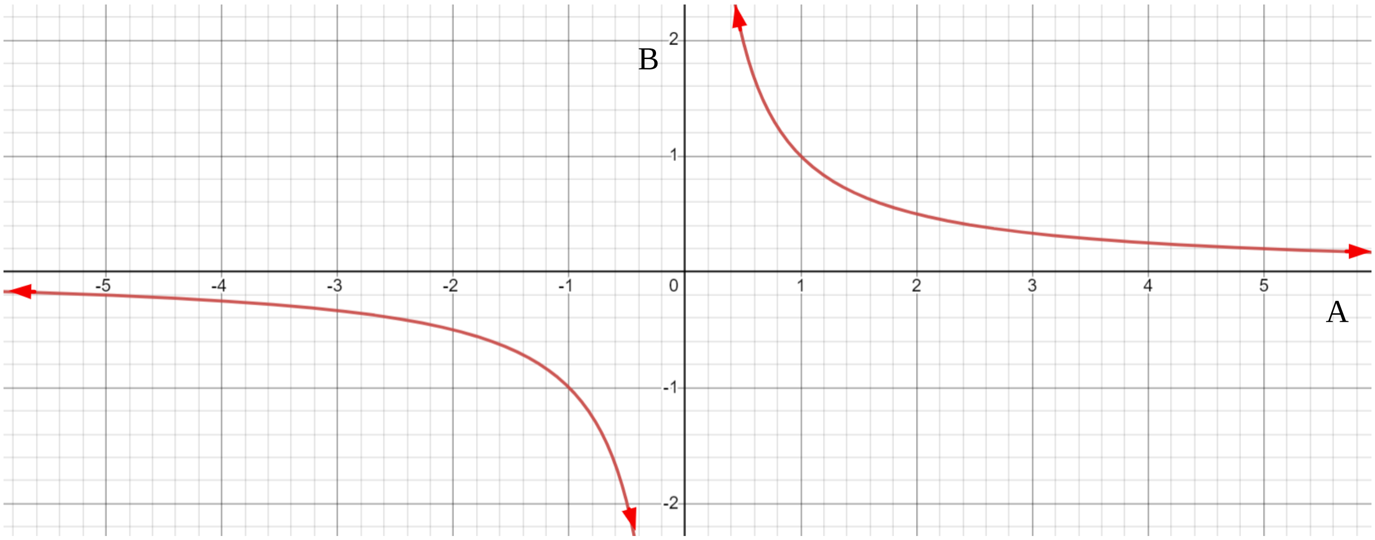
APPENDIX A

TASKS

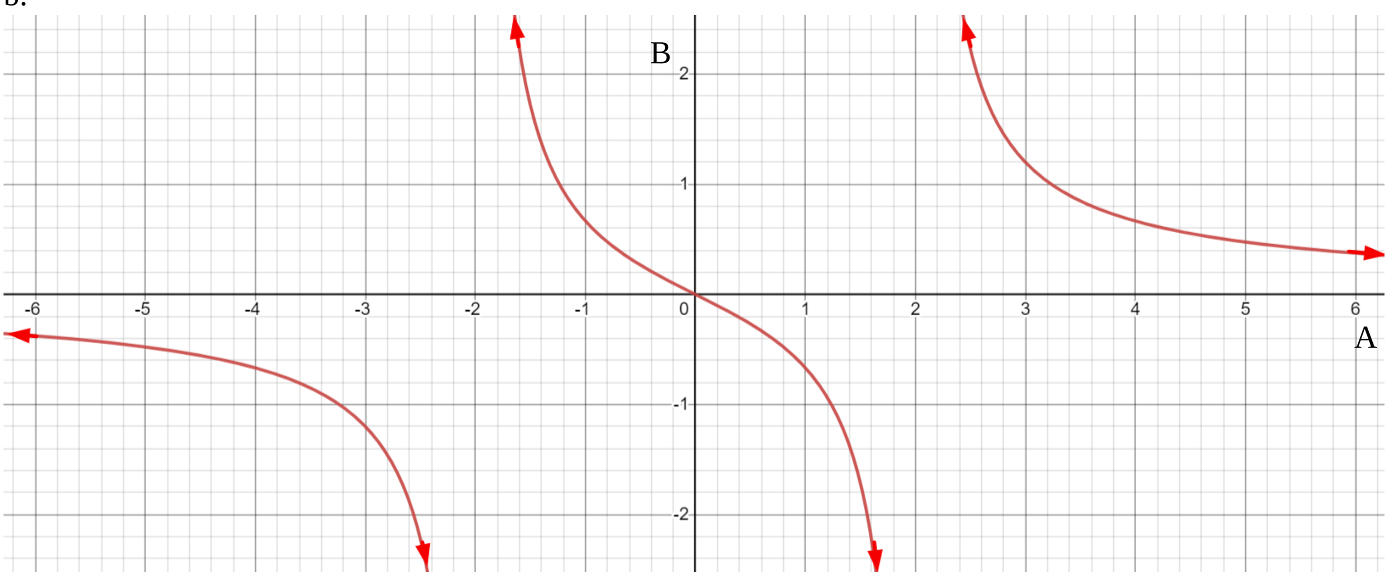
Part 1

Consider the following relationships between two Quantities A and B.

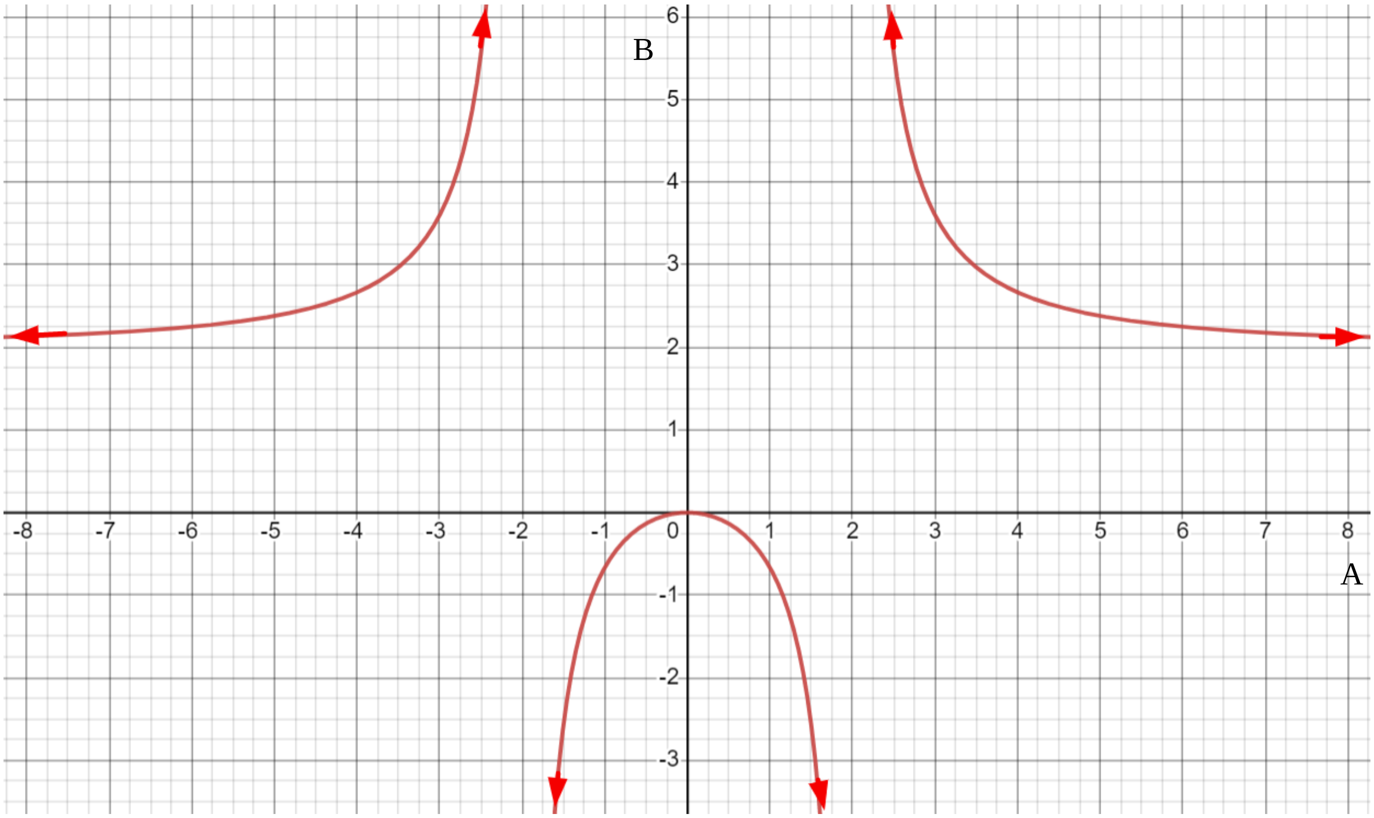
a.



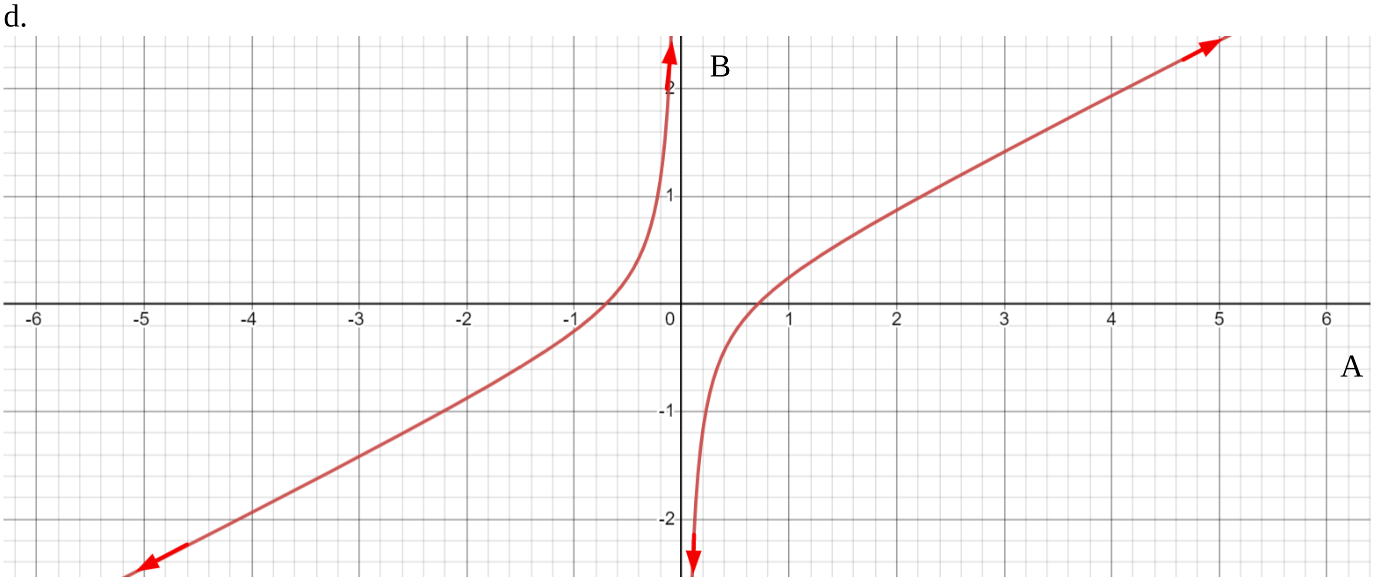
b.



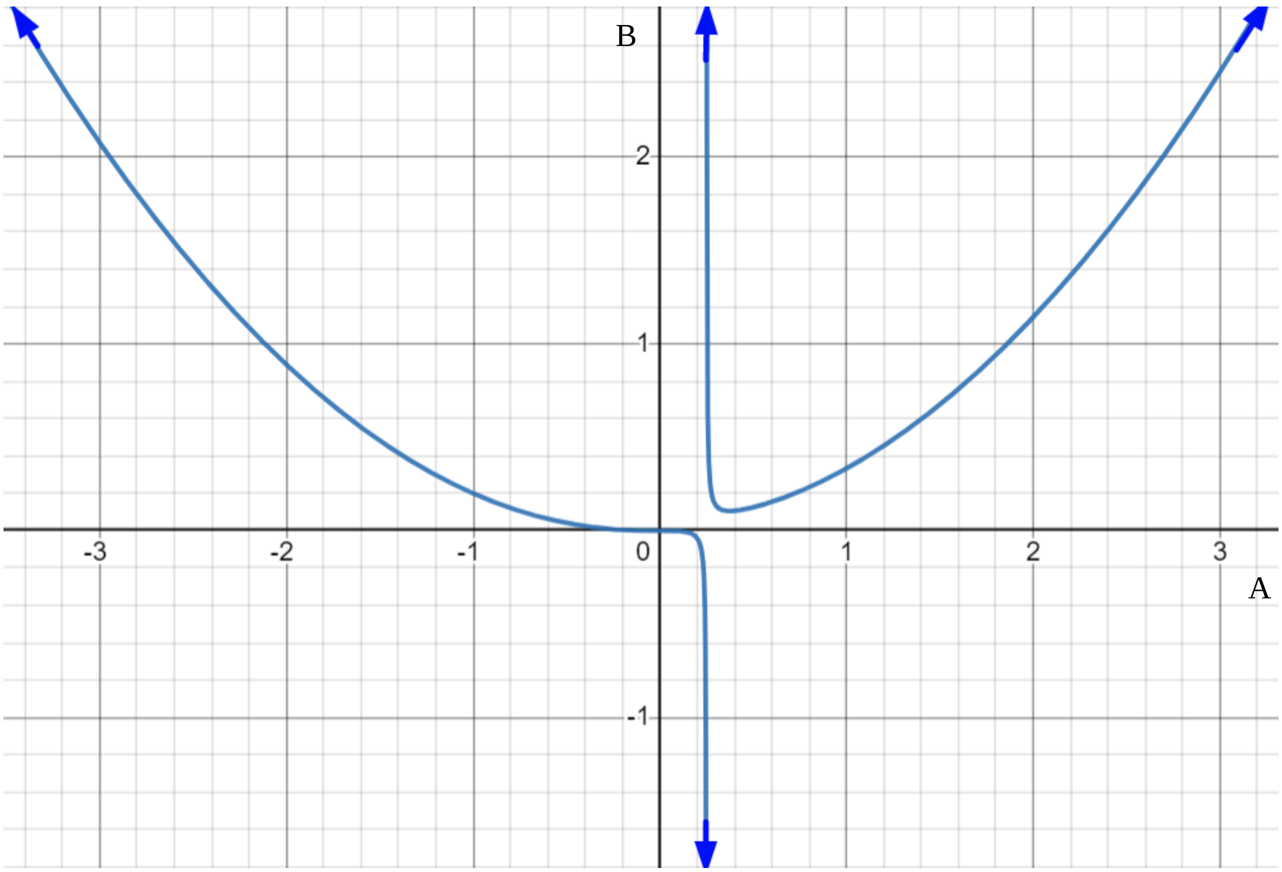
c.



d.



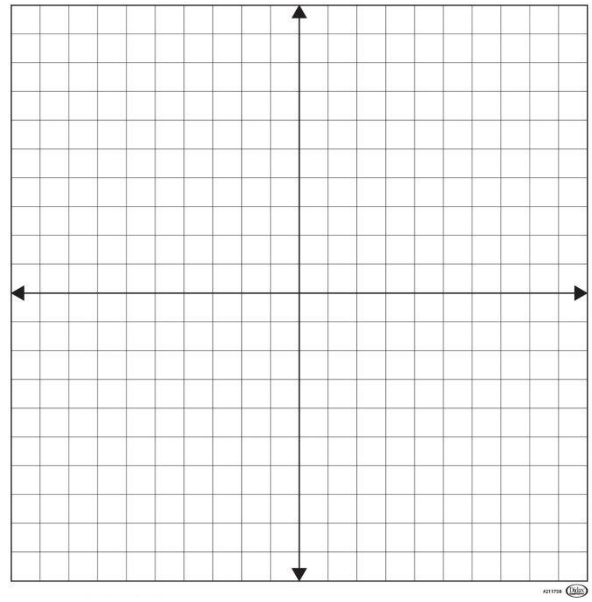
e.



Part 2
How could a graph show all the following relationship between Quantities A and B?

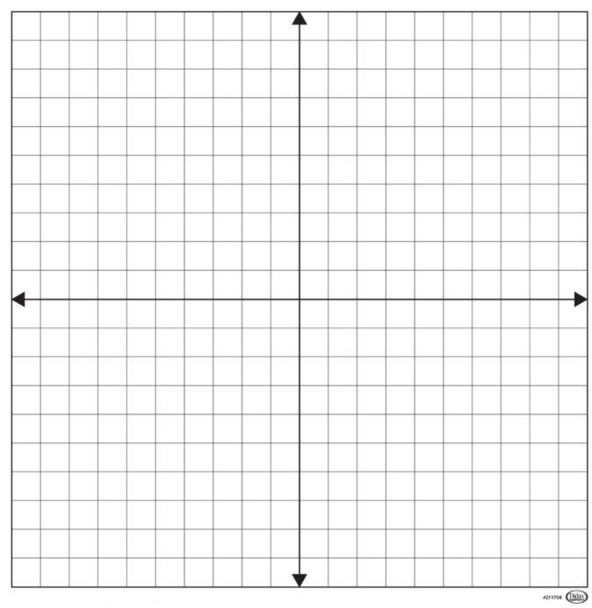
Graph 1

a. As Quantity A decreases without bound, Quantity B decreases to 1 by decreasing amounts of change.
b. As Quantity A increases without bound, the Quantity B increases to 1 by decreasing amounts of change.
c. As Quantity B increases without bound, Quantity A increases to 4 by increasing amounts of change.
d. As Quantity B increases without bound, Quantity A increases to 4 by increasing amounts of change.



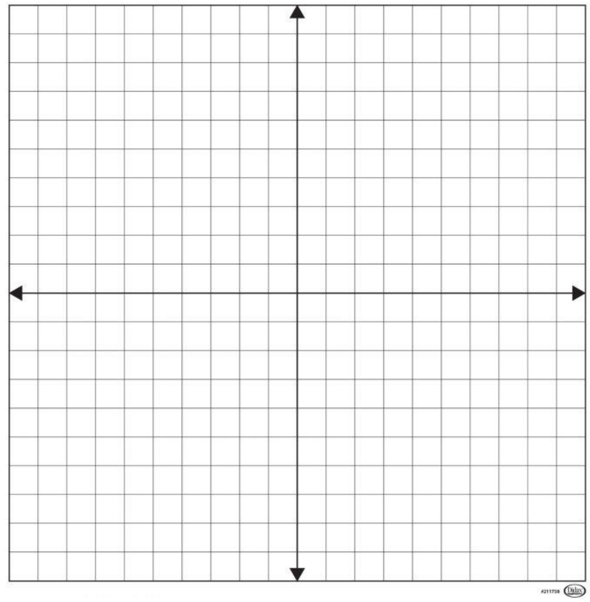
Graph 2

a. As Quantity A decreases without bound, the Quantity B decreases to 0 by decreasing amounts of change.
b. As Quantity A increases without bound, the Quantity B decreases to 0 by decreasing amounts of change.
c. As Quantity B increases without bound, the Quantity A increases to -5 and decreases to 4 by increasing amounts of change.
d. As Quantity B decreases without bound, the Quantity A decreases to -5 and increases to 4 by increasing amounts of change.



Graph 3

Write 3 characteristics and create a graph (or graphs) to capture those characteristics.



a.

b.

c.

Part 3 (adapted from Illustrative Mathematics)

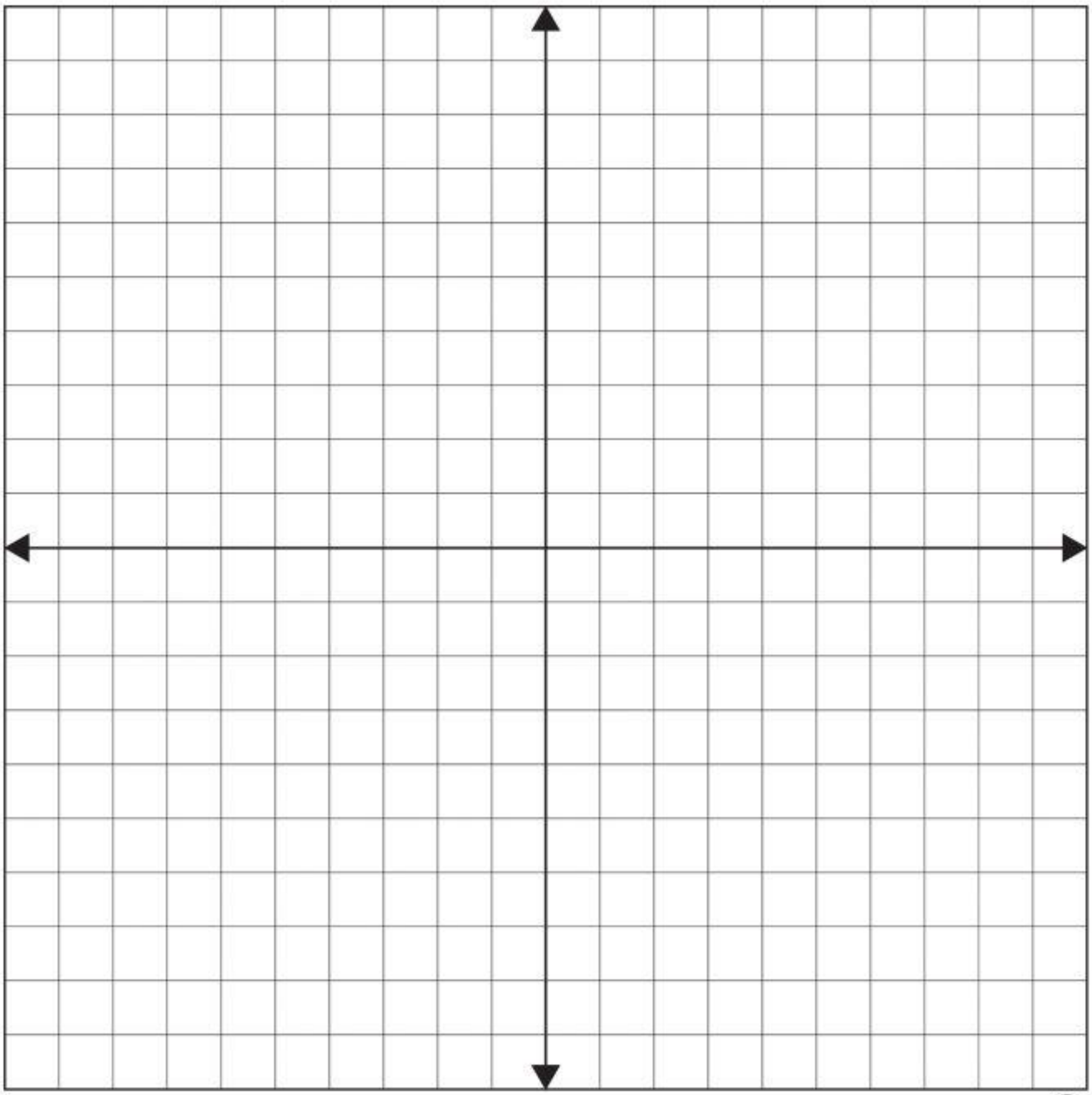
Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle. The function $T = \frac{30,000}{150-s}$ represents the time it will take Mike to reach his destination, where T is time in minutes and s is the speed of the current in feet per minute.

a. How can you interpret the function $T = \frac{30,000}{150-s}$? What does each part represent?

b. What happens if the speed of the current 150?

c. What happens when the speed of the current increases towards 150 ft/min?

d. What would happen if the speed of the current is greater than the speed Mike is able to paddle (150 ft/min)?



APPENDIX B

PRE-WRITTEN QUESTIONS

Questions to ask for Part 1 (to be expanded on during the interview):
Key phrases/Notes to self:

- Start generic and then work towards more specific ideas.
- The goal is insight, not teaching.
- Question covariation thinking.
- If they say something about an asymptote, consider questions like, “What makes you notice that? How does the relationship between the Quantities A and B make you think of asymptotes?”
- “Say more.”
- “I’m not quite sure what you mean, can you clarify that?”
- “Can you show me how you see that in the graph?”
- “Could you maybe draw me another graph that shows the same sort of thing?”

(a)

- What relationship do you notice between Quantities A and B? (How does the graph show the things being described) (The goal is to gain lots of insights into their initial reasoning)
- Can you describe what happens as the Quantity A decreases towards 0?
- (Anticipating some version of “the values are approaching 0”) What does it mean for the Quantities to “approach” 0? What is happening to Quantity A?
- Can you describe what happens as Quantity A increases towards 0?
- Can you describe what happens as Quantity B increases/decreases towards 0?
- Can you describe what happens as Quantity B increases without bound?
- Can you describe what happens as Quantity A increases without bound?

(b)

- What relationship do you notice between Quantities A and B? (How does the graph show the things being described) (The goal is to gain lots of insights into their initial reasoning)
- Can you describe what happens as Quantity A increases towards 2? -2?
- Can you describe what happens as Quantity A increases towards 0?
- Can you describe what happens as Quantity B increases/decreases towards 0?
- Can you describe what happens as Quantity B decreases without bound?
- Can you describe what happens as Quantity A increases without bound?
- When Quantity A is decreasing towards -2, what is happening to Quantity B?

(c)

- What relationship do you notice between Quantities A and B? (How does the graph show the things being described) (Does the student start to mimic key phrases from graphs A and B?)
- Can you describe what happens as Quantity A increases towards 2? -2?
- Can you describe what happens as Quantity B decreases without bound?
- Can you describe what happens as Quantity A increases without bound?
- When Quantity A is decreasing towards -2, what is happening to Quantity B?

(d)

- What relationship do you notice between Quantities A and B?
- Can you describe what happens as Quantity A increases (or decreases) towards 0?
- Can you describe what happens as Quantity B decreases without bound?
- Can you describe what happens as Quantity A increases without bound?

(e)

- What relationship do you notice between Quantities A and B?
- Can you describe what happens as Quantity B decreases without bound?
- Can you describe what happens as Quantity A increases without bound?
- Can you describe what happens to Quantity B as Quantity A increases without bound?

Summary/Reflective Questions:

- Looking across the graphs, what are similarities and differences?
- How is what is happening to Quantity B outputs as Quantity A in (e) different than (d)?

Based on the relationship in this graph and the other graphs, what does it mean for values to increases without bound?