

EXPLORATIONS IN SYMPLECTIC GEOMETRY: HOFER-ZEHNDER CONJECTURE AND LAGRANGIAN TORI IN $S^2 \times S^2$

by

HAN LOU

(Under the Direction of Michael Usher)

ABSTRACT

This dissertation is about two different aspects of symplectic geometry. The first part is about the Hofer-Zehnder conjecture. Arnold conjecture says that the number of 1-periodic orbits of a Hamiltonian diffeomorphism is greater than or equal to the dimension of the Hamiltonian Floer homology. In 1994, Hofer and Zehnder conjectured that there are infinitely many periodic orbits if the equality does not hold. We showed that the Hofer-Zehnder conjecture is true for semipositive symplectic manifolds with semisimple quantum homology. The second part is about Lagrangian submanifolds. In [Fuk+12], K. Fukaya, Y. Oh, H. Ohta, and K. Ono (FOOO) obtained the monotone symplectic manifold $S^2 \times S^2$ by resolving the singularity of a toric degeneration of a Hirzebruch surface. They identified a continuum of toric fibers in the resolved toric degeneration that are not Hamiltonian isotopic to the toric fibers of the standard toric structure on $S^2 \times S^2$. We provided a comprehensive classification: for any toric fiber in FOOO's construction of $S^2 \times S^2$, we determined whether it is Hamiltonian isotopic to a toric fiber of the standard toric structure of $S^2 \times S^2$.

INDEX WORDS: Hamiltonian Floer homology, Hofer-Zehnder conjecture, persistence module, boundary depth, Lagrangian submanifold, toric fiber, Hamiltonian isotopy, symmetric probe, displacement energy germ

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CHAPTER I

INTRODUCTION

The first part of this dissertation is from [AL23] and the second part is from [Lou24].

A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a nondegenerate closed 2-form. We call such a form ω a *symplectic form*. A *time-dependent Hamiltonian* is a smooth function

$$H : M \times S^1 \rightarrow \mathbb{R}$$

The *Hamiltonian vector field* X_H associated with H is defined by the equation

$$\omega(X_H, \cdot) = -dH_t$$

where $H_t(\cdot) = H(\cdot, t)$. The flow ϕ_H^t of X_H , assuming $\phi_H^0 = id$, is called the *Hamiltonian flow*. The time-1 map ϕ_H^1 is called the *Hamiltonian diffeomorphism*. Let x be a fixed point of ϕ_H^k for $k \in \mathbb{Z}$. Then a *k-period orbit* of X_H is defined as $x(t) := \phi_H^{kt}(x)$. We say x is *nondegenerate* if the linear map $(d\phi_H^1)_{x(0)}$ does not have eigenvalue 1. The Arnold conjecture [Arn65][Arn86] gives a lower bound for the number of the nondegenerate 1-periodic orbits, which is the sum of the Betti numbers of M , an invariant only depending on the topology of M . To prove the Arnold conjecture, A. Floer invented the Floer homology [Flo86][Flo87][Flo89].

The *quantum homology* of a symplectic manifold (M, ω) with dimension $2n$ is the singular homology with coefficients in Novikov ring. The quantum homology carries a special product defined by Gromov-Witten invariants. Let A be a homology class in $H_2(M, \mathbb{Z})$. Fix a Riemann surface Σ of genus g and let $\mathcal{M}^*(A; J)$ denote the set of all simple J -holomorphic maps $u : \Sigma \rightarrow M$ which represent the class A . By perturbing the almost complex structure J , the

moduli space $\mathcal{M}^*(A; J)$ can be a smooth orientable manifold of dimension

$$\dim \mathcal{M}^*(A; J) = n(2 - 2g) + 2c_1(A)$$

Denote by $\widetilde{\mathcal{M}}_{g,k}(A; J)$ the space of tuples $(u, z_1, \dots, z_k)$ consisting of an element $u \in \mathcal{M}(A; J)$ together with k pairwise distinct marked points $z_i \in S^2$. Then we focus on the case where $g = 0$. Denote by $\mathcal{M}_{0,k}(A; J)$ the quotient of $\widetilde{\mathcal{M}}_{0,k}(A; J)$ by the group $\mathrm{PSL}(2, \mathbb{C})$ and $\mathcal{M}_{0,k}^*(A; J)$ the subset for which u is simple. The evaluation map

$$\begin{aligned} \widetilde{\mathcal{M}}_{0,k}(A; J) &\rightarrow M^k \\ (u, z_1, \dots, z_k) &\mapsto (u(z_1), \dots, u(z_k)) \end{aligned}$$

descends to

$$\mathrm{ev} : \mathcal{M}_{0,k}^*(A; J) \rightarrow M^k$$

Then the Gromov-Witten invariants are obtained by taking the intersection of ev with cycles of complementary dimension in M^k . See [MS12] for details.

The Hofer-Zehnder conjecture states that if the inequality in Arnold conjecture is strict then there are infinitely many periodic orbits. One of the first results in the direction of the Hofer-Zehnder conjecture is given by J. Franks [Fra92][Fra96]. J. Franks proved that any time-one map of an area-preserving isotopy of the sphere with at least three fixed points must have infinitely many periodic orbits. E. Shelukhin proved the Hofer-Zehnder conjecture for closed monotone symplectic manifolds with semisimple even quantum homology, see [She22]. We generalized E. Shelukhin's result to the closed semipositive symplectic manifolds with semisimple even quantum homology, see [AL23]. S. Bai and G. Xu proved the Hofer-Zehnder conjecture for toric manifolds, see [BX23]. These recent works distinguish from each other. For example, the non-monotone blow-ups of $\mathbb{C}P^2$ are covered by [AL23], but not by [She22]. The four-point blow-up of $\mathbb{C}P^2$ for certain size of exceptional divisors is covered by [AL23], but not by [BX23]. Ostrover-Tyomkin's 8-dimensional monotone toric manifold whose quantum homology is not semisimple [OT09] is covered by [BX23], but not by [She22] or [AL23].

Now we state the main result in [AL23], which will be proved in Chapter 2.

Theorem A. *Let (M, ω) be a closed semipositive symplectic manifold with semisimple even quantum homology $QH_{\mathrm{ev}}(M; \Lambda_{\mathbb{K}, \mathrm{univ}})$ for a ground field $\mathbb{K}$. Then any Hamiltonian diffeomorphism ϕ with finitely many contractible fixed points such*

that

$$N(\phi, \mathbb{K}) = \sum_{x \in \text{Fix}(\phi)} \dim_{\mathbb{K}} HF^{\text{loc}}(\phi, x) > \dim_{\mathbb{K}} H_*(M; \mathbb{K})$$

must have infinitely many periodic points. If $\mathbb{K}$ has characteristic zero, then ϕ has a simple contractible p -periodic point for each sufficiently large prime p .

Next, we introduce the second part of the dissertation about the Lagrangian tori in $S^2 \times S^2$.

Let (M^{2n}, ω) be a symplectic manifold. A *Lagrangian submanifold* L is a submanifold of M with dimension n such that $\omega|_L = 0$. In [Arn86], Arnold defined a Lagrangian knot as a connected component of the space of Lagrangian embeddings in a fixed symplectic manifold. For Lagrangian embeddings $\mathbb{R}^2 \rightarrow \mathbb{R}^4$ (coinciding with embeddings of the plane $(z_1, 0)$ outside some sphere in the standard four-dimensional symplectic space), he also proposed the following two questions.

Question 1.0.1. [Arn86, section 6] Can any knot in the ordinary sense be realized by a Lagrangian one?

Question 1.0.2. [Arn86, section 6] Are there purely Lagrangian knots, that is, Lagrangian embeddings homotopic to the plane in the class of all embeddings, but non-homotopic in the class of Lagrangian embeddings?

Buliding on Arnold's questions, several significant works have deepened our understanding of Lagrangian embeddings. In [Cas86], Y. Chekanov constructed the *special tori* in $\mathbb{R}^{2n}$ that are not symplectomorphic to each other. These tori are examples of monotone Lagrangian tori that are Lagrangian isotopic but not Hamiltonian isotopic to an elementary torus. In [EP93], Y. Eliashberg and L. Polterovich considered if two Lagrangian embeddings are isotopic in smooth, Lagrangian, or Hamiltonian sense.

Considering the monotone $S^2 \times S^2$, G. Dimitroglou Rizell, E. Goodman, and A. Ivrii in [RGI16] showed that any two Lagrangian tori are Lagrangian isotopic. There are several different constructions of monotone Lagrangian tori in $S^2 \times S^2$ that are not Hamiltonian isotopic to the Clifford torus, the product of the equators. Using P. Biran's circle bundle construction in [Biro6] one can get such a Lagrangian torus. P. Albers and U. Frauenfelder in [AF07] constructed a nondisplaceable Lagrangian torus in T^*S^2 . Then one can get such a Lagrangian torus by an embedding from D^*S^2 , a disk subbundle of T^*S^2 , to $S^2 \times S^2$. M. Entov and L. Polterovich constructed a non-heavy monotone Lagrangian torus in [EP09, Example 1.22]. Y. Chekanov and F. Schlenk in [CS10] also constructed such a monotone Lagrangian torus. J. Oakley and M.

Usher in [OU16] showed that the above four Lagrangian tori are Hamiltonian isotopic to each other. A. Gable in [Gad13] also showed that the Lagrangian tori in [CS10] and [Biro6] are Hamiltonian isotopic.

We consider S^2 as the unit sphere in $\mathbb{R}^3$ with symplectic form ω_{std} such that the ω_{std} -area of S^2 is 4π . Then $(S^2, \frac{1}{2}\omega_{std}) \times (S^2, \frac{1}{2}\omega_{std})$ has a standard toric structure with the moment map

$$\mu : S^2 \times S^2 \rightarrow \mathbb{R}^2$$

$$((v_1, v_2, v_3), (w_1, w_2, w_3)) \mapsto \left(\frac{1}{2}v_1, \frac{1}{2}w_1 \right)$$

The moment polytope, as in Figure 1.1, is the square

$$P_1 = \left\{ (x, y) \in \mathbb{R}^2 \mid -\frac{1}{2} \leq x \leq \frac{1}{2}, -\frac{1}{2} \leq y \leq \frac{1}{2} \right\}$$

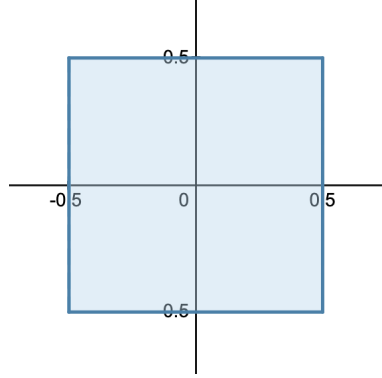


Figure 1.1: The moment polytope P_1 for the standard toric structure of $S^2 \times S^2$.

As in [Fuk+12], $S^2 \times S^2$ can be obtained by resolving the singularity of a toric degeneration. Now we recall the construction. The toric Hirzebruch surfaces $F_2(\alpha)$, $0 < \alpha < 1$, are toric manifolds with moment polytope

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 - 2y, 0 \leq y \leq 1 - \alpha\}$$

As $\alpha \rightarrow 0$, we obtain an orbifold $F_2(0)$ with a singularity of the form $\mathbb{C}^2/\pm$. The moment polytope of $F_2(0)$, as in Figure 1.2, is

$$P_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 - 2y, y \geq 0\}$$

and the preimage of the point $(0, 1)$ is the singularity. To resolve the singularity, we replace a neighborhood of it with a neighborhood of the zero sec-

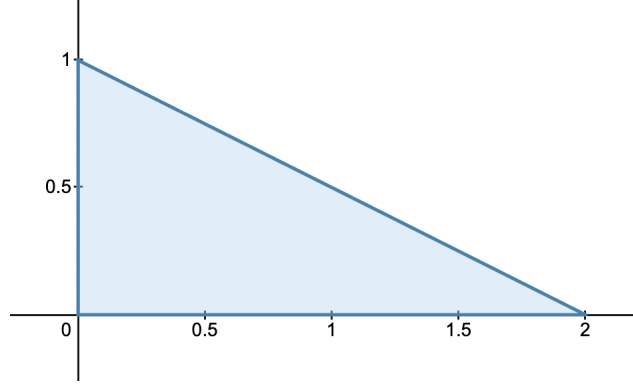


Figure 1.2: The moment polytope of $\hat{F}(0)$.

tion of the cotangent bundle T^*S^2 . The resulting complex surface is denoted by $\hat{F}(0)$. There is a well-known result that $F_2(\alpha)$ is symplectomorphic to $(S^2, \frac{1-\alpha}{2}\omega_{std}) \times (S^2, \frac{1+\alpha}{2}\omega_{std})$, and $\hat{F}(0)$ is symplectomorphic to $(S^2, \frac{1}{2}\omega_{std}) \times (S^2, \frac{1}{2}\omega_{std})$ [Fuk+12, Proposition 5.1]. We still say the moment polytope of $\hat{F}(0)$ is P_2 with the preimage of the point $(0, 1)$ being S^2 .

Given a point (x, y) in the segment $\{(x, y) \in P_2 \mid x + y = 1, \frac{1}{2} \leq y < 1\}$, K. Fukaya, Y. Oh, H. Ohta, and K. Ono in [Fuk+12] have shown the preimage of (x, y) under the moment map is a nondisplaceable Lagrangian torus using Lagrangian Floer homology. In particular, the preimage is not Hamiltonian isotopic to a toric fiber of the standard toric structure. This conclusion can be obtained from the fact that the only nondisplaceable toric fiber of the standard toric structure is the Clifford torus and from Theorem 1.1 in [Fuk+12]. J. Oakley and M. Usher in [OU16] showed that the preimage of $(\frac{1}{2}, \frac{1}{2})$ is Hamiltonian isotopic to the Lagrangian torus in [Biro6], [AF07], [EP09], and [CS10]. Then one can ask the following questions.

Question 1.0.3. Can this discussion be extended to encompass other interior points within the moment polytope P_2 ? Specifically, can we show if their preimages are Hamiltonian isotopic to toric fibers of the standard toric structure?

Question 1.0.4. Are there other Lagrangian tori, except those in [Fuk+12], that are not Hamiltonian isotopic to the product tori in $S^2 \times S^2$? We call a torus of the form $\alpha \times \beta$ with α and β are embedded curves in $S^1 \times \{pt\}$ and $\{pt\} \times S^1$ respectively a product torus.

We will prove the following theorems to answer the questions.

Theorem B. *Given an interior point (x, y) in P_2 with $x + y \neq 1$, let $L(x, y)$ be the preimage of (x, y) in $\hat{F}(0)$ and still denote by $L(x, y)$ its image under a*

symplectomorphism from $\hat{F}(0)$ to $S^2 \times S^2$. Then $L(x, y)$ is a Lagrangian torus Hamiltonian isotopic to a toric fiber. Furthermore, if we denote the preimage of an interior point (ξ, ζ) in P_1 by $T(\xi, \zeta)$, then $L(x, y)$ is Hamiltonian isotopic to

$$\begin{cases} T\left(\frac{1}{2} - y, \frac{3}{2} - 2y - x\right) & \text{for } 1 - y < x < 2 - 2y \\ T\left(-\frac{1}{2} + x, \frac{1}{2} - y\right) & \text{for } 0 < x < 1 - y \end{cases}$$

Remark 1.o.5. Theorem B does not depend on the choice of the symplectomorphism between $\hat{F}(0)$ and $S^2 \times S^2$ since every symplectomorphism of $S^2 \times S^2$ can be written as the composition of a Hamiltonian diffeomorphism and the diffeomorphism that switches the two factors of $S^2 \times S^2$. See [OU16] and [Gro85]. In addition, the toric fiber $T(\xi, \zeta)$ is Hamiltonian isotopic to $T(\zeta, \xi)$ by [Bre23].

Theorem C. *Let (x, y) be an interior point of P_2 and $x + y = 1$. Then $L(x, y)$ is not Hamiltonian isotopic to a product torus.*

Remark 1.o.6. For an interior point (x, y) in P_2 with $x + y = 1$ and $0 < y < \frac{1}{2}$, the preimage of (x, y) is a displaceable Lagrangian torus, so Theorem C does not follow from the techniques in [Fuk+12].

Theorem B and Theorem C will be proved in Chapter 3.

CHAPTER 2

ON THE HOFER-ZEHNDER CONJECTURE FOR SEMIPOSITIVE SYMPLECTIC MANIFOLDS

Let (M, ω) be a compact symplectic manifold. For $A \in \pi_2(M)$, let $v : S^2 \rightarrow M$ be a representative of A . Then

$$c_1(A) := \int_{S^2} v^* c_1$$

where $c_1 \in H^2(M)$ is the first Chern class, and

$$\omega(A) := \int_{S^2} v^* \omega.$$

In this chapter, we consider the compact semipositive symplectic manifolds.

Definition 2.0.1. A compact $2n$ -dimensional symplectic manifold (M, ω) is *semipositive* if for every $A \in \pi_2(M)$,

$$3 - n \leq c_1(A) < 0 \implies \omega(A) \leq 0$$

.

2.1 Hamiltonian Floer Homology

2.1.1 Novikov Ring

We follow [HS95] for the definition of Novikov ring. Let Γ be a group with a homomorphism $\varphi : \Gamma \rightarrow \mathbb{R}$ and let $\mathbb{F}$ be an integral domain. Define

$$\Lambda(\Gamma, \varphi; \mathbb{F}) := \{ \Gamma \rightarrow \mathbb{F} : A \mapsto \lambda_A \mid \# \{ A \in \Gamma : \lambda_A \neq 0, \varphi(A) < c \} < \infty, \forall c \in \mathbb{R} \}$$

with a ring structure given by the convolution

$$(\lambda * \theta)_A = \sum_{B \in \Gamma} \lambda_B \theta_{B^{-1}A}.$$

When φ is injective, Γ is isomorphic to $\mathbb{Z}^m$ for some positive integer m and

$$\varphi(k_1, \dots, k_m) = \sum_{j=1}^m w_j k_j$$

where the w_j are positive and rationally independent. Then we can identify $\Lambda(\Gamma, \varphi; \mathbb{F})$ with

$$\left\{ \sum_{k_1, \dots, k_m} a_{k_1, \dots, k_m} t_1^{k_1} \cdots t_m^{k_m} \mid \# \left\{ k : a_k \neq 0, \sum_{j=1}^m w_j k_j \leq c \right\} < \infty, \forall c \in \mathbb{R} \right\}$$

Example 2.1.1. Consider the following maps

$$\begin{aligned} \varphi_{c_1} : \pi_2(M) &\rightarrow \mathbb{Z} \\ A &\mapsto c_1(A) \end{aligned}$$

and

$$\begin{aligned} \varphi_\omega : \pi_2(M) &\rightarrow \mathbb{R} \\ A &\mapsto \omega(A) \end{aligned}$$

Define $\Gamma_0 := \frac{\ker \varphi_{c_1}}{\ker \varphi_{c_1} \cap \ker \varphi_\omega}$ and the Novikov ring $\Lambda_\omega = \Lambda(\Gamma_0, \varphi_\omega; \mathbb{Z})$.

Example 2.1.2 (Universal Novikov Ring). Let R be a commutative unital ring. The universal Novikov ring over R is defined as

$$\Lambda_{R,univ} = \left\{ \sum_{i=-K}^{\infty} a_i T^{\lambda_i} \mid a_i \in R, \lambda_i \nearrow \infty \right\}$$

By [HS95, Theorem 4.1], $\Lambda_{\mathbb{K},univ}$ is a field when $\mathbb{K}$ is a field, and we call it the *universal Novikov field* over $\mathbb{K}$.

Next, we show some properties of Novikov ring which will be used later for the semisimplicity of the even quantum homology.

Proposition 2.1.3 ([HS95], Theorem 4.2). *The ring $\Lambda_{\mathbb{Z},univ}$ is a PID. In particular, every nonzero prime ideal in $\Lambda_{\mathbb{Z},univ}$ is maximal.*

Lemma 2.1.4. *Let I be a prime ideal of $\Lambda_{\mathbb{Z},univ}$. If $\Lambda_{\mathbb{Z},univ}/I$ has characteristic p where p is a prime, then $I = \langle p \rangle$. In particular, $\Lambda_{\mathbb{Z},univ}/I = \Lambda_{\mathbb{Z}_p,univ}$*

Proof. Since $\Lambda_{\mathbb{Z},univ}$ is a PID, we can assume that $I = \langle f \rangle$ for some $f \in \Lambda_{\mathbb{Z},univ}$. Since $\Lambda_{\mathbb{Z},univ}/I$ has characteristic p , then $p(1 + \langle f \rangle) = \langle f \rangle$, which means $f|p$. Thus there is a $g \in \Lambda_{\mathbb{Z},univ}$ such that $fg = p$. In particular $p|(fg)$.

If $p|f$, then $f = pu$ where u is invertible. Thus $\langle f \rangle = \langle p \rangle$.

If $p \nmid f$, then $p|g$. Thus there is h such that $ph = g$. Then $fph = p$, which implies $fh = 1$. Thus f is invertible, which contradicts with that $\langle f \rangle$ is a prime ideal.

Thus $\Lambda_{\mathbb{Z},univ}/I = \Lambda_{\mathbb{Z},univ}/\langle f \rangle = \Lambda_{\mathbb{Z},univ}/\langle p \rangle = \Lambda_{\mathbb{Z}_p,univ}$. $\square$

Theorem 2.1.5. *$\Lambda_{\mathbb{Z}_p,univ}$ is perfect.*

Proof. Since $\Lambda_{\mathbb{Z}_p,univ}$ has characteristic p , we only need to show that for an element $f \in \Lambda_{\mathbb{Z}_p,univ}$ there is an element $g \in \Lambda_{\mathbb{Z}_p,univ}$ such that $f = g^p$.

Assume $f = \sum_{i=-N}^{\infty} a_i T^{\lambda_i}$. Then $a_i \in \mathbb{Z}_p$. Since $\mathbb{Z}_p$ is perfect, then $a_i = b_i^p$

for some $b_i \in \mathbb{Z}_p$. Take $g = \sum_{i=-N}^{\infty} b_i T^{\frac{\lambda_i}{p}}$. Then $g^p = f$. $\square$

Proposition 2.1.6 ([AL23], Proposition 3.1). *Let $\text{Frac}(\Lambda_{\mathbb{Z},univ})$ be the field of fractions of $\Lambda_{\mathbb{Z},univ}$. If $f \in \text{Frac}(\Lambda_{\mathbb{Z},univ})$, then it can be written as $\sum c_j T^{\mu_j}$, where $c_j \in \mathbb{Q}$ and only finitely many primes appear in the denominators of the coefficients of f .*

Proof. Note that if $h = \sum_{i=-N}^{\infty} h_i T^{\nu_i}$ is an element in $\Lambda_{\mathbb{Z}, \text{univ}}$, then $T^{-\nu-N} h$ only has non-negative exponents. Suppose $f \in \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$. We may assume without loss of generality that

$$f = \frac{a_0 + \sum_{i=1}^{\infty} a_i T^{\lambda_i}}{b_0 + \sum_{j=1}^{\infty} b_j T^{\theta_j}}$$

where $0 < \lambda_1 < \lambda_2 < \dots$ and $0 < \theta_1 < \theta_2 < \dots$. Set

$$A := a_0 + \sum_{i=1}^{\infty} a_i T^{\lambda_i} \quad \text{and} \quad B := b_0 + \sum_{j=1}^{\infty} b_j T^{\theta_j}$$

and let $g_0 = \frac{a_0}{b_0}$. Then

$$A - g_0 B = \sum_{i=1}^{\infty} a_i T^{\lambda_i} - \sum_{j=1}^{\infty} \frac{a_0 b_j}{b_0} T^{\theta_j}.$$

and the leading term has a non-negative exponent.

Case 1. $\lambda_1 < \theta_1$ The leading term in $A - g_0 B$ is $a_1 T^{\lambda_1}$. Define $g_1 = \frac{a_0}{b_0} + \frac{a_1}{b_0} T^{\lambda_1}$. Then

$$A - g_1 B = \sum_{i=2}^{\infty} a_i T^{\lambda_i} - \sum_{j=1}^{\infty} \frac{a_0 b_j}{b_0} T^{\theta_j} - \sum_{j=1}^{\infty} \frac{a_1 b_j}{b_0} T^{\lambda_1 + \theta_j}$$

which has the exponent of the leading term greater than λ_1 .

Case 2. $\lambda_1 = \theta_1$ The leading term in $A - g_0 B$ is $\left(a_1 - \frac{a_0 b_1}{b_0}\right) T^{\lambda_1}$. Define $g_1 = \frac{a_0}{b_0} + \left(\frac{a_1}{b_0} - \frac{a_0 b_1}{b_0^2}\right) T^{\lambda_1}$. Then

$$A - g_1 B = \sum_{i=2}^{\infty} a_i T^{\lambda_i} - \sum_{j=2}^{\infty} \frac{a_0 b_j}{b_0} T^{\theta_j} - \sum_{j=1}^{\infty} \left(\frac{a_1}{b_0} - \frac{a_0 b_1}{b_0^2}\right) b_j T^{\lambda_1 + \theta_j}$$

which has the exponent of the leading term greater than λ_1 .

Case 3. $\lambda_1 > \theta_1$ The leading term in $A - g_0B$ is $-\frac{a_0b_1}{b_0}T^{\theta_1}$. Define $g_1 = \frac{a_0}{b_0} - \frac{a_0b_1}{b_0^2}T^{\theta_1}$. Then

$$A - g_1B = \sum_{i=1}^{\infty} a_i T^{\lambda_i} - \sum_{j=2}^{\infty} \frac{a_0b_j}{b_0} T^{\theta_j} + \sum_{j=1}^{\infty} \frac{a_0b_1b_j}{b_0^2} T^{\theta_1+\theta_j}$$

which has the exponent of the leading term greater than θ_1 .

One can repeat this process to get g_n for $n \in \mathbb{N}$. It is easy to see that the exponent of the leading term of $A - g_nB$ strictly grows as n grows. Next we show that the exponent of the leading term of $A - g_nB$ grows to ∞ . Then we can conclude that $f = \lim g_n$. Let $\Sigma(A - g_nB)$ be the set of the linear combination of the exponents of the terms in $A - g_nB$ over nonnegative integers. Then

$$\cdots \subset \Sigma(A - g_{n+1}B) \subset \Sigma(A - g_nB) \subset \cdots \subset \Sigma(A - g_0B)$$

Thus the exponent of the leading term of $A - g_nB$ strictly grows in $\Sigma(A - g_0B)$. In particular it grows to ∞ .

Hence, the primes appearing in the denominators of the coefficients of f are the primes dividing b_0 , of which there are finitely many. $\square$

Remark 2.1.7. This result is not true for an arbitrary element in $\Lambda_{\mathbb{Q}, univ}$. Indeed, one can have elements of the form

$$\sum_{l=-K}^{\infty} \frac{1}{l!} T^l$$

which has infinitely many primes in the denominators.

2.1.2 Conley-Zehnder Index

We follow [AD13] for the definition of the Conley-Zehnder Index. Let H be a time-dependent Hamiltonian on M . Let $x(t)$ be a nondegenerate contractible 1-periodic orbit of X_H . Then there is a map $u : D^2 \rightarrow M$ such that $u(e^{it}) = x(t)$. We choose a trivialization of x^*TM such that it can be extended to a trivialization of u^*TM . For every t , the linear map $(d\phi_H^t)_{x(0)}$ can be considered as the matrix $\Phi(t)$ with respect to the bases of $T_{x(0)}M$ and $T_{x(t)}M$ determined by the trivialization. Then $\Phi(t)$ is a path in the symplectic group $Sp(2n)$.

Define

$$Sp(2n)^* = \{\Phi \in Sp(2n) \mid \det(I - \Phi) \neq 0\}$$

where I is the identity matrix.

Proposition 2.1.8 ([AD13], Proposition 7.1.4). *The open set $Sp(2n)^*$ has two connected components, which are*

$$Sp(2n)^+ = \{\Phi \in Sp(2n) \mid \det(I - \Phi) > 0\}$$

and

$$Sp(2n)^- = \{\Phi \in Sp(2n) \mid \det(I - \Phi) < 0\}$$

If $\det(I - \Phi(1)) > 0$, then let γ_Φ be a path connecting $\Phi(1)$ to $-I$ in $Sp(2n)^*$. If $\det(I - \Phi(1)) < 0$, then let γ_Φ be a path connecting $\Phi(1)$ to

$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ \hline 0 & & -I \end{array} \right]. \text{ Let } \gamma \text{ be the concatenated path of } \Phi(t) \text{ and } \gamma_\Phi.$$

Theorem 2.1.9 ([AD13], Theorem 7.1.3). *There exists a continuous map*

$$\rho : Sp(2n) \rightarrow S^1$$

satisfying the following properties:

(1) *Naturality: If Φ and T are in $Sp(2n)$, then*

$$\rho(T\Phi T^{-1}) = \rho(\Phi)$$

(2) *Product: If $\Phi \in Sp(2m)$ and $\Psi \in Sp(2n)$, then*

$$\rho \begin{bmatrix} \Phi & 0 \\ 0 & \Psi \end{bmatrix} = \rho(\Phi)\rho(\Psi)$$

(3) *Determinant: If $\Phi \in U(n)$, then*

$$\rho(\Phi) = \det(X + iY), \text{ where } \Phi = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$$

(4) *Normalization:* If $\text{Spec}(\Phi) \in \mathbb{R}$, then

$$\rho(\Phi) = (-1)^{m_0/2}$$

where m_0 is the total multiplicity of the negative real eigenvalues.

$$(5) \quad \rho(\Phi^T) = \rho(\Phi^{-1}) = \overline{\rho(\Phi)}$$

Finally let $\tilde{\gamma} : S^1 \rightarrow \mathbb{R}$ be a lift of $\rho \circ \gamma : S^1 \rightarrow S^1$. Then the *Conley-Zehnder index* of (x, u) is defined as

$$CZ(x, u) := \frac{\tilde{\gamma}(0) - \tilde{\gamma}(1)}{\pi}$$

Define an equivalence relation on the contractible 1-periodic orbits as the following:

$$(x, u_0) \equiv (y, u_1) \iff x = y, \int_{D^2} u_0^* c_1 = \int_{D^2} u_1^* c_1, \int_{D^2} u_0^* \omega = \int_{D^2} u_1^* \omega$$

Denote the equivalent class of (x, u) by $[x, u]$. Then the Conley-Zehnder index does not depend on the choice of the representative of $[x, u]$. Thus define $CZ([x, u]) := CZ(x, u)$.

An important property of Conley-Zehnder index is

$$CZ([x, u \# A]) = CZ([x, u]) - 2c_1(A)$$

where $u \# A$ is the connected sum of u with a spherical class $A \in H_2(M)$.

2.1.3 Transversality

We follow [HS95] to show that the moduli spaces used to define the Hamiltonian Floer homology are compact finite dimensional manifolds.

Definition 2.1.10. An *almost complex structure* on (M, ω) is a map

$$J : TM \rightarrow TM$$

such that $J^2 = -id$. An almost complex structure is *compatible* with the symplectic form ω if $\omega(\cdot, J\cdot)$ defines a Riemannian metric on M .

Definition 2.1.11. A *J-holomorphic sphere* is a smooth map $v : S^2 \rightarrow M$ such that

$$dv \circ i = J \circ dv$$

where i is the standard complex structure on $S^2 = \mathbb{C} \cup \{\infty\}$. A J -holomorphic curve v is called *simple* if $v = w \circ \phi$ with $\phi : S^2 \rightarrow S^2$ implies $\deg(\phi) = 1$.

The first step is to perturb the almost complex structure such that the moduli space of simple J -holomorphic spheres in a homotopy class is a finite dimensional manifold.

Proposition 2.1.12 ([HS95], Section 2). *Given $A \in \frac{\pi_2(M)}{\ker \varphi_{c_1} \cap \ker \varphi_\omega}$, denote by $\mathcal{M}(A; J)$ the space of J -holomorphic spheres in the class A and $\mathcal{M}_s(A; J) \subset \mathcal{M}(A; J)$ the subspace of simple J -holomorphic spheres. Then for a generic almost complex structure J compatible with ω , we have the following results.*

- (1) $\mathcal{M}_s(A; J)$ is a finite dimensional manifold with $\dim \mathcal{M}_s(A; J) = 2n + 2c_1(A)$.
- (2) If $c_1(A) < 0$ then $\mathcal{M}(A; J) = \emptyset$.
- (3) If $n = 2$ and $A \neq 0$ with $c_1(A) = 0$ then $\mathcal{M}(A; J) = \emptyset$.
- (4) If $n = 2$ and $c_1(A) = 1$ then the moduli space $\mathcal{M}(A; J)/G$ is a finite set, where $G = \text{PSL}(2, \mathbb{C})$ is the group of biholomorphic maps of S^2 .

Denote by $M_k(c; J)$ the set of points $x \in M$ such that there exists a non-constant J -holomorphic sphere v such that $c_1(v) \leq k$, $E(v) = \int_{S^2} v^* \omega \leq c$ and $x \in v(S^2)$.

Proposition 2.1.13 ([HS95], Section 2). *Assume $2n \geq 6$ and J is generic. Then the set $M_k(c; J)$ is compact for every $c > 0$ and every integer k .*

Next we perturb the Hamiltonian such that the moduli space of Floer trajectories satisfying a limit condition is a compact finite dimensional manifold.

Definition 2.1.14. Given a Hamiltonian $H : M \times S^1 \rightarrow \mathbb{R}$, the *Floer equation* is the partial differential equation

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} - \nabla H(t, u) = 0$$

where $u : \mathbb{R} \times S^1 \rightarrow M$, $(s, t) \mapsto u(s, t)$ and J is an almost complex structure compatible with ω .

Let $[x_\pm, u_\pm]$ be classes of contractible 1-periodic orbits of Hamiltonian H and J be a generic almost complex structure compatible with ω . Denote by

$\mathcal{M}([x_-, u_-], [x_+, u_+]; H, J)$ the moduli space of the solutions to the Floer equations which satisfy the limit condition

$$\lim_{x \rightarrow \pm\infty} u(s, t) = x_{\pm}(t)$$

and

$$[x_+, u_- \# u] = [x_+, u_+]$$

Theorem 2.1.15 ([HS95], Section 3). *Assume $2n \geq 4$. There is a generic Hamiltonian H such that the following results hold.*

- (1) *Every contractible 1-periodic orbit $x(t)$ is non-degenerate.*
- (2) *$x(t) \notin M_1(\infty; J)$ for every contractible 1-periodic orbit $x(t)$ and every $t \in \mathbb{R}$.*
- (3) *The moduli space $\mathcal{M}([x_-, u_-], [x_+, u_+]; H, J)$ is a finite dimensional manifold for all contractible 1-periodic orbits $[x_{\pm}, u_{\pm}]$ with*

$$\dim \mathcal{M}([x_-, u_-], [x_+, u_+]; H, J) = CZ([x_-, u_-]) - CZ([x_+, u_+]).$$
- (4) *$u(s, t) \notin M_0(\infty; J)$ for every $u \in \mathcal{M}([x_-, u_-], [x_+, u_+]; H, J)$ with $CZ([x_-, u_-]) - CZ([x_+, u_+]) \leq 2$ and every $(s, t) \in \mathbb{R} \times S^1$.*
- (5) *Denote by $M_k(c, H, J)$ the set of all points $x = u(s, t) \in M$ where $u \in \mathcal{M}([x_-, u_-], [x_+, u_+]; H, J)$ with energy*

$$E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} - X_H(t, u) \right|^2 \right) dt ds \leq c$$

and $CZ([x_-, u_-]) - CZ([x_+, u_+]) \leq k$. Then the sets $M_1(c; H, J)$ and $M_2(c; H, J)$ are compact for every $c > 0$

2.1.4 Hamiltonian Floer Homology

Definition 2.1.16. The *action functional* on the contractible 1-periodic orbits is defined as

$$\mathcal{A}_H([x, u]) = - \int_D u^* \omega + \int_0^1 H(t, x(t)) dt$$

Define the k -th chain group as

$$CF_k(H, J; \Lambda_\omega) = \left\{ \sum_j \lambda_j [x_j, u_j] \mid \# \{j : \lambda_j \neq 0, \mathcal{A}_H([x_j, u_j]) < c\} < \infty, \forall c > 0 \right\}$$

where $\lambda_j \in \Lambda_\omega$ and $[x_j, u_j]$ is a class of the contractible 1-periodic orbits with Conley-Zehnder index k . The action of λ_j on $[x_j, u_j]$ is defined as

$$\lambda_j [x_j, u_j] = \sum_{A \in \Gamma_0} \lambda_A [x_j, u_j \# A]$$

where $\lambda_j = \sum_{A \in \Gamma_0} \lambda_A T^{\omega(A)} \in \Lambda_\omega$.

We can extend the coefficient Λ_ω to the universal Novikov ring $\Lambda_{\mathbb{Z}, \text{univ}}$ by tensor product.

$$CF_k(H, J; \Lambda_{\mathbb{Z}, \text{univ}}) = CF_k(H, J; \Lambda_\omega) \otimes_{\Lambda_\omega} \Lambda_{\mathbb{Z}, \text{univ}}$$

Define the moduli space

$$\overline{\mathcal{M}}_k([x_-, u_-], [x_+, u_+]; H, J) := \mathcal{M}([x_-, u_-], [x_+, u_+]; H, J) / \mathbb{R}$$

where $CZ([x_-, u_-]) - CZ([x_+, u_+]) = k$ and the $\mathbb{R}$ -action on $\mathcal{M}([x_-, u_-], [x_+, u_+]; H, J)$ is given by translating the $\mathbb{R}$ -coordinate of the Floer trajectory connecting $[x_-, u_-]$ to $[x_+, u_+]$. By Section 2.1.3, we know $\overline{\mathcal{M}}_1([x_-, u_-], [x_+, u_+]; H, J)$ is a 0-dimensional compact manifold. Thus $\#\overline{\mathcal{M}}_1([x_-, u_-], [x_+, u_+]; H, J) < \infty$.

Define the differential of the chain complex as

$$d : CF_k(H, J; \Lambda_\omega) \rightarrow CF_{k-1}(H, J; \Lambda_\omega)$$

$$[x, u] \mapsto \sum_{\substack{[y, v] \\ CZ([x, u]) - CZ([y, v]) = 1}} \#\overline{\mathcal{M}}_1([x, u], [y, v]; H, J)$$

It was shown that $d^2 = 0$ using $\overline{\mathcal{M}}_2([x_-, u_-], [x_+, u_+]; H, J)$. See [HS95, Section 5]. Denote the homology of the chain complex $(CF_*(H, J; \Lambda_\omega), d)$ by $HF_*(H, J; \Lambda_\omega)$. The Floer homology $HF_*(H, J; \Lambda_\omega)$ does not depend on the generic choice of the Hamiltonian and the almost complex structure as shown in [HS95].

2.2 Quantum Homology

2.2.1 Quantum Homology

We follow [MS12] for the definition of the quantum homology. The quantum homology of (M, ω) is defined by $QH_*(M, \Lambda_{\mathbb{Z}, univ}) = H_*(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}, univ}$. There is a product on

$$QH_{ev}(M, \Lambda_{\mathbb{Z}, univ}) = \oplus_i QH_{2i}(M, \Lambda_{\mathbb{Z}, univ})$$

defined as follows. Choose an integer basis $e_0, \dots, e_N$ of the free part of $H_*(M, \mathbb{Z})$ such that $e_0 = [M] \in H_{2n}(M, \mathbb{Z})$ and each basis element e_ν has pure degree. Define the integer matrix $g_{\nu\mu}$ by

$$g_{\nu\mu} := \int_M PD(e_\nu) \cup PD(e_\mu),$$

and let $g^{\nu\mu}$ denote the inverse matrix. Then the product of $a, b \in H_{ev}(M, \mathbb{Z})$ is defined by

$$a * b := \sum_A \sum_{\nu, \mu} GW_{A,3}^M(a, b, e_\nu) g^{\nu\mu} e_\mu T^{\omega(A)}$$

The product of a and b can also be expressed as

$$a * b = \sum_A (a * b)_A T^{\omega(A)}$$

where

$$(a * b)_A := \sum_{\nu, \mu} GW_{A,3}^M(a, b, e_\nu) g^{\nu\mu} e_\mu \in H_{\deg(a) + \deg(b) + 2c_1(A) - 2n}(M, \mathbb{Z})$$

which is characterized by the condition

$$\int_M PD((a * b)_A) \cup c := GW_{A,3}^M(a, b, PD(c))$$

for $c \in H^*(M, \mathbb{Z})$.

2.2.2 PSS Isomorphism

We follow [PSS96] to define the PSS isomorphism between Hamiltonian Floer homology and quantum homology.

$$\text{PSS}_H : QH(M, \Lambda_{\mathbb{Z}, \text{univ}}) \rightarrow HF(H, J; \Lambda_{\mathbb{Z}, \text{univ}})$$

Let Σ be a Riemann surface of genus zero with l cylindrical ends $Z_i = \phi_i((0, \infty) \times S^1) \subset \Sigma$. Fix an almost complex structure j on Σ such that $\phi_i^* j$ agrees with the standard structure on the cylinders. Also fix l Hamiltonians $H_i : \mathbb{R} \times S^1 \times M \rightarrow \mathbb{R}$ such that $H_i(s, \cdot, \cdot)$ vanishes near $s = 0$ and independent of s for $s \geq 1$.

Given contractible 1-periodic orbits $\tilde{x}_i = [x_i, u_i]$ of X_{H_i} , define the Moduli space

$$\mathcal{M}_\Sigma(\tilde{x}_1, \dots, \tilde{x}_l, H_1, \dots, H_l, J)$$

as the space of all smooth maps $u : \Sigma \rightarrow M$ which satisfy the following conditions.

- (a) u is J -holomorphic on the complement $\Sigma - \cup_i Z_i$.
- (b) The maps $u_i = u \circ \phi_i$ satisfy

$$\begin{aligned} \frac{\partial u_i}{\partial s} + J(u) \frac{\partial u_i}{\partial t} - \nabla H_i(s, t, u_i) &= 0, \\ x_i(t) &= \lim_{s \rightarrow \infty} u_i(s, t) \end{aligned}$$

- (c) The map u capped off by the disks u_i (with opposite orientations) represents a torsion homology class in $H_2(M, \mathbb{Z})$.

Fix d distinct points $z_1, \dots, z_d \in \Sigma$ and homology classes $\alpha_1, \dots, \alpha_d \in H_*(M, \mathbb{Z})$ such that

$$\sum_{i=1}^l CZ([x_i, u_i]) = 2n - \sum_{\nu=1}^d (2n - \deg(\alpha_\nu)).$$

Define the moduli space

$$\mathcal{M}_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_l, H_1, \dots, H_l, J)$$

to be the set of all curves $u \in \mathcal{M}_\Sigma(\tilde{x}_1, \dots, \tilde{x}_l, H_1, \dots, H_l, J)$ with $u(z_\nu) \in \alpha_\nu$.

For generic choice of the Hamiltonians and the almost complex structure, $\mathcal{M}_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_l, H_1, \dots, H_l, J)$ is a finite set. Thus one can define a multi-linear map Ψ_Σ from

$$QH_*(M, \Lambda_{\mathbb{Z}, \text{univ}}) \otimes \dots \otimes QH_*(M, \Lambda_{\mathbb{Z}, \text{univ}})$$

to

$$HF(H_1, J; \Lambda_{\mathbb{Z}, \text{univ}}) \otimes \dots \otimes HF(H_l, J; \Lambda_{\mathbb{Z}, \text{univ}})$$

by

$$\Psi_\Sigma(\alpha_1, \dots, \alpha_d) = \sum_{\tilde{x}_i} \# \mathcal{M}_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_l, H_1, \dots, H_l, J)(\tilde{x}_1, \dots, \tilde{x}_l)$$

Example 2.2.1 (Pair-of-Pants Product). Take $d = 1$ and $\alpha_1 = [M] \in H_{2n}(M, \mathbb{Z})$. Take $l = 3$, $H_1 = H_2 = H$, and $H_3(t, \cdot) = -H(-t, \cdot)$. Then one can get the *pair-of-pants product* on the Floer homology.

$$HF_{2n-j}(H, J; \Lambda_{\mathbb{Z}, \text{univ}}) \otimes HF_{2n-k}(H, J; \Lambda_{\mathbb{Z}, \text{univ}}) \rightarrow HF_{2n-(j+k)}(H, J; \Lambda_{\mathbb{Z}, \text{univ}})$$

by Poincare duality.

Example 2.2.2 (PSS Isomorphism). Take $d = l = 1$. One can get the PSS isomorphism

$$\text{PSS}_H : QH(M; \Lambda_{\mathbb{Z}, \text{univ}}) \rightarrow HF(H, J; \Lambda_{\mathbb{Z}, \text{univ}})$$

2.2.3 Semisimplicity

Throughout this subsection, we assume that $QH_{ev}(M, \Lambda_{\mathbb{K}, \text{univ}})$ is semisimple where $\mathbb{K}$ is a field with characteristic 0 and .

Let R be a ring and $\mathfrak{p} \in \text{Spec}(R)$. Denote by $R_{\mathfrak{p}}$ the localization at $\mathfrak{p}$ and by $\mathfrak{k}(\mathfrak{p})$ the residue field $R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$.

The following theorem is Theorem 6.1 in [Ush11b].

Theorem 2.2.3. *Let R be a ring containing $\mathbb{Q}$ as a subfield and let A be a commutative R -algebra which, considered as an R -module, is finitely generated and free. Denote by $f : \text{Spec}(A) \rightarrow \text{Spec}(R)$ the morphism of schemes induced by the unique ring morphism $R \rightarrow A$ (sending r to $r \cdot 1$).*

(A) *The following are equivalent, for a point $\mathfrak{p} \in \text{Spec}(R)$:*

- (A1) The morphism f is unramified at every point in $f^{-1}(\{\mathfrak{p}\})$.
- (A2) There exists a field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ such that the map $\text{Spec}(A \otimes_R k) \rightarrow \text{Spec}(k)$ induced by the unique ring morphism $k \rightarrow A \otimes_R k$ is unramified.
- (A3) There exists a field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ such that $A \otimes_R k$ decomposes as a direct sum of field extensions of k .
- (A4) For every field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ the algebra $A \otimes_R k$ decomposes as a direct sum of field extensions of k .

Moreover, the set U_1 of points $\mathfrak{p} \in \text{Spec}(R)$ at which (A1) holds is open in $\text{Spec}(R)$.

(B) The following are equivalent, for a point $\mathfrak{p} \in \text{Spec}(R)$:

- (B1) There is some $\mathfrak{q} \in \text{Spec}(A)$ such that $f(\mathfrak{q}) = \mathfrak{p}$ and $f : \text{Spec}(A) \rightarrow \text{Spec}(R)$ is unramified at $\mathfrak{q}$.
- (B2) There exists a field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ such that the map $\text{Spec}(A \otimes_R k) \rightarrow \text{Spec}(k)$ induced by the unique ring morphism $k \rightarrow A \otimes_R k$ is unramified at some point $\mathfrak{q} \in \text{Spec}(A \otimes_R k)$.
- (B3) There exists a field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ and a direct sum splitting of k -algebras $A \otimes_R k = K \oplus S$ where $k \rightarrow K$ is a field extension.
- (B4) For every field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ there is a direct sum splitting of k -algebras $A \otimes_R k = K \oplus S$ where $k \rightarrow K$ is a field extension.

Moreover, the set U_2 of points $\mathfrak{p} \in \text{Spec}(R)$ at which (B1) holds is open in $\text{Spec}(R)$.

Remark 2.2.4. Since A is a finitely generated module in Theorem 2.2.3, $A \otimes_R k$ decomposes as a direct sum of algebraic extension of k in (A3) and (A4).

Corollary 2.2.5. $QH_{ev}(M, \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}}))$ is semisimple, where $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$ is the field of fractions of $\Lambda_{\mathbb{Z}, \text{univ}}$.

Proof. $\Lambda_{\mathbb{Q}, \text{univ}}$ contains $\mathbb{Q}$ as a subfield. The only prime ideal in $\Lambda_{\mathbb{Q}, \text{univ}}$ is $\{0\}$ and then $\Lambda_{\mathbb{Q}, \text{univ}}$ is the only residue field of itself. Because $\mathbb{K}$ has characteristic 0, it is a field extension of $\mathbb{Q}$. Then $\Lambda_{\mathbb{K}, \text{univ}}$ is a field extension of $\Lambda_{\mathbb{Q}, \text{univ}}$. Thus (A3) in Theorem 2.2.3 holds. Now consider the trivial extension of $\Lambda_{\mathbb{Q}, \text{univ}}$, we have $QH(M, \Lambda_{\mathbb{Q}, \text{univ}})$ is semisimple.

$\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$ is a field containing $\mathbb{Q}$ as a subfield. The only prime ideal in $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$ is $\{0\}$ and then $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$ is the only residue field of

itself. $\Lambda_{\mathbb{Q}, \text{univ}}$ is a field extension of $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$. Thus (A₃) in Theorem 2.2.3 holds. Now consider the trivial extension of $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$, we have $QH_{ev}(M, \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}}))$ is semisimple. $\square$

Corollary 2.2.6 ([AL23], Corollary 3.3). *Let $\overline{\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})}$ be the algebraic closure of $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$. Then $QH_{ev}(M, \overline{\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})})$ is semisimple.*

The following lemma is Lemma 3.6 in [Lor96]

Lemma 2.2.7. *Let A be a commutative ring, and let $P \subset A$ be a maximal ideal. Let $S \subset A \setminus P$ be a multiplicative set. Then the fields A/P and $S^{-1}A/S^{-1}P$ are isomorphic.*

The following lemma follows from Proposition 2.6.2, Lemma 2.6.4, Theorem 2.1.5 and Lemma 2.2.7.

Lemma 2.2.8. *Every residue field of $\Lambda_{\mathbb{Z}, \text{univ}}$ is perfect.*

Now we rewrite Theorem 6.1 in [Ush11b] to fit our context.

Theorem 2.2.9. *Let R be a ring such that every residue field of R is perfect and let A be a commutative R -algebra which, considered as an R -module, is finitely generated and free. Denote by $f : \text{Spec}(A) \rightarrow \text{Spec}(R)$ the morphism of schemes induced by the unique ring morphism $R \rightarrow A$ (sending r to $r \cdot 1$).*

(A) *The following are equivalent, for a point $\mathfrak{p} \in \text{Spec}(R)$:*

- (A₁) *The morphism f is unramified at every point in $f^{-1}(\{\mathfrak{p}\})$.*
- (A₂) *There exists an algebraic field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ such that the map $\text{Spec}(A \otimes_R k) \rightarrow \text{Spec}(k)$ induced by the unique ring morphism $k \rightarrow A \otimes_R k$ is unramified.*
- (A₃) *There exists an algebraic field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ such that $A \otimes_R k$ decomposes as a direct sum of field extensions of k .*
- (A₄) *For every algebraic field extension $\mathfrak{k}(\mathfrak{p}) \rightarrow k$ the algebra $A \otimes_R k$ decomposes as a direct sum of field extensions of k .*

Moreover, the set U_1 of points $\mathfrak{p} \in \text{Spec}(R)$ at which (A₁) holds is open in $\text{Spec}(R)$.

Definition 2.2.10. Let A be an R -algebra as in Theorem 2.2.9. We say that A is *generically semisimple* if the subset $U_1 \subset \text{Spec}(R)$ of Theorem 2.2.9(A) is nonempty.

Corollary 2.2.11. $QH_{ev}(M, \Lambda_{\mathbb{Z}, univ})$ is generically semisimple.

Proof. $Frac(\Lambda_{\mathbb{Z}, univ})$ is the residue field of $\Lambda_{\mathbb{Z}, univ}$ at the prime ideal $\{0\}$. By Corollary 2.2.5 and Theorem 2.2.9, $\Lambda_{\mathbb{Z}, univ}$ is generically semisimple. $\square$

Corollary 2.2.12. $QH_{ev}(M, \Lambda_{\mathbb{Z}_p, univ})$ is semisimple.

Proof. By Lemma 2.6.4 and Lemma 2.2.7, $\Lambda_{\mathbb{Z}_p, univ}$ is the residue field of $\Lambda_{\mathbb{Z}, univ}$ at the prime ideal $\langle p \rangle$. By Theorem 2.2.9, $QH_{ev}(M, \Lambda_{\mathbb{Z}_p, univ})$ is semisimple. $\square$

Since $QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})})$ is semisimple, let $\{\bar{e}_i\}_{i=1}^m$ be the idempotents of $QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})})$ such that

$$QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})}) = \bigoplus_{i=1}^m \bar{e}_i * QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})}).$$

Denote by

$$\bar{e}_i = \sum_{j=0}^n \bar{k}_{ij} h_j$$

where $\bar{k}_{ij} \in \overline{Frac(\Lambda_{\mathbb{Z}, univ})}$ and $h_j \in H_{2j}(M)$. Let $p_{ij}(x) \in Frac(\Lambda_{\mathbb{Z}, univ})[x]$ be the minimal polynomial of $\bar{k}_{ij}$ and $\{\alpha_{ij}^l\}$ be all the roots of $p_{ij}(x)$. Since

$$Frac(\Lambda_{\mathbb{Z}, univ})(\{\alpha_{ij}^l\}_{i,j,l})$$

is a finite extension, there is an element $\alpha \in \overline{Frac(\Lambda_{\mathbb{Z}, univ})}$ such that

$$Frac(\Lambda_{\mathbb{Z}, univ})(\{\alpha_{ij}^l\}_{i,j,l}) = Frac(\Lambda_{\mathbb{Z}, univ})(\alpha).$$

In particular, $\bar{k}_{ij} \in Frac(\Lambda_{\mathbb{Z}, univ})(\alpha)$ for any $i = 1, \dots, m$ and $j = 1, \dots, n$. Let $f(x)$ be the minimal polynomial of α . Denote by

$$f(x) = a_r x^r + a_{r-1} x^{r-1} + \dots + a_1 x + a_0$$

where $a_i \in Frac(\Lambda_{\mathbb{Z}, univ})$. Assume $a_i = \frac{a'_i}{a''_i}$ where $a'_i, a''_i \in \Lambda_{\mathbb{Z}, univ}$. Then

$$\prod_{i=1}^r a''_i \cdot f(x) \in \Lambda_{\mathbb{Z}, univ}[x],$$

still denote $\prod_{i=1}^r a_i'' \cdot f(x)$ by $f(x)$. Finally we have

$$\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})(\alpha) \cong \frac{\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})[x]}{\langle f \rangle}.$$

By reducing the coefficients of $f(x)$ to $\Lambda_{\mathbb{Z}_p, \text{univ}}$, we get

$$[f(x)]_p = g_1^{m_1}(x)g_2^{m_2}(x) \cdots g_s^{m_s}(x)$$

in $\Lambda_{\mathbb{Z}_p, \text{univ}}[x]$ where $g_i(x)$ are irreducible and distinct from each other.

Claim 2.2.13 ([AL23], Claim 3.4). $m_1 = m_2 = \cdots = m_s = 1$ for sufficiently large p .

Proof. Since $f(x)$ is irreducible over $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})[x]$, then

$$\gcd(f(x), f'(x)) = 1.$$

So,

$$r(x)f(x) + q(x)f'(x) = 1$$

for some $r(x), q(x) \in \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})[x]$. Let $\Theta \in \Lambda_{\mathbb{Z}, \text{univ}}$ be the product of the denominators of the coefficient of $r(x)$ and $q(x)$. Then

$$\widetilde{r(x)}f(x) + \widetilde{q(x)}f'(x) = \Theta \tag{2.2.1}$$

where $\widetilde{r(x)} = r(x) \cdot \Theta \in \Lambda_{\mathbb{Z}, \text{univ}}$ and $\widetilde{q(x)} = q(x) \cdot \Theta \in \Lambda_{\mathbb{Z}, \text{univ}}$. Denote by

$$\Theta = \theta_{-s}T^{\lambda-s} + \theta_{-s+1}T^{\lambda-s+1} + \cdots.$$

Then for all primes $p > |\theta_{-s}|$, Θ reduces to a nonzero element $[\Theta]_p$ in $\Lambda_{\mathbb{Z}_p, \text{univ}}$. Now by reducing the equation 2.2.1 for sufficiently large primes p , we have

$$[\widetilde{r(x)}]_p[f(x)]_p + [\widetilde{q(x)}]_p[f'(x)]_p = [\Theta]_p$$

Thus $\gcd([f(x)]_p, [f'(x)]_p) = 1$ in $\Lambda_{\mathbb{Z}_p, \text{univ}}[x]$, which means $m_1 = m_2 = \cdots = m_s = 1$. $\square$

Thus, for a sufficiently large prime p , $[f(x)]_p = g_1(x)g_2(x) \cdots g_s(x)$ where $g_i(x)$ are irreducible and distinct from each other.

Proposition 2.2.14. *For sufficiently large p , the idempotents $\bar{e}_i$ can be reduced to a nonzero element $[\bar{e}_i]_p$ in $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ where $\overline{\Lambda_{\mathbb{Z}_p, univ}}$ is the algebraic closure of $\Lambda_{\mathbb{Z}_p, univ}$.*

Proof. Recall that $\bar{e}_i = \sum_{j=1}^n \bar{k}_{ij} h_j$ where $\bar{k}_{ij} \in \text{Frac}(\Lambda_{\mathbb{Z}, univ})(\alpha)$ and

$$\text{Frac}(\Lambda_{\mathbb{Z}, univ})(\alpha) \cong \frac{\text{Frac}(\Lambda_{\mathbb{Z}, univ})[x]}{\langle f \rangle}.$$

Then we can write $\bar{k}_{ij}$ as $K_{ij}(x) + \langle f \rangle$ in $\frac{\text{Frac}(\Lambda_{\mathbb{Z}, univ})[x]}{\langle f \rangle}$.

Since $\bar{k}_{ij}$ is invertible, then there is an element $L_{ij}(x) \in \text{Frac}(\Lambda_{\mathbb{Z}, univ})[x]$ such that

$$K_{ij}(x)L_{ij}(x) + \langle f \rangle = 1 + \langle f \rangle, \text{ i.e. } K_{ij}(x)L_{ij}(x) = 1 + M_{ij}(x)f(x).$$

for $M_{ij}(x) \in \text{Frac}(\Lambda_{\mathbb{Z}, univ})[x]$. Let Υ_K, Υ_L and Υ_M be the product of the denominators of the coefficients of $K_{ij}(x), L_{ij}(x)$ and $M_{ij}(x)$ respectively. Then

$$\widetilde{K_{ij}(x)L_{ij}(x)}\Upsilon_M = \Upsilon_K\Upsilon_L\Upsilon_M + \widetilde{M_{ij}(x)}f(x)\Upsilon_K\Upsilon_L \quad (2.2.2)$$

where $\widetilde{K_{ij}(x)} = K_{ij}(x) \cdot \Upsilon_K \in \Lambda_{\mathbb{Z}, univ}[x]$, $\widetilde{L_{ij}(x)} = L_{ij}(x) \cdot \Upsilon_L \in \Lambda_{\mathbb{Z}, univ}[x]$ and $\widetilde{M_{ij}(x)} = M_{ij}(x) \cdot \Upsilon_M \in \Lambda_{\mathbb{Z}, univ}[x]$. For sufficiently large prime p , $[\Upsilon_K]_p \neq 0$, $[\Upsilon_L]_p \neq 0$, $[\Upsilon_M]_p \neq 0$ and $[f]_p \neq 0$. By reducing the equation 2.2.2, we have

$$[\widetilde{K_{ij}(x)}]_p[\widetilde{L_{ij}(x)}]_p[\Upsilon_M]_p = [\Upsilon_K]_p[\Upsilon_L]_p[\Upsilon_M]_p + [\widetilde{M_{ij}(x)}]_p[f(x)]_p[\Upsilon_K]_p[\Upsilon_L]_p \quad (2.2.3)$$

Thus $[\widetilde{K_{ij}(x)}]_p[\widetilde{L_{ij}(x)}]_p \neq 0$ in $\frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{[f(x)]_p}$. In particular $[\widetilde{K_{ij}(x)}]_p \neq 0$ in $\frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{[f(x)]_p}$. Then one can reduce $K_{ij}(x) + \langle f(x) \rangle$ to $[\widetilde{K_{ij}(x)}]_p[\Upsilon_K]_p^{-1} + \langle [f(x)]_p \rangle \in \frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle [f(x)]_p \rangle}$

Note that

$$\frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle [f(x)]_p \rangle} = \frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle g_1(x)g_2(x) \cdots g_s(x) \rangle} \cong \prod_{l=1}^s \frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle g_l(x) \rangle}$$

and each $\frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle g_l(x) \rangle}$ is a field because $g_l(x)$ is irreducible. Let

$$P_l : \frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle [f(x)]_p \rangle} \rightarrow \frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle g_l(x) \rangle}$$

be the projection. Then for at least one P_l such that

$$P_l(\widetilde{[K_{ij}(x)]_p}[\Upsilon_K]_p^{-1} + \langle [f(x)]_p \rangle) \neq 0.$$

Let

$$\iota_l : \frac{\Lambda_{\mathbb{Z}_p, univ}[x]}{\langle g_l(x) \rangle} \rightarrow \overline{\Lambda_{\mathbb{Z}_p, univ}}$$

be the inclusion. Then

$$\iota_l(P_l(\widetilde{[K_{ij}(x)]_p}[\Upsilon_K]_p^{-1} + \langle [f(x)]_p \rangle)) \neq 0$$

Thus we have reduced $\bar{k}_{ij}$ to a nonzero element in $\overline{\Lambda_{\mathbb{Z}_p, univ}}$. If we can reduce one of $\{\bar{k}_{ij}\}_{j=1}^n$ to a nonzero element in $\overline{\Lambda_{\mathbb{Z}_p, univ}}$, then $\bar{e}_i$ can be reduced to a nonzero element in $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$. But each $\bar{e}_i$ is a finite sum and there are finitely many $\bar{e}_i$, then one can choose sufficiently large p such that $\bar{k}_{ij}$ can be reduced to a nonzero element in $\overline{\Lambda_{\mathbb{Z}_p, univ}}$ for any i, j . $\square$

Remark 2.2.15. Actually, for any $l = 1, \dots, s$ and sufficiently large p ,

$$P_l(\widetilde{[K_{ij}(x)]_p}[\Upsilon_K]_p^{-1} + \langle [f(x)]_p \rangle) \neq 0.$$

Otherwise, suppose

$$P_l(\widetilde{[K_{ij}(x)]_p}[\Upsilon_K]_p^{-1} + \langle [f(x)]_p \rangle) = 0.$$

Then

$$\widetilde{[K_{ij}(x)]_p}[\Upsilon_K]_p^{-1} = g_l(x)\delta(x)$$

for some $\delta(x) \in \Lambda_{\mathbb{Z}, univ}[x]$. Now we can rewrite the equation 2.2.3 as:

$$\begin{aligned} & g_l(x)\delta(x)[\Upsilon_K]_p[\widetilde{[L_{ij}(x)]_p}[\Upsilon_M]_p \\ &= [\Upsilon_K]_p[\Upsilon_L][\Upsilon_M]_p + [\widetilde{[M_{ij}(x)]_p}g_1(x) \cdots g_l(x) \cdots g_s(x)][\Upsilon_K]_p[\Upsilon_L]_p \end{aligned} \tag{2.2.4}$$

Let η be a root of $g_l(x)$. Plug η into the equation 2.2.4, we get

$$[\Upsilon_K]_p[\Upsilon_L]_p[\Upsilon_M]_p = 0,$$

which contradicts with our choice of p .

Proposition 2.2.16 ([AL23], Proposition 3.6). *The reductions $[\bar{e}_i]_p$ are idempotents in $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}, univ}})$ for $i = 1, \dots, m$, $[\bar{e}_i]_p * [\bar{e}_j]_p = 0$ for $i \neq j$ and $\sum_{i=1}^m [\bar{e}_i]_p = 1$ where 1 is the multiplication identity in $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}, univ}})$.*

Proof. Since $QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})})$ is semisimple and $\{\bar{e}_i\}_{i=1}^m$ are idempotents such that

$$QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})}) = \bigoplus_{i=1}^m \bar{e}_i * QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})}),$$

we have the following equations:

$$\bar{e}_i * \bar{e}_i = \bar{e}_i$$

for $i = 1, \dots, m$,

$$\bar{e}_i * \bar{e}_j = 0, i \neq j$$

for $i, j = 1, \dots, m$,

$$\sum_{i=1}^m \bar{e}_i = 1$$

Then one can reduce the three equations to $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}, univ}})$. □

$$\text{Thus } QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}, univ}}) = \bigoplus_{i=1}^m [\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}, univ}}).$$

Since $QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})})$ is semisimple and $\{\bar{e}_i\}_{i=1}^m$ are idempotents, then each $\bar{e}_i * QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})})$ is an algebraic field extension of $\overline{Frac(\Lambda_{\mathbb{Z}, univ})}$. But $\overline{Frac(\Lambda_{\mathbb{Z}, univ})}$ is the algebraic closure of $Frac(\Lambda_{\mathbb{Z}, univ})$, so

$$\bar{e}_i * QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})}) \cong \overline{Frac(\Lambda_{\mathbb{Z}, univ})}.$$

Also,

$$\begin{aligned} QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})}) &= \bigoplus_{i=1}^m \bar{e}_i * QH_{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z}, univ})}) \\ &\cong \bigoplus_{i=1}^m \overline{Frac(\Lambda_{\mathbb{Z}, univ})}. \end{aligned}$$

Thus $m = rank(H_{ev}(M))$.

Since $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ is a free $\overline{\Lambda_{\mathbb{Z}_p, univ}}$ -module and $[\bar{e}_i]_p * QH_{ev}(M, \Lambda_{\mathbb{Z}, univ})$ is a submodule, then $[\bar{e}_i]_p * QH_{ev}(M, \Lambda_{\mathbb{Z}_p, univ})$ can be written as a direct sum of $\overline{\Lambda_{\mathbb{Z}_p, univ}}$. Since

$$m = rank(H_{ev}(M, \mathbb{Q})) = rank(H_{ev}(M, \mathbb{Z}_p))$$

for sufficiently large p and $[\bar{e}_i]_p$ is nonzero, we have

$$[\bar{e}_i]_p * QH_{ev}(M, \Lambda_{\mathbb{Z}_p, univ}) = \overline{\Lambda_{\mathbb{Z}_p, univ}}.$$

Thus $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ is semisimple and generated by the idempotents $\{[\bar{e}_i]_p\}_{i=1}^m$.

2.3 Persistence Module

2.3.1 Non-Archimedean valuation

Definition 2.3.1. A *non-Archimedean valuation* on a field Λ is a function

$$\nu : \Lambda \rightarrow \mathbb{R} \cup \{\infty\}$$

satisfying the following properties:

- (1) $\nu(x) = +\infty$ if and only if $x = 0$,
- (2) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in \Lambda$,
- (3) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in \Lambda$.

Furthermore, we set $\Lambda^0 = \nu^{-1}([0, +\infty))$ to be the subring of elements of nonnegative valuation.

Example 2.3.2. We defined the universal Novikov field in Example 2.1.2. Let $\mathbb{K}$ be a ground field. A non-Archimedean valuation on $\Lambda_{\mathbb{K}, univ}$ can be defined as

$$\nu \left(\sum_{i=-K}^{\infty} a_i T^{\lambda_i} \right) = \lambda_{-K} \quad (2.3.1)$$

on $\Lambda_{\mathbb{K}, \text{univ}} \setminus \{0\}$ and $\nu(0) = +\infty$.

We showed in Proposition 2.1.6 that every element in $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$ can be written as $\sum_{i=-K}^{\infty} c_i T^{\lambda_i}$ for $c_i \in \mathbb{Q}$. Thus we can define a valuation $\nu : \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}}) \rightarrow \mathbb{R} \cup \{\infty\}$ as in Equation 2.3.1.

Definition 2.3.3. A *non-Archimedean norm* on a field Λ is a map $|\cdot| : \Lambda \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties:

- (1) $|x| = 0$ if and only if $x = 0$,
- (2) $|xy| = |x||y|$ for all $x, y \in \Lambda$,
- (3) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \Lambda$.

Furthermore, Λ is said to be *complete* with respect to $|\cdot|$ if it is a complete metric space with respect to the induced topology.

Example 2.3.4. Given a non-Archimedean valuation ν , one can define a non-Archimedean norm by setting $|x| = e^{-\nu(x)}$. Conversely, if $|\cdot|$ is a non-Archimedean norm, a non-Archimedean valuation can be obtained by setting $\nu(x) = -\ln(|x|)$.

2.3.2 Extensions of valuations

Now we use the following results in [Cas86] to define filtration on the extensions of fields.

Lemma 2.3.5 ([Cas86], Chapter 7, Theorem 1.1). *Let $\mathfrak{k}$ be a field and complete with respect to a norm $|\cdot|$ and let $\mathfrak{K}$ be a finite extension of degree n . Then there is precisely one extension $\|\cdot\|$ of $|\cdot|$ to $\mathfrak{K}$. It is given by*

$$\|A\| = |N_{\mathfrak{K}/\mathfrak{k}}(A)|^{1/n}$$

where $A \in \mathfrak{K}$ and $N_{\mathfrak{K}/\mathfrak{k}}(A)$ is the determinant of the map $B \rightarrow AB$ for $B \in \mathfrak{K}$.

Furthermore, $\mathfrak{K}$ is complete with respect to $\|\cdot\|$.

Lemma 2.3.6 ([Cas86], Chapter 9, Lemma 2.1). *Let $\mathfrak{K} = \mathfrak{k}(A)$ be a separable extension and let $F(x) \in \mathfrak{k}[x]$ be the minimal polynomial for A . Let $\bar{\mathfrak{k}}$ be the completion of $\mathfrak{k}$ with respect to a norm $|\cdot|$. Let $F(x) = \phi_1(x) \cdots \phi_J(x)$ be the decomposition of $F(x)$ into irreducibles in $\bar{\mathfrak{k}}[x]$. Then the ϕ_j are distinct. Let $\mathfrak{K}_j = \bar{\mathfrak{k}}(B_j)$ where B_j is a root of $\phi_j(x)$. Then there is an injection*

$$\mathfrak{K} \hookrightarrow \mathfrak{K}_j \tag{2.3.2}$$

extending $\mathfrak{k} \hookrightarrow \bar{\mathfrak{k}}$ under which $A \mapsto B_j$. Denote by $|\cdot|_j$ the norm on $\mathfrak{K}$ induced by equation 2.3.2 and the unique norm on $\mathfrak{K}_j$ extending $|\cdot|$. Then the $|\cdot|_j$

$(1 \leq j \leq J)$ are precisely the extension of $|\cdot|$ to $\mathfrak{K}$. Furthermore, $\mathfrak{K}_j$ is the completion of $\mathfrak{K}$ with respect to $|\cdot|_j$.

Lemma 2.3.7 ([Cas86], Chapter 7, Corollary 1). *There is a unique extension of $|\cdot|$ to the algebraic closure $\bar{\mathfrak{k}}$ of $\mathfrak{k}$*

For each $\sum_{j=-K}^{\infty} a_j T^{\lambda_j} \in \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$ with $\lambda_{-K} < \lambda_{-K+1} < \dots$, the valuation $\nu \left(\sum_{j=-K}^{\infty} a_j T^{\lambda_j} \right) = \lambda_{-K}$. Then $|\cdot| = e^{-\nu(\cdot)}$ is a norm on $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$. Since $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$ has characteristic zero, it is perfect and then $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})(\alpha)$ is a separable extension. Now by Lemma 2.3.6 one can get a norm $|\cdot|$ on $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})(\alpha)$. Then $-\ln(|\cdot|)$ is a valuation on $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})(\alpha)$.

The valuation on $\Lambda_{\mathbb{Z}_p, \text{univ}}$ is $\nu \left(\sum_{j=-K}^{\infty} a_j T^{\lambda_j} \right) = \lambda_{-K}$ where $a_j \in \mathbb{Z}_p$ and $\lambda_{-K} < \lambda_{-K+1} < \dots$. Then $|\cdot| = e^{-\nu(\cdot)}$ is a norm on $\Lambda_{\mathbb{Z}_p, \text{univ}}$. Recall $f(x)$ is the minimal polynomial of α and, reducing to $\Lambda_{\mathbb{Z}_p, \text{univ}}[x]$,

$$[f(x)]_p = g_1(x)g_2(x) \cdots g_s(x)$$

where $g_i(x)$ are irreducible and distinct from each other. Then

$$|[f(0)]_p| = \prod_{i=1}^s |g_i(0)|.$$

Thus for some i ,

$$|g_i(0)| \leq |[f(0)]_p|^{1/s}.$$

Without loss of generality, assume $|g_1(0)| \leq |[f(0)]_p|^{1/s}$. There is an algebraic element γ over $\Lambda_{\mathbb{Z}_p, \text{univ}}$ such that $\frac{\Lambda_{\mathbb{Z}_p, \text{univ}}[x]}{\langle g_1(x) \rangle} = \Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)$. Since $\Lambda_{\mathbb{Z}_p, \text{univ}}$ is complete, by Lemma 2.3.5, one can get a norm $|\cdot|$ on $\Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)$ and then $-\ln(|\cdot|)$ is a valuation on $\Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)$.

Lemma 2.3.8 ([AL23], Lemma 4.1). *Let $b \in \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$. Then $|b| \geq |[b]_p|$ for sufficiently large p .*

Proof. Suppose $b = \sum_{i=-K}^{\infty} b_i T^{\lambda_i}$ with $\lambda_{-K} < \lambda_{-K+1} < \dots$. Then $\nu(b) = \lambda_{-K}$. Write $b_{-K} = \frac{b_{-K,0}}{b_{-K,1}}$ and then $[b_{-K}]_p = [b_{-K,0}]_p [b_{-K,1}]_p^{-1}$. If $[b_{-K}]_p =$

0, then $\nu(b) < \nu([b]_p)$. If $[b_{-K}]_p \neq 0$, then $\nu(b) = \nu([b]_p)$. Since $|\cdot| = e^{-\nu(\cdot)}$, then $|b| \geq |[b]_p|$. $\square$

Remark 2.3.9. The prime p in the above lemma depends on b and there is not a uniform p for all elements in $\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})$.

Proposition 2.3.10 ([AL23], Proposition 4.3). $|\gamma| \leq \max\{1, |f(0)|\}$.

Proof. Let $F_\gamma(x)$ be the characteristic polynomial of γ . Then

$$F_\gamma(x) = N_{\Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)/\Lambda_{\mathbb{Z}_p, \text{univ}}}(x - \gamma).$$

Since the degree of $\Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)$ over itself is 1, then $F_\gamma(x)$ is the minimal polynomial of γ , in particular, $F_\gamma(x) = g_1(x)$. Thus

$$N_{\Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)/\Lambda_{\mathbb{Z}_p, \text{univ}}}(\gamma) = (-1)^M g_1(0)$$

where M is the degree of $g_1(x)$. Then

$$|\gamma| = |N_{\Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)/\Lambda_{\mathbb{Z}_p, \text{univ}}}(\gamma)|^{1/M} = |g_1(0)|^{1/M}.$$

Since $|g_1(0)| \leq |[f(0)]_p|^{1/s} \leq |f(0)|^{1/s}$, then $|\gamma| \leq |f(0)|^{1/sM}$. Let N be the degree of $f(x)$. If $|f(0)| \leq 1$, then $|f(0)|^{1/sM} \leq 1$. If $|f(0)| > 1$, $|f(0)|^{1/sM} \leq |f(0)|$. Thus, $|\gamma| \leq \max\{1, |f(0)|\}$. $\square$

2.3.3 Non-Archimedean filtration

Definition 2.3.11. Let Λ be a field with a non-Archimedean valuation ν . Suppose C is a finite dimensional module over Λ . A *non-Archimedean filtration* is a function $l : C \rightarrow \mathbb{R} \cup -\infty$ satisfying the following properties:

- (1) $l(x) = -\infty$ if and only if $x = 0$
- (2) $l(\lambda x) = l(x) - \nu(\lambda)$ for all $\lambda \in \Lambda$ and $x \in C$
- (3) $l(x + y) \leq \max\{l(x), l(y)\}$

Proposition 2.3.12 (Proposition 2.3 in [UZ16]). *If $l(x) \neq l(y)$, then*

$$l(x + y) = \max\{l(x), l(y)\}$$

Definition 2.3.13. We call a Λ -basis $(x_1, \dots, x_N)$ of (C, l) *orthogonal* if

$$l\left(\sum \lambda_i x_i\right) = \max\{l(x_i) - \nu(\lambda_i)\}$$

for all $\lambda_i \in \Lambda$. It is called *orthonormal* if it also satisfies $l(x_i) = 0$ for all i .

Example 2.3.14 (Non-Archimedean filtration on Floer chain complex). We choose a $\Lambda_{\mathbb{K}, \text{univ}}$ -basis $\{x_1, \dots, x_N\}$ of $CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})$ and set the non-Archimedean filtration as

$$\begin{aligned} \mathcal{A} : CF(H, J; \Lambda_{\mathbb{K}, \text{univ}}) &\rightarrow \mathbb{R} \cup \{-\infty\} \\ \sum \lambda_i x_i &\mapsto \max\{\mathcal{A}_H(x_i) - \nu(\lambda_i)\} \end{aligned}$$

where $\mathcal{A}_H$ is the action functional. Equivalently, we declare $\{x_1, \dots, x_N\}$ to be an orthogonal basis of $(CF(H, J; \Lambda_{\mathbb{K}, \text{univ}}), \mathcal{A})$.

The basis given by $\{\tilde{x}_1, \dots, \tilde{x}_N\} := \{T^{\mathcal{A}(x_1)}x_1, \dots, T^{\mathcal{A}(x_N)}x_N\}$ is orthonormal.

We now consider the case where we extend the coefficients of $CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})$ to $\overline{\Lambda_{\mathbb{K}, \text{univ}}}$. For an orthogonal basis $\{x_1, \dots, x_N\}$ of $(CF(H, J; \Lambda_{\mathbb{K}, \text{univ}}), \mathcal{A})$, $\{x_1 \otimes 1, \dots, x_N \otimes 1\}$ is a basis of $CF(H, J; \overline{\Lambda_{\mathbb{K}, \text{univ}}})$. We define a non-Archimedean filtration $\overline{\mathcal{A}}$ on $CF(H, J; \overline{\Lambda_{\mathbb{K}, \text{univ}}})$ by setting

$$\overline{\mathcal{A}}\left(\sum \bar{\lambda}_i x_i \otimes 1\right) = \max\{\mathcal{A}_H(x_i) - \bar{\nu}(\bar{\lambda}_i)\}$$

where $\bar{\nu}$ is the extended valuation on $\overline{\Lambda_{\mathbb{K}, \text{univ}}}$ of the valuation ν on $CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})$. Then $\{x_1 \otimes 1, \dots, x_N \otimes 1\}$ is an orthogonal basis and $\{T^{\mathcal{A}(x_1)}x_1 \otimes 1, \dots, T^{\mathcal{A}(x_N)}x_N \otimes 1\}$ is an orthonormal basis of $CF(H, J; \overline{\Lambda_{\mathbb{K}, \text{univ}}})$.

Let $\{y_1, \dots, y_N\}$ be an orthonormal basis of $CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})$. Then it is related to the orthonormal basis $\{\tilde{x}_1, \dots, \tilde{x}_N\}$ by a matrix $A \in GL(B, \Lambda_{\mathbb{K}, \text{univ}}^0)$ in the sense that $A(\tilde{x}_i) = y_i$, furthermore,

$$\{y_1 \otimes 1, \dots, y_N \otimes 1\} = A\{\tilde{x}_1 \otimes 1, \dots, \tilde{x}_N \otimes 1\}$$

is an orthonormal basis of $CF(H, J; \overline{\Lambda_{\mathbb{K}, \text{univ}}})$.

We note that for any non-trivial $x \in CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})$ we have $\mathcal{A}(d(x)) < \mathcal{A}(x)$ where d is the differential of the Floer chain complex. Then, for a constant c that is not a critical value of action functional $\mathcal{A}_H$, $(CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<c}, d)$ is a chain complex where $CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<c} := \mathcal{A}^{-1}(-\infty, c)$ and denote by $HF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<c}$ the homology of it. We define the Floer complex in the action window $[a, b)$ as the quotient complex

$$CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{[a, b)} := \frac{CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<b}}{CF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<a}}$$

and denote by $HF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{[a, b)}$ the Floer homology of this quotient complex.

Example 2.3.15 (Non-Archimedean filtration on quantum homology). For each element $\sum f_i \alpha_i$ where $f_i \in \Lambda_{\mathbb{Z}, \text{univ}}$ and $\alpha_i \in H_*(M, \mathbb{K})$ define the filtration $l : QH(M) \rightarrow \mathbb{R} \cup \{-\infty\}$ to be $l(\sum f_i \alpha_i) = \max\{-\nu(f_i)\}$. Now, as in [PSS17][She22], for each element $\alpha \in QH(M, \Lambda_{\mathbb{K}, \text{univ}})$, we have a map

$$\alpha * : HF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<c} \rightarrow HF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<c+l(\alpha)}$$

defined by counting negative g -gradient trajectories $\gamma : (-\infty, 0] \rightarrow M$ of a Morse function f on M , for a Morse-Smale pair (f, g) , asymptotic to critical points of f as $s \rightarrow -\infty$, and with $\gamma(0)$ incident to Floer cylinders $u : \mathbb{R} \times S^1 \rightarrow M$ at $u(0, 0)$. This construction is reminiscent of the quantum cap product as in [PSS96][Schoo][Seio2][Flo89].

2.3.4 Spectral invariants

Definition 2.3.16. The *spectral invariant* of a nontrivial element $\alpha \in QH(M, \Lambda_{\mathbb{K}, \text{univ}})$ is defined as

$$c(\alpha, H) := \inf \{a \in \mathbb{R} \mid \text{PSS}_H(\alpha) \in \text{im}(HF(H, J; \Lambda_{\mathbb{K}, \text{univ}})^{<a} \rightarrow HF(H, J; \Lambda_{\mathbb{K}, \text{univ}}))\}$$

By [BC09], spectral invariants do not change under extension of coefficients. Spectral invariants enjoy a wealth of useful properties [Schoo] [Vit92] [Oho5] [Oho6] [Ush10] [Ush08]. We summarize some of their properties.

Proposition 2.3.17 ([AL23], Proposition 2.10). *The spectral invariants satisfy the following:*

(1) *Stability:*

$$\int_0^1 \min(H_t - G_t) f t \leq c(\alpha, H) - c(\alpha, G) \leq \int_0^1 \max(H_t - G_t) dt.$$

(2) *Triangle inequality:*

$$c(\alpha * \beta, H \# G) \leq c(\alpha, H) + c(\beta, G)$$

where

$$H \# G(t, x) := H(t, x) + G(t, (\phi_F^t)^{-1}(x))$$

(3) *Novikov action:*

$$c(\lambda\alpha, H) = c(\alpha, H) - \nu(\lambda)$$

(4) *Non-Archimedean property:*

$$c(\alpha + \beta, H) \leq \max\{c(\alpha, H), c(\beta, H)\}$$

We remark that by the stability property, the spectral invariants are defined for all Hamiltonians and all the above listed properties apply in this generality.

2.3.5 Barcode

In this subsection we follow [She22, Section 4.4.1] in order to define persistence modules and their associated barcodes and discuss the relation between them.

Let $\mathbb{K}$ be a field. Denote by $\text{Vect}_{\mathbb{K}}$ the category of finite dimensional $\mathbb{K}$ -vector spaces and by $(\mathbb{R}, \leq)$ the poset category of $\mathbb{R}$. A persistence module over $\mathbb{K}$ is a functor

$$V : (\mathbb{R}, \leq) \rightarrow \text{Vect}_{\mathbb{K}}$$

The collection of such functors together with their natural transformations form an abelian category $\text{Fun}((\mathbb{R}, \leq), \text{Vect}_{\mathbb{K}})$. We consider a full abelian subcategory

$$\mathbf{pmod} \subset \text{Fun}((\mathbb{R}, \leq), \text{Vect}_{\mathbb{K}})$$

which is defined by requiring that certain technical assumptions are satisfied. The following definition summarizes the data of such a persistence module.

Definition 2.3.18. A persistence module V in $\mathbf{pmod}$ consists of a family

$$\{V^a \in \text{Vect}_{\mathbb{K}}\}_{a \in \mathbb{R}}$$

of vector spaces and $\mathbb{K}$ -linear maps $\pi_V^{a,b} : V^a \rightarrow V^b$ for each $a \leq b$ such that $\pi_V^{a,a} = \text{id}_{V^a}$ and $\pi_V^{b,c} \circ \pi_V^{a,b} = \pi_V^{a,c}$ for all $a \leq b \leq c$. Furthermore, we require them to satisfy the following:

- (1) Support: $V^a = 0$ for all $a \ll 0$.
- (2) Finiteness: there exist a finite subset $S \subset \mathbb{R}$ such that for all a, b in the same connected components of $\mathbb{R} \setminus S$, the map $\pi_V^{a,b}$ is an isomorphism.

- (3) Continuity: for every two consecutive elements $s < s'$ of S , and any $a \in (s, s')$, the map $\pi_V^{a,s'}$ is an isomorphism.

We define $V^\infty = \lim_{a \rightarrow \infty} V^a$.

The normal form theorem [ZCo5][Cra15] implies that the isomorphism classes of a persistence module $V \in \mathbf{pmod}$ is determined by its *barcode*, that is, a multiset $\mathcal{B}(V) = \{(I_k, m_k)\}_{1 \leq k \leq N}$ of intervals $I_k \subset \mathbb{R}$ with multiplicities $m_k \in \mathbb{Z}_{>0}$. The intervals are of two types, $K = K(V)$ of them are finite, $I_k = (a_k, b_k]$, and $B = B(V) = N - K$ are infinite, $I_k = (a_k, \infty)$. The intervals are called bars and the bar-lengths are defined as $|I_k| = b_k - a_k$ in the finite case, and $|I_k| = +\infty$ otherwise.

The isometry theorem [Cha+16][BL15][Cha+09][CEHo6] shows that the barcode assignment map

$$\begin{aligned} \mathcal{B} : (\mathbf{pmod}, d_{inter}) &\rightarrow (\mathbf{barcodes}, d_{bottle}) \\ V &\mapsto \mathcal{B}(V) \end{aligned}$$

is an isometry. The *interleaving distance* is defined by setting

$$d_{inter}(V, W) = \inf\{\delta \geq 0 \mid \exists \delta - \text{interleaving}, f \in \text{hom}(V, W[\delta]), g \in \text{hom}(W, V[\delta])\},$$

where for $V \in \mathbf{pmod}$ and $c \in \mathbb{R}$, $V[c] \in \mathbf{pmod}$ is given by pre-composing with the functor $T_c : (\mathbb{R}, \leq) \rightarrow (\mathbb{R}, \leq), t \mapsto t + c$. We say that a pair $f \in \text{hom}(V, W[c]), g \in \text{hom}(W, V[c])$ is a *c-interleaving* if

$$g[c] \circ f = sh_{2\delta, V}, \quad f[c] \circ g = sh_{2\delta, W}$$

where for $c \geq 0$, $sh_{c, V} \in \text{hom}(V, V[c])$ is the natural transformation $id_{\mathbb{R}, \leq} \rightarrow T_c$. Note that, $d_{inter}(V, W) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, and it is finite if and only if $V^\infty \cong W^\infty$.

The *bottleneck distance* is defined as

$$d_{bottle}(\mathcal{B}, \mathcal{C}) = \inf\{\delta > 0 \mid \exists \delta - \text{matching between } \mathcal{B} \text{ and } \mathcal{C}\}$$

where a δ -matching between $\mathcal{B}, \mathcal{C}$ is defined as bijection $\sigma : \mathcal{B}^{2\delta} \rightarrow \mathcal{C}^{2\delta}$ between the sub-multisets $\mathcal{B}^{2\delta} \subset \mathcal{B}, \mathcal{C}^{2\delta} \subset \mathcal{C}$, which contains the bars of $\mathcal{B}, \mathcal{C}$, respectively, with bar-length greater than 2δ , such that if $\sigma((a, b]) = (c, d]$, then $\max\{|a - c|, |b - d|\} \leq \delta$. We have that $d_{bottle}(\mathcal{B}, \mathcal{C}) \in \mathbb{R} \cup \{\infty\}$, with it being finite if and only if $B(\mathcal{B}) = B(\mathcal{C})$.

Note that for each $c \in \mathbb{R}$ there is an isometry given by sending a barcode $\mathcal{B} = \{(I_k, m_k)\}$ to $\mathcal{B}[c] = \{(I_k - c, m_k)\}$. We can therefore consider the

quotient space $(\mathbf{barcodes}', d'_{bottle})$ by this isometric $\mathbb{R}$ -action, where

$$d'_{bottle}([\mathcal{B}], [\mathcal{C}]) = \inf_{c \in \mathbb{R}} d_{bottle}(\mathcal{B}, \mathcal{C}[c])$$

We observe that bar-lengths are well-defined in the quotient.

Following the discussion in [She22, Section 4.4], we describe two alternative descriptions of the bar-length spectrum, which coincide in the semipositive setting.

Consider the filtered Floer chain complex $(CF(H, J; \Lambda), d, \mathcal{A})$, where Λ is one of the following $\Lambda_{\mathbb{K}}, \overline{\Lambda_{\mathbb{K}, univ}}$, $\mathcal{A}$ is the non-Archimedean filtration, and d is the Floer differential. Then, by [UZ16], the complex $(CF(H, J; \Lambda), d)$ admits an orthogonal basis

$$E = \{\xi_1, \dots, \xi_B, \eta_1, \dots, \eta_K, \zeta_1, \dots, \zeta_K\}$$

such that $d\xi_j = 0$ for all $j \in \{1, \dots, B\}$, and $d\zeta_j = \eta_j$ for all $j \in \{1, \dots, K\}$. The lengths of the finite bars are given by

$$\beta_j = \beta_j(\phi_H^1, \mathbb{K}) = \mathcal{A}(\zeta_j) - \mathcal{A}(\eta_j)$$

which we can assume to satisfy $\beta_1 \leq \dots \leq \beta_K$. The length of the largest finite bar, is the boundary depth introduced by Usher [Ush11a][Ush13], and denoted by $\beta(\phi_H^1, \mathbb{K})$. There are B infinite bar-lengths corresponding to each ξ_i . This description yields the identity $N = B + 2K$, where N , B , and K can be computed by $N = \dim_{\Lambda} CF(H, J; \Lambda)$, $B = \dim_{\Lambda} HF(H, J; \Lambda)$, and $K = \dim_{\Lambda} \text{im}(d)$. We denote by

$$\beta_{tot}(\phi_H^1, \mathbb{K}) = \beta_1(\phi_H^1, \mathbb{K}) + \dots + \beta_K(\phi_H^1, \mathbb{K})$$

the total bar-length associated to the barcode.

Following [Fuk+13], the Floer differential d in the orthonormal basis has coefficients in Λ^0 . Therefore, one defines a Floer complex $(CF(H, J; \Lambda), d)$ whose homology is a finitely generated Λ^0 -module, and is therefore of the form $\mathcal{F} \oplus \mathcal{T}$, where $\mathcal{F}$ is a free Λ^0 -module and $\mathcal{T}$ is a torsion Λ^0 -module. The bar-lengths are given by the isomorphism

$$\mathcal{T} \cong \bigoplus_{1 \leq j \leq k} \Lambda^0 / (T^{\beta_j})$$

Combining the ideas in the proof of [She22, Lemma 16] and combining with Proposition 2.1.6 we show that the bar-length spectrum over $\mathbb{Z}_p$ coincide with that over $\mathbb{Q}$ for a sufficiently large p .

Lemma 2.3.19 ([AL23], Lemma 2.16). *Let p be sufficiently large and ϕ be a Hamiltonian diffeomorphism. The the bar-length spectrum*

$$0 < \beta_1(\phi, \mathbb{Z}_p) \leq \cdots \leq \beta_{K(\phi, \mathbb{Z}_p)}(\phi, \mathbb{Z}_p)$$

over $\mathbb{Z}_p$ coincides with the bar-length spectrum

$$0 < \beta_1(\phi, \mathbb{Q}) \leq \cdots \leq \beta_{K(\phi, \mathbb{Q})}(\phi, \mathbb{Q})$$

over $\mathbb{Q}$. In particular $\beta(\phi, \mathbb{Z}_p) = \beta(\phi, \mathbb{Q})$.

Proof. Let $\{\xi_1, \dots, \xi_B, \eta_1, \dots, \eta_K, \zeta_1, \dots, \zeta_K\}$ be an orthonormal singular value decomposition of $CF(H, J; \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}}))$ satisfying $d\xi_i = 0$ for all $i \in \{1, \dots, B\}$ and $d\zeta_j = T^{\beta_j} \eta_j$ for all $j \in \{1, \dots, K\}$, where β_j is the j -th bar-length in the spectrum. Note that there is a canonical orthonormal basis $\{[\tilde{x}_1, \tilde{u}_1], \dots, [\tilde{x}_N, \tilde{u}_N]\}$ where, $[\tilde{x}_i, \tilde{u}_i] = T^{\mathcal{A}([x_i, u_i])}[x_i, u_i]$ for all i , and $\{[x_i, u_i]\}_{i=1}^N$ is the collection of contractible fixed points of ϕ . These two orthonormal basis will be related by a matrix $Q \in GL(N, \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}}))$ whose coefficients have non-negative valuation, in particular, it is filtration-preserving. Since Q has only finitely many coefficients, Proposition 2.1.6 implies that for a sufficiently large p , it is possible to reduce Q to a matrix $[Q]_p \in GL(N, \Lambda_{\mathbb{Z}_p, \text{univ}})$, i.e. $[\det Q]_p \neq 0$. We can obtain a singular value decomposition $\{[\xi_1]_p, \dots, [\xi_B]_p, [\eta_1]_p, \dots, [\eta_K]_p, [\zeta_1]_p, \dots, [\zeta_K]_p\}$ of $CF(H, J; \Lambda_{\mathbb{Z}_p, \text{univ}})$, satisfying the same relations as before, by applying $[Q]_p$ to the canonical orthonormal basis given by the contractible fixed points of ϕ . In particular, it will have the same bar-length spectrum. On the other hand, $\{\xi_1 \otimes 1, \dots, \xi_B \otimes 1, \eta_1 \otimes 1, \dots, \eta_K \otimes 1, \zeta_1 \otimes 1, \dots, \zeta_K \otimes 1\}$ is an orthogonal singular value decomposition of $CF(H, J; \Lambda_{\mathbb{Q}, \text{univ}})$ with the same bar-length spectrum. $\square$

2.4 Upper Bound of Boundary Depth

Recall $\{\bar{e}_i = \sum_{j=0}^n \bar{k}_{ij} h_j\}_{i=1}^m$ are the idempotents of $QH_{ev}(M, \overline{\text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})})$ and $\{[\bar{e}_i]_p = \sum_{j=0}^n [\bar{k}_{ij}]_p h_j\}_{i=1}^m$ are the idempotents of $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ where $\bar{k}_{ij} \in \text{Frac}(\Lambda_{\mathbb{Z}, \text{univ}})(\alpha)$ and $[\bar{k}_{ij}]_p \in \Lambda_{\mathbb{Z}_p, \text{univ}}(\gamma)$. Suppose $\bar{k}_{ij} =$

$\sum_{s=0}^N b_{ijs} \alpha^s$. Then $[\bar{k}_{ij}]_p = \sum_{s=0}^N [b_{ijs}]_p \gamma^s$. Denote by $\Xi = \max_{i,j,s} \{|b_{ijs}|\}$. Thus

$$|[\bar{k}_{ij}]_p| \leq \max_{0 \leq s \leq N} \{|[b_{ijs}]_p| \gamma^s\} = \max_{0 \leq s \leq N} \{|[b_{ijs}]_p| |\gamma^s|\} \leq \Xi (\max\{1, |f(0)|\})^N.$$

Then

$$\nu([\bar{k}_{ij}]_p) \geq -\ln(\Xi (\max\{1, |f(0)|\})^N).$$

Thus

$$l([\bar{e}_i]_p) = \max_{0 \leq j \leq n} \{-\nu([\bar{k}_{ij}]_p)\} \leq \ln(\Xi (\max\{1, |f(0)|\})^N).$$

Now we have the following proposition.

Proposition 2.4.1 ([AL23], Proposition 4.4). *There is a number δ , independent of p , such that $l([\bar{e}_i]_p) \leq \delta$ for any $i = 1, \dots, m$.*

Given a Hamiltonian $H : S^1 \times M \rightarrow \mathbb{R}$, define

$$\begin{aligned} \bar{H} : S^1 \times M &\rightarrow \mathbb{R} \\ (t, x) &\mapsto -H(t, \phi_H^t(x)) \end{aligned}$$

Then $\phi_{\bar{H}}^t = (\phi_H^t)^{-1}$.

Define a bilinear pairing

$$\begin{aligned} \Delta : CF_*(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \times CF_*(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) &\rightarrow \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}} \\ (\sum a_i [x_i, u_i], \sum b_j [\bar{x}_j, \bar{u}_j]) &\mapsto \sum a_i b_i \end{aligned}$$

where the sums are finite and, for all k , $\bar{x}_k(t) = x_k(1 - t)$

Lemma 2.4.2 ([AL23], Lemma 2.11). *The bilinear pairing Δ is nondegenerate.*

Proof. Suppose

$$\Delta(\sum a_i [x_i, u_i], \cdot) = 0.$$

Then for every $[\bar{x}_i, \bar{u}_i]$,

$$\Delta(\sum a_i [x_i, u_i], [\bar{x}_i, \bar{u}_i]) = 0.$$

On the other hand,

$$\Delta\left(\sum a_i[x_i, u_i], [\bar{x}_i, \bar{u}_i]\right) = a_i.$$

Thus $a_i = 0$ for each i , i.e. $\sum a_i[x_i, u_i] = 0$. Similarly, if $\Delta(\cdot, \sum b_j[\bar{x}_j, \bar{u}_j]) = 0$, then $\sum b_j[\bar{x}_j, \bar{u}_j] = 0$. $\square$

Lemma 2.4.3 ([AL23], Lemma 2.12). $\nu\left(\Delta|_{CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})^{<\alpha} \times CF(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})^{-\alpha}}\right) > 0$

Proof. Suppose that

$$\left(\sum a_i[x_i, u_i], \sum b_i[\bar{x}_i, \bar{u}_i]\right) \in CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})^{<\alpha} \times CF(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})^{-\alpha}$$

Then

$$\begin{aligned} \mathcal{A}\left(\sum a_i[x_i, u_i]\right) &= \max\{\mathcal{A}(a_i[x_i, u_i])\} \\ &= \max\{\mathcal{A}_H([x_i, u_i]) - \nu(a_i)\} < \alpha. \end{aligned}$$

Thus for each i , $\mathcal{A}_H([x_i, u_i]) - \nu(a_i) < \alpha$, i.e. $\nu(a_i) > \mathcal{A}_H([x_i, u_i]) - \alpha$. Also,

$$\begin{aligned} \mathcal{A}\left(\sum b_i[\bar{x}_i, \bar{u}_i]\right) &= \max\{\mathcal{A}(b_i[\bar{x}_i, \bar{u}_i])\} \\ &= \max\{\mathcal{A}_H([\bar{x}_i, \bar{u}_i]) - \nu(b_i)\} \\ &= \max\{-\mathcal{A}_H([x_i, u_i]) - \nu(b_i)\} < -\alpha. \end{aligned}$$

Thus for each i , $-\mathcal{A}_H([x_i, u_i]) - \nu(b_i) < -\alpha$, i.e. $\nu(b_i) > -\mathcal{A}_H([x_i, u_i]) + \alpha$. Then

$$\begin{aligned} \nu(\Delta(\sum a_i[x_i, u_i], \sum b_i[\bar{x}_i, \bar{u}_i])) &= \nu(\sum a_i b_i) \\ &\geq \min\{\nu(a_i b_i)\} \\ &= \min\{\nu(a_i) + \nu(b_i)\} \\ &> \mathcal{A}_H([x_i, u_i]) - \alpha - \mathcal{A}_H([x_i, u_i]) + \alpha \\ &= 0 \end{aligned}$$

$\square$

Lemma 2.4.4 ([AL23], Lemma 2.13). $\Delta(d(a), b) = \pm\Delta(a, d(b))$ for $a \in CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$ and $b \in CF(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$.

Proof. Denote a basis of $CF_{k+1}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$ by $\{[x_i, u_i]\}$ and a basis of $CF_k(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$ by $\{[y_j, v_j]\}$. Then $\{[\bar{x}_i, \bar{u}_i]\}$ is a basis of $CF_{-k-1}(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$

and $\{[\bar{y}_j, \bar{v}_j]\}$ is a basis of $CF_{-k}(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$. Suppose $d([x_i, u_i]) = \sum a_j [y_j, v_j]$ and $d([\bar{y}_j, \bar{v}_j]) = \sum b_i [\bar{x}_i, \bar{u}_i]$. Then

$$\Delta(d([x_i, u_i]), [\bar{y}_j, \bar{v}_j]) = \Delta(\sum a_j [y_j, v_j], [\bar{y}_j, \bar{v}_j]) = a_j.$$

On the other hand

$$\Delta([x_i, u_i], d([\bar{y}_j, \bar{v}_j])) = \Delta([x_i, u_i], \sum b_i [\bar{x}_i, \bar{u}_i]) = b_i.$$

By definition, a_j is the number of the Floer trajectories connecting $[x_i, u_i]$ and $[y_j, v_j]$ and b_i is the number of the Floer trajectories connecting $[\bar{y}_j, \bar{v}_j]$ and $[\bar{x}_i, \bar{u}_i]$. Thus $a_j = b_i$, i.e.

$$\Delta(d([x_i, u_i]), [y_j, v_j]) = \Delta([x_i, u_i], d([y_j, v_j])).$$

Since Δ is bilinear, then

$$\Delta(d(a), b) = \Delta(a, d(b))$$

for any $a \in CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ and $b \in CF(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$. $\square$

Thus there is an induced pairing on homology.

$$\Delta : HF_*(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \times HF_*(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}$$

The proof of the following proposition is from Corollary 1.4 in [Ushio].

Proposition 2.4.5 ([AL23], Proposition 2.14). *Let $a \in QH(M, \overline{\Lambda_{\mathbb{Z}, \text{univ}}})$ be nontrivial. Then*

$$c(a, H) = -\inf\{c(b, \bar{H}) \mid b \in QH(M, \overline{\Lambda_{\mathbb{Z}, \text{univ}}}), \nu(\Delta(PSS_H(a), PSS_{\bar{H}}(b))) \leq 0\}.$$

Proof. First we show that

$$-c(a, H) \geq \inf\{c(b, \bar{H}) \mid b \in QH(M, \overline{\Lambda_{\mathbb{Z}, \text{univ}}}), \nu(\Delta(PSS_H(a), PSS_{\bar{H}}(b))) \leq 0\}.$$

Suppose that $\alpha < c(a, H)$. We have a short exact sequence of chain complexes

$$0 \rightarrow CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{<\alpha} \rightarrow CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{[\alpha, \infty)} \rightarrow 0$$

inducing an exact sequence

$$HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{<\alpha} \xrightarrow{i_\alpha} HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \xrightarrow{\pi_\alpha} HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{[\alpha, \infty)}$$

The fact that $\alpha < c(a, H)$ means that $\text{PSS}_H(a)$ is not represented by any chains of filtration level at most α , so that $\text{PSS}_H(a) \notin \text{im}(i_\alpha)$. Thus $\pi_\alpha(\text{PSS}_H(a)) \neq 0$.

Claim 2.4.6. *Fix a representative $\tilde{a}$ of $\text{PSS}_H(a)$. There is $b \in QH(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ such that*

$$\nu(\Delta(\text{PSS}_H(a), \text{PSS}_{\bar{H}}(b))) \leq 0$$

and

$$\mathcal{A}(\tilde{b}) = -\mathcal{A}(\tilde{a}) \leq -\alpha$$

where $\tilde{b}$ is a representatives of $\text{PSS}_{\bar{H}}(b)$ respectively.

Proof. Consider the dual vector space $CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^*$ of $CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$. Let $\{\xi_1, \dots, \xi_B, \eta_1, \dots, \eta_K, \zeta_1, \dots, \zeta_K\}$ be a singular value decomposition for the complex $CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$. We recall from [UZ16, Proposition 2.20] that there is an $\mathcal{A}^*$ -orthogonal dual basis $\{\xi_1^*, \dots, \xi_B^*, \eta_1^*, \dots, \eta_K^*, \zeta_1^*, \dots, \zeta_K^*\}$ of $CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^*$ such that $\mathcal{A}^*(\xi_i^*) = -\mathcal{A}(\xi_i)$, $\mathcal{A}^*(\eta_i^*) = -\mathcal{A}(\eta_i)$, and $\mathcal{A}^*(\zeta_i^*) = -\mathcal{A}(\zeta_i)$, where

$$\mathcal{A}^*(f^*) = \sup\{-\mathcal{A}(\theta) - \nu(f^*(\theta)) \mid 0 \neq \theta \in CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})\}$$

If $f = \sum a_i[x_i, u_i]$ is an element in $CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$, then we denote by $\bar{f}$ the element $\sum a_i[\bar{x}_i, \bar{u}_i]$ in $CF(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ and f^* the dual element of f in $CF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^*$. Then

$$\begin{aligned} \mathcal{A}(\bar{f}) &= \max\{\mathcal{A}_{\bar{H}}([\bar{x}_i, \bar{u}_i]) - \nu(a_i)\} \\ &= \max\{-\mathcal{A}_H([x_i, u_i]) - \nu(a_i)\} \\ &= \max\{-\mathcal{A}_H([x_i, u_i]) - \nu(f^*([x_i, u_i]))\} \\ &\leq \mathcal{A}^*(f^*) \end{aligned}$$

Next we check $d(\bar{\xi}_i) = 0$. Assume

$$d(\bar{\xi}_i) = \sum a_{ij}\bar{\xi}_j + \sum b_{ij}\bar{\eta}_j + \sum c_{ij}\bar{\zeta}_j$$

Then $a_{ij} = \Delta(\xi_j, d(\bar{\xi}_i)) = \Delta(d(\xi_j), \bar{\xi}_i) = \Delta(0, \bar{\xi}_i) = 0$. Similarly, $b_{ij} = c_{ij} = 0$. Thus $d(\bar{\xi}_i) = 0$

Assume $\tilde{a} = \sum a_i \xi_i$. Then $\mathcal{A}(\tilde{a}) = \max\{\mathcal{A}(\xi_i) - \nu(a_i)\} = \mathcal{A}(\xi_k) - \nu(a_k)$ for some k . Define $\tilde{b} := a_k^{-1} \bar{\xi}_k$. Then

$$\Delta(\tilde{a}, \tilde{b}) = a_k a_k^{-1} = 1$$

We have $\nu(\Delta(\tilde{a}, \tilde{b})) = 0$. It also means that $[\tilde{b}] \neq 0$ in $HF(\bar{H}, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$. In addition, $\mathcal{A}(\tilde{b}) = \mathcal{A}(\bar{\xi}_k) - \nu(a_k^{-1}) \leq \mathcal{A}^*(\xi_k^*) + \nu(a_k) = -\mathcal{A}(\xi_k) + \nu(a_k) = -\mathcal{A}(\tilde{a})$. Thus we can take $b := \text{PSS}_{\bar{H}}^{-1}([\tilde{b}])$ $\square$

Let $\alpha = c(a, H) - \epsilon$. Then

$$\mathcal{A}(\tilde{b}) \leq -c(a, H) + \epsilon$$

Take $\epsilon \rightarrow 0$. Then

$$\mathcal{A}(\tilde{b}) \leq -c(a, H)$$

We have

$$\inf\{c(b, \bar{H}) \mid b \in QH(M, \overline{\Lambda_{\mathbb{K}, univ}}), \nu(\Delta(\text{PSS}_H(a), \text{PSS}_{\bar{H}}(b))) \leq 0\} \leq \mathcal{A}(\tilde{b}) \leq -c(a, H)$$

Next we show that

$$-c(a, H) \leq \inf\{c(b, \bar{H}) \mid b \in QH(M, \overline{\Lambda_{\mathbb{K}, univ}}), \nu(\Delta(\text{PSS}_H(a), \text{PSS}_{\bar{H}}(b))) \leq 0\}.$$

Suppose that $\alpha > c(a, H)$. Thus there must be some cycle $c \in CF(\phi, \overline{\Lambda_{\mathbb{Z}_p, univ}})^{<\alpha}$ representing the class $\text{PSS}_H(a)$. If $b \in QH(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ is an arbitrary class satisfying $\nu(\Delta(\text{PSS}_H(a), \text{PSS}_{\bar{H}}(b))) \leq 0$, then by the definition of Δ it must hold that every representation $d \in CF(\bar{H}, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ of the class $\text{PSS}_{\bar{H}}(b)$ satisfies $\nu(\Delta(c, d)) \leq 0$. By Lemma 2.4.3, this can only be true if no representation d of b belongs to $CF(\bar{H}, \overline{\Lambda_{\mathbb{Z}_p, univ}})^{-\alpha}$, which amounts to the statement that $c(b, \bar{H}) \geq -\alpha$. Note that b was an arbitrary class with $\nu(\Delta(\text{PSS}_H(a), \text{PSS}_{\bar{H}}(b))) \leq 0$, while α was an arbitrary number exceeding $c(a, \phi)$, and so we obtain that

$$-c(a, H) \leq \inf\{c(b, \bar{H}) \mid b \in QH(M, \overline{\Lambda_{\mathbb{K}, univ}}), \nu(\Delta(\text{PSS}_H(a), \text{PSS}_{\bar{H}}(b))) \leq 0\}.$$

$\square$

On the other hand, we can define a pairing on the even quantum homology $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$.

$$\begin{aligned} \tilde{\Delta} : QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \times QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}}) &\rightarrow \overline{\Lambda_{\mathbb{Z}_p, univ}} \\ \left(\sum_{i=0}^{2n} a_i h_{2i}, \sum_{j=0}^{2n} b_j h_{2j} \right) &\mapsto \sum_{i+j=n} a_i b_j (h_{2i} \circ h_{2j}) \end{aligned}$$

where $a_i, b_i \in \overline{\Lambda_{\mathbb{Z}_p, univ}}$ and $h_{2i} \in H_{2i}(M, \mathbb{Z}_p)$ for $i = 0, \dots, n$.

The following result is from [EP03, Section 2.6.8] and [Ush23, Proposition 7.7]

Lemma 2.4.7. *Let $a, b \in QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$. Then*

$$\Delta(PSS_H(a), PSS_{\bar{H}}(b)) = \tilde{\Delta}(a, b) \quad (2.4.1)$$

As in [EP03, Section 2.3], we have the following lemma.

Lemma 2.4.8. $\tilde{\Delta}(a, b) = \tilde{\Delta}(a * b, [M])$. *In particular, $\Delta(PSS_H(a), PSS_{\bar{H}}(b)) = \Delta(PSS_H(a * b), PSS_{\bar{H}}([M]))$*

Definition 2.4.9. Suppose $QH_{ev}(M, \mathbb{K})$ is semisimple and $E = \{e_1, \dots, e_m\}$ are idempotents. Then define

$$\gamma_{e_j}(H, \mathbb{K}) = c(e_j, H, \mathbb{K}) + c(e_j, \overline{H}, \mathbb{K}),$$

$$\gamma_{e_j}(\phi, \mathbb{K}) = \inf_{\phi_H^1 = \phi} \{\gamma_{e_j}(H, \mathbb{K})\}$$

and

$$\gamma_E(\phi, \mathbb{K}) = \max_{1 \leq i \leq m} \{\gamma_{e_i}(\phi, \mathbb{K})\}.$$

Lemma 2.4.10 ([AL23], Lemma 4.6). *For $\theta \in QH_*(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$,*

$$c(\theta, 0, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \geq 0$$

$$\text{if } \nu(\Delta(PSS_H(\theta), PSS_{\bar{H}}([M]))) \leq 0.$$

Proof. Suppose that $\theta = \sum \theta_i h_i$ with $\theta_i \in \overline{\Lambda_{\mathbb{Z}_p, univ}}$ and $h_i \in H_*(M, \mathbb{Z}_p)$. Then

$$l(\theta) = \max\{-\nu(\theta_i)\} = -\min\{\nu(\theta_i)\}.$$

Observe that $\Delta(\text{PSS}_H(\theta), \text{PSS}_{\bar{H}}([M])) = \theta_j$ with $h_j = [pt]$. Then

$$\nu(\Delta(\text{PSS}_H(\theta), \text{PSS}_{\bar{H}}([M]))) = \nu(\theta_j) \geq -l(\theta) = -c(\theta, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}).$$

Since $\nu(\Delta(\text{PSS}_H(\theta), \text{PSS}_{\bar{H}}([M]))) \leq 0$, then $c(\theta, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \geq 0$. $\square$

Lemma 2.4.11 ([AL23], Lemma 4.7). *For $\theta \in QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ and $[\bar{e}_i]_p * \theta \neq 0$, we have that*

$$c(([\bar{e}_i]_p * \theta)^{-1}, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + c([\bar{e}_i]_p * \theta, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) = 2c([\bar{e}_i]_p, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}).$$

where the inversion is taken in the field $[\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$.

Proof. Because $[\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \cong \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}$, then there is $\delta \in \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}$ such that $[\bar{e}_i]_p * \theta = \delta[\bar{e}_i]_p$. Thus $([\bar{e}_i]_p * \theta)^{-1} = \delta^{-1}[\bar{e}_i]_p$.

We obtain that

$$\begin{aligned} & c(([\bar{e}_i]_p * \theta)^{-1}, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + c([\bar{e}_i]_p * \theta, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \\ &= c(\delta^{-1}[\bar{e}_i]_p, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + c(\delta[\bar{e}_i]_p, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \\ &= c([\bar{e}_i]_p, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) - \nu(\delta^{-1}) + c([\bar{e}_i]_p, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) - \nu(\delta) \\ &= 2c([\bar{e}_i]_p, 0, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}). \end{aligned}$$

$\square$

Proposition 2.4.12 ([AL23], Proposition 4.8). *For sufficiently large p , let $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ be semisimple, and $E_p = \{[\bar{e}_1]_p, \dots, [\bar{e}_m]_p\}$ be the idempotents such that*

$$QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) = \bigoplus_{i=1}^m [\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$$

and $[\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \cong \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}$. Then there is a constant D independent of p such that $\gamma_{E_p}(\phi) \leq D$ for each $\phi \in \text{Ham}(M, \omega)$

Proof. First by Proposition 2.4.5,

$$\begin{aligned} & c([\bar{e}_i]_p, H) + c([\bar{e}_i]_p, \bar{H}) \\ &= c([\bar{e}_i]_p, H) - \inf\{c(b, H) \mid b \in QH(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}), \nu(\Delta(\text{PSS}_H(b), \text{PSS}_{\bar{H}}([\bar{e}_i]_p))) \leq 0\} \end{aligned}$$

By Lemma 2.4.8

$$\begin{aligned}
& c([\bar{e}_i]_p, H) + c([\bar{e}_i]_p, \bar{H}) \\
&= c([\bar{e}_i]_p, H) - \inf \left\{ c(b, H) \left| \begin{array}{c} b \in QH(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}), \\ \nu(\Delta(\text{PSS}_H([M]), \text{PSS}_{\bar{H}}([\bar{e}_i]_p * b))) \leq 0 \end{array} \right. \right\} \\
&= \sup \left\{ c([\bar{e}_i]_p, H) - c(b, H) \left| \begin{array}{c} b \in QH(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}), \\ \nu(\Delta(\text{PSS}_H([M]), \text{PSS}_{\bar{H}}([\bar{e}_i]_p * b))) \leq 0 \end{array} \right. \right\}
\end{aligned}$$

By the triangle inequality of spectral invariants we have

$$\begin{aligned}
c([\bar{e}_i]_p, H) &\leq c([\bar{e}_i]_p * b, H) + c(([\bar{e}_i]_p * b)^{-1}, 0) \\
&\leq c([\bar{e}_i]_p, 0) + c(b, H) + c(([\bar{e}_i]_p * b)^{-1}, 0)
\end{aligned}$$

where the inverse $([\bar{e}_i]_p * b)^{-1}$ is taken in the field $[\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$. Since $c([\bar{e}_i]_p, 0) = l([\bar{e}_i]_p)$, then, by Proposition 2.4.1,

$$c([\bar{e}_i]_p, H) - c(b, H) \leq c(([\bar{e}_i]_p * b)^{-1}, 0) + \delta$$

. By Lemma 2.4.10,

$$c([\bar{e}_i]_p, H) - c(b, H) \leq c(([\bar{e}_i]_p * b)^{-1}, 0) + c([\bar{e}_i]_p * b, 0) + \delta$$

By Lemma 2.4.11

$$\begin{aligned}
c([\bar{e}_i]_p, H) - c(b, H) &\leq 2c([\bar{e}_i]_p, 0) + \delta \\
&= 2l([\bar{e}_i]_p) + \delta \\
&\leq 3\delta
\end{aligned}$$

□

The proof of the following proposition is the same as that of Proposition 12 in [She22].

Proposition 2.4.13 ([AL23], Proposition 4.9). *For sufficiently large p , let $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ be semisimple, and $E_p = \{[\bar{e}_1]_p, \dots, [\bar{e}_m]_p\}$ be the idempotents such that*

$$QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) = \bigoplus_{i=1}^m [\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$$

and $[\bar{e}_i]_p * QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \cong \overline{\Lambda_{\mathbb{Z}_p, univ}}$. Then

$$|\beta(\phi, \overline{\Lambda_{\mathbb{Z}_p, univ}}) - \beta(\psi, \overline{\Lambda_{\mathbb{Z}_p, univ}})| \leq \gamma_{E_p}(\phi\psi^{-1}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \delta$$

Proof. Given that $QH_{ev}(M, \Lambda_{\mathbb{K}, univ})$ is semisimple, we have shown that $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ is semisimple and $\{[\bar{e}_i]_p\}_{i=1}^m$ are the idempotents generating $QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$.

For any $a \in QH_{ev}(M, \overline{\Lambda_{\mathbb{Z}_p, univ}})$, there is a morphism by taking product with a

$$a* : HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})[l(a)]$$

Consider the composition property,

$$(a*) \circ (b*) : HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})[l(a) + l(b)]$$

Define $\delta(a, b) := l(a) + l(b) - l(a * b)$. If $\delta(a, b) \geq 0$ and $a * b \neq 0$, then

$$(a*) \circ (b*) = sh_{\delta(a, b)} \circ ((a * b)*)$$

If $a * b = 0$, then

$$(a*) \circ (b*) = 0.$$

Consider the additive property,

$$(a + b)* : HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})[\max\{l(a), l(b)\}]$$

Define $l(a, b) = \max\{l(a), l(b)\}$. If $a + b \neq 0$, then

$$sh_{l(a, b) - l(a + b)} \circ ((a + b)*) = sh_{l(a, b) - l(a)} \circ (a*) + sh_{l(a, b) - l(b)} \circ (b*)$$

If $a + b = 0$, then

$$(a + b)* = 0.$$

Now we consider persistence modules without finiteness conditions. A persistence module $\mathbb{V}$ over a field K is just a collection of functions $t \mapsto V_t$ from the poset category of $\mathbb{R}$ to the category of vector spaces over K .

Lemma 2.4.14. *The persistence modules $HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$ and $\bigoplus_{i=1}^m im([\bar{e}_i]_p*)$ are $l(E_p)$ -interleaved where $l(E_p) = \max_{1 \leq i \leq m} \{l([\bar{e}_i]_p)\}$. In particular, the inter-*

leaving distance between $HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$ and $\bigoplus_{i=1}^m \text{im}([\bar{e}_i]_p *)$ is at most $l(E_p)$

Proof. First we define a morphism

$$f_E : HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow \bigoplus_{i=1}^m \text{im}([\bar{e}_i]_p *) [l(E_p)]$$

as following. Consider the morphism

$$\begin{aligned} \tilde{f}_E : HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) &\rightarrow \bigoplus_{i=1}^m \text{im}([\bar{e}_i]_p *) \\ x &\mapsto ([\bar{e}_1]_p * x, \dots, [\bar{e}_m]_p * x). \end{aligned}$$

Take $f_E = sh_{l(E_p)} \circ \tilde{f}_E$.

Then we define a morphism

$$g_E : \bigoplus_{i=1}^m \text{im}([\bar{e}_i]_p *) \rightarrow HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) [l(E_p)].$$

as following. Let

$$\iota_E : \bigoplus_{i=1}^m \text{im}([\bar{e}_i]_p *) \rightarrow \bigoplus_{i=1}^m HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) [l([\bar{e}_i]_p)]$$

be the inclusion. Then define a shift map

$$\begin{aligned} sh_E : \bigoplus_{i=1}^m HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) [l([\bar{e}_i]_p)] &\rightarrow \bigoplus_{i=1}^m HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) [l(E_p)] \\ (a_1, \dots, a_m) &\mapsto (sh_{l(E_p)-l([\bar{e}_1]_p)}(a_1), \dots, sh_{l(E_p)-l([\bar{e}_m]_p)}(a_m)) \end{aligned}$$

and a sum map

$$\begin{aligned} \Sigma_E : \bigoplus_{i=1}^m HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) [l(E_p)] &\rightarrow HF_{ev}(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) [l(E_p)] \\ (a_1, \dots, a_m) &\mapsto \sum_{i=1}^m a_i \end{aligned}$$

Take $g_E = \Sigma_E \circ sh_E \circ \iota_E$.

Now we can compute

$$\begin{aligned}
g_E[l(E_p)] \circ f_E(x) &= g_E[l(E_p)] (sh_{l(E_p)}([\bar{e}_1]_p * x), \dots, sh_{l(E_p)}([\bar{e}_m]_p * x)) \\
&= \sum_{i=1}^m sh_{l(E_p)-l([\bar{e}_i]_p)} \circ sh_{l(E_p)}([\bar{e}_i]_p * x) \\
&= sh_{2l(E_p)}([M] * x) \\
&= sh_{2l(E_p)}(x).
\end{aligned}$$

Note that $\delta([\bar{e}_i]_p, [\bar{e}_i]_p) = l([\bar{e}_i]_p)$ and then $([\bar{e}_i]_p *) \circ ([\bar{e}_i]_p *) = sh_{l([\bar{e}_i]_p)} \circ ([\bar{e}_i]_p *)$

$$\begin{aligned}
&f_E[l(E_p)] \circ g_E([\bar{e}_1]_p * a_1, \dots, [\bar{e}_m]_p * a_m) \\
&= f_E[l(E_p)] \left(\sum_{i=1}^m sh_{l(E_p)-l([\bar{e}_i]_p)}([\bar{e}_i]_p * a_i) \right) \\
&= f_E[l(E_p)] \left(\sum_{i=1}^m [\bar{e}_i]_p * sh_{l(E_p)-l([\bar{e}_i]_p)}(a_i) \right) \\
&= sh_{l(E_p)} \left([\bar{e}_1]_p * \sum_{i=1}^m [\bar{e}_i]_p * sh_{l(E_p)-l([\bar{e}_i]_p)}(a_i), \dots, [\bar{e}_m]_p * \sum_{i=1}^m [\bar{e}_i]_p * sh_{l(E_p)-l([\bar{e}_i]_p)}(a_i) \right) \\
&= sh_{l(E_p)}([\bar{e}_1]_p * [\bar{e}_1]_p * sh_{l(E_p)-l([\bar{e}_1]_p)}(a_1), \dots, [\bar{e}_m]_p * [\bar{e}_m]_p * sh_{l(E_p)-l([\bar{e}_m]_p)}(a_m)) \\
&= sh_{l(E_p)}(sh_{l([\bar{e}_1]_p)} \circ [\bar{e}_1]_p * sh_{l(E_p)-l([\bar{e}_1]_p)}(a_1), \dots, sh_{l([\bar{e}_m]_p)} \circ [\bar{e}_m]_p * sh_{l(E_p)-l([\bar{e}_m]_p)}(a_m)) \\
&= sh_{l(E_p)}(sh_{l(E_p)}([\bar{e}_1]_p * a_1), \dots, sh_{l(E_p)}([\bar{e}_m]_p * a_m)) \\
&= sh_{2l(E_p)}([\bar{e}_1]_p * a_1, \dots, [\bar{e}_m]_p * a_m)
\end{aligned}$$

□

In great generality, one can define the boundary depth $\beta(\mathbb{V})$ of the persistence module $\mathbb{V}$ as the infimum of all $\lambda \in (0, \infty)$ with the property that, for all $s \in \mathbb{R}$,

$$\ker \left(V_s \rightarrow \varinjlim V_t \right) = \ker (V_s \rightarrow V_{s+\lambda})$$

Note the set on the left hand side is the same as the ascending union $\cup_{t \in [s, \infty)} \ker (V_s \rightarrow V_t)$. Depending on $\mathbb{V}$, there might be no λ with this property, in which case $\beta(\mathbb{V}) = \infty$. However, as in [UZ16], the Floer homology persistence module $HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})$ can easily be checked to have $\beta(\phi_H^1, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ equal to the length of the longest finite-length bar in the barcode (or zero if there are no finite-length bars). Note

also that, for the finite collection of persistence modules $\{\text{im}([\bar{e}_j]_p^*)\}$,

$$\beta(\oplus_j \text{im}([\bar{e}_i]_p^*)) = \max_j \beta(\text{im}([\bar{e}_i]_p^*)).$$

One has the following stability result:

Lemma 2.4.15. *The boundary depth $\beta(\oplus_j \text{im}([\bar{e}_i]_p^*))$ is finite and*

$$|\beta(\phi_H^1, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) - \beta(\oplus_j \text{im}([\bar{e}_i]_p^*))| \leq 2l(E_p)$$

Proof. Let $\beta > \beta(\phi_H^1, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ be arbitrary. Let

$$x \in \ker \left((\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s} \rightarrow \varinjlim (\oplus_j \text{im}([\bar{e}_i]_p^*))^{<t} \right),$$

say $x \mapsto 0$ under

$$(\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s} \rightarrow (\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s+\lambda};$$

it suffices to show that in fact $x \mapsto 0$ under

$$(\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s} \rightarrow (\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s+2l(E_p)+\beta}.$$

Since g_E is a morphism of persistence modules, it follows that $g_E(x) \mapsto 0$ under

$$HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{s+l(E_p)} \rightarrow HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{s+\lambda+l(E_p)}$$

From the definition of $\beta(\phi_H^1, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ we see that $g_E(x) \mapsto 0$ under

$$HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{s+l(E_p)} \rightarrow HF(H, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})^{s+\beta+l(E_p)}.$$

But then $f_E \circ g_E(x) \mapsto 0$ under

$$(\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s+2l(E_p)} \rightarrow (\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s+2l(E_p)+\beta}.$$

So since $f_E \circ g_E(x)$ is the image of x under

$$(\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s} \rightarrow (\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s+2l(E_p)}.$$

it follows that indeed $x \mapsto 0$ under

$$(\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s} \rightarrow (\oplus_j \text{im}([\bar{e}_i]_p^*))^{<s+2l(E_p)+\beta}.$$

□

Thus $\beta(\phi_H^1, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \leq \max \beta(\text{im}([\bar{e}_i]_p *)) + l(E_p)$. Next, we need to bound $\beta(\text{im}([\bar{e}_i]_p *))$

Let F and G be Hamiltonians such that $\phi_F^1 = \phi$ and $\phi_G^1 = \psi$. In the following discussion, F and G are not necessarily nondegenerate. Choose $\epsilon > 0$. Let F_ϵ and G_ϵ be perturbations of F and G respectively such that the following conditions hold.

1. $F_\epsilon, G_\epsilon, G_\epsilon \# \bar{F}_\epsilon$ and $F_\epsilon \# \bar{G}_\epsilon$ are nondegenerate.
2. $\|F_\epsilon - F\|_{C^2}, \|G_\epsilon - G\|_{C^2}, \|G_\epsilon \# \bar{F}_\epsilon - G \# \bar{F}\|_{C^2}$ and $\|F_\epsilon \# \bar{G}_\epsilon - F \# \bar{G}\|_{C^2}$ are far less than ϵ .

For $\epsilon_0 \ll \epsilon$, let $c_{G_\epsilon \# \bar{F}_\epsilon} \in CF_{2n}(G_\epsilon \# \bar{F}_\epsilon, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ be the an element representing $PSS_{G_\epsilon \# \bar{F}_\epsilon}([\bar{e}_i]_p)$ with

$$\mathcal{A}_{G_\epsilon \# \bar{F}_\epsilon}(c_{G_\epsilon \# \bar{F}_\epsilon}) \leq c([\bar{e}_i]_p, G_\epsilon \# \bar{F}_\epsilon, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_0.$$

Let $c_{F_\epsilon \# \bar{G}_\epsilon} \in CF_{2n}(F_\epsilon \# \bar{G}_\epsilon, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ be an element representing $PSS_{F_\epsilon \# \bar{G}_\epsilon}([\bar{e}_i]_p)$ with

$$\mathcal{A}_{F_\epsilon \# \bar{G}_\epsilon}(c_{F_\epsilon \# \bar{G}_\epsilon}) \leq c([\bar{e}_i]_p, F_\epsilon \# \bar{G}_\epsilon, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_0$$

When F (resp. G) is degenerate, $CF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}})$ (resp. $CF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}})$) is defined as the colimit of the chain complex of the perturbed Hamiltonian and almost complex structure. Then there are continuation maps

$$C_F : CF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}})^{<a} \rightarrow CF(F_\epsilon, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})^{<a+\mathcal{E}_+(F_\epsilon-F)}$$

and

$$C_{G_\epsilon} : CF(G_\epsilon, J; \overline{\Lambda_{\mathbb{Z}_p, univ}})^{<a} \rightarrow CF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}})^{<a+\mathcal{E}_+(G-G_\epsilon)}.$$

By taking product with $c_{G_\epsilon \# \bar{F}_\epsilon}$ we can get a chain map

$$\begin{aligned} C_{G_\epsilon \# \bar{F}_\epsilon} : CF(F_\epsilon, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) &\rightarrow CF(G_\epsilon, J; \overline{\Lambda_{\mathbb{Z}_p, univ}}) \\ x &\mapsto c_{G_\epsilon \# \bar{F}_\epsilon} * x \end{aligned}$$

Then define $C_{G \# \bar{F}} = C_{G_\epsilon} \circ C_{G_\epsilon \# \bar{F}_\epsilon} \circ C_F : CF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow CF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}})$. Similarly, one can define $C_{F \# \bar{G}} : CF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow CF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}})$.

We have

$$\begin{aligned}
\mathcal{A}_G(C_{G\#F}(x)) &= \mathcal{A}_{G_\epsilon}(c_{G_\epsilon\#F_\epsilon} * C_F(x)) + \mathcal{E}_+(G - G_\epsilon) \\
&\leq \mathcal{A}_{G_\epsilon\#F_\epsilon}(c_{G_\epsilon\#F_\epsilon}) + \mathcal{A}_{F_\epsilon}(C_F(x)) + \mathcal{E}_+(G - G_\epsilon) \\
&\leq c([\bar{e}_i]_p, G_\epsilon\#F_\epsilon, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \epsilon_0 + \mathcal{A}_{F_\epsilon}(C_F(x)) + \mathcal{E}_+(G - G_\epsilon) \\
&\leq c([\bar{e}_i]_p, G\#F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \mathcal{E}_+(G_\epsilon\#F_\epsilon - G\#F) + \epsilon_0 \\
&\quad + \mathcal{A}_F(x) + \mathcal{E}_+(F_\epsilon - F) + \mathcal{E}_+(G - G_\epsilon) \\
&= c([\bar{e}_i]_p, G\#F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \mathcal{A}_F(x) + \epsilon_1
\end{aligned}$$

Similarly, one can have $\mathcal{A}_F(C_{F\#G}(y)) \leq c([\bar{e}_i]_p, F\#G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \mathcal{A}_G(y) + \epsilon_1$. Thus there are induced morphisms

$$[C_{G\#F}] : HF_{ev}(F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow HF_{ev}(G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})[c([\bar{e}_i]_p, G\#F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \epsilon_1]$$

$$[C_{F\#G}] : HF_{ev}(G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow HF_{ev}(F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})[c([\bar{e}_i]_p, F\#G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \epsilon_1]$$

Denote by $[\bar{e}_i]_{p,0} \in CF(f, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$ a chain in the Morse complex computing $QH(M, \Lambda_{\mathbb{Z}_p, \text{univ}})$ representing $[\bar{e}_i]_p$ with $\mathcal{A}([\bar{e}_i]_{p,0}) = l([\bar{e}_i]_p)$.

Lemma 2.4.16. *There exist chain homotopies*

$$R_F : CF(F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow CF(F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})[1]$$

and

$$R_G : CF(G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow CF(G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})[1]$$

of degree 1, such that

$$C_{F\#G} \circ C_{G\#F} = (*[\bar{e}_i]_{p,0}) + dR_F - R_F d$$

$$C_{G\#F} \circ C_{F\#G} = (*[\bar{e}_i]_{p,0}) + dR_G - R_G d$$

$$\mathcal{A}_F(R_F(x)) \leq c([\bar{e}_i]_p, F\#G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + c([\bar{e}_i]_p, G\#F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \epsilon_2 + \mathcal{A}_F(x)$$

$$\mathcal{A}_G(R_G(x)) \leq c([\bar{e}_i]_p, F\#G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + c([\bar{e}_i]_p, G\#F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \epsilon_2 + \mathcal{A}_G(x)$$

for each $x \in CF(F, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$, $y \in CF(G, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$.

Proof. Let h be a small Morse function. Then there are two pair-of-pants products.

$$*' : CF(F_\epsilon \# \overline{G_\epsilon}, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \otimes CF(G_\epsilon \# \overline{F_\epsilon}, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow CF(h, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$$

$$*'' : CF(h, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \otimes CF(F, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \rightarrow CF(F, J; \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}})$$

By the associativity of pair of pants product, there is a chain homotopy $R''_{F,b,a}$ such that

$$b * (a * \cdot) = (b *' a) *'' (\cdot) + dR''_{F,b,a} + R''_{F,b,a}d$$

and

$$\mathcal{A}_F(R''_{F,b,a}(x)) \leq \mathcal{A}_{G_\epsilon \# \overline{F_\epsilon}}(a) + \mathcal{A}_{F_\epsilon \# \overline{G_\epsilon}}(b) + \epsilon_2 + \mathcal{A}_F(x)$$

Consider the PSS isomorphism $\Psi : CM(f) \rightarrow CF(h)$ and $\overline{\Psi} : CF(h) \rightarrow CM(f)$. It is a standard action estimate to show that

$$\mathcal{A}_h(\Psi(x)) \leq \mathcal{A}_h(x) + \epsilon_2$$

Moreover there exists a chain homotopy R_h such that

$$\Psi \circ \overline{\Psi} = 1 + dR_h + R_h d$$

and

$$\mathcal{A}_h(R_h(y)) \leq \mathcal{A}_h(y) + \epsilon_2$$

for each $y \in CF(h)$.

Finally, by gluing and homotopy of domain-dependent almost complex structures, we see that $\overline{\Psi}(a *' b)$ and $\Psi(x) *'' (\cdot)$ are homotopic to $a * b$ and $x * (\cdot)$ respectively, with homotopies $r_{a,b}$ and R_x that do not increase the action by more than ϵ_2 .

Now we proceed by noting first that by the compatibility of the pair-of-pants product and the quantum product under the PSS isomorphism,

$$[c_{F_\epsilon \# \overline{G_\epsilon}} * c_{G_\epsilon \# \overline{F_\epsilon}}] = [\overline{e}_i]_p * [\overline{e}_i]_p = [\overline{e}_i]_p = [\overline{e}_i]_{p,0}$$

Hence $g = c_{F_\epsilon \# \overline{G_\epsilon}} * c_{G_\epsilon \# \overline{F_\epsilon}} - [\overline{e}_i]_{p,0}$ is a boundary in the Morse complex of f . Moreover

$$\begin{aligned} \mathcal{A}(g) &\leq \max\{\gamma_{[\overline{e}_i]_p}(G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2, l([\overline{e}_i]_p)\} \\ &= \gamma_{[\overline{e}_i]_p}(G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2 \end{aligned}$$

since $\gamma_{[\overline{e}_i]_p}(G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \geq l([\overline{e}_i]_p)$. However, since $\beta(CM(f)) = 0$, there exists an element $r_F \in CF(f)$ with

$$\mathcal{A}(r_F) \leq \gamma_{[\overline{e}_i]_p}(G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2$$

and $g = c_{F_\epsilon \# \overline{G_\epsilon}} * c_{G_\epsilon \# \overline{F_\epsilon}} - [\overline{e}_i]_{p,0} = d(r_F)$.

Finally, by the Leibnitz rule

$$(r_F *) : CF(F) \rightarrow CF(F)[1]$$

gives a homotopy between $(c_{F_\epsilon \# \overline{G_\epsilon}} * c_{G_\epsilon \# \overline{F_\epsilon}}) * (\cdot)$ and $[\overline{e}_i]_{p,0} * (\cdot)$. Thus

$$\begin{aligned} R_F &= R''_{F, c_{F_\epsilon \# \overline{G_\epsilon}}, c_{G_\epsilon \# \overline{F_\epsilon}}} + (R_h(c_{G_\epsilon \# \overline{F_\epsilon}} *' c_{F_\epsilon \# \overline{G_\epsilon}})) *'' (\cdot) \\ &\quad + R_{\Psi(c_{G_\epsilon \# \overline{F_\epsilon}} *' c_{F_\epsilon \# \overline{G_\epsilon}})} + r_{c_{G_\epsilon \# \overline{F_\epsilon}}, c_{F_\epsilon \# \overline{G_\epsilon}}} * (\cdot) + (r_F *) \end{aligned}$$

gives the required homotopy. □

Then the following composition relations hold

$$\begin{aligned} sh_{\epsilon_2 - 2\epsilon_1} \circ [C_{F \# \overline{G}}][c([\overline{e}_i]_p, G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_1] \circ [C_{G \# \overline{F}}] \\ = sh_{\gamma_{[\overline{e}_i]_p}(G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2 - \mathcal{A}([\overline{e}_i]_{p,0})}([\overline{e}_i]_p *) \end{aligned}$$

$$\begin{aligned} sh_{\epsilon_2 - 2\epsilon_1} \circ [C_{G \# \overline{F}}][c([\overline{e}_i]_p, F \# \overline{G}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_1] \circ [C_{F \# \overline{G}}] \\ = sh_{\gamma_{[\overline{e}_i]_p}(G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2 - \mathcal{A}([\overline{e}_i]_{p,0})}([\overline{e}_i]_p *) \end{aligned}$$

In the first equation, the morphisms on both sides are from

$$HF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}})$$

to

$$HF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}})[\gamma_{[\overline{e}_i]_p}(G \# \overline{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2].$$

In the second equation, the morphisms on both sides are from

$$HF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}})$$

to

$$HF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}})[\gamma_{[\bar{e}_i]_p}(G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2].$$

Because $[C_{G \# \bar{F}}]$ and $[C_{F \# \bar{G}}]$ commute with $[\bar{e}_i]_p*$, then there are induced maps

$$\Phi'_{G \# \bar{F}} : I_m([\bar{e}_i]_p, F) \rightarrow I_m([\bar{e}_i]_p, G)(c([\bar{e}_i]_p, G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_1)$$

$$\Phi'_{F \# \bar{G}} : I_m([\bar{e}_i]_p, G) \rightarrow I_m([\bar{e}_i]_p, F)(c([\bar{e}_i]_p, F \# \bar{G}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_1)$$

where

$$I_m([\bar{e}_i]_p, F) = \text{im}([\bar{e}_i]_p* : HF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow HF(F, \overline{\Lambda_{\mathbb{Z}_p, univ}})[l([\bar{e}_i]_p)])$$

and

$$I_m([\bar{e}_i]_p, G) = \text{im}([\bar{e}_i]_p* : HF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}}) \rightarrow HF(G, \overline{\Lambda_{\mathbb{Z}_p, univ}})[l([\bar{e}_i]_p)])$$

Then the following relations hold.

$$sh_{\epsilon_2 - 2\epsilon_1} \circ \Phi'_{F \# \bar{G}}[c([\bar{e}_i]_p, G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_1] \circ \Phi'_{G \# \bar{F}} = sh_{\gamma_{[\bar{e}_i]_p}(G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2}$$

$$sh_{\epsilon_2 - 2\epsilon_1} \circ \Phi'_{G \# \bar{F}}[c([\bar{e}_i]_p, F \# \bar{G}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_1] \circ \Phi'_{F \# \bar{G}} = sh_{\gamma_{[\bar{e}_i]_p}(G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2}$$

Then define

$$\Phi_{G \# \bar{F}} = sh_{\frac{1}{2}c([\bar{e}_i]_p, F \# \bar{G}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) - \frac{1}{2}c([\bar{e}_i]_p, G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \frac{1}{2}\epsilon_2 - \epsilon_1}$$

$$\Phi_{F \# \bar{G}} = sh_{\frac{1}{2}c([\bar{e}_i]_p, G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) - \frac{1}{2}c([\bar{e}_i]_p, F \# \bar{G}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \frac{1}{2}\epsilon_2 - \epsilon_1}$$

The following relations hold

$$\Phi_{F \# \bar{G}}[\frac{1}{2}\gamma_{[\bar{e}_i]_p}(G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \frac{1}{2}\epsilon_2] \circ \Phi_{G \# \bar{F}} = sh_{\gamma_{[\bar{e}_i]_p}(G \# \bar{F}, \overline{\Lambda_{\mathbb{Z}_p, univ}}) + \epsilon_2}$$

$$\Phi_{G\#F}[\frac{1}{2}\gamma_{[\bar{e}_i]_p}(G\#\bar{F}, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \frac{1}{2}\epsilon_2] \circ \Phi_{F\#G} = sh_{\gamma_{[\bar{e}_i]_p}(G\#\bar{F}, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \epsilon_2}$$

Thus the persistence modules $I_m([\bar{e}_i]_p, F)$ and $I_m([\bar{e}_i]_p, G)$ are $\left(\frac{1}{2}\gamma_{[\bar{e}_i]_p}(G\#\bar{F}, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \frac{1}{2}\epsilon_2\right)$ -interleaved. By Lemma 2.4.15, $\beta(I_m([\bar{e}_i]_p), F)$ and $\beta(I_m([\bar{e}_i]_p), G)$ are finite and

$$|\beta(\phi_F^1, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) - \max \beta(I_m([\bar{e}_i]_p, F))| \leq 2l(E_p)$$

$$|\beta(\phi_G^1, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) - \max \beta(I_m([\bar{e}_i]_p, G))| \leq 2l(E_p)$$

$$|\max \beta(I_m([\bar{e}_i]_p, G)) - \max \beta(I_m([\bar{e}_i]_p, F))| \leq \gamma_{[\bar{e}_i]_p}(G\#\bar{F}, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) + \epsilon_2$$

□

By Proposition 2.4.12 and Proposition 2.4.13, we have the following theorem.

Theorem 2.4.17 ([AL23], Theorem 4.10). *Suppose that $QH_{ev}(M, \Lambda_{\mathbb{K}, \text{univ}})$ is semisimple. Then the boundary depth of each $\psi \in Ham(M, \omega)$ satisfies $\beta(\psi, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) \leq D + \delta$, where $D + \delta$ is independent of p .*

Since $\beta(\psi, \overline{\Lambda_{\mathbb{Z}_p, \text{univ}}}) = \beta(\psi, \Lambda_{\mathbb{Z}_p, \text{univ}})$, we have the following corollary.

Corollary 2.4.18 ([AL23], Theorem 4.11). *Suppose that $QH_{ev}(M, \Lambda_{\mathbb{K}, \text{univ}})$ is semisimple. Then the boundary depth of each $\psi \in Ham(M, \omega)$ satisfies $\beta(\psi, \Lambda_{\mathbb{Z}_p, \text{univ}}) \leq D + \delta$, where $D + \delta$ is independent of p .*

2.5 $\mathbb{Z}_p$ -equivariant Floer Homology

2.5.1 The $\mathbb{Z}_p$ -equivariant Floer Homology of $CF(\phi, \Lambda_{\mathbb{Z}_p}^0)^{\otimes p}$

The definition of the $\mathbb{Z}_p$ -equivariant Floer homology of $CF(\phi, \Lambda_{\mathbb{Z}_p}^0)^{\otimes p}$ is the same with that in [She22]. Let $\mathcal{K} = \mathbb{Z}_p[u^{-1}, u]$. Then $\Lambda_{\mathcal{K}}^0$ is a certain completion of $\Lambda_{\mathbb{Z}_p}^0 \otimes_{\mathbb{Z}_p} \mathcal{K}$. Define a $\mathbb{Z}_p$ -action on

$$CF(\phi; \Lambda_{\mathbb{Z}_p}^0)^{\otimes p} \otimes_{\Lambda_{\mathbb{Z}_p}^0} \Lambda_{\mathcal{K}}^0 \langle \theta \rangle,$$

where $\deg(u) = 2$, $\deg(\theta) = 1$ and $\theta^2 = 0$, by the $\Lambda_{\mathcal{K}}^0 \langle \theta \rangle$ -linear extension of

$$\tau(x_0 \otimes \cdots \otimes x_{p-1}) = (-1)^{|x_{p-1}|(|x_0| + \cdots + |x_{p-2}|)} x_{p-1} \otimes x_0 \cdots \otimes x_{p-2}$$

for $x_0 \otimes \cdots \otimes x_{p-1} \in CF(\phi, \Lambda_{\mathbb{Z}_p}^0)^{\otimes p}$. Define the differential by the $\Lambda_{\mathcal{K}}^0$ -linear extension of

$$d_{Tate}(x \otimes 1) = d^{(p)}x + \theta(1 - \tau)x$$

$$d_{Tate}(x \otimes \theta) = \theta d^{(p)}x + u(1 + \tau + \cdots + \tau^{p-1})x$$

where $d^{(p)}$ is the differential on $CF(\phi; \Lambda_{\mathbb{Z}_p}^0)^{\otimes p}$ induced by the Floer differential on $CF(\phi; \Lambda_{\mathbb{Z}_p}^0)$. We denote the induced homology by $H(\mathbb{Z}_p, CF(\phi; \Lambda_{\mathbb{Z}_p}^0)^{\otimes p})$.

2.5.2 The $\mathbb{Z}_p$ -equivariant Floer Homology of $CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0)$

We follow the ideas of Sugimoto in [Sug21] to define the $\mathbb{Z}_p$ -equivariant Floer homology of $CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0)$.

Let f be a $\mathbb{Z}_p$ -invariant Morse function on S^∞ , where the $\mathbb{Z}_p$ -action on S^∞ is given by the scalar multiplication by the p -th root of unity. Then for each degree $k, k \in \mathbb{N}$, there are p critical points denoted by $Z_k^m, m \in \{0, 1, \dots, p-1\}$. Then one can identify the Morse chain complex with $\mathbb{K}[\mathbb{Z}_p][\![u]\!] \langle \theta \rangle$. The critical points contained in S^{2k-1} are $\{Z_j^m\}$ with $j \in \{0, 1, \dots, 2k+1\}$ and $m \in \{0, 1, \dots, p-1\}$. Consider the following perturbed Cauchy-Riemann equation

$$\begin{cases} \bar{\partial}_J u + X_{H_t, w(s)}(u)^{0,1} = 0 \\ \partial_s w + \nabla f(w) = 0 \end{cases}$$

with limit conditions

$$\begin{cases} \lim_{s \rightarrow -\infty} (u(s, t), w(s)) = (x^+(t + m), Z_i^m) \\ \lim_{s \rightarrow \infty} (u(s, t), w(s)) = (x^-(t), Z_\alpha^0) \end{cases}$$

where α is 0 or 1 and $x^\pm$ are fixed points of ϕ^p . The almost complex structure in the equation are parametrized by S^∞ and satisfies the conditions given in Section 6.1 of [SZ21]. One can define a linear map

$$d_\alpha^{i,m} : CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0) \rightarrow CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0)$$

by counting the solutions to the above perturbed Cauchy-Riemann equation for $i \in \{0, 1, \dots, 2k+1\}$ and $m \in \{0, 1, \dots, p-1\}$. Take

$$d_\alpha^i = \sum_{m=0}^{p-1} d_\alpha^{i,m}.$$

We have the so-called X_k -module $(CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0) \otimes_{\Lambda_{\mathbb{Z}_p}^0} \Lambda_{\mathcal{K}}^0, \{d_\alpha^i\}_{i=0}^{2k+1})$ in [Sug21]. By [Sug21, Lemma 5.5], i.e. [Fuk+10, Lemma 7.2.184], we can have an X_∞ -module $(CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0) \otimes_{\Lambda_{\mathbb{Z}_p}^0} \Lambda_{\mathcal{K}}^0, \{d_\alpha^i\}_{i=0}^\infty)$. Then, we can define the differential on $CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0) \otimes_{\Lambda_{\mathbb{Z}_p}^0} \Lambda_{\mathcal{K}}^0$ as follows:

$$\begin{aligned} d_{\mathbb{Z}_p}(x \otimes 1) = & d_0^0(x) \otimes 1 + u d_0^2(x) \otimes 1 + u^2 d_0^4(x) \otimes 1 + \dots \\ & + d_0^1(x) \otimes \theta + u d_0^3(x) \otimes \theta + u^2 d_0^5(x) \otimes \theta + \dots \end{aligned}$$

$$\begin{aligned} d_{\mathbb{Z}_p}(x \otimes \theta) = & d_1^1(x) \otimes \theta + u d_1^3(x) \otimes \theta + u^2 d_1^5(x) \otimes \theta + \dots \\ & + u d_1^2(x) \otimes 1 + u^2 d_1^4(x) \otimes 1 + u^3 d_1^6(x) \otimes 1 + \dots \end{aligned}$$

We denote the homology by $H(\mathbb{Z}_p, CF(\phi^p; \Lambda_{\mathbb{Z}_p}^0))$.

2.5.3 The $\mathbb{Z}_p$ -equivariant Pair of Pants Product

Denote by S_p the $(p+1)$ -punctured sphere. Let $h : S_p \rightarrow \mathbb{R} \times S^1$ be the branched cover of $\mathbb{R} \times S^1$ at $(0, 0) \in \mathbb{R} \times S^1$ of ramification index p .

$$\epsilon_i^+ : [1, \infty) \times S^1 \rightarrow S_p, \quad i \in \{0, 1, \dots, p-1\}.$$

and

$$\epsilon_i^- : (-\infty, -1] \times S^1 \rightarrow S_p, \quad i \in \{0, 1, \dots, p-1\}.$$

are the trivialization of cylindrical ends as in [SZ21].

Given a fixed point x^- of ϕ^p and fixed points $\{x_0^+, x_1^+, \dots, x_{p-1}^+\}$ of ϕ , consider the following equation

$$\begin{cases} (du - Y_z) \circ j = J_{z,w} \circ (du - Y_z) \\ \partial_s w + \nabla f(w) = 0 \end{cases}$$

with limit conditions

$$\begin{cases} \lim_{s \rightarrow -\infty} (u(\epsilon_0^-(s, t)), w(s)) = (x^+(t), Z_\alpha^0) \\ \lim_{s \rightarrow \infty} (u(\epsilon_m^+(s, t)), w(s)) = (x_m^+(t), Z_i^m) \end{cases}$$

where α is 0 or 1 and $\{Z_i^m\}$ with $j \in \{0, 1, \dots, 2k+1\}$ are critical points in S^{2k-1} . The almost complex structure J and Hamiltonian perturbation Y in the equation satisfy the conditions in [SZ21, Section 8.1].

Then one can define a linear map

$$P_\alpha^{i,m} : CF(\phi, \Lambda_{\mathbb{Z}_p}^0)^{\otimes p} \rightarrow CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0)$$

by counting the solutions to the above perturbed Cauchy-Riemann equation for $i \in \{0, 1, \dots, 2k+1\}$ and $m \in \{0, 1, \dots, p-1\}$. Take

$$P_\alpha^i = \sum_{m=0}^{p-1} P_\alpha^{i,m}.$$

We have the so-called X_k -homomorphism $\{P_\alpha^i\}_{i=0}^{2k+1}$ in [Sug21]. Then, we can get the an X_∞ -homomorphism by [Sug21, Lemma 5.5]. Finally, define the equivariant pair of pants product as following.

$$\begin{aligned} P(x \otimes 1) = & P_0^0(x) \otimes 1 + uP_0^2(x) \otimes 1 + u^2P_0^4(x) \otimes 1 + \dots \\ & + P_0^1(x) \otimes \theta + uP_0^3(x) \otimes \theta + u^2P_0^5(x) \otimes \theta + \dots \end{aligned}$$

$$\begin{aligned} P(x \otimes \theta) = & P_1^1(x) \otimes \theta + uP_1^3(x) \otimes \theta + u^2P_1^5(x) \otimes \theta + \dots \\ & + uP_1^2(x) \otimes 1 + u^2P_1^4(x) \otimes 1 + u^3P_1^6(x) \otimes 1 + \dots \end{aligned}$$

One can define the equivariant pair of pants coproduct in the same way and then the equivariant pair of pants product is an isomorphism.

2.5.4 Total Bar Length

Once the equivariant Floer homology and equivariant pair of pants product are defined, we can prove the following theorem in the same way as in [She22].

Theorem 2.5.1 ([AL23], Theorem 5.1). *Let $\phi \in \text{Ham}(M, \omega)$ be a Hamiltonian diffeomorphism of a closed semipositive symplectic manifold (M, ω) . Sup-*

pose that $\text{fix}(\phi^p)$ is finite. Then

$$p \cdot \beta_{\text{tot}}(\phi, \mathbb{Z}_p) \leq \beta_{\text{tot}}(\phi^p, \mathbb{Z}_p).$$

First note that $HF(\phi, \Lambda_{\mathbb{Z}_p}^0) \otimes_{\Lambda_{\mathbb{Z}_p}^0} \Lambda_{\mathcal{K}}^0$ is isomorphic to $H(\mathbb{Z}_p, CF(\phi, \Lambda_{\mathbb{Z}_p}^0))$ since the $\mathbb{Z}_p$ -action on $CF(\phi, \Lambda_{\mathbb{Z}_p}^0)$ is trivial. Secondly, one can get the relation between bar lengths of $H(\mathbb{Z}_p, CF(\phi, \Lambda_{\mathbb{Z}_p}^0))$ and $H(\mathbb{Z}_p, CF(\phi, \Lambda_{\mathbb{Z}_p}^0)^{\otimes p})$ by quasi-Frobenius isomorphism $x \rightarrow x \otimes \cdots \otimes x$ on the chain level. Then equivariant pair of pants product gives the relation between bar lengths of $H(\mathbb{Z}_p, CF(\phi, \Lambda_{\mathbb{Z}_p}^0)^{\otimes p})$ and $H(\mathbb{Z}_p, CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0))$. Finally, using homological perturbation and cone map, one can get the relation of bar lengths between $H(\mathbb{Z}_p, CF(\phi^p, \Lambda_{\mathbb{Z}_p}^0))$ and $HF(\phi^p, \Lambda_{\mathbb{Z}_p}^0) \otimes_{\Lambda_{\mathbb{Z}_p}^0} \Lambda_{\mathcal{K}}^0$. In addition, due to the local equivariant Floer homology argument in [She22], it is not necessary to assume that ϕ is nondegenerate.

2.6 Degeneracy

2.6.1 Definition of local Floer homology

Let x be an isolated fixed point of a Hamiltonian diffeomorphism ϕ and ϕ^t be a Hamiltonian isotopy with $\phi^1 = \phi$. Then $x(t) = \phi^t(x)$ is an 1-periodic orbit. Let $\tilde{x} : S^1 \rightarrow S^1 \times M$ be the graph of x .

Take $\tilde{U}$ to be a small enough neighborhood of $\tilde{x}$ and $U = p_M(\tilde{U})$ where $p_M : S^1 \times M \rightarrow M$ is the projection. When x is a degenerate fixed point, we can take a sufficiently small non-degenerate perturbation ϕ_1 of ϕ with support in U such that the Floer trajectories connecting the 1-periodic orbits of ϕ_1 in U are contained in U . Thus every broken trajectory is also contained in U . Let $CF(\phi_1, x)$ be the vector space generated by the 1-periodic orbits of ϕ_1 in U over $\mathbb{K}$. Then we can define the Floer homology in U , which is independent of the choice of the perturbation and of the almost complex structure. We call this Floer homology in U the local Floer homology at x and denote by $HF^{\text{loc}}(\phi, x)$.

By the definition of local Floer homology, one can easily see that

$$HF^{\text{loc}}(\phi, x) \cong \mathbb{K}$$

generated by x .

Let $[x, u]$ and $[x, v]$ be two different capped periodic orbits. Then

$$CZ([x, u]) = CZ([x, v]) \pmod{2}.$$

Thus there is a well-defined $\mathbb{Z}/2$ -grading on $HF^{loc}(\phi, x)$. When x is non-degenerate, $HF^{loc}(\phi, x) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}, univ}$ contributes a copy of $\Lambda_{\mathbb{K}, univ}$ in $CF_k(\phi, \Lambda_{\mathbb{K}, univ})$.

For any two capped periodic orbits $[x, u]$ and $[y, v]$ of ϕ , there exists a crossing energy $2\epsilon_0 > 0$ such that all Floer trajectories, or product structures with $[x, u]$ and $[y, v]$ among their asymptotes carry energy at least $2\epsilon_0$.

Definition 2.6.1. An iteration ϕ^k of ϕ is *admissible at a fixed point x of ϕ* if $\lambda^k \neq 1$ for all eigenvalues $\lambda \neq 1$ of $d\phi_x$.

For example, when none of $\lambda \neq 1$ are roots of unity, ϕ^k is admissible for $k > 0$. Otherwise, ϕ^{p^n} is admissible for sufficiently large p and $n > 0$.

By Theorem 1.1 and Remark 1.2 in [GG10], we have the following theorem.

Theorem 2.6.2. *Let ϕ^k be an admissible iteration of ϕ at an isolated 1-periodic orbit x of ϕ . Then the k -iteration x^k of x is an isolated 1-periodic orbit of ϕ^k and $HF^{loc}(\phi^k, x^k) \cong HF^{loc}(\phi, x)$.*

Remark 2.6.3. Let ϕ^{k_1} and ϕ^{k_2} be admissible iterations of ϕ at an isolated 1-periodic orbit x of ϕ , then $HF^{loc}(\phi^{k_1}, x^{k_1}) \cong HF^{loc}(\phi^{k_2}, x^{k_2})$ by Theorem 2.6.2.

2.6.2 The construction of a canonical $\overline{Frac(\Lambda_{\mathbb{Z}, univ})}^0$ -complex

The contents in this section are from Section 3.4.7 in [She22]

Let ϕ be a Hamiltonian diffeomorphism. For each isolated 1-periodic orbit x of ϕ , there is a neighborhood U_x of x . Let ϕ_1 be a sufficiently small non-degenerate perturbation of ϕ . Because there are finitely many isolated 1-periodic orbits, one can choose $\phi_1 = \phi$ outside $\bigcup_{x \in Fix(\phi)} U_x$.

Theorem 2.6.4 ([AL23], Theorem 2.20). *There is a homotopically canonical $\Lambda_{\mathbb{K}, univ}^0$ -complex $CF(\phi, \Lambda_{\mathbb{K}, univ}^0)$ with the following properties.*

(1) *As a $\Lambda_{\mathbb{K}, univ}^0$ -module,*

$$CF(\phi, \Lambda_{\mathbb{K}, univ}^0) \cong \bigoplus_{x \in Fix(\phi)} HF^{loc}(\phi, x) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}, univ}^0.$$

(2) *Its differential is defined over $\Lambda_{\mathbb{K}, univ}^0$.*

(3) *The homology of*

$$CF(\phi, \Lambda_{\mathbb{K}, univ}) = CF(\phi, \Lambda_{\mathbb{K}, univ}^0) \otimes_{\Lambda_{\mathbb{K}, univ}^0} \Lambda_{\mathbb{K}, univ}$$

is isomorphic to $HF(\phi_1, \Lambda_{\mathbb{K}, univ})$.

(4) The bar length spectrum associated to $CF(\phi, \Lambda_{\mathbb{K}, \text{univ}})$, denoted by

$$\beta'_1(\phi, \Lambda_{\mathbb{K}, \text{univ}}) \leq \cdots \leq \beta'_{K(\phi, \Lambda_{\mathbb{K}, \text{univ}})}(\phi, \Lambda_{\mathbb{K}, \text{univ}}),$$

satisfies $\beta'_1(\phi, \Lambda_{\mathbb{K}, \text{univ}}) > \epsilon_0$ and is $2\delta_0$ -close to the part

$$\beta_{K'+1}(\phi_1, \Lambda_{\mathbb{K}, \text{univ}}) \leq \cdots \leq \beta_{K'+K(\phi, \Lambda_{\mathbb{K}, \text{univ}})}(\phi_1, \Lambda_{\mathbb{K}, \text{univ}})$$

of the bar length spectrum of ϕ_1 above ϵ_0 where

$$\beta_{K'}(\phi_1, \Lambda_{\mathbb{K}, \text{univ}}) < 2\delta_0 \ll \epsilon_0$$

and $\delta_0 \ll \epsilon_0$ is a small parameter converging to 0 as ϕ_1 converges to ϕ in the C^2 -topology.

(5) The $\beta'_j(\phi, \Lambda_{\mathbb{K}, \text{univ}})$ for $1 \leq j \leq K(\phi, \Lambda_{\mathbb{K}, \text{univ}})$ have a limit $\beta_j(\phi, \Lambda_{\mathbb{K}, \text{univ}})$ as the Hamiltonian perturbation goes to zero in the C^2 -topology.

Proof. We start with $CF(\phi_1, \Lambda_{\mathbb{K}, \text{univ}}^0)$. The differential d_{ϕ_1} can be written as

$$d_{\phi_1} = d_{\phi_1, \text{loc}} + T^{\epsilon_0} D_{\phi_1}$$

where $d_{\phi_1, \text{loc}}$ is the direct sum of the differentials of the local Floer complexes $CF^{\text{loc}}(\phi_1, x) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}, \text{univ}}^0$ over all 1-periodic orbits of ϕ . By crossing energy, D_{ϕ_1} is defined over $\Lambda_{\mathbb{K}, \text{univ}}^0$

Let $[x, u]$ be a capped 1-periodic orbits of ϕ . Then each 1-periodic orbits of ϕ_1 in U_x can inherit a capping from $[x, u]$

Let H_1 be a Hamiltonian generating ϕ_1 . Let x_1 be an 1-periodic orbits of ϕ_1 with capping u_1 inheriting from some $[x, u]$. Then

$$\mathcal{A}_{H_1}(x_1) - \mathcal{A}_{H_1}(d_{\phi_1, \text{loc}}(x_1)) < \delta_0$$

for some $\delta_0 \ll \epsilon_0$ where

$$\mathcal{A}_{H_1}(x_1) = \int_0^1 H_1(t, x_1(t)) dt - \int_{D^2} u_1^* \omega.$$

In fact, we can make the perturbation sufficiently small such that

$$|\mathcal{A}_H(x) - \mathcal{A}_{H_1}(x_1)| < \frac{\delta_0}{2}$$

and $\delta_0 \ll \epsilon_0$ where x_1 is any 1-periodic orbits of ϕ_1 in U_x .

As in [UZ16], there is a singular value decomposition for the complex

$$(CF(\phi_1, \Lambda_{\mathbb{K}, univ}^0), d_{\phi_1, loc}),$$

denoted by

$$(\xi_1, \dots, \xi_B, \eta_1, \dots, \eta_K, \zeta_1, \dots, \zeta_K)$$

with

$$d_{\phi_1, loc}(\xi_i) = 0, \quad d_{\phi_1, loc}(\zeta_j) = T^{\delta_j} \eta_j$$

for $i = 1, \dots, B$, $j = 1, \dots, K$. By the previous paragraph, one can see $\delta_j < \delta_0 \ll \epsilon_0$ for $j = 1, \dots, K$. If we denote

$$N = \sum_{x \in Fix(\phi)} \dim_{\Lambda_{\mathbb{K}, univ}} CF^{loc}(\phi_1, x) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}, univ},$$

then

$$K = \frac{N - B}{2}$$

and

$$B = \sum_{x \in fix(\phi)} \dim_{\Lambda_{\mathbb{K}, univ}} HF^{loc}(\phi, x) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}, univ}.$$

Let

$$X = \text{span}_{\Lambda_{\mathbb{K}, univ}^0} \{\xi_1, \dots, \xi_B\}$$

be the free part of the homology of $(CF(\phi_1, \Lambda_{\mathbb{K}, univ}^0), d_{\phi_1, loc})$. Let

$$\pi : CF(\phi_1, \Lambda_{\mathbb{K}, univ}^0) \rightarrow X$$

be the projection and

$$\iota : X \rightarrow CF(\phi_1, \Lambda_{\mathbb{K}, univ}^0)$$

be the inclusion. Define

$$\Theta : CF(\phi_1, \Lambda_{\mathbb{K}, univ}) \rightarrow CF(\phi_1, \Lambda_{\mathbb{K}, univ})$$

to be the linear map such that

$$\Theta(\xi_i) = 0,$$

$$\Theta(\eta_j) = T^{-\delta_j} \zeta_j$$

and

$$\Theta(\zeta_j) = 0$$

for $i = 1, \dots, B, j = 1, \dots, K$.

Now define a differential for X by the basic perturbation lemma in [Maroo].

Lemma 2.6.5. *Given chain complexes (M, d_M) , (N, d_N) , chain maps*

$$F : (M, d_M) \rightarrow (N, d_N),$$

$$G : (N, d_N) \rightarrow (M, d_M)$$

and a chain homotopy

$$H : M \rightarrow M$$

satisfying

$$(1) \quad Fd_M = d_N F$$

$$(2) \quad Gd_N = d_M G$$

$$(3) \quad GF - id_M = d_M H + H d_M$$

$$(4) \quad FG = id_N$$

$$(5) \quad HH = 0$$

$$(6) \quad HG = 0$$

$$(7) \quad GF = 0$$

$\tilde{d}_M$ is a perturbation of d_M on M . Then there are perturbations $\tilde{d}_N, \tilde{F}, \tilde{G}$ and $\tilde{H}$ of d_N, F, G and H that satisfy

$$(1) \quad \tilde{F}\tilde{d}_M = \tilde{d}_N\tilde{F}$$

$$(2) \quad \tilde{G}\tilde{d}_N = \tilde{d}_M\tilde{G}$$

$$(3) \quad \tilde{G}\tilde{F} - id_M = \tilde{d}_M\tilde{H} + \tilde{H}\tilde{d}_M$$

$$(4) \quad \tilde{F}\tilde{G} = id_N$$

The perturbation $(\tilde{d}_N, \tilde{F}, \tilde{G}, \tilde{H})$ is given by the following explicit formulas.

$$\tilde{d}_N = d_N + F(\partial_M + \partial_M H \partial_M + \partial_M H \partial_M H \partial_M + \partial_M H \partial_M H \partial_M H \partial_M + \dots)G$$

$$\tilde{F} = F + F(\partial_M + \partial_M H \partial_M + \partial_M H \partial_M H \partial_M + \partial_M H \partial_M H \partial_M H \partial_M + \dots)H$$

$$\tilde{G} = G + H(\partial_M + \partial_M H \partial_M + \partial_M H \partial_M H \partial_M + \partial_M H \partial_M H \partial_M H \partial_M + \dots)G$$

$$\tilde{H} = H + H(\partial_M + \partial_M H \partial_M + \partial_M H \partial_M H \partial_M + \partial_M H \partial_M H \partial_M H \partial_M + \dots)H$$

where $\partial_M := \tilde{d}_M - d_M$.

Now we apply the lemma to

$$(CF(\phi_1, \Lambda_{\mathbb{K}, univ}), d_{\phi_1, loc})$$

with perturbation

$$(CF(\phi_1, \Lambda_{\mathbb{K}, univ}), d_{\phi_1} = d_{\phi_1, loc} + T^{\epsilon_0} D_{\phi_1}),$$

$$(X \otimes_{\Lambda_{\mathbb{K}, univ}^0} \Lambda_{\mathbb{K}, univ}, 0),$$

$$\pi : CF(\phi_1, \Lambda_{\mathbb{K}, univ}) \rightarrow X \otimes_{\Lambda_{\mathbb{K}, univ}^0} \Lambda_{\mathbb{K}, univ},$$

$$\iota : X \otimes_{\Lambda_{\mathbb{K}, univ}^0} \Lambda_{\mathbb{K}, univ} \rightarrow CF(\phi_1, \Lambda_{\mathbb{K}, univ})$$

and

$$\Theta : CF(\phi_1, \Lambda_{\mathbb{K}, univ}) \rightarrow CF(\phi_1, \Lambda_{\mathbb{K}, univ}).$$

Then we get a differential d_ϕ on X where

$$d_\phi = \pi(T^{\epsilon_0} D_{\phi_1} + T^{2\epsilon_0} D_{\phi_1} \Theta D_{\phi_1} + T^{3\epsilon_0} D_{\phi_1} \Theta D_{\phi_1} \Theta D_{\phi_1} + \dots)\iota$$

is the perturbation of 0. Since $\delta_j < \delta_0 \ll \epsilon_0$, then $d_\phi, \tilde{\pi}, \tilde{\iota}$ and $T^{\delta_0} \tilde{\Theta}$ are defined over $\Lambda_{\mathbb{K}, univ}^0$ where $\tilde{\pi}, \tilde{\iota}$ and $\tilde{\Theta}$ are perturbations of π, ι and Θ respectively.

Take $CF(\phi, \Lambda_{\mathbb{K}, univ}^0)$ to be X and its differential to be d_ϕ . The conditions (1) and (2) in Theorem 2.6.4 are satisfied.

Since $\tilde{\Theta}$ is a chain homotopy between $CF(\phi_1, \Lambda_{\mathbb{K}, univ})$ and $X \otimes_{\Lambda_{\mathbb{K}, univ}^0} \Lambda_{\mathbb{K}, univ}$, the condition (3) in Theorem 2.6.4 is satisfied.

Because

$$d_\phi = T^{\epsilon_0}(\pi(D_{\phi_1} + T^{\epsilon_0} D_{\phi_1} \Theta D_{\phi_1} + T^{2\epsilon_0} D_{\phi_1} \Theta D_{\phi_1} \Theta D_{\phi_1} + \dots) \iota)$$

and

$$\pi(D_{\phi_1} + T^{\epsilon_0} D_{\phi_1} \Theta D_{\phi_1} + T^{2\epsilon_0} D_{\phi_1} \Theta D_{\phi_1} \Theta D_{\phi_1} + \dots) \iota$$

is defined over $\Lambda_{\mathbb{K}, univ}^0$, then the bar length $\beta'_i(\phi, \Lambda_{\mathbb{K}, univ})$ of $(CF(\phi, \Lambda_{\mathbb{K}, univ}), d_\phi)$ is always greater than ϵ_0 .

Since

$$\tilde{\pi} \tilde{\iota} = id_X$$

and

$$T^{\delta_0} \tilde{\iota} \tilde{\pi} - T^{\delta_0} id_{CF(\phi_1, \Lambda_{\mathbb{K}, univ}^0)} = d_{\phi_1} T^{\delta_0} \tilde{\Theta} + T^{\delta_0} \tilde{\Theta} d_{\phi_1},$$

then

$$\tilde{\pi}_* \tilde{\iota}_* = id$$

and

$$\tilde{\iota}_*(T^{\delta_0} \tilde{\pi})_* = (T^{\delta_0} id)_*$$

on the homology group. Thus $H(X, d_\phi)$ is isomorphic to the direct sum of the free part and torsion parts with torsion greater than ϵ_0 of $H(CF(\phi_1, \Lambda_{\mathbb{K}, univ}^0), d_{\phi_1})$. One can also see $CF(\phi, \Lambda_{\mathbb{K}, univ})$ and $CF(\phi_1, \Lambda_{\mathbb{K}, univ})$ are $\frac{\delta_0}{2}$ -quasiequivalence. Then by Corollary 8.8 in [UZ16],

$$|\beta'_j(\phi, \Lambda_{\mathbb{K}, univ}) - \beta_{K'+j}(\phi_1, \Lambda_{\mathbb{K}, univ})| \leq \delta_0 < 2\delta_0$$

for $j = 1, \dots, K(\phi, \Lambda_{\mathbb{K}, univ})$.

Note that $\beta_{K'+j}(\phi_1, \Lambda_{\mathbb{K}, univ})$ converges as ϕ_1 converges to ϕ and the limit only depends on ϕ . Then $\beta'_j(\phi, \Lambda_{\mathbb{K}, univ})$ converges and the limit only depends

on ϕ since

$$|\beta'_j(\phi, \Lambda_{\mathbb{K}, \text{univ}}) - \beta_{K'+j}(\phi_1, \Lambda_{\mathbb{K}, \text{univ}})| < 2\delta_0.$$

Denote the limit by $\beta_j(\phi, \Lambda_{\mathbb{K}, \text{univ}})$ for $j = 1, \dots, K(\phi, \Lambda_{\mathbb{K}, \text{univ}})$. We get condition (5) in Theorem 2.6.4. $\square$

2.7 Proof of Theorem A

First we consider the case where $\mathbb{K}$ has characteristic 0 and the Hamiltonian diffeomorphism ϕ and all of its iterates are nondegenerate. By the inequality

$$p \cdot \beta_{\text{tot}}(\phi, \mathbb{Z}_p) \leq \beta_{\text{tot}}(\phi^p, \mathbb{Z}_p)$$

and the simple observation that

$$\beta_{\text{tot}}(\psi, \mathbb{K}) \leq K(\psi, \mathbb{K}) \cdot \beta(\psi, \mathbb{K})$$

for any Hamiltonian diffeomorphism ψ and base field $\mathbb{K}$, we have

$$p \cdot \beta_{\text{tot}}(\phi, \mathbb{Z}_p) \leq \beta_{\text{tot}}(\phi^p, \mathbb{Z}_p) \leq K(\phi^p, \mathbb{Z}_p) \cdot \beta(\phi^p, \mathbb{Z}_p)$$

We note that for sufficiently large primes p we have by [She22, Lemma 16] the following equalities $N(\phi, \mathbb{Q}) = N(\phi, \mathbb{Z}_p)$, $\dim_{\mathbb{Q}} H_*(M; \mathbb{Q}) = \dim_{\mathbb{F}_p} H_*(M, \mathbb{F}_p)$, and $\beta(\phi, \mathbb{Q}) = \beta(\phi, \mathbb{F}_p)$ for any Hamiltonian diffeomorphism ϕ . If $\mathbb{K}$ has characteristic 0, it is a field extension of $\mathbb{Q}$, which by [She22, section 4.4.4] implies that $N(\phi, \mathbb{Q}) = N(\phi, \mathbb{K})$, $\dim_{\mathbb{Q}} H_*(M; \mathbb{Q}) = \dim_{\mathbb{K}} H_*(M; \mathbb{K})$, and $\beta(\phi, \mathbb{Q}) = \beta(\phi, \mathbb{K})$.

The assumption that $N(\phi, \mathbb{K}) > \dim_{\mathbb{K}} H_*(M; \mathbb{K})$ implies that the total bar length $\beta_{\text{tot}}(\phi, \mathbb{Z}_p)$ is positive for a sufficiently large prime p . Furthermore, the inequality

$$\beta(\phi^k, \mathbb{Z}_p) \leq C$$

yields

$$p \cdot \beta_{\text{tot}}(\phi, \mathbb{Z}_p) \leq C \cdot K(\phi^p, \mathbb{Z}_p)$$

which means that $K(\phi^p, \mathbb{K})$ grows at least linearly with respect to p . We now observe that in the nondegenerate setting the equation

$$N(\phi, \mathbb{K}) = \dim_{\mathbb{K}} H_*(M; \mathbb{K}) + 2K(\phi, \mathbb{K})$$

yields

$$\text{Fix}(\phi^p) = \dim_{\mathbb{K}} H_*(M; \mathbb{K}) + 2K(\phi^p, \mathbb{K})$$

which implies that ϕ must have infinitely many contractible periodic orbits.

In general, when the Hamiltonian diffeomorphism ϕ is degenerate, we can use the local equivariant Floer homology argument [She22, Section 7.4]. Furthermore, by the canonical complex whose properties are listed in Theorem 2.6.4, the upper bound for the boundary depth, which is also independent of p , continues to hold. Therefore, we can use the same argument as in the nondegenerate case to obtain the linear growth of $K(\phi^p, \mathbb{K})$ and, thus, of $N(\phi^p, \mathbb{K})$. To conclude the argument, we assume that p is large enough to guarantee ϕ^p is admissible iteration in the sense of Definition 2.6.1, it then follows by Theorem 2.6.2, that

$$HF^{loc}(\phi^{p_1}, x) \cong HF^{loc}(\phi^{p_2}, x)$$

for all $x \in \text{Fix}(\phi^{p_1})$ for any two primes $p_2 \geq p_1 \geq p$. In particular, there must be a new simple p' -periodic point for each prime $p' > p$. In fact, if $\text{Fix}(\phi^{p_1}) = \text{Fix}(\phi^{p_2})$ for $p_2 \geq p_1 \geq p$, then $N(\phi^{p_1}, \mathbb{K}) = N(\phi^{p_2}, \mathbb{K})$ contradicting the linear growth of $N(\phi^{p'}, \mathbb{K})$ for $p' \geq p$. A similar argument works when $\mathbb{K}$ has characteristic p the details of which can be found in [She22, Section 8].

CHAPTER 3

ON LAGRANGIAN TORI IN $S^2 \times S^2$

3.1 Proof of Theorem B

J. Oakley and M. Usher gave an explicit expression of the symplectomorphism from $\hat{F}(0)$ to $(S^2, \frac{1}{2}\omega_{std}) \times (S^2, \frac{1}{2}\omega_{std})$ in [OU16, Proof of Proposition 2.1]. The image of $L(x, y)$ under this symplectomorphism, still denoted by $L(x, y)$, is

$$\begin{aligned} L(x, y) &= \left\{ (v, w) \in S^2 \times S^2 \mid \frac{1}{2}|v + w| + \frac{1}{2}(v + w) \cdot e_1 = x, 1 - \frac{1}{2}|v + w| = y \right\} \\ &= \left\{ (v, w) \in S^2 \times S^2 \mid v_1 + w_1 = 2(x + y - 1), v \cdot w = 2(1 - y)^2 - 1 \right\} \end{aligned}$$

where (e_1, e_2, e_3) is an orthonormal basis for $\mathbb{R}^3$. Then we change the coordinates by

$$\begin{aligned} p &= x + y - 1 \\ q &= 1 - y \end{aligned}$$

Under the coordinates (p, q) , the moment polytope P_2 becomes

$$\{(p, q) \in \mathbb{R}^2 \mid -q \leq p \leq q, q \leq 1\}$$

and we still denote it by P_2 . The Lagrangian torus $L(x, y)$ can be written as

$$L_1(p, q) = \{(v, w) \in S^2 \times S^2 \mid v_1 + w_1 = 2p, v \cdot w = 2q^2 - 1\}$$

Similar to [OU16, Proof of Proposition 2.4], we can prove the following result.

Proposition 3.1.1 ([Lou24], Proposition 2.1). *For $(p, q) \in \text{Int}(P_2)$ the Lagrangian torus $L_1(p, q)$ is the orbit of an embedded curve Γ_1 in $S^2 \times S^2$ under the S^1 -action (R_t, R_t) , where*

$$R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix}$$

is the rotation around e_1 -axis by angle t .

Proof. Note that

$$(v_0, w_0) = \left(\left(p, \sqrt{1-q^2}, \sqrt{q^2-p^2} \right), \left(p, -\sqrt{1-q^2}, \sqrt{q^2-p^2} \right) \right)$$

is a point in $L_1(p, q)$. Then we rotate v_0 and w_0 around the vector $v_0 + w_0$ by angle θ to get an embedded curve

$$\Gamma_1 = \left\{ \left(\begin{bmatrix} p + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta) \\ \sqrt{1-q^2} \cos(\theta) \\ \sqrt{q^2-p^2} - \frac{p\sqrt{1-q^2}}{q} \sin(\theta) \end{bmatrix}, \begin{bmatrix} p - \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta) \\ -\sqrt{1-q^2} \cos(\theta) \\ \sqrt{q^2-p^2} + \frac{p\sqrt{1-q^2}}{q} \sin(\theta) \end{bmatrix} \right) \mid \theta \in [0, 2\pi] \right\}$$

Now we consider the Hamiltonians

$$(F_1, F_2) : \left(S^2, \frac{1}{2}\omega_{std} \right) \times \left(S^2, \frac{1}{2}\omega_{std} \right) \rightarrow \mathbb{R}^2$$

$$(v, w) \mapsto \left(-(v+w) \cdot e_1, \frac{v \cdot w}{4q} \right)$$

Then $L_1(p, q)$ is the regular level set $(F_1, F_2)^{-1}(-2p, \frac{2q^2-1}{4q})$. The Hamiltonian vector field X_{F_2} of F_2 is

$$X_{F_2}(v, w) = \left(\frac{v \times w}{2q}, \frac{w \times v}{2q} \right)$$

Take $(v, w) \in \Gamma_1$. Then one can compute

$$\begin{aligned} \left(\frac{v \times w}{2q}, \frac{w \times v}{2q} \right) &= \left(\begin{bmatrix} \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \cos(\theta) \\ -\sqrt{1-q^2} \sin(\theta) \\ -\frac{p\sqrt{1-q^2}}{q} \cos(\theta) \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \cos(\theta) \\ \sqrt{1-q^2} \sin(\theta) \\ \frac{p\sqrt{1-q^2}}{q} \cos(\theta) \end{bmatrix} \right) \\ &= \frac{d\gamma_1}{d\theta} \end{aligned}$$

where γ_1 is a parametrization of Γ_1 such that

$$\gamma_1(\theta) = \begin{pmatrix} \begin{bmatrix} p + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta) \\ \sqrt{1-q^2} \cos(\theta) \\ \sqrt{q^2-p^2} - \frac{p\sqrt{1-q^2}}{q} \sin(\theta) \end{bmatrix}, \begin{bmatrix} p - \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta) \\ -\sqrt{1-q^2} \cos(\theta) \\ \sqrt{q^2-p^2} + \frac{p\sqrt{1-q^2}}{q} \sin(\theta) \end{bmatrix} \end{pmatrix}$$

Thus Γ_1 is an integral curve of X_{F_2} . Since $\{F_1, F_2\} = 0$, then $L_1(p, q)$ is the orbit of the curve Γ_1 under the flow of F_1 , which gives the S^1 -action (R_t, R_t) . $\square$

As in [OU16, Proof of Proposition 2.4] and [Gad13, Lemma 2.4], the action (R_t, R_t) and (R_t, R_{-t}) are conjugate in $SO(3) \times SO(3)$, i.e.

$$(R_t, R_t) = (D_1, D_2)^{-1} (R_t, R_{-t}) (D_1, D_2)$$

where $D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Define $\Gamma_2 = (D_1, D_2) \Gamma_1$

and $L_2(p, q)$ to be the orbit of Γ_2 under the S^1 -action (R_t, R_{-t}) . Note that

$$(D_1, D_2) L_1(p, q) = L_2(p, q)$$

and (D_1, D_2) is a Hamiltonian diffeomorphism. Thus we have the following result.

Proposition 3.1.2 ([Lou24], Proposition 2.2). *$L_1(p, q)$ is Hamiltonian isotopic to $L_2(p, q)$.*

3.1.1 Case I: $0 < p^2 < q^4$

In this case neither of the components of Γ_2 passes through $-e_1$.

As in [OU16, Proof of Proposition 2.4], we consider the symplectomorphism

$$\begin{aligned}\psi_{-1} : (B^2(1), 2dx \wedge dy) &\rightarrow \left(S^2 \setminus \{-e_1\}, \frac{1}{2}\omega_{std}\right) \\ re^{i\theta} &\mapsto \left(1 - 2r^2, 2r\sqrt{1 - r^2} \cos(\theta), 2r\sqrt{1 - r^2} \sin(\theta)\right)\end{aligned}$$

where $B^2(1)$ is the open ball in $\mathbb{C}$ with radius 1. Note that

$$\psi_{-1}(e^{it} \cdot re^{i\theta}) = R_t \psi_{-1}(re^{i\theta}).$$

Let

$$\tilde{\Gamma}_2 = (\psi_{-1} \times \psi_{-1})^{-1}(\Gamma_2).$$

Then $L_2(p, q)$ is symplectomorphic to the Lagrangian torus $\tilde{L}_2(p, q)$ in

$$(B^2(1), 2dx \wedge dy) \times (B^2(1), 2dx \wedge dy),$$

that is the orbit of the curve $\tilde{\Gamma}_2$ under the S^1 -action (e^{it}, e^{-it}) .

Now we describe $\tilde{L}_2(p, q)$ in the way in [EP93]. Note that the S^1 -action (e^{it}, e^{-it}) is the Hamiltonian flow of the function

$$\begin{aligned}H : (\mathbb{C}^2, idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2) &\rightarrow \mathbb{R} \\ (z_1, z_2) &\mapsto |z_1|^2 - |z_2|^2\end{aligned}$$

Claim 3.1.3 ([Lou24], Claim 2.3). $H(\tilde{\Gamma}_2) = -p$.

Proof. Consider

$$\begin{aligned}h : S^2 \times S^2 &\rightarrow \mathbb{R} \\ ((v_1, v_2, v_3), (w_1, w_2, w_3)) &\mapsto -\frac{1}{2}(v_1 - w_1)\end{aligned}$$

Then

$$h \circ (\psi_{-1} \times \psi_{-1}) = H.$$

Take $(v, w) \in \Gamma_2$. Then

$$v_1 = p + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)$$

and

$$w_1 = -p + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta).$$

Thus $h(v, w) = -p$, furthermore, $H(\tilde{\Gamma}_2) = h(\Gamma_2) = -p$ $\square$

Consider the function

$$\begin{aligned} F : \mathbb{C}^2 &\rightarrow \mathbb{C} \\ (z_1, z_2) &\mapsto z_1 z_2 \end{aligned}$$

Let $\Gamma = F(\tilde{\Gamma}_2)$. It is easy to see that

$$\tilde{L}_2(p, q) \subset F^{-1}(\Gamma) \cap H^{-1}(\{-p\}).$$

By [EP93, Lemma 4.2 A], $F^{-1}(\Gamma) \cap H^{-1}(\{-p\})$ is a Lagrangian torus since Γ is an embedded curve in $B^2(1)$ and $p \neq 0$. Thus

$$\tilde{L}_2(p, q) = F^{-1}(\Gamma) \cap H^{-1}(\{-p\}).$$

Remark 3.1.4. Since the radius of $B^2(1)$ is 1, we can restrict F to $B^2(1) \times B^2(1) \rightarrow B^2(1)$.

Given a function $K : (B^2(1), 2dx \wedge dy) \rightarrow \mathbb{R}$, one can show that, for any $(z_1, z_2) \in B^2(1) \times B^2(1)$,

$$dF(X_{K \circ F}(z_1, z_2)) = \frac{1}{2}(|z_1|^2 + |z_2|^2) \left(-\frac{\partial K}{\partial y} \frac{\partial}{\partial x} + \frac{\partial K}{\partial x} \frac{\partial}{\partial y} \right)$$

by direct computation. Since $K \circ F$ is invariant under the flow of X_H , we have $dH(X_{K \circ F}) = 0$. Thus we can restrict $X_{K \circ F}$ to $H^{-1}(\{-p\})$. Then

$$dF(X_{K \circ F}|_{H^{-1}(\{-p\})}) = \frac{1}{2} \sqrt{p^2 + 4|z_1 z_2|^2} \left(-\frac{\partial K}{\partial y} \frac{\partial}{\partial x} + \frac{\partial K}{\partial x} \frac{\partial}{\partial y} \right)$$

Define the vector field $V^{p,K}$ on $B^2(1)$ by

$$V^{p,K}(z) = \frac{1}{2} \sqrt{p^2 + 4|z|^2} \left(-\frac{\partial K}{\partial y} \frac{\partial}{\partial x} + \frac{\partial K}{\partial x} \frac{\partial}{\partial y} \right)$$

We have

$$dF(X_{K \circ F}(z_1, z_2)) = V^{p,K}(F(z_1, z_2)), \text{ for } (z_1, z_2) \in H^{-1}(\{-p\}) \quad (3.1.1)$$

Lemma 3.1.5 ([Lou24], Lemma 2.5). *Let $\phi^{t,p,K}$ be the flow of $V^{p,K}$. Then*

$$\phi_{K \circ F}^t(\tilde{L}_2(p, q)) = F^{-1}(\phi^{t,p,K}(\Gamma)) \cap H^{-1}(\{-p\}).$$

Proof. By equation 3.1.1, we have

$$F \circ \phi_{K \circ F}^t(z_1, z_2) = \phi^{t,p,K} \circ F(z_1, z_2)$$

for $(z_1, z_2) \in H^{-1}(\{-p\})$. Take $(z_1, z_2) \in \tilde{L}_2(p, q)$. Then $F(z_1, z_2) \in \Gamma$ and $F \circ \phi_{K \circ F}^t(z_1, z_2) \in \phi^{t,p,K}(\Gamma)$. Thus $\phi_{K \circ F}^t(z_1, z_2) \in F^{-1}(\phi^{t,p,K}(\Gamma))$. Since $H(\phi_{K \circ F}^t(z_1, z_2)) = H(z_1, z_2)$, then $\phi_{K \circ F}^t(z_1, z_2) \in H^{-1}(\{-p\})$. Thus $\phi_{K \circ F}^t(\tilde{L}_2(p, q)) \subset F^{-1}(\phi^{t,p,K}(\Gamma)) \cap H^{-1}(\{-p\})$.

Since $-p \neq 0$, $F^{-1}(\phi^{t,p,K}(\Gamma)) \cap H^{-1}(\{-p\})$ is a Lagrangian torus for each t by [EP93]. Then $\phi_{K \circ F}^t$ is an embedding from torus $\tilde{L}_2(p, q)$ to torus $F^{-1}(\phi^{t,p,K}(\Gamma)) \cap H^{-1}(\{-p\})$. Thus

$$\phi_{K \circ F}^t(\tilde{L}_2(p, q)) = F^{-1}(\phi^{t,p,K}(\Gamma)) \cap H^{-1}(\{-p\}).$$

□

Proposition 3.1.6 ([Lou24], Proposition 2.6). *There is a smooth function K such that $\phi^{1,p,K}(\Gamma) = S^1(r)$ for some r where $S^1(r) = \{re^{i\theta} \in \mathbb{C} \mid 0 \leq \theta \leq 2\pi\}$.*

Proof. We consider the symplectic form

$$\omega^p = \frac{2r}{\sqrt{p^2 + 4r^2}} dr \wedge d\phi = \frac{2}{\sqrt{p^2 + 4x^2 + 4y^2}} dx \wedge dy$$

on $B^2(1)$. Since we assume $x + y \neq 1$, i.e. $p \neq 0$ in Theorem B, ω^p is defined at $(0, 0)$. One can show that, for any K , the vector field $V^{p,K}$ is the Hamiltonian vector field of K under the symplectic form ω^p . Then we choose r such that Γ and $S^1(r)$ enclose the same ω^p -area. This implies that Γ and $S^1(r)$ are Hamiltonian isotopic in $(B^2(1), \omega^p)$. Thus there is a Hamiltonian K such that $\phi^{1,p,K}(\Gamma) = S^1(r)$. □

Lemma 3.1.7 ([Lou24], Lemma 2.7). *The Lagrangian torus*

$$(\psi_{-1} \times \psi_{-1})(F^{-1}(S^1(r)) \cap H^{-1}(\{-p\}))$$

is the toric fiber

$$T\left(\frac{1+p-\sqrt{p^2+4r^2}}{2}, \frac{1-p-\sqrt{p^2+4r^2}}{2}\right).$$

Proof. Let (ξ, ζ) with $\xi = \frac{1+p-\sqrt{p^2+4r^2}}{2}$ and $\zeta = \frac{1-p-\sqrt{p^2+4r^2}}{2}$ be an interior point in P_1 , the moment polytope of the standard toric structure on $S^2 \times S^2$. Then the toric fiber over (ξ, ζ) is

$$T(\xi, \zeta) = \left\{ \left(\begin{bmatrix} \frac{2\xi}{\sqrt{1-4\xi^2}} \cos(\theta_1) \\ \frac{2\xi}{\sqrt{1-4\xi^2}} \sin(\theta_1) \end{bmatrix}, \begin{bmatrix} \frac{2\zeta}{\sqrt{1-4\zeta^2}} \cos(\theta_2) \\ \frac{2\zeta}{\sqrt{1-4\zeta^2}} \sin(\theta_2) \end{bmatrix} \right) \in S^2 \times S^2 \mid 0 \leq \theta_1, \theta_2 \leq 2\pi \right\}$$

Then

$$(\psi_{-1} \times \psi_{-1})^{-1}(T(\xi, \zeta)) = \left\{ \left(\sqrt{\frac{1-2\xi}{2}} e^{i\theta_1}, \sqrt{\frac{1-2\zeta}{2}} e^{i\theta_2} \right) \in \mathbb{C}^2 \mid 0 \leq \theta_1, \theta_2 \leq 2\pi \right\}$$

which is the fiber over $(1-2\xi, 1-2\zeta)$ under the moment map of $\mathbb{C}^2$.

One can easily check that

$$(\psi_{-1} \times \psi_{-1})^{-1}(T(\xi, \zeta)) \subset F^{-1}(S^1(r)) \cap H^{-1}(\{-p\}).$$

Let (z_1, z_2) be a point in $F^{-1}(S^1(r)) \cap H^{-1}(\{-p\})$. Write z_j as $r_j e^{i\theta_j}$ for $j = 1, 2$. Then $r_1 r_2 = r$ and $r_1^2 - r_2^2 = -p$. We can solve

$$r_1 = \sqrt{\frac{-p + \sqrt{p^2 + 4r^2}}{2}} = \sqrt{\frac{1-2\xi}{2}}$$

and

$$r_2 = \sqrt{\frac{p + \sqrt{p^2 + 4r^2}}{2}} = \sqrt{\frac{1-2\zeta}{2}}.$$

Thus $(z_1, z_2) \in (\psi_{-1} \times \psi_{-1})^{-1}(T(\xi, \zeta))$. □

Proposition 3.1.8 ([Lou24], Proposition 2.8). *The Lagrangian torus $L_1(x, y)$ is Hamiltonian isotopic to a toric fiber of the standard toric structure on $S^2 \times S^2$.*

Proof. By Lemma 3.1.5 and Proposition 3.1.6, $\tilde{L}_2(p, q)$ is Hamiltonian isotopic to $F^{-1}(S^1(r)) \cap H^{-1}(\{-p\})$ by $\phi_{K \circ F}^t$ where K is a Hamiltonian such that Γ and $S^1(r)$ are Hamiltonian isotopic in $(B^2(1), \omega^p)$. Then

$$L_2(p, q) = (\psi_{-1} \times \psi_{-1}) \left(\tilde{L}_2(p, q) \right)$$

is Hamiltonian isotopic to $(\psi_{-1} \times \psi_{-1})(F^{-1}(S^1(r)) \cap H^{-1}(\{-p\}))$ that is a toric fiber by Lemma 3.1.7. By Proposition 3.1.2, we have $L_1(x, y) = L_1(p, q)$ is Hamiltonian isotopic to a toric fiber. $\square$

Next we are going to figure out which fiber $L_1(x, y)$ is Hamiltonian isotopic to. By Proposition 3.1.6 Γ and $S^1(r)$ should enclose the same ω^p -area where $\omega^p = \frac{2r}{\sqrt{p^2+4r^2}} dr \wedge d\phi$.

Proposition 3.1.9 ([Lou24], Proposition 2.9). *The ω^p -area enclosed by Γ is $2\pi - 2\pi q$ for $0 < p^2 < q^4$.*

Proof. See Appendix. $\square$

Proposition 3.1.10 ([Lou24], Proposition 2.10). *The Lagrangian torus $L_1(p, q)$ is Hamiltonian isotopic to the toric fiber*

$$T(\xi, \zeta) = \begin{cases} T(q - \frac{1}{2}, q - p - \frac{1}{2}) & \text{for } 0 < p < q^2 \\ T(p + q - \frac{1}{2}, q - \frac{1}{2}) & \text{for } -q^2 < p < 0 \end{cases}$$

Proof. The explicit expression of the curve Γ is given in Appendix. Then we note that Γ rotates clockwise as θ changes from 0 to 2π . Thus we choose $S^1(r)$ to rotate clockwise. Then the ω^p -area of $S^1(r)$ is $\pi|p| - \pi\sqrt{p^2+4r^2}$. Since Γ and $S^1(r)$ have the same ω^p -area, we have

$$r^2 = (q - 1)^2 - |p|(q - 1)$$

According to Lemma 3.1.7 $L_1(p, q)$ is Hamiltonian isotopic to the fiber over the point (ξ, ζ) where $\xi = \frac{1+p-\sqrt{p^2+4r^2}}{2}$ and $\zeta = \frac{1-p-\sqrt{p^2+4r^2}}{2}$. Note that $0 < q < 1$. We have

$$\xi = \begin{cases} q - \frac{1}{2} & \text{for } 0 < p < q^2 \\ p + q - \frac{1}{2} & \text{for } -q^2 < p < 0 \end{cases} \quad \text{and} \quad \zeta = \begin{cases} q - p - \frac{1}{2} & \text{for } 0 < p < q^2 \\ q - \frac{1}{2} & \text{for } -q^2 < p < 0 \end{cases}$$

See Figure 3.1 and Figure 3.2. $\square$

3.1.2 Case 2: $p^2 \geq q^4$

We will use symmetric probes introduced by M. Abreu, M. Borman and D. McDuff in [ABM14], generalizing the definition of probes introduced by D. McDuff in [Maroo].

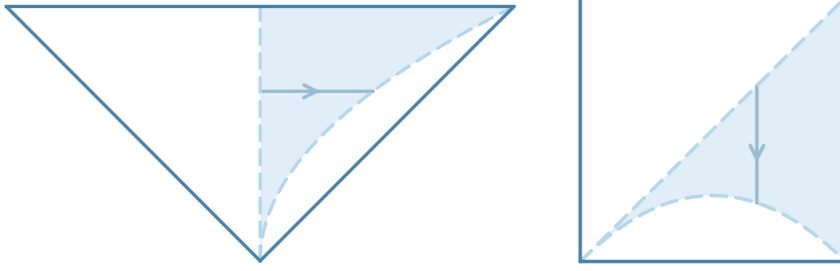


Figure 3.1: The case where $0 < p < q^2$
 Since $\xi = q - \frac{1}{2}$ and $\zeta = q - p - \frac{1}{2}$ then $-\xi^2 - \frac{1}{4} < \zeta < \xi$

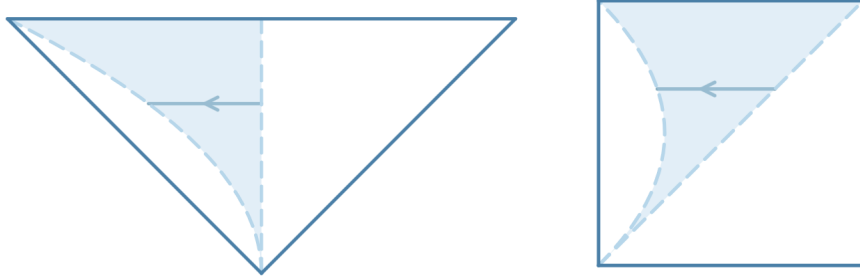


Figure 3.2: The case where $-q^2 < p < 0$
 Since $\xi = p + q - \frac{1}{2}$ and $\zeta = q - \frac{1}{2}$ then $-\zeta^2 - \frac{1}{4} < \xi < \zeta$

Definition 3.1.11. A *probe* P in a rational polytope $\Delta \subset \mathbb{R}^n$ is a directed rational line segment contained in Δ whose initial point b_P lies in the interior of a facet F_P of Δ and whose direction vector $v_P \in \mathbb{Z}^n$ is primitive and integrally transverse to the facet F_P . A probe P is *symmetric* if the endpoint e_P lies on the interior of a facet F'_P that is integrally transverse to v_P .

In [Bre23], J. Brendel proved that for two points x and x' in the symmetric probe, equidistant from the boundary of the probe, the toric fibers over these two points are Hamiltonian isotopic. His result was proven for the toric manifolds. In our case we will take a probe not passing through the singularity, then the result will also hold. We give the proof for the special case used to prove our result.

Proposition 3.1.12 ([Lou24], Proposition 2.12). *Given the symmetric probe $\sigma = \{p = a\}$, $-1 < a < 1$ and $a \neq 0$, in the polytope P_2 , let $(a, q_1), (a, q_2) \in \sigma$*

be at equal distance to the boundary of the symmetric probe, then $L_2(a, q_1)$ and $L_2(a, q_2)$ are Hamiltonian isotopic.

Proof. In [OU16], J. Oakley and M. Usher gave the moment map of $\hat{F}(0)$ as $(F + G, 1 - F)$ where $F(v, w) = \frac{1}{2}|v + w|$ and $G(v, w) = \frac{1}{2}(v_1 + w_1)$ with the moment polytope

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 - 2y, y \geq 0\}$$

Recall we changed the coordinates by $p = x + y - 1$ and $q = 1 - y$. Then the preimage Z of $\sigma \cap P_2$ under the moment map is

$$Z = \{(v, w) \in S^2 \times S^2 \mid v_1 + w_1 = 2a\}$$

We will apply toric reduction to Z as in [Bre23, Theorem 2.4]. Recall the moment polytope is in $\mathfrak{t}^* \cong \mathbb{R}^2$. We denote the two generators of $\mathfrak{t}^*$ by e_1^* and e_2^* . Let $K = \exp(e_1) \times \{1\}$. Then K acts freely on Z . In fact, given $\theta \in K$

$$\begin{aligned} & \theta \cdot ((v_1, v_2, v_3), (2a - v_1, w_2, w_3)) \\ &= \left(\begin{bmatrix} v_1 \\ v_2 \cos(\theta) - v_3 \sin(\theta) \\ v_2 \sin(\theta) + v_3 \cos(\theta) \end{bmatrix}, \begin{bmatrix} 2a - v_1 \\ w_2 \cos(\theta) - w_3 \sin(\theta) \\ w_2 \sin(\theta) + w_3 \cos(\theta) \end{bmatrix} \right) \end{aligned}$$

Then

$$\theta \cdot ((v_1, v_2, v_3), (2a - v_1, w_2, w_3)) = ((v_1, v_2, v_3), (2a - v_1, w_2, w_3))$$

implies that $\theta = 0$ or $v_2 = v_3 = w_2 = w_3 = 0$. In the later case $v_1 = \pm 1$ and $2a - v_1 = \pm 1$. Then $a = -1, 0$, or 1 contradicting with the assumption of a .

Thus Z/K is a toric manifold with toric action given by $\{1\} \times \exp(e_2)$ with moment polytope $\sigma \cap P_2$. Then Z/K is a sphere and the preimages of (a, q_1) and (a, q_2) under the moment map are two circles, denoted by S_{q_1} and S_{q_2} respectively. Since the two points (a, q_1) and (a, q_2) are at equal distance to the boundary of the symmetric probe, then S_{q_1} and S_{q_2} bound the disks with the same area on Z/K . Thus S_{q_1} and S_{q_2} are Hamiltonian isotopic. Then we lift the corresponding Hamiltonian to Z and extend it to the whole manifold $S^2 \times S^2$ by cutoff function, see [AM13], [Bre20]. $\square$

Now $L_1(p, q)$ is Hamiltonian isotopic to $L_1(p, 1 - q + |p|)$. By Proposition 3.1.10, $L_1(p, 1 - q + |p|)$ is Hamiltonian isotopic to

$$\begin{cases} T\left(\frac{1}{2} - q + p, \frac{1}{2} - q\right) & \text{for } q^2 \leq p < q \\ T\left(\frac{1}{2} - q, \frac{1}{2} - q - p\right) & \text{for } -q < p \leq -q^2 \end{cases}.$$

As in [Bre23], in the standard toric structure of $S^2 \times S^2$, the toric fiber

$$\begin{cases} T\left(\frac{1}{2} - q + p, \frac{1}{2} - q\right) & \text{for } q^2 \leq p < q \\ T\left(\frac{1}{2} - q, \frac{1}{2} - q - p\right) & \text{for } -q < p \leq -q^2 \end{cases}$$

is Hamiltonian isotopic to

$$\begin{cases} T\left(q - \frac{1}{2}, q - p - \frac{1}{2}\right) & \text{for } q^2 \leq p < q \\ T\left(q + p - \frac{1}{2}, q - \frac{1}{2}\right) & \text{for } -q < p \leq -q^2 \end{cases}.$$

See Figure 3.3 and Figure 3.4. Thus we have proven the following proposition.

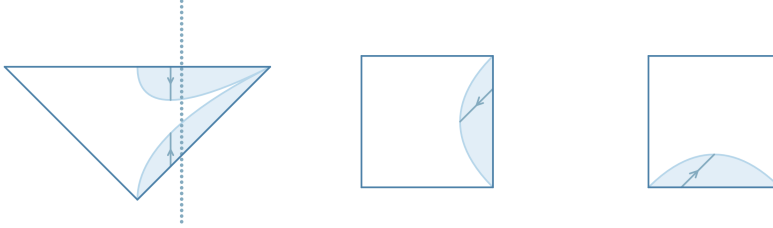


Figure 3.3: The case where $q^2 \leq p < q$
Since $\xi = q - \frac{1}{2}$ and $\zeta = q - p - \frac{1}{2}$ then $\xi - \zeta = p$

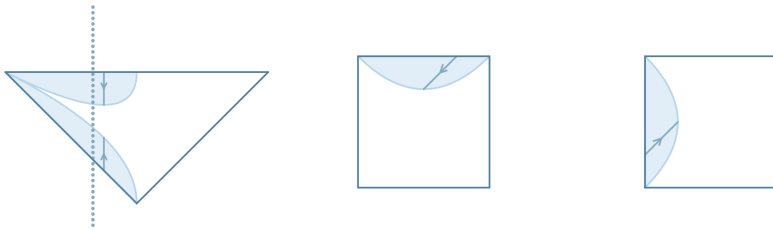


Figure 3.4: The case where $-q < p \leq -q^2$
Since $\xi = q + p - \frac{1}{2}$ and $\zeta = q - \frac{1}{2}$ then $\xi - \zeta = p$

Proposition 3.1.13 ([Lou24], Proposition 2.13). *The Lagrangian torus $L_1(p, q)$ is Hamiltonian isotopic to the toric fiber*

$$T(\xi, \zeta) = \begin{cases} T\left(q - \frac{1}{2}, q - p - \frac{1}{2}\right) & \text{for } q^2 \leq p < q \\ T\left(q + p - \frac{1}{2}, q - \frac{1}{2}\right) & \text{for } -q < p \leq -q^2 \end{cases}$$

Finally, when we change the coordinates (p, q) back to (x, y) , we get Theorem B.

3.2 Proof of Theorem C

In Theorem C, $x + y = 1$, which means $p = 0$. In [Fuk+12], they have proven that $L_1(0, q)$ with $0 < q \leq \frac{1}{2}$ are not Hamiltonian isotopic to toric fibers for the standard toric structure. Thus we focus on $L_1(0, q)$ with $\frac{1}{2} < q < 1$. Based on Theorem B, the moment polytope P_1 for the standard toric structure has been filled in by $L_1(p, q)$ with $p \neq 0$ except the diagonal. Thus if $L_1(0, q)$ is Hamiltonian isotopic to a fiber $T(\xi, \zeta)$ in the standard toric structure, it must be able to be Hamiltonian isotopic to some $T(\xi, \xi)$. Otherwise, assume $L_1(0, q)$ is Hamiltonian isotopic to some $T(\xi, \zeta)$ with $|\xi| \neq |\zeta|$. Denote the Hamiltonian isotopy by ϕ_S^t . By Weinstein's Lagrangian neighborhood theorem, there is a symplectomorphism from a neighborhood of $T(\xi, \zeta)$ to a neighborhood of the zero section of $T^*T(\xi, \zeta)$ which takes a Lagrangian torus C^1 -close to $T(\xi, \zeta)$ to the image of a closed 1-form in $T^*T(\xi, \zeta)$. Denote the 1-form corresponding to $\phi_S^1(L_1(\epsilon, q))$ by λ_1 for sufficiently small ϵ and the 1-forms corresponding to toric fibers $T(\xi', \zeta')$ by $\lambda_{\xi' - \xi, \zeta' - \zeta}$ for (ξ', ζ') close enough to (ξ, ζ) . Then there is toric fiber $T(\xi', \zeta')$ such that

$$[\lambda_{\xi' - \xi, \zeta' - \zeta}] = [\lambda_1] \in H^1(T(\xi, \zeta), \mathbb{R}).$$

Thus there is a smooth function

$$h : T(\xi, \zeta) \rightarrow \mathbb{R}$$

such that

$$\lambda_{\xi' - \xi, \zeta' - \zeta} - \lambda_1 = dh.$$

Then $\Pi \circ h$ generates a Hamiltonian isotopy between $\phi_S^1(L_1(\epsilon, q))$ and $T(\xi', \zeta')$ where $\Pi : T^*T(\xi, \zeta) \rightarrow T(\xi, \zeta)$ is the projection. On the other hand, $L_1(\epsilon, q)$ is Hamiltonian isotopic to $T\left(q - \frac{1}{2}, q - \epsilon - \frac{1}{2}\right)$ for positive ϵ by Theorem B.

If ϵ is small enough, $T(\xi', \zeta')$ is not Hamiltonian isotopic to $T(q - \frac{1}{2}, q - \epsilon - \frac{1}{2})$ by [Bre23]. We get a contradiction.

Then we will compute the displacement energy germ introduced in [CS10] to show that it is not possible that $L_1(0, q)$ is Hamiltonian isotopic to some $T(\xi, \xi)$.

Definition 3.2.1 ([CS10]). Let (M, ω) be a symplectic manifold and L be a closed embedded Lagrangian submanifold. The *displacement energy germ* is a function germ

$$S_L^e : H^1(L, \mathbb{R}) \rightarrow [0, \infty]$$

at the point $0 \in H^1(L, \mathbb{R})$ defined as

$$S_L^e(\delta) = e(L_\delta)$$

where L_δ is the image of a closed 1-form on L representing a sufficiently small class $\delta \in H^1(L, \mathbb{R})$ and $e(L_\delta)$ is the displacement energy of L_δ .

Proposition 3.2.2 ([CS10]). *For each symplectomorphism ψ we have*

$$S_{\psi(L)}^e = S_L^e \circ (\psi|_L)^*.$$

Note that the displacement energy of $T(\xi, \zeta)$ with $(\xi, \zeta) \neq (0, 0)$ is

$$\min \left\{ \frac{1}{2} - |\xi|, \frac{1}{2} - |\zeta| \right\}$$

since the displacement energy of the circle $v_1 = a (\neq 0)$ in S^2 is

$$\min \left\{ \frac{1}{2} - \frac{|a|}{2} \right\}.$$

Also see [Bre20, Example 4.1].

Now we compute the displacement energy germ of $L_1(0, q)$. Since

$$H^1(L_1(0, q), \mathbb{R}) \cong H_1(L_1(0, q), \mathbb{R}),$$

then a neighborhood of $0 \in H^1(L_1(0, q), \mathbb{R})$ can be identified with a neighborhood of the point $(0, q)$ in the moment polytope P_2 . Let $(\delta_1, \delta_2) \in \mathbb{R}^2$ be close enough to $(0, 0)$ and $\delta_1 \neq 0$. Then

$$S_{L_1(0, q)}^e(\delta_1, \delta_2) = e(L_1(\delta_1, q + \delta_2)).$$

First we consider the case where $\delta_1 > 0$. By Theorem B, $L_1(\delta_1, q + \delta_2)$ is Hamiltonian isotopic to the toric fiber $T(q + \delta_2 - \frac{1}{2}, q + \delta_2 - \delta_1 - \frac{1}{2})$. Thus the displacement energy is

$$e(L_1(\delta_1, q + \delta_2)) = \min \left\{ \frac{1}{2} - \left| q + \delta_2 - \frac{1}{2} \right|, \frac{1}{2} - \left| q + \delta_2 - \delta_1 - \frac{1}{2} \right| \right\}$$

Since $q > \frac{1}{2}$, we can choose δ_1 and δ_2 small enough such that $q + \delta_2 - \frac{1}{2} > 0$ and $q + \delta_2 - \delta_1 - \frac{1}{2} > 0$. Thus the displacement energy is

$$\begin{aligned} e(L_1(\delta_1, q + \delta_2)) &= \min \left\{ \frac{1}{2} - \left(q + \delta_2 - \frac{1}{2} \right), \frac{1}{2} - \left(q + \delta_2 - \delta_1 - \frac{1}{2} \right) \right\} \\ &= \min \{ 1 - q - \delta_2, 1 - q - \delta_2 + \delta_1 \} \\ &= 1 - q - \delta_2 \end{aligned}$$

The last equality is from $\delta_1 > 0$.

Then we consider the case where $\delta_1 < 0$. By Theorem B, $L_1(\delta_1, q + \delta_2)$ is Hamiltonian isotopic to the toric fiber $T(q + \delta_1 + \delta_2 - \frac{1}{2}, q + \delta_2 - \frac{1}{2})$. The displacement energy is

$$\begin{aligned} e(L_1(\delta_1, q + \delta_2)) &= \min \left\{ \frac{1}{2} - \left(q + \delta_1 + \delta_2 - \frac{1}{2} \right), \frac{1}{2} - \left(q + \delta_2 - \frac{1}{2} \right) \right\} \\ &= \min \{ 1 - q - \delta_1 - \delta_2, 1 - q - \delta_2 \} \\ &= 1 - q - \delta_2 \end{aligned}$$

The last equality is from $\delta_1 < 0$

Thus

$$S_{L_1(0,q)}^e(\delta_1, \delta_2) = 1 - q - \delta_2$$

when $\delta_1 \neq 0$

Next we compute the displacement energy germ of $T(\xi, \xi)$. Let $(\delta'_1, \delta'_2) \in \mathbb{R}^2$ be close enough to $(0, 0)$. If $\xi > 0$, then $\xi + \delta'_1 > 0$ and $\xi + \delta'_2 > 0$ for small enough δ'_1 and δ'_2 . Thus

$$\begin{aligned} S_{T(\xi,\xi)}^e(\delta'_1, \delta'_2) &= e(T(\xi + \delta'_1, \xi + \delta'_2)) \\ &= \min \left\{ \frac{1}{2} - \xi - \delta'_1, \frac{1}{2} - \xi - \delta'_2 \right\}. \end{aligned}$$

If $\xi < 0$, then $\xi + \delta'_1 < 0$ and $\xi + \delta'_2 < 0$ for small enough δ'_1 and δ'_2 . Thus

$$\begin{aligned} S_{T(\xi, \xi)}^e(\delta'_1, \delta'_2) &= e(T(\xi + \delta'_1, \xi + \delta'_2)) \\ &= \min \left\{ \frac{1}{2} + \xi + \delta'_1, \frac{1}{2} + \xi + \delta'_2 \right\}. \end{aligned}$$

The displacement energy germ of $T(\xi, \xi)$ is determined by two linearly independent functions but the displacement energy germ of $L_1(0, q)$ is determined by a single function when $\delta_1 \neq 0$. Thus there is not a linear isomorphism on $\mathbb{R}^2$ taking $S_{L_1(0, q)}^e(\delta_1, \delta_2)$ to $S_{T(\xi, \xi)}^e(\delta'_1, \delta'_2)$. Thus $L(0, q)$ and $T(\xi, \xi)$ are not symplectomorphic, in particular, not Hamiltonian isotopic.

APPENDIX A

AREA ENCLOSED BY A CURVE

Lemma A.o.i. *Let z be a point in the curve Γ . Then*

$$||z|| = \sqrt{\frac{1}{4} \left(1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - \frac{1}{4} p^2}$$

Proof. First we write down the explicit expression of the curve Γ

$$\Gamma = \left\{ \frac{1 - 2q^2 + p^2 - \frac{(q^2 - p^2)(1 - q^2)}{q^2} \sin^2(\theta) + 2i\sqrt{q^2 - p^2} \sqrt{1 - q^2} \cos(\theta)}{2\sqrt{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2}} \mid 0 \leq \theta \leq 2\pi \right\}$$

For a point $z = (x, y)$ in Γ ,

$$\begin{aligned} x &= \frac{1 - 2q^2 + p^2 - \frac{(q^2 - p^2)(1 - q^2)}{q^2} \sin^2(\theta)}{2\sqrt{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2}} \\ &= -\frac{1}{2} \sqrt{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} \\ &\quad + \frac{1 - q^2 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)}{\sqrt{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2}} \end{aligned}$$

$$\begin{aligned}
x^2 &= \frac{1}{4} \left(\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2 \right) \\
&\quad + \frac{\left(1 - q^2 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} \\
&\quad - 1 + q^2 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)
\end{aligned}$$

and

$$\begin{aligned}
y &= \frac{2\sqrt{q^2 - p^2} \sqrt{1 - q^2} \cos(\theta)}{2\sqrt{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2}} \\
y^2 &= \frac{(q^2 - p^2)(1 - q^2) \cos^2(\theta)}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2}
\end{aligned}$$

Note that

$$\frac{\left(1 - q^2 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} + \frac{(q^2 - p^2)(1 - q^2) \cos^2(\theta)}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} = 1 - q^2$$

Then

$$x^2 + y^2 = \frac{1}{4} \left(1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - \frac{1}{4} p^2$$

□

Proposition A.o.2. *The ω^p -area enclosed by Γ is $2\pi - 2\pi q$ for $0 < p^2 < q^4$.*

Proof. First we determine a 1-form σ such that $d\sigma = \omega^p$. Note that σ can be arranged to have the form

$\left(\frac{\sqrt{p^2 + 4r^2}}{2} + C \right) d\phi$. On one hand we can compute

$$\int_{B^2(1)} \frac{2r}{\sqrt{p^2 + 4r^2}} dr \wedge d\phi = \pi \sqrt{p^2 + 4} - \pi |p|$$

On the other hand,

$$\int_{B^2(1)} \frac{2r}{\sqrt{p^2 + 4r^2}} dr \wedge d\phi = \int_{S^1} \left(\frac{\sqrt{p^2 + 4 \cdot 1^2}}{2} + C \right) d\phi = \pi\sqrt{p^2 + 4} + 2\pi C$$

Thus $C = -\frac{|p|}{2}$ and $\sigma = \left(\frac{\sqrt{p^2 + 4r^2}}{2} - \frac{|p|}{2} \right) d\phi$.

Now we compute $\int_{\Gamma} \sqrt{p^2 + 4r^2} d\phi = \int_{\Gamma} \sqrt{p^2 + 4r^2} \frac{d\phi}{d\theta} d\theta$. Note that $1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) > 0$. By Lemma A.o.I, we have

$$\sqrt{p^2 - 4r^2} = 1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)$$

By taking the derivative with respect to θ on both sides of

$$\tan(\phi) = \frac{y}{x} = \frac{2\sqrt{q^2 - p^2} \sqrt{1 - q^2} \cos(\theta)}{1 - 2q^2 + p^2 - \frac{(q^2 - p^2)(1 - q^2)}{q^2} \sin^2(\theta)}$$

we can get

$$\begin{aligned} \frac{d\phi}{d\theta} &= \frac{\frac{-2\sqrt{q^2 - p^2} \sqrt{1 - q^2} \sin(\theta)}{q}}{\left[\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2 \right] \cdot \left[\left(1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2 \right]} \\ &\quad \cdot \left(\frac{p^2 - q^2 + p^2(1 - q^2)}{q} + \frac{(q^2 - p^2)(1 - q^2)}{q} \sin^2(\theta) \right) \\ &= -\frac{1}{2} \left(\frac{1}{\left(1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} - \frac{1}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} \right) \\ &\quad \cdot \left(\frac{p^2 - q^2 + p^2(1 - q^2)}{q} + \frac{(q^2 - p^2)(1 - q^2)}{q} \sin^2(\theta) \right) \end{aligned}$$

Now

$$\begin{aligned}
& \int_{\Gamma} \sqrt{p^2 + 4r^2} d\phi \\
&= \int_0^{2\pi} -\frac{1}{2} \frac{1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)}{\left(1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)\right)^2 - p^2} \\
&\quad \cdot \left(\frac{p^2 - q^2 + p^2(1 - q^2)}{q} + \frac{(q^2 - p^2)(1 - q^2)}{q} \sin^2(\theta) \right) d\theta \\
&\quad - \int_0^{2\pi} -\frac{1}{2} \frac{1 - \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)\right)^2 - p^2} \\
&\quad \cdot \left(\frac{p^2 - q^2 + p^2(1 - q^2)}{q} + \frac{(q^2 - p^2)(1 - q^2)}{q} \sin^2(\theta) \right) d\theta
\end{aligned}$$

In the first integral we replace θ with $2\pi - \theta$. Then

$$\begin{aligned}
& \int_{\Gamma} \sqrt{p^2 + 4r^2} d\phi \\
&= -\frac{1}{2} \int_0^{2\pi} \left(\frac{p^2 - q^2 + p^2(1 - q^2)}{q} + \frac{(q^2 - p^2)(1 - q^2)}{q} \sin^2(\theta) \right) \\
&\quad \cdot \frac{2 \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)\right)^2 - p^2} d\theta
\end{aligned}$$

Note that

$$\begin{aligned}
& \frac{p^2 - q^2 + p^2(1 - q^2)}{q} + \frac{(q^2 - p^2)(1 - q^2)}{q} \sin^2(\theta) \\
&= q \left(\frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) + 1 \right)^2 \\
&\quad + q \left[-p^2 - 2 \frac{q^2 - p^2}{q^2} - 2 \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right]
\end{aligned}$$

Then

$$\begin{aligned}
& \int_{\Gamma} \sqrt{p^2 + 4r^2} d\phi \\
&= -\frac{q}{2} \int_0^{2\pi} 2 \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) d\theta \\
&+ q \int_0^{2\pi} \left(\frac{q^2 - p^2}{q^2} + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right) \\
&\quad \cdot \frac{2 \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} d\theta \\
&= q \int_0^{2\pi} \left(\frac{q^2 - p^2}{q^2} + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right) \\
&\quad \cdot \frac{2 \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta)}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} d\theta
\end{aligned}$$

since $\int_0^{2\pi} 2 \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) d\theta = 0$. Note that

$$\begin{aligned}
& \left(\frac{q^2 - p^2}{q^2} + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right) \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \\
&= \left(\frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) + 1 \right)^2 - p^2 \\
&- \frac{p^2 + q^2}{q^2} \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) - 1 + p^2
\end{aligned}$$

We have

$$\begin{aligned}
\int_{\Gamma} \sqrt{p^2 + 4r^2} d\phi &= 2q \int_0^{2\pi} 1 d\theta + 2q \int_0^{2\pi} \frac{-\frac{p^2 + q^2}{q^2} \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) - 1 + p^2}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} d\theta \\
&= 4\pi q - 2q \int_0^{2\pi} \frac{\frac{p^2 + q^2}{q^2} \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) + 1 - p^2}{\left(1 + \frac{\sqrt{q^2 - p^2} \sqrt{1 - q^2}}{q} \sin(\theta) \right)^2 - p^2} d\theta
\end{aligned}$$

Now we focus on the integral

$$\begin{aligned}
& \int_0^{2\pi} \frac{\frac{p^2+q^2}{q^2} \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta) + 1 - p^2}{\left(1 + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta)\right)^2 - p^2} d\theta \\
&= \frac{(1+p)(q^2+p)}{2q^2} \int_0^{2\pi} \frac{1}{1+p + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta)} d\theta \\
&+ \frac{(1-p)(q^2-p)}{2q^2} \int_0^{2\pi} \frac{1}{1-p + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta)} d\theta
\end{aligned}$$

One can easily compute

$$\int_0^{2\pi} \frac{1}{1+p + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta)} d\theta = \frac{2\pi q}{q^2+p}$$

and

$$\int_0^{2\pi} \frac{1}{1-p + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta)} d\theta = \frac{2\pi q}{q^2-p}$$

Thus

$$\int_0^{2\pi} \frac{\frac{p^2+q^2}{q^2} \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta) + 1 - p^2}{\left(1 + \frac{\sqrt{q^2-p^2}\sqrt{1-q^2}}{q} \sin(\theta)\right)^2 - p^2} d\theta = \frac{2\pi}{q}$$

Then

$$\int_{\Gamma} \sqrt{p^2 + 4r^2} d\phi = 4\pi q - 4\pi$$

Next one can follow the same process to show that

$$\int_{\Gamma} \frac{|p|}{2} d\phi = 0$$

Thus if the curve Γ is reparametrized by replacing θ with $2\pi - \theta$, then the ω^p -area enclosed by Γ is $2\pi - 2\pi q$. $\square$

Remark A.o.3. If we write the 1-form $\frac{\sqrt{p^2+4r^2}}{2} d\phi$ as $\frac{-y\sqrt{p^2+4x^2+4y^2}}{2(x^2+y^2)} dx + \frac{x\sqrt{p^2+4x^2+4y^2}}{2(x^2+y^2)} dy$,

then it is easy to see that $\frac{\sqrt{p^2+4r^2}}{2} d\phi$ is not defined at $(0, 0)$. On the other hand

$\left(\frac{\sqrt{p^2+4r^2}}{2} - \frac{|p|}{2}\right) d\phi = \frac{-2y}{\sqrt{p^2+4x^2+4y^2}+|p|} dx + \frac{2x}{\sqrt{p^2+4x^2+4y^2}+|p|} dy$ is defined at $(0, 0)$.

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