Heegaard diagrams for 5-manifolds and contractible high-dimensional manifolds

by

Geunyoung Kim

(Under the Direction of David Gay)

Abstract

In this dissertation, we discuss smooth 4- and 5-manifolds, their interactions, and contractible smooth high-dimensional manifolds.

First, we study 5-dimensional cobordisms with 2- and 3-handles, 5-dimensional 3-handlebodies, and closed, orientable 5-manifolds via (5-dimensional) Heegaard diagrams. We show that every such smooth 5-manifold can be represented by a Heegaard diagram, and two Heegaard diagrams represent diffeomorphic 5-manifolds if and only if they are related by certain moves. As an application, we construct Heegaard diagrams for 5-dimensional cobordisms from the standard 4-sphere to the Gluck twists along knotted 2-spheres. This provides some equivalent statements regarding the Gluck twists being diffeomorphic to the standard 4-sphere.

Second, for any integer $n \geq 2$, we construct a contractible, compact, smooth $(n+3)$-manifold which is not homeomorphic to the standard $(n+3)$-ball, using a 0-handle, an n-handle, and an $(n+1)$-handle. The key step is the construction of an interesting knotted n-sphere in $S^n \times S^2$ generalizing the Mazur pattern. As a corollary, for any integer $n \geq 2$, there exists a smooth involution of S^{n+3} whose fixed point set is a non-simply connected homology $(n+2)$-sphere.

Index words: [Low-dimensional topology, high-dimensional topology, 4-manifolds, 5-manifolds, Heegaard diagrams, contractible manifolds]

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B.A., Pusan National University, 2015
M.S., Pusan National University, 2017

A Dissertation Submitted to the Graduate Faculty of the
University of Georgia in Partial Fulfillment of the Requirements for the Degree.

Doctor of Philosophy

Athens, Georgia

2024

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May 2024

Acknowledgments

I would like to express my deepest gratitude to my advisor, Dave Gay, for his steadfast support and guidance throughout my academic journey. Dave, you have always been there for me, never declining my random questions and consistently providing insightful answers. I also would like to express my appreciation to many topologists, including Akram Alishahi, Román Aranda, Swapnanil Banerjee, Sarah Blackwell, Eduardo Fernández, Devashi Gulati, Daniel Hartman, Seungwon Kim, Peter Lambert-Cole, Gordana Matić, Maggie Miller, Puttipong Pongtanapaisan, Mark Powell, Alex Tepper, and Terrin Warren.

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Chapter 1

Heegaard diagrams for 5-manifolds

1.1 Introduction

We work in the smooth category throughout. A (3-dimensional) ***Heegaard diagram*** is a triple (Σ, α, β), where Σ is a closed orientable surface, and each of α and β is an embedding of the disjoint union of circles. From $\Sigma \times [-1, 1]$, we can construct a 3-dimensional manifold by attaching 2-handles along $\alpha \times \{-1\}$ and $\beta \times \{1\}$, thereby building a 3-dimensional cobordism between two closed surfaces. One surface is the result of surgery on Σ along α, and the other is the result of surgery on Σ along β. If either surface is diffeomorphic to the 2-sphere S^2, capping it off yields a 3-manifold with one boundary component. If both surfaces are diffeomorphic to S^2, capping them off results in a closed 3-manifold. It is well-known that every 3-manifold can be represented by a Heegaard diagram, and two Heegaard diagrams represent diffeomorphic 3-manifolds if and only if they are related by certain moves [Rei33; Sin33].

Similar to the study of 3-manifolds via (3-dimensional) Heegaard diagrams, this approach extends to dimension 5. We begin by introducing 5-dimensional Heegaard diagrams.

Definition 1.1.1. An m-***component*** 2-***link*** α in a 4-manifold Σ is an embedding of the disjoint union of m-copies of 2-spheres in Σ. We say that α has ***trivial normal bundle*** if there exists an embedding $\phi : \coprod^m (S^2 \times B^2) \hookrightarrow \Sigma$ such that $\phi(\coprod^m (S^2 \times 0)) = \alpha$.

Definition 1.4.1. A (5-dimensional) ***Heegaard diagram*** is a triple (Σ, α, β) such that

1. Σ is a closed, connected, orientable 4-manifold,
2. $\alpha = \alpha_1 \cup \cdots \cup \alpha_m \subset \Sigma$ is an m-component 2-link with trivial normal bundle for some $m \in \mathbb{Z}$,
3. $\beta = \beta_1 \cup \cdots \cup \beta_n \subset \Sigma$ is an n-component 2-link with trivial normal bundle for some $n \in \mathbb{Z}$.

Definition 1.4.3. Let (Σ, α, β) be a Heegaard diagram. Define the following 5-manifolds:

1. Let
$$M_\alpha = (\Sigma \times [-1,0]) \cup (\coprod^m (B^3 \times B^2))$$
be the 5-manifold obtained from $\Sigma \times [-1,0]$ by attaching m 3-handles along $\alpha \times \{-1\} \subset \Sigma \times \{-1\}$. Here, $\partial M_\alpha = \Sigma \coprod \Sigma(\alpha)$, where $\partial_- M_\alpha = \Sigma \times \{0\} = \Sigma$ and $\partial_+ M_\alpha$ is diffeomorphic to the surgery $\Sigma(\alpha)$ of Σ along α.

2. If $\Sigma(\alpha) \cong \#^k(S^1 \times S^3)$, let
$$\widehat{M_\alpha} = M_\alpha \cup_g (\natural^k(S^1 \times B^4))$$
for some diffeomorphism $g : \#^k(S^1 \times S^3) \to \Sigma(\alpha)$. Here, $\partial \widehat{M_\alpha} = \Sigma \times \{0\} = \Sigma$.

3. Let
$$M_\beta = (\Sigma \times [0,1]) \cup (\coprod^n (B^3 \times B^2))$$
be the 5-manifold obtained from $\Sigma \times [0,1]$ by attaching n 3-handles along $\beta \times \{1\} \subset \Sigma \times \{1\}$. Here, $\partial M_\beta = \Sigma \coprod \Sigma(\beta)$, where $\partial_- M_\beta = \Sigma \times \{0\} = \Sigma$ and $\partial_+ M_\beta$ is diffeomorphic to the surgery $\Sigma(\beta)$ of Σ along β.

4. If $\Sigma(\beta) \cong \#^r(S^1 \times S^3)$, let
$$\widehat{M_\beta} = M_\beta \cup_h (\natural^r(S^1 \times B^4))$$
for some diffeomorphism $h : \#^r(S^1 \times S^3) \to \Sigma(\beta)$. Here, $\partial \widehat{M_\beta} = \Sigma \times \{0\} = \Sigma$.

5. Let
$$M_\alpha \cup_\Sigma M_\beta = M_\alpha \cup M_\beta,$$
where $M_\alpha \cap M_\beta = \Sigma \times \{0\} = \Sigma$, $\partial_-(M_\alpha \cup_\Sigma M_\beta) = \Sigma(\alpha)$, and $\partial_+(M_\alpha \cup_\Sigma M_\beta) = \Sigma(\beta)$.

6. If $\Sigma(\alpha) \cong \#^k(S^1 \times S^3)$, let
$$\widehat{M_\alpha} \cup_\Sigma M_\beta = \widehat{M_\alpha} \cup M_\beta,$$
where $\widehat{M_\alpha} \cap M_\beta = \Sigma \times \{0\} = \Sigma$ and $\partial(\widehat{M_\alpha} \cup_\Sigma M_\beta) = \Sigma(\beta)$.

7. If $\Sigma(\alpha) \cong \#^k(S^1 \times S^3)$ and $\Sigma(\beta) \cong \#^l(S^1 \times S^3)$, let
$$\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} = \widehat{M_\alpha} \cup \widehat{M_\beta},$$
where $\widehat{M_\alpha} \cap \widehat{M_\beta} = \Sigma \times \{0\} = \Sigma$.

We show that every 5-dimensional cobordism with only 2- and 3-handles, as well as every 5-dimensional 3-handlebody or closed, orientable 5-manifold, can be represented by a (5-dimensional) Heegaard diagram.

Theorem 1.4.6.

1. *Let X be a 5-dimensional cobordism with only 2- and 3-handles. Then X is diffeomorphic to $M_\alpha \cup_\Sigma M_\beta$ for some Heegaard diagram (Σ, α, β).*

2. *Let X be a 5-dimensional 3-handlebody. Then X is diffeomorphic to $\widehat{M_\alpha} \cup_\Sigma M_\beta$ for some Heegaard diagram (Σ, α, β).*

3. *Let X be a closed, orientable 5-manifold. Then X is diffeomorphic to $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta}$ for some Heegaard diagram (Σ, α, β).*

We recall that an ***n-dimensional*** k***-handlebody*** is a manifold obtained from B^n by attaching handles up to index k. We show that every n-dimensional k-handlebody has a product structure when $n \geq 2k+1$.

Theorem 1.3.11. *Let X be an n-dimensional k-handlebody. If $n \geq 2k+1$, then there exists an $(n-1)$-dimensional k-handlebody $Y \subset X$ such that $X \cong Y \times B^1$. More precisely, if $B^n = X_0 \subset X_1 \subset \cdots \subset X_k = X$ is a handle decomposition of X, then there exists a handle decomposition $B^{n-1} = Y_0 \subset Y_1 \subset \cdots \subset Y_k = Y$ of Y such that $X_i \cong Y_i \times B^1$ for every $0 \leq i \leq k$.*

The following corollary is immediate because $\widehat{M_\alpha}$ is a 5-dimensional 2-handlebody.

Corollary 1.4.5. Let (Σ, α, β) be a Heegaard diagram. If $\Sigma(\alpha)$ is diffeomorphic to $\#^k(S^1 \times S^3)$, then there exists a 4-dimensional 2-handlebody Y such that $\widehat{M_\alpha} \cong Y \times B^1$, and therefore, Σ is diffeomorphic to the double of Y.

We show that a Heegaard diagram for a 5-manifold is essentially unique, up to a sequence of certain moves.

Theorem 1.4.12.

1. *Let (Σ, α, β) and $(\Sigma', \alpha', \beta')$ be Heegaard diagrams. Then $M_\alpha \cup_\Sigma M_\beta \cong M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$ if and only if they are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.*

2. *Let (Σ, α, β) and $(\Sigma', \alpha', \beta')$ be Heegaard diagrams. Then $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} M_{\beta'}$ if and only if they are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.*

3. *Let (Σ, α, β) and $(\Sigma', \alpha', \beta')$ be Heegaard diagrams. Then $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} \widehat{M_{\beta'}}$ if and only if they are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.*

We recall that the Gluck twist of S^4 along a 2-knot is a surgery operation on S^4 where the regular neighborhood of the 2-knot is removed and then reattached in a non-trivial manner. Gluck showed that this operation yields a homotopy 4-sphere [Glu62], hence it is homeomorphic to S^4 [Fre82]. However, it remains unknown whether the Gluck twists are diffeomorphic to the standard S^4 in general. We construct a 5-dimensional cobordism from the standard S^4 to the Gluck twist along a 2-knot in S^4, with a single 2-handle and a single 3-handle. Additionally, we provide equivalent statements regarding the Gluck twists being diffeomorphic to the standard S^4.

Theorem 1.5.9. *Let $K \subset S^4$ be a 2-knot, and S^4_K be the Gluck twist of S^4 along K. Let $(\Sigma, \alpha, \beta) = (S^2 \tilde{\times} S^2, F, K\#F)$ be a Heegaard diagram, where F is a fiber of $S^2 \tilde{\times} S^2$. The following are equivalent:*

1. *S^4_K is diffeomorphic to S^4.*
2. *$W_{S^4,K} \cong M_\alpha \cup_\Sigma M_\beta$ is diffeomorphic to a twice-punctured $S^2 \tilde{\times} S^3$.*
3. *$(S^2 \tilde{\times} S^2, F, K\#F)$ and $(S^2 \tilde{\times} S^2, F, F)$ are related by isotopies, handle slides, stabilizations, and diffeomorphisms.*
4. *$(S^2 \tilde{\times} S^2, K\#F)$ is diffeomorphic to $(S^2 \tilde{\times} S^2, F)$.*

In section 1.2, we review basic handle decomposition theory, Kirby diagrams, (singular) banded unlink diagrams, and 1- and 2-surgery of 4-manifolds. In section 1.3, we show that every n-dimensional k-handlebody has a product structure when $n \geq 2k+1$. In section 1.4, we review 5-dimensional Heegaard diagrams and provide numerous examples. In section 1.5, we review the Gluck twist and construct a 5-dimensional cobordism from a 4-manifold X and the Gluck twist of X along a 2-knot.

1.2 Preliminaries

In subsection 1.2.1 we review handle decomposition theory of arbitrary dimensional smooth manifolds and a certain moves defined on handle decompositions of a manifold; see [Mil63; Mil15; Kos13] for more details. In subsection 1.2.2 we discuss handle decompositions of 4-manifolds via Kirby diagrams. A Kirby diagram is the union of a dotted unlink and a framed link in S^3; see [Kir06; Kir78; GS99; Akb16] for more details. In subsection 1.2.3 we review nice decompositions of a pair (X, F), where F is an embedded or immersed surface in a 4-manifold X, via (singular) banded unlink diagrams. A (singular) banded unlink diagram is a (singular) banded link in a Kirby diagram satisfying certain conditions; see [HKM20; HKM21] for more details. In subsection 1.2.4 we describe algorithms for finding Kirby diagrams for surgery of a 4-manifold along embedded 1-spheres or 2-spheres from a banded unlink diagram.

1.2.1 Handle decompositions

Definition 1.2.1. Let $B^k \subset \mathbb{R}^k$ be the standard unit k-ball. We call $h^k = B^k \times B^{n-k}$ an ***n-dimensional k-handle***, k the ***index*** of the handle, $B^k \times 0$ the ***core***, $0 \times B^{n-k}$ the ***cocore***, $S^{k-1} \times 0$ the ***attaching sphere***, $0 \times S^{n-k-1}$ the ***belt sphere***, $S^{k-1} \times B^{n-k}$ the ***attaching region***, and $B^k \times S^{n-k-1}$ the ***belt region***.

Definition 1.2.2. Let X be an n-manifold with boundary ∂X and $\phi : S^{k-1} \times B^{n-k} \hookrightarrow \partial X$ be an embedding. The quotient manifold

$$X \cup_\phi h^k = (X \coprod (B^k \times B^{n-k}))/x \sim \phi(x)$$

is called the ***manifold obtained from X by attaching an n-dimensional k-handle to ∂X along ϕ***. We call ϕ the ***attaching map of*** h^k.

Remark 1.2.3.

1. $X \cup_\phi h^k$ has corners, but there is a canonical way to smooth these corners, allowing us to assume that it is a smooth manifold [Kos13].
2. There is a canonical deformation retraction of $X \cup_\phi (B^k \times B^{n-k})$ onto $X \cup_{\phi|_{S^{k-1}\times 0}} (B^k \times 0)$ induced by the canonical deformation retraction of $B^k \times B^{n-k}$ onto $B^k \times 0$.
3. The diffeomorphism type of $X \cup_\phi h^k$ is determined by the isotopy class of ϕ.
4. By tubular neighborhood theorem [GP10], an embedding $\phi : S^{k-1} \times B^{n-k} \hookrightarrow \partial X$ is uniquely determined up to isotopy by the attaching sphere $S = \phi(S^{k-1} \times 0)$ of the k-handle and a framing of S which is an identification of the normal bundle νS of S with $S^{k-1} \times \mathbb{R}^{n-k}$. After choosing a fixed reference framing, the set of framings can be canonically identified with $\pi_{k-1}(GL(n-k)) \cong \pi_{k-1}(O(n-k))$.
5. If $S = \phi(S^{k-1} \times 0)$ is a framed $(k-1)$-sphere with the framing induced by ϕ, then we may write $X \cup_S h^k$ instead of $X \cup_\phi h^k$.
6. The attaching sphere of an n-dimensional 1-handle has two possible framings because $\pi_0(O(n-1)) \cong \mathbb{Z}_2$ when $n \geq 2$. There are two possible n-manifolds obtained by attaching a 1-handle, distinguished by whether they are orientable. In this dissertation, we are interested in orientable manifolds, so we can assume that there is a unique way to attach a 1-handle along a fixed attaching sphere.
7. For $k > 1$, we choose a base point in a connected component of $O(n-k)$ so $\pi_{k-1}(O(n-k)) \cong \pi_{k-1}(SO(n-k))$.
8. The attaching sphere of a 3-dimensional 2-handle has a unique framing because $\pi_1(SO(1)) \cong 1$.
9. The attaching sphere of a 4-dimensional 2-handle has integer framings because $\pi_1(SO(2)) \cong \mathbb{Z}$.
10. The attaching sphere of a 5-dimensional 2-handle has two framings because $\pi_1(SO(3)) \cong \mathbb{Z}_2$.
11. The attaching sphere of a 5-dimensional 3-handle has a unique framing because $\pi_2(SO(2)) \cong 1$.

Proposition 1.2.4 ([Mil63; Mil15]). Let $X' = (X \cup_\phi h^k) \cup_\psi h^l$ be an n-manifold obtained from an n-manifold X by attaching first a k-handle h^k and then an l-handle h^l, where $k \geq l$. Then X' can be obtained from X by attaching first an l-handle h^l and then a k-handle h^k, i.e.

$$(X \cup_\phi h^k) \cup_\psi h^l \cong (X \cup_{\psi'} h^l) \cup_{\phi'} h^k.$$

Proof. Let B^k be the belt sphere of h^k and A^l be the attaching sphere of h^l. Then $dim(B^k) = n - k - 1$ and $dim(A^l) = l - 1$ so $dim(B^k) + dim(A^l) = (n - k - 1) + (l - 1) = (n - 1) + (l - k - 1) < n - 1 = dim(\partial(X \cup_\phi h^k))$. We can now assume that A^l does not intersect B^k so A^l is isotopic into ∂X. Therefore, X' can be obtained from X by attaching first an l-handle h^l and then a k-handle h^k. □

Proposition 1.2.4 implies that handles can be attached in order of increasing index. Furthermore the attaching regions of handles with the same index can be disjoint each other in ∂X.

Definition 1.2.5. Let Y be an $(n-1)$-manifold and $\phi : S^{k-1} \times B^{n-k} \hookrightarrow Y$ be an embedding. We call

$$Y(\phi) = (Y \setminus int(\phi(S^{k-1} \times B^{n-k}))) \cup_{\phi|_{S^{k-1} \times S^{n-k-1}}} (B^k \times S^{n-k-1})$$

the ***manifold obtained from*** Y ***by surgery along*** ϕ or simply $(k-1)$-***surgery of*** Y. If $S = \phi(S^{k-1} \times 0)$ is a framed sphere with a framing induced by ϕ, we may denote $Y(\phi)$ by $Y(S)$ and refer to $Y(S)$ as the ***manifold obtained from*** Y ***by surgery along*** S.

Definition 1.2.6. Let X be an n-dimensional compact manifold with $\partial X = \partial_- X \coprod \partial_+ X$. A ***handle decomposition*** of X (relative to $\partial_- X$) is a sequence of manifolds

$$X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X$$

such that

1. $X_{-1} = \partial_- X \times [0, 1]$,
2. $\partial_- X_k = \partial_- X$,
3. X_k is obtained from X_{k-1} by attaching k-handles to $\partial_+ X_{k-1}$.

More precisely,

$$X_k = X_{k-1} \cup_{\phi_1} (B^k \times B^{n-k}) \cup_{\phi_2} \cdots \cup_{\phi_t} (B^k \times B^{n-k})$$

for some $\phi_i : S^{k-1} \times B^{n-k} \hookrightarrow \partial_+ X_{k-1}$ such that $\phi_i(S^{k-1} \times B^{n-k}) \cap \phi_j(S^{k-1} \times B^{n-k}) = \emptyset$ when $i \neq j \in \{1, \ldots, t\}$.

Proposition 1.2.7 ([Mil63; Mil15]). Every compact, smooth n-manifold X admits a handle decomposition (relative to $\partial_- X$).

Proof. By Morse theory, there exists a self-indexing Morse function $f : X \to [-1-\frac{1}{2}, n+\frac{1}{2}]$ such that $f^{-1}(-1-\frac{1}{2}) = \partial_- X$, $f^{-1}(n+\frac{1}{2}) = \partial_+ X$, and $f^{-1}(k)$ is the set of all index k non-degenerate critical points. Then $X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X$ is a handle decomposition of X (relative to ∂X), where $X_k = f^{-1}([-1-\frac{1}{2}, k+\frac{1}{2}])$. □

Remark 1.2.8.

1. X_0 is the disjoint union of X_{-1} and 0-handles.

2. If $\partial_- X = \emptyset$, then $X_{-1} = \emptyset$.

3. If there are no k-handles attached to $\partial_+ X_{k-1}$, then $X_{k-1} = X_k$.

4. Let X be a compact, connected manifold with $\partial_- X \neq \emptyset$ and $\partial_+ X \neq \emptyset$. Then X admits a handle decomposition without 0-handles and n-handles $X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$, where $X_{-1} = X_0$ and $X_{n-1} = X_n$.

5. Let X be a compact, connected manifold with $\partial_- X = \emptyset$. Then X admits a handle decomposition $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$, where X_0 is a single 0-handle. Here, X_k is called an ***n-dimensional k-handlebody***.

6. Let X be a closed manifold (i.e. compact with $\partial X = \emptyset$). Then X admits a handle decomposition $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n$, where X_0 is a single 0-handle and X_n is obtained from X_{n-1} by attaching a single n-handle.

7. $\partial_+ X_k$ is obtained from $\partial_+ X_{k-1}$ by $(k-1)$-surgery. Similarly, $\partial_+ X_{k-1}$ is obtained from $\partial_+ X_k$ by $(n-k-1)$-surgery.

8. Let $f : X \to [-1-\frac{1}{2}, n+\frac{1}{2}]$ be the Morse function in the proof of Proposition 1.2.7. Consider a function $g : X \to [-1-\frac{1}{2}, n+\frac{1}{2}]$ defined by $g(x) = n-1-f(x)$. Then $M_{-1} \subseteq M_0 \subseteq \cdots \subseteq M_n = X$ is a handle decomposition of X (relative to $\partial_+ X$), where $M_k = g^{-1}([-1-\frac{1}{2}, k+\frac{1}{2}])$. We call $M_{-1} \subseteq M_0 \subseteq \cdots \subseteq M_n = X$ the ***dual handle decomposition*** of the handle decomposition $X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X$.

9. We can read off the homology of $(X, \partial_- X)$ from a handle decomposition of $(X, \partial X)$. Let $C_k(X, \partial X)$ be the free abelian group generated by the oriented k-handles. The boundary map $\partial_k : C_k(X, \partial X) \to C_{k-1}(X, \partial X)$ is defined by $\partial_k(h^k) = (-1)^{k-1} \sum_i (A^k \cdot B_i^{k-1}) h_i^{k-1}$, where h_i^{k-1} is the indexed $(k-1)$-handle, and $A^k \cdot B_i^{k-1}$ is the algebraic intersection number between the attaching sphere A^k of h^k and the belt sphere B_i^{k-1} of h_i^{k-1}. See [DGK19] for more details.

Definition 1.2.9. Let $X \cup_\phi h_\alpha^k \cup_\psi h_\beta^k$ be an n-manifold obtained from X by attaching two k-handles to ∂X along ϕ and ψ. Let $X \cup_\phi h_\alpha^k \cup_{\psi'} h_{\beta'}^k$ be an n-manifold obtained from X by attaching two k-handles to ∂X along ϕ and ψ'. We say that $X \cup_\phi h_\alpha^k \cup_{\psi'} h_{\beta'}^k$ is obtained from $X \cup_\phi h_\alpha^k \cup_\psi h_\beta^k$ by a ***handle slide*** of h_β^k over h_α^k if there exists an embedding $F : (S^{k-1} \times B^{n-k}) \times [0,1] \to \partial(X \cup_\phi h_\alpha^k) \times [0,1]$ such that

1. $F(x,t) \subset \partial(X \cup_\phi h^k_\alpha) \times \{t\}$ for every $x \in S^{k-1} \times B^{n-k}$ and $t \in [0,1]$,
2. $F(x,0) = (\psi(x), 0)$ for every $x \in S^{k-1} \times B^{n-k}$,
3. $F(x,1) = (\psi'(x), 1)$ for every $x \in S^{k-1} \times B^{n-k}$,
4. $F((S^{k-1} \times 0) \times [0,1])$ and $B^k_\alpha \times [0,1]$ intersect transversely at one point, where B^k_α is the belt sphere of h^k_α.

Since a handle slide is an isotopy of an attaching map, we have the following:

Proposition 1.2.10. In Definition 1.2.9, $X \cup_\phi h^k_\alpha \cup_\psi h^k_\beta \cong X \cup_\phi h^k_\alpha \cup_{\psi'} h^k_{\beta'}$.

Definition 1.2.11. Let $N_1 \cup N_2 \subset M$ be an n-submanifold of an m-dimensional manifold M, where $m > n$. An $(n+1)$-submanifold $b \subset M$ is called a $(n+1)$***-dimensional*** 1***-handle connecting*** N_1 ***and*** N_2 if there exists an embedding $e : B^1 \times B^n \hookrightarrow M$ such that

1. $b = e(B^1 \times B^n)$,
2. $b \cap N_1 = e(\{-1\} \times B^n)$,
3. $b \cap M_2 = e(\{1\} \times B^n)$.

We call

$$N_1 \#_b N_2 = ((N_1 \cup N_2) \setminus e(\partial B^1 \times B^1)) \cup e(B^1 \times \partial B^n)$$

the ***manifold obtained from*** $N_1 \cup N_2$ ***by surgery along*** b or ***connected sum of*** N_1 ***and*** N_2 ***along*** b.

Remark 1.2.12. In Definition 1.2.9, the attaching sphere $A^k_{\beta'}$ of $h^k_{\beta'}$ is obtained by connected summing the push-off $\tilde{A}^k_\alpha$ (with respect to a given framing) of the attaching sphere A^k_α and the attaching sphere A^k_β for some k-dimensional 1-handle $b \subset \partial X$, i.e. $A^k_{\beta'} = \tilde{A}^k_\alpha \#_b A^k_\beta$.

Definition 1.2.13. Let $X \cup_\phi h^{k-1} \cup_\psi h^k$ be an n-manifold obtained from X by attaching a $(k-1)$-handle and a k-handle. If the attaching sphere of h^k and the belt sphere of h^{k-1} intersect transversely at a point in $\partial(X \cup_\phi h^{k-1})$, we call the pair (h^{k-1}, h^k) a ***cancelling*** $(k-1,k)$***-pair***. We say that $X \cup_\phi h^{k-1} \cup_\psi h^k$ is obtained from X by ***creation*** of a cancelling $(k-1,k)$-pair. We say that X is obtained from $X \cup_\phi h^{k-1} \cup_\psi h^k$ by ***annihilation*** of a cancelling $(k-1,k)$-pair.

Proposition 1.2.14 ([Mil15])**.** In Definition 1.2.13, $X \cup_\phi h^{k-1} \cup_\psi h^k \cong X$.

Theorem 1.2.15 ([Cer70])**.** *Any two handle decompositions of a compact smooth manifold* $(X, \partial_- X)$ *are related by isotopies, handle slides, and the creation/annihilation of cancelling pairs.*

Later we carefully interpret isotopies, handle slides, and the creation/annihilation of a cancelling pair in Theorem 1.2.15 in the context of Kirby diagrams for 4-manifolds in subsection 1.2.2 and Heegaard diagrams for 5-manifolds section 1.4.

1.2.2 Kirby diagrams for 4-manifolds

Definition 1.2.16. Let $K \subset S^3$ be a knot (the image of an embedding $e : S^1 \hookrightarrow S^3$). For any integer $m \in \mathbb{Z} \cong \pi_1(SO(2))$, there exists an embedding $\phi : S^1 \times B^2 \hookrightarrow S^3$ such that

1. $\phi(S^1 \times B^2) = \nu(K)$,
2. $\phi(S^1 \times \{(0,0)\}) = K$,
3. $lk(K, \phi(S^1 \times \{(1,0)\})) = m$,

where $\nu(K)$ is a closed regular neighborhood of K, and $lk(K, \phi(S^1 \times \{(1,0)\}))$ is the linking number between K and $\phi(S^1 \times \{(1,0)\})$. We call such a pair (K, ϕ) an m-***framed knot*** and simply draw the knot K with the integer m. Additionally, we define $\tilde{K} = \phi(S^1 \times \{(1,0)\})$ as the ***push-off*** of (K, ϕ). A link $L \subset S^3$ (the image of an embedding of the disjoint union of 1-spheres) is called a ***framed link*** if each component of L is a framed knot.

Definition 1.2.17. A knot $K \subset S^3$ is called the ***unknot*** if it bounds a 2-disk in S^3. An unknot $K \subset S^3$ with a dot is called a ***dotted unknot***. A dotted longitude $\tilde{K}$ on the boundary of the regular neighborhood $\partial\nu(K) \subset S^3$ of K with the linking number $lk(K, K') = 0$ is called a ***push-off*** of K. A link $L \subset S^3$ is called the ***unlink*** if it bounds the disjoint union of 2-disks in S^3. An unlink $L = K_1 \cup \cdots \cup K_n \subset S^3$ is called a ***dotted unlink*** if each K_i is a dotted unknot.

Definition 1.2.18. Let $L \subset S^3 = \partial B^4$ be the dotted unlink and $D \subset S^3$ be the disjoint union of 2-disks with $\partial D = L$. Let $D' \subset B^4$ be the set of the properly embedded disks obtained from D by pushing the interior of D into B^4. We define

$$M_L = \overline{B^4 - \nu(D')}$$

to be the closure of the exterior of the closed regular neighborhood $\nu(D')$ in B^4.

Remark 1.2.19.

1. $M_L \cong \natural^{|L|}(S^1 \times B^3)$, where $|L|$ is the number of components of L and $\natural^{|L|}(S^1 \times B^3)$ is the boundary connected sum of $|L|$ copies of $(S^1 \times B^3)$. Therefore, M_L can be considered as a manifold obtained from B^4 by attaching $|L|$ 1-handles.
2. $\partial M_L \cong \#^{|L|}(S^1 \times S^2)$, where $\#^{|L|}(S^1 \times S^2)$ is the connected sum of $|L|$ copies of $S^1 \times S^2$.

Definition 1.2.20. Let $\mathcal{K} = L_1 \cup L_2 \subset S^3$ be a link in S^3, where L_1 is the dotted unlink and $L_2 = \{(K_1, \phi_1), \ldots, (K_n, \phi_n)\}$ is a framed link. We call $\mathcal{K}$ a ***Kirby diagram***. Clearly, $L_2 \subset \partial M_{L_1}$ since $L_2 \subset S^3 - L_1$. We define

$$M_{\mathcal{K}} = M_{L_1 \cup L_2} = M_{L_1} \cup_{\phi_1} (B^2 \times B^2) \cup_{\phi_2} \cdots \cup_{\phi_n} (B^2 \times B^2)$$

to be the manifold obtained from M_{L_1} by attaching 2-handles along $\phi_i' s$.

Remark 1.2.21.

1. $M_{\mathcal{K}} = M_{L_1 \cup L_2}$ can be considered as the manifold obtained from B^4 by attaching $|L_1|$ 1-handles and $|L_2|$ 2-handles.

2. Let U be a dotted unknot and U' be a 0-framed unknot. Then $M_U \cong S^1 \times B^3 \ncong S^2 \times B^2 \cong M_{U'}$ but $\partial M_U \cong S^1 \times S^2 \cong \partial M_{U'}$. We note that M_U is obtained from $M_{U'}$ by surgery along $S^2 \times 0 \subset S^2 \times B^2$ and $M_{U'}$ is obtained from M_U by surgery along $S^1 \times 0 \subset S^1 \times B^3$.

3. Let $\mathcal{K}$ be a Kirby diagram and $\tilde{\mathcal{K}}$ be a Kirby diagram obtained from $\mathcal{K}$ by switching a dotted unlink to a 0-framed unlink. Then $\partial M_{\mathcal{K}} \cong \partial M_{\tilde{\mathcal{K}}}$.

4. Let $K \subset S^3$ be a dotted unknot and $D \subset S^3$ be the trivial 2-disk with $\partial D = K$. Then the disk $\tilde{D} = D \setminus int(\nu(K))$ can be considered as a visible part (hemisphere) of the belt sphere of the 1-handle. Therefore, we can see completely how 2-handles go over 1-handles by observing intersections between attaching spheres of 2-handles and $\tilde{D}$. See the left of Figure 1.1.

Definition 1.2.22. Let $\mathcal{K} = L_1 \cup L_2$ be a Kirby diagram with $\partial M_{\mathcal{K}} \cong \#^k(S^1 \times S^2)$ for some $k \geq 0$. Let $f : \#^k(S^1 \times S^2) \to \partial M_{\mathcal{K}}$ be a diffeomorphism. We define

$$\widehat{M_{\mathcal{K}}} = M_{\mathcal{K}} \cup_f (\natural^k(S^1 \times B^3))$$

to be the closed 4-manifold obtained from $M_{\mathcal{K}}$ by gluing $\natural^k(S^1 \times B^3)$.

Remark 1.2.23.

1. $\natural^k(S^1 \times B^3)$ can be considered as the union of a 0-handle and k 1-handles or the union of k 3-handles and a 4-handle.

2. $\widehat{M_{\mathcal{K}}}$ is uniquely determined up to diffeomorphism because every self-diffeomorphism of $\#^k(S^1 \times S^2)$ extends to a self-diffeomorphism of $\natural^k(S^1 \times B^3)$ [LP72]. Therefore, we do not need to draw k 3-handles and a single 4-handle in the diagram $\mathcal{K}$.

Proposition 1.2.24 (How to draw a Kirby diagram for the double DY of a 4-dimensional 2-handlebody Y)**.** Let Y be a 4-dimensional 2-handlebody and $\mathcal{K} = L_1 \cup L_2$ be a Kirby diagram for Y, where L_1 is a dotted unlink and L_2 is a framed link. The double of Y is $DY = Y \cup_{id} \overline{Y}$, and $\overline{Y}$ has the canonical handle decomposition with 2-handles, 3-handles, and a 4-handle by turning the handle decomposition of Y upside down. Here the attaching spheres of the 2-handles of $\overline{Y}$ are glued to the belt sphere of the 2-handles of Y. Therefore, we obtain a Kirby diagram $\mathcal{K}' = \mathcal{K} \cup J$, where J is the 0-framed meridians of L_2, i.e. $\widehat{M_{\mathcal{K}}} \cong DY$.

Example 1.2.25. The left of Figure 1.1 is a Kirby diagram for Mazur manifold and the middle of Figure 1.1 is a Kirby diagram for the double of Mazur manifold. (Ignore the red circle).

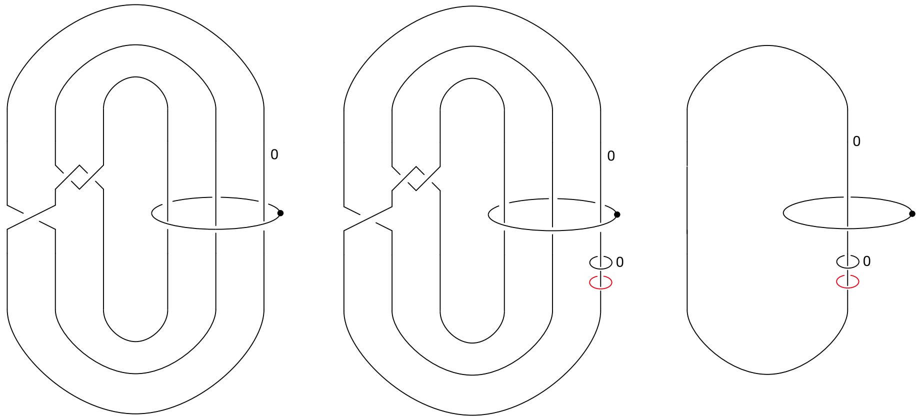

Figure 1.1: **Left**: A Kirby diagram for Mazur manifold M. The 0-framed attaching sphere K of the 2-handle intersects the belt sphere of the 1-handle geometrically three times and algebraically once. **Middle**: A Heegaard diagram for $M \times B^1$. This diagram is obtained from a Kirby diagram for the double of the left and by adding a red meridian. **Right**: Another Heegaard diagram for $M \times B^1$ obtained from the middle diagram after sliding K over 0-framed meridian to change the crossings of K. We can see that this Heegaard diagram represents B^5 after cancelling a $(1,2)$-pair, a $(2,3)$-pair, and performing a first destabilization.

The existence of a handle decomposition implies the following:

Theorem 1.2.26 ([Kir06]). *Every closed, connected, orientable, smooth 4-manifold is diffeomorphic to $\widehat{M_{\mathcal{K}}}$ for some Kirby diagram $\mathcal{K}$.*

We introduce some moves defined on Kirby diagrams, i.e., we interpret isotopies, handle slides, and cancelling pairs in Kirby diagrams.

Definition 1.2.27. Let $\mathcal{K} \subset S^3$ be a Kirby diagram. Let $K_i, K_j \subset \mathcal{K}$ be two knots and $\tilde{K}_j \subset \partial\nu(K_j)$ be a parallel push-off of K_j (where K_j is possibly a dotted unknot or a framed knot). A 2-dimensional submanifold $b \subset S^3$ is called a ***sliding band connecting*** K_i ***and*** $\tilde{K}_j$ if there exists an embedding $e : B^1 \times B^1 \hookrightarrow S^3$ such that

1. $b = e(B^1 \times B^1)$,
2. $b \cap K_i = e(\{-1\} \times B^1)$,
3. $b \cap K_j = e(\{1\} \times B^1)$,
4. $e((-1,1) \times B^1) \cap (\mathcal{K} \cup \nu(K_j)) = \emptyset$.

We call

$$K_i \#_b \tilde{K}_j = ((K_1 \cup K_2) \setminus e(\partial B^1 \times B^1)) \cup e(B^1 \times \partial B^1)$$

the ***manifold obtained from*** $K_i \cup \tilde{K_j}$ ***by surgery along*** b or ***connected sum of*** K_i ***and*** $\tilde{K_j}$ ***along*** b.

Definition 1.2.28. Let $\mathcal{K} = L_1 \cup L_2 \subset S^3$ be a Kirby diagram, where L_1 is a dotted unlink and L_2 is a framed link.

1. Let $K_i, K_j \subset L_1$ be two dotted knots. Let D be the disjoint union of disks with $\partial D = L_1$. Let $b \subset S^3$ be a sliding band connecting K_i and $\tilde{K_j}$ such that $b \cap int(D) = \emptyset$. We call $K_i \#_b \tilde{K_j}$ ***the result of sliding of*** K_i ***over*** K_j or 1-***handle slide over a*** 1-***handle***. If $\mathcal{K}'$ is another Kirby diagram, we say that $\mathcal{K}$ and $\mathcal{K}'$ are related by a 1-***handle slide over a*** 1-***handle*** if $\mathcal{K}' = (\mathcal{K} \backslash K_i) \cup (K_i \#_b \tilde{K_j})$. See the first row of Figure 1.2.

2. Let $K_i \subset L_2$ be an m_i-framed knot and $K_j \subset L_1$ be a dotted unknot. Let $b \subset S^3$ be a sliding band connecting K_i and $\tilde{K_j}$. We define $K_i \#_b \tilde{K_j}$ to be an $(m_i + 2lk(K_i, \tilde{K_j}))$-framed knot and call it ***the result of sliding of*** K_i ***over*** K_j or simply 2-***handle slide over a*** 1-***handle***. Here, $lk(K_i, \tilde{K_j})$ is calculated by orienting K_i and $\tilde{K_j}$ so that the orientation of $(K_i \cup \tilde{K_j}) \setminus b$ extends to the orientation of $K_i \#_b \tilde{K_j}$. If $\mathcal{K}'$ is another Kirby diagram, we say that $\mathcal{K}$ and $\mathcal{K}'$ are related by a 2-***handle slide over a*** 1-***handle*** if $\mathcal{K}' = (\mathcal{K} \setminus K_i) \cup (K_i \#_b \tilde{K_j})$. See the second row of Figure 1.2.

3. Let $K_i, K_j \subset L_2$ be m_i-framed knot and m_j-framed knot, respectively. Let $b \subset S^3$ be a sliding band connecting K_i and $\tilde{K_j}$. We define $K_i \#_b \tilde{K_j}$ to be an $(m_i + m_j + 2lk(K_i, \tilde{K_j}))$-framed knot and call it ***the result of sliding of*** K_i ***over*** K_j or simply 2-***handle slide over a*** 2-***handle***. Here, $lk(K_i, \tilde{K_j})$ is calculated by orienting K_i and $\tilde{K_j}$ so that the orientation of $(K_i \cup \tilde{K_j}) \setminus b$ extends to the orientation of $K_i \#_b \tilde{K_j}$. If $\mathcal{K}'$ is another Kirby diagram, we say that $\mathcal{K}$ and $\mathcal{K}'$ are related by a 2-***handle slide over a*** 2-***handle*** if $\mathcal{K}' = (\mathcal{K} \setminus K_i) \cup (K_i \#_b \tilde{K_j})$. See the third row of Figure 1.2.

Definition 1.2.29. Let $\mathcal{K} = L_1 \cup L_2 \subset S^3$ be a Kirby diagram, where L_1 is a dotted unlink and L_2 is a framed link.

1. Let $L = K_1 \cup K_2 \subset S^3 \setminus \mathcal{K}$ be a two-component link, where K_1 is a framed knot and K_2 is a dotted meridian of K_1. We call such L a ***cancelling*** $(1,2)$-***pair***. Let $\mathcal{K}' = \mathcal{K} \cup L$ be a Kirby diagram. We say that $\mathcal{K}'$ is obtained from $\mathcal{K}$ by ***creating*** a cancelling $(1,2)$-pair and that $\mathcal{K}$ is obtained from $\mathcal{K}'$ by ***annihilating*** a cancelling $(1,2)$-pair. See the left of Figure 1.3.

2. Let $B \subset S^3$ be a 3-ball such that $\mathcal{K} \cap B = \emptyset$. Let $U \subset B$ be a 0-framed unknot. We call such U a ***cancelling*** $(2,3)$-***pair***. Let $\mathcal{K}' = \mathcal{K} \cup U$ be a Kirby diagram. We say that $\mathcal{K}'$ is obtained from $\mathcal{K}$ by a ***creating*** a cancelling $(2,3)$-pair and that $\mathcal{K}$ is obtained from $\mathcal{K}'$ by ***annihilating*** a cancelling $(2,3)$-pair. See the right of Figure 1.3.

We note that for a cancelling $(2,3)$-pair, we do not draw a 3-handle which is cancelled with the 2-handle attached along the 0-framed unknot U, i.e. $M_{\mathcal{K}} \cong M_{\mathcal{K}'} \cup$ 3-handle. More precisely, the 3-handle is attached along the obvious 2-sphere $\{x_0\} \times S^2 \subset \partial M_{\mathcal{K}} \#(S^1 \times S^2) \cong \partial M_{\mathcal{K}'}$, where $x_0 \in S^1$.

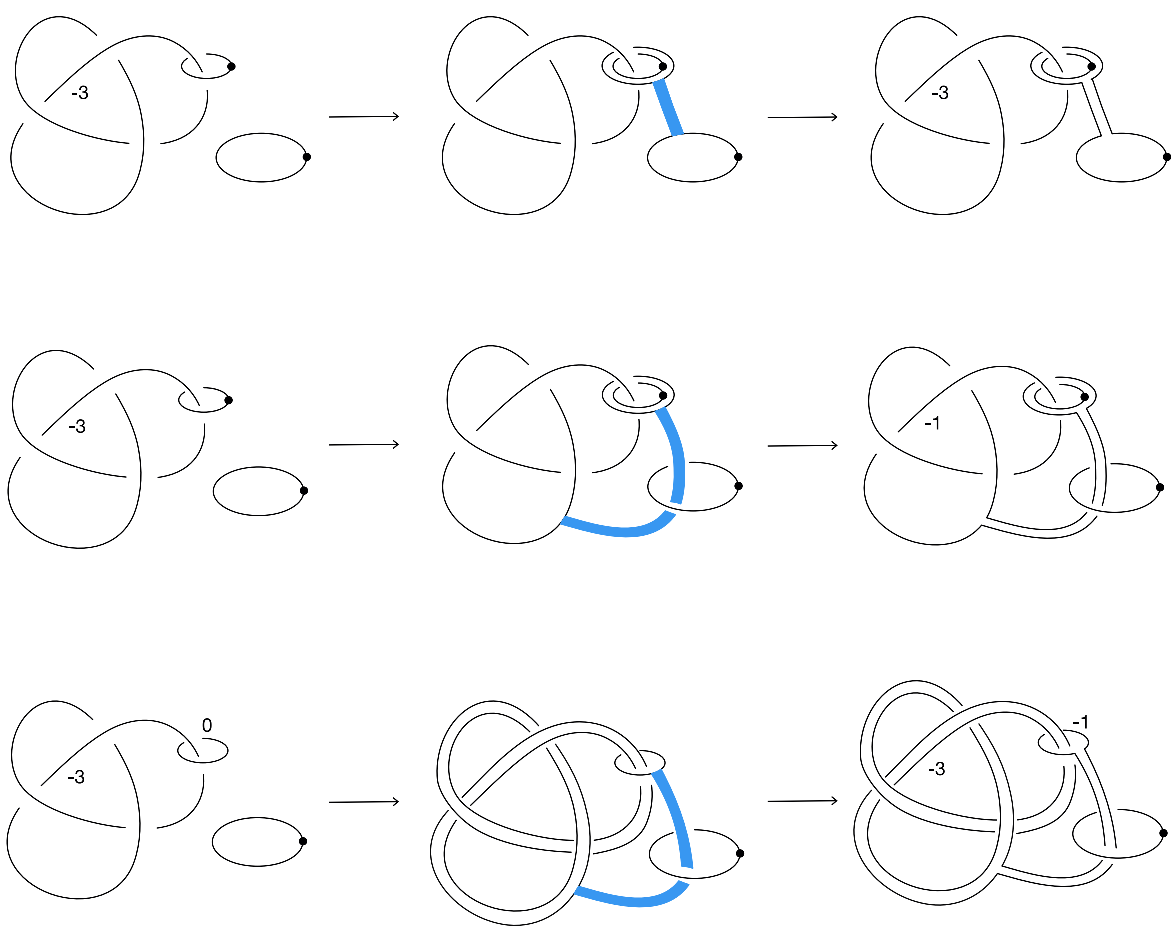

Figure 1.2: Three types of handle slides. **First row**: A 1-handle slide over a 1-handle. **Second row**: A 2-handle slide over a 1-handle. **Third row**: A 2-handle slide over a 2-handle.

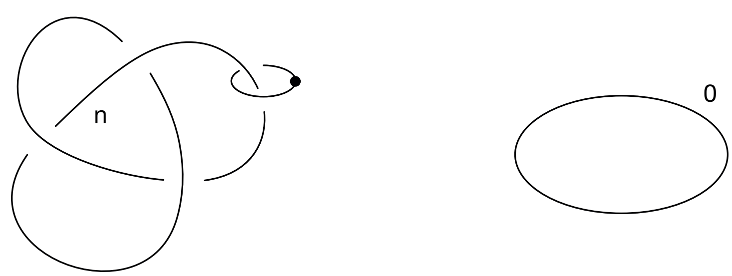

Figure 1.3: **Left**: A cancelling $(1, 2)$-pair. **Right**: A cancelling $(2, 3)$-pair.

Theorem 1.2.30 ([Kir06]). *Let $\mathcal{K}, \mathcal{K}'$ be Kirby diagrams for closed 4-manifolds. Then $\widehat{M_{\mathcal{K}}} \cong \widehat{M_{\mathcal{K}'}}$ if and only if they are related by isotopies, handle slides (1-handles over 1-handles, 2-handles over 1-handles, and 2-handles over 2-handles), and the creation/annihilation of cancelling pairs ((1, 2)-cancelling pairs and (2, 3)-cancelling pairs).*

We may refer to the moves defined on a Kirby diagram in Definition 1.2.28 and Definition 1.2.29 as ***Kirby moves***.

Definition 1.2.31. Let $\mathcal{K} \subset S^3$ be a Kirby diagram. Let $B \subset S^3$ be a 3-ball such that $\mathcal{K} \cap B$. Let $K \subset B$ be a ± 1-framed unknot. Let $\mathcal{K}' = \mathcal{K} \cup K$ be a Kirby diagram. We say that $\mathcal{K}'$ is obtained from $\mathcal{K}$ by ***blowing up*** and that $\mathcal{K}$ is obtained from $\mathcal{K}'$ by ***blowing down***.

We note that M_K is diffeomorphic to $\mathbb{C}P^2 \setminus int(B^4)$ or $\overline{\mathbb{C}P^2} \setminus int(B^4)$ so $\partial M_K \cong S^3$.

Theorem 1.2.32 ([Lic62]). *Every closed, connected, orientable, smooth 3-manifold is diffeomorphic to $\partial M_{\mathcal{K}}$ for some Kirby diagram $\mathcal{K}$. In particular, we can assume that $\mathcal{K}$ has no dotted unlink.*

Theorem 1.2.33 ([Kir78]). *Let $\mathcal{K}, \mathcal{K}'$ be Kirby diagrams. Let $\tilde{\mathcal{K}}, \tilde{\mathcal{K}}'$ be Kirby diagram obtained from K, K' by switching a dotted unlink to a 0-framed unlink, respectively. Then $\partial M_K \cong \partial M_{K'}$ if and only if $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}'$ are related by isotopies, 2-handle slides over 2-handles, and blow up/downs.*

Theorem 1.2.33 tells us how to check if a given Kirby diagram $\mathcal{K}$ represents a closed 4-manifold, i.e., if $\partial M_{\mathcal{K}} \cong \#^k(S^1 \times S^2)$ for some $k \geq 0$. If $\partial M_{\mathcal{K}} \cong \#^k(S^1 \times S^2)$, then the Kirby diagram $\tilde{\mathcal{K}}$ (obtained from $\mathcal{K}$ by switching a dotted unlink to a 0-framed unlink) and a k-component 0-framed unlink are related by isotopies, 2-handles slides over 2-handles, and blow up/downs.

1.2.3 Banded unlink diagrams for surfaces in 4-manifolds

Definition 1.2.34. A ***singular link*** L in a 3-manifold Z is the image of an immersion $i : \coprod^m S^1 \to Z$ which is injective except at isolated double points that are not tangencies. At every double point p, we include a small disks $v \cong B^2$ embedded in Z such that $(v, v \cap L) \cong (B^2, \{(x, y) \in B^2 | xy = 0\})$. We refer to these disks as the ***vertices*** of L.

Definition 1.2.35. A ***marked singular link*** (L, σ) in a 3-manifold Z is a singular link L along with decorations σ on the vertices of L, as follows: say that v is a vertex of L, with $\partial v \cap \overline{(L \setminus v)}$ consisting of the four points p_1, p_2, p_3, p_4 in cyclic order. Choose a co-orientation of the disk v. On the positive side of v, add an arc connecting p_1 and p_3. On the negative side of v, add an arc connecting p_2 and p_4. See the left of Figure 1.4.

Definition 1.2.36. Consider a marked singular link (L, σ) in a 3-manifold Z. Let v be a marked vertex of L; say that on the positive side of v, there is an arc with endpoints p_1 and p_3 and on the negative side of v, there is an arc with endpoints p_2 and p_4.

Let L^+ denote the link in Z obtained from (L, σ) by pushing the arc of L between p_1 and p_3 off v in the positive direction, and repeating for each vertex in L. We recall L^+ the ***positive resolution*** of (L, σ); see the top right of Figure 1.4.

Similarly, let L^- denote the link in Z obtained from (L, σ) by pushing the arc of L between p_1 and p_3 off v in the negative direction, and repeating for each vertex in L. We recall L^- the ***negative resolution*** of (L, σ); see the bottom right of Figure 1.4.

For each marked vertex v of L, these opposite push-offs form a bigon in a neighborhood of v, which bounds an embedded disk d_v. This disk d_v can be chosen so that its interior intersects L transversely in a single point near v. We call d_v a ***companion disk*** of v; see the middle right of Figure 1.4. For each marked vertex v, select such a disk d_v(ensuring that all of these disks are pairwise disjoint), and let D_L denote the union of all of these companion disks.

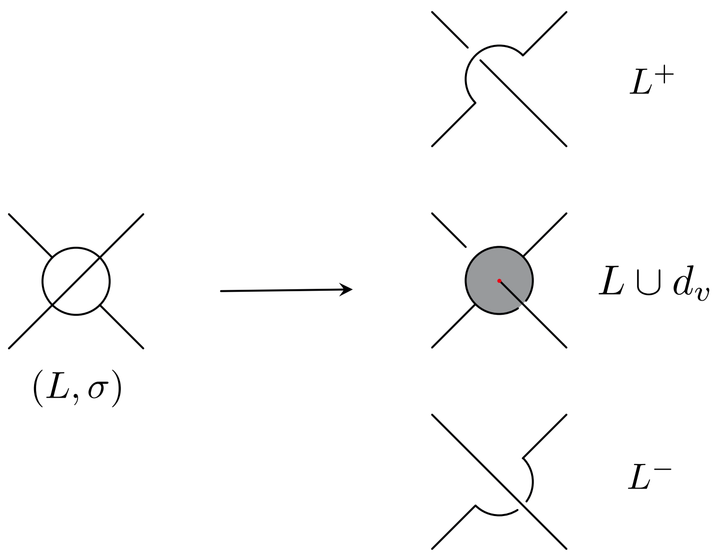

Figure 1.4: **Left**: A singular point v of L with a decoration σ. **Top right**: The positive resolution of (L, σ). **Middle right**: The union of L and a companion disk d_v. **Bottom right**: The negative resolution of (L, σ).

Definition 1.2.37. Let L be a marked singular link in Z, and let V_L denote the union of the vertices of L. A ***band*** b attached to L is the image of an embedding $\phi : B^1 \times B^1 \to Z \setminus V_L$ such that $b \cap L = \phi(B^1 \times \{-1, 1\})$. We call $\phi(B^1 \times \{0\})$ the ***core*** of the band b. Let L_b be the singular link defined by $L_b = (L - \phi(\{-1, 1\} \times B^1)) \cup \phi(B^1 \times \{-1, 1\})$.

We say that L_b is the result of performing ***band surgery*** to L along b. If B is a finite family of pairwise disjoint bands for L, then we will let L_B denote the link we obtain by performing band surgery along each of the bands in B. We say that L_B is the result of ***resolving*** the bands in B. Note that the self-intersections of L_B naturally correspond to those of L, so a choice of markings for L yields markings for L_B. A triple

(L, σ, B), where (L, σ) is a marked singular link and B is a family of disjoint bands for L, is called a ***marked singular banded link***. To ease notation, we may refer to the pair (L, B) as a ***singular banded link*** and implicitly remember that L is actually a marked singular link.

Definition 1.2.38. Let (L, B) be a singular banded link in 3-manifold Z. Let $B = \{b_1, \ldots, b_n\}$ be the set of bands attached to L. Let $V_L = \{v_1, \ldots, v_m\}$ be the set of marked vertices of L. Let $D_L = \{d_{v_1}, \ldots, d_{v_m}\}$ be the set of companion disks, where d_{v_i} is a companion disk of v_i.

We define a surface $S(L, B) \subset Z \times [0, 1]$ as follows:

$$S(L,B) \cap (Z \times \{t\}) = \begin{cases} L_B^+ \times \{t\} & t \in (\frac{2}{3}, 1] \\ (L^+ \cup B) \times \{t\} & t = \frac{2}{3} \\ L^+ \times \{t\} & t \in (\frac{1}{3}, \frac{2}{3}) \\ (L^- \cup D_L) \times \{t\} & t = \frac{1}{3} \\ L^- \times \{t\} & t \in [0, \frac{1}{3}). \end{cases}$$

Here, $S(L, B)$ is a surface properly immersed in $Z \times [0, 1]$ with two boundary components

$$(L^- \times \{0\}) \coprod (L_B^+ \times \{1\}) \subset Z \times \{0\} \coprod Z \times \{1\}$$

and with isolated transverse self-intersections all contained in $Z \times \{\frac{1}{3}\}$ corresponding to the marked vertices V_L. We refer to $S(L, B)$ as a ***surface segment realizing*** (L, B) in $Z \times [0, 1]$.

Remark 1.2.39. If $V_L = \emptyset$, we call (L, B) a ***banded link*** and $S(L, B)$ is clearly a properly embedded surface.

Definition 1.2.40. Let $\mathcal{K} = L_1 \cup L_2 \subset S^3$ be a Kirby diagram. Let (L, B) be a singular banded link in $S^3 \setminus \mathcal{K}$ with a set of disjoint bands $B \subset S^3 \setminus \mathcal{K}$ attached to L. A triple $(\mathcal{K}, L, B)$ is called a ***singular banded unlink diagram*** if L^- is the unlink in ∂M_{L_1} and L_B^+ is the unlink in $\partial M_{\mathcal{K}}$. We may refer to a singular banded unlink diagram without singular points as ***banded unlink diagram***.

Remark 1.2.41.

1. M_{L_1} is the 4-manifold obtained from B^4 by carving out properly embedded trivial disks in B^4 bounded by a dotted unlink L_1. Since $L_1 \cap L_2 = \emptyset$, we can see that L_2 can be naturally embedded in ∂M_{L_1}.
2. $M_{\mathcal{K}} = M_{L_1 \cup L_2}$ is obtained from M_{L_1} by attaching 2-handles along $L_2 \subset \partial M_{L_1}$.
3. The singular banded unlink (L, B) is in $S^3 \setminus \mathcal{K}$, so (L, B) can be naturally embedded in ∂M_{L_1} and $\partial M_{\mathcal{K}}$. Therefore, we can consider L^- in ∂M_{L_1} and L_B^+ in $\partial M_{\mathcal{K}}$.
4. We note that ∂M_{L_1} (resp, $\partial M_{\mathcal{K}}$) is diffeomorphic to $\#^k(S^1 \times S^2)$ for some $k \geq 0$.

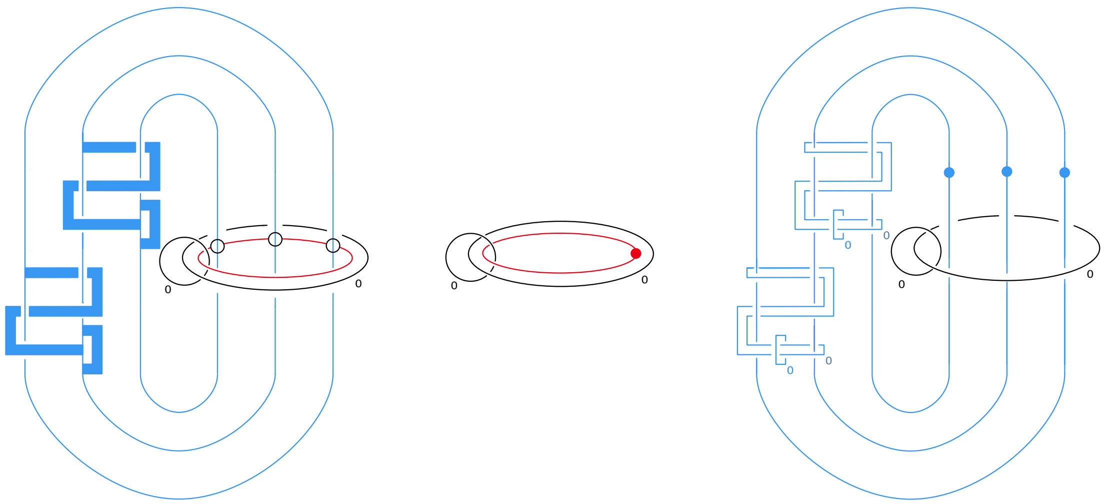

Figure 1.5: **Left**: A Heegaard diagram (Σ, α, β) for a 5-dimensional cobordism from S^4 to a non-simply connected homology 4-sphere with a 2-handle and a 3-handle, which are algebraically but not geometrically cancelled, or a contractible 5-manifold with a 0-handle, a 2-handle, and a 3-handle, which is not homeomorphic to B^5. Here, Σ (in black) represents $S^2 \times S^2$, α (in red) is the belt sphere of the 2-handle representing $\{x_0\} \times S^2 \subset S^2 \times S^2$, and β (in blue) is the attaching sphere of the 3-handle representing a 2-knot which is homotopic but not isotopic to $S^2 \times \{y_0\} \subset S^2 \times S^2$. **Middle**: A Kirby diagram for $\Sigma(\alpha)$ which is diffeomorphic to S^4. **Right**: A Kirby diagram for $\Sigma(\beta)$ which is diffeomorphic to the non-simply connected homology 4-sphere.

Example 1.2.42. Let $(\mathcal{K}, L, B) = (\mathcal{K}, J_1 \cup J_2, B_1 \cup B_2)$ be the diagram in the left of Figure 1.5, where $\mathcal{K}$ is the black 0-framed Hopf link, J_1 is the red unknot, $B_1 = \emptyset$, J_2 is the blue 3-component unlink, and B_2 is the set of blue bands attached to J_2. We can verify that

1. $(\mathcal{K}, J_1, B_1)$ is a banded unlink diagram,
2. $(\mathcal{K}, J_2, B_2)$ is a banded unlink diagram,
3. $(\mathcal{K}, L, B)$ is a singular banded unlink diagram.

Given a singular banded unlink diagram $(\mathcal{K}, L, B)$, we construct an immersed surface $S(\mathcal{K}, L, B)$ in $\widehat{M_\mathcal{K}}$. We view $\widehat{M_\mathcal{K}}$ slightly differently from Definition 1.2.22 as:

$$\widehat{M_\mathcal{K}} = M_{L_1} \cup (\partial M_{L_1} \times [0,1]) \cup_{L_2 \times \{1\}} (\coprod^{|L_2|}(B^2 \times B^2)) \cup_f (\natural^k(S^1 \times B^3)),$$

where $|L_2|$ is the number of components of L_2 and $f : \#^k(S^1 \times S^2) \to \partial M_\mathcal{K}$ is a diffeomorphism. We note that $M_{L_1} \cup (\partial M_{L_1} \times [0,1]) \cup_{L_2 \times \{1\}} (\coprod^{|L_2|}(B^2 \times B^2))$ is obtained from M_{L_1} by thickening the boundary ∂M_{L_1} and attaching 2-handles along $L_2 \times \{1\}$, so it is diffeomorphic to $M_\mathcal{K}$. Let $S(L, B)$ be a surface segment realizing (L, B) in $\partial M_{L_1} \times [0,1]$ described in Definition 1.2.38. Let $\Delta \subset M_{L_1}$ be the set of properly embedded trivial disks such that $\partial\Delta = L_- \times \{0\} \subset \partial M_{L_1} \times \{0\}$. Let $\Delta' \subset \#^k(S^1 \times B^3)$ be the set of properly embedded trivial disks such that $\partial\Delta' = L_B^+ \times \{1\} \subset \partial M_{L_1} \times \{1\}$. Note that $\partial\Delta'$ is embedded in ∂M_{L_2} as well.

Definition 1.2.43. A ***realizing surface*** corresponding to $(\mathcal{K}, L, B)$ is a surface

$$S(\mathcal{K}, L, B) = \Delta \cup S(L, B) \cup \Delta'$$

in $\widehat{M_{\mathcal{K}}}$.

Remark 1.2.44. The Euler characteristic is $\chi(S(\mathcal{K}, L, B)) = |L^-| - |B| + |L_B^+|$.

Definition 1.2.45. Let $(\mathcal{K}, L, B)$ and $(\mathcal{K}, L', B')$ be singular banded unlink diagrams, where $\mathcal{K} = L_1 \cup L_2$. We say that $(\mathcal{K}, L, B)$ and $(\mathcal{K}, L', B')$ are related by ***singular band moves*** if $(\mathcal{K}, L', B')$ is obtained from $(\mathcal{K}, L, B)$ by a sequence of moves in Figure 1.6 and Figure 1.7. The singular band moves (illustrated in Figure 1.6 and Figure 1.7) include:

1. Isotopy in $S^3 \setminus \mathcal{K}$,
2. Cup/cap moves,
3. Band slides,
4. Band swims,
5. Slides of bands over components of L_2,
6. Swims of bands about L_2,
7. Slides of unlinks and bands over L_1,
8. Sliding a vertex over a band,
9. Passing a vertex past the edge of a band,
10. Swimming a band through a vertex.

We may refer to moves $(1) - (7)$ (illustrated in Figure 1.6) as ***band moves*** (omitting the word "singular") since they do not involve the self-intersections of L. The remaining moves $(8) - (10)$ (illustrated in Figure 1.7) are specific to the interactions between points and bands.

Theorem 1.2.46 ([HKM20],[HKM21])**.** *Let (Y, F) be a pair, where Y is a closed, connected, orientable 4-manifold and $F \subset Y$ is an immersed surface. Then there exists a singular banded unlink diagram $(\mathcal{K}, L, B)$ such that (Y, F) is diffeomorphic to $(\widehat{M_{\mathcal{K}}}, S(\mathcal{K}, L, B))$.*

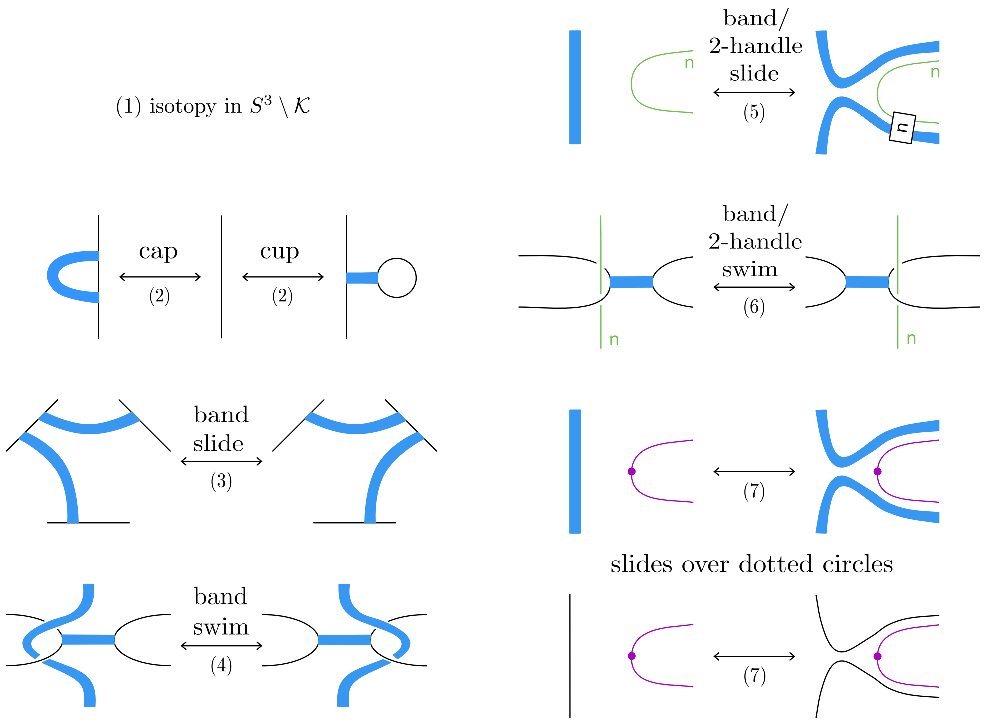

Figure 1.6: Singular band moves without singular points.

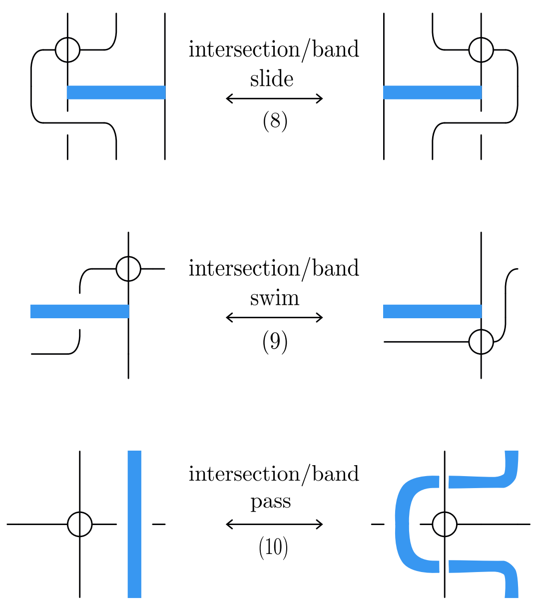

Figure 1.7: Singular band moves with singular points.

Theorem 1.2.47 ([HKM20],[HKM21]). *Let $(\mathcal{K}, L, B)$ and $(\mathcal{K}, L', B')$ be singular banded unlink diagrams. Then $S(\mathcal{K}, L, B)$ is isotopic to $S(\mathcal{K}, L', B')$ in $\widehat{M_{\mathcal{K}}}$ if and only if they are related by singular band moves.*

Theorem 1.2.48 ([HKM20],[HKM21]). *Let $(\mathcal{K}, L, B)$ and $(\mathcal{K}', L', B')$ be singular banded unlink diagrams. Then $(\widehat{M_{\mathcal{K}}}, S(\mathcal{K}, L, B))$ is diffeomorphic to $(\widehat{M_{\mathcal{K}'}}, S(\mathcal{K}', L', B'))$ if and only if they are related by Kirby moves and singular band moves.*

Example 1.2.49. $(S^2\tilde{\times}S^2, F) \cong (\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \mathbb{C}P^1\#\overline{\mathbb{C}P^1})$, where F is a fiber of $S^2\tilde{\times}S^2$.

Proof. The diagram on the left of Figure 1.8 is obtained from the right of Figure 1.8 by sliding the (-1)-framed unknot over the 1-framed unknot along the obvious band. □

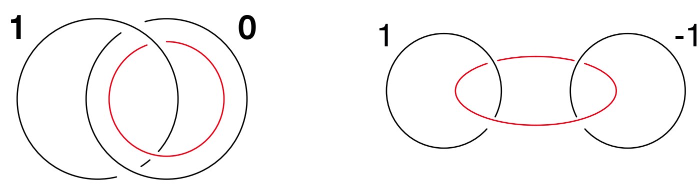

Figure 1.8: $(S^2\tilde{\times}S^2, F) \cong (\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \mathbb{C}P^1\#\overline{\mathbb{C}P^1})$, where F is a fiber of $S^2\tilde{\times}S^2$.

1.2.4 Kirby diagrams for 1- and 2- surgery

We illustrate how to obtain a Kirby diagram for 1- and 2-surgery of a 4-manifold. Note that 1-surgery is equivalent to attaching a 5-dimensional 2-handle, and 2-surgery is equivalent to attaching a 5-dimensional 3-handle.

We first explain how to obtain a Kirby diagram for 1-surgery of an arbitrary 4-manifold from a pair of a Kirby diagram and an embedded circle in the Kirby diagram.

Proposition 1.2.50 (A Kirby diagram for 1-surgery). Let (Y, γ) be a pair, where Y is a closed 4-manifold and $\gamma \subset Y$ is an embedded circle. Consider a pair $(\mathcal{K}, c)$, where $\mathcal{K}$ is a Kirby diagram for Y and $c \subset \mathcal{K}$ is an embedded circle representing γ. Then we can obtain a Kirby diagram $\mathcal{K}'$ for 1-surgery

$$Y(\gamma) = (Y \setminus int(\nu(\gamma))) \cup (B^2 \times S^2)$$

of Y along γ by following these steps:

1. Begin with the pair $(\mathcal{K}, c)$; see the top left of Figure 1.9.

2. Introduce a cancelling $(1,2)$-pair to $\mathcal{K}$, where the 2-handle is c with a framing (two possible framings) and the 1-handle is a dotted meridian m of c; see the top right of Figure 1.9.

3. Switch the dotted meridian to a 0-framed 2-handle; see the bottom left of Figure 1.9.

Introducing a cancelling $(1,2)$-pair in (2) still represents Y. The dot-zero exchange in (3) corresponds to 1-surgery of Y along γ, where $S^1 \times B^3$ is removed and $B^2 \times S^2$ is attached. Here, γ has two possible framings since $\pi_1(SO(3)) \cong \mathbb{Z}_2$. Another perspective is that any integer framing of c (representing γ) can be reduced to 0 or 1 by handle slides of c over the meridian m.

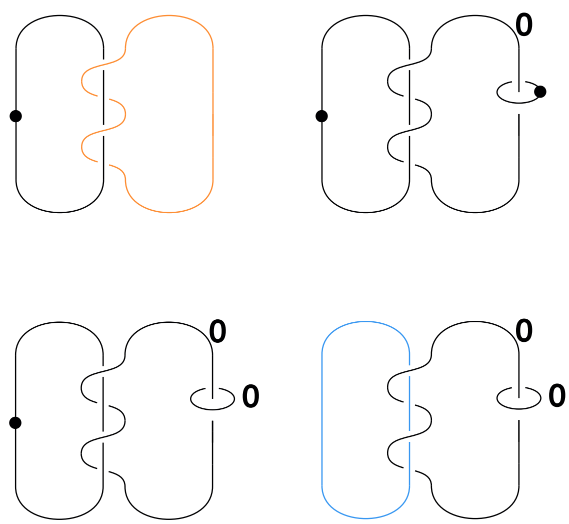

Figure 1.9: **Top left**: A Kirby diagram $\mathcal{K}$ for $S^1 \times S^3$ and an orange circle c representing $2 \in \mathbb{Z} \cong \pi_1(S^1 \times S^3)$. **Top right**: A new Kirby diagram for $S^1 \times S^3$ obtained by introducing a cancelling $(1,2)$-pair. **Bottom left**: A Kirby diagram $\mathcal{K}'$ for surgery of $S^1 \times S^3$ along γ representing $2 \in \mathbb{Z} \cong \pi_1(S^1 \times S^3)$ with the trivial framing. **Bottom right**: A banded unlink diagram obtained from the bottom left by switching the dotted circle to the blue circle. This banded unlink diagram represents a pair $(S^2 \times S^2, F)$, where F is a 2-knot representing $(2,0) \in \mathbb{Z} \oplus \mathbb{Z} \cong H_2(S^2 \times S^2)$. The Kirby diagram for the surgery of $S^2 \times S^2$ along F is the bottom left.

We now explain how to obtain a Kirby diagram for 2-surgery of an arbitrary 4-manifold from a banded unlink diagram.

Proposition 1.2.51 (A Kirby diagram for 2-surgery)**.** Let $(\mathcal{K}, L, B)$ be a banded unlink diagram for a pair (Y, F), where Y is a 4-manifold and $F \subset Y$ is a 2-knot with trivial normal bundle. Then we can obtain a Kirby diagram $\mathcal{K}'$ for 2-surgery

$$Y(F) = (Y \setminus int(\nu(F))) \cup (B^3 \times S^1)$$

of Y along F by following these steps:

1. Begin with the banded unlink diagram $(\mathcal{K}, L, B)$.

2. Switch the unlink L to a dotted unlink; see the top of Figure 1.10.

3. Switch the bands B to 0-framed 2-handles; see the bottom of Figure 1.10

Furthermore, if $\widehat{M_{\mathcal{K}}} = M_{\mathcal{K}} \cup (\natural^k(S^1 \times B^3))$, then $\widehat{M_{\mathcal{K}'}} = M_{\mathcal{K}'} \cup (\natural^{k+|L_B|}(S^1 \times B^3))$, where $|L_B|$ is the number of components of the surgery of L along B.

Proof. Let $(\mathcal{K}, L, B)$ be a banded unlink diagram for (Y, F), i.e. $(\widehat{M_{\mathcal{K}}}, S(\mathcal{K}, L, B)) = (Y, F)$. Let $\widehat{M_{\mathcal{K}}} = M_{\mathcal{K}} \cup (\natural^k(S^1 \times B^3))$ for some $k \geq 0$. Consider a banded unlink diagram $(\emptyset, L, B)$ from $(\mathcal{K}, L, B)$ by ignoring $\mathcal{K}$. Then $F' = S(\emptyset, L, B)$ is a 2-knot in $\widehat{M_{\emptyset}} = (0\text{-handle}) \cup (4\text{-handle}) = S^4$. By the construction of a realizing surface, we can assume that

$$Y \setminus int(\nu(F)) = \widehat{M_{\mathcal{K}}} \setminus int(\nu(S(\mathcal{K}, L, B)))$$

is obtained from $\widehat{M_{\emptyset}} \setminus int(\nu(S(\emptyset, L, B)))$ by the attaching 1-,2-, and 3-handles of $\widehat{M_{\mathcal{K}}}$ because (L, B) is embedded $S^3 \setminus \mathcal{K}$. There is a way to find a Kirby diagram for $\widehat{M_{\emptyset}} \setminus int(\nu(S(\emptyset, L, B)))$ (i.e. a Kirby diagram for the complement of a 2-knot in S^4); see Chapter 6.2 in [GS99]. The key idea is that each k-handle of F' induces a $(k+1)$-handle in the complement of F' in S^4. Here, the attaching sphere of the $(k+1)$-handle is $\partial(C \times B^1)$, where C is the core of h. More precisely, the unlink L, the bands B, and the result L_B of the surgery of L along B induce a dotted unlink, 0-framed 2-handles, and $(|L_B| - 1)$ 3-handles (not drawn), respectively, in the Kirby diagram for the complement of S^4 along F'; see Figure 1.10. Let $\mathcal{K}'$ be the Kirby diagram obtained from $(\mathcal{K}, L, B)$ by switching the unlink L to a dotted unlink and switching bands B to 0-framed 2-handles. Then the manifold obtained from $M_{\mathcal{K}'}$ by attaching the $(|L_B| - 1)$ 3-handles (induced by L_B) and k 3-handles of $\widehat{M_{\mathcal{K}}}$ is diffeomorphic to $Y \setminus int(\nu(F))$. Since we can consider $(B^3 \times S^1)$ in $Y(F) = (Y \setminus int(\nu(F))) \cup (B^3 \times S^1)$ as the union of a 3-handle and a 4-handle, $Y(F) \cong M_{\mathcal{K}'} \cup (\natural^{k+|L_B|(S^1 \times B^3)})$. It is worth noting that according to [LP72], there exists a unique way, up to diffeomorphism, to attach $(k + |L_B|)$ 3-handles and a 4-handle to $M_{\mathcal{K}'}$. Therefore, $\mathcal{K}'$ is a Kirby diagram for $Y(F)$. □

Example 1.2.52.

1. The middle of Figure 1.5 is a Kirby diagram for the surgery $\Sigma(\alpha)$ of Σ along α. Here, $(k + |L_B|)$ 3-handles are not drawn, with $k = 0$ and $|L_B| = 1$. After removing a cancelling $(1, 2)$-pair and a cancelling $(2, 3)$-pair, it becomes evident that the Kirby diagram represents S^4.

2. The right of Figure 1.5 is a Kirby diagram for the surgery $\Sigma(\beta)$ of Σ along β. Here, $(k + |L_B|)$ 3-handles are not drawn, with $k = 0$ and $|L_B| = 3$. In chapter 2, we demonstrate that this Kirby diagram represents a non-simply connected homology 4-sphere. Note that we can simplify this Kirby diagram by sliding the right two 1-handles over the other 1-handle and annihilating a cancelling $(1, 2)$-pair.

3. Considering the top left of Figure 1.10 as a banded unlink diagram for the unknotted 2-sphere U in S^4, the top right Figure 1.10 is a Kirby diagram for the surgery of S^4 along U, which is diffeomorphic to $S^1 \times S^3$.

4. In the left of Figure 1.8, we can derive a Kirby diagram for the surgery of $S^2 \tilde{\times} S^2$ along a fiber by switching the red circle to a dotted circle. We can confirm that the surgery is diffeomorphic to S^4 by removing a cancelling $(1, 2)$-pair and a cancelling $(2, 3)$-pair. Similarly, we can verify that the surgery of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ along $\mathbb{C}P^1 \# \overline{\mathbb{C}P^1}$ is diffeomorphic to S^4 in the right of Figure 1.8.

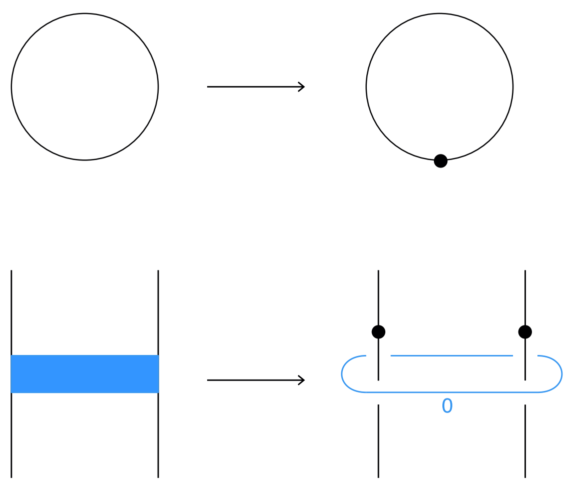

Figure 1.10: How to obtain a Kirby diagram for 2-surgery from a banded unlink diagram.

1.3 n-dimensional k-handlebodies $(n \geq 2k+1)$

We recall that an n-dimensional manifold X is called an ***n-dimensional k-handlebody*** if it admits a handle decomposition

$$B^n = X_0 \subset X_1 \subset \cdots \subset X_k = X,$$

where X_i is obtained from X_{i-1} by attaching i-handles. We note that an n-dimensional k-handlebody is not just the union of a 0-handle and k-handles. We say that an n-dimensional k-handlebody X admits a ***product structure*** if there exits a handle decomposition $B^{n-1} = Y_0 \subset Y_1 \subset \cdots \subset Y_k = Y$ of an $(n-1)$-dimensional k-handlebody $Y \subset X$ such that $X_i \cong Y_i \times B^1$ for every $0 \leq i \leq k$.

In this section, we show that every n-dimensional k-handlebody admits a product structure when $n \geq 2k+1$. It is true when $k = 0$ because an n-dimensional 0-handle is diffeomorphic to B^n and $B^n \cong B^{n-1} \times B^1$. For example, every 5-dimensional 2-handlebody is diffeomorphic to $Y \times B^1$ for some 4-dimensional 2-handlebody Y. As a corollary, the boundary of a 5-dimensional 2-handlebody is the double of a 4-dimensional 2-handlebody, not arbitrary 4-manifold.

A natural follow-up question is whether every n-dimensional k-handlebody admits a product structure when $n \leq 2k$. We can easily see counterexamples in the case where $k \geq 2$ and $k+1 \leq n \leq 2k$. Let X be the once-punctured $S^{n-k} \times S^k$, which is an n-dimensional k-handlebody obtained from B^n by attaching a single $(n-k)$-handle along the unknotted $(n-k-1)$-sphere A and a single k-handle along the unknotted $(k-1)$-handle B, both with trivial framing. Here, $A \cup B = (S^{n-k-1} \times 0) \cup (0 \times S^{k-1}) \subset (S^{n-k-1} \times B^k) \cup (B^{n-k} \times S^{k-1}) = \partial(B^{n-k} \times B^k) = \partial B^n = S^{n-1}$. A generates $H_{n-k-1}(S^{n-1} \setminus int(\nu(B)))$, and similarly, B generates $H_{k-1}(S^{n-1} \setminus int(\nu(A)))$. For instance, when $n = 4$ and $k = 2$, $A \cup B$ forms the Hopf link in S^3.

Clearly, X does not admit a product structure. If X did admit a product structure, then the attaching spheres $A \cup B$ should be isotopic to the equator S^{n-2} of S^{n-1}. This would allow us to construct an $(n-1)$-manifold Y obtained from B^{n-1} by attaching handles to S^{n-2} along $A \cup B$ such that $X \cong Y \times B^1$. However, if $A \cup B$ were isotopic to S^{n-2}, then the $(n-k-1)$-sphere A would be null homotopic in $S^{n-2} \setminus int(\nu(B)) \cong S^{n-k-2} \times B^k$, which is a contradiction since A is supposed to be a generator of $H_{n-k-1}(S^{n-1} \setminus int(\nu(B)))$. For example, the Hopf link in S^3 cannot be isotopic to the equator S^2 of S^3.

We first prove some lemmas to show that every n-dimensional k-handlebody admits a product structure when $n \geq 2k+1$.

Lemma 1.3.1. Let Y be an $(n-1)$-manifold with boundary. Let $\psi : S^{k-1} \times B^{n-k-1} \hookrightarrow \partial Y$ be an attaching map of an $(n-1)$-dimensional k-handle. Let $\Psi : (S^{k-1} \times B^{n-k-1}) \times B^1 \hookrightarrow \partial Y \times B^1 \subset \partial(Y \times B^1)$ be an attaching map of an n-dimensional k-handle such that $\Psi((x,y),t) = (\psi(x,y),t)$. Then $(Y \times B^1) \cup_\Psi ((B^k \times B^{n-k-1}) \times B^1) \cong (Y \cup_\psi (B^k \times B^{n-k-1})) \times B^1$.

Proof.
$$\begin{aligned}(Y \times B^1) \cup_\Psi ((B^k \times B^{n-k-1}) \times B^1) &= \frac{(Y \times B^1) \coprod ((B^k \times B^{n-k-1}) \times B^1)}{((x,y),t) \sim \Psi((x,y),t)} \\ &\cong \frac{(Y \times B^1) \coprod ((B^k \times B^{n-k-1}) \times B^1)}{((x,y),t) \sim (\psi(x,y),t)} \\ &\cong \frac{Y \coprod (B^k \times B^{n-k-1})}{(x,y) \sim \psi(x,y)} \times B^1 \\ &\cong (Y \cup_\psi (B^k \times B^{n-k-1})) \times B^1.\end{aligned}$$
□

Theorem 1.3.2 (Whitney Embedding Theorems [Whi36; Whi44; Ran02]).

1. *Let $f : M \to N$ be a smooth map. If $dim(N) \geq 2 \cdot dim(M) + 1$, then f is homotopic to an embedding $g : M \hookrightarrow N$.*

2. *Let $f, g : M \hookrightarrow N$ be homotopic embeddings. If $dim(N) \geq 2 \cdot dim(M) + 2$, then f and g are isotopic.*

Lemma 1.3.3. Let Y be an $(n-1)$-dimensional $(k-1)$-handlebody. Suppose $f : M \hookrightarrow Y$ is a proper embedding, where M is a $(k-1)$-manifold in Y. If $n \geq 2k+1$, then f is homotopic to a map $g : M \to \partial Y$.

Proof. Let $S \subset Y$ be a natural $(k-1)$-dimensional subcomplex (spine) of Y obtained from Y by collapsing the second factor of each handle to a point, i.e., Y deformation retracts to S. Since

$$n \geq 2k+1 \text{ and } dim(Y) - (dim(S) + dim(M)) = (n-1) - (2k-2) = n - 2k + 1 \geq 2,$$

we may assume that $f(M)$ and S do not intersect, so f is an embedding of M in the complement $Y \setminus \nu(S)$. Here, $Y \setminus \nu(S)$ is diffeomorphic to $\partial Y \times [0,1]$, where $\partial Y \times \{0\}$ is identified with ∂Y. Let $p : \partial Y \times [0,1] \to \partial Y \times \{0\} \cong \partial Y$ be the projection sending (y,t) to y. Define $g = p \circ f : M \to \partial Y$. Then f and g are homotopic. □

Lemma 1.3.4. Let Y be an $(n-1)$-dimensional $(k-1)$-handlebody. Suppose $f : M \hookrightarrow \partial(Y \times B^1) = (Y \times \{-1\}) \cup_{\partial Y \times \{-1\}} (\partial Y \times B^1) \cup_{\partial Y \times \{1\}} (Y \times \{1\})$ is an embedding, where M is a $(k-1)$-manifold. If $n \geq 2k+1$, then f is isotopic to an embedding $h : M \hookrightarrow \partial Y \times \{0\}$.

Proof. We may assume that f is transverse to $(\partial Y \times \{-1\}) \cup (\partial Y \times \{1\})$. Then the restriction map $f|_{f^{-1}(Y \times \{i\})} : f^{-1}(Y \times \{i\}) \hookrightarrow Y \times \{i\}$ is an embedding, where $i \in \{-1, 1\}$. By Lemma 1.3.3, each embedding $f|_{f^{-1}(Y \times \{i\})}$ is homotopic to a map $g_i : f^{-1}(Y \times \{i\}) \to \partial Y \times \{i\}$ so f is homotopic to a map $g : M \to \partial Y \times B^1$. Let $p : \partial Y \times B^1 \to \partial Y \times \{0\}$ be the projection sending (y,t) to $(y,0)$. Then f is homotopic to $p \circ f : M \to \partial Y \times \{0\}$. Here, $p \circ f$ is isotopic to an embedding $h : M \hookrightarrow \partial Y \times \{0\}$ by (1) in Theorem 1.3.2 since

$$n \geq 2k+1 \text{ and } dim(\partial Y \times \{0\}) = n - 2 \geq 2k - 1 = 2 \cdot dim(M) + 1.$$

We can consider $h : M \to \partial Y \times \{0\} \subset \partial(Y \times B^1)$ as an embedding of M in $\partial(Y \times B^1)$. Then f and h are isotopic by (2) in Theorem 1.3.2 since

$$n \geq 2k+1 \text{ and } dim(\partial(Y \times B^1)) = n - 1 \geq 2k = 2 \cdot dim(M) + 2.$$

□

Remark 1.3.5. In Lemma 1.3.4, any two homotopic embeddings $h, h' : M \hookrightarrow \partial Y \times \{0\}$ in $\partial Y \times \{0\}$ are isotopic in $\partial(Y \times B^1)$ by (2) in Theorem 1.3.2. For example, consider $h : S^1 \to \partial B^4 \times \{0\} = S^3 \times \{0\}$ such that $h(S^1) = U \times \{0\}$ and $h' : S^1 \hookrightarrow \partial B^4 \times \{0\} = S^3 \times \{0\}$ such that $h'(S^1) = K \times \{0\}$, where U is the unknot and K is the trefoil knot in S^3. Then h and h' are homotopic but not isotopic in $S^3 \times \{0\}$. However, h and h' are isotopic in $\partial(B^4 \times B^1) = S^4$.

Proposition 1.3.6. Let $i : O(l) \hookrightarrow O(l+1)$ be the natural inclusion of orthogonal groups. Consider the induced homomorphism $i_* : \pi_m(O(l)) \to \pi_m(O(l+1))$. Then i_* is an epimorphism if $m = l - 1$ and an isomorphism if $m < l - 1$.

Proof. Let $O(l) \to O(l+1) \to O(l+1)/O(l) \cong S^l$ be the canonical Serre fibration. Then there exists a natural long exact sequence

$$\cdots \to \pi_{m+1}(S^l) \longrightarrow \pi_m(O(l)) \xrightarrow{i_*} \pi_m(O(l+1)) \to \pi_m(S^l) \to \cdots .$$

If $m = l-1$, $\pi_{m+1}(S^l) \cong \mathbb{Z}$ and $\pi_m(S^l) = 1$ so i_* is an epimorphism. If $m < l-1$, $\pi_{m+1}(S^l) = 1$ and $\pi_m(S^l) = 1$ so i_* is an isomorphism. □

In Proposition 1.3.6, let $l = n-k-1$ and $m = k-1$. Then $m = l-1$ is equivalent to $n = 2k+1$, and $m < 2l-1$ is equivalent to $n > 2k+1$. Therefore, we have the following:

Remark 1.3.7. Let $i : O(n-k-1) \hookrightarrow O(n-k)$ be the natural inclusion. Consider the induced homomorphism $i_* : \pi_{k-1}(O(n-k-1)) \to \pi_{k-1}(O(n-k))$. Then i_* is an epimorphism if $n = 2k+1$ and an isomorphism if $n > 2k+1$.

Theorem 1.3.8 (Bott Periodicity Theorem [Bot57])**.** *If $l \geq m+2$,*

$$\pi_m(O(l)) = \begin{cases} 0 & \textit{if } m = 2,4,5,6 \textit{ (mod 8)} \\ \mathbb{Z}_2 & \textit{if } m = 0,1 \textit{ (mod 8)} \\ \mathbb{Z} & \textit{if } m = 3,7 \textit{ (mod 8).} \end{cases}$$

In Theorem 1.3.8, let $l = n-k$ and $m = k-1$. Then $l \geq m+2$ is equivalent to $n \geq 2k+1$. Therefore, we have the following:

Remark 1.3.9. If $n \geq 2k+1$,

$$\pi_{k-1}(O(n-k)) = \begin{cases} 0 & \text{if } k = 3,5,6,7 \text{ (mod 8)} \\ \mathbb{Z}_2 & \text{if } k = 1,2 \text{ (mod 8)} \\ \mathbb{Z} & \text{if } k = 0,4 \text{ (mod 8).} \end{cases}$$

Remark 1.3.9 tells us all possible framings of the attaching spheres of an n-dimensional k-handle when $n \geq 2k+1$.

Theorem 1.3.10. *Let Y be an $(n-1)$-dimensional $(k-1)$-handlebody. Let $\Phi : (S^{k-1} \times B^{n-k-1}) \times B^1 \hookrightarrow \partial(Y \times B^1)$ be an attaching map of an n-dimensional k-handle. If $n \geq 2k+1$, then there exists an embedding $\psi : S^{k-1} \times B^{n-k-1} \hookrightarrow \partial Y$ such that Φ is isotopic to Ψ, where $\Psi : (S^{k-1} \times B^{n-k-1}) \times B^1 \hookrightarrow \partial Y \times B^1 \subset \partial(Y \times B^1)$ is defined by $\Psi((x,y),t) = (\psi(x,y),t)$. In particular, $(Y \times B^1) \cup_\Phi ((B^k \times B^{n-k}) \times B^1) \cong (Y \cup_\psi (B^k \times B^{n-k-1})) \times B^1$.*

Proof. The attaching sphere $\Phi((S^{k-1} \times 0) \times \{0\})$ is isotopic to a sphere $F \times \{0\} \subset \partial Y \times \{0\}$ for some embedded $(k-1)$-sphere F in ∂Y by Lemma 1.3.4. Since $\nu(F \times \{0\}) \cong \nu(F) \times B^1$, we can assume that $\nu(F \times \{0\}) = \nu(F) \times B^1$, where $\nu(F \times \{0\})$ is a closed regular neighborhood of $F \times \{0\}$ in $\partial(Y \times B^1)$ and $\nu(F)$ is a closed regular neighborhood of F in ∂Y. We now can assume that $\Phi : (S^{k-1} \times B^{n-k-1}) \times B^1 \hookrightarrow \partial Y \times B^1 \subset \partial(Y \times B^1)$ is embedded in $\partial Y \times B^1$ such that $\Phi((S^{k-1} \times 0) \times \{0\}) = F \times \{0\}$ and $\Phi((S^{k-1} \times B^{n-k-1}) \times B^1) = \nu(F) \times B^1$. By Remark 1.3.7, a framing of $F \times \{0\} \subset \partial Y \times B^1$ is induced by a framing of $F \subset \partial Y$. Therefore, there exists an embedding $\psi : S^{k-1} \times B^{n-k-1} \hookrightarrow \partial Y$ such that

1. $\psi(S^{k-1} \times 0) = F$,
2. $\psi(S^{k-1} \times B^{n-k-1}) = \nu(F)$,
3. Φ is isotopic to Ψ,

where $\Psi : (S^{k-1} \times B^{n-k-1}) \times B^1 \hookrightarrow \partial Y \times B^1 \subset \partial(Y \times B^1)$ is defined by $\Psi((x, y), t) = (\psi(x, y), t)$. Thus, $(Y \times B^1) \cup_\Phi ((B^k \times B^{n-k}) \times B^1) \cong (Y \cup_\psi (B^k \times B^{n-k-1})) \times B^1$.

□

Theorem 1.3.11. *Let X be an n-dimensional k-handlebody. If $n \geq 2k+1$, then there exists an $(n-1)$-dimensional k-handlebody $Y \subset X$ such that $X \cong Y \times B^1$. More precisely, if $B^n = X_0 \subset X_1 \subset \cdots \subset X_k = X$ is a handle decomposition of X, then there exists a handle decomposition $B^{n-1} = Y_0 \subset Y_1 \subset \cdots \subset Y_k = Y$ of Y such that $X_i \cong Y_i \times B^1$ for every $0 \leq i \leq k$.*

Proof. Let $P(n, k)$ denote the statement that if $B^n = X_0 \subset X_1 \subset \cdots \subset X_k = X$ is a handle decomposition of X, then there exists a handle decomposition $B^{n-1} = Y_0 \subset Y_1 \subset \cdots \subset Y_k = Y$ of Y such that $X_i \cong Y_i \times B^1$ for every $0 \leq i \leq k$. We use induction to prove that $P(n, k)$ holds for all $k \geq 1$ and $n \geq 2k+1$.
Base Case. $P(n, 1)$ is true for every $n \geq 3$.
Let $B^{n-1} \times B^1 = B^n = X_0 \subset X_1 = X$ be a handle decomposition of an n-dimensional 1-handlebody X. Then $X = (B^{n-1} \times B^1) \cup_{\Psi_1} ((B^1 \times B^{n-2}) \times B^1) \cup_{\Psi_2} \cdots \cup_{\Psi_j} ((B^1 \times B^{n-2}) \times B^1)$ for some attaching maps Ψ_t of n-dimensional 1-handles, where $1 \leq t \leq j$. By Theorem 1.3.10, there exist attaching maps ϕ_t of $(n-1)$-dimensional 1-handles such that $X \cong (B^{n-1} \cup_{\phi_1} (B^1 \times B^{n-2}) \cup_{\phi_2} \cdots \cup_{\phi_j} (B^1 \times B^{n-2})) \times B^1$. Let $B^{n-1} = Y_0 \subset Y_1 = B^{n-1} \cup_{\phi_1} (B^1 \times B^{n-2}) \cup_{\phi_2} \cdots \cup_{\phi_j} (B^1 \times B^{n-2})$. Then $X_i \cong Y_i \times B^1$.
Inductive Step. If $P(n, k-1)$ is true when $k \geq 2$ and $n \geq 2k+1$, then $P(n, k)$ is true.
Assume that $P(n, k-1)$ is true when $k \geq 2$ and $n \geq 2k+1$. Let $B^n = X_0 \subset X_1 \subset \cdots \subset X_k = X$ is a handle decomposition of an n-dimensional k-handlebody X. By the assumption, there exists a handle decomposition $B^{n-1} = Y_0 \subset Y_1 \subset \cdots \subset Y_{k-1} = Y$ such that $X_i \cong Y_i \times B^1$ for every $0 \leq i \leq k-1$. Here, $X_k = (Y_{k-1} \times B^1) \cup_{\Psi_1} ((B^k \times B^{n-k-1}) \times B^1) \cup_{\Psi_2} \cdots \cup_{\Psi_j} ((B^k \times B^{n-k-1}) \times B^1)$ for some attaching maps Ψ_t of n-dimensional k-handles. By Theorem 1.3.10, there exists attaching maps ϕ_t of $(n-1)$-dimensional k-handles such that $X_k \cong (Y_{k-1} \cup_{\phi_1} (B^k \times B^{n-k-1}) \cup_{\phi_2} \cdots \cup_{\phi_j} (B^k \times B^{n-k-1})) \times B^1$. Let $Y_k = Y_{k-1} \cup_{\phi_1} (B^k \times B^{n-k-1}) \cup_{\phi_2} \cdots \cup_{\phi_j} (B^k \times B^{n-k-1})$. Therefore, $P(n, k)$ is true. □

For example, every 5-dimensional 2-handlebody is diffeomorphic to $Y \times B^1$ for some 4-dimensional 2-handlebody.

Remark 1.3.12. We note that there may exist Y and Y' such that $Y \times B^1 \cong X \cong Y' \times B^1$ but $Y \ncong Y'$. For example, let Y be Mazur manifold (contractible 4-manifold which is not homeomorphic to B^4) with a 0-handle, a 1-handle, and a 2-handle, and let Y' be a 4-manifold (diffeomorphic to B^4) with a 0-handle, and a cancelling $(1, 2)$-pair. Then $Y \times B^1 \cong B^5 \cong Y' \times B^1$ but $Y \ncong Y'$.

The boundary of an n-dimensional k-handlebody is a special type of closed $(n-1)$-manifold.

Corollary 1.3.13. Let X be an n-dimensional k-handlebody. If $n \geq 2k+1$, then ∂X is diffeomorphic to the double DY of Y for some $(n-1)$-dimensional k-handlebody Y.

Proof. By Theorem 1.3.11, there exists an $(n-1)$-dimensional handlebody Y such that $X \cong Y \times B^1$ so $\partial X = \partial(Y \times B^1) = Y \cup_{id} \overline{Y} = DY$, where id is the identity map of ∂Y. Therefore, ∂X is diffeomorphic to the double of Y. □

We can see that an n-dimensional k-handlebody is determined by a $2k$-dimensional k-handlebody when $n \geq 2k+1$.

Corollary 1.3.14. Let X be an n-dimensional k-handlebody. If $n \geq 2k+1$, then there exists a $2k$-dimensional k-handlebody Z such that $X \cong Z \times B^{n-2k}$.

Proof. Let $P(n)$ denote the statement above. We use induction to prove that $P(n)$ holds for all $n \geq 2k+1$.
Base Case. $P(2k+1)$ is true by Theorem 1.3.11.
Inductive Step. If $P(m)$ is true when $m \geq 2k+1$, then $P(m+1)$ is true.
Let X be an $(m+1)$-dimensional k-handlebody. By Theorem 1.3.11, there exists a m-dimensional k-handlebody W such that $X \cong W \times B^1$. By the assumption, there exists a $2k$-dimensional k-handlebody Y such that $W \cong Y \times B^{n-2k}$, so $X \cong W \times B^1 \cong Y \times B^{n-2k} \times B^1 \cong Y \times B^{n-2k+1}$. Therefore, $P(m+1)$ is true. □

Conjecture 1.3.15. Let X and X' be two $(2k+1)$-dimensional k-handlebodies. If ∂X and $\partial X'$ are diffeomorphic, then X and X' are diffeomorphic.

Remark 1.3.16. Let M be a closed orientable $(2k+1)$-manifold, and let $f : M \to [0, 2k+1]$ be a self-indexing Morse function with a single index 0 critical point and a single index $2k+1$ critical point. Then $M = W_1 \cup_\Sigma W_2$, where $W_1 = f^{-1}\left([0, \frac{2k+1}{2}]\right)$ and $W_2 = f^{-1}\left([\frac{2k+1}{2}, 2k+1]\right)$ are $(2k+1)$-dimensional k-handlebodies whose boundaries are diffeomorphic, denoted as $\Sigma = f^{-1}\left(\frac{2k+1}{2}\right)$. However, it is unknown whether W_1 and W_2 are always diffeomorphic. Note that Lawson[Law78] showed that every closed orientable 5-manifold M can be obtained by gluing two diffeomorphic 5-dimensional 2-handlebodies. However, this does not imply that W_1 and W_2 are always diffeomorphic in general. If Conjecture 1.3.15 is true, then Lawson's theorem clearly holds. W_1 and W_2 are always diffeomorphic when $k=1$, and we call such a decomposition $M = W_1 \cup W_2$ a ***Heegaard splitting*** of a closed 3-manifold M.

From now on, we focus on 5-dimensional 2-handlebodies. We rewrite Theorem 1.3.11 and Corollary 1.3.13 when $n = 5$ and $k = 2$.

Corollary 1.3.17. Let X be a 5-dimensional 2-handlebody. Then there exists a 4-dimensional 2-handlebody Y such that $X \cong Y \times B^1$, and consequently, ∂X is the double DY of Y. Furthermore, we can assume that the framings of 2-handles of Y are 0 or 1 due to $\pi_1(O(3)) \cong \mathbb{Z}_2$ and Remark 1.3.7.

Proposition 1.3.18. Let X be a 5-dimensional 2-handlebody with a 0-handle and 2-handles. Then X is diffeomorphic to $(\natural^a(S^2 \times B^3)) \natural (\natural^b(S^2 \tilde{\times} B^3))$ for some integers $a, b \geq 0$, and therefore, X is diffeomorphic to $\natural^k(S^2 \times B^3)$ or $\natural^k(S^2 \tilde{\times} B^3)$.

Proof. Let X be a 5-dimensional 2-handlebody with a 0-handle and k 2-handles. The boundary of 0-handle is diffeomorphic to S^4, and attaching spheres of k 2-handles are 1-links with k components. The 1-link is isotopic to the k-component unlink in S^4 because homotopy implies isotopy. Also, there are two possible framings for the attaching sphere of each 2-handle because $\pi_1(SO(3)) \cong \mathbb{Z}_2$. Therefore, X is diffeomorphic to $(\natural^a(S^2 \times B^3)) \natural (\natural^b(S^2 \tilde{\times} B^3))$, where a is the number of 2-handles with 0-framings, b is the number of 2-handles with 1-framings. If $b = 0$, $X \cong \natural^k(S^2 \times B^3)$. If $b > 0$, let C be the one of attaching spheres of 2-handles with 1-framing. We slide all 2-handles with 0-framings over C so that we have all framings of 2-handles are 1. Therefore, $X \cong \natural^k(S^2 \tilde{\times} B^3)$. □

1.4 Heegaard diagrams for 5-manifolds

Definition 1.4.1. A (5-dimensional) ***Heegaard diagram*** is a triple (Σ, α, β) such that

1. Σ is a closed, connected, orientable 4-manifold,
2. $\alpha = \alpha_1 \cup \cdots \cup \alpha_m \subset \Sigma$ is an m-component 2-link with trivial normal bundle for some $m \in \mathbb{Z}$,
3. $\beta = \beta_1 \cup \cdots \cup \beta_n \subset \Sigma$ is an n-component 2-link with trivial normal bundle for some $n \in \mathbb{Z}$.

Remark 1.4.2.

1. α and β may intersect each other transversely in Σ.
2. m and n do not have to be same.
3. We typically represent α in red and β in blue.

We now define some natural 5-manifolds constructed from a Heegaard diagram (Σ, α, β).

Definition 1.4.3. Let (Σ, α, β) be a Heegaard diagram. Define the following 5-manifolds:

1. Let

$$M_\alpha = (\Sigma \times [-1,0]) \cup (\coprod^m (B^3 \times B^2))$$

be the 5-manifold obtained from $\Sigma \times [-1,0]$ by attaching m 3-handles along $\alpha \times \{-1\} \subset \Sigma \times \{-1\}$. Here, $\partial M_\alpha = \Sigma \coprod \Sigma(\alpha)$, where $\partial_- M_\alpha = \Sigma \times \{0\} = \Sigma$ and $\partial_+ M_\alpha$ is diffeomorphic to the surgery $\Sigma(\alpha)$ of Σ along α.

2. If $\Sigma(\alpha) \cong \#^k(S^1 \times S^3)$, let

$$\widehat{M_\alpha} = M_\alpha \cup_g (\natural^k(S^1 \times B^4))$$

for some diffeomorphism $g : \#^k(S^1 \times S^3) \to \Sigma(\alpha)$. Here, $\partial \widehat{M_\alpha} = \Sigma \times \{0\} = \Sigma$.

3. Let

$$M_\beta = (\Sigma \times [0,1]) \cup (\coprod^n (B^3 \times B^2))$$

be the 5-manifold obtained from $\Sigma \times [0,1]$ by attaching n 3-handles along $\beta \times \{1\} \subset \Sigma \times \{1\}$. Here, $\partial M_\beta = \Sigma \coprod \Sigma(\beta)$, where $\partial_- M_\beta = \Sigma \times \{0\} = \Sigma$ and $\partial_+ M_\beta$ is diffeomorphic to the surgery $\Sigma(\beta)$ of Σ along β.

4. If $\Sigma(\beta) \cong \#^r(S^1 \times S^3)$, let

$$\widehat{M_\beta} = M_\beta \cup_h (\natural^r(S^1 \times B^4))$$

for some diffeomorphism $h : \#^r(S^1 \times S^3) \to \Sigma(\beta)$. Here, $\partial \widehat{M_\beta} = \Sigma \times \{0\} = \Sigma$.

5. Let

$$M_\alpha \cup_\Sigma M_\beta = M_\alpha \cup M_\beta,$$

where $M_\alpha \cap M_\beta = \Sigma \times \{0\} = \Sigma$, $\partial_-(M_\alpha \cup_\Sigma M_\beta) = \Sigma(\alpha)$, and $\partial_+(M_\alpha \cup_\Sigma M_\beta) = \Sigma(\beta)$.

6. If $\Sigma(\alpha) \cong \#^k(S^1 \times S^3)$, let

$$\widehat{M_\alpha} \cup_\Sigma M_\beta = \widehat{M_\alpha} \cup M_\beta,$$

where $\widehat{M_\alpha} \cap M_\beta = \Sigma \times \{0\} = \Sigma$ and $\partial(\widehat{M_\alpha} \cup_\Sigma M_\beta) = \Sigma(\beta)$.

7. If $\Sigma(\alpha) \cong \#^k(S^1 \times S^3)$ and $\Sigma(\beta) \cong \#^l(S^1 \times S^3)$, let

$$\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} = \widehat{M_\alpha} \cup \widehat{M_\beta},$$

where $\widehat{M_\alpha} \cap \widehat{M_\beta} = \Sigma \times \{0\} = \Sigma$.

Remark 1.4.4.

1. M_α is uniquely determined up to diffeomorphism by the isotopy class of α since the set of framings of a 2-handle is identified with $\pi_2(SO(2)) = 1$. Similarly for M_β.

2. $\widehat{M_\alpha}$ is uniquely determined up to diffeomorphism since any self-diffeomorphism of $\#^k(S^1 \times S^3)$ extends to a self-diffeomorphism of $\natural^k(S^1 \times B^4)$ [Ari+23; CH93]. We can consider $\widehat{M_\alpha}$ as a 5-manifold obtained from $\Sigma \times [-1, 0]$ by attaching $|\alpha|$ 3-handles, k 4-handles , and a 5-handle or as a 5-dimensional 2-handlebody i.e. the union of a 0-handle, k 1-handles, and m 2-handles. The Euler characteristic is $\chi(\widehat{M_\alpha}) = 1 - k + m$. Similarly for $\widehat{M_\beta}$.

3. $M_\alpha \cup_\Sigma M_\beta$ can be considered as a 5-dimensional cobordism with 2- and 3-handles.

4. $\widehat{M_\alpha} \cup_\Sigma M_\beta$ can be considered as a 5-dimensional 3-handlebody i.e. the union of a 0-handle, k 1-handles, m 2-handles, and n 3-handles. The Euler characteristic is $\chi(\widehat{M_\alpha} \cup M_\beta) = 1-k+m-n$.

5. $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta}$ can be considered as a closed 5-manifolds i.e. the union of the a 0-handle, k 1-handles, m 2-handles, n 3-handles, r 4-handles, and a 5-handle. The Euler characteristic is $\chi(\widehat{M_\alpha} \cup \widehat{M_\beta}) = 1 - k + m - n + r - 1$.

In the case where either M_α or M_β can be capped off, it implies that Σ is the double of some 4-dimensional 2-handlebody not just an arbitrary closed 4-manifold.

Corollary 1.4.5. Let (Σ, α, β) be a Heegaard diagram. If $\Sigma(\alpha)$ is diffeomorphic to $\#^k(S^1 \times S^3)$, then there exists a 4-dimensional 2-handlebody Y such that $\widehat{M_\alpha} \cong Y \times B^1$, and therefore, Σ is diffeomorphic to the double of Y.

Proof. $\widehat{M_\alpha}$ is obtained from $\Sigma \times [0, 1]$ by attaching 3-handles, 4-handles, and 5-handles, so it is a 5-dimensional 2-handlebody from the point of view of handle decomposition turned upside down. By Corollary 1.3.17, $\widehat{M_\alpha}$ is diffeomorphic to $Y \times B^1$ for some 4-dimensional 2-handlebody Y, and therefore, Σ is diffeomorphic to the double of Y. □

Every 5-dimensional cobordism with 2- and 3-handles or a 5-dimensional 3-handlebody or a closed 5-manifold admits a Heegaard diagram.

Theorem 1.4.6.

1. *Let X be a 5-dimensional cobordism with only 2- and 3-handles. Then X is diffeomorphic to $M_\alpha \cup_\Sigma M_\beta$ for some Heegaard diagram (Σ, α, β).*

2. *Let X be a 5-dimensional 3-handlebody. Then X is diffeomorphic to $\widehat{M_\alpha} \cup_\Sigma M_\beta$ for some Heegaard diagram (Σ, α, β).*

3. *Let X be a closed, orientable 5-manifold. Then X is diffeomorphic to $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta}$ for some Heegaard diagram (Σ, α, β).*

Proof.

1. Consider $f : X \to [\frac{3}{2}, \frac{7}{2}]$ as a self-indexing Morse function. Let A be the union of the ascending manifolds of the critical points of index 2, and let B be the union of the descending manifolds of the critical points of index 3. Then $(\Sigma, \alpha, \beta) = (f^{-1}(\frac{5}{2}), f^{-1}(\frac{5}{2}) \cap A, f^{-1}(\frac{5}{2}) \cap B)$ is a Heegaard diagram for $X = f^{-1}([\frac{3}{2}, \frac{7}{2}]) = f^{-1}([\frac{3}{2}, \frac{5}{2}]) \cup f^{-1}([\frac{5}{2}, \frac{7}{2}]) \cong M_\alpha \cup_\Sigma M_\beta$.

2. Consider $f : X \to [0, \frac{7}{2}]$ as a self-indexing Morse function with a single index 0 critical point. Let A be the union of the ascending manifolds of the critical points of index 2, and let B be the union of the descending manifolds of the critical points of index 3. Then $(\Sigma, \alpha, \beta) = (f^{-1}(\frac{5}{2}), f^{-1}(\frac{5}{2}) \cap A, f^{-1}(\frac{5}{2}) \cap B)$ is a Heegaard diagram for $X = f^{-1}([0, \frac{7}{2}]) = f^{-1}([0, \frac{5}{2}]) \cup f^{-1}([\frac{5}{2}, \frac{7}{2}]) \cong \widehat{M_\alpha} \cup_\Sigma M_\beta$.

3. Consider $f : X \to [0, 5]$ as a self-indexing Morse function with a single index 0 critical point and a single index 5 critical point. Let A be the union of the ascending manifolds of the critical points of index 2, and let B be the union of the descending manifolds of the critical points of index 3. Then $(\Sigma, \alpha, \beta) = (f^{-1}(\frac{5}{2}), f^{-1}(\frac{5}{2}) \cap A, f^{-1}(\frac{5}{2}) \cap B)$ is a Heegaard diagram for $X = f^{-1}([0, 5]) = f^{-1}([0, \frac{5}{2}]) \cup f^{-1}([\frac{5}{2}, 5]) \cong \widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta}$.

□

We introduce moves (isotopies, handle slides, stabilizations, and diffeomorphisms) defined on Heegaard diagrams. We first introduce isotopies.

Definition 1.4.7. Let (Σ, α, β) and $(\Sigma, \alpha', \beta')$ be Heegaard diagrams. We say that they are related by an ***isotopy*** if α is isotopic to α' and β is isotopic to β'.

We introduce handle slides.

Definition 1.4.8. Let (Σ, α, β) be a Heegaard diagram. Let $\alpha_i, \alpha_j \subset \alpha = \alpha_1 \cup \cdots \cup \alpha_m$ be two 2-knots and $\tilde{\alpha_j} \subset \partial\nu(\alpha_j)$ be the parallel push-off of α_j. A 3-dimensional submanifold $c \subset \Sigma$ is called a ***sliding cylinder connecting*** α_i **and** $\tilde{\alpha_j}$ if there exists an embedding $e : B^1 \times B^2 \hookrightarrow \Sigma$ such that

1. $c = e(B^1 \times B^2)$,
2. $c \cap \alpha_i = e(\{-1\} \times B^2)$,
3. $c \cap \tilde{\alpha_j} = e(\{1\} \times B^2)$,
4. $e((-1, 1) \times B^2) \cap (\alpha \cup \nu(\alpha_j)) = \emptyset$.

We define the cylinder sum

$$\alpha_i \#_c \tilde{\alpha_j} = (\alpha_1 \cup \alpha_2) \setminus int(e(\partial B^1 \times B^2)) \cup e(B^1 \times \partial B^2).$$

We call $\alpha_i \#_c \tilde{\alpha_j}$ a ***handle slide of*** α_i ***over*** α_j (along c). We say that (Σ, α) and (Σ, α') are related by a ***handle slide*** if $\alpha' = (\alpha \setminus \alpha_i) \cup (\alpha_i \#_c \alpha_j)$. We say that two Heegaard diagrams (Σ, α, β) and $(\Sigma, \alpha', \beta')$ are related by a ***handle slide*** if (Σ, α) and (Σ, α') are related by a handle slide and $\beta = \beta'$ or (Σ, β) and (Σ, β') are related by a handle slide and $\alpha = \alpha'$. See Figure 1.11.

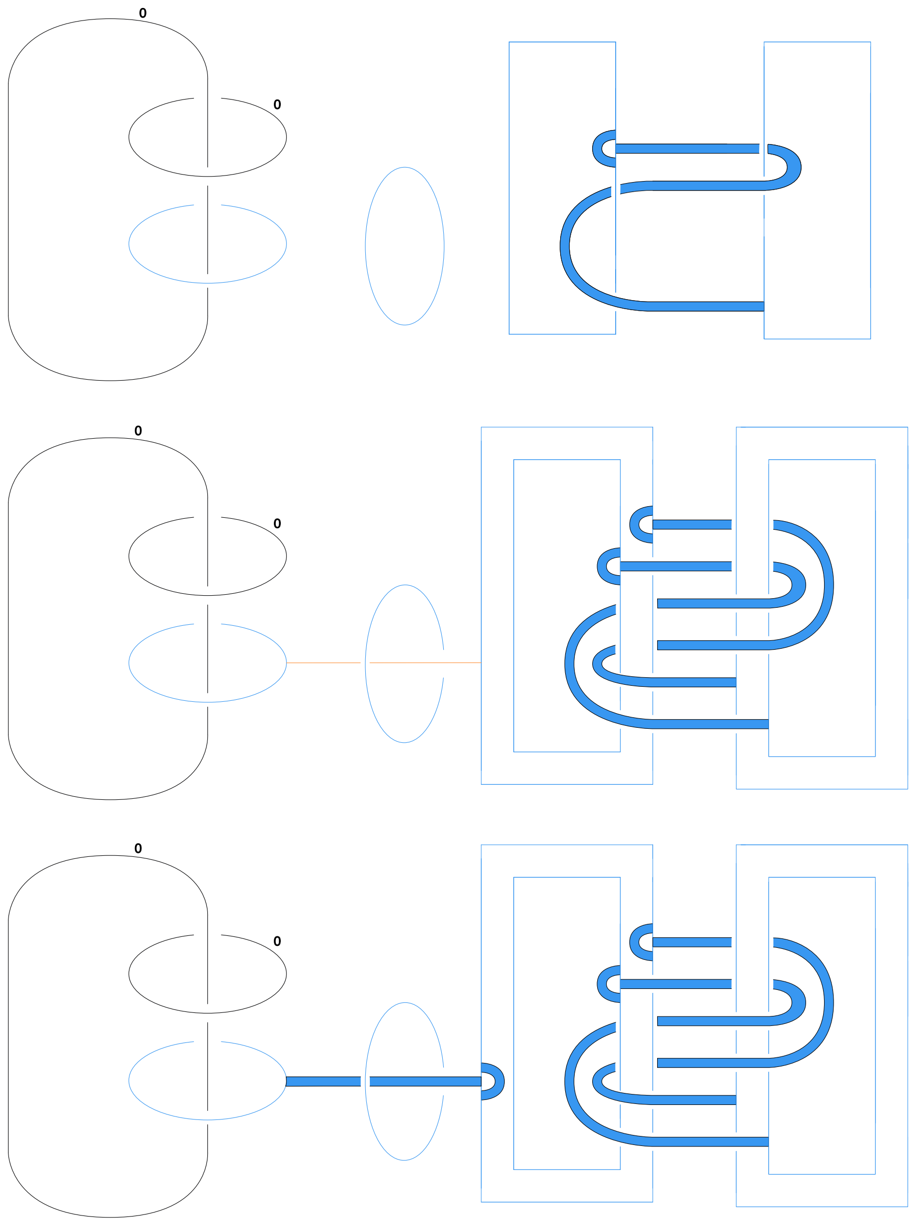

Figure 1.11: From **top** to **bottom**: A handle slide of $S^2 \times \{y_0\} \subset S^2 \times S^2$ over the spun trefoil along a sliding cylinder whose core is the orange arc. The orange arc is the core of a sliding cylinder (3-dimensional 1-handle) connecting $S^2 \times \{y_0\}$ and a parallel push-off of the spun trefoil. For the given orange arc, there are two possible sliding cylinders whose core is the orange arc; see Figure 1.12 for the banded unlink diagram for the surgery along the cylinder.

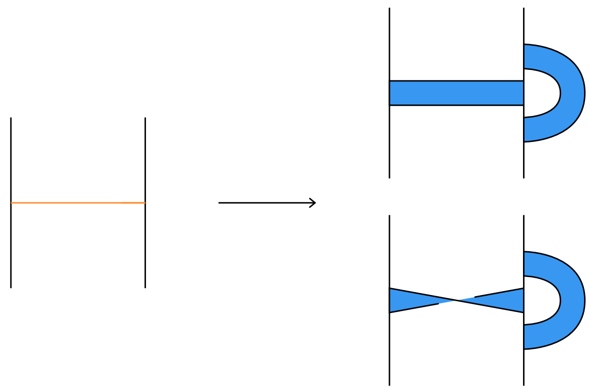

Figure 1.12: **Left**: An orange arc connecting two different components. **Right**: There are two possible banded unlink diagrams for the surgery along a sliding cylinder whose core is the orange arc. Surgery along $B^1 \times B^2$ is to remove $\{-1, 1\} \times B^2$ and glue $B^1 \times S^1$, where $B^1 \times \{0\}$ is identified with the orange arc. Here, $B^1 \times S^1 = B^1 \times (B^1_- \cup B^1_+) = (B^1 \times B^1_-) \cup (B^1 \times B^1_+)$, where $B^1 \times B^1_-$ and $B^1 \times B^1_+$ correspond to a long band and a rainbow band, respectively. If the long band is twisted, we can untwist the long band by sliding it over the rainbow band.

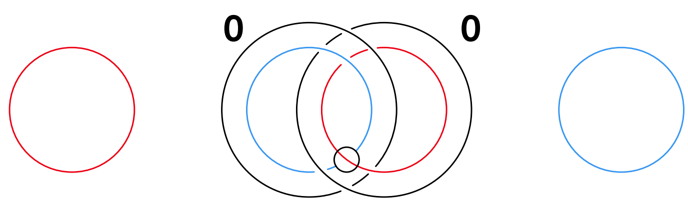

Figure 1.13: Three types of stabilizations. **Left**: A first stabilization. **Middle**: A second stabilization. **Right**: A third stabilization.

We introduce three types of stabilizations.

Definition 1.4.9. Let (Σ, α, β) be a Heegaard diagram.

1. A ***first stabilization*** of (Σ, α, β) is the Heegaard diagram $(\Sigma, \alpha', \beta) = (\Sigma, \alpha \cup U, \beta)$, where U is the unknotted 2-sphere in a 4-ball $B \subset \Sigma$ such that $B \cap \alpha = \emptyset$. We say that (Σ, α, β) and (Σ, α', β) are related by a ***first stabilization***. See the left of Figure 1.13.

2. A ***second stabilization*** of (Σ, α, β) is the Heegaard diagram $(\Sigma', \alpha', \beta') = (\Sigma \#(S^2 \times S^2), \alpha \cup (\{x_0\} \times S^2), \beta \cup (S^2 \times \{y_0\}))$ obtained by connected summing (Σ, α, β) with $(S^2 \times S^2, \{x_0\} \times S^2, S^2 \times \{y_0\})$, where $x_0, y_0 \in S^2$. We say that (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by a ***second stabilization***. See the middle of Figure 1.13.

3. A ***third stabilization*** of (Σ, α, β) is the Heegaard diagram $(\Sigma, \alpha, \beta') = (\Sigma, \alpha, \beta \cup U)$, where U is the unknotted 2-sphere in a 4-ball $B \subset \Sigma$ such that $B \cap \beta = \emptyset$. We say that (Σ, α, β) and (Σ, α, β') are related by a ***third stabilization***. See the right of Figure 1.13.

We note that for a first stabilization, we do not draw a 4-handle which is cancelled with the 3-handle attached along the unknotted 2-sphere U, i.e. $M_\alpha \cong M_{\alpha'} \cup$ 4-handle. More precisely, the 4-handle is attached along the obvious 3-sphere $\{x_0\} \times S^3 \subset \Sigma(\alpha)\#(S^1 \times S^3) \cong \Sigma(\alpha')$, where $x_0 \in S^1$. Similarly, for a third stabilization, we do not draw a 4-handle which is cancelled with the 3-handle attached along the unknotted 2-sphere U.

Remark 1.4.10. The definitions of isotopies, handle slides, and (first, second, and third) stabilizations defined on (5-dimensional) Heegaard diagrams correspond to the original definitions of isotopies, handle slides of handles, and cancelling $(1, 2)$-, $(2, 3)$-, and $(3, 4)$- pairs in dimension 5.

We introduce diffeomorphisms of Heegaard diagrams.

Definition 1.4.11. Let (Σ, α, β) and $(\Sigma', \alpha', \beta')$ be Heegaard diagrams. We say that they are related by a ***diffeomorphism*** if there exists a diffeomorphism $\phi : \Sigma \to \Sigma'$ sending α to α' and β to β'.

The following theorem tells us that studying 5-manifolds is equivalent to studying Heegaard diagrams.

Theorem 1.4.12.

1. *Let (Σ, α, β) and $(\Sigma', \alpha', \beta')$ be Heegaard diagrams. Then $M_\alpha \cup_\Sigma M_\beta \cong M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$ if and only if they are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.*

2. *Let (Σ, α, β) and $(\Sigma', \alpha', \beta')$ be Heegaard diagrams. Then $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} M_{\beta'}$ if and only if they are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.*

3. *Let (Σ, α, β) and $(\Sigma', \alpha', \beta')$ be Heegaard diagrams. Then $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} \widehat{M_{\beta'}}$ if and only if they are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.*

Proof.

1. $(\Rightarrow)$ Assume $M_\alpha \cup_\Sigma M_\beta \cong M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$. Let $\Phi : M_\alpha \cup_\Sigma M_\beta \to M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$ be a diffeomorphism. Then $(\Phi(\Sigma), \Phi(\alpha), \Phi(\beta))$ is a Heegaard diagram for $M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$, so $(\Phi(\Sigma), \Phi(\alpha), \Phi(\beta))$ and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, and (first, second, and third) stabilizations by Theorem 1.2.15 [Cer70]. Therefore, (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.
$(\Leftarrow)$ Assume two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, and (first, second, and third), stabilizations. Then $M_\alpha \cup_\Sigma M_\beta = M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$ by Theorem 1.2.15 [Cer70].

Assume two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by a diffeomorphism. Let $\phi : (\Sigma, \alpha, \beta) \to (\Sigma', \alpha', \beta')$ be a diffeomorphism such that $\phi(\alpha) = \alpha'$ and $\phi(\beta) = \beta'$. Then ϕ extends to a diffeomorphism $\Phi : \Sigma \times [-1, 1] \to \Sigma' \times [-1, 1]$ defined by $\Phi(x, t) = (\phi(x), t)$. Clearly, $\Phi(\alpha \times \{-1\}) = \alpha' \times \{-1\}$ and $\Phi(\beta \times \{1\}) = \beta' \times \{1\}$. We note that the isotopy class of the attaching map of a 5-dimensional 3-handle is determined by only its attaching sphere because $\pi_2(SO(2)) = 1$. Therefore, Φ extends to a diffeomorphism $\tilde{\Phi} : M_\alpha \cup_\Sigma M_\beta \to M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$ so $M_\alpha \cup_\Sigma M_\beta \cong M_{\alpha'} \cup_\Sigma M_{\beta'}$.

2. $(\Rightarrow)$ Assume $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} M_{\beta'}$. Let $\Phi : \widehat{M_\alpha} \cup_\Sigma M_\beta \to \widehat{M_{\alpha'}} \cup_{\Sigma'} M_{\beta'}$ be a diffeomorphism. Then $(\Phi(\Sigma), \Phi(\alpha), \Phi(\beta))$ is a Heegaard diagram for $\widehat{M_{\alpha'}} \cup_{\Sigma'} M_{\beta'}$ so $(\Phi(\Sigma), \Phi(\alpha), \Phi(\beta))$ and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, and (first, second, and third) stabilizations by Theorem 1.2.15 [Cer70]. Therefore, (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.
$(\Leftarrow)$ Assume two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, and (first, second, and third), stabilizations. Then $M_\alpha \cup_\Sigma M_\beta = M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$ by Theorem 1.2.15 [Cer70] . Since every self-diffeomorphism of $\natural^k(S^1 \times S^3)$ extends to a self-diffeomorphism of $\natural^k(S^1 \times B^4)$ [Ari+23; CH93], we can cap off $\Sigma(\alpha)$ and $\Sigma(\alpha')$ uniquely up to diffeomorphism so $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} M_{\beta'}$.
Assume two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by a diffeomorphism. Similarly to the proof above, there exists a diffeomorphism $\tilde{\Phi} : M_\alpha \cup_\Sigma M_\beta \to M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$. By [Ari+23; CH93], we can cap off $\Sigma(\alpha)$ and $\Sigma(\alpha')$ uniquely up to diffeomorphism so $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} M_{\beta'}$.

3. $(\Rightarrow)$ Assume $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} \widehat{M_{\beta'}}$. Let $\Phi : \widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \to \widehat{M_{\alpha'}} \cup_{\Sigma'} \widehat{M_{\beta'}}$ be a diffeomorphism. Then $(\Phi(\Sigma), \Phi(\alpha), \Phi(\beta))$ is a Heegaard diagram for $\widehat{M_{\alpha'}} \cup_{\Sigma'} \widehat{M_{\beta'}}$ so $(\Phi(\Sigma), \Phi(\alpha), \Phi(\beta))$ and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, and (first, second, and third) stabilizations by Theorem 1.2.15 [Cer70]. Therefore, (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, (first, second, and third) stabilizations, and diffeomorphisms.
$(\Leftarrow)$ Assume two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by isotopies, handle slides, and (first, second, and third), stabilizations. Then $M_\alpha \cup_\Sigma M_\beta = M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$ by Theorem 1.2.15 [Cer70] . By [Ari+23; CH93], we can cap off $\Sigma(\alpha), \Sigma(\beta), \Sigma(\alpha')$, and $\Sigma(\beta')$ uniquely up to diffeomorphism so $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} \widehat{M_{\beta'}}$.
Assume two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are related by a diffeomorphism. Similarly to the proof above, Similarly to the proof above, there exists a diffeomorphism $\tilde{\Phi} : M_\alpha \cup_\Sigma M_\beta \to M_{\alpha'} \cup_{\Sigma'} M_{\beta'}$. By [Ari+23; CH93], we can cap off $\Sigma(\alpha), \Sigma(\beta), \Sigma(\alpha')$, and $\Sigma(\beta')$ uniquely up to diffeomorphism so $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong \widehat{M_{\alpha'}} \cup_{\Sigma'} \widehat{M_{\beta'}}$.

□

Corollary 1.4.13. If two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ represent 5-manifolds that are diffeomorphic, then $\Sigma \#(\#^k(S^2 \times S^2))$ and $\Sigma' \#(\#^{k'}(S^2 \times S^2))$ are diffeomorphic for some $k, k' \geq 0$.

Proof. By Theorem 1.4.12, two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ representing diffeomorphic 5-manifolds are related by a sequence of second stabilizations. Then $\Sigma\#(\#^k(S^2\times S^2)) \cong \Sigma'\#(\#^{k'}(S^2\times S^2))$ for some $k, k' \geq 0$. □

Remark 1.4.14. Corollary 1.4.13 can be used to distinguish two 5-manifolds that are not diffeomorphic. In other words, if $\Sigma\#(\#^k(S^2 \times S^2)) \not\cong \Sigma'\#(\#^{k'}(S^2 \times S^2))$ for all $k, k' \geq 0$, then (Σ, α, β) and $(\Sigma', \alpha', \beta')$ represent non-diffeomorphic 5-manifolds. For example, since $\pi_1(\Sigma) \cong \pi_1(\Sigma\#(\#^k(S^2 \times S^2)))$ for all $k \geq 0$, if $\pi_1(\Sigma) \not\cong \pi_1(\Sigma')$, two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ represent non-diffeomorphic 5-manifolds. However, in dimension 3, any two Heegaard surfaces are diffeomorphic after some stabilizations (connected sum Heegaard surface with $S^1 \times S^1$) because every orientable surface is diffeomorphic to $\#^m(S^1 \times S^1)$ for some $m \geq 0$, so Heegaard surface cannot be used to distinguish 3-manifolds.

Theorem 1.4.15. *Let (Σ, α, β) be a Heegaard diagram for a 5-dimensional 3-handlebody or a closed 5-manifold X. Then $\pi_1(X) = \pi_1(\Sigma)$.*

Proof.

$$\begin{aligned}
\pi_1(X) &\cong \pi_1(M_\alpha) && (\pi_1 \text{ is determined by a 5-dimensional 2-handlebody.}) \\
&\cong \pi_1(Y \times B^1) && (\text{by Corollary 1.3.17.}) \\
&\cong \pi_1(Y) && (Y \times B^1 \text{ and } Y \text{ are homotopy equivalent.}) \\
&\cong \pi_1(DY) && (\text{by the construction of the double of a 4-dimensional 2-handlebody.}) \\
&\cong \pi_1(\Sigma) && (\text{by Corollary 1.4.5.})
\end{aligned}$$

□

Remark 1.4.16. We can read off the homology of a 5-manifold X from its Heegaard diagram (Σ, α, β) by (9) in Remark 1.2.8 because we can easily how α and β intersect each other in Σ.

We give a few examples of Heegaard diagrams for some 5-manifolds.

Example 1.4.17.

1. Let $(\Sigma, \alpha, \beta) = (\Sigma, \emptyset, \emptyset)$ be a Heegaard diagram. Then $M_\alpha \cup_\Sigma M_\beta \cong \Sigma \times [-1, 1]$.
2. Let $(\Sigma, \alpha, \beta) = (\Sigma, \alpha, \emptyset)$ be a Heegaard diagram. Then $M_\alpha \cup_\Sigma M_\beta \cong M_\alpha$.
3. Let $(\Sigma, \alpha, \beta) = (\Sigma, \emptyset, \beta)$ be a Heegaard diagram. Then $M_\alpha \cup_\Sigma M_\beta \cong M_\beta$.
4. Let $(\Sigma, \alpha, \beta) = (S^4, U, \emptyset)$ be a Heegaard diagram, where U is the unknotted 2-sphere in S^4. Then $M_\alpha \cup_\Sigma M_\beta \cong M_\alpha$ is diffeomorphic to the once-punctured $S^3 \times B^2$, where $\Sigma(\alpha) \cong S^1 \times S^3$. Let $\tilde{M}_\alpha = M_\alpha \cup (B^4 \times B^1)$ be the 5-manifold obtained from M_α by attaching a 4-handle along $\{x_0\} \times S^3 \subset S^1 \times S^3$. Then $\tilde{M}_\alpha$ is diffeomorphic to $S^4 \times [-1, 1]$, which corresponds to a first stabilization. Also, $\widehat{M}_\alpha \cup M_\beta \cong \widehat{M}_\alpha \cong B^5$ and $\widehat{M}_\alpha \cup \widehat{M}_\beta \cong S^5$. See the left of Figure 1.13.

5. Let $(\Sigma, \alpha, \beta) = (S^4, \emptyset, U)$ be a Heegaard diagram, where U is the unknotted 2-sphere in S^4. Then $M_\alpha \cup_\Sigma M_\beta \cong M_\beta$ is diffeomorphic to the once-punctured $S^3 \times B^2$, where $\Sigma(\beta) \cong S^1 \times S^3$. Let $\tilde{M}_\beta = M_\beta \cup (B^4 \times B^1)$ be the 5-manifold obtained from M_β by attaching a 4-handle along $\{x_0\} \times S^3 \subset S^1 \times S^3$. Then $\tilde{M}_\beta$ is diffeomorphic to $S^4 \times [-1, 1]$, which corresponds to a third stabilization. Also, $M_\alpha \cup \widehat{M_\beta} \cong \widehat{M_\beta} \cong B^5$ and $\widehat{M_\alpha} \cup \widehat{M_\beta} \cong S^5$. See the right of Figure 1.13.

6. Let $(\Sigma, \alpha, \beta) = (S^2 \times S^2, \{x_0\} \times S^2, \emptyset)$ be a Heegaard diagram. Then $M_\alpha \cup_\Sigma M_\beta \cong M_\alpha$ is diffeomorphic to the once-punctured $S^2 \times B^3$, where $\Sigma(\alpha) \cong S^4$ and $\Sigma(\beta) \cong S^2 \times S^2$. Also, $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong S^2 \times B^3$. See the left of Figure 1.8. (Switch the 1-framed unknot to a 0-framed unknot.)

7. Let $(\Sigma, \alpha, \beta) = (S^2 \tilde{\times} S^2, F, \emptyset)$ be a Heegaard diagram, where F is a fiber of $S^2 \tilde{\times} S^2$. Then $M_\alpha \cup_\Sigma M_\beta \cong M_\beta$ is diffeomorphic to the once-punctured $S^2 \tilde{\times} B^3$, where $\Sigma(\alpha) \cong S^4$ and $\Sigma(\beta) \cong S^2 \tilde{\times} S^2$. Also, $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong S^2 \tilde{\times} B^3$. See the left of Figure 1.8.

8. Let $(\Sigma, \alpha, \beta) = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \mathbb{C}P^1 \# \overline{\mathbb{C}P^1}, \emptyset)$ be a Heegaard diagram. In fact, there is a natural diffeomorphism between $(S^2 \tilde{\times} S^2, F)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \mathbb{C}P^1 \# \overline{\mathbb{C}P^1})$; see Figure 1.8. Therefore, $M_\alpha \cup_\Sigma M_\beta \cong M_\beta$ is diffeomorphic to the once-punctured $S^2 \tilde{\times} B^3$, where $\Sigma(\alpha) \cong S^4$ and $\Sigma(\beta) \cong S^2 \tilde{\times} S^2$. Also, $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong S^2 \tilde{\times} B^3$. See the right of Figure 1.8.

9. Let $(\Sigma, \alpha, \beta) = (S^2 \times S^2, \{x_0\} \times S^2, S^2 \times \{y_0\})$ be a Heegaard diagram. Then $M_\alpha \cup_\Sigma M_\beta$ is diffeomorphic to $S^4 \times [-1, 1]$, which corresponds to a second stabilization. Also, $\widehat{M_\alpha} \cup_\Sigma M_\beta \cong B^5$ and $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong S^5$. See the middle of Figure 1.13.

10. Let $(\Sigma, \alpha, \beta) = (S^2 \times S^2, \{x_0\} \times S^2, \beta)$ be the Heegaard diagram in the left of Figure 1.5, where β is a 2-knot in $S^2 \times S^2$ which is homotopic but not isotopic to $S^2 \times \{y_0\}$. The geometric intersection number between α and β is $|\alpha \cap \beta| = 3$ and the algebraic intersection number is $\alpha \cdot \beta = 1$. Here, $\widehat{M_\alpha} \cup_\Sigma M_\beta$ is contractible but not homeomorphic to B^5 by showing that $(\widehat{M_\alpha} \cup_\Sigma M_\beta) \times B^1 \cong B^6$ and $\partial(\widehat{M_\alpha} \cup_\Sigma M_\beta) = \Sigma(\beta)$ is non-simply connected. Then $M_\alpha \cup_\Sigma M_\beta$ is a 5-dimensional cobordism from the standard 4-sphere $\Sigma(\alpha) = S^4$ to a non-simply connected homology 4-sphere $\Sigma(\beta)$ with a single 2-handle and a single 3-handle, which are algebraically but not geometrically cancelled. We can easily read off the Kirby diagram for $\Sigma(\alpha)$ in the middle of Figure 1.5 and the Kirby diagram for $\Sigma(\beta)$ in the right of Figure 1.5 by switching the unlink to the dotted unlink and the bands to 0-framed 2-handles; see Proposition 1.2.51. The middle of Figure 1.5 represents S^4 after removing a cancelling $(1, 2)$-pair and a cancelling $(2, 3)$-pair. In the right of Figure 1.5, we can read off the fundamental group $\pi_1(\Sigma(\beta))$ directly (dotted 1-handles correspond to the generators and 2-handles correspond to the relations) and the author showed that there exists an epimorphism from $\pi_1(\Sigma(\beta))$ to the alternating group A_5 of order 5 in [Kim23]. We generalize this technique so that we construct contractible high-dimensional manifold which is not homeomorphic to the standard ball in chapter 2

11. Let $(\Sigma, \alpha, \beta) = (X \# S^2 \tilde{\times} S^2, F, K \# F)$ be a Heegaard diagram, where X is a closed 4-manifold, K is a 2-knot in X with trivial normal bundle, and F is a fiber of $S^2 \tilde{\times} S^2$. Then $M_\alpha \cup_\Sigma M_\beta$ is a 5-dimensional cobordism from X to the Gluck twist X_K of X along K with a single 2-handle and a single 3-handle, where $\Sigma(\alpha) \cong X$ and $\Sigma(\beta) \cong X_K$; see section 1.5 for more details.

Remark 1.4.18. Let (Σ, α, β) be a Heegaard diagram. By Theorem 1.2.46, there exists a singular banded unlink diagram $(\mathcal{K}, L, B) = (\mathcal{K}, L_1 \cup L_2, B_1 \cup B_2)$ such that

1. $(\mathcal{K}, L, B)$ is a singular banded unlink diagram for $(\Sigma, \alpha \cup \beta)$,
2. $(\mathcal{K}, L_1, B_1)$ is a banded unlink diagram for (Σ, α),
3. $(\mathcal{K}, L_2, B_2)$ is a banded unlink diagram for (Σ, β).

We may simply write $(\Sigma, \alpha, \beta) = (\mathcal{K}, L_1 \cup B_1, L_2 \cup B_2)$.

Proposition 1.4.19 (How to draw a Heegaard diagram for a 5-dimensional 2-handlebody)**.** Let X be a 5-dimensional 2-handlebody. Then there exists a 4-dimensional 2-handlebody Y such that $X \cong Y \times B^1$ by Corollary 1.3.17. Let $\mathcal{K}$ be a Kirby diagram for Y and $\mathcal{K}'$ be a Kirby diagram for the double DY of Y. By Proposition 1.2.24, $\mathcal{K}'$ is obtained from $\mathcal{K}$ by adding 0-framed meridians to each 2-handles in $\mathcal{K}$, 3-handles with the number of 1-handles in $\mathcal{K}$, and a single 4-handle. Add red circles α, each of which is parallel to each 0-framed meridian in $\mathcal{K}'$. Here, each red circle bounds a properly embedded trivial disk in the 0-handle and bounds the core disk of a 2-handle so it corresponds to a banded unlink diagram for the belt sphere of each 5-dimensional 2-handle of X. Then $(\Sigma, \alpha, \beta) = (\mathcal{K}', \alpha, \emptyset)$ is a Heegaard diagram for X that is $\widehat{M_\alpha} \cup M_\beta \cong \widehat{M_\alpha} \cong X$.

Example 1.4.20. Let M be a Mazur manifold. Then $M \times B^1$ is diffeomorphic to B^5.

Proof. See the right of Figure 1.1. □

Theorem 1.4.21. *Let Y be a 4-dimensional 2-handlebody and DY be the double of Y. Then there exists a 5-dimensional cobordism from $(\#^m(S^2 \times S^2))\#(\#^n(S^2 \tilde{\times} S^2))$ to DY with only 3-handles for some $m, n \geq 0$.*

Proof. Let $\mathcal{K} = L_1 \cup L_2$ be a Kirby diagram for Y and $\mathcal{K}' = L_1 \cup L_2 \cup J$ be the natural Kirby diagram for DY explained in Proposition 1.2.24, where J is the union of the 0-framed meridians of L_2; see the bottom left of Figure 1.9. By the construction of DY, $\widehat{M_{\mathcal{K}'}} = M_{\mathcal{K}'} \cup \natural^{|L_1|}(S^1 \times B^3)$. Switch the dotted link L_1 in $\mathcal{K}'$ to blue circles β; see the bottom left of Figure 1.9. Then $\mathcal{K}'' = \mathcal{K}' \setminus L_1$ is the union of some Hopf links, each of which has a 0-framed unknotted component so $\mathcal{K}''$ is a Kirby diagram for $(\#^m(S^2 \times S^2))\#(\#^n(S^2 \tilde{\times} S^2))$ for some $m, n \geq 0$. Here, $\widehat{M_{\mathcal{K}''}} = M_{\mathcal{K}''} \cup B^4$. Then $(\mathcal{K}'', \emptyset, \beta)$ is a Heegaard diagram for a 5-dimensional cobordism from $(\#^m(S^2 \times S^2))\#(\#^n(S^2 \tilde{\times} S^2))$ to DY because the Kirby diagram for the surgery of $(\#^m(S^2 \times S^2))\#(\#^n(S^2 \tilde{\times} S^2))$ along β is $\mathcal{K}'$ by Proposition 1.2.51. □

We now provide several examples of Heegaard diagrams for closed 5-manifold. According to [Law78], every closed 5-manifold M can be constructed as a twisted double $M = W \cup_f W$, where W is a 5-dimensional 2-handlebody and f is a self-diffeomorphism of ∂W. We first explain how to draw Heegaard diagrams for the double of a 5-dimensional 2-handlebody, specifically when $f = id$. Then we will discuss about Barden's classification of simply connected 5-manifolds and proceed to draw a Heegaard diagram for the Wu manifold which is a twisted double of 5-dimensional 2-handlebody, i.e. $f \neq id$. Here, the Wu manifold is the generator of 5-dimensional oriented cobordism group $\Omega_5^{SO} \cong \mathbb{Z}_2$.

Proposition 1.4.22 (How to draw a Heegaard diagram for the double of a 5-dimensional 2-handlebody**).** Let X be a 5-dimensional 2-handlebody X. By Proposition 1.4.19, $(\Sigma, \alpha, \beta) = (\mathcal{K}', \alpha, \emptyset)$ is a Heegaard diagram for X for some 4-dimensional 2-handlebody Y. Let M be the double of X. Then add blue circles β each of which is parallel to red circles α, i.e. we attach 5-dimensional 3-handles along the belt spheres of 5-dimensional 2-handles for X. Then $(\Sigma, \alpha, \beta) = (\mathcal{K}', \alpha, \alpha)$ is a Heegaard diagram for $M = DX$, i.e. $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong M$. See Figure 1.14.

Barden [Bar65] classified simply connected, closed, orientable, smooth 5-manifolds. The key idea is that every simply connected 5-manifold can be constructed by gluing two copies of a 5-dimensional 2-handlebody without 1-handles (a boundary connected sum of some copies of B^3-bundles over S^2), where the gluing map defined on the boundary is realized by an automorphism of the second homology group of the boundary.

There are two possible B^3-bundles over S^2; the trivial one is $S^2 \times B^3$ and the non-trivial one is $S^2 \tilde{\times} B^3$ since $\pi_1(SO(3)) \cong \mathbb{Z}_2$. We note that $\partial(S^2 \times B^3) = S^2 \times S^2$ and $\partial(S^2 \tilde{\times} B^3) = S^2 \tilde{\times} S^2 \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$; see Figure 1.8. Let $\{a, b\} = \{S^2 \times \{y_0\}, \{x_0\} \times S^2\}$ be the canonical generators of $H_2(S^2 \times S^2)$ and $\{c, d\} = \{\mathbb{C}P^1, \overline{\mathbb{C}P^1}\}$ be the generators of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. We also note that $\partial((S^2 \times B^3) \natural (S^2 \times B^3)) = (S^2 \times S^2) \# (S^2 \times S^2)$ and $\partial((S^2 \tilde{\times} B^3) \natural (S^2 \tilde{\times} B^3)) \cong (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \# (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$. Similarly, let $\{a_1, b_1, a_2, b_2\}$ be the canonical generators of $H_2((S^2 \times S^2) \# (S^2 \times S^2))$ and $\{c_1, d_1, c_2, d_2\}$ be the canonical generators of $H_2((\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \# (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}))$.

Consider the matrices

$$A(k) = \begin{pmatrix} 1 & 0 & 0 & -k \\ 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B(n) = \begin{pmatrix} 1 & n & -n & 0 \\ n & 1 & 0 & n \\ n & 0 & 1 & n \\ 0 & -n & n & 1 \end{pmatrix}.$$

By work of Wall [Wal64], there exist three diffeomorphisms

1. $f_k : (S^2 \times S^2) \# (S^2 \times S^2) \to (S^2 \times S^2) \# (S^2 \times S^2)$ such that the induced map $(f_k)_*$ on H_2 has a matrix representation $A(k)$,
2. $g_j : (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \# (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \to (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \# (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$ such that the induced map $(g_j)_*$ has a matrix representation $B(2^{j-1})$,

3. $g_{-1} : \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ such that the induced map $(g_j)_*$ on H_2 has a matrix representation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 1.4.23 ([Bar65]). We define some 5-manifolds.

1. $M_\infty = (S^2 \times B^3) \cup_{id} (S^2 \times B^3) = S^2 \times S^3$
2. $M_k = ((S^2 \times B^3) \natural (S^2 \times B^3)) \cup_{f_k} ((S^2 \times B^3) \natural (S^2 \times B^3))$
3. $X_{-1} = (S^2 \tilde{\times} B^3) \cup_{g_{-1}} (S^2 \tilde{\times} B^3) = SU(3)/SO(3)$
4. $X_0 = S^5$
5. $X_\infty = (S^2 \tilde{\times} B^3) \cup_{id} (S^2 \tilde{\times} B^3) = S^2 \tilde{\times} S^3$
6. $X_j = ((S^2 \tilde{\times} B^3) \natural (S^2 \tilde{\times} B^3)) \cup_{g_j} ((S^2 \tilde{\times} B^3) \natural (S^2 \tilde{\times} B^3))$

Theorem 1.4.24 ([Bar65]). *Every simply connected, closed, orientable, smooth 5-manifold is diffeomorphic to*

$$X_j \text{ or } X_j \# M_{k_1} \# \cdots \# M_{k_s},$$

where $-1 \leq j \leq \infty, 1 < k_1 \leq k_2 \leq \cdots \leq k_s$, and k_i divides k_{i+1} or $k_{i+1} = \infty$.

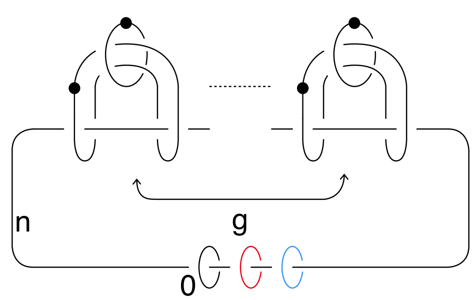

Figure 1.14: A S^3-bundle over a orientable genus g surface F_g. If we ignore the 0-framed meridian, the red meridian, and the blue meridian, then this is a Kirby diagram for a B^2-bundle over F_g with Euler number n. If we ignore the blue meridian, this is a Heegaard diagram for B^3-bundle over F_g by Proposition 1.4.19. This is a Heegaard diagram for a S^3-bundle over F_g by Proposition 1.4.22.

Example 1.4.25. Heegaard diagrams for closed 5-manifolds.

1. Three diagrams in Figure 1.13 are Heegaard diagrams for S^5.
2. The top right of Figure 1.10 is a Heegaard diagram for $S^1 \times S^4$.
3. Figure 1.14 is a Heegaard diagram for the trivial S^3-bundle over a genus g orientable surface F_g when n is even.

4. Figure 1.14 is a Heegaard diagram for the non-trivial S^3-bundle over a genus g orientable surface F_g when n is odd.

5. Figure 1.15 is a Heegaard diagram for Wu manifold (X_{-1} in Definition 1.4.23). The blue curve in Figure 1.15 is the image of the red curve in the right of Figure 1.8 under the map g_{-1}. Here we can calculate $H_2(X_{-1}) \cong \mathbb{Z}_2$ from $\alpha \cdot \beta = 2$.

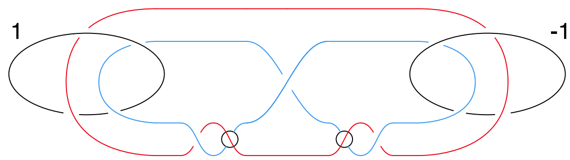

Figure 1.15: Wu manifold.

Example 1.4.26. $(S^2 \times S^3)\#(S^2 \tilde{\times} S^3) \cong (S^2 \tilde{\times} S^3)\#(S^2 \tilde{\times} S^3)$.

Proof. The bottom of Figure 1.16 (a Heegaard diagram for $(S^2 \tilde{\times} S^3)\#(S^2 \tilde{\times} S^3)$) is obtained from the top of the Figure 1.16 (a Heegaard diagram for $(S^2 \times S^3)\#(S^2 \tilde{\times} S^3)$) by handle slides along the orange guiding arcs. □

1.5 Gluck twists and Heegaard diagrams

Definition 1.5.1 ([Glu62])**.** Let X be a closed, connected, orientable 4-manifold. Let $K \subset X$ be a 2-knot in X with trivial normal bundle, i.e. $\nu(K) \cong S^2 \times B^2$. Let

$$\tau : S^2 \times S^1 = \partial(S^2 \times B^2) \to \partial(X \setminus int(\nu(K))) \cong S^2 \times S^1$$

be a diffeomorphism defined by $(x, \theta) \mapsto (rot_\theta(x), \theta)$, where rot_θ is a diffeomorphism that rotates $S^2 \subset \mathbb{R}^3$ by an angle of θ about the z-axis. The ***Gluck twist*** of X along K is the smooth 4-manifold

$$X_K = (X \setminus int(\nu(K))) \cup_\tau (S^2 \times B^2).$$

Remark 1.5.2.

1. If K is unknotted, then X_K is diffeomorphic to X.

2. If K is null-homotopic in X, then X and X_K are homotopy equivalent by [Glu62].

3. If X is a simply connected 4-manifold and K is null-homotopic in X, then X_K is homeomorphic to X by Freedman [Fre82].

4. Gluck twists of S^4 along non-trivial 2-knots may be potential counterexamples to the smooth 4-dimensional Poincaré conjecture. There are some families of 2-knots in S^4 so that Gluck twists of S^4 along them are known to be diffeomorphic to the standard S^4 [Glu62; Gor76; Mel77; Pao78; Lit79; NS12; NS22; GNS23].

5. If K is not null-homotopic, the diffeomorphism type may change. For example, the Gluck twist of $S^2 \times S^2$ along $\{x_0\} \times S^2$ is diffeomorphic to $S^2 \tilde{\times} S^2$.

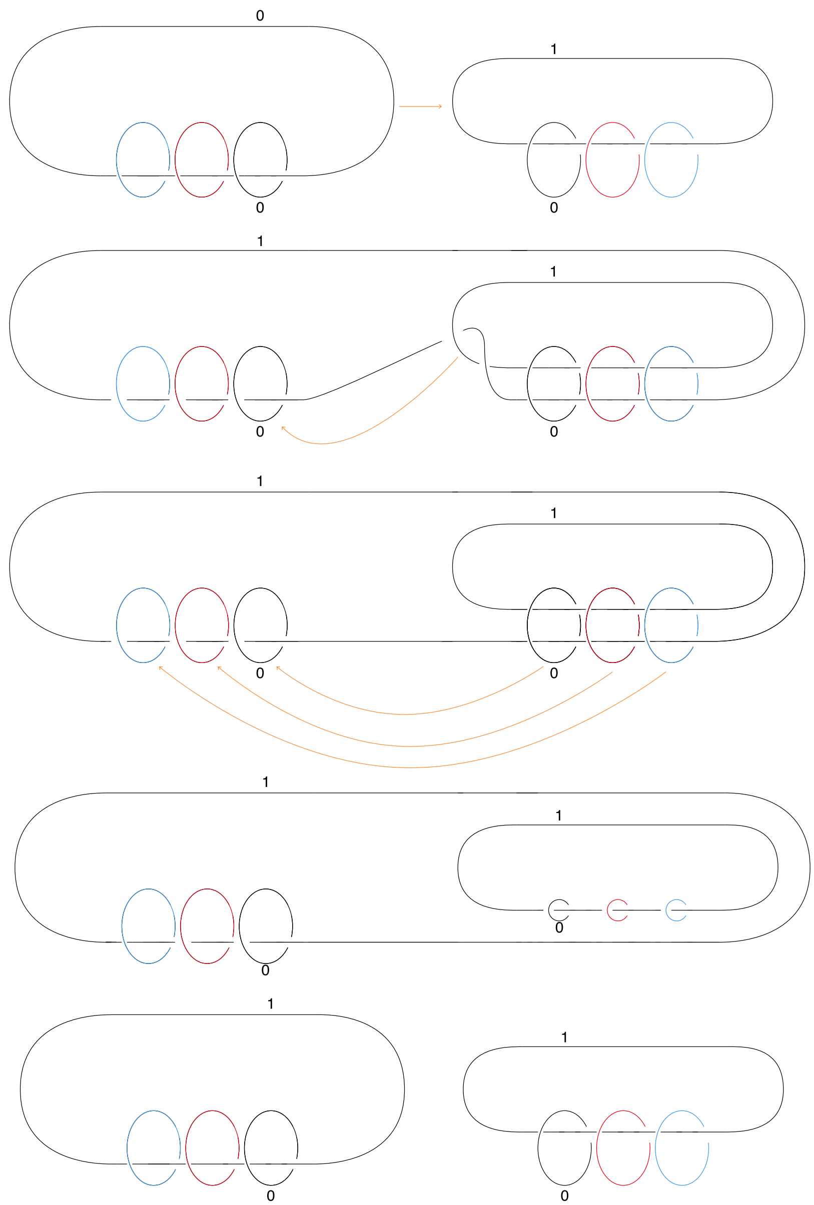

Figure 1.16: $(S^2 \times S^3)\#(S^2 \tilde{\times} S^3) \cong (S^2 \tilde{\times} S^3)\#(S^2 \tilde{\times} S^3)$.

Definition 1.5.3. Let X be a closed, connected, orientable 4-manifold. Let $K \subset X$ be a 2-knot in X with trivial normal bundle, i.e. $\nu(K) \cong S^2 \times B^2$ (K does not have to be null-homotopic). Let m_K be a meridian of K. We define

$$W_{X,K} = (X \times [0,1]) \cup_{m_k \times \{1\}} (B^2 \times B^3) \cup_{K \times \{1\}} (B^3 \times B^2)$$

to be the 5-manifold obtained from $X \times [0,1]$ by attaching a 2-handle along $m_K \times \{1\}$ with non-trivial framing and a 3-handle along $K \times \{1\}$.

Remark 1.5.4. There are two possible framings for a 5-dimensional 2-handle since $\pi_1(SO(3)) \cong \mathbb{Z}_2$, and there is a unique framing for a 5-dimensional 3-handle since $\pi_2(SO(2)) = 1$.

Theorem 1.5.5. *$W_{X,K}$ is a cobordism from X to X_K, i.e. $\partial_-(W_{X,K}) \cong X$ and $\partial_+(W_{X,K}) \cong X_K$.*

Proof. There exists an embedding $\phi : S^2 \times B^2 \hookrightarrow X$ such that $\phi(S^2 \times \{(0,0)\}) = K$ and $\phi(S^2 \times B^2) = \nu(K)$ because K has trivial normal bundle. We note that ϕ is unique up to isotopy because K has a unique framing. Then $X_K = (X \setminus int(\phi(S^2 \times B^2))) \cup_\tau (S^2 \times B^2)$. Since $(B^3 \times S^1) \setminus (B^3 \times S^1) = \emptyset$, $X_K = ((X \setminus int(\phi(S^2 \times B^2))) \cup_\phi (B^3 \times S^1)) \setminus (B^3 \times S^1) \cup_\tau (S^2 \times B^2)$. Here, we can consider $(X \setminus int(\phi(S^2 \times B^2))) \cup_\phi (B^3 \times S^1)$ as the surgery of X along K, which corresponds to attaching a 3-handle along K. Also, we can consider X_K as the surgery of $(X \setminus int(\phi(S^2 \times B^2))) \cup_\phi (B^3 \times S^1)$ along $\phi(\{pt\} \times S^1)$, which corresponds to attaching a 2-handle along the meridian m_K of K with non-trivial framing. Therefore, $W_{X,K}$ is a cobordism from X to X_K such that the bottom boundary $\partial_-(W_{X,K}) = X \times \{0\} \cong X$ and the top boundary $\partial_+(W_{X,K}) \cong X_K$. □

Note that $W_{X,K}$ is not an h-cobordism but it is simple in terms of number of handles, with only one 2-handle and one 3-handle.

Theorem 1.5.6. *Let $(\Sigma, \alpha, \beta) = (X\#(S^2 \tilde{\times} S^2), F, K\#F)$ be a Heegaard diagram, where F is a fiber of $S^2 \tilde{\times} S^2$ and $(X\#S^2 \tilde{\times} S^2, K\#F) = (X,K)\#(S^2 \tilde{\times} S^2, F)$ is the connected sum of pairs. Then $M_\alpha \cup_\Sigma M_\beta$ is diffeomorphic to $W_{X,K}$, and therefore, $\Sigma(\alpha) \cong \partial_-(W_{X,K}) \cong X$ and $\Sigma(\beta) \cong \partial_+(W_{X,K}) \cong X_K$.*

Proof. Let $W_{X,K} = (X \times [0,1]) \cup_{m_k \times \{1\}} (B^2 \times B^3) \cup_{K \times \{1\}} (B^3 \times B^2)$ in Definition 1.5.3. It suffices to show that $\partial_+(X \times [0,1] \cup_{m_K \times \{1\}} (B^2 \times B^3))$ is diffeomorphic to $X\#(S^2 \tilde{\times} S^2)$, the belt sphere of the 2-handle is F in $S^2 \tilde{\times} S^2 \subset X\#(S^2 \tilde{\times} S^2)$, and the attaching sphere of the 3-handle is $K\#F$ in $X\#(S^2 \tilde{\times} S^2)$. By construction of $W_{X,K}$, $\partial_+(X \times [0,1] \cup_{m_K \times \{1\}} (B^2 \times B^3))$ is diffeomorphic to the surgery of $X \times \{1\}$ along the meridian $m_K \times \{1\}$ with non-trivial framing. Since $m_K \times \{1\}$ is null-homologous, the surgery is diffeomorphic to $X\#(S^2 \tilde{\times} S^2)$ so the belt sphere of the 2-handle is a fiber F of $S^2 \tilde{\times} S^2 \subset X\#(S^2 \tilde{\times} S^2)$. Clearly the attaching sphere $K \times \{1\}$ of the 3-handle is embedded in $\partial_+(X \times [0,1] \cup_{m_K \times \{1\}} (B^2 \times B^3)) \cong X\#(S^2 \tilde{\times} S^2)$ and is isotopic to $K\#F$ in $X\#(S^2 \tilde{\times} S^2)$. □

Example 1.5.7. The left of Figure 1.17 is a Heegaard diagram for a cobordism from S^4 to the Gluck twist S^4_K of S^4 along the spun trefoil K. By [Glu62], the Gluck twist of S^4 along a spun knot is diffeomorphic to S^4. We can directly read off the boundary of this cobordism; see the middle and right of Figure 1.17.

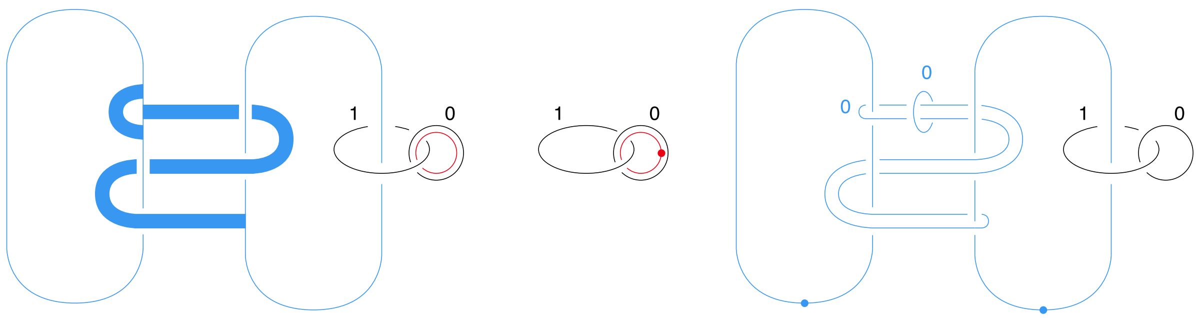

Figure 1.17: **Left**: A Heegaard diagram (Σ, α, β) for a cobordism $W_{X,K}$ from X to X_K, where $X = S^4$ and K is the spun trefoil. **Middle**: A Kirby diagram for X. **Right**: A Kirby diagram for X_K.

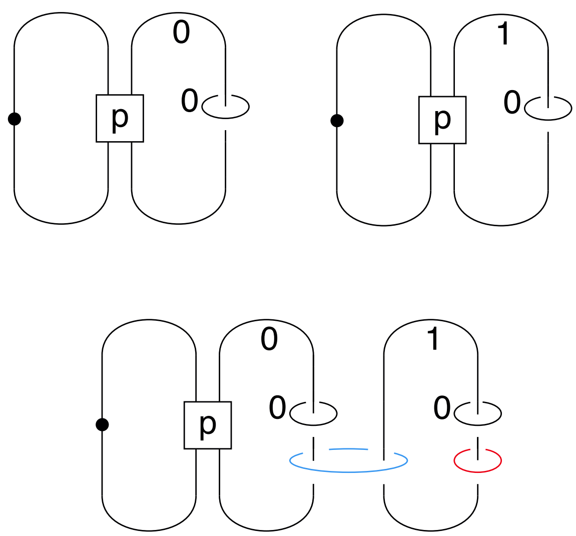

Figure 1.18: **Top left**: Surgery $M(\gamma_p)$ of $S^1 \times S^3$ along a circle γ_p representing $p \in \mathbb{Z} \cong \pi_1(S^1 \times S^3)$ with trivial framing. **Top right**: Surgery $M(\gamma_p)'$ of $S^1 \times S^3$ along a circle γ_p representing $p \in \mathbb{Z} \cong \pi_1(S^1 \times S^3)$ with non-trivial framing. **Bottom**: A Heegaard diagram for a cobordism $W_{X,K}$ from $X = M(\gamma_p)$ to $X_K = M(\gamma_p)'$, where $M(\gamma_p)'$ is the Gluck twist of $M(\gamma_p)$ along a meridian sphere S of γ_p. Here, the blue circle and the red circle represent $S \# F$ and F, respectively, where F is a fiber of $S^2 \tilde{\times} S^2$. Switching the red circle with a dotted circle produces the diagram in the top left, and similarly, switching the blue circle with a dotted circle yields the diagram in the top right.

Remark 1.5.8. Let $\gamma \subset X$ be a simple closed curve in X. Let $X(\gamma)$ be the surgery of X along γ with trivial framing. Let $X(\gamma)'$ be the surgery of X along γ with non-trivial framing. Then $X(\gamma)'$ is the Gluck twist of X along a meridian sphere S_γ of γ, where $S = \{x_0\} \times S^2 \subset S^1 \times S^2 = \partial(\gamma \times B^3) = \nu(\gamma)$. If $X = S^4$

and γ is any simple closed curve in X, then $X(\gamma)' = S^2 \tilde{\times} S^2$ is the Gluck twist of $X(\gamma) = S^2 \times S^2$ along a fiber $\{x_0\} \times S^2 \subset S^2 \times S^2$. See Figure 1.18 for another example.

Theorem 1.5.9. *Let $K \subset S^4$ be a 2-knot, and S^4_K be the Gluck twist of S^4 along K. Let $(\Sigma, \alpha, \beta) = (S^2 \tilde{\times} S^2, F, K\#F)$ be a Heegaard diagram, where F is a fiber of $S^2 \tilde{\times} S^2$. The following are equivalent:*

1. *S^4_K is diffeomorphic to S^4.*
2. *$W_{S^4,K} \cong M_\alpha \cup_\Sigma M_\beta$ is diffeomorphic to a twice-punctured $S^2 \tilde{\times} S^3$.*
3. *$(S^2 \tilde{\times} S^2, F, K\#F)$ and $(S^2 \tilde{\times} S^2, F, F)$ are related by isotopies, handle slides, stabilizations, and diffeomorphisms.*
4. *$(S^2 \tilde{\times} S^2, K\#F)$ is diffeomorphic to $(S^2 \tilde{\times} S^2, F)$.*

Proof. The equivalence of the statements is shown as follows.

$(1) \Rightarrow (2)$. We assume that S^4_K is diffeomorphic to S^4. By Theorem 1.5.5, $\partial_+(W_{S^4,K}) \cong S^4_K \cong S^4$. Consider a closed 5-manifold $\widehat{W_{S^4,K}}$ obtained from $W_{S^4,K}$ by gluing two 5-balls along the boundary components. By the classification of simply-connected 5-manifolds [Bar65] (see Theorem 1.4.24), $\widehat{W_{S^4,K}}$ is diffeomorphic to $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$ because $H_2(W_{S^4,K}) \cong \mathbb{Z}$. Since the middle level of $W_{S^4,K}$ (obtained from S^4 by surgery along the meridian m_K of K with non-trivial framing) is diffeomorphic to $S^2 \tilde{\times} S^2$, $\widehat{W_{S^4,K}}$ is diffeomorphic to $S^3 \tilde{\times} S^2$. Therefore, $W_{S^4,K}$ is diffeomorphic to twice-punctured $S^3 \tilde{\times} S^2$.

$(2) \Rightarrow (1)$. Assume that $W_{S^4,K}$ is diffeomorphic to a twice-punctured $S^2 \tilde{\times} S^3$. Then $\partial_+(W_{S^4,K})$ is diffeomorphic to S^4. By Theorem 1.5.5, $S^4_K \cong \partial_+(W_{S^4,K}) \cong S^4$.

$(2) \Leftrightarrow (3)$. $S^2 \tilde{\times} S^3$ is obtained by gluing together two copies of $S^2 \tilde{\times} B^3$ along the identity map on their common boundary, which is $S^2 \tilde{\times} S^2$. Here, $S^2 \tilde{\times} B^3$ is obtained from B^5 by attaching a 2-handle along the unknot with non-trivial framing. Consequently, the belt sphere of the 2-handle is a fiber F of $S^2 \tilde{\times} S^2$. Hence, $(S^2 \tilde{\times} S^2, F, F)$ is a Heegaard diagram not only for $S^2 \tilde{\times} S^3$ but also for the twice-punctured $S^2 \tilde{\times} S^3$ because $S^2 \tilde{\times} S^3$ is obtained from the twice-punctured $S^2 \tilde{\times} S^3$ by gluing two 5-balls. By Theorem 1.4.12, statements (2) and (3) are equivalent.

$(2) \Rightarrow (4)$. Consider $\widehat{M_\alpha} \cup_\Sigma \widehat{M_\beta} \cong S^2 \tilde{\times} S^3$ obtained from $M_\alpha \cup_\Sigma M_\beta$ by attaching two 5-balls. Here, $\widehat{M_\beta} \cong S^2 \tilde{\times} B^3$ is obtained from B^5 by attaching a 2-handle along the unknot with non-trivial framing. Consequently, $(S^2 \tilde{\times} S^2, K\#F)$ can be considered as a pair of the boundary of $\widehat{M_\beta}$ and the belt sphere of the 2-handle. Thus, $(S^2 \tilde{\times} S^2, K\#F)$ is diffeomorphic to $(S^2 \tilde{\times} S^2, F)$.

$(4) \Rightarrow (1)$. $S^4_K \cong \partial_+(W_{S^4,K})$ is diffeomorphic to the result of surgery on $S^2 \tilde{\times} S^2$ along F. Since $(S^2 \tilde{\times} S^2, K\#F)$ is diffeomorphic to $(S^2 \tilde{\times} S^2, F)$ and the surgery on $S^2 \tilde{\times} S^2$ along F is diffeomorphic to S^4, S^4_K is diffeomorphic to S^4. □

Note that this is analogous to, but distinct from Melvin's result [Mel77] that the statement (1) is equivalent to $(\mathbb{C}P^2, K\#\mathbb{C}P^1)$ being diffeomorphic to $(\mathbb{C}P^2, \mathbb{C}P^1)$.

Chapter 2

Contractible high-dimensional manifolds

2.1 Introduction

In this chapter, we prove the following two main theorems.

Theorem 2.3.16. *For any integer $n \geq 2$, there exists a contractible, compact, smooth $(n+3)$-manifold with boundary admitting a handle decomposition with a 0-handle, an n-handle and an $(n+1)$-handle which is not homeomorphic to the standard $(n+3)$-ball B^{n+3}.*

In order to prove Theorem 2.3.16, given $n \geq 2$, we first construct an interesting n-knot K^n in $S^n \times S^2$ (Definition 2.3.1) which is homotopic but not isotopic to $S^n \times \{y_0\}$, where $y_0 \in S^2$ (Proposition 2.3.2 and Corollary 2.3.7). Secondly we construct a contractible, compact, smooth $(n+3)$-manifold X_{K^n} (Definition 2.3.8) from $S^n \times B^3$ (0-handle $\cup$ n-handle) by attaching a single $(n+3)$-dimensional $(n+1)$-handle along the n-knot K^n in $S^n \times S^2 = \partial(S^n \times B^3)$. Finally we prove that X_{K^n} is contractible by showing that $X_{K^n} \times B^1$ is diffeomorphic to B^{n+4} (Proposition 2.3.10) and prove that X_{K^n} is not homeomorphic to B^{n+3} by showing that ∂X_{K^n} is a non-simply connected homology $(n+2)$-sphere (Corollary 2.3.13).

Theorem 2.3.18. *For any integer $n \geq 2$, there exists a smooth involution of S^{n+3} whose fixed point set is a non-simply connected homology $(n+2)$-sphere.*

In order to prove Theorem 2.3.18, given $n \geq 2$, we first show that the double $DX_{K^n} = X_{K^n} \cup_{id} \overline{X_{K^n}}$ of X_{K^n} is diffeomorphic to S^{n+3}, where $id : \partial X_{K^n} \to \partial X_{K^n}$ is an identity map (Lemma 2.3.17). We then define an involution $\phi : S^{n+3} \to S^{n+3}$ switching copies of X_{K^n} and fixing the non-simply connected homology $(n+2)$-sphere ∂X_{K^n}.

Remark 2.1.1. Here we discuss the relationship between our results and earlier results.

1. In [Maz61] Mazur proved Theorem 2.3.16 and Theorem 2.3.18 when $n = 1$. We can consider Mazur's 1-knot in $S^1 \times S^2$ as the result of surgery of three parallel copies of $S^1 \times \{y_0\} \subset S^1 \times S^2$ along two 2-dimensional 1-handles whose cores are trivial and with some twistings (See J_1 for Mazur's 1-knot and J_2 for another interesting 1-knot in Remark 2.2.6). However, we cannot generalize Mazur's 1-knot to an n-knot in $S^n \times S^2$ obtained from three parallel copies of $S^n \times \{y_0\} \subset S^n \times S^2$ by surgery along two $(n+1)$-dimensional 1-handles whose cores are trivial when $n \geq 2$ because the resulting n-knot is always isotopic to $S^n \times \{y_0\} \subset S^n \times S^2$. In Definition 2.3.1, we resolve this issue and find interesting $(n+1)$-dimensional 1-handles whose cores are non-trivial but very simple so that we construct the n-knot K^n in $S^n \times S^2$.

2. In [Sat91a] Sato proved Theorem 2.3.16 and Theorem 2.3.18 when $n = 2$. Sato constructed a 2-knot F in $S^2 \times S^2$ which is homotopic but not isotopic to $S^2 \times \{y_0\}$ by surgery along a simple closed curve in the complement of the 5-twist spun trefoil in S^4 in [Sat91b]. However, Sato's construction is not very explicit so it is difficult to visualize the 2-knot F. In particular, we do not know the geometric intersection number $|F \cap \{x_0\} \times S^2|$ directly and it is hard to see why $F \times \{0\} \subset S^2 \times S^2 \times B^1$ is isotopic to $S^2 \times \{y_0\} \times \{0\} \subset S^2 \times S^2 \times B^1$. Our construction of K^n resolves these issues.

3. In [Ker69], Kervaire proved the existence of contractible, compact, smooth $(n+3)$-manifolds which are not homeomorphic to B^{n+3}, where $n \geq 2$. However, the proof does not tell us much about the handle decomposition or give us a proof of Theorem 2.3.18.

Remark 2.1.2. Here we note some nice properties of our n-knot K^n in $S^n \times S^2$, and highlight some ways in which our construction is an improvement on the techniques used in the results described above.

1. The construction of K^n is very explicit and visualized for every $n \geq 2$ (Definition 2.3.1). We may construct infinitely many interesting n-knots in $S^n \times S^2$ by modifying the cores of $(n+1)$-dimensional 1-handles. For example, the cores used to construct K^n come from the case when $m_1 = 1, m_2 = -1$, and $m_3 = \cdots = m_{2i} = 0$ in the left of Figure 2.3. We may then construct infinitely many contractible, compact, smooth $(n+3)$-manifolds which are not homeomorphic to B^{n+3}.

2. K^n is the simplest example of this construction in the sense that the geometric intersection number $|K^n \cap (\{x_0\} \times S^2)| = 3$ and the algebraic intersection number $K^n \cdot (\{x_0\} \times S^2) = 1$, where $x_0 \in S^n$ (Proposition 2.3.2). Furthermore it is impossible to have $|F \cap \{x_0\} \times S^2| < 3$ for any n-knot F isotopic to K^n (Corollary 2.3.15).

3. A homotopy between $K^n \subset S^n \times S^2$ and $S^n \times \{y_0\} \subset S^n \times S^2$ is visualized (Proposition 2.3.2).

4. An isotopy between $K^n \times \{0\} \subset S^n \times S^2 \times B^1$ and $S^n \times \{y_0\} \times \{0\} \subset S^n \times S^2 \times B^1$ is visualized (Proposition 2.3.3). This isotopy is essential to proving that $X_{K^n} \times B^1$ is diffeomorphic to B^{n+4} (Proposition 2.3.10).

5. The construction of K^n gives an explicit handle decomposition of $S^n \times S^2 \setminus int(\nu(K^n))$ (Remark 2.3.5) so we can easily find the fundamental group $\pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$ (Proposition 2.3.6).

6. The construction of K^n gives an explicit handle decomposition of the non-simply connected homology $(n+2)$-sphere ∂X_{K^n} (Remark 2.3.11) so we can easily show that $\pi_1(X_{K^n}) \cong \pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$ (Proposition 2.3.12).

In section 2.2 we set up some standard notation and interpret Mazur's 1-knot in $S^1 \times S^2$ from the point of view of surgery as motivation for our construction of the n-knot K^n in $S^n \times S^2$. In section 2.3 we construct our n-knot K^n in $S^n \times S^2$ and the contractible $(n+3)$-manifold X_{K^n}, and then prove Theorem 2.3.16 and Theorem 2.3.18.

2.2 Preliminaries

We begin by explicitly describing the standard handle decomposition of $S^n \times S^2$ and the associated attaching maps.

Remark 2.2.1 (Standard handle decomposition of $S^n \times S^2$). Decompose $S^n = B^n_- \cup B^n_+$ into two n-dimensional balls and $S^2 = B^2_- \cup B^2_+$ into two 2-dimensional balls. Then $S^n \times S^2 = (B^n_- \cup B^n_+) \times (B^2_- \cup B^2_+) = (B^n_- \times B^2_-) \cup (B^n_- \times B^2_+) \cup (B^n_+ \times B^2_-) \cup (B^n_+ \times B^2_+)$ has a handle decomposition with a single 0-handle, a single 2-handle, a single n-handle, and a single $(n+2)$-handle. We can easily see the attaching sphere of the 2-handle (trivial 1-knot) and the attaching sphere of the n-handle (trivial $(n-1)$-knot) on the boundary of the 0-handle i.e., $(\{0\} \times S^1) \cup (S^{n-1} \times \{0\}) \subset \partial(B^n_- \times B^2_-) \cong \partial B^{n+2} = S^{n+1}$. For future reference, we parameterize the trivial 1-knot and the trivial $(n-1)$-knot in $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+1} \cup \{\infty\} \cong S^{n+1}$. The trivial 1-knot $\{0\} \times S^1$ corresponds to $A := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid {x_1}^2 + {x_2}^2 = 1, x_i = 0 \text{ for } i > 2\}$ and the trivial $(n-1)$-knot $S^{n-1} \times \{0\}$ corresponds to $B := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 = 0, (x_2+1)^2 + \sum_{i=3}^{n+1} {x_i}^2 = (\frac{1}{5})^2\}$. An $(\mathbb{R}^3 \times \{0\})$-slice $(A \cup B) \cap (\mathbb{R}^3 \times \{0\})$ of $A \cup B$ is the Hopf link in Figure 2.1, where $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R}^{n-2} \cong \mathbb{R}^{n+1}$. Again, $S^n \times S^2$ can be recovered from B^{n+2} by attaching a single 2-handle to $S^{n+1} \cong \mathbb{R}^{n+1} \cup \{\infty\}$ along A with 0-framing (product framing), a single n-handle with 0-framing (product framing), and a single $(n+2)$-handle (we don't draw the $(n+2)$-handle). The 0 in Figure 2.1 is shorthand for the obvious product framing. For example, Figure 2.1 is a Kirby diagram for $S^2 \times S^2$ when $n = 2$. A parallel copy $C_t := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 = 0, (x_2+1)^2 + \sum_{i=3}^{n+1} {x_i}^2 = (\frac{t}{4})^2\}$ of B bounds a properly embedded trivial n-ball D^-_t in B^{n+2} (C_t bounds a trivial n-ball in S^{n+1} and push the interior of the n-ball into the interior of B^{n+2}) and a copy of the core of the n-handle D^+_t so $C_t = D^-_t \cap D^+_t$ represents the equator of $S^n \times \{y_t\}$ and $D_t := D^-_t \cup D^+_t$ represents $S^n \times \{y_t\}$. From now on, we use red, blue and green for A, B, and C_t, respectively.

Next we establish terminology for handles embedded in an ambient manifold and attached to a submanifold.

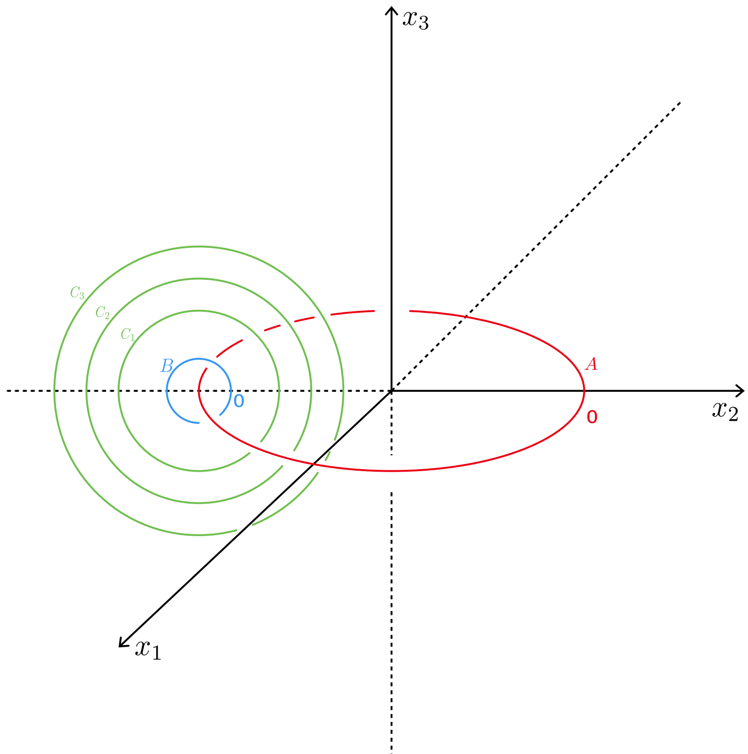

Figure 2.1: Standard handle decomposition of $S^n \times S^2$ and three parallel copies of $S^n \times \{y_0\}$. A is the attaching sphere of the 2-handle with 0-framing, B is the attaching sphere of n-handle with 0-framing, and C_t is the equator of $S^n \times \{y_t\}$.

Definition 2.2.2. Let $n \geq 1$. Let $N^n \subset M^{n+2}$ be a n-dimensional submanifold of $(n+2)$-dimensional manifold M. An $(n+1)$-dimensional submanifold $h \subset M$ is called a 1-***handle attached to*** N if there exists an embedding $e : B^1 \times B^n \hookrightarrow M$ such that $h = e(B^1 \times B^n)$ and $h \cap N = e(\partial B^1 \times B^n)$. We call $C_h := e(B^1 \times \{0\})$ the ***core*** of h and $N_h := (N \setminus int(e(\partial B^1 \times B^n))) \cup e(B^1 \times \partial B^n) \subset M$ ***the result of surgery*** of N along h in M. Here N_h is assumed to be oriented so that the orientation of $N \setminus int(e(\partial B^1 \times B^n))$ extends to the orientation of N_h.

Boyle proved the following when $n = 2$ and $M = S^4$. See [Boy88] for more details.

Proposition 2.2.3. Let $n \geq 2$. Let h and h' be 1-handles attached to $N^n \subset M^{n+2}$ with the same core C. Then there exists an ambient isotopy of M taking h to h' fixing N setwise. Furthermore, the results of surgery N_h and $N_{h'}$ are isotopic.

Proof. Following [Boy88], the difference between two such 1-handles with the same core gives a map $\theta : B^1 \to G_{n+1,n}$; the Grassmannian manifold of oriented n-planes in $\mathbb{R}^{n+1}$, with $\theta(-1) = \theta(1)$. Since $n \geq 2$ and $G_{n+1,n} \cong S^n$, we have $\pi_1(G_{n+1,n}) = 0$, so there exists an isotopy of M taking h to h' fixing N setwise. From this we see that the results of surgery N_h and $N_{h'}$ are isotopic. □

Corollary 2.2.4. Let $n \geq 2$. Let h and h' be 1-handles attached to $N^n \subset M^{n+2}$ with cores C_h and $C_{h'}$, respectively. If C_h and $C_{h'}$ are isotopic through arcs such that the boundary of each arc is in N and the interior of each arc doesn't intersect with N, then there exists an ambient isotopy of M taking h to h' fixing N setwise. In particular, the results of surgery N_h and $N_{h'}$ are isotopic.

Proof. By the tubular neighborhood theorem and Proposition 2.2.3, the isotopy taking C_h to $C_{h'}$ through arcs such that the boundary of each arc is in N and the interior of each arc does not intersect N can extend to an ambient isotopy of M taking h to h' fixing N setwise. From this we see that the results of surgery N_h and $N_{h'}$ are isotopic. □

Remark 2.2.5.

1. Corollary 2.2.4 is not true when $n = 1$ because $\pi_1(G_{n+1,n}) \cong \mathbb{Z}$. Therefore, different framings of a core may give non-isotopic 1-handles; see (4) in Remark 2.2.6.

2. When $n \geq 2$, a homotopy between arcs implies an isotopy between arcs and there is a unique framing of a core, so $(n+1)$-dimensional 1-handles are less complicated than 2-dimensional 1-handles.

We now analyze Mazur's knot J_1 and some other interesting knots J_2 and J_3 in $S^1 \times S^2$ from the point of view of surgery along 1-handles. Figure 2.2 illustrates these examples. In this figure we consider $S^1 \times S^2$ as the boundary of $S^2 \times B^2$, where $S^2 \times B^2$ is obtained from B^4 by attaching a 2-handle along the unknot with the 0-framing (product framing). Observe the following features of the knots constructed in Figure 2.2:

Remark 2.2.6.

1. J_1 is homotopic but not isotopic to $S^1 \times \{y_0\}$ in $S^1 \times S^2$; see (c) in Figure 2.2.

2. J_2 is homotopic but not isotopic to $S^1 \times \{y_0\}$ in $S^1 \times S^2$; see (f) in Figure 2.2.

3. J_3 is homotopic but not isotopic to the unknot in $S^1 \times S^2$; see (i) in Figure 2.2.

4. J_1, J_2 and J_3 are obtained from parallel copies of $S^1 \times \{y_0\}$ by surgery along 2-dimensional 1-handles in the figure. Here, we can see that handles are attached so that the results of surgery are oriented and depend on the cores and framings (or twistings) of the cores. (See the second column in Figure 2.2, and note that we may obtain different knots by twisting the bands more.)

5. In section 2.3 we will construct an n-knot in $S^n \times S^2$ from three parallel copies of $S^n \times \{y_0\}$ by surgery along two interesting $(n+1)$-dimensional 1-handles.

A natural question related to the construction of J_3 is whether one can construct an n-knot in $S^n \times S^2$ which is homotopic but not isotopic to the unknot from two parallel copies of $S^n \times \{y_0\}$ by surgery along a single $(n+1)$-dimensional 1-handle. However, the following theorem shows that this does not work for $n \geq 2$.

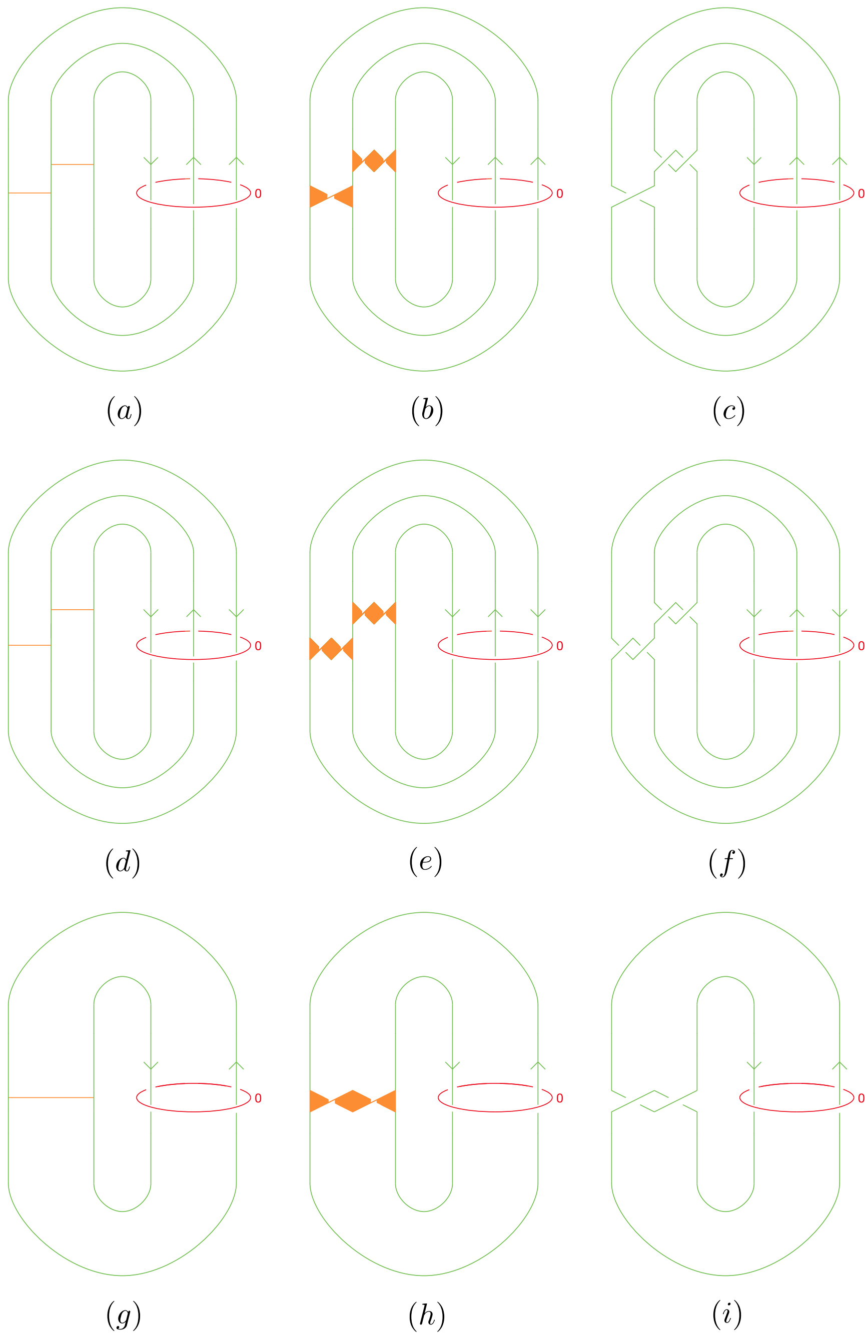

Figure 2.2: **First column**: Parallel copies of $S^1 \times \{y_0\}$ in $S^1 \times S^2$ with trivial cores of 2-dimensional 1-handles. **Second column**: 2-dimensional 1-handles determined by the cores and framings (or twistings). **Third column**: Results of surgery J_1, J_2, and J_3. **First row** : A process of obtaining J_1 by surgery. **Second row**: A process of obtaining J_2 by surgery. **Third row**: A process of obtaining J_3 by surgery.

Theorem 2.2.7. *Fix $n \geq 2$. Let $N = (S^n \times \{y_1\}) \cup (\overline{S^n \times \{y_2\}}) \subset S^n \times S^2$ with opposite orientations, where $y_1 \neq y_2 \in S^2$. Let $h = e(B^1 \times B^n)$ be a 1-handle attached to N for some embedding $e : B^1 \times B^n \hookrightarrow S^n \times S^2$ such that $(S^n \times \{y_1\}) \cap h = e(\{-1\} \times B^n)$ and $(\overline{S^n \times \{y_2\}}) \cap h = e(\{1\} \times B^n)$. Then the result of surgery N_h is isotopic to the unknot, i.e., N_h bounds an $(n+1)$-ball in $S^n \times S^2$.*

Proof. Consider the standard handle decomposition of $S^n \times S^2$ described in Remark 2.2.1 and Figure 2.1. Let $D_1 = S^n \times \{y_1\}$ and $D_2 = \overline{S^n \times \{y_2\}}$. Now consider a 1-handle h attached to $D_1 \cup D_2$. By Corollary 2.2.4, it suffices to consider the core of the 1-handle h. The core of the 1-handle can be isotoped into $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R}^{n-2} \subset \mathbb{R}^{n+1} \cup \{\infty\}$ and furthermore isotoped into the orange arc for some integers $m_1, \ldots, m_{2i}$ in the left of Figure 2.3 because a homotopy between arcs implies an isotopy of arcs in $S^n \times S^2$. We will show that the orange arc in the left of Figure 2.3 can be isotoped into the trivial arc in the right of Figure 2.3. Figure 2.4 illustrates how to isotope the orange arc into the trivial when $m_1 = 2$ and $m_2 = -1$. Repeating the process illustrated in this example shows how to do the general case. Therefore, the result of surgery N_h along h is isotopic to the result of surgery along the trivial 1-handle which is the unknot in $S^n \times S^2$ by Corollary 2.2.4. □

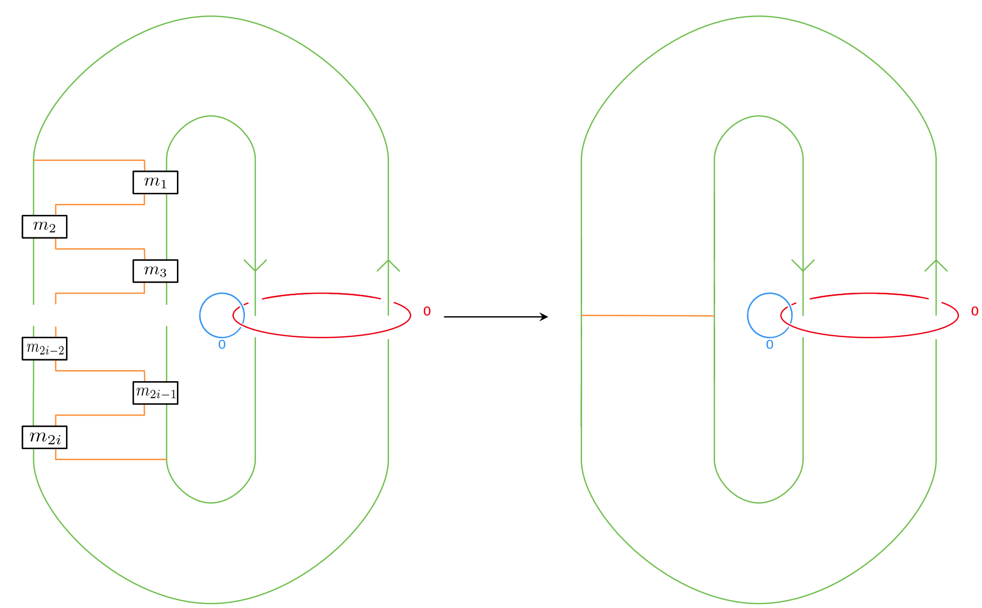

Figure 2.3: An isotopy between cores. An integer m in the box indicates m-full positive twist. C_i is the equator of D_i. **Left**: any arc attached to $C_1 \cup C_2$ is isotopic to the orange arc for some values of $m_1, \ldots, m_{2i}$. **Right**: the trivial arc attached to $C_1 \cup C_2$.

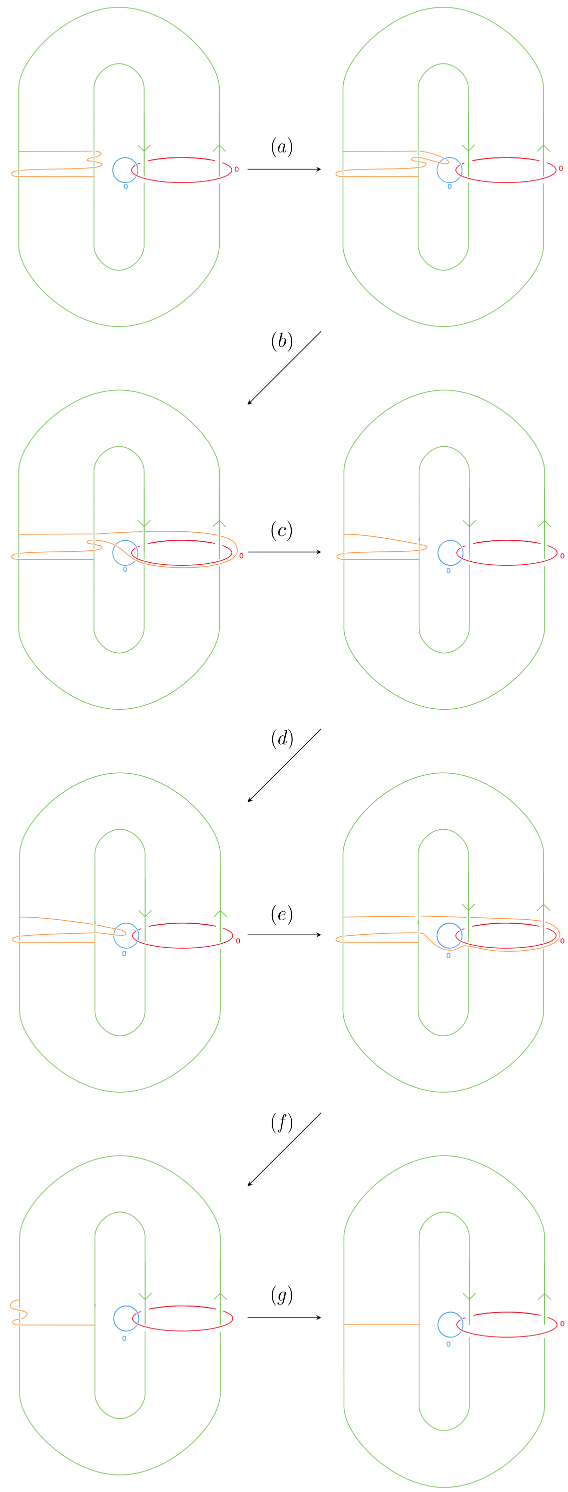

Figure 2.4: An isotopy between cores when $m_1 = 2$ and $m_2 = -1$. (a), (d): isotopies pushing the orange arc into B^{n+2} and pulling back. (b), (e): isotopies sliding the orange arc over 2-handle. (c), (f), (g): obvious isotopies.

2.3 Main theorem

Definition 2.3.1. Let $n \geq 2$. Let $N = (S^n \times \{y_1\}) \cup \overline{(S^n \times \{y_2\})} \cup (S^n \times \{y_3\}) \subset S^n \times S^2$, where $y_1, y_2, y_3 \in S^2$ are three distinct points. Let h_{12} be the 1-handle attached to $(S^n \times \{y_1\}) \cup \overline{(S^n \times \{y_2\})}$ whose core is in the top left of Figure 2.5. Let h_{23} be the 1-handle attached to $\overline{(S^n \times \{y_2\})} \cup (S^n \times \{y_3\})$ whose core is in the top left of Figure 2.5. We define the n-knot K^n in $S^n \times S^2$ to be the result of surgery of N along $h_{12} \cup h_{23}$.

We now see some properties of K^n.

Proposition 2.3.2. K^n is homotopic to $S^n \times \{y_0\}$ in $S^n \times S^2$, the geometric intersection number $|K^n \cap (\{x_0\} \times S^2)| = 3$, and the algebraic intersection number $K^n \cdot (\{x_0\} \times S^2) = 1$, where $x_0 \in S^n$.

Proof. There exists an isotopy between the union of the two cores in the top left of Figure 2.5 and the union of the two trivial cores in the top right of Figure 2.5 such that at one moment of the isotopy the arcs intersect the green spheres at four points (like crossing changes). The result of the surgery along the two 1-handles with the trivial cores in the top right of Figure 2.5 is isotopic to $S^n \times \{y_0\}$, so K^n is homotopic to $S^n \times \{y_0\}$. Clearly, $|K^n \cap (\{x_0\} \times S^2)| = 3$, and the algebraic intersection number $K^n \cdot (\{x_0\} \times S^2) = 1$ from the construction. □

Proposition 2.3.3. $K^n \times \{0\}$ in $S^n \times S^2 \times B^1$ is isotopic to $S^n \times \{y_0\} \times \{0\}$.

Proof. We can isotope each core of the 1-handles in the top left of Figure 2.5 to the trivial core in the top right of Figure 2.5 using the extra B^1 factor without intersections between arcs and green spheres. By Corollary 2.2.4, $K^n \times \{0\}$ is isotopic to $S^n \times \{y_0\} \times \{0\}$, which is isotopic to the result of the surgery along the trivial cores in the top right of Figure 2.5. □

Remark 2.3.4 (A handle decomposition of K^n)**.** K^n is obtained from three parallel copies of $S^n \times \{y_0\}$ by surgery along two $(n+1)$-dimensional 1-handles in the top left of Figure 2.5. Here, the green equator of each parallel copy bounds a properly embedded trivial n-ball (n-dimensional 0-handle) in B^{n+2} and a copy of the core (n-dimensional n-handle) of the $(n+2)$-dimensional n-handle of $S^n \times S^2$. Surgery along a 1-handle $B^1 \times B^n$ removes $(\{-1\} \times B^n) \cup (\{1\} \times B^n)$ and glues $B^1 \times S^{n-1}$ so $B^1 \times S^{n-1} = B^1 \times (B^n_- \cup B^n_+) = (B^1 \times B^{n-1}_-) \cup (B^1 \times B^{n-1}_+)$ consists of an n-dimensional 1-handle (yellow) and an $(n-1)$-handle (purple) in the bottom left of Figure 2.5. Therefore, K^n has a handle decomposition with three 0-handles, two 1-handles, two $(n-1)$-handles and three n-handles. For example, the bottom left of Figure 2.5 is a banded unlink diagram for K^2 in $S^2 \times S^2$ when $n = 2$. See the left of Figure 1.5.

Remark 2.3.5 (A handle decomposition of $S^n \times S^2 \setminus int(\nu(K^n))$)**.** Each i-handle of K^n determines an $(i+1)$-handle for $S^n \times S^2 \setminus int(\nu(K^n))$ when the codimension is 2. (See chapter 6.2 in [GS99].) We note that the number of $(n+1)$-handles for the complement induced by the n-handles for K^n is one less than the number of n-handles and a dotted $(n-1)$-sphere means carving a properly embedded trivial n-ball in B^{n+2} whose boundary is a dotted $(n-1)$-sphere, which is equivalent to attaching a 1-handle. Therefore, the handle decomposition of K^n in Remark 2.3.4 gives the handle decomposition of $S^n \times S^2 \setminus int(\nu(K^n))$ in the bottom right of Figure 2.5.

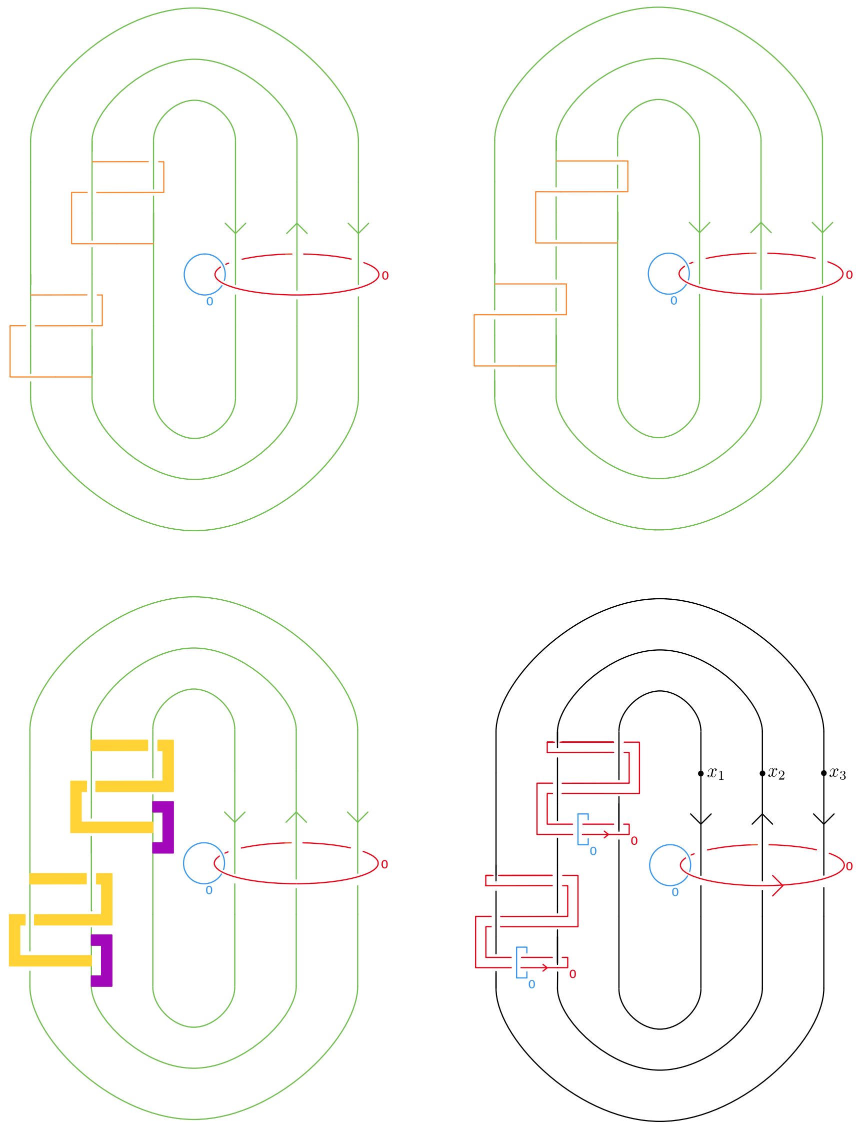

Figure 2.5: **Top left**: K^n is the result of surgery of $(S^n \times \{y_1\}) \cup (\overline{S^n \times \{y_2\}}) \cup (S^n \times \{y_3\})$ along two 1-handles h_{12} and h_{23} with orange cores. **Top right**: Isotoped cores (trivial cores) in $S^n \times S^2$ or $S^n \times S^2 \times B^1$. The isotopy looks like crossing changes in the $(\mathbb{R}^3 \times \{0\})$-slice. **Bottom left**: A handle decomposition of K^n in the handle decomposition of $S^n \times S^2$ consists of three n-dimensional 0-handle, two 1-handles (yellow), two $(n-1)$-handles (purple), and three n-handles (not drawn). **Bottom right**: A handle decomposition of $S^n \times S^2 \setminus int(\nu(K^n))$ consists of a single $(n+2)$-dimensional 0-handle, three 1-handles (black dotted $(n-1)$-spheres), three 2-handles (red 1-spheres), three n-handles (blue $(n-1)$-spheres) and two $(n+1)$-handles (not drawn).

Proposition 2.3.6. $\pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$ is non-trivial.

Proof. Consider the handle decomposition of the complement $S^n \times S^2 \setminus int(\nu(K^n))$ of K^n in $S^n \times S^2$ in the bottom right of Figure 2.5. A black dotted $(n-1)$-sphere and a red 1-sphere represent a 1-handle and a 2-handle, respectively, so the fundamental group $\pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$ has the following presentation:

$$\langle x_1, x_2, x_3 | x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1} = 1, x_2^{-1}x_3^{-1}x_2^{-1}x_3x_2x_3 = 1, x_1^{-1}x_2x_3^{-1} = 1 \rangle.$$

Delete x_3 by using the third relation $x_3 = x_1^{-1}x_2 \Leftrightarrow x^{-1}x_2x_3^{-1} = 1$:

$$\langle x_1, x_2 | x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1} = 1, x_2^{-2}x_1x_2^{-1}x_1^{-1}x_2^2x_1^{-1}x_2 = 1 \rangle.$$

Simplify the second relation by multiplying both sides by x_2 on the left and x_2^{-1} on the right:

$$\langle x_1, x_2 | x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1} = 1, x_2^{-1}x_1x_2^{-1}x_1^{-1}x_2^2x_1^{-1} = 1 \rangle.$$

Use the substitution $x_1 = ab^{-1}$ and $x_2 = b^2a^{-1}$, in other words let $a = x_1x_2x_1$ and $b = x_2x_1$:

$$\langle a, b | a^2b^{-3} = 1, ab^{-2}ab^{-1}ab^{-1}a^{-1}b^2a^{-1}b^2a^{-1}ba^{-1} = 1 \rangle.$$

Simplify the second relation by multiplying both sides by a^{-1} on the left and a on the right :

$$\langle a, b | a^2b^{-3} = 1, b^{-2}ab^{-1}ab^{-1}a^{-1}b^2a^{-1}b^2a^{-1}b = 1 \rangle.$$

Simplify the second relation by multiplying both sides by b on the left and b^{-1} on the right:

$$\langle a, b | a^2b^{-3} = 1, b^{-1}ab^{-1}ab^{-1}a^{-1}b^2a^{-1}b^2a^{-1} = 1 \rangle.$$

Include the relation $a^2 = b^3 = 1$:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1}ab^{-1}a^{-1}b^2a^{-1}b^2a^{-1} = 1 \rangle.$$

Multiply both sides by a on the right in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1}ab^{-1}a^{-1}b^2a^{-1}b^2 = a \rangle.$$

Multiply both sides by b on the right and use $b^3 = 1$ in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1}ab^{-1}a^{-1}b^2a^{-1} = ab \rangle.$$

Multiply both sides by a on the right in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1}ab^{-1}a^{-1}b^2 = aba \rangle.$$

Multiply both sides by b on the right and use $b^3 = 1$ in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1}ab^{-1}a^{-1} = abab \rangle.$$

Multiply both sides by a on the right in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1}ab^{-1} = ababa \rangle.$$

Multiply both sides by b on the right in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1}a = ababab \rangle.$$

Multiply both sides by a on the right and use $a^2 = 1$ in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}ab^{-1} = abababa \rangle.$$

Multiply both sides by b on the right in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1}a = abababab \rangle.$$

Multiply both sides by a on the right and use $a^2 = 1$ in the second relation:

$$\langle a, b | a^2 = b^3 = 1, b^{-1} = ababababa \rangle.$$

Multiply both sides by b on the right in the second relation:

$$\langle a, b | a^2 = b^3 = 1, 1 = ababababab \rangle.$$

Simplify the relations and get the following presentation:

$$\langle a, b | a^2 = b^3 = (ab)^5 = 1 \rangle,$$

which is isomorphic to the alternating group A_5 of degree 5. Therefore, $\pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$ is non-trivial. □

Corollary 2.3.7. K^n is not isotopic to $S^n \times \{y_0\}$ in $S^n \times S^2$.

Proof. Suppose that K^n is isotopic to $S^n \times \{y_0\}$. Then $S^n \times S^2 \setminus int(\nu(K^n))$ is diffeomorphic to $B^n \times S^2$, so $\pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$ is trivial, which is a contradiction to Proposition 2.3.6. □

We now construct a contractible $(n+3)$-manifold by using the n-knot K^n.

Definition 2.3.8. Let K^n be the n-knot in $S^n \times S^2$ in Definition 2.3.1. Let $\phi : S^n \times B^2 \hookrightarrow S^n \times S^2 = \partial(S^n \times B^3)$ be an embedding such that $\phi(S^n \times \{0\}) = K^n$. We define

$$X_{K^n} = (S^n \times B^3) \cup_\phi (B^{n+1} \times B^2)$$

to be the $(n+3)$-manifold obtained from $S^n \times B^3$ by attaching a single $(n+1)$-handle along ϕ.

Remark 2.3.9. There is a unique framing of the attaching sphere $\phi(S^n \times \{0\}) = K^n$ because $\pi_n(SO(2))$ is trivial when $n \geq 2$. Therefore, X_{K^n} is uniquely determined by the isotopy class of K^n.

Proposition 2.3.10. $X_{K^n} \times B^1$ is diffeomorphic to B^{n+4}.

Proof. Let $\phi : S^n \times B^2 \hookrightarrow S^n \times S^2$ be the embedding such that $\phi(S^n \times \{0\}) = K^n$ in Definition 2.3.1 and $\Phi : S^n \times B^2 \times B^1 \hookrightarrow S^n \times S^2 \times B^1$ be an embedding defined by $\Phi(x, y, t) = (\phi(x, y), t)$. Then $X_{K^n} \times B^1 = ((S^n \times B^3) \cup_\phi (B^{n+1} \times B^2)) \times B^1 \cong (S^n \times B^3 \times B^1) \cup_\Phi (B^{n+1} \times B^2 \times B^1)$. By Proposition 2.3.3, $\Phi(S^n \times \{0\} \times \{0\}) = K^n \times \{0\}$ is isotopic to $S^n \times \{y_0\} \times \{0\}$ in $S^n \times S^2 \times B^1 \subset \partial(S^n \times B^3 \times B^1)$ so the attaching sphere of the $(n+1)$-handle intersects the belt sphere of the n-handle geometrically once and $(S^n \times B^3 \times B^1) \cup_\Phi (B^{n+1} \times B^2 \times B^1) \cong B^{n+4}$. Therefore, $X_{K^n} \times B^1 \cong B^{n+4}$. □

Remark 2.3.11 (A handle decomposition of ∂X_{K^n})**.** ∂X_{K^n} is obtained from $S^n \times S^2$ by surgery along K^n, i.e., $\partial X_{K^n} = (S^n \times S^2 \setminus int(\nu(K^n))) \cup_{\phi|_{S^n \times S^1}} (B^{n+1} \times S^1)$, where $\phi(S^n \times B^2) = \nu(K^n)$. We can consider $B^{n+1} \times S^1$ as the union of an $(n+2)$-dimensional $(n+1)$-handle and an $(n+2)$-handle. Therefore, a handle decomposition of X_{K^n} is obtained from the handle decomposition of $S^n \times S^2 \setminus int(\nu(K^n))$ in Remark 2.3.5 by attaching an $(n+1)$-handle and an $(n+2)$-handle. Therefore, ∂X_{K^n} admits a handle decomposition with a 0-handle, three 1-handles, three 2-handles, three n-handles, three $(n+1)$-handles and an $(n+2)$-handle.

Proposition 2.3.12. $\pi_1(\partial X_{K^n}) \cong \pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$.

Proof. In Remark 2.3.11, $B^{n+1} \times S^1$ is the union of an $(n+1)$-handle and an $(n+2)$-handle, which does not affect the fundamental group of ∂X_{K^n}. Therefore, $\pi_1(\partial X_{K^n}) \cong \pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$. □

Corollary 2.3.13. ∂X_{K^n} is a non-simply connected homology $(n+2)$-sphere.

Proof. Clearly, ∂X_{K^n} is a homology $(n+2)$-sphere because X_{K^n} is a contractible manifold. Also, $\pi_1(\partial X_{K^n}) \cong \pi_1(S^n \times S^2 \setminus int(\nu(K^n)))$ is non-trivial by Proposition 2.3.6 and Proposition 2.3.12. □

Corollary 2.3.14. X_{K^n} is contractible but not homeomorphic to B^{n+3}.

Proof. X_{K^n} is contractible by Proposition 2.3.10 but not homeomorphic to B^{n+3} by Corollary 2.3.13. □

Corollary 2.3.15. There exists no n-knot F in $S^n \times S^2$ such that F is isotopic to K^n and $|F \cap (\{x_0\} \times S^2)| < 3$.

Proof. Suppose that there is an n-knot F in $S^n \times S^2$ such that F is isotopic to K^n and $|F \cap (\{x_0\} \times S^2)| = 1$. Since K^n and F are isotopic, X_{K^n} is diffeomorphic to X_F which is obtained from $S^n \times B^3$ by attaching a single $(n+1)$-handle along F. Since $|F \cap (\{x_0\} \times S^2)| = 1$, X_{K^n} is diffeomorphic to $B^{n+3} \cong X_F$, which is a contradiction to Corollary 2.3.14. Suppose that there is an n-knot F in $S^n \times S^2$ such that F is isotopic to K^n and $|F \cap (\{x_0\} \times S^2)| = 2$. Then the algebraic intersection number $F \cdot (\{x_0\} \times S^2)$ is 0 or ± 2, so $K^n \cdot (\{x_0\} \times S^2)$ is 0 or ± 2. This is a contradiction to $K^n \cdot (\{x_0\} \times S^2) = 1$. Therefore, there exists no n-knot F in $S^n \times S^2$ such that F is isotopic to K^n and $|F \cap (\{x_0\} \times S^2)| < 3$. □

We now prove our main theorems.

Theorem 2.3.16. *For any integer $n \geq 2$, there exists a contractible, compact, smooth $(n+3)$-manifold with boundary admitting a handle decomposition with a 0-handle, an n-handle and an $(n+1)$-handle which is not homeomorphic to the standard $(n+3)$-ball B^{n+3}.*

Proof. Let X_{K^n} be the $(n+3)$-manifold in Definition 2.3.1. X_{K^n} admits a handle decomposition with a 0-handle, an n-handle and an $(n+1)$-handle by Definition 2.3.1. X_{K^n} is contractible but not homeomorphic to B^{n+3} by Corollary 2.3.14. □

Lemma 2.3.17. The double $DX_{K^n} = X_{K^n} \cup_{id} \overline{X_{K^n}}$ of X_{K^n} is diffeomorphic to S^{n+3}, where $id : \partial X_{K^n} \to \partial X_{K^n}$ is an identity map.

Proof. $DX_{K^n} = X_{K^n} \cup_{id} \overline{X_{K^n}} \cong \partial(X_{K^n} \times B^1) \cong \partial(B^{n+4}) = S^{n+3}$ by Proposition 2.3.10. □

Theorem 2.3.18. *For any integer $n \geq 2$, there exists a smooth involution of S^{n+3} whose fixed point set is a non-simply connected homology $(n+2)$-sphere.*

Proof. By Lemma 2.3.17, $S^{n+3} \cong X_{K^n} \cup_{id} \overline{X_{K^n}}$. Define an involution $\phi : S^{n+3} \to S^{n+3}$ switching copies of X_{K^n} and fixing ∂X_{K^n}. □

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