

ON LOCALIZING SUBCATEGORIES OF LIE SUPERALGEBRA REPRESENTATIONS

by

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ABSTRACT

This dissertation studies the representation theory of classical Lie superalgebras from a categorical point of view. Given a classical Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, one can consider the category $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ of $\mathfrak{g}$ -supermodules which are semisimple as modules over the Lie algebra $\mathfrak{g}_{\bar{0}}$. Also, one can consider the full subcategory $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ of $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ which consists of the finite-dimensional supermodules. These are Frobenius categories, so one can form the stable categories $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ and $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ which are triangulated categories. The tensor product of supermodules gives $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ and $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ the structure of tensor triangulated categories, which raises many deep questions about the tensor structure.

Balmer associates to each essentially small tensor triangulated category $\mathcal{K}^c$ two topological spaces: $\text{Spc}(\mathcal{K}^c)$ and $\text{Spc}^h(\mathcal{K}^c)$, called the categorical (Balmer) spectrum and the homological spectrum respectively. When $\mathcal{K}^c$ is rigid, in all known examples the comparison map

$$\phi : \text{Spc}^h(\mathcal{K}^c) \rightarrow \text{Spc}(\mathcal{K}^c)$$

is a bijection. Balmer's Nerves-of-Steel Conjecture states that this is always the case. We prove the conjecture holds for $\mathcal{K}^c = \text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ for $\mathfrak{g}$ a Type A Lie superalgebra. The argument involves

using the detecting subalgebras introduced by Boe, Kujawa, and Nakano, as well as the stratification framework developed by Benson, Iyengar, and Krause. As a consequence, we are able to use the more recent h-stratification introduced by Barthel, Heard, Sanders, and Zou to classify localizing subcategories of $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$, again for Type A Lie superalgebras.

INDEX WORDS: [Classical Lie superalgebras, Detecting subalgebras, Tensor triangulated categories, Balmer spectrum, Homological spectrum, Stratification]

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CHAPTER 1

INTRODUCTION

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra over the complex numbers $\mathbb{C}$. The representation theory of such Lie superalgebras was studied via cohomology and support varieties in a series of early 2000s papers by Boe, Kujawa, and Nakano [BKN09; BKN10; BKN11; BKN17]. They showed, among other results, that under mild assumptions on the action of the algebraic group $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$, there exist subalgebras $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}} \subseteq \mathfrak{g}$, called detecting subalgebras. Detecting subalgebras are interesting in the sense that they have markedly simple representation theory, but they nonetheless determine the relative $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ -cohomology. In particular, [BKN10, Theorem 4.1.1] gives an isomorphism $H^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong H^{\bullet}(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})^N$, where N is a non-connected reductive group determined by a choice of detecting subalgebra $\mathfrak{f}$. Moreover, $H^{\bullet}(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \cong S^{\bullet}(\mathfrak{f}_{\bar{1}}^*)$, and the relative cohomology for detecting subalgebras are polynomial algebras, so relative $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ -cohomology is finitely generated.

One can consider the category $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ (resp. $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$) whose objects consist of all (resp. finite-dimensional) $\mathfrak{g}$ -supermodules which are semisimple as modules over $\mathfrak{g}_{\bar{0}}$ and whose morphisms consist of even morphisms between supermodules. These are abelian categories which have enough projective and injective objects. They are also Frobenius categories; i.e., projective and injective objects coincide, so one can form the stable module categories $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ and $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$. Objects in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ (resp. $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$) are the same as the objects in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ (resp. $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$), but

in the stable category morphisms consist of equivalence classes of morphisms where two morphisms are considered equivalent if their difference factors through a projective module.

The category $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ is a tensor triangulated category which is rigidly-compactly generated by the full tensor triangulated subcategory $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})$ of compact-rigid objects. In [BKN17] the authors consider the tensor triangular geometry of these categories. Techniques from geometric invariant theory are used to compute the Balmer spectrum for the detecting subalgebras and for the Lie superalgebra $\mathfrak{gl}(m|n)$. For the detecting subalgebras [BKN17, Theorem 4.5.4] gives a homeomorphism $\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{f}, \mathfrak{f}_0)})) \cong \text{Proj } H^\bullet(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C}) \cong \text{Proj } S^\bullet(\mathfrak{f}_1^*)$, but for $\mathfrak{g} = \mathfrak{gl}(m|n)$ the situation is a little different. The Balmer spectrum $\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}))$ is not homeomorphic to $\text{Proj } H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$, as one might initially suspect. Instead, one has to consider a stack quotient of $\text{Proj } H^\bullet(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C})$, where $\mathfrak{f} \subseteq \mathfrak{g}$ is a detecting subalgebra. This is the result of [BKN17, Theorem 5.2.2] which states that there is a homeomorphism $\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) \cong N\text{-Proj } H^\bullet(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C})$.

Often, the representation theory of classical Lie superalgebras over $\mathbb{C}$ resembles representation theory of finite groups in positive characteristic. If G is a finite group, and k is an algebraically closed field of characteristic p where p divides the order of G , then the group algebra kG is not semisimple and most of the time has wild representation type. The categories $kG\text{-Mod}$ and $kG\text{-mod}$ consisting of all (resp. finitely generated) left kG -modules are again abelian and Frobenius, so one can form the stable module categories $\text{StMod}(kG)$ and $\text{stmod}(kG)$ which are tensor triangulated categories. An analogous picture holds to Lie superalgebras in that $\text{StMod}(kG)$ is a compactly generated tensor triangulated category whose compact objects are precisely the objects in $\text{stmod}(kG)$. Benson, Carlson, and Rickard first studied the tensor triangular geometry of $\text{stmod}(kG)$ in the late 90s, and the main result of [BCR97] is a classification of the thick $\otimes$ -ideal subcategories. This in turn implies that there is a homeomorphism $\text{Spc}(\text{stmod}(kG)) \cong \text{Proj } H^\bullet(G, k)$. Later on in the 2000s, Benson, Iyengar, and Krause introduced the notion of a tensor triangulated category being stratified by the action of a graded-commutative ring R [BIK11a]. This was a fundamentally new idea that not only allowed for a computation of the Balmer spectrum, but that also had the

advantage of classifying $\otimes$ -ideal localizing subcategories. BIK's theory was applied to show that $\text{StMod}(kG)$ is stratified by the action of the cohomology ring $H^\bullet(G, k)$, a result which recovers Benson, Carlson, and Rickard's contributions, but also gives a classification of $\otimes$ -ideal localizing subcategories in terms of subsets of $\text{Proj } H^\bullet(G, k)$.

While stratification was introduced before Boe, Kujawa, and Nakano's computations of Balmer spectra for $\text{stab}(\mathcal{F}_{(\mathfrak{f}, \mathfrak{f}_0)})$ and $\text{stab}(\mathcal{F}_{(\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)_0)})$, the arguments given in [BKN17] do not rely on any stratification result. In fact, for most classical Lie superalgebras, e.g. $\mathfrak{gl}(m|n)$, the relative $(\mathfrak{g}, \mathfrak{g}_0)$ -cohomology ring fails to stratify. It was conjectured, however, that for a detecting subalgebra the relative cohomology ring $H^\bullet(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C})$ should stratify $\text{Stab}(\mathcal{C}_{(\mathfrak{f}, \mathfrak{f}_0)})$. This conjecture was proved in recent work by the author and Nakano [HN24] from the observation that, just as in the case of elementary abelian two groups in characteristic two, one can reduce the problem to proving a stratification result for the stable category of modules for the exterior algebra $\Lambda(\mathfrak{f}_1)$ viewed as a superalgebra by declaring the generators to be odd, which then comes down to a version of the classical BGG correspondence. In addition to giving a classification of the $\otimes$ -ideal localizing subcategories of $\text{Stab}(\mathcal{C}_{(\mathfrak{f}, \mathfrak{f}_0)})$, this result was also used in [HN24] to prove results about representations of Lie superalgebras concerning nilpotence and the newly developed homological spectra [Bal20]. Specifically it was shown that for $\mathfrak{g}$ a classical Lie superalgebra with a detecting subalgebra $\mathfrak{z} \subseteq \mathfrak{g}$ which is splitting in the sense of [SS23], and satisfying a natural assumption on realization of supports, then there is a homeomorphism $\text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) \cong N\text{-Proj}(H^\bullet(\mathfrak{z}, \mathfrak{z}_0; \mathbb{C}))$. This result led to a verification of Balmer's "Nerves-of-Steel" conjecture for $\mathfrak{g} = \mathfrak{gl}(m|n)$ [HN24, Theorem 7.2.1].

While the BIK approach to stratifying tensor triangulated categories represented a major breakthrough in the field, the drawback is that it requires that the category be equipped with an action by a graded-commutative Noetherian ring in order to construct the necessary support theory. For many TTCs (e.g. $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})$), a BIK stratifying ring is either not known or not readily available, rendering them outside of the scope of the BIK machinery. This prompted Barthel, Heard, and

Sanders in [BHS23b] to develop a theory of stratification called tensor triangular stratification or tt-stratification for short. Tt-stratification for compactly generated tensor triangulated categories is in terms of the support theory developed by Balmer and Favi [BF11] for “big” objects, and has the advantage that does not presuppose the action of a graded-commutative Noetherian ring. Instead, the supports are constructed using the more intrinsic Balmer spectrum of the full compact subcategory. A drawback to tt-stratification is that descent theorems still have to be handled one TTC at a time. Recently, in [Bar+24], Barthel, Heard, Sanders, and Zou developed a third theory of stratification, called h-stratification, which is in terms of the homological spectrum. It is also well behaved under descent. The relationship between tt-stratification and h-stratification becomes tight for TTCs that satisfy the Nerves-of-Steel Conjecture. These key insights allow us, in particular, to reduce the problem of studying $\mathfrak{gl}(m|n)$ to studying a splitting, detecting subalgebra, as well as subalgebras isomorphic to the queer Lie superalgebra $\mathfrak{q}(1)$.

The purpose of this thesis is to add categories of representations of Lie superalgebras to the growing list of tensor triangulated categories for which stratification results are known. We investigate the three notions of stratification for $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ when $\mathfrak{g}$ is a classical Lie superalgebra which has a splitting detecting subalgebra $\mathfrak{z} \subseteq \mathfrak{g}$. Specifically, our theorem is the following.

Theorem A. Let $\mathfrak{g}$ be a classical Lie superalgebra with a splitting, detecting subalgebra $\mathfrak{z} \subseteq \mathfrak{g}$ and which satisfies the realization condition of Assumption 7.3.2 [HN24, Theorem 7.2.1]. The tensor triangulated category $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ is tt-stratified by the Balmer spectrum $\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}))$, and tt-stratification is equivalent to h-stratification.

Classical Lie superalgebras which satisfy the hypotheses of Theorem A include classical Lie superalgebras of Type A. As a consequence of stratification, we obtain the classification of $\otimes$ -ideal localizing subcategories in terms of arbitrary subsets of the Balmer spectrum. This fact, combined with the Balmer spectrum computations of Boe, Kujawa, and Nakano, yields the following corollary.

Corollary B. Let $\mathfrak{g}$ be a classical Lie superalgebra with a splitting, detecting subalgebra $\mathfrak{z} \subseteq \mathfrak{g}$ and which satisfies the realization condition of Assumption 7.3.2. There is a bijection between the set of $\otimes$ -ideal localizing subcategories of $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ and subsets of $N\text{-Proj}(\mathbf{H}^\bullet(\mathfrak{z}, \mathfrak{z}_0; \mathbb{C}))$.

Our work is inspired in part by the strategy outlined for modular representation theory in [BIK11b] where BIK stratification was proven for $\text{StMod}(kG)$ where G is an arbitrary group by first proving the result for $\text{StMod}(kE)$, where E is an elementary abelian group, and then “bootstrapping” up by applying a version of Quillen’s stratification theorem and Chouinard’s theorem. Their argument involves several “change-of-categories” type arguments including a consideration of the homotopy category of injectives $\mathbf{K}(\text{Inj } kG)$. We make use of some of the analogous results in our setting, specifically for studying detecting subalgebras.

The thesis is organized as follows. Chapter 2 gives an overview of tensor triangular geometry with an aim towards introducing the language needed to state our results. We review the Balmer spectrum, as well as notion of support data and extending support data. In Chapter 3 we give an overview of the three stratification setups currently in the literature: BIK stratification, tt-stratification, and h-stratification. Chapter 4 covers background from Lie superalgebras. We explain how the tensor triangular theory of the first two chapters can be applied to study representations of Lie superalgebras. Doing so makes use of detecting subalgebras and splitting subalgebras, so we explain their construction, some aspects of their representation theory, and how they relate to projectivity conditions in Chapter 5. In Chapter 6 we explore stratification for the detecting subalgebras, explain how to construct homological residue fields, and consider various nilpotence theorems. Chapter 7 discusses the Nerves-of-Steel Conjecture, and the classification of tensor ideal localizing subcategories in Type A.

CHAPTER 2

TENSOR TRIANGULATED CATEGORIES

This chapter introduces tensor triangulated categories. We begin by discussing triangulated categories, giving the basic definitions, and explaining how the basic notions are modified in the presence of a tensor structure possessed by tensor triangulated categories. We discuss supports, the Balmer spectrum, and the concept of extending supports to big objects.

2.1 Triangulated categories

Triangulated categories were introduced by Verdier in his thesis [Ver96]. In some sense, their role is to clarify and extend notions from homological algebra. Recall that a triangulated category $\mathcal{T}$ is an additive category equipped with an auto-equivalence $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ called the *shift*, and a class of distinguished triangles:

$$M \rightarrow N \rightarrow Q \rightarrow \Sigma M$$

all subject to a list of axioms the reader can find in, for example, [Nee01, Ch. 1].

A non-empty, full, additive subcategory $\mathcal{S}$ of $\mathcal{T}$ is called a *triangulated subcategory* if (i) $M \in \mathcal{S}$ implies that $\Sigma^n M \in \mathcal{S}$ for all $n \in \mathbb{Z}$ and (ii) if $M \rightarrow N \rightarrow Q \rightarrow \Sigma M$ is a distinguished triangle

in $\mathcal{T}$, and two of $\{M, N, Q\}$ are objects in $\mathcal{S}$, then the third object is also in $\mathcal{S}$. A triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *thick* if $\mathcal{S}$ is closed under taking direct summands.

A triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called a *localizing subcategory* if $\mathcal{S}$ is closed under taking set-indexed coproducts. It follows from a version of the Eilenberg swindle that localizing subcategories are necessarily thick. Given a triangulated category, it is a central problem to classify its thick subcategories and localizing subcategories.

An object $C \in \mathcal{T}$ is called *compact* if $\mathrm{Hom}_{\mathcal{T}}(C, -)$ commutes with set-indexed coproducts. The full subcategory of compact objects in $\mathcal{T}$ is denoted by $\mathcal{T}^c$, and the triangulated category $\mathcal{T}$ is said to be *compactly generated* if the isomorphism classes of compact objects form a set, and if for each non-zero object $M \in \mathcal{T}$ there is an object $C \in \mathcal{T}^c$ such that $\mathrm{Hom}_{\mathcal{T}}(C, M) \neq 0$.

2.2 Tensor triangulated categories

A tensor triangulated category (TTC) is a triple $(\mathcal{K}, \otimes, \mathbb{1})$ consisting of a triangulated category $\mathcal{K}$, a symmetric, monoidal (tensor) product $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ which is exact in each variable, and a monoidal unit $\mathbb{1}$. For the remainder of this section, $\mathcal{K}$ denotes a TTC.

The usual paradigm involves the situation when $\mathcal{K}$ is rigidly-compactly generated as a TTC by its full subcategory of compact-rigid objects $\mathcal{K}^c$. By definition this means that (i) $\mathcal{K}$ is closed under set indexed coproducts, (ii) the tensor product preserves set-indexed coproducts, (iii) $\mathcal{K}$ is compactly generated as a triangulated category, (iv) the tensor product of compact objects is compact, (v) $\mathbb{1}$ is a compact object, and (vi) every compact object is rigid (i.e. dualizable).

The additional structure of the tensor product possessed by TTCs makes the problem of classifying localizing subcategories and thick ideals more tractable. When working in the context of a TTC however, one focuses on classifications for $\otimes$ -ideal localizing subcategories and thick $\otimes$ -ideal subcategories of compact objects. In many situations these match up with the purely triangular notions (c.f. [BIK11b, Section 4.1.4]). For example, if $\mathcal{K}$ is monogenic (generated by the unit object $\mathbb{1}$), as in the case of finite groups, then every localizing subcategory is automatically a $\otimes$ -

ideal. One difference between previously considered representation categories and our theory is that the categories of Lie superalgebra representations of interest in this paper are in general *not* monogenic. Because of this feature our available tools restrict us to a classification of $\otimes$ -ideal localizing subcategories rather than arbitrary localizing subcategories.

2.3 Support for objects in TTCs

The study of TTCs frequently involves considering various support spaces and notions of support for objects. The relevant definition here is that of a support datum, which was originally given in [Bal05]. The definition we give is slightly more general, and is suited to our paper. Let $\mathcal{K}$ be a TTC, X be a Zariski topological space ([BKN11, Section 2.3]), and let $\mathcal{X}$ denote the collection of all subsets of X . A *support datum* on $\mathcal{K}$ is an assignment $V : \mathcal{K} \rightarrow \mathcal{X}$ such that the following properties hold for M, N, M_i, Q objects in $\mathcal{K}$:

$$(2.3.1) \quad V(0) = \emptyset, \text{ and } V(\mathbb{1}) = X;$$

$$(2.3.2) \quad V(\oplus_{i \in I} M_i) = \cup_{i \in I} V(M_i) \text{ provided that } \oplus_{i \in I} M_i \text{ is an object of } \mathcal{K};$$

$$(2.3.3) \quad V(\Sigma M) = V(M)$$

$$(2.3.4) \quad \text{for any distinguished triangle } M \rightarrow N \rightarrow Q \rightarrow \Sigma M, V(N) \subseteq V(M) \cup V(Q);$$

$$(2.3.5) \quad V(M \otimes N) = V(M) \cap V(N);$$

Often useful are support data which satisfy the additional two properties:

$$(2.3.6) \quad V(M) = \emptyset \text{ if and only if } M = 0.$$

$$(2.3.7) \quad \text{for any closed subset } U \in \mathcal{X}, \text{ there exists an object } M \text{ in } \mathcal{K}^c \text{ such that } V(M) = U.$$

Property (2.3.7) is often called the *realization property*, and support data which satisfy Property 2.3.6 are called *faithful*.

2.4 The Balmer spectrum of a TTC

In his 2005 paper, [Bal05], Balmer uses the tensor product and unit object to associate to each TTC $\mathcal{K}$ a topological space known as the categorical (Balmer) spectrum $\mathrm{Spc}(\mathcal{K})$ in a way analogous to the construction of the prime spectrum of a commutative ring. Define a $\otimes$ -ideal in $\mathcal{K}$ as a full triangulated subcategory $\mathcal{I}$ of $\mathcal{K}$ such that $M \otimes N \in \mathcal{I}$ for all $M \in \mathcal{I}$ and $N \in \mathcal{K}$. A proper, thick $\otimes$ -ideal $\mathcal{P}$ of $\mathcal{K}$ is said to be prime if, for objects M and N in $\mathcal{K}$, $M \otimes N \in \mathcal{P}$ implies that $M \in \mathcal{P}$ or $N \in \mathcal{P}$. The Balmer spectrum is then defined as

$$\mathrm{Spc}(\mathcal{K}) := \{\mathcal{P} \subsetneq \mathcal{K} \mid \mathcal{P} \text{ is a prime ideal}\}.$$

The topology on $\mathrm{Spc}(\mathcal{K})$ is the familiar Zariski topology which has the closed sets given by

$$Z(\mathcal{C}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{C} \cap \mathcal{P} \neq \emptyset\},$$

where $\mathcal{C}$ is an arbitrary collection of objects in $\mathcal{K}$.

An important result of Balmer, [Bal05, Theorem 3.2], has to do with supports for essentially small TTCs constructed via the Balmer spectrum. One can construct a support datum on $\mathcal{K}^c$ as follows. Given an object $M \in \mathcal{K}^c$, define

$$\mathrm{supp}_{\mathrm{Bal}}(M) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c) \mid M \notin \mathcal{P}\}.$$

Balmer showed that the support datum given by $(\mathrm{Spc}(\mathcal{K}^c), \mathrm{supp}_{\mathrm{Bal}}(-))$ is universal in the sense that if (X, V) is any support datum on $\mathcal{K}^c$, then there exists a unique continuous map $f : X \rightarrow \mathrm{Spc}(\mathcal{K}^c)$ such that $V(M) = f^{-1}(\mathrm{supp}_{\mathrm{Bal}}(M))$.

2.5 Extending support to big objects

Let $\mathcal{K}$ be a rigidly-compactly generated TTC, and let $\mathcal{K}^c$ denote its full subcategory of compact-rigid objects. One might try to construct a universal notion of support for big TTCs which generalizes Balmer's construction for essentially small TTCs. Morally speaking, Balmer's construction fails to provide such a universal support for big objects because one expects that supports for big objects should be open, while Balmer supports are closed by definition. It turns out that no one has succeeded in constructing universal support for big objects, and it may be impossible to construct such a support datum in general (c.f. [BKS20]). Nonetheless, we are still interested in constructing supports for big objects which, though not universal, still prove useful in practice. The following definition is then motivated.

Definition 2.5.1. Let $\mathcal{K}$ be a rigidly-compactly generated TTC and let $\mathcal{K}^c$ denote the full subcategory of compact-rigid objects. Let (X, V) be a support datum on $\mathcal{K}^c$ where supports are closed subsets of X . The pair $(X, \mathcal{V})$ is said to be an *extension* of (X, V) if $\mathcal{V}$ is an assignment from $\mathcal{K}$ to arbitrary subsets of X satisfying the following:

- (a) $\mathcal{V}$ satisfies properties (2.3.1)-(2.3.5) for objects in $\mathcal{K}$;
- (b) $\mathcal{V}(M) = V(M)$ for all M in $\mathcal{K}^c$; and
- (c) if V satisfies (2.3.7) then $\mathcal{V}$ satisfies (2.3.7).

Notable examples of extension of support data include Balmer-Favi supports which are reviewed in Chapter 3 Section 3, as well as homological supports which are reviewed in Chapter 3 Section 5. See also [Bal20].

CHAPTER 3

STRATIFYING TENSOR TRIANGULATED CATEGORIES

In this chapter we introduce and discuss the various notions of stratification for tensor triangulated categories that exist in the literature today. Doing so involves considering different notions of support for big objects in TTCs. We begin with a discussion of the so-called BIK theory of supports via the action of a graded-commutative Noetherian ring. This theory naturally lends itself to BIK stratification. We then discuss so-called Balmer-Favi supports which were used by Barthel, Heard, and Sanders to develop tensor triangular stratification. Finally, we discuss homological supports and homological stratification, which were recently developed by Barthel, Heard, Sanders, and Zou. We caution the reader that absorbing the particular features of each theory are not necessary to understanding our main results.

3.1 BIK support

Let $\mathcal{K}$ be a rigidly-compactly generated TTC, and let $\mathcal{K}^c$ denote the full subcategory of compact-rigid objects. Aimed at obtaining tensor triangular classifications, Benson, Iyengar, and Krause defined an extension of Balmer's universal support datum for big objects. The additional assumption

needed throughout this subsection necessary to define these so-called BIK supports is that of an auxiliary canonical action of a graded-commutative ring R on $\mathcal{K}$. Let us explain what this means. One can always consider the graded-center $Z^\bullet(\mathcal{K})$ of $\mathcal{K}$ which is a graded-commutative ring whose degree n component is given by

$$Z^n(\mathcal{K}) = \{\eta : \text{Id}_{\mathcal{K}} \longrightarrow \Sigma^n \mid \eta\Sigma = (-1)^n \Sigma\eta\}.$$

An action of R on $\mathcal{K}$ is a homomorphism of graded-commutative rings $\phi : R \rightarrow Z^\bullet(\mathcal{K})$. If $\mathcal{K}$ admits an R -action, then $\mathcal{K}$ is called R -linear.

Given objects M and N in $\mathcal{K}$, set

$$\text{Hom}_{\mathcal{K}}^\bullet(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{K}}(M, \Sigma^i N).$$

Then $\text{Hom}_{\mathcal{K}}^\bullet(M, N)$ is a graded abelian group, and $\text{End}_{\mathcal{K}}^\bullet(M) := \text{Hom}_{\mathcal{K}}^\bullet(M, M)$ is a graded ring where the multiplication is given by applying the shift and then composing morphisms. Notice that $\text{Hom}_{\mathcal{K}}^\bullet(M, N)$ is a right $\text{End}_{\mathcal{K}}^\bullet(M)$ and a left $\text{End}_{\mathcal{K}}^\bullet(N)$ -bimodule. It follows that $\mathcal{K}$ being R -linear is equivalent to their being, for each object M in $\mathcal{K}$, an induced homomorphism of graded rings $\phi_M : R \rightarrow \text{End}_{\mathcal{K}}^\bullet(M)$ such that the induced R -module structures on $\text{Hom}_{\mathcal{K}}^\bullet(M, N)$ by ϕ_M and ϕ_N agree up to the usual sign.

Using the tensor product in $\mathcal{K}$ allows one to construct an action of the graded endomorphism ring $\text{End}_{\mathcal{K}}^\bullet(\mathbb{1})$ of the unit object via the maps defined by taking, for each M in $\mathcal{K}$, $\phi_M : \text{End}_{\mathcal{K}}^\bullet(\mathbb{1}) \rightarrow \text{End}_{\mathcal{K}}^\bullet(M)$ given by tensoring with M . Provided that $\text{End}_{\mathcal{K}}^\bullet(\mathbb{1})$ is Noetherian, any action on $\mathcal{K}$ induced via this action is called canonical.

BIK supports for objects in $\mathcal{K}$ are given in terms of the homogenous prime ideal spectrum $\text{Proj } R$ of R . For each $\mathfrak{p} \in \text{Proj } R$, a deep result in Bousfield localization allows one to construct an exact, local cohomology functor $\Gamma_{\mathfrak{p}} : \mathcal{K} \rightarrow \mathcal{K}$. Properties of these local cohomology functors

can be found in Section 3.1.2 of [BIK11b]. The space of BIK supports for $\mathcal{K}$ is defined to be

$$\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{K}) := \{\mathfrak{p} \in \mathrm{Proj}(R) \mid \Gamma_{\mathfrak{p}}(\mathcal{K}) \neq 0\}.$$

For an object $M \in \mathcal{K}$, the BIK support of M is defined as

$$\mathrm{Supp}_{\mathrm{BIK}}(M) := \{\mathfrak{p} \in \mathrm{Proj}(R) \mid \Gamma_{\mathfrak{p}}(M) \neq 0\}.$$

That $(\mathrm{Supp}_{\mathrm{BIK}}(\mathcal{K}), \mathrm{Supp}_{\mathrm{BIK}}(-))$ is a support data on $\mathcal{K}$ which extends $(\mathrm{Spc}(\mathcal{K}^c), \mathrm{supp}_{\mathrm{Bal}}(-))$ is the main content [BIK11b, Chapter 3].

3.2 BIK stratification

Let $\mathcal{K}$ be a rigidly-compactly generated TTC, and $\mathcal{K}^c$ denote the full subcategory of compact objects of $\mathcal{K}$. Let R be a graded-commutative Noetherian ring, and assume that $\mathcal{K}$ is R -linear. For this section only we follow the lead of BIK by assuming furthermore that $\mathcal{K}$ is *monogenic*; i.e., that $\mathcal{K}$ is compactly generated by the unit object $\mathbb{1}$ of $\mathcal{K}$.

As a first application of BIK supports, one can construct maps between the collection of $\otimes$ -ideal localizing subcategories of $\mathcal{K}$ and arbitrary subsets of $\mathrm{Proj} R$. The maps are defined as follows. Given a $\otimes$ -ideal localizing subcategory $\mathcal{C}$ of $\mathcal{K}$, set

$$\sigma(\mathcal{C}) = \mathrm{Supp}_{\mathrm{BIK}}(\mathcal{C}) = \{\mathfrak{p} \in \mathrm{Proj}(R) \mid \Gamma_{\mathfrak{p}}(\mathcal{C}) \neq 0\}.$$

Next, given a subset V of $\mathrm{Proj}(R)$, set

$$\tau(V) = \{M \in \mathcal{K} \mid \mathrm{Supp}_{\mathrm{BIK}}(M) \subseteq V\}.$$

BIK stratification has to do with two conditions which, when satisfied, guarantee that σ and τ provide mutually inverse bijections. The two conditions are the following.

(a) *The BIK local-to-global principle*: for each object M in $\mathcal{K}$,

$$\mathrm{Loc}_{\otimes} \langle \{M\} \rangle = \mathrm{Loc}_{\otimes} \langle \{\Gamma_{\mathfrak{p}} M \mid \mathfrak{p} \in \mathrm{Proj} R\} \rangle.$$

(b) *The BIK minimality condition*: for $\mathfrak{p} \in \mathrm{Supp}_{\mathrm{BIK}}(\mathcal{K})$, the subcategory $\Gamma_{\mathfrak{p}} \mathcal{K}$ is a minimal $\otimes$ -ideal localizing subcategory of $\mathcal{K}$.

When conditions (a) and (b) hold, $\mathcal{K}$ is said to be *stratified in the sense of BIK*. The following theorem [BIK11b, Theorem 4.19] gives the classification.

Theorem 3.2.1. If $\mathcal{K}$ is stratified in the sense of BIK, then the maps σ and τ provide mutually inverse bijections between the set of $\otimes$ -ideal localizing subcategories of $\mathcal{K}$ and subsets of $\mathrm{Proj} R$:

$$\{\otimes\text{-ideal localizing subcategories of } \mathcal{K}\} \xleftrightarrow[\tau]{\sigma} \{\text{subsets of } \mathrm{Supp}_{\mathrm{BIK}} \mathcal{K}\}.$$

A useful fact that applies in many practical situations is that the BIK local-to-global principle automatically holds in instances where the Krull dimension of $\mathrm{Proj} R$ is finite. This reduces much of the work involved with verifying BIK stratification to the minimality condition.

3.3 Balmer-Favi support

Throughout this section, let $\mathcal{K}$ be a rigidly-compactly generated TTC, and let $\mathcal{K}^c$ denote the full subcategory of compact-rigid objects. In [BF11], Balmer and Favi construct a support datum on $\mathcal{K}$ which extends Balmer’s universal support for $\mathcal{K}^c$. The key difference, from our point of view, between BIK supports and Balmer-Favi supports is that constructing Balmer-Favi supports does not make use of an auxiliary ring action. Instead, supports are constructed based on certain tensor idempotents which themselves are a tensor triangular abstraction of Rickard’s idempotent modules

from [Ric97]. Another important difference is that the space of supports is always the Balmer spectrum $\mathrm{Spc}(\mathcal{K}^c)$ of the compact subcategory, which is not necessarily the case for BIK supports. Throughout this section only we impose the additional hypothesis that $\mathrm{Spc}(\mathcal{K}^c)$ is Noetherian. One can actually get away with a weaker condition, namely assuming that $\mathrm{Spc}(\mathcal{K}^c)$ is weakly Noetherian (cf. [BHS23b, Section 1]). However, since all Balmer spectra we are concerned with are Noetherian, we work under a Noetherian hypothesis, and hope this provides some comfort to the reader. The driving force behind the scenes is again Bousfield localization, which guarantees that for every specialization closed subset $W \subseteq \mathrm{Spc}(\mathcal{K}^c)$, one can construct two $\otimes$ -idempotents $E(W) \cong E(W) \otimes E(W)$ and $F(W) \cong F(W) \otimes F(W)$ in $\mathcal{K}$ that fit into a distinguished triangle

$$E(W) \rightarrow \mathbb{1} \rightarrow F(W) \rightarrow \Sigma E(W).$$

The Noetherian hypothesis on $\mathrm{Spc}(\mathcal{K}^c)$ implies that every point $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$ is visible in the sense of [BF11, Section 7.9]. From this, [BF11, Lemma 7.8] gives that one can express each $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$ as $\mathcal{P} = Y_1 \cap Y_2^c$ for specialization closed subsets $Y_1, Y_2 \subseteq \mathrm{Spc}(\mathcal{K}^c)$. One then defines a $\otimes$ -idempotent $g(\mathcal{P}) := E(Y_1) \otimes F(Y_2)$ that depends only on $\mathcal{P}$ and not the choice of specialization closed subsets. For an object $M \in \mathcal{K}$, the Balmer-Favi support of M is defined to be

$$\mathrm{Supp}_{\mathrm{BF}}(M) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c) \mid M \otimes g(\mathcal{P}) \neq 0\}.$$

That $(\mathrm{Supp}_{\mathrm{BF}}(-), \mathrm{Spc}(\mathcal{K}^c))$ is a support datum on $\mathcal{K}$ that extends $(\mathrm{Spc}(\mathcal{K}^c), \mathrm{supp}_{\mathrm{Bal}}(-))$ is the content of [BF11, Prop. 7.18].

3.4 Tensor triangular stratification via Balmer-Favi supports

Balmer and Favi's original motivation that led to the definition of Balmer-Favi support was to transport methods from modular representation theory to algebraic geometry. A latent application

however, realized by Barthel, Heard, and Sanders in [BHS23b], is a stratification theory distinct from the theory developed by BIK which frees one from the requirement of a ring action, and which is universal in Noetherian contexts. Again, the stratification framework begins with defining maps that allow one to pass between $\otimes$ -ideal localizing subcategories of $\mathcal{K}$ and arbitrary subsets of $\mathrm{Spc}(\mathcal{K}^c)$. To that end, given a $\otimes$ -ideal localizing subcategory $\mathcal{C}$ of $\mathcal{K}$, set

$$\sigma(\mathcal{C}) = \bigcup_{M \in \mathcal{C}} \mathrm{Supp}_{\mathrm{BF}}(M).$$

Then, given a subset V of $\mathrm{Spc}(\mathcal{K}^c)$, set

$$\tau(V) = \{M \in \mathcal{K} \mid \mathrm{Supp}_{\mathrm{BF}}(M) \subseteq V\}.$$

The relevant conditions on Balmer-Favi supports which allow for tensor triangular classifications are the following.

(a) *The tt local-to-global principle:* for each object M in $\mathcal{K}$,

$$\mathrm{Loc}_{\otimes} \langle \{M\} \rangle = \mathrm{Loc}_{\otimes} \langle \{M \otimes g(\mathcal{P}) \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)\} \rangle.$$

(b) *The tt minimality condition:* for each $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}^c)$, the subcategory $\mathrm{Loc}_{\otimes} \langle \{g(\mathcal{P})\} \rangle$ is a minimal $\otimes$ -ideal localizing subcategory of $\mathcal{K}$.

When conditions (a) and (b) hold, $\mathcal{K}$ is said to be *tt-stratified*, and [BHS23b, Theorem 4.1] gives the classification. We record a condensed version here for completeness.

Theorem 3.4.1. If $\mathcal{K}$ is tt-stratified, then the maps σ and τ provide bijections between the set of $\otimes$ -ideal localizing subcategories of $\mathcal{K}$ and subsets of $\mathrm{Spc}(\mathcal{K}^c)$:

$$\{\otimes\text{-ideal localizing subcategories of } \mathcal{K}\} \xleftrightarrow[\tau]{\sigma} \{\text{subsets of } \mathrm{Spc}_{\mathrm{BF}} \mathcal{K}\}.$$

3.5 Homological support

The setup is again the same. Namely, we work in the context where $\mathcal{K}$ is a rigidly-compactly generated TTC, and the full subcategory of compact-rigid objects is denoted $\mathcal{K}^c$. Motivated by a desire for abstract nilpotence theorems in TTCs, Balmer defined a topological space called homological spectrum $\mathrm{Spc}^h(\mathcal{K}^c)$ (c.f. [Bal20]). A condensed review of the construction is as follows. Let $\mathbf{Ab}$ denote the category of abelian groups. The category $\mathrm{Mod}\text{-}\mathcal{K}^c$ of right $\mathcal{K}^c$ -modules is the category whose objects consist of additive functors $M : (\mathcal{K}^c)^{\mathrm{op}} \rightarrow \mathbf{Ab}$, and whose morphisms consist of natural transformations between functors. The module category $\mathrm{Mod}\text{-}\mathcal{K}^c$ is an abelian category and receives $\mathcal{K}^c$ via the Yoneda embedding which we denote by

$$h : \mathcal{K}^c \hookrightarrow \mathrm{Mod}\text{-}\mathcal{K}^c = \mathrm{Add}((\mathcal{K}^c)^{\mathrm{op}}, \mathbf{Ab})$$

$$M \mapsto \hat{M} := \mathrm{Hom}_{\mathcal{K}^c}(-, M)$$

$$f \mapsto \hat{f}.$$

Let $\mathcal{A}$ denote $\mathrm{Mod}\text{-}\mathcal{K}^c$. Day convolution gives $\mathcal{A}$ a tensor structure which is colimit-preserving in each variable and which makes h a monoidal functor. The tensor subcategory $\mathcal{A}^{\mathrm{fp}} := \mathrm{mod}\text{-}\mathcal{K}^c \subseteq \mathcal{A}$ of finitely presented objects is the Freyd envelope of $\mathcal{K}^c$. A homological prime of $\mathcal{K}^c$ is defined to be a maximal, proper, Serre $\otimes$ -ideal subcategory $\mathcal{B} \subseteq \mathcal{A}^{\mathrm{fp}}$, and the homological spectrum of $\mathcal{K}^c$ is defined to be the set of homological primes:

$$\mathrm{Spc}^h(\mathcal{K}^c) := \{\mathcal{B} \subsetneq \mathcal{A}^{\mathrm{fp}} \mid \mathcal{B} \text{ is a maximal Serre tensor ideal subcategory}\}.$$

One can define a support datum on $\mathcal{K}^c$ in terms of the homological spectrum by defining the homological support of an object M in $\mathcal{K}^c$ as

$$\mathrm{supp}^h(M) := \{\mathcal{B} \in \mathrm{Spc}^h(\mathcal{K}^c) \mid \hat{M} \notin \mathcal{B}\}.$$

One can view the collection $\text{supp}^h(M)$ as M ranges over all objects of $\mathcal{K}^c$ as a basis for the closed subsets of a topology on $\text{Spc}^h(\mathcal{K}^c)$. The universality of the Balmer spectrum and Balmer supports for objects of $\mathcal{K}^c$ implies the existence of a unique continuous map $\phi : \text{Spc}^h(\mathcal{K}^c) \rightarrow \text{Spc}(\mathcal{K}^c)$ called the comparison map. It is known to be surjective assuming the rigidity of $\mathcal{K}^c$. In all known examples, the comparison map is a bijection. This leads to the following.

Conjecture 3.5.1 (Nerves-of-Steel). Let $\mathcal{K}^c$ be rigid. The comparison map

$$\phi : \text{Spc}^h(\mathcal{K}^c) \rightarrow \text{Spc}(\mathcal{K}^c)$$

is a bijection.

The homological support for objects in $\mathcal{K}^c$ can be extended to a support datum on $\mathcal{K}$ via the following construction. From [Bal20] there is a pure-injective object $E_{\mathcal{B}}$ in $\mathcal{K}$ corresponding to each homological prime $\mathcal{B} \in \text{Spc}^h(\mathcal{K}^c)$. For objects M in $\mathcal{K}$, the extended homological support is defined as

$$\text{Supp}^h(M) := \{\mathcal{B} \in \text{Spc}^h(\mathcal{K}^c) \mid \text{hom}(M, E_{\mathcal{B}}) \neq 0\},$$

where $\text{hom}(-, -)$ denotes the internal hom in $\mathcal{K}$.

3.6 Stratification via homological support

In recent work, Barthel, Heard, Sanders, and Zou [Bar+24] developed a notion of stratification in terms of the homological spectrum and homological support. This theory of stratification, called homological stratification or h-stratification, has the advantage of satisfying a very general form of descent. Let $\mathcal{K}$ be a rigidly-compactly generated TTC, and let $\mathcal{K}^c$ denote the full subcategory of compact-rigid objects. Homological support determines natural maps σ and τ between $\otimes$ -ideal localizing subcategories of $\mathcal{K}$ and subsets of $\text{Spc}^h(\mathcal{K}^c)$. The maps are constructed in a similar way

to how they are constructed for tt-stratification. Given a $\otimes$ -ideal localizing subcategory $\mathcal{C}$ of $\mathcal{K}$, set

$$\sigma(\mathcal{C}) = \bigcup_{M \in \mathcal{C}} \text{Supp}^h(M).$$

Then, given a subset V of $\text{Spc}(\mathcal{K}^c)$, set

$$\tau(V) = \{M \in \mathcal{K} \mid \text{Supp}^h(M) \subseteq V\}.$$

As usual $\mathcal{K}$ is said to be homologically stratified if these maps give mutually inverse bijections. The theorem that gives sufficient and necessary conditions for homological stratification is given in [Bar+24]. We record a slightly modified form of the theorem here.

Theorem 3.6.1. The TTC $\mathcal{K}$ is homologically stratified if the following conditions hold.

(a) *The homological local-to-global principle:* for each object M in $\mathcal{K}$,

$$\text{Loc}_{\otimes}(\{M\}) = \text{Loc}_{\otimes}(\{M \otimes E_{\mathcal{B}} \mid \mathcal{B} \in \text{Spc}^h(\mathcal{K}^c)\}).$$

(b) *The homological minimality condition:* for each $\mathcal{B} \in \text{Spc}^h(\mathcal{K}^c)$, the subcategory $\text{Loc}_{\otimes}(\{E_{\mathcal{B}}\})$ is a minimal $\otimes$ -ideal localizing subcategory of $\mathcal{K}$.

3.7 Relationships between stratification theories, pros and cons, etc.

A developing theme in the area involves the comparison between the various notions of stratification and the implications that allow one to pass from one notion of stratification to another. As we will see sometimes it is relatively straightforward to obtain results for a TTC for a particular stratification theory, but for one reason or another it is convenient to be able to transfer the results to the other notions where other results can be applied to say something new. First let us consider the pros and cons.

Type of stratification	Pros	Cons
BIK	• Neeman’s theorem to work with • Many results in the literature available	• Requires ring action • $\text{End}^\bullet(1)$ may not help
Tensor triangular	• In terms of $\text{Spc}(\mathcal{K}^c)$ • No ring action required	• Need to understand $\text{Spc}(\mathcal{K}^c)$ • No general form of descent
Homological	• $\text{Spc}^h(\mathcal{K}^c)$ often concrete • Very general form of descent	• Often need Nerves-of-Steel • Local-global principle not trivial

The first implication result we want to highlight [BHS23b, Theorem D] which demonstrates the universality of tt-stratification in Noetherian contexts. This can be interpreted as “BIK stratification implies tt-stratification”.

Theorem 3.7.1. Let $\mathcal{K}$ be a rigidly-compactly generated TTC which is Noetherian and stratified in the sense of BIK by the action of a graded-commutative Noetherian ring R . Then the BIK space of supports $\text{Supp}_{\text{BIK}}(\mathcal{K})$ is canonically homeomorphic to $\text{Spc}(\mathcal{K}^c)$ and the BIK notion of support coincides with the Balmer-Favi notion of support.

The relationships between tt-stratification and h-stratification are not as clear-cut. This has to do with the fact that h-stratification does not rely on any point-set topological conditions on the homological spectrum, so h-stratification does not have a strong universality statement available. However, in weakly Noetherian situations, and when the Nerves-of-Steel Conjecture holds, the relationship is tight, as demonstrated by [Bar+24, Theorem E], which we record here.

Theorem 3.7.2. If $\mathcal{K}$ is a rigidly-compactly generated TTC with $\text{Spc}(\mathcal{K}^c)$ weakly Noetherian, then the following are equivalent:

- (a) $\mathcal{K}$ is tt-stratified;
- (b) $\mathcal{K}$ is h-stratified and the Nerves-of-Steel Conjecture holds for $\mathcal{K}$.

As the original authors note, this means that when the Nerves-of-Steel Conjecture holds, it suffices to consider only h-stratification.

Finally, we want to mention descent. Before h-stratification, descent results had to be verified one TTC at a time, and general methods were not available. The following theorem [Bar+24, Theorem A] demonstrates the power of h-stratification in contexts where natural restriction functors exist.

Theorem 3.7.3. Let $(f_i^* : \mathcal{K} \rightarrow \mathcal{S}_i)_{i \in I}$ be a family of exact, symmetric monoidal functors that preserve set-indexed coproducts and jointly detect when an object of $\mathcal{K}$ is zero. Suppose that $\mathcal{S}_i$ is tt-stratified for all $i \in I$. If $\mathcal{K}$ satisfies the Nerves-of-Steel Conjecture and has a weakly Noetherian spectrum, then the following are equivalent:

- (a) $\mathcal{K}$ is tt-stratified
- (b) $\mathcal{K}$ is h-stratified
- (c) $\mathcal{K}$ is generated by the images of the right adjoints $(f_i)_*$

This generalizes all the descent results in the literature, and provides a uniform approach for developing new results, as we will lay out for Lie superalgebras in the following sections.

CHAPTER 4

LIE SUPERALGEBRAS AND THEIR REPRESENTATIONS

This chapter gives an introduction to Lie superalgebras and their representation theory. Throughout this chapter the ground field is taken to be the complex numbers $\mathbb{C}$. We begin by describing the general idea of “super mathematics”. We define super vector spaces, explain basic constructions, and then define superalgebras, and supermodules including Lie superalgebras and Lie supermodules.

We go on to introduce certain abelian subcategories of Lie supermodules. These categories we describe are Frobenius, and from their data we explain how to define the stable categories which are tensor triangulated.

4.1 “Super mathematics”: the idea and first definitions

“Super mathematics” is an informal term to describe the study of algebraic structures equipped with a $\mathbb{Z}/2\mathbb{Z}$ -grading. The following definitions also serve as examples to illustrate the general concept.

Definition 4.1.1. A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $V = V_0 \oplus V_1$. Given a homogeneous element $v \in V_i$, the *degree* of v is denoted $\bar{v} = i$, where $i \in \mathbb{Z}/2\mathbb{Z}$. Homogeneous elements of V_0 are called *even*, and homogeneous elements of V_1 are called *odd*.

Let V and W be super vector spaces. Many of the usual constructions for vector spaces carry over to the super setting. For example, the vector space of linear transformations $\text{Hom}(V, W)$ is a super vector space by setting

$$\begin{aligned}\text{Hom}(V, W)_{\bar{0}} &= \{\phi \mid \phi(V_i) \subseteq W_i\} \\ \text{Hom}(V, W)_{\bar{1}} &= \{\phi \mid \phi(V_i) \subseteq W_{i+\bar{1}}\}\end{aligned}$$

where the subscripts are read modulo 2. The special case of this construction where $W = \mathbb{C}$ viewed as a super vector space concentrated in degree $\bar{0}$ makes the dual space V^* into a super vector space. Moreover, the usual vector space tensor product $V \otimes W$ can be viewed as a super vector space by taking

$$\begin{aligned}(V \otimes W)_{\bar{0}} &= (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}) \\ (V \otimes W)_{\bar{1}} &= (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}).\end{aligned}$$

We remind the reader that here Hom means $\text{Hom}_{\mathbb{C}}$ and $\otimes$ means $\otimes_{\mathbb{C}}$.

Definition 4.1.2. A *superalgebra* is a super vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ which is also a unital, associative algebra whose bilinear multiplication $A \times A \rightarrow A$ is such that $A_i A_j \subseteq A_{i+j}$, where $i, j \in \mathbb{Z}/2\mathbb{Z}$; i.e., the multiplication in A respects the $\mathbb{Z}/2\mathbb{Z}$ -grading.

Example 4.1.3. Let V be a super vector space. The endomorphisms of V denoted $\text{End}(V) := \text{Hom}(V, V)$ form a superalgebra under composition.

Definition 4.1.4. Given a superalgebra A , an *A -supermodule* is a super vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ which is also a left A -module such that $A_i M_j \subseteq M_{i+j}$ where $i, j \in \mathbb{Z}/2\mathbb{Z}$; i.e., the action of A on M respects the $\mathbb{Z}/2\mathbb{Z}$ -grading. Let M and N be A -supermodules. A *supermodule homomorphism* is a module homomorphism that preserves the $\mathbb{Z}/2\mathbb{Z}$ -grading.

4.2 Lie superalgebras and their representations

Lie superalgebras are a class of superalgebras that arise in physics as a tool to describe the mathematics of supersymmetry. They are the $\mathbb{Z}/2\mathbb{Z}$ (“super”) analog of ordinary Lie algebras. We record the definition here.

Definition 4.2.1. A Lie superalgebra is a super vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ together with a bilinear multiplication $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ called the *superbracket* which satisfies the following axioms:

1. Skew-supersymmetry: $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$
2. Super Jacobi identity: $(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{y}\bar{x}}[y, [z, x]] + (-1)^{\bar{z}\bar{y}}[z, [x, y]] = 0$

for homogeneous elements $x, y, z \in \mathfrak{g}$.

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called *classical* if there is a connected, reductive algebraic group $G_{\bar{0}}$ such that $\text{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$, and if there is an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$. If, in addition to being classical, $\mathfrak{g}$ has a nondegenerate, invariant, supersymmetric, even bilinear form, then $\mathfrak{g}$ is called *basic classical*. The basic classical Lie superalgebras were classified by Kac [Kac77].

Given a Lie superalgebra $\mathfrak{g}$, there is a universal enveloping superalgebra $U(\mathfrak{g})$ which satisfies a super analog of the PBW theorem for Lie algebras. The category of $\mathfrak{g}$ -supermodules has as objects all left $U(\mathfrak{g})$ -supermodules. This means that $\mathfrak{g}$ -supermodules are super vector spaces and that the $\mathfrak{g}$ -action is compatible with the $\mathbb{Z}/2\mathbb{Z}$ -grading. Morphisms between $\mathfrak{g}$ -supermodules are even (i.e. degree preserving) morphisms in $\text{Hom}_{\mathbb{C}}(M, M')$ which satisfy $f(xm) = (-1)^{\bar{f}\bar{x}}xf(m)$ for all $m \in M$ and all $x \in U(\mathfrak{g})$. This makes sense as stated only for homogenous elements, and should be extended via linearity in general. Given two $\mathfrak{g}$ -supermodules M, N , one can use the coproduct and antipode of $U(\mathfrak{g})$ to give $\mathfrak{g}$ -supermodule structures to the vector space tensor product $M \otimes N$, and, when M is finite-dimensional, the contragradient dual M^* . Denote the category of $\mathfrak{g}$ -supermodules as $\mathfrak{g}\text{-sMod}$. Because morphisms in $\mathfrak{g}\text{-sMod}$ are even, it is an abelian category. As

convention and notation we consider Lie algebras as Lie superalgebras concentrated in degree $\bar{0}$, and we refer to supermodules as modules when there is little room for confusion.

4.3 The categories $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ and $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ and their stable categories

Given a basic classical Lie superalgebra one can consider the $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ which is the category whose objects are $\mathfrak{g}$ -modules that, when viewed as modules for the Lie algebra $\mathfrak{g}_{\bar{0}}$, are semisimple; i.e., are direct sums of finite-dimensional simple $\mathfrak{g}_{\bar{0}}$ -modules. The full subcategory of $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ consisting of only finite-dimensional modules is denoted by $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$. Both the categories $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ and $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ are abelian categories. Moreover, $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ and $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ are also Frobenius categories. In other words, these categories have enough projective and injective objects, and projectives and injectives coincide. This implies that one can form the stable module categories $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ and $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$. Objects in the stable categories are the same as the objects in the underlying abelian categories from which they are formed, but morphisms in the stable categories are equivalence classes of morphisms where two morphisms are considered equivalent if their difference factors through a projective module. The stable categories are triangulated categories, and the tensor product of modules gives $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ and $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ the structure of tensor triangulated categories. Moreover, one has that $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ is a rigidly-compactly generated TTC with full subcategory of compact rigid objects $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$. Let $\mathcal{C}$ denote the category $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$, and let $\mathcal{F}$ denote the category $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$. Given modules M, N in $\mathcal{F}$, the group of degree n extensions, $\text{Ext}_{\mathcal{F}}^n(M, N)$ can be realized via relative Lie superalgebra cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$:

$$\text{Ext}_{\mathcal{F}}^n(M, N) \cong \text{Ext}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}^n(M, N) \cong H^n(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M^* \otimes N).$$

There exists an explicit Koszul type resolution that can be used to compute relative Lie superalgebra cohomology. An interesting feature that obtains when considering the relative cohomology ring $\text{Ext}_{\mathcal{F}}^{\bullet}(\mathbb{C}, \mathbb{C}) \cong H^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is that the cochains are uniformly zero. From this, BKN showed in

[BKN10, Section 2.5] that there is isomorphism

$$H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong S^\bullet(\mathfrak{g}_1^*)^{G_{\bar{0}}}$$

of graded rings, and that the relative cohomology is a polynomial algebra [BKN10, Section 8.11].

CHAPTER 5

DETECTING SUBALGEBRAS, SPLITTING SUBALGEBRAS, AND PROJECTIVITY

This chapter expands upon the previous one. We begin by introducing Boe, Kujawa, and Nakano’s detecting subalgebras. The detecting subalgebras are useful because they have a simplified representation theory compared to general classical Lie superalgebras, but they still “determine” cohomology in a sense that we will explain.

We proceed by defining Type I Lie superalgebras, and define the cohomological support variety. As in modular representation theory of finite groups, the cohomological support variety admits a “rank variety” description, and a modified version Dade’s Lemma holds.

Then, the condition of a Type I classical Lie superalgebra having an ample detecting subalgebra is discussed. The reason for this condition has to do with projectivity upon restriction, and it turns out that having an ample detecting subalgebra allows for a modified version of Chouinard’s theorem in modular representation theory.

Finally, we introduce Serganova and Sherman's splitting subalgebras, which allow us to make statements concerning projectivity for infinitely generated modules.

5.1 Detecting subalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic, classical Lie superalgebra. A remarkable contribution of Boe, Kujawa, and Nakano was to prove the existence of important subalgebras called detecting subalgebras $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}} \subseteq \mathfrak{g}$ which have much easier representation theory than $\mathfrak{g}$ but which nonetheless “detect” the relative $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ -cohomology. Detecting subalgebras are constructed by considering the action of the algebraic group $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$. We recall briefly the parts of their construction that are needed.

View the set $\mathfrak{g}_{\bar{1}}$ as an affine variety with the Zariski topology. A point $x \in \mathfrak{g}_{\bar{1}}$ is called *regular* if the orbit $G_{\bar{0}} \cdot x$ has maximum possible dimension, and *semisimple* if $G_{\bar{0}} \cdot x$ is closed in $\mathfrak{g}_{\bar{1}}$. The action of $G_{\bar{1}}$ is called *stable* if $\mathfrak{g}_{\bar{1}}$ has an open dense subset consisting of semisimple points. If there is an open dense subset of $\mathfrak{g}_{\bar{1}}$ such that the stabilizer subgroups of any two points in this set are conjugate subgroups of $G_{\bar{0}}$, then the stabilizer of such a point is called a stabilizer in general position. If the action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is stable, then such an open set exists in $\mathfrak{g}_{\bar{1}}$. Elements of such an open set are called generic. If the action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is stable then $\mathfrak{g}$ is said to be stable. If $\mathfrak{g}$ is stable, then there is necessarily a generic point $x_0 \in \mathfrak{g}_{\bar{1}}$. Let $H = \text{Stab}_{G_{\bar{0}}}(x_0)$ and $N = \text{Norm}_{G_{\bar{0}}}(H)$. Set $\mathfrak{f}_{\bar{1}} = \mathfrak{g}_{\bar{1}}^H$, and $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$. The Lie superalgebra $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$ is classical, and is a detecting subalgebra. The sense in which detecting subalgebras determine cohomology is as follows. The inclusion $\mathfrak{f} \subseteq \mathfrak{g}$ induces a restriction homomorphism $S^\bullet(\mathfrak{f}_{\bar{1}}^*) \rightarrow S^\bullet(\mathfrak{g}_{\bar{1}}^*)$ which induces an isomorphism

$$H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \rightarrow H^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})^N$$

in cohomology.

Remark 5.1.1. In [BKN10], the authors construct two families of detecting subalgebras $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}$. The subalgebras $\mathfrak{e}$ are not considered in this paper.

5.2 Type I Lie superalgebras:

A Lie superalgebra is *Type I* if it admits a $\mathbb{Z}$ -grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ concentrated in degrees -1 , 0 , and 1 with $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Examples of Type I Lie superalgebras include the general linear Lie superalgebra $\mathfrak{gl}(m|n)$, as well as the simple Lie superalgebras of types $A(m, n)$, $C(n)$ and $P(n)$. These all have stable actions of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which yields Type I detecting subalgebras.

For this dissertation, it will be important to distinguish Type I classical Lie superalgebras that contain a detecting subalgebra with favorable geometric properties.

Definition 5.2.1. Let $\mathfrak{g}$ be a Type I classical Lie superalgebra with a detecting subalgebra $\mathfrak{f} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_1$. Then $\mathfrak{f}$ is an *ample detecting subalgebra* if $\mathfrak{g}_j = G_0 \cdot \mathfrak{f}_j$ for $j = -1, 1$.

As we will see in next section, there is an abundance of examples of Type I Lie superalgebras with ample detecting subalgebras that encompass many cases of simple Lie superalgebras over $\mathbb{C}$.

5.3 Examples of Lie Superalgebras with Ample Detecting Subalgebras

In this section, we will provide examples of Type I classical Lie superalgebras that contain an ample detecting subalgebra. Many of these actions involving G_0 on $\mathfrak{g}_{\pm 1}$ arise naturally in the context of linear algebra. For a more detailed description of these actions, the reader is referred to [BKN11, Section 3.8].

General Linear Superalgebras:

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. As a vector space this is isomorphic to the set of $m + n$ by $m + n$ matrices. For a basis, one can take the elementary matrices $E_{i,j}$ where $1 \leq i, j \leq m + n$. The degree zero component is $\mathfrak{g}_0 \cong \mathfrak{gl}(m) \times \mathfrak{gl}(n)$ with corresponding reductive group $G_0 \cong GL(m) \times GL(n)$.

Constructions of detecting subalgebras for classical Lie superalgebras are explicitly described in [BKN10, Section 8]. Set $r = \min(m, n)$. A detecting subalgebra is given by $\mathfrak{f} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_1$ where $\mathfrak{f}_{-1}$ is the span of $\{E_{m+i,i} : i = 1, 2, \dots, r\}$, $\mathfrak{f}_1$ is the span of $\{E_{i,m+i} : i = 1, 2, \dots, r\}$, and $\mathfrak{f}_0 = [\mathfrak{f}_1, \mathfrak{f}_1]$.

The action of G_0 on $\mathfrak{g}_{-1}$ is given by $(A, B).X = BXA^{-1}$ and on $\mathfrak{g}_1$ by $(A, B).X = AXB^{-1}$. It is a well-known fact from linear algebra that the orbits representatives are the matrices of a given rank in $\mathfrak{g}_{\pm 1}$. It follows that $\mathfrak{g}_{\pm 1} = G_0 \cdot \mathfrak{f}_{\pm 1}$, and $\mathfrak{f}$ is an ample detecting subalgebra.

Other Type A Lie Superalgebras:

The other Type A Lie superalgebras $\mathfrak{g}$ are all Type I, and they all have $\mathfrak{g}_{\pm 1} \cong \mathfrak{gl}(m|n)_{\pm 1}$. Furthermore, one has $\mathfrak{f}$ as given above for $\mathfrak{gl}(m|n)$ as a subalgebra of $\mathfrak{g}$ [BKN11, Sections 3.8.2 and 3.8.3].

When $m \neq n$, $\mathfrak{g} = \mathfrak{sl}(m|n) \subseteq \mathfrak{gl}(m|n)$ consists of the matrices of supertrace zero, and

$$G_0 = \{(A, B) \in GL(m) \times GL(n) \mid \det(A)\det(B)^{-1} = 1\}.$$

The G_0 -orbits are the same as the $GL(m) \times GL(n)$ -orbits, and $\mathfrak{f}$ is an ample detecting subalgebra.

For the Lie superalgebra $\mathfrak{sl}(n|n)$ has a one dimensional center given by scalar multiples of the identity matrix, and one has

$$G_0 \cong \{(A, B) \in GL(n) \times GL(n) \mid \det(A)\det(B)^{-1} = 1\}.$$

For elements of $\mathfrak{g}_{\pm 1}$ with rank strictly less than n , the G_0 -orbits coincide with the $GL(n) \times GL(n)$ -orbits. The orbits of full rank matrices form a one parameter family with each orbit containing a unique matrix which is a scalar multiple of the identity. The orbit theory for the $\mathfrak{g} = \mathfrak{psl}(n|n)$ case is analogous to $\mathfrak{sl}(n|n)$. Consequently, in both these setting the algebra $\mathfrak{f}$ is ample.

Type C Lie Superalgebras which are Type I:

In this case $\mathfrak{g} = \mathfrak{osp}(2|2n)$ with $G_{\bar{0}} \cong \mathbb{C}^\times \times Sp(2n)$. One has $\mathfrak{g}_1 \cong V_{2n}$, the natural module for $Sp(2n)$. The action of $Sp(2n)$ is transitive on $V_{2n} \setminus \{0\}$. One has an explicit detecting subalgebra $\mathfrak{f} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_1$ where $\dim \mathfrak{f}_{\pm 1} = 1$. The transitivity of the action of $G_{\bar{0}}$ on $\mathfrak{g}_1$ shows that $\mathfrak{g}_1 = G_{\bar{0}} \cdot \mathfrak{f}_1$. A similar argument demonstrates that $\mathfrak{g}_{-1} = G_{\bar{0}} \cdot \mathfrak{f}_{-1}$.

Type P Lie Superalgebras:

For Type P Lie superalgebras $\mathfrak{g} = \tilde{\mathfrak{p}}(n)$ and $\mathfrak{g} = \tilde{\mathfrak{p}}(n)$ one has an explicit detecting subalgebra $\mathfrak{f} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_1$ where $\mathfrak{f}_{\pm 1}$ contains matrices of all possible ranks.

Let $\mathfrak{g} = \tilde{\mathfrak{p}}(n)$. Then $G_{\bar{0}} \cong GL(n)$ and $\mathfrak{g}_{-1} \cong \Lambda^2(V^*)$ and $\mathfrak{g}_1 \cong S^2(V)$ as $G_{\bar{0}}$ -modules, where V denotes the natural $GL(n)$ -module. There are a finite number of orbits given again by the condition on rank, and their closure relation forms a chain. This shows that $\mathfrak{f}$ is ample.

Now let $\mathfrak{g} = \mathfrak{p}(n) = [\tilde{\mathfrak{p}}(n), \tilde{\mathfrak{p}}(n)]$ be the simple Lie superalgebra of type $P(n-1)$. One has $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are as above but $G_{\bar{0}} \cong SL(n)$. This case follows the paradigm of $\mathfrak{sl}(n|n)$. The $GL(n)$ -orbits corresponding to matrices of rank less than n in $\mathfrak{g}_{\pm 1}$ are also $G_{\bar{0}}$ -orbits. The matrices of rank n yield a one parameter family of orbits that have orbit representatives in $\mathfrak{f}_{\pm 1}$, which demonstrates the ampleness of $\mathfrak{f}$.

5.4 Cohomological and Rank Varieties

We review the constructions in [BKN11, Section 3.2] for Type I Lie superalgebras. Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Type I Lie superalgebra. Then $\mathfrak{g}_{\pm 1}$ are abelian Lie superalgebras. Consequently, $U(\mathfrak{g}_{\pm 1})$ identifies with an exterior algebra, and the cohomology ring for these superalgebras identifies with the symmetric algebra on the dual of $\mathfrak{g}_{\pm 1}$. Set $R_{\pm 1} = H^\bullet(\mathfrak{g}_{\pm 1}, \mathbb{C}) \cong S^\bullet(\mathfrak{g}_{\pm 1}^*)$. Let M be a finite-

dimensional $U(\mathfrak{g}_{\pm 1})$ -module and let

$$J_M = \{r \in R_{\pm 1} \mid r.m = 0 \text{ for all } m \in \text{Ext}_{U(\mathfrak{g}_{\pm 1})}^\bullet(M, M)\}$$

The (cohomological) *support variety* of M is defined as

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \text{MaxSpec}(R_{\pm 1}/J_M).$$

Moreover, the support variety $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ is canonically isomorphic to the following rank variety:

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}^{\text{rank}}(M) = \{x \in \mathfrak{g}_{\pm 1} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}.$$

These varieties satisfy many of the important properties of support theory that include (i) the detection of projectivity over $U(\mathfrak{g}_{\pm 1})$ and (ii) the tensor product property.

For a detecting subalgebra $\mathfrak{f} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_1$, one can apply the prior construction to obtain support varieties for $M \in \mathcal{F}_{(\mathfrak{f}, \mathfrak{f}_0)}$, namely $\mathcal{V}_{\mathfrak{f}_{\pm 1}}(M)$ and $\mathcal{V}_{\mathfrak{f}_{\pm 1}}^{\text{rank}}(M)$.

5.5 Projectivity for Type I Lie Superalgebras

For Type I classical Lie superalgebras, one can construct Kac and dual Kac modules (cf. [BKN11, Section 3.1]). A module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}$ is *tilting* if and only if it has both a Kac and a dual Kac filtration. The use of these filtrations was a key idea in proving the following criteria for projectivity in the category $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}$ (see [BKN11, Section 3]).

Theorem 5.5.1. Let $\mathfrak{g}$ be a Type I classical Lie superalgebra and $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}$. The following are equivalent.

- (a) M is a projective module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}$
- (b) M is a tilting module

$$(c) \mathcal{V}_{\mathfrak{g}_1}(M) = \{0\} \text{ and } \mathcal{V}_{\mathfrak{g}_{-1}}(M) = \{0\}$$

$$(d) \mathcal{V}_{\mathfrak{g}_1}^{\text{rank}}(M) = \{0\} \text{ and } \mathcal{V}_{\mathfrak{g}_{-1}}^{\text{rank}}(M) = \{0\}$$

It should be noted that for an arbitrary (infinitely generated) module $M \in \mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$, one can have projectivity over $U(\mathfrak{g}_{\pm 1})$, but M may not be projective in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$. For example, if $\mathfrak{g} = \mathfrak{gl}(1|1)$, one can take an infinite coproduct of projective modules in the principal block $P = \bigoplus_{m \in \mathbb{Z}} P(m| -m)$. By making suitable identifications, one can form an (infinite) “zigzag module” (of radical length 2) that has a Kac and dual Kac filtration, which is projective upon restriction to $U(\mathfrak{g}_{\pm 1})$. The zigzag module is not projective in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ because it has radical length less than 4. This construction can also be performed for projective modules in the principal block for the restricted enveloping algebra of $\mathfrak{sl}_2$ (cf. [Pol67]), and has been observed in other situations by Cline, Parshall and Scott [CPS88, Example 3.2].

5.6 Projectivity via Ample Detecting Subalgebras

The following theorem allows us connect projectivity of a module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ to projectivity when restricting the module to the detecting subalgebra.

Theorem 5.6.1. Let $\mathfrak{g}$ be a Type I classical Lie superalgebra with an ample detecting subalgebra $\mathfrak{f}$. Let $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$. Then M is projective in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ if and only if M is projective in $\mathcal{F}_{(\mathfrak{f}, \mathfrak{f}_{\bar{0}})}$.

Proof. Let M be projective in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$. Then by Theorem 5.5.1, $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \{0\}$. It follows that $\mathcal{V}_{\mathfrak{f}_{\pm 1}}(M) = \{0\}$ and by Theorem 5.5.1, M is a projective module in $\mathcal{F}_{(\mathfrak{f}, \mathfrak{f}_{\bar{0}})}$.

Conversely, assume that M is a projective module in $\mathcal{F}_{(\mathfrak{f}, \mathfrak{f}_{\bar{0}})}$. Then $\mathcal{V}_{\mathfrak{f}_{\pm 1}}(M) = \{0\}$. Let $y \in \mathcal{V}_{\mathfrak{g}_1}(M)$. Then $y = g \cdot x$ where $g \in G_{\bar{0}}$ and $x \in \mathfrak{f}_1$ since $\mathfrak{f}$ is an ample detecting subalgebra. Since M is a rational $G_{\bar{0}}$ -module, $\mathcal{V}_{\mathfrak{g}_1}(M)$ is $G_{\bar{0}}$ -stable. This implies that $x \in \mathcal{V}_{\mathfrak{g}_1}(M)$ and $x \in \mathcal{V}_{\mathfrak{f}_1}(M)$, and $x = 0$. Consequently, $\mathcal{V}_{\mathfrak{g}_1}(M) = \{0\}$, and by the same reasoning $\mathcal{V}_{\mathfrak{g}_{-1}}(M) = \{0\}$. One can now conclude by Theorem 5.5.1 that M is a projective module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$. $\square$

5.7 Splitting subalgebras

In order to make use of the Balmer spectrum computations from [BKN17], and the Nerves-of-Steel result from [HN24], we need to ensure that our detecting subalgebras satisfy an additional condition. We need that they are so-called *splitting subalgebras*. The idea of splitting subalgebras was introduced by Serganova and Sherman in [SS23]. In the original paper the authors work in the context of the ambient algebraic supergroup, but for our purposes it will be useful to rephrase the definitions somewhat into the context of Lie superalgebras. The following definition is from [HN24] and is equivalent to the original definition.

Definition 5.7.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra and G be an algebraic supergroup scheme with $\text{Lie } G = \mathfrak{g}$. Moreover, let $Z \leq G$ be a subsupergroup with $\mathfrak{z} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{z}_{\bar{1}}$ being classical and $\text{Lie } Z = \mathfrak{z}$. Then $\mathfrak{z}$ is a *splitting subalgebra* if and only if the trivial module $\mathbb{C}$ is a direct summand of $\text{ind}_Z^G \mathbb{C}$.

The following theorem summarizes results in [SS23, Section 2]. The approach presented here is slightly different and uses the work for BBW parabolic subgroups by D. Grantcharov, N. Grantcharov, Nakano and Wu [Gra+21].

Theorem 5.7.2. Let $\mathfrak{g}$ be a classical Lie superalgebra and $\mathfrak{z}$ be a splitting subalgebra. Let M, N be modules in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$.

- (a) $R^j \text{ind}_Z^G \mathbb{C} = 0$ for $j > 0$.
- (b) M is projective in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ if and only if M when restricted to $\mathfrak{z}$ is projective in $\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})}$.
- (c) For all $n \geq 0$, $\text{Ext}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}^n(M, N \otimes \text{ind}_Z^G \mathbb{C}) \cong \text{Ext}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})}^n(M, N)$.
- (d) For all $n \geq 0$, the restriction map $\text{res} : \text{Ext}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}^n(M, N) \rightarrow \text{Ext}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})}^n(M, N)$ is injective.

Remark 5.7.3. Serganova and Sherman proved that the detecting subalgebra $\mathfrak{f}$ for classical Lie algebras of Type A are splitting subalgebras. [SS23, Theorem 1.1].

CHAPTER 6

NILPOTENCE THEOREMS FOR LIE SUPERALGEBRA REPRESENTATIONS

6.1 Stratification for detecting subalgebras

We now turn to the problem of showing BIK stratification for the detecting subalgebras. In fact, we are able to work in a slightly more general setting. Specifically, let $\mathfrak{z} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{z}_{\bar{1}}$ be a classical Lie superalgebra such that $\mathfrak{z}_{\bar{0}}$ is a torus, and $[\mathfrak{z}_{\bar{0}}, \mathfrak{z}_{\bar{1}}] = 0$. The Lie superalgebras satisfying these hypotheses encompass all of the detecting subalgebras introduced in the previous section.

Let $R := H^\bullet(\mathfrak{z}, \mathfrak{z}_{\bar{0}}; \mathbb{C})$. In [BKN17], the authors point out that $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$ is an R -linear triangulated category. Also, since $\dim \text{Proj } R$ is finite, the local-global principle holds. It was conjectured by BKN that R stratifies $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$. This result would recover their computation of the Balmer spectrum, but instead other methods were used. The goal of this section is to pursue the stratification avenue, and to prove the following theorem.

Theorem 6.1.1. The tensor-triangulated category $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$ is stratified in the sense of BIK by the action of the relative Lie superalgebra cohomology ring $H^\bullet(\mathfrak{z}, \mathfrak{z}_{\bar{0}}; \mathbb{C}) \cong S^\bullet(\mathfrak{z}_{\bar{1}}^*)$.

Proof. As notation let $\mathcal{K} = \text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)})$, let $\mathcal{C} = \mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)}$, and let $R = H^\bullet(\mathfrak{z}, \mathfrak{z}_0; \mathbb{C})$. Because of the previous remarks, it remains only to check the minimality condition. It turns out though, that it is not convenient to check minimality for $\mathcal{K}$ directly. The following argument follows closely that of [BIK11b, Section 5.2] where the category $\text{Stab}(kE)$ is considered, where k is an algebraically closed field of characteristic two, and E is an elementary abelian 2-group. The idea is to reduce the problem to a different TTC where the result is known via a version of Neeman's theorem.

As a first step in this direction we connect $\mathcal{C}$ to the category of supermodules for the superalgebra $\Lambda^\bullet(\mathfrak{z}_1)$ which is the exterior algebra on $\mathfrak{z}_1$ viewed as a superalgebra by declaring the generators to be odd. This is done by observing that because $\mathfrak{z}_0$ is a torus which commutes with $\mathfrak{z}$, the weight space decomposition for a $\mathfrak{z}$ -supermodule viewed as a module over $\mathfrak{z}_0$ is a decomposition as $\mathfrak{z}$ -supermodules. This gives a decomposition of the category $\mathcal{C} = \bigoplus_{\lambda \in \mathfrak{z}_0^*} \mathcal{C}_\lambda$.

The principal block $\mathcal{C}_0$ consists of modules which are annihilated by the ideal I of $U(\mathfrak{z})$ generated by $U(\mathfrak{z}_0)$. Therefore, since $U(\mathfrak{z})/I \cong \Lambda^\bullet(\mathfrak{z}_1)$, there is an isomorphism of categories $\mathcal{C}_0 \cong \Lambda^\bullet(\mathfrak{z}_1)\text{-sMod}$, where again $\Lambda^\bullet(\mathfrak{z}_1)$ is the exterior algebra on $\mathfrak{z}_1$ viewed as a superalgebra by declaring the generators to be odd. This equivalence passes to an equivalence at the level of the stable module categories: $\mathcal{K}_0 \cong \text{Stab}(\Lambda^\bullet(\mathfrak{z}_1)\text{-sMod})$. But from this equivalence one sees that it suffices to classify localizing subcategories for the principal block because there is a natural bijection between localizing subcategories for $\mathcal{K}$ and localizing subcategories for $\mathcal{K}_0$.

Next, we observe that a similar problem obtains as the one that occurs for elementary abelian groups in modular representation theory. Namely, the graded endomorphism ring of the unit, $\mathbb{C}$, in $\mathcal{C}$ is not the cohomology ring R . Instead, it is an analog of the Tate cohomology ring, which is typically not Noetherian. To get around this problem we instead consider the homotopy category of injectives $\mathbf{K}(\text{Inj } \mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)})$. In [Kra05], the author introduces a recollement

$$\mathbf{K}_{\text{ac}}(\text{Inj } \mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbf{K}(\text{Inj } \mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbf{D}(\mathcal{C}_{\mathfrak{g}, \mathfrak{g}_0})$$

that gives an equivalence of tensor triangulated categories between the full subcategory of $\mathbf{K}(\text{Inj } \mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)})$ consisting of acyclic complexes $\mathbf{K}_{\text{ac}}(\text{Inj } \mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)}) \simeq \text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)})$.

Now consider the universal enveloping superalgebra of the quotient $U(\mathfrak{z}/\mathfrak{z}_0)$. Then by the previous paragraph there is an isomorphism of $\mathbb{C}$ -algebras $U(\mathfrak{z}/\mathfrak{z}_0) \cong \Lambda^\bullet(\mathfrak{z}_1^*)$. Therefore, there is an isomorphism of rings $\Lambda^\bullet(\mathfrak{z}_1^*) \cong \mathbb{C}[z_1, \dots, z_r]/(z_i^2)$. Choose a basis $\{y_1, \dots, y_r\}$ of $\mathfrak{z}_1^*$ so that $R \cong \mathbb{C}[y_1, \dots, y_r]$ is an isomorphism of rings, and view R as a differential graded algebra with zero differential and $|y_i| = 1$ for each i .

The $\mathbb{C}$ -algebra $U(\mathfrak{z}/\mathfrak{z}_0) \otimes_{\mathbb{C}} R$ is graded with degree i component $U(\mathfrak{z}/\mathfrak{z}_0) \otimes_{\mathbb{C}} R^i$ and with multiplication defined by $(a \otimes s)(b \otimes t) = ab \otimes st$. Consider $U(\mathfrak{z}/\mathfrak{z}_0) \otimes_{\mathbb{C}} R$ as a differential graded algebra with zero differential. The degree one element δ defined as

$$\delta = \sum_{i=1}^r z_i \otimes_{\mathbb{C}} y_i.$$

satisfied $\delta^2 = 0$. Let J denote the differential graded module over $U(\mathfrak{z}/\mathfrak{z}_0) \otimes_{\mathbb{C}} R$ with graded module structure and differential given by

$$J = U(\mathfrak{z}/\mathfrak{z}_0) \otimes_{\mathbb{C}} R, \quad d(e) = \delta e.$$

Since J is a differential graded module over $U(\mathfrak{z}/\mathfrak{z}_0) \otimes_{\mathbb{C}} R$, for each differential graded module M over $U(\mathfrak{z}/\mathfrak{z}_0)$ there is an induced structure of a differential graded R -module on $\text{Hom}_{U(\mathfrak{z}/\mathfrak{z}_0)}(J, M)$. Then the functor

$$\text{Hom}_{U(\mathfrak{z}/\mathfrak{z}_0)}(J, -) : \mathbf{K}(\text{Inj } \mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)}) \rightarrow \mathbf{D}(R)$$

to the derived category is an equivalence of triangulated categories.

To see this first observe that as a complex, J consists of injective $U(\mathfrak{z}/\mathfrak{z}_0)$ -modules. This follows from the fact that $U(\mathfrak{z}/\mathfrak{z}_0)$ is self-injective. Finally, applying a version of Neeman's Theorem to $\mathbf{D}(R)$ (c.f. [BIK11b, Section 5.2.3]) yields the result $\square$

Theorem 6.1.1 gives the following corollary.

Corollary 6.1.2. Tensor ideal localizing subcategories of $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}\bar{0})})$ are in bijection with subsets of the Balmer spectrum $\text{Spc}(\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}\bar{0})})) \cong \text{Proj } H^\bullet(\mathfrak{z}, \mathfrak{z}\bar{0}; \mathbb{C})$.

6.2 Homological Residue Fields

In this section, we recall Balmer’s construction [Bal20] of homological residue fields. One of the main questions in tensor triangular geometry is to find the appropriate tensor triangular analog to ordinary fields in commutative algebra. In particular, given $\mathcal{K}$, how does one construct functors $F : \mathcal{K} \rightarrow \mathcal{F}$ to its “residue fields”? This question is explored in [BKS19], and some major takeaways are that there are several important properties one would like the notion of field to have. Moreover, there are many examples of tensor triangulated categories that should be considered as tensor triangulated fields. However, it is not clear exactly what the definition should be. The following definition was proposed in [BKS19, Definition 1.1], and will be the running definition in this work.

Definition 6.2.1. A non-trivial (big) tensor triangulated category $\mathcal{F}$ is a *tensor triangulated field* if every object of $\mathcal{F}$ is a coproduct of compact-rigid objects of $\mathcal{F}^c$, and if every non-zero object in $\mathcal{F}$ is tensor-faithful.

While this definition encapsulates many of the desired properties of fields, there is not yet a purely tensor triangular construction of them analogous to extracting residue fields in commutative algebra. Instead, Balmer uses the homological spectrum to construct homological tensor functors to abelian categories:

Definition 6.2.2. Given a homological prime $\mathcal{B} \in \text{Spc}^h(\mathcal{K})$, the *homological residue field* corresponding to $\mathcal{B}$ is the functor

$$\bar{h}_{\mathcal{B}} = Q_{\mathcal{B}} \circ h : \mathcal{K} \hookrightarrow \mathcal{A} = \text{Mod-}\mathcal{K} \twoheadrightarrow \bar{\mathcal{A}}(\mathcal{K}, \mathcal{B}) := \frac{\text{Mod-}\mathcal{K}}{\langle \mathcal{B} \rangle}$$

composed of the Yoneda embedding followed by the Gabriel quotient.

A natural question at this point is whether or not homological residue fields are related to the tensor triangular fields of Definition 6.2.2. The answer is yes, and an explicit connection useful for the computation of homological residue fields in examples is the content of the following theorem stated in [BC21, Lemma 2.2].

Theorem 6.2.3. Given a big tensor-triangulated category $\mathcal{K}$, a tensor-triangulated field $\mathcal{F}$, and a monoidal exact functor $F : \mathcal{K} \rightarrow \mathcal{F}$ with right adjoint U , one has the following diagram:

$$\begin{array}{ccc}
 \mathcal{K} & \xrightarrow{h} & \mathbf{Mod}\text{-}\mathcal{K}^c \\
 \uparrow F & & \uparrow \hat{F} \\
 \mathcal{F} & \xrightarrow{h} & \mathbf{Mod}\text{-}\mathcal{F}^c \\
 \downarrow U & & \downarrow \hat{U}
 \end{array}
 \begin{array}{c}
 \swarrow R \\
 \searrow Q \\
 \text{Mod-}\mathcal{K}^c / \text{Ker}(\hat{F}) = \overline{\mathcal{A}}_{\mathcal{B}} \\
 \swarrow \overline{F} \\
 \searrow \overline{U}
 \end{array}$$

where $\hat{F}$ is the exact cocontinuous functor induced by F , the functor Q is the Gabriel quotient with respect to $\text{Ker}(\hat{F})$ and the functor $\overline{F}$ is induced by the universal property, hence $\hat{F} = \overline{F}Q$ and $\overline{F}$ is exact and faithful.

The adjunctions $F \dashv U$, $\hat{F} \dashv \hat{U}$, $\overline{F} \dashv \overline{U}$, and $Q \dashv R$, are depicted with $\hat{F}h = hF$ and $\hat{U}h = hU$. Moreover, $\mathcal{B} := \text{Ker}(\hat{F}) \cap \mathcal{A}^{\text{fp}}$ is a homological prime and $\text{Ker}(\hat{F}) = \langle \mathcal{B} \rangle$ and $\overline{h}_{\mathcal{B}} = Q \circ h : \mathcal{K} \rightarrow \overline{\mathcal{A}}_{\mathcal{B}}$ is a homological residue field of $\mathcal{K}$.

6.3 Nilpotence and Colimits

In this section we clarify the notions of nilpotence in the stable categories of Lie superalgebra representations and relate them to colimit constructions in module categories and homotopy colimits in the stable categories. We first discuss the concept of nilpotence. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra.

Definition 6.3.1. Let M and N be modules in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$.

- (a) A map $f : M \rightarrow N$ is called *null* if $f = 0$ in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$; i.e., f is null if and only if f factors through a projective module.
- (b) A map $f : M \rightarrow N$ is called *tensor nilpotent* if there exists some $n \in \mathbb{Z}_{\geq 0}$ such that the tensor power $f^{\otimes n} : M^{\otimes n} \rightarrow N^{\otimes n}$ is null.

In the case when M is compact, one can transform the condition of the nilpotence of the map f to the adjoint map.

Lemma 6.3.2. Let M be a compact object. A map $f : M \rightarrow N$ is tensor nilpotent if and only if the adjoint map $\hat{f} : \mathbb{C} \rightarrow M^* \otimes N$ is tensor nilpotent.

Proof. Since f is tensor nilpotent, there exists some n such that $f^{\otimes n} : M^{\otimes n} \rightarrow N^{\otimes n}$ factors through a projective. But since tensor products of projective modules are projective, tensoring with $(M^*)^{\otimes n}$ gives that $\hat{f}^{\otimes n}$ factors through a projective; i.e., that $\hat{f}$ is tensor nilpotent.

For the converse, if $\hat{f}$ is tensor nilpotent, then $\hat{f}$ factors through a projective. By the same reasoning, tensoring with $M^{\otimes n}$ gives a factorization of f via a projective. $\square$

Next we need to recall the definition of a colimit in the category $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)}$ and a homotopy colimit in its stable module category.

Definition 6.3.3. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a classical Lie superalgebra.

- (a) Let

$$\theta : N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \xrightarrow{f_3} \dots$$

be a system of modules and homomorphisms in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)}$. Let $\gamma : \bigoplus_{i=1}^{\infty} N_i \rightarrow \bigoplus_{i=1}^{\infty} N_i$ be defined by $\gamma(m) = m - f_i(m)$ whenever $m \in N_i$. The *colimit* of the system is the module given by $\text{coker } \gamma$.

- (b) Let

$$\theta : X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots$$

be a system of modules and homomorphisms in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$. The *homotopy colimit* of the system is the module obtained by completing the map

$$\bigoplus X_i \xrightarrow{1-f} \bigoplus X_i$$

to a triangle:

$$\bigoplus X_i \xrightarrow{1-f} \bigoplus X_i \rightarrow \text{hocolim}(X_i) \rightarrow .$$

The following lemmas are given in [Ric97], the second of which is modified here for Lie superalgebra representations.

Lemma 6.3.4. Let $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots$ be a sequence of maps in a triangulated category with countable direct sums. If for each $i > 0$ there exists $k > i$ such that $\alpha_1 \dots \alpha_k = 0$, then $\text{hocolim}(X_i) \cong 0$.

The next lemma clarifies the relationship between homotopy colimits in the stable category with colimits in the ordinary module category.

Lemma 6.3.5. Let $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots$ be a sequence of modules and homomorphisms in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$. The colimit $\text{colim}(X_i)$ in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)}$ is isomorphic in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ to the homotopy colimit $\text{hocolim}(X_i)$.

These two lemmas together allow one to derive the analog of [BC18, Lemma 2.3]

Theorem 6.3.6. A map $f : k \rightarrow N$ is $\otimes$ -nilpotent if and only if the colimit of

$$\theta : k \xrightarrow{f} N \xrightarrow{f \otimes 1} N \otimes N \xrightarrow{f \otimes 1 \otimes 1} N \otimes N \otimes N \longrightarrow \dots$$

is projective.

Proof. First suppose that $f : k \rightarrow N$ is $\otimes$ -nilpotent. Then there exists some n for which $\hat{f} : k \rightarrow N^{\otimes n}$ is null. Therefore, Lemma 6.3.4 implies that the homotopy colimit of the system viewed in

the stable category is isomorphic to zero, which is to say that the colimit of the system is projective by Lemma 6.3.5.

Now suppose that the colimit of the system is projective. Since our category is Frobenius, the colimit is injective as well, and the map from k to the colimit factors through the injective hull $I(k)$ of k which is a finite-dimensional projective module. But then the map from $I(k)$ to the colimit must factor through some finite stage of the system. In other words, for some $n > 0$ the composition $f : k \rightarrow N^{\otimes n}$ is null, which implies that f is $\otimes$ -nilpotent. $\square$

6.4 Nilpotence Theorems

Nilpotence theorems have played an important role in cohomology and representation theory. Devinatz, Hopkins, and Smith showed in [DHS88] that a map between finite spectra which gets annihilated by all Morava K -theories must be tensor-nilpotent. Neeman [NB92] and Thompson [Tho97] proved nilpotence theorems for maps in derived categories using ordinary residue fields, and Benson, Carlson, and Rickard [BCR97] proved nilpotence theorems in modular representation theory, where the residue fields are given by cyclic shifted subgroups, or, in the case of finite group schemes, π -points [FP07]. With these examples in mind, Balmer used homological residue fields to present a unified treatment that applies to all tensor triangulated categories [Bal20]. In the case where $\mathcal{K}^c$ sits inside of a big rigidly-compactly generated tensor triangulated category $\mathcal{K}$, one can make a connection to the homological spectrum. In particular, he proved the following theorem [Bal20, Corollary 4.7]:

Theorem 6.4.1. Let $\mathcal{K}$ be a rigidly-compactly generated “big” tensor-triangulated category with full subcategory of compact rigid objects $\mathcal{K}^c$. Let $f : x \rightarrow Y$ be a morphism in $\mathcal{K}$ with $x \in \mathcal{K}^c$ and Y arbitrary. If $\bar{h}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}^c; \mathcal{B})$ for every homological residue field $\bar{h}_{\mathcal{B}}$, then there exists $n \geq 1$ such that $f^{\otimes n} = 0$ in $\mathcal{K}$.

The nilpotence theorem stated above can be combined with the theory of detecting subalgebras developed by Boe, Kujawa, and Nakano, to the study of nilpotence in the stable categories of Lie superalgebra representations. The following nilpotence theorem via homological residue fields is a direct translation of Theorem 6.4.1 in the context of superalgebra representations.

Theorem 6.4.2. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a classical Lie superalgebra, and let $f : M \rightarrow N$ be a morphism in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ with $M \in \text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})$. Suppose that $\bar{h}_{\mathcal{B}}(f) = 0$ for all $\mathcal{B} \in \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}))$. Then there exists $n \geq 1$ such that $f^{\otimes n} = 0$ in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$.

Proof. This is immediate since $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})$ sits inside of $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ as the compact objects: $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}) = (\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)}))^c$. Moreover, the compact objects and the rigid objects coincide and generate $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ as a tensor-triangulated category. This is the setup of Theorem 6.4.1. $\square$

6.5 A Nilpotence Theorem via Detecting Subalgebra

The salient feature first discovered about detecting subalgebras was that these subalgebras detect nilpotence in cohomology. We will now show that a remarkable feature for classical Lie subalgebras with a splitting subalgebras is that nilpotence of arbitrary maps in the stable module category is governed by nilpotence when restricting the map to a splitting subalgebra. In particular, to show that a morphism $f : M \rightarrow N$ is nilpotent in the big stable module category where M is compact, it is enough to check vanishing on those homological residue fields constructed via homological primes from the stable categories of modules over the splitting subalgebra.

Theorem 6.5.1. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a classical Lie superalgebra with a splitting subalgebra $\mathfrak{z} = \mathfrak{z}_0 \oplus \mathfrak{z}_1 \subseteq \mathfrak{g}$. Let $f : M \rightarrow N$ be a morphism in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ with $M \in \text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})$. Let $\text{res} : \text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)}) \rightarrow \text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)})$ be the usual restriction functor. If $\bar{h}_{\mathcal{B}}(\text{res}(f)) = 0$ for all $\mathcal{B} \in \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_0)}))$, then there exists $n \geq 1$ such that $f^{\otimes n} = 0$ in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$.

Proof. By our hypothesis, $\bar{h}_{\mathcal{B}}(\text{res}(f)) = 0$ for all $\mathcal{B} \in \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_0)}))$. Theorem 6.4.2 implies that $\text{res}(f)$ is tensor nilpotent in $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)})$.

It follows that $\text{res}(\widehat{f}) : \mathbb{C} \rightarrow M^* \otimes N$ is tensor nilpotent in $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}\bar{0})})$, and by Theorem 6.3.6 its associated colimit is projective in $\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}\bar{0})}$. Therefore, the colimit as an object in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}\bar{0})}$ is projective by part (b) of Theorem 5.7.2. Invoking Theorem 6.3.6 again implies that $\widehat{f}$ is tensor nilpotent, thus f is tensor nilpotent. $\square$

CHAPTER 7

THE NERVES-OF-STEEL CONJECTURE AND LOCALIZING SUBCATEGORIES IN TYPE A

The goals of this section are to determine the homological spectrum for $\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$ and $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$, where $\mathfrak{g}$ is a classical Lie superalgebra with splitting subalgebra $\mathfrak{z}$. We also consider the comparison map defined in Section 3.5.

7.1 Stratification and the Comparison Map

The first result we state involves using the classification of localizing subcategories for the detecting subalgebras from the previous section in order to verify the Nerves-of-Steel Conjecture in this setting. We state the theorem with the most general hypothesis.

Theorem 7.1.1. Let $\mathfrak{z} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{z}_{\bar{1}}$ be a classical Lie superalgebra with $\mathfrak{z}_{\bar{0}}$ a torus and $[\mathfrak{z}_{\bar{0}}, \mathfrak{z}_{\bar{1}}] = 0$. Then the comparison map

$$\phi : \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})) \rightarrow \text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})}))$$

is a bijection.

Proof. Since $\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$ is rigid, the map ϕ is surjective. The injectivity of ϕ follows from the argument outlined in [Bal20, Example 5.13]. The main point is to use the classification of localizing subcategories of $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$ and the existence of pure injective objects. See also [BKS19, Corollary 4.26]. $\square$

Remark 7.1.2. A more general argument showing that BIK stratification and tt-stratification imply the Nerves-of-Steel Conjecture can be found in [BHS23a, Theorem 4.7].

7.2 Identifying the Homological Spectrum

Let $\mathfrak{g}$ be a classical Lie superalgebra and $\mathfrak{z}$ be a detecting subalgebra in $\mathfrak{g}$. We will need to work with a field extension K of $\mathbb{C}$ such that the transcendence degree is larger than the dimension of $\mathfrak{z}$. Note that this is the analogous setup as in [BC21, Example 3.9]. The stable module categories involved will be viewed over the field extension K . Let $\mathcal{P}_x$ be the prime ideal in $\text{Proj}(S^\bullet(\mathfrak{z}_{\bar{1}}^*))$ associated with the “generic point” x (cf. [BCR96, Sections 2 and 3] for an explanation of this terminology).

For $x \in \mathfrak{z}_{\bar{1}}$, with $\mathfrak{z}_{\bar{1}}$ viewed as a vector space over K , let $\langle x \rangle$ denote the Lie subsuperalgebra generated by x . One has $U(\langle x \rangle)$ is either $K[x]/(x^2)$ or $U(\mathfrak{q}(1))$. In either case, the blocks are either semisimple or have finite representation type, and one can verify that $\text{Stab}(\mathcal{C}_{(\langle x \rangle, \langle x \rangle_{\bar{0}})})$ is a tensor triangular field. For $x \in \mathfrak{z}_{\bar{1}}$, one has two monoidal exact functors given by restriction:

$$\pi_x^{\mathfrak{g}} : \text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}) \rightarrow \text{Stab}(\mathcal{C}_{(\langle x \rangle, \langle x \rangle_{\bar{0}})}) \quad (7.2.1)$$

$$\pi_x^{\mathfrak{z}} : \text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})}) \rightarrow \text{Stab}(\mathcal{C}_{(\langle x \rangle, \langle x \rangle_{\bar{0}})}) \quad (7.2.2)$$

Let $\text{res} : \text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}) \rightarrow \text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$ be the natural functor obtained by restricting $\mathfrak{g}$ -modules to $\mathfrak{z}$ -modules. Then $\pi_x^{\mathfrak{g}} = \pi_x^{\mathfrak{z}} \circ \text{res}$ for all $x \in \mathfrak{z}_{\bar{1}}$.

Now one can apply Theorem 6.2.3 (where $F = \pi_x^{\mathfrak{g}}$ and $\pi_x^{\mathfrak{z}}$), to obtain $\mathcal{B}_x$ a homological prime (resp. $\mathcal{B}'_x$) associated to $\pi_x^{\mathfrak{g}}$ (resp. $\pi_x^{\mathfrak{z}}$). Similarly, let $\bar{\mathbf{h}}_{\mathcal{B}_x}$ (resp. $\bar{\mathbf{h}}_{\mathcal{B}'_x}$) be the homological residue field corresponding to $\mathcal{B}_x$ (resp. $\mathcal{B}'_x$).

The goal now is to show that $\{\mathcal{B}_x\}_{x \in \mathfrak{z}_{\bar{1}}}$ contains all of the homological primes. Our main tool is the following result given by [Bal20, Theorem 5.4].

Theorem 7.2.1. Let $\mathcal{K}$ be a big tensor-triangulated category which is rigidly-compactly generated by the full subcategory of compact objects $\mathcal{K}^c$. Consider a family $\mathcal{E} \subseteq \mathrm{Spc}^h(\mathcal{K}^c)$ of points in the homological spectrum. Suppose that the corresponding functors

$$\{\bar{\mathbf{h}}_{\mathcal{B}} : \mathcal{K} \rightarrow \bar{\mathcal{A}}(\mathcal{K}^c; \mathcal{B})\}_{\mathcal{B} \in \mathcal{E}}$$

collectively detect $\otimes$ -nilpotence in the following sense: If $f : x \rightarrow Y$ in $\mathcal{K}$ is such that $x \in \mathcal{K}^c$ and $\bar{\mathbf{h}}_{\mathcal{B}}(f) = 0$ for all $\mathcal{B} \in \mathcal{E}$ then $f^{\otimes n} = 0$ for some $n \geq 1$. Then we have $\mathcal{E} = \mathrm{Spc}^h(\mathcal{K}^c)$.

We are now ready to provide conditions on when one can identify a collection of homological primes that detect nilpotence on $\mathrm{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$.

Theorem 7.2.2. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra and $\mathfrak{z} \subseteq \mathfrak{g}$ be a Lie subsuperalgebra. Denote by G , $G_{\bar{0}}$ and Z the associated supergroup (schemes) such that $\mathfrak{g} = \mathrm{Lie} G$, $\mathfrak{g}_{\bar{0}} = \mathrm{Lie} G_{\bar{0}}$ and $\mathfrak{z} = \mathrm{Lie} Z$. Set $N = \mathrm{Norm}_{G_{\bar{0}}}(\mathfrak{z}_{\bar{1}})$. Assume that

- (a) $\mathfrak{z} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{z}_{\bar{1}}$ with $\mathfrak{z}_{\bar{0}}$ a torus and $[\mathfrak{z}_{\bar{0}}, \mathfrak{z}_{\bar{1}}] = 0$;
- (b) Z is a splitting subgroup of G .

Then $\mathcal{E}/N = \{\mathcal{B}_x : x \in \mathfrak{z}_{\bar{1}}\}/N$ (i.e., a set of N -orbit representatives) detects nilpotence in $\mathrm{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$.

Proof. The idea of the proof is to find a set of homological primes $\mathcal{E}$ that detects nilpotence in $\mathrm{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$. Then one can apply Theorem 7.2.1 (e.g., [Bal20, Theorem 5.4].)

The first step is to compare homological residue fields for $\mathfrak{g}$ and $\mathfrak{z}$. If $f : M \rightarrow N$ is in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ with M compact then one can compare the diagrams for $\bar{h}_{\mathcal{B}_x}$ and $\bar{h}_{\mathcal{B}'_x}$ to conclude the following.

(1) *If $\bar{h}_{\mathcal{B}_x}(f) = 0$ then $\bar{h}_{\mathcal{B}'_x}(\text{res}(f)) = 0$ for $x \in \mathfrak{z}_{\bar{1}}$.*

Now one can apply the stratification result for detecting subalgebras, Theorem 7.1.1, to conclude that $\{\mathcal{B}'_x : x \in \mathfrak{z}_{\bar{1}}\}$ are the homological primes for $\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$. Therefore, by (1) and Theorem 6.4.2, one has

(2) *If $\bar{h}_{\mathcal{B}'_x}(\text{res}(f)) = 0$ for all $x \in \mathfrak{z}_{\bar{1}}$ then $\text{res}(f) : M \rightarrow N$ is $\otimes$ -nilpotent in $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})})$.*

Applying Theorem 6.5.1 since $\mathfrak{z}$ is a splitting subalgebra of $\mathfrak{g}$, one can conclude that $f : M \rightarrow N$ is $\otimes$ -nilpotent in $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$. Let $\mathcal{E} = \{\mathcal{B}_x : x \in \mathfrak{z}_{\bar{1}}\}/N$. Since M is a $G_{\bar{0}}$ -module, it follows that the functors $\pi_x^{\mathfrak{g}}$ (resp. $\pi_{nx}^{\mathfrak{g}}$) will provide the same decomposition of M in $\text{Stab}(\mathcal{C}_{(\langle x \rangle, \langle x \rangle_{\bar{0}})})$ (resp. $\text{Stab}(\mathcal{C}_{(\langle nx \rangle, \langle nx \rangle_{\bar{0}})})$). By considering Theorem 6.2.3, it follows that $\bar{h}_{\mathcal{B}_x}(f) = 0$ if and only if $\bar{h}_{\mathcal{B}_{nx}}(f) = 0$. Therefore, $\mathcal{E}/N$ detects nilpotence.

□

In the previous theorem, one can state that $\mathcal{E}/N = \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}))$. However, with the definition of $\mathcal{E}/N$ there are certain homological primes that might be identified in the set. We will show in the following section that different N -orbit representatives yield different elements in $\text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}))$.

7.3 The Nerves-of-Steel Conjecture

There are noticeable differences between the stable module category for finite group schemes versus the stable module category for Lie superalgebras. For example, the map:

$$\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})) \rightarrow \text{Proj}(\text{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})) \quad (7.3.1)$$

is not always a homeomorphism (e.g., when $\mathfrak{g} = \mathfrak{gl}(m|n)$). In general, the cohomology ring $H^\bullet(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C})$ does not stratify $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$, and there are many examples where the support theory does not detect projectivity. This is the main reason one needs to use the cohomology of the detecting subalgebra to realize the homological spectrum and the Balmer spectrum.

Boe, Kujawa and Nakano [BKN17] showed that for $\mathfrak{g} = \mathfrak{gl}(m|n)$, one has a homeomorphism:

$$\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) \cong N\text{-Proj}(H^\bullet(\mathfrak{f}, \mathfrak{f}_0; \mathbb{C})) \quad (7.3.2)$$

where $\mathfrak{f}$ is a detecting (splitting) subalgebra of $\mathfrak{g}$ and N is the normalizer of $\mathfrak{f}_1$ in G_0 . From this example, it is clear that in order to compute the Balmer spectrum for Lie superalgebras one needs to find a suitable replacement for the cohomology ring.

From Section 3.5, when one has a splitting subalgebra $\mathfrak{z}$ of $\mathfrak{g}$, one can compute the homological spectrum and show there is a surjection:

$$\text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) \rightarrow N\text{-Proj}(H^\bullet(\mathfrak{z}, \mathfrak{z}_0, \mathbb{C})) \quad (7.3.3)$$

In other words, since $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}$ is rigid, the comparison map

$$\phi : \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) \rightarrow \text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) \quad (7.3.4)$$

is surjective. Our goal is to use the prior calculation of the homological spectrum to give conditions on when the Nerves-of-Steel Conjecture holds (i.e., when ϕ is bijective).

We can now identify the homological spectrum and the Balmer spectrum for classical Lie superalgebras with a splitting subalgebra under a suitable condition on realization of supports.

Theorem 7.3.1. Let $\mathfrak{g}$ be a classical Lie superalgebra with a splitting, detecting subalgebra $\mathfrak{z} \cong \mathfrak{z}_0 \oplus \mathfrak{z}_1$. Assume that

- (i) $\mathfrak{z} = \mathfrak{z}_0 \oplus \mathfrak{z}_1$ where $\mathfrak{z}_0$ is a torus and $[\mathfrak{z}_0, \mathfrak{z}_1] = 0$.

- (ii) Given W an N -invariant closed subvariety of $\text{Proj}(S^\bullet(\mathfrak{z}_1^*))$, there exists $M \in \text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})$ with $V_{(\mathfrak{z}, \mathfrak{z}_0)}(M) = W$.

Then

- (a) There exists a 1-1 correspondence

$$\{\text{thick tensor ideals of } \text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})\} \leftrightarrow \mathcal{X}_{sp}$$

where $X = N\text{-Proj}(S^\bullet(\mathfrak{z}_1))$, and $\mathcal{X}_{sp}$ is the set of specialization closed sets of X .

- (b) There exists a homeomorphism $\eta : N\text{-Proj}(S^\bullet(\mathfrak{z}_1)) \rightarrow \text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}))$.

- (c) The comparison map $\phi : \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) \rightarrow \text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}))$ is bijective.

Proof. (a) and (b) follow by [BKN17, Theorems 3.4.1, 3.5.1]. For part (c), let $\rho = \eta^{-1}$ which is given by a concrete description in [NVY24, Corollary 6.2.4]. Consider the following diagram of topological spaces:

$$\begin{array}{ccccc} \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_0)})) & \xrightarrow{\phi'} & \text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{z}, \mathfrak{z}_0)})) & \xrightarrow{\rho'} & \text{Proj}(S^\bullet(\mathfrak{z}_1)) \\ \downarrow \theta & & \downarrow \hat{\pi} & & \downarrow \pi \\ \text{Spc}^h(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) & \xrightarrow{\phi} & \text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})) & \xrightarrow{\rho} & N\text{-Proj}(S^\bullet(\mathfrak{z}_1)) \end{array} \quad (7.3.5)$$

One has that ρ' is a homeomorphism and ϕ' is a bijection for $\mathfrak{z}$. From part (b), the map ρ is a homeomorphism. The maps π and $\hat{\pi}$ are surjections. The map θ sends $\mathcal{B}_x$ to $\mathcal{B}'_x$ in $\mathcal{E}/N$. Suppose that $\phi(\mathcal{B}_{x_1}) = \phi(\mathcal{B}_{x_2})$. Then using the commutativity, one has $\mathcal{P}_{x_1} = \mathcal{P}_{x_2}$ in $N\text{-Proj}(S^\bullet(\mathfrak{z}_1))$ which means that x_1 and x_2 are N -conjugate. This proves that $\mathcal{B}_{x_1} = \mathcal{B}_{x_2}$ in $\mathcal{E}/N$.

□

We remark that the verification of the Nerves-of-Steel Conjecture in the previous theorem uses stratification results only for $\text{Stab}(\mathcal{C}_{(\mathfrak{z}, \mathfrak{z}_0)})$, unlike the the case for finite group schemes where the

stratification is needed for $\text{StMod}(kG)$. Because the assumption needed to prove the Nerves-of-Steel Conjecture will come up again in the next section, we state it here.

Assumption 7.3.2. Suppose $\mathfrak{g}$ be a classical Lie superalgebra with a splitting, detecting subalgebra $\mathfrak{z} \cong \mathfrak{z}_{\bar{0}} \oplus \mathfrak{z}_{\bar{1}}$. Assume that

- (i) $\mathfrak{z} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{z}_{\bar{1}}$ where $\mathfrak{z}_{\bar{0}}$ is a torus and $[\mathfrak{z}_{\bar{0}}, \mathfrak{z}_{\bar{1}}] = 0$.
- (ii) Given W an N -invariant closed subvariety of $\text{Proj}(S^\bullet(\mathfrak{z}_{\bar{1}}^*))$, there exists $M \in \text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ with $V_{(\mathfrak{z}, \mathfrak{z}_{\bar{0}})}(M) = W$.

Theorem 7.3.3. Assumption 7.3.2 holds for Lie superalgebras of Type A.

Proof. For $\mathfrak{g} = \mathfrak{gl}(m|n)$ this was done by Boe, Kujawa, and Nakano in [BKN17, Theorem 7.21.1]. The case $\mathfrak{g} = \mathfrak{sl}(m|n)$ is [HN24, Theorem 8.3.2]. $\square$

7.4 Stratification for Type A Lie superalgebras

Now, we turn our attention to $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$, where $\mathfrak{g}$ is a classical Lie superalgebra which satisfies Assumption 7.3.2. Boe, Kujawa, and Nakano computed the Balmer spectrum to be $N\text{-Proj}(\mathbf{H}^\bullet(\mathfrak{z}, \mathfrak{z}_{\bar{0}}; \mathbb{C}))$, and showed that $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ is not stratified in the sense of BIK by the cohomology ring $\mathbf{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$. The natural question then is whether or not $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ is tt-stratified or h-stratified. That for $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ these notions are equivalent and satisfied is Theorem A, which we restate now for convenience.

Theorem A. Let $\mathfrak{g}$ be a classical Lie superalgebra with a splitting, detecting subalgebra $\mathfrak{z} \subseteq \mathfrak{g}$ and which satisfies the realization condition of Assumption 7.3.2. The tensor triangulated category $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})})$ is tt-stratified by the Balmer spectrum $\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}))$, and tt-stratification is equivalent to h-stratification.

Proof. First note that by Theorem 7.3.1, the Nerves-of-Steel Conjecture holds for $\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)})$. Therefore, since $\text{Spc}(\text{stab}(\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_0)}))$ is Noetherian, the equivalence of tt-stratification and h-stratification of $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ follows from Theorem 3.7.2.

Next, in order to show that the equivalent conditions of tt-stratification and h-stratification hold for $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$, we show that $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ is h-stratified. For this, our tool is Theorem 3.7.3. For this step, we need to work over an extension field K of $\mathbb{C}$ such that the transcendence degree is larger than the dimension of $\mathfrak{z}$ (c.f. [HN24, Section 6.2] or [BC21, Example 3.9]). Consider the family of monoidal, exact functors

$$\left\{ \pi_x^{\mathfrak{g}} : \text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)}) \rightarrow \text{Stab}(\mathcal{C}_{(\langle x \rangle, \langle x \rangle_0)}) \right\}_{x \in \mathfrak{z}_{\bar{1}}}, \quad (7.4.1)$$

where $\langle x \rangle$ denotes the Lie subsuperalgebra generated by x and is either a one-dimensional abelian Lie subsuperalgebra, or a subsuperalgebra isomorphic to the queer Lie superalgebra $\mathfrak{q}(1)$. In any case, the relevant fact is that for each $x \in \mathfrak{z}_{\bar{1}}$, $\text{Stab}(\mathcal{C}_{(\langle x \rangle, \langle x \rangle_0)})$ is BIK stratified, and therefore tt-stratified and h-stratified. Moreover, the collection of functors in 7.4.1 jointly detect when an object of $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ is zero. Thus, in order to prove that $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ is h-stratified, we need to show that $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ is generated by the images of the right adjoints of the functors in 7.4.1.

It turns out to be convenient to work in the context of the ambient algebraic supergroup scheme. Let G denote the ambient algebraic supergroup scheme such that $\text{Lie}(G) = \mathfrak{g}$, and let $Z \leq G$ be a subsupergroup scheme such that $\text{Lie}(Z) = \mathfrak{z}$. By [Gra+21] the categories $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ and $\text{Rep}(G)$ are equivalent. The corresponding family of functors to consider is

$$\left\{ \text{res}_X^G : \text{Stab}(\text{Rep}(G)) \rightarrow \text{Stab}(\text{Rep}(X)) \right\}_{X \in Z_{\bar{1}}} \quad (7.4.2)$$

which has right adjoints given by induction:

$$\left\{ \text{ind}_X^G : \text{Stab}(\text{Rep}(X)) \rightarrow \text{Stab}(\text{Rep}(G)) \right\}_{X \in Z_{\bar{1}}}. \quad (7.4.3)$$

We need to show that the images of the functors in 7.4.3 generate $\text{Stab}(\text{Rep}(G))$. Notice that by transitivity of induction we have

$$\text{ind}_X^G(-) = \text{ind}_Z^G \text{ind}_X^Z(-).$$

Since $Z \leq G$ is a splitting subgroup the image of $\text{ind}_Z^G(-)$ generates $\text{Stab}(\text{Rep}(G))$. To see this let M be a module in $\text{Stab}(\text{Rep}(G))$, because Z is splitting, M is a direct summand of $\text{ind}_Z^G \text{res}_Z^G M$.

It only remains to show that the images $(\text{ind}_X^Z(-))_{X \in Z_1}$ generate $\text{Stab}(\text{Rep}(Z))$. To prove this, we again appeal to Theorem 3.7.3, but in a different way. This time, we use the fact that $\text{Stab}(\text{Rep}(Z))$ being generated by the images of $(\text{ind}_X^Z(-))_{X \in Z_1}$ is equivalent to $\text{Stab}(\text{Rep}(Z))$ being tt-stratified. But $\text{Stab}(\text{Rep}(Z))$ is BIK stratified by 6.1.1 which implies tt-stratification. $\square$

As a consequence, we obtain the classification of tensor ideal localizing subcategories of $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$, the content of Corollary B, which we repeat here.

Corollary B. Let $\mathfrak{g}$ be a classical Lie superalgebra with a splitting, detecting subalgebra $\mathfrak{z} \subseteq \mathfrak{g}$ and which satisfies the realization condition of Assumption 7.3.2. There is a bijection between the set of $\otimes$ -ideal localizing subcategories of $\text{Stab}(\mathcal{C}_{(\mathfrak{g}, \mathfrak{g}_0)})$ and subsets of $N\text{-Proj}(\mathbf{H}^\bullet(\mathfrak{z}, \mathfrak{z}_0; \mathbb{C}))$.

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