Some Results on Point-Plane Incidences in $\mathbb{R}^3$

by

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(Under the Direction of Giorgis Petridis)

Abstract

In this expository thesis, the focus will be on bounds for point-plane incidences. We begin by stating and proving two results about point-line incidences in $\mathbb{R}^2$. We then prove two results about point-plane incidences in $\mathbb{R}^3$, one of which uses reformulations of the two results on point-line incidences. We also examine constructions for each point-plane result in $\mathbb{R}^3$, which show that the proven bounds are sharp.

Index words: [Combinatorial geometry, incidence structures]

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Chapter 1

Introduction

Incidence structures are important objects of study in combinatorics. For example, in combinatorial geometry, one incidence structure that has been studied concerns the number of incidences between points and lines in $\mathbb{R}^2$, which we will define rigorously in the next chapter. Regarding this problem, Erdős and Purdy [8] conjectured that for any positive integers m and n, the number of incidences between a point set $\mathcal{P}$ with n points and a line set $\mathcal{L}$ of size m is $O(m^{3/2}n^{3/2} + m + n)$. This was proven by Szemerédi and Trotter in 1983 [12]. In 1997, Széleky [11] gave a more elegant proof, which we present in Chapter 2. While incidence problems are natural combinatorial problems and are interesting to study in their own right, the results and techniques used to study them can also be used to tackle a number of seemingly unrelated problems. Two such problems are the *unit distances problem* and the *distinct distances problem*. These two related problems were introduced in a 1946 paper of Erdős [7]. Both problems start with a set of n points in a plane. The unit distances problem asks for an upper bound on the number of pairs of points that could be a unit distance from each other. The distinct distances problem, on the other hand, asks for a lower bound on the number of distinct distances that can be determined by the points. The techniques developed in Chapter 2 are useful in studying both of these problems. More surprising is the application of incidence geometry to the *sum-product problem*. To state this problem, suppose A is a set with n real numbers and define

$$A + A = \{a + b : a, b \in A\}, \qquad \text{and} \qquad AA = \{ab : a, b \in A\}.$$

Erdős and Szemerédi [9] conjectured that for any $\epsilon > 0$,

$$\max\{|A + A|, |AA|\} = \Omega_\epsilon(n^{2-\epsilon}).$$

Using the ideas in Chapter 2, Elekes [4] was able to show that

$$\max\{|A + A|, |AA|\} = \Omega(n^{5/4}).$$

A natural generalization of point-line incidences is to study incidences in higher dimensional space. Thus in Chapter 3, we will generalize the problem to studying point-plane incidences in $\mathbb{R}^3$ and discuss some results by Elekes and Tóth [6] and by Brass and Knauer [3]. Before we tackle these results, we begin with some definitions and notation that will be used throughout the thesis.

1.1 Notation and Definitions

Let $\mathcal{P}$ denote a set of points either in $\mathbb{R}^2$ or $\mathbb{R}^3$. Usually the context will make clear whether we are considering a point set in $\mathbb{R}^2$ or $\mathbb{R}^3$; however, if there is ambiguity, we will specify. Let $\mathcal{L}$ be a set of a lines in $\mathbb{R}^2$ and $\mathcal{Q}$ be a set of planes in R^3. Given $\mathcal{P}$ and $\mathcal{L}$ in $\mathbb{R}^2$ or $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^3$, we can define incidence.

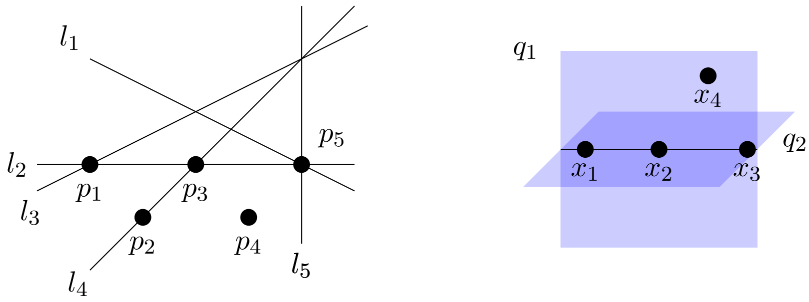

Figure 1.1: (a) Example of point-line incidences in $\mathbb{R}^2$ and (b) of point-plane incidences in $\mathbb{R}^3$.

Definition 1.1.1 (Incidence)**.** An incidence is a pair (p, l) or (p, Q) where $p \in \mathcal{P}$, $l \in \mathcal{L}$, and $Q \in \mathcal{Q}$ such that the point p is contained in the line l or the plane Q.

Incidences of the form $(p, l) \in \mathcal{P} \times \mathcal{L}$ in $\mathbb{R}^2$ are called *point-line incidences*, and incidences of the form $(p, Q) \in \mathcal{P} \times \mathcal{Q}$ in $\mathbb{R}^3$ are called *point-plane incidences*. We will denote the number of incidences as $I(\mathcal{P}, \mathcal{L})$ and $I(\mathcal{P}, \mathcal{Q})$, respectively. As an example of point-line incidences, let $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5\}$ and let $\mathcal{L} = \{l_1, l_2, l_3, l_4, l_5\}$ in $\mathbb{R}^2$ as shown in Figure 1.1(a). This configuration of five lines and five points has 8 incidences, i.e., $I(\mathcal{P}, \mathcal{L}) = 8$. In Figure 1.1(b), we have $\mathcal{P} = \{x_1, x_2, x_3, x_4\}$, $\mathcal{Q} = \{q_1, q_2\}$, and $I(\mathcal{P}, \mathcal{Q}) = 7$.

We remark here that there is dual incidence structure. Given a pair (p, l) or (p, Q) rather than thinking of the point as lying on the line l or plane Q, we can instead think of the line or plane as lying on the point p. This duality allows us to interchange the point set and line set (or plane set).

We will use standard asymptotic notation throughout. Let $f(n)$ and $g(n)$ be functions. We write $f(n) = O(g(n))$ if there are constants c and N such that $f(n) \leq cg(n)$ whenever $n \geq N$. If there are constants c and N such that $f(n) \geq cg(n)$ for any $n \geq N$, then we say $f(n) = \Omega(g(n))$. If $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, then we'll write $f(n) = \Theta(g(n))$. Finally, we say $f(n) = o(g(n))$ if $f(n)/g(n) \to 0$ as $n \to \infty$. We will on occasion write $f(n) = O_\alpha(g(n))$ by which we mean the constant depends on some parameter α.

In order to look at some examples of asymptotic notation, let $f(n) = 3n^2 + n + 5$. We can write $f(n) = O(n^2)$ since $3n^2 + n + 5 \leq 3n^2 + n^2 + 5n^2 = 9n^2$ whenever $n \geq 1$. Similarly, we can write $f(n) = \Omega(n^2)$ since $3n^2 + n + 5 \geq 3n^2$ for all $n \geq 1$. Because $f(n) = O(n^2)$ and $f(n) = \Omega(n^2)$, then we can write $f(n) = \Theta(n^2)$. Now take $g(n) = 3n + 4$, then we have $g(n) = o(n^2)$ since $(3n + 4)/n^2 \to 0$ as $n \to \infty$.

Chapter 2

Point-Line Incidences in $\mathbb{R}^2$

The material in this section is based on Chapter 1 of Sheffer's book [10]. Our goal in this section will be to prove two theorems about point-line incidences in $\mathbb{R}^2$. The first of these results is a theorem due to Szemerédi and Trotter [12]. To give a proof, we will first prove the Crossing Lemma which concerns vertex-edge graphs. The second result in this chapter is due to Beck [2]. We will also define rich lines and rich points, which we will use to reformulate the theorems of Szemerédi-Trotter and Beck, which will be used in Chapter 3.

2.1 The Crossing Lemma

Our goal in this section is to prove the Crossing Lemma for vertex-edge graphs. To that end, we begin by stating some definitions and giving examples of graphs before preceding to the proof of the lemma. Given a set V, which we call the vertex set, we can define a vertex-edge graph.

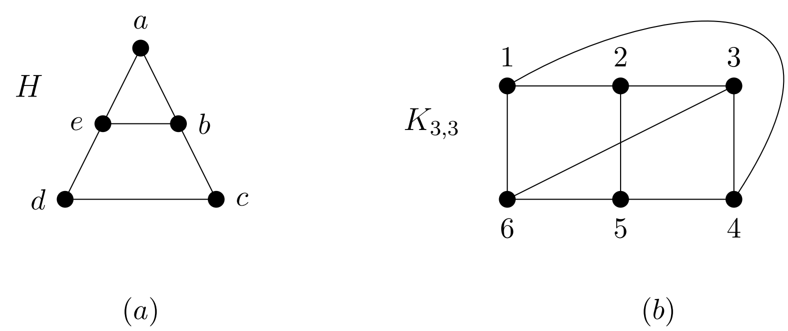

Figure 2.1: (a) A graph H and (b) a drawing of $K_{3,3}$ with one crossing.

Definition 2.1.1 (Vertex-Edge Graph)**.** A vertex-edge graph, G, is an ordered pair, $G = (V, E)$ where V is a set of vertices called the vertex set, and E is a collection of two-element subsets of V whose elements are called edges.

Unless stated otherwise, we will refer to a vertex-edge graph as a graph. At times, we may write $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. The graph represented in figure 2.1 (a) has vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{b, e\}, \{c, d\}, \{d, e\}\}$, and the graph represented in figure 2.1 (b) has vertex set $V = \{1, 2, 3, 4, 5, 6\}$, and edge set $E = \{\{1, 2\}, \{1, 4\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 6\}, \{4, 5\}, \{5, 6\}\}$. Having defined a graph, we can now define statistics on G, which we can study. Of particular interest is the crossing number, cr(G).

Definition 2.1.2 (Crossing Number)**.** The crossing number of a graph $G = (V, E)$, denoted cr(G), is the smallest non-negative integer k such that G can be drawn in the plane with k edge crossings.

In order to define the next idea, we need to discuss some basic concepts in graph theory. A pair of vertices $u, v \in V$ are called *connected* if there exists a walk from u to v. By a *walk*, we mean a sequence of edges which join a sequence of vertices. If every pair of vertices of G are connected, then we say G is a *connected graph*. Finally, if a graph, G, has a crossing number of 0, i.e., $\mathrm{cr}(G) = 0$, then we say G is *planar*, which means G can be drawn in the plane without having edges cross.

We now illustrate these terms using the two graphs in Figure 2.1. The vertex pair (a, d) in H is connected since the sequence of edges $(\{a, b\}, \{b, c\}, \{c, d\})$ is a walk between vertex a and vertex b. Similarly, the vertex pair $(1, 4)$ is connected in $K_{3,3}$ by the walk $(\{1, 4\})$. Moreover, all the vertices in H and $K_{3,3}$ are connected, so these are examples of connected graphs. In terms of planarity, we can see that Graph (a) is planar (and so it has crossing number 0) while $K_{3,3}$ does not appear to be. At first glance, one may think that it is possible to redraw $K_{3,3}$ to avoid a crossing; however, it is known that the crossing number of $K_{3,3}$ is 1, so any drawing of $K_{3,3}$ in the plane must contain at least 1 crossing.

If we consider a connected planar graph G with e edges and v vertices, then the faces of a drawing of G are the largest 2-dimensional regions that are bounded by edges, including the outer region. For example, the graph in figure 2.1 (a) has two bounded and one unbounded region. If we let f denote the number of faces of G, then we can state *Euler's formula* for a connected planar graph G as

$$v + f = e + 2. \tag{2.1}$$

If G is not connected, then we have $v + f > e + 2$. Thus for a general planar graph, we have $v + f - 2 \geq e$. Note that every edge of G is the boundary of two faces or both sides of the edge are the boundary of the same face. Additionally if $e \geq 3$, every face has a boundary consisting of at least 3 edges. Thus we have $2e \geq 3f$. Note that if G is the path graph on two or three vertices then the unbounded face does not have at least 3 edges on the boundary; however, when

G is the path graph on three vertices, the inequality is satisfied. Plugging this into the modified version of equation 2.1, we can get an inequality that holds for any planar graph G that is not the path graph on two vertices. Namely,

$$e \leq v + f - 2 \leq v + \frac{2e}{3} - 2$$

which we can rewrite using $|E| = e$ and $|V| = v$ as

$$|E| \leq 3|V| - 6. \tag{2.2}$$

With this inequality, we can give a lower bound on $\mathrm{cr}(G)$ as stated in the next lemma.

Lemma 1. *For any graph $G = (V, E)$ that is not the path graph on two vertices, we have $cr(G) \geq |E| - 3|V| + 6$.*

Proof. Consider a drawing of graph G in the plane that minimizes the number of crossings. Take $E' \subseteq E$ to be a maximum subset of edges such that no two edges of E' intersect. This edge set determines a new graph $G' = (V, E')$ which is planar by construction. Substituting this into equation 2.2, we have that $|E'| \leq 3|V| - 6$. Additionally, by construction, every edge of $(E \setminus E')$ intersects E' at least once and $|E| - |E'| \geq |E| - 3|V| + 6$. Therefore there are at least $|E| - 3|V| + 6$ crossings in the drawing. Since we drew G to minimize the number of crossings, we can conclude $\mathrm{cr}(G) \geq |E| - 3|V| + 6$. $\square$

Using this lemma and a probabilistic argument, we can now prove the Crossing Lemma.

Lemma 2 (The Crossing Lemma). *Let $G = (V, E)$ be a graph with $|E| \geq 4|V|$. Then $cr(G) = \Omega(|E|^3/|V|^2)$.*

Proof. Consider a drawing of G with $\text{cr}(G)$ crossings drawn such that G does not have any three edges crossing at the same point. Set $p = \frac{4|V|}{|E|}$. By assumption, $0 < p \leq 1$. Construct a subgraph $G' = (V', E')$ by removing each vertex of V (and adjacent edges) with probability $1 - p$, and let c' be the number of crossings of G that have both of their edges in E'.

We denote the expectation of a random variable as $\mathbb{E}[\cdot]$. Since every vertex of V remains in G' with probability p, we have $\mathbb{E}[|V'|] = p|V|$. Since each edge contains two vertices, any edge survives only if its endpoints survive. Thus $\mathbb{E}[|E'|] = p^2|E|$. Finally, each crossing remains if the two edges remain (and thus the four endpoints) which means $\mathbb{E}[c'] = p^4\text{cr}(G)$. (Note, here we're relying on the fact that G was drawn such that only two edges determine a crossing). By lemma 1, we have $c' - |E'| + 3|V'| \geq 6$ for G'. So using linearity of expectation, we have

$$\begin{aligned}
\mathbb{E}[c' - |E'| + 3|V'|] &= p^4\text{cr}(G) - p^2|E| + 3p|V| \\
&= \frac{4^4|V|^4}{|E|^4}\text{cr}(G) - \frac{4^2|V|^2}{|E|^2}|E| + 3\frac{4|V|}{|E|}|V| \\
&= \frac{4^4|V|^4}{|E|^4}\text{cr}(G) - \frac{4|V|^2}{|E|}.
\end{aligned}$$

Since this is the expected value, there must exist a graph $G^* = (V^*, E^*)$ with c^* (the crossings from G remaining in G^*) such that

$$c^* - |E^*| + 3|V^*| \leq \mathbb{E}[c' - |E'| + 3|V'|] = \frac{4^4|V|^4}{|E|^4}\text{cr}(G) - \frac{4|V|^2}{|E|}. \quad (2.3)$$

Combining Inequality 2.3 with the bound implied by lemma 1, we have

$$0 < 6 \leq c^* - |E^*| + 3|V^*| \leq \frac{4^4|V|^4}{|E|^4}\mathrm{cr}(G) - \frac{4|V|^2}{|E|}.$$

Rearranging this inequality gives us the lower bound stated in the lemma. $\square$

Having proven the Crossing Lemma, we now turn our attention to a theorem of Szemerédi and Trotter which gives us an upper bound on the number of point-line incidences in $\mathbb{R}^2$.

2.2 Szemerédi-Trotter Theorem

Theorem 3 (Szemerédi-Trotter [12]). *Let $\mathcal{P}$ be a set of n points, and let $\mathcal{L}$ be a set of m lines, both in $\mathbb{R}^2$. Then $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n^{2/3} + m + n)$.*

Proof. Write $\mathcal{L} = \{l_1, l_2, \ldots, l_m\}$, and let n_i denote the number of points of $\mathcal{P}$ that are on l_i and so $I(\mathcal{P}, \mathcal{L}) = \sum_{i=1}^{m} n_i$. Since we are interested in counting the number of point-line incidences, we can discard any l_i for which $n_i = 0$ since such a line does not contribute to $I(\mathcal{P}, \mathcal{L})$. The main idea of this proof is to enumerate $I(\mathcal{P}, \mathcal{L})$ in two different ways by creating an abstract graph G from the given point-line configuration and by considering the given point-line configuration as a drawing of the graph.

To construct G, we take the vertex set V to correspond to the points of $\mathcal{P}$. To construct edges of G, consider two vertices $u, v \in V$. We have $\{u, v\} \in E$ if the points in $\mathcal{P}$ corresponding to u and v are consecutive points (fixing a direction) on one of the $l_i \in \mathcal{L}$. See Figure 2.2 for an example. So each l_i contributes exactly $n_i - 1$ edges to E since if there are n_i points, there are $n_i - 1$ consecutive pairs of points. Thus $|V| = n$ and $|E| = \sum_{i=1}^{m}(n_i - 1) = I(\mathcal{P}, \mathcal{L}) - m$.

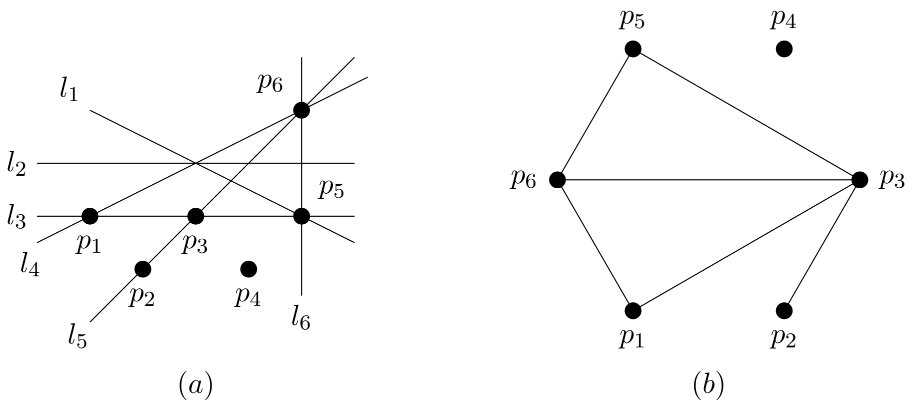

Figure 2.2: (a) A point-line configuration and (b) its corresponding graph.

If $|E| < 4|V|$, then $I(\mathcal{P}, \mathcal{L}) - m < 4n$ and so $I(\mathcal{P}, \mathcal{L}) < 4n + m$. Thus we have $I(\mathcal{P}, \mathcal{L}) = O(n + m)$. If $|E| \geq 4|V|$, then we can apply Lemma 2, and we get

$$\mathrm{cr}(G) = \Omega\left(\frac{|E|^3}{|V|^2}\right) = \Omega\left(\frac{(I(\mathcal{P}, \mathcal{L}) - m)^3}{n^2}\right). \tag{2.4}$$

Now we draw G according to the given point-line configuration, i.e., every vertex is at the corresponding point and every edge is the corresponding line segment. (We may have to redraw edges of G such that no three edges intersect at the same point.) Thus $V(G)$ corresponds to the points of $\mathcal{P}$ and $E(G)$ corresponds to the line segments between point, and every crossing in this drawing of G corresponds to an intersection of two lines in $\mathcal{L}$. Since every two lines intersect at most once, we have $\mathrm{cr}(G) \leq \binom{m}{2} = {}^1\!/\!_2(m^2 - m) = O(m^2)$ and so $\mathrm{cr}(G) = O(m^2)$. Combining this with 2.4, we have

$$\Omega\left(\frac{(I(\mathcal{P}, \mathcal{L}) - m)^3}{n^2}\right) = \mathrm{cr}(G) = O(m^2).$$

Solving for $I(\mathcal{P}, \mathcal{L})$, we find $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n^{2/3} + m)$. Combining this result with the result for when $|E| < 4|V|$, we get $I(\mathcal{P}, \mathcal{L}) = O(n^{2/3}m^{2/3} + n + m)$ as desired. $\square$

In fact, the bound given in Theorem 3 is tight. To see this, we present a construction by Elekes [5]. When $n = \Omega(m^2)$, we can take n points on a single line and obtain n incidences. Similarly, if $m = \Omega(n^2)$, we can take m lines through a single point and obtain m incidences. So what remains is to construct a configuration with $\Theta(n^{2/3}m^{2/3})$ incidences when $n = O(m^2)$ and $m = O(n^2)$. Let $r = (n^2/4m)^{1/3}$ and $s = (2m^2/n)^{1/3}$ and for simplicity, assume that these are integers. Set

$$\mathcal{P} = \{\ (i, j)\ :\ 1 \leq i \leq r \quad \text{and} \quad 1 \leq j \leq 2rs\ \}$$

and

$$\mathcal{L} = \{\ y = ax + b\ :\ 1 \leq a \leq s \quad \text{and} \quad 1 \leq b \leq rs\ \}$$

as seen in Figure 2.3. Notice that we have $|\mathcal{P}| = n$ and $|\mathcal{L}| = m$ since

$$|\mathcal{P}| = 2r^2 s = 2 \cdot \left(\frac{n^2}{4m}\right)^{2/3} \cdot \left(\frac{2m^2}{n}\right)^{1/3} = n,$$

and

$$|\mathcal{L}| = rs^2 = \left(\frac{n^2}{4m}\right)^{1/3} \cdot \left(\frac{2m^2}{n}\right)^{2/3} = m.$$

Now if we consider a line $l \in L$, for any $x \in \{1, \ldots, r\}$, there exists a unique $y \in \{1, \ldots, 2rs\}$ such that (x, y) and l are incident. This means that

$$I(\mathcal{P}, \mathcal{L}) = r \cdot |\mathcal{L}| = \left(\frac{n^2}{4m}\right)^{1/3} \cdot m = 2^{-2/3}n^{2/3}m^{2/3}.$$

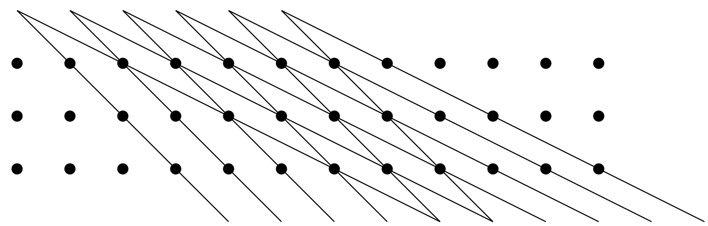

Figure 2.3: Elekes's construction rotated by $90°$ with $r = 3$ and $s = 2$

Combining the various cases together, we have a construction that show the upper bound on $I(\mathcal{P}, \mathcal{L})$ given by Theorem 3 is sharp.

2.3 Rich Points

For the results in Chapter 3 on point-plane incidences, we will need a reformulation of the two main theorems of this chapter (Szemerédi-Trotter and Beck) in terms of rich points. In this section, we reformulate Theorem 3. After we have proven Beck's Theorem, we will state and prove a reformulation of that theorem as well.

Definition 2.3.1 (k-rich line/plane)**.** Given a point set $\mathcal{P} \subseteq \mathbb{R}^d$ and a positive integer k, we say a line is ℓ is *k-rich* if $|\ell \cap \mathcal{P}| \geq k$. Similarly, we say a plane Q is *k-rich* if $|Q \cap \mathcal{P}| \geq k$.

We can also talk about points being k-rich. In this context, a k-rich point is one that is incident to at least k lines or k planes depending on if we are discussing point-line incidences or point-plane incidences. For example in Figure 1.1, line l_2 is 3-rich and plane q_1 is 4-rich. We also note that in the same figure, point p_5 is 3-rich and point x_2 is 2-rich.

We introduce some notation that will be helpful when discussing k-rich lines. Let $x \in \mathcal{P}$ be a point and $\ell \in \mathcal{L}$ be a line, then define $i_{\mathcal{P}}(\ell) = |\mathcal{P} \cap \ell|$ to be the number of points in $\mathcal{P}$ incident to ℓ and $i_{\mathcal{L}}(x) = |x \cap \mathcal{L}|$ to be the number of lines in $\mathcal{L}$ incident to x. Additionally, we let

$$L_k = \{\ell : i_{\mathcal{P}}(\ell) = k\},$$

which is the collection of lines incident to exactly k points of $\mathcal{P}$, and we can denote the collection of lines incident to at least k points of $\mathcal{P}$ by $\mathbb{L}_k$ which can be written as the set $\{\ell : i_{\mathcal{P}}(\ell) \geq k\}$. Using this notation we can write

$$\mathbb{L}_k = \bigcup_{j \geq k} L_j.$$

We note that $\mathbb{L}_k$ is precisely the set of k-rich lines.

We now restate Theorem 3 using this notion of k-rich lines. Below, we show that Theorem 3 implies Theorem 4; however, the converse is also true, and thus the two theorems are equivalent.

Theorem 4 (Szemerédi-Trotter Reformulation)**.** *Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$ and let k be a positive integer. Then the number of k-rich lines,*

$$|\mathbb{L}_k| = O\left(\frac{n^2}{k^3} + \frac{n}{k}\right).$$

In particular, it is possible to obtain

$$|\mathbb{L}_k| \leq 8^4 \left(\frac{n^2}{k^3} + \frac{n}{k}\right).$$

Proof. First note that since every line in $\mathbb{L}_k$ is incident to at least k points of $\mathcal{P}$, we have $k|\mathbb{L}_k| \leq I(\mathcal{P}, \mathbb{L}_k)$. By Theorem 3, we have the following inequality

$$k|\mathbb{L}_k| \leq I(\mathcal{P}, \mathbb{L}_k) = O\left(n^{2/3}|\mathbb{L}_k|^{2/3} + n + |\mathbb{L}_k|\right).$$

We now consider three cases.

Case 1: $k|\mathbb{L}_k| = O\left(n^{2/3}|\mathbb{L}_k|^{2/3}\right)$.

In this case, we have for some constant C_1 that $k|\mathbb{L}_k| \leq C_1 n^{2/3}|\mathbb{L}_k|^{2/3}$ which we can rearrange and find that $|\mathbb{L}_k| \leq C_1^3 n^2/k^3$ and thus $|\mathbb{L}_k| = O(n^2/k^3)$.

Case 2: $k|\mathbb{L}_k| = O(n)$.

We can divide both sides by k, and we get $|\mathbb{L}_k| = O(n/k)$.

Case 3: $k|\mathbb{L}_k| = O(|\mathbb{L}_k|)$.

This means that there is some positive constant C_2 such that $k|\mathbb{L}_k| \leq C_2|\mathbb{L}_k|$ and thus $k \leq C_2$. Additionally, we know $|\mathbb{L}_k|$ is bounded above by the number of distinct lines which is at most n^2. Hence,

$$|\mathbb{L}_k| \leq n^2 = O\left(\frac{n^2}{C_2}\right) = O\left(\frac{n^2}{k^3}\right).$$

The final remark in the theorem follows, since it is possible to take the constant in Theorem 3 to be 8^4. $\square$

2.4 Beck's Theorem

Our next theorem concerns the number of line determined by a set of points in $\mathbb{R}^2$. We say that a set of points $\mathcal{P}$ *determines* a line ℓ if there are two distinct points of $\mathcal{P}$ on ℓ. It is natural to seek a lower bound on the number of lines determined by a planar point set with n points. Let us consider some simple examples. If all n points are collinear, then only one line is determined. If instead $n - 1$ are collinear, then there are n lines determined. The situation seems to change if no line contains more than $n/2$ points, in which case it is challenging to find points sets that determine $o(n^2)$ lines.

Beck [2] proved an at-first-sight unexpected result which roughly states that every point set either has a near maximum number of collinear points or determines a near maximum number of lines.

Theorem 5 (Beck)**.** *Any $n \geq 4$ points on the plane either contain at least $2^{-16}n$ collinear points or determine at least $2^{-36}n^2$ lines.*

To prove this theorem, we'll show that if no line contains $2^{-16}n$ points, then we have at least $2^{-36}n^2$ lines that can be made using the n points. To accomplish this, we seek to bound $|\mathbb{L}_2| = \sum_{k>1} |L_k|$ from below.

Proof. Our main goal is to show that for an appropriate constant K (which we'll take to be 2^{16}) and assuming no line contains n/K points, that we have

$$\sum_{2K<k} k(k-1)|L_k| \leq \frac{n^2}{2}.$$

Begin by noticing that since two points uniquely determine a line, there are $|L_2|$ lines that have exactly two point-line incidences, and if we're counting ordered pairs of points, we can weight this with $2(1)$. Similarly for three point-line incidences, there are exactly $|L_3|$ lines with $3(2)$ ordered pairs of points. Continuing in this, way we get $n(n-1) = \sum_{k>1} k(k-1)|L_k|$. Now note that for $K > 1$,

$$\begin{aligned}
\sum_{k>K} 2k|\mathbb{L}_k| &= \sum_{k>K} 2k\Big(|L_k| + |L_{k+1}| + \cdots\Big) \\
&= 2(K+1)\Big(|L_{K+1}| + \cdots\Big) + 2(K+2)\Big(|L_{K+2}| + \cdots\Big) \\
&= |L_{K+1}|\Big[2(K+1)\Big] + |L_{K+2}|\Big[2(K+1) + 2(K+2)\Big] + \cdots \\
&= \sum_{k>K}\left(\sum_{j=K+1}^{k} 2j\right)|L_k|. \qquad (2.5)
\end{aligned}$$

We can further simplify this to get rid of the double sum by noting that

$$\begin{aligned}
\sum_{j=K+1}^{k} 2j = 2\left(\sum_{j=1}^{k} j - \sum_{j=1}^{K} j\right) &= 2\left(\frac{(k+1)k}{2} - \frac{(K+1)K}{2}\right) \\
&= k^2 + k - K^2 - K.
\end{aligned}$$

Substituting this back into equation 2.5, we see that

$$\sum_{k>K} 2k|\mathbb{L}_k| = \sum_{k>K}(k^2 + k - K^2 - K)|L_k|.$$

Bounding this from below, we get

$$\begin{aligned}
\sum_{k>K}(k^2 + k - K^2 - K)|L_k| &= \sum_{k>K}[(k^2 - K^2) + (k - K)]|L_k| \\
&\geq \sum_{k>K}(k^2 - K^2)|L_k|.
\end{aligned}$$

We can take this sum over fewer elements ($k > 2K$), and under this assumption, we have

$$\sum_{k>K}(k^2 - K^2)|L_k| \geq \sum_{k>2K}(k^2 - K^2)|L_k| \geq \sum_{k>2K}(k^2 - (k/2)^2)|L_k|$$
$$= \frac{3}{4}\sum_{k>2K} k^2|L_k|$$

where we have used that if $k > 2K$, then $k/2 > K$ and thus $-K > -k/2$. Tracing the logic, we have found

$$\sum_{k>K} 2k|\mathbb{L}_k| \geq \frac{3}{4}\sum_{k>2K} k^2|L_k|$$

which implies that

$$\frac{8}{3}\sum_{k>K} k|\mathbb{L}_k| \geq \sum_{k>2K} k^2|L_k| \geq \sum_{k>2K} k(k-1)|L_k|. \tag{2.6}$$

Since we are looking for an upper bound, we continue bounding the left side of equation 2.6 from above. We can do that as follows

$$\frac{8}{3}\sum_{k>K} k|\mathbb{L}_k| \leq 4\sum_{k>K} k|\mathbb{L}_k| = 4\sum_{k=K+1}^{\lfloor n/K \rfloor} k|\mathbb{L}_k|$$

since $8/3 < 4$ and $k > K$ means that $k \geq K+1$ and by assumption, no line is incident to at least n/K points. So $|\mathbb{L}_k| = 0$ for $k > n/K$ and thus do not contribute to the sum.

Using Theorem 4, we have $|\mathbb{L}_k| \leq 8^4\left(\dfrac{n^2}{k^3} + \dfrac{n}{k}\right)$, and so

$$4\sum_{k=K+1}^{\lfloor n/K \rfloor} k|\mathbb{L}_k| \leq 2^{14}\sum_{k=K+1}^{\lfloor n/K \rfloor} k\left(\frac{n^2}{k^3} + \frac{n}{k}\right) = 2^{14}\sum_{k=K+1}^{\lfloor n/K \rfloor}\left(\frac{n^2}{k^2} + n\right). \tag{2.7}$$

We'll now bound this above by splitting the sum into two part and bounding each part from above. First note,

$$\sum_{k=K+1}^{\lfloor n/K \rfloor} \frac{n^2}{k^2} = n^2 \sum_{k=K+1}^{\lfloor n/K \rfloor} \frac{1}{k^2} \leq n^2 \int_K^\infty \frac{1}{x^2} dx = \frac{n^2}{K}.$$

Additionally, we have

$$\sum_{k=K+1}^{\lfloor n/K \rfloor} n = n\left(\frac{n}{K} - (K+1) + 1\right) = n\left(\frac{n}{K} - K\right) \leq \frac{n^2}{K}.$$

Combining these together with Inequality 2.7, we see that

$$4\sum_{k=K+1}^{\lfloor n/K \rfloor} k|\mathbb{L}_k| \leq 2^{14}\left(\frac{n^2}{K} + \frac{n^2}{K}\right) = \frac{2^{15} n^2}{K}.$$

Tracing the work up to this step, we have shown that

$$\sum_{k>2K} k(k-1)|L_k| \leq \frac{2^{15} n^2}{K}$$

or equivalently,

$$-\sum_{k>2K} k(k-1)|L_k| \geq -\frac{2^{15} n^2}{K}.$$

Now suppose that $K = 2^{16}$, then using the equality above in addition to splitting a sum over $k > 1$ into two sums one with $1 < k \leq 2^{17}$ and the other with $k > 2^{17}$, we get

$$\sum_{1<k\leq 2^{17}} k(k-1)|L_k| = \sum_{k>1} k(k-1)|L_k| - \sum_{k>2^{17}} k(k-1)|L_k|$$
$$\geq n^2 - n - \frac{n^2}{2}$$
$$= \frac{n^2}{2} - n.$$

To complete the proof, we need one last observation

$$2^{34} \sum_{1<k\leq 2^{17}} |L_k| = \sum_{1<k\leq 2^{17}} 2^{17} \cdot 2^{17}|L_k| \geq \sum_{1<k\leq 2^{17}} k(k-1)|L_k|.$$

Using this fact, we have for $n \geq 4$

$$\sum_{k>1}|L_k| \geq \sum_{1<k\leq 2^{17}} |L_k| \geq 2^{-34} \sum_{1<k\leq 2^{17}} k(k-1)|L_k|$$
$$\geq 2^{-34}\left(\frac{n^2}{2} - n\right) \geq 2^{-36}n^2. \qquad \square$$

We now reformulate and prove Theorem 5 using the language of rich points. As with the reformulation of Szemerédi-Trotter, we only prove one direction of the implication; however, Theorem 5 and Theorem 6 are equivalent.

Theorem 6 (Beck Reformulation)**.** *Let $\mathcal{P}$ be a finite set with $n \geq 4$ points in $\mathbb{R}^2$. If no line $\ell \in \mathcal{L}$ contains $2^{-17}n$ points of $\mathcal{P}$, then there are at least $2^{-19}n$ points of p that are $(2^{-19}n)$-rich.*

Proof. In the proof of Theorem 5 above, we showed that for $n \geq 4$

$$\sum_{1<k\leq 2^{17}} k(k-1)|L_k| \geq \frac{n^2}{2} - n \geq \frac{n^2}{4}.$$

Since no line in $\mathcal{L}$ is incident to $2^{-17}n$ points, we have by definition of L_k

$$I(\mathcal{P}, \mathcal{L}) = \sum_{1<k\leq 2^{17}} k|L_k|.$$

Because $k \leq 2^{17}$, we know $k-1 \leq 2^{17}-1$. Using this observation with the bound derived at the beginning, we get the following lower bound.

$$\sum_{1<k\leq 2^{17}} k|L_k| \geq \sum_{1<k\leq 2^{17}} \frac{k(k-1)}{2^{17}-1}|L_k| = \frac{1}{2^{17}-1} \sum_{1<k\leq 2^{17}} k(k-1)|L_k| \geq \frac{n^2}{2^{19}}.$$

We will now partition the point set into the points that are $(2^{-19}n)$-rich and those that are not. We'll call the set of $(2^{-19}n)$-rich points $\mathcal{P}_1$, and the other set of points $\mathcal{P}_2$. Thus we've defined

$$\mathcal{P}_1 = \left\{x \in \mathcal{P} : i_{\mathcal{L}}(x) \geq \frac{n}{2^{19}}\right\}$$

and

$$\mathcal{P}_2 = \left\{x \in \mathcal{P} : i_{\mathcal{L}}(x) < \frac{n}{2^{19}}\right\}.$$

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint sets, we can write $I(\mathcal{P}, \mathcal{L}) = I(\mathcal{P}_1, \mathcal{L}) + I(\mathcal{P}_2, \mathcal{L})$. Our goal will be to put an upper bound on $I(\mathcal{P}_2, \mathcal{L})$. We will then use this bound to show that $2^{-19}n^2 \leq I(\mathcal{P}_1, \mathcal{L}) \leq n|\mathcal{P}_1|$. From this we get $|\mathcal{P}_1| \geq 2^{-19}n$ as desired.

To proceed, note that each point of $\mathcal{P}_2$ is incident to at most $2^{19}n$ points and thus

$$I(\mathcal{P}_2, \mathcal{L}) = \sum_{x\in\mathcal{P}_2} i_{\mathcal{L}}(x) \leq |\mathcal{P}_2|\frac{n}{2^{19}} \leq \frac{n^2}{2^{19}}$$

since $\mathcal{P}_2 \subseteq \mathcal{P}$ and $|\mathcal{P}| = n$. Now since $I(\mathcal{P}_1, \mathcal{L}) = I(\mathcal{P}, \mathcal{L}) - I(\mathcal{P}_2, \mathcal{L})$, and we have appropriate bounds on $I(\mathcal{P}, \mathcal{L})$ and $I(\mathcal{P}_2, \mathcal{L})$, we have

$$I(\mathcal{P}_1, \mathcal{L}) \geq \frac{n^2}{2^{18}} - \frac{n^2}{2^{19}} = \frac{n^2}{2^{19}}.$$

Finally, since each point of $\mathcal{P}_1$ is incident to at most $n - 1$ lines determined by $\mathcal{P}$, and so to at most n lines of $\mathcal{L}$, we have

$$\frac{n^2}{2^{19}} \leq I(\mathcal{P}_1, \mathcal{L}) = \sum_{x \in \mathcal{P}_1} i_{\mathcal{L}}(x) \leq n|\mathcal{P}_1|. \qquad \square$$

Chapter 3

Point-Plane Incidences in $\mathbb{R}^3$

In the previous chapter, we examined point-line incidences in $\mathbb{R}^2$. Using these results, we will prove some results on point-plane incidences in $\mathbb{R}^3$. Following Elekes and Tóth [6], we make the following definitions. Note Elekes and Tóth proved results in a general d-dimension setting; however, since our focus is on point-plane incidences in $\mathbb{R}^3$, we will take $d = 3$ and modify the definitions to apply in this specific case.

3.1 Preliminaries

Definition 3.1.1 (γ-saturated)**.** Given a point set $\mathcal{P}$, a plane Q in $\mathbb{R}^3$, and $\gamma > 0$. We say that Q is *γ-saturated* if the point set $Q \cap \mathcal{P}$ spans at least $\gamma \cdot |Q \cap \mathcal{P}|^2$ distinct lines.

Definition 3.1.2 (α-degenerate)**.** Given a point set $\mathcal{P}$, a plane Q in $\mathbb{R}^3$, and $\alpha > 0$. We say that Q is *α-degenerate* if $Q \cap \mathcal{P}$ is non-empty and at most $\alpha \cdot |Q \cap \mathcal{P}|$ points of $Q \cap \mathcal{P}$ lie in a line.

Theorem 7. *(Erdős-Beck [2]) Let x and k be integers with $0 < x \leq k$. There is an absolute constant c_0 such that if P is a set of k points in the plane, at most $k - x$ of which are collinear, then P spans at least $c_0 xk$ lines.*

This theorem is similar to Theorem 6 and follows a similar proof. As a special case of Theorem 7, we can take $x = \max(\lfloor (1-\alpha)k \rfloor, 1)$, and thus for $\alpha < 1$, we have every α-degenerate plane is $c_0(1-\alpha)$-saturated.

Definition 3.1.3 (w-strongly incident)**.** Given a point set $\mathcal{P} \subseteq \mathbb{R}^3$ and a plane Q, a point $p \in \mathcal{P} \cap Q$ is *w-strongly incident* to Q if Q contains at least w points $q_1, q_2, \ldots, q_w \in \mathcal{P} \setminus \{p\}$ such that the lines pq_i are all distinct for $i = 1, 2, \ldots, w$.

Lemma 8. *For a point set $\mathcal{P}$ and a k-rich γ-saturated plane Q, at least $\gamma k/2$ points are $(\gamma k/2)$-strongly incident to Q.*

Proof. Consider a γ-saturated plane Q. Trimming down, we can assume $k = |Q \cap \mathcal{P}|$. Since we are interested in points that are $(\gamma k/2)$-strongly incident to Q, each of these points must be on at least $\gamma k/2$ lines. Thus we can delete all points of $Q \cap \mathcal{P}$ that are incident to fewer than $\gamma k/2$ lines spanned by $Q \cap \mathcal{P}$. We deleted points from at most $k(\gamma \cdot k/2) = (\gamma k^2/2)$ lines spanned by $Q \cap \mathcal{P}$ since there are k points in $Q \cap \mathcal{P}$ and each deleted point is incident to at most $\gamma \cdot k/2$ lines. Since Q is γ-saturated, we know $Q \cap \mathcal{P}$ spans at least γk^2 lines, and thus we have not deleted any point from at least $\gamma k^2/2$ lines. Therefore the remaining points of $Q \cap \mathcal{P}$ still span $\gamma k^2/2$ distinct lines. Finally, because every point is incident to at most k lines of Q, there are at least $(\gamma k^2/2)/k = \gamma k/2$ points which are $(\gamma k/2)$-strongly incident to Q. $\square$

3.2 Elekes-Tóth Theorem

Theorem 9 (Elekes-Tóth [6]). *For any $\alpha < 1$ and for any set of n points in $\mathbb{R}^3$, the number of k-rich α-degenerate planes is*

$$O_\alpha\left(\frac{n^3}{k^4} + \frac{n^2}{k^2}\right).$$

The general idea of the proof is as follows, we would like to reduce the question of point-plane incidences to a question about point-line incidences which will allow us to use Theorem 4. To accomplish this we will choose a plane and project points of $\mathcal{P}$ from a point with a lot of strong point-plane incidences.

Proof. Recall that every α-degenerate plane is γ-saturated where $\gamma = c_0(1-\alpha)$. Let $\mathcal{P}$ be a set of n points, and let q denote the number of γ-saturated k-rich planes. Applying Lemma 8 to each such plane S, we find $\gamma k/2$ points that are $(\gamma k/2)$-strongly incident to S. Since this is true for each of the s planes, we have at least $s\gamma k/2$ strong point-plane incidences. Additionally, since we have n points, we must have at least one point, p_1, with the property that p_1 is $(\gamma k/2)$-strongly incident to at least $s\gamma k/(2n)$ planes.

Now choose π to be a plane not incident to p_1. We project every point of $\mathcal{P}$ from p_1 to π (see Figure 3.1). The image $\mathcal{P}'$ of this projection contains at most $n-1$ distinct points. Because this mapping projects planes to lines, $\mathcal{P}'$ has the property that there are $s\gamma k/(2n)$ distinct lines, each containing at least $\gamma k/2$ distinct points of $\mathcal{P}'$.

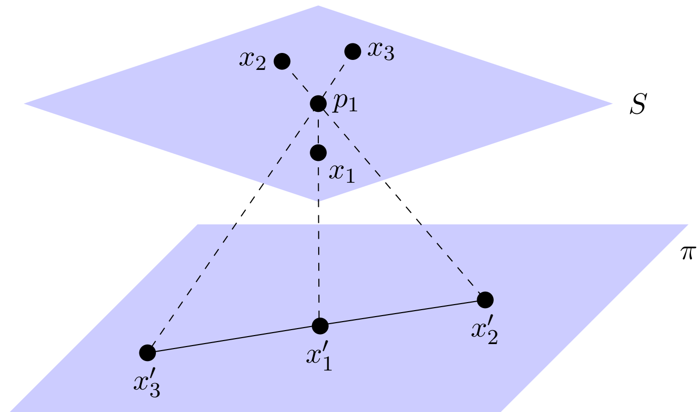

Figure 3.1: An example of the projection from the point p_1 on S to the plane π.

By Theorem 4, we have

$$\frac{s\gamma k}{2n} = O\left(\frac{n^2}{(\gamma k/2)^3} + \frac{n}{\gamma k/2}\right).$$

Solving for s, we get that

$$s = O_\gamma\left(\frac{n^3}{k^4} + \frac{n^2}{k^2}\right) = O_\alpha\left(\frac{n^3}{k^4} + \frac{n^2}{k^2}\right). \qquad \square$$

We now give a construction of Elekes-Tóth and show that the bound given above is tight. For this construction, assume k is even and n is an integer multiple of $2k$. We will give two constructions which depend on whether $k \leq \sqrt{n}$ or $k \geq \sqrt{n}$.

For $k \leq \sqrt{n}$, let $\mathcal{P}$ be the $\frac{k}{2} \times 2 \times \frac{n}{k}$ grid

$$\mathcal{P} = \left\{(x, y, z) \in \mathbb{Z}^3 : 1 \leq x \leq \frac{k}{2}, 1 \leq y \leq 2, 1 \leq z \leq \frac{n}{k}\right\}.$$

If we focus on the two planes $y = 1$ and $y = 2$, we notice that in either plane, every line

$$z = m_i x + b_i \qquad \text{where} \qquad 1 \leq m_i \leq \frac{n}{k^2}, 1 \leq b_i \leq \frac{n}{2k},$$

passes through the points whose x-coordinate ranges over $1, 2, \ldots, k/2$. Thus each plane contains many $(k/2)$-rich lines. If we choose a line from each plane such that the two lines are parallel, i.e.,

$$\{y = 1, z = m_i x + b_i\} \quad \text{and} \quad \{y = 2, z = m_i x + b_j\}, \tag{3.1}$$

then we can define a plane π which contains these two lines. We take the plane set $\mathcal{Q}$ to be the collection of all such planes π. Note that π intersects $\mathcal{P}$ in k points and is $1/2$-degenerate (see Figure 3.2). Since we get a plane π for each triple (m_i, b_i, b_j), the number of distinct planes is

$$\frac{n}{k^2} \cdot \frac{n}{2k} \cdot \frac{n}{2k} = \frac{n^3}{4k^4}.$$

For $k \geq \sqrt{n}$, we use the same point set $\mathcal{P}$ defined above; however, in this case, we consider two horizontal lines

$$\{y = 1, z = h_i\} \quad \text{and} \quad \{y = 2, z = h_j\} \tag{3.2}$$

where $h_i, h_j \in \{1, \ldots, n/k\}$. In this case we can define a plane π as the plane containing these two horizontal lines (see Figure 3.3). There are $(n/k)^2 = n^2/k^2$ such planes (one for each pair of h_i and h_j). Furthermore, each of these planes is $1/2$-degenerate since each horizontal line passes through $k/2$ points

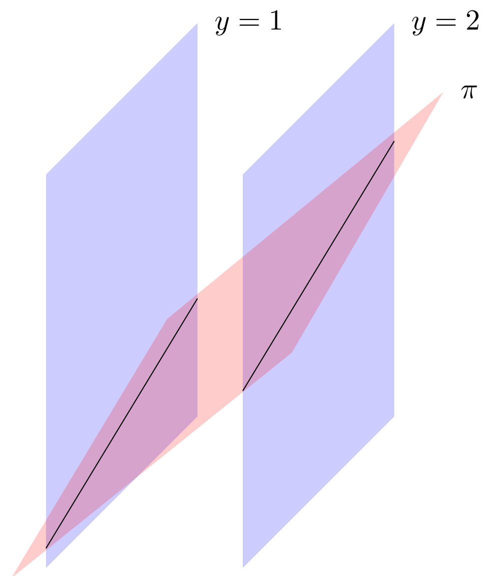

Figure 3.2: An example of such a plane π satisfying equation 3.1.

of $\mathcal{P}$ and thus the planes determined by two lines pass through k points. We take the plane set $\mathcal{Q}$ to be the set of all planes defined in this way.

Combining the two cases together, we have the bound given in Theorem 9 is sharp.

3.3 Brass-Knauer Theorem

In the previous section, we saw an upper bound on the number of point-plane incidences for arbitrary points sets. In this section, we show a lower bound by constructing an infinite family of $(\mathcal{P}, \mathcal{Q})$ that have "many" incidences. The usage of "many" will be clarified below. We begin with a definition that will be used in the proof of the main theorem in this section.

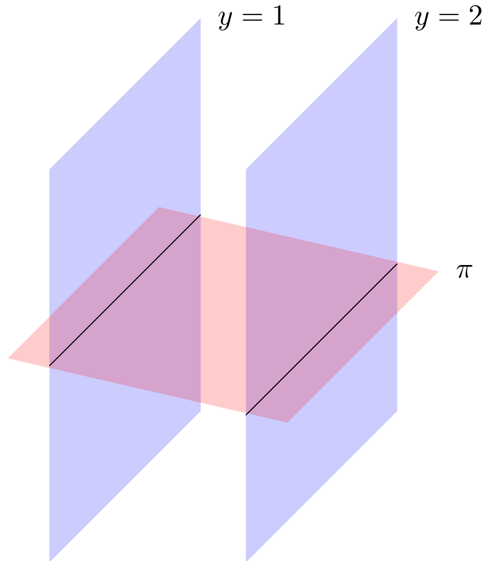

Figure 3.3: An example of such a plane π satisfying equation 3.2.

Definition 3.3.1 (Primitive vector)**.** We say a point (x, y, z) in $\mathbb{Z}^3$ is primitive if x, y, and z have no common factors. We define a *primitive vector* as the vector between the origin and a primitive point.

Additionally, in order to state the main theorem, we define incidence graph in the context of point-plane incidences.

Definition 3.3.2 (Incidence graph)**.** An *incidence graph* is a bipartite graph where the vertex set is the disjoint union of a plane set $\mathcal{Q}$ and a point set $\mathcal{P}$ and the edge (Q, p) is included in the edge set if $Q \in \mathcal{Q}$, $p \in \mathcal{P}$, and $p \in Q$.

Theorem 10 (Brass-Knauer [3])**.** *For infinitely many m, n, there is a set of n points and m planes in $\mathbb{R}^3$ such that there are $\Omega((nm)^{7/10})$ incidences and the incidence graph does not contain a $K_{3,2}$.*

Proof. We begin by constructing our point set X and a vector set Y. We will use these to define the set of planes $\mathcal{Q}$ as those that have normal vectors in Y

and contain at least one point of X and then argue that this point and plane set has $\Omega((nm)^{7/10})$ incidences using a theorem of Bárány et al. [1].

Let $X = \{1, \ldots, \nu\}^3$. We'll choose Y from the primitive vectors of a 3-dimensional lattice cube $\{1, \ldots, \mu\}^3$ such that every 2-dimensional linear subspace contains at most 2 vectors of Y. We'll take $\mathcal{Q}$ to be the set of all planes that have normal vectors from Y and contain at least one point of X.

There are at most $3\mu\nu$ distinct planes for each vector in Y since the inner product of a point from X and a vector from Y is at most $3\mu\nu$. Thus there are $|X|$ points and at most $3\mu\nu|Y|$ planes. There are $|X||Y|$ incidences since given $x_0 \in X$ and $y_0 \in Y$ there is a unique plane determined by $x \cdot y_0 = x_0 \cdot y_o$. If $x, x_0 \in X$ lie on the same plane with normal vector y_0, then the generated plane will be the same, but we get a new incidence.

Under this construction the incidence graph does not contain a $K_{3,2}$ since any three planes with non-empty intersection must have three distinct normal vectors from Y. These normal vector span a space of dimension at least 3 by the way Y was chosen, so the intersection is an affine space of dimension at most 0, a point.

To get the desired bound, we need to know the maximum number of primitive vector from $\{1, \ldots, \mu\}^3$ such that any 2-dimensional linear subspace contains at most two vectors. We will denote this quantity $f^{\text{lin}}(\mu, 3, 2, 3)$. Note this determines the (maximum) size of Y. Clearly, $|X| = \nu^3$ as there are no restrictions on how X was chosen.

Bárány and colleagues found that $f^{\text{lin}}(\mu, 3, 2, 3) = \Theta(\mu^{3/2})$ and thus we have $C_1\mu^{3/2} \leq |Y| \leq C_2\mu^{3/2}$ for some constants C_1 and C_2. By removing points, we may assume $|X| = C_1\mu^{3/2}$. Since we know $I(X, \mathcal{Q}) = |X||Y|$, we can substitute and write $I(X, \mathcal{Q}) = C_1\nu^3\mu^{3/2}$. We would like to relate this bound back to $|\mathcal{Q}|$. Using the bounds constructed above as well as the upper bound implied by Bárány et al., we get

$$|\mathcal{Q}| \leq 3\mu\nu|Y| = 3C_1\nu\mu^{5/2}.$$

We can rearrange this to get a lower bound on $\mu^{3/2}$

$$\mu^{3/2} \geq \frac{|\mathcal{Q}|^{3/5}}{(3C_1)^{3/5}\nu^{3/5}}.$$

Substituting this back into the expression for $I(X, Q)$, we find

$$I(X, \mathcal{Q}) = C_1\nu^3\mu^{3/2} \geq \frac{C_1\nu^3}{(3C_1)^{3/5}\nu^{3/5}}|\mathcal{Q}|^{3/5}.$$

Simplifying and using that $\nu^{12/5} = \nu^{3(4/5)} = |X|^{4/5}$ we get

$$I(X, \mathcal{Q}) \geq K|X|^{4/5}|\mathcal{Q}|^{3/5}$$

for a constant K.

To finish, let $|X| = n$ and $|\mathcal{Q}| = m$. We consider two cases, when $n \geq m$ and $n < m$. In the first case, we have

$$I(X, \mathcal{Q}) \geq Kn^{7/10}n^{1/10}m^{6/10} \geq Kn^{7/10}m^{1/10}m^{6/10} = K(nm)^{7/10}.$$

In the second case, we can repeat the arguments above using the dual construction, which would give us $I(X, \mathcal{Q}) \geq K'm^{4/5}n^{3/5}$ and for a similar reason as case one, this implies $I(X, \mathcal{Q}) \geq K'(mn)^{7/10}$. Thus we can conclude $I(X, \mathcal{Q}) = \Omega((nm)^{7/10})$. $\square$

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