Christoffel Words and Associated Algorithms

by

Jason Emanuel George

(Under the Direction of Leonard Chastkofsky)

Abstract

This thesis discusses Christoffel words which are a subset of the free monoid on two letters and algorithms which produce them. It covers Christoffel morphisms, standard factorizations, the Burrows-Wheeler matrix, and continued fractions. The four algorithms included serve as a way to demonstrate these ideas in relation to Christoffel words. Code for these algorithms is included which can be used in the GAP programming language. The thesis concludes with an examination of the Hebrew and Gregorian calendar systems and their relationship to Christoffel words.

Index words: Christoffel words, Free monoids, Continued Fractions, Calendar Systems

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Jason Emanuel George

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Jason Emanuel George

Major Professor: Leonard Chastkofsky

Committee: Paul Pollack

William Graham

Electronic Version Approved:

Ron Walcott

Dean of the Graduate School

The University of Georgia

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Contents

1 Introduction

Let $\{x, y\}^*$ denote the free monoid on letters x and y. A string of concatenated letters is then called a word. For example, $w = c_1c_2...c_n$, where each $c_i \in \{x, y\}^*$. The binary operation is concatenation, so if $w_1 = c_1c_2...c_m$ and $w_2 = d_1d_2...d_n$ are elements in $\{x, y\}^*$, then $w_1w_2 = c_1c_2...c_md_1d_2...d_n$. If none of c_i, d_i are the identity elements, then $c_1c_2...c_md_1d_2...d_n$ is in its most simplified form as there are no inverses in monoids. Furthermore, the property of associativity is observed; for example, $(yx)x = y(xx) = yx^2$. For a word w and an $n \in \mathbb{N}$, it is common to write the concatenation of n instances of w as $w \dots w = w^n$. Two words w_1 and w_2 are defined to be conjugate if $w_1 = uv$ and $w_2 = vu$ for some $v, u \in \{x, y\}^*$. The reversal of a word $w = c_1c_2...c_n$ is denoted as $\tilde{w} = c_nc_{n-1}...c_2c_1$. Lastly, the number of instances of x and y in a word w are denoted w_x and w_y, respectively.

This thesis concerns itself with a subset of $\{x, y\}^*$ called the Christoffel words and, in particular, algorithms which produce Christoffel words. Each section other than Section 6 contains one of these algorithms and a proof explaining why it works. Section 6 contains an application of Christoffel words in the context of calendar systems. Many of the theorems, lemmas, and examples have graphical interpretations, and they have been included wherever applicable.

2 Basics of Christoffel Words

This section contains the definition of Christoffel words and some of their properties. Foundational ideas for later theorems and algorithms are presented here as well as the first of the Christoffel word-producing algorithms. The material included is based on the initial papers on the subject from the late 1800's ([7],[8],[12],[13],[14],[18]), and the geometric perspective is based on ([2],[3],[15]).

2.1 Definition of Christoffel Words

Take a and b to be relatively prime, nonnegative integers, and consider the line of slope $\frac{b}{a}$ in the Cartesian coordinate plane. The **lower Christoffel path** of slope $\frac{b}{a}$ is defined as the path on the $\mathbb{Z} \times \mathbb{Z}$ gridded integer lattice from $(0,0)$ to (a,b) satisfying two conditions:

1. The path lies under the line of slope $\frac{b}{a}$.
2. The region enclosed by the line and the path contains no integer points other than those on the boundary.

The **lower Christoffel word** of slope $\frac{b}{a}$ is then constructed by encoding the path as a sequence of unit steps. Let (i,j) be an integer point on the Christoffel path. Then, a step from (i,j) to $(i+1,j)$ is encoded as an x, and a step from (i,j) to $(i,j+1)$ is encoded as a y. The Christoffel word is formed by putting each unit step in order starting from the origin. Similarly, upper Christoffel words are defined using the path that lies above the line of slope $\frac{b}{a}$. Going forward, the terms "Christoffel path" and "Christoffel word" will mean "lower Christoffel path" and "lower Christoffel word" unless otherwise stated.

Since every positive rational number can be expressed as the unique ratio of two relatively prime integers, there is a unique Christoffel word associated with a given slope. A word w is defined to be primitive if for all words u, there is no integer $k > 1$ such that $u^k = w$.

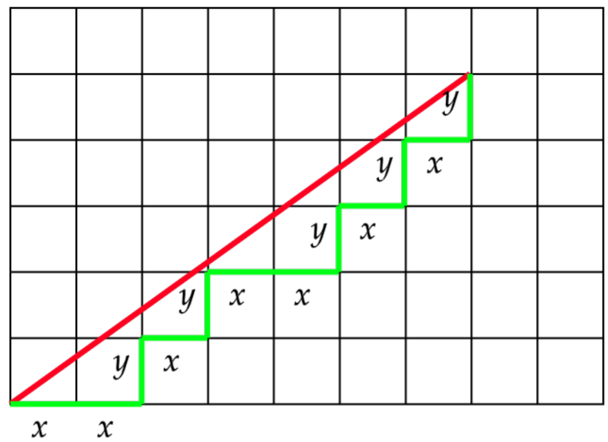

Figure 1: The red line segment is the line of slope $\frac{5}{7}$, and the green path is the Christoffel path. The resulting Christoffel word is $xxyxyxxyxyxy$.

If $w = u^k$ where w is the Christoffel word of slope $\frac{b}{a}$ and $k > 1$, then $w_x = a = ku_x$ and $w_y = b = ku_y$, which implies that the slope of w is $\frac{kb}{ka}$ which is not reduced. Thus, w is not a Christoffel word. By similar logic, every conjugate of a Christoffel word is primitive as well. It is important to note that there are exactly two Christoffel words of length 1.

1. The Christoffel word of slope 0 is x.

2. The Christoffel word of slope ∞ is y.

These are considered trivial Christoffel words. Note that there is no Christoffel word of length 0, as $0 \not\perp 0$. For the purposes of this text, the empty word is not considered a Christoffel word.

2.2 Label of a Point

Definition 2.1. Let a and b be relatively prime, positive integers. Define the **label** of a point (i, j) on the Christoffel path of slope $\frac{b}{a}$ to be the number $\frac{ib-ja}{a}$. This number is also the vertical distance from (i, j) to the line of slope $\frac{b}{a}$.

The label of a point can be used to uniquely identify points on a Christoffel path for a given slope, which is shown in the next lemma.

Lemma 2.1. Let a and b be relatively prime, positive integers, and let $(\frac{s}{a}, \frac{t}{a})$ be the labels of consecutive points on the Christoffel path from $(0,0)$ to (a,b). Then $t \equiv s+b \pmod{a+b}$. Moreover, t takes as value each of $0,1,2,...,a+b-1$ exactly once as t ranges over all consecutive pairs of labels.

Proof. Suppose $\frac{s}{a}$ is the label for (i,j). If $s < a$, then the next point on the Christoffel path is $(i+1,j)$, which has label $\frac{ib+b-ja}{a}$, and $t \equiv ib+b-ja \equiv s+b \mod (a+b)$. If $s \geq a$, then the next point on the Christoffel path is $(i,j+1)$, which has label $\frac{ib-ja-a}{a}$, and $t \equiv ib-ja-a = s-a \equiv s+b \mod a+b$. Since b is relatively prime to $a+b$, t must be equivalent to each of $0,1,2,...,a+b-1$ exactly once over all points on the Christoffel path. $\square$

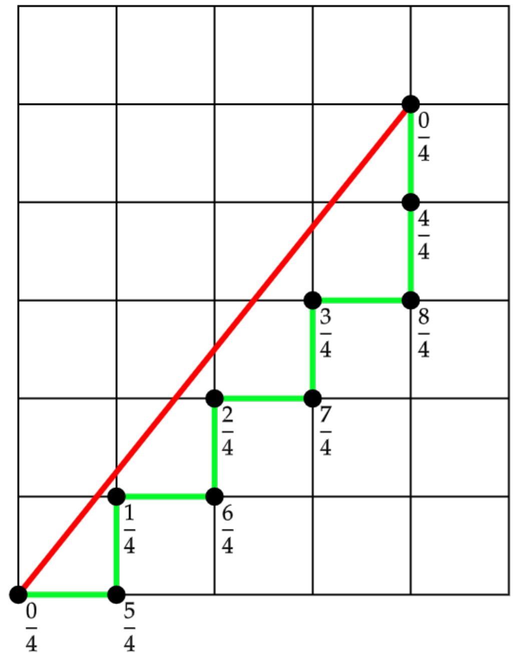

Figure 2: The Christoffel path of slope $\frac{5}{4}$ with labels at each integer point.

2.3 GAP Implementation of the Basic Algorithm

We will now present the first of the Christoffel word algorithms, the basic algorithm.

```
cw := function(a) # returns the Christoffel Word for of slope a
    local r, w, x, y, w_length, d, i;
    r := a; # any positive, rational number
    w := [0]; # 0 represents an x. 1 represents a y
    x := 1;
    y := 0;
    w_length := DenominatorRat(r) + NumeratorRat(r);
    for i in [2..w_length] do
        d := r * x - y - 1; # calculates the numerator of the label-1
     if d < 0 then # adds x if (numerator of the label)-1<1, else adds y
            x := x + 1;
            Append(w, [0]);
        else
            y := y + 1;
```

```
            Append(w, [1]);
        fi;
    od;
    return w;
end;
```

The algorithms in this text return the appropriate Christoffel word as a list of 0's and 1's, with 0 and 1 representing x and y, respectively. The basic algorithm works by checking if the numerator of the label of a point is less than 1, in which case, an x is appended to the word. Otherwise, a y is appended. The algorithm starts at the origin and continues until the length of the word is $a + b$, producing the correct Christoffel word.

2.4 Intersection Characterization

Definition 2.2. Let a and b be relatively prime, positive integers. Let L be the open line segment of slope $\frac{b}{a}$ from $(0, 0)$ to (a, b), and let Z denote the $\mathbb{Z} \times \mathbb{Z}$ gridded integer lattice. Define

$$Int(a, b) := \{(x_i, y_i) \in L \cap Z \mid x_i < x_{i+1} \; \forall \; i \in \{1, 2, \ldots, a + b - 2\}\}.$$

In other words, $Int(a, b)$ denotes the sequence of points between $(0, 0)$ and (a, b) on L which intersect the gridded integer lattice. This definition can be used to construct a characterization of Christoffel words.

Theorem 2.2. Let a and b be relatively prime, positive integers, and let $(x_i, y_i) \in Int(a, b)$ with $i = 1, ..., a + b - 2$. Construct the word w as follows:

1. Let $u \in \{x, y\}^*$ with length $a + b - 2$. For $i = 1, ..., a + b - 2$, if $x_i \in \mathbb{Z}$, then the i-th letter of u is x. If $y_i \in \mathbb{Z}$, then the i-th letter of u is y.

2. Set $w = xuy$.

w is the Christoffel word of slope $\frac{b}{a}$.

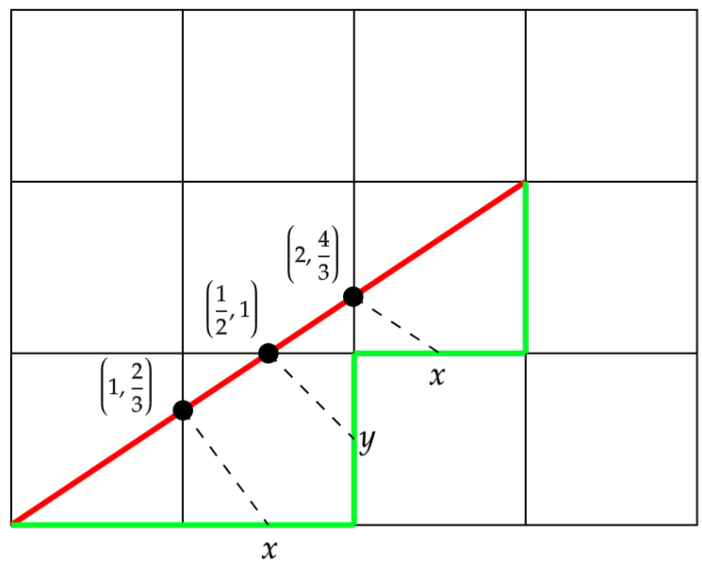

Figure 3: The intersection points in $Int(3, 2)$ correspond with steps on the Christoffel path. The correspondence depends on which coordinate is an integer.

Proof. Let $(x_i, y_i) \in Int(a, b)$. If $x_i \in \mathbb{Z}$, then the point $(x_i, \lfloor y_i \rfloor)$ must be on the Christoffel path C of slope $\frac{b}{a}$. Also, the point $(x_i + 1, \lfloor y_i \rfloor)$ must be on C, as the label of $(x_i, \lfloor y_i \rfloor)$ is less than 1. In this manner, points of this type correspond to horizontal unit steps on C. Similarly, if $y_i \in \mathbb{Z}$, then the point $(\lceil x_i \rceil, y_i)$ must be on the C, as well as the point $(\lceil x_i \rceil, y_i - 1)$, since the label of $(\lceil x_i \rceil, y_i - 1)$ must be greater than 1. Points of this type correspond to vertical unit steps on C.

To finish the proof, since the sequence $Int(a, b)$ follows the line of slope $\frac{a}{b}$ from the origin, the points in $Int(a, b)$ correspond to unit steps on C in the same order as the Christoffel word w, which is the word u. The only unit steps on C not accounted for are the first and last steps, so $w = xuy$. $\square$

This characterization is anticipated in the earliest papers by Christoffel and Smith in [7],[18]. Section 5 and Section 6 will expand on this and present connections to new ideas.

2.5 Christoffel Morphisms

Definition 2.3. A Christoffel morphism is an endomorphism on the free monoid $\{x, y\}^*$ that sends each Christoffel word to a Christoffel word or a conjugate of a Christoffel word.

Note that if G and H are Christoffel Morphisms, then $G \circ H$ is also a Christoffel morphism. Since G is an endomorphism, G is determined by how G transforms x and y, which we will denote as $G = (G(x), G(y))$. Using this notation, we will define five endomorphisms:

$$\mathbf{G} = (x, xy),\ \mathbf{D} = (yx, x)$$

$$\tilde{\mathbf{G}} = (x, yx),\ \tilde{\mathbf{D}} = (xy, y)$$

$$\mathbf{E} = (y, x).$$

It must be shown that these five endomorphisms are injective.

Lemma 2.3. The morphisms $\mathbf{G}$, $\mathbf{D}$, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{D}}$, and $\mathbf{E}$ are injective.

Proof. For $\mathbf{E}$, note that $\mathbf{E} \circ \mathbf{E}$ is the identity morphism $\Rightarrow \mathbf{E}$ is an involution. Use induction to prove the lemma for $\mathbf{G}$. Let $\mathbf{G}(u) = \mathbf{G}(v)$ with $|\mathbf{G}(u)| = |\mathbf{G}(v)| = 1$ for some $u, v \in \{x, y\}^*$. Then $\mathbf{G}(u) = \mathbf{G}(v) = x \Rightarrow u = v = x$. Let $|\mathbf{G}(u)| = |\mathbf{G}(v)| = n$ and assume the lemma holds for all $1 \le i \le n - 1$. Write $\mathbf{G}(u) = c_1 c_2 \dots c_n = \mathbf{G}(v)$ where each $c_i \in \{x, y\}$. There are two cases: either $c_n = x$ or $c_{n-1} c_n = xy$. If $c_n = x$, let

$$\mathbf{G}(u) = \mathbf{G}(w_1 x) = c_1 c_2 \dots c_{n-1} x = \mathbf{G}(w_2 x) = \mathbf{G}(v) \text{ for some } w_1, w_2 \in \{x, y\}^*.$$

By the induction hypothesis, $w_1 = w_2 \Rightarrow u = v$. If $c_{n-1} c_n = xy$, let

$$\mathbf{G}(u) = \mathbf{G}(w_1 y) = c_1 c_2 \dots c_{n-2} xy = \mathbf{G}(w_2 y) = \mathbf{G}(v) \text{ for some } w_1, w_2 \in \{x, y\}^*.$$

Again, by the induction hypothesis, $w_1 = w_2 \Rightarrow u = v$. The proofs for $\mathbf{D}$, $\tilde{\mathbf{G}}$, and $\tilde{\mathbf{D}}$ are similar. $\square$

Now, we will show that the five endomorphisms are Christoffel morphisms.

Lemma 2.4. The morphisms $\mathbf{G}$ and $\tilde{\mathbf{D}}$ take Christoffel words of slope $\frac{b}{a}$ to Christoffel words of slope $\frac{b}{a+b}$ and $\frac{a+b}{a}$, respectively.

Proof. First, prove the result for $\mathbf{G}$. Suppose $a \perp b$, and consider the Christoffel path C of slope $\frac{b}{a}$. The Christoffel path encodes the Christoffel word w of slope $\frac{b}{a}$ as a series of unit vectors $e_1 = (i,j) \to (i+1,j)$ and $e_2 = (i,j) \to (i,j+1)$. The morphism $\mathbf{G}$ sends $y \to xy$, thus $\mathbf{G}$ sends e_2 to steps e_1 and e_2. Now, let $\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $\mathcal{G}(c,d) = (c, c+d)$. Since $\mathcal{G}$ is a linear transformation of $\mathbb{R}^2$, the region R between the line of slope $\frac{b}{a}$ and C maps to the region $\mathcal{G}(R)$ between the line of slope $\frac{b}{a+b}$ and $\mathcal{G}(C)$.

Note that $(c, c+d)$ is an integer point if and only if (c,d) is an integer point. Since R contains no integer points, neither does $\mathcal{G}(R)$. Also, the region between vectors e_1, e_2, and $e_1 + e_2$ also contains no integer points. Extend the region $\mathcal{G}(R)$ by replacing each instance of $e_1 + e_2$ with steps e_1 and e_2. The resulting region contains no integer points. The path along the boundary of that region under the line of slope $\frac{b}{a+b}$ contains no integer points, so it must be the Christoffel path of slope $\frac{b}{a+b}$. Since this path is the same as the path encoded by $\mathbf{G}(w)$, $\mathbf{G}$ takes Christoffel words of slope $\frac{b}{a}$ to Christoffel words of slope $\frac{b}{a+b}$.

The proof for $\tilde{\mathbf{D}}$ is similar. Define $\mathcal{D} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathcal{D}(c,d) = (c+d, d)$, and the same argument works by replacing instances of e_1 with steps e_1 and e_2. $\square$

Working backwards through the proof gives us the next result.

Corollary 2.5. If w is a Christoffel word of slope less than 1, then there exists a Christoffel word u such that $\mathbf{G}(u) = w$. Similarly, if w is a Christoffel word of slope greater than 1, then there exists a Christoffel word u such that $\tilde{\mathbf{D}}(u) = w$.

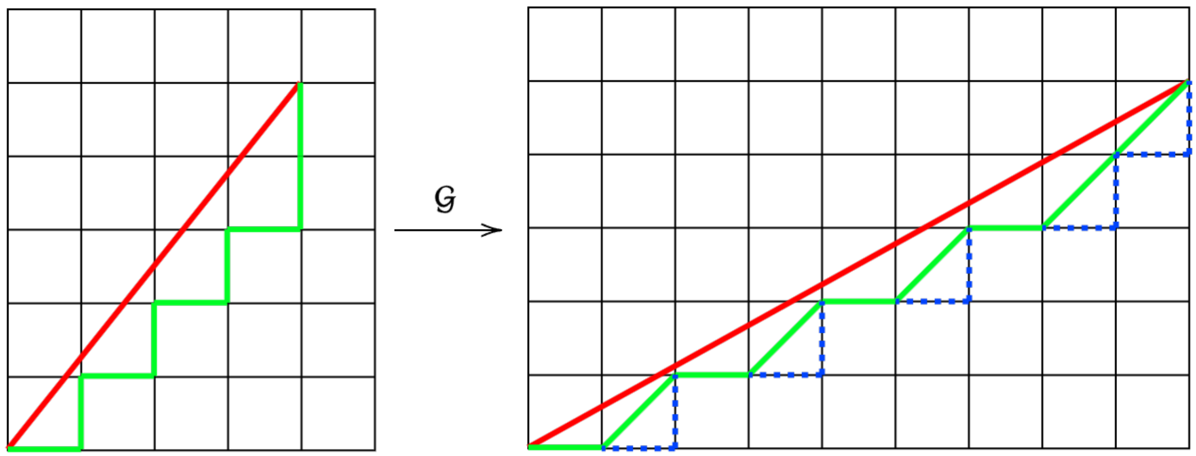

Figure 4: The graph on the left shows the Christoffel path of slope $\frac{5}{4}$. Under $\mathcal{G}$, the graph is transformed to the one on the right. The dotted line indicates where instances of $e_1 + e_2$ are replaced with steps e_1 and e_2, which creates the Christoffel path of slope $\frac{5}{9}$.

Proof. Suppose w is a Christoffel word of slope less than 1. The path created by replacing e_1 and e_2 with $e_1 + e_2$ is the image of some path transformed by $\mathcal{G}$. This path must be a Christoffel path since the region enclosed by the path and the corresponding line contains no integer points. Thus, there is a Christoffel word u such that $\mathbf{G}(u) = w$. The proof for $\tilde{\mathbf{D}}$ is analogous. $\square$

Lemma 2.6. For every word $w \in \{x, y\}^*$, there exists a word $u \in \{x, y\}^*$ such that $\mathbf{G}(w) = xu$ and $\tilde{\mathbf{G}}(w) = ux$ and a word $v \in \{x, y\}^*$ such that $\mathbf{D}(w) = yv$ and $\tilde{\mathbf{D}}(w) = vy$.

Proof. Use induction on the length of w. In the case where $|w| = 1$, either $w = x$ or $w = y$. If $w = x$, let $u = \epsilon$ and $v = x$. Then $\mathbf{G}(x) = x$, $\tilde{\mathbf{G}}(x) = x$, $\mathbf{D}(x) = yx$, and $\tilde{\mathbf{D}}(x) = xy$. If $w = y$, let $u = y$ and $v = \epsilon$. Then $\mathbf{G}(y) = yx$, $\tilde{\mathbf{G}}(y) = xy$, $\mathbf{D}(y) = y$, and $\tilde{\mathbf{D}}(y) = y$.

Assume the lemma is true for all words of length less than n. Let $w = aw'$ with $a \in \{x, y\}$. Then w' satisfies the induction hypothesis, so $\mathbf{G}(w') = xu'$ and $\tilde{\mathbf{G}}(w') = u'x$ for some

$u' \in \{x,y\}^*$ and $\mathbf{D}(w') = yv'$ and $\tilde{\mathbf{D}}(w') = v'y$ for some $v' \in \{x,y\}^*$. If $a = x$,

$$\mathbf{G}(w) = \mathbf{G}(xw') = xxu' \text{ and } \tilde{\mathbf{G}}(w) = \tilde{\mathbf{G}}(xw') = xu'x$$
$$\mathbf{D}(w) = \mathbf{D}(xw') = yxyv' \text{ and } \tilde{\mathbf{D}}(w) = \tilde{\mathbf{D}}(xw') = xyv'y.$$

Choosing $u = xu'$ and $v = xyv'$ satisfies the conditions. If $a = y$,

$$\mathbf{G}(w) = \mathbf{G}(yw') = xyxu' \text{ and } \tilde{\mathbf{G}}(w) = \tilde{\mathbf{G}}(yw') = yxu'x$$
$$\mathbf{D}(w) = \mathbf{D}(yw') = yyv' \text{ and } \tilde{\mathbf{D}}(w) = \tilde{\mathbf{D}}(yw') = yv'y.$$

Choosing $u = yxu'$ and $v = yv'$ satisfies the conditions and the proof is done.

□

Now, we shall prove the following theorem.

Theorem 2.7. The morphisms $\mathbf{G}$, $\mathbf{D}$, $\tilde{\mathbf{G}}$, and $\tilde{\mathbf{D}}$ are Christoffel morphisms.

Proof. Lemma 2.4 gives that $\mathbf{G}$ and $\tilde{\mathbf{D}}$ take Christoffel words to Christoffel words. For $\tilde{\mathbf{G}}$, note that for every Christoffel word w, Lemma 2.6 guarantees a word u with $\mathbf{G}(w) = xu$ and $\tilde{\mathbf{G}}(w) = ux$. Since $\mathbf{G}$ takes Christoffel words to Christoffel words, xu is a Christoffel word and hence, $ux = \tilde{\mathbf{G}}(w)$ is a conjugate of a Christoffel word. A similar proof works for $\mathbf{D}$. Lemma 2.6 guarantees a word v with $\mathbf{D}(w) = yv$ and $\tilde{\mathbf{D}}(w) = vy$. Since vy is a Christoffel word, $yv = \mathbf{D}(w)$ is a conjugate of a Christoffel word. □

The next two lemmas and the subsequent theorem will show that $\mathbf{E}$ is a Christoffel morphism.

Lemma 2.8. The morphism $\mathbf{E}$ takes lower Christoffel words of slope $\frac{b}{a}$ to upper Christoffel words of slope $\frac{a}{b}$.

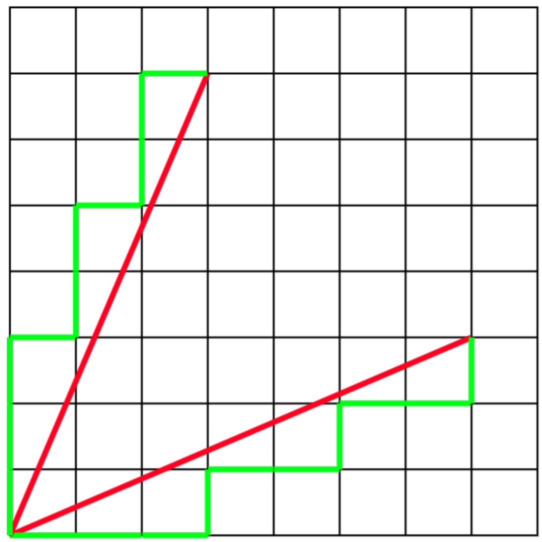

Figure 5: The Christoffel word of slope $\frac{7}{3}$ is constructed by swapping the horizontal and vertical unit steps, as described in Lemma 2.8.

Proof. The technique from the proof of Lemma 2.4 is used. Consider the Christoffel path C of slope $\frac{b}{a}$ as a sequence of unit vectors e_1 and e_2. The morphism $\mathbf{E}$ can be thought of as swapping e_1 and e_2. Let $\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with $\mathcal{E}(c, d) = (d, c)$. Note that (c, d) is an integer point if and only if (d, c) is an integer point, and that $\mathcal{E}$ takes the line of slope $\frac{b}{a}$ to the line of slope $\frac{a}{b}$. Let R be the region between the line of slope $\frac{b}{a}$ and C. Since R contains no integer points, neither does $\mathcal{E}(R)$. Thus, $\mathcal{E}(C)$ must be the upper Christoffel path of slope $\frac{a}{b}$. $\square$

Lemma 2.9 (Cohn [9], de Luca, Mignosi [10])**.** The upper Christoffel word and lower Christoffel word of slope $\frac{b}{a}$ are conjugate.

Proof. Let w be the Christoffel word of slope $\frac{b}{a}$. The word ww then encodes the path from $(0, 0)$ to $(2a, 2b)$. Consider the point on the path with maximum label, that is, the point with maximum vertical distance from L, the line of slope $\frac{b}{a}$. Since each possible label is attained

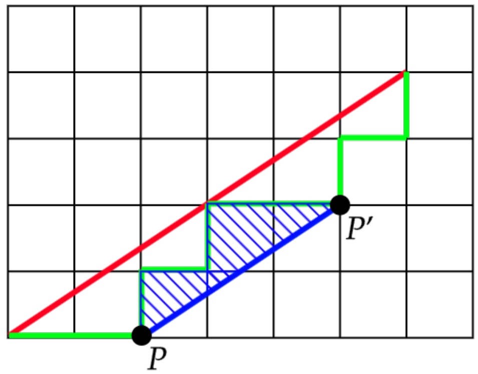

Figure 6: This is the path encoded by ww where w is the Christoffel word of slope $\frac{2}{3}$. The shaded region between the path and the line segment from P to P' contains no integer points, so the path from P to P' is the upper Christoffel word of slope $\frac{2}{3}$.

exactly once on the path for w, say at point P, there are two such points on the path for ww: P and P'. By Lemma 2.1, the label for P and P' is $\frac{a+b-1}{a}$. The line segment from P to P' must have slope $\frac{b}{a}$ since both points are vertically equidistant from L. Construct the word u by following the path encoded by ww from P to P'.

Claim: u is the upper Christoffel word of slope $\frac{b}{a}$. It must be shown that there are no integer points contained in the region R enclosed by the line segment from P to P' and the path encoded by u. Let (i, j) be a point on ww and assume that the integer point $(i, j-1)$ is in R and not on the path ww. Then the point $(i-1, j)$ must be on the path encoded by u. The vertical distance from $(i, j-1)$ to L is

$$1 + \frac{ib - ja}{a} = \frac{ib - (j-1)a}{a} \leq \frac{a+b-1}{a} \Leftrightarrow \frac{(i-1)b - ja}{a} \leq -\frac{1}{a}.$$

However, $\frac{(i-1)b-ja}{a}$ is the vertical distance from $(i-1, j)$ to L, which cannot be negative. By contradiction, R contains no integer points. Note that $|u| = |w|$ and u is a factor of ww, so u is a conjugate of w. $\square$

Theorem 2.10. The endomorphisms $\mathbf{G}$, $\mathbf{D}$, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{D}}$, and $\mathbf{E}$ are Christoffel morphisms.

Proof. By Theorem 2.7, $\mathbf{G}$, $\mathbf{D}$, $\tilde{\mathbf{G}}$, and $\tilde{\mathbf{D}}$ are Christoffel Morphisms. If w is a Christoffel word, then $\mathbf{E}(w)$ is an upper Christoffel word by Lemma 2.8 which is the conjugate of a lower Christoffel word by Lemma 2.9. $\square$

2.6 Generators of the Monoid of Christoffel Morphisms

The set of Christoffel morphisms under composition is a monoid. The morphisms studied in the previous section generate this monoid, and this section will prove this fact. A short lemma is required.

Lemma 2.11. If w is a Christoffel word or a conjugate of a Christoffel word, then xx and yy cannot both be factors of w.

Proof. Use the intersection characterization. Let $\frac{b}{a}$ be the slope of w, and assume that xx and yy are both factors of w. Since xx is a factor, for some $(x_i, y_i) \in Int(a,b)$, $|y_{i+1} - y_i| < 1$, which forces $\frac{b}{a} < 1$. Similarly, since yy is a factor, for some $(x_i, y_i) \in Int(a,b)$, $|x_{i+1} - x_i| < 1$, which forces $\frac{b}{a} > 1$. This is a contradiction. $\square$

Theorem 2.12. The monoid of Christoffel morphisms is generated by $\mathbf{G}$, $\mathbf{D}$, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{D}}$, and $\mathbf{E}$.

Proof. The proof takes five steps.

Step 1: Show that any Christoffel morphism $\mathbf{H}$ is nonerasing, i.e. show that $|\mathbf{H}(w)| > |w|$ for any Christoffel word w. Suppose $\mathbf{H}$ is erasing. Then either $\mathbf{H}(x) = \epsilon$ or $\mathbf{H}(y) = \epsilon$. In the first case, $\mathbf{H}(xyy) = yy$, which is not primitive. In the second case, $\mathbf{H}(xxy) = xx$, which is also not primitive. In either case, the resulting word is not a Christoffel word or a conjugate of a Christoffel word since neither can be primitive.

Step 2: If $\mathbf{H}$ is a nonidentity Christoffel morphism, then $\mathbf{H}(x)$ and $\mathbf{H}(y)$ must begin or end with the same letter. Suppose $\mathbf{H}(x)$ begins with an x and $\mathbf{H}(y)$ begins with a y. There are two scenarios:

(i): $\mathbf{H}(x)$ ends with an x. If $\mathbf{H}(y)$ ends with a y, then xx is a factor of $\mathbf{H}(xxy) = (x \dots x)(x \dots x)(y \dots y)$. By the lemma, yy cannot also be a factor of $\mathbf{H}(xxy)$. In particular, yy cannot be a factor of $\mathbf{H}(x)$ or $\mathbf{H}(y)$. Similarly, $\mathbf{H}(xyy) = (x \dots x)(y \dots y)(y \dots y)$, and the lemma gives that xx cannot be a factor of $\mathbf{H}(x)$ or $\mathbf{H}(y)$. This forces $\mathbf{H}(x) = (xy)^i x$ and $\mathbf{H}(y) = (yx)^j y$ for some integers i and j. Also, $\mathbf{H}(xy) = (xy)^i x (yx)^j y = (xy)^{i+j+1}$. Since $\mathbf{H}(xy)$ is primitive, then $i + j + 1 = 1$. But then $\mathbf{H}$ is the identity morphism, which is a contradiction. Thus, $\mathbf{H}(y)$ ends with an x.

(ii): $\mathbf{H}(x)$ ends with a y. If $\mathbf{H}(y)$ ends with an x, then $\mathbf{H}(xy) = (x \dots y)(y \dots x)$. So xx and yy are factors of every conjugate of $\mathbf{H}(xy)$ sans potentially $\mathbf{H}(y)\mathbf{H}(x) = (y \dots x)(x \dots y)$. This contradicts Lemma 2.11, since $\mathbf{H}(xy)$ is a conjugate of a Christoffel word. Hence, $\mathbf{H}(y)$ ends with a y.

In the case where $\mathbf{H}(x)$ begins with a y and $\mathbf{H}(y)$ begins with an x, study $\mathbf{E} \circ \mathbf{H}$ instead.

Step 3: If $\mathbf{M}$ is a nonidentity Christoffel morphism, then there exists a morphism $\mathbf{H} : \{x, y\}^* \to \{x, y\}^*$ and an $\mathbf{H'} \in \{\mathbf{G}, \mathbf{D}, \tilde{\mathbf{G}}, \tilde{\mathbf{D}}\}$ such that $\mathbf{M} = \mathbf{H'} \circ \mathbf{H}$. Let $w \neq \epsilon$ be a word in $\{x, xy\}^*$, that is, w is a concatenation of xs and xys. Clearly, yy cannot be a factor of w. On the contrary, if yy is not a factor of w, then w must be a concatenation of xs and xys, hence $w \in \{x, xy\}^*$. Similar definitions hold for the free monoids $\{x, yx\}^*$, $\{xy, y\}^*$, and $\{yx, y\}^*$. The claim is that the image of $\mathbf{M}$ is in one of these four monoids. To prove the claim, use the fact that $\mathbf{G}$, $\mathbf{D}$, $\tilde{\mathbf{G}}$, and $\tilde{\mathbf{D}}$ are injective. The image of each of the morphisms are in $\{x, xy\}^*$, $\{yx, y\}^*$, $\{x, yx\}^*$, and $\{xy, y\}^*$, respectively, so the composition of $\mathbf{H'}^{-1} \circ \mathbf{M} = \mathbf{H}$.

By the lemma, xx and yy cannot both be factors of $\mathbf{M}(xy)$. Assume yy is not a factor, which implies yy is also not a factor of $\mathbf{M}(x)$ or $\mathbf{M}(y)$. Now, use Step 2, and there are five cases to check:

(i): Assume $\mathbf{M}(x)$ and $\mathbf{M}(y)$ both begin with an x. Then the image of $\mathbf{M}$ is a subset of $\{x, xy\}^*$. Thus, $\mathbf{G}^{-1} \circ \mathbf{M} = \mathbf{H}$ is a morphism on $\{x, y\}^*$.

(ii): Assume $\mathbf{M}(x)$ and $\mathbf{M}(y)$ both begin with a y. By the lemma, neither may end with a y as otherwise that would contradict our assumption that yy is not a factor of $\mathbf{M}(xy)$. Therefore, both $\mathbf{M}(x)$ and $\mathbf{M}(y)$ end with an $x \Rightarrow$ the images of $\mathbf{M}(x)$ and $\mathbf{M}(y)$ are in the monoid $\{x, yx\}^*$. Thus $\tilde{\mathbf{G}}^{-1} \circ \mathbf{M} = \mathbf{H}$ is a morphism on $\{x, y\}^*$.

(iii): If $\mathbf{M}(x)$ and $\mathbf{M}(y)$ both end with an x. By similar logic, the images of $\mathbf{M}(x)$ and $\mathbf{M}(y)$ are in the monoid $\{x, yx\}^*$, and $\tilde{\mathbf{G}}^{-1} \circ \mathbf{M} = \mathbf{H}$ is a morphism on $\{x, y\}^*$.

(iv): If $\mathbf{M}(x)$ and $\mathbf{M}(y)$ both end with a y. By similar logic, the images of $\mathbf{M}(x)$ and $\mathbf{M}(y)$ are in the monoid $\{x, xy\}^*$, and $\mathbf{G}^{-1} \circ \mathbf{M} = \mathbf{H}$ is a morphism on $\{x, y\}^*$.

(v): If xx is not a factor of $\mathbf{M}(xy)$ instead of yy, use the same cases above using $\mathbf{D}$, $\tilde{\mathbf{D}}$, $\{yx, y\}^*$, and $\{xy, y\}^*$.

Step 4: Show that the morphism $\mathbf{H}$ obtained in Step 3 is a Christoffel morphism. Using the decomposition, $\mathbf{M} = \mathbf{H}' \circ \mathbf{H}$ where $\mathbf{H}' \in \{\mathbf{G}, \mathbf{D}, \tilde{\mathbf{G}}, \tilde{\mathbf{D}}\}$. Here is the case where $\mathbf{H}' = \mathbf{G}$. From Corollary 2.5, if $\mathbf{G}(w)$ is a Christoffel word, then w is also a Christoffel word. Suppose $\mathbf{G}(w)$ is a conjugate of a Christoffel word, and let $\mathbf{G}(w) = uv$ where vu is a Christoffel word. Since $\mathbf{G}(w) \in \{x, xy\}^* \Rightarrow yy$ is not a factor of u or v, and v begins with and x and u ends with a y, then u and v must also be in $\{x, xy\}^*$. Thus $\mathbf{G}^{-1}(u)$ and $\mathbf{G}^{-1}(v)$ are in $\{x, y\}^*$, and $w = \mathbf{G}^{-1}(u)\mathbf{G}^{-1}(v) \Rightarrow \mathbf{G}^{-1}(v)\mathbf{G}^{-1}(u)$ is a Christoffel word. The proofs for $\mathbf{D}$, $\tilde{\mathbf{G}}$, and $\tilde{\mathbf{D}}$ are all similar.

Step 5: Show that $\mathbf{M} = \mathbf{H'}_1 \circ \cdots \circ \mathbf{H'}_n$ such that each $\mathbf{H'}_i \in \{\mathbf{G}, \mathbf{D}, \tilde{\mathbf{G}}, \tilde{\mathbf{D}}\}$. By Step 4, $\mathbf{M} = \mathbf{H}' \circ \mathbf{H}$ for some $\mathbf{H}' \in \{\mathbf{G}, \mathbf{D}, \tilde{\mathbf{G}}, \tilde{\mathbf{D}}\}$ where $\mathbf{H}$ is a Christoffel morphism. Thus, $|\mathbf{M}(x)| + |\mathbf{M}(y)| > |\mathbf{H}(x)| + |\mathbf{H}(y)|$, and using induction on $|\mathbf{M}(x)| + |\mathbf{M}(y)|$ completes the proof. □

This decomposition will prove useful in later sections, particularly in the proof of the Box Algorithm. In fact, the morphisms $\mathbf{D}$ and $\tilde{\mathbf{G}}$ are not required to generate the monoid of Christoffel morphisms, as $\mathbf{D} = \mathbf{E} \circ \mathbf{G} \circ \mathbf{E}$ and $\tilde{\mathbf{G}} = \mathbf{E} \circ \tilde{\mathbf{D}} \circ \mathbf{E}$, but they are included in order to simplify the proof of Theorem 2.12.

2.7 Standard Factorization

Every nontrivial Christoffel word can be factored into a product of two Christoffel words in a unique way. Doing so leads to the construction of the Christoffel tree and will allow us to build Christoffel words using a new algorithm. Many of the results are due to Jean-Pierre Borel and François Laubie [4].

Definition 2.4. Let a and b be relatively prime with $a, b > 0$, and let (i, j) be the point with label $\frac{1}{a}$ on the Christoffel path of slope $\frac{b}{a}$. The **standard factorization** of the Christoffel word of slope $\frac{b}{a}$ is the factorization $w = (w_1, w_2)$ with w_1 encoding the portion from $(0, 0)$ to (i, j) and w_2 encoding the portion from (i, j) to (a, b).

The point with label $\frac{1}{a}$ is the point on the Christoffel path with minimum, nonzero vertical distance to the line of slope $\frac{b}{a}$. Since each label is attained exactly once on the Christoffel path, this factorization is unique. It can then be shown that this factorization is the only way to factor a Christoffel word as two Christoffel words.

Theorem 2.13. If (w_1, w_2) is the standard factorization of a nontrivial Christoffel word, then w_1 and w_2 are Christoffel words.

Proof. Let $a \perp b$ and let (i, j) be the point on the Christoffel path from $(0, 0)$ to (a, b) with label $\frac{1}{a}$. Let L be the line segment from $(0, 0)$ to (a, b), L_1 be the line segment from $(0, 0)$ to (i, j), and L_2 be the line segment from (i, j) to (a, b). Suppose there is an integer point on L_1 or L_2. Such a point must be on the Christoffel path, but then (i, j) would not be the point with minimum label. The regions enclosed by L_1 and the Christoffel path and L_2 and the Christoffel path also contain no integer points, since they are both a subset of the region enclosed by L and the Christoffel path. This causes $i \perp j$, and $(a - i) \perp (b - j)$. Therefore, the words w_1 and w_2 obtained in the standard factorization are the Christoffel words of $\frac{j}{i}$ and $\frac{b-j}{a-i}$, respectively. $\square$

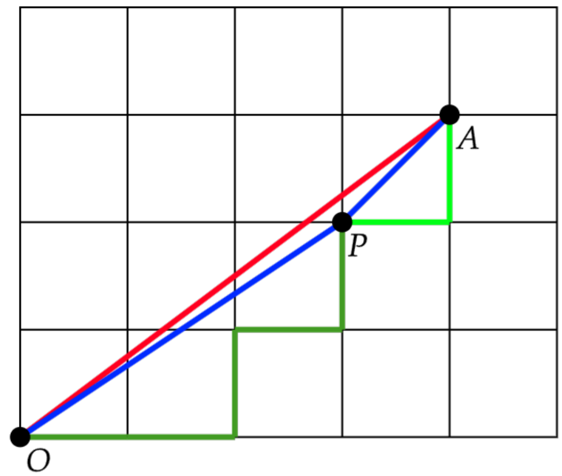

Figure 7: Point P is the point on the Christoffel path closest to the line of slope $\frac{3}{4}$. As such the factorization (w_1, w_2) is obtained where w_1 is the portion of the path from O to P and w_2 is the portion of the path from P to A. Both of the words are Christoffel words since there are no integer points in the region contained by the Christoffel path and the lines OP and PA.

The proof of the next theorem is due to Hugh Thomas.

Theorem 2.14. (Borel, Laubie [4]). A nontrivial Christoffel word w has a unique factorization $w = w_1 w_2$ where w_1 and w_2 are Christoffel words.

Proof. Let (w_1, w_2) be the standard factorization of a Christoffel word w of slope $\frac{b}{a}$. By definition, the word w is split at the point $P = (i, j)$ on the Christoffel path with label $\frac{1}{a}$. Suppose there was a different factorization of w into Christoffel words (w_3, w_4). Such a factorization would split at a point $Q = (k, l)$ on the Christoffel path. Let $O = (0, 0)$ and $A = (a, b)$, and consider the two triangles ΔOPA and ΔOQA. From the proof of Theorem 2.13, ΔOPA contains no integer points in its interior or on its boundary, save for its vertices. Since w_3 and w_4 are also Christoffel words, ΔOQA also has no integer points in its interior or on its boundary, save for its vertices. Since each label occurs exactly once on the Christoffel path for w, the label of (k, l) is not $\frac{1}{a}$. By Pick's theorem, the area of

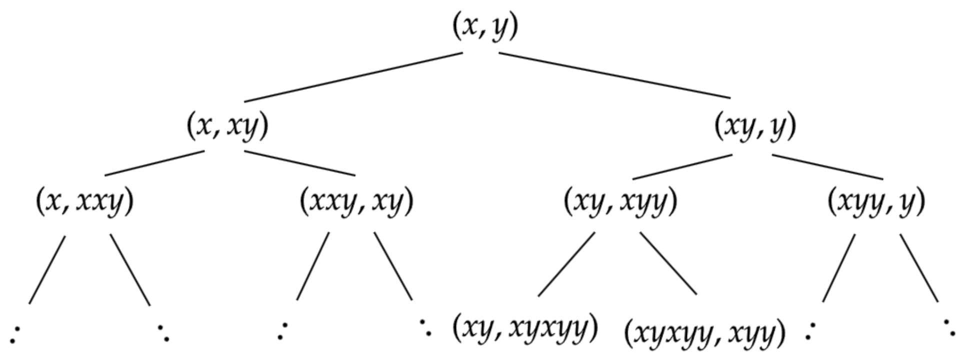

Figure 8: The first few layers of the Christoffel tree are shown here.

$\Delta OPA = \Delta OQA = 0 + \frac{1}{2}(3) - 1 = \frac{1}{2}$. Thus, the triangles ΔOPA and ΔOQA have the same base and the same area but different heights. This is a contradiction. $\square$

2.8 The Christoffel Tree

We can construct a complete, binary tree using $\mathbf{G}$ and $\tilde{\mathbf{D}}$. To do this, set (x, y) as the root. For any pair of words (u, v) in the tree, the left branch is $(u, v) \circ \mathbf{G} = (u, uv)$ and the right branch is $(u, v) \circ \tilde{\mathbf{D}} = (uv, v)$. A pair in the tree (u, v) is the morphism which takes (x, y) to (u, v). These rules were introduced by Gerard Rauzy in [17]. The Christoffel tree is also described in ([4],[11]).

Theorem 2.15. The Christoffel tree contains the standard factorization of each lower Christoffel word exactly once.

Proof. The proof is in three steps.

Step 1: Show that each vertex on the Christoffel tree has the property that u, v, and uv are Christoffel words. This follows from the definition. Each node (u, v) on the tree is a composition of $\mathbf{G}$s and $\tilde{\mathbf{D}}$s by construction, and since $\mathbf{G}$ and $\tilde{\mathbf{D}}$ take Christoffel words to Christoffel words, $(u, v)(x)$, $(u, v)(y)$, and $(u, v)(xy)$ are all Christoffel words.

Step 2: Show that a vertex (u, v) is the standard factorization of the Christoffel word uv. Since u and v are Christoffel words by Step 1, (u, v) must be the standard factorization of uv by Theorem 2.14.

Step 3: Let w be a Christoffel word. Show that the standard factorization of a Christoffel word (w_1, w_2) occurs exactly once in the Christoffel tree. In other words, show that $(w_1, w_2) = (\mathbf{H}_1 \circ \cdots \circ \mathbf{H}_n)(x, y)$ for a unique sequence of $\mathbf{H}_i \in \{\mathbf{G}, \tilde{\mathbf{D}}\}$ and an $n \in \mathbb{N}$. First, suppose that the slope of $w = (u, v)$ is less than 1. We want to show that $(\mathbf{G}^{-1}(u), \mathbf{G}^{-1}(v))$ is the standard factorizaton of $\mathbf{G}^{-1}(uv)$. From the proof of Lemma 2.11, since yy is not a factor of uv, it cannot be a factor of either u or v, which means that u and v are in $\{x, xy\}^*$, which contains the image of $\mathbf{G}$. Thus, $\mathbf{G}^{-1}(u)$ and $\mathbf{G}^{-1}(v)$ are Christoffel words by Corollary 2.5, and hence must be the standard factorization of $\mathbf{G}^{-1}(uv)$ by Theorem 2.14. If the slope of w is greater than 1, use the same argument with $\tilde{\mathbf{D}}$ instead. Thus, each $\mathbf{H}_i \in \{\mathbf{G}, \tilde{\mathbf{D}}\}$.

All that remains is to show that the standard factorization occurs exactly once in the Christoffel tree. This fact follows from the nature of binary trees. Being a left branch on the tree corresponds with a precomposition with $\mathbf{G}$ and being a right branch corresponds with a precomposition with $\tilde{\mathbf{D}}$. Thus, if (u, v) is at two nodes on the Christoffel tree, then

$$(u, v) = (\mathbf{H}_1 \circ \cdots \circ \mathbf{H}_m)(x, y)$$
$$(u, v) = (\mathbf{H'}_1 \circ \cdots \circ \mathbf{H'}_n)(x, y)$$

where $m, n \in \mathbb{N}$ and $\mathbf{H}_i, \mathbf{H'}_i \in \{\mathbf{G}, \tilde{\mathbf{D}}\}$. To finish off the proof, note that the only Christoffel word in the image of both $\mathbf{G}$ and $\tilde{\mathbf{D}}$ is xy, which is the root of the tree. For all other words, the slope must be either greater than 1 or less than 1, which forces $\mathbf{H}_1 = \mathbf{H'}_1, \ldots, \mathbf{H}_m = \mathbf{H'}_n$ and $m = n$. Thus, the two routes down the tree are identical. $\square$

2.9 Christoffel Morphisms and Standard Factorizations

The following theorem will be important in the proof of the box algorithm, and it neatly ties together much of the preliminary material.

Theorem 2.16. A Christoffel morphism f maps Christoffel words to Christoffel words if and only if $f = (w_1, w_2)$ where (w_1, w_2) is the standard factorization of some Christoffel word.

Proof. ($\Leftarrow$): Suppose $f = (w_1, w_2)$ where (w_1, w_2) is the standard factorization of some Christoffel word. By Theorem 2.15, $f = \mathbf{H}_1 \circ \cdots \circ \mathbf{H}_n$ where $n \in \mathbb{N}$ and $\mathbf{H}_i \in \{\mathbf{G}, \tilde{\mathbf{D}}\}^* \Rightarrow f$ is composed of $\mathbf{G}$'s and $\tilde{\mathbf{D}}$'s, so f takes Christoffel words to Christoffel words.

($\Rightarrow$): Suppose f takes Christoffel words to Christoffel words. Then

$$w_1 w_2 = f(x)f(y) = f(xy) = w$$

where w, w_1, and w_2 are Christoffel words. Since the standard factorization of w is unique, $f = (w_1, w_2)$. $\square$

3 Box Algorithm

3.1 Box Algorithm Definition and Proof

This section introduces the box algorithm which is used to find Christoffel words of a particular slope by using standard factorizations. We begin with the definition and then prove that the algorithm works. Finally, there will be a few examples.

Definition 3.1. Let $\frac{b}{a}$ be a reduced, positive rational number such that $a, b \geq 1$. Let $H_i : \{x, y\}^* \to \{x, y\}^*$ be morphisms. Define the box algorithm as follows:

1. Let $a_0 = a$ and $b_0 = b$.

2. If $a_i = 1$, set $n = i$ and $H_i(w) = (xy^{b_i - 1}, y)(w)$ and proceed to step 6. If $b_i = 1$, set $n = i$ and $H_i(w) = (x, x^{a_i - 1}y)(w)$ and proceed to step 6. Otherwise, proceed to step 3.

3. Use the division algorithm to find $q_i, r_i \in \mathbb{Z}$ such that $a_i = b_i q_i + r_i$ with $0 \leq r_i < b_i$. If $r_i \geq b_i - r_i$, proceed to step 4. If $r_i < b_i - r_i$, proceed to step 5.

4. Set H_i to be $H_i(w) = (x^{q_i + 1}y, x^{q_i}y)(w)$. Set $a_{i+1} = r_i$ and $b_{i+1} = b_i - r_i$ Repeat from step 2.

5. Set H_i to be $H_i(w) = (x^{q_i}y, x^{q_i + 1}y)(\tilde{w})$. Set $a_{i+1} = b_i - r_i$ and $b_{i+1} = r_i$. Repeat from step 2.

6. Set $w = H_0 \circ H_1 \circ ... \circ H_{n-1} \circ H_n(xy)$.

Theorem 3.1. The box algorithm produces the Christoffel word of a given slope $\frac{b}{a}$.

Proof. The proof requires two parts. First, show that $H_0, ..., H_n$ each take Christoffel words to Christoffel words. Second, show that the word w obtained in step 6 of the algorithm is the correct Christoffel word.

Part 1: Take cases depending on each type of H_i. Let $i = 0, .., n$.

Case (i), $a_i = 1$ or $b_i = 1$: If $a_i = 1$, then $H_i(w) = (xy^{b_i-1}, y)(w)$. (xy^{b_i-1}, y) is the standard factorization of the Christoffel word of slope $\frac{b_i}{1}$, hence H_i takes Christoffel words to Christoffel words by Theorem 2.16. Similarly, if $b_i = 1$, then $H_i(w) = (x, x^{a_i-1}y)(w)$. $(x, x^{a_i-1}y)$ is the standard factorization of the Christoffel word of slope $\frac{1}{a_i}$, hence H_i takes Christoffel words to Christoffel words by Theorem 2.16.

Case (ii), $r_i \geq b_i - r_i$: In this case, $H_i(w) = (x^{q_i+1}y, x^{q_i}y)(w)$. Note that

$$(\mathbf{G})^q \circ \tilde{\mathbf{D}} \circ (x, y) = (\mathbf{G})^q \circ (xy, y) = (x^{q_i+1}y, x^{q_i}y) = H_i,$$

so H_i is a composition of $\mathbf{G}$s and $\tilde{\mathbf{D}}$s, thus H_i takes Christoffel words to Christoffel words.

Case (iii), $r_i < b_i - r_i$: In this case, $H_i(w) = (x^{q_i}y, x^{q_i+1}y)(\tilde{w})$. Let $F = (x^{q_i}y, x^{q_i+1}y)$. Consider

$$H_i(w) = F(\tilde{w}) = F \circ \mathbf{E} \circ \mathbf{E}^{-1}(\tilde{w}).$$

Since $\mathbf{E} \circ \mathbf{E} = (y, x) \circ (y, x) = (x, y)$, $\mathbf{E} = \mathbf{E}^{-1}$, so

$$H_i(w) = F \circ \mathbf{E} \circ \mathbf{E}(\tilde{w}),$$

thus

$$F \circ \mathbf{E} = (x^{q_i}y, x^{q_i+1}y) \circ (y, x) = (x^{q_i+1}y, x^{q_i}y).$$

As seen in case (i), $F \circ \mathbf{E}$ takes Christoffel words to Christoffel words. It remains to be shown that $\mathbf{E}(\tilde{w})$ takes Christoffel words to Christoffel words.

Let w_0 be a Christoffel word. Consider the lower Christoffel path for w_0 of slope $\frac{d}{c}$. Reversing the word w_0 and switching the xs and ys is analogous to constructing the word w_0' by following the Christoffel path for w starting at (c, d) and ending at $(0, 0)$ and considering each vertical step an x and each horizontal step a y. Since the region between the Christoffel path and the line of slope $\frac{d}{c}$ contains no integer points, and w_0 ends in a $y \Rightarrow w_0'$ starts with an x, w_0' is a lower Christoffel word. Thus, $\mathbf{E}(\tilde{w_0})$ is a Christoffel word, so H_i takes Christoffel words to Christoffel words.

Part 2: In order to use induction on n, it is necessary to show that the algorithm eventually terminates. Of course, if $a = 1$ or $b = 1$, that algorithm terminates after one iteration. Suppose $a_i, b_i \neq 1$. For any $a_i \perp b_i$, the division algorithm yields $q_i, r_i \in \mathbb{Z}$ such that $a_i = b_i q_i + r_i$ with $0 \leq r_i < b_i$. $a_i \perp b_i \Rightarrow r_i \neq 0$. Then both r_i and $b_i - r_i$ are less than $b_i \Rightarrow$ both a_{i+1} and b_{i+1} are less than b_i.

We need to show that r_i and $b_i - r_i$ are relatively prime. Let k divide both r_i and $b_i - r_i$. Then k divides any linear combination of r_i and $b_i - r_i$, in particular,

$$k | q_i(b_i - r_i) + (q_i + 1)r_0 = b_i q_i + r_i = a_i \text{ and } k | b_i - r_i + r_i = b_i \Rightarrow k = 1.$$

Thus, $gcd(r_i, b_i - r_i) = 1 \Rightarrow gcd(a_{i+1}, b_{i+1}) = 1$. Note that for $a_i, b_i > 1$, $r_i \neq 0$, otherwise $a_i = b_i q_i \Rightarrow gcd(a_i, b_i) = b_i > 1$, which is a contradiction. Since r_i also cannot be equal to b_i, a_{i+1} and b_{i+1} cannot be 0. By induction, $a_i = 1$ or $b_i = 1$ for some $0 \leq i \in \mathbb{Z}$. The algorithm can then be applied to $\frac{r_i}{b_i - r_i}$ or $\frac{b_i - r_i}{r_i}$, and the algorithm will eventually terminate.

Now, use induction on n. In the case where $n = 0$, $w = H_0(xy) = xy^{b_0} = xy^b$ or $w = H_0(xy) = x^{a_0}y = x^a y$. In either case, w is the correct Christoffel word. Let $n > 0$ and assume the theorem holds for $0 \leq i < n$. Since $n > 2$, $a_0, b_0 \neq 1$. Since the algorithm takes $n-1$ iterations to find the Christoffel word of slope $\frac{b_1}{a_1}$, the resulting word $w_1 = H_1 \circ ... \circ H_n(xy)$ is correct by hypothesis.

If $r_0 \geq b_0 - r_0$, $w = H_0(w_0) = (x^{q_0+1}y, x^{q_0}y)(w_0)$. w_0 contains r_0 instances of x and $b_0 - r_0$ instances of y, so w contains

$$r_0(q_0 + 1) + (b_0 - r_0)q_0 = r_0 q_0 + r_0 + b_0 q_0 - r_0 q_0 = b_0 q_0 + r_0 = a_0 = a$$

instances of x and

$$r_0(1) + (b_0 - r_0)(1) = b_0 = b$$

instances of y. If $r_0 < b_0 - r_0$, then $w = H_0(w_0) = (x^{q_0}y, x^{q_0+1}y)(\tilde{w_0})$. w_0 and $\tilde{w_0}$ both contain $b_0 - r_0$ instances of x and r_0 instances of y, so w contains

$$(b_0 - r_0)q_0 + r_0(q_0 + 1) = a_0 = a$$

instances of x and

$$(b_0 - r_0)(1) + r_0(1) = b_0 = b$$

instances of y. In either case, H_0 takes Christoffel words to Christoffel words by Part 1, so $w = H_0(w_0)$ is the Christoffel word of slope $\frac{b}{a}$. $\square$

3.2 Box Algorithm Examples

Example 3.1. $a = 9$ and $b = 5$.

First iteration: $a_0 = 9$, $b_0 = 5$. The division algorithm yields $q_0 = 1$ and $r_0 = 4$. $4 \geq 5 - 4 = 1$, so $a_1 = r_0 = 4$ and $b_1 = b_0 - r_0 = 1$, thus, $H_0(w) = (x^2y, xy)$.

Second iteration: $a_1 = 4$, $b_1 = 1$. Then $H_1(w) = (x, x^3y)(w)$.

Finally, $w = H_0 \circ H_1(xy) = H_0(x^4y) = (x^2y)^4xy = xxyxxyxxyxxyxy$, which is indeed the Christoffel word of slope $\frac{5}{9}$.

Example 3.2. $a = 6$ and $b = 13$.

First iteration: $a_0 = 6$, $b_0 = 13$. Then $q_0 = 0$ and $r_0 = 6$. $6 < 13 - 6 = 7$, so $a_1 = 7$ and $b_1 = 6$. $H_0(w) = (y, xy)(\tilde{w})$.

Second iteration: $a_1 = 7$, $b_1 = 6$. Then $q_1 = 1$ and $r_1 = 1$. $1 < 6 - 1 = 5$, so $a_2 = 5$ and $b_2 = 1$. $H_2(w) = (xy, x^2y)(\tilde{w})$.

Third iteration: $a_2 = 5$, $b_2 = 1$. Then $H_2(w) = (x, x^4y)(w)$.

Finally, $w = H_0 \circ H_1 \circ H_2(xy) = H_0 \circ H_1(x^5y) = H_0(x^2y(xy)^5) = (xyy)^5xyy^2 = xyyxyyxyyxyyxyyxyyy$, which is the Christoffel word of slope $\frac{13}{6}$.

These examples can be reworked in a similar way which showcases more clearly how this algorithm builds Christoffel words from Christoffel words.

Example 3.3 (Example 3.1 reworked)**.** $a = 9$ and $b = 5$. So there are 9 xs and 5 ys.

First iteration: Take 5 boxes and place a y in each box. Then distribute the xs evenly among the boxes by placing an x in each box until there are 4 left, and put each of the 4 left in one of the boxes. Call boxes with the extra x A, and call the box without the extra x B. Then there are 4 boxes A containing xxy and 1 box B containing xy. Repeat the process with the As and Bs taking the place of the xs and ys, respectively.

Second iteration: Take a box and put the lone B in it. To distribute the As evenly, they must all go in the lone box with B, call it box C. Now, "unpack" the boxes:

$$C = AAAAB = xxyxxyxxyxxyxy,$$

which is the Christoffel word of slope $\frac{5}{9}$.

The box algorithm can be thought of as "packing" the xs and ys of a into boxes. Evenly distributing the xs among the ys is analogous to using the division algorithm in step 3. Since the number of boxes of each type is determined in the same way as step 3, those boxes can then be packed into boxes. This process repeats until there is only one of a type of box, at which point the boxes are then "unpacked," revealing the Christoffel word. The previous example dealt with the case when $r_i \geq b_i - r_i$, in which case, subsequent iterations consider the boxes with the extra x as the xs for the next iteration and the boxes without the extra x the ys for the next iteration. This reflects how the homomorphism $H_i(w) = (x^{q_i+1}y, x^{q_i}y)(w)$ contains an extra x in the first input. The next example deals with the case where $r_i < b_i - r_i$.

Example 3.4. (Example 3.2 reworked) $a = 6$ and $b = 13$. So there are 6 xs and 13 ys.

First iteration: Take 13 boxes and place a y in each box. Then distribute the xs evenly among the boxes by placing each of them in a box. Call boxes with an x A, and call the box without an x B. Then there are 6 boxes A containing xy and 7 boxes B containing y. Repeat the process with the Bs and As taking the place of the xs and ys, respectively.

Second iteration: Take 6 boxes and place an A in each box. Then distribute the Bs evenly like the previous iteration. Then there is 1 box D containing ABB and 5 boxes C containing AB. Repeat the process with the Ds and Cs taking the place of the Bs and As, respectively.

Third iteration: Since there is only 1 D, the box E contains all the Cs and Ds. Now, "unpack" the boxes:

$$E = CCCCCD = ABABABABABABB = xyyxyyxyyxyyxyyxyyy.$$

3.3 GAP Implementation of the Box Algorithm

This algorithm takes a rational number p and produces the Christoffel word of slope p using the box algorithm.

```
AppendPower := function(w, l, n) # Appends l instances of n to w
    local i;
    for i in [1..l] do
        Append(w,[n]);
    od;
    return(w);
end;

cwbox := function(p) # Box Algorithm, p > 0
    local r, q, a, b, w, wi, j, f, i;
    a := DenominatorRat(p); #Step 1
    b := NumeratorRat(p);
    r := a mod b; # Division algorithm, Step 3
    q := (a-r)/b;
    if (a > 1 and b > 1) then
        if (r < b-r) then
            wi := cwbox(r/(b-r)); # Step 5
        else
            wi := cwbox((b-r)/r); # Step 4
        fi;
    else # Step 2
        w := [];
        AppendPower(w,a,0);
        AppendPower(w,b,1);
        return w;
    fi;
    w := []; # Step 6, the remaining code builds the word
    f := Length(wi);
    if (r >= b - r) then
        for i in [1..f] do
            if (wi[i] = 0) then
                Append(w, [0]);
            fi;
            AppendPower(w,q,0);
            Append(w, [1]);
        od;
    else
        for i in [1..f] do
```

```
            if (wi[f - (i-1)] = 1) then
                Append(w, [0]);
            fi;
            AppendPower(w,q,0);
            Append(w, [1]);
        od;
    fi;
    return w;
end;
```

4 Nearest Word Algorithm

In this section, we develop the next algorithm: the nearest word algorithm. This algorithm does not produce the Christoffel word of a given slope, but it does produce a particular conjugate. In order to understand this algorithm we will first gain an understanding of how the conjugates of a Christoffel word are related via the Burrows-Wheeler matrix. This construction was introduced by Burrows and Wheeler in [6].

4.1 Burrows-Wheeler Matrix

Let w be a Christoffel word. The Burrows-Wheeler matrix of w is constructed by listing w and its conjugates in lexicographic order (i.e. the ordering in a dictionary). For our purposes, we only need for x to come before y in the ordering. As an example, here is the Burrows-Wheeler matrix for the Christoffel word of slope $\frac{3}{4}$,

$$\begin{matrix} x & x & y & x & y & x & y \\ x & y & x & x & y & x & y \\ x & y & x & y & x & x & y \\ x & y & x & y & x & y & x \\ y & x & x & y & x & y & x \\ y & x & y & x & x & y & x \\ y & x & y & x & y & x & x \end{matrix}$$

By studying BMW, certain patterns become apparent. These are summarized as properties in the following theorem

Theorem 4.1. Let w be the Christoffel word of slope $\frac{b}{a}$. Let $BWM(w)$ denote the Burrows-Wheeler matrix for w. The following properties hold:

1. The first and last rows of $BWM(w)$ are, respectively, the lower and upper Christoffel words of slope $\frac{b}{a}$.

2. Any two consecutive rows of $BWM(w)$ differ in exactly two consecutive positions.

3. The t-th row of $BWM(w)$ is the reversal of the $(a+b-1-t)$-th row of $BWM(w)$.

4. If the first row of $BWM(w)$ is xuy, the last row is yux.

Proof. Proof of (1) and (2): Let w be the Christoffel word of slope $\frac{b}{a}$, and let w_t denote the word obtained by reading $|w|$ words along the path given by ww starting at the lattice point T with label $\frac{t}{a}$. In particular, $w_0 = w$, and note that $0 \leq t < a+b$. Define $n_t(k)$ to be the numerator of the label of the point k steps after T along the path for w_t. That is, $n_t(k) = t + bk \bmod (a+b)$. By Lemma 2.1, $n_0(k)$ and thus $n_t(k)$ takes on each of $n = 0, 1, ..., a+b-1$ exactly once as k ranges over $\{0, \ldots, a+b-1\}$. Suppose w_{t-1} and w_t are two consecutive conjugates. Observe that

$$n_t(k) = t + kb \bmod (a+b) \text{ and } n_{t-1}(k) = (t-1) + kb \bmod (a+b)$$
$$\Rightarrow n_t(k) = n_{t-1}(k) + 1 \bmod (a+b).$$

Since $n_t(k) = 0 \bmod (a+b)$ at exactly one instance of k, say k', then for $k \neq k'$, $n_{t-1}(k) \bmod (a+b) < n_t(k) \bmod (a+b)$ and

$$0 = n_t(k') \bmod (a+b) < n_{t-1}(k') = a+b-1 \bmod (a+b).$$

Recall that the $(k+1)$-st letter of a Christoffel word is x if and only if the label at the k-th point on the Christoffel path is less than 1. Then the $(k+1)$-st letter of w_t is x if and only if $n_t(k) \bmod (a+b) < a$. By using this fact, a correspondence can be made between $n_t(k)$ and w_t. If $0 < n_t(k) < a$, then $0 \leq n_{t-1}(k) < a-1 \Rightarrow$ the $(k+1)$-st letters of w_t and w_{t-1} are both x. If $a < n_t(k) < a+b$, then $a \leq n_{t-1}(k) < a+b-1 \Rightarrow$ the $(k+1)$-st letters of w_t and w_{t-1} are both y. In the case that $n_t(k) = 0$, $k = k'$ and $n_{t-1}(k') = a+b-1$, so the $(k+1)$-st letter of w_t is x and the $(k+1)$-st of w_{t-1} is y. All that remains is $n_t(k) = a$, in which case the $(k+1)-st$ letter of w_t is y and the $(k+1)-st$ of w_{t-1} is x. Observe that $n_t(k'-1) = t+(k'-1)b = t+k'b-b = n_t(k')-b = -b = a$ indicating that this case occurs at $k = (k'-1)$. Thus, for any $t = 1, 2, ..., a+b-1$, $w_{t-1} = uxyv$ and $w_t = uyxv$ for some $u, v \in \{x, y\}^*$.

Using this representation, w_t is after w_{t-1} in lexicographic order, thus the $(t+1)$-st row of $BWM(w)$ is w_t, finishing the proof for Part 2. To finish off Part 1, $w_0 = w$ by the definition of w_t. Recall from the proof of Lemma 2.9 that the conjugate path for the upper Christoffel word of slope $\frac{b}{a}$ begins at the point with label $\frac{a+b-1}{a}$. It follows that w_{a+b-1} is the upper Christoffel word.

Proof of (3): Suppose w is the Christoffel word of slope $\frac{b}{a}$. Define $B\tilde{W}M(w)$ to be the Burrows-Wheeler matrix of w with each row reversed. Then $E(B\tilde{W}M(w))$ is the Burrows-Wheeler matrix for the Christoffel word of slope $\frac{a}{b}$, w'. Next, define $\rho(BWM(w))$ to be the Burrows-Wheeler matrix of w with the $(t+1)$-th row swapped with the $(a+b-1-t)$-th row. Since the reversal of w is w_{a+b-1}, $E(\rho(BWM(w)))$ is also the Burrows-Wheeler matrix for w'. Then $E(B\tilde{W}M(w)) = E(\rho(BWM(w))) \Leftrightarrow B\tilde{W}M(w) = \rho(BWM)(w)$.

Proof of (4): Observe that

$$n_0(k) = kb \bmod (a+b) \text{ and } n_{a+b-1}(k) = (-1) + kb \bmod (a+b)$$
$$\Rightarrow n_0(k) = n_{a+b-1}(k) + 1 \bmod (a+b)$$

and proceed as in the proof for (1) and (2). $\square$

Using these facts, it is easy to show that the central rows of $BWM(w)$ have a special property.

Corollary 4.2. If $a+b$ is odd, the middle row which is the $(\frac{a+b-1}{2})$-nd row of $BWM(w)$ is a palindrome.

Proof. Using Part (3) of Theorem 4.1, the $(\frac{a+b-1}{2})$-nd row is the reversal of the $(a+b-1-\frac{a+b-1}{2})$-nd row. Observe that $a+b-1-\frac{a+b-1}{2}=\frac{2a+2b-2}{2}-\frac{a+b-1}{2}=\frac{a+b-1}{2}$, and the proof is done. $\square$

A similar result is found when $a+b$ is even, and although no conjugates are palindromes a certain pair of conjugates which are reversals of each other are close to being palindromes.

Corollary 4.3. If $a+b$ is even, the $(\frac{a+b-2}{2})$-nd and $(\frac{a+b}{2})$-nd row of $BWM(w)$ are respectively of the form $uxy\tilde{u}$ and $uyx\tilde{u}$ for some $u \in \{x,y\}^*$.

Proof. By Part (3) of Theorem 4.1, the $(\frac{a+b-2}{2})$-nd row of $BWM(w)$ is the reversal of the $(\frac{a+b}{2})$-nd row. Observe that

$$n_{\frac{a+b-2}{2}}(\frac{a+b-2}{2})=\frac{a+b-2}{2}+\frac{a+b-2}{2}b=-1-b=a-1, \text{ and}$$
$$n_{\frac{a+b}{2}}(\frac{a+b-2}{2})=\frac{a+b}{2}+\frac{a+b-2}{2}b=-b=a.$$

Thus, the x and y at the $(\frac{a+b-2}{2})$-nd and $(\frac{a+b}{2})$-nd position of the $w_{\frac{a+b-2}{2}}$ and $w_{\frac{a+b}{2}}$ swap. This swap is guaranteed to occur only once between two consecutive rows of $BWM(w)$ by Part (2). Note that the positions of the swapping are the two in the middle of the word, so $w_{\frac{a+b-2}{2}}=uxyv$ and $w_{\frac{a+b}{2}}=uyxv$ where $u,v\in\{x,y\}^*$ and $|u|=|v|$. But Part (3) gives that $\tilde{v}yx\tilde{u}=uyxv$, so $\tilde{u}=v$. $\square$

4.2 Nearest Word Algorithm

The next algorithm does not produce a Christoffel word, but it either produces a conjugate of a Christoffel word which is a palindrome, or it produces two conjugates of a Christoffel word which are reversals of one another. Interestingly, this algorithm arises from producing a sequence of unit steps on the gridded integer lattice which lie as ”close” to the line of slope $\frac{b}{a}$ as possible.

Definition 4.1. Let a and b be relatively prime, positive integers. Define the nearest word algorithm as follows:

1. Set $i = j = 0$. Let w be an empty word.

2. Repeat steps 2 through 4 until $i + j = a + b$.

   (i) If $|j - \frac{b}{a}(i+1)| < |(j+1) - \frac{b}{a}(i)|$, proceed to step 3.

   (ii) If $|j - \frac{b}{a}(i+1)| > |(j+1) - \frac{b}{a}(i)|$, proceed to step 4.

   (iii) If $|j - \frac{b}{a}(i+1)| = |(j+1) - \frac{b}{a}(i)|$, proceed to either step 3 or step 4.

3. Append an x to w and set $i = i + 1$.

4. Append a y to w and set $j = j + 1$.

Theorem 4.4. Let a and b be relatively prime, positive integers. The nearest word algorithm on a and b produces a conjugate(s) of the Christoffel word of slope $\frac{b}{a}$. If $a + b$ is odd, this conjugate is a palindrome. If $a + b$ is even, the algorithm produces two conjugates which are reversals of each other.

Proof. Let a and b be relatively prime, positive integers, and let w be the Christoffel word of slope $\frac{b}{a}$. Let L be the line of slope $\frac{b}{a}$ passing through the origin. Recall from the proof of Lemma 2.9 that there is a unique point P on the Christoffel path C of w which has maximum vertical distance from L. This is the point with label $\frac{a+b-1}{a}$, and let L' be the line of slope $\frac{b}{a}$ passing through point P. As explained in the proof of Lemma 2.9, there are no integer points in the region R between L and L' inclusive which are not on C. The proof is now split in two cases.

Case 1, $a + b$ is odd: Let $\bar{L}$ be the line $y = \frac{b}{a}x - \frac{a+b-1}{2}$. Note that $\bar{L}$ is exactly halfway between L and L'. The line $\bar{L}$ also passes through the point P. Suppose $P = (x_0, y_0)$, and for an integer point (i', j') let $(i, j) = (i', j') - P$. Suppose $i' \geq x_0$ and $j' \geq y_0$, and then compare the vertical distance of $(i'+1, j')$ and $(i', j'+1)$ to $\bar{L}$. Observe that

$$\begin{aligned}
|(j'+1) - \frac{b}{a}i'| = |(j+y_0+1) - \frac{b}{a}(i+x_0)| &= |(j+y_0+1) - \frac{b}{a}i - y_0)| \\
&= |(j+1) - \frac{b}{a}i|, \text{ and} \\
|j' - \frac{b}{a}(i'+1)| = |(j+y_0) - \frac{b}{a}(i+x_0+1)| &= |(j+y_0) - \frac{b}{a}(i+1) - x_0)| \\
&= |j - \frac{b}{a}(i+1)|.
\end{aligned}$$

Now, consider what happens when (i', j') is a point on the path encoded by ww. In this scenario, either $(i'+1, j')$ is on the path or $(i', j'+1)$ is on the path. Suppose $(i'+1, j')$ is on the path which forces $(i', j'+1)$ to be above the line L. Thus, $|(j+1) - \frac{b}{a}i| > \frac{a+b-1}{2a}$ and $|j - \frac{b}{a}(i+1)| \leq \frac{a+b-1}{2a}$ since $(i'+1, j')$ is on the path and therefore between L and L'. Consequently, $|j - \frac{b}{a}(i+1)| < |(j+1) - \frac{b}{a}i|$. The step from (i', j') to $(i'+1, j')$ encodes an x.

A similar inequality arises in the case when $(i', j'+1)$ is on the path. In this case, $(i'+, j'-1)$ is below the line L'. Thus, $|(j+1) - \frac{b}{a}i| \leq \frac{a+b-1}{2a}$ and $|j - \frac{b}{a}(i+1)| > \frac{a+b-1}{2a}$ since $(i', j'+1)$ is on the path and hence between L and L'. Thus, $|j - \frac{b}{a}(i+1)| > |(j+1) - \frac{b}{a}i|$. The step from (i', j') to $(i', j'+1)$ encodes a y.

Define the sequence $(i', j')_k$ with $0 \leq k < a+b$ with $(i', j')_0 = P$. Let $(i, j)_k = (i', j')_k - P$, thus $(i, j)_0 = (0, 0)$. Let $(i', j')_{k+1} = (i'+1, j')$ if $|j - \frac{b}{a}(i+1)| < |(j+1) - \frac{b}{a}i|$, and $(i', j')_{k+1} = (i', j'+1)$ if $|j - \frac{b}{a}(i+1)| > |(j+1) - \frac{b}{a}i|$. By the above inequalities, $(i', j')_k$ forms, in order, the integer points on the path of ww from P to the second point with label $\frac{a+b-1}{2a}$. This is precisely the conjugate of w beginning at P which is the $(\frac{a+b-1}{2}+1)$-st row of $BWM(w)$ which is a palindrome by Corollary 4.2.

Case 2, $a+b$ is even: There are two possible starting points which will necessarily be considered: the point on the path with label $\frac{a+b-2}{2a}$ and the point with label $\frac{a+b}{2a}$. Let $\bar{L}$ be the line of slope $\frac{b}{a}$ which passes through the point P with label $\frac{a+b-2}{2a}$. Suppose $P = (x_0, y_0)$, and for an integer point (i', j') define $(i, j) = (i', j') - P$. As before, suppose $i' \geq x_0$ and $j' \geq y_0$.

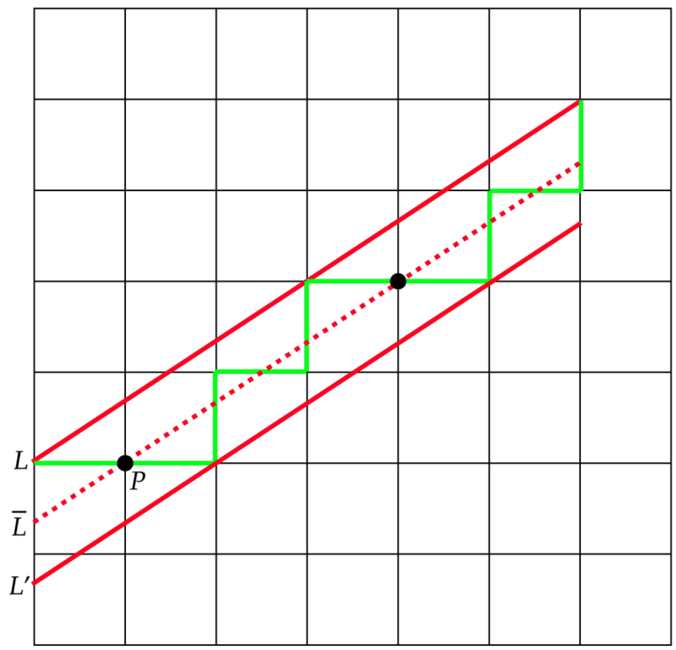

Figure 9: The point P has label $\frac{2}{3}$, and the line $\bar{L}$ which passes through P evenly splits L and L'. The Christoffel path stays between L and L'. The nearest word algorithm produces the portion of the path from P to the next integer point intersecting $\bar{L}$.

The same procedure from Case 1 will be used to find vertical distances to $\bar{L}$; however, there is one situation which will be accounted for separately.

Suppose (i', j') is a point on w with $(i', j'+1)$ lying above L. In addition, assume the vertical distance from $(j'+1, i')$ to L is greater than $\frac{1}{a}$. Recall that $(i'+1, j')$ is on the path, and the vertical distance from $(i'+1, j')$ to $\bar{L}$ is less than or equal to $\frac{a+b}{2a}$. Then

$$\begin{aligned}
|(j'+1) - \frac{b}{a}i'| &= |(j+y_0+1) - \frac{b}{a}(i+x_0)| = |(j+y_0+1) - \frac{b}{a}i - y_0)| \\
&= |(j+1) - \frac{b}{a}i| > \frac{a+b-2}{2a} + \frac{1}{a} = \frac{a+b}{2a}, \text{ and} \\
|j' - \frac{b}{a}(i'+1)| &= |(j+y_0) - \frac{b}{a}(i+x_0+1)| = |(j+y_0) - \frac{b}{a}(i+1) - x_0)| \\
&= |j - \frac{b}{a}(i+1)| \leq \frac{a+b}{2a}.
\end{aligned}$$

Thus, $|j - \frac{b}{a}(i+1)| < |(j+1) - \frac{b}{a}i|$. In the case that $(j'+1, i')$ has vertical distance of exactly $\frac{1}{a}$ to $\bar{L}$, the equation

$$|(j+1) - \frac{b}{a}i| = \frac{a+b-2}{2a} + \frac{1}{a} = \frac{a+b}{2a} \text{ and}$$
$$|j - \frac{b}{a}(i+1)| = 1 - \frac{a+b-2}{2a} - \frac{1}{a} + \frac{b}{a} = \frac{a+b}{2a}$$

is obtained. Thus, $|(j+1) - \frac{b}{a}i| = |j - \frac{b}{a}(i+1)|$. Note that this only occurs when the label of (i', j') is $\frac{a-1}{a}$, since $1 - \frac{1}{a} = \frac{a-1}{a}$. Call P' the point with label $\frac{a-1}{a}$. This lines up with the $(\frac{a+b-2}{2})$-nd step of the $(\frac{a+b-2}{2})$-nd row of $BWM(w)$ because $n_{\frac{a+b-2}{2}}(\frac{a+b-2}{2}) = \frac{a+b-2}{a} + \frac{a+b-2}{2}b = -b-1 = a-1 \mod (a+b)$.

In the case where $(i'+1, j')$ is below L', then

$$|j' - \frac{b}{a}(i'+1)| = |j - \frac{b}{a}(i+1)| > \frac{a+b}{2a} \text{ and}$$
$$|(j'+1) - \frac{b}{a}i'| = |(j+1) - \frac{b}{a}i| \leq \frac{a+b}{2a} \Rightarrow$$
$$|(j+1) - \frac{b}{a}i| < |j - \frac{b}{a}(i+1)|.$$

Define the sequence $(i', j')_k$ with $0 \leq k < a+b$ such that $(i', j')_0 = P$. Let $(i, j)_k = (i', j')_k - P$, thus $(i, j)_0 = (0, 0)$. Let $(i', j')_{k+1} = (i'+1, j')$ if $|j - \frac{b}{a}(i+1)| < |(j+1) - \frac{b}{a}i|$, and $(i', j')_{k+1} = (i', j'+1)$ if $|j - \frac{b}{a}(i+1)| > |(j+1) - \frac{b}{a}i|$. At $(i', j')'_k = P'$, $|j - \frac{b}{a}(i+1)| = |(j+1) - \frac{b}{a}i|$, in which case $(i', j')_{k'+1}$ can be either $(i'+1, j')$ or $(i', j'+1)$. The choices eventually reunite at $(i'+1, j'+1)$, which is on the path. If $(i'+1, j')$ is chosen, then $(i', j')_{k'+2} = (i'+1, j'+1)$ because $|(j_k+1) - \frac{b}{a}(i_k+1)| \leq \frac{a+b}{2a}$ since $(i'+1, j'+1)$ is between L and L' and $|j_k - \frac{b}{a}(i_k+2)| > \frac{a+b}{2a}$ since $(i'+1, j'+1)$ is on the path $\Rightarrow$ $(i'+2, j')$ is not on the path. If $(i', j'+1)$ is chosen, then $(i', j')_{k'+2} = (i'+1, j'+1)$ since $|(j_k+2) - \frac{b}{a}i_k| > \frac{1}{a} + \frac{a+b-2}{2a} = \frac{a+b}{2a}$.

In summary, there are two possible paths defined by $(i', j')_k$. One choice follows the Christoffel path encoding the word $w_{\frac{a+b-2}{2}}$, and the other choice is obtained by swapping the

x and y in the $(\frac{a+b-2}{2})$-nd and $(\frac{a+b}{2})$-nd places of $w_{\frac{a+b-2}{2}}$, which produces $w_{\frac{a+b}{2}}$. By Corollary 4.2, the $(\frac{a+b-2}{2})$-nd and $(\frac{a+b}{2})$-nd row of $BWM(w)$ are reversals of each other, and the proof is complete. $\square$

Here are a few examples:

Example 4.1. Let $a = 7$ and $b = 2$. Let w be the empty word.

1. $i = j = 0$, $|0 - \frac{7}{2}(1)| = \frac{7}{2} > 1 = |1 - \frac{7}{2}(0)|$. $w = y$, and $i = 0$, $j = 1$.

2. $i = 0$, $j = 1$, $|1 - \frac{7}{2}(1)| = \frac{5}{2} > 2 = |2 - \frac{7}{2}(0)|$. $w = yy$, and $i = 0$, $j = 2$.

3. $i = 0$, $j = 2$, $|2 - \frac{7}{2}(1)| = \frac{3}{2} < 3 = |3 - \frac{7}{2}(0)|$. $w = yyx$, and $i = 1$, $j = 2$.

4. $i = 1$, $j = 2$, $|2 - \frac{7}{2}(2)| = 5 > \frac{1}{2} = |3 - \frac{7}{2}(1)|$. $w = yyxy$, and $i = 1$, $j = 3$.

5. $i = 1$, $j = 3$, $|3 - \frac{7}{2}(2)| = 4 > \frac{1}{2} = |4 - \frac{7}{2}(1)|$. $w = yyxyy$, and $i = 1$, $j = 4$.

6. $i = 1$, $j = 4$, $|4 - \frac{7}{2}(2)| = 3 > \frac{3}{2} = |5 - \frac{7}{2}(1)|$. $w = yyxyyy$, and $i = 1$, $j = 5$.

7. $i = 1$, $j = 5$, $|5 - \frac{7}{2}(2)| = 2 < \frac{5}{2} = |6 - \frac{7}{2}(1)|$. $w = yyxyyyx$, and $i = 2$, $j = 5$.

8. $i = 2$, $j = 5$, $|5 - \frac{7}{2}(3)| = \frac{11}{2} > 1 = |6 - \frac{7}{2}(2)|$. $w = yyxyyyxy$, and $i = 2$, $j = 6$.

9. $i = 2$, $j = 6$, $|6 - \frac{7}{2}(3)| = \frac{9}{2} > 0 = |7 - \frac{7}{2}(2)|$. $w = yyxyyyxyy$, and $i = 2$, $j = 7$. $i + j = 9 = a + b$, and the algorithm is complete.

Indeed, $w = yyxyyyxyy$ is the correct conjugate.

The next example involves the case when $a + b$ is even.

Example 4.2. Let $a = 3$ and $b = 5$. Let w be the empty word.

1. $i = j = 0$, $|0 - \frac{3}{5}(1)| = \frac{3}{5} < 1 = |1 - \frac{3}{5}(0)|$. $w = x$, and $i = 1$, $j = 0$.

2. $i = 1$, $j = 0$, $|0 - \frac{3}{5}(2)| = \frac{6}{5} > \frac{2}{5} = |1 - \frac{3}{5}(1)|$. $w = xy$, and $i = 1$, $j = 1$.

3. $i = 1$, $j = 1$, $|1 - \frac{3}{5}(2)| = \frac{1}{5} < \frac{7}{5} = |2 - \frac{3}{5}(1)|$. $w = xyx$, and $i = 2$, $j = 1$.

4. $i = 2$, $j = 1$, $|1 - \frac{3}{5}(3)| = \frac{4}{5} = \frac{4}{5} = |2 - \frac{3}{5}(2)|$. Going forward there are two possible words: $w = xyxx$ and $w' = xyxy$, and either $i = 3$ and $j = 1$ or $i = 2$ and $j = 2$.

5. If $i = 3$, $j = 1$, $|1 - \frac{3}{5}(4)| = \frac{7}{5} > \frac{1}{5} = |2 - \frac{3}{5}(3)|$. Then $w = xyxxy$. If $i = 2$, $j = 2$, $|2 - \frac{3}{5}(3)| = \frac{4}{5} < \frac{11}{5} = |3 - \frac{3}{5}(2)|$. Then $w' = xyxyx$. In either case, $i = 3$, $j = 2$.

6. $i = 3$, $j = 2$, $|2 - \frac{3}{5}(4)| = \frac{2}{5} < \frac{4}{5} = |3 - \frac{3}{5}(3)|$. $w = xyxxyx$ or $w' = xyxyxx$, and $i = 4$, $j = 2$.

7. $i = 4$, $j = 2$, $|2 - \frac{3}{5}(5)| = 1 > \frac{3}{5} = |3 - \frac{3}{5}(4)|$. $w = xyxxyxy$ or $w' = xyxyxxy$, and $i = 4$, $j = 3$.

8. $i = 4$, $j = 3$, $|3 - \frac{3}{5}(5)| = 0 < \frac{8}{5} = |4 - \frac{3}{5}(4)|$. $w = xyxxyxyx$ or $w' = xyxyxxyx$. $i + j = 8 = a + b$, and the algorithm ends.

The words w and w' are produced, which are the correct conjugates. Also, $\tilde{w} = w'$, and taking $u = xyx$ gives that $w = uxy\tilde{u}$ and $w' = uyx\tilde{u}$.

There is another way to define Step 2 of the nearest word algorithm which has an identical output. Instead of checking vertical distance, this version checks horizontal distance, and the proof is analogous to the proof of the vertical case.

4.3 GAP Implementation of Nearest Word Algorithm

In the case when $a+b$ is even, the program produces the word which is the $(\frac{a+b-2}{2})$-nd row of $BWM(w)$.

```
NearestWord := function(r)
    local l, word, x, y, i, a, b;
    a := DenominatorRat(r);
    b := NumeratorRat(r);
    l := a + b;
    word := []; # Step 1
    x := 0;
    y := 0;
    for i in [1..l] do # Step 2
        if (AbsoluteValue(y-(b/a)*(x+1))<=AbsoluteValue((y+1)-(b/a)*x)) then
            Append(word,[0]); # Step 3
            x := x + 1;
        else
            Append(word,[1]); # Step 4
            y := y + 1;
        fi;
    od;
    return(word);
end;
```

5 Continued Fractions

This section observes the connection between Christoffel words and continued fractions. Much of the work discussed is based on a note written by Henry J. Smith in 1876 [18], and his explanation presents a connection to the integer characterization which will be used in Section 6. An algorithm which produces Christoffel words based on continued fraction representations is included.

5.1 Definition of Continued Fractions

Definition 5.1. Let $\alpha = \frac{b}{a}$ be a rational number. Construct a sequence as follows:

1. Set $b_0 = \alpha$ and $a_0 = \lfloor b_0 \rfloor$.
2. If $i > 0$ and $a_{i-1} \neq b_{i-1}$, then set
$$b_i = \frac{1}{b_{i-1} - a_{i-1}} \text{ and } a_i = \lfloor b_i \rfloor.$$
3. If $i > 0$ and $a_{i-1} = b_{i-1}$, the recursion terminates.

The resulting sequence $[a_0, ..., a_n]$ is called the simple continued fraction representation of α and can also be written
$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}.$$
Since α is a rational number, the sequence will always terminate. Furthermore, for $0 \leq i \leq n$ the i-th continuant is the rational number obtained from the sequence $[a_0, ..., a_i]$.

It is important to note that every rational number has two continued fraction representations, $[a_0, ..., a_n]$ and $[a_0, ..., a_n - 1, 1]$. Using continued fractions, Smith formulated the following characterization of Christoffel words.

Theorem 5.1. (Smith [18]) A word $w = xuy$ is a Christoffel word if and only if $uxy = s_n$ or $uyx = s_n$ where s_n is defined recursively as $s_{-1} = x$, $s_0 = y$, and $s_{i+1} = s_i^{c_i} s_{i-1}$ for $0 \leq i < n$, where $[c_0, \ldots, c_n]$ is the continued fraction representation of $\frac{w_y}{w_x}$.

The following example uses both continued fraction representations.

Example 5.1. Let w be the Christoffel word of slope $\frac{8}{3}$. Begin by finding the continued fraction representations.

$$\begin{aligned} b_0 &= \frac{8}{3},\ a_0 = \left\lfloor \frac{8}{3} \right\rfloor = 2 \\ b_1 &= \frac{1}{\frac{8}{3} - 2} = \frac{1}{\frac{2}{3}} = \frac{3}{2},\ a_1 = \left\lfloor \frac{3}{2} \right\rfloor = 1 \\ b_2 &= \frac{1}{\frac{3}{2} - 1} = \frac{1}{\frac{1}{2}} = 2,\ a_2 = \lfloor 2 \rfloor = 2 \end{aligned}$$

Thus, the two representations are $[2, 1, 2]$ and $[2, 1, 1, 1]$. Using the shorter representation, the Christoffel word can be obtained as such:

$$s_{-1} = x,\ s_0 = y$$

$$s_1 = s_0^2 s_{-1} = y^2 x$$

$$s_2 = s_1^1 s_0 = (y^2 x)^1 y = y^2 xy$$

$$s_3 = s_2^2 s_1 = (y^2 xy)^2 y^2 x = yyxyyyxyyyx.$$

Thus, $u = yyxyyyxyy \Rightarrow w = xuy = xyyxyyyxyyy$, which is indeed the correct Christoffel word. Using the other continued fraction representation yields

$$s_{-1} = x,\ s_0 = y$$
$$s_1 = s_0^2 s_{-1} = y^2 x$$
$$s_2 = s_1^1 s_0 = (y^2x)^1 y = y^2xy$$
$$s_3 = s_2^1 s_1 = (y^2xy)^1 y^2 x = y^2xy^3x$$
$$s_4 = s_3^1 s_2 = (y^2xy^3x)^1 y^2xy = yyxyyyxyyxy.$$

The word u obtained is the same as before, as is w; notably, the yx at the end of s_3 in the first changed to an xy at the end of s_4.

Remark If n is odd, s_n ends in yx. If n is even, s_n ends in xy. This is because $s_{n+1} = s_n^{c_n} s_{n-1}$ ends with the same letter as s_{n-1}. Since $s_{-1} = x$ and $s_0 = y$, an induction on n shows this remark to be true.

This theorem can be used in the other direction to find the continued fraction representation of a rational number from the Christoffel word. Let $v = uxy$. Let c_0 be the highest power of y which is a prefix of v. Suppose that $s_1, s_2, \ldots, s_{i+1}$ and $c_0, c_1 \ldots, c_i$ have already been found. Then c_{i+1} is the highest integer k such that $s_{i+1}^k s_i$ is a prefix of v. If $v = uxy$ does not work, try $v = uyx$. Here is an example using the Christoffel word of slope $\frac{8}{3}$.

Example 5.2. Suppose $w = xyyxyyyxyyy$, and let $v = yyxyyyxyyxy$. The highest power of y which is a prefix of v is 2, so $c_0 = 2$. From the theorem, $s_{-1} = x$, $s_0 = y$, and $s_{i+1} = s_i^{c_i} s_{i-1}$. Thus, $s_1 = s_0^2 s_{-1} = y^2x$. The highest power of $s_1 = y^2x$ which is a prefix of v is $1 = c_1$. Then, $s_2 = s_1^1 s_0 = y^2xy$, and the highest power of s_2 which is a prefix of v is $2 = c_2$. However, $s_3 = s_2^2 s_1 = yyxyyyxyyyx \neq v$, but this can be fixed be setting $v = yyxyyyxyyyx$ and repeating the same calculations. Hence, the continued fraction representation of $\frac{8}{3}$ is $[2, 1, 2]$, and the second representation must be $[2, 1, 1, 1]$.

5.2 Recursive Continued Fractions Algorithm

The following algorithm is related to Theorem 5.2, and it also creates a relationship to the standard factorization of Christoffel words. This algorithm is called the recursive continued fractions algorithm, which is abbreviated as RCFA.

Theorem 5.2. Suppose $\frac{b}{a}$ is a reduced rational number and has continued fraction representation $[c_0, \ldots, c_n]$. If $n = 0$, define $w_0 = xy^{c_0}$. If the continued fraction is empty, by convention $n = -1$ and $w_{-1} = y$. Suppose $n > 0$. Let w_{n-2} be the Christoffel word of ratio with convergent $[c_0, \ldots, c_{n-2}]$ and w_{n-1} the Christoffel word of ratio with convergent $[c_0, \ldots, c_{n-1}]$. The Christoffel word w_n of slope $\frac{b}{a}$ is $w_n = w_{n-2}w_{n-1}^{c_n}$ if n is even and $w_n = w_{n-1}^{c_n}w_{n-2}$ if n is odd.

Proof. A proof by induction will suffice. Clearly, the base cases $n = -1$ and $n = 0$ work, so assume that w_{n-2} is the Christoffel word of ratio with convergent $[c_0, \ldots, c_{n-2}]$ and w_{n-1} is the Christoffel word of ratio with convergent $[c_0, \ldots, c_{n-1}]$. There are two cases based on the parity of n.

Case 1, n is odd: By the remark, s_{n-1} ends with xy and s_n ends with yx. From Theorem 5.1, $s_{n+1} = s_n^{c_n}s_{n-1}$, $s_{n-1} = u_{n-1}xy$, and $s_n = u_nyx$ for some words u_{n-1}, u_n. Then $s_{n+1} = (u_nyx)^{c_n}(u_{n-1}xy) \Rightarrow w_n = x(u_nyx)^{c_n}u_{n-1}y = (xu_ny)^{c_n}xu_{n-1}y = w_{n-1}^{c_n}w_{n-2}$.

Case 2, n is even: By the remark, s_{n-1} ends with yx and s_n ends with xy. From Theorem 5.1, $s_{n+1} = s_n^{c_n}s_{n-1}$, $s_{n-1} = u_{n-1}yx$, and $s_n = u_nxy$ for some words u_{n-1}, u_n. Then $s_{n+1} = (u_nxy)^{c_n}(u_{n-1}yx) \Rightarrow w = x(u_nxy)^{c_n}u_{n-1}y$. Thus, the upper Christoffel word of $\frac{b}{a}$ is $w' = y(u_nxy)^{c_n}u_{n-1}x = (yu_nx)^{c_n}yu_{n-1}x$. By Theorem 4.1, $\tilde{w'} = w \Rightarrow x\tilde{u}_{n-1}y(x\tilde{u}_ny)^{c_n}$, and thus $x\tilde{u}_{n-1}y = xu_{n-1}y = w_{n-2}$ and $x\tilde{u}_ny = xu_ny = w_{n-1}$. Thus, $w = w_{n-2}w_{n-1}^{c_n}$. □

Corollary 5.3. Let $n > 0$. If $[c_0, \ldots, c_n]$ is the continued fraction representation of $\frac{b}{a}$ and $c_n = 1$, then the words w_{n-2} and w_{n-1} obtained in Theorem 5.2 form the standard factorization of w_n, the Christoffel word of slope $\frac{b}{a}$. If n is even, $w_n = w_{n-2}w_{n-1}$. If n is odd, $w = w_{n-1}w_{n-2}$.

Proof. The words w_{n-2} and w_{n-1} are Christoffel words by Theorem 5.1. Since $c_n = 1$, either $w_n = w_{n-2}w_{n-1}$ or $w_n = w_{n-1}w_{n-2}$. Either way, since factorization into two Christoffel words is unique, this must be the standard factorization. $\square$

If the convergent $[c_0, \ldots, c_n]$ does not end with $c_n = 1$, then a separate factorization $w_n = w_{n-2}w_{n-1}^{c_n}$ or $w_n = w_{n-1}^{c_n}w_{n-2}$ is obtained. Here, w is factored into more than two Christoffel words.

Example 5.3. Find the Christoffel word of slope $\frac{8}{3}$ using Theorem 5.2. The continued fraction representations of $\frac{8}{3}$ are $[2, 1, 2]$ and $[2, 1, 1, 1]$. Use the second representation.

1. $n = -1$, $w_{-1} = y$ and $n = 0$, $w_0 = xy^2$.

2. $n = 1$, $w_1 = w_0^{c_1}w_{-1} = (xy^2)^1y = xy^3$

3. $n = 2$, $w_2 = w_0w_1^{c_2} = xy^2(xy^3)^1 = xy^2xy^3$

4. $n = 3$, $w_3 = w_2^{c_3}w_1 = (xy^2xy^3)^1xy^3 = xyyxyyyxyyy$.

This is indeed the correct Christoffel word, and (w_2, w_1) is also the standard factorization of w_n by Corollary 5.3.

Using the factorizations in Theorem 5.2, we can observe a well known fact about continued fractions.

Corollary 5.4. Let $[c_0, \ldots, c_n]$ be the continued fraction representation of a rational number α with $n \geq 0$. Let $0 \leq i < n$. If i is even, the i-th continuant of α is less than α. If i is odd, the i-th continuant of α is greater than α. Furthermore, as i increases, the even continuants increase and the odd continuants decrease.

Proof. Use induction on n. For $n = 0$, the statement is vacuously true. For $n = 1$, $[c_0, c_1] = c_0 + \frac{1}{c_1} > c_0 = [c_0]$. Suppose that the statement holds for $0 \leq k < n$. We will handle the proof for the even case, with the odd case being analogous. Let $[c_0, \ldots, c_n]$ be the continued fraction representation of α with n even. By Theorem 5.2, the Christoffel word obtained used the RCFA is $w_n = w_{n-2}w_{n-1}^{c_n}$. The slope of w_{n-2} must be less than the slope of w_n since w_{n-2} encodes the path from the origin to some point below the line of slope α. The slope of w_{n-1} must be greater than the slope of w_n since $(w_{n-1})^{c_n}$ encodes the path from some point below the line of slope α to a point intersecting the line of slope α. Since w_{n-2} has slope the convergent $[c_0, \ldots, c_{n-2}]$ and w_{n-1} has slope the convergent $[c_0, \ldots, c_{n-1}]$, then $[c_0, \ldots, c_{n-2}] < [c_0, \ldots, c_n] < [c_0, \ldots, c_{n-1}]$. By the induction hypothesis,

$$[c_0] < [c_0, c_1, c_2] < \cdots < [c_0, \ldots, c_{n-2}] < [c_0, \ldots, c_n] \text{ and}$$

$$[c_0, \ldots, c_n] < [c_0, \ldots, c_{n-1}] < \cdots < [c_0, c_1, c_2, c_3] < [c_0, c_1].$$

$\square$

5.3 GAP Implementation of RCFA

To use the algorithm, type

```
rcfa(r)
```

where r is a reduced, rational number.

```
cf:=function(a,b) # Finds the continued fraction representation
    local x;
    x:=Indeterminate(Integers);
    return ContinuedFractionExpansionOfRoot(a*x-b,30);
end;

evenConv:=function(w_1,w_2,n) # Builds w if n is even
    local i, w;
    w := w_1;
    for i in [1..n] do
        w := Concatenation(w,w_2);
    od;
    return(w);
end;

oddConv:=function(w_1,w_2,n) # Builds w if n is odd
    local i, w;
    w := w_2;
    for i in [1..(n-1)] do
        w := Concatenation(w,w_2);
    od;
    w := Concatenation(w,w_1);
    return(w);
end;

rcfa:=function(r)
    local a, b, c, w, l, w_1, w_2, i, j;
    if (r = 0) then
        return([0]);
    fi;
    a := DenominatorRat(r);
    b := NumeratorRat(r);
    c := cf(a,b);
    w := [];
```

```
    l := Length(c);
    w_1 := [1];
    w_2 := [0];
    for i in [1..c[1]] do # Case when n = 0
        Append(w_2,[1]);
    od;
    w := w_2;
    for j in [2..Length(c)] do
        if (IsEvenInt(j+1)) then
            w := evenConv(w_1,w_2,c[j]);
        else
            w := oddConv(w_1,w_2,c[j]);
        fi;
        w_1 := w_2;
        w_2 := w;
    od;
    return(w);
end;
```

6 Calendar Systems

We conclude this thesis with an application. In order to reconcile periodic systems present in astronomy, scientists and philosophers attempted to accurately align the calendar year with natural events. To do this they had to understand how multiple periodic systems interacted.

6.1 Periodic Phenomena

There are multiple phenomena on which ancient societies based their calendar systems: the tropical year, the lunar month, and the spring and autumn equinoxes. These periodic phenomena do not exactly coincide on a regular basis, but astronomers in that time created systems which would keep their calendar year aligned with natural cycles. In order to maintain this alignment, scientists would have superimposed two periodic phenomena, which is described in the following theorem.

Theorem 6.1. Suppose p and q are relatively prime, positive integers. Set $P = \{ip \mid 0 < i < q\}$ and $Q = \{jq \mid 0 < j < p\}$. Write $P \cup Q$ as $A = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2, \cdots < a_n$ and $n = p+q-2$. Then the word $xw_1w_2 \ldots w_ny$ where $w_i = x$ if $a_i \in P$ and $w_i = y$ if $a_i \in Q$ is the Christoffel word of slope $\frac{q}{p}$.

Proof. Let L be the line segment from $(0,0)$ to (a,b). Divide L in two ways. In the first way, divide L into p segments $P_1, \ldots, P_p$ of equal length. In the second way, divide L into q segments $Q_1, \ldots, Q_q$ of equal length. Let S denote the set of endpoints of the line segments P_i, Q_j, not including the endpoints of L. Using this construction, $a_i \in A$ corresponds with the i-th point in S from $(0,0)$. Also, $a_i \in P \Leftrightarrow a_i$ corresponds to an endpoint of some P_k. The endpoints of the P_k are the points on L which intersect vertical lines at integer intervals, which are the points in $Int(p,q)$ which correspond to horizontal steps on the Christoffel path. The argument for when $a_i \in Q$ is similar. $\square$

An average tropical year is about 365.24 days. Some cultures, both modern and ancient, base their calendar systems on the lunar month and solar year. However, twelve lunar months is slighter shorter than a tropical year, about 354.36 days, and thirteen lunar months is slightly longer, about 383.90 days. To reconcile this, there is a cycle of twelve-month and thirteen-month years. In the 5th century BC, Meton of Athens judged the cycle to last 6,940 days, which is approximately 19 years, and this pattern is called a Metonic Cycle.

6.2 The Hebrew Calendar

The Hebrew calendar is a lunisolar calendar, and there is a 19-year cycle with 12-month regular years and 13-month leap years on years 3, 6, 8, 11, 14, 17, and 19. By letting x represent a 12-month year and y represent a 13-month year, the word $xxyxxyxyxxyxxyxxyxy$ describes this cycle. This is not a Christoffel word, but it is a conjugate of the Christoffel word of ratio $\frac{7}{12}$, which is $xxyxxyxxyxyxxyxxyxy$. Before the Hebrew Calendar was standardized, scientists relied on observation to determine when to insert a leap year. As such, the cycle is not exactly a Christoffel word, because the ratio does not exactly approximate the astronomical ratio. The advantages were that the calendar did not drift and that religious observances were on the proper dates. When the system was standardized, the 19-year cycle was established, which caused the calendar to drift slightly.

Others have proposed using continued fractions and Christoffel words to create more accurate calendar systems. For example, there was proposal to change the current Hebrew calendar to a 353-year system with 130 leap years, the distribution of which is determined by the Christoffel word associated with the ratio $\frac{130}{223}$. This ratio is a closer approximation to the astronomical ratio than $\frac{7}{12}$.

6.3 The Gregorian Calendar

The Gregorian calendar is a solar calendar with regular 365-day years and 366-day leap years. The rules for their inclusion are as follows: if the year is divisible by 4, it is a leap except years which are both divisible by 100 and not divisible by 400. Representing regular years with an x and leap years with a y, the word $w' = (((x^3y)^{24}x^4))^3(x^3y)^{25}$ represents the pattern of leap years. Note that this system involves 303 regular years and 97 leap years, and the Christoffel word of ratio $\frac{97}{303}$ can be calculated using the RCFA. The convergent of $\frac{97}{303}$ is $[0, 3, 8, 12]$, and thus the Christoffel word is $w = (x(x^3y)^8)^{12}x^3y$. Clearly, $w' \neq w$ and w' is not a conjugate of w since $(x^3y)^24$ is not a factor of w. This explains why the solstices and equinoxes are not as consistent in the Gregorian calendar system. For example, the spring equinox has occurred as early as March 19th and as late as March 21st.

However, this calculation is related to a truer approximation of the astronomical cycle. Consider the 2nd continuant of $[0, 3, 8, 12]$, which is 8/25. If there are 8 leap years and 25 regular years, the average year length is $((8 \cdot 366)+(25 \cdot 365))/33 \approx 365.2424$ days. If there are 97 leap years and 303 regular years, the average year length is $((97 \cdot 366) + (303 \cdot 365))/33 =$ 365.2425 days. The actual length of the tropical year is currently about 365.2422, thus the 33-year cycle would create a more accurate system than the 400-year cycle of the Gregorian calendar in terms of tracking celestial phenomena. However, the division-by-4 rule makes the Gregorian calendar much easier to remember than the 33-year system.

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