Optimal Strategies for Pairs Trading under Geometric Brownian Motion and Mean-Reversion Models with Regime Switching

by

Emily Beatrice Crawford Das

(Under the Direction of Jingzhi Tie and Qing Zhang)

Abstract

This dissertation explores several questions on the topic of pairs trading. The idea of pairs trading is to simultaneously trade a pair of securities, typically stocks. The purpose is to hedge the risk associated with buying and holding shares of a single stock by selling shares of a second stock. This method can be beneficial, because it has the potential to be profitable under any market circumstances. That is to say, it can be profitable even when prices are not going up. The strategy is to track and compare the relative strengths of the prices of two stocks over time. When their prices diverge, the plan is to go long in the weaker stock and go short in the stronger stock. This technique bets on the eventual reversal of their price strengths. The objective is to trade the pairs over time to maximize an overall return with a fixed commission cost for each transaction. The optimal policy is then characterized by threshold curves obtained by solving the Hamilton-Jacobi-Bellman (HJB) equations that arise from following a dynamic programming approach.

Index words: [Geometric Brownian Motion, Mean-reversion Model, Regime Switching, Hamilton-Jacobi-Bellman Equation]

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Dedication

To my grandmother, Beatrice Mixon Cornell, in loving memory.

To my mother, Theresa Crawford, for all the years of love and support.

To my aunt, Sonja Crawford, who once dreamed of earning a PhD in Mathematics.

To my husband, Bikash.

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Chapter 1

Introduction to Pairs Trading

1.1 Introduction

This dissertation explores several questions in the field of mathematical finance, specifically focusing on identifying optimal strategies for pairs trading. Traditional stock trading strategies encourage investors to buy low and sell high in order to secure a profit. However, this is only possible when prices go up, which cannot be guaranteed. To alleviate this, the practice of pairs trading was introduced by Gerry Bamberger and pioneered by quantitative analysts in Nunzio Tartaglia's group at Morgan Stanley in the 1980s. The idea of pairs trading is to hedge the risk associated with buying and holding shares of a single stock by enacting trades involving a second, usually strongly correlated, stock. The benefit of this method is that in can be profitable under any market circumstances, due to its market neutral nature. For related literature and detailed discussions on the subject, we refer the reader to the paper by Gatev et al. [**10**], the book by Vidyamurthy [**22**], and references therein.

Pairs of stocks are typically chosen when their prices follow roughly the same trajectory over time, i.e. when they are cointegrated; see Gatev et al. [**10**] and Liu and Timmermann [**16**] for further discussion. When there is a divergence of the stock prices to a certain level, the pairs trade would be triggered: to short the stronger stock and to long the weaker one, betting on the eventual convergence of the prices. This is the strategy we seek to model in this dissertation. Another similar strategy bets on the eventual divergence of the prices. When the difference between the prices decreases to a certain level, the pairs trade is entered by longing the stronger stock and shorting the weaker one.

Mathematical trading rules, including pairs trading rules, have been studied for several decades. Traditional pairs trading uses mean-reversion models, and closed-form solutions are often derived. However, another commonly used model for stock price movements involves geometric Brownian motion. For example, Zhang [**25**] considered a selling rule determined by two threshold levels: a target price and a cut-loss limit. In [**25**], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [**11**] studied the optimal selling rule under a model with switching geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. Note that these papers are only concerned with geometric Brownian motion type models. Chapters 2 and 3 of this dissertation are concerned with pairs trading under the assumption of geometric Brownian motion.

The latter part of this dissertation also considers pairs trading strategies when a mean-reversion model is assumed. Mean-reversion models are commonly used to depict price movements that tend to move toward an equilibrium level. We refer the reader to Cowles and Jones [**6**], Fama and French [**8**], and Gallagher and Taylor [**9**], among others, for studies in connection with mean-reversion and stock returns. Mean-reversion models also find applications beyond stock markets. They are utilized for stochastic interest rates, as explored by Vasicek [**21**] and Hull [**14**], stochastic volatility, as studied by Hafner and Herwartz [**12**], and energy markets, as examined by Blanco and Soronow [**2**]. There are also relevant findings for options pricing involving mean-reversion assets, as demonstrated by Bos, Ware, and Pavlov [**3**]. We also introduce regime-switching to the mean-reversion model. Regime-switching models complicate the modeling problem, since the Markov chain incorporates another source of uncertainty into the models.

Market models with regime switching are important to market analysis. Regime-switching models are often used to better reflect a random market environment. In a mean-reversion model, the rate of reversion, the mean (equilibrium), and the volatility are all subject to change in the long run. One way to capture these changes is to introduce a switching process dictating sudden changes in system parameters. The models incorporate parameters to describe the trends of the market which switches among a finite number of states, for instance, the uptrend (bull market) and the downtrend (bear market). Regime-switching models were first introduced by Hamilton [**13**] in 1989 to describe time series. The models have also been employed by Zhang [**25**] for optimal stock selling rules, Yin and Zhang [**23**] for applications in portfolio management, and Yin and Zhou [**24**] for dynamic Markowitz problems. Unlike these papers, this dissertation does not introduce regime-switching in the context of geometric Brownian motions. A mean-reverting Itô diffusion of the form $\mathrm{d}Z_t = \theta(\alpha_t)[\mu(\alpha_t) - Z_t]\mathrm{d}t + \sigma(\alpha_t)\mathrm{d}W_t$ is used instead.

In many optimal trading problems, Hamilton-Jacobi-Bellman (HJB) equations are derived. Various techniques in stochastic control theory have been employed to solve these equations, such as ordinary differential equations (ODE), partial differential equations (PDE), smooth fitting, and viscosity solution methods. However, the associated HJB equations may involve highly complicated PDEs for which classical solutions are very hard to obtain. To avoid solving these complicated HJB equations, stochastic approximation methods can be used. Recent references on stochastic approximation can be found in [4], [15]. However, in this dissertation, we only consider the ODEs and PDEs under smooth-fitting conditions.

This dissertation is organized as follows: Chapter 2 is concerned with one round-trip pairs trade for a pair of stocks whose prices follow geometric Brownian motions. We assume that the initial pairs position may be either long or flat. We derive the associated HJB equations for the value functions and solve them to find closed-form solutions and an optimal trading rule. Chapter 3 extends the round-trip pairs trading problem from Chapter 2 to include the possibility that the initial pairs position may be long, flat, or short. This results in a new set of value functions and, hence, a new set of HJB equations. We are able to solve the HJB equations in closed form and obtain an optimal trading rule. Chapter 4 is once again concerned with pairs trading, but now we assume the prices of the stocks follow a mean-reversion process. We introduce regime switching to incorporate the possiblity of different market modes. The quasi-variational inequalities for the value functions provide a set of sufficient conditions for the optimality of the trading strategy.

1.2 Problem One: Round-Trip Pairs Trading under Geometric Brownian Motions

One typical model for daily stock price movements is the following stochastic differential equation,

$$\mathrm{d}\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix} \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \mathrm{d} \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \right], \tag{1.1}$$

where $\{X_t^1, t \geq 0\}$ denote the prices of some stock $\mathbf{S}^1$, and $\{X_t^2, t \geq 0\}$ denote the prices of some stock $\mathbf{S}^2$, μ_i, $i = 1, 2$ are the return rates, σ_{ij}, $i, j = 1, 2$ are the volatility constants, and (W_t^1, W_t^2) is a 2-dimensional standard Brownian motion. One benefit of this model, is that it does not specify any relationship between the pairs of stocks or require them to satisfy any measure of correlation, thus allowing for greater possibilities in the choice of pairs [**20**]. The Brownian motion, whose sample path is a random walk, encodes the assumption that it is impossible to accurately predict the change in the price of a stock from day to day. We consider a pairs position $\mathbf{Z}$ where holding one share of $\mathbf{Z}$ means being long one share in stock $\mathbf{S}^1$ and being short one share in stock $\mathbf{S}^2$. We allow that the initial position of $\mathbf{Z}$ may be either long ($i = 1$) or flat ($i = 0$).

To the above stochastic differential equation (1.1), we assign the following partial differential operator

$$\mathcal{A} = \frac{1}{2}\left\{a_{11}x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} + a_{22}x_2^2\frac{\partial^2}{\partial x_2^2}\right\} + \mu_1 x_1\frac{\partial}{\partial x_1} + \mu_2 x_2\frac{\partial}{\partial x_2}, \tag{1.2}$$

where $a_{11} = \sigma_{11}^2 + \sigma_{12}^2$, $a_{12} = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}$, $a_{22} = \sigma_{21}^2 + \sigma_{22}^2$, and x_1, x_2 are the initial prices of stocks $\mathbf{S}^1$ and $\mathbf{S}^2$, respectively [**18**]. We then go about solving the Hamilton-Jacobi-Bellman equations

$$\begin{cases} \min\left\{\rho v_0(x_1, x_2) - \mathcal{A}v_0(x_1, x_2),\ v_0(x_1, x_2) - v_1(x_1, x_2) + \beta_{\mathrm{b}}x_1 - \beta_{\mathrm{s}}x_2\right\} = 0, \\ \min\left\{\rho v_1(x_1, x_2) - \mathcal{A}v_1(x_1, x_2),\ v_1(x_1, x_2) - \beta_{\mathrm{s}}x_1 + \beta_{\mathrm{b}}x_2\right\} = 0, \end{cases}$$

where $\rho > 0$ is a given discount factor (the rate at which the value of money decreases over time), β_b and β_s are the transaction fees associated with buying and selling, and v_i are candidate solutions for supremums of reward functions of the form

$$\begin{aligned} J_0(x_1, x_2, (\tau_1, \tau_2)) &= \mathbb{E}\big[e^{-\rho\tau_2}\left(\beta_s X_{\tau_2}^1 - \beta_b X_{\tau_2}^2\right)\mathbb{I}_{\{\tau_2 < \infty\}} - e^{-\rho\tau_1}\left(\beta_b X_{\tau_1}^1 - \beta_s X_{\tau_1}^2\right)\mathbb{I}_{\{\tau_1 < \infty\}}\big], \\ J_1(x_1, x_2, \tau_0) &= \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X_{\tau_0}^1 - \beta_b X_{\tau_0}^2\right)\mathbb{I}_{\{\tau_0 < \infty\}}\right], \end{aligned}$$

for times $\tau_0 \geq 0$ and $\tau_2 \geq \tau_1 \geq 0$. To solve this system, we must find thresholds k_1 and k_2 for buying and selling, as in [**20**].

1.3 Problem Two: Round-Trip Pairs Trading under Geometric Brownian Motions with Reversible Initial Positions

Having previously allowed the initial pairs position to be long or flat, a natural next question to consider is the short side of pairs trading. So, we begin again with the same stochastic differential equation as in (1.1) and the same partial differential operator as in (1.2), but now we allow our intial pairs position to be flat ($i = 0$), long ($i = 1$), or short ($i = -1$). If initially we are short in **Z**, we will buy one share of **Z**, i.e. buy one share of $\mathbf{S}^1$ and sell one share of $\mathbf{S}^2$, at some time $\tau_0 \geq 0$, which will conclude our trading activity. If initially we are long in **Z**, we will sell one share of **Z**, i.e. sell $\mathbf{S}^1$ and buy $\mathbf{S}^2$ at some time $\tau_0 \geq 0$, which will conclude our trading activity. Otherwise, if initially we are flat, we can either go long or short one share in **Z** at some time $\tau_1 \geq 0$. Depending on our activity at time τ_1, we would then either sell $\mathbf{S}^1$ and buy $\mathbf{S}^2$ (if long) or buy $\mathbf{S}^1$ and sell $\mathbf{S}^2$ (if short) at some time $\tau_2 \geq \tau_1$, thus concluding our trading activity. Hence, for $x_1, x_2 > 0$, the HJB equations become

$$\begin{cases} \min\Big\{\rho v_1(x_1,x_2) - \mathcal{A}v_1(x_1,x_2),\ v_1(x_1,x_2) - \beta_{\rm s}x_1 + \beta_{\rm b}x_2\Big\} = 0, \\ \min\Big\{\rho v_{-1}(x_1,x_2) - \mathcal{A}v_{-1}(x_1,x_2),\ v_{-1}(x_1,x_2) + \beta_{\rm b}x_1 - \beta_{\rm s}x_2\Big\} = 0, \\ \min\Big\{\rho v_0(x_1,x_2) - \mathcal{A}v_0(x_1,x_2),\ v_0(x_1,x_2) - v_1(x_1,x_2) + \beta_{\rm b}x_1 - \beta_{\rm s}x_2, \\ \qquad\qquad v_0(x_1,x_2) - v_{-1}(x_1,x_2) - \beta_{\rm s}x_1 + \beta_{\rm b}x_2\Big\} = 0, \end{cases}$$

where ρ, β_s, and β_b are as in Problem One, and v_i are candidate solutions for supremums of reward functions of the form

$$\begin{aligned} J_{-1}(x_1,x_2,\tau_0) =& \mathbb{E}\left[-e^{-\rho\tau_0}\left(\beta_b X^1_{\tau_0} - \beta_s X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right], \\ J_0(x_1,x_2,\tau_1,\tau_2) =& \mathbb{E}\Big[\Big\{e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\Big\}\mathbb{I}_{\{u=1\}} \\ &+ \Big\{e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}} - e^{-\rho\tau_2}\left(\beta_b X^1_{\tau_2} - \beta_s X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\Big\}\mathbb{I}_{\{u=-1\}}\Big], \\ J_1(x_1,x_2,\tau_0) =& \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]. \end{aligned}$$

We seek thresholds k_1, k_2, k_3, and k_4 for buying and selling **Z**. Let k_1 indicate the price at which we will sell one share of **Z** when the net position is flat. Similarly, we will denote by k_2 the threshold for selling one share of **Z** when the net position is long. Next, k_3 will indicate the price at which we will buy one share of **Z** when the net position is short. Finally, the threshold for buying one share of **Z** when the net position is flat will be denoted by k_4. We define the function u as follows.

$$u(x_1, x_2, i) := \begin{cases} -1, & \text{for } i = 0 \text{ and } x_2 \leq x_1 k_1, \\ -1, & \text{for } i = 1 \text{ and } x_2 \leq x_1 k_2, \\ 1, & \text{for } i = -1 \text{ and } x_2 \geq x_1 k_3, \\ 1, & \text{for } i = 0 \text{ and } x_2 \geq x_1 k_4. \end{cases}.$$

Note the dependence of the reward function J_0 on this function u.

After investigating this problem numerically, we were surprised to discover that choosing $k_1 = k_2$ and $k_3 = k_4$ leads to a valid solution to the HJB equations, and we could prove the uniqueness of these thresholds by application of a special implicit function theorem [17]. This leads us to using the term reversible to describe the initial positions due to the apparent symmetry between going one-share long in **Z** and going one-share short in **Z** with the roles of $\mathbf{S}^1$ and $\mathbf{S}^2$ interchanged.

1.4 Problem Three: Pairs Trading under a Mean-Reversion Model with Regime Switching

Another typical model for stock price movements is the mean-reverting (Ornstein-Uhlenbeck) process. In this joint work with Dr. Phong Luu, Dr. Jingzhi Tie, and Dr. Qing Zhang, this model was coupled with a two-state Markov chain, a switching process that reacts to sudden changes in system parameters that might occur when a bear market becomes a bull market and vice versa. To focus on closed-form solutions, we only consider the Markov chain, which we denote $\alpha_t, t = 1, 2$, with an absorbing state. The absorbing state assumption is reasonable, because markets tend to stay in one state for a significant period of time. As before, we consider two stocks $\mathbf{S}^1$ and $\mathbf{S}^2$. Let X_t^1 and X_t^2 denote their prices, respectively, at time t. The corresponding pairs position consists of a long position in $\mathbf{S}^1$ and short position in $\mathbf{S}^2$. For simplicity, we include one share of $\mathbf{S}^1$ and K_0 shares of $\mathbf{S}^2$ (for some $K_0 > 0$) in the pairs position. The price of the position is given by $Z_t = X_t^1 - K_0 X_t^2$, which is a stochastic process governed by

$$\mathrm{d}Z_t = \theta(\alpha_t)[\mu(\alpha_t) - Z_t]\mathrm{d}t + \sigma(\alpha_t)\mathrm{d}W_t, Z_0 = x,$$

where θ, μ, and σ are functions of a two-state Markov chain $\alpha_t \in \{1, 2\}$, and W_t is a standard Brownian motion independent of α_t.

We consider the Markov chain with the absorbing state $\alpha = 2$. In particular, its generator is $Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$, for some $\lambda > 0$. Let $\mathcal{A}_\alpha$, $\alpha = 1, 2$, denote the generator of (Z_t, α_t). Then,

$$\begin{aligned}
\mathcal{A}_1 v(x,1) &= \frac{\sigma_1^2}{2} \cdot \frac{\mathrm{d}^2 v(x,1)}{\mathrm{d}x^2} + \theta_1(\mu_1 - x)\frac{\mathrm{d}v(x,1)}{\mathrm{d}x} + \lambda(v(x,2) - v(x,1)), \\
\mathcal{A}_2 v(x,2) &= \frac{\sigma_2^2}{2} \cdot \frac{\mathrm{d}^2 v(x,2)}{\mathrm{d}x^2} + \theta_2(\mu_2 - x)\frac{\mathrm{d}v(x,2)}{\mathrm{d}x}.
\end{aligned}$$

The associated HJB equations are given by:

$$\begin{aligned}
&\min\Big\{[\rho - \mathcal{A}_1]v_0(x,1),\ v_0(x,1) - v_1(x,1) + x + K\Big\} = 0, \\
&\min\Big\{[\rho - \mathcal{A}_1]v_1(x,1),\ v_1(x,1) - v_0(x,1) - x + K\Big\} = 0, \\
&\min\Big\{[\rho - \mathcal{A}_2]v_0(x,2),\ v_0(x,2) - v_1(x,2) + x + K\Big\} = 0, \\
&\min\Big\{[\rho - \mathcal{A}_2]v_1(x,2),\ v_1(x,2) - v_0(x,2) - x + K\Big\} = 0,
\end{aligned}$$

where $\rho > 0$ is a discount factor and K is a fixed percentage transaction cost. For this problem, one share long in the pairs position $\mathbf{Z}$ means the combination of a one-share long position in $\mathbf{S}^1$ and a K_0-share short position in $\mathbf{S}^2$. Note that the value of the pairs position Z_t may be negative.

Let $0 \le \tau_1^b \le \tau_1^s \le \tau_2^b \le \tau_2^s \le \cdots$ denote a sequence of stopping times. A buying decision is made at τ_n^b and a selling decision at τ_n^s, $n = 1, 2, \ldots$. We consider the case that the net position at any time can be either long (with one share of $\mathbf{Z}$) or flat (no stock position of either $\mathbf{S}^1$ or $\mathbf{S}^2$). Let $i = 0, 1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell $\mathbf{Z}$ before acquiring any future shares. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$.

Likewise, if initially the net position is flat ($i = 0$), then one should start by buying a share of **Z**. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$.

Thus, the v_i above are candidate solutions for supremums of reward functions of the form:

$$J_i(x, \alpha, \Lambda_i) = \begin{cases} \mathbb{E}\left\{ \sum_{n=1}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \infty\}} \right\}, & \text{if } i = 0, \\ \mathbb{E}\left\{ e^{-\rho\tau_1^s}(Z_{\tau_1^s} - K) \right. \\ \quad \left. + \sum_{n=2}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \infty\}} \right\}, & \text{if } i = 1, \end{cases} \tag{1.3}$$

where the term $\mathbb{E} \sum_{n=1}^{\infty} \xi_n$ is interpreted as $\limsup_{N\to\infty} \mathbb{E} \sum_{n=1}^{N} \xi_n$ for given random variables ξ_n.

1.5 Mathematical Preliminaries

This section summarizes a number of established results that are used in this dissertation. These results and their proofs can be found in [1], [5], [18].

1.5.1 Stochastic Processes

Definition 1.5.1 (Stochastic Process). *A stochastic process is a collection of random variables* $\{X(t)\}_{t\in\Lambda}$ *defined on the same probability space* $(\Omega, \mathcal{F}, P)$*, where* Λ *is some indexing set.*

Typically, Λ is either the non-negative integers $\Lambda = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ or the half line $\Lambda = \mathbb{R}_+ = [0, \infty)$. When $\Lambda = \mathbb{Z}_+$, we call such a process a discrete-time stochastic process. When $\Lambda = \mathbb{R}_+$, we call it a continuous-time stochastic process. Also, $X(t)(\omega)$ is sometimes written as $X_t(\omega)$ or $X(t, \omega)$ for notational convenience.

Definition 1.5.2 (Brownian Motion). *A standard one-dimensional Brownian motion is a process* $\{B(t)\}_{t\in\mathbb{R}_+}$ *such that*

(i) $B(0) = 0$*, almost surely*

(ii) $B(t)$ *has independent increments, i.e., if* $0 < t_1 < t_2 < \ldots < t_n$ *then the random variables* $B(t_1) - B(0), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ *are independent.*

(iii) *For all* $s \geq 0, B(t+s) - B(s)$ *is equal in distribution to a normal random variable with mean* 0 *and variance t, i.e., a random variable with density*

$$p(x) = \frac{1}{\sqrt{2\pi t}} e^{\dfrac{-x^2}{2t}}$$

(iv) $t \to B(t)$ *is continuous, almost surely*

Definition 1.5.3 (Itô Diffusion). *A (time-homogeneous) Itô diffusion is a stochastic process* $X_t(\omega) = X(t, \omega) : [0, \infty] \to \mathbb{R}^n$ *satisfying a stochastic differential equation of the form*

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}B(t), \quad t \geq s, \quad X(s) = x \tag{1.4}$$

where $B(t)$ *is* m*-dimensional Brownian motion, and* $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ *satisfy*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|, \quad x, y \in \mathbb{R}^n,$$

i.e., $b(\cdot)$ *and* $\sigma(\cdot)$ *are Lipschitz continuous.*

For some fixed s, we will denote by $X^{s,x}(t)$, for $t \geq s$, the solution to (1.4) with initial condition $X(s) = x$, almost surely. If $s = 0$, we write $X^x(t)$ for $X^{s,x}(t)$.

Let Q^x be the probability law of a given Itô diffusion $\{X(t)\}_{t \in \Lambda}$ when its initial value is $X(0) = x \in \mathbb{R}^n$. The expectation with respect to Q^x is denoted by $\mathbb{E}^x[\cdot]$. Hence, we have

$$\mathbb{E}^x \left[f_1 \left(X \left(t_1 \right) \right) \cdots f_k \left(X \left(t_k \right) \right) \right] = \mathbb{E} \left[f_1 \left(X^x \left(t_1 \right) \right) \cdots f_k \left(X^x \left(t_k \right) \right) \right]$$

for all bounded Borel functions $f_1, \cdots, f_k$ and all times $t_1, \cdots, t_k \geq 0, k = 1, 2, \ldots$.

Theorem 1.5.1 (Markov Property for Itô Diffusions). *Let f be a bounded Borel function from $\mathbb{R}^n \to \mathbb{R}$. Then for $t, h \geq 0$,*

$$\mathbb{E}^x\left[f(X(t+h)) \mid \mathcal{F}_t\right](\omega) = \mathbb{E}^{X(t,\omega)}[f(X(h)].$$

Definition 1.5.4 (Filtration). *A filtration of the σ-algebra $\mathcal{F}$ is an increasing sequence of sub-σ-algebras $\{\mathcal{F}_t\}_{t\in\Lambda}$, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$. A stochastic process $\{X(t)\}_{t\in\Lambda}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\in\Lambda}$ if for each $t \in \Lambda$, $X(t)$ is $\mathcal{F}_t$-measurable.*

Definition 1.5.5 (Stopping Time/Markov Time). *Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}$. A function (random variable) $\tau : \Omega \to [0, \infty]$ is called a stopping time with respect to (adapted to) $\{\mathcal{F}_t\}$ if*

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$$

for all $t \geq 0$.

If $H \subset \mathbb{R}^n$ is any set, we define τ_H, the first exit time from H, as follows

$$\tau_H = \inf\left\{t > 0 : X_t \notin H\right\}.$$

Note that τ_H is a stopping time for any Borel set H.

Definition 1.5.6. *Suppose τ is a stopping time adapted to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, and let $\mathcal{F}_\infty$ denote the smallest σ-algebra containing the whole collection $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$. Define the σ-algebra $\mathcal{F}_\tau$ to be the σ-algebra generated by all sets of the form $B \cap \{\tau \leq t\}$ where $B \in \mathcal{F}_\infty$ and $t \in \mathbb{R}_+$.*

Theorem 1.5.2 (Srong Markov Property for Itô Diffusions). *Let f be a bounded Borel function from $\mathbb{R}^n$ to $\mathbb{R}$ and τ be a stopping time with respect to $\{\mathcal{F}_\tau\}$, $\tau < \infty$, almost surely. Then for all $h \geq 0$,*

$$\mathbb{E}^x\left[f(X(\tau+h)) \mid \mathcal{F}_\tau\right] = \mathbb{E}^{X(\tau)}\left[f(X(h))\right].$$

Definition 1.5.7 (Generator of an Itô Diffusion). *Let $\{X(t)\}$ be a (time-homogeneous) Itô diffusion in $\mathbb{R}^n$. The (infinitesimal) generator $\mathcal{A}$ of $X(t)$ is defined by*

$$\mathcal{A}f(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}^x[f(X(t))] - f(x)}{t}, x \in \mathbb{R}^n.$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_{\mathcal{A}}(x)$, while $\mathcal{D}_{\mathcal{A}}$ denotes the set of functions for which the limit exists for all $x \in \mathbb{R}^n$.

Definition 1.5.8 (Generator of an Itô Diffusion). *Let $X(t)$ be the Itô diffusion satisfying*

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}B(t).$$

If $f \in C_0^2(\mathbb{R}^n)$, then $f \in \mathcal{D}_A$ and

$$\mathcal{A}f(x) = \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\left(\sigma\sigma^T\right)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Theorem 1.5.3 (Dynkin's Formula). *Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time with $\mathbb{E}[\tau] < \infty$. Then*

$$\mathbb{E}^x[f(X(\tau))] = f(x) + \mathbb{E}^x\left[\int_0^\tau \mathcal{A}f(X(s))\mathrm{d}s\right].$$

1.5.2 Martingales

Definition 1.5.9 (Martingale/Martingale Difference). *An n-dimensional stochastic process $\{X(t)\}_{t\in\mathbb{R}_+}$ is said to be a martingale on $(\Omega, \mathcal{F}, P)$ with respect to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ if*

(i) $X(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$,

(ii) $\mathbb{E}[\|X(t)\|] < \infty$ for all t, and

(iii) $\mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s)$ with probability 1 for all $t \geq s$.

The sequence $\{X(t)\}_{t\in\mathbb{Z}_+}$ is called a martingale difference sequence if the condition (iii) above is replaced by $\mathbb{E}[X(t) \mid \mathcal{F}_{t-1}] = 0$ with probability 1.

Definition 1.5.10 (The Martingale Problem). *If $\mathrm{d}X(t) = b(X(t)) + \sigma(X(t))\mathrm{d}B(t)$ is an Itô diffusion in $\mathbb{R}^n$ with generator $\mathcal{A}$, and if $f \in C_0^2(\mathbb{R}^n)$ and $X(0) = x$, almost surely, then*

$$f(X(t)) = f(x) + \int_0^t \mathcal{A}f(X(s))\mathrm{d}s + \int_0^t \nabla f^T(X(s))\sigma(X(s))\mathrm{d}B(s)$$

Define $M_f(t) = f(X(t)) - \int_0^t \mathcal{A}f(X(s))\mathrm{d}s$.
We say that $X(t)$ *solves the martingale problem for generator* $\mathcal{A}$ *if* $M_f(t)$ *is a martingale for each* f *in* $C_0^2(\mathbb{R}^n)$.

Theorem 1.5.4. $M_f(t)$ *is a* $\mathcal{F}_t$*-martingale, where* $\mathcal{F}_t = \sigma(\{X(s), s \leq t\})$.

Chapter 2

Round-Trip Pairs Trading under Geometric Brownian Motions

2.1 Introduction

This chapter is concerned with an optimal strategy for simultaneously trading a pair of stocks. The purpose of pairs trading is to hedge the risk associated with buying and holding shares of one stock by selling shares of a related stock. The idea of pairs trading is to track the prices of two stocks that follow roughly the same trajectory over time. A pairs trade is triggered by the divergence of their prices and consists of a pair of positions to short the strong stock and to long the weak one. Such a strategy bets on the reversal of their price strengths. Pairs trading, which was pioneered by quantitative researchers at brokerage firms in the 1980s, is beneficial, because it can be profitable under any market circumstances [**10**]. A round-trip trading strategy refers to opening and then closing a pair of security positions.

Some typical pairs-trading models assume the difference of the stock prices satisfies a mean-reversion equation. However, we consider the optimal pairs-trading problem by allowing the stock prices to follow general geometric Brownian motions as in [**20**]. One benefit of this model is that it does not specificy any relationship between the pairs of stocks or require them to satisify any measure of correlation, thus allowing for greater possibilities in the choice of pairs. The Brownian motion, whose sample path is a random walk, encodes the assumption that it is impossible to accurately predict the change in the price of a stock from day to day. Our objective is to trade the pairs over time to maximize an overall return with a fixed commission cost for each transaction. In this chapter, we allow the initial pairs position to be either long or flat. The optimal policy is characterized by threshold curves obtained by solving the associated Hamilton-Jacobi-Bellman (HJB) equations.

2.2 Problem Formulation

Consider two stocks, $\mathbf{S}^1$ and $\mathbf{S}^2$. Let $\{X_t^1, t \geq 0\}$ denote the prices of the stock $\mathbf{S}^1$, and let $\{X_t^2, t \geq 0\}$ denote the prices of the stock $\mathbf{S}^2$. They satisfy the following stochastic differential equation:

$$\mathrm{d}\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix} \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \mathrm{d}\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \right], \tag{2.1}$$

where μ_i, $i = 1, 2$ are the return rates, σ_{ij}, $i, j = 1, 2$ are the volatility constants, and (W_t^1, W_t^2) is a 2-dimensional standard Brownian motion.

In this chapter, we consider a round-trip pairs trading strategy. We assume the pairs position, which we will denote $\mathbf{Z}$, consists of a one-share long position in stock $\mathbf{S}^1$ and a one-share short position in stock $\mathbf{S}^2$. We consider the case that the net position may initially be long (with one share of $\mathbf{Z}$) or flat (with no stock holdings of either $\mathbf{S}^1$ or $\mathbf{S}^2$). Let $i = 0, 1$ denote the initial net positions of long and flat, respectively. If initially we are long ($i = 1$), we will close the pairs position $\mathbf{Z}$ at some time $\tau_0 \geq 0$ and conclude our trading activity. Otherwise, if initially we are flat ($i = 0$), we will first obtain one share of $\mathbf{Z}$ at some time $\tau_1 \geq 0$, and then close pairs position $\mathbf{Z}$ at some time $\tau_2 \geq \tau_1$, thus concluding our trading activity.

Let K denote the fixed percentage of transaction costs associate with buying or selling of stocks and $\rho > 0$ be a discount factor. To further simplify the notation, we set $\beta_b = 1 + K$ and $\beta_s = 1 - K$. Then given the initial state (x_1, x_2), the initial net position $i = 0, 1$, and the decision sequences $\Lambda_1 = (\tau_0)$ and $\Lambda_0 = (\tau_1, \tau_2)$, the resulting reward functions are

$$\begin{aligned} J_0(x_1, x_2, \Lambda_0) &= \mathbb{E}\big[e^{-\rho\tau_2}\left(\beta_s X_{\tau_2}^1 - \beta_b X_{\tau_2}^2\right)\mathbb{I}_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X_{\tau_1}^1 - \beta_s X_{\tau_1}^2\right)\mathbb{I}_{\{\tau_1<\infty\}}\big], \\ J_1(x_1, x_2, \Lambda_1) &= \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X_{\tau_0}^1 - \beta_b X_{\tau_0}^2\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]. \end{aligned}$$

Let $V_0(x_1, x_2) = \sup_{\Lambda_0} J_0(x_1, x_2, \Lambda_0)$ and $V_1(x_1, x_2) = \sup_{\Lambda_1} J_1(x_1, x_2, \Lambda_1)$ be the associated value functions.

2.3 Properties of the Value Functions

In this section, we establish basic properties of the value functions.

Lemma 1. *For all x_1, $x_2 > 0$, we have*

$$0 \leq V_0(x_1, x_2) \leq 2x_1 + 2x_2,$$
$$\beta_s x_1 - \beta_b x_2 \leq V_1(x_1, x_2) \leq x_1.$$

Proof. Note that for all $x_1, x_2 > 0, V_1(x_1, x_2) \geq J_1(x_1, x_2, \Lambda_1) = \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]$. In particular,

$$V_1(x_1, x_2) \geq J_1(x_1, x_2, 0) = \beta_s x_1 - \beta_b x_2.$$

For all $\tau_0 \geq 0$, $J_1(x_1, x_2, \Lambda_1)$

$$\begin{aligned}
&= \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\leq \mathbb{E}\left[e^{-\rho\tau_0}\left(X^1_{\tau_0} - X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&= x_1 + \mathbb{E}\left[\int_0^{\tau_0}(-\rho+\mu_1)\,e^{-\rho t}X^1_t \mathrm{d}t\,\mathbb{I}_{\{\tau_0<\infty\}}\right] - x_2 - \mathbb{E}\left[\int_0^{\tau_0}(-\rho+\mu_2)\,e^{-\rho t}X^2_t \mathrm{d}t\,\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\leq x_1 - x_2 - \mathbb{E}\left[\int_0^{\tau_0}(-\rho+\mu_2)\,e^{-\rho t}X^2_t \mathrm{d}t\,\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\leq x_1 - x_2 + \mathbb{E}\left[\int_0^{\infty}(\rho-\mu_2)\,e^{-\rho t}X^2_t \mathrm{d}t\right] \\
&= x_1.
\end{aligned}$$

Also, for all $x_1, x_2 > 0$,

$$\begin{aligned}
V_0(x_1, x_2) &\geq J_0(x_1, x_2, \Lambda_0) \\
&= \mathbb{E}\Big[e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}.
\end{aligned}$$

Clearly, $V_0(x_2, x_2) \geq 0$ by definition and taking $\tau_1 = \infty$. Now, for all $0 \leq \tau_1 \leq \tau_2$,

$$\begin{aligned}
&J_0(x_1, x_2, \Lambda_0) \\
&\quad = \mathbb{E}\big[e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\big] - \mathbb{E}\big[e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\big] \\
&\quad \leq \mathbb{E}\big[e^{-\rho\tau_2} X^1_{\tau_2}\mathbb{I}_{\{\tau_2<\infty\}}\big] - \mathbb{E}\big[e^{-\rho\tau_2} X^2_{\tau_2}\mathbb{I}_{\{\tau_2<\infty\}}\big] - \mathbb{E}\big[e^{-\rho\tau_1} X^1_{\tau_1}\mathbb{I}_{\{\tau_1<\infty\}}\big] + \mathbb{E}\big[e^{-\rho\tau_1} X^2_{\tau_1}\mathbb{I}_{\{\tau_1<\infty\}}\big] \\
&\quad \leq x_1 - \mathbb{E}\left[x_2\mathbb{I}_{\{\tau_2<\infty\}}\right] + \mathbb{E}\left[\int_0^{\tau_2}(\rho-\mu_2)\, e^{-\rho t} X^2_t \mathrm{d}t\, \mathbb{I}_{\{\tau_2<\infty\}}\right] \\
&\qquad + x_2 - \mathbb{E}\left[x_1\mathbb{I}_{\{\tau_1<\infty\}}\right] + \mathbb{E}\left[\int_0^{\tau_1}(\rho-\mu_1)\, e^{-\rho t} X^1_t \mathrm{d}t\, \mathbb{I}_{\{\tau_1<\infty\}}\right].
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_1}(\rho-\mu_1)\, e^{-\rho t} X^1_t \mathrm{d}t\, \mathbb{I}_{\{\tau_1<\infty\}}\right] &\leq \mathbb{E}\left[\int_0^{\infty}(\rho-\mu_1)\, e^{-\rho t} X^1_t \mathrm{d}t\right] \\
&= (\rho-\mu_1)\int_0^{\infty} e^{-\rho t} x_1 e^{\mu_1 t}\mathrm{d}t \\
&= x_1.
\end{aligned}$$

Similarly,

$$\mathbb{E}\left[\int_0^{\tau_2}(\rho-\mu_2)\, e^{-\rho t} X^2_t \mathrm{d}t\, \mathbb{I}_{\{\tau_2<\infty\}}\right] \leq x_2.$$

Thus, for all Λ_0, we have $J_0(x_1, x_2, \Lambda_0) \leq 2x_1 + 2x_2$. □

2.4 HJB Equations

In this section, we study the associated HJB equations. To the above stochastic differential equation (2.1), we assign the following partial differential operator. Let

$$\mathcal{A} = \frac{1}{2}\left\{a_{11}x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + a_{22}x_2^2\frac{\partial^2}{\partial x_2^2}\right\} + \mu_1 x_1\frac{\partial}{\partial x_1} + \mu_2 x_2\frac{\partial}{\partial x_2}, \tag{2.2}$$

where $a_{11} = \sigma_{11}^2 + \sigma_{12}^2,\ a_{12} = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}$, and $a_{22} = \sigma_{21}^2 + \sigma_{22}^2$ **[18]**. The associated HJB equations have the form: For $x_1, x_2 > 0$,

$$\begin{cases} \min\Big\{\rho v_0(x_1,x_2) - \mathcal{A}v_0(x_1,x_2),\ v_0(x_1,x_2) - v_1(x_1,x_2) + \beta_{\mathrm{b}}x_1 - \beta_{\mathrm{s}}x_2\Big\} = 0, \\ \min\Big\{\rho v_1(x_1,x_2) - \mathcal{A}v_1(x_1,x_2),\ v_1(x_1,x_2) - \beta_{\mathrm{s}}x_1 + \beta_{\mathrm{b}}x_2\Big\} = 0. \end{cases}$$

To solve the above HJB equations, we first convert them into single variable equations. Let $y = x_2/x_1$ and $v_i(x_1,x_2) = x_1 w_i(x_2/x_1)$, for some function $w_i(y)$ and $i = 0, 1$. Then,

$$\begin{aligned} \frac{\partial v_i}{\partial x_1} &= \frac{\partial}{\partial x_1}\left[x_1 w_i\left(\frac{x_2}{x_1}\right)\right] = x_1\frac{\partial}{\partial x_1}\left[w_i\left(\frac{x_2}{x_1}\right)\right] + w_i\left(\frac{x_2}{x_1}\right)\frac{\partial}{\partial x_1}[x_1] \\ &= x_1 w_i'\left(\frac{x_2}{x_1}\right)\cdot\left(-\frac{x_2}{{x_1}^2}\right) + w_i\left(\frac{x_2}{x_1}\right) \\ &= w_i(y) - y w_i'(y), \end{aligned}$$

$$\begin{aligned} \frac{\partial v_i}{\partial x_2} &= \frac{\partial}{\partial x_2}\left[x_1 w_i\left(\frac{x_2}{x_1}\right)\right] = x_1\frac{\partial}{\partial x_2}\left[w_i\left(\frac{x_2}{x_1}\right)\right] \\ &= x_1 w_i'\left(\frac{x_2}{x_1}\right)\cdot\left(\frac{1}{x_1}\right) \\ &= w_i'(y), \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 v_i}{\partial {x_1}^2} &= \frac{\partial}{\partial x_1}\left[w_i\left(\frac{x_2}{x_1}\right) - \left(\frac{x_2}{x_1}\right)\cdot w_i'\left(\frac{x_2}{x_1}\right)\right] \\ &= \frac{\partial}{\partial x_1}\left[w_i\left(\frac{x_2}{x_1}\right)\right] - \frac{\partial}{\partial x_1}\left[\left(\frac{x_2}{x_1}\right)\cdot w_i'\left(\frac{x_2}{x_1}\right)\right] \\ &= w_i'\left(\frac{x_2}{x_1}\right)\left(-\frac{x_2}{{x_1}^2}\right) - \left[\left(\frac{x_2}{x_1}\right) w_i''\left(\frac{x_2}{x_1}\right)\left(-\frac{x_2}{{x_1}^2}\right) + w_i'\left(\frac{x_2}{x_1}\right)\left(-\frac{x_2}{{x_1}^2}\right)\right] \\ &= \frac{y^2 w_i''(y)}{x_1} + \frac{y w_i'(y)}{x_1} - \frac{y w_i'(y)}{x_1} \\ &= \frac{y^2 w_i''(y)}{x_1}, \end{aligned}$$

$$\frac{\partial^2 v_i}{\partial {x_2}^2} = \frac{\partial}{\partial x_2}\left[w_i'\left(\frac{x_2}{x_1}\right)\right] = w_i''\left(\frac{x_2}{x_1}\right)\cdot\left(\frac{1}{x_1}\right)$$

$$= \frac{w_i''(y)}{x_1},$$

$$\frac{\partial^2 v_i}{\partial x_1 x_2} = \frac{\partial}{\partial x_1}\left[w_i'\left(\frac{x_2}{x_1}\right)\right] = w_i''\left(\frac{x_2}{x_1}\right)\cdot\left(-\frac{x_2}{{x_1}^2}\right)$$
$$= -\frac{y w_i''(y)}{x_1}.$$

Write $\mathcal{A}v_i$ in terms of w_i to obtain

$$\begin{aligned}
\mathcal{A}v_i &= \frac{1}{2}\left\{a_{11}{x_1}^2\left(\frac{y^2 w_i''(y)}{x_1}\right) + 2a_{12}x_1x_2\left(-\frac{y w_i''(y)}{x_1}\right) + a_{22}{x_2}^2\left(\frac{w_i''(y)}{x_1}\right)\right\} \\
&\quad + \mu_1 x_1\left(w_i(y) - y w_i'(y)\right) + \mu_2 x_2\left(w_i'(y)\right) \\
&= \frac{1}{2}a_{11}x_1y^2w_i''(y) - a_{12}x_1y^2w_i''(y) + \frac{1}{2}a_{22}x_1y^2w_i''(y) + \mu_1x_1w_i(y) + \mu_2x_1yw_i'(y) \\
&\quad - \mu_1x_1yw_i'(y) \\
&= x_1\left\{\frac{1}{2}\left[a_{11} - 2a_{12} + a_{22}\right]y^2w_i''(y) + (\mu_2 - \mu_1)yw_i'(y) + \mu_1w_i(y)\right\}.
\end{aligned}$$

Let $\mathcal{L}w_i(y) = \lambda y^2 w_i''(y) + (\mu_2 - \mu_1)yw_i'(y) + \mu_1 w_i(y)$, where $\lambda = \dfrac{a_{11} - 2a_{12} + a_{22}}{2}$.
So $\mathcal{A}v_i = x_1\mathcal{L}w_i$. Note that $\lambda \geq 0$ since

$$\begin{aligned}
\lambda &= \frac{1}{2}\left[\sigma_{11}^2 + \sigma_{12}^2 - 2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) + \sigma_{21}^2 + \sigma_{22}^2\right] \\
&= \frac{1}{2}\left[\sigma_{11}^2 - 2\sigma_{11}\sigma_{21} + \sigma_{21}^2 + \sigma_{12}^2 - 2\sigma_{12}\sigma_{22} + \sigma_{22}^2\right] \\
&= \frac{1}{2}\left[(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2\right].
\end{aligned}$$

Here we only consider the case when $\lambda \neq 0$. If $\lambda = 0$, the problem reduces to a first order case and can be treated accordingly. The HJB equations can be given in terms of y and w_i as follows:

$$\begin{cases}
\min\left\{x_1\left(\rho w_0(y) - \mathcal{L}w_0(y)\right),\ x_1(w_0(y) - w_1(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}}y)\right\} = 0, \\
\min\left\{x_1\left(\rho w_1(y) - \mathcal{L}w_1(y)\right),\ x_1(w_1(y) - \beta_{\mathrm{s}} + \beta_{\mathrm{b}}y)\right\} = 0,
\end{cases}$$

or equivalently,

$$\begin{cases} \min\left\{(\rho-\mathcal{L})w_0(y),\ w_0(y)-w_1(y)+\beta_{\rm b}-\beta_{\rm s}y\right\}=0, \\ \min\left\{(\rho-\mathcal{L})w_1(y),\ w_1(y)-\beta_{\rm s}+\beta_{\rm b}y\right\}=0. \end{cases} \tag{2.3}$$

To solve (2.3), we first consider the equations $(\rho-\mathcal{L})w_i(y)=0, i=0,1,$ which can be rewritten as

$$-\lambda y^2 w_i''(y)-(\mu_2-\mu_1)yw_i'(y)+(\rho-\mu_1)w_i(y)=0.$$

Clearly, these are the Euler equations and their solutions are of the form y^δ, for some δ. Substitute this into the equation $(\rho-\mathcal{L})w_i=0$ to obtain

$$\begin{aligned} & -\lambda y^2[\delta(\delta-1)y^{\delta-2}]-(\mu_2-\mu_1)y[\delta y^{\delta-1}]+(\rho-\mu_1)y^\delta=0 \\ \Longrightarrow\ & -\lambda\delta(\delta-1)y^\delta-(\mu_2-\mu_1)\delta y^\delta+(\rho-\mu_1)y^\delta=0 \\ \Longrightarrow\ & \left[-\lambda\delta^2+\lambda\delta+(\mu_1-\mu_2)\delta+(\rho-\mu_1)\right]y^\delta=0 \\ \Longrightarrow\ & \left[\delta^2-\delta-\left(\frac{\mu_1-\mu_2}{\lambda}\right)\delta-\frac{\rho-\mu_1}{\lambda}\right]y^\delta=0 \\ \Longrightarrow\ & \left[\delta^2-\delta\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)-\frac{\rho-\mu_1}{\lambda}\right]y^\delta=0. \end{aligned}$$

Then since $y^\delta \neq 0$, it must be that

$$\delta^2-\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)\delta-\frac{\rho-\mu_1}{\lambda}=0.$$

This equation has two roots, δ_1 and δ_2, given by

$$\begin{aligned} \delta_1 &= \frac{1}{2}\left(1+\frac{\mu_1-\mu_2}{\lambda}+\sqrt{\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)^2+\frac{4\rho-4\mu_1}{\lambda}}\right), \\ \delta_2 &= \frac{1}{2}\left(1+\frac{\mu_1-\mu_2}{\lambda}-\sqrt{\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)^2+\frac{4\rho-4\mu_1}{\lambda}}\right). \end{aligned} \tag{2.4}$$

These roots are both real since we assume $\rho > \mu_1$. We also assume $\rho > \mu_2$, so

$$\begin{aligned}
&\frac{4\rho - 2\mu_1 - 2\mu_2}{\lambda} > \frac{4\mu_2 - 2\mu_1 - 2\mu_2}{\lambda} = \frac{2\mu_2 - 2\mu_1}{\lambda} \\
&\implies 1 + \left(\frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{2(\mu_1 - \mu_2)}{\lambda} + \frac{4\rho - 4\mu_1}{\lambda} > 1 + \left(\frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{2(\mu_2 - \mu_1)}{\lambda} \\
&\implies \left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda} > \left(1 + \frac{\mu_2 - \mu_1}{\lambda}\right)^2 \\
&\implies \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}} > \sqrt{\left(1 + \frac{\mu_2 - \mu_1}{\lambda}\right)^2} \\
&\implies \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}} > \left|1 + \frac{\mu_2 - \mu_1}{\lambda}\right| = \left|1 - \frac{\mu_1 - \mu_2}{\lambda}\right| \geq 1 - \frac{\mu_1 - \mu_2}{\lambda} \\
&\implies 1 + \frac{\mu_1 - \mu_2}{\lambda} + \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}} > 2 \\
&\implies \frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\lambda} + \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}}\right) > 1 \\
&\implies \delta_1 > 1.
\end{aligned}$$

Also, since

$$\begin{aligned}
1 + \frac{\mu_1 - \mu_2}{\lambda} \leq \left|1 + \frac{\mu_1 - \mu_2}{\lambda}\right| &= \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2} \\
&< \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}},
\end{aligned}$$

we must have $\delta_2 < 0$.

We conclude that the general solution of $(\rho - \mathcal{L})w_i(y) = 0$ should be of the form: $w_i(y) = c_{i1}y^{\delta_1} + c_{i2}y^{\delta_2}$, for some constants c_{i1} and c_{i2}, $i = 0, 1$. Note that as $y \to 0$, $y^{\delta_2} \to \infty$, and as $y \to \infty$, $y^{\delta_1} \to \infty$. Also note the following identities in δ_1 and δ_2:

$$-\delta_1\delta_2 = -\frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\lambda} + \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}}\right)$$

$$
\begin{aligned}
&\cdot \frac{1}{2}\left(1+\frac{\mu_1-\mu_2}{\lambda}-\sqrt{\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)^2+\frac{4\rho-4\mu_1}{\lambda}}\right)\\
&= -\frac{1}{4}\left[\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)^2-\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)^2-\frac{4\rho-4\mu_1}{\lambda}\right]\\
&= \frac{\rho-\mu_1}{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
\delta_1+\delta_2 &= \frac{1}{2}\left(1+\frac{\mu_1-\mu_2}{\lambda}+\sqrt{\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)^2+\frac{4\rho-4\mu_1}{\lambda}}\right)\\
&\quad+\frac{1}{2}\left(1+\frac{\mu_1-\mu_2}{\lambda}-\sqrt{\left(1+\frac{\mu_1-\mu_2}{\lambda}\right)^2+\frac{4\rho-4\mu_1}{\lambda}}\right)\\
&= 1+\frac{\mu_1-\mu_2}{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
(\delta_1-1)(1-\delta_2) &= \delta_1-\delta_1\delta_2-1+\delta_2 = \delta_1+\delta_2-1-\delta_1\delta_2\\
&= 1+\frac{\mu_1-\mu_2}{\lambda}-1+\frac{\rho-\mu_1}{\lambda}\\
&= \frac{\rho-\mu_2}{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
\frac{-\delta_1\delta_2}{(\delta_1-1)(1-\delta_2)} &= \frac{\rho-\mu_1}{\lambda}\cdot\frac{\lambda}{\rho-\mu_2}\\
&= \frac{\rho-\mu_1}{\rho-\mu_2}.
\end{aligned}
$$

Now, the second part of the HJB equation

$$
\min\Big\{(\rho-\mathcal{L})w_1(y),\ w_1(y)-\beta_{\mathrm{s}}+\beta_{\mathrm{b}}y\Big\}=0
$$

is independent of w_0 and can be solved first. We must find thresholds k_1 and k_2 for buying and selling, as in [**20**].

First, we need to find k_1 so that on the interval $(0,k_1]$, $w_1(y) = \beta_{\mathrm{s}}-\beta_{\mathrm{b}}y$, and on the interval (k_1,∞), $w_1(y)=C_2y^{\delta_2}$. Then the smooth-fit conditions determine k_1 and C_2.

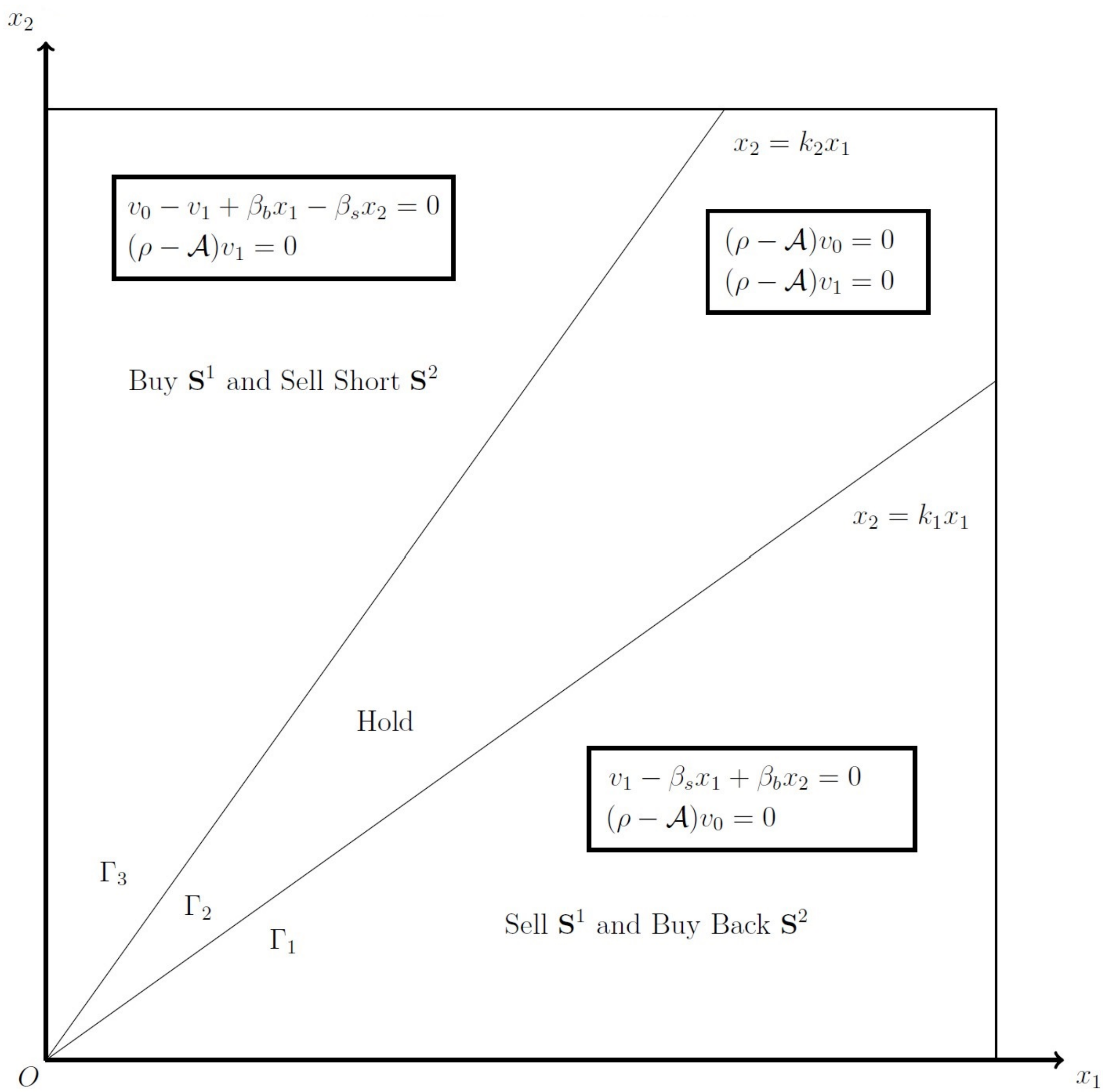

Figure 2.1: Thresholds for buying and selling regions

Necessarily, the continuity of w_1 and its first order derivative at $y = k_1$ imply

$$\beta_{\mathrm{s}} - \beta_{\mathrm{b}} k_1 = C_2 k_1^{\delta_2} \quad \text{and} \quad -\beta_{\mathrm{b}} = C_2 \delta_2 k_1^{\delta_2 - 1}.$$

From this system of equations, we can see

$$\begin{aligned}
-\frac{\beta_{\mathrm{b}}}{\delta_2} \cdot k_1 &= C_2 {k_1}^{\delta_2} = \beta_{\mathrm{s}} - \beta_{\mathrm{b}} k_1 \\
\implies \left(-\frac{\beta_{\mathrm{b}}}{\delta_2} + \frac{\beta_{\mathrm{b}}\delta_2}{\delta_2} \right) k_1 &= \beta_{\mathrm{s}} \\
\implies \frac{-1+\delta_2}{\delta_2} \cdot k_1 &= \frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \\
\implies k_1 &= \frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \cdot \frac{-\delta_2}{1-\delta_2}.
\end{aligned}$$

Also,

$$\begin{aligned}
C_2 &= \frac{\beta_{\mathrm{b}}}{-\delta_2} {k_1}^{1-\delta_2} \\
&= \frac{\beta_{\mathrm{b}}}{-\delta_2} \left(\frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \cdot \frac{-\delta_2}{1-\delta_2} \right)^{1-\delta_2} \\
&= \frac{{\beta_{\mathrm{s}}}^{1-\delta_2}}{(1-\delta_2)^{1-\delta_2}} \cdot \frac{\beta_{\mathrm{b}}}{-\delta_2} \cdot \frac{{\beta_{\mathrm{b}}}^{\delta_2 - 1}}{(-\delta_2)^{\delta_2 - 1}} \\
&= \left(\frac{\beta_{\mathrm{s}}}{1-\delta_2} \right)^{1-\delta_2} \left(\frac{\beta_{\mathrm{b}}}{-\delta_2} \right)^{\delta_2}.
\end{aligned}$$

We obtain the function

$$w_1(y) = \begin{cases} \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y, & \text{for } y \le k_1, \\ C_2 y^{\delta_2}, & \text{for } y > k_1, \end{cases}$$

with k_1 and C_2 given above. Next we need to solve the first part of HJB equation:

$$\min \Big\{ (\rho - \mathcal{L}) w_0(y),\ w_0(y) - w_1(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y \Big\} = 0.$$

We need to find k_2 so that on the interval $(0, k_2)$, $w_0(y) = C_1 y^{\delta_1}$, and on the interval $[k_2, \infty)$, $w_0(y) = w_1(y) - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y = C_2 y^{\delta_2} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y$. Then the continuity of w_0 and its first order derivative at $y = k_2$ yield

$$C_1 k_2^{\delta_1} = C_2 k_2^{\delta_2} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2 \quad \text{and} \quad C_1 \delta_1 k_2^{\delta_1 - 1} = C_2 \delta_2 k_2^{\delta_2 - 1} + \beta_{\mathrm{s}}.$$

Take the ratio of the above two equations and get

$$\frac{k_2}{\delta_1} = \frac{C_2 k_2^{\delta_2} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2}{C_2 \delta_2 k_2^{\delta_2 - 1} + \beta_{\mathrm{s}}}.$$

This implies

$$\begin{aligned}
& k_2[C_2 \delta_2 k_2^{\delta_2 - 1} + \beta_{\mathrm{s}}] = \delta_1 [C_2 k_2^{\delta_2} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} k_2] \\
\Longrightarrow\ & \delta_1 C_2 k_2^{\delta_2} - \delta_2 C_2 k_2^{\delta_2} + \delta_1 \beta_{\mathrm{s}} k_2 - \beta_{\mathrm{s}} k_2 - \beta_{\mathrm{b}} \delta_1 = 0 \\
\Longrightarrow\ & C_2(\delta_1 - \delta_2) k_2^{\delta_2} + \beta_{\mathrm{s}}(\delta_1 - 1) k_2 - \beta_{\mathrm{b}} \delta_1 = 0.
\end{aligned}$$

We get an equation of k_2:

$$f(k_2) := C_2(\delta_1 - \delta_2) k_2^{\delta_2} + \beta_{\mathrm{s}}(\delta_1 - 1) k_2 - \beta_{\mathrm{b}} \delta_1 = 0.$$

Consider

$$f(y) := C_2(\delta_1 - \delta_2) y^{\delta_2} + \beta_{\mathrm{s}}(\delta_1 - 1) y - \beta_{\mathrm{b}} \delta_1.$$

Note that as $y \to \infty$, $f(y) \to \beta_{\mathrm{s}}(\delta_1 - 1)y - \beta_{\mathrm{b}}\delta_1$, since $\delta_2 < 0$. That is, as $y \to \infty$, $f(y) \to \infty$, since $\beta_{\mathrm{s}} > 0$, $\delta_1 - 1 > 0$. Also, as $y \to 0^+$, $f(y) \to C_2(\delta_1 - \delta_2) y^{\delta_2} - \beta_{\mathrm{b}}\delta_1$. That is, as $y \to 0^+$, $f(y) \to \infty$, since $C_2 > 0$, $\delta_1 - \delta_2 > 0$, and $\delta_2 < 0$. Now,

$$\begin{aligned}
f'(y) &= C_2 \delta_2 (\delta_1 - \delta_2) y^{\delta_2 - 1} + \beta_{\mathrm{s}}(\delta_1 - 1) \\
f''(y) &= C_2 \delta_2 (\delta_2 - 1)(\delta_1 - \delta_2) y^{\delta_2 - 2} = C_2(-\delta_2)(1 - \delta_2)(\delta_1 - \delta_2) y^{\delta_2 - 2}.
\end{aligned}$$

Note then that $f''(y) > 0$ for all $y > 0$ since $C_2 > 0$, $(-\delta_2) > 0$, $(1 - \delta_2) > 0$, and $(\delta_1 - \delta_2) > 0$. Hence f is convex for all $y > 0$. Then

$$\begin{aligned} f'(y) = 0 &\iff C_2\delta_2(\delta_1 - \delta_2)y^{\delta_2 - 1} + \beta_{\mathrm{s}}(\delta_1 - 1) = 0 \\ &\iff y^{\delta_2 - 1} = \frac{\beta_{\mathrm{s}}(\delta_1 - 1)}{C_2(-\delta_2)(\delta_1 - \delta_2)} \\ &\iff y = \left[\frac{\beta_{\mathrm{s}}(\delta_1 - 1)}{C_2(-\delta_2)(\delta_1 - \delta_2)}\right]^{\frac{1}{\delta_2 - 1}} \\ &\iff y = \left[\frac{C_2(-\delta_2)(\delta_1 - \delta_2)}{\beta_{\mathrm{s}}(\delta_1 - 1)}\right]^{\frac{1}{1-\delta_2}} > 0. \end{aligned}$$

Hence f attains its global minimum at $y_c = \left[\frac{\beta_{\mathrm{s}}(\delta_1 - 1)}{C_2(-\delta_2)(\delta_1 - \delta_2)}\right]^{\frac{1}{\delta_2 - 1}}$. We will show that $f(y) = 0$ has two solutions and take the larger one to be k_2. Since we already know C_2, once we find k_2, we can express C_1 using the relationship above:

$$\begin{aligned} C_1 = \frac{C_2\delta_2 k_2^{\delta_2 - 1} + \beta_{\mathrm{s}}}{\delta_1 k_2^{\delta_1 - 1}} &= \left(\frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \cdot \frac{-\delta_2}{1 - \delta_2}\right)^{1-\delta_2} \frac{\beta_{\mathrm{b}}}{-\delta_2} \frac{\delta_2 k_2^{\delta_2 - 1}}{\delta_1 k_2^{\delta_1 - 1}} + \frac{\beta_{\mathrm{s}}}{\delta_1 k_2^{\delta_1 - 1}} \\ &= -\left(\frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}}\right)^{-\delta_2}\left(\frac{-\delta_2}{1 - \delta_2}\right)^{1-\delta_2}\left(\frac{\beta_{\mathrm{s}}}{\delta_1}\right)\frac{k_2^{\delta_2 - 1}}{k_2^{\delta_1 - 1}} + \frac{\beta_{\mathrm{s}}}{\delta_1 k_2^{\delta_1 - 1}} \\ &= \left[1 - \left(\frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}}\right)^{-\delta_2}\left(\frac{-\delta_2}{1 - \delta_2}\right)^{1-\delta_2} k_2^{\delta_2 - 1}\right]\left(\frac{\beta_{\mathrm{s}}}{\delta_1}\right) k_2^{1-\delta_1}. \end{aligned}$$

We show that $f(y_c) < 0$, thus implying the existence of k_2. We compute $f(y_c)$ as follows:

$$\begin{aligned} f(y_c) &= C_2(\delta_1 - \delta_2)y_c^{\delta_2} + \beta_{\mathrm{s}}(\delta_1 - 1)y_c - \beta_{\mathrm{b}}\delta_1 \\ &= C_2(\delta_1 - \delta_2)\left[\frac{\beta_{\mathrm{s}}(\delta_1 - 1)}{-\delta_2 C_2(\delta_1 - \delta_2)}\right]^{\frac{\delta_2}{\delta_2 - 1}} + \beta_{\mathrm{s}}(\delta_1 - 1)\left[\frac{\beta_{\mathrm{s}}(\delta_1 - 1)}{-\delta_2 C_2(\delta_1 - \delta_2)}\right]^{\frac{1}{\delta_2 - 1}} - \beta_{\mathrm{b}}\delta_1 \\ &= C_2^{\frac{1}{1-\delta_2}}(\delta_1 - \delta_2)^{\frac{1}{1-\delta_2}}[\beta_{\mathrm{s}}(\delta_1 - 1)]^{\frac{\delta_2}{\delta_2 - 1}}(-\delta_2)^{-\frac{\delta_2}{\delta_2 - 1}} + \frac{[\beta_{\mathrm{s}}(\delta_1 - 1)]^{\frac{\delta_2}{\delta_2 - 1}}}{[-\delta_2 C_2(\delta_1 - \delta_2)]^{\frac{1}{\delta_2 - 1}}} - \beta_b\delta_1 \\ &= C_2^{\frac{1}{1-\delta_2}}(\delta_1 - \delta_2)^{\frac{1}{1-\delta_2}}[\beta_{\mathrm{s}}(\delta_1 - 1)]^{\frac{\delta_2}{\delta_2 - 1}}[(-\delta_2)^{-\frac{\delta_2}{\delta_2 - 1}} + (-\delta_2)^{\frac{1}{1-\delta_2}}] - \beta_{\mathrm{b}}\delta_1. \end{aligned}$$

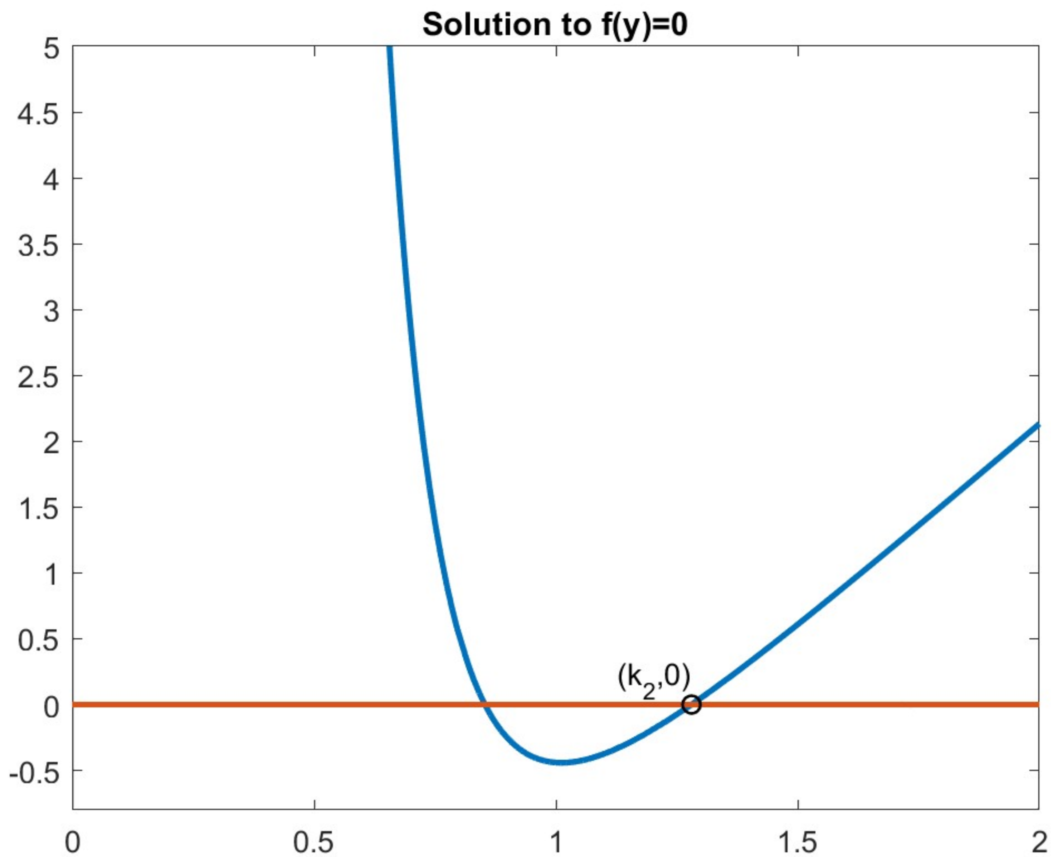

Figure 2.2: Example of solution to $f(k_2) = 0$.

Next we insert $C_2 = \left(\frac{\beta_{\text{s}}}{1-\delta_2}\right)^{1-\delta_2} \cdot \left(\frac{\beta_{\text{b}}}{-\delta_2}\right)^{\delta_2}$ into $f(y_c)$ to get

$$
\begin{aligned}
f(y_c) &= \left(\frac{\beta_{\text{s}}}{1-\delta_2}\right)\left(\frac{\beta_{\text{b}}}{-\delta_2}\right)^{\frac{\delta_2}{1-\delta_2}} (\delta_1-\delta_2)^{\frac{1}{1-\delta_2}} [\beta_{\text{s}}(\delta_1-1)]^{\frac{\delta_2}{\delta_2-1}} [(-\delta_2)^{-\frac{\delta_2}{\delta_2-1}} + (-\delta_2)^{\frac{1}{1-\delta_2}}] - \beta_{\text{b}}\delta_1 \\
&= \beta_{\text{s}}^{1+\frac{\delta_2}{\delta_2-1}} \beta_{\text{b}}^{\frac{\delta_2}{1-\delta_2}} \frac{(\delta_1-\delta_2)^{\frac{1}{1-\delta_2}} (\delta_1-1)^{\frac{\delta_2}{\delta_2-1}} [(-\delta_2)^{-\frac{\delta_2}{\delta_2-1}} + (-\delta_2)^{\frac{1}{1-\delta_2}}]}{(1-\delta_2)(-\delta_2)^{\frac{\delta_2}{1-\delta_2}}} - \beta_{\text{b}}\delta_1 \\
&= \beta_{\text{b}} \left[\left(\frac{\beta_{\text{s}}}{\beta_{\text{b}}}\right)^{1+\frac{-\delta_2}{1-\delta_2}} \frac{(\delta_1-\delta_2)^{\frac{1}{1-\delta_2}} (\delta_1-1)^{\frac{\delta_2}{\delta_2-1}} [(-\delta_2)^{-\frac{\delta_2}{\delta_2-1}} + (-\delta_2)^{\frac{1}{1-\delta_2}}]}{(1-\delta_2)(-\delta_2)^{\frac{\delta_2}{1-\delta_2}}} - \delta_1 \right]
\end{aligned}
$$

$$= \beta_{\mathrm{b}} \left[\left(\frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \right)^{1+\frac{-\delta_2}{1-\delta_2}} (\delta_1 - \delta_2)^{\frac{1}{1-\delta_2}} (\delta_1 - 1)^{\frac{\delta_2}{\delta_2 - 1}} - \delta_1 \right].$$

Since $\delta_2 < 0$, we let $\delta_2 = -r$ with $r > 0$ and $\beta = \dfrac{\beta_b}{\beta_s} > 1$. This will imply

$$\begin{aligned} f(y_c) &= \beta_{\mathrm{b}} \left[\left(\frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \right)^{1+\frac{r}{1+r}} (\delta_1 + r)^{\frac{1}{1+r}} (\delta_1 - 1)^{\frac{r}{1+r}} - \delta_1 \right] \\ &= \beta_{\mathrm{b}} \delta_1 \left[\beta^{-1-\frac{r}{1+r}} \left(1 + \frac{r}{\delta_1} \right)^{\frac{1}{1+r}} \left(1 - \frac{1}{\delta_1} \right)^{\frac{r}{1+r}} - 1 \right]. \end{aligned}$$

The necessary and sufficient condition for the existence of k_2 is $f(y_c) \le 0$, and this is equivalent to

$$\left(1 + \frac{r}{\delta_1} \right)^{\frac{1}{1+r}} \left(1 - \frac{1}{\delta_1} \right)^{\frac{r}{1+r}} \le \beta^{\frac{1+2r}{1+r}}.$$

We apply the geometric-arithmetic mean inequality

$$A^\theta B^{1-\theta} \le \theta A + (1-\theta) B \text{ with } \theta = \frac{1}{1+r},\ A = 1 + \frac{r}{\delta_1} \text{ and } B = 1 - \frac{1}{\delta_1}$$

to the left hand side of the above inequality to get

$$\left(1 + \frac{r}{\delta_1} \right)^{\frac{1}{1+r}} \left(1 - \frac{1}{\delta_1} \right)^{\frac{r}{1+r}} \le \left(1 + \frac{r}{\delta_1} \right) \cdot \frac{1}{1+r} + \left(1 - \frac{1}{\delta_1} \right) \cdot \frac{r}{1+r} = 1.$$

This implies $f(y_c) \le 0$ if

$$1 < \beta^{\frac{1+2r}{1+r}} \quad \Longleftrightarrow \quad 1 < \beta.$$

This obviously holds since $\beta > 1$. So we establish the existence of k_2.

Theorem 1. *Let δ_i be given by (2.4) and k_i be as described. Then the following functions w_1, w_0 satisfy the HJB equations (2.3):*

$$w_1(y) = \begin{cases} \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y, & \textit{for } 0 < y \le k_1, \\ \left(\dfrac{\beta_{\mathrm{s}}}{1-\delta_2} \right)^{1-\delta_2} \left(\dfrac{\beta_{\mathrm{b}}}{-\delta_2} \right)^{\delta_2} y^{\delta_2}, & \textit{for } y > k_1, \end{cases}$$

$$w_1(y) = \begin{cases} \left[1 - \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{-\delta_2} \left(\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} k_2^{\delta_2 - 1}\right] \left(\frac{\beta_\mathrm{s}}{\delta_1}\right) k_2^{1-\delta_1} y^{\delta_1}, & \textit{for } 0 < y < k_2, \\ \left(\frac{\beta_\mathrm{s}}{1-\delta_2}\right)^{1-\delta_2} \left(\frac{\beta_\mathrm{b}}{-\delta_2}\right)^{\delta_2} y^{\delta_2} - \beta_\mathrm{b} + \beta_\mathrm{s} y, & \textit{for } y \geq k_2. \end{cases}$$

Proof. Note that it is clear that $C_2 = \left(\frac{\beta_\mathrm{s}}{1-\delta_2}\right)^{1-\delta_2} \cdot \left(\frac{\beta_\mathrm{b}}{-\delta_2}\right)^{\delta_2} > 0$. We also wish to establish $C_1 = \left[1 - \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{-\delta_2} \left(\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} k_2^{\delta_2 - 1}\right] \left(\frac{\beta_\mathrm{s}}{\delta_1}\right) k_2^{1-\delta_1} > 0$. Consider,

$$\begin{aligned} C_1 > 0 &\iff C_2 \delta_2 k_2^{\delta_2 - 1} + \beta_\mathrm{s} > 0 \\ &\iff \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}} \cdot \frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} \frac{\beta_\mathrm{b}}{-\delta_2} \delta_2 k_2^{\delta_2 - 1} + \beta_\mathrm{s} > 0 \\ &\iff \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{-\delta_2} \left(\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} \frac{\beta_\mathrm{s}}{\beta_\mathrm{b}} \cdot \frac{\beta_\mathrm{b}}{-\delta_2} \delta_2 k_2^{\delta_2 - 1} + \beta_\mathrm{s} > 0 \\ &\iff -\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{-\delta_2} \left(\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} \beta_\mathrm{s} k_2^{\delta_2 - 1} + \beta_\mathrm{s} > 0 \\ &\iff \beta_\mathrm{s} > \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{-\delta_2} \left(\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} \beta_\mathrm{s} k_2^{\delta_2 - 1} \\ &\iff 1 > \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{-\delta_2} \left(\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} k_2^{\delta_2 - 1} \\ &\iff k_2^{1-\delta_2} > \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{-\delta_2} \left(\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} \\ &\iff k_2 > \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}} \left(\frac{-\delta_2}{1-\delta_2}\right). \end{aligned}$$

Note then that if $f\left(\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}} \left(\frac{-\delta_2}{1-\delta_2}\right)\right) < 0$, we establish $C_1 > 0$.

$$\begin{aligned} f\left(\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}} \left(\frac{-\delta_2}{1-\delta_2}\right)\right) &= C_2(\delta_1 - \delta_2) \left[\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}} \left(\frac{-\delta_2}{1-\delta_2}\right)\right]^{\delta_2} \\ &\quad + \beta_\mathrm{s}(\delta_1 - 1) \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}} \left(\frac{-\delta_2}{1-\delta_2}\right) - \beta_\mathrm{b} \delta_1 \end{aligned}$$

$$
\begin{aligned}
&= C_2(\delta_1-\delta_2)\left[\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-1}{1-\delta_2}}\left(\frac{-\delta_2}{1-\delta_2}\right)\right]^{\delta_2} \\
&\quad+\beta_\mathrm{s}(\delta_1-1)\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\left(\frac{-\delta_2}{1-\delta_2}\right)-\beta_\mathrm{b}\delta_1 \\
&= C_2(\delta_1-\delta_2)\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\delta_2}\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\left(\frac{-\delta_2}{1-\delta_2}\right)^{\delta_2} \\
&\quad+\beta_\mathrm{s}(\delta_1-1)\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\left(\frac{-\delta_2}{1-\delta_2}\right)-\beta_\mathrm{b}\delta_1 \\
&= \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\Bigg[C_2(\delta_1-\delta_2)\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\delta_2}\left(\frac{-\delta_2}{1-\delta_2}\right)^{\delta_2} \\
&\quad+\beta_\mathrm{s}(\delta_1-1)\left(\frac{-\delta_2}{1-\delta_2}\right)\Bigg]-\beta_\mathrm{b}\delta_1 \\
&= \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\Bigg[\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\cdot\frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2}\frac{\beta_\mathrm{b}}{-\delta_2}(\delta_1-\delta_2)\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\delta_2}\left(\frac{-\delta_2}{1-\delta_2}\right)^{\delta_2} \\
&\quad+\beta_\mathrm{s}(\delta_1-1)\left(\frac{-\delta_2}{1-\delta_2}\right)\Bigg]-\beta_\mathrm{b}\delta_1 \\
&= \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\Bigg[\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)\left(\frac{-\delta_2}{1-\delta_2}\right)\left(\frac{\beta_\mathrm{b}}{-\delta_2}\right)(\delta_1-\delta_2) \\
&\quad+\beta_\mathrm{s}(\delta_1-1)\left(\frac{-\delta_2}{1-\delta_2}\right)\Bigg]-\beta_\mathrm{b}\delta_1 \\
&= \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\left(\frac{\beta_\mathrm{s}}{1-\delta_2}\right)[(\delta_1-\delta_2)+(\delta_1-1)(-\delta_2)]-\beta_\mathrm{b}\delta_1 \\
&= \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\left(\frac{\beta_\mathrm{s}}{1-\delta_2}\right)[\delta_1-\delta_2-\delta_1\delta_2+\delta_2]-\beta_\mathrm{b}\delta_1 \\
&= \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\left(\frac{\beta_\mathrm{s}}{1-\delta_2}\right)[\delta_1(1-\delta_2)]-\beta_\mathrm{b}\delta_1 \\
&= \left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{\frac{-\delta_2}{1-\delta_2}}\cdot\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\beta_\mathrm{b}\delta_1-\beta_\mathrm{b}\delta_1 \\
&= \beta_\mathrm{b}\delta_1\left[\left(\frac{\beta_\mathrm{s}}{\beta_\mathrm{b}}\right)^{1+\frac{-\delta_2}{1-\delta_2}}-1\right] \\
&< 0,
\end{aligned}
$$

since $\left(\frac{\beta_{\rm s}}{\beta_{\rm b}}\right)^{1+\frac{-\delta_2}{1-\delta_2}} < \frac{\beta_{\rm s}}{\beta_{\rm b}} < 1$. Hence we have shown that $C_1 > 0$. Now we consider the following intervals:

$$\begin{aligned}\Gamma_1 &= (0, k_1] \\ \Gamma_2 &= (k_1, k_2) \\ \Gamma_3 &= [k_2, \infty).\end{aligned}$$

We have chosen k_1, k_2 such that we establish the following equalities:

$$\begin{aligned}\Gamma_1 :\ & w_1(y) - \beta_{\rm s} + \beta_{\rm b} y = 0\,, \\ & (\rho - \mathcal{L}) w_0(y) = 0,\end{aligned}$$

$$\begin{aligned}\Gamma_2 :\ & (\rho - \mathcal{L}) w_1(y) = 0\,, \\ & (\rho - \mathcal{L}) w_0(y) = 0,\end{aligned}$$

$$\begin{aligned}\Gamma_3 :\ & (\rho - \mathcal{L}) w_1(y) = 0\,, \\ & w_0(y) - w_1(y) + \beta_{\rm b} - \beta_{\rm s} y = 0,\end{aligned}$$

for solutions of the form

$$w_0(y) = \begin{cases} C_1 y^{\delta_1}, & \text{for } y \in \Gamma_1, \\ C_1 y^{\delta_1}, & \text{for } y \in \Gamma_2, \\ C_2 y^{\delta_2} - \beta_{\rm b} + \beta_{\rm s} y, & \text{for } y \in \Gamma_3, \end{cases}$$

$$w_1(y) = \begin{cases} \beta_{\rm s} - \beta_{\rm b} y, & \text{for } y \in \Gamma_1, \\ C_2 y^{\delta_2}, & \text{for } y \in \Gamma_2, \\ C_2 y^{\delta_2}, & \text{for } y \in \Gamma_3. \end{cases}$$

We now proceed to establish the following variational inequalities, thus confirming that we have solved the HJB equation:

$$\Gamma_1 : (\rho - \mathcal{L})w_1(y) \geq 0,$$
$$w_0(y) - w_1(y) + \beta_{\text{b}} - \beta_{\text{s}} y \geq 0,$$

$$\Gamma_2 : w_1(y) - \beta_{\text{s}} + \beta_{\text{b}} y \geq 0,$$
$$w_0(y) - w_1(y) + \beta_{\text{b}} - \beta_{\text{s}} y \geq 0,$$

$$\Gamma_3 : w_1(y) - \beta_{\text{s}} + \beta_{\text{b}} y \geq 0,$$
$$(\rho - \mathcal{L})w_0(y) \geq 0.$$

Γ_1:

Using $(\rho - \mathcal{L})w_0(y) = 0$ and $w_1(y) = \beta_{\text{s}} - \beta_{\text{b}} y$, we obtain

$$\begin{aligned} w_0(y) - w_1(y) + \beta_{\text{b}} - \beta_{\text{s}} y &= C_1 y^{\delta_1} - \beta_{\text{s}} + \beta_{\text{b}} y + \beta_{\text{b}} - \beta_{\text{s}} y \\ &= C_1 y^{\delta_1} + (\beta_{\text{b}} - \beta_{\text{s}})(y + 1) \\ &\geq 0, \end{aligned}$$

since $C_1 > 0$, $\beta_{\text{b}} > \beta_{\text{s}}$, and $y > 0$. Also,

$$\begin{aligned} (\rho - \mathcal{L})w_1(y) &= (\rho - \mathcal{L})(\beta_{\text{s}} - \beta_{\text{b}} y) \\ &= \rho\beta_{\text{s}} - \rho\beta_{\text{b}} y + \mathcal{L}\beta_{\text{b}} y - \mathcal{L}\beta_{\text{s}} \\ &= \rho\beta_{\text{s}} - \rho\beta_{\text{b}} y + \mu_2\beta_{\text{b}} y - \mu_1\beta_{\text{s}} \\ &= (\rho - \mu_1)\beta_{\text{s}} - (\rho - \mu_2)\beta_{\text{b}} y \\ \implies (\rho - \mathcal{L})w_1(y) \geq 0 &\iff (\rho - \mu_1)\beta_{\text{s}} - (\rho - \mu_2)\beta_{\text{b}} y \geq 0 \end{aligned}$$

$$\begin{aligned}
&\iff (\rho-\mu_1)\beta_{\mathrm{s}} \ge (\rho-\mu_2)\beta_{\mathrm{b}} y \\
&\iff \frac{(\rho-\mu_1)\beta_{\mathrm{s}}}{(\rho-\mu_2)\beta_{\mathrm{b}}} \ge y \\
&\iff \frac{(\rho-\mu_1)\beta_{\mathrm{s}}}{(\rho-\mu_2)\beta_{\mathrm{b}}} \ge k_1
\end{aligned}$$

since $k_1 \ge y$ for all $y \in \Gamma_1$. But note that

$$\begin{aligned}
\frac{(\rho-\mu_1)\beta_{\mathrm{s}}}{(\rho-\mu_2)\beta_{\mathrm{b}}} \ge k_1 &\iff \frac{-\delta_1\delta_2}{(\delta_1-1)(1-\delta_2)} \cdot \frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \ge k_1 \\
&\iff \frac{\delta_1}{(\delta_1-1)} \cdot k_1 \ge k_1,
\end{aligned}$$

which obviously holds since $\delta_1 > \delta_1 - 1 > 0$. Thus we have established the variational inequalities for the region Γ_1.

Γ_3:

Using $(\rho-\mathcal{L})w_1(y) = 0$ and $w_1(y) = w_0(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y$, we obtain

$$\begin{aligned}
w_1(y) - \beta_{\mathrm{s}} + \beta_{\mathrm{b}} y &= w_0(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y - \beta_{\mathrm{s}} + \beta_{\mathrm{b}} y \\
&= C_2 y^{\delta_2} - \beta_{\mathrm{b}} + \beta_{\mathrm{s}} y + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y - \beta_{\mathrm{s}} + \beta_{\mathrm{b}} y \\
&= C_2 y^{\delta_2} + \beta_{\mathrm{b}} y - \beta_{\mathrm{s}}.
\end{aligned}$$

Note that the continuity of w_1 and w_1' at k_1 ensure that

$$\begin{aligned}
&C_2 k_1^{\delta_2} + \beta_{\mathrm{b}} k_1 - \beta_{\mathrm{s}} = 0, \\
&C_2 \delta_2 k_1^{\delta_2 - 1} + \beta_{\mathrm{b}} = 0.
\end{aligned}$$

Let $g(y) = C_2 y^{\delta_2} + \beta_{\mathrm{b}} y - \beta_{\mathrm{s}}$. Then $g'(y) = C_2\delta_2 y^{\delta_2-1} + \beta_{\mathrm{b}}$. Note that

$$g'(y) \ge 0 \iff C_2\delta_2 y^{\delta_2-1} + \beta_{\mathrm{b}} \ge 0 \iff \beta_{\mathrm{b}} \ge C_2(-\delta_2) y^{\delta_2-1}$$

$$
\begin{aligned}
&\iff \frac{\beta_{\mathrm{b}}}{C_2(-\delta_2)} \geq y^{\delta_2 - 1} \\
&\iff \frac{C_2(-\delta_2)}{\beta_{\mathrm{b}}} \leq y^{1-\delta_2} \\
&\iff k_1^{1-\delta_2} \leq y^{1-\delta_2} \\
&\iff k_1 \leq y.
\end{aligned}
$$

Thus $g(y) = C_2 y^{\delta_2} + \beta_{\mathrm{b}} y - \beta_{\mathrm{s}}$ is increasing for all $y \geq k_1$. In particular, since $C_2 k_1^{\delta_2} + \beta_{\mathrm{b}} k_1 - \beta_{\mathrm{s}} = 0$, we must have $C_2 y^{\delta_2} + \beta_{\mathrm{b}} y - \beta_{\mathrm{s}} \geq 0$ for all $y \geq k_1$. Thus $C_2 y^{\delta_2} + \beta_{\mathrm{b}} y - \beta_{\mathrm{s}} = w_1(y) - \beta_{\mathrm{s}} + \beta_{\mathrm{b}} y \geq 0$ for all $y \in \Gamma_2 \cup \Gamma_3$. Also,

$$
\begin{aligned}
(\rho - \mathcal{L}) w_0(y) &= (\rho - \mathcal{L})(w_1(y) + \beta_{\mathrm{s}} y - \beta_{\mathrm{b}}) \\
&= (\rho - \mathcal{L})(w_1(y)) + (\rho - \mathcal{L})(\beta_{\mathrm{s}} y - \beta_{\mathrm{b}}) \\
&= 0 + \rho \beta_{\mathrm{s}} y - \rho \beta_{\mathrm{b}} + \mathcal{L} \beta_{\mathrm{b}} - \mathcal{L} \beta_{\mathrm{s}} y \\
&= \rho \beta_{\mathrm{s}} y - \rho \beta_{\mathrm{b}} + \mu_1 \beta_{\mathrm{b}} - \mu_2 \beta_{\mathrm{s}} y \\
&= (\rho - \mu_2) \beta_{\mathrm{s}} y - (\rho - \mu_1) \beta_{\mathrm{b}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\rho - \mathcal{L}) w_0(y) \geq 0 &\iff (\rho - \mu_2) \beta_{\mathrm{s}} y - (\rho - \mu_1) \beta_{\mathrm{b}} \geq 0 \\
&\iff (\rho - \mu_2) \beta_{\mathrm{s}} y \geq (\rho - \mu_1) \beta_{\mathrm{b}} \\
&\iff y \geq \frac{(\rho - \mu_1) \beta_{\mathrm{b}}}{(\rho - \mu_2) \beta_{\mathrm{s}}} \\
&\iff k_2 \geq \frac{(\rho - \mu_1) \beta_{\mathrm{b}}}{(\rho - \mu_2) \beta_{\mathrm{s}}},
\end{aligned}
$$

since $k_2 \leq y$ for all $y \in \Gamma_3$. Note that $\dfrac{(\rho - \mu_1) \beta_{\mathrm{b}}}{(\rho - \mu_2) \beta_{\mathrm{s}}} = \dfrac{-\delta_1 \delta_2}{(\delta_1 - 1)(1 - \delta_2)} \cdot \dfrac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}}$ and consider

$$
\begin{aligned}
f\left(\frac{-\delta_1 \delta_2}{(\delta_1 - 1)(1 - \delta_2)} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}} \right) = C_2(\delta_1 - \delta_2) &\left(\frac{-\delta_1 \delta_2}{(\delta_1 - 1)(1 - \delta_2)} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}} \right)^{\delta_2} \\
&+ \beta_{\mathrm{s}}(\delta_1 - 1) \left(\frac{-\delta_1 \delta_2}{(\delta_1 - 1)(1 - \delta_2)} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}} \right) - \beta_{\mathrm{b}} \delta_1
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\beta_{\mathrm{s}}}{\beta_{\mathrm{b}}} \cdot \frac{-\delta_2}{1-\delta_2}\right)^{1-\delta_2} \left(\frac{\beta_{\mathrm{b}}}{-\delta_2}\right) (\delta_1 - \delta_2) \left(\frac{\delta_1}{\delta_1 - 1} \cdot \frac{-\delta_2}{1-\delta_2} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}}\right)^{\delta_2} \\
&\quad + \beta_{\mathrm{b}} \left(\frac{-\delta_1 \delta_2}{1-\delta_2}\right) - \beta_{\mathrm{b}} \delta_1 \\
&= \frac{\delta_1 - \delta_2}{1-\delta_2} \left(\frac{\delta_1}{\delta_1 - 1}\right)^{\delta_2} \beta^{2\delta_2 - 1} \beta_{\mathrm{b}} + \beta_{\mathrm{b}} \delta_1 \left(\frac{-\delta_2}{1-\delta_2} - 1\right).
\end{aligned}
$$

Now, let $\delta_2 = -r$ with $r > 0$. Then

$$
f\left(\frac{-\delta_1 \delta_2}{(\delta_1 - 1)(1-\delta_2)} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}}\right) = \left(\frac{\delta_1 + r}{1+r}\right) \left(\frac{\delta_1 - 1}{\delta_1}\right)^{r} \beta^{-2r-1} \beta_{\mathrm{b}} + \beta_{\mathrm{b}} \delta_1 \left(\frac{r}{1+r} - 1\right).
$$

Hence

$$
\begin{aligned}
f\left(\frac{-\delta_1 \delta_2}{(\delta_1 - 1)(1-\delta_2)} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}}\right) < 0 &\iff \left(\frac{\delta_1 + r}{1+r}\right) \left(\frac{\delta_1 - 1}{\delta_1}\right)^{r} \beta^{-2r-1} \beta_{\mathrm{b}} < \beta_{\mathrm{b}} \delta_1 \left(\frac{-r+1+r}{1+r}\right) \\
&\iff \left(\frac{r+1}{\delta_1}\right) \left(\frac{\delta_1 + r}{1+r}\right) \left(\frac{\delta_1 - 1}{\delta_1}\right)^{r} < \beta^{2r+1} \\
&\iff \left(\frac{\delta_1 + r}{\delta_1}\right)^{\frac{1}{r+1}} \left(\frac{\delta_1 - 1}{\delta_1}\right)^{\frac{r}{r+1}} < \beta^{\frac{2r+1}{r+1}} \\
&\iff \left(1 + \frac{r}{\delta_1}\right)^{\frac{1}{r+1}} \left(1 - \frac{1}{\delta_1}\right)^{\frac{r}{r+1}} < \beta^{\frac{2r+1}{r+1}}.
\end{aligned}
$$

Applying the arithmetic-geometric mean inequality to the left-hand side yields

$$
\begin{aligned}
\left(1 + \frac{r}{\delta_1}\right)^{\frac{1}{r+1}} \left(1 - \frac{1}{\delta_1}\right)^{\frac{r}{r+1}} &\leq \left(\frac{1}{r+1}\right) \left(1 + \frac{r}{\delta_1}\right) + \left(\frac{r}{r+1}\right) \left(1 - \frac{1}{\delta_1}\right) \\
&= \frac{1}{r+1} + \frac{r}{r+1} \cdot \frac{1}{\delta_1} + \frac{r}{r+1} - \frac{r}{r+1} \cdot \frac{1}{\delta + 1} \\
&= \frac{r+1}{r+1} = 1 < \beta < \beta^{\frac{2r+1}{r+1}}.
\end{aligned}
$$

So, $f\left(\frac{-\delta_1 \delta_2}{(\delta_1 - 1)(1-\delta_2)} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}}\right) < 0$ holds. That is, $k_2 > \frac{(\rho - \mu_1)}{(\rho - \mu_2)} \cdot \frac{\beta_{\mathrm{b}}}{\beta_{\mathrm{s}}}$, which establishes $(\rho - \mathcal{L}) w_0(y) \geq 0$ for all $y \in \Gamma_3$.

Γ_2:

On Γ_2, we have $w_1(y) - \beta_s + \beta_b y = C_2 y_2^\delta - \beta_s + \beta_b y$. Note that we have already shown that $C_2 y_2^\delta - \beta_s + \beta_b y \geq 0$ for all $y \in \Gamma_2 \cup \Gamma_3$. Hence, $w_1(y) - \beta_s + \beta_b y \geq 0$ for all $y \in \Gamma_2$.
We also have $w_0(y) - w_1(y) + \beta_b - \beta_s y = C_1 y_1^\delta - C_2 y_2^\delta + \beta_b - \beta_s y$. Let

$$\phi(y) = C_1 y^{\delta_1} - C_2 y^{\delta_2} + \beta_b - \beta_s y.$$

Hence

$$\begin{aligned}
\phi'(y) &= C_1\delta_1 y^{\delta_1-1} + C_2(-\delta_2) y^{\delta_2-1} - \beta_s \\
\phi''(y) &= C_1\delta_1(\delta_1-1) y^{\delta_1-2} - C_2(-\delta_2)(1-\delta_2) y^{\delta_2-2}.
\end{aligned}$$

By continuity of w_0, we know $C_1 k_2^{\delta_1} - C_2 k_2^{\delta_2} + \beta_b - \beta_s k_2 = 0$. That is, we know $\phi(k_2) = 0$.
By continuity of w_0', we know $C_1\delta_1 k_2^{\delta_1-1} + C_2(-\delta_2) k_2^{\delta_2-1} - \beta_s = 0$. That is, we know $\phi'(k_2) = 0$.
By continuity of w_1, we know $C_2 k_1^{\delta_2} = \beta_s - \beta_b k_1$. Hence, $C_1 k_1^{\delta_1} - C_2 k_1^{\delta_2} + \beta_b - \beta_s k_1 = C_1 k_1^{\delta_1} - \beta_s + \beta_b k_1 + \beta_b - \beta_s k_1 = C_1 k_1^{\delta_1} + (k_1+1)(\beta_b - \beta_s) \geq 0$. That is, we know $\phi(k_1) \geq 0$.
By continuity of w_1', we know $-C_2(-\delta_2) k_1^{\delta_2-1} = -\beta_b$. Hence, $C_1\delta_1 k_1^{\delta_1-1} + C_2(-\delta_2) k_1^{\delta_2-1} - \beta_s = C_1\delta_1 k_1^{\delta_1-1} + \beta_b - \beta_s \geq 0$. That is, we know $\phi'(k_1) \geq 0$. Now,

$$\begin{aligned}
\phi''(y) &= C_1\delta_1(\delta_1-1) y^{\delta_1-2} - C_2(-\delta_2)(1-\delta_2) y^{\delta_2-2} \\
&= \left(\frac{C_2\delta_2 k_2^{\delta_2-1} + \beta_s}{\delta_1 k_2^{\delta_1-1}}\right)\delta_1(\delta_1-1) y^{\delta_1-2} - C_2(-\delta_2)(1-\delta_2) y^{\delta_2-2} \\
&= \frac{C_2\delta_2(\delta_1-1) k_2^{\delta_2-1} k_2^{-1}}{k_2^{\delta_1-1} k_2^{-1}} \cdot y^{\delta_1-2} + \frac{\beta_s(\delta_1-1) k_2^{-1}}{k_2^{\delta_1-1} k_2^{-1}} \cdot y^{\delta_1-2} - \frac{C_2(-\delta_2)(1-\delta_2) k_2^{\delta_2-2}}{k_2^{\delta_2-2}} \cdot y^{\delta_2-2} \\
&= -C_2(-\delta_2) k_2^{\delta_2-2}\left[(\delta_1-1)\left(\frac{y}{k_2}\right)^{\delta_1-2} + (1-\delta_2)\left(\frac{y}{k_2}\right)^{\delta_2-2}\right] + \beta_s(\delta_1-1) k_2^{-1}\left(\frac{y}{k_2}\right)^{\delta_1-2}.
\end{aligned}$$

Hence $\phi''(k_2) = \beta_s(\delta_1-1) k_2^{-1} - C_2(-\delta_2)(\delta_1-\delta_2) k_2^{\delta_2-2}$. Then note that

$$k_2 > \left[\frac{\beta_s(\delta_1-1)}{C_2(\delta_1-\delta_2)(-\delta_2)}\right]^{\frac{1}{\delta_2-1}} \implies k_2^{\delta_2-1} < \frac{\beta_s(\delta_1-1)}{C_2(\delta_1-\delta_2)(-\delta_2)},$$

since $\delta_2 - 1 < 0$. Thus,

$$
\begin{aligned}
&(k_2^{\delta_2-1})k_2^{-1}(-C_2)(-\delta_2)(\delta_1-\delta_2) > \left(\frac{\beta_{\mathrm{s}}(\delta_1-1)}{C_2(\delta_1-\delta_2)(-\delta_2)}\right)k_2^{-1}(-C_2)(-\delta_2)(\delta_1-\delta_2) \\
&\implies (k_2^{\delta_2-2})(-C_2)(-\delta_2)(\delta_1-\delta_2) > -\beta_{\mathrm{s}}(\delta_1-1)k_2^{-1} \\
&\implies \beta_{\mathrm{s}}(\delta_1-1)k_2^{-1} - C_2(-\delta_2)(\delta_1-\delta_2)k_2^{\delta_2-2} > 0.
\end{aligned}
$$

That is, $\phi''(k_2) > 0$.

Consider the equation $\phi''(y) = 0$.

$$
\begin{aligned}
\phi''(y) = 0 &\iff C_1\delta_1(\delta_1-1)y^{\delta_1-2} - C_2(-\delta_2)(1-\delta_2)y^{\delta_2-2} = 0 \\
&\iff C_1\delta_1(\delta_1-1)y^{\delta_1-2} = C_2(-\delta_2)(1-\delta_2)y^{\delta_2-2} \\
&\iff y^{\delta_1-\delta_2} = \frac{C_2(-\delta_2)(1-\delta_2)}{C_1\delta_1(\delta_1-1)} \\
&\iff y = \left(\frac{C_2(-\delta_2)(1-\delta_2)}{C_1\delta_1(\delta_1-1)}\right)^{\frac{1}{\delta_1-\delta_2}}.
\end{aligned}
$$

Note then that $\phi''(y) = 0$ has a unique solution in $[k_1, k_2]$.

Observe that ϕ, ϕ', and ϕ'' are continuous on $[k_1, k_2]$. Since $\phi(k_2) = \phi'(k_2) = 0$ and $\phi''(k_2) > 0$, there exists $\varepsilon_1 > 0$ such that ϕ is nonnegative, decreasing, and convex over the interval $(k_2 - \varepsilon_1, k_2)$. Since $\phi(k_1) \geq 0$ and $\phi'(k_1) \geq 0$, there exists $\varepsilon_2 > 0$ such that ϕ is nonnegative and increasing on $(k_1, k_1 + \varepsilon_2)$; moreover, $k_1 + \varepsilon_2 < k_2 - \varepsilon_1$. Suppose, if possible, there exists $y \in (k_1 + \varepsilon_2, k_2 - \varepsilon_1)$ such that $\phi(y) < 0$. Note that $\phi\left(k_1 + \frac{\varepsilon_2}{2}\right) > 0$. Then by Intermediate Value Theorem, there exists $y_1 \in \left(k_1 + \frac{\varepsilon_2}{2}, y\right)$ such that $\phi(y_1) = 0$. Similarly, since $\phi\left(k_2 - \frac{\varepsilon_1}{2}\right) > 0$, there exists $y_2 \in \left(y, k_2 - \frac{\varepsilon_1}{2}\right)$ such that $\phi(y_2) = 0$. Note also that $\phi'\left(k_1 + \frac{\varepsilon_2}{2}\right) > 0$ and $\phi'(y_1) < 0$. So, by Intermediate Value Theorem, there exists $\widetilde{y_1} \in \left(k_1 + \frac{\varepsilon_2}{2}, y_1\right)$ such that $\phi'(\widetilde{y_1}) = 0$. Similarly, since $\phi'(y_2) > 0$, there exists $\widetilde{y_2} \in (y_1, y_2)$ such that $\phi'(\widetilde{y_2}) = 0$. Also, since $\phi'\left(k_2 - \frac{\varepsilon_1}{2}\right) < 0$, there exists $\widetilde{y_3} \in \left(y_2, k_2 - \frac{\varepsilon_1}{2}\right)$ such that $\phi'(\widetilde{y_3}) = 0$. Finally, since $\phi'(\widetilde{y_1}) = \phi'(\widetilde{y_2}) = 0$, by Rolle's Theorem, there exists $y_1^* \in (\widetilde{y_1}, \widetilde{y_2})$ such that $\phi''(y_1^*) = 0$. Similarly, since $\phi'(\widetilde{y_3}) = 0$, there exists $y_2^* \in (\widetilde{y_2}, \widetilde{y_3})$ such that $\phi''(y_2^*) = 0$. But this is a contradiction, because $y_1^* \in [k_1, k_2]$, $y_2^* \in [k_1, k_2]$, but $y_1^* \neq y_2^*$; whereas the equation $\phi''(y) = 0$ has exactly one solution in the interval $[k_1, k_2]$.

Hence, $\phi(y) = C_1 y^{\delta_1} - C_2 y^{\delta_2} + \beta_{\rm b} - \beta_{\rm s} y \geq 0$ on Γ_2. That is, $w_0(y) - w_1(y) + \beta_{\rm b} - \beta_{\rm s} y \geq 0$ for all $y \in \Gamma_2$. □

2.5 A Verification Theorem

Theorem 2. *We have* $v_i(x_1, x_2) = x_1 w_i \left(\frac{x_1}{x_2}\right) = V_i(x_1, x_2)$, $i = 0, 1$. *Moreover, if initially* $i = 0$, *let* $\Lambda_0^* = (\tau_1^*, \tau_2^*)$ *be such that*

$$\tau_1^* = \inf\{t \geq 0 \,|\, (X_t^1, X_t^2) \in \Gamma_3\}, \tau_2^* = \inf\{t \geq \tau_1^* \,|\, (X_t^1, X_t^2) \in \Gamma_1\}.$$

Similarly, if initially $i = 1$, *let* $\Lambda_1^* = (\tau_0^*)$ *be such that*

$$\tau_0^* = \inf\{t \geq 0 \,|\, (X_t^1, X_t^2) \in \Gamma_1\}.$$

Then Λ_0^* *and* Λ_1^* *are optimal.*

Proof. The proof is divided into 4 steps.

Step 1: $v_0(x_1, x_2) \geq 0$.

Recall that $C_1 > 0, C_2 > 0$ has previously been established. Now,

$$v_0(x_1, x_2) = x_1 w_0 \left(\frac{x_2}{x_1}\right) = \begin{cases} C_1 x_2^{\delta_1} x_1^{1-\delta_1}, & \text{for } (x_1, x_2) \in \Gamma_1 \cup \Gamma_2, \\ C_2 x_2^{\delta_2} x_1^{1-\delta_2} - \beta_b x_1 + \beta_s x_2, & \text{for } (x_1, x_2) \in \Gamma_3. \end{cases}$$

Hence to show $v_0(x_1, x_2) \geq 0$, it suffices to show $w_0(y) \geq 0$ on Γ_3. The continuity of w_0 and w_0' yield $w_0(k_2) = C_2 k_2^{\delta_2} - \beta_b + \beta_s k_2 = C_1 k_2^{\delta_1} > 0$ and $w_0'(k_2) = C_2 \delta_2 k_2^{\delta_2 - 1} + \beta_s = C_1 \delta_1 k_2^{\delta_1 - 1} > 0$. Also, $w_0''(y) = C_2 \delta_2 (\delta_2 - 1) y^{\delta_2 - 2} > 0$ for all $y > 0$. In particular, since $w_0''(y) > 0$ for all $y \in \Gamma_3$, we know $w_0'(y)$ is increasing on Γ_3. And since $w_0'(k_2) > 0$, it must be that $w_0'(y) > 0$ for all $y \in \Gamma_3$. This in turn implies that $w_0(y)$ is increasing on Γ_3. Since we know $w_0(k_2) > 0$, it must be that $w_0(y) > 0$ for all $y \in \Gamma_3$.

Step 2**:** $-Ax_1 - Bx_2 \le v_i(x_1,x_2) \le Ax_1 + Bx_2$**,** $i = 0,1$**.**

Let $i = 0$. On $\Gamma_1 \cup \Gamma_2$, we have $0 \le v_0(x_1,x_2) = C_1 x_1^{1-\delta_1} x_2^{\delta_1} \le C_1 x_1 k_2^{\delta_1}$. On Γ_3, $-\beta_b x_1 + \beta_s x_2 \le v_0(x_1,x_2) = C_2 x_1^{1-\delta_2} x_2^{\delta_2} - \beta_b x_1 + \beta_s x_2 \le C_2 x_1 k_1^{\delta_2} - \beta_b x_1 + \beta_s x_2$. Hence we can choose suitable A and B so the inequalities hold when $i = 0$. Then let $i = 1$. On $\Gamma_2 \cup \Gamma_3$, we have $0 \le v_1(x_1,x_2) = C_2 x_1^{1-\delta_2} x_2^{\delta_2} \le C_2 x_1 k_1^{\delta_2}$. On Γ_1, $-\beta_b x_2 \le v_1(x_1,x_2) = \beta_s x_1 - \beta_b x_2 \le \beta_s x_1$. So again we can choose suitable A and B so the inequalities hold when $i = 1$.

Step 3**:** $v_i(x_1,x_2) \ge J_i(x_1,x_2,\Lambda_i)$**.**

The functions v_0 and v_1 are continuously differentiable on the entire region $\{x_1 > 0,\ x_2 > 0\}$ and twice continuously differentiable on the interior of Γ_i, $i = 1,2,3$. In addition, they satisfy

$$\begin{aligned} 0 &\le (\rho - \mathcal{L}) w_0(y)\,, \\ 0 &\le (\rho - \mathcal{L}) w_1(y), \\ -\beta_b + \beta_s y \le w_0(y) - w_1(y) &\le w_0(y) - \beta_s + \beta_b y\,. \end{aligned}$$

In particular, $\rho v_i(x) - \mathcal{A} v_i(x) \ge 0$, $i = 0,1$, whenever they are twice continuously differentiable. Using these inequalities, Dynkin's formula, and Fatou's Lemma, as in Øksendal [**18**], we have $\mathbb{E}\left[e^{-\rho(\theta_1 \wedge N)} v_i(X^1_{\theta_1\wedge N}, X^2_{\theta_1\wedge N})\right] \ge \mathbb{E}\left[e^{-\rho(\theta_2 \wedge N)} v_i(X^1_{\theta_2\wedge N}, X^2_{\theta_2\wedge N})\right]$ for any stopping times $0 \le \theta_1 \le \theta_2$, almost surely, and any N.

For each $j = 1,2$,

$$\begin{aligned} &\mathbb{E}\left[e^{-\rho(\theta_j \wedge N)} v_i(X^1_{\theta_j\wedge N}, X^2_{\theta_j\wedge N})\right] \\ &= \mathbb{E}\left[e^{-\rho(\theta_j \wedge N)} v_i(X^1_{\theta_j\wedge N}, X^2_{\theta_j\wedge N}) \mathbb{I}_{\{\theta_j < \infty\}}\right] + \mathbb{E}\left[e^{-\rho(\theta_j \wedge N)} v_i(X^1_{\theta_j\wedge N}, X^2_{\theta_j\wedge N}) \mathbb{I}_{\{\theta_j = \infty\}}\right] \\ &= \mathbb{E}\left[e^{-\rho(\theta_j \wedge N)} v_i(X^1_{\theta_j\wedge N}, X^2_{\theta_j\wedge N}) \mathbb{I}_{\{\theta_j < \infty\}}\right] + \mathbb{E}\left[e^{-\rho N} v_i(X^1_N, X^2_N) \mathbb{I}_{\{\theta_j = \infty\}}\right]. \end{aligned}$$

In view of Step 2, the second term on the right hand side converges to zero because both $\mathbb{E}\left[e^{-\rho N} X^1_N\right]$ and $\mathbb{E}\left[e^{-\rho N} X^2_N\right]$ go to zero as $N \to \infty$. Also, $e^{-\rho(\theta_j\wedge N)} v_i(X^1_{\theta_j\wedge N}, X^2_{\theta_j\wedge N}) \mathbb{I}_{\{\theta_j<\infty\}} \to e^{-\rho\theta_j} v_i(X^1_{\theta_j}, X^2_{\theta_j}) \mathbb{I}_{\{\theta_j<\infty\}}$ almost surely as $N \to \infty$.

By showing the existence of $\gamma_i, i=1,2$ such that

$$\sup_n \mathbb{E}\left[\left(e^{-\rho(\theta_j\wedge N)}X^1_{\theta_j\wedge N}\right)^{\gamma_1}\right] < \infty,$$
$$\sup_n \mathbb{E}\left[\left(e^{-\rho(\theta_j\wedge N)}X^2_{\theta_j\wedge N}\right)^{\gamma_2}\right] < \infty,$$

we can show that both $\left\{e^{-\rho(\theta_j\wedge N)}X^1_{\theta_j\wedge N}\right\}$ and $\left\{e^{-\rho(\theta_j\wedge N)}X^2_{\theta_j\wedge N}\right\}$ are uniformly integrable. Hence we obtain the uniform integrability of $\left\{e^{-\rho(\theta_j\wedge N)}v_i(X^1_{\theta_j\wedge N}, X^2_{\theta_j\wedge N})\right\}$ and send N to ∞ to obtain $\mathbb{E}\left[e^{-\rho\theta_1}v_i(X^1_{\theta_1}, X^2_{\theta_1})\mathbb{I}_{\{\theta_1<\infty\}}\right] \geq \mathbb{E}\left[e^{-\rho\theta_2}v_i(X^1_{\theta_2}, X^2_{\theta_2})\mathbb{I}_{\{\theta_2<\infty\}}\right]$, for $i=0,1$.

Given $\Lambda_0 = (\tau_1, \tau_2), \Lambda_1 = (\tau_0)$,

$$\begin{aligned}
v_0(x_1,x_2) &\geq \mathbb{E}\left[e^{-\rho\tau_1}v_0(X^1_{\tau_1}, X^2_{\tau_1})\mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_1}\left(v_1(X^1_{\tau_1}, X^2_{\tau_1}) - \beta_b X^1_{\tau_1} + \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&= \mathbb{E}\left[e^{-\rho\tau_1}v_1(X^1_{\tau_1}, X^2_{\tau_1})\mathbb{I}_{\{\tau_1<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} + \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_2}v_1(X^1_{\tau_2}, X^2_{\tau_2})\mathbb{I}_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} + \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} + \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&= J_0(x_1,x_2,\Lambda_0), \\
v_1(x_1,x_2) &\geq \mathbb{E}\left[e^{-\rho\tau_1}v_1(X^1_{\tau_0}, X^2_{\tau_0})\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&= J_1(x_1,x_2,\Lambda_1).
\end{aligned}$$

Step 4: $v_i(x_1,x_2) = J_i(x_1,x_2,\Lambda_i^*)$.

Let $i=0$. Define $\tau_1^* = \inf\{t\geq 0 \mid (X^1_t, X^2_t)\in\Gamma_3\}$, $\tau_2^* = \inf\{t\geq \tau_1^* \mid (X^1_t, X^2_t)\in\Gamma_1\}$. We apply Dynkin's formula and notice that, for each n, $v_0(x_1,x_2) = \mathbb{E}\left[e^{-\rho(\tau_1^*\wedge n)}v_0(X^1_{\tau_1^*\wedge n}, X^2_{\tau_1^*\wedge n})\right]$. Note also that $\lim_{n\to\infty} \mathbb{E}\left[e^{-\rho(\tau_1^*\wedge n)} v_0(X^1_{\tau_1^*\wedge n}, X^2_{\tau_1^*\wedge n})\right] = \mathbb{E}\left[e^{-\rho\tau_1^*} v_0(X^1_{\tau_1^*}, X^2_{\tau_1^*})\mathbb{I}_{\{\tau_1^*<\infty\}}\right]$. It follows that

$$\begin{aligned} v_0(x_1, x_2) &= \mathbb{E}\left[e^{-\rho\tau_1^*} v_0(X^1_{\tau_1^*}, X^2_{\tau_1^*})\mathbb{I}_{\{\tau_1^*<\infty\}}\right] \\ &= \mathbb{E}\left[e^{-\rho\tau_1^*}\left(v_1(X^1_{\tau_1^*}, X^2_{\tau_1^*}) - \beta_b X^1_{\tau_1^*} + \beta_s X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\right]. \end{aligned}$$

We also have

$$\begin{aligned} \mathbb{E}\left[e^{-\rho\tau_1^*} v_1(X^1_{\tau_1^*}, X^2_{\tau_1^*})\mathbb{I}_{\{\tau_1^*<\infty\}}\right] &= \mathbb{E}\left[e^{-\rho\tau_2^*} v_1(X^1_{\tau_2^*}, X^2_{\tau_2^*})\mathbb{I}_{\{\tau_2^*<\infty\}}\right] \\ &= \mathbb{E}\left[e^{-\rho\tau_2^*}\left(\beta_s X^1_{\tau_2^*} - \beta_b X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\right]. \end{aligned}$$

Combine these to obtain

$$\begin{aligned} v_0(x_1, x_2) &= \mathbb{E}\left[e^{-\rho\tau_2^*}\left(\beta_s X^1_{\tau_2^*} - \beta_b X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}} - \left(\beta_b X^1_{\tau_1^*} + \beta_s X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\right] \\ &= J_0(x_1, x_2, \Lambda_0^*). \end{aligned}$$

Let $i = 1$. Define $\tau_0^* = \inf\{t \geq 0 \mid (X^1_t, X^2_t) \in \Gamma_1\}$. We apply Dynkin's formula and notice that, for each n, $v_1(x_1, x_2) = \mathbb{E}\left[e^{-\rho(\tau_0^*\wedge n)} v_1(X^1_{\tau_0^*\wedge n}, X^2_{\tau_0^*\wedge n})\right]$. Note also that $\lim_{n\to\infty} \mathbb{E}\left[e^{-\rho(\tau_0^*\wedge n)} v_1(X^1_{\tau_0^*\wedge n}, X^2_{\tau_0^*\wedge n})\right] = \mathbb{E}\left[e^{-\rho\tau_0^*} v_1(X^1_{\tau_0^*}, X^2_{\tau_0^*})\mathbb{I}_{\{\tau_0^*<\infty\}}\right]$. It follows that

$$\begin{aligned} v_1(x_1, x_2) &= \mathbb{E}\left[e^{-\rho\tau_0^*} v_1(X^1_{\tau_0^*}, X^2_{\tau_0^*})\mathbb{I}_{\{\tau_0^*<\infty\}}\right] \\ &= \mathbb{E}\left[e^{-\rho\tau_0^*}\left(\beta_s X^1_{\tau_0^*} - \beta_b X^2_{\tau_0^*}\right)\mathbb{I}_{\{\tau_0^*<\infty\}}\right] \\ &= J_1(x_1, x_2, \Lambda_1^*). \end{aligned}$$

$\square$

2.6 A Numerical Example

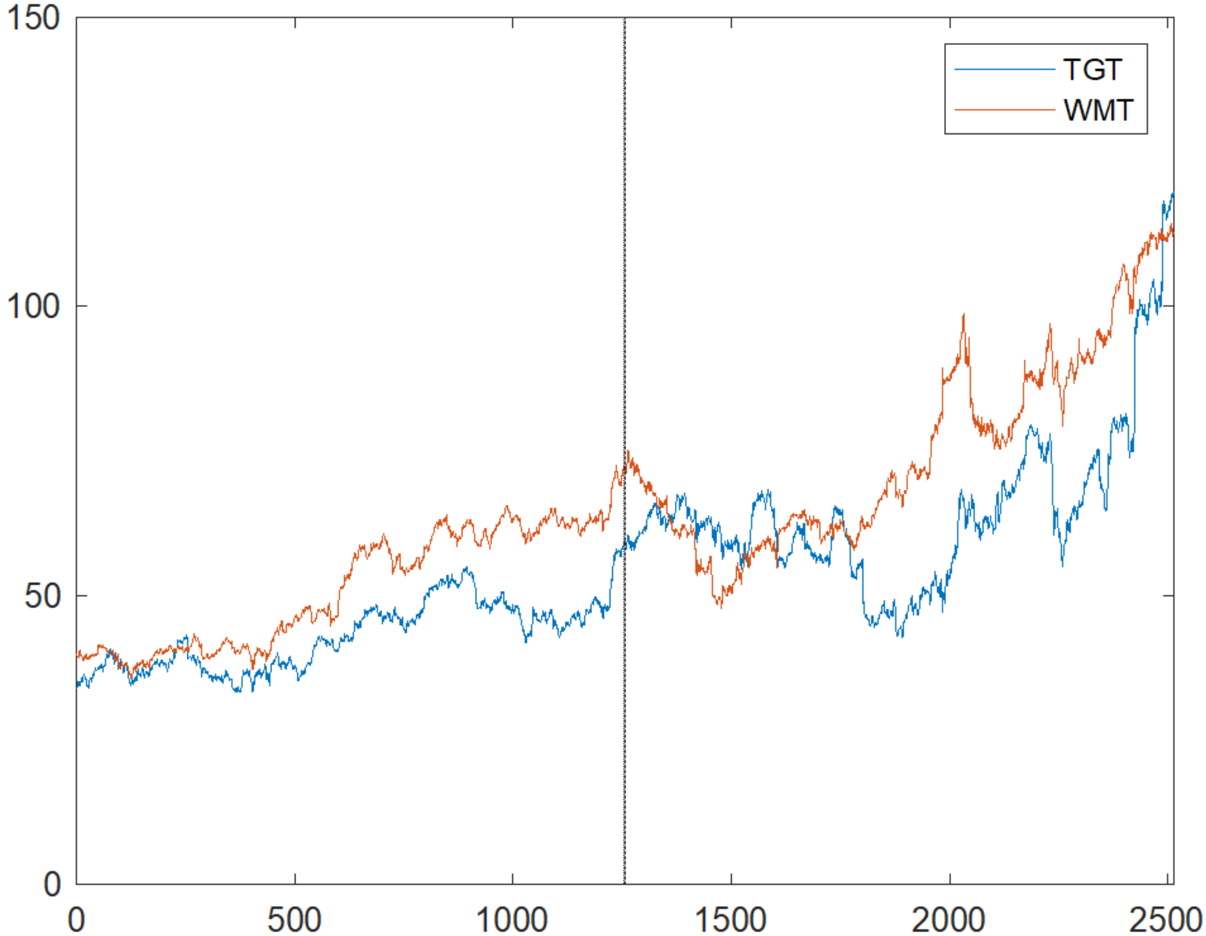

Figure 2.3: Closing Prices of TGT and WMT from 2010 to 2020

We consider adjusted closing price data for Walmart (WMT) and Target (TGT) from 2010 to 2020. The first half of the data is used to calibrate the model, and the second half is used to test the results. Using a least-squares method, we obtain the following parameters: $\mu_1 = 0.09696$, $\mu_2 = 0.14347$, $\sigma_{11} = 0.19082$, $\sigma_{12} = 0.04036$, $\sigma_{21} = 0.04036$, and $\sigma_{22} = 0.13988$. We specify $K = 0.001$ and $\rho = 0.5$. Then we find $k_1 = 0.85527$, and $k_2 = 1.28061$.

Next we examine the dependence of k_1 and k_2 on the parameters by varying each. In Table 2.1, we see that k_1 and k_2 both decrease in μ_1. This leads to a larger buying region, Γ_3.

Table 2.1: k_1 and k_2 with varying μ_1

μ_1	-0.00304	0.04696	0.09696	0.14696	0.19696
k_1	0.91380	0.89057	0.85527	0.80194	0.72644
k_2	1.54188	1.41541	1.28061	1.12891	0.96334

On the other hand, both k_1 and k_2 increase in μ_2, as indicated in Table 2.2. This creates a larger Γ_1 and, hence, encourages early exit.

Table 2.2: k_1 and k_2 with varying μ_2

μ_2	0.04347	0.09347	0.14347	0.19347	0.24347
k_1	0.76457	0.81341	0.85527	0.88736	0.91037
k_2	0.98771	1.12128	1.28061	1.47155	1.72474

When varying σ_{11} and σ_{22}, as in Table 2.3 and Table 2.4, we find that k_2 increases while k_1 decreases, in both σ_{11} and σ_{22}. This leads to a smaller buying zone, Γ_1, due to the increased risk, as well as a smaller selling zone, Γ_3, because there is more price movement overall.

Table 2.3: k_1 and k_2 with varying σ_{11}

σ_{11}	0.09082	0.14082	0.19082	0.24082	0.29082
k_1	0.92069	0.89220	0.85527	0.81532	0.77497
k_2	1.21691	1.24468	1.28061	1.32066	1.36327

Table 2.4: k_1 and k_2 with varying σ_{22}

σ_{22}	0.03988	0.08988	0.13988	0.18988	0.23988
k_1	0.88356	0.87601	0.85527	0.82593	0.79206
k_2	1.25304	1.26036	1.28061	1.30985	1.34491

However, as $\sigma_{12} = \sigma_{21}$ increases, we find that k_2 decreases, while k_1 increases (Table 2.5). The greater correlation leads to a larger Γ_1, and hence more opportunity for buying, as well as a larger Γ_3, and hence more opportunity for selling.

Table 2.5: k_1 and k_2 with varying $\sigma_{12} = \sigma_{21}$

σ_{12}	-0.05964	-0.00964	0.04036	0.09036	0.14036
k_1	0.73242	0.79189	0.85527	0.92029	0.97527
k_2	1.41132	1.34509	1.28061	1.21730	1.15901

Since ρ represents the rate at which money loses value over time, k_2 decreases in ρ, while k_1 increases in ρ, as in Table 2.6, reflecting the fact that we are less likely to want to hold in this case.

Table 2.6: k_1 and k_2 with varying ρ

ρ	0.4	0.45	0.5	0.55	0.6
k_1	0.84068	0.84858	0.85527	0.86105	0.86611
k_2	1.36281	1.31541	1.28061	1.25387	1.23262

Finally, larger transaction costs discourage trading. Naturally, Table 2.7 shows that as K increases, k_2 increases and k_1 decreases.

Table 2.7: k_1 and k_2 with varying K

K	0.0000	0.0005	0.0010	0.0015	0.0020
k_1	0.85698	0.85613	0.85527	0.85442	0.85356
k_2	1.27670	1.27866	1.28061	1.28254	1.28447

Using the stock prices of WMT ($\mathbf{S}^1$) and TGT ($\mathbf{S}^2$) from 2015 to 2020, we backtest the pairs trading rule. We found the pair $(k_1, k_2) = (0.85527, 1.28061)$ using the parameters obtained based on the historical price data from 2010 to 2015. Since we assume that we are initially flat ($i = 0$), a pairs trade (long $\mathbf{S}^1$ and short $\mathbf{S}^2$) is triggered when (X_t^1, X_t^2) enters Γ_3. The position is closed when (X_t^1, X_t^2) enters Γ_1. Initially, we allocate the trading capital \$100 K. When the first long signal is triggered, we use

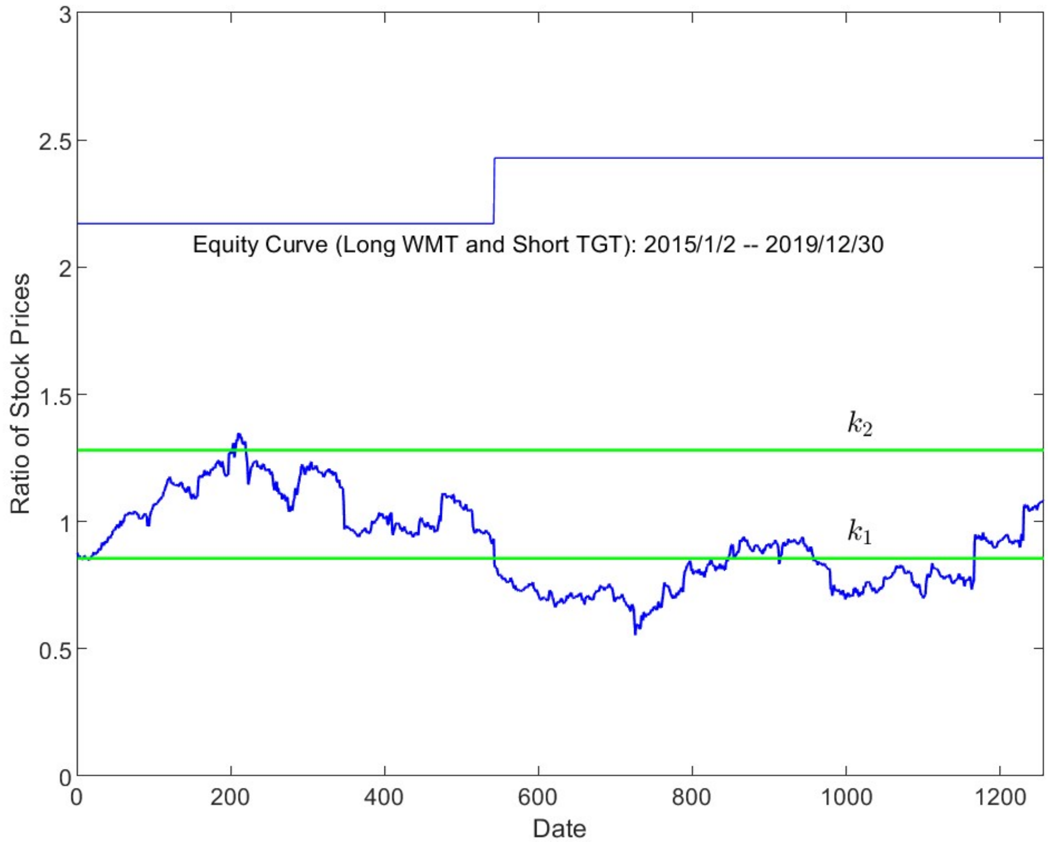

Figure 2.4: $\mathbf{S}^1 = \text{WMT}, \mathbf{S}^2 = \text{TGT}$ with threshold levels k_1, k_2

half of our capital to purchase WMT stocks and short the same amount of TGT, reversing these trades when the short signal is triggered. Each pairs transaction is charged \$5 commission. In Figure 2.4, the ratio of the stock prices is plotted against the thresholds k_1 and k_2. The equity curve indicates the date at which the round trip trade is finished and the proportion of profit earned.

We can also interchange the roles by taking $\mathbf{S}^1 = \text{TGT}$ and $\mathbf{S}^2 = \text{WMT}$. The new thresholds will be $(\widetilde{k}_1, \widetilde{k}_2) = \left(\frac{1}{k_2}, \frac{1}{k_1}\right) = (0.78087, 1.16922)$. In Figure 2.5, the ratio of the stock prices is plotted against the thresholds $\widetilde{k}_1$ and $\widetilde{k}_2$. At the conclusion of our first round trip, we can initiate a second round trip the next time (X_t^1, X_t^2) enters Γ_3, closing the position on the last trading day, $12/30/2019$. The

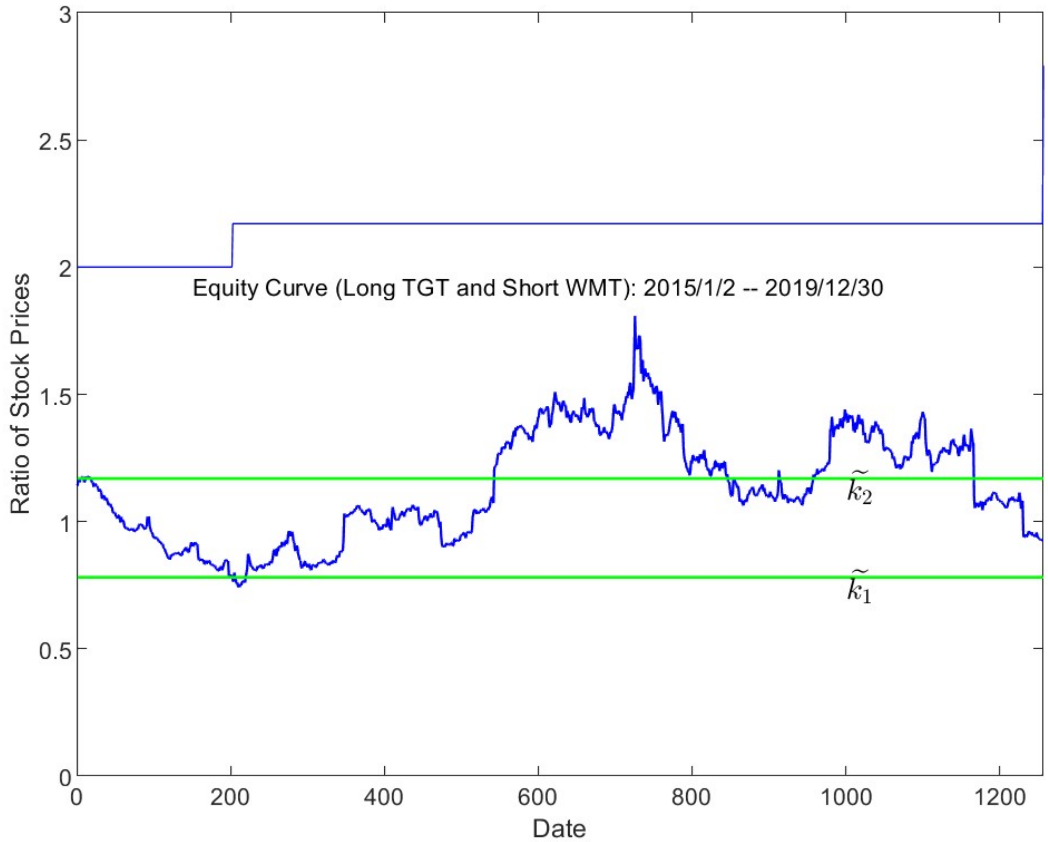

Figure 2.5: $\mathbf{S}^1 = \text{TGT}, \mathbf{S}^2 = \text{WMT}$ with threshold levels $\widetilde{k}_1, \widetilde{k}_2$

equity curve indicates the date at which each round trip trade is finished and the proportion of profit earned. Note that both types of trades have no overlap and, hence, they can be executed simultaneously without overextending our capital.

On the final trading day, there is $179,253 in the account. The grand total profit is $79,253, an increase of 79.25% in a five year span. Since only six trades are executed, the capital remains in cash most of the time and will earn interest or can be used for short-term trading, giving us the opportunity to further increase our capital.

Chapter 3

Round-Trip Pairs Trading under Geometric Brownian Motions with Reversible Initial Positions

3.1 Introduction

Having previously allowed the initial pairs position to be long or flat, a natural next question to consider is the short side of pairs trading. So, we begin again with the same stochastic differential equation as in (2.1) and the same partial differential operator as in (2.2), but now we allow our intial pairs position to be flat ($i = 0$), long ($i = 1$), or short ($i = -1$). As before, our initial trading decision will depend on the initial position. If intially we are long, we must sell one share of **Z** and conclude our trading activity. Whereas, if initially we are short, we must buy one share of **Z** and conclude our trading activity. However, if initially we are flat, we can either buy or sell one share of **Z**. Depending on that choice, our next trading move would be to sell or buy, respectively, after which we would conclude our trading activity. We use the term reversible to describe the initial positions due to the apparent symmetry between going one-share long in **Z** and going one-share short in **Z** with the roles of $\mathbf{S}^1$ and $\mathbf{S}^2$ interchanged.

3.2 Problem Formulation

As in Chapter 2, we consider two stocks, $\mathbf{S}^1$ and $\mathbf{S}^2$. We let $\{X_t^1, t \geq 0\}$ denote the prices of the stock $\mathbf{S}^1$, and let $\{X_t^2, t \geq 0\}$ denote the prices of the stock $\mathbf{S}^2$. They satisfy the following stochastic differential equation:

$$\mathrm{d}\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix} \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \mathrm{d}\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \right]$$

where μ_i, $i = 1, 2$ are the return rates, σ_{ij}, $i, j = 1, 2$ are the volatility constants, and (W_t^1, W_t^2) is a 2-dimensional standard Brownian motion.

We assume the pairs position, which we will denote $\mathbf{Z}$, consists of a one-share long position in stock $\mathbf{S}^1$ and a one-share short position in stock $\mathbf{S}^2$. We consider the case that the net position may initially be short (with one share short in $\mathbf{Z}$), long (with one share long in $\mathbf{Z}$), or flat (with no stock holdings of either $\mathbf{S}^1$ or $\mathbf{S}^2$). Let $i = -1, 0, 1$ denote the initial net positions of short, long, and flat, respectively. If initially we are short in $\mathbf{Z}$ ($i = -1$), we will buy one share of $\mathbf{Z}$, i.e. buy one share of $\mathbf{S}^1$ and sell one share of $\mathbf{S}^2$, at some time $\tau_0 \geq 0$, which will conclude our trading activity. If initially we are long in $\mathbf{Z}$ ($i = 1$), we will sell one share of $\mathbf{Z}$, i.e. sell $\mathbf{S}^1$ and buy $\mathbf{S}^2$ at some time $\tau_0 \geq 0$, which will conclude our trading activity. Otherwise, if initially we are flat ($i = 0$), we can either go long or short one share in $\mathbf{Z}$ at some time $\tau_1 \geq 0$. Depending on our activity at time τ_1, we would then either sell $\mathbf{S}^1$ and buy $\mathbf{S}^2$ (if long) or buy $\mathbf{S}^1$ and sell $\mathbf{S}^2$ (if short) at some time $\tau_2 \geq \tau_1$, thus concluding our trading activity.

We seek thresholds k_1, k_2, k_3, and k_4 for buying and selling $\mathbf{Z}$. Let k_1 indicate the price at which we will sell one share of $\mathbf{Z}$ when the net position is flat. Similarly, we will denote by k_2 the threshold for selling one share of $\mathbf{Z}$ when the net position is long. Next, k_3 will indicate the price at which we will buy one share of $\mathbf{Z}$ when the net position is short. Finally, the threshold for buying one share of $\mathbf{Z}$ when the net position is flat will be denoted by k_4. Then define the following function:

$$u(x_1, x_2, i) = \begin{cases} -1, & \text{for } i = 0 \text{ and } x_2 \leq x_1 k_1, \\ -1, & \text{for } i = 1 \text{ and } x_2 \leq x_1 k_2, \\ 1, & \text{for } i = -1 \text{ and } x_2 \geq x_1 k_3, \\ 1, & \text{for } i = 0 \text{ and } x_2 \geq x_1 k_4. \end{cases}$$

Let K denote the fixed percentage of transaction costs associated with buying or selling of stocks and $\rho > 0$ be a discount factor. As in Chapter 2, let $\beta_b = 1 + K$ and $\beta_s = 1 - K$. Then given the initial state (x_1, x_2), the initial net position $i = -1, 0, 1$, and the decision sequences $\Lambda_{-1} = (\tau_0)$, $\Lambda_1 = (\tau_0)$ and $\Lambda_0 = (\tau_1, \tau_2)$, the resulting reward functions are

$$\begin{aligned} J_{-1}(x_1, x_2, \tau_0) =& \mathbb{E}\left[-e^{-\rho\tau_0}\left(\beta_b X^1_{\tau_0} - \beta_s X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right], \\ J_0(x_1, x_2, \tau_1, \tau_2, u) =& \mathbb{E}\Big[\Big\{e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}} \\ & - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\Big\}\mathbb{I}_{\{u=1\}} \\ & + \Big\{e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}} \\ & - e^{-\rho\tau_2}\left(\beta_b X^1_{\tau_2} - \beta_s X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\Big\}\mathbb{I}_{\{u=-1\}}\Big], \\ J_1(x_1, x_2, \tau_0) =& \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]. \end{aligned}$$

For $i = -1, 0, 1$, let $V_i(x_1, x_2)$ denote the value functions with initial state $(X^1_0, X^2_0) = (x_1, x_2)$ and initial net positions $i = -1, 0, 1$. That is, $V_i(x_1, x_2) = \sup_{\Lambda_i} J_i(x_1, x_2, \Lambda_i)$.

3.3 Properties of the Value Functions

In this section, we establish basic properties of the value functions.

Lemma 2. *For all x_1, $x_2 > 0$, we have*

$$\beta_s x_1 - \beta_b x_2 \leq V_1(x_1, x_2) \leq x_1,$$

$$\beta_s x_2 - \beta_b x_1 \le V_{-1}(x_1, x_2) \le x_2, \text{ and}$$

$$0 \le V_0(x_1, x_2) \le 4x_1 + 4x_2.$$

Proof. Note that for all $x_1, x_2 > 0, V_1(x_1, x_2) \ge J_1(x_1, x_2, \tau_0) = \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]$. In particular,

$$V_1(x_1, x_2) \ge J_1(x_1, x_2, 0) = \beta_s x_1 - \beta_b x_2.$$

Similarly, $V_{-1}(x_1, x_2) \ge J_{-1}(x_1, x_2, \tau_0) = \mathbb{E}\left[-e^{-\rho\tau_0}\left(\beta_b X^1_{\tau_0} - \beta_s X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]$. In particular,

$$V_{-1}(x_1, x_2) \ge J_{-1}(x_1, x_2, 0) = \beta_s x_2 - \beta_b x_1.$$

Finally,

$$\begin{aligned} V_0(x_1, x_2) &\ge J_0(x_1, x_2, \tau_1, \tau_2, u) \\ &= \mathbb{E}\big[\big\{e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\big\}\mathbb{I}_{\{u=1\}} \\ &\quad + \big\{e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1\infty\}} - e^{-\rho\tau_2}\left(\beta_b X^1_{\tau_2} - \beta_s X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\big\}\mathbb{I}_{\{u=-1\}}\big]. \end{aligned}$$

Clearly, $V_0(x_2, x_2) \ge 0$ by definition and taking $\tau_1 = \infty$. So we establish the desired lower bounds.

Now, for all $\tau_0 \ge 0$,

$$\begin{aligned} J_1(x_1, x_2, \tau_0) &= \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &\le \mathbb{E}\left[e^{-\rho\tau_0}\left(X^1_{\tau_0} - X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &= \mathbb{E}\left[e^{-\rho\tau_0}X^1_{\tau_0}\mathbb{I}_{\{\tau_0<\infty\}}\right] - \mathbb{E}\left[e^{-\rho\tau_0}X^2_{\tau_0}\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &= x_1 + \mathbb{E}\left[\int_0^{\tau_0}\left(-\rho+\mu_1\right)e^{-\rho t}X^1_t \mathrm{d}t\,\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &\quad - x_2 - \mathbb{E}\left[\int_0^{\tau_0}\left(-\rho+\mu_2\right)e^{-\rho t}X^2_t \mathrm{d}t\,\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &\le x_1 - x_2 - \mathbb{E}\left[\int_0^{\tau_0}\left(-\rho+\mu_2\right)e^{-\rho t}X^2_t \mathrm{d}t\,\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &\le x_1 - x_2 + \mathbb{E}\left[\int_0^{\infty}\left(\rho-\mu_2\right)e^{-\rho t}X^2_t \mathrm{d}t\right] \\ &= x_1 - x_2 + (\rho-\mu_2)\int_0^{\infty} e^{-\rho t}x_2 e^{\mu_2 t}\mathrm{d}t \end{aligned}$$

$$= x_1 - x_2 + x_2$$

$$= x_1.$$

Also, for all $\tau_0 \geq 0$,

$$\begin{aligned}
J_{-1}(x_1, x_2, \tau_0) &= \mathbb{E}\left[-e^{-\rho\tau_0}\left(\beta_b X^1_{\tau_0} - \beta_s X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\leq \mathbb{E}\left[-e^{-\rho\tau_0}\left(X^1_{\tau_0} - X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&= \mathbb{E}\left[e^{-\rho\tau_0}X^2_{\tau_0}\mathbb{I}_{\{\tau_0<\infty\}}\right] - \mathbb{E}\left[e^{-\rho\tau_0}X^1_{\tau_0}\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&= x_2 + \mathbb{E}\left[\int_0^{\tau_0}\left(-\rho+\mu_2\right)e^{-\rho t}X^2_t \mathrm{d}t\, \mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\quad - x_1 - \mathbb{E}\left[\int_0^{\tau_0}\left(-\rho+\mu_1\right)e^{-\rho t}X^1_t \mathrm{d}t\, \mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\leq x_2 - x_1 - \mathbb{E}\left[\int_0^{\tau_0}\left(-\rho+\mu_1\right)e^{-\rho t}X^1_t \mathrm{d}t\, \mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&\leq x_2 - x_1 + \mathbb{E}\left[\int_0^{\infty}\left(\rho-\mu_1\right)e^{-\rho t}X^1_t \mathrm{d}t\right] \\
&= x_2 - x_1 + \left(\rho-\mu_1\right)\int_0^{\infty} e^{-\rho t}x_1 e^{\mu_1 t}\mathrm{d}t \\
&= x_2 - x_1 + x_1 \\
&= x_2.
\end{aligned}$$

And, for all $0 \leq \tau_1 \leq \tau_2$,

$$\begin{aligned}
&J_0(x_1, x_2, \tau_1, \tau_2, u) \\
&= \mathbb{E}\Big[e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\Big] - \mathbb{E}\Big[e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\Big] \\
&\quad + \mathbb{E}\Big[e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\Big] - \mathbb{E}\Big[e^{-\rho\tau_2}\left(\beta_b X^1_{\tau_2} - \beta_s X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\Big] \\
&\leq \mathbb{E}\Big[e^{-\rho\tau_2}X^1_{\tau_2}\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\Big] - \mathbb{E}\Big[e^{-\rho\tau_2}X^2_{\tau_2}\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\Big] \\
&\quad - \mathbb{E}\Big[e^{-\rho\tau_1}X^1_{\tau_1}\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\Big] + \mathbb{E}\Big[e^{-\rho\tau_1}X^2_{\tau_1}\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\Big] \\
&\quad + \mathbb{E}\Big[e^{-\rho\tau_1}X^1_{\tau_1}\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\Big] - \mathbb{E}\Big[e^{-\rho\tau_1}X^2_{\tau_1}\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\Big] \\
&\quad - \mathbb{E}\Big[e^{-\rho\tau_2}X^1_{\tau_2}\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\Big] + \mathbb{E}\Big[e^{-\rho\tau_2}X^2_{\tau_2}\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\Big]
\end{aligned}$$

$$\le x_1 - \mathbb{E}\left[x_2 \mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\right] + \mathbb{E}\left[\int_0^{\tau_2} (\rho-\mu_2)\, e^{-\rho t} X_t^2 \mathrm{d}t\, \mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\right]$$
$$+ x_2 - \mathbb{E}\left[x_1 \mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right] + \mathbb{E}\left[\int_0^{\tau_1} (\rho-\mu_1)\, e^{-\rho t} X_t^1 \mathrm{d}t\, \mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right]$$
$$+ x_1 - \mathbb{E}\left[x_2 \mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right] + \mathbb{E}\left[\int_0^{\tau_1} (\rho-\mu_2)\, e^{-\rho t} X_t^2 \mathrm{d}t\, \mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right]$$
$$+ x_2 - \mathbb{E}\left[x_1 \mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\right] + \mathbb{E}\left[\int_0^{\tau_2} (\rho-\mu_1)\, e^{-\rho t} X_t^1 \mathrm{d}t\, \mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\right].$$

Now,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_1} (\rho-\mu_1)\, e^{-\rho t} X_t^1 \mathrm{d}t\, \mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right] &\le \mathbb{E}\left[\int_0^{\tau_1} (\rho-\mu_1)\, e^{-\rho t} X_t^1 \mathrm{d}t\, \mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&\le \mathbb{E}\left[\int_0^{\infty} (\rho-\mu_1)\, e^{-\rho t} X_t^1 \mathrm{d}t\right] \\
&= (\rho-\mu_1)\int_0^{\infty} e^{-\rho t} x_1 e^{\mu_1 t}\mathrm{d}t \\
&= x_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_2} (\rho-\mu_2)\, e^{-\rho t} X_t^2 \mathrm{d}t\, \mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\right] &\le x_2, \\
\mathbb{E}\left[\int_0^{\tau_1} (\rho-\mu_2)\, e^{-\rho t} X_t^2 \mathrm{d}t\, \mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right] &\le x_2, \text{ and} \\
\mathbb{E}\left[\int_0^{\tau_2} (\rho-\mu_1)\, e^{-\rho t} X_t^1 \mathrm{d}t\, \mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\right] &\le x_1.
\end{aligned}$$

Thus, for all $0 \le \tau_1 \le \tau_2$, $J_0(x_1, x_2, \tau_1, \tau_2, u) \le 4x_1 + 4x_2$. □

3.4 HJB Equations

In this section, we study the associated HJB equations. Let

$$\mathcal{A} = \frac{1}{2}\left\{a_{11}x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} + a_{22}x_2^2\frac{\partial^2}{\partial x_2^2}\right\} + \mu_1 x_1 \frac{\partial}{\partial x_1} + \mu_2 x_2 \frac{\partial}{\partial x_2},$$

where $a_{11} = \sigma_{11}^2 + \sigma_{12}^2,\ a_{12} = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22},$ and $a_{22} = \sigma_{21}^2 + \sigma_{22}^2$. The associated HJB equations have the form, for $x_1, x_2 > 0$:

$$\begin{cases} \min\Big\{\rho v_1(x_1,x_2) - \mathcal{A}v_1(x_1,x_2),\ v_1(x_1,x_2) - \beta_{\rm s}x_1 + \beta_{\rm b}x_2\Big\} = 0, \\ \min\Big\{\rho v_{-1}(x_1,x_2) - \mathcal{A}v_{-1}(x_1,x_2),\ v_{-1}(x_1,x_2) + \beta_{\rm b}x_1 - \beta_{\rm s}x_2\Big\} = 0, \\ \min\Big\{\rho v_0(x_1,x_2) - \mathcal{A}v_0(x_1,x_2),\ v_0(x_1,x_2) - v_1(x_1,x_2) + \beta_{\rm b}x_1 - \beta_{\rm s}x_2, \\ \qquad\qquad v_0(x_1,x_2) - v_{-1}(x_1,x_2) - \beta_{\rm s}x_1 + \beta_{\rm b}x_2\Big\} = 0. \end{cases}$$

As in Chapter 2, the HJB equations can be reduced to an ODE problem by applying the following substitution. Let $y = x_2/x_1$ and $v_i(x_1,x_2) = x_1 w_i(x_2/x_1)$, for some function $w_i(y)$ and $i = -1, 0, 1$. The HJB equations can be given in terms of y and w_i as follows:

$$\begin{cases} \min\Big\{\rho w_1(y) - \mathcal{L}w_1(y),\ w_1(y) - \beta_{\rm s} + \beta_{\rm b}y\Big\} = 0, \\ \min\Big\{\rho w_{-1}(y) - \mathcal{L}w_{-1}(y),\ w_{-1}(y) + \beta_{\rm b} - \beta_{\rm s}y\Big\} = 0, \\ \min\Big\{\rho w_0(y) - \mathcal{L}w_0(y),\ w_0(y) - w_1(y) + \beta_{\rm b} - \beta_{\rm s}y,\ w_0(y) - w_{-1}(y) - \beta_{\rm s} + \beta_{\rm b}y\Big\} = 0. \end{cases}$$

We would like to open pairs position **Z** when the price of $\mathbf{S}^2$ is large relative to the price of $\mathbf{S}^1$ (k_3 and k_4) and close pairs position **Z** when the price of $\mathbf{S}^2$ is small relative to the price of $\mathbf{S}^1$ (k_1 and k_2). Additionally, we would be more willing to open pairs position **Z** when the net position is short than when the net position is flat, since when the net position is short we experience the risk of holding one share of $\mathbf{S}^2$ while borrowing one share of $\mathbf{S}^1$. Similarly, we would be more willing to close pairs position **Z** when the net position is long than when the net position is flat, since when the net position is long we experience the risk of borrowing one share of $\mathbf{S}^2$ while holding one share of $\mathbf{S}^1$. This suggests that we should expect $k_1 \le k_2 \le k_3 \le k_4$.

w_1 and k_1:

The first equation

$$\min\Big\{\rho w_1(y) - \mathcal{L}w_1(y),\ w_1(y) - \beta_{\rm s} + \beta_{\rm b}y\Big\} = 0$$

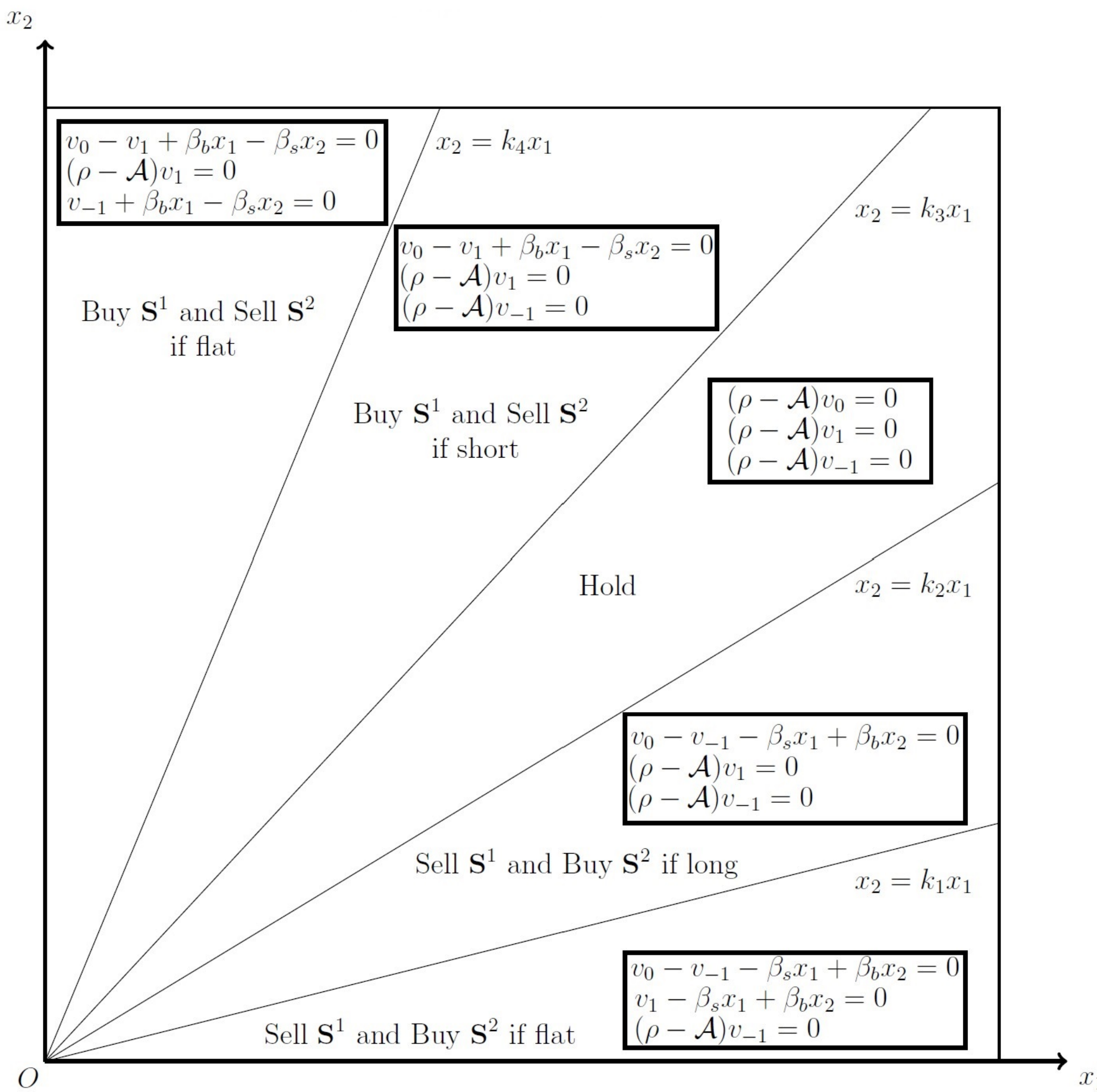

Figure 3.1: Thresholds for buying and selling regions

has solution

$$w_1(y) = \begin{cases} \beta_s - \beta_b y, & \text{for } \ 0 < y \leq k_1, \\ C_2 y^{\delta_2}, & \text{for } \ y > k_1, \end{cases}$$

as in Chapter 2. Then the smooth-fitting conditions yield

$$\beta_s - \beta_b k_1 = C_2 k_1^{\delta_2} \quad \text{and} \quad -\beta_b = C_2 \delta_2 k_1^{\delta_2 - 1}.$$

This will imply

$$(\beta_s - \beta_b k_1)\delta_2 = -\beta_b k_1 \implies k_1 = \frac{-\delta_2}{1-\delta_2} \cdot \frac{\beta_s}{\beta_b},$$

and

$$C_2 = \frac{\beta_b}{-\delta_2} \cdot k_1^{1-\delta_2} = \left(\frac{-\delta_2}{\beta_b}\right)^{-\delta_2} \left(\frac{\beta_s}{1-\delta_2}\right)^{1-\delta_2}.$$

w_{-1} and k_4:

Also, the second equation

$$\min\left\{\rho w_{-1}(y) - \mathcal{L} w_{-1}(y),\ w_{-1}(y) + \beta_{\mathrm{b}} - \beta_{\mathrm{s}} y\right\} = 0$$

has solution

$$w_{-1}(y) = \begin{cases} C_1 y^{\delta_1}, & \text{for } 0 < y < k_4, \\ \beta_s y - \beta_b, & \text{for } y \geq k_4. \end{cases}$$

Then the smooth-fitting conditions yield

$$C_1 k_4^{\delta_1} = \beta_s k_4 - \beta_b \quad \text{and} \quad C_1 \delta_1 k_4^{\delta_1 - 1} = \beta_s.$$

This will imply

$$(\beta_s k_4 - \beta_b)\delta_1 = \beta_s k_4 \implies k_4 = \frac{\delta_1}{\delta_1 - 1} \cdot \frac{\beta_b}{\beta_s},$$

and

$$C_1 = \frac{\beta_s}{\delta_1} \cdot k_4^{1-\delta_1} = \left(\frac{\beta_s}{\delta_1}\right)^{\delta_1} \left(\frac{\delta_1 - 1}{\beta_b}\right)^{\delta_1 - 1}.$$

w_0, k_2, and k_3:

Additionally, the third equation

$$\min\left\{\rho w_0(y) - \mathcal{L}w_0(y),\ w_0(y) - w_1(y) + \beta_{\rm b} - \beta_{\rm s}y,\ w_0(y) - w_{-1}(y) - \beta_{\rm s} + \beta_{\rm b}y\right\} = 0$$

has solution

$$w_0(y) = \begin{cases} C_1 y^{\delta_1} + \beta_{\rm s} - \beta_{\rm b}y, & \text{for } 0 < y \le k_2, \\ B_1 y^{\delta_1} + B_2 y^{\delta_2}, & \text{for } k_2 < y < k_3, \\ C_2 y^{\delta_2} - \beta_{\rm b} + \beta_{\rm s}y, & \text{for } y \ge k_3. \end{cases}$$

Then the smooth-fitting conditions yield

$$\begin{aligned} &C_1 k_2^{\delta_1} + \beta_{\rm s} - \beta_{\rm b}k_2 = B_1 k_2^{\delta_1} + B_2 k_2^{\delta_2}, \\ &C_1\delta_1 k_2^{\delta_1-1} - \beta_{\rm b} = B_1\delta_1 k_2^{\delta_1-1} + B_2\delta_2 k_2^{\delta_2-1}, \\ &B_1 k_3^{\delta_1} + B_2 k_3^{\delta_2} = C_2 k_3^{\delta_2} - \beta_{\rm b} + \beta_{\rm s}k_3, \\ &B_1\delta_1 k_3^{\delta_1-1} + B_2\delta_2 k_3^{\delta_2-1} = C_2\delta_2 k_3^{\delta_2-1} + \beta_{\rm s}. \end{aligned}$$

There are four equations and four parameters, B_1, B_2, k_2, and k_3, that need to be found. These equations can be written in the matrix form:

$$\begin{pmatrix} k_2^{\delta_1} & k_2^{\delta_2} \\ \delta_1 k_2^{\delta_1-1} & \delta_2 k_2^{\delta_2-1} \end{pmatrix} \begin{pmatrix} B_1 - C_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 & -k_2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_{\rm s} \\ \beta_{\rm b} \end{pmatrix},$$

and

$$\begin{pmatrix} k_3^{\delta_1} & k_3^{\delta_2} \\ \delta_1 k_3^{\delta_1-1} & \delta_2 k_3^{\delta_2-1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 - C_2 \end{pmatrix} = \begin{pmatrix} k_3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_{\rm s} \\ \beta_{\rm b} \end{pmatrix}.$$

We introduce a new matrix

$$\Phi(r) = \begin{pmatrix} r^{\delta_1} & r^{\delta_2} \\ \delta_1 r^{\delta_1-1} & \delta_2 r^{\delta_2-1} \end{pmatrix} \quad \text{and its inverse} \quad \Phi(r)^{-1} = \frac{1}{\delta_1 - \delta_2} \begin{pmatrix} -\delta_2 r^{-\delta_1} & r^{1-\delta_1} \\ \delta_1 r^{-\delta_2} & -r^{1-\delta_2} \end{pmatrix},$$

for $r \neq 0$. Returning to the smooth-fit conditions above, we have

$$\begin{pmatrix} B_1 - C_1 \\ B_2 \end{pmatrix} = \Phi(k_2)^{-1} \begin{pmatrix} 1 & -k_2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_\mathrm{s} \\ \beta_\mathrm{b} \end{pmatrix},$$

and

$$\begin{pmatrix} B_1 \\ B_2 - C_2 \end{pmatrix} = \Phi(k_3)^{-1} \begin{pmatrix} k_3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_\mathrm{s} \\ \beta_\mathrm{b} \end{pmatrix}.$$

This implies

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} + \Phi(k_2)^{-1} \begin{pmatrix} 1 & -k_2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_\mathrm{s} \\ \beta_\mathrm{b} \end{pmatrix} = \begin{pmatrix} 0 \\ C_2 \end{pmatrix} + \Phi(k_3)^{-1} \begin{pmatrix} k_3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_\mathrm{s} \\ \beta_\mathrm{b} \end{pmatrix}.$$

The second equality yields two equations of k_2 and k_3 that we can rewrite as

$$\left[\Phi(k_3)^{-1} \begin{pmatrix} k_3 & -1 \\ 1 & 0 \end{pmatrix} - \Phi(k_2)^{-1} \begin{pmatrix} 1 & -k_2 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} \beta_\mathrm{s} \\ \beta_\mathrm{b} \end{pmatrix} = \begin{pmatrix} C_1 \\ -C_2 \end{pmatrix}.$$

The matrix in $[\cdot]$ above is

$$\frac{1}{\delta_1 - \delta_2} \begin{pmatrix} (1-\delta_2) k_3^{1-\delta_1} + \delta_2 k_2^{-\delta_1} & \delta_2 k_3^{-\delta_1} + (1-\delta_2) k_2^{1-\delta_1} \\ -(1-\delta_1) k_3^{1-\delta_2} - \delta_1 k_2^{-\delta_2} & -\delta_1 k_3^{-\delta_2} - (1-\delta_1) k_2^{1-\delta_2} \end{pmatrix}.$$

The two equations involving k_2 and k_3 are

$$\frac{1}{\delta_1 - \delta_2} \begin{pmatrix} (1-\delta_2) k_3^{1-\delta_1} + \delta_2 k_2^{-\delta_1} & \delta_2 k_3^{-\delta_1} + (1-\delta_2) k_2^{1-\delta_1} \\ (1-\delta_1) k_3^{1-\delta_2} + \delta_1 k_2^{-\delta_2} & \delta_1 k_3^{-\delta_2} + (1-\delta_1) k_2^{1-\delta_2} \end{pmatrix} \begin{pmatrix} \beta_\mathrm{s} \\ \beta_\mathrm{b} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Recall that

$$C_1 = \frac{\beta_s}{\delta_1} \cdot k_4^{1-\delta_1} = \left(\frac{\beta_s}{\delta_1} \right)^{\delta_1} \left(\frac{\delta_1 - 1}{\beta_b} \right)^{\delta_1 - 1} \text{ and } C_2 = \frac{\beta_b}{-\delta_2} \cdot k_1^{1-\delta_2} = \left(\frac{-\delta_2}{\beta_b} \right)^{-\delta_2} \left(\frac{\beta_s}{1-\delta_2} \right)^{1-\delta_2}.$$

The system of equations for k_2 and k_3 is

$$\frac{(1-\delta_2)k_3^{1-\delta_1}+\delta_2 k_2^{-\delta_1}}{\delta_1-\delta_2}\beta_{\mathrm{s}}+\frac{\delta_2 k_3^{-\delta_1}+(1-\delta_2)k_2^{1-\delta_1}}{\delta_1-\delta_2}\beta_{\mathrm{b}}=\frac{\beta_s}{\delta_1}\cdot k_4^{1-\delta_1},$$
$$\frac{(1-\delta_1)k_3^{1-\delta_2}+\delta_1 k_2^{-\delta_2}}{\delta_1-\delta_2}\beta_{\mathrm{s}}+\frac{\delta_1 k_3^{-\delta_2}+(1-\delta_1)k_2^{1-\delta_2}}{\delta_1-\delta_2}\beta_{\mathrm{b}}=\frac{\beta_b}{-\delta_2}\cdot k_1^{1-\delta_2}.$$

We are looking for solutions (k_2, k_3) in the triangular region

$$T=\{(r,s):\ k_1\le r<s\le k_4\}\subset\mathbb{R}_+^2.$$

Let $\gamma=\dfrac{\beta_b}{\beta_s}$. Then we can reduce the system to

$$F_1(k_2,k_3):=\frac{(1-\delta_2)k_3^{1-\delta_1}+\delta_2 k_2^{-\delta_1}}{\delta_1-\delta_2}+\frac{\delta_2 k_3^{-\delta_1}+(1-\delta_2)k_2^{1-\delta_1}}{\delta_1-\delta_2}\gamma-\frac{k_4^{1-\delta_1}}{\delta_1}=0, \tag{3.1}$$
$$F_2(k_2,k_3):=\frac{(1-\delta_1)k_3^{1-\delta_2}+\delta_1 k_2^{-\delta_2}}{\delta_1-\delta_2}+\frac{\delta_1 k_3^{-\delta_2}+(1-\delta_1)k_2^{1-\delta_2}}{\delta_1-\delta_2}\gamma-\frac{\gamma k_1^{1-\delta_2}}{-\delta_2}=0. \tag{3.2}$$

Note that, by application of a special implicit function theorem [17], (k_1, k_4) is the unique solution to the system, since:

$$\begin{aligned}
F_1(k_1,k_4)&=\frac{(1-\delta_2)k_4^{1-\delta_1}+\delta_2 k_1^{-\delta_1}}{\delta_1-\delta_2}+\frac{\delta_2 k_1^{-\delta_1}+(1-\delta_2)k_4^{1-\delta_1}}{\delta_1-\delta_2}\gamma-\frac{k_4^{1-\delta_1}}{\delta_1}\\
&=\frac{k_4^{-\delta_1}}{\delta_1-\delta_2}\left[(1-\delta_2)k_4-\frac{\delta_1-\delta_2}{\delta_1}k_4+\delta_2\gamma\right]+\frac{k_1^{-\delta_1}}{\delta_1-\delta_2}\left[\delta_2+(1-\delta_2)\gamma k_1\right]\\
&=\frac{k_4^{-\delta_1}}{\delta_1-\delta_2}\left[\frac{(1-\delta_2)\delta_1}{\delta_1-1}\gamma-\frac{\delta_1-\delta_2}{\delta_1-1}\gamma+\delta_2\gamma\right]+\frac{k_1^{-\delta_1}}{\delta_1-\delta_2}\left[\delta_2+(-\delta_2)\right]\\
&=\frac{k_4^{-\delta_1}}{\delta_1-\delta_2}\left[\frac{-\delta_2(\delta_1-1)}{\delta_1-1}\gamma+\delta_2\gamma\right]\\
&=0,
\end{aligned}$$

and

$$F_2(k_1,k_4)=\frac{(1-\delta_1)k_4^{1-\delta_2}+\delta_1 k_1^{-\delta_2}}{\delta_1-\delta_2}+\frac{\delta_1 k_4^{-\delta_2}+(1-\delta_1)k_1^{1-\delta_2}}{\delta_1-\delta_2}\gamma-\frac{\gamma k_1^{1-\delta_2}}{-\delta_2}$$

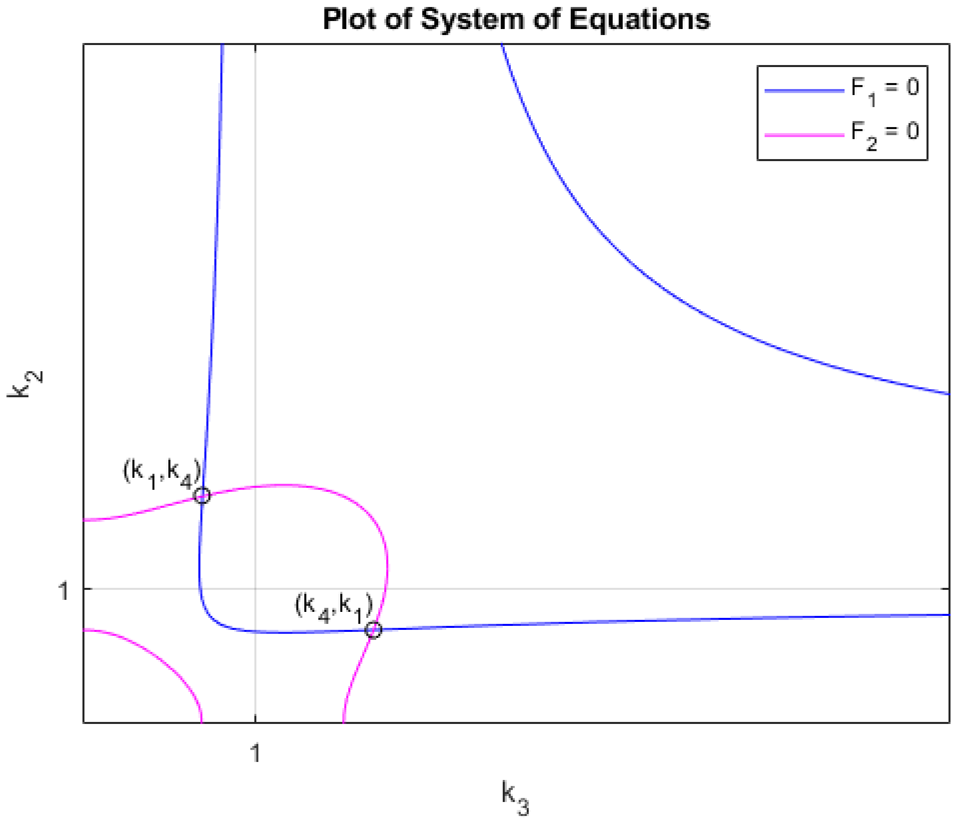

Figure 3.2: Numerical solution to system of equations in (3.1) and (3.2).

$$
\begin{aligned}
&= \frac{k_4^{-\delta_2}}{\delta_1 - \delta_2} \left[(1-\delta_1)k_4 - \delta_1\gamma\right] + \frac{k_1^{-\delta_2}}{\delta_1 - \delta_2} \left[\delta_1 + (1-\delta_1)\gamma k_1 + \frac{\delta_1 - \delta_2}{\delta_2}\gamma k_1\right] \\
&= \frac{k_4^{-\delta_2}}{\delta_1 - \delta_2} \left[-\delta_1\gamma + \delta_1\gamma\right] + \frac{k_1^{-\delta_2}}{\delta_1 - \delta_2} \left[\delta_1 + \frac{(1-\delta_1)(-\delta_2)}{1-\delta_2} - \frac{\delta_1 - \delta_2}{1-\delta_2}\right] \\
&= \frac{k_1^{-\delta_2}}{\delta_1 - \delta_2} \left[\delta_1 + \frac{\delta_1\delta_2 - \delta_1}{1-\delta_2}\right] \\
. &= \frac{k_1^{-\delta_2}}{\delta_1 - \delta_2} \left[\delta_1 - \frac{\delta_1(1-\delta_2)}{1-\delta_2}\right] \\
&= 0.
\end{aligned}
$$

Now, recall that the smooth-fit conditions for w_0 can be written as:

$$\begin{cases} (B_1 - C_1)k_2^{\delta_1} + B_2 k_2^{\delta_2} = \beta_s - \beta_b k_2, \\ \\ (B_1 - C_1)\delta_1 k_2^{\delta_1 - 1} + B_2 \delta_2 k_2^{\delta_2 - 1} = -\beta_b, \end{cases}$$

and

$$\begin{cases} B_1 k_3^{\delta_1} + (B_2 - C_2)k_3^{\delta_2} = \beta_s k_3 - \beta_b, \\ \\ B_1 \delta_1 k_3^{\delta_1 - 1} + (B_2 - C_2)\delta_2 k_3^{\delta_2 - 1} = \beta_s. \end{cases}$$

From these we obtain:

$$\begin{aligned} \frac{(B_1 - C_1)k_2^{\delta_1}}{(B_1 - C_1)\delta_1 k_2^{\delta_1 - 1}} = \frac{\beta_s - \beta_b k_2 - B_2 k_2^{\delta_2}}{-\beta_b - B_2 \delta_2 k_2^{\delta_2 - 1}} &\implies \frac{k_2}{\delta_1} = \frac{\beta_s - \beta_b k_2 - B_2 k_2^{\delta_2}}{-\beta_b - B_2 \delta_2 k_2^{\delta_2 - 1}} \\ &\implies \beta_s \delta_1 - \beta_b \delta_1 k_2 - B_2 \delta_1 k_2^{\delta_2} = -\beta_b k_2 - B_2 \delta_2 k_2^{\delta_2} \\ &\implies B_2 k_2^{\delta_2}(\delta_1 - \delta_2) = \beta_s \delta_1 + \beta_b (1 - \delta_1) k_2 \\ &\implies B_2 = \frac{\beta_s \delta_1 k_2^{-\delta_2} - \beta_b (\delta_1 - 1) k_2^{1 - \delta_2}}{\delta_1 - \delta_2}. \end{aligned}$$

Also,

$$\begin{aligned} &\frac{B_2 k_2^{\delta_2}}{B_2 \delta_2 k_2^{\delta_2 - 1}} = \frac{\beta_s - \beta_b k_2 - (B_1 - C_1)k_2^{\delta_1}}{-\beta_b - (B_1 - C_1)\delta_1 k_2^{\delta_1 - 1}} \\ &\implies \frac{k_2}{\delta_2} = \frac{\beta_s - \beta_b k_2 - (B_1 - C_1)k_2^{\delta_1}}{-\beta_b - (B_1 - C_1)\delta_1 k_2^{\delta_1 - 1}} \\ &\implies \beta_s \delta_2 - \beta_b \delta_2 k_2 - (B_1 - C_1)\delta_2 k_2^{\delta_1} = -\beta_b k_2 - (B_1 - C_1)\delta_1 k_2^{\delta_1} \\ &\implies (B_1 - C_1)k_2^{\delta_1}(\delta_1 - \delta_2) = -\beta_b (1 - \delta_2) k_2 - \beta_s \delta_2 \\ &\implies B_1 - C_1 = \frac{\beta_s(-\delta_2) k_2^{-\delta_1} - \beta_b (1 - \delta_2) k_2^{1 - \delta_1}}{\delta_1 - \delta_2}, \end{aligned}$$

$$\frac{B_1 k_3^{\delta_1}}{B_1 \delta_1 k_3^{\delta_1 - 1}} = \frac{\beta_s k_3 - \beta_b - (B_2 - C_2)k_3^{\delta_2}}{\beta_s - (B_2 - C_2)\delta_2 k_3^{\delta_2 - 1}}$$

$$
\begin{aligned}
&\implies \frac{k_3}{\delta_1} = \frac{\beta_s k_3 - \beta_b - (B_2 - C_2)k_3^{\delta_2}}{\beta_s - (B_2 - C_2)\delta_2 k_3^{\delta_2 - 1}} \\
&\implies \beta_s \delta_1 k_3 - \beta_b \delta_1 - (B_2 - C_2)\delta_1 k_3^{\delta_2} = \beta_s k_3 - (B_2 - C_2)\delta_2 k_3^{\delta_2} \\
&\implies (B_2 - C_2)k_3^{\delta_2}(\delta_1 - \delta_2) = \beta_s(\delta_1 - 1)k_3 - \beta_b \delta_1 \\
&\implies B_2 - C_2 = \frac{\beta_s(\delta_1 - 1)k_3^{1-\delta_2} - \beta_b \delta_1 k_3^{-\delta_2}}{\delta_1 - \delta_2},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{(B_2 - C_2)k_3^{\delta_2}}{(B_2 - C_2)\delta_2 k_3^{\delta_2 - 1}} = \frac{\beta_s k_3 - \beta_b - B_1 k_3^{\delta_1}}{\beta_s - B_1 \delta_1 k_3^{\delta_1 - 1}} &\implies \frac{k_3}{\delta_2} = \frac{\beta_s k_3 - \beta_b - B_1 k_3^{\delta_1}}{\beta_s - B_1 \delta_1 k_3^{\delta_1 - 1}} \\
&\implies \beta_s \delta_2 k_3 - \beta_b \delta_2 - B_1 \delta_2 k_3^{\delta_1} = \beta_s k_3 - B_1 \delta_1 k_3^{\delta_1} \\
&\implies B_1 k_3^{\delta_1}(\delta_1 - \delta_2) = \beta_s(1 - \delta_2)k_3 + \beta_b(\delta_2) \\
&\implies B_1 = \frac{\beta_s(1 - \delta_2)k_3^{1-\delta_1} - \beta_b(-\delta_2)k_3^{-\delta_1}}{\delta_1 - \delta_2}.
\end{aligned}
$$

Note then that if $k_2 = k_1$, we have

$$
\begin{aligned}
&k_2 = \frac{\beta_s}{\beta_b} \cdot \frac{-\delta_2}{1 - \delta_2} \\
&\implies \beta_s(-\delta_2) = \beta_b(1 - \delta_2)k_2 \\
&\implies \beta_s(-\delta_2)k_2^{-\delta_1} = \beta_b(1 - \delta_2)k_2^{1-\delta_1} \\
&\implies \beta_s(-\delta_2)k_2^{-\delta_1} - \beta_b(1 - \delta_2)k_2^{1-\delta_1} = 0 \\
&\implies \frac{\beta_s(-\delta_2)k_2^{-\delta_1} - \beta_b(1 - \delta_2)k_2^{1-\delta_1}}{\delta_1 - \delta_2} = 0 \\
&\implies B_1 - C_1 = 0 \\
&\implies B_1 = C_1.
\end{aligned}
$$

Also, if $k_3 = k_4$, we have

$$
\begin{aligned}
&k_3 = \frac{\beta_b}{\beta_s} \cdot \frac{\delta_1}{\delta_1 - 1} \\
&\implies \beta_b \delta_1 = \beta_s(\delta_1 - 1)k_3
\end{aligned}
$$

$$
\begin{aligned}
&\implies \beta_b\delta_1 k_3^{-\delta_2} = \beta_s(\delta_1 - 1)k_3^{1-\delta_2}\\
&\implies \beta_s(\delta_1 - 1)k_3^{1-\delta_2} - \beta_b\delta_1 k_3^{-\delta_2} = 0\\
&\implies \frac{\beta_s(\delta_1 - 1)k_3^{1-\delta_2} - \beta_b\delta_1 k_3^{-\delta_2}}{\delta_1 - \delta_2} = 0\\
&\implies B_2 - C_2 = 0\\
&\implies B_2 = C_2.
\end{aligned}
$$

Hence, in this case

$$
w_0(y) = \begin{cases} C_1 y^{\delta_1} + \beta_\mathrm{s} - \beta_\mathrm{b} y, & \text{for } \ 0 < y \le k_1,\\ C_1 y^{\delta_1} + C_2 y^{\delta_2}, & \text{for } \ k_1 < y < k_4,\\ C_2 y^{\delta_2} - \beta_\mathrm{b} + \beta_\mathrm{s} y, & \text{for } \ y \ge k_4. \end{cases}
$$

Let us relabel these threshholds as

$$
k_1^* := k_1 = k_2 = \frac{-\delta_2}{1-\delta_2} \cdot \frac{\beta_s}{\beta_b}, \tag{3.3}
$$

$$
k_2^* := k_3 = k_4 = \frac{\delta_1}{\delta_1 - 1} \cdot \frac{\beta_b}{\beta_s}. \tag{3.4}
$$

Then we have the following.

Theorem 3. *Let δ_i be given by (2.4) and k_i^* be given by (3.3), (3.4). Then the following functions w_1, w_{-1}, and w_0 satisfy the HJB equations (3.4):*

$$
w_1(y) = \begin{cases} \beta_s - \beta_b y, & \textit{for } \ 0 < y \le k_1^*,\\ \left(-\dfrac{\delta_2}{\beta_b}\right)^{-\delta_2} \left(\dfrac{\beta_s}{1-\delta_2}\right)^{1-\delta_2} y^{\delta_2}, & \textit{for } \ y > k_1^*, \end{cases}
$$

$$
w_{-1}(y) = \begin{cases} \left(\dfrac{\beta_s}{\delta_1}\right)^{\delta_1} \left(\dfrac{\delta_1 - 1}{\beta_b}\right)^{\delta_1 - 1} y^{\delta_1}, & \textit{for } \ 0 < y < k_2^*,\\ \beta_s y - \beta_b, & \textit{for } \ y \ge k_2^*, \end{cases}
$$

$$w_0(y) = \begin{cases} \left(\frac{\beta_s}{\delta_1}\right)^{\delta_1} \left(\frac{\delta_1 - 1}{\beta_b}\right)^{\delta_1 - 1} y^{\delta_1} + \beta_s - \beta_b y, & \textit{for } \ 0 < y \leq k_1^*, \\ \left(\frac{\beta_s}{\delta_1}\right)^{\delta_1} \left(\frac{\delta_1 - 1}{\beta_b}\right)^{\delta_1 - 1} y^{\delta_1} + \left(-\frac{\delta_2}{\beta_b}\right)^{-\delta_2} \left(\frac{\beta_s}{1 - \delta_2}\right)^{1 - \delta_2} y^{\delta_2}, & \textit{for } \ k_1^* < y < k_2^*, \\ \left(-\frac{\delta_2}{\beta_b}\right)^{-\delta_2} \left(\frac{\beta_s}{1 - \delta_2}\right)^{1 - \delta_2} y^{\delta_2} - \beta_b + \beta_s y, & \textit{for } \ y \geq k_2^*. \end{cases}$$

Proof. We divide the first quadrant of the plane into 3 regions,

$$\Gamma_1 : 0 < y \leq k_1^*, \ \ \Gamma_2 : k_1^* < y < k_2^*, \ \ \Gamma_3 : k_2^* \leq y.$$

Thus, to establish that we have found a solution to the HJB equations, we must establish the following list of variational inequalities:

$$\begin{cases} (\rho - \mathcal{L}) w_1(y) \geq 0, & \text{for } \ y \in \Gamma_1, \\ w_1(y) - \beta_s + \beta_b y \geq 0, & \text{for } \ y \in \Gamma_2 \cup \Gamma_3, \\ w_{-1}(y) + \beta_b - \beta_s y \geq 0, & \text{for } \ y \in \Gamma_1 \cup \Gamma_2, \\ (\rho - \mathcal{L}) w_{-1}(y) \geq 0, & \text{for } \ y \in \Gamma_3, \\ (\rho - \mathcal{L}) w_0(y) \geq 0, & \text{for } \ y \in \Gamma_1 \cup \Gamma_3, \\ w_0(y) - w_1(y) + \beta_b - \beta_s y \geq 0, & \text{for } \ y \in \Gamma_1 \cup \Gamma_2, \\ w_0(y) - w_{-1}(y) - \beta_s + \beta_b y \geq 0, & \text{for } \ y \in \Gamma_2 \cup \Gamma_3. \end{cases}$$

On Γ_1,

$$\begin{aligned} (\rho - \mathcal{L}) w_1(y) &= (\rho - \mathcal{L})(\beta_s - \beta_b y) \\ &= \rho \beta_s - \rho \beta_b y - \mathcal{L} \beta_s + \mathcal{L} \beta_b y \\ &= \rho \beta_s - \mu_1 \beta_s + \mu_1 \beta_b y + (\mu_2 - \mu_1) \beta_b y - \rho \beta_b y \\ &= (\rho - \mu_1) \beta_s - (\rho - \mu_2) \beta_b y. \end{aligned}$$

Hence,

$$\begin{aligned}(\rho-\mathcal{L})w_1(y)\geq 0 &\iff (\rho-\mu_1)\beta_s \geq (\rho-\mu_2)\beta_b y\\ &\iff y\leq \frac{\rho-\mu_1}{\rho-\mu_2}\cdot\frac{\beta_s}{\beta_b}\\ &\iff y\leq \frac{\delta_1}{\delta_1-1}\cdot\frac{-\delta_2}{1-\delta_2}\cdot\frac{\beta_s}{\beta_b}\\ &\iff y\leq \frac{\delta_1}{\delta_1-1}\cdot k_1^*,\end{aligned}$$

which holds, since $y \leq k_1^* \leq \dfrac{\delta_1}{\delta_1-1}\cdot k_1^*$.

On $\Gamma_2\cup\Gamma_3$,

$$w_1(y)-\beta_s+\beta_b y = C_2 y^{\delta_2}-\beta_s+\beta_b y.$$

Hence

$$w_1(y)-\beta_s+\beta_b y\geq 0 \iff C_2 y^{\delta_2}-\beta_s+\beta_b y\geq 0.$$

Let $f(y)=C_2y^{\delta_2}-\beta_s+\beta_b y$. Then

$$\begin{aligned}f'(y)\geq 0 &\iff C_2\delta_2 y^{\delta_2-1}+\beta_b\geq 0\\ &\iff C_2(-\delta_2)y^{\delta_2-1}\leq \beta_b\\ &\iff y^{\delta_2-1}\leq \frac{\beta_b}{C_2(-\delta_2)}=(k_1^*)^{\delta_2-1}\\ &\iff y^{1-\delta_2}\geq (k_1^*)^{1-\delta_2}\\ &\iff y\geq k_1^*,\end{aligned}$$

which clearly holds. Hence $f(y)$ is increasing for $y>k_1^*$. Since $f(k_1^*)=0$, it must be that $w_1(y)-\beta_s+\beta_b y\geq 0$ on $\Gamma_2\cup\Gamma_3$.

On $\Gamma_1\cup\Gamma_2$,

$$w_{-1}(y)+\beta_b-\beta_s y = C_1 y^{\delta_1}+\beta_b-\beta_s y.$$

Hence

$$w_{-1}(y) + \beta_b - \beta_s y \geq 0 \iff C_1 y^{\delta_1} + \beta_b - \beta_s y \geq 0.$$

Let $g(y) = C_1 y^{\delta_1} + \beta_b - \beta_s y$. Then

$$\begin{aligned} g'(y) \leq 0 &\iff C_1 \delta_1 y^{\delta_1 - 1} - \beta_s \leq 0 \\ &\iff C_1 \delta_1 y^{\delta_1 - 1} \leq \beta_s \\ &\iff y^{\delta_1 - 1} \leq \frac{\beta_s}{C_1 \delta_1} = (k_2^*)^{\delta_1 - 1} \\ &\iff y \leq k_2^*, \end{aligned}$$

which clearly holds. Hence $g(y)$ is decreasing for $y < k_2^*$. Since $g(k_2^*) = 0$, it must be that $w_{-1}(y) + \beta_b - \beta_s y \geq 0$ on $\Gamma_1 \cup \Gamma_2$.

On Γ_3,

$$\begin{aligned} (\rho - \mathcal{L}) w_{-1}(y) &= (\rho - \mathcal{L})(\beta_s y - \beta_b) \\ &= \rho \beta_s y - \rho \beta_b - \mathcal{L} \beta_s y + \mathcal{L} \beta_b \\ &= \rho \beta_s y - \mu_2 \beta_s y + \mu_1 \beta_b - \rho \beta_b \\ &= (\rho - \mu_2) \beta_s y - (\rho - \mu_1) \beta_b. \end{aligned}$$

Hence,

$$\begin{aligned} (\rho - \mathcal{L}) w_{-1}(y) \geq 0 &\iff (\rho - \mu_2) \beta_s y \geq (\rho - \mu_1) \beta_b \\ &\iff y \geq \frac{\rho - \mu_1}{\rho - \mu_2} \cdot \frac{\beta_b}{\beta_s} \\ &\iff y \geq \frac{-\delta_2}{1 - \delta_2} \cdot \frac{\delta_1}{\delta_1 - 1} \cdot \frac{\beta_b}{\beta_s} \\ &\iff y \geq \frac{-\delta_2}{1 - \delta_2} \cdot k_2^*, \end{aligned}$$

which holds, since $y \geq k_2^* \geq \dfrac{-\delta_2}{1-\delta_2} \cdot k_2^*$.

On Γ_1,

$$
\begin{aligned}
(\rho - \mathcal{L})w_0(y) &= (\rho - \mathcal{L})(w_{-1}(y) + w_1(y)) \\
&= (\rho - \mathcal{L})w_{-1}(y) + (\rho - \mathcal{L})w_1(y) \\
&= 0 + (\rho - \mathcal{L})w_1(y),
\end{aligned}
$$

and we have already established that $(\rho - \mathcal{L})w_1(y) \geq 0$ on Γ_1.

On Γ_3,

$$
\begin{aligned}
(\rho - \mathcal{L})w_0(y) &= (\rho - \mathcal{L})(w_1(y) + w_{-1}(y)) \\
&= (\rho - \mathcal{L})w_1(y) + (\rho - \mathcal{L})w_{-1}(y) \\
&= 0 + (\rho - \mathcal{L})w_{-1}(y),
\end{aligned}
$$

and we have already established that $(\rho - \mathcal{L})w_{-1}(y) \geq 0$ on Γ_3.

On Γ_1,

$$
\begin{aligned}
w_0(y) - w_1(y) + \beta_b - \beta_s y &= C_1 y^{\delta_1} + \beta_s - \beta_b y - \beta_s + \beta_b y + \beta_b - \beta_s y \\
&= C_1 y^{\delta_1} + \beta_b - \beta_s y,
\end{aligned}
$$

and we have already established that $C_1 y^{\delta_1} + \beta_b - \beta_s y \geq 0$ on Γ_1.

On Γ_2,

$$
\begin{aligned}
w_0(y) - w_1(y) + \beta_b - \beta_s y &= C_1 y^{\delta_1} + C_2 y^{\delta_2} - C_2 y^{\delta_2} + \beta_b - \beta_s y \\
&= C_1 y^{\delta_1} + \beta_b - \beta_s y,
\end{aligned}
$$

and we have already established that $C_1 y^{\delta_1} + \beta_b - \beta_s y \geq 0$ on Γ_2.

On Γ_2,

$$\begin{aligned} w_0(y) - w_{-1}(y) - \beta_s + \beta_b y &= C_1 y^{\delta_1} + C_2 y^{\delta_2} - C_1 y^{\delta_1} - \beta_s + \beta_b y \\ &= C_2 y^{\delta_2} - \beta_s + \beta_b y, \end{aligned}$$

and we have already established that $C_2 y^{\delta_2} - \beta_s + \beta_b y \geq 0$ on Γ_2.

On Γ_3,

$$\begin{aligned} w_0(y) - w_{-1}(y) - \beta_s + \beta_b y &= C_2 y^{\delta_2} - \beta_b + \beta_s y - \beta_s y + \beta_b - \beta_s + \beta_b y \\ &= C_2 y^{\delta_2} - \beta_s + \beta_b y, \end{aligned}$$

and we have already established that $C_2 y^{\delta_2} - \beta_s + \beta_b y \geq 0$ on Γ_3.

□

3.5 A Verification Theorem

Theorem 4. *We have* $v_i(x_1, x_2) = x_1 w_i\left(\frac{x_2}{x_1}\right) = V_i(x_1, x_2)$, $i = -1, 0, 1$. *If initially* $i = -1$, *let* $\tau_0^* = \inf\{t \geq 0 : (X_t^1, X_t^2) \in \Gamma_3\}$. *If initially* $i = 1$, *let* $\tau_0^* = \inf\{t \geq 0 : (X_t^1, X_t^2) \in \Gamma_1\}$. *Finally, if initially* $i = 0$, *let* $\tau_1^* = \inf\{t \geq 0 : (X_t^1, X_t^2) \notin \Gamma_2\}$. *If* $\left(X_{\tau_1^*}^1, X_{\tau_1^*}^2\right) \in \Gamma_1$, *then* $u^* = -1$ *and* $\tau_2^* = \inf\{t \geq \tau_1^* : (X_t^1, X_t^2) \in \Gamma_3\}$. *Otherwise, if* $\left(X_{\tau_1^*}^1, X_{\tau_1^*}^2\right) \in \Gamma_3$, *then* $u^* = 1$ *and* $\tau_2^* = \inf\{t \geq \tau_1^* : (X_t^1, X_t^2) \in \Gamma_1\}$.

Proof. Given $(\rho - \mathcal{A})v_i(x_1, x_2) \geq 0$, $i = -1, 0, 1$, and applying Dynkin's formula and Fatou's Lemma as in Øksendal [**18**], we have for any stopping times $0 \leq \tau_1 \leq \tau_2$, almost surely,

$$\mathbb{E}e^{-\rho\tau_1} v_i\left(X_{\tau_1}^1, X_{\tau_1}^2\right) \geq \mathbb{E}e^{-\rho\tau_2} v_i\left(X_{\tau_2}^1, X_{\tau_2}^2\right).$$

Hence, we have

$$
\begin{aligned}
v_0(x_1,x_2) &\geq \mathbb{E}\left[e^{-\rho\tau_1} v_0\left(X^1_{\tau_1}, X^2_{\tau_1}\right)\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_1} v_0\left(X^1_{\tau_1}, X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_1}\left(v_1\left(X^1_{\tau_1}, X^2_{\tau_1}\right) - \beta_b X^1_{\tau_1} + \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right] \\
&\quad + \mathbb{E}\left[e^{-\rho\tau_1}\left(v_{-1}\left(X^1_{\tau_1}, X^2_{\tau_1}\right) + \beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right] \\
&= \mathbb{E}\left[e^{-\rho\tau_1} v_1\left(X^1_{\tau_1}, X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right] \\
&\quad + \mathbb{E}\left[e^{-\rho\tau_1} v_{-1}\left(X^1_{\tau_1}, X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right] \\
&\quad + \mathbb{E}\left[e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right] \\
&\quad - \mathbb{E}\left[e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_2} v_1\left(X^1_{\tau_2}, X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\right] \\
&\quad + \mathbb{E}\left[e^{-\rho\tau_2} v_{-1}\left(X^1_{\tau_2}, X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\right] \\
&\quad + \mathbb{E}\left[e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right] \\
&\quad - \mathbb{E}\left[e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=1\}}\right] \\
&\quad - \mathbb{E}\left[e^{-\rho\tau_2}\left(\beta_b X^1_{\tau_2} - \beta_s X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\mathbb{I}_{\{u=-1\}}\right] \\
&\quad + \mathbb{E}\left[e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=-1\}}\right] \\
&\quad - \mathbb{E}\left[e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\mathbb{I}_{\{u=1\}}\right] \\
&= \mathbb{E}\Big[\left\{e^{-\rho\tau_2}\left(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}\left(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}}\right\}\mathbb{I}_{\{u=1\}} \\
&\quad + \left\{e^{-\rho\tau_1}\left(\beta_s X^1_{\tau_1} - \beta_b X^2_{\tau_1}\right)\mathbb{I}_{\{\tau_1<\infty\}} - e^{-\rho\tau_2}\left(\beta_b X^1_{\tau_2} - \beta_s X^2_{\tau_2}\right)\mathbb{I}_{\{\tau_2<\infty\}}\right\}\mathbb{I}_{\{u=-1\}}\Big] \\
&= J_0\left(x_1, x_2, \tau_1, \tau_2, u\right),
\end{aligned}
$$

for all $0 \leq \tau_1 \leq \tau_2$. This implies $v_0\left(x_1, x_2\right) \geq V_0\left(x_1, x_2\right)$. Also,

$$
\begin{aligned}
v_1(x_1,x_2) &\geq \mathbb{E}\left[e^{-\rho\tau_0} v_1\left(X^1_{\tau_0}, X^2_{\tau_0}\right)\right] \\
&\geq \mathbb{E}\left[e^{-\rho\tau_0} v_1\left(X^1_{\tau_0}, X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\
&= \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^2_{\tau_0} - \beta_b X^1_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]
\end{aligned}
$$

$$= \mathbb{E}\left[-e^{-\rho\tau_0}\left(\beta_b X^1_{\tau_0} - \beta_s X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right]$$
$$= J_1\left(x_1, x_2, \tau_0\right),$$

and

$$\begin{aligned} v_{-1}(x_1,x_2) &\geq \mathbb{E}\left[e^{-\rho\tau_0} v_{-1}\left(X^1_{\tau_0}, X^2_{\tau_0}\right)\right] \\ &\geq \mathbb{E}\left[e^{-\rho\tau_0} v_{-1}\left(X^1_{\tau_0}, X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &= \mathbb{E}\left[e^{-\rho\tau_0}\left(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}\right)\mathbb{I}_{\{\tau_0<\infty\}}\right] \\ &= J_{-1}\left(x_1, x_2, \tau_0\right), \end{aligned}$$

for all $0 \leq \tau_0$. Hence $v_1\left(x_1, x_2\right) \geq V_1\left(x_1, x_2\right)$ and $v_{-1}\left(x_1, x_2\right) \geq V_{-1}\left(x_1, x_2\right)$.

Now define $\tau_1^* = \inf\left\{t \geq 0 : (X^1_t, X^2_t) \notin \Gamma_2\right\}$. If $\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right) \in \Gamma_1$, then $\tau_2^* = \inf\left\{t \geq \tau_1^* : (X^1_t, X^2_t) \in \Gamma_3\right\}$. Otherwise, if $\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right) \in \Gamma_3$, then $\tau_2^* = \inf\left\{t \geq \tau_1^* : (X^1_t, X^2_t) \in \Gamma_1\right\}$. Using Dynkin's formula, we obtain

$$v_0(x_1,x_2) = \mathbb{E}\left[e^{-\rho\tau_1^*} v_0\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\right],$$

$$\mathbb{E}\left[e^{-\rho\tau_1^*} v_1\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\right] = \mathbb{E}\left[e^{-\rho\tau_2^*} v_1\left(X^1_{\tau_2^*}, X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\right],$$

and

$$\mathbb{E}\left[e^{-\rho\tau_1^*} v_{-1}\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\right] = \mathbb{E}\left[e^{-\rho\tau_2^*} v_{-1}\left(X^1_{\tau_2^*}, X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\right].$$

Thus,

$$\begin{aligned} v_0(x_1,x_2) &= \mathbb{E}\left[e^{-\rho\tau_1^*} v_0\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\right] \\ &= \mathbb{E}\left[e^{-\rho\tau_1^*}\left(v_1\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right) - \beta_b X^1_{\tau_1^*} + \beta_s X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=1\}}\right] \\ &\quad + \mathbb{E}\left[e^{-\rho\tau_1^*}\left(v_{-1}\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right) + \beta_s X^1_{\tau_1^*} - \beta_b X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=-1\}}\right] \\ &= \mathbb{E}\left[e^{-\rho\tau_1^*} v_1\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=1\}}\right] \\ &\quad + \mathbb{E}\left[e^{-\rho\tau_1^*} v_{-1}\left(X^1_{\tau_1^*}, X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=-1\}}\right] \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\left[e^{-\rho\tau_1^*}\left(\beta_s X^1_{\tau_1^*}-\beta_b X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=-1\}}\right]\\
&-\mathbb{E}\left[e^{-\rho\tau_1^*}\left(\beta_b X^1_{\tau_1^*}-\beta_s X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=1\}}\right]\\
=&\,\mathbb{E}\left[e^{-\rho\tau_2^*}v_1\left(X^1_{\tau_2^*},X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\mathbb{I}_{\{u^*=1\}}\right]\\
&+\mathbb{E}\left[e^{-\rho\tau_2^*}v_{-1}\left(X^1_{\tau_2^*},X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\mathbb{I}_{\{u^*=-1\}}\right]\\
&+\mathbb{E}\left[e^{-\rho\tau_1^*}\left(\beta_s X^1_{\tau_1^*}-\beta_b X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=-1\}}\right]\\
&-\mathbb{E}\left[e^{-\rho\tau_1^*}\left(\beta_b X^1_{\tau_1^*}-\beta_s X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=1\}}\right]\\
\geq&\,\mathbb{E}\left[e^{-\rho\tau_2^*}\left(\beta_s X^1_{\tau_2^*}-\beta_b X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\mathbb{I}_{\{u^*=1\}}\right]\\
&-\mathbb{E}\left[e^{-\rho\tau_2^*}\left(\beta_b X^1_{\tau_2^*}-\beta_s X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\mathbb{I}_{\{u^*=-1\}}\right]\\
&+\mathbb{E}\left[e^{-\rho\tau_1^*}\left(\beta_s X^1_{\tau_1^*}-\beta_b X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=-1\}}\right]\\
&-\mathbb{E}\left[e^{-\rho\tau_1^*}\left(\beta_b X^1_{\tau_1^*}-\beta_s X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\mathbb{I}_{\{u^*=1\}}\right]\\
=&\,\mathbb{E}\Bigg[\left\{e^{-\rho\tau_2^*}\left(\beta_s X^1_{\tau_2^*}-\beta_b X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}-e^{-\rho\tau_1^*}\left(\beta_b X^1_{\tau_1^*}-\beta_s X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}\right\}\mathbb{I}_{\{u^*=1\}}\\
&+\left\{e^{-\rho\tau_1^*}\left(\beta_s X^1_{\tau_1^*}-\beta_b X^2_{\tau_1^*}\right)\mathbb{I}_{\{\tau_1^*<\infty\}}-e^{-\rho\tau_2^*}\left(\beta_b X^1_{\tau_2^*}-\beta_s X^2_{\tau_2^*}\right)\mathbb{I}_{\{\tau_2^*<\infty\}}\right\}\mathbb{I}_{\{u^*=-1\}}\Bigg]\\
=&\,J_0\left(x_1,x_2,\tau_1^*,\tau_2^*,u^*\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
v_1(x_1,x_2)=&\,\mathbb{E}\left[e^{-\rho\tau_0^*}v_1\left(X^1_{\tau_0^*},X^2_{\tau_0^*}\right)\mathbb{I}_{\{\tau_0^*<\infty\}}\right]\\
=&\,\mathbb{E}\left[e^{-\rho\tau_0^*}\left(\beta_s X^2_{\tau_0^*}-\beta_b X^1_{\tau_0^*}\right)\mathbb{I}_{\{\tau_0^*<\infty\}}\right]\\
=&\,J_1\left(x_1,x_2,\tau_0^*\right),
\end{aligned}
$$

$$
\begin{aligned}
v_{-1}(x_1,x_2)=&\,\mathbb{E}\left[e^{-\rho\tau_0^*}v_{-1}\left(X^1_{\tau_0^*},X^2_{\tau_0^*}\right)\mathbb{I}_{\{\tau_0^*<\infty\}}\right]\\
=&\,\mathbb{E}\left[e^{-\rho\tau_0^*}\left(\beta_s X^1_{\tau_0^*}-\beta_b X^2_{\tau_0^*}\right)\mathbb{I}_{\{\tau_0^*<\infty\}}\right]\\
=&\,J_{-1}\left(x_1,x_2,\tau_0^*\right).
\end{aligned}
$$

□

3.6 A Numerical Example

As in Chapter 2, we consider adjusted closing price data for Walmart (WMT) and Target (TGT) from 2010 to 2020. The first half of the data is used to calibrate the model, and the second half is used to test the results. Using a least-squares method, we obtain the following parameters: $\mu_1 = 0.09696$, $\mu_2 = 0.14347$, $\sigma_{11} = 0.19082$, $\sigma_{12} = 0.04036$, $\sigma_{21} = 0.04036$, and $\sigma_{22} = 0.13988$. We specify $K = 0.001$ and $\rho = 0.5$. Then we find $k_1^* = 0.85527$, and $k_2^* = 1.32175$.

Next we examine the dependence of k_1^* and k_2^* on the parameters by varying each. In Table 3.1, we see that k_1^* and k_2^* both decrease in μ_1. This leads to a larger buying region, Γ_3.

Table 3.1: k_1^* and k_2^* with varying μ_1

μ_1	-0.00304	0.04696	0.09696	0.14696	0.19696
k_1^*	0.91380	0.89057	0.85527	0.80194	0.72644
k_2^*	1.54402	1.42682	1.32175	1.23477	1.17006

On the other hand, both k_1^* and k_2^* increase in μ_2, as indicated in Table 3.2. This creates a larger Γ_1 and, hence, encourages early exit.

Table 3.2: k_1^* and k_2^* with varying μ_2

μ_2	0.04347	0.09347	0.14347	0.19347	0.24347
k_1^*	0.76457	0.81341	0.85527	0.88736	0.91037
k_2^*	1.15468	1.21883	1.32175	1.48176	1.72581

When varying σ_{11} and σ_{22}, as in Table 3.3 and Table 3.4, we find that k_1^* decreases while k_2^* increases, in both σ_{11} and σ_{22}. This leads to a smaller buying zone, Γ_1, due to the increased risk, as well as a smaller selling zone, Γ_3, because there is more price movement overall.

Table 3.3: k_1^* and k_2^* with varying σ_{11}

σ_{11}	0.09082	0.14082	0.19082	0.24082	0.29082
k_1^*	0.92069	0.89220	0.85527	0.81532	0.77497
k_2^*	1.22784	1.26704	1.32175	1.38652	1.45871

Table 3.4: k_1^* and k_2^* with varying σ_{22}

σ_{22}	0.03988	0.08988	0.13988	0.18988	0.23988
k_1^*	0.88356	0.87601	0.85527	0.82593	0.79206
k_2^*	1.27943	1.29045	1.32175	1.36871	1.42724

However, as $\sigma_{12} = \sigma_{21}$ increases, we find that k_1^* increases, while k_2^* decreases (Table 3.5). The greater correlation leads to a larger Γ_1, and hence more opportunity for buying, as well as a larger Γ_3, and hence more opportunity for selling.

Table 3.5: k_1^* and k_2^* with varying $\sigma_{12} = \sigma_{21}$

σ_{12}	-0.05964	-0.00964	0.04036	0.09036	0.14036
k_1^*	0.73242	0.79189	0.85527	0.92029	0.97527
k_2^*	1.54345	1.42754	1.32175	1.22837	1.15911

Since ρ represents the rate at which money loses value over time, k_1^* increases in ρ, while k_2^* decreases in ρ, as in Table 3.6, reflecting the fact that we are less likely to want to hold in this case.

Table 3.6: k_1^* and k_2^* with varying ρ

ρ	0.4	0.45	0.5	0.55	0.6
k_1^*	0.84068	0.84858	0.85527	0.86105	0.86611
k_2^*	1.40518	1.35725	1.32175	1.29425	1.27222

Finally, larger transaction costs discourage trading. Naturally, Table 3.7 shows that as K increases, k_1^* decreases and k_2^* increases.

Table 3.7: k_1^* and k_2^* with varying K

K	0.0000	0.0005	0.0010	0.0015	0.0020
k_1^*	0.85698	0.85613	0.85527	0.85442	0.85356
k_2^*	1.31911	1.32043	1.32175	1.32307	1.32439

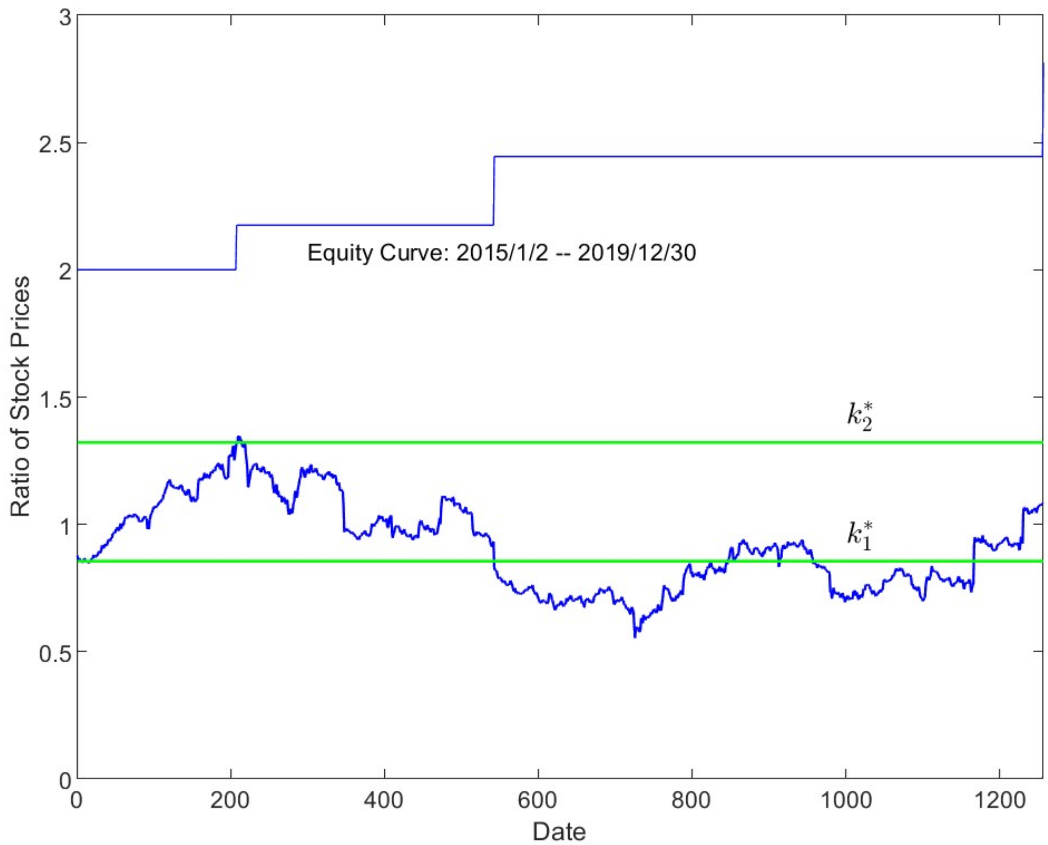

Figure 3.3: $\mathbf{S}^1 = \text{WMT}, \mathbf{S}^2 = \text{TGT}$ with threshold levels k_1^*, k_2^*

Using the stock prices of WMT ($\mathbf{S}^1$) and TGT ($\mathbf{S}^2$) from 2015 to 2020, we backtest the pairs trading rule. We found the pair $(k_1, k_2) = (0.85527, 1.32177)$ using the parameters obtained based on the historical prices from 2010 to 2015. Since we assume that we are initially flat ($i = 0$), a pairs trade is triggered when (X_t^1, X_t^2) enters Γ_1 (short $\mathbf{S}^1$ and long $\mathbf{S}^2$) or Γ_3 (long $\mathbf{S}^1$ and short $\mathbf{S}^2$). Depending on which occurs first, the pairs position is reversed when (X_t^1, X_t^2) enters Γ_3 or Γ_1, respectively. Initially,

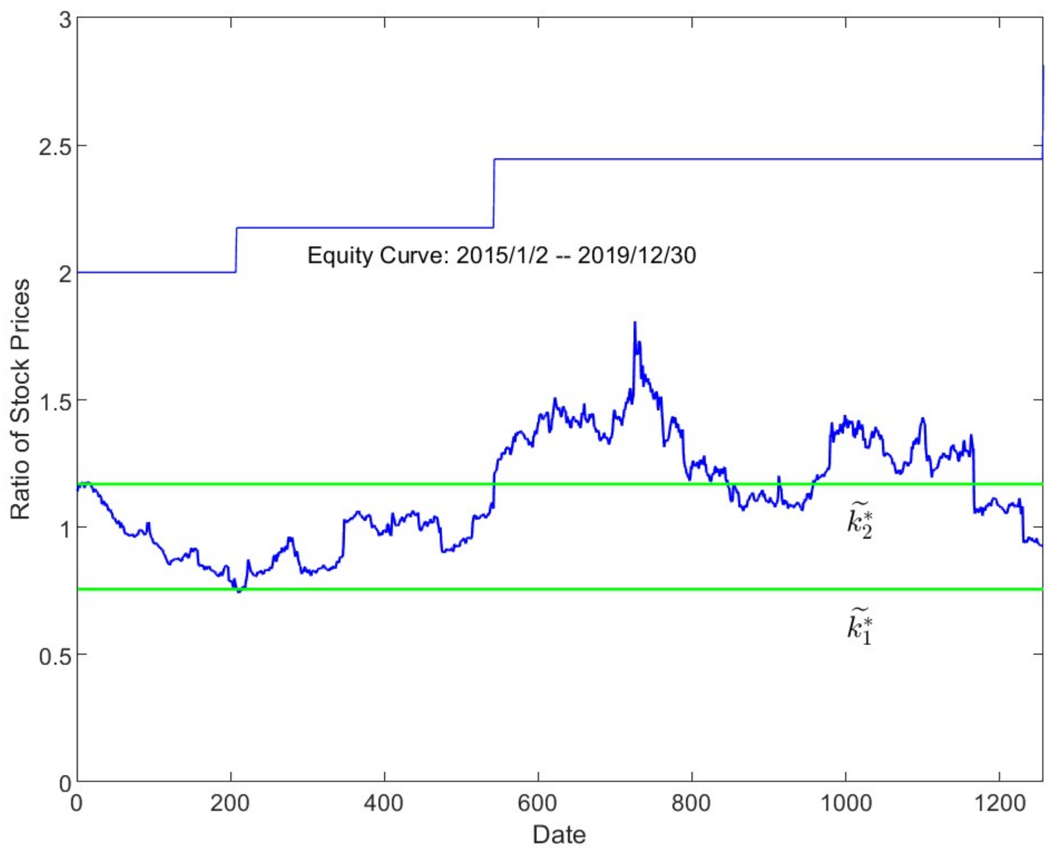

Figure 3.4: $\mathbf{S}^1 = \text{TGT}, \mathbf{S}^2 = \text{WMT}$ with threshold levels $\widetilde{k_1^*}, \widetilde{k_2^*}$

we allocate the trading capital $\$100$ K. When the first short signal is triggered, we simulate the short sale of $\$50$ K in WMT stocks and the purchasing of the same amount of TGT and reverse these trades when the long signal is triggered. Each pairs transaction is charged $\$5$ commission. In Figure 3.3, the ratio of the stock prices is plotted against the thresholds k_1^* and k_2^*. A second round trip can be initiated the next time (X_t^1, X_t^2) is in Γ_1 or Γ_3 and will proceed accordingly. The final round trip will be closed on the last trading day, $12/30/2019$. The equity curve indicates the date at which each round trip trade is finished and the proportion of profit earned.

We can also interchange the roles by taking $\mathbf{S}^1 = \text{TGT}$ and $\mathbf{S}^2 = \text{WMT}$. The new thresholds will be $(\widetilde{k}_1^*, \widetilde{k}_2^*) = \left(\frac{1}{k_2^*}, \frac{1}{k_1^*}\right) = (0.75656, 1.16922)$. In Figure 3.4, the ratio of the stock prices is plotted against the thresholds $\widetilde{k}_1^*$ and $\widetilde{k}_2^*$. Note that this results in the exact same sequence of trades as when the roles were reversed. Hence, there is no need to consider this scenario.

On the final trading day, there is \$181,351 in the account. The grand total profit is \$81,351, an increase of 81.35% in a five year span. Since only six trades are executed, the capital remains in cash most of the time and will earn interest or can be used for short-term trading, giving us the opportunity to further increase our capital.

Chapter 4

Pairs Trading under a Mean-Reversion Model with Regime Switching

4.1 Introduction

This is joint work with Dr. Phong Luu, Dr. Jingzhi Tie, and Dr. Qing Zhang. This chapter delves further into the mathematics of pairs trading. Specifically, this chapter focuses on the scenario where the difference between a pair follows a mean-reversion model. Mean-reversion models are commonly employed in financial markets to capture price movements that tend to gravitate towards an equilibrium level. We also introduce the problem of regime switching. Market models with regime switching are important in market analysis. In a mean-reversion model, the rate of reversion, the mean (equilibrium), and the volatility are all subject to change in the long run. One way to capture these changes is to introduce a switching process dictating sudden changes in system parameters.

The main purpose of this chapter is to study pairs trading rules under mean-reversion models coupled with a two-state Markov chain. In particular, we consider an optimal pairs trading rule in which a pairs (long-short) position consists of a long position of one stock and a short position of the other. The pair's value Z_t is defined as a difference of the stock prices. The state processes (Z_t, α_t) are modeled so that Z_t is mean-reversion coupled with a two-state Markov chain, α_t. To focus on closed-form solutions, we only consider the Markov chain with an absorbing state. The objective is to initiate (buy) and close (sell) the pairs positions sequentially to maximize a discounted payoff function. A fixed (commission or slippage) cost will be imposed to each transaction. We study the problem following a dynamic programming approach and establish the associated HJB equations for the value functions. We show that the corresponding optimal stopping times can be determined by four threshold levels x_1, x_2, x_3, and x_4. These key levels can be obtained by solving a set of algebraic-like equations. In addition, we provide a set of sufficient conditions that guarantee the optimality of our pairs trading rule. We also examine the dependence of these threshold levels on various parameters in a numerical example.

4.2 Problem formulation

We consider two stocks $\mathbf{S}^1$ and $\mathbf{S}^2$. Let X_t^1 and X_t^2 denote their prices, respectively, at time t. The corresponding pairs position consists of a long position in $\mathbf{S}^1$ and short position in $\mathbf{S}^2$. For simplicity, we include one share of $\mathbf{S}^1$ and K_0 shares of $\mathbf{S}^2$ (for some $K_0 > 0$) in the pairs position. The price of the position is given by $Z_t = X_t^1 - K_0 X_t^2$. We assume that Z_t is a mean-reverting (Ornstein-Uhlenbeck) process governed by

$$\mathrm{d}Z_t = \theta(\alpha_t)[\mu(\alpha_t) - Z_t]\mathrm{d}t + \sigma(\alpha_t)\mathrm{d}W_t, Z_0 = x,$$

where θ, μ, and σ are functions of a two-state Markov chain $\alpha_t \in \{1, 2\}$, and W_t is a standard Brownian motion independent of α_t. In this chapter, we consider the Markov chain with the absorbing state $\alpha = 2$. In particular, its generator is $Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$, for some $\lambda > 0$.

Remark 4.2.1. Our main focus is the full characterization of the solution in closed form. In view of this, we limit our attention to the above setup. Generalization of the HJB equations to the case with more than two states is possible, but their closed-form solutions are difficult to obtain. As for the absorbing state condition, it will not much affect the applicability of the results in practice, because pairs trading typically involves short-term actions, while switching in market modes is of longer term. The Markov chain with an absorbing state will help to capture a major portion of the switching effects under our discounted reward functions.

In this chapter, one share long in the pairs position $\mathbf{Z}$ means the combination of a one-share long position in $\mathbf{S}^1$ and a K_0-share short position in $\mathbf{S}^2$. Note that the value of the pairs position Z_t may be negative. Let $0 \le \tau_1^b \le \tau_1^s \le \tau_2^b \le \tau_2^s \le \cdots$ denote a sequence of stopping times. A buying decision is made at τ_n^b and a selling decision at τ_n^s, $n = 1, 2, \ldots$.

We consider the case that the net position at any time can be either long (with one share of $\mathbf{Z}$) or flat (no stock position of either $\mathbf{S}^1$ or $\mathbf{S}^2$). Let $i = 0, 1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell $\mathbf{Z}$ before acquiring any future shares. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should start by buying a share of $\mathbf{Z}$. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$.

Let $K > 0$ denote the fixed transaction cost (e.g., slippage and/or commission) associated with buying or selling of $\mathbf{Z}$. Given the initial state $(Z_0, \alpha_0) = (x, \alpha)$, initial net position $i = 0, 1$, and the decision sequences, Λ_0 and Λ_1, the corresponding reward functions are

$$J_i(x, \alpha, \Lambda_i) = \begin{cases} \mathbb{E}\left\{ \sum_{n=1}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \infty\}} \right\}, & \text{if } i = 0, \\ \mathbb{E}\left\{ e^{-\rho\tau_1^s}(Z_{\tau_1^s} - K) \right. \\ \quad \left. + \sum_{n=2}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \infty\}} \right\}, & \text{if } i = 1, \end{cases}$$

where $\rho > 0$ is a given discount factor. In this paper, the term $\mathbb{E} \sum_{n=1}^{\infty} \xi_n$ is interpreted as $\limsup_{N \to \infty} \mathbb{E} \sum_{n=1}^{N} \xi_n$ for given random variables ξ_n.

4.3 Properties of the Value Functions

Let $V_i(x, \alpha)$ denote the value functions with the initial state $(Z_0, \alpha_0) = (x, \alpha)$ and initial net positions $i = 0, 1$. That is,

$$V_i(x, \alpha) = \sup_{\Lambda_i} J_i(x, \alpha, \Lambda_i).$$

It can be shown as in Song and Zhang [**19**] the following inequalities hold:

$$V_0(x,\alpha) \geq V_1(x,\alpha) - x - K,\ V_1(x,\alpha) \geq V_0(x,\alpha) + x - K,$$

and, for some constants C_1 and C_2,

$$0 \leq V_0(x,\alpha) \leq C_1|x| + C_2,\ \text{and}\ x - K \leq V_1(x,\alpha) \leq C_1|x| + C_2. \tag{4.1}$$

4.4 HJB equations

Let $\mathcal{A}_\alpha$, $\alpha = 1, 2$, denote the generator of (Z_t, α_t). Then,

$$\begin{aligned}
\mathcal{A}_1 v(x,1) &= \frac{\sigma_1^2}{2} \cdot \frac{\mathrm{d}^2 v(x,1)}{\mathrm{d}x^2} + \theta_1(\mu_1 - x)\frac{\mathrm{d}v(x,1)}{\mathrm{d}x} + \lambda(v(x,2) - v(x,1)), \\
\mathcal{A}_2 v(x,2) &= \frac{\sigma_2^2}{2} \cdot \frac{\mathrm{d}^2 v(x,2)}{\mathrm{d}x^2} + \theta_2(\mu_2 - x)\frac{\mathrm{d}v(x,2)}{\mathrm{d}x}.
\end{aligned}$$

The associated HJB equations are given by:

$$\begin{aligned}
&\min\Big\{[\rho - \mathcal{A}_1]v_0(x,1),\ v_0(x,1) - v_1(x,1) + x + K\Big\} = 0, \\
&\min\Big\{[\rho - \mathcal{A}_1]v_1(x,1),\ v_1(x,1) - v_0(x,1) - x + K\Big\} = 0. \\
&\min\Big\{[\rho - \mathcal{A}_2]v_0(x,2),\ v_0(x,2) - v_1(x,2) + x + K\Big\} = 0, \\
&\min\Big\{[\rho - \mathcal{A}_2]v_1(x,2),\ v_1(x,2) - v_0(x,2) - x + K\Big\} = 0.
\end{aligned} \tag{4.2}$$

These HJB equations are equivalent to the corresponding set of variational inequalities outlined in Øksendal [**18**]. Each equation consists of two parts. The continuation region is determined by the first part, while a buy/sell action is dictated by the second part.

To simplify the notation, we let

$$u_j(x) = v_j(x,1) \quad \text{and} \quad w_j(x) = v_j(x,2)\ \text{for}\ j = 0, 1.$$

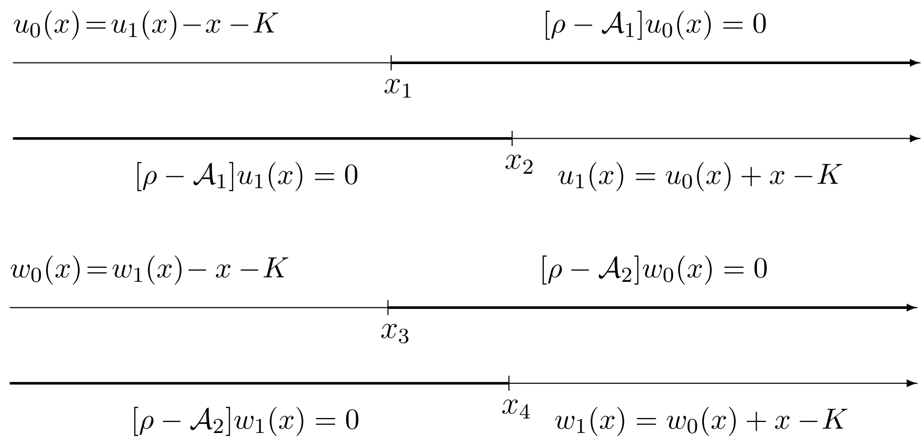

Figure 4.1: Continuation Regions (darkened intervals)

The HJB equations can be written in terms of these functions:

$$
\begin{aligned}
&\min\Big\{[\rho - \mathcal{A}_1]u_0(x),\ u_0(x) - u_1(x) + x + K\Big\} = 0,\\
&\min\Big\{[\rho - \mathcal{A}_1]u_1(x),\ u_1(x) - u_0(x) - x + K\Big\} = 0.\\
&\min\Big\{[\rho - \mathcal{A}_2]w_0(x),\ w_0(x) - w_1(x) + x + K\Big\} = 0,\\
&\min\Big\{[\rho - \mathcal{A}_2]w_1(x),\ w_1(x) - w_0(x) - x + K\Big\} = 0.
\end{aligned}
\tag{4.3}
$$

Intuitively, the optimal strategy should be of the buy-low-and-sell-high type as in [**19**]. One would expect threshold levels x_1, x_2, x_3, and x_4 (with $x_1 < x_2$ and $x_3 < x_4$) as in Figure 4.1: if $\alpha_t = 1$ buy when $Z_t \leq x_1$ and sell when $Z_t \geq x_2$; and if $\alpha_t = 2$ buy when $Z_t \leq x_3$ and sell when $Z_t \geq x_4$.

Note that the last two equations in (4.3) are independent of $\alpha_t = 1$ due to the absorbing state. We can solve for them separately. To this end, we first start with the equation $[\rho - \mathcal{A}_2]w_j(x) = 0$, which is

$$
\frac{\sigma_2^2}{2} \cdot \frac{\mathrm{d}^2 w_j(x)}{\mathrm{d}x^2} + \theta_2(\mu_2 - x)\frac{\mathrm{d}w_j(x)}{\mathrm{d}x} - \rho w_j(x) = 0. \tag{4.4}
$$

The equation for $w_j(x)$ is homogeneous. As shown in Eloe at al. [7], it has two linearly independent solutions given by

$$\psi_1(x) = \int_0^\infty \eta_2(t) e^{-\kappa_2(\mu_2 - x)t} \mathrm{d}t \quad \text{and} \quad \psi_2(x) = \int_0^\infty \eta_2(t) e^{\kappa_2(\mu_2 - x)t} \mathrm{d}t,$$

where $\kappa_2 = \sqrt{2\theta_2}/\sigma_2$, $\beta_2 = \rho/\theta_2$, and $\eta_2(t) = t^{\beta_2 - 1}\exp(-t^2/2)$. Note that $\psi_1(x) \to 0$ as $x \to -\infty$ and $\psi_2(x) \to 0$ as $x \to \infty$.

In view of Figure 4.1, the solution for the equation $\min\Big\{[\rho - \mathcal{A}_2]w_0(x),\ w_0(x) - w_1(x) + x + K\Big\} = 0$ has the form $w_0(x) = w_1(x) - x - K$ for $x < x_3$; and $[\rho - \mathcal{A}_2]w_0(x) = 0$ for $x > x_3$. The linear growth conditions (4.1) on the value functions imply, for some A_2, $w_0(x) = A_2\psi_2(x)$ for $x > x_3$.

Similarly, the solution for the equation $\min\Big\{[\rho - \mathcal{A}_2]w_1(x),\ w_1(x) - w_0(x) - x + K\Big\} = 0$ has the form $w_1(x) = w_0(x) + x - K$ for $x > x_4$; and $[\rho - \mathcal{A}_2]w_1(x) = 0$ for $x < x_4$. The linear growth conditions (4.1) imply, for some A_1, $w_1(x) = A_1\psi_1(x)$ for $x < x_4$.

Therefore, we have

$$w_0(x) = \begin{cases} A_1\psi_1(x) - x - K & \text{for } x < x_3, \\ A_2\psi_2(x) & \text{for } x \geq x_3, \end{cases} \quad \text{and} \quad w_1(x) = \begin{cases} A_1\psi_1(x) & \text{for } x < x_4, \\ A_2\psi_2(x) + x - K & \text{for } x \geq x_4. \end{cases}$$

Then the smooth-fit conditions at x_3 and x_4 yield

$$\begin{cases} A_1\psi_1(x_3) - x_3 - K = A_2\psi_2(x_3), \\ A_1\psi_1'(x_3) - 1 = A_2\psi_2'(x_3), \end{cases} \quad \text{and} \quad \begin{cases} A_1\psi_1(x_4) = A_2\psi_2(x_4) + x_4 - K, \\ A_1\psi_1'(x_4) = A_2\psi_2'(x_4) + 1. \end{cases}$$

We can rewrite the above system in matrix form to get

$$\begin{pmatrix} \psi_1(x_3) & \psi_2(x_3) \\ \psi_1'(x_3) & \psi_2'(x_3) \end{pmatrix} \begin{pmatrix} A_1 \\ -A_2 \end{pmatrix} = \begin{pmatrix} x_3 + K \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \psi_1(x_4) & \psi_2(x_4) \\ \psi_1'(x_4) & \psi_2'(x_4) \end{pmatrix} \begin{pmatrix} A_1 \\ -A_2 \end{pmatrix} = \begin{pmatrix} x_4 - K \\ 1 \end{pmatrix}.$$

This implies x_3 and x_4 have to satisfy

$$\begin{pmatrix} A_1 \\ -A_2 \end{pmatrix} = \begin{pmatrix} \psi_1(x_3) & \psi_2(x_3) \\ \psi_1'(x_3) & \psi_2'(x_3) \end{pmatrix}^{-1} \begin{pmatrix} x_3+K \\ 1 \end{pmatrix} = \begin{pmatrix} \psi_1(x_4) & \psi_2(x_4) \\ \psi_1'(x_4) & \psi_2'(x_4) \end{pmatrix}^{-1} \begin{pmatrix} x_4-K \\ 1 \end{pmatrix}. \tag{4.5}$$

Once we find x_3 and x_4, we can then find A_1 and A_2.

Next, we move on to solve the first two equations in (4.3). First, note that the homogeneous equations $[\rho - \mathcal{A}_1]u_j(x) = 0$ are given by

$$\frac{\sigma_1^2}{2}\frac{\mathrm{d}^2 u_j(x)}{\mathrm{d}x^2} + \theta_1(\mu_1 - x)\frac{\mathrm{d}u_j(x)}{\mathrm{d}x} - (\rho+\lambda)u_j(x) = -\lambda w_j(x).$$

The lemma below is about the solution of the above non-homogeneous ODE.

Lemma 3. *The general solution of*

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + \theta(\mu - x)\frac{\mathrm{d}f(x)}{\mathrm{d}x} - (\rho+\lambda)f(x) = -\lambda g(x) \tag{4.6}$$

is of the form

$$f(x) = C_1\int_0^\infty \eta(t)e^{-\kappa(\mu-x)t}\mathrm{d}t \; + C_2\int_0^\infty \eta(t)e^{\kappa(\mu-x)t}\mathrm{d}t \; + \frac{\lambda}{\sigma\sqrt{\pi\theta}}\int_{-\infty}^\infty g(y)K(x,y,\mu)\mathrm{d}y,$$

for constants C_1 and C_2. Here.

$$K(x,y,\mu) = \int_0^1 u^{\frac{\rho+\lambda}{\theta}-1}(1-u^2)^{-\frac{1}{2}}\exp\left\{-\frac{\theta}{\sigma^2}\frac{[(x-\mu)u + (\mu-y)]^2}{1-u^2}\right\}\mathrm{d}u,$$

and κ and $\eta(t)$ are given by

$$\kappa = \frac{\sqrt{2\theta}}{\sigma}, \textit{ and } \beta = \frac{\rho+\lambda}{\theta};\; \eta(t) = t^{\beta-1}\exp(-t^2/2).$$

Proof. To find the general solution of (4.6), we only need to find a special solution. We use the method of Fourier transform to reduce the second order equation of x to a first order equation of its dual variable,

ξ. Define the Fourier transform with respect to x as

$$\widehat{u}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x) \mathrm{d}x.$$

Then its inverse is given by

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \widehat{u}(\xi) \mathrm{d}\xi.$$

We consider the case when the solution of (4.6) has decay properties

$$\lim_{|x| \to \infty} x f(x) = 0.$$

This yields

$$\begin{aligned}
\widehat{f'}(\xi) &= \int_{-\infty}^{\infty} e^{-ix\xi} f'(x) \mathrm{d}x \\
&= e^{-ix\xi} f(x) \Big|_{-\infty}^{\infty} + i\xi \int_{-\infty}^{\infty} e^{-i\xi x} f(x) \mathrm{d}x \\
&= i\xi \int_{-\infty}^{\infty} e^{-i\xi x} f(x) \mathrm{d}x
\end{aligned}$$

and

$$\widehat{x f(x)}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} x f(x) \mathrm{d}x = i \frac{\mathrm{d}}{\mathrm{d}\xi} \widehat{f}(\xi).$$

Applying the Fourier transform to Equation (4.6) and using the above properties, we obtain

$$i\theta\mu\xi \widehat{f}(\xi) - i\theta \frac{\mathrm{d}}{\mathrm{d}\xi}[i\xi \widehat{f}(\xi)] - \frac{\sigma^2}{2}\xi^2 \widehat{f}(\xi) - (\rho + \lambda)\widehat{f}(\xi) = -\lambda \widehat{g}(\xi).$$

We rewrite the above equation as follows

$$\theta \frac{\mathrm{d}}{\mathrm{d}\xi}[\xi \widehat{f}(\xi)] + [i\theta\mu\xi - (\rho + \lambda) - \frac{\sigma^2}{2}\xi^2]\widehat{f}(\xi) = -\lambda \widehat{g}(\xi).$$

Let $u(\xi) = \xi \widehat{f}(\xi)$. Then we have a first order linear equation

$$\theta \frac{\mathrm{d}u}{\mathrm{d}\xi} + \left[i\theta\mu - \frac{\rho + \lambda}{\xi} - \frac{\sigma^2}{2}\xi \right] u = -\lambda \widehat{g}.$$

This equation can be written in the standard form of the first order linear equation:

$$\frac{\mathrm{d}u}{\mathrm{d}\xi}+\left[i\mu-\frac{\rho+\lambda}{\theta\xi}-\frac{\sigma^2}{2\theta}\xi\right]u=-\frac{\lambda}{\theta}\widehat{g}.$$

This linear equation has a multiplier

$$m(\xi)=\exp\left\{\int\left(i\mu-\frac{\rho+\lambda}{\theta\xi}-\frac{\sigma^2}{2\theta}\xi\right)\mathrm{d}\xi\right\}=\xi^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\xi-\frac{\sigma^2}{4\theta}\xi^2}.$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[u(\xi)\xi^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\xi-\frac{\sigma^2}{4\theta}\xi^2}\right]=-\frac{\lambda}{\theta}\widehat{g}(\xi)\xi^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\xi-\frac{\sigma^2}{4\theta}\xi^2}.$$

By integrating both sides, we find

$$u(\xi)\xi^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\xi-\frac{\sigma^2}{4\theta}\xi^2}=-\frac{\lambda}{\theta}\int\widehat{g}(\xi)\xi^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\xi-\frac{\sigma^2}{4\theta}\xi^2}\mathrm{d}\xi+c,$$

where c is an arbitrary constant. So the solution $u(\xi)$ in integral form is given by

$$u(\xi)=-\frac{\lambda}{\theta}\xi^{\frac{\rho+\lambda}{\theta}}e^{-i\mu\xi+\frac{\sigma^2}{4\theta}\xi^2}\int_{\xi_0}^{\xi}\widehat{g}(\eta)\eta^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\eta-\frac{\sigma^2}{4\theta}\eta^2}\mathrm{d}\eta+c\xi^{\frac{\rho+\lambda}{\theta}}e^{-i\mu\xi+\frac{\sigma^2}{4\theta}\xi^2}.$$

This implies

$$\widehat{f}(\xi)=-\frac{\lambda}{\theta}\xi^{\frac{\rho+\lambda}{\theta}-1}e^{-i\mu\xi+\frac{\sigma^2}{4\theta}\xi^2}\int_{\xi_0}^{\xi}\widehat{g}(\eta)\eta^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\eta-\frac{\sigma^2}{4\theta}\eta^2}\mathrm{d}\eta+c\xi^{\frac{\rho+\lambda}{\theta}-1}e^{-i\mu\xi+\frac{\sigma^2}{4\theta}\xi^2}.$$

We want to find a special solution with certain decay properties, so we take $\xi_0=\infty$ and $c=0$. Hence we have a closed form:

$$\widehat{f}(\xi)=\frac{\lambda}{\theta}\xi^{\frac{\rho+\lambda}{\theta}-1}e^{-i\mu\xi+\frac{\sigma^2}{4\theta}\xi^2}\int_{\xi}^{\infty}\widehat{g}(\eta)\eta^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\eta-\frac{\sigma^2}{4\theta}\eta^2}\mathrm{d}\eta.$$

Introducing a new variable $s=\eta/\xi$, we then have $\eta=\xi s$ and

$$\widehat{f}(\xi)=\frac{\lambda}{\theta}e^{-i\mu\xi+\frac{\sigma^2}{4\theta}\xi^2}\int_{1}^{\infty}\widehat{g}(s\xi)s^{-\frac{\rho+\lambda}{\theta}}e^{i\mu\xi s-\frac{\sigma^2}{4\theta}\xi^2s^2}\mathrm{d}s.$$

Substitute $\widehat{g}(\xi s) = \int_{-\infty}^{\infty} e^{-i\xi sy} g(y)\mathrm{d}y$ into the last integral and we obtain

$$\widehat{f}(\xi) = \frac{\lambda}{\theta} e^{-i\mu\xi + \frac{\sigma^2}{4\theta}\xi^2} \int_1^\infty \int_{-\infty}^{\infty} g(y) s^{-\frac{\rho+\lambda}{\theta}} e^{i\mu\xi s - i\xi sy - \frac{\sigma^2}{4\theta}\xi^2 s^2} \mathrm{d}y\mathrm{d}s.$$

Applying the inverse Fourier transform, we have

$$\begin{aligned} f(x) &= \frac{\lambda}{2\pi\theta} \int_{-\infty}^{\infty} \int_1^\infty \int_{-\infty}^{\infty} e^{ix\xi - i\mu\xi + i\mu\xi s - i\xi sy - \frac{\sigma^2}{4\theta}(s^2-1)\xi^2} g(y) s^{-\frac{\rho+\lambda}{\theta}} \mathrm{d}y\mathrm{d}s\mathrm{d}\xi \\ &= \frac{\lambda}{2\pi\theta} \int_{-\infty}^{\infty} g(y) \int_1^\infty s^{-\frac{\rho+\lambda}{\theta}} \int_{-\infty}^{\infty} e^{ix\xi - i\mu\xi + i\mu\xi s - i\xi sy - \frac{\sigma^2}{4\theta}(s^2-1)\xi^2} \mathrm{d}\xi\mathrm{d}s\mathrm{d}y. \end{aligned}$$

The integral with respect to ξ can computed explicitly by apply the following formula:

$$\int_{-\infty}^{\infty} e^{-ix\xi - \frac{\theta}{2}\xi^2} \mathrm{d}\xi = \sqrt{\frac{2\pi}{\theta}} e^{-\frac{x^2}{2\theta}}.$$

This yields

$$\int_{-\infty}^{\infty} e^{ix\xi - i\mu\xi(1-s) - i\xi sy - \frac{\sigma^2}{4\theta}(s^2-1)\xi^2} \mathrm{d}\xi = \frac{2}{\sigma} \cdot \sqrt{\frac{\pi\theta}{s^2-1}} \exp\left\{ -\frac{\theta[x-\mu+(\mu-y)s]^2}{\sigma^2(s^2-1)} \right\}.$$

Hence we obtain

$$f(x) = \frac{\lambda}{\sigma\sqrt{\pi\theta}} \int_{-\infty}^{\infty} g(y) \int_1^\infty s^{-\frac{\rho+\lambda}{\theta}} (s^2-1)^{-1/2} \exp\left\{ -\frac{\theta[x-\mu+(\mu-y)s]^2}{\sigma^2(s^2-1)} \right\} \mathrm{d}s\mathrm{d}y.$$

Let $K_0(x, y, \mu)$ be the inside integral:

$$K_0(x, y, \mu) = \int_1^\infty s^{-\frac{\rho+\lambda}{\theta}} (s^2-1)^{-1/2} \exp\left\{ -\frac{\theta[x-\mu+(\mu-y)s]^2}{\sigma^2(s^2-1)} \right\} \mathrm{d}s.$$

Let $u = 1/s$ in the previous integral. Then we have

$$K(x, y, \mu) = \int_0^1 u^{\frac{\rho+\lambda}{\theta}-1} (1-u^2)^{-\frac{1}{2}} \exp\left\{ -\frac{\theta}{\sigma^2} \frac{[(x-\mu)u + (\mu-y)]^2}{1-u^2} \right\} \mathrm{d}u.$$

$\square$

In view of Lemma 3 , the solutions of the homogeneous equations $[\rho - \mathcal{A}_1]u_j(x) = 0$ from the first two equations in (4.3) are of the form, for $j = 0, 1$,

$$u_j(x) = B_1\phi_1(x) + B_2\phi_2(x) + \gamma_j(x),$$

for some constants B_1 and B_2, where

$$\begin{aligned}
\phi_1(x) &= \int_0^\infty \eta_1(t)e^{-\kappa_1(\mu_1 - x)t}\mathrm{d}t\,, \\
\phi_2(x) &= \int_0^\infty \eta_1(t)e^{\kappa_1(\mu_1 - x)t}\mathrm{d}t\,, \\
\gamma_j(x) &= \frac{\lambda}{\sigma_1\sqrt{\pi\theta_1}}\int_{-\infty}^\infty w_j(y)K(x, y, \mu_1)\mathrm{d}y,
\end{aligned}$$

with

$$\kappa_1 = \frac{\sqrt{2\theta_1}}{\sigma_1},\ \beta_1 = \frac{\rho + \lambda}{\theta_1},\ \eta_1(t) = t^{\beta_1 - 1}\exp(-t^2/2),$$

and

$$K(x, y, \mu_1) = \int_0^1 u^{\frac{\rho+\lambda}{\theta_1} - 1}(1 - u^2)^{-\frac{1}{2}}\exp\left\{-\frac{\theta_1}{\sigma_1^2}\frac{[(x - \mu_1)u + (\mu_1 - y)]^2}{1 - u^2}\right\}\mathrm{d}u.$$

Again, in view of the linear growth conditions (4.1), it follows that

$$u_0(x) = \begin{cases} B_1(x)\phi_1(x) + \gamma_1(x) - x - K, & \text{for } x < x_1, \\ B_2\phi_2(x) + \gamma_0(x), & \text{for } x \geq x_1, \end{cases}$$

and

$$u_1(x) = \begin{cases} B_1\phi_1(x) + \gamma_1(x), & \text{for } x < x_2, \\ B_2\phi_2(x) + \gamma_0(x) + x - K, & \text{for } x \geq x_2. \end{cases}$$

Then, the smooth-fit conditions at x_1 and x_2 yield

$$\begin{cases} B_1\phi_1(x_1) + \gamma_1(x_1) - x_1 - K = B_2\phi_2(x_1) + \gamma_0(x_1), \\ B_1\phi_1'(x_1) + \gamma_1'(x_1) - 1 = B_2\phi_2'(x_1) + \gamma_0'(x_1), \end{cases}$$

$$\begin{cases} B_1\phi_1(x_2) + \gamma_1(x_2) = B_2\phi_2(x_2) + \gamma_0(x_2) + x_2 - K, \\ B_1\phi_1'(x_2) + \gamma_1'(x_2) = B_2\phi_2'(x_2) + \gamma_0'(x_2) + 1. \end{cases}$$

These can be written in matrix form as follows:

$$\begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) \\ \phi_1'(x_1) & \phi_2'(x_1) \end{pmatrix} \begin{pmatrix} B_1 \\ -B_2 \end{pmatrix} = \begin{pmatrix} x_1 + K + \gamma_0(x_1) - \gamma_1(x_1) \\ 1 + \gamma_0'(x_1) - \gamma_1'(x_1) \end{pmatrix},$$
$$\begin{pmatrix} \phi_1(x_2) & \phi_2(x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) \end{pmatrix} \begin{pmatrix} B_1 \\ -B_2 \end{pmatrix} = \begin{pmatrix} x_2 - K + \gamma_0(x_2) - \gamma_1(x_2) \\ 1 + \gamma_0'(x_2) - \gamma_1'(x_2) \end{pmatrix}.$$

It follows that

$$\begin{aligned} \begin{pmatrix} B_1 \\ -B_2 \end{pmatrix} &= \begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) \\ \phi_1'(x_1) & \phi_2'(x_1) \end{pmatrix}^{-1} \begin{pmatrix} x_1 + K + \gamma_0(x_1) - \gamma_1(x_1) \\ 1 + \gamma_0'(x_1) - \gamma_1'(x_1) \end{pmatrix} \\ &= \begin{pmatrix} \phi_1(x_2) & \phi_2(x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) \end{pmatrix}^{-1} \begin{pmatrix} x_2 - K + \gamma_0(x_2) - \gamma_1(x_2) \\ 1 + \gamma_0'(x_2) - \gamma_1'(x_2) \end{pmatrix}. \end{aligned} \tag{4.7}$$

The last equality can be used to determine x_1 and x_2 and then B_1 and B_2.

To summarize, the solutions of the HJB equations (4.3) have the following forms:

$$u_0(x) = \begin{cases} B_1\phi_1(x) + \gamma_1(x) - x - K, & \text{for } x < x_1, \\ B_2\phi_2(x) + \gamma_0(x), & \text{for } x \geq x_1, \end{cases}$$
$$u_1(x) = \begin{cases} B_1\phi_1(x) + \gamma_1(x), & \text{for } x < x_2, \\ B_2\phi_2(x) + \gamma_0(x) + x - K, & \text{for } x \geq x_2, \end{cases}$$

and

$$w_0(x) = \begin{cases} A_1\psi_1(x) - x - K, & \text{for } x < x_3, \\ A_2\psi_2(x), & \text{for } x \geq x_3, \end{cases}$$
$$w_1(x) = \begin{cases} A_1\psi_1(x), & \text{for } x < x_4, \\ A_2\psi_2(x) + x - K, & \text{for } x \geq x_4. \end{cases}$$

Variational inequalities for w_0 and w_1

Note that the HJB equations (4.3) consist of both equalities and inequalities. Next, we focus on the inequality parts. We first consider $w_0(x)$ and $w_1(x)$. Recall that, on $(-\infty, x_3)$, $[\rho - \mathcal{A}_2]w_1(x) = 0$ and $w_0(x) = w_1(x) - x - K$. The corresponding inequalities are given by

$$[\rho - \mathcal{A}_2]w_0(x) \geq 0 \quad \text{and} \quad w_1(x) \geq w_0(x) + x - K.$$

Since $w_1(x) = A_1\psi_1(x)$ and $w_0(x) = A_1\psi_1(x) - x - K$, it follows that

$$\begin{aligned} 0 \leq [\rho - \mathcal{A}_2]w_0(x) &= [\rho - \mathcal{A}_2](w_1(x) - x - K) \\ &= -(\rho - \mathcal{A}_2)(x + K) = -\rho(x + K) + \mathcal{A}_2(x + K) \\ &= -\rho(x + K) + \theta_2(\mu_2 - x) = \theta_2\mu_2 - \rho K - (\rho + \theta_2)x \end{aligned}$$

is equivalent to

$$x \leq \frac{\theta_2\mu_2 - \rho K}{\rho + \theta_2} \quad \text{for } x \leq x_3,$$

which is equivalent in turn to

$$x_3 \leq \frac{\theta_2\mu_2 - \rho K}{\rho + \theta_2}. \tag{4.8}$$

The other inequality $w_1(x) \geq w_0(x) + x - K$ holds since $w_1(x) = w_0(x) + x + K > w_0(x) + x - K$.

Next, on (x_3, x_4), the corresponding inequalities are

$$x - K \leq w_1(x) - w_0(x) \leq x + K \quad \Longleftrightarrow \quad |w_1(x) - w_0(x) - x| \leq K$$

with $w_0(x) = A_2\psi_2(x)$ and $w_1(x) = A_1\psi_1(x)$. This implies

$$|A_1\psi_1(x) - A_2\psi_2(x) - x| \leq K, \quad \text{for} \quad x \in (x_3, x_4). \tag{4.9}$$

Finally, on (x_4, ∞), $[\rho - \mathcal{A}_2]w_0(x) = 0$ and $w_1(x) = w_0(x) + x - K$. The corresponding inequalities are

$$[\rho - \mathcal{A}_2]w_1(x) \geq 0 \quad \text{and} \quad w_0(x) \geq w_1(x) - x - K.$$

Since $w_0(x) = A_2\psi_2(x)$ and $w_1(x) = A_2\psi_2(x) + x - K$, we have

$$0 \le [\rho - \mathcal{A}_2]w_1(x) = [\rho - \mathcal{A}_2](w_0(x) + x - K)$$
$$= (\rho - \mathcal{A}_2)(x - K) = (\rho + \theta_2)x - (\rho K + \theta_2\mu_2)$$

is equivalent to

$$x \ge \frac{\theta_2\mu_2 + \rho K}{\rho + \theta_2}, \quad \text{for } x \ge x_4,$$

which is equivalent also to

$$x_4 \ge \frac{\theta_2\mu_2 + \rho K}{\rho + \theta_2}. \tag{4.10}$$

The other inequality $w_0(x) \ge w_1(x) - x - K$ holds since $w_0(x) = w_1(x) - x + K > w_1(x) - x - K$.

Variational inequalities for u_0 and u_1

We next consider the inequalities for $u_0(x)$ and $u_1(x)$ on the intervals $(-\infty, x_1)$, (x_1, x_2) and (x_2, ∞). First, on $(-\infty, x_1)$, we have $u_0(x) = u_1(x) - x - K$ and $[\rho - \mathcal{A}_1]u_1(x) = 0$; and the corresponding inequalities are $[\rho - \mathcal{A}_1]u_0(x) \ge 0$ and $u_1(x) \ge u_0(x) + x - K$. The second inequality holds since $u_1(x) = u_0(x) + x + K \ge u_0(x) + x - K$. To simplify the notation, let $\mathcal{A}_1^0 = \frac{\sigma_1^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \theta_1(\mu_1 - x)\frac{\mathrm{d}}{\mathrm{d}x}$. Then, we have

$$0 \le [\rho - \mathcal{A}_1]u_0(x)$$

is equivalent to

$$0 \le (\rho - \mathcal{A}_1^0)u_0 - \lambda(w_0(x) - u_0(x))$$
$$= (\rho + \lambda)u_0(x) - \mathcal{A}_1^0 u_0(x) - \lambda w_0(x).$$

Note that

$$(\rho - \mathcal{A}_1)u_1(x) = 0 \quad \Longleftrightarrow \quad (\rho + \lambda)u_1(x) - \mathcal{A}_1^0 u_1(x) = \lambda w_1(x).$$

Combine these to obtain

$$
\begin{aligned}
0 \le (\rho - \mathcal{A}_1)u_0(x) &= (\rho+\lambda)u_0(x) - \mathcal{A}_1^0 u_0(x) - \lambda w_0(x) \\
&= (\rho+\lambda)[u_1(x) - x - K] - \mathcal{A}_1^0[u_1(x) - x - K] - \lambda w_0(x) \\
&= (\rho+\lambda)u_1(x) - \mathcal{A}_1^0 u_1(x) - (\rho+\lambda)(x+K) + \mathcal{A}_1^0(x+K) - \lambda w_0(x) \\
&= \lambda[w_1(x) - w_0(x)] - (\rho+\lambda)(x+K) + \theta_1(\mu_1 - x) \\
&= \lambda[w_1(x) - w_0(x)] + \theta_1\mu_1 - (\rho+\lambda+\theta_1)x - (\rho+\lambda)K,
\end{aligned}
$$

which is equivalent to

$$
\lambda[w_1(x) - w_0(x)] + \theta_1\mu_1 - (\rho+\lambda+\theta_1)x - (\rho+\lambda)K \ge 0, \text{ for } x < x_1. \tag{4.11}
$$

Next, on (x_1, x_2), the corresponding inequalities are

$$
u_0(x) \ge u_1(x) - x - K \quad \text{and} \quad u_1(x) \ge u_0(x) + x - K,
$$

which are equivalent to $|u_1(x) - u_0(x) - x| \le K$. Recall that $u_1(x) = B_1\phi_1(x) + \gamma_1(x)$ and $u_0(x) = B_2\phi_2(x) + \gamma_0(x)$. It follows that

$$
|B_1\phi_1(x) + \gamma_1(x) - B_2\phi_2(x) - \gamma_0(x) - x| \le K, \text{ for } x_1 < x < x_2. \tag{4.12}
$$

Finally, on (x_2, ∞), we have $[\rho - \mathcal{A}_1]u_0(x) = 0$ and $u_1(x) = u_0(x) + x - K$; and the corresponding inequalities are

$$
u_0(x) \ge u_1(x) - x - K \quad \text{and} \quad [\rho - \mathcal{A}_1]u_1(x) \ge 0.
$$

The first inequality holds since $u_0(x) = u_1(x) - x + K > u_1(x) - x - K$. For the second inequality, we note that

$$
[\rho - \mathcal{A}_1]u_0(x) = 0 \quad \Longleftrightarrow \quad (\rho+\lambda)u_0(x) - \mathcal{A}_1^0 u_0(x) = \lambda w_0(x)
$$

and

$$0 \le [\rho - \mathcal{A}_1]u_1(x)$$

is equivalent to

$$\begin{aligned}0 &\le [\rho - \mathcal{A}_1]u_1 - \lambda(w_1(x) - u_1(x)) \\ &= (\rho + \lambda)u_1(x) - \mathcal{A}_1^0 u_1(x) - \lambda w_1(x).\end{aligned}$$

Combine these to obtain

$$\begin{aligned}0 &\le (\rho - \mathcal{A}_1)u_1(x) \\ &= (\rho + \lambda)u_1(x) - \mathcal{A}_1^0 u_1(x) - \lambda w_1(x) \\ &= (\rho + \lambda)[u_0(x) + x - K] - \mathcal{A}_1^0[u_0(x) + x - K] - \lambda w_1(x) \\ &= (\rho + \lambda)u_0(x) - \mathcal{A}_1^0 u_0(x) + (\rho + \lambda)(x - K) - \mathcal{A}_1^0(x - K) - \lambda w_1(x) \\ &= \lambda[w_0(x) - w_1(x)] + (\rho + \lambda)(x - K) - \theta_1(\mu_1 - x) \\ &= \lambda[w_0(x) - w_1(x)] - \theta_1\mu_1 + (\rho + \lambda + \theta_1)x - (\rho + \lambda)K,\end{aligned}$$

which is equivalent to

$$\lambda[w_0(x) - w_1(x)] - \theta_1\mu_1 + (\rho + \lambda + \theta_1)x - (\rho + \lambda)K \ge 0, \text{ for } x > x_2. \tag{4.13}$$

To summarize the results obtained so far, we have

Theorem 4.4.1. *Let x_1 and x_2 be given in (4.7) and x_3 and x_4 in (4.5). Assume the inequalities (4.8), (4.9),(4.10), (4.11), (4.12), and (4.13) hold. Then, the functions*

$$v_0(x,1) = u_0(x) = \begin{cases} B_1\phi_1(x) + \gamma_1(x) - x - K & \text{for } x < x_1, \\ B_2\phi_2(x) + \gamma_0(x) & \text{for } x \geq x_1, \end{cases}$$
$$v_1(x,1) = u_1(x) = \begin{cases} B_1\phi_1(x) + \gamma_1(x) & \text{for } x < x_2, \\ B_2\phi_2(x) + \gamma_0(x) + x - K & \text{for } x \geq x_2, \end{cases}$$

and

$$v_0(x,2) = w_0(x) = \begin{cases} A_1\psi_1(x) - x - K & \text{for } x < x_3, \\ A_2\psi_2(x) & \text{for } x \geq x_3, \end{cases}$$
$$v_1(x,2) = w_1(x) = \begin{cases} A_1\psi_1(x) & \text{for } x < x_4, \\ A_2\psi_2(x) + x - K & \text{for } x \geq x_4, \end{cases}$$

satisfy the HJB equations (4.2).

4.5 A Verification Theorem

We state a verification theorem next. Its proof can be given similarly as in Song and Zhang [**19**].

Theorem 4.5.1. *Assume the conditions of the previous theorem and $v_0(x,\alpha) \geq 0$. Then, $v_i(x,\alpha)$ are the value functions, i.e., $v_i(x,\alpha) = V_i(x,\alpha)$, for $i = 0,1$, $\alpha = 1,2$, and x. Let $D_b = \{(x,1) : x > x_1\} \cup \{(x,2) : x > x_3\}$ and $D_s = \{(x,1) : x < x_2\} \cup \{(x,2) : x < x_4\}$. If initially $i = 0$, let $\Lambda_0^* = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$ with $\tau_1^b = \inf\{t : (Z_t, \alpha_t) \notin D_b\}$, $\tau_1^s = \inf\{t \geq \tau_1^b : (Z_t, \alpha_t) \notin D_s\}$, $\tau_2^b = \inf\{t \geq \tau_1^s : (Z_t, \alpha_t) \notin D_b\}$, and so on. If initially $i = 1$, let $\Lambda_1^* = (\tau_1^s, \tau_2^b, \tau_2^s, \ldots)$ with $\tau_1^s = \inf\{t : (Z_t, \alpha_t) \notin D_s\}$, $\tau_2^b = \inf\{t \geq \tau_1^s : (Z_t, \alpha_t) \notin D_b\}$, $\tau_2^s = \inf\{t \geq \tau_2^b : (Z_t, \alpha_t) \notin D_s\}$, and so forth. Then, Λ_0 and Λ_1 are optimal.*

4.6 A Numerical Example

In this section, we consider a numerical example with the following specifications:

$$\mu_1 = 0, \mu_2 = 0.5, \theta_1 = 1, \theta_2 = 1, \sigma_1 = 0.5, \sigma_2 = 0.5, \lambda = 3, \rho = 0.1, K = 0.003.$$

We use Newton's method to solve the equations in (4.5) to obtain $x_3 = 0.295314$ and $x_4 = 0.498290$. Then we use these x_3 and x_4 to solve the equations in (4.7) to get $x_1 = -0.118156$ and $x_2 = 0.132020$. Here the inequalities (4.8) and (4.10) are used to limit the range for x_3 and x_4 and can be verified directly with the values of x_3 and x_4 respectively. Also, the solutions for (4.5) and (4.7) consist of a set of pairs. The rest of the inequalities (4.8)-(4.13) are used to select the pairs that satisfy all of them. Here each of the inequalities (4.9), (4.11)-(4.13) can be rearranged in the form $f(x) \geq 0$, and be verified by the minimum of $f(x)$ being non-negative on the corresponding interval. The corresponding value functions u_0, u_1, w_0, and w_1 are plotted in Figure 4.2.

We next vary one of the parameters at a time and examine the dependence of (x_1, x_2, x_3, x_4). First we examine the dependence of (x_1, x_2, x_3, x_4) on μ_1. As can be seen in Table 4.1, both x_1 and x_2 increase in μ_1. This is because μ_1 is the mean level when $\alpha = 1$. As μ_1 rises, it raises the trading band corresponding to $\alpha = 1$. Note that in this case, neither x_3 nor x_4 is affected due to the fact that $\alpha = 2$ is absorbing.

Table 4.1: x_1, x_2, x_3, x_4 with varying μ_1

μ_1	x_1	x_2	x_3	x_4
-0.2	-0.282615	-0.032631	0.295314	0.498290
-0.1	-0.200239	0.048728	0.295314	0.498290
0	-0.118156	0.132020	0.295314	0.498290
0.1	-0.035907	0.215687	0.295314	0.498290
0.2	0.048433	0.299286	0.295314	0.498290

Similarly, as we vary μ_2, x_3 and x_4 exhibit similar behavior, while x_1 and x_2 barely change. This can be seen in Table 4.2.

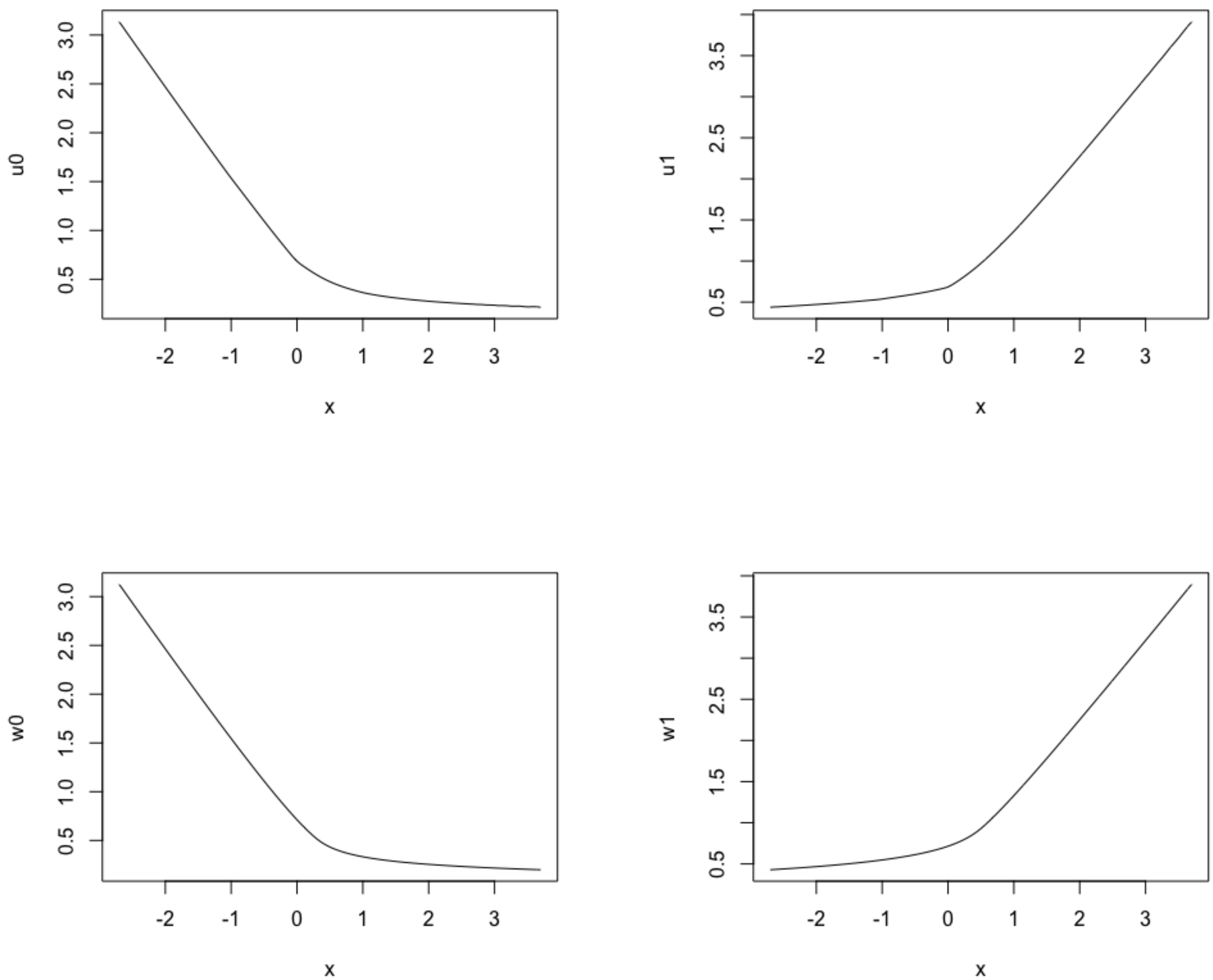

Figure 4.2: Value Functions $u_0(x)$, $u_1(x)$, $w_0(x)$, and $w_1(x)$

Table 4.2: x_1, x_2, x_3, x_4 with varying μ_2

μ_2	x_1	x_2	x_3	x_4
0.3	-0.117226	0.133386	0.136663	0.339629
0.4	-0.117777	0.132596	0.216005	0.418976
0.5	-0.118156	0.132020	0.295314	0.498290
0.6	-0.118395	0.131627	0.374582	0.577565
0.7	-0.118533	0.131391	0.453802	0.656792

Next, we vary θ_1. As θ_1 increases, x_1 increases while x_2 decreases (Table 4.3). This is because θ_1 is the mean reversion rate when $\alpha = 1$. The larger the θ_1, the more forceful the Z_t is pulled back to its mean, resulting in a smaller trading band around the mean level.

Table 4.3: x_1, x_2, x_3, x_4 with varying θ_1

θ_1	x_1	x_2	x_3	x_4
0.8	-0.123438	0.139573	0.295314	0.498290
0.9	-0.120738	0.135682	0.295314	0.498290
1	-0.118156	0.132020	0.295314	0.498290
1.1	-0.115716	0.128606	0.295314	0.498290
1.2	-0.113420	0.125437	0.295314	0.498290

Similar behavior is observed in Table 4.4 for x_3 and x_4 as θ_2 varies.

Table 4.4: x_1, x_2, x_3, x_4 with varying θ_2

θ_2	x_1	x_2	x_3	x_4
0.8	-0.118137	0.132050	0.284158	0.501107
0.9	-0.118148	0.132033	0.290217	0.499713
1	-0.118156	0.132020	0.295314	0.498290
1.1	-0.118163	0.132010	0.299681	0.496885
1.2	-0.118169	0.132001	0.303479	0.495519

In Tables 4.5 and 4.6, we vary σ_1 and σ_2 separately. Larger volatility corresponds to a wider price range. As a result, we see a wider trading band (smaller x_1 and bigger x_2 or smaller x_3 and bigger x_4).

Table 4.5: x_1, x_2, x_3, x_4 with varying σ_1

σ_1	x_1	x_2	x_3	x_4
0.3	-0.083771	0.095985	0.295314	0.498290
0.4	-0.101716	0.114706	0.295314	0.498290
0.5	-0.118156	0.132020	0.295314	0.498290
0.6	-0.133533	0.148252	0.295314	0.498290
0.7	-0.148118	0.163604	0.295314	0.498290

Table 4.6: x_1, x_2, x_3, x_4 with varying σ_2

σ_2	x_1	x_2	x_3	x_4
0.3	-0.118162	0.132014	0.323132	0.467952
0.4	-0.118160	0.132016	0.308772	0.483890
0.5	-0.118156	0.132020	0.295314	0.498290
0.6	-0.118151	0.132026	0.282604	0.511634
0.7	-0.118146	0.132033	0.270515	0.524185

Next, we vary K. A larger K discourages frequent trading. This can be seen in Table 4.7 by decreasing x_1 (and x_3) and increasing x_2 (and x_4), respectively.

Table 4.7: x_1, x_2, x_3, x_4 with varying K

K	x_1	x_2	x_3	x_4
0.001	-0.083112	0.087305	0.327876	0.468124
0.002	-0.103943	0.112881	0.308834	0.485869
0.003	-0.118156	0.132020	0.295314	0.498290
0.004	-0.129196	0.148099	0.284446	0.508168
0.005	-0.138306	0.162345	0.275191	0.516505

Finally, we vary λ (with $\mu_1 = 0.5, \mu_2 = 0, \theta_1 = \theta_2 = 1, \sigma_1 = \sigma_2 = 0.5, \rho = 0.1$, and $K = 0.003$). Recall that (x_3, x_4) is associated with the absorbing state $\alpha = 2$ and therefore independent of λ. As λ increases to infinity, the corresponding (x_1, x_2) decreases and approaches to (x_3, x_4). This trend can be seen in Table 4.8.

Table 4.8: x_1, x_2, x_3, x_4 with varying λ

λ	x_1	x_2	x_3	x_4
1	0.317598	0.573764	−0.101480	0.101480
5	0.229788	0.469172	−0.101480	0.101480
10	0.104981	0.319730	−0.101480	0.101480
20	0.042377	0.207961	−0.101480	0.101480
50	−0.036283	0.083097	−0.101480	0.101480

Appendix A

MATLAB Codes

Codes for Chapter 2: Round-Trip Pairs Trading under GBM

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computing Least-Squares Parameters for Stock 1 and Stock 2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

format longg
T2=readmatrix('WMT1015.csv');   %%% input Stock 1 price data
T1=readmatrix('TGT1015.csv');   %%% input Stock 2 price data
S=[T1(:,6),T2(:,6)];
N=size(S(:,1));
M=size(S(:,2));
N=N(1);
M=M(1);
N=min(N,M);

for i=1:N-1
    u1(i,1)=log(S(i,1)/S(i+1,1));
    u2(i,1)=log(S(i,2)/S(i+1,2));
```

```
end

u1_bar=sum(u1(:,1))/N;
u2_bar=sum(u2(:,1))/N;

for i=1:N-1
    bb1(i,1)=(u1(i,1)-u1_bar)*(u1(i,:)-u1_bar);
    bb2(i,1)=(u2(i,1)-u2_bar)*(u2(i,:)-u2_bar);
    bb12(i,1)=(u1(i,1)-u1_bar)*(u2(i,:)-u2_bar);
end

bb1=sum(bb1(:,1))/(N-1);
bb2=sum(bb2(:,1))/(N-1);
bb12=sum(bb12(:,1))/(N-1);

sigma1=sqrt(bb1*252);
sigma2=sqrt(bb2*252);
sigma1sigma2=(bb12*252);

b0=min(sigma1,sigma2);
minB=1000;

for j=0:10000
    b=b0*j/10000;
    aa=abs(b*(sqrt(sigma1*sigma1-b*b)...
        +sqrt(sigma2*sigma2-b*b))-sigma1sigma2);
    if aa<minB
        minB=aa;
        b_star=b;
```

```
    end
end

b=b_star;
s12=b_star;   %%% output Stock 1/Stock 2 correlation constant
s21=b_star;   %%% output Stock 1/Stock 2 correlation constant
s11=sqrt(sigma1*sigma1-b_star*b_star);
                %%% output Stock 1 volatility constant
s22=sqrt(sigma2*sigma2-b_star*b_star);
                %%% output Stock 2 volatility constant

for k=1:N
    X(k)=k/252;
    Y(k,1)=log(S(k,1));
    Y(k,2)=log(S(k,2));
end

for k=1:N
    AA0(k,1)=X(k)*X(k);
    BB0(k,1)=X(k);
    CC0(k,1)=X(k)*Y(k,1);
    CC1(k,1)=X(k)*Y(k,2);
    DD0(k,1)=Y(k,1);
    DD1(k,1)=Y(k,2);
end

AA01=sum(AA0(:,1));
BB01=sum(BB0(:,1));
CC01=sum(CC0(:,1));
```

```
DD01=sum(DD0(:,1));
CC11=sum(CC1(:,1));
DD11=sum(DD1(:,1));

A1=(CC01-BB01*DD01/N)/(AA01-BB01*BB01/N);
B1=(DD01-A1*BB01)/N;

A2=(CC11-BB01*DD11/N)/(AA01-BB01*BB01/N);
B2=(DD11-A2*BB01)/(N);

m1=A1+0.5*sigma1*sigma1;   %%% output Stock 1 return rate
m2=A2+0.5*sigma2*sigma2;   %%% output Stock 1 return rate

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computing k_1, C_2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

K=0.001;   %%% input transaction costs
r=0.5;   %%% input discount factor

Bb=1+K;
Bs=1-K;
a11=s11^2+s12^2;
a12=s11*s21+s12*s22;
a22=s21^2+s22^2;
l=(a11-2*a12+a22)*0.5;

d1=0.5*(1+(m1-m2)/l+((1+(m1-m2)/l)^2+(4*r-4*m1)/l)^(0.5));
                                                        %%% output delta_1
```

```
d2=0.5*(1+(m1-m2)/l-((1+(m1-m2)/l)^2+(4*r-4*m1)/l)^(0.5));
                                                          %%% output delta_2

k1=(Bs/Bb)*(-d2/(1-d2))   %%% output k_1
C2=(Bb/(-d2))*k1^(1-d2);   %%% output C_2

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting Equation to Show Existence of k_2, C_1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

syms x
eqnLeft = C2*(d1-d2)*x^(d2)+Bs*(d1-1)*x-Bb*d1;
eqnRight = 0;
fplot([eqnLeft eqnRight])
hold on

axis([0,2,-0.8,5])
k2 = vpasolve(eqnLeft == eqnRight, x, 1.5)   %%% output k_2
plot(k2,0,'ko')
labels={'(k_2,0)'};
text(k2,0,labels,'VerticalAlignment','bottom',...
    'HorizontalAlignment','right');
title('Solution to f(y)=0')

C1=(C2*d2*k2^(d2-1)+Bs)/(d1*k2^(d1-1));   %%% output C_1

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Stock Trading Simulation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
format longg
T4=readmatrix('WMT1520.csv');   %%% input Stock 1 price data
T3=readmatrix('TGT1520.csv');   %%% input Stock 2 price data
SS=[T3(:,5),T4(:,5)];
NN=size(SS(:,1));
MM=size(SS(:,2));
NN=NN(1);
MM=MM(1);
NN=min(NN,MM);

XX=zeros(size(SS(:,1)));
for i=1:NN
    XX(i,1)=SS(i,1)./SS(i,2);   %%% y = x_2 / x_1
end

YY=zeros(size(SS));
for i=1:NN
    YY(i,1)=k2;
    YY(i,2)=k1;
end

count=1;
for i=1:NN
    if XX(i,1)>=k2
        G3(count,1)=i;   %%% Dates on which y lies in Gamma_3
        count=count+1;
    end
end
```

```
count=1;
for i=1:NN
    if XX(i,1)<=k1
        G1(count,1)=i;   %%% Dates on which y lies in Gamma_1
        count=count+1;
    end
end

p=100000
p1=p+(p/2)/SS(G1(1,1),1)*SS(G3(1,1),1)...
    -((p/2)/SS(G1(1,1),2)*SS(G3(1,1),2))-20
                        %%% profit from first round trip

for i=1:G3(1,1)-1
    eq2(i,1)=p;
end
for i=G3(1,1):NN
    eq2(i,1)=p1;
end

count=1;
for i=1:size(G1)
    if G1(i,1)>G3(1,1)
        G1(count,1)=G1(i,1);
        count=count+1;
    end
end
```

```
p2=p1+(p1/2)/SS(G3(1,1),2)*SS(G1(1,1),2)...
    -((p1/2)/SS(G3(1,1),1)*SS(G1(1,1),1))-20
                        %%% profit from second round trip

for i=1:G1(1,1)-1
    eq1(i,1)=p1;
end
for i=G1(1,1):NN
    eq1(i,1)=p2;
end

count=1;
for i=1:size(G1)
    if G1(i,1)>G3(1,1)
        G1(count,1)=G1(i,1);
        count=count+1;
    end
end

count=1;
for i=1:size(G3)
    if G3(i,1)>G1(1,1)
        G3(count,1)=G3(i,1);
        count=count+1;
    else
        G3(count,1)=NN;
    end
end
```

```
p3=p2+(p2/2)/SS(G1(1,1),1)*SS(G3(1,1),1)...
    -((p2/2)/SS(G1(1,1),2)*SS(G3(1,1),2))-20
                        %%% profit from third round trip

for i=G3(1,1):NN
    eq2(i,1)=p3;
end

profit=p3-100000   %%% total profit

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the First Equity Curve
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure
plot(1+eq1/100000,'b'), hold on;
text(150,2.1,['Equity Curve (Long WMT and Short TGT):' ...
    ' 2015/1/2 -- 2019/12/30']);
fontsize(8,"points")
plot(XX,'b-','LineWidth', 0.85)
hold on
plot(YY,'g','LineWidth', 1)
text(1000,YY(1000,1)+0.1,'$$k_2$$', 'Interpreter', 'LaTeX');
text(1000,YY(1000,2)+0.1,'$$k_1$$', 'Interpreter', 'LaTeX');
axis([0,1257,0,3])
xlabel('Date')
ylabel('Ratio of Stock Prices')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Interchanging the Roles of Stock 1 and Stock 2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

k4=1/k1
k3=1/k2

WW=zeros(size(SS(:,1)));
for i=1:NN
    WW(i,1)=SS(i,2)./SS(i,1);
end

ZZ=zeros(size(SS));
for i=1:NN
    ZZ(i,1)=k4;
    ZZ(i,2)=k3;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the Second Equity Curve
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure(2)
plot(1+eq2/100000,'b'), hold on;
text(150,1.925,['Equity Curve (Long TGT and Short WMT):' ...
    ' 2015/1/2 -- 2019/12/30']);
fontsize(8,"points")
plot(WW,'b-','LineWidth', 0.85)
hold on
plot(YY,'g','LineWidth', 1)
```

```
text(1000,ZZ(1000,1)-0.05,'$$\widetilde{k_2}$$', 'Interpreter', 'LaTeX');
text(1000,ZZ(1000,2)-0.05,'$$\widetilde{k_1}$$', 'Interpreter', 'LaTeX');
axis([0,1257,0,3])
xlabel('Date')
ylabel('Ratio of Stock Prices')
```

Codes for Chapter 3: Round-Trip Pairs Trading under GBM with Reversible Initial Positions

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computing Least-Squares Parameters for Stock 1 and Stock 2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

format longg
T2=readmatrix('WMT1015.csv');   %%% input Stock 1 price data
T1=readmatrix('TGT1015.csv');   %%% input Stock 2 price data
S=[T1(:,6),T2(:,6)];
N=size(S(:,1));
M=size(S(:,2));
N=N(1);
M=M(1);
N=min(N,M);

for i=1:N-1
    u1(i,1)=log(S(i,1)/S(i+1,1));
    u2(i,1)=log(S(i,2)/S(i+1,2));
end

u1_bar=sum(u1(:,1))/N;
```

```
u2_bar=sum(u2(:,1))/N;

for i=1:N-1
    bb1(i,1)=(u1(i,1)-u1_bar)*(u1(i,:)-u1_bar);
    bb2(i,1)=(u2(i,1)-u2_bar)*(u2(i,:)-u2_bar);
    bb12(i,1)=(u1(i,1)-u1_bar)*(u2(i,:)-u2_bar);
end

bb1=sum(bb1(:,1))/(N-1);
bb2=sum(bb2(:,1))/(N-1);
bb12=sum(bb12(:,1))/(N-1);

sigma1=sqrt(bb1*252);
sigma2=sqrt(bb2*252);
sigma1sigma2=(bb12*252);

b0=min(sigma1,sigma2);
minB=1000;

for j=0:10000
    b=b0*j/10000;
    aa=abs(b*(sqrt(sigma1*sigma1-b*b)...
        +sqrt(sigma2*sigma2-b*b))-sigma1sigma2);
    if aa<minB
        minB=aa;
        b_star=b;
    end
end
```

```
b=b_star;
s12=b_star;   %%% output Stock 1/Stock 2 correlation constant
s21=b_star;   %%% output Stock 1/Stock 2 correlation constant
s11=sqrt(sigma1*sigma1-b_star*b_star);
                %%% output Stock 1 volatility constant
s22=sqrt(sigma2*sigma2-b_star*b_star);
                %%% output Stock 2 volatility constant

for k=1:N
    X(k)=k/252;
    Y(k,1)=log(S(k,1));
    Y(k,2)=log(S(k,2));
end

for k=1:N
    AA0(k,1)=X(k)*X(k);
    BB0(k,1)=X(k);
    CC0(k,1)=X(k)*Y(k,1);
    CC1(k,1)=X(k)*Y(k,2);
    DD0(k,1)=Y(k,1);
    DD1(k,1)=Y(k,2);
end

AA01=sum(AA0(:,1));
BB01=sum(BB0(:,1));
CC01=sum(CC0(:,1));
DD01=sum(DD0(:,1));
CC11=sum(CC1(:,1));
DD11=sum(DD1(:,1));
```

```
A1=(CC01-BB01*DD01/N)/(AA01-BB01*BB01/N);
B1=(DD01-A1*BB01)/N;

A2=(CC11-BB01*DD11/N)/(AA01-BB01*BB01/N);
B2=(DD11-A2*BB01)/(N);

m1=A1+0.5*sigma1*sigma1;   %%% output Stock 1 return rate
m2=A2+0.5*sigma2*sigma2;   %%% output Stock 1 return rate

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computing k_1, k_4, C_1, C_2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

K=0.001;   %%% input transaction costs
r=0.5;   %%% input discount factor

Bb=1+K;
Bs=1-K;
a11=s11^2+s12^2;
a12=s11*s21+s12*s22;
a22=s21^2+s22^2;
l=(a11-2*a12+a22)*0.5;

d1=0.5*(1+(m1-m2)/l+((1+(m1-m2)/l)^2+(4*r-4*m1)/l)^(0.5));
                                                        %%% output delta_1
d2=0.5*(1+(m1-m2)/l-((1+(m1-m2)/l)^2+(4*r-4*m1)/l)^(0.5));
                                                        %%% output delta_2
```

```
k1=(Bs/Bb)*(-d2/(1-d2))   %%% output k_1
C2=(Bb/(-d2))*k1^(1-d2);   %%% output C_2

k4=(Bb/Bs)*(d1/(d1-1))   %%% output k_4
C1=(Bs/(d1))*k4^(1-d1);   %%% output C_1

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting System of Equations to Estimate k_2, k_3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

syms x y
eqn1 = ((1-d2)*x^(1-d1)+d2*y^(-d1))/(d1-d2)...
    +(((d2)*x^(-d1)+(1-d2)*y^(1-d1))*G)/(d1-d2)-(k4^(1-d1))/d1 == 0;
eqn2 = ((1-d1)*x^(1-d2)+d1*y^(-d2))/(d1-d2)...
    +(((d1)*x^(-d2)+(1-d1)*y^(1-d2))*G)/(d1-d2)-(G*(k1^(1-d2)))/(-d2) == 0;
a = axes;
F1=fimplicit(eqn1,[0,5],'b');
hold on
grid on
F2=fimplicit(eqn2,[0,5],'m');
hold on
M1 = "F_1 = 0";
M2 = "F_2 = 0";
legend([F1,F2], [M1, M2]);
L=sym(-4:5:6);
a.XTick=double(L);
a.YTick=double(L);
M=arrayfun(@char,L,'UniformOutput',false);
a.XTickLabel=M;
```

```
a.YTickLabel=M;
title('Plot of System of Equations');
plotx=[k1, k4];
ploty=[k4, k1];
xlabel('k_3');
ylabel('k_2');
labels={'(k_1,k_4)', '(k_4,k_1)'};
plot(plotx,ploty,'ko','HandleVisibility','off');
text(plotx,ploty,labels,'VerticalAlignment',...
    'bottom','HorizontalAlignment','right');
S=solve(eqn1,eqn2,'ReturnConditions',true);

k2=k4;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Stock Trading Simulation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
format longg
T4=readmatrix('WMT1520.csv');   %%% input Stock 1 price data
T3=readmatrix('TGT1520.csv');   %%% input Stock 2 price data
SS=[T3(:,5),T4(:,5)];
NN=size(SS(:,1));
MM=size(SS(:,2));
NN=NN(1);
MM=MM(1);
NN=min(NN,MM);

XX=zeros(size(SS(:,1)));
for i=1:NN
```

```
    XX(i,1)=SS(i,1)./SS(i,2);   %%% y = x_2 / x_1
end

YY=zeros(size(SS));
for i=1:NN
    YY(i,1)=k2;
    YY(i,2)=k1;
end

count=1;
for i=1:NN
    if XX(i,1)>=k2
        G3(count,1)=i;   %%% Dates on which y lies in Gamma_3
        count=count+1;
    end
end

count=1;
for i=1:NN
    if XX(i,1)<=k1
        G1(count,1)=i;   %%% Dates on which y lies in Gamma_1
        count=count+1;
    end
end

p=100000;
if G1(1,1)<G3(1,1)
    p1=p+(p/2)/SS(G1(1,1),1)*SS(G3(1,1),1)...
        -((p/2)/SS(G1(1,1),2)*SS(G3(1,1),2))-20;
```

```
                    %%% profit from first round trip
    for i=1:G3(1,1)-1
        eq(i,1)=p;
    end
    for i=G3(1,1):NN
        eq(i,1)=p1;
    end
end

count=1;
for i=1:size(G1)
    if G1(i,1)>G3(1,1)
        G1(count,1)=G1(i,1);
        count=count+1;
    end
end

p2=p1+(p1/2)/SS(G3(1,1),2)*SS(G1(1,1),2)...
    -((p1/2)/SS(G3(1,1),1)*SS(G1(1,1),1))-20;
                    %%% profit from second round trip
for i=G1(1,1):NN
    eq(i,1)=p2;
end

count=1;
for i=1:size(G3)
    if G3(i,1)>G1(1,1)
        G3(count,1)=G3(i,1);
        count=count+1;
```

```
    else
        G3(count,1)=NN;
    end
end

p3=p2+(p2/2)/SS(G1(1,1),1)*SS(G3(1,1),1)...
    -((p2/2)/SS(G1(1,1),2)*SS(G3(1,1),2))-20;
                        %%% profit from third round trip
for i=G3(1,1):NN
    eq(i,1)=p3;
end

profit=p3   %%% total profit

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the First Equity Curve
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure
plot(1+eq/100000,'b'), hold on;
text(300,2.075,'Equity Curve: 2015/1/2 -- 2019/12/30');
fontsize(8,"points")
plot(XX,'b-','LineWidth', 0.85)
hold on
plot(YY,'g','LineWidth', 1)
text(1000,YY(1000,1)+0.1,'$$k_2^*$$','Interpreter','LaTeX');
text(1000,YY(1000,2)+0.1,'$$k_1^*$$','Interpreter','LaTeX');
axis([0,1257,0,3])
xlabel('Date')
```

```
ylabel('Ratio of Stock Prices')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Interchanging the Roles of Stock 1 and Stock 2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

k4=1/k1
k3=1/k2

WW=zeros(size(SS(:,1)));
for i=1:NN
    WW(i,1)=SS(i,2)./SS(i,1);
end

ZZ=zeros(size(SS));
for i=1:NN
    ZZ(i,1)=k4;
    ZZ(i,2)=k3;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the Second Equity Curve
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure(2)
plot(1+eq/100000,'b'), hold on;
text(300,2.075,'Equity Curve: 2015/1/2 -- 2019/12/30');
fontsize(8,"points")
plot(WW,'b-','LineWidth', 0.85)
```

```
hold on
plot(ZZ,'g','LineWidth', 1)
text(1000,ZZ(1000,1)-0.15,'$$\widetilde{k_2^*}$$','Interpreter','LaTeX');
text(1000,ZZ(1000,2)-0.15,'$$\widetilde{k_1^*}$$','Interpreter','LaTeX');
axis([0,1257,0,3])
xlabel('Date')
ylabel('Ratio of Stock Prices')
```

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