PROSPECTIVE TEACHERS' BELIEFS ABOUT MATHEMATICAL AUTHORITY AND THE INFLUENCE STUDENT TEACHING HAS ON THEIR BELIEFS

by

MICHAEL HAMILTON

(Under the Direction of AnnaMarie Conner)

ABSTRACT

This descriptive and interpretive multi-case study investigated secondary prospective teachers' (PTs') beliefs about mathematical authority and how their student teaching experience(s) influenced their beliefs. The theoretical framework guiding this study was comprised of a novel definition for mathematical authority I developed, a conception of beliefs as informing an individual's actions (Rokeach, 1968), and a theorization of how individuals hold beliefs in a system of beliefs (Green, 1971). Six PTs who student taught during the Fall 2021 semester were selected to participate in the study, four of whom are discussed in this dissertation. For each PT, data sources included three semi-structured interviews prior to student teaching, a lesson plan interview during student teaching, three semi-structured interviews after student teaching, PT generated mathematical authority diagrams prior to and after student teaching, and artifacts from their student teaching practicum (e.g., lesson plans and weekly reflections). To understand each PT's beliefs, all data was initially coded with broad codes informed by prior research on mathematics teachers' beliefs and my theoretical framework. Through an iterative process of coding and employing the constant comparative method (Strauss & Corbin, 1998), new codes emerged and the coding scheme was refined. Coded data informed my written narratives of each

PT's beliefs, which are presented as the results of this study. The participating PTs believed numerous sources can be mathematical authorities in the classroom, but they mainly focused on the teacher and students as mathematical authorities. The PTs had varying beliefs about the teacher and students as mathematical authorities, but across the four PTs, each believed the teacher is the main mathematical authority in the classroom and believed all students can be mathematical authorities. Conducting a cross-case analysis of the PTs' belief systems enabled me to make inferences for differences in what each PT believed, particularly their beliefs about students as mathematical authorities. Student teaching seemed to reinforce the beliefs of each PT and provided a continued context for each PT to develop a pedagogy informed by their beliefs. Implications for mathematics teachers, mathematics teacher educators, and mathematics education researchers are discussed.

INDEX WORDS: Mathematical Authority, Teachers' Beliefs, Prospective Teachers, Teacher Education, Student Teaching

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## DEDICATION

I dedicate this dissertation and all the work required to make this dissertation possible to my wife Lane and my children: Roan, Bennett, Henry, and Heidi. Through all of the successes and failures, you have supported me and motivated me to persevere through these challenging yet rewarding five years. Without your support and unconditional love, writing this dissertation would not have been possible.

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# CHAPTER 1

# BACKGROUND AND INTRODUCTION

The community of mathematics education researchers and teacher educators have long advocated for instructional strategies that have been shown to be effective in mathematics classrooms (National Council of Teachers of Mathematics [NCTM], 2014; Association of Mathematics Teacher Educators [AMTE], 2017). These strategies include, among others, engaging students in high-quality tasks that promote problem solving, reasoning with mathematical concepts and procedures, and discussions of student-generated solutions and ideas (NCTM, 2014). In contrast to more traditional modes of instruction, ones that tend to be teacher-centered and focused on the application of prescribed procedures with accuracy, student-centered instructional strategies require students to exercise their agency in the classroom, actively engage with mathematics, and build "his or her own mathematical knowledge from personal experiences" (NCTM, 2014, p. 9). The clear focus of these instructional strategies is on privileging and leveraging *students'* ideas, *students'* solutions, and *students'* experiences to support students as they construct their own mathematical meanings and understandings. From my own experiences as a high school mathematics teacher and teacher educator, I know first-hand that enacting such student-centric instructional strategies is neither an easy nor a straight-forward endeavor. Stein et al. (2008) captured this challenge well when they stated,

> since the advent of more student-centered, inquiry-based forms of instructional practice, teachers have struggled with how to orchestrate discussions in ways that both engage

students' sense-making in authentic ways and move the class as a whole toward the development of important and worthwhile ideas in the discipline (p. 332).

Researchers have argued that how authority operates in mathematics classrooms, particularly related to how students are positioned as mathematical authorities, has profound implications for the mathematics teaching and learning that occurs in those classrooms. Scholars have asserted that, rather than being passive consumers of mathematical facts and procedures, students should be positioned as legitimate contributors to the mathematics constructed in the classroom and ones who play a critical role in determining the validity and reasonableness of mathematical ideas, concepts, and procedures (Engle & Conant, 2002; Stein et al., 2008). Put differently, they argue that students should be positioned as mathematical authorities in the classroom. Other researchers have claimed that students' mathematical outcomes are improved as a result of being positioned as mathematical authorities in the classroom. For instance, Gresalfi and Cobb (2006) argued when students are positioned as mathematical authorities in the classroom, they engage in rigorous mathematical content and high levels of mathematical argumentation. Similarly, Kinser-Traut and Turner (2020) contended that positioning students as mathematical authorities is "key to ambitious mathematics teaching that builds on *students'* [emphasis added] knowledge and experiences" (p. 10). Hence, in contrast to modes of teaching where the teacher is positioned as the sole authority in the classroom, when teachers enact instructional strategies that position students as mathematical authorities, students have more and higher-quality opportunities to engage in mathematical discussions, problem solving, and reasoning. Moreover, when students are frequently viewed as mathematical authorities in the classroom, they are empowered "to argue, evaluate, and confirm the validity of *their* [emphasis added] mathematical ideas" (Dunleavy, 2015, pp. 63-64).

Other researchers have also maintained it is essential that teachers position students as mathematical authorities but have done so from a different perspective. Rather than considering the mathematical ideas and content students develop or engage in, these researchers have argued that positioning students as mathematical authorities can benefit students beyond the mathematics classroom. For instance, Wilson and Lloyd (2000) posited that an implication of positioning students as mathematical authorities in the classroom is that they take much of the responsibility for their own learning. Likewise, Depaepe et al. (2012) contended that students have "possibilities to act autonomously" (p. 224) when teachers enact more reform-oriented, student-centered practices. Hence, when teachers enact instructional practices that position students as mathematical authorities, students then have opportunities to develop into self-directed learners of mathematics who also, as they learn and mature, take ownership of their own learning.

As researchers have used an authority lens to examine and understand mathematics classrooms, some have also found that positioning students as mathematical authorities can lead to more equitable student outcomes in mathematics. For instance, Boaler and Staples (2008) found that as teachers consistently enacted reform-oriented practices, students achieved at higher levels on standardized tests and developed more positive dispositions towards mathematics. Similarly, Dunleavy (2015) reported how a high school mathematics teacher enacted instructional strategies that positioned students as mathematical authorities, which then led to more equitable student participation in her classroom. While the results reported by Dunleavy and Boaler and Staples validate some researchers' claims of the benefits of positioning students as mathematical authorities, evidence of teachers consistently and productively positioning students as mathematical authorities is scarce. For instance, other researchers have shown that mathematics teachers do not consistently enact instructional strategies that position students as mathematical authorities in their classrooms (e.g., Amit & Fried, 2005; Hamm & Perry, 2002). Likewise, researchers have documented instances in which teachers appear to struggle to consistently enact student-centered instructional practices that position students as mathematical authorities (e.g., Swan, 2007). Amit and Fried suggested the struggle teachers face when it comes to consistently and productively positioning students as mathematical authorities may be due to teachers feeling as if enacting reform-oriented practices requires "the diminution of teachers' authority" (p. 146).

In order to support teachers as they implement practices that productively position students as mathematical authorities, researchers and educators must first understand teachers' perceptions of mathematical authority relations in their classrooms (Wagner & Herbel-Eisenmann, 2014b). However, few researchers have considered teachers' perspectives of authority relations in the classroom. Studies that have investigated teachers' views of mathematical authority in classrooms did not investigate those teachers' beliefs about mathematical authority (e.g., Wagner & Herbel-Eisenmann, 2014a, 2014b; Wilson & Lloyd, 2000). Yet, if practices that productively position students as mathematical authorities are to be realized and sustained in mathematics classrooms, then understanding teachers' beliefs about mathematical authority is a crucial first step (Cross, 2009). A primary reason that understanding both in-service and prospective teachers' (PTs') beliefs is such an integral part in substantive, lasting change to instructional practice is that, as Rokeach (1968) put it, "all beliefs are predispositions to action" (p. 113). In other words, the beliefs that an individual holds lead to and inform any action they take. An implication, then, to understanding PTs' beliefs about mathematical authority, in particular, is that it can help mathematics teacher educators (MTEs) and mathematics education researchers understand why beginning teachers enact, or do not enact, instructional practices that position students as mathematical authorities. Such understandings can aid researchers and MTEs as they educate, develop, and support both PTs and novice teachers as they attempt to enact equitable, reform-oriented instruction in classrooms.

One significant aspect of understanding the beliefs PTs hold is understanding how their beliefs change over the course of their teacher education programs. Numerous scholars and educators have posited that one of the primary goals of teacher education should be to challenge and promote change to the beliefs PTs hold (e.g., Cooney, 1999; Green, 1971; Llinares, 2002). Following these suggestions, educators have designed courses and engaged PTs in field experiences in hopes of promoting productive changes to PTs' beliefs (e.g., Conner et al., 2011; Liljedahl et al., 2007; Mewborn, 1999). In many cases, researchers have studied changes to PTs' beliefs as a result of taking mathematics pedagogy and content courses, yet most of this research has been conducted with elementary PTs (e.g., Ambrose, 2004; Grootenboer, 2008; Liljedahl et al., 2007). Few researchers have investigated the change in PTs' beliefs as a result of taking at least one course (e.g., Conner et al., 2011; Cooney et al., 1998) and even fewer (cf. Vacc & Bright, 1999) have investigated the influence that a PT's student teaching practicum has on their beliefs.

Initially, the purpose of this study was to investigate practicing teachers' beliefs about mathematical authority and how those beliefs relate to their instructional practice. However, due to the onset of the COVID-19 pandemic and corresponding protocols in schools aimed to stop the spread of the virus, I decided to investigate prospective teachers' beliefs, rather than practicing teachers' beliefs. Thus, the primary purpose of this study was to examine the beliefs secondary PTs hold pertaining to mathematical authority relations in classrooms. After shifting the focus of this study to PTs' beliefs about mathematical authority, I also decided to investigate the influence a PT's student teaching practicum has on their beliefs. Hence, a secondary goal of this study was to extend the research on secondary mathematics PTs' beliefs by investigating how student teaching influences PTs' beliefs about mathematical authority. Specifically, the following research questions guided this study:

1) What do secondary mathematics PTs believe regarding who or what can be a mathematical authority in the classroom?
    a. To what extent or in what instances do PTs believe students can be mathematical authorities?
2) How do secondary mathematics PTs' beliefs about mathematical authority relate to their beliefs about mathematics, teaching mathematics, and students?
3) How does a PT's student teaching practicum influence their beliefs about mathematical authority?

In the next chapter, I review the literature pertinent to this study by reviewing the extant research on authority and mathematical authority in classrooms, then by reviewing the literature on mathematics teachers' beliefs. As I review these two bodies of literature, I also argue for this study's need and describe the theoretical framework that guided the design of this study and data analysis.

# CHAPTER 2

# LITERATURE REVIEW AND THEORETICAL FRAMEWORK

The primary aim of this study was to examine PTs' beliefs about mathematical authority and how those beliefs relate to beliefs about mathematics, teaching mathematics, and students. For this reason, this review of the literature is broken into two main sections. In the *Authority and Mathematical Authority* section I review research on authority and mathematical authority in educational settings and end by arguing for and providing a new definition of *mathematical authority*. In the *Conceptions of Beliefs and Research on Teachers' Beliefs* section, I consider differing conceptions of beliefs, review empirical research on mathematics teachers' beliefs, and end by discussing Leatham's (2006) sensible systems framework, which I used to conduct this study. Finally, I end this chapter summarizing these two sections and further argue for why studying PTs' beliefs about mathematical authority will practically and theoretically benefit the mathematics education community.

## Authority and Mathematical Authority

### Weberian Authority

Max Weber (1925/1947) provided a conception of authority that has been used and drawn upon by numerous education scholars. Weber, who was a German sociologist and political economist, defined authority, or *imperative control*, as "the probability that a command with a given specific content will be obeyed by a given group of persons" (p. 152). He claimed that a true authority relation exists when one party or individual, the subordinate, chooses to obey the commands or orders of another individual, the superordinate, when the superordinate is seen to have a valid claim to legitimacy by the subordinate. Put differently, for an authority relation to exist, the superordinate must have a legitimate claim to give orders or commands, and the subordinate must see the superordinate's claim to legitimacy as valid, choosing to obey the superordinate's orders on the basis of this legitimacy.

It is important to emphasize that Weber (1925/1947) perceived authority as a *relationship* between two parties that hinges on the *legitimacy* of the one who gives orders. Weber did not perceive authority as an inherent personal attribute or as a direct consequence of one's title (e.g., boss, teacher, Mayor). Although, as will be discussed shortly, the attributes of or title given to an individual can play a role in their right or ability to give orders that are obeyed, these alone do not determine an authority relation. Amit and Fried (2005) summarized Weber's conception of authority well when they wrote that an authority relation "has as much to do with those who obey it as it does with those who command it: a relationship of authority is a quasi-reciprocal one" (p. 147). In contrast, Weber defined *power* as being able to impose one's will against another, despite the belief or resistance of the other. Power, then, unlike authority, does not require the subordinate's belief in the legitimacy of the superordinate. Rather, power is something that someone can attain, accumulate, and exercise regardless of whether the subordinate believes the superordinate has a valid claim to legitimacy.

Because Weber's (1925/1947) conception of authority hinges on the notion of legitimacy, he defined three types of authority that "claims to legitimacy may be based on" (p. 328). The first type of authority, *legal-rational authority*, is based on established systems, rules, and laws, which uphold rational order and values. When citizens, as an example, follow the orders of a traffic cop, they may do so because they believe the cop is giving directives to maintain order on public roadways. The citizen views the legitimacy of the cop as valid based on the established laws and norms of the city or state in which the citizen lives and, thus, willingly follows the cop's orders. *Traditional authority*, Weber's second type of authority, is characterized by a personal loyalty to another on "an established belief in the sanctity of immemorial traditions" (p. 328). One example of traditional authority is that of a devout Catholic following the direction and guidance of the Pope. A devout Catholic follows such direction and guidance based on their religious belief which includes a traditional view of the Pope as the sacred figure head and leader of the Catholic Church. In Weber's final type of authority, termed *charismatic authority*, one who is viewed as exceptional is followed or obeyed on the basis of their heroism, desirable qualities, or exceptional ideas. The relationship between Martin Luther King Jr. and those who followed his example of nonviolent protest during the Civil Rights movement of the 1960s might be characterized as a charismatic authority relation.

Just as Weber (1925/1947) defined multiple types of authority, many education researchers investigating authority also define and study different types of authority. In the following section, I review authority research that has been carried out in mathematics classrooms. Before doing so, however, I want to note that mathematics education researchers who have investigated authority have done so by either studying *authority* and *types of authority*, like those defined by Weber, in mathematics classrooms *or* by studying *mathematical authority* in classrooms. I highlight this distinction to note there are subtle differences in research that has investigated *authority* relations in mathematics classrooms when compared to research that has explored *mathematical authority* relations. I highlight some of these difference in the following paragraphs by first reviewing the research that examines *authority* relations in mathematics classrooms while also attending to the constructs researchers defined and used. I then review research on *mathematical authority* and provide arguments for why I developed a novel definition of mathematical authority to carry out this study.

**Authority Research in Mathematics Education**

Researchers who study authority relations in mathematics classrooms have done so from either the students' perspective or from the teacher perspective. Examining authority relations from the students' perspective, Amit and Fried (2005) studied who students viewed as authorities as they engaged in and learned mathematics. As part of their theoretical framework, Amit and Fried used the three types of authority described by Weber (1925/1947; i.e., legal-rational, traditional, charismatic) and added a fourth type of authority, *expert authority*, which Amit and Fried described as "the authority of those who 'know their subject'" (p. 148). Amit and Fried found that students relied on numerous authorities, including their friends, parents, siblings, as well as their teacher. These various sources formed a hierarchy of authorities, and the teacher was unquestionably at the top. Furthermore, Amit and Fried found that students turned to the teacher as an authority on traditional, expert, and even charismatic grounds. In their study, Amit and Fried argued that students positioning the teacher as the primary or sole authority in the classroom made it "very easy for students to use concepts unreflectively" (p. 160). Rather than reflecting on and reasoning with mathematical concepts and procedures, students simply followed the procedures, instructions, and suggestions provided by the teacher. Hence, a key finding of Amit and Fried's study is that when the teacher is viewed by students as the sole or ultimate authority in the classroom, students have little need to consider and develop mathematical concepts and relationships for themselves, rather they need only to follow and apply the guidance and instruction of the teacher.

Also from the students' perspective, Langer-Osuna (2011, 2016) has examined authority relations among mathematics students as they engage in collaborative group work. In her 2011 study, Langer-Osuna never defined or described authority. However, in her 2016 study, Langer-Osuna defined *intellectual authority* and *directive authority*, but did not draw on Weber (1925/1947) to do so. *Intellectual authority* was understood "in terms of positioning students as credible sources of information pertinent to the particular task at hand" (Engle & Conant, 2002, as cited in Langer-Osuna, 2016, p. 109) and *directive authority* as an individual being able to give orders or directives that are followed by peers in a group setting.

In the two studies, Langer-Osuna (2011, 2016) found that authority relations among students during collaborative work influences student learning and engagement. In one study, Langer-Osuna (2016) investigated how authority relations developed as two students worked "in small group collaborative mathematics problem solving in elementary classrooms" (p. 107). Langer-Osuna found that one student was positioned as both an intellectual and directive authority in the pair of students and this positioning "constrained" the other student's "opportunities to engage in math sense making" (pp. 120-121). Additionally, she found that students' perceptions of "teacher evaluations of student ideas and behaviors" (p. 120) were highly influential in two ways: 1) when it came to how directive authority and intellectual authority were constructed among students and 2) how stable those authority relations became over time. In other words, the teacher's position as an authority in the classroom, particularly in relation to the two students, had great bearing on the development and stability of the authority relations between the two focal students.

In her other study, Langer-Osuna (2011) found that the development of two students' mathematical identities was linked to how their peers positioned them, or did not position them, as authorities. As one student, a male named Kofi, was increasingly positioned as an authority by his peers, he also engaged more in group work and, by the end of the year, identified as "a learner of mathematics and a leader" (p. 223). In contrast, a female student named Brianna in the same group, who was initially positioned as an authority by her peers, was later positioned as "bossy" because her peers felt she "was overstepping her authority by assigning tasks or trying to keep students [her peers] on task" (p. 212). In subsequent group projects, Brianna was no longer positioned as an authority by her peers, which led to a drastic decrease in her engagement during group work. In contrast with Kofi's identity, Langer-Osuna claimed that Brianna's identity as a student at the end of the year "was not focused on mathematics at all" (p. 221). Instead, Brianna identified as a "workaholic" who needed "to take more of a back seat in her education" (p. 223). As a result of this study, Langer-Osuna claimed that it may be important "to not only legitimize particular students' mathematical contributions but also to legitimize their displays of authority" (p. 223). Langer-Osuna stated that doing so may lead to increased engagement and the development of productive mathematical identities for some students, especially those whose "authority displays are often rejected as inappropriate" (p. 223).

Collectively, the studies conducted by Amit and Fried (2005) and Langer-Osuna (2011, 2016) reveal the influences authority relations can potentially have on students' engagement in mathematics. Students positioning one individual, whether that be the teacher or another student, as the ultimate or sole authority can have negative, likely unintended, consequences for student learning. In Amit and Fried's study, the teacher was positioned as the sole authority in the classroom which led to students using mathematical concepts and procedures in an unreflective manner. Similarly, Langer-Osuna (2016) showed how the position of one student as the primary authority during group work greatly restricted their peers' "opportunities to engage in math sense making" (pp. 120-121). These studies, then, serve as cautionary accounts of what can happen when one individual is positioned as the sole or ultimate authority by students in the classroom. Additionally, Langer-Osuna's (2011) findings underscore how the development, or in Brianna's case the deterioration, of authority relations can influence students' mathematical engagement in group work and, over time, whether they develop productive mathematical identities. However, it is important to point out that none of these studies considered the teacher's perceptions or beliefs about authority relations in their classrooms. While Langer-Osuna (2016) claimed that the classroom teacher "made particular discursive moves meant to hand intellectual authority to students" (pp. 121-122), it is unclear whether this was the actual intent of the teacher or if this was Langer-Osuna's inference about what the teacher intended. Without an understanding of the teacher's beliefs about authority in their classrooms, including the intent of certain instructional moves, authority relations may continue to have the negative, unintended consequences to student learning that Amit and Fried and Langer-Osuna revealed.

I now turn to authority research in mathematics classrooms that was conducted from the teacher's perspective. Similar to research from the student's perspective, research on authority conducted from the teacher's perspective is scarce. Thus, I review these studies by examining how researchers defined authority and summarize the key findings of each study. I then look across the studies, explicating common findings and note research areas that have yet to be considered through an authority lens.

Kinser-Traut and Turner (2020) used the notion of authority "to investigate patterns and shifts in one novice teacher's understandings and practices related to CMT (children's mathematical thinking) and CFoK (linguistic, cultural, and family funds of knowledge)" (p. 10). To conduct the study, Kinser-Traut and Turner drew on Weber (1925/1947) to define authority as “the right, or power, an individual has, either given or assumed, to shape learning and events within the classroom” (p. 9). Additionally, drawing on Gerson and Bateman (2010), they understood *sharing authority* “to mean when students feel empowered to engage more fully in learning” (Kinser-Traut & Turner, 2020, p. 9). They found that the teacher in their study shared authority with her students by privileging students’ mathematical ideas and using their ideas as the basis of classroom discussions. Through these practices, Kinser-Traut and Turner claimed the teacher was able to consistently connect to CMT. However, they found that the teacher did not share authority with students when it came to determining what to include or exclude from the mathematics curriculum, including what contexts to use in problems. Hence, the teacher did not draw on students’ experiences or interests while planning or enacting lessons and, as a result, did not often connect to CFoK. Kinser-Traut and Turner concluded that sharing authority with students was a “generative practice” that led to the enactment of more “ambitious mathematics teaching practices” (p. 29). The results of their study reveal that teachers may need further support when it comes to viewing their students as authorities who are “capable of generating mathematical ideas and bringing cultural and family experiences that support mathematics learning” (p. 29).

Wagner and Herbel-Eisenmann (2014a, 2014b) also conducted research on authority relations in classrooms from the teacher’s perspective. In their research, they used the definition of authority provided by Pace and Hemmings (2007): “a social relationship in which some people are granted the legitimacy to lead and others agree to follow” (p. 6). It should be noted that Pace and Hemmings leaned heavily on Weber’s (1925/1947) definitions of authority to construct their own. In addition to this definition of authority, Wagner and Herbel-Eisenmann defined and operationalized four authority structures in one of their studies (2014a). The first authority structure was *personal authority*, described as "the expectation that students follow the authority of their teacher " (p. 873). They described *discourse as authority* as suggesting that a discipline or set of rules, which had been formed "from outside personal relationships" (p. 873) were to be followed. Next, *discursive inevitability* was described as one assuming their next step, action, or thought was predetermined or inevitable. Finally, *personal latitude* was understood as one being positioned as an authority and, thus, able to make decisions for themselves.

In one study, Wagner and Herbel-Eisenmann (2014a) investigated how authority relations played out in one high school mathematics teacher's classroom. They found that all four authority structures, described in the previous paragraph, were operating in the teacher's classroom at different moments. Nevertheless, the *personal authority* structure was most prominent in the teacher's classroom. In other words, there was often an expectation that students followed the direction and guidance of the teacher without questioning. In one case, a student even relied on the teacher as an authority to determine "what to write and what not to write in his notes" (p. 877). However, there was a moment when the teacher, frustrated by "his students' lack of mathematical agency" (p. 880), had an explicit conversation with his students about authority. Wagner and Herbel-Eisenmann found that this move led to students taking ownership over their learning as well as increased engagement during class. Wagner and Herbel-Eisenmann claimed that these changes were evidenced by students seeking alternative solution strategies to problems and posing their own problems they wished to be explored. Thus, the teacher's explicit move to position students as authorities in the classroom led to students relying less on the teacher as the primary or sole authority in the classroom and "a shift toward students sharing authority for their own learning" (p. 881).

In their other study, Wagner and Herbel-Eisenmann (2014b) took a different approach to investigating authority relations in classrooms by studying how teachers represented authority in their classrooms. The primary method used in their study was teacher-generated diagrams of how they viewed authority operating in their classrooms. Teachers' diagrams were a valuable data source because they revealed how the participating teachers were thinking about authority and, as Wagner and Herbel-Eisenmann put it, "understanding how teachers think about authority must be the basis of teacher educators' work with teachers on issues of authority" (p. 203). They found that the 34 participating teachers represented authority in their classroom using very different diagrams and these diagrams included 65 "sources of authority" in total (p. 211). Included as sources of authority were individuals such as the teacher, students, and family; communities or organizations including NCTM, the Department of Education, and professional learning communities; and inanimate objects such as textbooks, calculators, and the mathematics curriculum. Hence, Wagner and Herbel-Eisenmann highlighted that there can be numerous potential sources of authority operating in one classroom. Moreover, the variety of sources of authority represented in the diagrams suggests that extant research on authority, which has focused solely on authority relations between teachers and students, may not be capturing or considering the various authority relations operating within mathematics classrooms. Wagner and Herbel-Eisenmann's study was unique from other investigations using an authority lens because they considered teachers' perceptions and thoughts about authority in their classrooms.

Across the studies that examined authority relations in mathematics classrooms from the teacher's perspective, there were similar outcomes. For instance, Kinser-Traut and Turner (2020) and Wagner and Herbel-Eisenmann (2014a) reported positive outcomes when the teacher positioned students as authorities in their classroom. Specifically, Kinser-Traut and Turner reported how a teacher was able to connect to student's mathematical thinking as a result of positioning students as authorities when it came to generating, discussing, and reasoning with mathematical concepts. In Wagner and Herbel-Eisenmann's study, students became more engaged and started to take more ownership over their learning after the teacher explicitly addressed authority with his students and started to position them as authorities in the classroom. However, similar to the authority studies from the student's perspective, neither of these studies examined what the teacher perceived, thought, or believed about authority relations in their classrooms. Wagner and Herbel-Eisenmann's (2014b) study was an exception, because they sought to understand how teachers thought about authority operating in their classrooms through the use of teacher-generated diagrams. Still, this study, along with each of the studies reviewed so far, did not examine the teacher's beliefs nor how the teacher's thoughts or perceptions related to authority may have influenced their instructional practice.

The existing authority research in mathematics classrooms has yielded some promising findings, namely that students who are positioned as authorities may engage more in mathematics and develop identities as doers of mathematics. Yet, these studies have also produced some troubling findings. For instance, when students are not frequently positioned as authorities, they may not engage as fully in mathematics or may engage in an unreflective manner (e.g., Amit & Fried, 2005; Wagner & Herbel-Eisenmann, 2014a). Both the promising and troubling findings of these studies highlight the importance of authority relations in mathematics classrooms and, thus, the importance of further understanding the instructional strategies teachers implement to position students as authorities as well as *why* teachers position certain individuals or objects as authorities in their classroom at different moments. Studying teachers' *beliefs* about authority relations in mathematics classrooms can build on the extant authority research in mathematics classrooms and further our understanding of teachers' instructional practices related to authority and more broadly. I describe the importance of understanding teachers' beliefs about mathematical authority in the *Mathematics Teacher Beliefs Research* section.

Before moving on to research that investigated *mathematical authority* in classrooms, I want to point out some common features of the constructs defined in authority research in mathematics education. None of the definitions of authority described in this section are limited to the mathematics classroom. While each of the scholars chose to study authority relations in mathematics classroom contexts, they could just as easily have studied authority, and the different types of authority, in the context of an English or Science classroom. In fact, Wagner and Herbel-Eisenmann (2014b) implicitly acknowledged this when they wrote, "we see mathematics education as a paradigm context for studying issues of authority in education" (p. 202). In other words, their conception of authority could be studied in a variety of educational settings, but they viewed mathematics education as a unique context for studying authority relations. I contend that the construct of *mathematical authority* stands as a unique type of authority, as I will describe in subsequent paragraphs, due to its limited applicability to those engaging in some mathematical activity. In the next two sections, I first review the studies that have investigated mathematical authority in classrooms, critically examine how researchers have defined or described mathematical authority, and provide my own, novel definition of mathematical authority.

## Mathematical Authority Research

When examining studies that investigated mathematical authority, there is not as much diversity when compared to those investigating other types of authority. For instance, researchers studying authority relations in mathematics classrooms have investigated student-to-student authority relations, student-teacher authority relations, and how teachers thought about authority operating in their classrooms. In contrast, research on *mathematical authority* has primarily investigated how, or if, teachers *shared* (e.g., Wilson & Lloyd, 2000), *delegated* (e.g., Dunleavy, 2015), or *distributed* (e.g., Depaepe et al., 2012) their mathematical authority with students. Similarly, Hamm and Perry (2002) studied how teachers *granted* students with mathematical authority. Because researchers studying mathematical authority operationalize these terms in similar manners, in this section I refer to a teacher *sharing* mathematical authority with students rather than referring to the construct according to how it was used by each author(s).

Researchers who have studied mathematical authority have found that, even when teachers are attempting to share mathematical authority with their students, the teacher is often still positioned as the ultimate or sole mathematical authority. Hamm and Perry (2002), for instance, found that one out of six participating teachers enacted practices that shared mathematical authority with their students, but the enactment of those practices was infrequent. Overall, they found that the six teachers usually enacted practices that established themselves as the sole or primary mathematical authority in the classroom. Similarly, Depaepe et al. (2012) found, “although students in one classroom were often bestowed authority they actually received insufficient opportunities to exercise that authority” (p. 232). As evidence, Depaepe et al. pointed to the teacher being the “sole evaluator” (p. 229) of students’ suggestions, ideas, and solutions, and not giving students opportunities to consider and critique the idea and solutions of their peers. In some regard, these findings are like some described by Amit and Fried (2005), Langer-Osuna (2016), and Wagner and Herbel-Eisenmann (2014a), namely that many students will often view the teacher as the sole or ultimate authority in the mathematics classroom. What studies such as Hamm and Perry's and Depaepe et al.'s reveal is that mathematics teachers may play a substantial role in developing classroom conditions where students rely heavily on the teacher as an authority. Teachers enacting practices or establishing classroom structures or norms that consistently position themselves as the ultimate or sole mathematical authority in the classroom may leave students with no option but to continually turn to the teacher as the sole mathematical authority in the room. Hence, Hamm and Perry's and Depaepe et al.'s findings complement others' findings (e.g., Amit & Fried, 2005) by showing that students are inclined to use mathematical concepts and procedures prescribed by the teacher in an unreflective manner if the teacher is positioned as the sole mathematical authority in the classroom.

In a more positive light, Dunleavy (2015) reported how a high school teacher shared mathematical authority through the enactment of more student-centered instructional strategies. Dunleavy argued that the teacher sharing mathematical authority with her students resulted in more equitable student participation and students engaging in "high-level and algebraic thinking and learning" (p. 78). Dunleavy, then, highlighted some potential positive outcomes when teachers do position their students as authorities in the classroom. Furthermore, the results of Dunleavy's study add to the promising results of other studies, such as Wagner and Herbel-Eisenmann's (2014a) study. Together, these studies show that when teachers position students as authorities in the classroom, specifically when engaging in and learning mathematics, students may participate more equitably in the classroom, have access to higher quality mathematical instruction, and begin to take ownership over their own learning.

Similar to research on authority relations in mathematics classrooms, research investigating mathematical authority in classrooms is scant. As discussed in this section, the results of mathematical authority research are similar to the results, generally speaking, of research on authority relations in mathematics classrooms. Specifically, there are unfavorable, likely unintended, consequences to students' learning and engagement in mathematics when the teacher is positioned as the ultimate or sole mathematical authority in the classroom. However, researchers conducting these studies have also shown that positioning students as mathematical authorities in the classroom can have positive and productive mathematical outcomes for students. Thus, these studies further support my contention that investigating how instructional practices that position students as mathematical authorities can be enacted in more classrooms and enacted consistently is a worthwhile endeavor. This endeavor, I believe, must start with the beliefs teachers hold about mathematical authority.

I now turn to the construct of mathematical authority. In the following section, I examine how *mathematical authority* has been described or defined by other scholars and argue that a new conception of mathematical authority is needed. I then develop a new definition of mathematical authority and describe how the construct was operationalized for this study.

**Defining Mathematical Authority**

The concept and study of mathematical authority is relatively new to mathematics education research. Consequently, few conceptions of mathematical authority exist, and most of them are more descriptions than definitions. For instance, Wilson and Lloyd (2000) never defined mathematical authority, but they provided one of the first descriptions of mathematical authority as an individual being able to explore, discuss, and choose mathematical ideas or strategies as one engages in mathematical problem solving. Wilson and Lloyd went on to describe mathematical authority as something that one can *use* and teachers can *share* with their students. An implication stemming from Wilson and Lloyd's description of mathematical authority is that mathematical authority is something one can *have* or *exercise*. Stein et al.

(2008), similarly, did not define mathematical authority but described students' mathematical authority as being able to engage in mathematics on their own, being publicly recognized for their mathematical ideas, and ultimately developing into local authorities when it comes to the discipline of mathematics. Stein et al. went on to describe mathematical authority as something that teachers can *give* to students as well as something that can be *nurtured* and *supported*. Hence, like Wilson and Lloyd, they described mathematical authority as something one *has*, or can have, but needs to be *encouraged*, akin to one's self-confidence. The most often cited description of mathematical authority comes from Gresalfi and Cobb (2006). They described mathematical authority as,

> the degree to which students are given opportunities to be involved in decision making and whether they have a say in establishing priorities in task completion, method, or pace of learning. Thus authority is not about 'who's in charge' in terms of classroom management but 'who's in charge' in terms of making mathematical contributions (p. 51).

Moreover, they described mathematical authority as something that is *distributed*, always by the teacher to students and, thus, as something that one *has* and can *exercise*.

One attribute of these descriptions of mathematical authority that differentiates them from the types of authority mentioned previously is the specific focus to the teaching and learning of mathematics. Even so, I argue the descriptions are limited in their use when studying mathematical authority in classrooms. One limitation is that, from each description, mathematical authority is described as something one can have or exercise. Going back to Weber's (1925/1947) conceptions of authority and power, it would seem that Wilson and Lloyd (2000), Stein et al. (2008), and Gresalfi and Cobb (2006) are describing something more like mathematical *power* than mathematical *authority*. Each of the descriptions lacks mention of relationships between two parties and, moreover, describes mathematical authority as one being in charge, to some extent, of some mathematical activity. Additionally, in each of the descriptions provided, mathematical authority is assumed to be something that one individual can have and exercise, even to the point of exercising it over others. For instance, Gresalfi and Cobb gave a hypothetical example of a teacher being the only one having mathematical authority in a classroom and solely determining the validity of others' mathematical contributions. One might be able to quickly imagine such a teacher, maybe from their classroom experiences, and recall the teacher undermining the validity of one student's contribution to the resistance or opposition of the student. In such a case, it seems the teacher would be exercising their mathematical power or the power of their position as a teacher, rather than establishing or leveraging a mathematical authority relation with their students. Although likely not their intent, by describing mathematical authority as something one can have and exercise and not describing mathematical authority as a form of relationship between two parties, based on the legitimacy of one of the parties, I contend these descriptions of mathematical authority do not accurately capture what is meant when one considers an authority relation.

Another constraint of the descriptions of mathematical authority mentioned above is that each excludes several authority relations that could potentially be operating in mathematics classrooms. As Wagner and Herbel-Eisenmann (2014b) reported in their study of teachers' representations of authority in their classrooms, items such as textbooks, calculators, manipulatives, and the mathematics curriculum were mentioned by teachers as sources of authority operating in classrooms. Although all the items listed by the teachers in Wagner and Herbel-Eisenmann's study may not have been sources of mathematical authority, but of a different type of authority, it would seem reasonable to claim that at least some items listed could be potential sources of mathematical authority, given the sources were mentioned by practicing mathematics teachers. Thus, mathematical authority, it would seem, is more complex than something that operates only between students and teachers in the classroom. The descriptions of mathematical authority provided by Wilson and Lloyd (2000), Stein et al. (2008), and Gresalfi and Cobb (2006) exclude many inanimate objects, as well as individuals other than teachers and students, from being potential sources of authority in mathematics classrooms. Hence, I argue that a new description or definition of mathematical authority is needed, one that is constructed in a way that is not limited to just teachers and students but allows for numerous potential sources of mathematical authority.

In developing my definition of mathematical authority, I drew from tenets of Weber's (1925/1947) conception of authority and Gresalfi and Cobb's (2006) conception of mathematical authority. The tenets of Weber's definition of authority that I wanted to draw from were that of a *relationship* between two parties hinging on the *legitimacy* of one of the parties. However, in the case of mathematical authority, the notions of giving commands or orders that are obeyed does not seem fitting. Instead of one party being a legitimate source of giving orders or commands, I argue it is more appropriate that a mathematical authority relation depend upon the legitimacy of one individual, object, or community as a source of mathematical knowledge or mathematical reasoning. Drawing slightly from Gresalfi and Cobb, another aspect of mathematical authority I believe is important is, as a consequence of being viewed as a legitimate source of mathematical knowledge or reasoning, one is also viewed as able to make meaningful mathematical contributions in the classroom. Finally, as previously mentioned, I want to define mathematical authority in a way that is open to many potential people or objects as sources of authority relations. Bringing these important aspects together, I define mathematical authority as *a relationship in which at least one individual views another subject (i.e., a person, community, textbook) as a legitimate source of mathematical knowledge or mathematical reasoning and, thus, able to make meaningful mathematical contributions*.

One clear and pragmatic constraint of my definition is concerned with how to discuss mathematical authority in classrooms. If mathematical authority is something that someone *has* or can *exercise*, then it is easy to discuss mathematical authority in the classroom and to what extent someone *has* mathematical authority or is *exercising* their mathematical authority. For pragmatic and clarifying purposes, I will consider someone or something to be *a* mathematical authority if they are positioned as a legitimate source of mathematical knowledge and/or reasoning by at least one other individual in the context at hand. Additionally, if discussing certain actions or strategies, then one can say that such actions or strategies are implemented in hopes of promoting or developing mathematical authority relations. For instance, when a teacher designs a lesson in which students develop their own ideas, understanding, or procedures in relation to a task and engage in mathematical argumentation, one could say that the teacher was positioning students as mathematical authorities and hoping to promote student-to-student mathematical authority relations. In other words, the teacher may be implicitly, or explicitly, communicating that students should position their peers as legitimate sources of mathematical knowledge or reasoning as they engage in certain activities and, as a result, mathematical authority relations are being promoted between students. Finally, if someone is being positioned as a mathematical authority by at least one other individual, and the mathematical authority makes claims or suggestions that the other(s) take up as legitimate or true, whether implicitly or explicitly, then I will say the mathematical authority is *leveraging* their position as a mathematical authority.

Having reviewed the authority research that has been conducted in mathematics classrooms and having defined mathematical authority, I now turn to the literature on mathematics teachers' beliefs. As a starting point, I examine different conceptions and theories of beliefs. I then review the empirical literature on mathematics teachers' beliefs and end my review of the beliefs literature by describing the affordances of Leatham's (2006) sensible systems framework.

## Conceptions of Beliefs and Research on Teachers' Beliefs

### What Is a Belief?

Defining *belief* can be a challenging endeavor as scholars have acknowledged that parsing out the difference between *beliefs* and *knowledge* is no simple task. Knowledge has been viewed by some as a belief with certainty (Philipp, 2007) and requiring general or group consensus (Nespor, 1987 as cited in Pajares, 1992). Beliefs, on the other hand, have been understood to be more idiosyncratic and disputable. Yet, due to their highly personal nature, beliefs are often less open to "evaluation and critical examination" (Pajares, 1992, p. 311) when compared to knowledge. Even though beliefs may be more personal and disputable when compared to knowledge, Nespor (1987, as cited in Pajares, 1992) argued, "beliefs are far more influential than knowledge in determining how individuals organize and define tasks and problems and are stronger predictors of behavior" (p. 311).

When it comes to defining a belief, no common definition has been used across the literature. However, the differences in definitions are not as stark when compared to the various definitions of authority (as detailed in the previous section). Definitions provided by Rokeach

(1968) and Philipp (2007) are ones that are often cited by researchers studying mathematics teachers' beliefs. Rokeach defined a belief as "any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase, 'I believe that…'" (p. 113). Rokeach went on to assert that "all beliefs are predispositions to action" (p. 113). Philipp, on the other hand, defined beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (p. 259) and went on to describe beliefs as "lenses that affect one's view of some aspect of the world or as dispositions toward action" (p. 259). While there are slight differences in the two definitions, one important point of agreement between Philipp and Rokeach is that an individual's beliefs lead them to act in certain ways.

For the purposes of this study, I use the definition of beliefs provided by Rokeach (1968). The reasons for using this definition are twofold. First, I agree with Rokeach's acknowledgement that the beliefs an individual holds may be consciously or unconsciously held. In other words, an individual may be able to clearly articulate some of their beliefs but not others. Second, and related to the first reason, Rokeach's definition implies that inferences about what a person believes can be made from what that person says and does. In alignment with Rokeach, I, too, maintain that a teacher's beliefs can be inferred by both their actions and their words.

Another construct that is salient across the mathematics teachers' beliefs literature is that of belief systems. Rokeach (1968) is, again, one that is often cited when researchers define and discuss belief systems. However, Green's (1971) theory of belief systems is more prominent and, in my view, more developed. Green theorized that individuals organize their beliefs in systems that have three characteristics. First, Green contended that beliefs are held, not necessarily in a purely logical manner (i.e., holding 'Belief A' implies an individual will hold 'Belief B'), but in what Green called a “quasi-logical structure” (pp. 44-45). By “quasi-logical” Green meant that the beliefs an individual holds are held in a manner that is logical to the individual, but this logical relationship between beliefs may or may not seem logical to someone else. Moreover, in this “quasi-logical” structure, some beliefs are considered primary and others derivative, meaning an individual may hold a certain belief “because he thinks it is derivable [derivative] from some other [primary] belief” (p. 45). The second characteristic of belief systems that Green posited was that beliefs can be held with varying levels of psychological strength. That is, for example, one may believe that dogs are better pets than cats, although this belief may not be held as strongly as their belief in the extraordinary, intrinsic value of all human life. Green considered beliefs that are held more strongly to be “psychologically central, or core, beliefs” (p. 46). The third characteristic of Green’s conception of belief systems is that beliefs are held in clusters. A possibility of beliefs being held in clusters is that a certain cluster of beliefs may be isolated from other clusters. Hence, beliefs in one cluster may not relate to or influence beliefs held in other clusters.

Together, Rokeach’s (1968) definition of beliefs along with Green’s (1971) theory of belief systems comprise my conception of beliefs and how they are held. In subsequent chapters I refer to these conceptions, especially Green’s, to describe how I studied PTs’ beliefs and my inferences regarding how a PT’s beliefs relate to each other in a system of beliefs. I now turn to the literature on mathematics teachers’ beliefs by examining empirical research and explicating how beliefs about mathematical authority may be situated in the extant literature.

## Mathematics Teacher Beliefs Research

Mathematics teachers’ beliefs have received considerable attention from the mathematics education research community. This attention is likely due to the general understanding that beliefs have great influence on an individual's behavior (Pajares, 1992; Philipp, 2007). While it has been shown that beliefs are not the only factor that influences teachers' behavior (e.g., Borko et al., 1992; Cross, 2009), the impact of teachers' beliefs on their instructional practices has been shown to be profound. Moreover, scholars have claimed that beliefs serve as a filter that influences the way teachers learn (Conner et al., 2011), perceive (Kagan, 1992), and "characterize phenomena" (Pajares, 1992, p. 310). For instance, in summarizing research on teachers' beliefs related to curriculum, Philipp (2007) stated that curriculum materials are not used as intended if a teacher's beliefs about mathematics, teaching mathematics, and learning mathematics are not in alignment with the "beliefs that serve as the foundation of the reform-oriented curriculum" (p. 291).

Researchers have investigated beliefs held by prospective teachers (e.g., Conner & Singletary, 2021; Cooney et al., 1998), elementary teachers (e.g., Bobis et al., 2016; Sztajn, 2003), secondary teachers (e.g., Aguirre & Speer, 1999; Stockero et al., 2020; Yurekli et al., 2020), and even college instructors (e.g., Speer, 2008). Traditionally researchers have investigated three main categories of mathematics teachers' beliefs: beliefs about mathematics, teaching mathematics, and learning mathematics. Some researchers have shown that teachers' beliefs about mathematics might be most influential because these beliefs appear to inform teachers' beliefs about learning and teaching mathematics. For instance, Thompson (1984) found that one teacher believed mathematics was a static body of rules and facts found in nature, and Thompson claimed this belief influenced the teacher's belief that teaching mathematics meant passing this body of rules and facts from teacher to student. Thompson also found that another teacher believed mathematics was dynamic, "continuously expanding," and, when studied, "sharpens one's ability to reason logically and rigorously" (p. 113). Furthermore, Thompson implied that this teacher's beliefs about learning mathematics, namely "that students learn best by doing and reasoning about mathematics on their own" (p. 120), proceeded from their beliefs about mathematics. Similarly, Cross (2009) posited that across the five teachers who participated in her study, "beliefs about teaching and learning appeared to stem from beliefs about the epistemology of mathematics" (p. 338). To make this claim, Cross drew on Green's (1971) theory of belief systems to develop "hypothesized models" (p. 326), representing how the teachers' beliefs were related. Thus, by drawing on Green's concept of belief systems, particularly their "quasi-logical structure" (pp. 44-45), Cross was able to show how the teacher's beliefs about mathematics influenced the teacher's beliefs about learning mathematics and teaching mathematics.

Regarding the beliefs PTs hold, research has largely focused on the three main categories of beliefs described above. Calderhead and Robson (1991), for instance, studied the beliefs about teaching and learning held by twelve elementary PTs as they began their teacher education programs. They found that beliefs PTs hold "can influence what they find relevant and useful in the course, and how they analyse their own and others' practice" (p. 7). Other studies have investigated changes to PTs' beliefs as a result of taking a course or sequence of courses. However, much of this research has been conducted with elementary PTs. Liljedahl et al. (2007), for instance, found elementary PTs' beliefs about mathematics, teaching mathematics, and learning mathematics productively changed as a result of engaging in a mathematics methods course. Liljedahl et al. claimed that two key aspects of the methods course led to the changes in beliefs: consistently engaging the PTs in problem solving and having them reflect in reflective journals on "what mathematics is, and what it means to teach and learn mathematics" (p. 328). Similarly, Grootenboer (2008) studied changes to elementary PTs' beliefs about teaching and learning mathematics as a result of taking a mathematics education course. Grootenboer found that the beliefs of the PTs enrolled in the course did change, but the changes in the PTs' beliefs were not evidenced in their classroom teaching. Referring to Green's (1971) conception of belief systems, he also claimed that the PTs held "primary beliefs about mathematics teaching and learning that appeared to be grounded in their prior experiences as school students" (p. 493).

Studies investigating changes to secondary mathematics PTs' beliefs are fewer but have found results similar to those who studied elementary PTs' beliefs. For instance, Conner et al. (2011) and Jao (2017) found that engagement in mathematics education courses (a sequence of two courses in Conner et al.'s case) led to productive changes in secondary PTs' beliefs about teaching mathematics. Moreover, and in contrast to some studies of elementary PTs' beliefs, Conner et al. found their "participants' beliefs about mathematics and proof remained relatively consistent" (p. 498). Cooney et al. (1998) also investigated changes to secondary PTs' beliefs as the PTs progressed through their mathematics education program, which included student teaching. Cooney et al. not only described changes to four secondary PTs' beliefs but also proposed four belief structures that described the PTs' propensity to change their beliefs. For instance, one belief structure they proposed was *reflective connectionist*, which they described as one who "integrates voices, analyzes the merits of various positions, and comes to terms with what he or she believes in a committed way" (p. 330).

Researchers have also designed studies to describe changes to PTs' beliefs as a result of engaging in field experiences, yet the majority of these studies are, again, conducted with elementary PTs. For instance, Ambrose (2004) studied changes to elementary PTs' beliefs that resulted from their engagement in an intensive field experience during the first semester of their teacher education program. In the field experience, pairs of student teachers "worked with individual children using specific tasks and activities designed to elicit children's thinking" (p. 98). Ambrose found that engaging in this field experience led to some productive changes to the PTs' beliefs about teaching and learning mathematics. Moreover, she found that the PTs held on to some of their previously held beliefs about teaching and learning and claimed this was evidence "that prospective teachers do not let go of old beliefs while they are forming new ones" (pp. 116-117). Vacc and Bright (1999) also studied changes to elementary PTs' beliefs, as measured by the Cognitively Guided Instruction Belief Scale (Peterson et al., 1989 as cited in Vacc & Bright, 1999). Although they studied changes to the PTs' beliefs at various points throughout their program, part of their study did examine the PTs' beliefs prior to student teaching and again after student teaching. They found that the PTs' beliefs about teaching and learning mathematics changed "significantly during student teaching" (p. 108). Furthermore, they noted that the extent of belief changes varied among the participating PTs and suggested that some of this variation may be explained by "the amount of consistency that exists among the philosophical perspectives of the teacher educators [e.g., instructors, supervisors, mentor teachers] with whom preservice teachers work" (pp. 108-109).

As I have noted, much of the research on PTs' beliefs has been conducted with elementary PTs. Specifically, research that has investigated changes to PTs' beliefs as a result of engaging in a field experience, including student teaching, has largely been with elementary PTs. I agree with Conner et al. (2011), who claimed that secondary teachers are likely to differ from elementary teachers, "particularly in their mathematical experiences and understandings" (p. 484). Consequently, student teaching may influence secondary PTs' beliefs in different ways when compared to elementary PTs. Studies that further investigate how student teaching influences secondary PTs' beliefs may help mathematics teacher educators, mentor teachers, and student teaching supervisors further support student teachers as they transition from their formal education to becoming practicing teachers (Richardson, 1996).

Other studies on practicing and prospective teachers' beliefs have extended research on the three traditional categories of beliefs (i.e., beliefs about mathematics, teaching mathematics, and learning mathematics) by studying more specific or nuanced teacher beliefs or beliefs that go beyond mathematics. Examples of investigations of more nuanced beliefs include Bobis et al.'s (2016) investigation into teachers' beliefs about student engagement in mathematics classrooms, Diamond's (2019) research on mathematics teachers' beliefs about students' transfer of knowledge, and Conner and Singletary's (2021) study on PTs' beliefs about collective argumentation and proof. While the beliefs that Bobis et al., Diamond, and Conner and Singletary studied are more specific when compared to general beliefs about mathematics, teaching mathematics, and learning mathematics, these more nuanced beliefs tended to be subsets of the traditional categories. For instance, I consider many of the beliefs Diamond found, such as "students transfer their learning to a novel situation when the novel situation prompts the use of a learned association or procedure" (p. 469), as a subset of teachers' beliefs about learning mathematics. Conner and Singletary found that two PTs held beliefs about proof that were considered part of their beliefs about developing mathematical knowledge and, thus, a subset of their beliefs about mathematics. Moreover, they found that the PTs held differing beliefs about the role of a math teacher, and these beliefs were a subset of their beliefs about teaching mathematics.

Regarding beliefs beyond mathematics, both Sztajn (2003) and Skott (2001) described how teachers' beliefs about their students' affective and future needs played an influential role in shaping the teachers' practices. Skott revealed how one teacher's beliefs about the importance of developing his students' self-confidence influenced his interactions with groups of students. Initially, Skott thought the teacher's actions with one group of students appeared to differ from his interactions with the other group. However, Skott highlighted the teacher's beliefs about raising students' self-confidence and concerns with classroom management influencing the teacher interacting differently in the two instances. Sztajn, while studying two teachers attempting to enact "recommendations for change in mathematics teaching" (p. 54), found the two teachers implemented these recommendations differently. These differences in implementation, Sztajn argued, were due to "the way they [the two teachers] perceive students who come from different socioeconomic backgrounds" (p. 70). Specifically, the two teachers' practices were influenced by what they believed their students' current and future needs were, which were influenced by a combination of the teachers' "value-laden visions of students, of parents, and of society" (p. 70). Together, the studies conducted by Skott and Sztajn indicate that mathematics teachers may hold beliefs, especially related to their students, that may not fit in the categories of mathematics, teaching mathematics, and learning mathematics and these beliefs can have great influence on a teacher's practice.

Even though there has been extensive research on mathematics teachers' beliefs, investigations into what teachers believe about authority or mathematical authority are scarce. That studies on teachers' beliefs about mathematical authority are scarce is surprising, given that many studies have uncovered teachers' beliefs that seem to be related to authority. For instance, one of the teachers in Thompson's (1984) study believed "that it was her responsibility to direct and control all classroom activities" (p. 119). This teacher may have believed that the teacher should be viewed as the sole mathematical authority in the classroom, and authority in general, and that students should not be positioned as mathematical authorities in the classroom. Other researchers have reported beliefs that the teacher should serve as a guide that supports students as they reason with concepts and develop mathematical understandings on their own (Beswick, 2007; Cross, 2009). In these cases, the teachers may have believed that students should be positioned as mathematical authorities in classrooms in order to develop their own mathematical knowledge, however it is unclear whether the teachers did indeed hold this belief about mathematical authority. That studies on teachers' beliefs about mathematical authority are scarce is also surprising given the increased interest in authority relations in classrooms among mathematics education researchers and educators.

Investigations into what PTs believe about mathematical authority can lead to novel insights into not only the beliefs that PTs may hold and how they are held (i.e., in a system of beliefs), but also the instructional practices of beginning teachers. As Conner et al. (2011) put it, the beliefs PTs hold provide "a backdrop for their eventual practice" (p. 485). Thus, insights into PTs' beliefs about mathematical authority may be a critical aspect in the work of supporting novice teachers to consistently enact practices that position students as mathematical authorities. Wagner and Herbel-Eisenmann (2014b) articulated this view when they claimed,

> In the same way that many mathematics education scholars would argue that understanding how students think about rational number or problem solving can help to improve the teaching of rational number or problem solving, we argue that understanding how teachers think about authority must be the basis of teacher educators' work with teachers on issues of authority (p. 203).

In other words, if we are to support teachers in enacting practices that consistently position students as mathematical authorities in a lasting and substantive manner, then investigating the beliefs teachers hold concerning mathematical authority is an essential first step.

Up to this point I have reviewed the empirical research on mathematics teachers' beliefs and have discussed how research on beliefs about mathematical authority and how student teaching influences PTs' beliefs may extend the extant literature. Throughout this review, I have alluded to the influence teachers' beliefs have on their practice but have yet to explore what extant research has to say about this influence. In the subsequent paragraphs I examine what researchers have found regarding the relationships between the beliefs teachers hold and their practice. Furthermore, I use this review to then argue for the importance and value of Leatham's (2006) sensible systems framework and why this framework was used to conduct this study.

**Why The Sensible Systems Framework?**

There are several researchers who have found teachers' beliefs to be internally consistent as well as consistent with their practice. By internally consistent I mean the teacher held beliefs that were compatible with each other in a system of beliefs (Green, 1971). Beswick (2007), for instance, stated that one teacher's beliefs "clearly formed a highly integrated system" (p. 106), and their beliefs were consistent with their instructional practices. Furthermore, there are instances in the teachers' beliefs literature in which a teacher's beliefs initially seemed inconsistent with their practice. Yet, due to certain methodological and theoretical approaches, researchers were able to develop inferences that the teachers held beliefs that were both internally consistent and consistent with their instructional practice (e.g., Conner & Singletary, 2021; Cross Francis, 2015; Speer, 2008). Speer, for example, claimed her use of videoclip interviews as a method afforded fine grained analysis that enabled her to understand how the participating teacher's beliefs related to their practice in a consistent manner.

Although there are many researchers who have found teachers' beliefs to be internally consistent and consistent with their practice, there are others who have claimed teachers hold inconsistent beliefs. In such cases, researchers have claimed that other factors were at play and led to the inconsistencies. Specifically, researchers have argued that insufficient content knowledge (Borko et al., 1992), concerns over classroom management (Thompson, 1984), and time constraints (Raymond, 1997) have all been factors that have prohibited teachers from enacting practices that were consistent with their beliefs. However, I argue that to make such claims, researchers have positioned themselves as being able to determine when beliefs should be considered consistent or inconsistent with teachers' practices. Consider, as an example, Cross's (2009) claim that "individuals may hold beliefs that are contradictory since they are not perceived by the individual to be conflicting" (p. 341). Raymond (1997), in another instance, claimed that the teacher participating in her study "was unaware of the many inconsistencies between her professed beliefs and actual practice" (p. 568). In both cases, the following question arises: *In whose eyes are these beliefs inconsistent or contradictory?* From both Cross's and Raymond's claims, it seems the researcher was the one who determined when a teachers' beliefs were inconsistent, both with their practice and in relation to their other beliefs. Put differently, the researcher was the one to determine the accuracy, or *consistency*, when it came to the model of each teacher's beliefs system and the model describing how certain beliefs should influence or inform a teacher's practice.

As I have hinted to this point, to assume that a teacher can hold beliefs that are internally inconsistent or are inconsistent with their practice is, I contend, an epistemological and ontological issue. Epistemologically, assuming that inconsistencies can exist—both in a teacher's system of belief and between their beliefs and practice—positions the researcher as being able to employ methods that enable them to ascertain "pure representations of teachers' cognition" (Speer, 2005, p. 387). Instead, I agree with Philipp (2007) when he suggested that "when we observe apparent contradictions, we would assume that the inconsistencies exist only in our minds, not within the teachers" (p. 276). In other words, I contend that as researchers, we should not assume that we can know with certainty what a teacher actually believes, nor should we assume that we can determine how certain beliefs influence or are enacted in a teacher's instruction. Interestingly, Philipp did not consider his position an ontological one, stating, "I do not suggest that inconsistencies are nonexistent or can all be explained away" (p. 276). On this point, I disagree with Philipp and, instead, agree with Leatham's (2006) ontological position that teachers are sensible, rational individuals and, thus, the beliefs they hold are sensible, rational, and consistent to them. An implication of viewing teachers and their beliefs in this light is that "beliefs seen as contradictory to an external observer are not likely to be seen as contradictory to the one holding those beliefs" (p. 94). Hence, taking this theoretical position, teachers' beliefs are always assumed to be held in a system that is internally consistent to them.

As a consequence of his view that teachers' beliefs should be viewed as rational and sensible, Leatham (2006) developed the sensible systems framework. Leatham's sensible systems framework was motivated, in part, by beliefs researchers taking "a positivistic approach to belief structure" (p. 91) and assuming the beliefs they inferred are the ones that teachers hold. As I mentioned in the previous paragraph, Leatham discredited this position and instead argued that teachers should be viewed as sensible, rational beings and, as such, they "develop beliefs into organized systems that make sense to them" (p. 93). Thus, if using the sensible systems framework, researchers can never assume that teachers hold inconsistent beliefs or that teachers can hold beliefs that are inconsistent with their practice. Furthermore, taking this perspective implies that researchers can only make *inferences* as to what a teacher believes based on the teacher's words and actions. No matter how compelling the evidence for the inferences a researcher makes, they can never claim with certainty what a teacher actually believes. Leatham described the limits of researchers' inferences as follows:

> Teacher actions, therefore, do not prove our belief inferences. When a teacher acts in a way that is consistent with the beliefs we have inferred, we have evidence that we may be on track, but we do not know what belief or beliefs the teacher was actually acting on at the time. When a teacher acts in a way that seems inconsistent with the beliefs we have inferred, we look deeper, for we must have either misunderstood the implications of that belief, or some other belief took precedence in that particular situation. (p. 95)

In sum, by using the sensible systems framework I assume that I am not able to know or infer a teacher's beliefs with complete confidence and that teachers organize and hold their beliefs in a system that is rational and consistent to them.

## Review of Literature Summary

In this review of the literature, I have shown that studies investigating teachers' beliefs about mathematical authority have yet to be carried out. Furthermore, I have argued that both authority relations and the beliefs teachers hold can influence the instruction in mathematics classrooms and, hence, students' mathematical outcomes. To this end, I agree with Cross's (2009) assertion:

> It is believed that for there to be improvements in mathematics achievement, classroom practices must reflect reform recommendations. This would require change in the instructional practices of many mathematics teachers; a change that can only be actualized if we come to a better understanding of not only the types of beliefs these teachers have but also how these beliefs are related to each other and practice. (p. 326)

Therefore, this study merges these, up until now, two distinct lines of research, namely research on mathematical authority in classrooms and teachers' beliefs, specifically PTs' beliefs, with the aim of further understanding and improving the mathematics instruction to which students have access.

The theoretical framework guiding this study is composed of three parts. The first part of my framework is my definition of mathematical authority as a relationship in which at least one individual views another subject (i.e., a person, community, textbook) as a legitimate source of mathematical knowledge or mathematical reasoning and, thus, able to make meaningful mathematical contributions. Second, I use Rokeach's (1968) definition of beliefs and Green's (1971) theory of belief systems to inform how I conceptualize what a belief is and how one holds and organizes their beliefs. Lastly, the sensible systems framework developed by Leatham (2006), comprises the third and final part of my theoretical framework. In his development of the sensible systems framework, Leatham argued that the framework has both theoretical and methodological implications. To this point, I have described many of the theoretical implications. In the next chapter I explicate many of the methodological implications that come with using the sensible system framework, along with the other aspects of my theoretical framework, as I describe the methods used to carry out this study.

# CHAPTER 3

## RESEARCH DESIGN AND METHODS

One of the theoretical assumptions of Leatham's (2006) sensible systems framework is that individuals may not be able to clearly articulate their beliefs or even be aware they hold certain beliefs. A methodological implication stemming from this assumption, and others, is "in order to infer a person's beliefs with any degree of believability, one needs numerous and varied resources from which to draw those inferences" (Leatham, 2006, p. 93). Furthermore, to infer and describe the beliefs PTs hold and infer how their beliefs are internally related (e.g., in a system of beliefs), varied qualitative research methods are required. Hence, this study is a multiple-case study (Yin, 2003) with varied data sources that enabled me to develop a "rich and holistic account" (Merriam, 1998, p. 41) of each PT's beliefs. The purpose of this multiple-case study is both descriptive and interpretive in nature (Merriam, 1998, pp. 38-39). The study is descriptive in that the data collected and ensuing analysis enabled me to write thick descriptions of what each participant believed, particularly about mathematical authority. This is an important outcome of the study given the lack of research, and hence knowledge, thus far concerning teachers' beliefs about mathematical authority. Also, I was able to develop inferences about how a PT's student teaching experience influenced their beliefs. Consequently, this study is interpretive as two outcomes of this study are 1) inferred models of how a PT's beliefs relate internally and 2) inferences regarding how each PT's student teaching practicum changed or reinforced their beliefs about mathematical authority.

The multiple-case study design of the study afforded some of the analysis I was able to conduct and, consequently, the results of the study. Miles and Huberman (1994) described some of these affordances as follows:

> Multiple-case sampling adds *confidence* to findings. By looking at a range of similar and contrasting cases, we can understand a single-case finding, grounding it by specifying *how* and *where* and, if possible, *why* it carries on as it does. We can strengthen the precision, the validity, and stability of the findings… If a finding holds in one setting and, given its profile, also holds in a comparable setting but does not in a contrasting case, the finding is more robust (p. 29; emphasis in original).

The multiple-case study design of this study allowed me to develop robust understandings regarding the "complexity, and situational uniqueness" (Stake, 2006, p. 6) of each individual PT's beliefs about mathematical authority by comparing each PT's beliefs with the beliefs held by the other PTs. Studying the beliefs held by multiple PTs also enabled me to see if any beliefs were salient across multiple teachers and develop inferences for why this might be the case.

## Participants

As previously mentioned, to make strong inferences about what an individual believes as well as how their beliefs form a sensible, consistent system, extensive data along with detailed and in-depth analyses are required. For instance, Cross Francis (2015), who also used Leatham's (2006) sensible systems framework, initially perceived one teacher's beliefs to be inconsistent. However, Cross Francis viewed this perceived inconsistency as a site for further exploration with the goal "to better understand the broader set of beliefs that could be influential in the teachers' decision making and behavior" (p. 191). Due to the detailed analyses and multiple passes through the data required when using Leatham's sensible systems framework, I decided it was imperative to keep the number of participants small. However, to strengthen the findings of this study, I wanted the number of participants large enough so there would be similarities and differences across cases (Miles & Huberman, 1994). To increase the possibility of capturing contrasting beliefs across multiple teachers and to capture beliefs about mathematical authority that are salient across multiple teachers, I recruited six PTs to participate in this study.

The six PTs who participated in this study were all undergraduate students at a large, public university in the Southeastern United States. All six PTs were part of a larger cohort of secondary mathematics education students, all of whom student taught during the Fall 2021 semester. Prior to their student teaching semester, the cohort of PTs took a sequence of three mathematics pedagogy courses and a sequence of three corresponding courses focused on mathematics content knowledge for teaching. During the first and third of the pedagogy courses, I served as a Teaching Assistant and, as a result, developed positive, professional relationships with many of the PTs in the cohort. To recruit the six PTs who participated in this study, I sent an email to the entire cohort announcing that I was seeking PTs to take part in a research study that would examine their beliefs. My aim was to recruit six participants and six PTs responded to the email, saying they would like to take part in the study. Each of the six PTs that responded to the email participated in the study for its duration.

Although all six participants took part in all aspects of the study, only four (Chris, Hannah, Grace, and Simon) were chosen for discussion in this dissertation. I decided to not include the other two participants, Lydia and Jack, for discussion in this dissertation for different reasons. Due to schedule conflicts, Lydia's sixth interview and the development of her second mathematical authority diagram took place in mid-January 2022, rather than during December 2021. This proved problematic as Lydia began teaching full-time in early January and this clearly influenced how she thought about and developed her second mathematical authority diagram as well as her responses in the sixth and final interviews. Specifically, Lydia grounded the development of her mathematical authority diagram and her responses in those interviews in her experiences as a novice full-time teacher and not in her student teaching experiences. Jack, on the other hand, took part in all interviews as originally scheduled. Yet, as I analyzed data from Jack's interviews, it seemed Jack often responded to questions with an idealized, theoretical vision of the mathematics classroom and mathematics pedagogy, rather than grounding his responses in his experience(s) as a student teacher. Because it was difficult to discern when Jack was discussing his in-the-classroom experiences as a student teacher from when he was discussing his idealized vision of the mathematics classroom, it was challenging to answer my research questions, thus I decided not to discuss his beliefs in this dissertation. Yet, Jack's story may be important to discuss in a different study with a different set of research questions.

The four PTs whom I discuss in this dissertation each student taught in a high school located in the Southeastern United States. Chris student taught in accelerated and on-level Algebra 1 courses. Hannah student taught in on-level Geometry courses and on-level Algebra 1 courses. Simon and Grace student taught in on-level and support Geometry courses. Coincidentally, Chris, Grace, and Simon student taught at the same suburban high school and Hannah student taught at a rural high school in a different school district. I provide other pertinent details about each participant in Chapter 4, before the narrative describing each of their beliefs.

## Data Sources

In order to conduct a quality case study, the case study researcher must collect and consider multiple forms of data (Merriam, 1998; Yin, 2003). When it comes to studying teachers' beliefs, using various data collection strategies is also imperative because individuals are often not able to clearly articulate or "accurately represent their beliefs" (Pajares, 1992, p. 314). Moreover, individuals are not consciously aware of some of the beliefs they hold. For instance, how often are we consciously aware of the belief that a bridge will hold the weight of our car or the belief that the "Nutrition Facts" listed on the cereal we buy are accurate? Thus, to develop strong inferences for the beliefs my participating PTs held, multiple and varied sources of data were collected. To understand the PTs' beliefs about and related to mathematical authority prior to student teaching, I conducted three interviews with each PT the summer before their student teaching semester (Summer 2021). During the second of those interviews, each PT also developed a diagram of how they thought mathematical authority operates in classrooms (see Wagner & Herbel-Eisenmann, 2014b). During the PTs' student teaching semester, I collected lesson plans and weekly reflections they submitted to their student teaching supervisor, and I conducted an interview in which we discussed a lesson plan of each PT's choosing. After student teaching, I conducted three more interviews with each PT. Before the sixth interview, each PT developed a second mathematical authority diagram and recorded a video of them explaining their diagram. The first interview protocol used for all participants can be found in Appendix A and Grace's interview protocols for interviews two through seven can be found in Appendix B. In what follows, I further describe the varied data collection methods I used and how the data generated from these methods helped me answer the research questions guiding this study.

**Initial Interview**

The initial interview was a semi-structured interview conducted at the beginning of Summer 2021 (see Table 1). This interview was designed to provide each PT opportunities to discuss their beliefs about mathematics, teaching mathematics, learning mathematics, and students (see Appendix A for protocol). Numerous researchers have suggested that a teacher's beliefs about mathematics influence their beliefs about teaching mathematics or learning mathematics (e.g., Cross, 2009; Thompson, 1984). Hence, this interview began by having each PT discuss their views of mathematics and why they chose to become a mathematics teacher. Each PT was then asked questions that elicited their beliefs about learning mathematics, teaching mathematics, and students. While beliefs about learning mathematics were not a main focus of this study, I wanted to provide PTs opportunities to discuss some of their beliefs about learning mathematics in case these beliefs appeared to inform or be related to their beliefs about mathematical authority. Hence, when I asked questions that focused on learning mathematics, I asked follow-up questions only if it seemed that a response would elicit or relate to some of their beliefs about mathematical authority.

Going into the initial interview, I anticipated many of the PTs' beliefs about mathematical authority would be related to or subsets of their beliefs about teaching mathematics and students. For this reason, the interview protocol ended with questions that helped form my initial understanding concerning what each PT believed about the teacher as a mathematical authority, teaching mathematics, and students as mathematical authorities. Overall, the first interview generated responses that helped me answer my first two research questions and these responses informed subsequent interview protocols for each PT.

**Second Interview**

The second interview was conducted approximately three weeks after the first interview and included multiple components. The first two parts of this interview were tailored to each teacher with the purpose of further exploring each PT's beliefs about mathematics, teaching mathematics, and students, particularly as these beliefs related to their beliefs about mathematical authority. Hence, this interview helped me explore answers to my first two research questions. The time between interviews was intentional as it allowed me to begin analyzing data from the first interview, which informed the second interview protocol. Specifically, these analyses enabled me to determine responses from the first interview to follow-up on, aspects of their beliefs to explore further, and responses to keep in mind as they developed their mathematical authority diagrams during the third part of this interview.

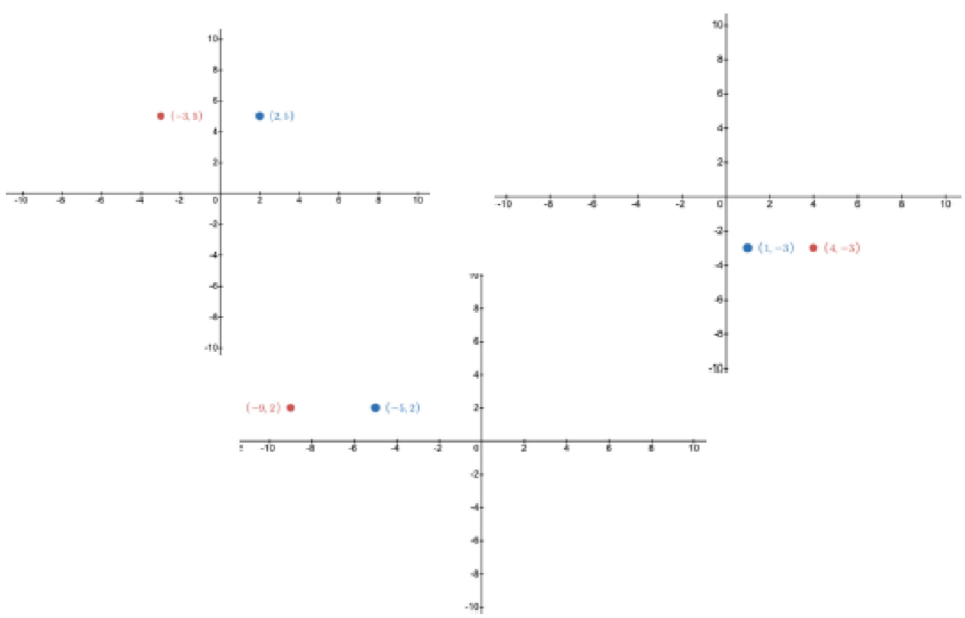
The first part of the interview included questions and prompts that I asked or posed to every PT and questions/prompts that were tailored to each teacher. Questions that I asked every participant were intended to elicit each PT's beliefs about the teacher and students as mathematical authorities. For instance, two questions I asked each participant were: *What comes to mind when I say that the teacher is a mathematical authority in the classroom?* and *What if I were to say that students or a student is a mathematical authority in the classroom, what comes to mind?* Informed by their responses in the first interview, I also asked each participant questions that elicited their beliefs about technology and other objects as mathematical authorities in the classroom. Aspects of the second interview protocol that were common for all PTs can be found in Appendix B.

The second part of this interview included two hypothetical classroom situations that I described to each participant and asked them to engage in as if they were the teacher. The two hypothetical scenarios that were discussed are presented in Excerpt 1 and Excerpt 2. The purpose of these situations was to elicit responses that may reveal each PT's beliefs about the teacher as a mathematical authority and whether, or in what instances, students can or should be positioned as mathematical authorities. Specifically, having PTs discuss how they would act in the situations elicited responses about what each PT would do in situations where students provided mathematical claims they perceived to be correct and in situations where students provided mathematical claims they perceived to be incorrect. Additionally, discussing the scenarios elicited responses regarding each PT's beliefs regarding who (i.e., the teacher or students) is responsible to give mathematical explanations when correct ideas are presented and when incorrect ideas are presented.

**Excerpt 1**

*First Hypothetical Scenario Discussed with Each PT*

***Background***: This is an on-level sixth-grade classroom. Students have not yet received instruction regarding how to add or subtract integers. Also, there is a number line visible to the students in the classroom, that does from -15 to 15. The goal of the lesson is for students to be able to determine the horizontal distance between two points when given their coordinate pairs. Students have just placed points on the coordinate plane with a horizontal distance of 5 units, 3 units, and 4 units respectively and are discussing the points as a ***whole class***. The points (see below) are recorded on the front board.



$$(-3, 5) \ \& \ (2, 5)$$

$$(-3, 5) \ \& \ (2, 5)$$

$$(-3, 5) \ \& \ (2, 5)$$

**T: Alright, what if we couldn't see the coordinate plane, is there a way to determine the distance between these points if we just had this list of coordinate pairs?**

Jose: Yes, I think we can just look at the larger number and count backwards to the smaller number.

Zack: [Only student in the room with his hand raised]

**T: Okay, good, let's go on to the… Oh wait, Zack do you want to add something to what we just said?**

Zack: Kind of, but not really. I was just going to say that you can also just take the absolute value of the two first numbers and add them together.

**T: You mean, take the absolute value of the $x$–coordinates and add them together?**

Zack: Yeah.

**Excerpt 2**

*Second Hypothetical Scenario Discussed with Each PT*

***Background:*** This is an Honors Pre-Calculus class with a mix of Juniors and Seniors. Students have begun investigating transformations of functions and have developed general rules for various transformations (e.g., $f(x) + a$ : shift the original function $f(x)$ up $a$ units, assuming $a$ is positive). They also have gone over the definitions for Domain and Range.

**T: With your partner, take 4 minutes and sketch the graph of $f(x) = \sqrt{-x}$ and determine the function's domain and range.**

Marcos: [Right after you finish talking, calls out] Wait! That's impossible! You can't take the square root of a negative number!

Finally, during the last part of this interview, each PT produced a diagram of how they thought mathematical authority operates in the classroom. Wagner and Herbel-Eisenmann (2014b) used authority diagrams in their study on teachers' perceptions of authority in the classroom and argued that teachers' authority diagrams can "open up understanding of how educators think about authority" (p. 223). In their study, Wagner and Herbel-Eisenmann first put a filled-in dot on a piece of paper to represent the teacher, and then "instructed them [the teacher] to use symbols, lines, words, or whatever they needed to show how authority works in their classroom to complete the diagram" (p. 208). Similarly, I instructed each PT to represent themselves with a filled-in dot and then draw a diagram representing how they believe mathematical authority operates in the mathematics classroom. I encouraged each PT to think aloud as they produced their diagram. Because they already discussed some of their beliefs about what sources could be mathematical authorities in the classroom, I looked and listened for aspects of their diagrams and explanations in which they reiterated or expanded upon those beliefs. In some cases, these diagrams revealed individuals, objects, communities, etc. that a PT considered to be a mathematical authority that we had not yet discussed or that I had not considered as a mathematical authority. Thus, having the PTs produce these diagrams was instrumental in that it led to modifications to or strengthened some of the inferences I made about what each PT believed about mathematical authority at this point in the study and it revealed some new beliefs about mathematical authority, which I was able to explore further in subsequent interviews.

I should note that each PT constructed their diagrams via the online medium Jamboard (https://jamboard.google.com). The purpose for using Jamboard was twofold. First, it provided a context in which each PT was able to construct their diagram and talk about aspects of their diagram while both of us viewed their diagram via the *Share Screen* feature of Zoom (https://zoom.us). Second, using Jamboard allowed me to give each PT access to develop and edit their diagram during the interview, then restrict their access to the diagram after they completed the diagram. The importance of restricting access to their diagram after the second interview will be made clear in the description of the sixth interview.

**Third Interview**

The third interview took place two to four weeks after the second interview. The time between the second and third interviews allowed me to analyze the second interview data and develop inferences about the beliefs each PT held, including which beliefs appeared to be strongly held by each PT. During the third interview I engaged each PT in a statement sort activity (see Cooney, 1985; Singletary, 2012) that helped me infer which beliefs about mathematics, teaching mathematics, students, and mathematical authority were important to each PT. Prior to the third interview, I recorded statements from the first two interviews that seemed to relate to their beliefs about mathematics, teaching mathematics, students, and mathematical authority. The statements were numbered, compiled in a PDF file, and emailed to each PT three days prior to the interview (an example can be seen in Excerpt 3). In that email I asked each PT to do the following:

1. Read each of the statements first.

2. As you read, consider which statements you would say represent important aspects of your beliefs.
3. Then, in the PDF, highlight those statements that represent important aspects of your beliefs. Feel free to highlight as many or as few statements as you would like.
4. Respond to this e-mail and attach the document with your highlighted statements.

I also asked each participant to send their highlighted statements a day before the interview so that I could review their highlighted statements prior to the interview.

**Excerpt 3**

*Subset of Statements Sent to Grace Prior to Third Interview*

(1) “if you're learning mathematics, you should be able to understand and visualize the concepts, apply them in multiple contexts, and be able to see something new and take those same concepts and apply them in a new context”

(2) I: “Where do you think these procedures that students follow or individuals who do mathematics follow, where do those procedures come from?”

Grace: “They come from someone else who was able to problem solve through the problem and figure out how you're supposed to do it. And then they were copied, and regurgitated, and it kept going.”

(3) “as a student, you just have to hope that the teacher truly has your best interest at heart. And if they know they messed up that they'll go back and fix it later… And that's what you really have to rely on.”

(4) “asking them questions about their process would tell me a lot about what they understood, and what they thought was correct, or what they thought was incorrect but was correct.”

To begin the third interview, I asked each PT if there was anything they were looking for as they considered which statements to highlight. I also asked questions about a few of their highlighted statements as a way to further explore aspects of their beliefs about mathematical authority. The selection of these statements was informed by my preliminary analysis of the first and second interviews. That is, discussing these statements with the PTs enabled me to further explore aspects of their beliefs that I was unsure of or that seemed particularly important. I then asked each PT to sort all the statements into groups of their choosing. I told them that if there were statements that did not represent their beliefs, how they were thinking about the mathematics classroom, or how they were thinking about mathematical authority in the classroom, then they could discard those statements. I also encouraged them to talk aloud as they were sorting the statements and that they should label each of their groups at some point during the statement sort activity. After each PT completed the activity, for each group of statements I asked them to "explain why you grouped these particular statements together and why you chose this label." I also asked them to explain differences between groups of statements that seemed similar and if they thought there were any relationships between groups. Engaging each PT in the statement sort activity provided insights into not only the beliefs that each PT considered to be important and influential at the time (i.e., before student teaching), but also how they might perceive the relationships between different groups of beliefs.

**Weekly Reflections**

At the end of almost each week during the PTs' student teaching practicum, they responded to a reflection prompt provided by their student teaching supervisor. I gave each participant the option to share, or not share, their weekly reflections with me as part of their involvement in the study, and each PT opted to share their weekly reflections with me.

Reflection prompts varied across supervisors but included prompts that instructed PTs to discuss their philosophy of teaching, their pedagogical areas of strength and growth, what makes a lesson "good," and how they knew their students understood the mathematics taught in a given week. Some of the PTs' reflections included descriptions of their beliefs about or related to mathematical authority. Hence, some of the PTs' weekly reflection responses informed aspects of their beliefs to further explore in the interviews after student teaching or yielded data that reinforced some of the inferences I made about their beliefs related to mathematical authority.

**Weekly Lesson Plans**

During their student teaching practicum, each PT was required to submit a detailed lesson plan for at least one lesson each week. I asked each PT to share their weekly detailed lesson plans with me and gave them the option to stop sending their lesson plans at any point during the study. To my knowledge, each PT sent me all the detailed lesson plans they developed during student teaching. Lesson plans developed by the four PTs provided insights into some of their instructional practices, including the tasks they engaged students in, their use of technology, and the different grouping strategies and thus discussions they implemented in the classroom (e.g., whole class, small group, turn-and-talk). These insights helped form some of my inferences regarding the PTs' beliefs about teaching mathematics, learning mathematics, as well as what sources could be mathematical authorities in the classroom (e.g., students, graphing calculators).

**Fourth Interview – Lesson Plan Interview**

During the second half of the PTs' student teaching practicum, I had each PT choose one lesson plan to form the basis of the fourth interview. The fourth interview, which I refer to as the lesson plan interview from here on, was loosely structured because the main purpose of the interview was to provide each PT the opportunity to talk through their lesson and their instructional decision making. The interview was conducted during the second half of student teaching so that each PT was more established in their student teaching classroom, had time to get to know their students, and was more comfortable with lesson planning. Prior to the interview I reviewed all the detailed lesson plans they submitted to that point to look for trends in their instruction as well as their use of tasks and/or technology. I then reviewed the lesson plan they chose to discuss in the interview and developed questions for each aspect of that lesson plan. Some questions were intended to explicate the PT's thinking as they were developing that specific lesson plan. The intentions for other questions were to have them discuss trends I noticed across all lesson plans they developed or to discuss aspects of instruction that were present in other lessons plans but not the one they chose. For instance, across lesson plans, I noticed multiple PTs planned for students to discuss and work on tasks or problems in small groups, thus I asked what they viewed as the purpose(s) of engaging students in small groups. In other instances, if I noticed a PT often planned to use technology such as Desmos (https://www.desmos.com) or GeoGebra (https://www.geogebra.org), but the use of technology was absent in the lesson they chose to discuss, then I would ask them to describe how they engaged students in technology and what role technology played in those other lessons. Hence, the lesson plan interview provided insights into their beliefs about teaching mathematics, students as mathematical authorities, and objects as mathematical authorities (e.g., Desmos, GeoGebra, other manipulatives).

**Fifth Interview**

The fifth interview took place shortly after the conclusion of each PT's student teaching practicum. Prior to this interview I conducted preliminary analyses on all data collected during their student teaching and used these analyses to inform areas of exploration in this interview.

For instance, across the four participants, I asked about specific instances from previous interviews, their lesson plans, or their weekly reflections that helped me better understand their beliefs about the teacher as a mathematical authority and students as mathematical authorities. Regarding their beliefs about students as mathematical authorities, I asked questions that elicited their beliefs about which students could be positioned as mathematical authorities and/or instructional practices they mentioned previously that seemed to position students as mathematical authorities. Additionally, in previous interviews, each of the four participants described their role in the classroom as a facilitator and described asking assessing and advancing questions (Smith & Sherin, 2019) as they supported student learning. Hence, at the beginning of each participant's fifth interview, I provided them several analogies to choose from that best described their role as a facilitator. This provided insights into their beliefs about teaching mathematics, the teacher as a mathematical authority, and, in some cases, students as mathematical authorities. I also asked each participant to define assessing and advancing questions, why they would use those questions when interacting with students, and how they would describe "the authority between you and the student(s)" when asking those questions. Questions and prompts that were asked/posed to each participant in this interview, including the analogies that best described their role as a facilitator, can be found in Appendix B.

**Sixth Interview**

The sixth interview took place three weeks after the fifth interview. Prior to the sixth interview, I asked each PT to develop a new mathematical authority diagram like the one they developed in the second interview. I e-mailed each participant instructions on how to create the diagram, asked them to create a short video (no longer than 5 minutes) in which they described the different aspects of their diagram, and asked them to send me their diagram and video at least a day before the sixth interview. To develop the sixth interview protocol for each participant, I analyzed data from their previous interviews, examined their second mathematical authority diagram and corresponding video to identify aspects of their diagram to explore, and compared their two mathematical authority diagrams side-by-side. Analyzing their second mathematical authority diagram alongside their first mathematical authority diagram and in tandem with their previous interviews enabled me to develop questions about their second mathematical authority diagram that helped me better understand their beliefs about mathematical authority. For each participant, I started the sixth interview by asking questions about their second mathematical authority diagram and some of their descriptions in their corresponding video. I then placed their two diagrams side-by-side and asked if they perceived any substantive differences in the two diagrams and, if so, how they might explain those differences. If the PT did not identify some of the differences I perceived, I then pointed out those perceived differences and asked how they would explain or describe them. Discussing each PT's second mathematical authority diagram and comparing the two diagrams they developed provided instrumental insights into their beliefs about mathematical authority, particularly their beliefs about what sources can be mathematical authorities or how some sources influence the teacher or students as mathematical authorities, as well as how their student teaching practicum influenced their beliefs.

**Final Interview**

The final interview took place at the beginning of the Spring 2022 semester. This gave me at least three weeks between the sixth interview and this final interview. The primary purpose of the interview was to provide a final opportunity to further investigate any of their beliefs about mathematical authority, particularly related to students as mathematical authorities. Having at least three weeks between the sixth interview and this final interview provided time to continue preliminary analyses and pinpoint specific responses or instances from the previous interviews to further explore. A secondary purpose of this interview was to understand each PT's educational background. Hence, at the beginning of the final interview, I asked each PT to describe their high school and collegiate education experiences, and if any of their high school teachers or college professors were especially influential or inspired them to become a math teacher.

**Mentor Teacher Interviews**

I planned to interview each PT's mentor teacher during the Fall 2021 semester with the purpose of gaining further insight into each PT's teaching context, including understanding some of the beliefs and perceptions of their mentor teacher. Of the four PTs I discuss in this dissertation, only two of their mentor teachers consented and took part in the mentor teacher interview. The interviews with these two mentor teachers did not yield data that further supported my inferences regarding the two respective PTs' beliefs about mathematical authority. Hence, although I did conduct some preliminary analyses of the two mentor teacher interviews, I did not include the data generated from these interviews in the data corpus that I coded and, thus, the data generated in these interviews did not inform the narratives I wrote regarding the PTs' beliefs about mathematical authority.

I include this description of the mentor teacher interviews here for two reasons. The first reason is for methodological transparency, that is, to make clear that I intended to interview each PT's mentor teachers, but I was able to only interview two mentor teachers and ultimately concluded those two interviews did not generate data that supported or furthered my understanding of the two respective PTs' beliefs. The second reason is to make recommendations for future studies investigating the influence student teaching has on prospective teachers. I contend that interviews with mentor teachers can provide meaningful insights into PTs' student teaching experiences by helping researchers further understand PTs' student teaching context and the potential influences their mentor teacher can have on their beliefs and/or practice. Yet, as was the case with this study, one mentor teacher interview is likely not sufficient to further understand a PT's student teaching context or to develop defensible inferences for the beliefs about mathematics, teaching mathematics, or learning mathematics a mentor teacher may hold. Thus, researchers who plan to consider the student teacher-mentor teacher relationship plays a role in shaping a PTs' beliefs and/or instructional practice, then researchers likely need to interview the mentor teacher on multiple occasions.

**Summary**

Table 1 contains a timeline of when I conducted each interview and collected other data sources. As mentioned in the previous section, there were numerous weeks between each interview and this time afforded initial analyses of the data generated. These analyses significantly influenced the questions, prompts, and activities I asked each PT to engage in or respond to in successive interviews. Researchers who have used Leatham's (2006) sensible systems framework have found their participating teachers' beliefs to be internally consistent and consistent with their instruction, but that these beliefs were complex and, at times, exceptionally nuanced (e.g., Conner & Singletary, 2021; Cross Francis, 2015). This was also the case with each of the PTs who participated in this study. The time between interviews afforded me the time necessary to consider some of the complexity and nuances of each PT's beliefs about mathematical authority and develop questions, prompts, and activities that generated parsimonious data that helped me better understand these complexities and nuances. In the following section, I detail how I analyzed the data generated from these numerous data sources, both during data collection and after all data was collected.

**Table 1**

*Data Collection Description and Timeline*

| Data Source | Brief Description | Date(s) Data Was Collected | Research Question(s) Addressed |
|---|---|---|---|
| Interview 1 | Elicit beliefs about mathematics, teaching mathematics, and mathematical authority | June 1st – June 24th | RQs 1 & 2 |
| Interview 2 | Further explore responses from Interview 1; Responses to instructional situations; Develop mathematical authority diagrams | June 23rd – July 14th | RQs 1 & 2 |
| Interview 3 | Statement sort activity | July 18th – August 10th | RQs 1 & 2 |
| Weekly Reflections | Prompts and questions provided by student teaching supervisor | Throughout student teaching | RQs 1 – 3 |
| Weekly Lesson Plans | Detailed lesson plans required each week during student teaching | Throughout student teaching | RQs 1 – 3 |
| Interview 4 | Discuss one detailed lesson plan of their choosing; Discuss trends in other lesson plans | October 11th – 25th | RQs 1 – 3 |
| Interview 5 | Further explore aspects of beliefs about mathematical authority; Elicit responses that may yield evidence of changes in beliefs | November 16th – 22nd | RQs 1 – 3 |
| Interview 6 | Construct a new mathematical authority diagram; Compare and contrast this diagram with the one they constructed prior to student teaching | December 7th – 15th | RQs 1 & 3 |
| Final Interview | Follow-up on any PST's responses from Interviews 5 and 6: Receive feedback on the research process | February 8th – 28th | RQs 1 – 3 |

## Data Analysis

Data analysis has been described by Merriam (1998) as a process of "making sense out of the data" as well as one of "meaning making" (p. 178). When considering how to make meaning out of the data collected for this study, I did so in two different manners. First, I considered how to make meaning of each PT's words, descriptions, gestures, and diagrams during the data collection phase of the study. I drew upon my previous experiences interviewing teachers multiple times, in which I found analyzing data as it was collected helped me develop interview protocols that elicited teachers' responses that built upon, extended, and supported my understanding of the data. As Merriam (1998) put it, "data that have been analyzed while being collected are both parsimonious and illuminating" (p. 163). Second, I considered how, after collecting all the data, to analyze the entire data corpus and make sense of it all. In other words, I contemplated how to focus on each PT individually, analyze all collected data, develop strong inferences for the beliefs each PT holds, then compare their beliefs with the other PTs' beliefs. In the following sections I describe how I made sense of the data collected for this multi-case study by first describing how I analyzed the data during data collection. I then describe how I analyzed the data after all data was collected, both within each case and across cases.

### Within Case Analysis While Collecting Data

Merriam (1998) described data analysis and data collection as "a *simultaneous* activity in qualitative research" (p. 151; emphasis in original). As I alluded to in the *Data Sources* section, preliminary data analysis began immediately after the initial interview and continued as new data was collected. I transcribed all interviews and began analyzing the data as I transcribed each interview. As I transcribed, I reflected on what each PT said during the interview, took note of seemingly important statements, instances, or claims made by each PT, and jotted down questions that arose, which I revisited as I developed subsequent interview protocols.

After transcribing the initial interview, I used the qualitative data software MAXQDA (VERBI Software, 2021) to code the data as a means "to derive meaning and insight from the word usage and frequency pattern found in the texts" (Yin, 2003, p. 110). For my preliminary analysis, I coded the data using the following broad codes: *mathematics*, *teaching mathematics*, *learning mathematics, students,* and *mathematical authority.* Because my definition of mathematical authority is open to a variety of individuals, objects, entities, and communities being positioned as mathematical authorities, under the *mathematical authority* code I also included the subcodes *students as authority, teacher as authority, object or item as authority*, and *community or discipline as authority*. As I analyzed the first interview, I applied the broad codes when it seemed the PT was describing aspects of their beliefs aligned with the code. For instance, if there were moments where a PT described a student, or students, as a legitimate source of mathematical knowledge or reasoning or described instances in which students could make meaningful mathematical contributions, I applied the *students as authority* code. Similar to when I transcribed the interview, as I coded the interview data, I jotted notes about questions I could ask in the following interview or ideas for prompts/activities to include in subsequent interview protocols.

After I coded the first interview, for each PT I generated a coding summary for each broad code. The coding summary for a code included all statements from the first interview coded with the respective code. For instance, the coding summary for Grace's *teacher as authority* code included all of Grace's statements from the first interview on which I applied the *teacher as authority* code. I analyzed each coding summary by looking for themes to emerge within each code. Looking for emergent themes within each code enabled me to make sense of each PT's statements or claims related to each code, which helped me form working hypotheses of each PT's beliefs about or related to mathematical authority. This analysis also focused my attention on aspects of each PT's beliefs that I needed to further explore in subsequent interviews. I used the themes that emerged in this analysis to inform the second protocol for each PT. Prior to conducting each PT's second interview, I reviewed the emergent themes to remind myself of what each PT said related to each code, which influenced which responses or parts of responses I followed up on during the second interview.

After conducting the second interview with each PT, the preliminary analysis that followed was similar to the analysis that followed the first interview. More specifically, I transcribed each PT's second interview, coded the transcript using the previously mentioned broad codes, and generated summary reports for each broad code. As I transcribed and coded the second interview, I continued to note any statements or moments in the interview that, if further explored, could help me answer my research question. This time, coding summaries for each PT's codes included instances from both the first and second interviews. I again looked for themes to emerge as I analyzed each coding summary for each participant. These themes informed the questions I asked at the beginning of the third interview and also informed which statement I chose for the statement sort. That is, I chose statements that, in my view, seemed representative of the themes that emerged during this analysis and, consequently, their beliefs about and related to mathematical authority.

During the third interview, each PT chose statements they considered to be important aspects of their beliefs and placed all statements into groups of their choosing. Additionally, they labeled each group of statements and explained how they grouped the statements. The analysis that followed the third interview was similar to the analyses that followed the first and second interviews. Analysis also included analyzing the statements each PT claimed were important aspects of their beliefs and how they grouped their statements. For instance, one PT created a group of statements in which all but one of the statements was highlighted (i.e., represented an important aspect of her beliefs), which helped me develop inferences for which of her beliefs about and related to mathematical authority she held most strongly. Analyzing each PT's coding summaries after the third interview in tandem with their grouped statements informed my hypotheses about which of their beliefs may be strongly held and which beliefs may inform other beliefs. That is, I was able to develop a preliminary, hypothesized model of each PT's system of beliefs (Green, 1971). I was also able to determine aspects of each PT's beliefs to further explore in the lesson plan interview and interviews after student teaching.

During their student teaching, I reviewed each PT's weekly reflections and lesson plans as they sent them to me. I made memos in each lesson plan and weekly reflection of instances that seemed to be related to their beliefs about mathematical authority. For instance, if a PT planned for students to present their idea or solution to the whole class or engage in a whole class discussion of student-generated ideas, then I made a memo related to those instances. These memos informed some of the lesson plan interview protocol, fifth interview protocol, as well as the inferences I developed of each PT's beliefs after all data was collected.

The analyses that followed the lesson plan interview, fifth interview, and sixth interview were similar to the analyses that followed the first three interviews. I transcribed each interview, coded each interview using the broad codes, and generated coding summaries for each PT's codes. As I analyzed each coding summary, I looked for themes to emerge and these emergent themes informed subsequent interview protocols. Additionally, I made note of any similarities I noticed across participants. For instance, I noticed each of the PTs described asking assessing and advancing questions as they supported students in the classroom, and each PT would, with varying frequency, script assessing and advancing questions in their lesson plans. Hence, this informed the fifth interview protocol, in which I asked each participant questions about their use of assessing and advancing questions and how they would describe authority operating when they asked students those questions.

Finally, after each PT's final interview, I transcribed the interview and coded the interview with the initial broad codes. After the final interview was conducted, I completed an initial round of coding all collected data. After this round of coding was complete, I reviewed notes and memos taken throughout the data collection phase of the study and used these to inform new codes. For instance, one PT frequently discussed how practical aspects of teaching influenced his planning and instruction. Thus, I added the code *Practicalities of Teaching* to the initial coding scheme.

With a modified coding scheme, informed by my data analysis during data collection, I then shifted my focus from preliminary data analysis for the purpose of generating data that was "both parsimonious and illuminating" (Merriam, 1998, p. 163), to analyzing data of each case (i.e., each PT) with the end goal of "conveying an understanding of the case" (Merriam, 1998, p. 193). In the subsequent paragraphs, I describe how I carried out the within-case analysis, ultimately leading to a narrative describing each PT's system of beliefs related to and about mathematical authority and how student teaching influenced their beliefs.

**Within Case Analysis After Data Collection**

I began the within case data analysis by coding all interviews with all participants using the modified coding scheme developed after all data was collected. After this round of coding was completed, for each PT, I employed the constant comparative method (Strauss & Corbin, 1998) by searching for emergent themes within each code; comparing those themes and, when necessary, combining themes to create a more generalized theme; and comparing the resulting themes within each code to themes in related codes. For instance, I compared all themes that emerged for all codes under the *students as authorities* umbrella code to see if there were similar themes that emerged and thus could be combined. I then used these emergent themes to generate new codes and wrote memos describing all codes. It was at this point that I decided not to discuss Jack's and Lydia's beliefs in this dissertation. I coded each of the remaining four PTs' interviews with the new coding scheme and, while doing so, looked for instances that could not be captured with the existing codes. When there were instances that could not be captured by the coding scheme, I applied a broad code (e.g., *students as mathematical authorities*) to capture that instance. After this round of coding was complete, there were several instances that were not captured by the existing codes. But the low number of those instances and the PTs' statements in those instances did not warrant the inclusion of new codes. Thus, I coded those instances with the corresponding parent code (e.g., *students as mathematical authorities*, *teacher as mathematical authority*, *teaching mathematics*).

After coding all interviews a third time with the final coding scheme, I then started to use the coded data to write narratives of what each PT believed. My coding scheme was tiered so that there were parent codes with subcodes under those parent codes. For instance, one parent code was *teacher as authority*, which had numerous subcodes, and some subcodes also had subcodes. For each PT, I generated coding summaries of all codes and again looked for themes to emerge within each code. However, this time I focused on one parent code at a time to generate a narrative about what each PT believed related to that parent code. For instance, as I was focusing on all subcodes under the *teacher as authority* parent code, I first looked for themes to emerge from the coding summary of one subcode and described how that theme related to their beliefs about teaching mathematics. As I looked for themes to emerge in the coding summaries of other subcodes, I either merged those themes with similar, already existing themes related to the PT's beliefs about the teacher as a mathematical authority, or I added the emergent themes to the already existing themes related to the PT's beliefs about the teacher as a mathematical authority. The result of this process was numerous themes related to the PT's beliefs about the teacher as a mathematical authority and data excerpts supporting those themes. I then used these themes and supporting data to write a narrative about the PT's beliefs about the teacher as a mathematical authority. I repeated this process for all other beliefs about and related to mathematical authority and for all participants.

I should note that within the coding summaries for all codes, I was able to mark which data was generated before student teaching, during student teaching, and after student teaching. Thus, as I looked for themes to emerge for each code, I was also able to note if that theme was applicable before student teaching, after student teaching, or for the duration of the study. This was an important part of my analysis because it enabled me to develop inferences about the influence each PT's student teaching experience(s) had on their beliefs.

After I wrote narratives for each PT's beliefs, I diagrammed a hypothesized model of their beliefs. These models helped me visualize the relationships between beliefs and answer the following questions about their system of beliefs: *Which beliefs were primarily held beliefs? Which beliefs were strongly held beliefs? What beliefs about mathematics, teaching mathematics, learning mathematics, or students were not beliefs about mathematical authority but seemed to inform or interact with their beliefs about mathematical authority?* Answering these questions helped me revise each PT's narrative and also write about relationships between beliefs that appeared to be particularly salient or impactful. I present the narrative of each PT's beliefs in the following chapter (Chapter 4).

**Across Case Analysis**

To conduct the cross-case analysis, I looked for similarities and differences across the four PTs' specific beliefs and system of beliefs (Green, 1971). I reviewed both the narratives of each PT's beliefs and the hypothesized diagrams of their systems of beliefs to determine similarities and differences across the four PTs' beliefs about mathematical authority. As I reviewed the narratives and diagrams, I also looked for aspects of their beliefs or relations among beliefs that were similar or could explain differences across the participants. After I identified similarities, differences, and conjectures for why there were differences, I then wrote the results of that analysis according to their beliefs about the teacher as a mathematical authority, beliefs about students as mathematical authorities, beliefs about other sources as mathematical authorities, and the relationships between beliefs. I present the results of the cross-case analysis in the Conclusions chapter (Chapter 5).

The cross-case analysis also furthered my understanding of each individual PT's beliefs and systems of beliefs (Green, 1971). As I conducted the within-case analysis and went through the iterative process of refining the narratives of what each PT believed, I could not help but consider how their beliefs were similar to or differed from the other PTs' beliefs. For each PT, this analysis enabled me to consider unique and subtle differences in their beliefs about and related to mathematical authority and develop explanations for those differences. Furthermore, comparing the beliefs held across participants accentuated which beliefs were salient for each individual PT. Thus, the cross-case analysis supported some of my inferences for how strongly each PT held their beliefs as well as inferences for which of their beliefs prominently influenced other beliefs (i.e., were primary beliefs).

# CHAPTER 4

# RESULTS

In this chapter, I present four separate narratives describing each of the four PTs' beliefs related to and about mathematical authority. I first report Simon's beliefs, then Grace's, then Chris's, and close by reporting Hannah's beliefs. At the beginning of each PT's narrative, I provide some background information of the participant, then I describe their beliefs about mathematics, teaching mathematics, learning mathematics, and students. Next, I report the PT's beliefs about mathematical authority by describing their beliefs about the teacher as a mathematical authority, students as mathematical authorities, and other sources as mathematical authorities. I close each narrative by describing the influence their student teaching had on their beliefs about mathematical authority and highlight salient relationships among their beliefs. As I report each PT's beliefs, I draw on Green's (1971) conception of belief systems. Specifically, I describe beliefs as *primary* if they seemed to inform other beliefs, as *derivative* if they seemed to be informed by other beliefs, and as *core* or *strongly held* if the PT believed something strongly. Additionally, for the purpose of clarity, I italicize all beliefs each individual PT held.

## Simon's Beliefs

Simon described himself as "good at math" and said that growing up he wanted to either be a math teacher or a biomedical engineer. Before choosing to major in Mathematics Education, Simon had opportunities to tutor students in mathematics; in those experiences, he was able to help students understand mathematics which was, according to Simon in the first interview, "a really phenomenal feeling to me." Ultimately, Simon chose to teach mathematics because doing so enabled him to leverage his understanding of mathematics to help students. Simon earned his degree in Mathematics Education at the conclusion of the Fall 2021 semester and planned to begin teaching full time at the start of the 2022-2023 academic year.

Simon described enjoying his time as a student teacher because he was able to develop relationships with students, put what he read in his courses into practice, and learn how to practically manage the different aspects of being a teacher. This is not surprising because Simon came across as a very pragmatic individual who frequently considered practical matters when discussing his teaching. I describe how his consideration of the practical aspects of teaching influenced his beliefs about teaching mathematics and related to his beliefs about students as mathematical authorities.

## Simon's Beliefs About Mathematics, Teaching Math, Learning Math, and Students

### *Simon's Beliefs About Mathematics*

Simon strongly believed *mathematics is a body of knowledge that has been discovered and is continuing to grow through new discoveries*. Specifically, Simon believed mathematics is a body of knowledge that has been proven true and describes much, if not all, of the physical world. At various points in the study, Simon described mathematics as "discovered" and an "approximation of the rules," the "mechanics of the world," "the basis of all interactions of physical things," and "what we all agree is true and can be demonstrated in the real world." Simon seemed to intentionally use the word "discovered" when discussing how mathematics has been and continues to be developed. For instance, Simon contended that if all our current mathematical knowledge was "lost," individuals would eventually be able to "discover" that knowledge again as they engage with and attempt to understand nature. A consequence of his belief that mathematics is a body of knowledge that is inherent in the natural world is that he viewed this body of knowledge as correct mathematics.

Simon also strongly believed *mathematics entails "a way of thinking, a way of approaching problems and solving them."* During the final interview, Simon described mathematical thinking as, "you make theories, you attempt to try to find things that are true using other [prior] knowledge, and you build upon your knowledge base doing so." Simon went on to say that mathematical thinking entails hitting a "roadblock" when solving a problem and considering other potential solution paths to the problem. Moreover, Simon suggested mathematical thinking is unique to the discipline of mathematics and a form of thinking in which individuals can become increasingly proficient as they become further "trained" in the productive approaches or practices of mathematics. For instance, Simon implied that mathematics professors have undergone extensive "training" in mathematics and consequently know how to best "approach" mathematics. Hence, Simon believed mathematics was not only a body of "true" knowledge, but also a set of practices or approaches that lead to the discovery or development of mathematical knowledge. That is, Simon did not view his beliefs about mathematics as contradictory, but as complementary—he believed mathematics was a body of knowledge and that there are practices or approaches that are productive and necessary for the discovery of that knowledge. Moreover, Simon's beliefs about mathematics seemed to be primarily held beliefs because they were related to and influenced many other beliefs Simon held, including his beliefs about learning mathematics.

***Simon's Beliefs About Learning Mathematics***

Derived from Simon's beliefs about mathematics, Simon believed *learning mathematics entailed learning how and when to use or apply correct mathematics as well as how to* *productively approach mathematics*. On multiple occasions throughout the study, Simon contended that an individual has learned mathematics when they understand how mathematical ideas and procedures can be applied in novel situations or scenarios and are able to apply those ideas or procedures, particularly outside of the classroom. For instance, during the first interview, Simon claimed, "learning [mathematics] is when you have a skill that stays with you, that you can demonstrate at a later time when it is useful." In other instances, Simon implied that student learning was evidenced when students understand connections between "what you [a student] learn in the classroom and real life." Simon also believed learning mathematics entailed learning productive approaches to or practices of mathematics. Simon claimed, on multiple occasions, that as students learn new mathematics, they will experience failure but by persevering, will be able to develop understanding and procedural fluency. Moreover, Simon claimed, "being able to look at failure and how they struggle through failure and towards success is a very key trait [when learning mathematics] that needs to be taught to students more." Hence, Simon suggested that approaching mathematics with a disposition that expects "failure" and is willing to try new things is something that *should*, and thus *can* be taught. Simon also implied that such a disposition is essential when engaging in and learning new mathematics.

Simon believed *students learn mathematics best when they "discover" or develop the intended mathematics on their own and/or in collaboration with their peers*. Simon consistently claimed that students retain mathematical knowledge and have a heightened sense of "ownership" over their mathematical knowledge when they are able to develop that knowledge for themselves. During the second interview, Simon explicated this belief when he stated, "I think that [discovering mathematics] is actually the best way to learn those [mathematical] truths and to discover them on your own [unclear utterance] and why it's helpful." During the fifth interview, Simon described a Desmos (https://www.desmos.com) task he implemented that led to students developing the distance formula for two points in the coordinate plane. Simon claimed the Desmos task, and others like it, foster students' "ownership of the material" and, because students develop the mathematics on their own as they engage in such tasks, "they will more likely remember it, they will more likely be able to use it in context and understand when to use it in context." Although Simon believed students learn best when they "discover" the mathematics, he also believed students could learn mathematics by the teacher "imparting" knowledge (i.e., via lecturing) to students and then providing students time to practice applying that knowledge. Simon consistently claimed that some mathematical concepts were not practical for students to "discover" in the context of the mathematics classroom and that students could learn such concepts by him "giving" them those ideas and then providing students time to practice working with the ideas he "gave" students. For instance, in the first interview, Simon claimed, "certain concepts you [the teacher] need to demonstrate in order for them [students] to start understanding."

Because Simon believed that students learn mathematics best when they "discover" the mathematics on their own, one might expect Simon would plan and enact lessons in which students were provided opportunities to "discover" the intended mathematical ideas or procedures of a given lesson. However, Simon held a prominent belief about teaching in schools that often influenced, at least in Simon's descriptions, how he planned and enacted instruction. I start the *Simon's Beliefs About Teaching Mathematics* section by describing this belief.

### *Simon's Beliefs About Teaching Mathematics*

Simon strongly believed *there are many practical aspects of teaching and learning mathematics that teachers need to consider and when considered, significantly influence* *teachers' lesson planning and instruction*. Throughout his time in the study, Simon described how the state Department of Education (DoE) and school administration's expectations, time constraints in the classroom, as well as the time and effort required to develop lessons impacted his planning and enacted instruction. In this section I detail Simon's description of how time constraints in the classroom influenced one aspect of his instruction, and I report how other practical aspects influenced his instruction in the *Simon's Beliefs About Mathematical Authority* sections. In this section I detail, in part, how Simon described time constraints in the classroom influencing his planning and instruction to highlight Simon's belief that there are practical aspects of teaching and learning mathematics that influence teachers' instruction.

Related to his belief that students learn best when they "discover" the mathematics themselves, Simon insisted, throughout the study, that students could "discover" all mathematics in the prescribed curriculum (i.e., content standards) on their own if given enough time. Yet, due to time constraints in the classroom and the school year, Simon did not think it was practical to provide students opportunities to discover the intended mathematics for every lesson. Hence, Simon's belief concerning the practical aspects of teaching led to him enacting lessons in which he fostered students' learning more directly, and thus in ways that may not best support student learning (i.e., via students' discovering the intended mathematics). During the third interview, Simon claimed that because teachers "only have a certain amount of [time] per day to have the students in that learning environment [i.e., the classroom]," there are times where the teacher needs to "expedite that learning process…allowing them to still tinker with the mechanics of it as much as possible." Notice Simon implied the limited time students are in the classroom each day means teachers need to "expedite" students learning the intended mathematics for a given lesson, meaning it may not always be practical for students to learn the intended mathematics by “discovering” that mathematics. After student teaching, Simon provided an example of how he expedited student learning when he taught factoring polynomials by grouping:

> that process [factoring by grouping], given enough time, students would eventually find that, if you're given a few months to look at specifically polynomial factoring, I'm sure eventually some of them would run into that. But in the case where you have four days to impart this knowledge to the students, the teachers can kind of assume that they wouldn't have the time to develop that knowledge naturally and, instead, choose to give them that knowledge directly.

Simon claimed some students would eventually be able to explore, develop, and understand factoring by grouping if given enough time (“a few months”), but giving students sufficient time to explore and develop that understanding is not practical in the K-12 classroom context. Consequently, Simon decided to expedite students learning the process of factoring polynomials by grouping by choosing to “give them that knowledge directly.” Simon, then, conceded that students learning factoring by grouping in a way he believed would be best (i.e., via discovery) was not practical due to the time constraints within which he and students had to operate.

Before describing Simon’s other salient belief about teaching mathematics, I want to acknowledge and address one potential question the reader may have in relation to the previous paragraph. I am intentionally claiming it was Simon’s *view* or *perception* of time constraints that influenced his instruction. I am not suggesting there are not restrictions when it comes to the time teachers, including Simon, have with their students on a given day and within a given year. Rather, I claim that teachers view or perceive those time constraints and operate within them differently. In Simon’s case, he consistently described how time constraints influenced his instructional planning as well as how he operated in the classroom. Thus, I infer that Simon believed time constraints were one of the key practical aspects he needed to consider as a math teacher, and this belief often interacted with other beliefs he held to inform his instruction. I describe some of those interactions in the *Influence of Student Teaching on Simon's Beliefs and Some Relations Among His Beliefs* section.

Simon held another, less prominent belief about teaching mathematics, namely that *mathematics teachers need extensive content and pedagogical content knowledge (Shulman, 1986) to teach mathematics effectively*. Throughout his time in the study, Simon described how he leveraged his content and pedagogical content knowledge to select tasks, enact instructional structures, and develop "intentionally designed lessons" that would foster students' understanding of the intended mathematics. During the third interview, Simon explicated his view of the relation between his knowledge of content and pedagogy and designing lessons when he explained, "the teacher having been trained in mathematics, but also in teaching, will know the best way to intentionally design the lesson so that the student can get the most out of it when they complete that lesson." After student teaching, Simon suggested that he relied on his content knowledge and pedagogical content knowledge to determine which concepts or procedures students could "discover" given the limited time in the classroom and which concepts or procedures he would need to teach using more direct or traditional methods. Thus, Simon believed his knowledge of correct mathematics and how to teach mathematics was essential in developing lessons that would foster students' learning of the intended mathematics given the limited time he had with students for a given lesson.

### *Simon's Beliefs About Students*

Simon had one salient belief about students that related to his beliefs about mathematical authority, namely that *students will never be as "trained" in mathematics as their teachers*.

Simon's belief about students was derived from his belief that an aspect of mathematics is engaging in productive practices and approaches. The way Simon often described student and teacher roles in the classroom, including how students relate to teachers and vice versa, highlighted his assumption that math teachers are always, with rare exception, more "trained" in mathematics than their students. For instance, as Simon was talking through his second mathematical authority diagram, he claimed that students are "untrained" in mathematics and have not been "exposed to the math" for as long as teachers. Because Simon's belief that students will never be as trained in mathematics as their teachers has implications for his beliefs about the teacher and students as mathematical authorities in the classroom, I expand on this belief in the following sections.

## Simon's Beliefs About Mathematical Authority

### *Simon's Beliefs About the Teacher as a Mathematical Authority*

Simon believed *the teacher should be and is positioned as the ultimate mathematical authority in the classroom by students* and the reason for that positioning was multi-faceted. Two primary reasons Simon believed the teacher is the ultimate mathematical authority derive from Simon's beliefs about mathematics. Specifically, Simon believed the teacher is the ultimate mathematical authority in the classroom because of their extensive, correct content knowledge and because they are "trained" in the practices or approaches of mathematics. Simon believed the foremost reason the teacher is the ultimate mathematical authority in the classroom is because they know more correct mathematics than their students. When Simon was developing his first mathematical authority diagram (see Figure 1), he drew a unidirectional arrow going from the *teacher* to *students* and claimed, "the teacher has direct mathematical authority over the students because they are more familiar with the knowledge." Simon made similar claims about the teacher's position as the ultimate mathematical authority due to their content knowledge in other interviews before student teaching as well as after student teaching. For instance, as Simon was developing his second diagram (see Figure 2), he asserted, "the mathematics always wins and all mathematics [sic] authority stems from correct mathematics." He then added, "at the end of the day, in the classroom, I am the highest mathematical authority as a teacher." Taking these two quotes together, Simon implied the teacher is positioned as the ultimate mathematical authority in the classroom by students because they know more correct mathematics when compared to their students.

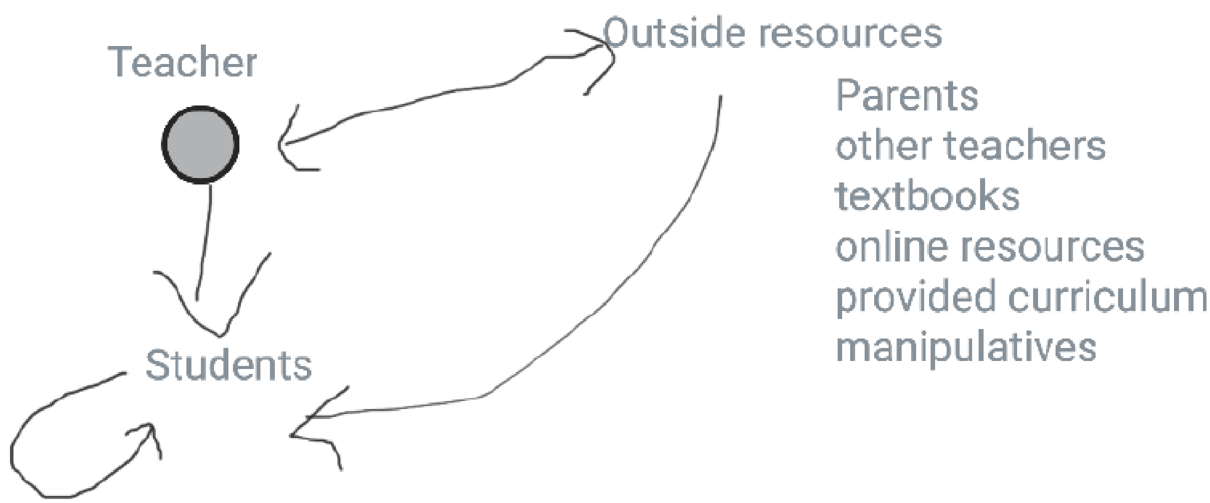



*Figure 1*. Simon's First Mathematical Authority Diagram

Related to Simon's beliefs about students, he also described teachers being the ultimate mathematical authority due to being more "trained in mathematics" than students. For instance, during the third interview, Simon claimed, "math knowledge can come from most anywhere and most anyone, but there are people that are trained in mathematics that are more likely to have correct math knowledge." He went on to imply that, in the classroom, the teacher is the individual who is "trained in mathematics," and thus a more legitimate mathematical authority than students. From Simon's claim that teachers have been "trained in mathematics," I conjecture he meant that teachers are more familiar with (i.e., trained in) the productive practices and approaches of mathematics. During the fifth interview, Simon made a distinction between mathematical content knowledge and understanding how to approach mathematics, and implied both influence one's position as a mathematical authority: "They [students] always have some mathematical authority, because they *know some math* and they *know how to approach it* [emphasis added]…but we [teachers] just do have more because we are more trained with it." Here and elsewhere, Simon implied that understanding more correct mathematics and being further "trained" in the acceptable practices of mathematics when compared to students, are the two primary reasons the teacher is the most legitimate mathematical authority in the classroom.

A final aspect that Simon believed led to him being positioned as the ultimate mathematical authority in the classroom by students was his pedagogical content knowledge (Shulman, 1986). On multiple occasions, Simon implied that because he was "trained in teaching," students would position him as the most legitimate mathematical authority when they had questions or needed help progressing on a task or problem. For instance, during the sixth interview, Simon claimed, "every single person can be a mathematical authority, but I am now more of a mathematical authority because of my advanced training in both mathematics and education…I have learned all these skills about how to demonstrate mathematics to students." In this instance, Simon suggested that his position as the ultimate mathematical authority is multi-faceted, and one facet is his pedagogical content knowledge. In other words, Simon, as the teacher, knew how to best "demonstrate mathematics to students," and this is one factor that leads students to view him as the most legitimate mathematical authority in the classroom who can help and support their learning. Still, Simon believed the legitimacy of the teacher depended primarily on his more extensive content knowledge and being further “trained” in mathematics when compared to students.

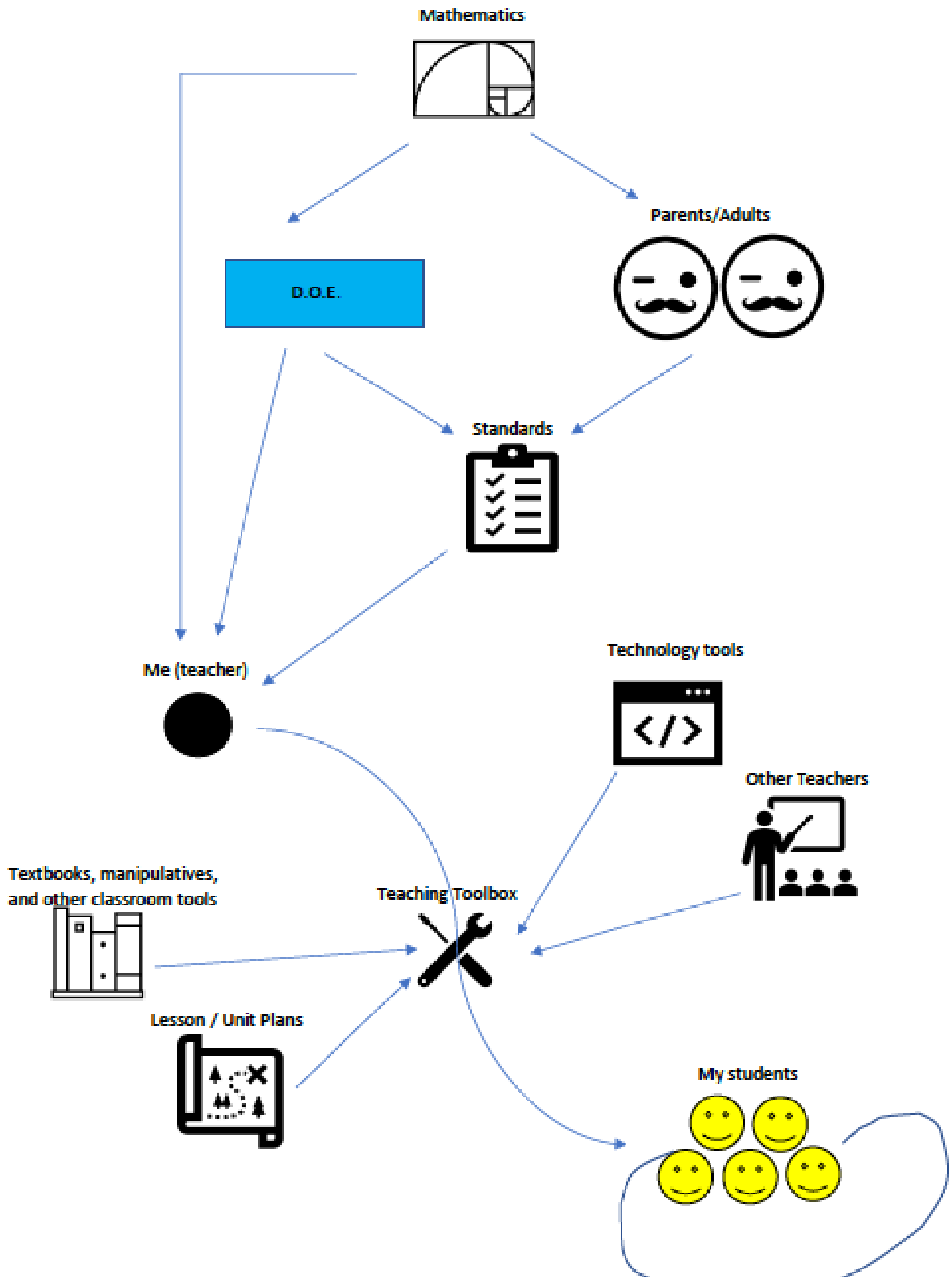



*Figure 2*. Simon’s Second Mathematical Authority Diagram

As the ultimate mathematical authority, Simon believed *his primary role was to support students' learning of correct mathematics*. Conversely, Simon often described his role as ensuring students did not develop what he called "bad math." Simon, during the second interview, explicated the relation between his position as the ultimate mathematical authority and his role of making sure students did not exhibit or learn incorrect mathematics when he stated the teacher is to make sure a student's incorrect idea, procedure, or suggestion, "doesn't go too far and is corrected before it spreads too much…that's why the teacher is the ultimate [mathematical] authority, to make sure all input [e.g., an idea, procedure, suggestion] is beneficial." After student teaching, Simon continued to believe his primary role was to ensure students learned correct mathematics and continued to relate that role to his position as the ultimate mathematical authority. For instance, as Simon described his second mathematical authority diagram (see Figure 2), he stated,

> especially with untrained students who haven't been exposed to the math as long, I am definitely a much stronger mathematical authority than they would be in any particular math case, and therefore, although they can use their own mathematical authority to discover new things, and that is the ideal way to teach, if they do go down a wrong path, then my mathematical authority will outweigh theirs and I can hopefully steer them back in the right direction.

Hence, Simon suggested, in this instance and others, that as the ultimate mathematical authority, his primary role was to facilitate students understanding of correct mathematics, and this often meant leveraging his position to "steer" students away from developing incorrect mathematics.

Simon's belief that he, as the ultimate mathematical authority, was to support students learning of correct mathematics influenced his instruction in different ways depending on his view of the time constraints in the classroom. In other words, Simon's two beliefs—that he was to support students' learning of correct mathematics and that there are practical aspects of teaching mathematics that influence math teachers' instruction—often interacted to inform the different ways he supported student learning. Moreover, Simon suggested there was a continuum of interventions or methods he could use to support student learning that ranged from less direct to more direct, and that he preferred to use less direct forms of intervention because they fostered students' ownership of their learning. In the subsequent paragraphs, I detail some methods Simon described using to support student learning and provide evidence for my inference that his belief about the practical aspects of teaching influenced which methods he used.

Simon's preferred methods to support student learning were assessing and advancing questions (Smith & Sherin, 2019). Moreover, Simon related his use of assessing and advancing questions to his position as the ultimate mathematical authority in the classroom. Throughout interviews before student teaching, Simon described, in general terms, how he would use assessing and advancing questions to support students in learning the intended mathematics. In the fifth interview, Simon provided more specific descriptions of assessing and advancing questions as well as how mathematical authority operates when asking such questions. He described assessing questions as "a formative assessment, and very quick and dirty pulse that you can take to see where the student is, to then see if they need an advancing question." When I asked Simon to describe the mathematical authority between him and students when asking assessing questions, he replied, "that's me making sure…they understand the concepts, and that is…me flexing my math authority and saying, 'Hey, I am using my facilitator approach to make sure that you are on the right track.'" Hence, Simon viewed assessing questions as a strategy he used to leverage his position as the ultimate mathematical authority to ensure students were developing correct mathematics (i.e., they were "on the right track").

Simon described advancing questions as "me [Simon] deflecting them [students] as a bumper in the right way." Moreover, Simon insisted he was leveraging his position as the ultimate mathematical authority in the classroom when asking advancing questions because, "I do know the math better than them and I know what about the math would be most beneficial for them to discover in order to keep going on that path." Simon went on to say that when asking advancing questions, he was pushing students "in the right direction so that they can discover the math on their own, or at least think they discovered the math on their own." Considering these two quotes in tandem, Simon viewed advancing questions as an effective instructional strategy he could use as the ultimate mathematical authority because he knew the questions that, when asked, would lead students to either reconciling their incorrect mathematics or making further, correct progress as they considered his questions. Simon later said that advancing questions were an important instructional strategy, particularly if students were following an unproductive solution path or train of thought, because "I still want them to feel like they walked that path, at least a majority on their own, unassisted." Thus, Simon consistently described advancing questions as one of the primary and less direct strategies he used in the classroom as the ultimate mathematical authority to support students' learning of mathematics and, simultaneously, have students' *feel* as if they were discovering the intended mathematics in a given lesson.

Although Simon preferred to support student learning by asking assessing and advancing questions, he suggested that because he was the ultimate mathematical authority in the classroom, he could support students' learning more directly if he perceived there was insufficient instructional time for students to discover the intended mathematics. For instance, in the first interview, Simon claimed the teacher knows the mathematics they want students to "take away" and the teacher can support students learning that mathematics, "with assessing and advancing questions, leading them towards those conclusions. Or, if absolutely necessary, they can just show the students those conclusions if, for instance, they run out of time." In this instance, Simon suggested assessing and advancing questions were less direct interventions and, consequently, not as time efficient when compared to "showing" students the intended, correct mathematics. After student teaching, Simon continued to explain the need to consider time constraints in the classroom when considering how to intervene and support student learning, and provided examples of intervening more directly in cases where students' approaches to problems were inefficient or not in line with his learning goals. In those instances, Simon claimed, "if there are time constraints, that would make me need to step in like that…there are times where my mathematical authority is still more than theirs even if they're not necessarily doing incorrect mathematics." Both instances included in this paragraph highlight not only Simon's belief that, as the ultimate mathematical authority, he was to ensure students learn correct mathematics, but also how his belief about the practical aspects of teaching and learning mathematics in classrooms influenced how he supported student learning. Notably, Simon described how the time constraints of the classroom meant he was not always able to support student learning using assessing and advancing questions, which led to him intervening more directly.

Simon also believed that, as the ultimate mathematical authority, *he determined when a student learned or understood mathematics*. At multiple times during the first interview, Simon implied that the teacher determines when students learn mathematics and claimed the teacher "is essentially the final say on which math is correct." During the third interview, Simon claimed that when students demonstrate understanding via their responses to assessing questions, "that's the way they prove their understanding, is showing the teacher that they have the understanding." Notice that Simon indicated the teacher is the one who certifies whether students "proved" or "showed" they understood the intended mathematics. When Simon talked through his second mathematical authority diagram, he explicated his belief that the teacher is the one who certifies when students learn mathematics, and directly related this role to the teacher being the ultimate mathematical authority in the classroom:

> At the end of the day, in the classroom, I am the highest mathematical authority as a teacher…it's my job to make sure that all the mathematics that the students leave with and that the students end up understanding is correct mathematics, and therefore I am the highest [mathematical] authority in the room.

Hence, Simon believed that, as the ultimate mathematical authority, he certified when students came to the point of learning or understanding the intended mathematical concepts, ideas, or procedures, and students solely turned to him to verify they learned the intended mathematics.

***Simon's Beliefs About Students as Mathematical Authorities***

Simon believed that anyone's position as a mathematical authority was primarily rooted in correct mathematics. Relating this to students, Simon believed *some students will be viewed as more legitimate mathematical authorities, both by Simon and their peers, because they understand more mathematics in comparison to their peers*, and this more extensive understanding meant they could teach or, in Simon's words, "impart" that mathematics to their peers. In Simon's first mathematical authority diagram, he drew an arrow going from *students* that looped back to *students* (see Figure 1), and explained the inclusion of that arrow in his diagram by stating students can be mathematical authorities to their peers, "if you've [students have] understood the concept and hopefully well enough and true enough to be able to pass it on, a student can take the role of a teacher where they show another student that concept." Notice Simon implied that students being able to communicate or teach correct mathematics to a peer plays a role in their position as a mathematical authority but to "pass it on" to a peer, a certain level of understanding of the mathematics is required. Throughout the other interviews, Simon described students as mathematical authorities when they understood correct mathematics and were then able to explain that mathematics to their peers. During the final interview, for instance, I asked Simon what makes students more legitimate mathematical authorities when compared to their peers, to which Simon replied, "if they are more consistently correct with mathematics in general. If they can see where and when to utilize different mathematics in an effective way to get a desired outcome, that would make them a more definite mathematical authority." Simon continued and stated, "Also, if you're talking about specifically mathematical authority, then it's not necessarily just about being correct, but also demonstrating and imparting knowledge."

Although Simon believed some students will be more legitimate mathematical authorities when compared to their peers, he believed *all students can be legitimate sources of mathematical knowledge and/or reasoning*. On multiple occasions, Simon claimed that all students are mathematical authorities because they can discover mathematics on their own and be a source of help for their peers in certain situations. For instance, during the fifth interview, Simon claimed any student "can be a source of knowledge about math and their understanding of math, even though it's not all encompassing, is still valid and is still helpful when you're trying to discover new things." Simon went on to imply that students who are less legitimate mathematical authorities may not contribute meaningful mathematics, correct mathematics in Simon's view, as often as their peers or the teacher, but "they [students] know some math and they know how to approach it." Thus, Simon suggested that all students can be legitimate sources of mathematical knowledge or reasoning, particularly when it comes to the mathematics they understand or know how to approach.

Related to Simon's beliefs that all students can be mathematical authorities and his belief that students learn best when they "discover" the mathematics, prior to student teaching Simon consistently discussed how he would intentionally design lessons so that students would discover the intended mathematics as they progress through the lesson. For instance, during the third interview, Simon explained that as students engaged in lessons he developed, "the student is discovering it [the mathematical knowledge] but the teacher's [sic] facilitating that discovery in a way that the teacher knows is going to be the best for the students." In other instances prior to student teaching, Simon implied that students can use their prior knowledge to engage in lessons and discover the new, intended mathematical concepts or procedures. Thus, prior to student teaching, Simon described, in broad terms, how he would position students as mathematical authorities by providing them time and space to use their mathematical knowledge and reasoning to engage in tasks or problems (i.e., lessons) on their own and, consequently, discover new mathematics. Related to his belief that some students are more legitimate mathematical authorities, Simon also implied that students who understand more mathematics or more readily understand the mathematics of a given lesson will make more progress through a lesson, when compared to their peers, before needing support from the teacher, most often in the form of assessing and advancing questions.

Though Simon described how he would position students to discover mathematics on their own prior to student teaching, he also explained the practical aspects of teaching influenced how he planned to teach certain mathematical ideas, concepts, or procedures. Moreover, the influence of "practical realities" on Simon's lesson planning and students' positioning as mathematical authorities became more apparent during the lesson plan interview and interviews after student teaching. Across the lesson plans Simon submitted during his time as a student teacher, there were a mix of lessons in which Simon planned to "give" students the ideas, concepts, or procedures, and lessons in which he planned for students to use their prior knowledge to explore a task and develop some new mathematics. Regardless of the type of lesson, Simon consistently planned for students to work through and discuss tasks or problems with their peers in small groups. However, the reasons for having students engage in small groups were different for the different types of lessons (i.e., more direct teaching vs. exploratory). I contend his reasons for engaging students in small groups for the different types of lessons illustrate how he positioned students as mathematical authorities and how he expected students to position their peers as mathematical authorities as students engaged in the different types of lessons. In the following paragraphs I describe these differences and relate his descriptions of students as mathematical authorities in these lessons to his beliefs that all students can be mathematical authorities, but some will be more legitimate mathematical authorities.

On multiple occasions after student teaching, Simon described the purpose of having students work in small groups when engaging in more exploratory lessons. For instance, during the fifth interview, Simon described a lesson in which students engaged in a Desmos (https://www.desmos.com) task that fostered students' discovery of the distance formula for two points in the coordinate plane by extending their knowledge of the Pythagorean Theorem. I asked Simon why he planned for students to engage in small groups during this lesson, to which Simon replied,

> there's an exploratory aspect where students can bounce ideas off each other as well, in this kind of lesson where there's more of a group collaboration and building of that

> concept [i.e., the distance formula]. That isn't the case when you have, like for the factoring [polynomials by grouping] thing, you kind of have just give it to them…I like these tasks [e.g., the Desmos task] a lot more because you can have this engagement with students and they actually feel like…they are discovering themselves by talking with each other, by exploring it themselves. They've become more of a mathematical authority…because they are exploring it.

Thus, Simon acknowledged that a distinct aspect of exploratory lessons was that students would position their peers as mathematical authorities in the "building of that concept." Simon expanded on what he meant when he said there was more "building of that concept," claiming that all members of a small group can make meaningful contributions that the other group members consider. Moreover, he claimed, "that banter and that give-and-take can engender a lot of good mathematical thoughts and hopefully, lead to a solid conclusion that they [the group of students] made." In other words, Simon positioned all students as able to make meaningful mathematical contributions when engaging in small groups and expected students to position each of their peers as legitimate sources of mathematical knowledge and reasoning who can contribute ideas and insights that support the discovery or development of the intended mathematics. Even so, Simon claimed some students would be viewed as more legitimate mathematical authorities in their group because, as they engage in a task, they understand and can explain to their peers "the next step" or a relation between ideas their peers may not understand.

When Simon planned to engage students in lessons where he "gave" students the ideas, concepts, or procedures, Simon's description of students' positions as mathematical authorities, both by him and their peers, changed. As Simon implied in the excerpt included in the previous paragraph, when he planned to engage students in non-exploratory lessons, he did not position students to develop the intended mathematical concept, idea, or procedure of the given lesson. Consequently, Simon's primary purpose for having students engage in small groups in these lessons differed from the exploratory lessons. Specifically, Simon described planning for students to work on problems in small groups so that students who understood the mathematical procedure, concept, or definitions provided by the teacher, could help their peers who did not understand. For instance, during the fifth interview, Simon explained a primary reason he planned for students to work on problems in small groups was so that students who "understood the material" can help their peers by explaining a step in a mathematical procedure, clarifying a definition, or verifying that a peer's solution or answer was correct. Simon's stated purpose for having students engage in small groups during non-exploratory lessons implies he was positioning some students, those who better or further understood the mathematics at hand, as more legitimate mathematical authorities who their peers could turn to if they had questions or needed help understanding. Moreover, Simon suggested that students would turn to peers who they perceived as more legitimate mathematical authorities for help or to validate their solutions. Simon provided one specific example of a student, Macy, who assumed her peer, Sofia, was always correct if they disagreed or arrived at two different answers even though, in Simon's view, Macy was correct as often as Sofia.

As I described earlier, Simon viewed advancing questions as a key instructional strategy he used as the ultimate mathematical authority to support students' learning and have them *feel* as if they were discovering new mathematics. Although Simon often described his use of advancing questions in the context of discovery-based lessons, he also included advancing questions in non-exploratory lesson plans. When Simon described why and how he asked advancing questions, his intention was to position students as legitimate sources of mathematical knowledge and reasoning who can reason with his questions and either reconcile their incorrect mathematics or continue to make progress towards the intended mathematics on their own or with their peers. For instance, during the third interview, Simon described a hypothetical situation in which students may be considering multiple approaches to a problem, “some of which aren’t necessarily mathematically accurate.” Simon then said he would ask advancing questions that have students consider “specific cases where they could see for themselves, if they plugged in these values, then the outputs won't make sense.” In this hypothetical situation, Simon was positioning students as able to use their mathematical knowledge and reasoning, while considering his question(s), to determine “for themselves” which approaches to the problem were invalid. In another instance during the lesson plan interview, Simon claimed that if students get to a certain point and are unsure how to continue making progress on a task or problem, an advancing question can “kind of help them get guided in the right direction so that they can take that leap for themselves but still feel like they did it themselves and I didn't just give it to them.” Put differently, as students consider and reason with Simon’s advancing question(s), they will continue to make progress on the problems or task without further and more direct support from the teacher. Thus, even though Simon viewed advancing questions as a strategy that enabled him to leverage his position as the ultimate mathematical authority to support student learning, he also viewed it as a strategy he could use to position students as legitimate sources of mathematical knowledge or reasoning.

### ***Simon’s Beliefs About Other Sources as Mathematical Authorities***

Simon believed *the Department of Education (DoE) is a mathematical authority because the DOE is a legitimate source of pedagogical content knowledge (Shulman, 1986) that* *determines the mathematics to be taught and learned in schools.* At multiple times throughout the study, Simon described the DoE as the entity that determines and distributes the content standards for a given course, and, in doing so, sets the expectations for what mathematics is to be taught and learned in schools. For instance, as Simon described his inclusion of the *D.O.E.* in his second diagram (see Figure 2), he explained that individuals at the DoE are, "extremely good mathematical authorities that then use their authority to tweak the standards they want us to teach because…they know that it's best to teach these certain things over others and, because of that, they've made these standards." Simon also claimed that the standards "reflect directly what I will be teaching." Thus, Simon believed individuals at the DoE who developed the content standards were more legitimate mathematical authorities than himself who used their mathematical knowledge and pedagogical content knowledge to determine what mathematics was best for students to learn and, consequently, teachers to teach. Moreover, Simon primarily positioned the DoE as a mathematical authority by viewing the content standards they develop as the "dictated mathematics" he was to teach.

Simon also believed *textbooks and other curricular resources were mathematical authorities because they are legitimate sources of mathematical content knowledge and pedagogical content knowledge* (Shulman, 1986). Prior to student teaching, Simon indicated that he viewed textbooks for a particular course or topic area to be the collection of all necessary and correct mathematics for that course or topic area. Hence, on one hand, Simon positioned textbooks as legitimate sources of mathematical knowledge to which he, and others, could turn to increase his/their content knowledge. On the other hand, and more so after student teaching, Simon described textbooks and other curricular resources (e.g., free online curriculum) as sources to which he and other teachers turn to choose tasks, problems, or activities while lesson planning. As Simon was talking through his second diagram, he explained that he would draw from curricular resources (e.g., textbooks, technology, online curriculum), “in order to teach my students and, hopefully…to show them correct mathematics.” During the sixth interview, Simon explained how he leveraged his position as the teacher to sift through curricular resources and determine which one would be the best or most legitimate, source of pedagogical content knowledge: “I pick which one [curricular resource] I believe demonstrates the most correct mathematics or the mathematics that pushes students towards understanding…down an avenue that I'm particularly trying to guide them down or that demonstrates a particular concept more efficiently.” Simon’s emphasis on resources that engender “the most correct mathematics” and ones that do so “more efficiently” highlights his belief that curricular resources can be sources of pedagogical content knowledge because they can foster students’ understanding of correct mathematics and some resources are more legitimate mathematical authorities because they foster students’ understanding of more correct mathematics and/or foster that understanding efficiently.

As a special kind of curricular resource, Simon believed *that manipulatives, including some technologies, are true instantiations of correct mathematics and thus mathematical authorities.* As Simon developed his first mathematical authority diagram (see Figure 1), he included *manipulatives* under *Outside Resources* that students and teachers view as mathematical authorities. Simon included calculators, graphs produced by technology such as Desmos, algebra tiles, and other objects under the heading of *manipulatives*. When I asked Simon why he included *manipulatives* in his diagram, he stated, “manipulatives are a great tool for students to come to a mathematical concept of their own.” Simon continued by saying that as students are interacting with a manipulative, “the manipulative is a demonstration of the mathematical law that is here…the learning potential of the manipulative is that it shows that mathematical law." In Simon's descriptions of various manipulatives, the accuracy or legitimacy of a manipulative was never questioned. For instance, after we discussed the hypothetical scenario of students attempting to graph $f(x) = \sqrt{-x}$ in the second interview, Simon claimed, "when you're looking at $y = \sqrt{-x}$, the calculator would be a manipulative that a student can look at afterwards to understand that the concept, the truth that they've found is correct and verifiable." In other words, students could unquestionably trust the calculator as a legitimate source of mathematical knowledge and turn to those sources to verify their ideas. Simon acknowledged some manipulatives would demonstrate concepts "in particular ways that…students will latch on to more," and thus, implied students will more easily see and understand a "mathematical law" by interacting with some manipulatives over others.

**Influence of Student Teaching on Simon's Beliefs and Some Relations Among His Beliefs**

Simon's beliefs about mathematical authority appeared to stay consistent throughout his time in the study. Hence, I conjecture that Simon's student teaching did not influence changes to his beliefs about mathematical authority, but rather reinforced his beliefs. Moreover, his student teaching practicum seemed to provide a continued, practical context in which his beliefs informed specific actions (i.e., were enacted). For instance, Simon believed, both prior to and after student teaching, that some students are more legitimate mathematical authorities when compared to their peers. Yet, how he positioned more legitimate mathematical authorities in the classroom depended on the type of lesson (i.e., exploratory vs. non-exploratory) he enacted. When enacting non-exploratory lessons, for example, he viewed some students as more legitimate mathematical authorities if they were able to explain a step of a procedure or clarify a definition to their peers. Thus, student teaching afforded Simon an opportunity to form a personalized pedagogy that was informed by and aligned with his beliefs about and related to mathematical authority.

In the previous sections I described how some of Simon's beliefs related to other beliefs, for instance that his beliefs about mathematics informed his beliefs about learning mathematics. In the following paragraphs, I describe other relations among Simon's beliefs, particularly how some beliefs interacted with each other to inform his instruction, at least how Simon described his instruction.

Simon's most strongly held and primary beliefs were his beliefs about mathematics. This may have been evidenced most clearly when Simon was describing his second mathematical authority diagram and claimed, "all mathematical authority is, of course, based in mathematics itself. You can't be a mathematical authority if you have incorrect mathematics…the mathematics always wins and all mathematical authority stems from correct mathematics." Many of Simon's beliefs about mathematical authority were derived from his beliefs that mathematics was discovered in the physical world and that there are productive practices or "approaches" when engaging in or discovering mathematics. For instance, Simon's belief that the teacher is the ultimate mathematical authority was influenced by his beliefs about mathematics as well as his belief about students, namely that students will never be as trained in mathematics as the teacher. Simon's belief that some students are more legitimate mathematical authorities when compared to their peers was also influenced by his beliefs about mathematics. Specifically, Simon believed students are more legitimate mathematical authorities if they understand more correct mathematics than their peers. Finally, Simon's belief that manipulatives can be demonstrations of a "mathematical law" and, consequently, a mathematical authority seemed to be related to his belief that mathematics was and is discovered in the physical world. In sum, Simon's beliefs about mathematics were not only strongly held, but seemed to be primary beliefs that related to many of his beliefs about mathematical authority.

Another one of Simon's most strongly held beliefs was his belief that there are practical aspects of teaching that influence a teacher's instruction. I already described how this belief interacted with his beliefs about learning mathematics to inform his planning and enactment of lessons, which in turn had implications for how students were positioned as mathematical authorities. Still, I want to close the description of Simon's beliefs by highlighting how strongly held and primary this belief was for Simon. He believed that students should be afforded opportunities to "discover" the mathematics included in the curriculum and he believed that students *could* discover the mathematics on their own and in collaboration with their peers. Yet, when Simon described how he planned and enacted lessons, he would often explain how a practical aspect of teaching influenced his planning and/or how he supported student learning in the classroom. The influence of these practical aspects of teaching often led to Simon planning lessons in which he fostered students' learning by "giving" or "imparting" mathematical knowledge to his students. This was still consistent with his belief that one can learn mathematics through the transmission of knowledge, yet he did not believe "imparting" knowledge to his students was the best way to foster their learning of mathematics. Thus, I contend there was not inconsistency between Simon's beliefs—namely about learning and students as mathematical authorities—and his descriptions of his instructional practice. Rather, it seemed that Simon strongly held his belief about the practical aspects of teaching and this belief significantly influenced his planning and instruction, such that he often did not plan to foster students learning of mathematics in ways he though was best (i.e., via "discovery").

## Grace's Beliefs

Grace always considered teaching mathematics as a career option, but some of her experiences as a high school student ultimately led Grace to pursue teaching. Grace took part in a three-year "teacher pathway" offered by her high school, through which she was afforded opportunities to intern in elementary classrooms. Grace claimed she enjoyed interacting with and supporting students daily during those experiences, which cemented her decision to become a teacher. From those experiences, Grace knew she did not want to teach in elementary schools but claimed to be torn between wanting to teach science and mathematics. However, Grace described having "an awesome math teacher for three years in a row in high school," with whom she had a great relationship. Grace claimed this teacher was instrumental in her choice to teach mathematics. Entering college, Grace initially chose to major in Mathematics, but quickly switched her major to Mathematics Education when she realized the major was offered by her university. Additionally, Grace stated that her high school math teacher implemented "hands-on, high-level activities" that enabled Grace and her peers "to discover" the intended mathematics, which Grace said she also attempted to implement during her time as a student teacher.

Notable about how Grace discussed her mathematics background is that she was consistently encouraged as a math student. Grace claimed that growing up she did not enjoy mathematics but was consistently told that she was "good at math." She went on to say that being consistently told she was good at math led to her feeling, "I can do this [mathematics] because someone's told me I'm good at it." Even though Grace did not always enjoy mathematics, she always felt capable to "do mathematics" due to others' encouraging and supporting Grace as a math student. Moreover, this encouragement and support, over time, fostered her positive disposition towards mathematics as well as her inclination to teach math, and it seemed to have an impact on her beliefs about teaching mathematics, learning mathematics, and students.

## Grace's Beliefs about Mathematics, Teaching Math, Learning Math, and Students

### *Grace's Beliefs About Mathematics*

Grace believed *mathematics is a discipline that one can understand and apply to multiple aspects of their lives*. Grace viewed mathematics as unique compared to other subjects or disciplines because one can come to understand concepts and how to apply mathematical procedures, particularly when solving problems. In multiple instances, Grace suggested that one can understand and solve problems in multiple aspects of their life by using mathematics. For instance, in the fifth interview, Grace claimed students were "going to use math the rest of their life," and that mathematics is "something that affects their [students'] daily life." Hence, Grace believed mathematics is not merely a subject to be taught and learned in the confines of classrooms, but it is also a discipline that affects students' lives beyond the classroom and one that individuals use to problem solve and make sense of various aspects of their lived realities.

### *Grace's Beliefs About Learning Mathematics*

Derived from her beliefs about mathematics, Grace believed *the best form of learning mathematics occurs when students can accurately enact procedures and develop understanding of concepts and procedures*. In the first interview, Grace conceded that an individual can "memorize steps" and still learn mathematics, but she wanted students' learning of mathematics to go beyond just "following the recipe." When I asked how she would contrast "learning mathematics the way you envision versus just memorizing steps," Grace replied:

> *So if you're learning mathematics, you should be able to understand and visualize the concepts, apply them in multiple contexts, and be able to see something new and take*

> *those same concepts and apply them in a new context.* But if you're just memorizing steps, and you're just memorizing how to do that type of problem, then as soon as you see something slightly different from what you were memorizing, then you're stuck and you don't know what to do because…you don't know what to do when it looks different, or when it's reversed, or when it's a different piece of information.

The italicized portion of this excerpt was included in the statements I provided Grace prior to the third interview. Grace highlighted that statement, and others like it, as one that represented an important aspect of her beliefs and, during the statement sort, included it in a group labeled "Learning and Doing Mathematics." Hence, Grace implied that learning mathematics can take on various forms, but a particular form of learning mathematics was best. Namely, learning that fosters rich understanding of concepts and procedures is superior because it enables individuals to apply their knowledge to varied and future contexts or problems.

One of Grace's primary beliefs about learning mathematics was that *having positive experiences in the classroom enhances learning*. Throughout her time in the study, Grace implied students having fun and experiencing success in the classroom positively affects their learning of mathematics. During the statement sort in the third interview, Grace labeled one group of statements "Prior Knowledge/Experience" and another group as "Learning and Doing Mathematics." Grace claimed these two groups are related because, "if you're repeatedly told you're bad at math, you're not going to want to try repeatedly to do something you've been told you're bad at. So those prior experiences really affect how you learn and do things later on." Here Grace implied that a student's negative experiences in the classroom, and possibly outside the classroom, will negatively impact their engagement in the classroom and, consequently, their learning. Conversely, Grace believed positive experiences with mathematics can positively affect students' learning of mathematics. Grace claimed she learned the most in classes where she experienced success and that students she taught during student teaching became increasingly engaged in the classroom, and consequently learned more, as they received positive feedback from Grace and Grace's mentor teacher. In sum, Grace believed when students had positive experiences in her classroom or in previous math teachers' classrooms, those experiences positively influenced their engagement and, as a result, their learning.

***Grace's Beliefs About Students***

Grace's most salient and strongly held belief about students was that *students have unique, valuable insights and experiences to contribute in the classroom.* Throughout her time in the study, Grace claimed students have unique perspectives, experiences, and backgrounds that she wanted to consider and privilege in the classroom. While developing her first mathematical authority diagram in the second interview, Grace implied that she could learn from each of her students because they bring their own, unique perspectives and backgrounds to the classroom. After student teaching, Grace implied that her experiences and perspectives were not superior to her students' experiences, and "the only way I [Grace] would feel comfortable teaching is if I'm willing to listen to my students and incorporate what they have to say." Thus, Grace believed that no individual's experiences were superior, students have varied experiences and perspectives, and thus all students had unique insights and approaches they could offer in the classroom.

Grace also believed *students' affect and dispositions towards mathematics influences their engagement in the classroom*. On multiple occasions, Grace claimed that students often enter the classroom with negative dispositions towards mathematics and learning mathematics and she implied these dispositions affect students' engagement. For instance, during the first interview, Grace claimed, "students will come in with a bad attitude [towards math], and you can't always change that. You probably won't for 95% of your students and I know that." Grace went on to say students' "bad attitudes" towards math is a "huge challenge" for teachers. Grace also consistently claimed that a student's confidence and other aspects of their "personality" can prominently influence how they engage in the classroom. Grace claimed that during her time student teaching, the students who were more confident and outgoing were the ones "participating all the time and try[ing] to put themselves out there." In this moment and others, Grace linked aspects of a student's personality and/or affect to their propensity to engage in the classroom. Grace's belief about students' affect and dispositions influencing their engagement in the classroom related to one of her beliefs about *Teaching Mathematics* and *Students as Mathematical Authorities,* which I begin to detail in the following paragraphs.

***Grace's Beliefs About Teaching Mathematics***

Grace strongly believed that *when teaching mathematics, it is imperative to foster students' positive dispositions about mathematics and learning, as well as themselves as doers of mathematics*. Moreover, this belief was derived from her beliefs that positive experiences enhance learning and that students' affect and/or dispositions influence their engagement in the classroom. Throughout Grace's time in the study, she suggested one of her main roles as a teacher was to help students develop positive views of mathematics and of themselves as doers and learners of mathematics. As I mentioned in the preceding paragraph, she viewed this role as challenging because students often "come in with a bad attitude about the subject in general," yet necessary because developing positive dispositions towards mathematics and learning would support students' learning and benefit them in their future mathematics courses and beyond the mathematics classroom. For instance, during the lesson plan interview, Grace claimed because her students would continue taking other math courses, it was important for her students to be “reassured that what they’re doing is good work,” because doing so would increase their engagement and learning. Grace went on to say that she hoped her students would go into their future math classes “having the attitude of, ‘Okay, I did something right in Coach W’s and Ms. G’s class. Maybe I can do something right here, as well.’” In these instances and others, Grace suggested that, when teaching mathematics, it is important to foster positive experiences for students to enhance their learning in their current class and so they can reference and build upon their positive experiences in their future math classes.

Grace also frequently described supporting students’ development of what she termed “soft skills,” which included problem-solving skills, productive struggle, a growth mindset, and students’ mathematical self-confidence. Grace claimed that developing these “soft skills” would benefit students “in the long run, even outside of the math class.” Out of these soft skills, Grace discussed developing students’ mathematical confidence the most. When I asked Grace to describe the relationships between “students having positive experiences, building their self-confidence, and taking ownership of their learning,” she provided the following description:

> I think it's really hard to feel confident in something when you're having a bad time. So, making sure they're having positive learning experiences and highlighting their progress, making it very explicit when they're doing good work, helps build their confidence. And the more confident they're feeling about a subject, the more likely they're going to want to take ownership of it, show interest in the material, and feel like mathematics is something they want to own. I think a lot of them right now do not want to own mathematics. But with that increased confidence and feeling like, “Oh, this is something I'm good at,” or “This is something that's important to me,” that can really help with building that concept of ownership.

Notice that Grace thought developing students' mathematical confidence would lead to students developing positive views of mathematics and wanting to take ownership over their learning. Grace later explained that students taking ownership of their learning means "thinking of themselves as learners of math, as doers of math." Hence, Grace implied that developing students' mathematical confidence and students taking ownership of their learning were related to dispositions she wanted students to develop: a disposition towards mathematics as important and interesting and a disposition of themselves as doers and learners of mathematics.

Informed by her beliefs about learning mathematics, Grace believed *she best fostered students' learning by providing them opportunities to extend their prior knowledge to "discover" and understand novel concepts or procedures*. Grace espoused this belief prior to student teaching, but this belief was most clearly evidenced in the lesson plans Grace submitted during student teaching and her descriptions of her instruction after student teaching. In Grace's lesson plans, she consistently planned to engage students in tasks that fostered their development and understanding of the intended mathematical concepts and procedures, whole class discussions where students shared and explained their conceptions and procedures, and practice enacting the procedures discussed as a class. For instance, in a representative lesson plan Grace submitted towards the end of her student teaching, she planned for students to "discover the equation for a circle by applying the distance formula and…explain how the distance formula relates to the equation for a circle." In the sixth interview, Grace described the task in which she initially engaged students during that lesson:

> the context was a cell tower and this new house you wanted to buy, and you want to make sure you had good cell service. So, you had to make sure you're within the radius of

the cell tower to have good cell service…And then it had them graph every house that was within four miles of the cell tower.

After giving students time to engage in the task, Grace assigned each group of students "a number [in the task] to teach to the class," then provided students a worksheet where they practiced writing equations of circles in the coordinate plane. In other words, Grace planned to enact a lesson structure within which students had time to "discover" or develop the intended mathematical conceptions or procedures in groups, discuss those conceptions/procedures as a whole class, then practice applying what they discussed.

## Grace's Beliefs about Mathematical Authority

### *Grace's Beliefs About the Teacher as a Mathematical Authority*

Grace believed *the teacher is positioned by students as the primary mathematical authority in the classroom due to the teacher having more content knowledge in relation to their students and because of the teacher being an authority on traditional grounds* (Weber, 1925/1947). Grace consistently indicated students will predominantly position the teacher as the primary mathematical authority because teachers know more correct mathematics than students. For instance, in the second interview, Grace claimed, "the teacher serves as an educational authority, first and foremost, like your students are trusting you to teach them correct mathematics, and they're trusting that what you're saying is accurate. And that trust creates an authority figure." This instance highlights Grace's belief that the teacher knowing more correct mathematics than their students leads to students trusting the mathematics the teacher is teaching, and in turn "creates an authority figure." Grace espoused this belief after student teaching as well. As Grace was describing her second mathematical authority diagram (see Figure 3**)**, she explained why she included the yellow arrow going from the *Teacher* to the *Students*: "as a teacher, you show mathematical authority in your classroom, obviously, by teaching your students and by being someone that they can trust with the math content." Thus, Grace continued to believe students position the teacher as the primary mathematical authority in the classroom due to students being able to "trust" the mathematics taught by the teacher is correct.

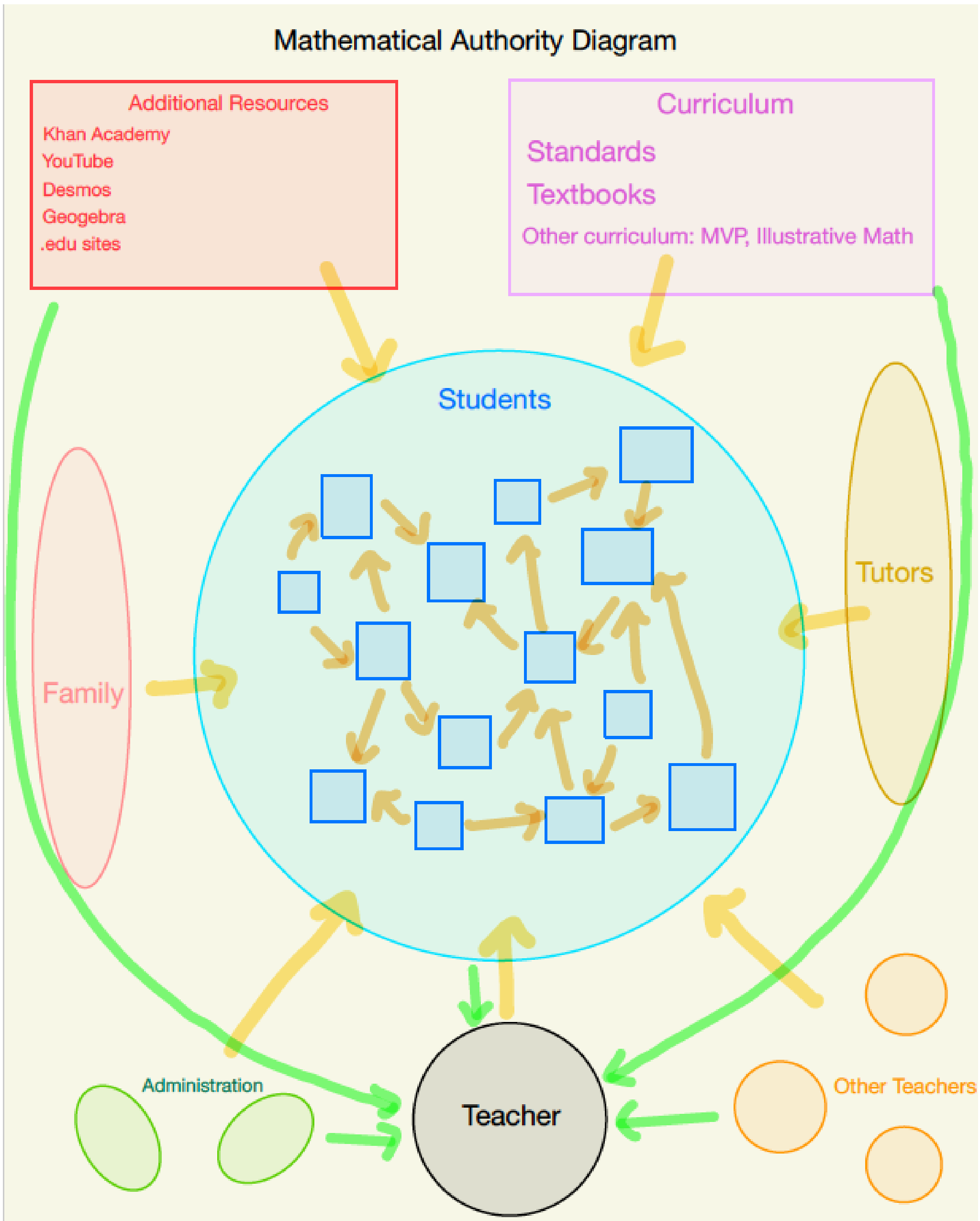



*Figure 3*. Grace's Second Mathematical Authority Diagram

Grace believed an additional, yet related aspect of a teacher's position as the primary mathematical authority was on traditional grounds (Weber, 1925/1947). As mentioned above, Grace consistently discussed students "trusting" the teacher, particularly trusting the content they were teaching was correct. From Grace emphasizing students' "trust" in the teacher on multiple occasions, I infer Grace believed students assume the teacher knows more correct mathematics than them and will teach correct mathematics because the teacher is the sole individual in the classroom who holds the position of *teacher*. Put differently, Grace believed that in any subject the teacher will be "an educational authority," simply because they are *the teacher* in the classroom. Thus, Grace believed that in the mathematics classroom, the adult who holds the position of *teacher* is assumed to know more mathematics and is able to teach that mathematics correctly: hence, that adult (the teacher) is positioned as the primary mathematical authority in the classroom.

As the primary mathematical authority, Grace strongly believed i*t was her responsibility to ensure students made adequate progress towards meeting the learning goals for the day and, consequently, developed the correct, intended mathematics*. Most often Grace described ensuring students progressed towards meeting the learning goals for a lesson by addressing students' mistakes or misconceptions. Grace consistently described focusing on the mistakes or misconceptions she perceived students were making/developing and stressed the importance of students not holding to those mistakes/misconceptions as they left the classroom. For instance, the excerpt below comes from the second interview when Grace described students' conceptions she would look and listen for as students discussed the graph and key features of $f(x) = \sqrt{-x}$.

Int: How do you as a teacher then determine, because it sounds like you're listening for certain things to then direct the conversation. How do you as the teacher determine those things to listen for?

Grace: I try to think like a student would think and… predict their misconceptions or their moments of discovery. So like, if I was going through this, in this Pre-Calc class, then I'd be like, "Alright, well, oh, this is interesting, I don't think we can use negative [infinity] to infinity as our domain because it's not gonna [sic] work for all those." Then I'm listening for comments that are similar to that. Or I'm listening for things that sound so wrong that I know we need to talk about them as a class too.

Int: And why is that last part important?

Grace: Because I don't want students to move on from this activity if they have a deep-seated misconception about the content.

Notice Grace thought it was *her* responsibility to identify and then address students' mistakes or "deep-seated misconceptions" before moving on to the next activity or section of a lesson. Thus, Grace implied that, as the primary mathematical authority, she needed to ensure students did not develop "misconceptions" as they engaged in a task or activity before she could then facilitate students' learning of the correct, intended mathematics. Grace explicated these complementary roles as the primary mathematical authority during the fifth interview when she related her role as a facilitator to the banks of a river and explained that when students exhibit a misconception,

> that's when the river is flooding and we're off the rails, we're no longer headed where we're supposed to be. So that's where I would feel like I need to step in as banks of a river to say, "Okay, let's correct that. Let's get back on track. Oh, that part of what you're

saying, super valid, great observation, but we need to look at it this way." Or, have you noticed this part that would change their misconception. So yeah, definitely being those…banks that are going to keep us in line and making progress.

Again, Grace emphasizing her need to "step in" (notice her use of *I* in the excerpt) highlights her view that addressing students' mistakes and misconceptions was an essential role as the primary mathematical authority and necessary to support students' development of the correct, intended mathematics of the lesson (i.e., meet the learning goals).

Grace described questioning as her preferred strategy to make sure students were making adequate progress on tasks and problems. Moreover, Grace's description of the purpose of her questions, specifically assessing and advancing questions (Smith & Sherin, 2019), seemed to be consistent with her belief that students learn best when they extend their prior knowledge to "discover" new concepts or procedures. Before student teaching, Grace described assessing and advancing questions as working in tandem to either support students' progress on a task or problem, or to help students recognize aspects of their ideas or solutions that were incorrect. During the lesson plan interview, Grace claimed the overarching purpose of the assessing and advancing questions she scripted in her lesson plans and asked in the classroom, was to first understand students' ideas or solutions, then ask an advancing question "that could progress them towards a solution." During the fifth interview, I asked Grace how she would describe the mathematical authority between her and students when asking advancing questions, to which she replied, "I'm more of the authority but it's an authority where I'm empowering them still to discover the math. So they know that I'm asking the right question and it's going to help them, but the mathematics is still their own." Interestingly, Grace claimed students *knew* she would ask "the right question" that would help them make progress on a task or confront their incorrect ideas. Hence, Grace suggested that because students viewed her as the primary mathematical authority in the classroom, they knew it was *her* questions that, when considered, would help them make progress on the task or problem at hand. Grace's descriptions of assessing and advancing questions also related to her beliefs about students as mathematical authorities, hence I further detail her descriptions of assessing and advancing questions in the *Grace's Beliefs About Students as Mathematical Authorities* section.

As the primary mathematical authority, Grace also believed that *she determined when a student learned the intended mathematics*. When Grace discussed learning in the classroom context prior to student teaching, she described teachers as the ones who determine and certify when students understand concepts and procedures or are able to correctly enact procedures. During the first interview, Grace claimed students know the concepts and procedures they learn are correct because, "hopefully the teacher told them [they are correct] at some point, they got a good grade on a test, and now they assume that it was correct." Student teaching seemed to reinforce this belief, evidenced by her emphasis on assessment use in the lesson plan interview and interviews after student teaching. On multiple occasions during the last four interviews, Grace described using formative assessments to determine what students have learned and have yet to learn. For instance, when I asked Grace in the fifth interview to describe how assessments informed her instruction, Grace explained she started each class with what she decided "either needed review or remediation." Grace suggested she was the one who determined the mathematics students had yet to understand and, thus, needed to review or remediate. By implication, Grace also determined what mathematics students had learned and thus did not need to review at the beginning of class. In other moments after student teaching, Grace implied she was the sole individual in the classroom who certified when students developed correct understandings of concepts and when they enacted procedures or processes correctly. Because Grace believed she, as the teacher, was the sole individual in the classroom who determined when students learned, I infer that she believed determining when students learn correct mathematics was a critical role of hers as the primary mathematical authority in the classroom.

Finally, Grace not only believed it was imperative to foster students' positive dispositions, she also believed that *as the primary mathematical authority in the classroom she was uniquely positioned by students to develop their mathematical self-confidence and increase the likelihood a student would be positioned as a mathematical authority by their peers*. Starting in the lesson plan interview, Grace began to emphasize the importance of the affirmation and encouragement she provided students and how such affirmation coming from her, the teacher, developed students' self-confidence. In the lesson plan interview, Grace claimed it was important to encourage positive dispositions and affirm students' engagement and effort in the classroom because it would help them be confident in their abilities and conceptions in future math classes. In the interviews after student teaching, Grace suggested the affirmation and encouragement she provided students was most impactful and meaningful to students because it came from *her*, the teacher. For instance, on two occasions in the last interview, Grace explained how her affirmation could lead to a student being positioned as a mathematical authority by their peers. In one of those instances, Grace claimed,

> giving a student the opportunity to answer a question well [i.e., productively] allows other students to see, "Oh, he knows this. They know this. If I have this question again, maybe I can ask them about this particular thing, because Ms. G just affirmed his answer." Just giving them that public recognition of, "Hey, that was a great observation," and affirming their thoughts that way.

In this instance, Grace suggested her affirmation and encouragement carries a certain weight in students' eyes, which can change how they perceive their peers. Similarly, Grace explained that her affirmation and encouragement could change students' perceptions of themselves, particularly their mathematical self-confidence. Because Grace believed students positioned her as the primary mathematical authority in the classroom, she believed her affirmation and encouragement was especially impactful, in that it could change students' perceptions of themselves (i.e., their self-confidence) and the perception of their peers (e.g., begin to see their peers as mathematical authorities).

***Grace's Beliefs About Students as Mathematical Authorities***

Both prior to and after student teaching, Grace strongly believed, *from her perspective as the teacher, all students can be legitimate sources of mathematical knowledge and/or reasoning*. Furthermore, this belief was one of Grace's more strongly held beliefs and was derived from her belief that students have unique valuable insights and experiences to contribute in the classroom. In the two mathematical authority diagrams Grace developed (see Figures 3 and 4), she drew bidirectional arrows between the teacher and students and explained her inclusion of these arrows, claiming students could consider and solve problems in ways that she would never consider. When Grace talked through her second diagram, she contended,

> my students have really valuable mathematical thoughts that I trust and want to hear more about. So, I find a lot of authority in my students and know that they have great ideas that I would never be able to think of because I'm only one person.

Grace suggested, in this instance and others, any student can be a legitimate source of mathematical knowledge and/or reasoning because they have experiences and perspectives that were different from hers and, hence, any student could think about or approach a problem or task in ways that Grace may not have considered before. Consequently, Grace believed, from her perspective as the teacher, all students can be mathematical authorities because she can learn something about mathematics from her students as they contribute their own, unique mathematical perspectives.

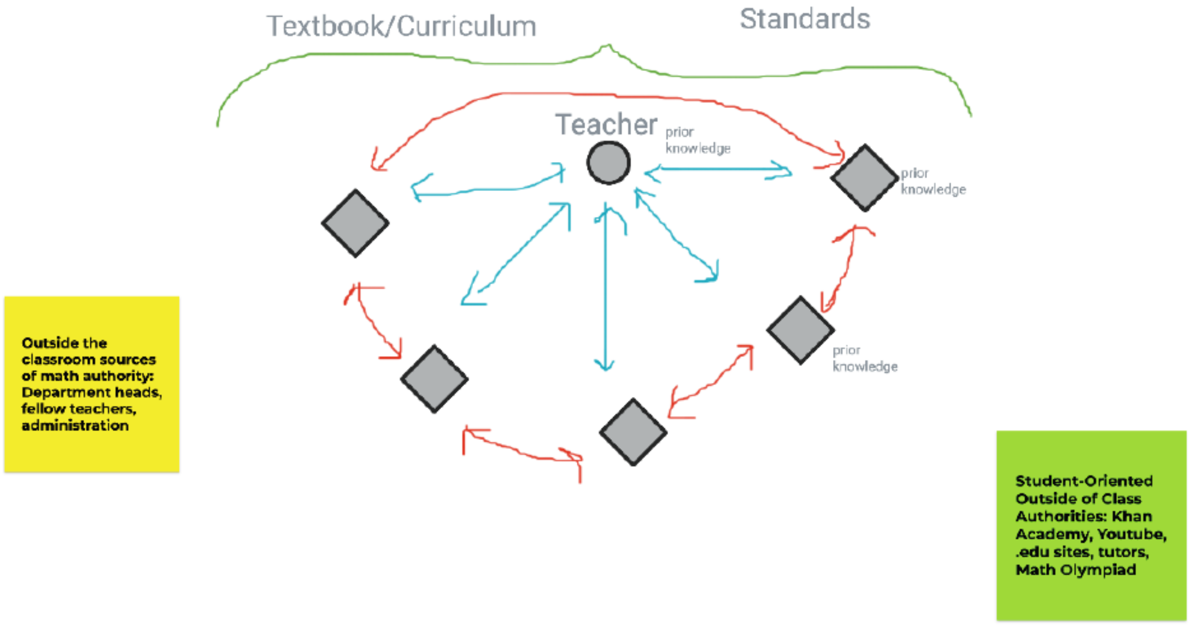



*Figure 4*. Grace’s First Mathematical Authority Diagram

Grace believed that *a student’s position as a mathematical authority, whether by the teacher or their peers, depended upon them developing or exhibiting correct mathematical conceptions, insights, or solutions, and clearly communicating their ideas*. In other words, Grace believed correct mathematics was necessary for students to be positioned as mathematical authorities, but correct mathematics was not sufficient: students also needed to communicate their conceptions or solutions to be positioned as mathematical authorities. Grace espoused this belief during the third interview when she was discussing students who are “just quieter in general” and claimed, “they'll be seen as less of a mathematical authority, even if they're getting every question right, just because part of being an authority is being able to interact with other people, and discuss, and build knowledge." In cases where students developed and communicated mathematics that Grace determined was partially correct or correct for some but not all cases, Grace still viewed a student as a mathematical authority if they were able to communicate their mathematical idea or solution. For instance, when Grace and I discussed the scenario in which Zack suggests a method for finding the horizontal distance between two points that does not always work, Grace explained she still viewed Zack as a "legitimate source of [mathematical] reasoning and knowledge because his method works, clearly on the coordinates that he was thinking about." Grace added that another component to her viewing Zack as a mathematical authority was his ability to clearly express his idea to the whole class. Hence, Grace viewed Zack as a mathematical authority because she viewed his method as valid (i.e., it "works") for a subset of coordinates, even though he did not "consider other cases where it doesn't work," and he was able to communicate his idea in a way the whole class could understand and consider. An important implication of this belief Grace held, as alluded to above, is that students who were "quieter in general" were less frequently positioned as mathematical authorities when compared to peers who were more apt to share their conceptions, ideas, or solutions.

Regarding students positioning their peers as mathematical authorities, Grace's beliefs seemed to shift because of her student teaching experience(s). Prior to student teaching, Grace believed *all students would be positioned as mathematical authorities by their peers, even though some would be viewed as more legitimate mathematical authorities.* In her first authority diagram, Grace represented students with four gray squares and drew bidirectional arrows between the squares (see Figure 4). Grace explained she intentionally drew bidirectional arrows between students because, “they [students] each have their own skill set or knowledge that their fellow students can respect…Like, either they trust their knowledge on that topic, or they find value in how they're thinking about the math.” Grace suggested every student will have moments in the classroom in which they contribute (i.e., communicate) mathematical insights, ideas, or knowledge that their peers “trust” and see as legitimate mathematical contributions. In other words, Grace implied students will be positioned as mathematical authorities by their peers in situations when they are able to communicate their conceptions and their peers “trust their knowledge on that topic.” Still, Grace claimed some students will be viewed as more legitimate mathematical authorities by their peers, particularly in whole class settings, because those students “are consistently contributing valuable math.”

After student teaching, Grace continued to believe that some students would be viewed as more legitimate mathematical authorities by their peers and believed that *some, but not all, students would be viewed as mathematical authorities by their peers in the classroom*. After student teaching, Grace highlighted students who were confident and/or outgoing as being consistently positioned as mathematical authorities, and this emphasis was influenced by Grace’s student teaching experience(s). In her second diagram (see Figure 3), Grace represented students with different sized rectangles and unidirectional arrows going between the rectangles. Grace explained why she represented students this way, stating that a student’s personality and/or confidence can lead to them being “seen as bigger mathematical authorities than others.” As Grace expanded on her choice to represent students with different sized rectangles during the sixth interview, Grace contrasted outgoing students with a specific student from her time student teaching who was not as outgoing:

> There were certainly students in our classes that consistently had more mathematical authority than others, for whatever reason. I think most of the time it was personality differences. So, we had one super-brilliant student but he's…really just doesn't like talking in front of the whole class. He’ll have great conversations with you one-on-one but doesn't love speaking up in whole class discussions. So, most students did not look to him as an authority, like on a regular day-to-day basis, even though I know he really…has great ideas. But we have other students that are really outgoing, want to participate, are people pleasers, and just are constantly talking and answering questions, and students would ask them for help, for sure, before they would ask the other student just because of their personality. So yeah, just differences among students and how they behave can certainly affect how much [mathematical] authority they have in a classroom.

Interestingly, Grace went on to say that she viewed the “super-brilliant student” as a mathematical authority because “I [Grace] totally understood what they were saying.” Hence, Grace suggested she still viewed all students, regardless of their personality or confidence, as mathematical authorities after student teaching because every student had unique mathematical ideas, perspectives, and insights they could then *communicate to her* in a way that she could understand. However, Grace implied that some students may never be viewed as mathematical authorities by their peers in the classroom and related this to her second diagram when she claimed, “there could also be students that don’t have any arrows going [from] them at all.” In other words, Grace suggested that some students may never share, explain, or contribute their ideas or solutions in discussions with their peers and, consequently, would not be viewed as a mathematical authority by their peers. Additionally, Grace suggested students who frequently shared or explained their conceptions were often viewed as more legitimate mathematical authorities by their peers, although this was not always the case.

After student teaching, Grace also believed *students' relationships with or perceptions of their peers influenced whether they would view their peers as mathematical authorities*. In the sixth and final interviews, Grace frequently mentioned one of her students, Barry, who she "worked with" a lot by helping him focus his energy and activity towards engaging in the mathematics of a given lesson. Grace claimed that she viewed Barry as a mathematical authority because he would often contribute or explain his correct ideas or solutions in the classroom. Yet, when it came to Barry's peers viewing him as a mathematical authority, Grace said, "I know some students do, I know not all the students do, because some of them are still going to get caught up on his behavior in other classes, his past grades in other classes, or the fact that he'll tell them when he does terribly on a test." Grace then used Barry's particular case to generalize how relationships can influence one's position as a mathematical authority by claiming, "I think every person is going to value different things as far as building that idea of mathematical authority." From this general statement, and explanations of her second diagram, Grace insinuated that the factors one considers when determining who they view as legitimate sources of mathematical knowledge or reasoning differ from person-to-person. In the specific case with Barry, for instance, Grace implied some students' past experiences with Barry, academically and/or behaviorally, were barriers to them viewing Barry as a mathematical authority in any setting. Thus, Grace suggested that an individual's perceptions and prior experiences with peers can play a significant role in who they position as legitimate sources of mathematical knowledge and/or reasoning. Notable about this belief is it became more perceptible after student teaching. I do not claim Grace did not hold this belief before student teaching, but that her student teaching experience(s) led to this belief becoming more salient after student teaching.

Because Grace believed all students can be legitimate sources of mathematical knowledge and/or reasoning in the classroom, she strongly believed *it was important to teach mathematics in a way that consistently privileges students' conceptions and solutions in the classroom*. Grace's descriptions of her instructional practices suggested that she enacted strategies she thought afforded students opportunities to develop mathematical ideas on their own or in collaboration with peers, as well as strategies that leveraged students' ideas to form the basis of classroom discussion. Two sets of strategies Grace described highlight how she privileged students' ideas as she supported their learning and as she facilitated whole class discussions: 1) her use of questions, mainly assessing and advancing questions (Smith & Sherin, 2019), and 2) having students present their ideas or solutions in the whole class setting.

Grace consistently described how she attempted to support students' learning the intended mathematics of a lesson and, simultaneously, privilege students' ideas and/or solutions through her use of assessing and advancing questions (Smith & Sherin, 2019). During the fifth interview, Grace claimed that when asking assessing questions, she was positioning students as mathematical authorities because as students responded to assessing questions, they were "answering what they know about the mathematics in that specific situation." In this instance and elsewhere, Grace implied that she crafted and asked assessing questions to provide students opportunities to share the mathematical ideas or solutions *they* developed as they engaged in tasks or problems. In other words, as Grace asked assessing questions, she attempted to position her students as mathematical authorities and understand their conceptions, perspectives, and solutions.

As I described in the *Grace's Beliefs About the Teacher as a Mathematical Authority* section, Grace viewed herself as "more of the [mathematical] authority" when asking advancing questions. Still, Grace viewed her use of advancing questions as moments where she leveraged her position as the primary mathematical authority to empower students and support them in developing the mathematical ideas/solutions on their own. In the fifth interview, Grace explained that when asking students advancing questions, "I'm not telling them anything new... the mathematics is still completely their own and they're still developing it and thinking through their own thoughts." Notice Grace believed students were still developing their own mathematics when she asked them advancing questions because she was not "telling them anything new." Hence, Grace viewed advancing questions as a strategy she could implement to help students further develop their own ideas or solutions and in a way that students would be progressing towards the intended mathematics for a given lesson. Thus, Grace implied, in this instance and elsewhere, that when asking advancing questions, she leveraged her position as the primary mathematical authority and, simultaneously, positioned students as legitimate sources of mathematical knowledge or reasoning who could develop the mathematics in a way that was "still completely their own."

Related to her belief that students who clearly communicate their mathematical conceptions or solutions are positioned as mathematical authorities, Grace explained the main instructional strategy she implemented to privilege students' conceptions in the whole class setting, and position them as mathematical authorities, was having students present or explain their solutions and ideas to the whole class. In the fifth interview, Grace claimed when students present their ideas or solutions at the board, "it's their steps, it's how they wrote it out, they explained it to the class, and they get ownership of that method." She went on to describe the value of having students explain their ideas in the whole class setting:

> to explain it their own way and continue referencing that, really emphasizes the fact that we see them as important, they're making valuable contributions to class, and that everyone in the room is capable of making contributions to the class.

Thus, Grace used this strategy to consistently privilege students' ideas or solutions in whole class discussions of mathematics and to communicate that she valued all students' ideas and contributions. During the final interview, Grace even claimed she began numbering the questions she gave students, "so that we had enough questions where every student had to work out one on the board." Hence, Grace's explanations of how and why she had students present their ideas or solutions to the whole class suggests that she wanted to position students as mathematical authorities by consistently privileging students' ideas and ways of engaging in problems or tasks, and that it was important to, over time, privilege all of her students' ideas/solutions in whole class discussions.

***Grace's Beliefs About Other Sources of Mathematical Authority***

Grace believed *the content standards provided by the state are mathematical authorities because they are legitimate sources of pedagogical content knowledge* (Shulman, 1986). Specifically, Grace believed *content standards determine the mathematics that is taught in classrooms*. When developing her first mathematical authority diagram (see Figure 4), Grace placed *Standards* above the teacher and students and explained her placement of *Standards*: "the teacher and students are trusting the standards to be mathematically correct and I think a big part of it is like mathematically valuable, where it's content that they are trusting is worth learning and teaching." Grace's placement and description of *Standards* in her first diagram highlights her belief that the standards outline the mathematics students need to learn and teachers need to teach within a given course or subject. Hence, rather than the teacher determining the mathematics their students should learn, Grace believed the teacher can turn to the standards for that knowledge. Grace demonstrated and espoused a similar view of the standards after student teaching, evidenced in her second diagram by her drawing arrows from the standards to the teacher and students (see Figure 3) and claiming the standards informed the content she taught during student teaching.

Related to the content standards provided for a given course, Grace believed *textbooks and other curricular resources can be legitimate sources of pedagogical content knowledge* (Shulman, 1986). Throughout her time in the study, Grace explicitly described textbooks, curricular resources (e.g., Illustrative Mathematics [https://illustrativemathematics.org], Open Up Resources [https://openupresources.org]), and tasks on Desmos (https://www.desmos.com) as "the interpretation of the standards." For instance, in the second interview, when I asked Grace what sources she would consult to determine what needs to be taught, she referenced Illustrative Mathematics (IM) and explained IM provided "activities and applications that takes the standard from just saying here's what students understand to actually what types of problems do those look like." Thus, Grace viewed IM and other curricular resources as legitimate sources of pedagogical content knowledge that provided tasks and activities that were aligned with the content standards and fostered students' understanding of mathematics. After student teaching, Grace espoused this same view and described implementing task or problems provided by curricular resources that "were always tied back to a standard." Hence, Grace continued to believe textbooks and other curricular resources were legitimate sources of pedagogical content knowledge that provided interpretations of the standards and thus legitimate ways the teacher could address the content standards students were expected to meet.

From the teacher's perspective, Grace believed *other teachers can be legitimate sources of both mathematical content knowledge and pedagogical content knowledge* (Shulman, 1986). Prior to student teaching, Grace described department chairs and more experienced teachers being legitimate sources of mathematical knowledge but did so in ambiguous terms. After student teaching, however, Grace explained that as a student teacher, she would turn to other teachers, primarily her mentor teacher, if she did not adequately understand aspects of the mathematics she was teaching or how the mathematical topics she was teaching were related. In the sixth interview, Grace explained she often discussed the mathematics she was to teach with her mentor teacher and implied that her mentor teacher helped her understand mathematics that she did not fully understand. Grace also explained that during student teaching, she would turn to other teachers as legitimate sources of pedagogical content knowledge. For instance, when explaining her second diagram, Grace claimed she would turn to other teachers to "get advice on how best to teach things, or what order to cover content." During the sixth interview, Grace described a specific instance in which she implemented a task introducing circles in the coordinate plane that was developed by another teacher at her placement school. By implementing this task and considering it as one that would foster students' understanding and one that aligned with the standards, Grace implied that she viewed the other math teacher as a legitimate source of pedagogical content knowledge. Overall, Grace believed other teachers can be legitimate sources of mathematical knowledge that supported her understanding of mathematics, mathematics pedagogy, and consequently her enacted instruction.

**Influence of Student Teaching on Grace's Beliefs and Some Relations Between Her Beliefs**

Grace's beliefs about mathematical authority prior to and after student teaching were largely unchanged. The one subtle change in Grace's beliefs that I inferred was concerning students positioning their peers as mathematical authorities in the classroom. Before student teaching Grace believed that all students would be positioned by their peers as mathematical authorities but after student teaching, Grace no longer believed this would be the case. When Grace and I discussed similarities and difference across her two mathematical authority diagrams, Grace consistently claimed that the changes to how she represented students in her second diagram, when compared to her first diagram, were due to her student teaching experience(s). Specifically, in her first diagram, Grace represented students with uniformly sized squares and bidirectional arrows between all students, but in her second diagram, Grace represented students with different sized rectangles and unidirectional arrows between students. Grace related the difference in the way she represented students (i.e., different sixed rectangles vs. squares) to her student teaching experience(s) when she stated, "there were certainly students in our classes that consistently had more mathematical authority than others, for whatever reason. I think most of the time it was personality differences." Grace also related her use of unidirectional arrows in her second diagram to her student teaching experience(s) when she stated, "in my [teaching] experience, with student relationships among each other, there are always going to be some students that find authority in others but it doesn't go both ways." Although this was one subtle change to Grace's beliefs about mathematical authority, I highlight this change due to it being informed by Grace's experiences with students during her student teaching practicum and her reflection on those experiences.

Other than this small change, Grace's beliefs seemed to be reinforced by her student teaching experience(s). Additionally, I conjecture that Grace's student teaching practicum provided a practical, continued context where Grace was able to enact her beliefs and develop a personalized pedagogy that was informed by her beliefs. This is most evidenced by Grace's descriptions of her questioning practices and how she attempted to position students as mathematical authorities by having them present their ideas and/or solutions in the whole class setting. For instance, both prior to and after student teaching, Grace believed that all students can be mathematical authorities in the classroom, but her descriptions of *how* she attempted to position students as mathematical authorities in the classroom via questioning and having them present their ideas or solutions became more salient after student teaching. Hence, student teaching provided Grace a context in which she was able to develop her own, personal pedagogical approach (i.e., instructional practice) that was informed by her beliefs that all students can be mathematical authorities, as well as her other beliefs.

Thus far I have described how Grace's beliefs related to each other and highlighted some of Grace's more strongly held beliefs. To close Grace's narrative, I focus on two of Grace's strongly held beliefs—that it is important to consistently privilege students' ideas, conceptions, and solutions in the classroom and that she was to ensure students developed correct mathematics—and how Grace described these two beliefs interacted to inform her instruction.

Grace strongly believed it was important to privilege students' ideas, conceptions, and solutions in the classroom, but she also strongly believed it was her responsibility, as the primary mathematical authority, to ensure students were learning the correct, intended mathematics. When Grace described some aspects of her pedagogy, these two beliefs seemed to interact, and this interaction appeared to be a source of tension for Grace. This tension was most clearly evidenced during the fifth interview when Grace related her role as a facilitator in the classroom to the banks of a river.

Int: I'm going to put a number of things in the chat. And my question is, when you think of your role as a math teacher, which of those analogies best describes that role?

Grace: I think I'm gonna [sic] go with either banks of a river or bowling bumpers... In my head I'm thinking about them kind of the same. So, I tried to be a facilitator more than anything. Which, as far as math goes, like, ideally, if the students are the river, then the math is theirs, they're coming up with things on their own, they should be involved in like, what direction we're taking in the classroom, but then I can restrict that and make sure they're going in the right direction.

Int: Okay, so can you expand more on like, the students are the river, you're…the banks of the river, how does that translate to you as a math teacher?

Grace: So like, I would be putting the limits on, I don't want to say limits, but kind of like, limits on where class is going. So, if a lot of our mathematical ideas are student found and student discovered, student led, and it's like, the students should be talking more than I'm talking to the whole class, in my classroom, then I'm providing the guidance to say, "Okay, let's make sure we're going in this direction,"…to make sure that we're on track and we're ultimately getting where we need to be, as far as hitting our learning goals and addressing our standards. So, yeah. So, students would be the ones, ideally generating the majority of the math through whatever tasks I can provide them. And then I would just be

> facilitating that, keeping them in line maybe, and making sure they're on track to make good progress towards our learning goal.

Notice that Grace was hesitant to say that she would "be putting the limits…on where class is going" in this instance. Grace strongly believed that students' ideas and solutions should form the basis of classroom activity and discussions, and, in this moment, it seemed as if she did not want to say that she would put a "limit" on the ideas and solutions students generated. However, Grace also strongly believed she was to ensure, "we're on track and we're ultimately getting where we need to be, as far as hitting out learning goals and addressing our standards." The instructional strategy that seemed to enable Grace to embrace this tension in a way that was consistent with these two beliefs was her use of assessing and advancing questions. As I reported above, Grace described assessing and advancing questions working in tandem and, moreover, an instructional strategy with which she could help students either confront their incorrect ideas or approaches to a problem or continue progressing on a task or problem when they were "stuck." Yet, Grace also suggested that when she supported student learning by asking assessing questions then advancing questions, students were able to make progress on a task or problem in a way that was "still completely their own." Hence, Grace viewed her questioning practices as a set of pedagogically powerful strategies, as they were instructional strategies Grace could implement in the classroom in a manner that was consistent with two of her more strongly held beliefs about mathematical authority.

## Chris's Beliefs

As Chris was growing up, he imagined he would always become a teacher. Chris explained that as he got older, he was drawn to the teaching profession because he could both make a positive impact on students' lives and coach basketball as a teacher. When I asked him why he chose teaching over other professions “where you can help people,” Chris responded that as he considered his future career, “you want to find the intersection of where you’re gifted and where your interests are. And so, I loved math, and I love basketball, and I love coaching, so teaching is the perfect avenue for that.” Chris, at multiple moments throughout the study, described math as his “favorite subject” and the subject, when compared to other school subjects, he “genuinely enjoy[ed].” Thus, Chris chose to major in Mathematics Education at his university and graduated with his degree at the end of the Fall 2021 semester.

Although Chris claimed he always wanted to become a teacher, the months leading up to his student teaching semester, he began to question whether teaching was the right profession for him. Chris claimed he did not feel as if he was “doing a good job,” and that he was not giving his best effort when it came to the teacher education courses he took prior to student teaching. However, Chris recalled that prior to student teaching, he decided to enter his student teaching practicum with a mindset of, “‘I’m gonna [sic] do this to the best of my ability. I’m gonna [sic] leave it all out on the floor and if I still suck [at teaching], then I’ll figure something out.’” Chris later said that the outcome of this approach to student teaching—giving his best effort—was that he felt a renewed “love” for teaching and for students in his classroom. Moreover, Chris claimed that the effort he put into student teaching and how he felt as a result, influenced the way he approached teaching. Chris summarized this influence when he stated,

> I knew my confidence [as a teacher] came from how hard I worked in the controllables [sic]. So, I wanted my students that were willing to buy-in and put what they had into it, I wanted them to feel that same level of accomplishment.

Chris's renewed confidence as a teacher and how his confidence informed his approach to teaching seemed to have an impact on Chris's beliefs about teaching mathematics, students, and students as mathematical authorities, which I describe in the following sections.

## Chris's Beliefs about Mathematics, Teaching Math, Learning Math, and Students

### *Chris's Beliefs About Mathematics*

Chris believed that *mathematics is a subject, or discipline, that is unique because one can understand mathematical concepts and "why things work."* In the first interview, Chris described how understanding concepts was unique to mathematics:

> in other subjects, there's not as much of that [understanding concepts]. Like English, it's just kind of like, you write or you read, and you just do it. And social studies is more memorizing stuff. And there is a little bit of that in science, but like in math you can really dive deep into why things work…And I really enjoy that, like finding out why.

In another instance, Chris claimed he enjoyed mathematics as a student because he had "lightbulb moments" in which he was able to make sense of and understand mathematical concepts and procedures, and not just how to accurately apply procedures. Chris suggested that these "lightbulb moments" only occurred in his math classes because mathematics is a unique discipline in which he was able to understand the concepts that are foundational to the knowledge that makes up the discipline of mathematics.

### *Chris's Beliefs About Learning Mathematics*

Related to his beliefs about mathematics, Chris believed *one can learn mathematics in different ways, but ultimately learning mathematics entails understanding the underlying concepts and being able to accurately enact procedures*. In multiple instances, Chris suggested the best form of learning mathematics occurs when students develop rich conceptual understanding, then leverage their conceptual understanding to understand how to enact procedures. However, during the first interview, Chris suggested that one can “figure out the procedure” first, then develop conceptual understanding as they “work backwards and figure out why this procedure works.” Still, Chris emphasized that learning mathematics entailed both conceptual understanding *and* being able to enact procedures accurately. For instance, as Chris reflected on the Calculus I course he took early in his undergraduate experience, he claimed that, for various reasons, he had to focus on knowing how to enact the procedures in the class and did not understand many concepts presented in the course. Chris went on to say that because he did not understand the concepts, “I [Chris] don't understand what even Calculus is.” Hence, in this instance Chris suggested that he was able to correctly enact the procedures he was exposed to in his Calculus I course, but that he did not truly *learn* the mathematics in his Calculus I course because he did not “understand the big concepts” in the course.

One of Chris’s primary and strongly held beliefs about learning mathematics was that *individuals learn best when they actively develop or understand the mathematics on their own*. Prior to student teaching, Chris described wanting students to discover the mathematics on their own and believed this discovery enhanced students’ learning. In the first interview, Chris claimed, “students retain it the best that way [if they discover on their own]. And I think it forms the conceptual understanding and I think if you form that first, for most students, the procedures will come.” Put differently, Chris believed if students discovered the mathematics themselves, they would develop conceptual understanding of the mathematics and retain that understanding longer than if they did not discover the mathematics on their own. Chris echoed this view of students’ discovering the mathematics in the lesson plan interview when he recalled telling his students the importance of discovering the mathematics on their own: “I’ve told them like, ‘When *you* realize something, it sticks [emphasis added]. When I tell you something, it doesn't stick.’” Chris espoused his belief that students learn best through discovery after student teaching as well; I describe those moments in the *Chris’s Beliefs About Students as Mathematical Authorities* section.

### *Chris’s Beliefs About Teaching Mathematics*

One of Chris’s strongly held beliefs about teaching was that *the teacher is the ultimate authority, mathematical and otherwise, in the classroom*. As the traditional authority (Weber, 1925/1947) in the classroom, Chris believed he was the one who established and reinforced the expectations and norms in the classroom. Chris acknowledged some classroom norms were co-constructed by students, but claimed he would encourage and foster those norms if they were productive and if a norm was unproductive, then “it’s the teacher’s responsibility to not let it be a norm.” Additionally, Chris viewed establishing clear expectations and productive norms as necessary to foster students’ engagement and learning of mathematics. As part of his first mathematical authority diagram (see Figure 5), Chris included a box labeled *Clear Expectations* under the teacher with an arrow to several boxes under *Students* and explained his inclusion of *Clear Expectations* by stating, “if you have clear expectations or understand what is expected of them, and that'll help them to be better as students.” One example of such expectations Chris, as the traditional authority, implemented for the purpose of fostering students’ engagement and learning was his use of seating charts. Throughout his time in the study Chris described implementing an intentional seating chart for the purpose of fostering students’ engagement and learning because he was able to place students with peers who worked well together or who could help them understand the mathematics at hand. Moreover, it was assumed by Chris that students would sit in the seat assigned to them by him and engage in group discussions (e.g., turn-and-talks) when instructed to do so. Although one example, Chris's description of seating charts is a prominent example of how Chris believed he could leverage his position as the ultimate authority in the classroom to foster students' engagement and learning.

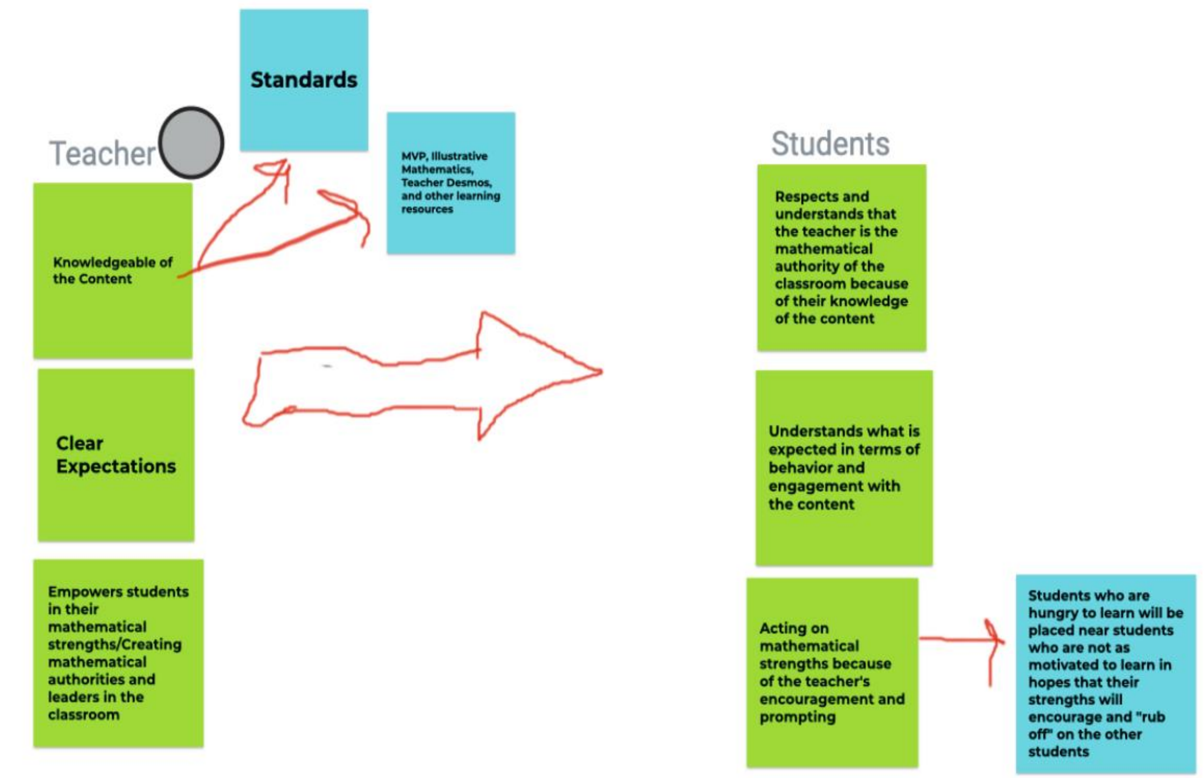



*Figure 5*. Chris's First Mathematical Authority Diagram

Chris also believed *the learning goals for a lesson play a prominent role in shaping much of the activity in the classroom*. Chris claimed that when he planned lessons, he would develop the learning goals he wanted students to meet each day and would use those goals to inform the activities and tasks he engaged students in as well as the types of questions he would ask students. For instance, during the fifth interview, I asked Chris how he determined the questions he asked students as they engaged in tasks, to which Chris replied, "ultimately, the direction is the content end goal or the learning goals, so I would say it's based off of that." During the lesson plan interview, Chris suggested that the learning goals he included in each of his lesson plans informed what he planned to emphasize as he enacted lessons. For instance, in the lesson plan interview, Chris and I discussed a lesson in which Chris had the following learning goal: "Students will understand function composition notation." Chris claimed that if he were to teach this lesson again, he would modify this goal to also include an understanding of the "definition of composition," because his students "were really struggling with that idea of composition." Hence, Chris implied that if an understanding of the "definition of composition" was included in his learning goal(s) for this lesson, then he would make sure that students understood the definition and how it related to the notation for function composition. Chris discussed learning goals in other interviews and how they informed many of his actions as a teacher, but these instances related to his position as a mathematical authority and, thus, I describe some of those instances in the *Chris's Beliefs About the Teacher as a Mathematical Authority* section.

Finally, Chris believed *mathematics teachers need extensive content knowledge to teach effectively*. Chris believed to plan effectively, ask quality questions, and support students in developing the intended mathematics for a lesson, the teacher needed to understand the mathematics "at a high level." Chris mentioned teachers' content knowledge in each of his interviews, in multiple reflections, and while talking through both diagrams. During the fifth interview, Chris described how his content knowledge prominently influenced multiple aspects of his instruction:

> because I had greater knowledge, I was able to present it. Like, I knew what was coming next and so I could ask questions that would sequence really well. I knew the task at a higher level, so I knew at this point I'm gonna [sic] ask this question, this point I'm going to push them to think on their own because they have everything they need to think in

> this way. Like, I can better prepare the timing of lessons, better prepare when we're going to turn-and-talk, when we're going to use this teacher talk move, when I'm going to have them work for groups, when I'm going to tell them, “Hey, I want you to work on this by yourself for three minutes.” Like, I was better, I was more equipped to do that because my content knowledge was there.

This excerpt demonstrates why Chris placed such emphasis on a teacher’s content knowledge, as he believed it influenced the quality of a teacher’s instruction in multiple ways.

### *Chris’s Beliefs About Students*

Chris believed that *students’ content knowledge will never surpass that of the teacher*. Chris acknowledged there might be exceptions for students who are “brilliant” but claimed that, in general, students will not “learn and know more than you as the teacher.” In his second diagram recording, Chris described how this belief about students influenced his belief about content knowledge when he claimed,

> We have to start knowing the standards and above, so students can meet us where they're at, because they're going to start less than we are because their math content knowledge is going to be less than us. So we have to set the bar and know the content really well, so that they can meet us where they’re at.

In other words, Chris suggested what a math teacher knows will set the upper bound for the mathematical knowledge students will be able to learn in that teacher’s classroom. After student teaching, Chris claimed students would learn from certain curricular resources because he learned from them. Chris seemed to assume that because he had extensive content knowledge and learned as he engaged with those curricular sources, then students, who do not have as much familiarity with the content, can learn as they engage with those sources as well. In sum, a belief

Chris held about students was that they would never surpass his content knowledge as the teacher and, thus, he had to develop deep, conceptual understanding of the mathematics he was to teach if his students were to also develop conceptual understanding.

Chris also strongly believed that *students can "control" their effort in the classroom as well as their dispositions towards learning and mathematics*. Chris's views of what students could control started to become evident as he engaged in the statement sort in the third interview. During that activity, Chris created a group of statements that he labeled "The Students' Controllables [sic]." When I asked Chris to say more about this group, he explained, "these are things that students can control…I mean, these are things that I as the teacher can encourage and motivate, but at the end of the day they have to choose to apply it." Within this group, Chris placed statements that described students who "have a hunger to learn," "believe they can do it [discover the math] on their own," and are willing to "keep going" when they do not "get it on the first try." The statements Chris included in this group and the way Chris described the group of statements, suggests Chris believed students' effort and dispositions towards learning and mathematics are things that students can choose. In other words, students could either choose to believe they were capable of learning mathematics or not and could choose to put forth effort in the classroom to understand mathematics. Chris reiterated this belief about students in the final interview when he referenced students' "controllables [sic]" and claimed, "Anybody can work hard. Like, anyone can control that." He went on to say, "willingness to share, willing to make mistakes…Definitely a self-advocate, advocate for yourself as a student, like those are the things I want to encourage…I think all of those things you can control."

### Chris's Beliefs About Mathematical Authority

#### *Chris's Beliefs About the Teacher as a Mathematical Authority*

As I described earlier, Chris believed *the teacher is the ultimate authority in the classroom, and this included the teacher being positioned by students as the ultimate mathematical authority*. Related to his beliefs that the teacher needs extensive content knowledge and students will never surpass the content knowledge of the teacher, Chris believed the teacher being the most knowledgeable one in the classroom is what leads students to position them as the ultimate mathematical authority. For instance, while developing his first mathematical authority diagram, Chris explained, "I think as a teacher…the first thing that you're looking at is your, through your knowledge of the content. If you don't know that…if you don't know the content, why would they [students] respect you as an authority?" Chris assumed that students would always position the teacher as the most legitimate source of mathematical knowledge and reasoning in the classroom, and often this positioning by students was implicit. In the following paragraphs I provide further evidence of Chris's belief that the teacher is the ultimate mathematical authority and describe Chris's view that students would always, and often implicitly, position him as the ultimate mathematical authority.

Chris believed that *as the ultimate mathematical authority, students position the teacher as the one who certifies when they learned the intended mathematics of a lesson*. This belief seemed to interact often with his belief about the learning goals prominently influencing multiple aspects of classroom instruction. Specifically, Chris implied that because he was the one who established the learning goals for a lesson and because he was viewed as the ultimate mathematical authority, he was uniquely positioned to determine when students met the learning goals for a lesson or were making adequate progress towards those goals. For instance, Chris implied that students generally accept the grades and other forms of feedback teachers provide students on assignments. Prior to student teaching, Chris claimed as a student he knew he was correct, incorrect, or on the right track based on his teachers' feedback, which included grades. On multiple occasions after student teaching, Chris described grades as an indication of what students did or did not learn. For instance, during his second mathematical authority diagram recording, Chris described a discussion he had with a student who had a failing grade on a quiz. Through Chris's telling of this instance, the student acknowledged that the quiz grade established that he did not understand the content assessed on the quiz. Put differently, the student positioned Chris as the ultimate mathematical authority and accepted the grade assigned by Chris as an indication they did not sufficiently understand the concepts and/or how to apply the procedures assessed. While this positioning of Chris, the teacher, as the ultimate mathematical authority does depend on students accepting their grades, or accepting the feedback from the teacher in other cases, every time Chris described students' grades, he assumed students would accept the grades they were given. Thus, Chris believed students would position him, implicitly, as the ultimate mathematical authority by viewing the grades Chris assigned, and other forms of feedback, as indications of what mathematics they did or did not learn.

Chris also believed *as the ultimate mathematical authority, students viewed the teacher as able to determine the ideas or procedures that would help them make progress towards meeting the learning goals*. Throughout his time in the study, Chris described students positioning him as able to help them when they were "stuck" or not on the "right track." For instance, in the first two interviews Chris described helping students who were "stopped, can't go any further" through his questioning until "they started getting the ball rolling to where they didn't need me to finish that step, or to go on to the next one, or finish that thought, that's…when

I stepped away." In this instance and others, Chris implied students would always consider and attempt to answer his questions and, thus, position Chris as the mathematical authority in the classroom who could help them get back on the "right track" through his questions.

Chris also described instances in which he would choose students to present their solutions because doing so would help other students in the class make progress on the task or problems at hand and, thus, towards meeting the learning goals. Starting in the lesson plan interview and continuing in the fifth interview, Chris described a specific instance in which he had one student, Sarah, present her solution to a problem on one of the front white boards. In the lesson plan interview, Chris described circulating throughout the room as students were working on a task and noticed that many students were having "a hard time with [part] 'b'." Chris then explained that he noticed Sarah "figured it [part 'b'] out" and had her "show her work" on one of the front boards. When I asked Chris why he chose Sarah to present her solution on the front board, he explained,

> it's not uncommon for a student to go up and write on the board. And so they [students] know like, "Okay, Sarah's writing, so I probably should look." So, most of them, I don't really have to give them a direction, they'll just start looking and then when she's finished, I'm like, "Alright, everybody look at what Sarah did." And then they'll look, and…I'm pretty sure someone even made a comment of, "Oh, we've done this before." And I was like, "Okay, well talk me through it since you've done it before."

In the way Chris described why he had Sarah present her solution and the norm of having students present their ideas or solution, Chris implied that he would intentionally select students to present their solution or ideas on the front board. Moreover, Chris suggested there was a norm in the class that when Chris selected a student to present, that student was correct and their solution would highlight mathematics other students needed to consider. In other words, Chris expected other students in the class would position the student presenting as a legitimate source of mathematical knowledge because he, as the teacher, verified the student was correct and determined their solution, or an aspect of their solution, would help the other students make progress on the task. These instances are complex because Chris is positioning the student presenting, Sarah in the instance above, as a legitimate source of mathematical knowledge or reasoning because she “figured it out” and expects the student’s peers to also position them as a mathematical authority. Yet, Chris suggested that the student presenting is being positioned as a mathematical authority by their peers because *he* determined they are correct and should be positioned as a mathematical authority. In sum, this instance and the norm of having students present to the whole class highlights another way in which Chris believed he leveraged his position as the ultimate mathematical authority to determine the mathematical ideas or procedures to which students need to be exposed in order to develop the intended mathematics (i.e., meet the learning goals), even when a student is the one who presents those ideas or procedures.

Important to note about Chris’s belief about the teacher being positioned as the ultimate mathematical authority is that he believed questioning was the best way he could leverage his positioning to support students’ learning. Related to his belief that individuals learn best when they develop or understand the mathematics on their own as well as his belief about learning goals, Chris described asking students assessing question, then advancing questions (Smith & Sherin, 2019) until they were able to make progress on a task or generate ideas on their own. In the fifth interview, when relating his role as a facilitator in the classroom to a GPS for a car, Chris described how he viewed his use of questions by saying, “through my questioning, I'm guiding them back to the right track. And because I know the end goal as the mathematical authority…I established the learning goal for that lesson…So, I established where they need to go." In this instance, Chris not only described, in broad strokes, how he viewed the role of his questions when interacting with students who might be incorrect or following an unproductive train of thought, but how he viewed questions in relation to his role as the ultimate mathematical authority in the classroom. Because he is the only one in the classroom who established the learning goals and what correct mathematics students were to develop or be able to do within a given lesson, Chris believed he could leverage his position to ask questions that guide students back to a correct or more productive way of thinking or solving a problem as a means to meet the learning goal(s) for the day. Moreover, in all instances throughout the study in which Chris described his questioning, Chris assumed that students would consider his questions as they attempted to make sense of or continue progressing on a task or problem. Hence, Chris believed students would always position him as the ultimate source of mathematical knowledge and reasoning in the classroom, and the best way he could leverage that positioning to help students develop the intended mathematics on their own was through his questioning.

### *Chris's Beliefs About Students as Mathematical Authorities*

Both prior to and after student teaching, Chris believed *all students could be viewed as mathematical authorities in the classroom*. Chris's belief that all students could be mathematical authorities was related to his belief that students learn best when they are able to develop the mathematical ideas or relationships on their own. In his reflection after the second week of student teaching, Chris wrote, "My goal is for students to be empowered and believe in themselves. Allowing them the opportunity to discover mathematics for themselves gives them ample opportunities to grow into their student and mathematical identities." As evidenced in this reflection and in the activities/structures he discussed and included in his lesson plans, Chris not only believed that students learned best when they were able to develop the mathematics on their own, he believed students *could* develop the intended mathematics on their own. In other words, Chris believed all students could be legitimate sources of math knowledge and reasoning who could make progress on problems or tasks on their own, contribute to classroom discussions, and develop the intended mathematics for a given lesson. Later in this section I further describe Chris's belief that all students could be mathematical authorities in the classroom along with the instructional strategies he planned to implement to position all students as such. Still, that Chris believed all students could be positioned as mathematical authorities does not imply that Chris believed all students *would* or that all students would consistently be positioned as mathematical authorities in the classroom. In the following paragraphs I describe what Chris believed was required for a student to be positioned as a mathematical authority.

Chris believed *for a student to be positioned as a mathematical authority they need to generate or develop some idea or solution and be able to share, explain, or teach that idea/solution to someone else*. In almost all cases, the idea or solution the student developed needed to be correct to be positioned by Chris or the student's peers as a mathematical authority. In the second interview, when Chris discussed the scenario in which Jose suggests finding the horizontal distance between two points by counting backwards, Chris said that after finding the horizontal distance between the first two coordinate pairs on the board as a class, he would have students determine the distance between the next two sets of points in groups of two or three. Chris explained he would have students do this in their groups because, "they may not know it individually, but it's probably rare the person next to them… it's pretty rare that between two or three students that none of them could be able to figure it out." In this instance, Chris implied that at least one of the two or three students would understand how to find the horizontal distance using Jose's correct method and then be able to help their peers understand by explaining Jose's method. Moreover, Chris suggested the student who understood Jose's method and explained that method to their peers was a legitimate source of mathematical knowledge, and he assumed their peers would position them in a similar manner. Similarly, in the fifth interview, Chris claimed that students will be inclined to turn to their peers for help when they believe their peer is explaining correct mathematics. In one instance, Chris described a hypothetical situation in which this interaction might occur: "if Julie, assuming that she did teach it to him and it was correct and accurate and made sense to Brad, I would say after that, Brad probably is asking Julie more questions." Hence, Chris implied that Brad would position Julie as a mathematical authority, as evidenced by Brad asking Julie more questions, after she was able to teach Brad correct mathematics in a way that he could understand.

Chris suggested that the only exception for correct mathematics being necessary for a student to be positioned as a mathematical authority was when students were engaging in whole class discussions. Chris believed it was important for students' ideas to form the basis of class discussions, even when those ideas were incorrect or incomplete. For instance, during the fifth interview, Chris claimed that after students engaged in turn-and-talks, he would initiate a whole class discussion by saying, "'What group, what pair feels like sharing?'…And I always say, 'Just share something with me. It doesn't have to be the full answer. Doesn't have to be, share something that you know. Get me started.'" In other instances prior to student teaching, Chris indicated that students' incorrect ideas could be discussed in the whole class setting and leveraged, often by being compared to correct ideas, to ultimately reach a correct consensus. In both cases, Chris implied that a student who has generated incomplete or incorrect mathematics can still make meaningful mathematical contributions to a whole class discussion because those ideas can help the whole class come to understand correct mathematics. Put differently, Chris believed students whose ideas were incomplete or incorrect could be positioned as legitimate sources of mathematical knowledge or reasoning because they could make meaningful contributions to the whole class discussion, because their ideas could be reasoned with and support their peers' understanding of the correct, intended mathematics.

Even though Chris believed all students could be positioned as mathematical authorities in the classroom, he believed *some students are more legitimate mathematical authorities than others*. Chris consistently suggested some students would more frequently be viewed as a legitimate source of mathematical knowledge or reasoning and, furthermore, more frequently make meaningful mathematical contributions in small group and whole class discussions. Chris described students who were more legitimate mathematical authorities as ones who "controlled what they could control." These were students who persevered when working on problems or tasks, would ask questions when they "got stuck," willingly and consistently offered their ideas and solutions in the whole class setting, and were not only able to teach or explain their mathematical conceptions or solutions to their peers, but were *willing* to teach or explain to their peers. In the sixth interview, Chris described how he viewed these students differently than other students:

> When I look back at the classes I taught, the [more legitimate] mathematical authorities I would have picked…were not the highest grades. There were some who had higher [grades]…but it wasn't like I looked at the grades and boom, boom, boom, these are the three. And what separates to me, or not even separates, I think one is a higher quality of mathematical authority is the willingness to help and to teach others, but you have to

> know what you're teaching to do it effectively. So at a minimum, they have to know the content at a level to teach, which in my opinion is very high. So, it is likely that those students are higher achieving students but that's not always true and I saw that in my class.

Hence, Chris viewed students who were highly engaged, demonstrated a desire to learn, and willingly helped their peers as more legitimate mathematical authorities. Additionally, as exhibited in the excerpt above, students who Chris viewed as more legitimate mathematical authorities also understood the intended, correct mathematics for any given lesson. Chris suggested that these students may not have the highest grades in the class, and thus may not have been viewed as the highest achieving students by the teacher and their peers, but they took advantage of the various opportunities provided to them to understand the intended concepts or procedures for any given lesson and were positioned as the "person in their group that helped [their peers]."

One might question whether students who are highly engaged and demonstrate a desire to learn and teach what they understand to their peers means they are viewed as a more legitimate source of mathematical knowledge or reasoning or more able to make meaningful mathematical contributions in the classroom. However, Chris demonstrated that he believed these students were more legitimate mathematical authorities than their peers in several ways, none more salient than when he described his use of seating charts. In multiple lesson plans, Chris wrote that he, along with his mentor teacher and co-teacher, intentionally developed seating charts for his classes. During the fifth interview, Chris expanded on his use of seating charts by saying, "we put kids who really wanted to answer all the time, so we put them around kids who had a lot of questions and struggled a little bit." In another instance during the fifth interview, Chris claimed, “I will be intentional with seating charts, and placing students, maybe a stronger with a weaker or two students who work really well together.” Hence, Chris designed seating charts intentionally with the goal of placing at least one student who he viewed as a more legitimate mathematical authority (i.e., a “stronger” student) in each group. Thus, Chris implied that he viewed some students as more legitimate mathematical authorities when compared to their peers and intentionally ensured there was one of those students in each group. Chris’s descriptions of his use of seating charts implies that he believed the more legitimate mathematical authorities’ peers would often position these students as mathematical authorities as well by turning to them with their questions or when they needed help understanding a particular idea or concept.

Related to Chris’s beliefs that all students could be viewed as mathematical authorities and that individuals learn best when they develop or understand the mathematics on their own, Chris believed *it is important to provide activities or structures that enable students to generate their own ideas, reason with others’ ideas, and explain their ideas or solutions to others* (i.e., to Chris and their peers). Chris consistently described engaging students in group work and turn-and-talks as two ways he positioned students to generate their own ideas and reason with their peers’ ideas. Chris planned for students to be seated in groups of two or three and, in nearly every lesson plan he submitted, planned for students to work on tasks or problems in their groups because he wanted all students to consider and discuss the mathematical ideas or procedures that were the focus of each lesson. Even prior to student teaching Chris viewed group work as a strategy he could leverage to have all students engage with and discuss mathematical ideas. For instance, in the second interview, when discussing what Chris would do after Marcos claimed that the graph for $f(x) = \sqrt{-x}$ does not exist, Chris said he would,

> use that and be like, "Alright guys, turn-and-talk and talk about what Marcos just said and how you feel about it, what you think about it." …Get them all thinking about what Marcos said, because…that's a valuable thing to contribute. So, you want to use that in the right way and leverage it in that way.

Initially, Chris said he would have Marcos explain to the whole class why graphing $f(x) = \sqrt{-x}$ is impossible but changed his response because he thought Marcos's ideas was a "valuable" contribution and he wanted *all* students to consider and reason with Marcos's statement. In other words, Chris believed all students could be legitimate sources of mathematical reasoning who could reason with Marcos's claim and, for this reason, wanted all students to reason with Marcos's ideas by having them discuss with a peer, rather than just have Marcos explain his idea to the whole class.

During and after student teaching, Chris continued to describe turn-and-talks and group work as strategies he could implement and that would afford students opportunities to generate ideas, explain those ideas to peers, and reason with others' ideas. However, reflecting on his experience(s) during student teaching, Chris believed that in order for all or most students to be positioned as mathematical authorities, students needed additional structures in the classroom. Specifically, Chris believed students needed independent think/work time before engaging in their groups so that students could generate and contribute their own ideas to small group discussions. During the lesson plan interview, Chris claimed that when students went straight into their groups to discuss an idea or work on a task or set of problems, he noticed, "working in groups, there typically is a group leader…they just lean on and I don't like that, because they just lean on them and they're not understanding it well enough on the assessments." In the fifth interview, Chris further described why he wanted students to think independently before engaging in groups and stated the primary reason was, “at some point, you have to be able to understand it [the mathematics] on your own. And while turn-and-talks are great to help you get there, at a certain point you have to be able to understand it on your own.” Thus, Chris implied he believed all students could understand the mathematics on their own and potentially contribute to group discussions, but some students needed additional time and space to generate their own thoughts or solutions prior to engaging in their groups. Chris continued to believe all students could be legitimate sources of mathematical knowledge or reasoning, but after reflecting on his experience(s) during student teaching, acknowledged some students may need more time to generate approaches or ideas on their own that they could then contribute in their small group discussions.

I want to note that the students Chris referred to as “a group leader” in the instance above, were the same students who he viewed as more legitimate mathematical authorities. Chris believed these students would be consistently positioned as mathematical authorities by their peers, yet he wanted the other students, ones who were not “group leaders,” to generate their own ideas, contribute to their small group discussions, and, most importantly, not rely solely on the more legitimate mathematical authority(ies) for explanations of ideas or how to enact procedures. During student teaching, Chris seemed to notice that a subset of his students were not consistently thinking about the mathematics on their own, generating their own ideas or solutions, and consequently were not being viewed as mathematical authorities in their groups. Moreover, Chris implied that a result of these students solely relying on the more legitimate mathematical authorities was that they were not understanding the mathematics and not doing well on assessments. Thus, Chris implied that providing students time to generate ideas and approaches to problems or tasks on their own, could be a remedy to these issues and lead to more students being viewed as legitimate sources of mathematical knowledge or reasoning in small group discussions.

In addition to turn-and-talks, there were other structures/activities Chris implemented to provide students opportunities to explain or teach ideas to others and be positioned as mathematical authorities, yet his own position as the ultimate mathematical authority often played a mediating role. In the *Chris's Beliefs About the Teacher as a Mathematical Authority* section, I described moments in which Chris positioned students as mathematical authorities by having them present their ideas or solutions on the front board, but the norm in the classroom was that he, as the ultimate mathematical authority, already determined the students presenting were correct. Similarly, starting in the lesson plan interview and continuing after student teaching, Chris described moments where students would come to him with a question, he would help students understand the idea or procedure through his own line of questioning, and he then had that student "teach" their peers who had similar questions. In the lesson plan interview, Chris summarized these moments:

> I always try to leverage it because she [the student with the initial question] understood all of that. She went from not understanding any of it, how to even get started on finding the max height, to teaching it to someone else within five or ten minutes. So I always try to give them the expo marker, give them the iPad, and let them show the class.

During the fifth interview, Chris further described how he leveraged these instances:

> It was in their groups that the same misconception would occur, and so it'd be like one girl I helped with the misconception, and then I come back like five-ten minutes later, or within the minute that I was helping this girl, they're like, "Oh, are you on number two?

I'm stuck on it." And I'm like, "Oh, well, perfect. She figured it out where she was stuck. It looks like you're stuck in the same spot. Hey, you can help her."

Similar to the instance in which Chris chose Sarah to present her solution, these moments are complex because Chris suggested that he positioned the students he initially helped as mathematical authorities who could help or teach their peers and, moreover, he believed the students' peers would and should also position them as legitimate sources of mathematical knowledge or reasoning. Yet, the legitimacy of the student being positioned as a mathematical authority stemmed from Chris's position as the ultimate mathematical authority. Notice in Chris's descriptions, he first verified the student he initially helped had developed the correct idea or solution before having them teach their peers when he claimed they "understood all of that" or "figured it out." In moments like these, Chris suggested that he leveraged his position as the ultimate mathematical authority to not only certify the student was correct but also communicate to their peers that he verified they were correct and, thus, could be viewed as a legitimate source of mathematical knowledge or reasoning who could help them with the problem or question at hand.

Other instances in which Chris's position as the ultimate mathematical authority played a role in positioning students as mathematical authorities were when he would structure students' engagement in tasks. Chris often planned for students to engage in structured tasks or activities that, as students progressed through the task/activity, would lead to students meeting the learning goal(s) for that day. Moreover, the instructions he provided students before they engaged in an activity or task often structured students' engagement and supported students meeting the lesson's learning goal(s). For instance, both prior to and after student teaching Chris described his use of "Desmos slider activities" (https://www.desmos.com) that allowed students to “explore” function transformations in the coordinate plane. As an example, Figure 6 is a portion of the student materials from a lesson Chris developed in which students explored, among other things, transformations of absolute value functions in the coordinate plane. As shown in Figure 6, Chris planned for students to build the slider activity themselves and planned to give students “about ten minutes to explore these transformations and write down their observations.” In the lesson plan, Chris went on to write, “The goal is for students to discover these connections with the tools given to them.” In the lesson plan interview, I asked Chris to expand on how he planned for students to discover the transformations and connections using Desmos. Chris explained, “with the sliders… If they take the directions that, we explained it, like look for specific values, they should see the transformations happen…Like, if they take those directions and they apply them to what they see for specific values.” On multiple occasions, Chris claimed he used Desmos slider activities often “because it gives them [students] the ability to be the authority to discover.” However, from the excerpts above, Chris positioned students as able to “see” or discover the connections or transformations if they followed the instructions *he* provided. In other words, Chris believed students could be positioned as mathematical authorities to “see” or “discover” the intended transformations, yet Chris claimed for students to discover these transformations, they needed to position him as a legitimate source of mathematical reasoning and follow his directions for engaging in the activity.

3. Go to desmos.com/calculator. In the website, type in the function

$$f(x) = a|bx - h| + k.$$

You will want to click "add all sliders." You should then have a screen that looks like the following to the right. Experiment with the different values and observe what happens. Fill in the chart below.

| | |
|---|---|
| $a$ | |
| $b$ | |
| $h$ | |
| $k$ | |

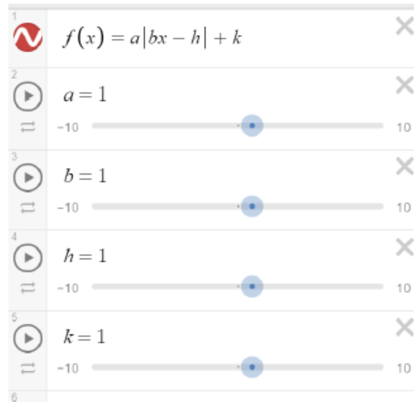



*Figure 6*. Desmos Slider Activity Instructions Provided to Chris's Students

### ***Chris's Beliefs About Other Sources as Mathematical Authorities***

Chris believed *the content standards provided by the state department of education are mathematical authorities because they are legitimate sources of pedagogical content knowledge (Shulman, 1986) that largely dictate the mathematics learned and taught in schools*. Throughout his time in the study, Chris claimed the content standards for the course he was teaching were one of the first resources he consulted when lesson planning. For instance, when engaging in the statement sort in the third interview, Chris claimed "a norm and expectation is that the course will be built around the standards, which is the math, like the standards are the math knowledge that's expected for students to learn." Put differently, Chris positioned the standards as largely determining the mathematical concepts and procedures he, and other teachers, was to teach and the depth to which students were to understand those concepts and procedures. However, Chris believed there were occasions in which the teacher did not have to teach content that was included in a course's standards. Chris believed the teacher could choose to teach mathematical ideas that were not included in the standards if he felt students needed to understand those ideas for their next course or endeavor. Chris not only espoused this belief prior to student teaching, during student teaching he also planned to teach students ideas that were not included in the accelerated Algebra 1 standards. In a lesson plan Chris submitted about halfway through student teaching, he planned to introduce accelerated Algebra 1 students to the concept of limits and acknowledged that limits were not part of the accelerated Algebra 1 standards. Within the lesson plan, Chris provided rationale for introducing limits to his students by stating,

> Students in this course are projected to take AP Calculus in the next two years. This is a great opportunity to accelerate and push them to think outside of Algebra and build connections for future math courses as Limits are the foundation of Calculus.

Hence, Chris believed the standards for a given course were mathematical authorities because they determine most, but not all of the topics, concepts, procedures that were to be taught and learned in the mathematics classroom.

Related to the content taught and learned in the classroom, prior to student teaching Chris also believed *some curricular resources are legitimate sources of pedagogical content knowledge* (Shulman, 1986). Chris believed resources such as Open Up Resources (https://openupresources.org) and Illustrative Mathematics (https://illustrativemathematics.org) provided interpretations of the content standards through the tasks and/or problems included in those resources. In other words, Chris positioned these sources as mathematical authorities that provided tasks or problems that were aligned with the content standards and that he could include in his lessons to foster student learning. While Chris implemented at least one task from Open Up Resources during his time as a student teacher, as evidenced in the lesson plans he submitted, it is unclear whether or not Chris believed these curricular resources could be positioned as mathematical authorities after student teaching, as he seldom mentioned curricular resources in the interviews after student teaching.

Chris also believed that *technology, such as graphs generated by Desmos, can be positioned as legitimate sources of mathematical knowledge*. As I described in the *Chris's Beliefs About Students as Mathematical Authorities* section, Chris often engaged students in Desmos (https://www.desmos.com) slider activities when teaching function transformations, and he positioned students as mathematical authorities to develop the mathematical "connections" on their own while engaging in these activities. However, Chris also believed Desmos was a legitimate source of mathematical knowledge for the graphs it generated, and assumed students would also position Desmos as such. Chris described the role Desmos played when students engaged in the slider activities as follows:

> The role Desmos plays is the function, it creates the function with the sliders. And so you can increase and decrease the sliders, put them at zero, put them at negative, whatever. And you can see where the sliders are and you can see the instant, how it changes the function. And so, you can see the transformation taking place, like if it goes from a positive to a negative, if it multiplies and how that affects the whatever. But yeah, you can see it, Desmos plays the role of the visual and the instant transformation… Desmos is probably the teacher.

Note that Chris did not question the accuracy of the transformations produced by Desmos. Moreover, Chris implied that students could, and should, consider the graphs generated by Desmos as accurate and record their observations as they moved the sliders and observed the corresponding transformations. Thus, Chris believed Desmos was a legitimate source of mathematical knowledge because he positioned Desmos to accurately perform the transformations as students engaged with the slider activities and he expected students to also position Desmos as an unquestioned mathematical authority as they attempted to "discover" the transformations on their own.

Related to his beliefs that mathematics teachers need extensive content knowledge and that content knowledge is what makes a teacher the ultimate mathematical authority in the classroom, Chris believed *other teachers can be mathematical authorities who help him better understand content he did not sufficiently understand*. During and after student teaching, Chris acknowledged there were moments during student teaching where his knowledge of the content he was to teach was not sufficient. For instance, while reflecting on his instruction during student teaching in the fifth interview, Chris stated,

> I don't feel like I understood Algebra at a high level. And so, the first two units I was struggling with teaching, not because I had bad management, or my questions weren't even bad for the knowledge that I knew, but I didn't know it well enough. And so, then the way I prepared for lessons was different.

In the reflections Chris wrote during student teaching and in the last three interviews, he described turning to his mentor teacher and other teachers whom he observed at his placement school to help him better understand the mathematics he was to teach. Moreover, Chris claimed he became more confident in his content knowledge as he positioned and sought out other teachers as mathematical authorities. Still, Chris didn't turn to just any math teacher in his placement school. Instead, Chris turned to teachers who he observed teach, such as his mentor teacher, that Chris claimed demonstrated, "they knew the content at a high enough level to help me [Chris]." Chris believed these teachers were mathematical authorities because they knew the mathematics at a "higher level" than Chris. Moreover, Chris believed positioning these teachers as mathematical authorities would help increase his content knowledge and lead to him planning for and enacting higher quality instruction.

**Influence of Student Teaching on Chris's Beliefs and Some Relations Among His Beliefs**

The beliefs Chris held before student teaching seemed to remain consistent during and after student teaching. Hence, I conjecture that for Chris, his student teaching practicum provided a continued context in which he was able to reflect on his practice and beliefs, and through this reflection, develop a pedagogy that was personal to him and aligned with the beliefs I reported in the previous sections. This conjecture may be supported most by Chris's explanations for why he planned to provide students time to think and generate ideas or solutions strategies before engaging in small groups. Chris suggested that, during student teaching, he did not initially provide students time to generate ideas on their own, but he noticed this practice was not supporting all students' engagement, learning, nor their performance on assessments. Chris reflected on this observation and decided to make a small modification to his use of turn-and-talks and students working in small groups. Moreover, this modification seemed to be informed by his beliefs that a) individuals learn best when they develop and understand the mathematics on their own, b) all students can be legitimate sources of mathematical knowledge and/or reasoning, and c) it is important to provide activities or structures that enable students to generate ideas on their own. In other words, Chris held to those three beliefs and enacted a classroom structure that was consistent with those beliefs. Furthermore, Chris's student teaching practicum provided a context in which he could reflect on his instruction along with students' engagement and consider instructional strategies that he could implement in the classroom that were consistent with his beliefs.

In the previous sections I have reported the beliefs Chris held about mathematical authority and how those beliefs related to Chris's other beliefs. In the subsequent paragraphs I highlight two of Chris's beliefs—one about the teacher and one about students—that were not only strongly held by Chris but seemed to inform or interact with many of his other beliefs and, consequently these beliefs seemed to have prominent implications for his instructional practice.

Chris's belief that he was the ultimate authority, mathematical and otherwise, was one of Chris's most strongly held beliefs and interacted with many of his beliefs about mathematical authority and teaching mathematics. For instance, Chris's belief that he was positioned as the ultimate mathematical authority in the classroom influenced his belief that the teacher is the one who certifies when students have learned. Additionally, Chris's belief about his position as the ultimate authority in the classroom also played a significant role in the activities or structures Chris described implementing to position students as mathematical authorities. I already described some instances in which this interaction occurred (e.g., selecting students to present their ideas or solutions to the whole class), however Chris's beliefs about his position as the ultimate authority also influenced how he engaged students in small groups and turn-and-talks. Specifically, Chris's descriptions of how he developed and used seating charts seemed to prominently influence his preference to consistently engage students in small groups and/or turn-and-talks. That is, Chris seemed to prefer to engage students in turn-and-talks and group work because he knew, due to his implementation of a seating chart, that there was at least one student in each group who could serve as a legitimate source of mathematical knowledge/or reasoning and help their peers who were "stuck" or did not yet understand the mathematics at hand. Hence, Chris suggested that he leveraged his position as the ultimate authority to manufacture peer interactions (e.g., turn-and-talks) in a way that would support students' learning and to do so in a way that positioned students, both by Chris and their peers, as mathematical authorities.

Chris's most strongly held and primary belief about students was that they could "control" their effort and dispositions in the classroom. This belief directly influenced Chris's belief that some students are more legitimate mathematical authorities when compared to their peers. Chris consistently described students who "controlled what they can control" as more legitimate mathematical authorities and, relatedly, it was evident that Chris held these students in high regard. Moreover, how Chris viewed these students seemed to influence some of his instructional practices. For instance, Chris suggested that students whom he would encourage to "teach" their peers were often the ones he viewed as more legitimate mathematical authorities. Additionally, his belief that these students were more legitimate mathematical authorities who could help and, in many cases, lead their peers as they engaged in problems or tasks informed how he planned to group students in the class. Specifically, Chris planned to have at least one student who he viewed as a more legitimate mathematical authority in each group. Consequently, Chris's view of these students had implications for the seating charts he developed and, as I previously described, how he viewed small groups and turn-and-talks as productive instructional strategies. In sum, Chris's two primary and strongly held beliefs—that the teacher is the ultimate authority and that some students are more legitimate mathematical authorities than their peers—seemed to significantly influence many aspects of his instructional practice.

### Hannah's Beliefs

Hannah explained that her desire to become a teacher started when she was a young child and was affirmed when she was a junior in high school. She claimed that, as a junior in high school, "I [Hannah] really realized the impact a teacher can have [on students]," and this realization cemented her decision to become a teacher. Hannah also explained that she chose to teach mathematics over other subjects because mathematics was the subject that, when compared to other school subjects, she was “passionate about” and “always enjoyed.” Moreover, Hannah described having a “really incredible” AP Calculus teacher her junior year in high school, and the impact this teacher had on Hannah seemed to inform Hannah’s desire to teach high school mathematics and thus major in Mathematics Education at her university.

Hannah often referred to the impact her high school math teachers had on, not only her decision to teach mathematics, but also her approach to teaching mathematics. For instance, Hannah described one of her high school teachers fostering Hannah’s understanding of the concepts in the curriculum, not just the procedures, and that she hoped to foster her students’ conceptual understanding in similar ways. Hannah also described her AP Calculus teacher as one who “really focused on building relationships with students and always made people feel welcome, like coming in for help, or whatever they needed.” Hannah went on to imply that her AP Calculus teacher provided a model of what empowering students and making all students feel valued could look and sound like in the secondary mathematics classroom. For instance, in the third interview, Hannah claimed, “I've seen, in the past, teachers do this well and what my hope is to do this well, like making students feel loved and important as people.” Hannah’s desire to empower her students, like her high school teachers did for her, seemed to have implications for her beliefs about teaching mathematics, students, and students as mathematical authorities, which I describe in the following sections.

## Hannah's Beliefs about Mathematics, Teaching Math, Learning Math, and Students

### *Hannah's Beliefs About Mathematics*

Hannah believed *mathematics involves logically reasoning and being able to clearly justify one's reasoning or thinking*. During the first interview, Hannah implied that as a high school student and initially as a college student, she believed mathematics was more objective than other subjects and required enacting procedures correctly to arrive at a correct answer to a problem. However, Hannah's beliefs about mathematics and what it means to engage in mathematics began to shift as she engaged in her college math courses. Specifically, Hannah claimed that her views of mathematics began to change because "a lot of the math classes at [her university], that I took where it was very proof heavy or focus on justification or the reasoning behind things. That's what I started liking more about math instead of just the final answer." In the third interview, Hannah further explicated her beliefs about different ways an individual can engage in mathematics and claimed, "there's so many different components to math and not just…being able to know the answer quickly and right away, but also being able to defend and justify your answers is important and clearly communicating your idea." Hence, Hannah claimed individuals can engage in mathematics in multiple ways, but she emphasized the importance of logically reasoning and being able to justify and communicate one's ideas when engaging in mathematics.

### *Hannah's Beliefs About Learning Mathematics*

Derived from her belief about mathematics, Hannah believed *learning mathematics required one to understand and reason with "processes" (i.e., procedures) and concepts, not just know how to accurately enact procedures*. Similar to how her beliefs about mathematics shifted, Hannah's beliefs about learning mathematics shifted as she engaged in her collegiate math courses. Hannah claimed as a high school student, she "was definitely all about getting to the final answer, as long as I got it right." However, as she engaged in collegiate math courses, she started to view learning mathematics as, "less about memorizing a bunch of steps and more understanding the reasoning why…understanding the reasoning and not just getting down to a solution." Hannah's belief that learning mathematics entailed understanding and reasoning with concepts and processes was also evidenced in her descriptions of her lesson planning. During the final interview, Hannah claimed that she often planned for students to first develop conceptual understanding, then encouraged students to develop understanding of procedures by "connecting back to the conceptual understanding." Hannah implied she did not want students to just learn definitions and step-by-step procedures, rather she wanted her students to learn mathematics by reasoning with and understanding the concepts and procedures of a given lesson.

One of Hannah's primary beliefs about learning mathematics was that *learning is an active endeavor that is supported as students relate their own ideas and solutions to others' ideas or solutions*. At multiple times in the study, Hannah implied that students could learn how to accurately enact procedures to obtain correct answers if they were directly shown (e.g., lectured) how to perform a procedure and then practiced applying the procedure. However, aligned with Hannah's belief about what learning mathematics entailed, she believed reasoning with and understanding concepts and procedures required students to be actively engaged, to reason with others' ideas and/or solutions, and to relate other's conceptions to their own. For instance, during the first interview, I asked Hannah how she thought students learned best, to which she replied, "definitely a lot of group and partner work…because that not only allows you to see different perspectives, but also practice explaining your thinking." In this instance and others, Hannah implied that students learned as they actively thought about, explained, and discussed the mathematical concepts or procedures at hand. After student teaching, Hannah suggested the students she taught learned most when they were "figuring it out" with their peers in small groups or when they were making sense of others' ideas in whole class discussions. Thus, Hannah implied that as students engage in tasks and discussion with their peers, they reason with and attempt to make sense of mathematical concepts and/or procedures. In other words, Hannah believed that actively engaging students in tasks and discussions supported students' reasoning with and understanding of mathematical ideas because through discussions, students were actively thinking about and making sense of the mathematics at hand.

A primary and strongly held belief Hannah had about learning mathematics was that *learning mathematics is enhanced when students feel supported and empowered*. Throughout her time in the study, Hannah extrapolated from her experiences as a high school student to explain that one of the biggest impacts on students' learning she could make was empowering them as doers of mathematics and communicating her belief that all students could engage in and understand mathematics. For instance, during the first interview, Hannah explained,

> I feel like I've learned more, and people I've talked to, when their teacher actually believed in them, because when your teacher doesn't believe in you it's like, "Okay, well, nothing I do is gonna [sic] be effective."…that idea of like, "Well, if nothing I do is actually going to affect my learning then I'm just not going to try."

During the final interview, I asked Hannah if there were past teachers who influenced her view of teaching, and Hannah provided an example of one of her high school math teachers who "empowered people to take responsibility for their own learning and their own work," which enhanced Hannah's and Hannah's peers' learning of mathematics. Influenced by her experiences as a high school student, Hannah believed that when students feel supported and empowered in any classroom, especially the math classroom, they engage more, take more ownership over their learning, and, consequently, learn more.

### *Hannah's Beliefs About Students*

Hannah held two primary and core beliefs about students, one being that *all students have unique perspectives and backgrounds they bring to the classroom and, consequently, different needs that need to be met*. Hannah consistently emphasized viewing students as human beings who have varied needs, learning and otherwise, as well as unique ideas and perspectives they can contribute in the classroom. After Hannah completed the statement sort in the third interview, she referenced one group she labeled "Empowering/building self-efficacy" (see Figure 7) and noted the statements in the group were "less about the math side and more seeing students as, just in general, as people that have important ideas and empowering them to where they start believing in themselves…that they have important ideas and abilities." During the fifth interview, Hannah related her role as a facilitator in the classroom to a shepherd and claimed she needed to plan her instruction to meet the needs of the class as a collective while also tailoring her instruction to meet students' individual needs, which she did by "continuously building a relationship with them [students]." Hence, Hannah implied she viewed relationships with students as necessary to meet their varied needs and to best understand the backgrounds and perspectives they bring into the classroom.

**Group 1: Empowering / building self-efficacy**

- (1) "I feel like I've learned more, and people I've talked to, when their teacher actually believed in them"
- (10) "when you feel like you came to a discovery, then you feel like you have ownership of like, 'Oh, yeah, I understand that because I came to that conclusion on my own'"
- (13) "it first starts with teachers creating an environment where everyone's ideas are welcome and respected"
- (19) "treating your students like human beings first and…understanding that there's things outside the classroom that affect them and understanding that the relationships that they build are important."
- (23) "I want them (students) to feel empowered as, like they have important ideas and they can have discussions with each other and it's not just them asking me questions and me telling them answers"
- (34) "empowering every student to believe that they're, whether or not they like math, that they're capable of doing it."
- (41) "leveraging students voices in the classroom and asking questions to where they're the ones doing the explaining, and not just you, as the teacher lecturing the whole time or answering every single question, but turning things back to students"
- (46) "your ability to do math is not correlated with whether or not you like it or have fun doing it."

*Figure 7*. Hannah's Statement Group Labeled "Empowering / building self-efficacy"

*Note.* Highlighted statements are statements Hannah indicated as "important aspects of her beliefs" prior to the third interview.

Hannah's other primary and strongly held belief about students was that *students' affect prominently influences their engagement in the classroom*. Specifically, Hannah often described students' self-efficacy and confidence as factors that influence their engagement in the classroom. During the third interview, when discussing a statement about students participating and feeling "comfortable and willing to share," Hannah claimed, "as students feel empowered and their self-efficacy grows, then they will…see their own voice as valuable and make them more willing to share. Like, all those things increase together." Hannah also explained she was cognizant of students' affect as she considered various strategies she could implement in the classroom. For instance, during the fifth interview, Hannah discussed her thoughts about "cold calling" (i.e., calling on students without them volunteering) and stated she did not often cold call because, "if you call on a student and then they have no idea, then that's just gonna [sic] make them embarrassed and then they're like, ‘Okay, well, I'm never talking again in this class.’” Thus, Hannah considered students’ affect as she thought about the instructional strategies she would implement in the classroom because she believed multiple facets of a student’s affect can influence their willingness or inclination to engage in the math classroom.

***Hannah’s Beliefs About Teaching Mathematics***

Hannah held two beliefs about teaching mathematics that, in tandem, influenced many of the instructional practices she described implementing. Hannah’s most strongly held belief about teaching mathematics was that *it is important to empower students as doers and learners of mathematics and to develop their mathematical self-efficacy.* Moreover, this belief was derived from her beliefs that students’ affect influences their engagement and that learning is enhanced when students feel supported and empowered. Hannah consistently explained that communicating to her students, both implicitly and explicitly, that she cared for them as people was imperative if students were to be empowered as doers of mathematics in the classroom. During the statement sort in the third interview, Hannah included eight statements, seven of which she highlighted prior to the interview indicating they were important aspects of her beliefs, in the group labeled “Empowering/building self-efficacy” (see Figure 7). After Hannah completed the statement sort, I asked Hannah why she thought all but one of the statements in this group were highlighted, to which Hannah replied,

> Because that's an important part of my beliefs, even though these specifically don't relate to mathematics, but I've seen in the past, teachers do this well and my hope is to do this well, like making students feel loved and important as people and that’ll trickle down to specifically in math…So like, if I felt like they [previous teachers] didn't see me or classmates as having important ideas, that led me to be discouraged in the mathematics.

During the fifth interview, Hannah again explained that establishing a classroom culture in which "students are comfortable to participate," was necessary if students were to develop their self-efficacy and feel empowered to develop mathematical ideas or solutions on their own. Moreover, Hannah's continued emphasis on fostering a positive classroom culture was informed by her belief that it is imperative to empower students as doers of mathematics and help develop their mathematical self-efficacy.

Hannah also strongly believed that *privileging students' ideas, solutions, and voice in the classroom would best foster students' understanding of concepts and procedures.* This belief was derived from multiple beliefs, two of which were her beliefs about learning mathematics. Moreover, based on Hannah's descriptions of her instruction, this belief seemed to interact often with her other belief about teaching to inform her instruction. In the *Hannah's Beliefs About Learning Mathematics* section, I described Hannah's belief that learning mathematics is fostered as students actively engage in the classroom, particularly in discussions of mathematics. As I alluded to in that section, Hannah believed students' learning was supported in discussions in which they considered and reasoned with their *peers'* ideas and solutions, not just their own and not just the teacher's. Hence, Hannah believed it was essential to privilege students' voice when facilitating mathematical discussions because doing so would best foster students' learning of mathematics. Hannah also consistently described implementing tasks students could engage in on their own and that would foster discussions of student-generated conceptions, perspectives, and solutions. For instance, in the second interview, Hannah explained that engaging in "problems that have higher cognitive demand and not just the straightforward plug-and-chug ones," supported students' learning of mathematics because as students engaged in such tasks, they were able to approach and engage in mathematics in their own and varied ways, not just one correct way, and make connections across their peers' varied conceptions and approaches. Moreover, Hannah implied that providing students opportunities to develop their *own* ideas and relate *their* ideas to their peers empowered them as doers of mathematics. After student teaching, Hannah described implementing tasks that students could approach on their own and in multiple ways and claimed she attempted to continually privilege students' ideas, solutions, and voice as they discussed approaches and solutions to tasks.

Hannah's stated reasons for engaging students in "higher cognitive demand" tasks and student-centered discussions suggest that these practices were informed by her beliefs about teaching mathematics. These practices, along with others, were also informed by some of her beliefs about mathematical authority. Thus, I further describe Hannah's descriptions of her instructional practices in the *Hannah's Beliefs About the Teacher as a Mathematical Authority* and *Hannah's Beliefs About Students as Mathematical Authorities* sections.

## Hannah's Beliefs About Mathematical Authority

### *Hannah's Beliefs About the Teacher as a Mathematical Authority*

Hannah believed *the teacher should be viewed as the mathematical authority who leads the classroom community (i.e., the leading mathematical authority), and should not be viewed as the ultimate mathematical authority by students*. Across all interviews and both diagrams, Hannah insisted that the teacher is a source of mathematical authority in the classroom but is not "the top or…one that can never be wrong." As Hannah began constructing her first mathematical authority diagram (see Figure 8), she explicated how she thought about the teacher as a mathematical authority:

> As I'm thinking about the teacher and how authority operates in drawing a diagram with students and everything, I wouldn't want the teacher to be at the top of some kind of

> chain or…when I think diagram I think of arrows and boxes. So, not something where all the boxes flow from the teacher or all the arrows point back to the teacher, because, again, as we've talked about, I don't think the teacher is *the* authority [emphasis added].

Consequently, in her first diagram, Hannah drew bidirectional arrows between the teacher and the students and placed the teacher in the middle of the students. During the statement sort in the third interview, Hannah referenced the teacher's position in her first diagram and stated, "[I am] not saying that you're [the teacher] the only mathematical authority or even the most important one, but there is some sense of like, you're the final one, you make the final say of things [e.g., discussion points]." As I considered Hannah's statements with her positioning of the teacher in her first diagram, I inferred that Hannah was suggesting the teacher should not be viewed as the ultimate mathematical authority in the classroom but is the mathematical authority who facilitates (i.e., leads) discussions and often has the final say in terms of what mathematics to discuss or consider as well as in matters where there are disagreements between students. After student teaching, Hannah continued to espouse her belief that the teacher should be the leading mathematical authority and not the ultimate mathematical authority in the classroom. In multiple instances during interviews after student teaching, Hannah said she encouraged students to not view her as the only or primary source of mathematical knowledge in the classroom and, as I describe later, encouraged all students to view themselves as mathematicians and legitimate sources of mathematical knowledge and reasoning. Hence, throughout her time in the study, Hannah maintained that the teacher should be viewed as *a* mathematical authority in the classroom, but not the only one, the ultimate one, or even the most legitimate one.

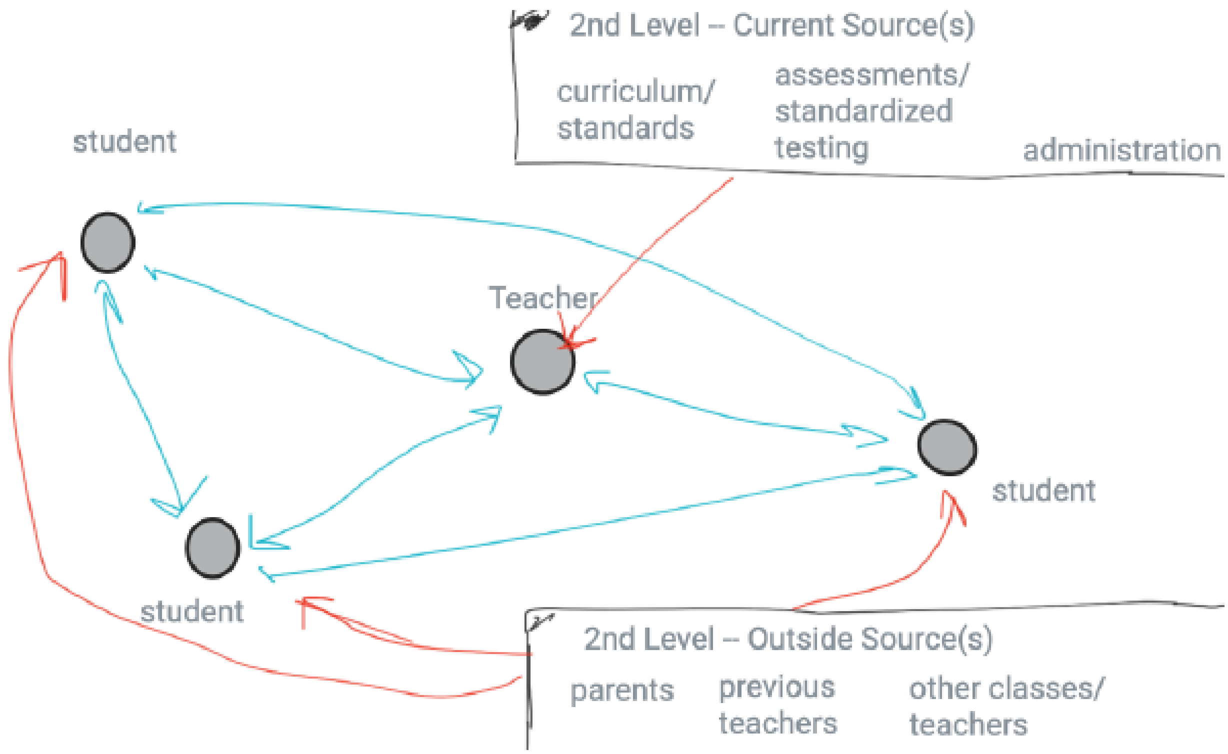



*Figure 8*. Hannah's First Mathematical Authority Diagram

Hannah held a notable belief about students that was closely related to her beliefs about the teacher as a mathematical authority, namely that *students are likely to view the teacher as the ultimate mathematical authority in the classroom*. As I described above, Hannah did not want her students to view her as the ultimate mathematical authority, but she acknowledged students often do view the teacher that way. During the second interview, when I asked Hannah what came to mind when I said, "a teacher is a mathematical authority in the classroom," she replied, "it's not saying they're the only one, but I think a lot of times that's what students think." In the sixth interview, Hannah claimed she actively attempted to change students' views of the teacher as the ultimate mathematical authority, encouraging them to not view her as "the source of all mathematical knowledge in the world," and instead view themselves as legitimate sources of mathematical knowledge and reasoning. Hannah implied she did not want her students to view her as the ultimate mathematical authority because it had negative implications for their engagement and view of themselves (i.e., mathematical self-efficacy). Hence, Hannah consistently attempted to disrupt her students' view of her, the teacher, as the ultimate mathematical authority and, as I describe later, continually attempted to position students as mathematical authorities in the classroom.

Regarding the legitimacy of the teacher as the leading mathematical authority, Hannah believed it was largely due to their correct content knowledge. Prior to student teaching, Hannah described moments where her previous math teachers had been incorrect and suggested they were not viewed as legitimate sources of mathematical knowledge, by Hannah and her peers, in those moments. For instance, in the second interview, Hannah provided an example in which one of her high school teachers would post answer keys that would occasionally have incorrect answers. Hannah stated that the teacher having incorrect answers led to her not "trusting" the teacher and she suggested that when a teacher makes mistakes or demonstrates incorrect mathematics, students may not view them as legitimate sources of mathematical knowledge in those moments. After student teaching, Hannah suggested the teacher is positioned as a mathematical authority by students because they demonstrate or espouse correct mathematics. For instance, when describing how she wanted students to view her as a source of mathematical authority but not the only source in the classroom, Hannah implied that students often view the teacher as "the source of all mathematical knowledge." Hence, Hannah suggested that students often view the teacher as a, if not *the*, mathematical authority in the classroom because they view the teacher as the most legitimate source of mathematical knowledge in the classroom.

Throughout her time in the study, Hannah described the teacher's position as the leading mathematical authority operating in multiple ways in the classroom. In most cases, Hannah described how she attempted to leverage her position as a mathematical authority to enact instructional practices that would support students' learning of mathematics and, at the same time, position her students as mathematical authorities. Consequently, I detail how Hannah discussed those practices in the *Hannah's Beliefs About Students as Mathematical Authorities* section. In this section, however, I report Hannah's salient beliefs about how she was to operate in the classroom as the leading mathematical authority, starting with the teacher's role of assessing students' conceptions to determine whether students learned or understood mathematics.

Hannah believed *as the leading mathematical authority, she determined whether students understood or learned mathematics*. During the first interview, Hannah claimed she determined when students learned by "seeing their work on assessments, or hearing them explain things in class, either to you or overhearing their explanations to students." In other instances prior to student teaching, Hannah suggested that staying attentive while students engaged in mathematical discussions and examining their work on assessments were instances in which she leveraged her position as the leading mathematical authority to determine whether and when students understood the mathematics of a given lesson. During the lesson plan interview and interviews after student teaching, Hannah continued to espouse her belief that she determined whether students learned mathematics. For instance, during the final interview, Hannah described her use of assessments during student teaching:

> Assessing will help figure out where students are in relation to the learning goal, but also, students' voice is important in that because, for example…if I give an assessment and there's a clear misunderstanding on a question, then I can use that to address in the next lesson like, "Hey, it looks like a lot of people were confused on this and that's not even

your fault. I just probably didn't explain it that well. Or maybe we just need to spend more time on this."

Notice here that Hannah, as the teacher, was the one who determined when there was "a clear misunderstanding" and whether students were meeting the learning goal of a lesson. Hence, after student teaching, Hannah continued to believe one of her roles as the leading mathematical authority was to assess students' understandings (or "misunderstandings") of mathematics and determine whether students learned the intended mathematics.

Hannah also believed that *as the leading mathematical authority, she played a primary role in establishing productive sociomathematical norms (Yackel & Cobb, 1996) in the classroom*. This belief seemed to interact with her belief about mathematics to inform the sociomathematical norms she wanted to privilege in the classroom. Throughout her time in the study, Hannah suggested that what she emphasized or valued when it came to doing mathematics would influence what students valued. During the statement sort in the third interview, Hannah created a group titled "Important practices/qualities of effective teachers" that included statements describing the importance of teachers focusing on "justification or the reasoning behind things," "understanding…and not just getting down to a solution," and "having a growth mindset" as learners of mathematics. As she was sorting these statements, Hannah claimed, "as a teacher, if you encourage or focus on justification and reasoning and put value on that, then students will learn to do that as well." In this instance, Hannah implied the teacher needs to privilege and value certain norms and practices in the classroom because students will follow their lead and begin to take up those norms and practices. In other words, because the teacher is the leading mathematical authority in the classroom, the sociomathematical norms they value and prioritize in their classroom are the ones students are likely to adopt. Hannah continued to suggest the teacher plays a primary role in establishing sociomathematical norms after student teaching as well. For instance, during the sixth interview, Hannah claimed that she did not want to solely privilege and value students getting the right answer, but she wanted to place value on students "contributing something that's helpful for learning overall." She went on to say that she wanted to highlight students' questions or insights that relate current concepts to previously learned concepts or bring up the limited applicability of certain ideas. Hence, Hannah suggested that she communicated to students what she viewed as legitimate mathematical contributions in the classroom and contributions that would support their learning. Thus, Hannah implied that she attempted to foster sociomathematical norms in her classroom that she believed would best support her students' learning of productive mathematics.

Informed by Hannah's beliefs about the importance of empowering students in the classroom, Hannah also believed *she could leverage her position as the leading mathematical authority to empower students and help increase their mathematical self-efficacy*. At multiple times during the study, Hannah implied that the positive feedback and affirmation she provided students led to students feeling empowered and confident in their mathematical conceptions. During the third interview, I asked Hannah how she would help students see each of their peers as "relatively equal and able to contribute" in the classroom, to which Hannah replied,

> maybe if there's something that you hear over here in discussion, and you can either talk to that student, or bring it up without talking to them, but like, "Oh, this person had a really good idea. Do you want to share what you said?" framing it so they already know, "Okay, well, my teacher has confirmed what I said." And that will give them a little bit of confidence.

Hence, Hannah implied that her feedback and affirmation, as the teacher, was particularly impactful because it could lead to students feeling empowered as a doer of mathematics and, consequently, to an increase in their mathematical self-efficacy. In other words, because the teacher is the leading mathematical authority, their positive feedback and affirmation is more impactful to students when compared to the feedback and affirmation of other students in the classroom. Hannah continued to hold this belief after student teaching as well. For instance, during the fifth interview, Hannah provided examples of students sharing solutions or asking questions that she affirmed or signified as important and claimed that her determining a student's solution or question was important would "make them [students], likely feel important and build confidence." Thus, Hannah not only believed that it was important to empower students as learners of mathematics and help develop students' self-efficacy, but, as the leading mathematical authority in the classroom, she was best positioned to do so.

***Hannah's Beliefs About Students as Mathematical Authorities***

Derived from her belief that all students have unique perspectives and backgrounds they bring to the classroom, Hannah strongly believed *all students are legitimate sources of mathematical knowledge or reasoning simply because they are human beings*. On multiple occasions, Hannah implied or explicitly claimed that all students are legitimate sources of mathematical knowledge or reasoning because they have brains and thus can think, reason, rationalize, and draw upon their unique experiences and conceptions. For instance, during the first interview, I asked Hannah if students can be legitimate sources of math knowledge or reasoning, to which she replied, "Absolutely…the math knowledge that exists isn't just in textbooks or teachers' lessons. Like, students have brains that think in different ways, and you could come up with a solution in a different way than would ever be in a textbook." In the second interview and again in the sixth interview, Hannah made the distinction between a student "being a mathematical authority" and a student "exercising" or "asserting" their mathematical authority. From these instances and others, I infer that Hannah believed all students are legitimate sources of mathematical knowledge or reasoning who *can* make meaningful contributions in the classroom. However, for students to transition from those who *can* make meaningful mathematical contributions to those that *are making* or *have made* meaningful contributions (i.e., "exercise" or "assert" their mathematical authority), they need to share or communicate their knowledge in the classroom. In other words, Hannah believed all students *are* legitimate sources of mathematical knowledge and reasoning, but to be viewed as mathematical authorities within the classroom community, particularly by their peers, students need to communicate, share, or contribute their mathematical ideas or conceptions.

Hannah believed *students are viewed as mathematical authorities in the classroom when they share or communicate their mathematical conceptions, justifications, or reasoning*. When Hannah distinguished between a student "being a mathematical authority" and a student "exercising" their mathematical authority, she claimed, "part of exercising it [mathematical authority] is participating and being willing to share your thoughts and contribute to brainstorming or whatever is the case." Consistent with Hannah's belief about mathematics, Hannah explained students can communicate their reasoning for a solution, justification for an idea, or their understanding of a concept or procedure and be viewed as a mathematical authority by their peers and by Hannah. Moreover, Hannah claimed students did not need to be "completely correct" to be viewed as a legitimate source of mathematical knowledge or reasoning. Rather, Hannah suggested that so long as students were communicating or explaining *their* understanding of the mathematics at hand, whether completely correct or not, along with the reasoning or justification for their ideas or understanding, she would view them as a mathematical authority and expected their peers to view them as mathematical authorities as well. After student teaching, Hannah continually made statements that were further evidence of her belief that students who communicate their conceptions, reasoning, and/or justifications are viewed as mathematical authorities in the classroom. For instance, during the final interview, Hannah provided the following contrast between two hypothetical students:

> A component of mathematical authority has to do with students' voice in the classroom. So if I, as a student, hate talking in front of everyone, and I think about all these things really deeply, and make all these connections, and even if I see myself…as a source of information, but then another student with all those same things, but then they're willing to answer questions and ask questions and be involved in discussion, then…they're using their authority more because they're using it to interact with other people and talk about the mathematics.

In this instance, I infer that Hannah continued to make the distinction between students who she believes are legitimate sources of mathematical knowledge and reasoning and students who are going to be *viewed* as mathematical authorities by the classroom community (i.e., the teacher and other students) because they are willing and able to communicate and contribute their conceptions, solutions, reasoning, and/or justifications. Still, Hannah acknowledged there were factors that prohibited some students from readily making mathematical contributions in the classroom, which consequently influenced how they were viewed in the classroom.

Related to her belief that students' affect influences their engagement, Hannah believed *students' familial and educational background and experiences influence their propensity to contribute ideas in the classroom and, hence, their position as a mathematical authority.* In both authority diagrams (Figures 8 and 9), Hannah included *parents/guardians*, *previous teachers*, *current teacher*, and *previous math experiences* with unidirectional arrows going from those sources to *student[s]*. Hannah explained those sources are not “active” mathematical authorities in the classroom, rather they influence students and their position as mathematical authorities in the classroom. During the sixth interview, Hannah explicated how she thought about the influence of those sources on students:

> If they [students] have previous teachers that were not super engaging, didn't really ask good questions to elicit their thinking, just taught and expected students to learn things, but not a whole lot of engagement…so the student didn't feel comfortable asking questions or participating in the class, and just kind of like sat there and wrote down things. And then also, say they have parents at home that always shoot down their ideas, don't really engage with them at all, aren't really building them up and encouraging them, then coming into the classroom they might be more likely to just sit there, be checked out, not be super engaged…I think that could work in the reverse too. Like, if you have students that have had really supportive teachers and really encouraging parents at home, and then they come into the classroom really built up and eager to participate, but then they have a teacher that always shoots down their questions and never calls on them, then that can kind of lower their view of themselves and kind of inhibit their mathematical authority.

Hence, Hannah viewed students’ familial and educational backgrounds as factors that can influence students’ positions as mathematical authorities and factors that she, as the teacher, needed to be cognizant of when interacting with and supporting students.

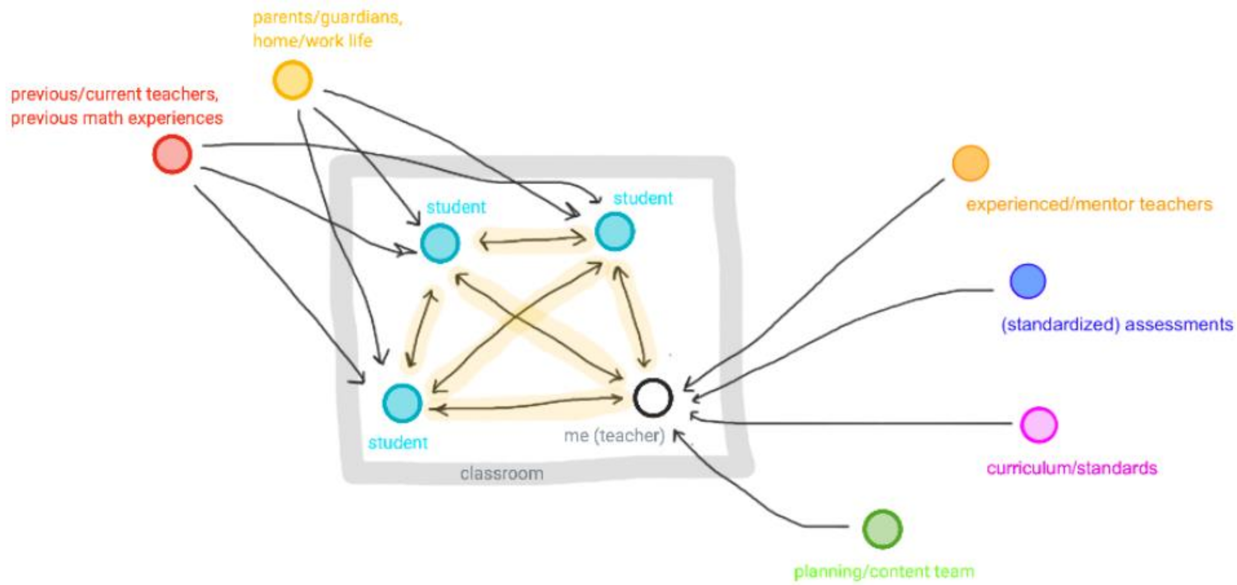



*Figure 9*. Hannah's Second Mathematical Authority Diagram

Even though Hannah believed all students are legitimate sources of mathematical knowledge and reasoning, she realized that students in her classroom may, at least initially, only view the teacher as a mathematical authority. Additionally, she believed some students may not readily contribute their mathematical ideas, solutions, insights in the classroom due to their familial and/or educational experiences. Consequently, Hannah continually sought ways to empower students and communicate, implicitly and explicitly, that students' ideas were important, valuable, and should be privileged in the classroom. In other words, Hannah believed, as the leading mathematical authority, she was to support students' learning of mathematics and do so in ways that continually positioned students as mathematical authorities in her classroom. In the subsequent paragraphs I detail how Hannah described doing this in the classroom, both before and after student teaching.

In the *Hannah's Beliefs About Teaching Mathematics* section, I provided evidence for my inference that Hannah's strongly held belief that *privileging students' ideas, solutions, and voice* *in the classroom would best foster students' understanding of concepts and procedures* was derived, in part, from her beliefs about learning mathematics. This belief also seemed to be partially derived from Hannah's belief that all students are legitimate sources of mathematical knowledge and reasoning as well as her belief that students often position the teacher as the ultimate mathematical authority in the classroom. Hannah consistently described enacting instructional strategies and teacher moves (Smith & Sherin, 2019) that privileged students' ideas, solutions, and voice with the following purposes: position students as mathematical authorities, empower students as doers of mathematics, and change students' views of the teacher as the ultimate mathematical authority. In the subsequent paragraphs, I describe some of the instructional strategies and accompanying teacher moves Hannah discussed implementing with these purposes in mind.

Throughout her time in the study, Hannah described having students work in small groups to engage in tasks because she believed all students are mathematical authorities who can generate ideas and solutions. When discussing the scenario with Marcos in the second interview, Hannah claimed she would have students discuss Marcos's claim that it's impossible to graph $f(x) = \sqrt{-x}$ with their peers in small groups, "so that they're more intimately involved in it [the discussion] instead of just a whole class [discussion]…I think in this case, I would turn it back to the partner so that everyone is having to think through this." Hannah went on to say that as students discussed Marcos's claim in small groups, she would circulate but would not answer students' questions initially because she wanted them to consider their peers' ideas and ask each other questions. In this instance, Hannah's reasons for having students discuss Marcos's claim in small groups, and not just as a whole class, suggests that she believed all students could reason with Marcos's claim, answer peers' questions, and learn from their peers as they considered their ideas about Marcos's claim. After student teaching, Hannah described her intentions for students engaging in small groups in a similar manner. For instance, during the fifth interview, Hannah explained that she frequently had students discuss problems or tasks in small groups before discussing as a whole class because, "students are more comfortable talking amongst themselves first, or thinking about their own ideas, and then talking about it in a smaller setting, and then sharing [in the whole class setting]." Hannah's explanation in this moment highlights how she viewed students engaging in small groups as a strategy that afforded students the time and space to consider a question or problem, generate ideas on their own and in collaboration with their peers, and then share their ideas in the whole class setting. Hence, this moment and the one described earlier illustrate how Hannah viewed students engaging in small groups as a strategy she implemented to position *all* students as legitimate sources of mathematical knowledge and reasoning who were able to develop their own ideas or solutions to tasks and reason with the ideas or solutions of their peers.

Hannah claimed that as students worked on problems or tasks in small groups, the main way she supported their learning and progress on tasks or problems was through questioning. Hannah discussed questioning as an integral aspect of her practice prior to student teaching, but descriptions of her questions and the intent of her questions became more salient after student teaching. In the interviews after student teaching, Hannah described asking assessing and advancing questions (Smith & Sherin, 2019) as the main way she supported student learning as they worked through tasks or problems on their own or in small groups. Moreover, Hannah consistently described her commitment to improving her questioning practices because, in Hannah's view, asking effective questions would empower students and lead to them being consistently positioned as mathematical authorities. For instance, during the fifth interview, Hannah described how she viewed mathematical authority operating between her and students when asking advancing questions, by claiming,

> It depends on how you ask the questions. But, if you're asking questions that are really going to help students move from where they're currently at, to where you're trying to get them to be, I think both you have part of the mathematical authority because you're making that connection in your mind and trying to figure out what question to ask them. But then you're also putting it on them because the way they respond, or if they start writing things down based off what you asked, then they also have part of the mathematical authority because they're figuring that part out on their own or engaging in a discussion with you about whatever question you just asked.

Thus, Hannah implied students would position her, the teacher, as a mathematical authority when she asked advancing questions because considering her questions would help them make progress on a problem or task. Yet, Hannah aimed to "turn it back" to students by asking advancing questions that would empower students to develop conceptual relationships or make progress on tasks without further support from her. During each of the interviews after student teaching, Hannah claimed that, as a student teacher, she worked to improve her questioning practices because "good," effective questions, when asked, consistently position students as mathematical authorities. During the sixth interview, Hannah explicated the relation between the questions she asked and positioning students as mathematical authorities: "if I really see them as a mathematical authority, that means they could be thinking of things, and probably are thinking of things, that I've never even thought of, or a connection I haven't even made before. So, I can develop that by the questions that I ask." Hence, Hannah was committed to the craft of asking "good" questions because when she asked good questions, she could consistently position students as legitimate sources of mathematical knowledge and/or reasoning and empower them to make progress on tasks on their own.

Throughout her time in the study, Hannah described facilitating mathematical discussions in a way that would empower students, lead to them viewing their peers as mathematical authorities, and lead to them not viewing her, the teacher, as the ultimate mathematical authority in the classroom. During the statement sort in the third interview, Hannah explicated her view of how engaging in discussions would lead to students viewing their peers, and not just her, as mathematical authorities when she claimed,

> I want students to see each other as mathematical authorities and not just me and so then that [engaging in discussions] will lead them to value each other's perspectives and solutions instead of just whatever I'm saying. But then again, the more they have these discussions [small group and whole class] and see these different perspectives and connections, then that'll lead to them seeing each other as math authorities.

After student teaching, Hannah continued to view small group and whole class discussions as strategies that would empower students and lead to them viewing their peers as mathematical authorities. For instance, during the final interview, Hannah claimed that engaging in discussions in which students' ideas and solutions were privileged would empower students as doers of mathematics because, "they're [students are] like, 'Oh, someone else in the class figured [it] out, so I might be able to.' Instead of, 'Oh well, she decided, she's a teacher, she's gone to college, so she probably knows that stuff better.'" In this instance, Hannah implied that as students considered their peers' ideas, they positioned their peers as mathematical authorities and seeing their peers make meaningful contributions in the classroom might lead to them feeling empowered to also contribute their ideas. Moreover, in this instance and others, Hannah suggested that engaging in such discussions could play a role in eradicating their view of the teacher as the ultimate mathematical authority in the classroom. Thus, Hannah viewed whole class and small group discussions as powerful instructional strategies because if students' ideas and solutions formed the basis of those discussions, then this could empower students as doers of mathematics, lead to students viewing peers as legitimate sources of mathematical knowledge and reasoning, and change students' view of the teacher as the ultimate mathematical authority.

Both prior to and after student teaching, Hannah detailed the talk moves (Smith & Sherin, 2019) she incorporated as she facilitated whole class discussions with the goal of continually privileging students' perceptions, ideas, and solutions. During the first interview, Hannah claimed she saw one of her roles in the classroom as a facilitator who bounces students' ideas off each other. To start the second interview, I asked Hannah to further describe what "bouncing students' ideas off of one another" looks and sounds like, to which Hannah replied:

> It's like a lot of the talk moves that we've looked at [in pedagogy courses] of like, "Oh, can someone else restate what this person has said?" or, "Do you agree or disagree? And why?" Not just like, "Oh yeah, same," but like, "Why do you agree with that?" or like, "Can you explain more [about] something that this person said?" So, creating that discussion, instead of the teacher being like, "Okay, well this person said this and this is what they meant," like letting students do that for each other and asking those questions to where they start to do it more naturally.

Hence, Hannah described talk moves she could implement in whole class discussions to not only consistently position students as legitimate sources of mathematical knowledge and reasoning but also work towards eliminating students' view that the teacher is the only one who provides explanations and interprets or clarifies students' descriptions. After student teaching, Hannah explained how she incorporated talk moves to consistently position students as mathematical authorities in the whole class setting. For instance, during the fifth interview, Hannah claimed that over her time student teaching, she worked to incorporate more student voice in the classroom by increasingly affording students opportunities to explain relationships between concepts or ideas, relate their own ideas to the ideas of their peers, and reason with their peers' ideas. Moreover, Hannah attributed incorporating more student voice in the classroom to reflecting on an article she read during one of her pedagogy courses in which the author encourages teachers to "never say anything a kid can say" (Reinhart, 2000). Thus, Hannah implied her uses of talk moves were not merely strategies to increase students' engagement, but they were strategies she implemented because she believed students could reason with and explain relationships between concepts, relationships between their ideas and their peers' ideas, and aspects of ideas that were correct or incorrect. In other words, Hannah viewed the talk moves she used when facilitating class discussions as strategies that consistently positioned students as mathematical authorities.

### *Hannah's Beliefs About Other Sources as Mathematical Authorities*

Hannah believed *the content standards for a given course are mathematical authorities because they are legitimate sources of pedagogical content knowledge* (Shulman, 1986). Throughout her time in the study, Hannah suggested the content standards largely determine the mathematical content that students should learn by the end of the course. During the first interview, Hannah claimed that content standards, "define the end goal of what students should be able to do and know and understand…[and] are also things that are prerequisite knowledge before moving to the next course." Hannah, then, viewed the standards as a legitimate source of pedagogical content knowledge that determine the concepts students should learn and procedures or skills students should be proficient in by the end of a particular math course. During the third interview, Hannah added that she consulted the standards as one source to determine the mathematical content to teach, but that the standards were not the only source. In addition to the standards, Hannah described "things that students are interested in, or things that have come up in their [students'] discussion," when considering the mathematics to teach. Hence, Hannah believed the content standards were a legitimate source of pedagogical content knowledge that played a role in determining the mathematics to be taught and learned in the classroom, but the standards did not solely determine that mathematics. After student teaching, Hannah espoused similar views of the content standards as mathematical authorities, most evidenced by an arrow she included in her second mathematical authority diagram (see Figure 9) going from *curriculum/standards* to the *teacher*. Explaining the inclusion of that arrow, Hannah implied the standards largely informed the mathematics she taught while a student teacher.

Hannah also believed *textbooks and other curricular sources were a legitimate source of pedagogical content knowledge (Shulman, 1986), similar to the content standards*. Throughout her time in the study, Hannah described textbooks and other curricula as sources of tasks or problems she could implement in her lessons. Hannah indicated she would initially consult a variety of curricular resources but, over time, begin to view certain curricular resources as mathematical authorities if she observed that when implemented, the curriculum supported students' learning. As Hannah was describing the inclusion of *curriculum* in her first mathematical authority diagram (see Figure 8), she stated, "I would use those things as a base, but then as I am going, see what works or what doesn't work with students." She went on to say curricular resources that "work with students" are a legitimate source of mathematical knowledge "because it [the curriculum] allowed students to progress and learn." Hannah, while describing her second diagram, espoused a similar view of the curriculum as a mathematical authority when she claimed that the curriculum she chooses will "affect the way I facilitate discussion or the kinds of questions that I ask because I know…the curriculum that I'm trying to teach to." Hence, Hannah believed curricular resources, particularly those that she believed supported students' learning, were legitimate sources of pedagogical content knowledge that provided tasks or problems she could use to foster students' learning and, moreover, those curricular resources would play a role in informing aspects of her instruction (e.g., questions, discussions).

Hannah also believed *other mathematics teachers can be mathematical authorities because they can be legitimate sources of pedagogical content knowledge* (Shulman, 1986). Throughout her time in the study, Hannah described consulting other teachers regarding how to teach specific topics or support students in meeting certain content standards. In the second interview, Hannah claimed she would consult teachers whose "teaching style you believe was effective," on how they structured their lessons and sequenced the topics within a particular course. Hence, for Hannah to consider a teacher as a legitimate source of pedagogical content knowledge, she needed to deem that teacher's approach to teaching mathematics as effective. In all of Hannah's descriptions of her mentor teacher and student teaching supervisor, she implied that she viewed their approaches to teaching as effective. Thus, in the lesson planning interview and subsequent interviews, Hannah described multiple instances of positioning her mentor teacher and student teaching supervisor as mathematical authorities. For instance, during the sixth interview, she claimed her mentor teacher would often provide insights into the types of tasks that fostered students' rich understanding of mathematics and aspects of tasks or mathematical concepts and procedures students typically struggled to understand. Similarly, Hannah described using her supervisor’s feedback regarding which concepts students may struggle to understand or how to “reframe” discussions of concepts so that such discussions would support students’ understandings of concepts they would encounter in future courses. Thus, Hannah consistently positioned her mentor teacher and student teaching supervisor as mathematical authorities she could turn to for advice and feedback on her instruction, planning, and how to meet the mathematical learning needs of her students.

**Influence of Student Teaching on Hannah’s Beliefs and Some Relations Among Her Beliefs**

Hannah’s beliefs about mathematical authority seemed to be the same prior to and then after student teaching. Hannah claimed that early in her student teaching semester she was most concerned with becoming a legitimate teacher in the eyes of her students and the other teachers at her placement school, and this led to her planning and enacting lessons differently than she did by the end of her student teaching semester. For instance, during the fifth interview, Hannah claimed that at the beginning of the semester, “I would always highlight that [certain] observation, or I would always come up with it and then use that more to start the discussion.” Yet, as the semester progressed, she described how she attempted to incorporate more student voice:

> I tried to transition more that way [incorporating more student voice], like, whenever there was a key understanding I was wanting them to develop, either asking questions or having them work in groups on certain things, and then asking groups to share out instead of me saying that final thing.

Thus, I infer that, for Hannah, student teaching provided a context in which Hannah developed her own, personal pedagogy and one that was informed by (i.e., aligned with) her beliefs as she reflected upon her instructional practice.

In the preceding sections I reported some inferred relationships between the beliefs Hannah held. For instance, I inferred that one of Hannah's beliefs about learning mathematics was derived from her belief about mathematics that I reported. In the subsequent paragraph, I highlight Hannah's beliefs that appeared to be most primary and core in her system of beliefs. That is, I highlight her beliefs that Hannah believed strongly and ones that seemed to inform other beliefs held by Hannah.

Hannah's beliefs that seemed to be primary *and* most strongly held by Hannah were her beliefs about students. I reported Hannah's beliefs about students in multiple sections, so I summarize her beliefs about students in Table 2. That Hannah held these beliefs strongly was most evidenced in the third interview when she developed the group of statements labeled "Empowering / self-efficacy" (see Figure 7). As I previously reported, Hannah highlighted seven of the eight statements in that group, which indicated those statements represented important aspects of her beliefs. Additionally, at various moments in the study Hannah emphasized she wanted her students to feel "loved," "valued," "supported," and "empowered." Thus, I conjecture Hannah's beliefs about students that I reported were the ones Hannah held most strongly in her system of beliefs. Additionally, Hannah's beliefs about students seemed to be primary beliefs, in that all of them seemed to inform, to some degree, some of her beliefs about teaching mathematics and mathematical authority. When Hannah discussed her beliefs about teaching or her instructional practice, she often related those beliefs or practice to her beliefs about students. Thus, Hannah's beliefs about students seemed to be the beliefs Hannah held that most prominently influenced her instructional practice.

**Table 2**

*Hannah's Beliefs About Students*

| Hannah's beliefs about students | Section belief is reported |
|---|---|
| All students have unique perspectives and backgrounds they bring to the classroom and, consequently, different needs that need to be met | Beliefs About Students |
| Students' affect prominently influences their engagement in the classroom | Beliefs About Students |
| Students are likely to view the teacher as the ultimate mathematical authority in the classroom | Beliefs About the Teacher as a Mathematical Authority |
| All students are legitimate sources of mathematical knowledge or reasoning simply because they are human beings | Beliefs About Students as Mathematical Authorities |
| Students' familial and educational background and experiences influence their propensity to contribute ideas in the classroom and, hence, their position as a mathematical authority | Beliefs About Students as Mathematical Authorities |

# CHAPTER 5

# CONCLUSIONS

In the previous chapter I reported each of the four PT's beliefs about and related to mathematical authority, described how student teaching influenced their beliefs, and highlighted some salient relations among each PT's beliefs. In this chapter I describe similarities and notable differences across the four prospective teachers' beliefs about mathematical authority and provide explanations for some of the differences across the four teachers. The structure of this chapter mirrors the structure of each participant's narrative of their beliefs about mathematical authority that I presented in Chapter 4. That is, I report conclusions from the cross-case analysis starting with beliefs about the teacher as a mathematical authority, then students as mathematical authorities, then other sources as mathematical authorities, and close with conclusions about the relations among the PTs' beliefs and how student teaching influenced the PTs' beliefs. In each section, I describe commonalities across the four PTs' beliefs and provide some explanations for those commonalities. One of the affordances of conducting this multi-case study of PTs' beliefs about mathematical authority is that it enabled me to compare the different beliefs held by the four participants and, consequently, better understand some of the beliefs held by an individual participant (Miles & Huberman, 1994; Stake, 2006). Thus, I also describe some notable differences in the beliefs held across the four PTs and explain those differences.

## Conclusions Concerning Beliefs About the Teacher as a Mathematical Authority

Each of the four prospective teachers held similar, yet distinct beliefs about the teacher's position as a mathematical authority in the classroom. Namely, each of them believed the teacher is nearly always positioned by students as the main (i.e., ultimate, leading, primary) mathematical authority in the classroom. The reasons why each participant believed the teacher is viewed as the main mathematical authority (i.e., their legitimacy as a mathematical authority) differed across participants, yet each participant believed that one aspect of the teacher's legitimacy was their deeper or more extensive content knowledge when compared to students. For instance, each of the PTs believed the only time when students may not view the teacher as a mathematical authority is when the teacher demonstrates or teaches incorrect mathematics. Each PT's belief that the teacher is the main mathematical authority in the classroom was primarily held (Green, 1971), meaning it seemed to inform other beliefs about mathematical authority held by each PT. Even though each PT believed, in a primary manner, the teacher is positioned as the main mathematical authority in the classroom, how strongly this belief was held differed across the four PTs. In the cases of Hannah and Grace, their beliefs that the teacher should be the leading or primary mathematical authority in the classroom, respectively, were not strongly held beliefs. In contrast, Simon and Chris strongly believed the teacher should be and is positioned as the ultimate mathematical authority in the classroom. The strength with which a PT believed the teacher was positioned as the main mathematical authority in the classroom seemed to have implications for how each PT described how they interacted with students and supported students' learning. I describe some of these implications in the subsequent paragraphs.

Regardless of how strongly each PT believed the teacher is and/or should be the main mathematical authority in the classroom, for each PT this belief informed two other beliefs about mathematical authority that were subsets of their beliefs about teaching mathematics. Namely, each PT believed that as the main mathematical authority they 1) were to support students' learning of the intended and/or correct mathematics and 2) certified when students learned mathematics.

Each PT believed that one of their primary roles as the teacher and main mathematical authority in the classroom was to support student learning of productive mathematics. The degree to which the PTs' beliefs about their role of supporting students learning interacted with their beliefs about students as mathematical authorities varied across the four PTs and seemed to be related to how strongly they believed the teacher is the main mathematical authority in the classroom. In Hannah's case, for instance, her belief about the teacher being the main mathematical authority in the classroom was not strongly held. As the leading mathematical authority in the classroom, she believed one of her roles was to support student learning, but her descriptions of how she supported student learning were more related to her beliefs about students as mathematical authorities. In other words, Hannah believed one of her roles was to support students' learning of mathematics, but she also believed it was best to support students in ways that empowered them and positioned them as mathematical authorities. In contrast, Simon strongly believed he was positioned as the ultimate mathematical authority in the classroom and believed he was to support student learning by steering students away from developing and/or exhibiting "bad math." When Simon described how he supported student learning, he described his position as the ultimate mathematical authority "outweighing" students' positions as mathematical authorities in instances where he would "steer them [students] back in the right direction."

The degree to which each PT's belief that they were to support student learning as the main mathematical authority interacted with their beliefs about students as mathematical authorities influenced their descriptions of the instructional practices they enacted to support student learning. Across participants, there was some variation in the instructional practices they described enacting to support students' learning, yet each participant described assessing and advancing questions (Smith & Sherin, 2019) as instrumental instructional practices they used to support students' learning. I conjecture descriptions of assessing and advancing questions were salient across the four PTs because Smith and Sherin's (2019) *The 5 Practices in Practice* was a foundational text used in all three of the participants' mathematics pedagogy courses. Yet, how each participant viewed their use of assessing and advancing questions differed, and I conjecture this difference can be explained, in part, by the beliefs about mathematical authority each PT strongly held. Chris's and Simon's descriptions of assessing and advancing questions were more related to their position as the main mathematical authority, whereas Grace and Hannah acknowledged they were positioned as a mathematical authority by students when they asked assessing and advancing questions, but they also described asking such questions in a way that would position students as mathematical authorities. In Chris's case, he believed that as the ultimate mathematical authority, he was able to support students' learning via advancing questions because he was the one who established the learning goals for the lesson and, consequently, knew how to best support students' progress toward meeting those goals through his questioning. On the other hand, although Grace viewed herself as "more of the [mathematical] authority" when asking advancing questions, she also claimed that when she supported students' learning by asking advancing questions, "the mathematics is still completely their own and they're still developing it and thinking through their own thoughts." Hence, Grace attempted to leverage her position as the primary mathematical authority in the classroom to support students' learning by asking advancing questions, but she described doing so in a way that would also position students as legitimate sources of mathematical knowledge and/or reasoning.

Another key aspect of Chris's, Hannah's, and Grace's descriptions of their practice that informed how they supported student learning was the learning goals they established for each lesson. The influence of learning goals was most salient in Chris's descriptions of how he supported student learning; however, Grace and Hannah also described how the learning goal(s) for a given lesson informed their instruction. For instance, during the fifth interview Grace claimed that she would need to act as the "banks of a river" to guide students in whole class discussions, "to make sure that we're on track and we're ultimately getting where we need to be, as far as hitting our learning goals and addressing our standards." I conjecture that Chris, Hannah, and Grace described learning goals informing how they supported student learning because they read about and discussed learning goals (Smith & Sherin, 2019) in their first of three pedagogy courses and the learning goals for a lesson were continually referenced in their subsequent pedagogy courses.

Each PT also believed that, as the main mathematical authority, they were positioned by students to certify when students learned correct and/or the intended mathematics. Although I did not observe the four PTs' enacted practice, I argue that it is reasonable to infer that what each PT certified as learning mathematics depended upon their beliefs about learning mathematics. For instance, if a teacher believes that they determine when students learn mathematics and they believe learning mathematics entails only enacting procedures with accuracy, then it would be reasonable to assume that the teacher would only communicate to a student, whether explicitly or implicitly, that they learned mathematics when they were able to enact the procedure(s) at hand with accuracy. As this relates to the four PTs, each of the four PTs had slightly different beliefs about what constitutes learning mathematics. Correspondingly, I argue that what each PT certified as learning mathematics during their student teaching, and what they will certify as learning as they begin their teaching careers, likely varied and will vary depending on what they believe is required to learn mathematics. In Chris's case, for instance, he believed learning mathematics entailed understanding the underlying concepts and being able to accurately enact procedures. Hence, I argue that it is reasonable to assume that Chris, during student teaching, certified that a student learned mathematics when they were able to enact the procedure(s) at hand correctly and when they were able to provide sufficient explanations, in Chris's view, of the underlying concepts. This connection between PTs' beliefs about learning mathematics and their belief that, as the main mathematical authority, they certify when students learn mathematics has implications for teacher education that I discuss in Chapter 6.

Hannah and Grace held similar beliefs about the teacher as the main mathematical authority that were distinct from Simon's and Chris's beliefs. Specifically, they believed they were uniquely positioned in the classroom to empower students and foster students' development of their mathematical self-efficacy (in Hannah's case) or their mathematical self-confidence (in Grace's case). Both Hannah and Grace strongly held beliefs about teaching mathematics that related to empowering students and fostering the development of students' productive dispositions. In Hannah's case, she strongly believed it is important to empower students as doers and learners of mathematics and to develop their mathematical self-efficacy. In Grace's case, she strongly believed it is imperative to foster students' positive dispositions about mathematics and learning, as well as themselves as doers of mathematics. These beliefs seemed to interact with both Hannah's and Grace's beliefs about their position as the main mathematical authority to inform their beliefs that they could leverage their position as the leading or primary mathematical authority to empower students, foster students' positive views of themselves, and, in Grace's case, increase the likelihood that a student would be positioned as a mathematical authority by their peers. In other words, both believed that it was important to empower students as doers of mathematics or as mathematicians and believed they were uniquely positioned in the classroom by students such that their feedback and affirmation carried a particular weight and led to students feeling empowered in the classroom. I argue that this is evidence that both Hannah and Grace considered how they could leverage their unique positions as the leading and primary mathematical authority, respectively, in the classroom in a way that would support students in the classroom beyond learning mathematics. That is, they were cognizant of and reflected on students' affect in the classroom and attempted to leverage their position in a way that they could have a positive impact on their students' dispositions towards mathematics, their view of themselves, and even how their peers viewed them in the classroom.

Comparing the beliefs held across participants accentuated some beliefs that were uniquely held by individual PTs and unique interactions among beliefs. In other words, analyzing the beliefs about the teacher as a mathematical authority held across the four PTs, highlighted some of the idiosyncratic beliefs held by an individual PT. Although I did not observe the four PTs' classroom instruction, these idiosyncrasies may provide novel insights into each PT's unique, personal approach to teaching and how they reflect on their practice. For instance, in Simon's case, his belief concerning the many practical aspects of teaching and learning in the classroom and how, through his descriptions, this belief interacted with his belief about supporting students' learning became particularly salient as I analyzed the beliefs held across participants. It seemed that Simon's beliefs about the practical aspects of teaching prominently influenced how he leveraged his position as the ultimate mathematical authority to support students' learning in the classroom. Hence, Simon's belief about the practical aspects of teaching and how this belief uniquely interacted with his beliefs about the teacher as the ultimate mathematical authority, suggests that his belief about the practical aspects of teaching played a prominent role in shaping his approach to teaching and how he operated as the ultimate mathematical authority in the classroom. In the case of Hannah, she believed the teacher *should* be viewed as the *leading* mathematical authority, but she believed students *often* position the teacher as the *ultimate* mathematical authority in the classroom. Hannah was the only PT to espouse beliefs about how the teacher should be viewed as a mathematical authority by their students and how this contrasted with how students often view the teacher as a mathematical authority. Hannah demonstrated she was cognizant of students' inclination to position the teacher as the ultimate mathematical authority, and the mismatch between her belief about how students *should* position the teacher and how students *often do* position the teacher as a mathematical authority seemed to be a prominent influence on her instructional practices. That is, Hannah seemed to reflect on how students positioned her as a mathematical authority in the classroom and used this reflection to inform some of her instructional decisions.

**Conclusions Concerning Beliefs About Students as Mathematical Authorities**

All four PTs believed all students can be positioned as mathematical authorities in the classroom. This was particularly the case when considering the teacher positioning students as mathematical authorities. That is, from each of the PT's perspectives as teachers, they believed students could be legitimate sources of mathematical knowledge and reasoning in the classroom. In the cases of Chris and Simon, their beliefs that all students can be positioned as mathematical authorities was not a strongly held belief and did not seem to be related to other beliefs they held about students. In contrast, Hannah and Grace both strongly believed students can be viewed as mathematical authorities in the classroom. In both cases—Grace's and Hannah's—their belief that all students can be mathematical authorities was related to their strongly held beliefs that all students have unique backgrounds, experiences, and perspectives they bring into the classroom. As this related to mathematics, both Grace and Hannah valued students' unique mathematical backgrounds and perspectives and, on multiple occasions, claimed they could learn and gain new mathematical insights from their students. Hence, Hannah's and Grace's strongly held beliefs that all students can be legitimate sources of mathematical knowledge and reasoning in the classroom were rooted in their other strongly held beliefs about students.

Interestingly, all four PTs believed that in order for a student to be viewed as a mathematical authority in the classroom—both by their peers and the teacher—they needed to develop a productive mathematical conception, idea, or solution and communicate that conception/idea/solution to the teacher or their peers. In Hannah's case, she believed any mathematical idea, conception, or solution a student developed and shared was productive, regardless of whether she viewed it as correct or not, because others (i.e., their peers or the teacher) could consider and learn from such mathematical contributions. In the cases of Grace, Chris, and Simon, they believed students needed to develop, exhibit, or understand correct mathematics and then communicate that mathematics to at least one other individual to be viewed as a mathematical authority. Notable here is that Grace, Chris, and Simon believed that developing correct mathematics was a necessary but not sufficient condition for students to be viewed as mathematical authorities in the classroom; students also needed to communicate, share, or teach that mathematics to others in the classroom. This emphasis on students communicating, sharing, or teaching their correct mathematics had clear implications for which students Grace, Chris, and Simon described as more legitimate mathematical authorities in their classrooms and, I conjecture, informed some of the instructional practices they described implementing to position students as mathematical authorities.

Grace, Chris, and Simon all believed that some students will be viewed as more legitimate mathematical authorities in the classroom when compared to their peers. In Grace's case, she believed some students would be viewed as more legitimate mathematical authorities by their peers because they consistently contribute correct mathematical ideas, conceptions, or solutions in the whole class setting. However, Grace did not seem to believe, from her perspective as the teacher, that any student was a more legitimate mathematical authority than their peers. Yet, Chris and Simon believed that, from their perspective as teachers, some students will be more legitimate mathematical authorities than their peers. In both Chris's and Simon's cases, they believed students are more legitimate mathematical authorities when they more frequently make meaningful mathematical contributions in small group and whole class discussions. Chris believed students who "controlled what they could control" are more legitimate mathematical authorities because these students are the ones who would consistently contribute their ideas, conceptions, and solutions in discussions. Simon, on the other hand, believed the students who are more legitimate mathematical authorities are ones who understand more correct mathematics when compared to their peers. Simon also suggested these students would be able to teach their peers, or "impart" mathematical knowledge because of their better understanding of mathematics. Hence, in the cases of Grace, Chris, and Simon, their belief about the students who will be viewed as more legitimate mathematical authorities, whether that be by their peers or the teacher, was related to their beliefs about what makes a student a mathematical authority in the first place. They believed the following when it came to students as mathematical authorities: 1) students who understood correct mathematics and communicated, shared, or taught that mathematics are viewed as mathematical authorities and 2) the students who communicate, share, or teach their correct mathematical conceptions most often are the ones who are viewed as more legitimate mathematics authorities in the classroom when compared to their peers.

Hannah, Grace, and Chris also believed it was important to engage students in activities and/or structures that consistently privileged students' ideas or conceptions. In each of Hannah's, Grace's, and Chris's cases, this belief was derived, in part, from their beliefs that all students can be mathematical authorities. This suggests that in order for PTs to believe that it is important to privilege students' ideas, conceptions, and solutions in the classroom, they first need to believe that students can be mathematical authorities in the classroom. Although this may seem trivial, it highlights the importance of teachers needing to believe that all students can be legitimate sources of mathematical knowledge or reasoning in the classroom if they are to believe that students' ideas, conceptions, solutions should be privileged in the math classroom.

Each PT described instructional strategies they implemented to position students as mathematical authorities. In each case, the PT's descriptions of how they attempted to position students as mathematical authorities was consistent with their beliefs that students needed to communicate, share, or teach mathematics to another individual to be viewed as a mathematical authority. For instance, Grace and Chris both explained that one way they positioned students as mathematical authorities in the classroom was by having them present their ideas or solutions to the whole class. After student teaching, Grace described intentionally planning to engage students in enough problems so that *all* students had opportunities to present or post their solutions at the front of the class for the whole classroom community to consider. Hannah described, among other strategies, teacher talk moves she used when facilitating whole class discussions (Smith & Sherin, 2019) to continually "turn it back" to students and have them explain or reason with others' ideas or contribute their own ideas. Moreover, she viewed whole class discussions, and her facilitation of such discussions, as strategies that would disrupt students' view of the teacher as the ultimate mathematical authority and help students view their peers as legitimate sources of mathematical knowledge or reasoning. Simon described planning and engaging students in exploratory lessons and activities in which students could collaboratively develop or "discover" the intended mathematical concept or procedure. When describing students' engagement in such lessons or activities, he claimed, "that banter and that give-and-take can engender a lot of good mathematical thoughts and hopefully, lead to a solid conclusion that they [the group of students] made." In each of the four PTs' cases, they described instructional strategies that enabled students to contribute their mathematical ideas or solutions and explained that students contributing had great potential for students to view their peers as legitimate sources of mathematical knowledge and/or reasoning.

For both Chris and Hannah, their belief that students' ideas, conceptions, and solutions should be privileged in the classroom was also informed by some of their beliefs about learning mathematics. It seemed that Chris and Hannah consistently attempted to enact instruction that positioned students as mathematical authorities *and* aligned with their beliefs about how to best foster students' learning in the classroom. In Chris's case, because he believed all students can be legitimate sources of mathematical knowledge or reasoning in the classroom, he not only believed students learn best when they actively develop or understand the mathematics on their own, he believed students *can* develop or understand the mathematics on their own. Similarly, Hannah not only believed student learning is fostered as they relate their own ideas and solutions to others' ideas or solutions, she also believed students *can* relate their ideas/solutions to others' ideas/solutions. The result was that Hannah and Chris were both acutely aware of how different grouping strategies would or would not 1) position students as mathematical authorities and 2) support students' learning of mathematics. For instance, in Chris's case, he consistently described instances where he would engage students in turn-and-talks or small group discussions, rather than whole class discussions, because doing so engaged all students and positioned them as legitimate sources of mathematical knowledge and reasoning.

In Chris's and Simon's descriptions of their instructional planning, their strongly held beliefs about the teacher as a mathematical authority seemed to interact often with their beliefs about students as mathematical authorities. For instance, both described engaging students in tasks that would enable students to "discover" or develop the intended mathematics but suggested they—Chris and Simon—intentionally structured students' engagement so that students would develop the intended conceptions or procedures. Simon explicated this relationship when he claimed that he would develop lessons such that "the student is discovering it [the mathematical knowledge] but the teacher's facilitating that discovery in a way that the teacher knows is going to be the best for the students." Such explicit explanations of the teacher intentionally structuring tasks or lessons were absent from Hannah's and Grace's descriptions of their practice. Hannah and Grace likely intentionally designed lessons and activities that would support students' learning of the intended mathematics for a given lesson. However, they did not seem to view intentionally structuring students' activity in the classroom as instances in which they leveraged their position as the main mathematical authority in the classroom. Hence, Chris's and Simon's beliefs that they were the ultimate mathematical authority in the classroom seemed to influence not only how they supported students' learning in the classroom, but also how they planned to structure students' engagement such that students would stay on "the right track" and meet the learning goals for the given lesson.

In the *Conclusions Concerning Beliefs About the Teacher as a Mathematical Authority* section, I recounted the four PTs' descriptions of assessing and advancing questions to support student learning. Here I want to note that Grace and Hannah, in particular, viewed advancing questions as questions they could ask to position students as mathematical authorities. Hannah and Grace both acknowledged that when asking advancing questions they were being positioned by students as mathematical authorities because, as Grace put it, "they [students] know that I'm asking the right question and it's going to help them." Yet, they both believed they could leverage this positioning by students to, in turn, position their students as mathematical authorities and empower them as doers of mathematics. For instance, Hannah claimed that when asking students advancing questions,

> you're also putting it on them [students] because the way they respond, or if they start writing things down based off what you asked, then they also have part of the mathematical authority because they're figuring that part out on their own or engaging in a discussion with you about whatever question you just asked.

Hence, Grace and Hannah suggested that when they asked students advancing questions, they were able to subtly leverage their position as the primary or leading mathematical authority, respectively, to productively empower students and position them as legitimate sources of mathematical knowledge and/or reasoning.

After comparing the beliefs about students as mathematical authorities held across the four PTs, some beliefs held by individual PTs were illuminated as distinct. The distinct beliefs about students as mathematical authorities held by an individual PT highlighted some of the complexities and challenges each PT faced when attempting to productively position students as mathematical authorities in the classroom. In Grace's case, after student teaching she believed that a student's confidence or personality can influence their position as a mathematical authority, as can the relationships and prior experiences they have with peers. Grace seemed to be attuned to the relational aspects of mathematical authority and appeared to have a growing awareness that a student viewing one of their peers as a mathematical authority is not always reduced to that peer developing and communicating correct mathematics. In other words, Grace seemed to become increasingly aware of how complex mathematical authority relations can be in classrooms. In Hannah's case, one belief about students as mathematical authorities that was accentuated by the cross-case analysis was her belief that all students are legitimate sources of mathematical knowledge or reasoning simply because they are human. Hannah explained that she wanted to privilege and incorporate students' voices in the classroom but doing so was challenging, especially early in her student teaching semester. Yet, she intentionally worked on incorporating more students' voices in the classroom throughout her student teaching semester. Hannah's unique belief that all students are legitimate sources of mathematical knowledge or reasoning seemed to highlight how Hannah perceived the challenge of productively positioning students as mathematical authorities in the classroom and how she aimed to address that challenge.

### Conclusions Concerning Beliefs About Other Sources as Mathematical Authorities

Each of the PTs believed the content standards for given courses are mathematical authorities because they are legitimate sources of pedagogical content knowledge (Shulman, 1986). Moreover, each PT believed that the standards strictly or largely determine the mathematics that is to be taught and learned in classrooms. Interestingly, Simon explicitly described the content standards as an extension of the Department of Education (DoE). That is, Simon believed the DoE is a mathematical authority who determines the mathematics that needs to be taught and learned in schools through creating and disseminating the content standards. Hannah and Chris both believed the content standards largely determine the mathematics to be taught and learned in classrooms, but they also believed other mathematics—mathematics not included in the standards—could be discussed in the classroom. Consistent with her strong beliefs about students as mathematical authorities, Hannah described being open to discussing students' interests or ideas that may not be aligned with the standards in the classroom. Similarly, and consistent with his strongly held belief about the teacher being the ultimate authority in the classroom, Chris believed mathematics not included in the standards, but that he considered important for students to learn to best prepare them for future endeavors, should also be taught and learned in classrooms.

Each of the four PTs also believed that textbooks and other curricular resources can be legitimate sources of pedagogical content knowledge (Shulman, 1986). Related to their beliefs about the content standards being mathematical authorities, each PT believed that textbooks and curricular resources provide problems and tasks that they could engage students in to foster their learning and thus meet the content standards of the courses they were teaching. Grace most clearly explicated this relationship when she claimed, "these resources are interpretations of the standards." Hence, rather than having to develop problems and tasks that would support students meeting the content standards all on their own, each PT believed they could turn to textbooks and other resources as legitimate sources of pedagogical content knowledge for such tasks and problems.

Chris, Grace, and Hannah believed that other mathematics teachers can be legitimate sources of mathematical knowledge and/or pedagogical content knowledge (Shulman, 1986). Grace and Chris described, on multiple occasions, turning to other teachers at their student teaching placement school if they did not sufficiently understand the mathematics of a given lesson or unit, or if they had questions about certain concepts or procedures. Thus, they believed that other math teachers are legitimate sources of mathematical content knowledge. That is not to say that Hannah did not believe other teachers are legitimate sources of mathematical content knowledge, but she did not describe turning to teachers in this way. Grace and Hannah also believed other math teachers can be legitimate sources of pedagogical content knowledge. For instance, Grace and Hannah described turning to other math teachers for suggestions regarding how to best teach certain mathematical concepts or how to sequence concepts across lessons in a way that would support students' understanding. In sum, Chris, Grace, and Hannah believed other math teachers can be mathematical authorities that, through either furthering their understanding of mathematical concepts or suggesting approaches to teach certain topics, can indirectly influence the mathematics that is taught and learned in classrooms.

Chris and Simon believed that technology and, in Simon's case, manipulatives can be mathematical authorities in the classroom. Both suggested, on multiple occasions, that technology, such as the Desmos graphing calculator (https://www.desmos.com), provide true instantiations of the mathematics at hand. Both suggested that students, and presumably the teacher, can always consider the calculations performed or graphs generated by a calculator to be true. Interestingly, Simon considered technology as one of many manipulatives that can be used in the mathematics classrooms and claimed, "the manipulative is a demonstration of the mathematical law that is here…the learning potential of the manipulative is that it shows that mathematical law." Simon's claim highlights that Simon—and, similarly, Chris—believed technology and other manipulatives provide taken-as-true representations of the mathematical concept or relationship at hand, and thus are legitimate sources of mathematical knowledge in the classroom.

## Conclusions Concerning the Influence of Student Teaching and Relations Among Beliefs

Across the four PTs, the beliefs about mathematical authority they held prior to student teaching were, with little exception, the beliefs they held after student teaching. Student teaching seemed to only influence changes to Grace's beliefs about mathematical authority, particularly her beliefs about students positioning their peers as mathematical authorities. This result was, in my view, surprising and runs contrary to some prior research on the influence of student teaching on PTs' beliefs. For instance, Vacc & Bright (1999) found that elementary PTs' beliefs about teaching and learning mathematics changed "significantly during student teaching" (p. 108). Because student teaching did not inform changes to the PTs' beliefs, with one exception, I concluded that student teaching seemed to reinforce the PTs' beliefs about mathematical authority. In other words, in almost all cases, the PTs' student teaching experience(s) did not perturb the beliefs about mathematical authority each PT held the summer prior to their student teaching semester. Inferring the reasons for why each PT's beliefs about mathematical authority, with the exception of Grace's beliefs about students as mathematical authorities, remained unchanged after student teaching are beyond the scope of this current study, but may be an area of exploration for future research.

Each PT's student teaching experience(s) provided a prolonged, practical context that afforded each PT opportunities to develop their own, personalized pedagogy. Although the PTs, along with their other cohort members, had field experiences prior to student teaching, their student teaching practicum was the first opportunity for each PT to plan and enact instruction across more than three consecutive lessons. Hence, I contend that student teaching provided each PT numerous opportunities to reflect on how to productively plan for and enact instruction and do so in ways that was consistent with their beliefs about mathematical authority. That student teaching provided the PTs a meaningful opportunity to form their own, personalized pedagogy was evidenced in multiple ways. For instance, after student teaching, Simon explained how he developed some exploratory lessons that enabled students to "discover" the intended mathematics and some lessons in which he "gave" students definitions, procedures, or concepts due to his perceptions of time constraints in the classroom. Chris also explained that after reflecting on how his students engaged in small groups, he planned to provide students time to think and generate ideas on their own before having students discuss and collaborate on problems or tasks. Moreover, Chris planning to give students independent think time appeared to be informed, in part, by his beliefs that all students can be legitimate sources of mathematical knowledge and/or reasoning *and* it is important to provide activities or structures that enable students to generate ideas on their own. Thus, student teaching provided Chris an opportunity to reflect on his instruction and his students' engagement in the classroom and consider how he could modify his instruction in ways that were consistent with his beliefs.

Analyzing the four PTs' systems of beliefs (Green, 1971) afforded insights regarding how important it is to consider and understand relations among teachers' beliefs. As I have reported, the four PTs held many common or similar beliefs about mathematical authority. For instance, all four PTs believed that all students can be mathematical authorities in the classroom. Yet, there were stark contrasts in how each PT's beliefs about and related to mathematical authority formed a coherent system of beliefs. It seemed that isolated beliefs (i.e., one belief) did not inform particular aspects of their instruction, rather each PT's instruction seemed to be informed by numerous beliefs at the same time. Moreover, the strength with which these beliefs were held and the relationships among these beliefs prominently influenced each PT's instruction. To illustrate the importance of understanding not only what PTs believe but how those beliefs are held in a system of beliefs, I contrast Hannah's and Simon's belief systems.

Simon and Hannah held similar beliefs about the teacher as a mathematical authority and beliefs about students as mathematical authorities. Specifically, both believed all students can be viewed as mathematical authorities in the classroom when they communicate productive mathematics, and they believed that the teacher is positioned as the main mathematical authority in the classroom. Yet, the strength with which Hannah and Simon held their beliefs about the teacher and students as mathematical authorities, and which beliefs about mathematics, learning mathematics, or students related to their beliefs about mathematical authority, seemed to explain differences in how Simon and Hannah described their classroom instruction. In particular, the difference in Simon's and Hannah's belief systems (Green, 1971) provides insights into their different descriptions of how they operated as the main mathematical authority in the classroom and how they positioned students as mathematical authorities in the classroom. For instance, in Simon's case, his belief that mathematics is a body of correct knowledge that has been discovered seemed to significantly influence his beliefs about mathematical authority as well as his instruction. Simon's beliefs about mathematics seemed to partially inform his strongly held belief that the teacher is and should be the ultimate mathematical authority in the classroom. Also, Simon consistently emphasized needing to ensure students did not develop "bad math" and that anyone's or anything's position as a mathematical authority was contingent upon demonstrating or developing correct mathematics. In contrast, Hannah's belief that students have unique perspectives and backgrounds they can draw upon to contribute in the classroom prominently influenced her beliefs about mathematical authority. This belief about students seemed to inform her strongly held belief that all students can be legitimate sources of mathematical knowledge and reasoning because they are human beings. Furthermore, her belief that all students can be legitimate sources of mathematical knowledge or reasoning seemed to significantly influence her instruction, particularly her descriptions of how she attempted to disrupt students' views of the teacher as the ultimate mathematical authority and her emphasis on empowering students as doers of mathematics. Hence, comparing Hannah's and Simon's belief systems in the cross-case analysis accentuated how their beliefs about students and mathematics, respectively, played a significant role in informing their strongly held beliefs about mathematical authority (i.e., beliefs about the teacher as a mathematical and students as mathematical authorities), and their strongly held beliefs about mathematical authority seemed to significantly inform their instruction.

# CHAPTER 6

## DISCUSSION AND IMPLICATIONS

The purpose of this study was to investigate prospective teachers' (PTs') beliefs about and related to mathematical authority and how PTs' student teaching practicum influenced their beliefs about mathematical authority. My interest in studying teachers' beliefs about mathematical authority was informed by my experiences as a high school mathematics teacher and teacher educator, interest in the relationships that operate within classrooms and influence the teaching and learning of mathematics (e.g., authority relations), and my view of beliefs as "predispositions to action" (Rokeach, 1968, p. 113). Initially, I planned to investigate practicing teachers' beliefs about mathematical authority and how those beliefs relate to their instructional practice. Yet, due to the COVID-19 pandemic and corresponding COVID-19 protocols in schools, I shifted the focus of this study to investigate prospective teachers' beliefs about mathematical authority. With this shift in focus, and informed by my experiences supervising student teachers, I decided to capitalize on the opportunity to investigate PTs' beliefs by studying how their student teaching practicum influenced their beliefs about mathematical authority. Thus, this study builds upon and extends previous research on mathematics teachers' beliefs (e.g., Conner & Singletary, 2021; Cross, 2009; Diamond, 2019; Liljedahl et al., 2007; Philipp, 2007; Thompson, 1984), mathematical authority relations in classrooms from the teacher's perspective (e.g., Amit & Fried, 2005; Dunleavy, 2015; Wagner & Herbel-Eisenmann, 2014a, 2014b; Wilson & Lloyd, 2000), and the influence student teaching has on prospective teachers (e.g., Vacc & Bright, 1999). The following research questions guided this study:

1) What do secondary mathematics PTs believe regarding who or what can be a mathematical authority in the classroom?
    a. To what extent or in what instances do PTs believe students can be mathematical authorities?
2) How do secondary mathematics PTs' beliefs about mathematical authority relate to their beliefs about mathematics, teaching mathematics, and students?
3) How does a PT's student teaching practicum influence their beliefs about mathematical authority?

To carry out this study I developed a novel definition of mathematical authority that, in tandem with a theory of beliefs, informed the design of the study and data analysis. I drew on Gresalfi and Cobb's (2006) conception of mathematical authority and Weber's (1925/1947) conception of authority in general, to define mathematical authority as a relationship in which at least one individual views another subject (i.e., a person, community, textbook) as a legitimate source of mathematical knowledge or mathematical reasoning and, thus, able to make meaningful mathematical contributions. Additionally, the theoretical understanding of beliefs that informed this study was comprised of three parts: 1) Rokeach's (1968) definition of beliefs, 2) Green's (1971) conceptualization of belief systems, and 3) Leatham's (2006) sensible systems framework. Green's conceptualization of belief systems prominently informed how I developed inferences for, not only specific beliefs each PT held about mathematical authority, but also the relationships between each PT's beliefs about and related to mathematical authority. Using Leatham's sensible systems framework, I developed inferences for how each PT's beliefs formed a belief system that was coherent in the view of the PT.

Varied qualitative research methods afforded the generation of data that enabled me to address my research questions by developing a “rich and holistic account” (Merriam, 1998, p. 41) of each PT’s beliefs about and related to mathematical authority and how student teaching influenced those beliefs. The main data sources for this study were seven interviews with each participant: three interviews prior to student teaching, one interview during student teaching, and three interviews after student teaching. Participant-developed mathematical authority diagrams—developed by each participant both prior to and after student teaching—provided novel opportunities for each participant to explicate their beliefs about the different sources of mathematical authority operating in classrooms and the relationships between those different sources. I used an iterative coding process, in which I used the constant comparative method (Strauss & Corbin, 1998) to modify and refine the coding scheme used to analyze all data. Coding summaries of the resulting codes were then used to further analyze the data and develop narratives regarding each PT’s beliefs about and related to mathematical authority and how student teaching influenced their beliefs.

The results of this study were narratives describing each of the four PT’s beliefs related to and about mathematical authority. In each case, I first reported each PT’s beliefs about mathematics, teaching mathematics, learning mathematics, and students that I inferred were related to their beliefs about mathematical authority. I then described each PT’s beliefs about the teacher as a mathematical authority, students as mathematical authorities, and other sources as mathematical authority. Finally, I described my conjectures for how each PT’s student teaching practicum influenced their beliefs about mathematical authority and highlighted prominent relationships between some of the beliefs held by each PT. Addressing the first research question, I found that each PT believed the teacher is and should be positioned as the main mathematical authority in the classroom, all students can be mathematical authorities in the classroom, and there are several other sources of mathematical authority operating in mathematics classrooms (e.g., the content standards, textbooks, technology). I also found that each PT believed that in order for a student to be viewed as a mathematical authority in the classroom, they needed to develop productive mathematics and communicate that mathematics to the teacher or their peers. Additionally, each PT described instructional practices or structures they enacted during student teaching that positioned students as mathematical authorities in the classroom.

Addressing the second research question, I found that each PT held beliefs about mathematics, teaching mathematics, learning mathematics, and students that were related to their beliefs about mathematical authority. Which of these beliefs seemed to inform or interact with beliefs about mathematical authority differed across the four participants. Moreover, how each PT's beliefs about mathematics, teaching mathematics, learning mathematics, and students related to their beliefs about mathematical authority seemed to explain some of the differences across the four PTs' beliefs about the teacher as a mathematical authority, beliefs about students as mathematical authorities, and descriptions of how they planned to position students as mathematical authorities in the classroom.

Finally, addressing the third research question, I found that student teaching, with one exception, seemed to reinforce the beliefs about mathematical authority each PT held during the summer prior to their student teaching practicum. This finding contrasts with Vacc and Bright's (1999) finding that elementary PTs' beliefs about teaching and learning mathematics changed "significantly during student teaching" (p. 108). Moreover, for me, this result was surprising, as I expected student teaching to influence more subtle shifts across the four PTs' beliefs about mathematical authority. In other words, I did not expect to find *significant* shifts in each PT's beliefs about mathematical authority, rather I expected to find more instances of subtle changes to the PTs' beliefs about mathematical authority, as was the case with Grace.

In the subsequent sections, I discuss the limitations of this study then discuss implications of the results of this study.

## Limitations

The primary purpose of this study was to investigate prospective secondary mathematics teachers' beliefs about and related to mathematical authority. However, due to COVID-19 protocols in schools at the time of this study, I was unable to observe the four PTs' classroom instruction during their student teaching semester and, consequently, I was unable to develop inferences for how their beliefs related to their instructional practice. Although I used varied qualitative methods to infer each PT's beliefs about and related to mathematical authority, without observing their classroom instruction I can only hypothesize how their beliefs informed their instructional practice during student teaching. Moreover, observing each PT's instructional practice may have provided additional insights into their beliefs. For instance, observing PTs' instruction could afford discussions during interviews about specific instances from their instructional practice. These discussions could provide further evidence for the beliefs a PT holds or reveal aspects of their beliefs about or related to mathematical authority to explore further in subsequent interviews.

A secondary purpose of this study was to investigate how student teaching influences PTs' beliefs about mathematical authority. Interviewing each PT three times during the summer prior to their student teaching semester enabled me to develop inferences for what each PT believed right before they began student teaching. However, I did not investigate how the beliefs each PT held prior to student teaching were formed. For the purposes of this study, I did not investigate what each PT believed before starting their teacher education programs, how their teacher education program influenced their beliefs, or what they viewed as the purpose of student teaching. Investigating these aspects of PTs' beliefs could provide further insights into PTs' beliefs about and related to mathematical authority as well as how student teaching influences their beliefs. For instance, understanding what PTs view as the purpose of student teaching could explain, at least in part, how student teaching influences their beliefs. If, for example, a PT views their student teaching practicum as the context in which they attempt to put into practice what they have reflected on and learned during their prior coursework, then it may be reasonable to expect student teaching may not inform substantive changes to that PT's beliefs.

Due to only interviewing mentor teachers once, I was unable to develop inferences about the mentor teachers' beliefs or the influence each mentor teacher had on their student teacher's instruction or beliefs. Further understanding mentor teachers' beliefs and perceptions of their student teachers' instruction could provide further insights into how student teaching influences PTs' beliefs and/or instruction. For instance, mentor teachers likely view PTs' instruction through a different lens when compared to how a PT views their own instruction. To understand how mentor teachers view their relationship with student teachers as well as their student teachers' instruction, multiple interviews that seek to understand mentor teachers' beliefs about mathematics, teaching mathematics, learning mathematics, and the nature of the mentor teacher-student teacher relationship are likely needed. Similarly, interviewing PTs' student teaching supervisors could provide novel insights into PTs' beliefs and instruction during student teaching.

Another limitation of this study was that each of the participants engaged in the same teacher education program and were members of the same cohort of prospective secondary mathematics teachers. It is likely the case that PTs in other teacher education programs have different experiences—both in terms of coursework and early field experiences—than the PTs discussed in this dissertation. Hence, PTs that go through other teacher education programs may hold beliefs about and related to mathematical authority that vary from the beliefs the PTs in this study held. Moreover, PTs from other teacher education programs may have very different student teaching experiences when compared to the participants of this study. Consequently, student teaching may influence other PTs' beliefs differently when compared to the PTs who participated in this study. Also, PTs that go through the same teacher education program but are members of a different cohort are likely to have different experiences as they engage in their coursework, early field experiences, and possibly student teaching. For instance, PTs from another cohort may have different instructors in their teacher education courses and are likely to have different mentor teachers and student teacher supervisors. It could be the case that these differences could lead to PTs from other cohorts holding beliefs that vary from the PTs discussed in this dissertation or that student teaching leads to more substantive shifts to the beliefs held by PTs from other cohorts.

In the remaining sections I discuss implications for teachers, mathematics teacher educator, and future research. As I discuss implications for future research, I also address some of the limitations of this study I described in the preceding paragraphs.

## Implications for Teachers

All four of the PTs discussed in this dissertation believed that the teacher should be and is positioned by students as the main mathematical authority in the classroom. As I reflected on this result and, more generally, teacher-student relationships in the mathematics classroom, I concluded that the teacher's position as the main mathematical authority in the classroom is unavoidable. Prior studies that have investigated how authority operates in mathematics classrooms from the teacher's perspective have largely studied how teachers position their students as mathematical authorities in the classroom (e.g., Depaepe et al., 2012; Dunleavy, 2015; Hamm & Perry, 2002; Kinser-Traut & Turner, 2020), and they have argued that teachers need to consistently position their students as mathematical authorities in the classroom. While I firmly agree it is imperative for teachers to consistently and productively position their students as mathematical authorities in the classroom, reflecting on the results of this study I argue it is also important for teachers to acknowledge they are positioned as the main mathematical authority in the classroom by their students and to reflect upon the implications of that positioning. As the main mathematical authority, teachers have profound influence in their classrooms and can undoubtedly leverage their position to positively influence students' learning and development of productive dispositions towards mathematics. I contend that teachers acknowledging and reflecting upon their position as the main mathematical authority in the classroom can be a powerful first step towards teachers considering how to productively leverage their position to support their students' development of productive dispositions and learning of mathematics. As teachers reflect on their position as the main mathematical authority in the classroom, they can, for example, consider how they can affirm students' mathematics or provide students feedback in the classroom, play a leading role in the co-construction of productive sociomathematical norms (Yackel & Cobb, 1996) in the classroom, or steer students away from developing unproductive mathematical conceptions and towards more productive mathematical conceptions. Teachers reflecting on their position as the main mathematical authority can also include reflecting on how they can leverage their position to productively position their students as mathematical authorities.

Each of the four PTs believed that all students can be positioned as mathematical authorities in the classroom. Moreover, the four PTs described instructional strategies they used or instances in which they leveraged their position as the main mathematical authority in the classroom to then position students as mathematical authorities. Often these instances were complex because the teacher was being positioned as a mathematical authority by the students and, simultaneously, the teacher was positioning students as mathematical authorities. For instance, Hannah and Grace described asking advancing questions as moments where students were positioning them as a mathematical authority because students knew the teacher's question would help them make progress on the problem or task at hand, yet Grace and Hannah claimed they were positioning their students as mathematical authorities who, as they considered the question asked, could make progress or generate ideas on their own. While advancing questions are just one strategy, there are numerous ways teachers can subtly leverage their position in the classroom to productively position their students as mathematical authorities. Teachers, then, can acknowledge that many instances in which they attempt to position students as mathematical authorities are complex because they, as the teacher, are leveraging their position as a mathematical authority in the classroom, while concurrently positioning their students as mathematical authorities. Moreover, teachers can reflect on their instruction and consider varied instructional strategies that enable them to subtly leverage their position as the main mathematical authority to productively position their students as legitimate sources of mathematical knowledge and reasoning. Such instructional strategies teachers may consider—and one's the participants of this study described—include using teacher talk moves while facilitating whole class discussions (Smith & Sherin, 2019) and choosing students to present their idea or solutions to the whole class.

Two of the participating PTs—Chris and Simon—also believed technology and manipulatives (in the case of Simon) can be mathematical authorities in the classroom. Chris and Simon both suggested that technology and manipulatives provide taken-as-true representations of mathematical relationships or concepts. In both cases, technology and manipulatives did not seem to be considered as tools students used to reason with mathematics across different representations (e.g., graphical, algebraic, contextual) but were tools to which students could turn to observe specific mathematical relationships or verify their solution or answer was correct. An implication for teachers is, if they believe technology and/or manipulatives can be legitimate sources of mathematical knowledge in the classroom, then they need to consider how they can encourage students to engage with those legitimate sources of mathematical knowledge (i.e., technology or manipulatives). In other words, teachers can reflect on what they communicate to students about technology and whether they communicate to students that they should reason with the mathematics demonstrated by technology or manipulatives and reason with the mathematics represented across varied representations. That is, it may be productive for teachers to encourage students to position technology and manipulatives as mathematical authorities that can be used to reasoned with mathematical concepts or ideas across multiple representations, rather than viewing technology and manipulatives as tools that produce true instantiations of mathematics or that can be used to simply validate one's answer to a problem.

## Implications For Mathematics Teacher Educators

The results of this study have numerous implications for MTEs and mathematics teacher education programs. Some of these implications relate directly to the implications for teachers I described in the previous section. For instance, although teachers are assuredly capable of reflecting on their own regarding how they can leverage their position as the main mathematical authority in the classroom, MTEs can engage prospective and practicing teachers in such reflections and support teachers as they engage in these reflections. Encouraging both practicing and prospective teachers to reflect on their instruction using a mathematical authority lens can provide teachers a novel perspective through which they can reflect on their instructional practice. For instance, if working with prospective teachers, MTEs could ask PTs if there are instances in which a mathematics teacher is not viewed as a mathematical authority in the classroom and highlight that the teacher is always positioned as the main mathematical authority in the classroom with rare exceptions (e.g., demonstrating incorrect mathematics). MTEs could then facilitate PTs' discussions about productive and less productive ways teachers can leverage their position as the main mathematical authority in the classroom. As MTEs facilitate these discussions, they could have PTs consider not only how teachers leverage their position as the main mathematical authority to support student learning, MTEs could also have PTs consider how their feedback and affirmation can profoundly influence students' affect and dispositions towards mathematics. Such discussions could lead to prospective teachers critically reflecting on how they will operate as the main mathematical authority in their future classrooms in ways that will best support their students' learning of productive mathematics and development of productive dispositions towards mathematics.

Each of the four PTs discussed in this dissertation believed that the content standards along with textbooks and other curricular resources are mathematical authorities because they are legitimate sources of pedagogical content knowledge (Shulman, 1986). An implication for MTEs then, is to elicit and understand teachers' beliefs about content standards and curricular resources as mathematical authorities. Assuming other teachers hold similar beliefs as the four PTs in this dissertation, MTEs should engage teachers in critical reflection of the mathematics described in the standards as well as the mathematics that students can learn or develop as they engage in problems or tasks provided by curricular resources. When it comes to the content standards, MTEs could have PTs describe what students would understand and be able to do if they were to meet a particular set of standards. An MTE could then facilitate discussions in which PTs compare their responses to their peers and consider what mathematical conceptions may be most productive for their students to learn. Moreover, MTEs could have PTs discuss how they might critically examine the mathematics outlined in the standards when they become full-time teachers. Related to the textbook and other curricular resources, MTEs can engage PTs in task analyses in which PTs reflect on the conceptual or procedural understanding(s) tasks or problems promote and consider how, or if, they would modify the tasks/problems to foster students' productive understandings of mathematics. To promote PTs' continued reflection, and productive shifts in their beliefs, MTEs may need to engage PTs in critical reflection of content standards and task analyses throughout their teacher education program, including—and maybe most especially—during their student teaching practicum, in which PTs develop their own, personalized pedagogy and practically consider how to modify their instruction to meet the learning needs of the students in their classroom.

Across the four PTs in this study, all believed that as the main mathematical authority, the teacher is to 1) support students' learning of productive mathematics and 2) certify when students learn mathematics. This finding suggests that a teacher's beliefs about learning mathematics will prominently influence how a teacher leverages their position as the main mathematical authority in the classroom. For instance, if a teacher believes that learning mathematics entails memorizing definitions and enacting prescribed procedures with accuracy, one can imagine that the teacher would not support students' conceptual understanding and would leverage their position as the main mathematical authority to verify that a student learned mathematics only when the student was able to enact the procedure(s) at hand with no mistakes. Thus, an implication for MTEs and teacher education programs is that they should elicit and understand PTs' beliefs about learning mathematics and develop activities that promote productive beliefs about what it means to learn mathematics. As an example, MTEs could promote PTs' beliefs that "mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse" (NCTM, 2014, p. 11). Engaging PTs in numerous discussions about what it means to learn mathematics and how to effectively assess when students have learned mathematics, may be one productive way MTEs can elicit and understand PTs' beliefs about learning mathematics and even foster PTs' critical reflection of their beliefs about learning mathematics.

If a goal of mathematics teacher education is for teachers to *strongly* believe that all students can be positioned as mathematical authorities in the classroom, just promoting the belief that all students can be mathematical authorities may not be sufficient. All four PTs discussed in this dissertation believed that all students can be mathematical authorities in the classroom, but Hannah and Grace were the only two that held this belief strongly. Moreover, in both Hannah's and Grace's cases, their strongly held beliefs that all students have unique backgrounds, experiences, and perspectives they bring into the classroom directly informed their strong belief that all students can be mathematical authorities in the classroom. Hence, an implication of this relationship in Grace's and Hannah's cases for MTEs is, if teachers are to strongly believe that all students can be legitimate sources of mathematical knowledge and reasoning, then MTEs may need to engage teachers in critical reflection on their beliefs about students in general. Specifically, MTEs may need to consider how they can engage teachers in activities, assignments, and/or discussions that foster an asset-based view of all students (Celedón-Pattichis et al., 2018). Moreover, the cases of Hannah and Grace suggest that it may be productive for MTEs to foster an asset-based view of all students, in general terms—that is, not just in the context of mathematics classrooms—then discuss the implications of having an asset-view of students for the mathematics classroom. In sum, if teachers are to strongly believe that all students can be mathematical authorities in the classroom, teachers may first need to consider students as whole individuals with unique and valued backgrounds, perspectives, and experiences, then consider how such a belief about students translates to the context of the mathematics classroom.

Wagner & Herbel-Eisenmann (2014b) suggested that authority diagrams can be tools that meaningfully support teachers' reflections on how authority operates in their classroom. I echo their suggestion because the mathematical authority diagrams each participant of this study developed seemed to provide a novel medium for them to not only describe the varied sources of mathematical authority they believe operate in the mathematics classroom but also the relationships between those varied sources. Having the four PTs develop mathematical authority diagrams revealed they thought about sources of mathematical authority operating in mathematics classrooms that I did not consider prior to this study (e.g., manipulatives, home/work life, tutors). Thus, if MTEs want mathematics teachers to reflect on the varied sources of mathematical authority that may be operating in their classrooms and how those sources relate to other sources—particularly the students and teacher in the classroom—then having teachers develop mathematical authority diagrams may be a powerful tool that foregrounds *teachers'* perspectives of mathematical authority. Moreover, and as was the case with Grace, having PTs develop mathematical authority diagrams at multiple points in their teacher education program may provide MTEs novel insights into how PTs' views or beliefs about mathematical authority shift as they progress through their programs. Having practicing teachers develop mathematical authority diagrams at multiple points during continued, authority-focused professional development may provide similar insights.

Finally, one of the main and surprising findings on this study was that student teaching, for the most part, did not influence changes to the four PTs' beliefs about mathematical authority but reinforced their beliefs. The only subtle shift across the four PTs' beliefs about mathematical authority occurred in Grace's beliefs about students viewing their peers as mathematical authorities in the classroom. Researchers have shown that secondary mathematics PTs' university coursework and field experiences (not including student teaching) can promote productive changes to PTs' beliefs about teaching (e.g., Conner et al., 2011). I argue the results of this prior research on secondary mathematics PTs' beliefs and the results of this study suggest that, not only can MTEs promote productive shifts to PTs' beliefs as they engage in their teacher education programs, but if MTEs are to foster productive shifts to PTs' beliefs, they may need to do so before PTs begin their student teaching practicum. I argue the implications of these results for MTEs are twofold. First, if MTEs are to foster productive changes to PTs' beliefs, then MTEs need to elicit and begin to understand PTs' beliefs as they begin their teacher education programs. With such understandings, MTEs can design and implement activities and assignments in methods courses, content courses, and corresponding field experiences that problematize the beliefs PTs hold as they enter their teacher education programs and support PTs as they reflect on their beliefs. Secondly, if student teaching reinforces secondary mathematics

PTs' beliefs, then it is imperative for MTEs to understand what PTs believe as they enter their student teaching semester. MTEs understanding what PTs believe as they enter their student teaching semester will enable MTEs to support student teachers and suggest modifications to student teachers' pedagogy that align with the student teachers' beliefs. Hence, if MTEs understand the beliefs PTs hold as they enter their student teaching semester, they may be able to promote productive shifts in PTs' instructional practice and, simultaneously, support PTs as they develop their own, personalized pedagogy during student teaching that is consistent with their beliefs.

## Implications For Future Research

To better understand PTs' beliefs about mathematical authority as well as the instructional practices PTs implement in their classrooms to position student as mathematical authorities, research that investigates the relationship between PTs' beliefs about mathematical authority and their instructional practice is needed. Conner et al. (2011) claimed that the beliefs PTs hold provide "a backdrop for their eventual practice" (p. 485). Studies that investigate PTs' beliefs about mathematical authority and how those beliefs relate to their practice will provide MTEs more clear and detailed descriptions of the instructional practices novice teachers are likely to enact in their classrooms to position students as mathematical authorities and—by understanding their beliefs that inform such practices—robust explanations for their instructional practices. Furthermore, all four participants of this study described instructional practices they implemented to position students as mathematical authorities that were consistent with their beliefs about what students needed to do to be viewed as a mathematical authority in the classroom. Thus, research that investigates how teachers' beliefs about mathematical authority relate to their instructional practice may want to focus on how teachers' beliefs about the legitimacy of students as mathematical authorities informs how teachers plan for and enact instruction that positions students as mathematical authorities in the classroom.

All the participants of this study were prospective secondary mathematics teachers. The beliefs about and related to mathematical authority that the four PTs discussed in this dissertation held may be very different from the beliefs held by practicing teachers. For instance, some of the authority diagrams developed by practicing teachers reported by Wagner and Herbel-Eisenmann (2014b) are drastically different when compared to the diagrams developed by the participants of this study. Thus, future research should investigate practicing teachers' beliefs about mathematical authority and how those beliefs relate to their practice. Research that recruits teachers with varying levels of experience, teaching different math courses, and teaching in different school contexts (e.g., rural, urban, Title I) will provide needed insights into the beliefs about mathematical authority held by teachers at various career stages and in varied school contexts. Moreover, this research would further the field of mathematics education's limited understanding of the instructional practices teachers implement to position students as mathematical authorities. Such understanding would be foundational for any MTE or researcher who aims to support practicing teachers' continued professional development while using a mathematical authority lens (Wagner & Herbel-Eisenmann, 2014b).

Wagner and Herbel-Eisenmann (2014b) argued that "understanding how teachers think about authority must be the basis of teacher educators' work with teachers on issues of authority" (p. 203). Thus, as researchers better understand practicing teachers' beliefs about mathematical authority and how those beliefs relate to their instructional practice, researchers can then leverage these understandings to conduct longitudinal, mathematical-authority-focused studies with practicing teachers. As an example, researchers might conduct studies in which they engage teachers in multi-year professional learning communities with a focus on mathematical authority in classrooms. Such studies would provide the field of mathematics education novel and valuable insights into not only instructional strategies that teachers plan and enact to position students as mathematical authorities in the classroom but also how practicing teachers' beliefs about and related to mathematical authority shift as they engage in these continued professional learning communities.

As I mentioned in the previous section, one of the surprising results of this study was that student teaching, with one exception, did not influence changes to the four PTs' beliefs about mathematical authority. However, the four PTs discussed in this dissertation went through the same secondary mathematics teacher education program and all four were part of the same cohort of prospective teachers. One might question if the beliefs held by prospective secondary mathematics teachers from a different cohort or from a different teacher education program are more likely to shift as PTs engage in their student teaching practicum. Thus, further research is needed that investigates how student teaching influences prospective secondary mathematics teachers' beliefs. Studies that investigate the influence of the mentor teacher-student teacher relationship, the student teaching supervisor-student teacher relationship, and how PTs reflect on their practice during student teaching would provide novel and needed insights into how PTs engage during their student teaching practicum as well as what aspects of their student teaching practicum are most impactful. Such insights could inform how MTEs structure PTs' student teaching practicum, including the targeted supports they provide PTs during student teaching and how they encourage or prompt PTs to reflect on their planning and instruction during student teaching.

Related to one of the implications I discussed for MTEs, research that investigates the beliefs prospective secondary mathematics teachers hold as they begin their teacher education programs is needed. Calderhead and Robson (1991) investigated beliefs about teaching and learning mathematics held by elementary PTs as they began their teacher education programs and found these beliefs "can influence what they [PTs] find relevant and useful in the course, and how they analyse their own and others' practice" (p. 7). Yet, studies that investigate secondary mathematics PTs' beliefs as they enter their teacher education programs are scant. If student teaching largely reinforces secondary mathematics PTs' beliefs, then MTEs only have two or three semesters (depending on their teacher education program) to foster productive shifts in the beliefs PTs hold. Studies that investigate the beliefs secondary mathematics PTs hold as they enter their teacher education programs would provide MTEs baseline understandings of the beliefs the PTs in their classrooms may hold, which could inform how they engage PTs in their courses and corresponding field experiences. Thus, such studies may enable MTEs to promote more productive shifts to PTs' beliefs as they engage in their teacher education coursework and field experiences prior to student teaching.

Finally, I close by echoing a methodological implication for future beliefs research: namely, that teacher-generated diagrams are powerful methodological tools that can "open up understanding of how educators think about authority" (Wagner & Herbel-Eisenmann, 2014b, p. 223). In this study, the mathematical authority diagrams generated by the four PTs provided each of them a medium in which they were able to explicate how *they* were thinking about mathematical authority in classrooms as well as the relationships between different sources of mathematical authority. Rather than solely responding to my interview questions, having the PTs generate mathematical authority diagrams enabled them to express their views and beliefs about how mathematical authority operates in classrooms without having to consider how they would respond to my questions. Hence, having each PT develop mathematical authority diagrams elicited responses that are not as easy to elicit using other methods (e.g., semi-structured interviews). Additionally, having each PT develop mathematical authority diagrams prior to and after student teaching, and having them compare their two diagrams, generated responses that either confirmed or disconfirmed some of my inferences for how student teaching influenced their beliefs. In the case of Grace, comparing her two diagrams and how she represented students differently in the two diagrams afforded discussions that strengthened my inference for how student teaching informed changes to her beliefs about students as mathematical authority. Thus, researchers who plan to investigate changes to either practicing or prospective teachers' beliefs over time, should consider how they can include teacher-generated diagrams as part of their methods because doing so may yield invaluable insights into how the teachers' beliefs shift, or do not shift, and the reasons for those shifts or lack of shifts.

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# APPENDIX A: INITIAL INTERVIEW PROTOCOL

***Interview Script (Italics are interviewer notes and/or potential follow-up questions)*:**

Welcome [participant name]. Again, thank you for agreeing to take part in this study and for taking time to do this interview. Over the next hour I will ask you questions about mathematics and aspects of the mathematics classroom. Know that I am not looking for "right," "good," or "on-track" responses. Rather, I want to better understand how you are thinking about mathematics and the mathematics classroom. If you feel that a question needs to be clarified, please do not hesitate to ask me to do so. I will be recording this interview using the recording features of Zoom as well as with an audio recorder. Before we get started, do you have any questions or concerns?

***Introductory Questions***

- Out of the various degree fields, why did you choose mathematics education as your major?
    - *Don't focus so much on 'mathematics' here.*
    - *Instead, follow-up on why they chose teaching versus other professions.*

***Elicit Beliefs Related to Mathematics***

- What about mathematics made you want to teach mathematics instead of another subject like Science, English, or History/Social Studies?
    - *If they talk about what they like about mathematics, in contrast to the other subjects, take that up. Revoice elements of math they said they like, then ask if there are other aspects about mathematics that they like.*

- What do you dislike about mathematics?

- Are there important characteristics or traits that one needs when it comes to doing mathematics?
    - *If they talk about following rules or procedures, then ask where the rules and/or procedures come from.*
    - *If they talk about being creative and open minded (or the like) then ask if everyone can engage in mathematics in those ways (i.e., creatively) or just certain individuals.*

***Elicit Beliefs About Learning Mathematics***

- From your experiences, what are some ways you think students/individuals learn mathematics best?
    - How can you tell whether a student is learning?
- Is it possible for students to get the wrong answer *and* have a robust understanding of the content? Why or why not?
    - What about the other way around: Is it possible for a student to get the right answer to a mathematics problem and still not understand the problem or the mathematics involved? Why or why not?

***Elicit Beliefs About Teaching Mathematics***

- I am going to put a list of mathematics courses currently offered in Georgia high schools in the Zoom chat. Out of the courses listed which one would you most look forward to teaching? Why?
    - Which course would you least look forward to teaching? Why?

- *Courses that I will list: AP Stats, AP Calc, Algebra I, Geometry, Algebra II, Pre-Calculus, Mathematics of Finance*

- As a prospective classroom teacher, in what ways do you anticipate having an impact on students' learning of mathematics?
- What does it take to be a good math teacher?
    - *Might talk about characteristics, if so, ask the next follow-up questions.*
    - If we were to walk into a great math teacher's classroom, what instructional practices would we see?
    - How would you describe the role of a teacher in a math class?
        - *Ask them to expand and be specific. For instance, if they describe the role as a 'facilitator' ask them to say more about what it means for the teacher to be a facilitator in the classroom. What are they facilitating?*

***Elicit Beliefs About Students as Mathematical Authorities***

- What role do students play, or should play in a mathematics classroom?
    - *Similar as above, ask follow-up questions that encourage them to be specific.*
- How do students know how to solve a problem in mathematics?
- How do students know something is right in mathematics?
    - *If they describe students' reasoning/sense-making but also external sources such as the calculator, textbook, teacher, etc., ask them if there are situations or circumstances in which students rely on different "sources."*
    - Related to these last two questions, what or who do you believe students view as legitimate sources of mathematical knowledge or reasoning?

# APPENDIX B: GRACE'S INTERVIEW PROTOCOLS

Below are the interview protocols used for Grace's second through seventh interviews. For all interview protocols, italicized text indicates notes for the interviewer and/or potential follow-up questions. Bold text indicates an aspect of the interview protocol that was common for all participants.

## Grace's Second Interview Protocol

**Welcome [*Participant Name*]. Thank you for taking time out of your busy schedule to do this interview. Over the next 90 minutes I will ask you questions about mathematics and aspects of the mathematics classroom. Specifically, during this interview we will discuss classroom interactions and authority in the mathematics classroom. We will also look at some hypothetical classroom scenarios and talk about what you would do as the teacher. As with the first interview, I am not looking for "right", "good", or "on-track" responses. If you feel that a question needs to be clarified, please do not hesitate to ask me to do so. I will be recording this interview using the recording features of Zoom as well as with an audio recorder.**

**Before we get started, do you have any questions or concerns?**

Part 1 of Interview – Follow-Up to First Interview

- **We're going to start with some questions similar to the last interview and some that will follow-up to your responses from the first interview.**

- **What was a high-point in your mathematics experience? Think about a specific experience of event and include what happened as well as who was involved?**
  - ***Why does this experience stick-out compared to other experiences with mathematics?***
  - ***What role did the other individual(s)/object(s) play in this moment?***
  - ***If they say they were by themselves, then ask if there were resources or individuals that played an instrumental role leading up to this moment.***

Related to Their Beliefs About Mathematics and Mathematical Authority

- **How do you know what is right and what to do in mathematics?**
  - ***If they mention teachers as either telling them what to do or providing feedback that they were right, ask if that's the case for all students, in general?***
  - ***If they mention their own reasoning/ability to determine what to do or if something is correct, then ask how they developed those abilities/skills and if students, generally speaking, can develop those same skills/abilities.***
- In the last interview you mentioned that you like math because you can "be given some mess of something and be able to take it apart and create something." You also said that a lot of procedures have been passed down over time from those in the past who were able to problem solve "and figure out how you're supposed to do it." How do these two ideas relate or play out when someone is problem solving or doing mathematics?

- *If she brings up the idea of recognizing certain concepts in the problem and then choosing a procedure based off the concepts, then ask her to elaborate on what she means by "concept."*
  - *Are the procedures you choose to enact ones that have been previously discovered and passed down over time?*
- In the last interview you brought up an example of forgetting a formula on one of the AP Calc tests but being able to develop the formula in the moment. You went on to say that giving "students the skills to get back to the procedures can be really helpful." Can you talk more about what you meant by this? Specifically, when you talk about students having skills to then develop procedures, what do you mean by skills?
  - *Would you say that students need certain levels of skills, knowledge, or ability to develop procedures?*

Related to Their Beliefs About Teaching Mathematics, Learning Mathematics, and Mathematical Authority

- When I asked how students learn best you said through "conceptual, discovery activities where you can… invent the mathematics on your own with a framework" and then practicing the procedures once students understand the underlying concepts. When you said "invent the mathematics on your own with a framework," what did you mean by the word "framework" there?
  - *Does the teacher provide the framework for students and then students engage in the problem and "invent the mathematics"?*

        - *Can students provide/develop/suggest the framework?*
    - *Through engaging in these activities, you said students "will be able to feel like they've found it themselves." Can you expand on what you meant by that?*
- You brought up assessment multiple times as an important thing teachers do in the math classroom. Why is it important for teacher to assess students frequently throughout a lesson or over a sequence of lessons?
    - *How does the teacher determine the direction students should go?*
    - *What should the teacher do if, through these assessments, they think students aren't going in the direction they should be going?*
        - *So, if students are not on-track, then it's the teacher's role to steer them back on the right path?*
- Related to what we just talked about, what sources do you turn to when considering what should or needs to be taught in the classroom?
    - *Why those sources and not others?*
        - *Would you say, from your perspective as a teacher, that those sources are credible in terms of knowing what math students should learn or be exposed to?*
        - *From your perspective then, would you consider more experienced math teachers as legitimate sources of math knowledge?*
- **When I say the teacher is an authority in the classroom, what comes to mind?**
    - ***They may talk about the teacher as a "behavioral" authority in the classroom. If so, then quickly transition to the next question.***

- **What about an authority in the classroom when it comes to mathematics. Put differently, what comes to mind when I say that the teacher is a mathematical authority in the classroom?**
    - ***Can you provide an example or examples where the teacher is a mathematical authority?***
        - ***If they bring up the notion of being an intermediary between the mathematical community/discipline of mathematics and students, ask them to say more about this. What is the relationship between the teacher and the mathematical community/discipline and how does that influence their relationship with students (when it comes to the mathematics)?***
- **What if I were to say that students or a student is a mathematical authority in the classroom, what come to mind?**
- **When you think about the role of the teacher, are there moments where the teacher is not a mathematical authority?**
    - ***Push them to be specific. For instance, if they say the teacher is not a mathematical authority when students are working in groups, or using a manipulative/technology, ask them why that is the case and if the teacher is not a mathematical authority in that moment, is someone or something else a mathematical authority?***
- **Are there any other people or objects that you think can be or are a source of math knowledge or reasoning in the classroom?**

Related to Their Beliefs About Objects as Mathematical Authorities

- At the end of the last interview, you brough up the internet as something that students turn to as a legitimate source of mathematical knowledge or reasoning. You specifically mentioned "Khan Academy" or websites ending in with ".edu." Why do you consider those sources to be legitimate?
    - *Are there other sources on the internet you would include?*
    - *Are there internet sources you would not include?*
    - *When you say the internet as a source, it sounds like you are saying "people who have a website or who post on the internet." Are those individuals the sources you have in mind?*
        - *What makes those individuals or sites legitimate?*
        - *What makes other sources not as legitimate?*
        - *Do these same criteria hold for a textbook? Or is it different?*

Related to Their Beliefs About Students and Mathematical Authority

- Multiple times you brought up the importance of students sharing their work and ideas in the class. Can you provide some examples of what that looks or sounds like in the math classroom? And why is students sharing their work in the math classroom important to you?
    - *You said that when student are presenting it's important to not say something is correct or incorrect right away. Why is that the case?*
        - *How does a student know that what is presented is correct?*

        - *If multiple strategies are presented, what does a teacher then do with those multiple strategies?*
            - *How do students know which one is right, or best, or the most efficient?*
            - *What if some are incorrect? How is that determination made?*

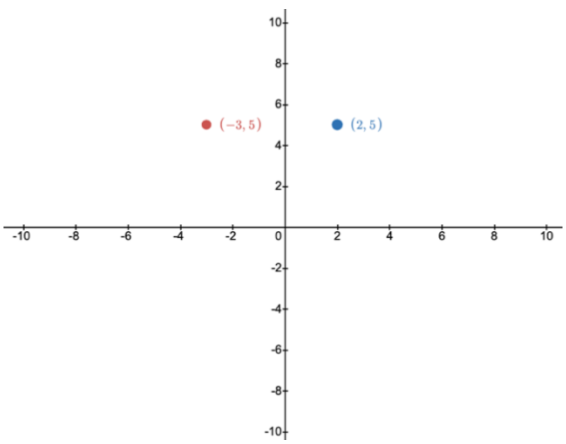
- **When students get stuck on a problem or don't know how to start, to whom or what do students turn to as sources of mathematical knowledge or reasoning?**
    - ***Do you think students should turn to a peer for this help?***
        - ***If so, does it depend on the peer or no?***
        - ***If it does, then when would a student know they can turn to a peer as a legitimate source of math knowledge or reasoning?***
        - ***If she brings up parents again – Can a student turn to their parent all the time?***

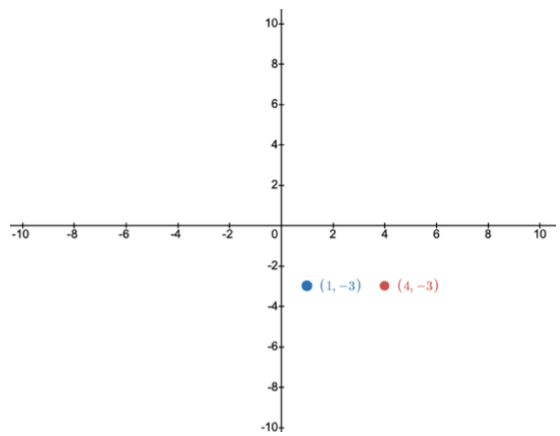
Part 2 of Interview – Classroom Scenarios

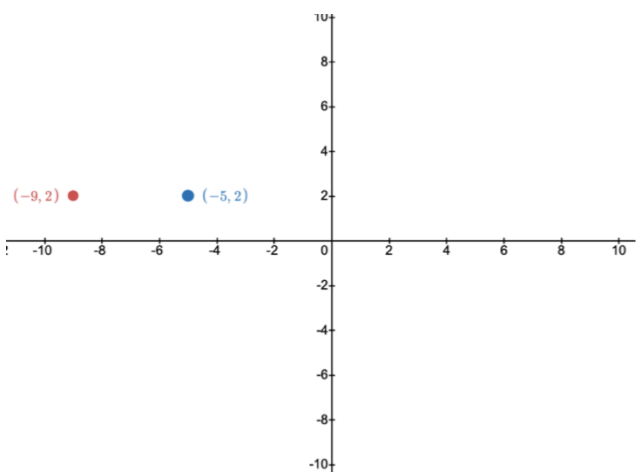
- **Now we're going to look at 2 classroom scenarios and discuss what you would do as the teacher. Each scenario has some background information and then a short transcript. After you consider the math and we read through the transcript, I will ask you a question for you to consider what you would do as the teacher. I will share my screen with the scenario so that we can both see it.**
- **Scenario #1**

  ***Background*: This is an on-level sixth-grade classroom. Students have not yet received instruction regarding how to add or subtract integers. Also, there is a**

**number line visible to the students in the classroom, that does from -15 to 15. The goal of the lesson is for students to be able to determine the horizontal distance between two points when given their coordinate pairs. Students have just placed points on the coordinate plane with a horizontal distance of 5 units, 3 units, and 4 units respectively and are discussing the points as a *whole class*. The points (see below) are recorded on the front board.**




$$(-3,5)\ \&\ (2,5)$$

$$(4,-3)\ \&\ (1,-3)$$

$$(-9,2)\ \&\ (-5,2)$$

---

**T: Alright, what if we couldn't see the coordinate plane, is there a way to determine the distance between these points if we just had this list of coordinate pairs?**

Jose: Yes, I think we can just look at the larger number and count backwards to the smaller number.

- **Keeping in mind this is a whole-class discussion, how would you respond?**
    - ***Ask specifics: Who is talking? Who is explaining? If a question is asked, is it being asked to a particular student or all students?***

Zack: [Only student in the room with his hand raised]

**T: Okay, good, let's go on to the… Oh wait, Zack do you want to add something to what we just said?**

Zack: Kind of, but not really. I was just going to say that you can also just take the absolute value of the two first numbers and add them together.

**T: You mean, take the absolute value of the $x$ −coordinates and add them together?**

Zack: Yeah.

- **How would you respond, or what would you do next as the teacher?**
    - ***Ask specifics: Who is talking? Who is explaining? If a question is asked, is it being asked to a particular student or all students***
- **Would you say that your response to Zack (differed/was the same) when compared to your response to Jose? Why was that?**

- **Scenario #2**

***Background:*** This is an Honors Pre-Calculus class with a mix of Juniors and Seniors. Students have begun investigating transformations of functions and have developed general rules for various transformations (e.g. $f(x) + a$ : shift the original function $f(x)$ up $a$ units, assuming $a$ is positive). They also have gone over the definitions for Domain and Range.

- **Do you have a mental picture of what this classroom looks like?**
- **What resources do students have available to them?**

---

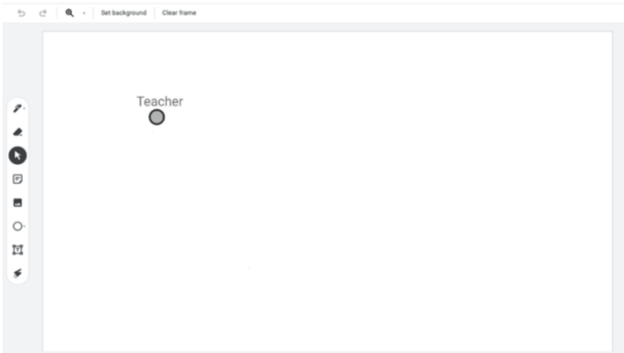
**T: With your partner, take 4 minutes and sketch the graph of $f(x) = \sqrt{-x}$ and determine the function's domain and range."**

Marcos: [Right after you finish talking, calls out] Wait! That's impossible! You can't take the square root of a negative number!

- **What would you do next as the teacher?**
    - ***Ask specifics: Who is talking? Who is explaining? If a question is asked, is it being asked to a particular student or all students? What is the anticipated outcome of the instructional move/strategy?***
    - ***What role does the teacher play if the teacher asks students to discuss Marcos's comments?***

Part 3 of Interview – Authority Diagrams

- I will have a Jamboard created with a dot labeled "Teacher" already on the board (see below).




- **Now, you are going to develop a diagram of how you think authority operates in the mathematics classroom. I have already represented the teacher with a black dot, so you do not need to think about how you will represent yourself. However, you can move the dot wherever you would like. Feel free to use any of the features of Jamboard, such as lines, arrows, words, sticky notes, images, whatever you want to represent how you think authority operates. Also, as you are developing your diagram, feel free to describe your diagram and your thinking behind it. It may be easiest to use your tablet or phone to draw the diagram. Also, I will continue ot share my screen and keep Jamboard on the screen as we discuss your diagram. Before we get started, do you have any questions about the diagram.**
    - ***Pay attention to all sources of authority and any arrows, lines, etc. they might use to represent relationships between different sources***
    - ***If they do not include a source or aspect of authority they mentioned previously, then I will ask them about this.***
    - *In the first interview you mentioned that students might turn to the textbook, but I'm not seeing that in the diagram. Do you view that as a source of authority in the mathematics classroom? Why or why not?*

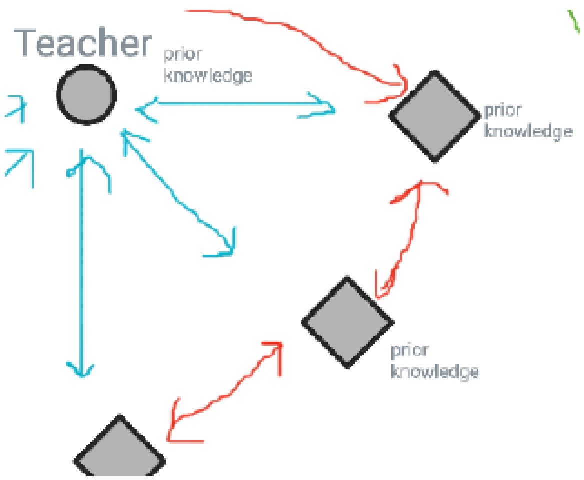
## Grace's Third Interview Protocol

**Welcome [*Participant Name*]. Thank you for taking time out of your busy schedule to do this interview. Over the next hour we will talk about the statements you highlighted and engage in a statement sort activity. As with the previous two interviews, please feel free to discuss your ideas, conceptions, and beliefs openly, as I am not looking for "right", "good", or "on-track" responses. If you feel that a question needs to be clarified, please do not hesitate to ask me to do so. I will be recording this interview using the recording features of Zoom as well as with an audio recorder.**

**Before we get started, do you have any questions or concerns?**

Discussion of The Statements They Highlighted

- **Can you describe why you considered these highlighted statements as important aspects of your beliefs?**
  - ***As you were reading through the statements, was there anything you were looking for to include as an important aspect of your beliefs?***
- While developing your diagram you said, in statement 37, "because you can't really separate a person from their own educational experiences. So those stick with them and those affect how they're seen as authority in a class or not," and you put "prior knowledge" on the teacher and the students. Can you say more about what you meant by this statement and/or what the "prior knowledge" in your diagram represents or means?




- *Is this something that is developed by the individual or student?*
- *Is it reliant on teachers or other sources?*
- *And how does this prior knowledge "affect how they're seen as authority in a class or not"?*

- In statement 21 you said "I use that question a lot, to be honest, for both correct and incorrect…because I don't want them to clue in that it's incorrect. So I know I asked that question probably more than anything else last semester of, 'Does anyone else see another way to do it?' Because I think it's still valuable, even if it is correct, that there's multiple ways to do things." Can you provide an example of this when a student suggested or presented something that was correct?
    - *In this case, why do you think it was valuable to ask this question and see multiple ways to do things?*
    - *Can you provide an example when then suggestion/idea was incorrect?*
        - *Do you see this question playing a different role/purpose when a student suggests an idea that's not correct?*
- Finally, in statement 12 you said "every single little thing that I can do in a classroom, it's going to affect my students." Can you expand on what you meant by that?

Statement Sort and Grouping Activity

- **Before you do anything with the statements, let me explain this activity. The purpose of this activity is to better understand your beliefs and how you are thinking about mathematics, the mathematics classroom, and particularly how authority operates in the math classroom.**
    - **You will place statements into groups, and you can group them in any way you would like. In fact, if there are statements that you think do not belong in a group, in other words if there are statements that do not represent your beliefs or how you are thinking about the mathematics classroom and/or authority in the math classroom, then you can leave those out, cross through them, or put them in a "discard" group.**
    - **Then, after you have grouped all cards, I will ask you to come up with a 'label' to describe each group of cards. Do you have any questions about what you will be doing during this activity?**
    - **As you sort the cards, please feel free to talk aloud about what you are thinking. Also, know that I might ask some questions as you are moving the cards or thinking aloud.**
        - ***If they are not doing so, ask them to describe what they are considering when grouping the cards.***
        - ***If they put a card in one group and then put it into a different group, ask them to explain why they decided to do that.***

- **After all cards are sorted.**
    - **For each set of cards:**
        - **Can you explain why you grouped these particular cards together and why you chose this label?**
            - ***If, as they are describing, there seems to be other cards that could fit in this group or if there are cards in the group that do not appear to belong, ask why those statements were included/excluded.***
            - ***Ask them to explain their inclusion/exclusion criteria for each group.***
            - ***Also, if some labels seem to be similar, ask them to describe the difference between those labels.***
        - **What would you say is/are the common aspect(s)/feature(s) of these statements?**
    - **Would you say that there is any relationship between any of the groups of statements/cards?**
        - ***For instance, is the group of cards related at all to this other group of cards? If so, how would you describe that relationship? If not, why not?***

**Grace's Lesson Plan Interview**

**Welcome [participant's name]. Thank you for taking time out of your busy schedule to do this interview. Over the next hour or so we will talk about your "Introduction to Angles" lesson plan and I will ask some questions about other lesson plans you have developed, your instruction during student teaching, as well as your approach to lesson planning. If you feel that a question or prompt needs to be clarified, please do not hesitate to ask me to do so. I will be recording this interview using the recording features of Zoom as well as with an audio recorder. Before we get started, do you have any questions or concerns?**

When Discussing the Goals of the Lesson

- **In general, is there any information that you consider when it comes to lesson planning?**
- Can you expand on what you mean by 'explore' in this goal?

When Discussing the "Launch" Section

- You planned for a 10-minute work session. What were students doing during those ten minutes?
    - *What did your interactions with students sound and look like during this time?*
- It looks like you wanted to have two students who did "different things" explain how they proved the angles were congruent. Can you recall the students' explanations?
    - *What were other students doing as the students were explaining?*
    - *What were you doing as they were explaining? And what did you do after they explained?*

- *Why did you want two students who approached the task differently to explain their work/thinking?*

- You have a conclusion stated in this section. Was this conclusion stated or emphasized during this section? If so, how did that look/sound in the classroom?

When Discussing the "Desmos and Notes" Section

- Can you walk me through how this section played out in the classroom?
- In the "Instructions" for numbers 1 and 2 you emphasized that students needed to be able to 'argue' why their answer was correct. Why was that important for this task?
    - *How did students' arguments sound?*
        - *What happened after students provided their arguments?*
            - *Did other students have questions? If so, how were those questions addressed?*
- With the card sort in number 3, how were students engaging with the card sort during this time?
    - *What were you doing during this time?*
    - *You wanted a number of students to share their groupings. Why did you plan to have multiple students share their groupings?*
        - *Did this play out in the classroom as you planned?*
        - *Can you recall what a students' explanation sounded like?*
            - *After a student provided an explanation, what happened?*
- With the "Notes" subsection, you planned to "go over" these terms. What does it mean to "go over"?

- *What were students supposed to be doing during this time?*

- Then you go into a "Free work session". Can you describe how students engaged during this work session?
    - *What were you doing during this time?*
        - *Can you recall what a student-teacher interaction looked and sounded like during this time?*
    - *You planned to "give feedback to students work individually as to whether they are 'correct.'"*
        - *First, can you remember why you have 'correct' in quotations there?*
        - *Second, what did this feedback sound like?*

When Discussing the "Anticipated Misconceptions or Difficulties" Section

- Generally speaking, when you lesson plan, can you describe how you plan to address or counter students' difficulties or misconceptions?
    - *Can you recall a moment in this lesson or a more recent one where a student had a misconception or experienced some difficulty? How did you approach that student?*

Use of Technology in Lessons

- During this lesson you have students engage with Desmos and you had students engage with Desmos in the "Dilation mini-golf" lesson. When you have students engage with Desmos, is the purpose always the same or does it differ?
    - *If it's the same, then what is the purpose of using Desmos?*

    - *If it's different, can you describe the different ways you have students use Desmos?*
        - *If it is a Desmos "exploration" task, then what is the role of the students as they work with that task?*
        - *What do you typically do as students are engaged in the Desmos task?*

Changes to Lesson Planning or Instruction Since the Beginning of the Semester

- **Would you say that your lesson plans or your approach to lesson planning has changed since the beginning of you student teaching?**
    - ***What has led to those changes? OR You said no, why do you think your approach has stayed the same?***
- **What about your instruction, how would you describe any changes in your instruction, big or small, since the beginning of the semester?**
    - ***What has led to those changes in your instruction?***

**Grace's Fifth Interview Protocol**

**Welcome [participant name]! Thank you for taking time out of your busy schedule to do this interview. Over the next hour or so we will continue our conversation about mathematics, the mathematics classroom, as well as classroom interactions and authority in the mathematics classroom. As with the previous interviews, remember I am interested in better understanding your beliefs, thoughts, and perceptions, so do not feel any pressure to provide a "right" or "on-track" response. If you feel that a question needs to be clarified, please do not hesitate to ask me to do so. I will be recording this interview using the recording features of Zoom as well as with an audio recorder. Before we get started, do you have any questions or concerns?**

Intro Questions/Prompts

- **Can you recall one highlight moment from your student teaching that made an impression on you?**
  - **What about a low moment from your student teaching?**
- **When you think of your role as a math teacher, which analogy best describes that role?**

  **Shepherd Tour Guide Banks of A River Court Judge**

  **Marriage Counselor Sports Coach Bowling Bumpers Medical Doctor**

  **GPS for a car Camp Counselor Orchestra Conductor Band Director**

  **Captain of A Ship**
  - **Which aspects of a [*insert analogy*] are you seeing as related to your role as a math classroom?**

        - **Are there any aspects of [*insert analogy*] that are not related to your role as a math classroom?**
- *Likely related to her role as the one who determine what math is correct*
    - In the LP interview you brought up helping students make "progress towards a solution" or make more adequate progress towards a certain goal. How does that relate to your role as a [insert analogy]?
        - *In that same interview you brought up assessing and advancing questions as instructional strategies you would use to help students make that progress. Were there any other instructional strategies you also used to help students progress towards a solution or goal?*
            - *If so, how did those help students make progress?*
    - In a previous interview you mentioned that when students are working on their own or with their peers, you need to listen for things that students may be saying that is wrong because "I don't want students to, like move on from this activity, if they have a like, deep-seated misconception about the content."
        - *So, how does this relate to the analogy you chose?*
        - *Can you provide examples from your student teaching where you helped students stay on track or avoid developing a 'deep-seated misconception'?*

Assessing and Advancing Questions

- **A couple things that come up often in your lesson plans, in the interviews we've done so far, and your reflections are assessing and advancing questions. I'm curious, what is your definition of an assessing question?**

    - ***Given that definition, why would you want to use assessing questions when interacting with students?***
    - ***Based off your definition/description, how would you describe the authority between you and the student(s) when you are asking assessing questions?***
- **Similarly, what is your definition of an advancing question?**
    - ***Given that definition, why would you want to use advancing questions when interacting with students?***
    - ***Based off your definition/description, how would you describe the authority between you and the student(s) when you are asking advancing questions?***

Multiple Students Presenting or Explaining

- Often in the previous interviews and in your lesson plans you mention having students work in groups and presenting their work/thinking to their peers at the board. When I asked why you have students present at the board and not you, one reason you gave was that it gives them ownership over their knowledge.
    - Can you say more about you mean by "ownership of their knowledge" and why that's important?
        - *Why is it important for students to view themselves as doers/learners of math?*
        - *Why is it important for students to feel confident or proud in their work?*
        - *How does presenting at the board help foster that ownership of their knowledge?*

- *Can you think of other instructional strategies that you implement in hopes that students take ownership of their knowledge?*
  - *How does strategies or 'moves' help students take ownership of their knowledge?*
- Can you think of any other reasons for having students present or explain their thinking?

Students as Authorities

- **When it comes to students as mathematical authorities, were there any students or moments from your student teaching that come to mind?**
  - ***What about those students and/or moments led to them being a mathematical authority?***
    - ***Do you think these students viewed themselves as mathematical authorities? Why or why not?***
  - ***For students who weren't mathematical authorities in those moments, what do you think it would take for them to become mathematical authorities?***
    - ***What about from their perspective? What would it take for them to view themselves as mathematical authorities?***
- In the LP interview, you said that you wanted to foster or promote positive experiences for your students. For instance, you said "I think it's an important motivator to feel like you're making progress and to feel like you're learning and you're not totally lost, or you don't totally suck at this. I think it's so important that our students are getting positive

feedback and are being reassured that what they're doing is good work, even if it's not perfect"

- *Can you describe the relationships between these positive experiences and students' mathematical self-confidence?*
  - *Is there any relationship between these positive experiences, boosting students' self-confidence and their position as a mathematical authority?*
    - *If so, can you describe that relationship?*

Influence of Mentor Teacher

- **How would you say your mentor teacher and/or UGA supervisor influenced your instruction over the course of your student teaching?**
  - ***When you think about your future classroom and self as a teacher, do you anticipate teaching in differently when compared to your student teaching?***
    - ***Why or why not?***
    - ***What specific changes do you anticipate making?***

## Grace's Sixth Interview Protocol

**Welcome [participant name]! Thank you for taking time out of your busy schedule to do this interview. Over the next hour or so we will continue our conversation about mathematics, the mathematics classroom, as well as classroom interactions and authority in the mathematics classroom, and we will take a look at your authority diagrams. As with the previous interviews, remember I am interested in better understanding your beliefs, thoughts, and perceptions, so do not feel any pressure to provide a "right" or "on-track" response. If you feel that a question needs to be clarified, please do not hesitate to ask me to do so. I will be recording this interview using the recording features of Zoom as well as with an audio recorder. Before we get started, do you have any questions or concerns?**

- **What influenced your instruction the most during student teaching? Rank the top three-to-fice influences.**

  **Time for planning  Instructional materials available  Mentor Teacher**

  **Time for instruction  UGA Supervisor  Pacing guide  Students**

  **Content knowledge  EOC Tests  School expectations  Content team (team of teachers)**

  - **In what specific ways did those factors influence your instruction?**
    - ***Did the influence of [factor] influence how you interacted with students? In what specific ways?***
    - ***If [factor] was not a factor influencing your instruction, how would your instruction look different?***

- **_If they bring up students – How did your students influence the way you taught?_**
    - **_Would your instruction look different, or did it look different, with a different group of students?_**
        - **_What about that different group of students led to, or would lead to those changes?_**

Students as Mathematical Authorities

- You said that "there are a lot of factors to recognizing individual students as mathematical authorities" and mentioned their personality, confidence, and willingness to participate or contribute in the mathematics classroom. Maybe it's not one of the factors I just mentioned, but which factor is the biggest in terms of students being mathematical authorities in the classroom?
    - *Given that, how do students become mathematical authorities in the classroom?*
    - *Can students be mathematical authorities in all instances, or during all moments in the classroom?*
- You have an arrow from "Additional Resources" to the students. How do those additional resources play a role in a student's status or position as a mathematical authority?
- You said that you "find [mathematical] authority in your students" but that a lot of teachers may not share that same view or belief. Why do you think that is?
    - *What do you think is different about you or your beliefs that differentiates you from other teachers?*
        - *How has that influenced your instruction and interaction with students?*

"Other Teachers" as Mathematical Authorities

- When talking about "Other Teachers" as mathematical authorities, you mentioned getting advice on how to best teach things, sequence content, and/or how ideas/concepts connect together. How do you see other teachers' mathematical authority manifesting or showing up in your classroom?
  - *Can you provide an example from your student teaching?*
  - *In what specific ways do you see their influence changing or enhancing your instruction?*
  - *Do you anticipate that throughout your teaching career you will always go to other teachers as mathematical authorities?*
    - *Why or why not?*
    - *If not, when do you anticipate that shift occurring?*

"Curriculum" as Mathematical Authority

- You mentioned that you trust the things you have in the "Curriculum" box as mathematical authorities. Can you say more about that? Why do you see them as mathematical authorities?
  - How do those sources, as mathematical authorities, influence what is happening in the classroom?
  - *Are all "textbooks" or "other curriculum" mathematical authorities? Why or why not?*
  - *Which curricular resources did you often turn to as mathematical authorities during student teaching?*

        - *How did that influence your instruction?*
    - In what ways do the sources in your "Curriculum" box inform or determine the mathematical knowledge and reasoning that is taught or learned in the classroom?
        - *Can you provide an example where a task really helped students develop productive mathematics?*
        - *What about the other way around. Were there any tasks you used that you thought did not foster students developing productive mathematics?*

Before Talking About Changes from the First Mathematical Authority Diagram to the Second Diagram

- When looking at the arrows you drew, you only have one arrow going out from the "Teacher" and the "Students" respectively and those arrows go to each other. Can you talk about why that is the case?
- **Is this how mathematical authority operated in the classroom during your student teaching? Why or why not?**

Changes from First Mathematical Authority Diagram to Second Diagram

- **Now, I'm going to put this diagram next to the diagram you created back in the summer.**
    - **Obviously the diagrams look different, but as you look over these two diagrams, do you see any substrative differences in the diagrams? And if so, how would you describe those differences?**
    - There are a few things that stick out to me

        - Subtle, but in the first diagram each of you students are the same shape and are oriented the same way. In the second diagram you intentionally made the students different sizes and shapes. Why do you think there was that difference when representing the students?
            - *Why do you think you changed the way you represented students in the second diagram?*
            - *Were there interactions or experiences from student teaching that you think led to those changes?*
            - *Also, in the first diagram you had bidirectional arrows between students and in the second diagram all of the arrows between students are unidirectional. Why do you think that was the case?*
        - The green sticky note in the first diagram is "student-oriented" but now can be mathematical authorities in relation to both students and teachers. Is there a reason(s) you had the arrow from "Additional Resources" to "Teacher" in the second diagram?
            - Were there experiences from student teaching that led to that change?
- **When you reflect back to before student teaching, would you expect to see any specific changes in the second diagram?**
    - ***What changes?***
    - ***Why did you think there would be those changes?***

## Grace's Final Interview Protocol

**Welcome [participant name]. Thank you for taking time out of your busy schedule to do this interview. Over the next hour or so we will continue/finish our conversation about mathematics, the mathematics classroom, as well as classroom interactions and authority in the mathematics classroom. As with the previous interviews, remember I am interested in better understanding your beliefs, thoughts, and perceptions, so do not feel any pressure to provide a "right" or "on-track" response. If you feel that a question needs to be clarified, please do not hesitate to ask me to do so. I will be recording this interview using the recording features of Zoom as well as with an audio recorder. Before we get started, do you have any questions or concerns?**

Educational Background

- **Can you start by describing your high school? Where was it located and would you consider it an urban, suburban, or rural school?**
- **Throughout your high school and college experience, were there any teachers that were especially influential?**
    - **Were there any particular individuals that inspired you to become a math teacher?**
        - ***What about that teacher or those teachers stick out to you?***
        - ***How have they influenced the way you think about teaching mathematics?***
    - **What degree(s) did you graduate with this past Spring?**
- **In terms of your family, did/do you have any family members that were educators?**

- **If so, did that influence your decision to become a teacher? Why or why not?**
- **How would you say your family influenced your experience in school?**

Students as Mathematical Authorities

- Back in the third interview you said "you can't really take a person and separate them from their experiences. So whether or not they're seen as an authority in the class is directly related to how they present themselves and how their prior experiences have led to how they present themselves in this current math classroom" First, would you say that you agree with this fully, partly, or disagree?
    - Second, can you talk more about the relationship(s) between students' previous experiences, how they present themselves in the classroom, and them as a mathematical authority in the classroom?
    - What if students' past experiences aren't productive or positive? How does that influence their position as a mathematical authority in the classroom?
        - Is there anything you, as their current math teacher, can do to position such a student as a mathematical authority in the classroom?
- In the most recent interview you said that students' confidence was the biggest factor in terms of students being viewed as a mathematical authority. Can you talk about how students come to develop that confidence?
    - *Is it their mathematical understanding? And if so, how do they know they've developed or correct or productive mathematical understandings?*

- You also said that some of it might come down to students' personalities. For those that were positioned as more of a mathematical authority, what about their personality led to that?
    - *Do you think that would be the case for those students in other content areas as well? Why or why not?*

- In the LP interview you said "I think having that positive disposition leads to better motivation and productive struggle in the long run, even outside of the math class"
  - How does a positive disposition relate to students' confidence and being a mathematical authority in the classroom?
    - Is positive disposition a prerequisite for being confident as a mathematics student?
- When I asked about assessing questions and a student's position as a mathematical authority, you said that the student is a mathematical authority "because they are answering what they know about the mathematics in that specific situation". How does that make them a mathematical authority in that interaction when asking them assessing questions?
  - When I asked about advancing questions you said "in that case, I would say I'm more of the authority but it's like an authority where I'm empowering them still to discover the math." Can you say more about what you mean by that?
    - *What do you mean by "empowering them still to discover the math"?*
      - *So when asking advancing questions, what is the relationship between your position as a mathematical authority and your students' position?*

- When discussing the diagrams in the last interview you said , "I think in a perfect world, students are seeing each other as equal mathematical authorities. So in theory, that would be phenomenal and that would be the ideal. But in practice, there's always going to be some differentiation going on where the students are not seeing each other as authorities." Can you recall what you meant when you said "there's always going to be some differentiation going on where the students are not seeing each other as authorities"?
    - *You said "in a perfect world, students are seeing each other as equal mathematical authorities." Do you think that is feasible, attainable in an actual mathematics classroom? Why or why not?*
        - *What conditions would have to be in place for that "perfect world" scenario to take place?*
- You have consistently said that you can learn so much from your students and that there is "such value in giving students a voice in the classroom." I'm curious, can you talk about where this view comes from? Were there experiences from high school or college that influenced this view? Or do you think it stems from something else?

Objects and Community as Authorities

- In the third interview you said, "department heads or fellow teachers…heavily affect what a teacher is teaching and potentially how they're teaching." Would you still agree with that statement?
    - *How would you say that department heads and other teachers influenced your instruction during student teaching?*
        - *Do you anticipate that being the same when you start teaching in the Fall?*